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Method of Averaging for Differential Equations on an Infinite Interval

Theory and Applications

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Dedicated to Natalia, Irina, and Alexei

Contents

Preface

In a review of books on the method of averaging, J.A. Murdock [1999] has written: "The subject of averaging is vast, and it is possible to read four or five books entirely devoted to averaging and find very little overlap in the material which they cover."

One more book on averaging is presented to the reader. It has little in common with other books devoted to this subject.

Bogoliubov's small book (Bogoliubov [1945]) laid the foundation of the theory of averaging on the infinite interval. The further development of the theory is contained in the books Bogoliubov, N.N., and Mitropolskiy, A.Yu. [1961] and Malkin I.G. [1956].

In recent years many new results have been obtained, simpler proofs of known theorems have been found, and new applications of the method of averaging have been specified.

In this book the author has tried to state rigorously the theory of the method of averaging on the infinite interval in a modern form and to provide a better understanding of some results in the application of the theory.

The book has two parts. The first part is devoted to the theory of averaging of linear differential equations with almost periodic coefficients. The theory of stability for solutions of linear differential equations with near to constants coefficients is stated. Shtokalo's method is described in more exact and modernized form. The application of the theory to a problem of a parametric resonance is considered. A separate chapter is devoted to application of ideas of the method of averaging to construction of asymptotics for linear differential equations with oscillatory decreasing coefficients. In the last chapter some properties of solutions of linear singular perturbed differential equations with almost periodic coefficients are considered.

At the same time in the first part the basis for construction of the nonlinear theory is laid.

The second part is devoted to nonlinear equations.

In the first four chapters the systems in standard form are considered when the right-hand side of the system is proportional to a small parameter. The first chapter is devoted to construction of the theory of averaging on the infinite interval in the first order averaging. In particular, some results are stated that have been obtained in recent years. In the second chapter we describe the first applications of theorems on averaging on the infinite interval. The majority of applied problems considered here are traditional. Use of a method of averaging allows one to perform a rigorous treatment of all results on existence and stability of periodic and almost periodic solutions. In the third chapter the method of averaging is applied to the study of the stability of equilibriums of various pendulum systems with an oscillating pivot. First, the history of research related to the problem of stabilization of the upper equilibrium of a pendulum with an oscillating pivot is stated. Then the stability of the equilibriums of a pendulum with an almost periodically oscillating pivot are investigated. Some modern results are stated. For example, the problems of stabilization of Chelomei's pendulum and a pendulum with slowly decreasing oscillations of the pivot are considered. In the fourth chapter the higher order approximations of the method of averaging are constructed, and the conditions of their justification on the infinite interval in the periodic and almost periodic cases are established. The existence and stability of the rotary motions of a pendulum with an oscillating pivot are studied. A critical case of an autonomous system when the stability of the trivial equilibrium is related to the bifurcations is considered. In the fifth chapter theorems similar to Banfi's theorem are proved: the uniform asymptotic stability of solutions of averaged equation implies closeness of solutions of exact and averaged equations with close initial conditions on an infinite interval. The approach to these problems as proposed by the author is developed. This approach is based on special theorems on stability under constantly acting perturbations. Some applications are considered.

The subsequent three chapters are devoted to systems with rapidly rotating phase. Here we consider the problems of closeness of solutions of exact and averaged equations on the infinite interval, existence and stability of resonance periodic solutions in two-dimensional systems with rapidly rotating phase, existence and stability of almost periodic solutions in two-dimensional systems with rapidly rotating phase and slowly varying coefficients.

The book contains a number of exercises. These exercises are located in chapters that are devoted to applications of theory to problems of theory oscillations. The exercises should help to develop application technique of the method of averaging for the study of applied problems.

The book has three appendices. The first appendix contains useful facts about almost periodic functions. This is the main class of functions that are used throughout the book. In the second appendix some facts on the stability theory are stated in the form in which they are used in the book. The third appendix contains descriptions of some elementary facts of functional analysis.

The book is addressed to the broad audience of mathematicians, physicists, and engineers who are interested in asymptotic methods of the theory of nonlinear oscillations. It is accessible to graduate students.

I would like to thank Alex Bourd for invaluable help in the typesetting of this manuscript.

Part I

Averaging of Linear Differential Equations

Chapter 1

Periodic and Almost Periodic Functions. Brief Introduction

In this chapter, we describe in brief the main classes of functions that we shall use in what follows. The functions of these classes are determined for all $t \in (-\infty, \infty)$ (we shall write $t \in \mathcal{R}$).

1.1 Periodic Functions

We shall associate each periodic function $f(t)$ with the period T (it is not necessary that the function be continuous) a Fourier series

$$f(t) \sim a_0 + \sum_{k=1}^{\infty} a_k \cos \frac{2\pi}{T} kt + b_k \sin \frac{2\pi}{T} kt.$$

It is often convenient to write the Fourier series in the complex form. Presenting $\cos \frac{2\pi}{T} kt$ and $\sin \frac{2\pi}{T} kt$ in the complex form and assuming $c_k = \frac{1}{2}(a_k - ib_k)$, $c_{-k} = \frac{1}{2}(a_k - ib_k)$, we obtain

$$f(t) \sim \sum_{k=-\infty}^{\infty} c_k e^{i \frac{2\pi}{T} kt},$$

where $c_0 = a_0$. The number a_0 is determined by the formula

$$a_0 = \frac{1}{T} \int_0^T f(t) dt$$

and is called the mean value of a periodic function. From this point on, the mean value will be of priority. We emphasize the following property of a periodic function. Let an indefinite integral of the periodic function $f(t)$ (accurate within a constant) be

$$\int f(t) dt = c_0 t + g(t),$$

where $g(t)$ is a periodic function. The Fourier series of the function $g(t)$ is obtained by integrating termwise the Fourier series of the function $f(t)$.

If we introduce the norm

$$||f(t)|| = \max_{t\in[0,T]} |f(t)|,$$

the continuous periodic functions generate a complete normalized linear space (a Banach space) that we denote by P_T.

Along with continuous periodic functions, we also consider the periodic functions with a finite number of simple discontinuities (jumps) on a period, as well as the generalized periodic functions that are the derivatives of such periodic functions. We shall represent such functions by the Fourier series. As is known (see Schwartz [1950]), every generalized periodic function $f(t)$ is a sum of a trigonometric series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt \quad n = 0, 1, \ldots$$

The trigonometric series converges, in the generalized sense, if and only if

$$\frac{a_n}{n^k} \to 0, \quad \frac{b_n}{n^k} \to 0 \quad n \to \infty$$

for some integer $k \geq 0$. Hence, we can perform various analytical operations on the Fourier series. For instance, differentiating a saw-tooth periodic function $f(t)$ corresponding to the Fourier series

$$f(t) = \frac{1}{2} + \sum_{k=-\infty,\, k\neq 0}^{\infty} \frac{\cos 2\pi kt}{2\pi k}$$

yields a periodic function (δ-periodic function)

$$\sum_{k=-\infty}^{\infty} \delta(t-n) = 1 + 2\sum_{k=-\infty,\, k\neq 0}^{\infty} \cos 2\pi kt = \sum_{k=-\infty}^{\infty} e^{i2\pi kt},$$

where $\delta(t)$ is Dirac's δ-function. Here, the equalities are understood in terms of the generalized function theory. The sine series (see Antosik, Mikusinski, Sikorski [1973])

$$\sum_{k=1}^{\infty} \sin kt$$

converges, in the generalized sense, to the function $\frac{1}{2}\cot t$. Therefore, the series

$$\sum_{k=1}^{\infty} \sin(2k-1)t$$

converges, in the generalized sense, to the function $\frac{1}{2\sin t}$.

Let the T-periodic function $f(t)$ be differentiable everywhere except the points t_k, where it has jumps α_0. Then, in the generalized sense

$$\dot{f}(t) = \{\dot{f}(t)\} + \alpha_0 \delta_T(t - t_0),$$

where $\{\dot{f}(t)\}$ is the classical part of the derivative, and

$$\delta_T(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{ik\frac{2\pi}{T}t}.$$

1.2 Almost Periodic Functions

Let a trigonometric polynomial be expressed as

$$T_n(t) = \sum_{k=1}^{n} a_k \cos \omega_k t + b_k \sin \omega_k t, \tag{1.1}$$

where a_k, b_k, ω_k are real numbers. It is convenient to write expression (1.1) in the complex form

$$T_n(t) = \sum_{k=1}^{n} c_k e^{i\lambda_k t},$$

where λ_k are real numbers.

There exist the trigonometric polynomials that are not periodic functions. Consider the polynomial $f(t) = e^{it} + e^{i\pi t}$, for example. Assume that $f(t)$ is a periodic function having some period ω. The identity $f(t+\omega) = f(t)$ then takes the form

$$(e^{i\omega} - 1)e^{it} + (e^{i\pi\omega} - 1)e^{i\pi t} \equiv 0.$$

Because the functions e^{it} and $e^{i\pi t}$ are linearly independent, we have

$$e^{i\omega} - 1 = 0, \quad e^{i\pi\omega} - 1 = 0.$$

Hence, $\omega = 2k\pi$ and $\pi\omega = 2h\pi$, where k and h are integers. These equalities cannot hold simultaneously.

Definition 1.1. The function $f(t)$ determined for $t \in \mathcal{R}$ will be called **almost periodic** if this function is a limit of uniform convergence on the entire real axis of the sequence $T_n(t)$ of the trigonometric polynomials in the form (1.1). That is, for any $\varepsilon > 0$, there is a positive integer N such that for $n > N$

$$\sup_{-\infty<t<\infty} |f(t) - T_n(t)| < \varepsilon.$$

It is evident that any continuous periodic function will be almost periodic from the standpoint of Definition 1.1. Let us describe the properties of the further required almost periodic functions.

1) Each almost periodic function is uniformly continuous and bounded on the entire real axis.

2) If $f(t)$ is an almost periodic function and c is a constant, then $cf(t)$, $f(t+c)$, $f(ct)$ are almost periodic functions.

3) If $f(t)$ and $g(t)$ are almost periodic functions, then $f(t) \pm g(t)$ and $f(t) \cdot g(t)$ are almost periodic functions.

It follows from 3) that if $P(z_1, z_2, \ldots, z_k)$ is a polynomial of variables $z_1, z_2, \ldots, z_k$, and $f_1(t), f_2(t), \ldots, f_k(t)$ are almost periodic functions, then the function $F(t) = P(f_1, f_2, \ldots, f_k)$ are also almost periodic.

4) If $f(t)$ and $g(t)$ are almost periodic functions and $\sup_{-\infty<t<\infty} |g(t)| > 0$, then $\frac{f(t)}{g(t)}$ is an almost periodic function.

5) The limit of a uniformly convergent sequence of almost periodic functions is an almost periodic function.

Let $\Phi(z_1, z_2, \ldots, z_n)$ be a function uniformly continuous on a closed bounded set Π in a n-dimensional space. Let $f_1(t), \ldots, f_n(t)$ be almost periodic functions and $(f_1(t), \ldots, f_n(t) \in \Pi$ for $t \in \mathcal{R}$. Then it follows from 5) that $F(t) = \Phi(f_1(t), \ldots, f_n(t)$ is an almost periodic function.

The following property of an almost periodic function is particularly important.

6) For an almost periodic function $f(t)$, there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_a^{a+T} f(t)dt = \langle f(t) \rangle$$

uniformly with respect to a. The number $\langle f(t) \rangle$ is independent of the choice of a and is called the mean value of the almost periodic function $f(t)$.

Let $f(t)$ be a periodic function with the period ω. We represent the real number T as $T = n\omega + \alpha_n$, where n is an integer and α_n obeys the inequality

$$0 \le \alpha_n \le \omega.$$

$T \to \infty$ implies $n \to \infty$. We calculate the mean value of the function $f(t)$:

$$\langle f(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T f(t)dt = \lim_{n\to\infty} \frac{1}{n\omega + \alpha_n} \int\limits_0^{n\omega+\alpha_n} f(t)dt =$$

$$\lim_{n\to\infty}\frac{1}{n\omega+\alpha_n}\left\{\sum_{i=0}^{n-1}\int\limits_{i\omega}^{(i+1)\omega} f(t)dt+\int\limits_{n\omega}^{n\omega+\alpha_n} f(t)dt\right\}=$$

$$=\lim_{n\to\infty}\frac{1}{n\omega+\alpha_n}\left\{n\int\limits_0^{\omega} f(t)dt+\int\limits_0^{\alpha_n} f(t)dt\right\}=\frac{1}{\omega}\int\limits_0^{\omega} f(t)dt.$$

Thus, the mean value that we introduced for the periodic functions coincides with an ordinary mean value of a periodic function.

The existence of the mean value allows constructing a Fourier series for an almost periodic function. Let $f(t)$ be an almost periodic function. Because the function $e^{i\lambda t}$ is periodic for any real λ, we see that the product $f(t)e^{i\lambda t}$ is an almost periodic function. Therefore, there exists the mean value

$$a(\lambda)=\langle f(t)e^{i\lambda t}\rangle.$$

Of fundamental importance is the fact that the function $a(\lambda)$ may be non-zero for a countable set of λ at most. The numbers $\lambda_1,\ldots,\lambda_n,\ldots$ are called the Fourier exponent, and the numbers $a_1,\ldots,a_n,\ldots$ are the Fourier coefficients of the function $f(t)$.

Thus, each almost periodic function $f(t)$ corresponds to the Fourier series:

$$f(t)\sim\sum_n a_n e^{i\lambda_n t}.$$

We can perform formal operations on the Fourier series. Let $f(t)$ and $g(t)$ be the almost periodic functions and

$$f(t)\sim\sum_n a_n e^{i\lambda_n t}=\sum_\lambda a(\lambda)e^{i\lambda t},$$

$$g(t)\sim\sum_n b_n e^{i\mu_n t}=\sum_\lambda b(\lambda)e^{i\lambda t}.$$

Then:

1) $kf(t)\sim\sum_n ka_n e^{i\lambda_n t}$ (k=constant),
2) $e^{i\lambda t}f(t)\sim\sum_n a_n e^{i(\lambda_n+\lambda)t}$,
3) $f(t+\alpha)\sim\sum_n a_n e^{i\lambda\alpha}e^{i\lambda_n t}$ $(\alpha\in\mathcal{R})$,
4) $\bar{f}(t)\sim\sum_n \bar{a}_n e^{-i\lambda_n t}$,
5) $f(t)+g(t)\sim\sum_\lambda(a(\lambda)+b(\lambda))e^{i\lambda t}$,
6) $f(t)\cdot g(t)\sim\sum_n c_n e^{i\nu_n t}$, where

$$c_n=\sum_{\lambda_p+\mu_q=\nu_n} a_p b_q.$$

If the derivative of an almost periodic function $f(t)$ is an almost periodic function, then its Fourier series is obtained from the Fourier series of $f(t)$

by a termwise differentiation. If the indefinite integral of an almost periodic function $f(t)$ is an almost periodic function, then

$$F(t) = \int_0^t f(t)dt \sim c + \sum_n \frac{a_n}{i\lambda_n} e^{i\lambda_n t} (\lambda_n \neq 0).$$

Let us consider in detail the integration of the almost periodic functions. If $f(t)$ is a periodic function with non-zero mean value, then the following equality is valid

$$\int_0^t f(t)dt = \langle f \rangle t + g(t),$$

where $g(t)$ is a periodic function. For the almost periodic functions, the latter equality, generally speaking, does not hold. There exist almost periodic functions with zero mean value such that their integral is unbounded and thus is not an almost periodic function, such as

$$f(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} e^{i \frac{1}{k^2} t}.$$

We shall call the almost periodic function $f(t)$ **correct** if the equality

$$\int_0^t f(t)dt = \langle f \rangle t + g(t),$$

where $g(t)$ is an almost periodic function, holds. The function $f(t)$ is correct if it is a trigonometric polynomial. If the Fourier exponents of an almost periodic function are separated from zero, $\lambda_n \geq \delta > 0$, then this function will also be correct.

If there exists a finite set of numbers $\omega_1, \omega_2, \ldots, \omega_m$ such that each Fourier exponent of an almost periodic function is a linear combination of these numbers

$$\lambda_n = n_1\omega_1 + \cdots + n_m\omega_m,$$

where $n_1, \ldots, n_m$ are integers, then this almost periodic function is called **quasi-periodic**. The quasi-periodic functions can be obtained from the periodic functions of many variables. For example, let $F(x, y)$ be a function periodic in each of its variables with the period 2π and continuous. Then $F(\omega_1 t, \omega_2 t)$ is a quasi-periodic function if the numbers ω_1, ω_2 are incommensurable.

The above properties of the almost periodic functions imply that these functions generate a linear space. By introducing the norm

$$||f(t)|| = \sup_{-\infty < t < \infty} |f(t)|,$$

we make this space into a Banach space (a complete normalized linear space). This space is denoted by B. It is easy to see that the mean value is a linear functional on this space, that is, the mean value has the following properties:

1) $\langle cf(t)\rangle = c\langle f(t)\rangle$ (c=constant),

2) $\langle (f(t) + g(t))\rangle = \langle f(t)\rangle + \langle g(t)\rangle$,

3) If the sequence of the almost periodic functions $f_1(t), \ldots, f_n(t), \ldots$, for $t \in \mathcal{R}$, converges uniformly to the almost periodic function $f(t)$, then

$$\lim_{n\to\infty} \langle f_n(t)\rangle = \langle f(t)\rangle.$$

We defined an almost periodic function as a uniform limit on an infinite interval of a sequence of trigonometric polynomials. This definition served the basis in the book of Corduneanu [1989]. Historically, H. Bohr was the first who defined almost periodic functions but we do not cite his definition here. Often, the following definition by S. Bohner is convenient.

Definition 1.2. *A function $f(t)$ continuous on the real axis is called almost periodic if from each infinite sequence of functions*

$$f(t+h_1),\ f(t+h_2), \ldots,\ f(t+h_k), \ldots$$

it is possible to choose a subsequence such that it converges uniformly on the entire real axis.

1.3 Vector-Matrix Notation

Later on, we shall use a vector-matrix notation. By $y = (y_1, \ldots, y_n)$ we denote a vector, and $y_1, \ldots, y_n$ are the components of the vector. If the components are the functions of the variable t, then we obtain a vector-function $y(t)$, which will simply be called a function (a function with values in a n-dimensional space) unless that causes misunderstanding. We naturally define the vector-functions as follows

$$\frac{dy}{dt} = \left(\frac{dy_1}{dt}, \ldots, \frac{dy_n}{dt}\right), \int y(t)dt = \left(\int y_1(t)dt, \ldots, \int y_n(t)dt\right).$$

In a set of n-dimensional vectors, we introduce a norm by the formula

$$||y|| = \sum_{i=1}^{n} |y_i|$$

or the formula

$$||y|| = \sqrt{\sum_{i=1}^{n} y_i^2}.$$

The norm has the following properties:

$$||x + y|| \leq ||x|| + ||y||, \quad ||cy|| = |c|||y|| \quad (c = const).$$

Another frequently used inequality is worth mentioning:

$$||\int_a^b y(t)dt|| \leq \int_a^b ||y(t)||dt.$$

In what follows, we shall denote the vector norm by $|\cdot|$. Let A be a square matrix of order n with the elements a_{ij}. We introduce the norm of the matrix using the formula

$$||A|| = \sum_{i,j=1}^{n} |a_{ij}|.$$

For the norm of a matrix we could have used other definitions, such as, for example,

$$||A|| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|.$$

It is easy to see that

$$||A + B|| \leq ||A|| + ||B||,$$

$$||cA|| = |c| \cdot ||A|| \quad (c = const),$$

$$||Ax|| \leq ||A|| \cdot ||x|| \quad (x - vector),$$

$$||AB|| \leq ||A|| \cdot ||B||.$$

Naturally, we introduce the $\frac{dA}{dt}$ and $\int A(t)dt$. The inequality

$$||\int_a^b A(t)dt|| \leq \int_a^b ||A(t)||dt$$

holds true. Further, we shall denote the norm of the matrix by $|\cdot|$.

We shall call the vector-function $f(t) = (f_1(t), \ldots, f_n(t))$ almost periodic if its components $f_i(t)$ are the almost periodic functions. It is easy to see that the almost periodic vector-functions possess all the properties of the scalar almost periodic functions as described in the previous clause. The almost periodic vector-functions constitute a Banach space on introduction of the norm

$$||f(t)|| = \sup_{-\infty < t < \infty} |f(t)|,$$

where $|f(t)|$ is the norm of the vector $f(t)$. We shall denote this space by $\mathbf{B_n}$.

The periodic vector-functions with the period T constitute a Banach space on introduction of the norm

$$||f(t)|| = \max_{0 \le t \le T} |f(t)|.$$

We denote this space by $\mathbf{P_T}$.

We shall call the matrix-function almost periodic if its elements are the almost periodic functions. Finally, the vector-function $T(t)$ will be called a trigonometric polynomial if its components are the trigonometric polynomials.

Chapter 2

Bounded Solutions

2.1 Homogeneous System of Equations with Constant Coefficients

Consider a system of differential equations

$$\frac{dx}{dt} = Ax, \tag{2.1}$$

where A is a constant square matrix of order n. Let us recall some properties of system (2.1), that will be necessary later on. The general solution of system (2.1) can be written as

$$x(t) = e^{tA}x(0),$$

where $x(0)$ is a vector of initial conditions, the matrix exponent e^{tA} is determined by the matrix series

$$e^{tA} = \sum_{m=0}^{\infty} \frac{t^m A^m}{m!}.$$

The behavior of the solutions of system (2.1) as $t \to \pm\infty$ is entirely determined by the positioning of eigenvalues of the matrix A. If all eigenvalues of the matrix A have non-zero real parts, then system (2.1) has no solutions bounded for all $t \in \mathcal{R}$, except zero solutions. If all eigenvalues of the matrix A have negative real parts, then there exist constants $M > 0, \gamma > 0$ such that the following inequality holds

$$|e^{tA}| \leq Me^{-\gamma t},\ t \geq 0.$$

If all eigenvalues of the matrix A have positive real parts, then there exist constants $M, \gamma > 0$ such that the following inequality holds

$$|e^{tA}| \leq Me^{\gamma t},\ t \leq 0.$$

2.2 Bounded Solutions of Inhomogeneous Systems

Consider a system of inhomogeneous linear differential equations

$$\frac{dx}{dt} = Ax + f(t), \tag{2.2}$$

where A is a constant square matrix of order n, $f(t)$ is a vector-function bounded for all t, i.e.

$$\sup_{-\infty<t<\infty} |f(t)| \leq K < \infty.$$

We shall consider a problem when system (2.2) has a unique solution $x(t)$ bounded for $t \in \mathcal{R}$. Evidently, if system (2.2) has two bounded solutions, then their difference is a bounded solution of homogeneous equation (2.1). Hence, the necessary condition for the existence of a unique bounded solution of system (2.2) at the bounded function f(t) is absence of bounded solutions of system (2.1) except the trivial ones. Therefore, we shall assume that the matrix A has no eigenvalues with the zero real part. We shall show that under this assumption, system (2.2) has a unique bounded solution.

Using a linear transformation

$$x = Py,$$

where P is a constant invertible matrix, we transform (2.2) into

$$\frac{dy}{dt} = P^{-1}APy + P^{-1}f(t). \tag{2.3}$$

We can choose the matrix P so that the matrix $P^{-1}AP$ has a block-diagonal form

$$P^{-1}AP = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 is a matrix of order k such that its eigenvalues have negative real parts, and A_2 is a matrix of order $(n-k)$ such that its eigenvalues have positive real parts. Evidently, the function $P^{-1}f(t)$ is bounded. The problem of the bounded solutions of system (2.2) is equivalent to the problem of the solutions of system (2.3). Therefore, we assume that the matrix A takes the form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

In this case, we can rewrite system (2.2) as

$$\begin{aligned} \tfrac{dx_1}{dt} &= A_1x_1 + f_1(t), \\ \tfrac{dx_2}{dt} &= A_2x_1 + f_2(t), \end{aligned} \tag{2.4}$$

where $x = (x_1, x_2)$, x_1 and x_2 are the k-dimensional and $n-k$-dimensional vectors, respectively; $f_1(t)$ are the first k components of the vector $f(t)$, and $f_2(t)$ are the last $(n-k)$ components of the vector $f(t)$. The general solution of system (2.4) can be written as

$$\begin{aligned} x_1(t) &= e^{tA_1}x_1(0) + \int_0^t e^{(t-s)A_1} f_1(s)ds, \\ x_2(t) &= e^{tA_2}x_2(0) + \int_0^t e^{(t-s)A_2} f_2(s)ds. \end{aligned} \tag{2.5}$$

Since the eigenvalues of A_1 have negative real parts, there exist constants $M_1, \gamma_1 > 0$ such that the following inequality holds true

$$|e^{tA_1}| \le M_1 e^{-\gamma_1 t}, \quad t \ge 0. \tag{2.6}$$

We multiply both parts of the first equality of system (2.5) by the matrix e^{-tA_1} and obtain

$$e^{-tA_1}x_1(t) = x_1(0) + \int_0^t e^{-sA_1} f_1(s)ds. \tag{2.7}$$

Assume that $x_1(t)$ is a bounded function. Then, we pass on to the limit in equality (2.7) as $t \to -\infty$ with allowance for estimate (2.6) and arrive at

$$0 = x_1(0) + \int_0^{-\infty} e^{-sA_1} f_1(s)ds.$$

Consequently, if $x_1(t)$ is a bounded function, then

$$x_1(0) = \int_{-\infty}^0 e^{-sA_1} f_1(s)ds.$$

Therefore, if $x_1(t)$ is a bounded function, then it is determined by the formula

$$x_1(t) = e^{tA_1}x_1(0) + \int_0^t e^{(t-s)A_1} f_1(s)ds =$$

$$\int_{-\infty}^0 e^{(t-s)A_1} f_1(s)ds + \int_0^t e^{(t-s)A_1} f_1(s)ds = \int_{-\infty}^t e^{(t-s)A_1} f_1(s)ds.$$

Once all eigenvalues of the matrix A_2 have positive real parts, there exist constants $M_2, \gamma_2 > 0$, such that the inequality below holds true

$$|e^{tA_2}| \le M_2 e^{\gamma_2 t}, \quad t \le 0. \tag{2.8}$$

Multiplying both parts of the second equality of system (2.5) by e^{-tA_2} produces

$$e^{-tA_2}x_2(t) = x_2(0) + \int_0^t e^{-sA_2} f_2(s)ds.$$

On the assumption that $x_2(t)$ is a bounded function, we pass on to the limit as $t \to \infty$ with allowance for estimate (2.8) and obtain

$$x_2(0) = -\int_0^\infty e^{-sA_2} f_2(s)ds.$$

So, if $x_2(t)$ is a bounded function, then

$$x_2(t) = -\int_0^\infty e^{(t-s)A_2} f_2(s)ds + \int_0^t e^{(t-s)A_2} f_2(s)ds = -\int_t^\infty e^{(t-s)A_2} f_2(s)ds.$$

Hence, if system (2.2) has a bounded solution, then this solution is represented by the formulas

$$\begin{aligned} x_1(t) &= \int_{-\infty}^{t} e^{(t-s)A_1} f_1(s)ds, \\ x_2(t) &= -\int_{t}^{\infty} e^{(t-s)A_2} f_2(s)ds. \end{aligned} \tag{2.9}$$

We show that formulas (2.9) indeed determine bounded functions. It follows from estimates (2.6) and (2.8) that

$$|x_1(t)| \leq \int_{-\infty}^{t} |e^{(t-s)A_1}||f_1(s)|ds \leq M_1 \int_{-\infty}^{t} e^{-\gamma_1(t-s)}ds \sup_{-\infty<t<\infty} |f_1(t)| =$$

$$\frac{M_1}{\gamma_1} \sup_{-\infty<t<\infty} |f_1(t)|,$$

$$|x_2(t)| \leq \int_{t}^{\infty} |e^{(t-s)A_2}||f_2(s)|ds \leq M_2 \int_{t}^{\infty} e^{\gamma_2(t-s)}ds \sup_{-\infty<t<\infty} |f_2(t)| =$$

$$\frac{M_2}{\gamma_2} \sup_{-\infty<t<\infty} |f_2(t)|.$$

Therefore, system (2.2) has a unique bounded solution

$$x(t) = (x_1(t), x_2(t)) = (\int_{-\infty}^{t} e^{(t-s)A_1} f_1(s)ds, -\int_{t}^{\infty} e^{(t-s)A_2} f_2(s)ds) =$$

$$= \int_{-\infty}^{\infty} G(t-s)f(s)ds,$$

where the matrix-function $G(t)$ is determined by the formula

$$G(t) = \begin{cases} \begin{pmatrix} e^{tA_1} & 0 \\ 0 & 0 \end{pmatrix}, & t \geq 0, \\ -\begin{pmatrix} 0 & 0 \\ 0 & e^{tA_2} \end{pmatrix}, & t < 0. \end{cases} \tag{2.10}$$

This solution meets the estimate

$$|x(t)| \leq M_3 \sup_{-\infty<t<\infty} |f(t)|,$$

where $M_3 = \max\left(\frac{M_1}{\gamma_1}, \frac{M_2}{\gamma_2}\right)$.

It is worthy of note that the matrix-function $G(t)$ is continuous everywhere except the point $t = 0$, where it undergoes a simple discontinuity (jump)

$$G(t+0) - G(t-0) = I,$$

where I is an identity matrix. For $G(t)$ the following estimate holds true

$$|G(t)| \leq Me^{-\gamma|t|}, \quad t \in \mathcal{R}, \tag{2.11}$$

where $M > 0$ and $\gamma > 0$ are some constants. $G(t)$ is differentiable in all $t \neq 0$, and for the matrix $\frac{dG}{dt}$ an estimate of the form (2.10) holds true. The function $G(t)$ is called the Green's function for the problem of bounded solutions, or the Green's function for the bounded boundary-value problem.

Let us state the obtained result as a theorem.

Theorem 2.1. *Let all eigenvalues of the matrix A have non-zero real parts. Then for each bounded function $f(t)$ there exists a unique bounded solution of system (2.2) and this solution is determined by the formula*

$$x(t) = \int_{-\infty}^{\infty} G(t-s)f(s)ds. \tag{2.12}$$

Corollary 2.1. *Let $f(t)$ be an almost periodic function. Then the solution determined by formula (2.12) is almost periodic. If $f(t)$ is a periodic function, then the corresponding solution is periodic.*

Proof. If $f(t)$ is a trigonometric polynomial, then it is easy to verify that $x(t)$ is a trigonometric polynomial. Now let $f_n(t)$ be a sequence of trigonometric polynomials that for all $t \in \mathcal{R}$ converges uniformly to the almost periodic function $f(t)$. Then the sequence $x_n(t)$ uniformly converges to the function $x(t)$ for $t \in \mathcal{R}$, which follows from the inequality

$$|x(t) - x_n(t)| \le \int_{-\infty}^{\infty} |G(t-s)||f(s) - f_n(s)|ds \le$$

$$\le M \int_{-\infty}^{\infty} e^{-\gamma|t-s|}ds \sup_{-\infty<t<\infty} |f(t) - f_n(t)|.$$

Thus, $x(t)$ is an almost periodic function. If $f(t)$ is a periodic function with the period T, then it is easy to see that $x(t)$ is a periodic function with the period T.

Remark 2.1. If $f(t)$ is a T-periodic function, then the requirement that Theorem 2.1 imposes on the matrix A to have no eigenvalues with zero real part is unnecessary. A unique T-periodic solution of system (2.2) exists if the following condition holds true:

the matrix A has neither zero eigenvalue nor purely imaginary eigenvalues in the form $i\frac{2\pi}{T}k$, where k is an integer.

We shall call this Pi**-condition**.

In this case the homogeneous system

$$\frac{dx}{dt} = Ax$$

has no T-periodic solutions, except the zero solution. It is easy to write out the Green's function of the T-periodic boundary-value problem

$$\frac{dx}{dt} = Ax + f(t), \quad x(0) = x(T).$$

Apparently, if $x(t)$ is the solution of the above inhomogeneous system, then the function $x(t+T)$ is also its solution. Therefore, the condition of periodicity of the solution $x(t+T) = x(t)$ follows from the condition $x(0) = x(T)$ (the solutions $x(t)$ and $x(t+T)$ coincide at the initial instant $t = 0$ and, thus, coincide for all t). From the formula for a general solution of the inhomogeneous system

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}f(s)ds,$$

we obtain

$$x(T) = e^{TA}x(0) + \int_0^T e^{(T-s)A}f(s)ds = x(0).$$

Hence, the initial condition of the periodic solution takes the form

$$x(0) = \left(I - e^{TA}\right)^{-1} \int_0^T e^{(T-s)A} f(s)ds = \left(e^{-TA} - I\right)^{-1} \int_0^T e^{-sA} f(s)ds,$$

where I is an identity matrix. Invertibility of the matrix $\left(e^{-TA} - I\right)$ is a result of the condition Π. Substituting the value $x(0)$ into the formula of the general solution yields the formula for determining a unique T-periodic solution

$$x(t) = \int_0^T G(t-s) f(s)ds, \tag{2.13}$$

where the periodic Green's function takes the form

$$G(t-s) = \begin{cases} \left(\left(e^{-TA} - I\right)^{-1} + I\right) e^{(t-s)A}, & \text{if } s < t \\ \left(e^{-TA} - I\right)^{-1} e^{(t-s)A}, & \text{if } s > t. \end{cases}$$

Sometimes, instead of representing a T-periodic solution in the form (2.13), it is convenient to write it as

$$x(t) = e^{tA} \left[e^{-TA} - I\right] \int_t^{t+T} e^{-sA} f(s)ds =$$

$$\int_t^{t+T} \left\{e^{uA} \left[e^{-TA} - I\right] e^{-tA}\right\}^{-1} f(u)du.$$

It follows from the latter formula that for a unique periodic solution $x(t)$ of system (2.2) the following inequality holds true

$$|x(t)| \leq K \int_0^T |f(u)|du,$$

where

$$K = \sup_{0 \leq t \leq T} \sup_{t \leq \tau \leq t+T} |\{e^{\tau A}[e^{-TA} - I]e^{-tA}\}^{-1}|$$

is a constant independent of $f(t)$ and dependent only on T and e^{tA}. Finally, we wish to present one more T-periodic solution of an inhomogeneous system

$$x(t) = \int_0^T \left(e^{-TA} - I\right)^{-1} e^{-uA} f(t+u)du. \tag{2.14}$$

2.3 The Bogoliubov Lemma

Consider an inhomogeneous system

$$\frac{dx}{dt} = Ax + f(\frac{t}{\varepsilon}), \tag{2.15}$$

where $f(t)$ is an almost periodic function, $\varepsilon > 0$ is a small scalar parameter. Let all eigenvalues of the matrix A have nonzero real parts. Then system (2.15) for each ε, by virtue of Corollary 2.1, has a unique almost periodic solution $x(t,\varepsilon)$. We shall be interested in the condition under which $|x(t,\varepsilon)|$ as $\varepsilon \to 0$ tends to zero uniformly with respect to $t \in \mathcal{R}$. Such conditions result from the following lemma that is due to N. N. Bogoliubov. It is convenient to formulate the lemma in a different way as it is in the books of Bogoliubov [1945], and Bogoliubov and Mitropolskiy [1961].

The Bogoliubov Lemma. *Let the mean value of the function $f(t)$ equal zero, i.e.*

$$\langle f \rangle = \lim_{t\to\infty} \frac{1}{T} \int_0^T f(s)ds = 0. \tag{2.16}$$

Then

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |x(t,\varepsilon)| = 0.$$

Proof. According to Theorem 2.1, the solution $x(t,\varepsilon)$ of system (2.15) takes the form

$$x(t,\varepsilon) = \int_{-\infty}^{\infty} G(t-s)f(\frac{s}{\varepsilon})ds = \int_{-\infty}^{t} G(t-s)f(\frac{s}{\varepsilon})ds + \int_{t}^{\infty} G(t-s)f(\frac{s}{\varepsilon})ds.$$

We make a change of variables $s = t+u$ and obtain

$$x(t,\varepsilon) = \int_{-\infty}^{0} G(-u)f\left(\frac{t+u}{\varepsilon}\right)du + \int_{0}^{\infty} G(-u)f\left(\frac{t+u}{\varepsilon}\right)du =$$

$$\int_{-\infty}^{0} G(-u)\frac{d}{du}\left(\int_{t}^{t+u} f\left(\frac{\sigma}{\varepsilon}\right)\right)d\sigma + \int_{0}^{\infty} G(-u)\frac{d}{du}\left(\int_{t}^{t+u} f\left(\frac{\sigma}{\varepsilon}\right)\right)d\sigma.$$

Integrating each term in the right-hand side by parts yields

$$x(t,\varepsilon) = -\int_{-\infty}^{0} \frac{dG(-u)}{du}\left(\int_{t}^{t+u} f\left(\frac{\sigma}{\varepsilon}\right)d\sigma\right)du-$$

$$
-\int_0^\infty \frac{dG(-u)}{du} \left(\int_t^{t+u} f\left(\frac{\sigma}{\varepsilon}\right) d\sigma \right) du.
$$

When $u \neq 0$, inequality (2.11) implies the estimate

$$
\left| \frac{dG(-u)}{du} \right| \leq M_1 e^{-\gamma_1 |u|},
$$

where M_1, γ_1 are positive constants. We denote

$$
\sup_{-\infty<t<\infty} |f(t)|
$$

by M_2. Then, for an arbitrary $T > 0$, we obtain the inequality

$$
|x(t, \varepsilon)| \leq M_1 M_2 \int_{-\infty}^{-T} e^{-\gamma_1 |u|} |u| du + M_1 M_2 \int_T^\infty e^{-\gamma_1 |u|} |u| du +
$$

$$
+M_1 \int_{-T}^{T} e^{-\gamma_1 |u|} \left| \int_t^{t+u} f\left(\frac{\sigma}{\varepsilon}\right) d\sigma \right| du.
$$

Assume $\eta > 0$. The latter inequality, in view of the convergence of the first two integrals in its right-hand side, implies the existence of $T > 0$ such that

$$
|x(t, \varepsilon)| < \frac{\eta}{2} + \frac{2M_1}{\gamma_1} \sup_{|\tau-s|\leq T} \left| \int_s^\tau f\left(\frac{\sigma}{\varepsilon}\right) d\sigma \right|.
$$

However,

$$
\lim_{\varepsilon\to 0} \sup_{|\tau-s|\leq T} \left| \int_s^\tau f\left(\frac{\sigma}{\varepsilon}\right) d\sigma \right| = 0.
$$

Indeed,

$$
\begin{aligned}
&\sup\nolimits_{|\tau-s|\leq T} \left| \int_s^\tau f\left(\tfrac{\sigma}{\varepsilon}\right) d\sigma \right| = \sup\nolimits_{|\tau-s|\leq T} \left| \frac{\tau-s}{\frac{\tau-s}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{\tau}{\varepsilon}} f(u) du \right| \leq \\
&\leq T \sup\nolimits_{|\tau-s|\leq T} \left| \frac{1}{\frac{\tau-s}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{\tau}{\varepsilon}} f(u) du \right|.
\end{aligned}
\tag{2.17}
$$

As $\varepsilon \to 0$, the right-hand side of inequality (2.17) tends to zero as a result of the fact that condition (2.16) holds and the almost periodic function has a uniform mean value. Hence, for sufficiently small ε

$$
\frac{2M_1}{\gamma_1} \sup_{|\tau-s|\leq T} \left| \int_s^\tau f\left(\frac{\sigma}{\varepsilon}\right) d\sigma \right| < \frac{\eta}{2}
$$

and

$$|x(t, \varepsilon)| < \eta, \quad t \in \mathcal{R}.$$

The lemma is proved.

Remark. By changing the time $\tau = \varepsilon t$, we transform system (2.15) into

$$\frac{dx}{d\tau} = \varepsilon Ax + \varepsilon f(\tau).$$

We could have stated the Bogoliubov lemma for this system.

Chapter 3

Lemmas on Regularity and Stability

3.1 Regular Operators

Consider a differential operator

$$Lx = \frac{dx}{dt} + A(t)x,$$

where $A(t)$ is an $n \times n$ matrix composed of almost periodic functions. The operator L is defined on the set of differentiable almost periodic functions with values in n-dimensional space $\mathcal{R}^n$.

Definition 3.1. *We shall call operator L regular, if for every almost periodic function $f(t)$, a system of differential equations*

$$Lx = f(t)$$

has a unique almost periodic solution $x(t)$.

Due to Banach's Inverse Mapping Theorem (see Appendix C) the regularity of the operator L implies the existence of a continuous inverse L^{-1}:

$$x(t) = L^{-1}f(t)$$

in the space B_n of almost periodic vector-functions. Theorem 2.1 says that a system

$$\frac{dx}{dt} = Ax + f(t)$$

has a unique solution $x(t) \in B_n$ for any given $f(t) \in B_n$, if all eigenvalues of the matrix A have non-zero real parts. Thus, if A satisfies this condition, the operator

$$Lx = \frac{dx}{dt} - Ax$$

is regular. The inverse L^{-1} is given by

$$x(t) = L^{-1}f(t) = \int_{-\infty}^{\infty} G(t-s)f(s)ds,$$

where G(t) is Green's function for the problem of bounded solutions. Similarly, one can define a regular operator

$$Lx = \frac{dx}{dt} + A(t)x,$$

where $A(t)$ is a matrix whose elements are T-periodic functions. In this case, one needs to require that for any given T-periodic vector function $f(t)$, the system

$$Lx = f(t)$$

would have a unique T-periodic solution $x(t)$. Using Remark 2.1 we obtain that the operator

$$Lx = \frac{dx}{dt} - Ax$$

is regular if the condition Π is satisfied, i.e., the matrix A has neither zero nor imaginary eigenvalues of the form $i\frac{2\pi}{T}k$, where k is an integer.

3.2 Lemma on Regularity

We consider a family of differential operators L_ε of the form

$$L_\varepsilon x = \frac{dx}{dt} - A(t,\varepsilon)x,$$

that depend on a parameter ε.

Definition 3.2. We shall call the operator L_ε *uniformly regular* if it is regular for all $\varepsilon \in (0, \varepsilon_0)$, and there exists a constant $K > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ the norm of L_ε^{-1} in the space B_n is bounded by the constant K

$$||L_\varepsilon^{-1}|| \leq K.$$

We consider a family of operators

$$L_\varepsilon x = \frac{dx}{dt} - A(\frac{t}{\varepsilon})x \tag{3.1}$$

that depend on a parameter $\varepsilon \in (0, \varepsilon_0)$. We now obtain a condition of uniform regularity of operator (3.1) for small ε.

Lemma on Regularity. *Let the operator*

$$L_0 x = \frac{dx}{dt} - A_0 x, \tag{3.2}$$

where

$$A_0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T A(t)dt,$$

is regular. Then, for sufficiently small ε, operator (3.1) is uniformly regular.

Proof. We have to prove that, for any given function $f(t) \in B_n$, the system

$$\frac{dx}{dt} - A(\frac{t}{\varepsilon})x = f(t) \tag{3.3}$$

has a unique solution $x(t) \in B_n$, for sufficiently small ε. Consider a matrix

$$H(t,\varepsilon) = \int_{-\infty}^{\infty} G_0(t-s)[A(\frac{t}{\varepsilon}) - A_0]ds,$$

where $G_0(t-s)$ is Green's function of the problem of bounded solutions for the operator L_0, i.e., a matrix-function in

$$x(t) = \int_{-\infty}^{\infty} G_0(t-s)f(s)ds,$$

where $x(t)$ is a solution of the system

$$\frac{dx}{dt} - A_0x = f(t),$$

which is bounded in $t \in \mathcal{R}$. The matrix $H(t,\varepsilon)$ is a solution, which is bounded in $t \in \mathcal{R}$, of the nonhomogeneous system

$$\frac{dH}{dt} = A_0H + [A(\frac{t}{\varepsilon}) - A_0]. \tag{3.4}$$

Since the matrix $(A(\frac{t}{\varepsilon}) - A_0)$ has zero mean value, Bogoliubov lemma implies that

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |H(t,\varepsilon)| = 0. \tag{3.5}$$

We make the following change of variables in system (3.3)

$$x(t) = y(t) + H(t,\varepsilon)y(t). \tag{3.6}$$

Clearly, for sufficiently small ε, change (3.6) is invertible, and, $y(t) \in B_n$ implies that $x(t) \in B_n$. A simple calculation utilizing (3.4) shows that $y(t)$ should be defined as an almost periodic solution of a system

$$\frac{dy}{dt} - A_0y + D(t,\varepsilon)y = [I + H(t,\varepsilon)]^{-1}f(t), \tag{3.7}$$

where

$$D(t,\varepsilon) = -[I + H(t,\varepsilon)]^{-1}\left\{-H(t,\varepsilon)A_0 + [A(\frac{t}{\varepsilon}) - A_0]H(t,\varepsilon)\right\}. \qquad (3.8)$$

Limit equality (3.5) implies that

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |D(t,\varepsilon)| = 0. \qquad (3.9)$$

The problem of the existence of a unique almost periodic solution of system (3.3), for sufficiently small ε, is equivalent to the problem of the existence of a unique almost periodic solution of system (3.7). The latter is, in turn, equivalent to the problem of the existence of a unique solution in B_n of a system of integral equations

$$y(t) = \int_{-\infty}^{\infty} G_0(t-s)[-D(s,\varepsilon)y(s) + (I + H(t,\varepsilon))^{-1}f(s)]ds. \qquad (3.10)$$

We now estimate the B_n-norm of the operator

$$S(\varepsilon)y = \int_{-\infty}^{\infty} G_0(t-s)D(s,\varepsilon)y(s)ds.$$

Recall that the properties of Green's function $G_0(t)$ imply the existence of constants $M, \gamma > 0$, such that

$$|G_0(t-s)| \le Me^{-\gamma|t-s|}, \quad -\infty < t, s < \infty.$$

We obtain

$$||S(\varepsilon)|| \le \sup_{-\infty<t<\infty} \int_{-\infty}^{\infty} |G_0(t-s)||D(s,\varepsilon)||y(s)|ds \le$$

$$\le \int_{-\infty}^{\infty} e^{-\gamma(t-s)}ds \sup_{-\infty<t<\infty} |D(s,\varepsilon)|||y|| \le \frac{M}{\gamma} \sup_{-\infty<t<\infty} |D(s,\varepsilon)|||y||.$$

From (3.9) we get that

$$||S(\varepsilon)y|| \le \alpha(\varepsilon)||y||,$$

where

$$\lim_{\varepsilon\to 0} \alpha(\varepsilon) = 0.$$

This implies that, for sufficiently small ε, the operator $I - S(\varepsilon)$ (here I is the identity operator) has a continuous inverse in B_n that can be represented as a Neumann's series (see Appendix C.)

$$(I - S(\varepsilon))^{-1} = \sum_{k=0}^{\infty} S^k(\varepsilon).$$

System (3.10) can be written as a nonhomogeneous operator equation in B_n

$$[I - S(\varepsilon)]y = g(t),$$

where an almost periodic function $g(t)$ is defined by

$$g(t) = \int_{-\infty}^{\infty} G_0(t-s)[I + H(s,\varepsilon)]^{-1} f(s)ds.$$

Therefore, for sufficiently small ε, system (3.10) has a unique solution $y(t) \in B_n$. Thus, for sufficiently small ε, operators L_ε are regular. To complete the proof, we ought to show that the norms of operators L_ε^{-1} are uniformly bounded for small ε, i.e., there exist constants K and ε_1 such that

$$||L_\varepsilon^{-1}|| \leq K, \quad 0 < \varepsilon < \varepsilon_1. \tag{3.11}$$

Indeed,

$$L_\varepsilon^{-1} = [L_\varepsilon - L_0 + L_0]^{-1} = L_0^{-1}[(L_\varepsilon - L_0)L_0^{-1} + I]^{-1}.$$

Operator $L_\varepsilon - L_0$ has a form

$$L_\varepsilon - L_0 = A(\frac{t}{\varepsilon}) - A_0.$$

Therefore, operators $D_\varepsilon = (L_\varepsilon - L_0)L_0^{-1}$ are uniformly bounded which implies the inequality (3.11).

Remark 3.1. In the process of the proof we had to establish that the system of differential equations

$$\frac{dx}{dt} - A(\frac{t}{\varepsilon})x = f(t), \tag{3.12}$$

for sufficiently small ε, has a unique almost periodic solution $x(t)$ for any given almost periodic function $f(t)$. We introduce a new time in (3.12) via $\tau = \frac{t}{\varepsilon}$ to get the system

$$\frac{dx}{d\tau} - \varepsilon A(\tau)x = \varepsilon f(\varepsilon\tau). \tag{3.13}$$

Thus, the lemma on regularity can be obtained from the fact that the system (3.13) has a unique almost periodic solution for any almost periodic function $f(\tau)$.

We will use Remark 3.1 to prove the lemma on regularity assuming that the elements of the matrix $A(\tau)$ are correct almost periodic functions. In this case Bogoliubov lemma will not be needed. In (3.13) we make a change of variables

$$x = y + \varepsilon Y(\tau)y, \tag{3.14}$$

where the almost periodic matrix $Y(\tau)$ will be defined later. Substituting (3.14) into (3.13) yields

$$(I + \varepsilon Y(\tau))\frac{dy}{d\tau} + \varepsilon\frac{dY}{d\tau}y = \varepsilon A(\tau)y + \varepsilon^2 A(\tau)Y(\tau)y + \varepsilon f(\varepsilon\tau). \tag{3.15}$$

We choose an almost periodic matrix $Y(\tau)$ with zero mean value using

$$\frac{dY}{d\tau} = A(\tau) - A_0,$$

where the constant matrix A_0 is composed of mean values of elements of $A(\tau)$. Then, the matrix $I + \varepsilon Y(\tau)$ is invertible, for sufficiently small ε, and, system (3.15) can be written as

$$\frac{dy}{d\tau} = \varepsilon A_0 y + \varepsilon^2 (I+\varepsilon Y(\tau))^{-1}[-Y(\tau)A_0 + A(\tau)Y(\tau)]y + (I+\varepsilon Y(\tau))^{-1}\varepsilon f(\varepsilon\tau).$$

The remaining proof of the existence of solution of (3.13) in B_n is the same as the proof of the lemma on regularity.

3.3 Lemma on Regularity for Periodic Operators

We now concentrate on the problem of regularity of operators with periodic coefficients. Namely, we consider a family of operators

$$L_\varepsilon x = \frac{dx}{dt} - \varepsilon A(t)x, \tag{3.16}$$

where elements of the matrix $A(t)$ are T-periodic functions of t, and $\varepsilon > 0$ is a small parameter. We would like to study the problem of the existence of a unique T-periodic solution of the system

$$\frac{dx}{dt} = \varepsilon A(t)x + f(t) \tag{3.17}$$

for any given T-periodic vector-function $f(t)$.

Lemma on Regularity, periodic case.
If the matrix A_0

$$A_0 = \frac{1}{T}\int_0^T A(s)ds$$

does not have a zero eigenvalue, then operator (3.16) is regular, for sufficiently small ε*.*

Proof. In system (3.17) we make a change of variables

$$x = y + \varepsilon Y(t)y,$$

where $Y(t)$ is a T-periodic matrix with zero mean value defined by

$$\frac{dY}{dt} = A(t) - A_0.$$

Then, system (3.17) becomes

$$\frac{dy}{dt} = \varepsilon A_0 y + \varepsilon^2 F(\tau, \varepsilon)y + (I + \varepsilon Y(t))^{-1} f(t). \tag{3.18}$$

where

$$F(t, \varepsilon) = A(t)Y(t) - Y(t)(I + \varepsilon Y(t))^{-1} A_0.$$

If the matrix A_0 does not have a zero eigenvalue, then, for sufficiently small ε, the matrix εA_0 has neither a zero eigenvalue nor imaginary eigenvalues in the form $i\frac{2\pi}{T}k$, where k is an integer. Therefore, the operator

$$L_0 x = \frac{dx}{dt} - \varepsilon A_0 x$$

is regular in the space of T-periodic vector-functions P_T. The rest of the proof is essentially the same as the proof of the lemma on regularity.

We also note that

$$||L_0^{-1} f(t)|| \leq \frac{K}{\varepsilon} ||f||,$$

where K is a constant, the norms are taken in the space P_T. This follows from the representation

$$L_0^{-1} f = \int_0^T \varepsilon [e^{-\varepsilon A_0 T} - I]^{-1} e^{-\varepsilon A_0 s} \frac{1}{\varepsilon} f(t+s) ds$$

and the limit equality

$$\lim_{\varepsilon \to 0} \varepsilon [e^{-\varepsilon A_0 T} - I]^{-1} = -\frac{1}{T} A_0^{-1}.$$

The last statement is concerned with T-periodic functions that have a finite number of simple discontinuities (jumps). Assume that the matrix A_0 satisfies the conditions of the lemma on regularity. Clearly, system (3.17) has a unique T-periodic solution, for sufficiently small ε if the elements of the matrix $A(t)$ and vector-functions $f(t)$ are T-periodic with a finite number of simple discontinuities (jumps) on period.

3.4 Lemma on Stability

We have shown above that, for sufficiently small ε, the regularity of the operator L_ε follows from the regularity of the operator L_0. It turns out that there is connection between the stability of solutions of the system

$$\frac{dx}{dt} - A(\frac{t}{\varepsilon})x = 0 \tag{3.19}$$

and the stability of solutions of

$$\frac{dx}{dt} - A_0 x = 0, \tag{3.20}$$

where the constant matrix A_0 is composed of mean values of the elements of the matrix $A(t)$.

We will make use of a well known lemma on integral inequalities by Gronwall-Bellman:

Gronwall-Bellman Lemma.

Let a non-negative continuous scalar function $u(t)$ satisfy the integral inequality

$$u(t) \leq c + \alpha \int_0^t u(\tau) d\tau,$$

where $c, \alpha \geq 0$. Then

$$u(t) \leq ce^{\alpha t} \quad (t \geq 0).$$

Lemma on Stability

Let the eigenvalues of the matrix A_0 have non-zero real parts. Then, for sufficiently small ε, the trivial solution of system (3.19) is asymptotically stable if all eigenvalues of A_0 have negative real parts. The trivial solution of system (3.19) is unstable, for sufficiently small ε, if matrix A_0 has at least one eigenvalue with a positive real part.

Proof. We make a change of variables (3.6) in system (3.19) to get

$$\frac{dy}{dt} - A_0 y + D(t, \varepsilon) y = 0, \tag{3.21}$$

where matrix $D(t, \varepsilon)$ is defined by (3.8). Then, for sufficiently small ε, the problems of the stability of the trivial solutions of systems (3.19) and (3.21) are equivalent. First, let all eigenvalues of A_0 have negative real parts. Then there exist constants $M_1, \gamma_1 > 0$ such that

$$|e^{tA_0}| \le M_1 e^{-\gamma_1 t}, \quad t \ge 0. \tag{3.22}$$

The solution of system (3.21) satisfies a system of integral equations

$$y(t) = e^{tA_0} y(0) + \int_0^t e^{(t-s)A_0} D(s,\varepsilon) y(s) ds.$$

Taking into consideration (3.22), we get

$$|y(t)| \le M_1 e^{-\gamma_1 t} |y(0)| + M_1 p(\varepsilon) \int_0^t e^{-\gamma_1 (t-s)} |y(s)| ds, \tag{3.23}$$

where

$$p(\varepsilon) = \sup_{-\infty < t < \infty} |D(t,\varepsilon)|.$$

Letting $u(t) = e^{\gamma_1 t}|y(t)|$ and using (3.23) yields the following inequality

$$u(t) \le M_1 |y(0)| + M_1 p(\varepsilon) \int_0^t u(s) ds.$$

The Gronwall-Bellman lemma implies that

$$|y(t)| \le M_1 |y(0)| e^{(-\gamma_1 + M_1 p(\varepsilon))t}.$$

We note that, due to (3.9), we have

$$\lim_{\varepsilon \to 0} p(\varepsilon) = 0.$$

We choose a sufficiently small ε such that $M_1 p(\varepsilon) \le \frac{\gamma_1}{2}$. Then

$$|y(t)| \le M_1 |y(0)| e^{-\frac{\gamma_1}{2} t}, \quad t \ge 0.$$

The last inequality shows that, for sufficiently small ε, the trivial solution of the system (3.19) is asymptotically stable.

We now consider a matrix A_0 that has at least one eigenvalue with a positive real part. Without loss of generality, we assume that the matrix A_0 has k eigenvalues with negative real parts and $(n-k)$ eigenvalues with positive real parts. Again, without loss of generality, we assume that A_0 has a block-diagonal form

$$A_0 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{3.24}$$

where A_1 is a $k \times k$ matrix whose eigenvalues have negative real parts, and A_2 is a $(n-k) \times (n-k)$ matrix whose eigenvalues have positive real parts. We make the change of variables (3.6) in (3.18) to get

$$\frac{dy}{dt} - A_0 y + D(t, \varepsilon) y = 0. \tag{3.25}$$

For system (3.25) we consider the problem of the existence of solutions which are bounded on the half-axis $[0, \infty)$. Thanks to the representation (3.24) this problem is equivalent to the problem of the existence of solutions, which are bounded on the half-axis $[0, \infty)$, of systems

$$\frac{dy_1}{dt} - A_1 y_1 + (D(t, \varepsilon) y)_1 = 0, \tag{3.26}$$

$$\frac{dy_2}{dt} - A_2 y_2 + (D(t, \varepsilon) y)_2 = 0, \tag{3.27}$$

where y_1 is a k-dimensional vector, y_2 is a $(n-k)$-dimensional vector, $(D(t, \varepsilon)y)_1$ are the first k components of the vector $D(t, \varepsilon)y$, and $(D(t, \varepsilon)y)_2$ are the last $(n-k)$ components of the vector $D(t, \varepsilon)y$. The properties of matrices A_1 and A_2 imply the existence of constants M_1,γ_1,M_2, $\gamma_2 > 0$ such that

$$|e^{tA_0}| \leq M_1 e^{-\gamma_1 t}, \quad t \geq 0, \tag{3.28}$$

$$|e^{tA_2}| \leq M_2 e^{\gamma_2 t}, \quad t \leq 0. \tag{3.29}$$

First, we consider a problem of the existence of solutions, which are bounded on $[0, \infty)$, of the nonhomogeneous system

$$\frac{dy_1}{dt} = A_1 y_1 + f_1(t), \tag{3.30}$$

$$\frac{dy_2}{dt} = A_2 y_2 + f_2(t), \tag{3.31}$$

where $f_1(t)$, $f_2(t)$ are bounded on $[0, \infty)$. The estimate (3.28) implies that all solutions of system (3.30) are bounded on $[0, \infty)$ and are defined by

$$y_1(t) = e^{tA_1} y_1(0) + \int_0^t e^{(t-s)A_1} f_1(s) ds.$$

System (3.31) cannot have more than one solution that is bounded on $[0, \infty)$ because the homogeneous system

$$\frac{dy_2}{dt} = A_2 y_2$$

does not have a non-trivial solution, which is bounded on $[0, \infty)$, due to (3.29). The general solution of system (3.31) has the form

$$y_2(t) = e^{tA_2} y_2(0) + \int_0^t e^{(t-s)A_2} f_2(s) ds. \tag{3.32}$$

By multiplying (3.32) by matrix e^{-tA_2} and taking a limit for $t \to \infty$ we obtain that the initial condition of a solution that is bounded on $[0, \infty)$ should have the form

$$y_2(0) = \int_0^\infty e^{-sA_2} f_2(s) ds, \tag{3.33}$$

Therefore, a solution that is bounded on $[0, \infty)$ should have the representation

$$y_2(t) = -\int_t^\infty e^{(t-s)A_2} f_2(s) ds. \tag{3.34}$$

The estimate (3.29) shows that (3.34) does indeed yield a unique solution, which is bounded on $[0, \infty)$, of system (3.31).

We come back to the problem of the existence of solutions, which are bounded on $[0, \infty)$, of (3.26) and (3.27). We obtain that solutions, which are bounded for $t \geq 0$, of systems (3.26) and (3.27) are the solutions of systems of integral equations

$$y_1(t) = e^{tA_1} y_1(0) + \int_0^t e^{(t-s)A_1} (D(s,\varepsilon) y(s))_1 ds,$$

$$y_2(t) = -\int_t^\infty e^{(t-s)A_2} (D(s,\varepsilon) y(s))_2 ds.$$

Thus, solutions, which are bounded on the positive half-axis, of system (3.25) are the solutions of a system of integral equations

$$y(t) = \begin{pmatrix} e^{tA_1} & 0 \\ 0 & 0 \end{pmatrix} y(0) - \int_0^t \left(\begin{pmatrix} e^{(t-s)A_1} & 0 \\ 0 & 0 \end{pmatrix} D(s,\varepsilon) y(s) ds + \right.$$

$$+ \int_t^\infty \left(\begin{pmatrix} 0 & 0 \\ 0 & e^{(t-s)A_2} \end{pmatrix} D(s,\varepsilon) y(s) ds.\right.$$

Let $y^1(t)$ and $y^2(t)$ be two solutions of system (3.25), such that $y_1^1(0) = y_1^2(0)$, $|y^1(t)|, |y^2(t)| \leq r_0$ for $t \geq 0$. Then, utilizing (3.28) and (3.29) yields

$$|y^1(t) - y^2(t)| \leq M_1 \int_0^t e^{-\gamma_1(t-s)} |y^1(s) - y^2(s)| ds \sup_{0 \leq t < \infty} |D(t,\varepsilon)| +$$

$$+M_2 \int\limits_t^\infty e^{\gamma_2(t-s)}|y^1(s) - y^2(s)|ds \sup_{0\le t<\infty} |D(t,\varepsilon)|.$$

Therefore,

$$|y^1(t) - y^2(t)| \le \left[\frac{2M_1}{\gamma_1} + \frac{M_2}{\gamma_2}\right] p(\varepsilon) \sup_{0\le t<\infty} |y^1(t) - y^2(t)|, \tag{3.35}$$

where $p(\varepsilon) = \sup_{0\le t<\infty} |D(t,\varepsilon)|$ tends to zero as $\varepsilon \to 0$. We can select an ε such that

$$\left[\frac{2M_1}{\gamma_1} + \frac{M_2}{\gamma_2}\right] p(\varepsilon) < \frac{1}{2}.$$

Then the inequality (3.35) implies that

$$y^1(t) \equiv y^2(t).$$

This means that among the solutions of the system (3.35) with fixed $y_1(0)$ (i.e., with fixed first k components of the initial condition) there is no more than one solution, that is bounded on $[0,\infty)$, whose norm is less than r_0. Thus, there exist infinitely many initial conditions from any neighborhood of the origin, such that the corresponding solutions of (3.35) leave the ball $|y| \le r_0$ for some $t > 0$. Thus, the trivial solution of (3.35) is unstable.

Remark 3.2. The lemma on stability is often applied to systems (3.19) that are represented in a different form. Namely, we introduce a new time $t = \varepsilon\tau$ in (3.19) and obtain a system

$$\frac{dx}{d\tau} - \varepsilon A(\tau)x = 0. \tag{3.36}$$

The averaged system in time τ has the form

$$\frac{dx}{d\tau} - \varepsilon A_0 x = 0.$$

Evidently, for (3.36) the assertion of the lemma on stability does not change. If the elements of $A(\tau)$ are correct almost periodic functions then Bogoliubov lemma is not needed for the proof of the lemma on stability. To prove this assertion we would have to make a change of variables in system (3.36)

$$x = y + \varepsilon Y(\tau)y,$$

where an almost periodic matrix $Y(\tau)$ with zero mean value is defined by

$$\frac{dY}{d\tau} = A(\tau) - A_0.$$

Here the constant matrix A_0 is composed of the mean values of the corresponding elements of $A(\tau)$.

Remark 3.3 Using similar arguments we could obtain the lemma on stability for a more general system

$$\frac{dx}{d\tau} - \varepsilon A(\tau, \varepsilon)x = 0,$$

where $A(\tau, \varepsilon)$ is continuous in ε uniformly with respect to $t \in \mathcal{R}$ and is almost periodic in t uniformly with respect to ε. The matrix A_0 is defined by

$$A_0 = \lim_{T \in \infty} \frac{1}{T} \int_0^T A(s, 0)ds.$$

Remark 3.4. It is easy to see that the lemma on stability takes place if the elements of the matrix $A(t)$ are T-periodic functions that have a finite number of simple discontinuities (jumps) on the period.

Chapter 4

Parametric Resonance in Linear Systems

4.1 Systems with One Degree of Freedom. The Case of Smooth Parametric Perturbations

As an example of the applications of the lemma on stability, we consider the problem of parametric resonance for the equation

$$\frac{d^2x}{dt^2} + \omega^2[1 + \varepsilon f(t)]x = 0, \tag{4.1}$$

where ω is a real parameter, $\varepsilon > 0$ is a small parameter, $f(t)$ is an almost periodic or periodic function. If the parameter ω is such that the zero solution of equation (4.1) is unstable, then this equation has unbounded solutions. In order to find such values of the parameter ω, we use the lemma on stability. We shall assume that $f(t) = A\cos\lambda t$, i.e. we consider the Mathieu equation

$$\frac{d^2x}{dt^2} + \omega^2[1 + \varepsilon A\cos\lambda t]x = 0. \tag{4.2}$$

Rewrite equation (4.2) as

$$\frac{d^2x}{dt^2} + \omega^2 x = F(t), \tag{4.3}$$

where $F(t) = -\varepsilon\omega^2 Ax\cos\lambda t$. By means of a change

$$\begin{aligned} x &= a\cos\nu t + b\sin\nu t, \\ \tfrac{dx}{dt} &= -a\nu\sin\nu t + b\nu\cos\nu t, \end{aligned} \tag{4.4}$$

where a, b are new variables, the frequency ν being chosen later, we transform equation (4.3) into the system

$$\begin{aligned} &\tfrac{da}{dt}\cos\nu t + \tfrac{db}{dt}\sin\nu t = 0 \\ &-\tfrac{da}{dt}\nu\sin\nu t + \tfrac{db}{dt}\nu\cos\nu t - \nu^2(a\cos\nu t + b\sin\nu t) = -\omega^2(a\cos\nu t + b\sin\nu t) \\ &-\varepsilon\omega^2 A\cos\lambda t)(a\cos\nu t + b\sin\nu t). \end{aligned}$$

Solving the latter system with respect to the derivatives $\frac{da}{dt}$, $\frac{db}{dt}$, we obtain

$$\begin{aligned}
&\tfrac{da}{dt} = (\tfrac{\omega^2-\nu^2}{\nu})(a\cos\nu t\sin\nu t + b\sin^2\nu t)+ \\
&+\tfrac{\varepsilon\omega^2}{\nu}A\cos\lambda t(a\cos\nu t\sin\nu t + b\sin^2\nu t) \\
&\tfrac{db}{dt} = -(\tfrac{\omega^2-\nu^2}{\nu})(a\cos^2\nu t + b\cos\nu t\sin\nu t)- \\
&-\tfrac{\varepsilon\omega^2}{\nu}A\cos\lambda t(a\cos^2\nu t + b\cos\nu t\sin\nu t).
\end{aligned} \tag{4.5}$$

Assume that $\nu^2 - \omega^2 = \varepsilon h$, where h is a constant. Then system (4.5) takes the form of system (3.36), to which we can apply the lemma on stability. Letting $\nu = \frac{\lambda}{2}$ and averaging the right-hard side of system (4.5) over t yields the averaged system with constant coefficients

$$\begin{aligned}
\tfrac{d\bar{a}}{dt} &= \varepsilon\left(-\tfrac{h}{2\nu} - \tfrac{A\omega^2}{4\nu}\right)\bar{b}, \\
\tfrac{d\bar{b}}{dt} &= \varepsilon\left(\tfrac{h}{2\nu} - \tfrac{A\omega^2}{4\nu}\right)\bar{a}
\end{aligned} \tag{4.6}$$

The eigenvalues of the matrix of system (4.6) are determined from the equation

$$s^2 + \varepsilon^2\left(\frac{h^2}{4\nu^2} - \frac{A^2\omega^4}{16\nu^2}\right) = 0.$$

If the following inequality holds

$$\frac{h^2}{4\nu^2} - \frac{A^2\omega^4}{16\nu^2} < 0, \tag{4.7}$$

then the eigenvalues of the matrix of the averaged system are real and have different signs. Therefore, the zero solution of system (4.6) will be unstable. By virtue of the lemma on stability, for sufficiently small ε, the zero solution of system (4.5) is also unstable, and, thus, for sufficiently small ε, the zero solution of system (4.1) is also unstable. Substituting the value of h into inequality (4.7), we obtain

$$\frac{A^2\omega^4}{4} > \frac{(\nu^2-\omega^2)^2}{\varepsilon^2},$$

or

$$\frac{\varepsilon|A|\omega^2}{2} > |\nu^2 - \omega^2|.$$

We write the latter inequality as

$$\omega^2\left(1 - \frac{|A|\varepsilon}{2}\right) < \nu^2 < \omega^2\left(1 + \frac{|A|\varepsilon}{2}\right).$$

Since $2\nu = \lambda$, we have the following inequality that holds up to the accuracy of the terms of order ε:

$$2\omega\left(1 - \frac{|A|\varepsilon}{4}\right) < \lambda < 2\omega\left(1 + \frac{|A|\varepsilon}{4}\right). \tag{4.8}$$

Thus, if the excitation frequency λ is within the interval (4.8), then the system undergoes a principal resonance with the amplitude of oscillations rising by an exponential law. This resonance results from the periodic change of one of the parameters of an oscillating system and is therefore called a **parametric resonance**. Inequality (4.8) defines a zone of instability, within which the equilibrium $x = 0$ in equation (4.1) is unstable.

Now consider an equation with damping

$$\frac{d^2x}{dt^2} + \varepsilon\delta\frac{dx}{dt} + \omega^2[1 + \varepsilon A\cos\lambda t]x = 0, \tag{4.9}$$

where $\delta > 0$. By making a change

$$x = a\cos\nu t + b\sin\nu t,$$
$$\tfrac{dx}{dt} = -a\nu\sin\nu t + b\nu\cos\nu t,$$

where a, b are the new variables, the frequency ν being chosen later, assuming $2\nu = \frac{\lambda}{2}$ and averaging over t, we obtain the averaged system

$$\begin{aligned} \tfrac{d\bar{a}}{dt} &= \varepsilon\left(-\tfrac{h}{2\nu} - \tfrac{A\omega^2}{4\nu}\right)\bar{b} - \tfrac{\delta}{2}\bar{a}, \\ \tfrac{d\bar{b}}{dt} &= \varepsilon\left(\tfrac{h}{2\nu} - \tfrac{A\omega^2}{4\nu}\right)\bar{a} - \tfrac{\delta}{2}\bar{b}. \end{aligned} \tag{4.10}$$

The eigenvalues of the matrix in system (4.10) are determined from the equation $s^2 + \varepsilon\delta s + \varepsilon^2\left(\frac{\delta^2}{4} + \frac{h^2}{4\nu^2} - \frac{A^2\omega^4}{16\nu^2}\right) = 0$. The zero solution of system (4.10) is unstable if the inequality holds

$$\frac{A^2\omega^4}{16\nu^2} - \frac{\delta^2}{4} > \frac{h^2}{4\nu^2}.$$

Since $\varepsilon h^2 = \nu^2 - \omega^2$, we have the inequality

$$\frac{\varepsilon^2A^2\omega^4}{4\nu^4} - \frac{\varepsilon^2\delta^2}{\nu^2} > \left(1 - \frac{\omega^2}{\nu^2}\right)^2.$$

The latter inequality can be rewritten to the accuracy of the terms of order ε

$$1 - \sqrt{\frac{\varepsilon^2A^2}{4} - \frac{4\varepsilon^2\delta^2}{\lambda^2}} < \left(\frac{2\omega}{\lambda}\right)^2 < 1 + \sqrt{\frac{\varepsilon^2A^2}{4} - \frac{4\varepsilon^2\delta^2}{\lambda^2}}. \tag{4.11}$$

By virtue of the lemma on stability, this is just the inequality that defines the zone of instability of the zero solution $x = 0$ of equation (4.9).

It is worthy of note that for the existence of an instability zone, the following additional inequality should hold

$$\frac{\varepsilon^2A^2}{4} - \frac{4\varepsilon^2\delta^2}{\lambda^2} > 0.$$

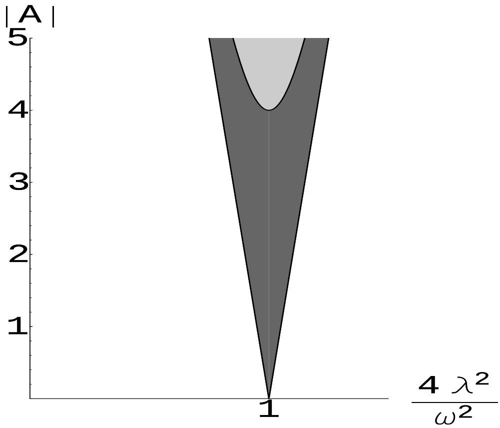

FIGURE 4.1: Zone of Instability.

Therefore, under damping, the instability zone appears at more intense parametric excitation.

Figure 4.1 presents the instability zones constructed as per inequalities (4.8) (the dark and light domains) and (4.11) (the lighter domain).

Exercise 4.1. Find the instability zones for equation (4.1) if $f(t) = A_1 \cos t + A_2 \cos \sqrt{2}t$.

4.2 Parametric Resonance in Linear Systems with One Degree of Freedom. Systems with Impacts

We consider a problem of parametric resonance for the equation

$$\frac{d^2x}{dt^2} + \omega^2[1 + \varepsilon f(t)]x = 0, \tag{4.12}$$

where ω is a real parameter, $\varepsilon > 0$ is a small parameter, $f(t)$ is a generalized periodic function with the zero mean value. The function $f(t)$ is a generalized derivative of a piecewise continuous periodic function with a finite number of simple discontinuities (jumps) on the period. By $g(t)$ we denote this periodic function, i.e. $g'(t) = f(t)$. Transform equation (4.12) into a system of two equations. We introduce a variable y by the formula

$$\frac{dx}{dt} = y - \varepsilon\omega^2 g(t)x. \tag{4.13}$$

Taking equation (4.12) into account, we obtain

$$\frac{dy}{dt} = -\omega^2 x + \varepsilon\omega^2 g(t)y - \varepsilon^2\omega^4 g^2(t)x. \tag{4.14}$$

The system of equations (4.13), (4.14) is equivalent to the initial equation (4.12). We introduce the new variables a, b as follows:

$$\begin{aligned} x &= a\cos\nu t + b\sin\nu t, \\ y &= -a\nu\sin\nu t + b\nu\cos\nu t, \end{aligned} \tag{4.15}$$

where the frequency ν will be chosen later. Substituting (4.15) into system (4.13), (4.14) yields the system of equations

$$\begin{aligned} &\tfrac{da}{dt}\cos\nu t + \tfrac{db}{dt}\sin\nu t = -\varepsilon\omega^2 g(t)(a\cos\nu t + b\sin\nu t), \\ &-\tfrac{da}{dt}\nu\sin\nu t + \tfrac{db}{dt}\nu\cos\nu t - \nu^2[a\cos\nu t + b\sin\nu t] = -\omega^2[a\cos\nu t + b\sin\nu t] + \\ &\varepsilon\omega^2 g(t)(-a\nu\sin\nu t + b\nu\cos\nu t) + O(\varepsilon^2). \end{aligned}$$

Solving the latter system for $\frac{da}{dt}$, $\frac{db}{dt}$ according to the Cramer's rule, we obtain the system of equations

$$\begin{aligned} \tfrac{da}{dt} &= -\tfrac{\nu^2-\omega^2}{\nu}\left[\tfrac{a}{2}\sin 2\nu t + b\sin^2\nu t\right] - \varepsilon\omega^2 g(t)[a(\cos^2\nu t - \sin^2\nu t) \\ &+ b\sin 2\nu t] + O(\varepsilon^2), \\ \tfrac{db}{dt} &= \tfrac{\nu^2-\omega^2}{\nu}\left[a\cos^2\nu t + \tfrac{b}{2}\sin 2\nu t\right] - \varepsilon\omega^2 g(t)(a\sin 2\nu t + b(-\cos^2\nu t \\ &+ \sin^2\nu t) + O(\varepsilon^2). \end{aligned} \tag{4.16}$$

We shall assume that $\nu^2 - \omega^2 = \varepsilon h$, where h is a constant. Then system (4.16) takes the form of system (3.35). If the frequency ν is commensurate with the principal frequency of the Fourier series for the function $g(t)$, then the right-hand side of system (4.16) appears to be a periodic vector-function with a finite number of the first-kind discontinuities on the period. As was noted in the previous section (Remark 3.4), the lemma on stability is applicable to such a system as well.

Now, averaging the right-hand side of the obtained system over t implies the system with constant coefficients. Using this system, we can investigate the stability of the solutions of system (4.16) for sufficiently small ε.

For instance, as the generalized function, we take a δ-periodic function corresponding to the Fourier series

$$f(t) \sim \sum_{k=1}^{\infty} \cos(2k-1)t.$$

Then the function $g(t)$ conforms to the Fourier series

$$g(t) \sim \sum_{k=1}^{\infty} \frac{\sin(2k-1)}{2k-1}.$$

Let $\nu = \frac{2k-1}{2}$. We average the right-hand side of system (4.16) over t, and the averaged equations have the form

$$\begin{aligned} \frac{d\bar{a}}{dt} &= -\varepsilon \left[\left(\frac{h}{2\nu} + \frac{\omega^2}{2(2k-1)} \right) \bar{b} \right], \\ \frac{d\bar{b}}{dt} &= \varepsilon \left[\left(\frac{h}{2\nu} - \frac{\omega^2}{2(2k-1)} \right) \bar{a} \right]. \end{aligned} \tag{4.17}$$

It follows from the inequality

$$\frac{h^2}{4\nu^2} - \frac{\omega^4}{4(2k-1)^2} < 0 \tag{4.18}$$

that the eigenvalues of the matrix in system (4.17) are real and have different signs. Hence, the zero solution of system (4.17) is unstable. By virtue of the lemma on stability, for sufficiently small ε, the zero solution of system (4.16) is also unstable. Therefore, for sufficiently small ε, the zero solution of equation (4.11) will be unstable as well. Substituting the value of h into inequality (4.18), we arrive at

$$|\nu^2 - \omega^2| < \frac{\varepsilon\omega^2}{2}.$$

Up to the accuracy of the terms of order ε, we obtain the inequality

$$2\omega\left(1 - \frac{\varepsilon}{4}\right) < 2k - 1 < 2\omega\left(1 + \frac{\varepsilon}{4}\right). \tag{4.19}$$

Thus, if the excitation frequency $2k-1$ is within the interval (4.19), then the system undergoes a resonance with the oscillation amplitude rising by the exponential law. Inequality (4.19) defines the instability zone within which the equilibrium $x = 0$ in equation (4.12) is unstable.

If excitation is assumed to be the piecewise continuous function $f(t)$ with the Fourier series

$$f(t) \sim \sum_{k=1}^{\infty} \frac{\sin(2k-1)}{2k-1},$$

then the instability zone shrinks for $k > 1$ and has the form

$$2\omega\left(1 - \frac{\varepsilon}{4(2k-1)}\right) < 2k - 1 < 2\omega\left(1 + \frac{\varepsilon}{4(2k-1)}\right).$$

This zone is even smaller for the differentiable function

$$f(t) = \sum_{k=1}^{\infty} \frac{\sin(2k-1)}{(2k-1)^2}.$$

However, for the zone of the principal resonance ($k = 1$), all inequalities coincide.

The rise of parametric oscillations in the system with damping under the δ-periodic excitation is investigated similarly.

Note that parametric oscillations in the systems with impacts have been investigated in many works (see, e.g., Babitskii and Krupenin [1978, 2001], Krupenin [1979, 1981], Nagaev and Khodzhaev [1973]).

4.3 Parametric Resonance in Linear Systems with Two Degrees of Freedom. Simple and Combination Resonance

Consider the problem of the parametric resonance in the following system with two degrees of freedom

$$\begin{aligned} \tfrac{d^2x_1}{dt^2} + \lambda^2\omega_1^2 x_1 + \varepsilon\lambda^2 A x_2 \cos 2t = 0, \\ \tfrac{d^2x_2}{dt^2} + \lambda^2\omega_2^2 x_2 + \varepsilon\lambda^2 B x_1 \cos 2t = 0, \end{aligned} \tag{4.20}$$

where $\varepsilon > 0$ is a small parameter, λ, ω_1, ω_2, A, B are real parameters. System (4.20) demonstrates all features of the parametric resonance in the general systems with many degrees of freedom. We only note that a linear Hamiltonian system with two degrees of freedom corresponding to system (4.20) takes the form

$$\frac{dy_i}{dt} = \frac{\partial H}{\partial x_i}, \quad \frac{dx_i}{dt} = -\frac{\partial H}{\partial y_i}, \quad i = 1, 2 \tag{4.21}$$

with the Hamiltonian

$$H = \frac{1}{2}(y_1^2 + y_2^2) - \frac{1}{2}[\lambda^2(\omega_1^2 x_1^2 + \omega_2^2) x_2^2] - \varepsilon\lambda^2 A(\cos 2t) x_1 x_2.$$

System (4.21) is special of system (4.20) with $A = B$.

In system (4.20), we should distinguish between the two cases - **a simple resonance** and **a combination resonance**.

Simple resonance. The simple resonance corresponds to the closeness of the natural vibration frequencies $\lambda\omega_1$ and $\lambda\omega_2$.

We make a change

$$\begin{aligned} x_1 &= a_1 \cos \nu_1 t + b_1 \sin \nu_1 t, \\ \dot{x}_1 &= -a_1\nu_1 \sin \nu_1 t + b_1\nu_1 \cos \nu_1 t, \\ x_2 &= a_2 \cos \nu_2 t + b_2 \sin \nu_2 t, \\ \dot{x}_2 &= -a_2\nu_2 \sin \nu_2 t + b_2\nu_2 \cos \nu_2 t, \end{aligned} \tag{4.22}$$

where a_1, b_1, a_2, b_2 are the new variables, the frequencies ν_1, ν_2 being chosen later, and transform system (4.20) into a system of four first-order equations. Solving the resultant system for $\dot{a}_1$, $\dot{b}_1$, $\dot{a}_2$, $\dot{b}_2$ by the Cramer's rule, we obtain the system

$$\begin{aligned}
&\dot{a}_1 = (\lambda^2\omega_1^2 - \nu_1^2)\left(\tfrac{a_1}{2}\sin 2\nu_1 t + b_1 \sin^2 \nu_1 t\right) + \\
&+\varepsilon\lambda^2 A \cos 2t (a_2 \cos \nu_2 t + b_2 \sin \nu_2 t) \sin \nu_1 t, \\
&\dot{b}_1 = (\nu_1^2 - \lambda^2\omega_1^2)\left(a_1 \cos^2 \nu_1 t + \tfrac{b_1}{2}\sin 2\nu_1 t\right) - \\
&-\varepsilon\lambda^2 A \cos 2t (a_2 \cos \nu_2 t + b_2 \sin \nu_2 t) \cos \nu_1 t, \\
&\dot{a}_2 = (\lambda^2\omega_2^2 - \nu_2^2)\left(\tfrac{a_2}{2}\sin 2\nu_2 t + b_2 \sin^2 \nu_2 t\right) + \\
&+\varepsilon\lambda^2 B \cos 2t (a_1 \cos \nu_1 t + b_1 \sin \nu_1) \sin \nu_2 t, \\
&\dot{b}_2 = (\nu_2^2 - \lambda^2\omega_2^2)\left(a_2 \cos^2 \nu_2 t + \tfrac{b_2}{2}\sin 2\nu_2 t\right) - \\
&-\varepsilon\lambda^2 B \cos 2t (a_1 \cos \nu_1 t + b_1 \sin \nu_1) \cos \nu_2 t.
\end{aligned} \tag{4.23}$$

Assuming $\nu_1 = \nu_2 = 1$, we transform system (4.23) into

$$\begin{aligned}
&\dot{a}_1 = (\lambda^2\omega_1^2 - 1)\left(\tfrac{a_1}{2}\sin 2t + b_1 \sin^2 t\right) + \\
&+\varepsilon\lambda^2 A \cos 2t \left(\tfrac{a_2}{2}\sin 2t + b_2 \sin^2 t\right), \\
&\dot{b}_1 = (1 - \lambda^2\omega_1^2)\left(a_1 \cos^2 t + \tfrac{b_1}{2}\sin 2t\right) - \\
&-\varepsilon\lambda^2 A \cos 2t \left(a_2 \cos^2 t + \tfrac{b_2}{2}\sin 2t\right), \\
&\dot{a}_2 = (\lambda^2\omega_2^2 - 1)\left(\tfrac{a_2}{2}\sin 2t + b_2 \sin^2 t\right) + \\
&+\varepsilon\lambda^2 B \cos 2t \left(\tfrac{a_1}{2}\sin 2t + b_1 \sin^2 t\right), \\
&\dot{b}_2 = (1 - \lambda^2\omega_2^2)\left(a_2 \cos^2 t + \tfrac{b_2}{2}\sin 2t\right) - \\
&-\varepsilon\lambda^2 B \cos 2t \left(a_1 \cos^2 t + \tfrac{b_1}{2}\sin 2t\right).
\end{aligned} \tag{4.24}$$

Let

$$1 - \lambda^2\omega_1^2 = \varepsilon h_1, \quad 1 - \lambda^2\omega_2^2 = \varepsilon h_2,$$

where h_1, h_2 are constants. Then the lemma on stability is applicable to system (4.24). Averaging system (4.24) yields the averaged system

$$\begin{aligned}
\dot{\bar{a}}_1 &= \varepsilon\left[-\tfrac{h_1}{2}\bar{b}_1 - \lambda^2 \tfrac{A}{4}\bar{b}_2\right], \\
\dot{\bar{b}}_1 &= \varepsilon\left[\tfrac{h_1}{2}\bar{a}_1 - \lambda^2 \tfrac{A}{4}\bar{a}_2\right], \\
\dot{\bar{a}}_2 &= \varepsilon\left[-\tfrac{h_2}{2}\bar{b}_2 - \lambda^2 \tfrac{B}{4}\bar{b}_1\right], \\
\dot{\bar{b}}_2 &= \varepsilon\left[\tfrac{h_2}{2}\bar{a}_2 - \lambda^2 \tfrac{B}{4}\bar{a}_1\right].
\end{aligned} \tag{4.25}$$

The eigenvalues of the matrix in system (4.25) are determined from the equation

$$s^4 - \varepsilon^2\left(\lambda^4\frac{AB}{8} - \frac{h_1^2 + h_2^2}{4}\right)s^2 + \varepsilon^4\left(\lambda^4\frac{AB}{16} - \frac{h_1 h_2}{4}\right)^2 = 0. \tag{4.26}$$

Equation (4.26) has roots lying in the right-hand half-plane of the complex plane if the following inequality holds

$$\lambda^4\frac{AB}{8} - \frac{h_1^2 + h_2^2}{4} > 0. \tag{4.27}$$

Allowing for the formulas of the detunings h_1, h_2, we obtain the inequality

$$\lambda^4 \frac{AB}{2} > \frac{(1-\lambda^2\omega_1^2)^2(1-\lambda^2\omega_2^2)^2}{\varepsilon^4}. \tag{4.28}$$

If inequality (4.28) holds, then the zero solution of the averaged system (4.25) is unstable. By virtue of the lemma on stability, for sufficiently small ε, the zero solution of system (4.23) is unstable, and therefore, for sufficiently small ε, the zero solution of system (4.20) will also be unstable.

We would like to note that inequality (4.28) holds only if the numbers A and B have the same sign. In particular, it holds for the Hamiltonian system (4.21). Hence, for this system, there exists an instability zone defined by the resonance relations $\lambda\omega_1 = \lambda\omega_2 = 1$.

Combination resonance. Return to system (4.20) and again make change (4.22). The combination resonances are defined by the relations

$$\lambda\omega_1 = \pm\lambda\omega_2 + 2. \tag{4.29}$$

For the definiteness, we shall assume that the resonance relation

$$\lambda(\omega_1 + \omega_2) = 2 \tag{4.30}$$

holds. In system (4.23), we suppose $\nu_1 = 2 - \lambda\omega_2$, $\nu_2 = 2 - \lambda\omega_1$ and introduce the detunings

$$\varepsilon h_1 = (2-\lambda\omega_2)^2 - \lambda^2\omega_1^2, \quad \varepsilon h_2 = (2-\lambda\omega_1)^2 - \lambda^2\omega_2^2.$$

Now we can apply the lemma on stability to system (4.24). We average system (4.24), keeping in mind that $\nu_1 + \nu_2 = 4 - \lambda(\omega_1 + \omega_2) = 2$. We obtain a system precisely the same as system (4.25). Therefore, the zero solution of system (4.20) for sufficiently small ε will be unstable provided inequality (4.27) that under resonance (4.30) takes the form

$$\lambda^4 \frac{AB}{2} > \frac{[(2-\lambda\omega_2)^2 - \lambda^2\omega_1^2]^2[(2-\lambda\omega_1)^2 - \lambda^2\omega_2^2]^2}{\varepsilon^4}. \tag{4.31}$$

It follows from inequality (4.31) that, similarly to the case of the simple resonance, the numbers A and B must have the same signs.

Exercise 4.2. Let the resonance relation

$$\lambda(\omega_1 - \omega_2) = 2$$

hold. Show that in this case the parametric resonance takes place if the numbers A and B have opposite signs. Find the inequality that defines the instability zone.

Chapter 5

Higher Approximations. The Shtokalo Method

5.1 Problem Statement

The lemma on stability works if all eigenvalues of the averaged system have non-zero real parts. We assume that matrix A_0 (composed of the mean values of the corresponding elements of $A(t)$) has no eigenvalues with positive real parts, but has eigenvalues with zero real parts. In this case the study of the stability of the trivial solution of a system

$$\frac{dx}{dt} = \varepsilon A(t)x$$

becomes a more complicated task.

In this chapter we present the method developed by I.Z. Shtokalo [1946, 1961] for the investigation of the stability of systems with almost periodic coefficients that are close to constants.

We consider a problem of the stability of the trivial solution of a system

$$\frac{dx}{dt} = \left(A + \sum_{k=1}^{m} \varepsilon^k A_k(t) + \varepsilon^{m+1} F(t,\varepsilon) \right) x, \tag{5.1}$$

where $\varepsilon > 0$ is a small parameter, A is a constant $n \times n$ matrix, $A_k(t)$, $k = 1, \ldots, m$ are $n \times n$ matrices that are represented as

$$A_k(t) = \sum_{l=1}^{r} C_{kl} e^{i\lambda_l t}.$$

Here C_{kl} are constant matrices, and λ_l are real numbers. In other words, the elements of the matrices $A_k(t)$ are trigonometric polynomials with arbitrary frequencies λ_l, $l = 1, \ldots, r$.

We shall say that such matrices $A_k(t)$ belong to the class Σ. The elements of the matrix $F(t, \varepsilon)$ are functions that are almost periodic in t uniformly with respect to ε and continuous in ε uniformly with respect to $t \in \mathcal{R}$.

We shall assume that all eigenvalues of the matrix A have non-positive real parts and there exists at least one eigenvalue with zero real part.

5.2 Transformation of the Basic System

We present an algorithm, that is due to I.Z. Shtokalo, for investigating the stability of the trivial solution of the system (5.1). It is natural to look for a change of variables that would transform system (5.1) into

$$\frac{dy}{dt} = \left(A + \sum_{k=1}^{m} \varepsilon^k B_k + \varepsilon^{m+1} G(t, \varepsilon) \right) y, \tag{5.2}$$

where B_k are constant matrices and the matrix $G(t, \varepsilon)$ has the same properties as $F(t, \varepsilon)$.

We shall assume that the matrix A is in Jordan canonical form. Then, without loss of generality, we can assume that all eigenvalues of A are real. Indeed, if A has complex eigenvalues, we can make a change of variables in (5.1)

$$x = e^{iRt} z,$$

where R is a diagonal matrix that is composed of imaginary parts of eigenvalues of A. This change of variables is bounded in t. It transforms matrix A into matrix $(A - iR)$, all of whose eigenvalues are real. Such change of variables does not affect the stability of the solutions of the system under consideration and the new matrices $A_k(t)$ also belong to the class Σ.

According to the method of Bogoliubov-Shtokalo we should look for a change of variables that transforms (5.1) into (5.2), of the following form

$$x = \left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t) \right) y, \tag{5.3}$$

where I is an identity matrix, $Y_k(t)$ $(k = 1, 2, \ldots, m)$ are $n \times n$ matrices that belong to Σ. By substituting (5.3) into (5.1) and replacing $\frac{dy}{dt}$ with the right-hand side of the system (5.2) we get

$$\left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t) \right) \left(A + \sum_{k=1}^{m} \varepsilon^k B_k \right) y + \varepsilon^{m+1} \left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t) \right) F(t, \varepsilon) y +$$
$$+ \left(\sum_{k=1}^{m} \varepsilon^k \frac{dY_k(t)}{dt} \right) y = \left(A + \sum_{k=1}^{m} \varepsilon^k A_k(t) \right) \left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t) \right) y +$$
$$+ \varepsilon^{m+1} U(t, \varepsilon) y,$$

where

$$U(t, \varepsilon) = F(t, \varepsilon) \left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t) \right).$$

Equalizing the coefficients of powers of ε we obtain the matrix equations that determine the matrices B_i, $Y_i(t)\,(i = 1, 2, \dots m)$

$$
\begin{array}{l}
\frac{dY_1}{dt} + Y_1 A - AY_1 = A_1(t) - B_1, \\
\frac{dY_2}{dt} + Y_2 A - AY_2 = A_2(t) - Y_1(t)B_1 + A_1(t)Y_1(t) - B_2, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\frac{dY_m}{dt} + Y_m A - AY_m = A_m(t) - \sum_{k=1}^{m-1} Y_{m-k}(t)B_k + \\
+\sum_{k=1}^{m-1} A_k(t)Y_{m-k}(t) - B_m.
\end{array}
\tag{5.4}
$$

Consider the first of the matrix equations. We select B_1 to be the matrix that is composed of the mean values of the elements of $A_1(t)$. Then the matrix $Y_1(t)$ is uniquely determined as a matrix from the class Σ with zero mean value. Indeed, we look for $Y_1(t)$ in the form

$$
Y_1(t) = \sum_{l=1}^{r} D_l e^{i\lambda_l t},
$$

where $\lambda_l \neq 0$ and the matrices D_l are determined below. By substituting the last expression into the matrix equation we get the equations for determining the matrices D_l:

$$
(i\lambda_l I - A)D_l + D_l A = C_{1l}. \tag{5.5}
$$

Since the matrix A has only real eigenvalues the intersection of the spectral of matrices $(i\lambda_l I - A)$ and A is empty. Therefore, the matrix equation (5.5) has a unique solution (see, for example, Gantmacher [1959], Daleckii and Krein [1974]). All subsequent matrix equations have the same structure. Matrices $B_i\,(i = 2, \dots, m)$ can be determined as mean values of the right-hand side of the corresponding matrix equations. To determine the elements of matrices $Y_i(t)$ that belong to Σ and have zero mean value, we obtain the matrix equations in the form (5.5). Thus, the matrices B_i, $Y_i(t)\,(i = 1, 2, \dots, m)$ can be uniquely determined. It is now easy to show that the change of variables (5.3) transforms the system (5.1) into (5.2). Substituting (5.3) into (5.1) yields

$$
\left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t)\right) \frac{dy}{dt} + \left(\sum_{k=1}^{m} \varepsilon^k \frac{dY_k(t)}{dt}\right) =
$$

$$
\left(A + \sum_{k=1}^{m} \varepsilon^k A_k(t)\right) \left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t)\right) y + \varepsilon^{m+1} U(t, \varepsilon) y,
$$

The last equality can be written as

$$
\left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t)\right) \left(\frac{dy}{dt} - (A + \sum_{k=1}^{m} \varepsilon^k B_k) y\right) =
$$

$$
\left(A + \sum_{k=1}^{m} \varepsilon^k A_k(t)\right) \left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t)\right) y - \left(\sum_{k=1}^{m} \varepsilon^k \frac{dY_k}{dt}\right) y +
$$

$$+\varepsilon^{m+1}U(t,\varepsilon)y - \left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t)\right)\left(A + \sum_{k=1}^{m} \varepsilon^k B_k\right) y$$

Taking into consideration the matrix equalities (5.4) we obtain

$$\left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t)\right)\left(\frac{dy}{dt} - (A + \sum_{k=1}^{m} \varepsilon^k B_k)y\right) = \varepsilon^{m+1} S(t,\varepsilon)y,$$

where $S(t,\varepsilon)$ has the same properties as $F(t,\varepsilon)$. Therefore, for sufficiently small ε, using the change of variables (5.3) system (5.1) can be transformed into

$$\frac{dy}{dt} = \left(A + \sum_{k=1}^{m} \varepsilon^k B_k + \varepsilon^{m+1} G(t,\varepsilon)\right) y, \tag{5.6}$$

where

$$G(t,\varepsilon) = \left(I + \sum_{k=1}^{m} \varepsilon^k Y_k(t)\right)^{-1} S(t,\varepsilon)y.$$

5.3 Remark on the Periodic Case

We shall denote by Σ_T the class of $n \times n$ matrices whose elements are continuous T-periodic functions and by Σ_T^0 the subset of Σ_T that consists of matrices having zero mean value.

If the original system (5.1) has the form

$$\frac{dx}{dt} = \left(A + \sum_{k=1}^{m} \varepsilon^k A_k(t) + \varepsilon^{m+1} F(t,\varepsilon)\right) x,$$

where A is a constant matrix, the matrices $A_k(t)$, $k = 1, 2, \ldots, m$ belong to Σ_T, the elements of $F(t,\varepsilon)$ are T-periodic in t and continuous in all variables. Then we could transform system (5.1) into (5.2) using a change of variables (5.3), where matrices $Y_k(t)$, $k = 1, 2, \ldots, m$ belong to Σ_T and have zero mean value.

In this case we can relax the conditions on the constant matrix A. Namely, we shall say that A satisfies to condition Γ if *all eigenvalues of A are such that* $\lambda_j - \lambda_k \neq \frac{2\pi i l}{T}$*,*$j \neq k$*,* $l = \pm 1, \pm 2, \ldots$*.*

The proof given below is due to P.N. Nesterov.

The statement will be proved if we can show that the matrix differential equation

$$\frac{dY}{dt} + YA - AY = G(t), \tag{5.7}$$

has a unique solution $Y(t) \in \Sigma_T^0$, where A is a constant matrix that satisfies the condition Γ, and $G(t) \in \Sigma_T^0$ is a given matrix.

We can assume that A is in Jordan canonical form. Moreover, we first assume that A is a diagonal matrix. Then the system (5.7) can be considered as n^2 scalar equations

$$\dot{y}_{kk} = g_{kk}(t) \tag{5.8}$$

and

$$\dot{y}_{kl} + (\lambda_k - \lambda_l) y_{kl} = g_{kl}(t), \quad k \neq l, \tag{5.9}$$

where λ_k, λ_l are eigenvalues of A. Clearly, equation (5.8) has a unique T-periodic solution with zero mean value. The nonhomogeneous equation (5.9) has a unique T-periodic solution if the homogeneous equation does not have non-trivial T-periodic solutions. It is easy to see that the homogeneous equation has this property if

$$\lambda_k - \lambda_l \neq \frac{2\pi i m}{T}, \quad m = \pm 1, \pm 2, \dots.$$

We now consider that case when the Jordan canonical form of A is not a diagonal matrix. We take, for instance, an eigenvalue λ_k and assume that it has the corresponding Jordan block of size 2:

$$J_k = \begin{pmatrix} \lambda_k & 1 \\ 0 & \lambda_k \end{pmatrix}$$

We denote by Y_k a 2×2 matrix with elements $y_{k1}, y_{k2}, y_{k+1,1}, y_{k+1,2}$. Then system

$$\frac{dY_k}{dt} + J_k Y_k - Y_k J_k = G_k,$$

where $G_k \in \Sigma_t^0$ is a 2×2 matrix can be considered as four scalar equations

$$\begin{aligned} &\dot{y}_{k1} + y_{k+1,1} = g_{k1}, \quad \dot{y}_{k2} + y_{k+1,2} - y_{k1} = g_{k2}, \\ &\dot{y}_{k+1,1} = g_{k+1,1}, \quad \dot{y}_{k+1,2} + y_{k+1,1} = g_{k+1,2}. \end{aligned}$$

We can uniquely determine the value $y_{k+1,1}$ that belongs to Σ_T^0 from the third equation. This allows us to find y_{k1} and $y_{k+1,2}$ (that belong to Σ_T^0) using the first and the fourth equations, respectively. Finally, we determine $y_{k2} \in \Sigma_T^0$ using the second equation.

The considered case demonstrates the general approach.

Let the matrix A be composed of Jordan blocks $J_{\lambda_1}, J_{\lambda_2}, \dots, J_{\lambda_k}$ of sizes $n_1 \times n_1, n_2 \times n_2, \dots, n_k \times n_k$, respectively. We divide the matrix $Y(t)$ into the blocks of sizes $n_i \times n_j$, i.e.,

$$Y(t) = \begin{pmatrix} \overbrace{Y_{11}}^{n_1} & \overbrace{Y_{12}}^{n_2} & \dots & \overbrace{Y_{1k}}^{n_k} & \}n_1 \\ Y_{21} & Y_{22} & \dots & Y_{2k} & \}n_2 \\ \vdots & \vdots & \ddots & \vdots & \\ Y_{k1} & Y_{k2} & \dots & Y_{kk} & \}n_k \end{pmatrix}.$$

Thus, the block matrix Y_{ij} has the dimensions $n_i \times n_j$, where $i, j = 1, \ldots, k$. We do a similar subdivision for the matrix $G(t)$. Using the well known rules on operations with block matrices (see, for example, Gantmacher [1959]) we conclude that the system (5.7) can be replaced with k^2 independent systems which in turn determine matrices Y_{ij}:

$$\dot{Y}_{ij} + Y_{ij} J_{\lambda_j} - J_{\lambda_i} Y_{ij} = G_{ij}(t). \tag{5.10}$$

This is a system of $n_i n_j$ scalar equations that determine the coefficients $y_{ps}^{(ij)}$ of the matrix Y_{ij}, where $p = 1, \ldots, n_i$ and $s = 1, \ldots, n_j$. For the sake of brevity we omit the dependence of the element y_{ps} on i and j. Let y_{ps} be an element that belongs to the p-th row and the s-th column of the matrix Y_{ij}. We note that

$$J_{\lambda_l} = \lambda_l I_{n_l} + E_{n_l}, \qquad l = 1, \ldots, k,$$

where I_{n_l} is an identity $n_l \times n_l$ matrix, and E_{n_l} is an $n_l \times n_l$ whose only non-zero elements are located on a diagonal above the main diagonal. We can rewrite the system (5.10) as

$$\dot{Y}_{ij} + (\lambda_j - \lambda_i) Y_{ij} + U_{ij} = G_{ij}(t),$$

where

$$U_{ij} = Y_{ij} E_{n_j} - E_{n_i} Y_{ij}.$$

A simple calculation shows that

$$U_{ij} = \begin{pmatrix} -y_{21} & y_{11} - y_{22} & \cdots & y_{1(n_j-1)} - y_{2n_j} \\ -y_{31} & y_{21} - y_{32} & \cdots & y_{2(n_j-1)} - y_{3n_j} \\ \vdots & \vdots & \cdots & \vdots \\ -y_{n_i 1} & y_{(n_i-1)1} - y_{n_i 2} & \cdots & y_{(n_i-1)(n_j-1)} - y_{n_i n_j} \\ 0 & y_{n_i 1} & \cdots & y_{n_i (n_j-1)} \end{pmatrix}.$$

Since $u_{n_i 1} = 0$ we obtain a scalar equation for determining $y_{n_i 1}$:

$$\dot{y}_{n_i 1} + (\lambda_j - \lambda_i) y_{n_i 1} = f_{n_i 1}(t).$$

This equation has a solution in Σ_T^0 due to the condition Γ. Further, for $y_{n_i 2}$ we have

$$\dot{y}_{n_i 2} + (\lambda_j - \lambda_i) y_{n_i 2} = g_{n_i 2}(t) - y_{n_i 1}.$$

Since we have already determined the value $y_{n_i 1}$ that belongs to Σ_T^0, the function $g_{n_i 2}(t) - y_{n_i 1}$ also belongs to Σ_T^0. Using the same arguments we find all elements of the last row of the matrix Y_{ij}. Now consider the penultimate row. All components of the $(n_i - 1)$-th row of the matrix U_{ij} that have a minus sign were obtained during the previous step. By moving left to right, we can subsequently determine all elements of the $(n_i - 1)$-th row of the matrix Y_{ij}. By moving up a row we subsequently find all elements of the matrix Y_{ij}.

5.4 Stability of Solutions of Linear Differential Equations with Near Constant Almost Periodic Coefficients

Evidently, the problem of the stability of the trivial solutions of systems (5.1) and (5.6) are equivalent.

It seems natural to think that the asymptotic stability (instability) of the trivial solution of the system

$$\frac{dy}{dt} = \left(A + \sum_{k=1}^{m_0} \varepsilon^k B_k \right) y, \quad m_0 < m$$

implies the asymptotic stability (instability) of the trivial solution of system

$$\frac{dy}{dt} = \left(A + \sum_{k=1}^{m} \varepsilon^k B_k \right) y.$$

This is, however, not always true as demonstrated by the following example (see Kolesov and Mayorov [1974b]).

Consider a two-dimensional system of differential equations

$$\frac{dx}{dt} + \varepsilon A(\varepsilon) x = 0, \tag{5.11}$$

where

$$A(\varepsilon) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{j=2}^{\infty} \varepsilon^j \begin{pmatrix} -(-1)^j & 0 \\ (-1)^j & 0 \end{pmatrix}.$$

Let

$$A_1(\varepsilon) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.12}$$

and, for $k \geq 2$, let

$$A_k(\varepsilon) = A_1(\varepsilon) + \sum_{j=2}^{k} \varepsilon^j \begin{pmatrix} -(-1)^j & 0 \\ (-1)^j & 0 \end{pmatrix}. \tag{5.13}$$

We shall denote by λ_{1k} and λ_{2k} $(k = 1, 2, \dots)$ the eigenvalues of the matrices (5.12) and (5.13), respectively. It is easy to verify that

$$\lambda_{11} = \lambda_{21} = \varepsilon,$$

and, for $k \geq 2$, that

$$\lambda_{1k}(\varepsilon) = 2\varepsilon + o(\varepsilon), \quad \lambda_{2k}(\varepsilon) = -(-1)^k 2\varepsilon^k + o(\varepsilon^k).$$

Therefore, the solutions of a differential equation (5.11) are asymptotically stable for odd k and unstable for even k, for sufficiently small ε.

This example shows that we ought to impose additional restrictions when investigating the stability of the trivial solution of systems

$$\frac{dx}{dt} = A_m(\varepsilon)x \tag{5.14}$$

where $A_m(\varepsilon) = \sum_{k=1}^{m} \varepsilon^k A_k$ and A_k $(k = 1, 2, \ldots, m)$ are constant matrices.

Definition. We shall say that a system of equations (5.14) is *strongly stable (unstable)* if for any $R > 0$ one can find a $\varepsilon(R) > 0$ such that, for any $0 < \varepsilon \leq \varepsilon(R) > 0$, the solutions of each of the systems

$$\frac{dx}{dt} = A_m(\varepsilon)x + \varepsilon^{m+1}Dx$$

are stable (unstable) for any matrix D, such that $||D|| \leq R$. We shall say that the algorithm for the investigation of the stability can be *completed* if there exists an m_0 such that the system of differential equations

$$\frac{dx}{dt} = A_{m_0}(\varepsilon)x \tag{5.15}$$

is either strongly stable or strongly unstable.

We note that each of these properties is invariant with respect to the selection of m_0. This follows directly from the definitions above. We now state the main theorem.

Theorem 5.1. *Assume that the algorithm for the investigation of the stability can be completed. Then there exists* $\varepsilon_0 > 0$*, such that for any* $0 < \varepsilon < \varepsilon_0$ *the trivial solution of the system of the differential equations (5.1) is asymptotically stable or unstable depending on the stability properties of the trivial solution of the corresponding system of differential equations (5.15).*

We conclude by noting that we can use any criterion of the stability of solution of a system of differential equations with constant coefficients for determining the strong stability or strong instability of system (5.14). Additionally, we have to pay attention that the conclusions about strong stability or strong instability are not changed due to the terms of order $(m + 1)$ with respect to ε.

In case of periodic coefficients, as we have seen above, the condition on the spectrum of A of system (5.2) can be relaxed. Namely, we do not have to require that all eigenvalues of A are real. We only need to require the condition Γ defined above. The detailed description of the algorithm of investigating the stability of the trivial solution of system (5.1) for the case of T-periodic matrices $A_k(t)$ can be found in Roseau [1966].

A weak side of the method of I.Z. Shtokalo is the necessity of performing the change of variables that transforms the original unperturbed matrix A into a matrix with real eigenvalues. This makes it difficult to generalize for differential equations with distributed parameters.

Kolesov and Mayorov [1974a] suggested a new efficient algorithm for investigating the stability of system (5.1). This algorithm does not require transforming the matrix A into a matrix with real eigenvalues.

5.5 Example. Generalized Hill's Equation

We consider as an example the equation

$$\frac{d^2x}{dt^2} + \varepsilon c \frac{dx}{dt} + \varepsilon f(t, \varepsilon)x = 0, \tag{5.16}$$

where ε is a small positive parameter, c is a positive constant, $f(t, \varepsilon) = q(t) - \varepsilon\rho^2$. Here, the function $q(t) = \sum_{k=1}^{r} \cos \omega_k t$ is a trigonometric polynomial with zero mean value, and ρ^2 is a positive constant. Consider the problem of the stability of the trivial solution of equation (5.16), for sufficiently small ε. We introduce a parameter $\mu = \sqrt{\varepsilon}$ and rewrite equation (5.16) as a system

$$\begin{aligned} \tfrac{dx}{dt} &= \mu y \\ \tfrac{dy}{dt} &= -\mu q(t)x - \mu^2 cy + \mu^3\rho^2 x. \end{aligned} \tag{5.17}$$

We can think of system (5.17) as a system (5.1) with $A = 0$. Its averaged system of the first approximation is

$$\frac{dz}{dt} = \mu B_1 z, \tag{5.18}$$

where the matrix B_1 has the form

$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{5.19}$$

The matrix B_1 has a zero eigenvalue, so the lemma on stability does not apply. Instead, we use Shtokalo's algorithm. For the matrix $Y_1(t)$ from system (5.4) we obtain the equation

$$\frac{dY_1}{dt} = A_1(t) - B_1,$$

where

$$A_1(t) = \begin{pmatrix} 0 & 1 \\ -q(t) & 0 \end{pmatrix}.$$

This implies, as we have seen before, that the matrix B_1 has the form (5.19). Then the matrix $Y_1(t)$ is

$$Y_1(t) = \begin{pmatrix} 0 & 0 \\ -\tilde{q}(t) & 0 \end{pmatrix},$$

where

$$\tilde{q}(t) = \sum_{k=1}^{r} \frac{a_k \sin \omega_k t}{\omega_k}.$$

From system (5.4) we obtain that the matrix B_2 can be determined as the mean value of the matrix

$$A_1(t)Y_1(t) + Y_1(t)B_1 + A_2(t),$$

where

$$A_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}$$

is a constant matrix. It is easy to see that $B_2 = A_2(t)$, because the mean value of $\tilde{q}(t)$ is zero, and,

$$B_1 + \mu B_2 = \begin{pmatrix} 0 & 1 \\ 0 & -\mu c \end{pmatrix}.$$

This matrix also has a zero eigenvalue. Therefore, we have to determine matrix B_3. First, we compute the matrix $Y_2(t)$. Because

$$A_1(t)Y_1(t) = \begin{pmatrix} -\tilde{q}(t) & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\frac{dY_2(t)}{dt} = A_1(t)Y_1(t) + Y_1(t)B_1 + A_2(t),$$

we get

$$Y_2(t) = \begin{pmatrix} \bar{q}(t) & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\bar{q}(t) = \frac{a_k \cos \omega_k t}{\omega_k^2}.$$

Due to (5.4) we obtain that B_3 is determined as a mean value of the matrix

$$A_3(t) - Y_2(t)B_1 - Y_1(t)B_2 + A_1(t)Y_2(t) + A_2(t)Y_1(t). \tag{5.20}$$

It is easy to see that the matrix $A_2(t)Y_1(t)$ is zero, and the mean values of the second and the third terms are zeroes as well. Then

$$B_3 = \begin{pmatrix} 0 & 0 \\ \rho^2 - q_0 & 0 \end{pmatrix},$$

where $q_0 = \sum_{k=1}^{r} \frac{a_k^2}{2\omega_k^2}$. The matrix $B_1 + \mu B_2 + \mu^2 B_3$ has the form

$$\begin{pmatrix} 0 & 1 \\ \mu^2(\rho^2 - q_0) & -\mu c \end{pmatrix}.$$

This matrix has a negative trace. The eigenvalues of the matrix $B_1 + \mu B_2 + \mu^2 B_3$ would have negative real parts if the determinant of this matrix were positive, i.e., if the following inequality holds

$$-\rho^2 + \sum_{k=1}^{r} \frac{a_k^2}{2\omega_k^2} > 0. \tag{5.21}$$

Therefore, the trivial solution of the system

$$\frac{dz}{dt} = (\mu B_1 + \mu^2 B_2 + \mu^3 B_3)z$$

is asymptotically stable if (5.21) holds. By computing B_4 we will see that the additional term of a higher degree appears in the second row of the matrix $B_1 + \mu B_2 + \mu^2 B_3 + \mu^3 B_4$. This perturbation does not affect the property of asymptotic stability. For any subsequent step of the algorithm, the additional terms can only appear on the second row of the matrix which likewise would not affect the stability. Thus, the trivial solution of (5.16) is asymptotically stable, for sufficiently small ε, if (5.21) holds, and is unstable if

$$-\rho^2 + \sum_{k=1}^{r} \frac{a_k^2}{2\omega_k^2} < 0.$$

Equation (5.16) can also be transformed into a system more naturally, without the introduction of a small parameter μ. In this case we obtain the system

$$\begin{aligned} \tfrac{dx}{dt} &= y, \\ \tfrac{dy}{dt} &= -q(t)x - \varepsilon c y + \varepsilon^2 \rho^2 x, \end{aligned}$$

which has the form

$$\frac{dz}{dt} = (A + \varepsilon A_1(t) + \varepsilon^2 A_2(t))z.$$

The matrix A has the form

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

and, again, the lemma on stability is not applicable. We obtain a system of matrix equations for determining B_1 and $Y_1(t)$:

$$\frac{dY_1}{dt} + Y_1 A - A Y_1 = A_1(t) - B_1.$$

The matrix B_1 is the mean value of $A_1(t)$:

$$B_1 = \begin{pmatrix} 0 & 0 \\ 0 & -c \end{pmatrix}.$$

To find the elements $y_{ij}(t)$ of the matrix $Y_1(t)$ we obtain the differential equations

$$y'_{11} - y_{21} = 0, \quad y'_{12} + y_{11} = 0, \quad y'_{21} + q(t) = 0, \quad y'_{22} + y_{21} = 0. \tag{5.22}$$

Solving these, and choosing the solutions in the form of almost periodic polynomials with zero mean value, we get

$$Y_1(t) = \begin{pmatrix} \tilde{q}(t) & -\bar{q}(t) \\ -\tilde{q}(t) & \bar{q}(t) \end{pmatrix}.$$

It is now easy to see that the matrix B_2 is determined as the mean value of $A_2(t) + A_1(t)Y_1(t)$ (other terms have zero mean value) or that

$$B_2 = \begin{pmatrix} 0 & 0 \\ \rho^2 - q_0 & 0 \end{pmatrix}.$$

The condition for the asymptotic stability of the trivial solution of system

$$\frac{dy}{dt} = (A + \varepsilon B_1 + \varepsilon^2 B_2)y$$

is the same as inequality (5.21).

We note that in the last case it was sufficient to find the second approximation. However, we had to solve differential equations (5.22).

Exercise 5.1. Investigate the stability of the trivial solution of the system

$$\ddot{x} + \sigma_1^2 x = 2\varepsilon \sin \omega_1 t,$$
$$\ddot{y} + \sigma_2^2 y = 2\varepsilon \cos \omega_2 t.$$

Here σ_1, σ_2 are real non-zero constants, and $\varepsilon > 0$ is a small parameter.

5.6 Exponential Dichotomy

For the next topic we have to introduce a concept that is concerned with the behavior of solutions of a system of linear homogeneous differential equations as $t \to \infty$.

First, we consider a system with constant coefficients,

$$\frac{dx}{dt} = Ax. \tag{5.23}$$

We assume that all eigenvalues of the matrix A have non-zero real parts, and A has a block-diagonal form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tag{5.24}$$

where A_1 is a $k \times k$ matrix whose eigenvalues have negative real parts, and A_2 is a $(n-k) \times (n-k)$ matrix whose eigenvalues have positive real parts. Then the n-dimensional space $\mathcal{R}^n$ of initial conditions of solutions of (5.23) can be represented as a direct sum of subspaces

$$\mathcal{R}^n = E_+ + E_-$$

where E_+ is a k-dimensional space that consists of n-dimensional vectors whose first k elements are non-zero, and E_- is a $(n-k)$-dimensional space that consists of n-dimensional vectors whose last $(n-k)$ elements are non-zero. The corresponding space of solutions X can be represented as a direct sum of subspaces

$$X = X_+ + X_-,$$

and there exist positive constants $M_+, M_-, \gamma_+, \gamma_-$, such that

$$|x(t)| \leq M_+ e^{-\gamma_+(t-s)}|x(s)|, \quad -\infty < s \leq t < \infty \quad (x(t) \in X_+), \tag{5.25}$$

and

$$|x(t)| \leq M_- e^{\gamma_-(t-s)}|x(s)|, \quad -\infty < t \leq s < \infty \quad (x(t) \in X_-). \tag{5.26}$$

The norms of solutions that belong to X_+ tend to zero as $t \to \infty$, while the norms of solutions that belong to X_- tend to zero as $t \to -\infty$. We shall say that, for solutions of system (5.23), there takes place **an exponential dichotomy of solutions on $\mathcal{R}$**. In this definition the inequality (5.26) can be replaced by

$$|x(t)| \geq M_-^* e^{\gamma_-^*(t-s)}|x(s)|, \quad -\infty < s \leq t < \infty \quad (x(t) \in X_-)$$

for some positive constants M_-^*, γ_-^*. Often a dichotomy is defined not in terms of inequalities for the solutions, but in terms of inequalities for the norm of the fundamental matrix of the system, i.e.,

$$|e^{(t-s)A_1}| \leq M_+ e^{-\gamma_+(t-s)}, \quad -\infty < s \leq t < \infty, \tag{5.27}$$

$$|e^{(t-s)A_2}| \leq M_- e^{\gamma_-(t-s)}, \quad -\infty < t \leq s < \infty. \tag{5.28}$$

If the matrix A cannot be represented in the block-diagonal form (5.24), and does not have eigenvalues with a zero real part, then, evidently, system (5.23) has an exponential dichotomy of solutions. However, the subspaces of initial conditions E_+,E_- and the subspaces of solutions X_+,X_- are different from before. Denote by $P_+, P_- (P_+ + P_- = I)$ the operators which project $\mathcal{R}^n$ onto the subspaces in which all the eigenvalues of the matrix A have either positive or negative real parts, respectively. Then, the inequalities (5.27) and (5.28) of an exponential dichotomy can be written as

$$|e^{tA}P_+e^{-sA}| \leq M_+e^{-\gamma_+(t-s)}, \quad -\infty < s \leq t < \infty, \tag{5.29}$$

$$|e^{tA}P_-e^{-sA}| \leq M_-e^{\gamma_-(t-s)}, \quad -\infty < t \leq s < \infty. \tag{5.30}$$

We also note that in the case of an exponential dichotomy of solutions the operator $Lx = \frac{dx}{dt} - Ax$ is regular.

We now define an exponential dichotomy on $\mathcal{R}$ for the solutions of the system of differential equations

$$\frac{dx}{dt} = A(t)x, \tag{5.31}$$

where elements of matrices $A(t)$ are almost periodic functions.

A more detailed description of the corresponding theory is contained in Krasnosel'kii, Burd, and Kolesov [1973] or Coppel [1978].

If the space of initial conditions $\mathcal{R}^n$ of system (5.31) can be represented as a direct sum of subspaces E_+, E_-, the space of solutions X can be represented as a direct sum of subspaces X_+, X_- so that the initial values of solutions of system (5.31) belong to E_+ and E_-, respectively, and (5.25) as well as (5.26) hold, then for system (5.31) an exponential dichotomy of solution on $\mathcal{R}$ takes place.

From this definition the fact immediately follows that if subspace X_- is empty, then the trivial solution of system (5.31) is asymptotically stable. If, however, X_- is not empty, then the trivial solution of (5.31) is unstable.

It turns out that an exponential dichotomy of solutions of system (5.31) is equivalent to regularity of the operator $Lx = \frac{dx}{dt} - A(t)x$, and constants M_+, M_-,γ_+, γ_- depend only on the norm of the inverse operator L^{-1} in the space of almost periodic vector-functions and on the norm of the matrix $A(t)$.

To state the definition of a dichotomy in terms of the estimates of the norm of the fundamental matrix $U(t)$ we would need to consider estimates on the norms $|U(t)P_+U^{-1}(s)|$ and $|U(t)P_-U^{-1}(s)|$ in inequalities (5.29) and (5.30).

Thus, if for system (5.31) an exponential dichotomy takes place, then the nonhomogeneous system

$$\frac{dx}{dt} = A(t)x + f(t)$$

has a unique almost periodic solution for any almost periodic vector-function $f(t)$, and that solution is defined by

$$x(t) = \int_{-\infty}^{+\infty} G(t,s)f(s)ds,$$

where the matrix function $G(t,s)$ has the form

$$G(t,s) = \begin{cases} U(t)P_+U^{-1}(s), & t \geq s, \\ U(t)P_-U^{-1}(s), & t < s. \end{cases}$$

The matrix function $G(t,s)$ is called Green's function for the almost periodic boundary value problem. This function satisfies

$$|G(t,s)| \leq Me^{-\gamma|t-s|}, \quad -\infty < t, s < \infty,$$

where M,γ are positive constants.

In case we are interested only in the stability properties of solutions, it is sufficient to consider an exponential dichotomy on a positive half-axis $\mathcal{R}_+$. We shall say that system (5.31) has an exponential dichotomy on $\mathcal{R}_+$ if there exist projections P_+, P_- and positive constants M, γ such that

$$|U(t)P_+U^{-1}(s)| \leq Me^{-\gamma(t-s)}, \quad t \geq s \geq 0,$$

$$|U(t)P_-U^{-1}(s)| \leq Me^{-\gamma(s-t)}, \quad s \geq t \geq 0.$$

5.7 Stability of Solutions of Systems with a Small Parameter and an Exponential Dichotomy

Let $U(t,\varepsilon)$ be a fundamental matrix of the linear system

$$\frac{dx}{dt} = A(t,\varepsilon)x, \tag{5.32}$$

where the matrix $A(t,\varepsilon)$ is periodic (almost periodic) in $t \in \mathcal{R}$ uniformly with respect to a real parameter $\varepsilon \in (0,\varepsilon_0)$ and is a sufficiently smooth function of ε.

The following definition for a positive half-axis $\mathcal{R}_+$ is contained in Hale and Pavlu [1983].

Definition. For system (5.32) an exponential dichotomy of order k takes place if there exist a projection P_ε that is continuous for $\varepsilon \in (0,\varepsilon)$, positive constants K_+, K_-, and functions $\alpha_1(\varepsilon) = c_1\varepsilon^k$, $\alpha_2(\varepsilon) = c_2\varepsilon^k$, $c_1, c_2 > 0$ such that

$$|U(t,\varepsilon)P_\varepsilon U^{-1}(s,\varepsilon)| \leq K_+e^{-\alpha_1(\varepsilon)(t-s)}, \quad -\infty < s \leq t < \infty,$$

$$|U(t,\varepsilon)(I - P_\varepsilon)U^{-1}(s,\varepsilon)| \leq K_-e^{\alpha_2(\varepsilon)(t-s)}, \quad -\infty < t \leq s < \infty.$$

The property of an exponential dichotomy of order k is equivalent to the following estimate of the norm of the inverse operator of

$$L_\varepsilon x = \frac{dx}{dt} - A(t,\varepsilon)x$$

in the space of periodic (almost periodic) vector-functions

$$||L_\varepsilon^{-1}|| \leq C\varepsilon^{-k},$$

where C is a positive constant. Clearly, if system (5.32) has an exponential dichotomy of order k and $P_\varepsilon = I$, then the trivial solution of system (5.32) is asymptotically stable. Otherwise, an exponential dichotomy of order k and $P_\varepsilon \neq I$ imply that the trivial solution is unstable. We can now restate Theorem 5.1 in different terms.

Theorem 5.2. *Let a linear system of differential equations with constant coefficients that depend on a parameter* ε*,*

$$\frac{dx}{dt} = A(\varepsilon)x \tag{5.33}$$

have an exponential dichotomy of order k*. Let matrix* $B(t,\varepsilon)$ *be almost periodic in* t *uniformly with respect to* $\varepsilon \in (0, \varepsilon_0)$*. If* $\sup_{t\in\mathcal{R}} ||B(t,\varepsilon)|| = O(\varepsilon^N)$ *for* $N \geq k+1$*, then the trivial solution of the perturbed system*

$$\frac{dx}{dt} = A(\varepsilon)x + B(t,\varepsilon)x \tag{5.34}$$

is asymptotically stable for $P_\varepsilon = I$ *and is unstable for* $P_\varepsilon \neq I$*.*

Proof of this theorem is almost the same as the proof of the lemma on stability. The estimates for matrix exponent is replaced with the corresponding estimates for the fundamental matrix of system (5.33) which follow from the properties of an exponential dichotomy of order k. We can also show that system (5.34) has an exponential dichotomy of order k.

Verification of the conditions of Theorem 5.2 in concrete situations is, usually, straightforward. To investigate the stability of the system of differential equations

$$\frac{dx}{dt} = (A + \varepsilon B_1 + \cdots + \varepsilon^k B_k)x$$

we can apply any of the known criteria of the stability for systems with constant coefficients. We must, however, make sure that the higher order terms (in ε) do not affect the stability.

If the matrix A has a simple zero eigenvalue and all its other eigenvalues have negative real parts, then the matrix $A(\varepsilon) = A + \varepsilon B_1 + \cdots + \varepsilon^k B_k$, for sufficiently small ε, has a simple real eigenvalue $\lambda(\varepsilon) = a_1\varepsilon + a_2\varepsilon^2 + \ldots$. The stability properties of the system of differential equations (5.1) depend only on the sign of the first non-zero coefficient a_{j_0} of the eigenvalue $\lambda(\varepsilon)$ under the assumption that $j_0 \leq k$ (see Krasnosel'skii, Burd, and Kolesov [1973]).

We mention without a proof some sufficient conditions for an exponential dichotomy of order $k \leq N$ (see Hale and Pavlu [1983]). These conditions are equivalent to the condition of a strong k-hyperbolicity that were introduced by Murdock and Robinson [1980a] (see also Murdock and Robinson [1980b]).

If all eigenvalues of the matrix A are distinct, and if the eigenvalues $\lambda_i(\varepsilon)$, $(i = 1, 2, \ldots, n)$ of the matrix $A + \varepsilon B_1 + \ldots, \varepsilon^N B_N$ for an appropriate numbering satisfy

$$\Re\lambda_i(\varepsilon) < -c\varepsilon^k, \quad i = 1, 2, \ldots, r$$

$$\Re\lambda_i(\varepsilon) > c\varepsilon^K, \quad i = r + 1, \ldots, n$$

for some $k \leq N$ and some positive constant c, then the system of equations

$$\frac{dx}{dt} = (A + \varepsilon B_1 + \cdots + \varepsilon^N B_N)x$$

has an exponential dichotomy of order $k \leq N$.

5.8 Estimate of Inverse Operator

We concentrate on the estimation in space B_n of the inverse operator for operator

$$L(\varepsilon)x = \frac{dx}{dt} - A(\varepsilon)x,$$

where the matrix $A(\varepsilon)$ is an analytic function in ε.

Let the matrix $A(0)$ have the eigenvalues with zero real part. All eigenvalues of the matrix $A(\varepsilon)$, for sufficiently small $\varepsilon > 0$, have non-zero real parts. The operator $L(\varepsilon)$ has continuous inverse in B_n. We will describe the corresponding result that is due to Yu. S. Kolesov and V.V. Mayorov (see Mischenko, Yu. Kolesov, A. Kolesov, and Rozov [1994]).

Consider expressions

$$\sup_{-\infty<\omega<\infty} ||[A(\varepsilon) + i\omega I]^{-1}|| \tag{5.35}$$

and $||L^{-1}(\varepsilon)||$ as the functions of ε. Here i is the imaginary unity, I is an identity matrix. The point $\varepsilon = 0$ is a singularity of the function $||L^{-1}(\varepsilon)||$ as the operator $L(0)$ has no inverse. It turns out that the functions (5.35) and $||L^{-1}(\varepsilon)||$ have poles of the same order at the $\varepsilon = 0$. From this statement follows that under $\varepsilon > 0$ we have estimate

$$||L^{-1}(\varepsilon)|| \leq \frac{M}{\varepsilon^\alpha}, \quad M, \alpha > 0. \tag{5.36}$$

To define the order of the pole in estimate (5.36) we need to consider determinant

$$det[A(\varepsilon) + i\omega I]$$

and select ω so that this determinant had the maximum order in ε. We shall represent determinant in the form

$$det[A(\varepsilon) + i\omega I] = \prod_{j=1}^{n} (\lambda_j(\varepsilon) + i\omega),$$

where $\lambda_j(\varepsilon)$ is eigenvalues of matrix $A(\varepsilon)$. It is enough to consider only those eigenvalues of the matrix $A(\varepsilon)$ which have zero real parts at $\varepsilon = 0$.

For example, we consider the operator

$$L(\mu)x = \frac{dx}{dt} - A(\mu)x$$

from 5.5. Here

$$A(\mu) = \begin{pmatrix} 0 & 1 \\ \mu^2(q - \rho_0) & -\mu c \end{pmatrix},$$

where $c > 0$, q, ρ_0 are real numbers. It is easy to see that in this case $\alpha = 2$ if $q - \rho_0$ 0, i.e.,

$$||L^{-1}(\mu)|| \leq \frac{M}{\mu^2}.$$

Chapter 6

Linear Differential Equations with Fast and Slow Time

The data presented in this section can be found in Burd [1979a].

6.1 Generalized Lemmas on Regularity and Stability

Consider a system of differential equations

$$\frac{dx}{dt} = A(\frac{t}{\varepsilon}, t)x, \tag{6.1}$$

where $\varepsilon > 0$ is a small parameter, $A(\frac{t}{\varepsilon}, t)$ is a square matrix of order n. System (6.1) has two time scales - fast and slow. After changing the time $t = \varepsilon\tau$, system (6.1) transforms into a system in the standard form

$$\frac{dx}{d\tau} = \varepsilon A(\tau, \varepsilon\tau)x. \tag{6.2}$$

We shall assume that the elements $a_{ij}(t, s)$ of the matrix $A(t, s)$ are defined for $-\infty < t, s < \infty$, are almost periodic in the variable t uniformly with respect to s and almost periodic in s uniformly with respect to t. Then there exists the mean value of the matrix $A(t, s)$ over the first variable

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T A(t, s)dt = \bar{A}(s).$$

Averaging system (6.2) over the fast time yields the system

$$\frac{dx}{d\tau} = \varepsilon\bar{A}(\varepsilon\tau)x,$$

which, at the initial time, takes the form

$$\frac{dx}{dt} = \bar{A}(t)x. \tag{6.3}$$

Now the following questions arise. Does the regularity of the operator

$$Lx = \frac{dx}{dt} - \bar{A}(t)x$$

imply the regularity of the operator

$$L_\varepsilon x = \frac{dx}{dt} - A(\frac{t}{\varepsilon}, t)x$$

for sufficiently small ε? Are the properties of the stability of solutions of the homogeneous equations

$$\frac{dx}{dt} = A(\frac{t}{\varepsilon}, t)x$$

and

$$\frac{dx}{dt} = \bar{A}(t)x$$

interrelated? System (6.3) is a system with variable coefficients. However, it is simpler than the original system (6.1).

First, we bring in the generalized Bogoliubov lemma. Let the operator

$$Lx = \frac{dx}{dt} - A(t)x,$$

where $A(t)$ is an almost periodic matrix, be regular. Suppose the vector-function $f(t,\varepsilon)$ is determined for $-\infty < t < \infty$ and $0 < \varepsilon < \varepsilon_0$. Let $f(t,\varepsilon)$ be almost periodic in t for every fixed ε. Then the system

$$Lx = f(t,\varepsilon)$$

will have a unique almost periodic solution $x(t,\varepsilon)$. The generalized Bogoliubov lemma gives conditions under which this solution tends to zero as $\varepsilon \to 0$ uniformly with respect to t.

We say that $f(t,\varepsilon)$ as $\varepsilon \to 0$ converges properly to zero if

$$\lim_{\varepsilon\to 0} \sup_{|t-s|\le T} \left| \int_s^t f(\tau,\varepsilon)d\tau \right| = 0$$

for each $T > 0$ and $||f(t,\varepsilon)||_{B^n} < m < \infty$ for $0 < \varepsilon < \varepsilon_0$.

Lemma 6.1. *If $f(t,\varepsilon)$ converges properly to zero, then the almost periodic solution $x(t,\varepsilon)$ of the system*

$$\frac{dx}{dt} = A(t)x + f(t,\varepsilon)$$

satisfies the limit equality

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |x(t,\varepsilon)| = 0.$$

Proof. As we noted in Section 5.6, the solution $x(t,\varepsilon)$ is determined by the formula

$$x(t,\varepsilon) = \int_{-\infty}^{\infty} G(t,s)f(s,\varepsilon)ds. \tag{6.4}$$

Making the change $s = t+\tau$, we transform (6.4) into

$$x(t,\varepsilon) = \int_{-\infty}^{\infty} G(t,t+\tau)f(t+\tau,\varepsilon)d\tau.$$

We rewrite the latter equality

$$x(t,\varepsilon) = \int_{-\infty}^{0} G(t,t+\tau)d_\tau\left[\int_t^{t+\tau} f(\sigma,\varepsilon)d\sigma\right] + \int_0^{\infty} G(t,t+\tau)d_\tau\left[\int_t^{t+\tau} f(\sigma,\varepsilon)d\sigma\right].$$

Integrating by parts of each summand in the right-hand side yields

$$x(t,\varepsilon) = -\int_{-\infty}^{0} \frac{\partial G(t,t+\tau)}{\partial\tau}\left[\int_t^{t+\tau} f(\sigma,\varepsilon)d\sigma\right]d\tau-$$

$$-\int_0^{\infty} \frac{\partial G(t,t+\tau)}{\partial\tau}\left[\int_t^{t+\tau} f(\sigma,\varepsilon)d\sigma\right]d\tau.$$

Since

$$G(t,s) = \begin{cases} U(t)P_+U^{-1}(s), & t \geq s, \\ U(t)P_-U^{-1}(s), & t < s, \end{cases}$$

we have that for $\tau \neq 0$, the estimate is valid

$$\left|\frac{\partial G(t,t+\tau)}{\partial\tau}\right| \leq m_1 e^{-\gamma|\tau|}, \quad (-\infty < t < \infty), \tag{6.5}$$

where m_1, γ_1 are positive constants. It follows from (6.5) and the conditions imposed on $f(t,\varepsilon)$ that for an arbitrary $T > 0$

$$|x(t,\varepsilon)| \leq mm_1 \int_{-\infty}^{-T} e^{-\gamma|\tau|}|\tau|d\tau + m_1\int_{-T}^{T} e^{-\gamma|\tau|}\left|\int_t^{t+\tau} f(\sigma,\varepsilon)d\sigma\right|d\tau+$$

$$+mm_1\int_T^{\infty} e^{-\gamma|\tau|}\tau d\tau.$$

Assume $\eta > 0$. The latter inequality implies the existence of $T > 0$ such that

$$|x(t,\varepsilon)| < \frac{\eta}{2} + \frac{2m_1}{\gamma_1} \sup_{|t-s|\le T} \left| \int_s^t f(\sigma,\varepsilon)d\sigma \right|. \tag{6.6}$$

It follows from (6.6), in view of the properly convergent $f(t,\varepsilon)$, that

$$|x(t,\varepsilon)| < \eta \quad (-\infty < t < \infty)$$

for small $\varepsilon \in (0,\varepsilon_0)$. The lemma is proved.

Lemma 6.2. *Let the vector-function $f(t,s)$ be determined for $-\infty < t, s < \infty$, almost periodic in t uniformly with respect to s and almost periodic in s uniformly with respect to t. Suppose*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t,s)dt = 0.$$

Then the vector-function

$$\varphi(t,\varepsilon) = f(\frac{t}{\varepsilon}, t)$$

converges properly to zero for $\varepsilon \to 0$.

Proof. If the second variable of the vector-function $f(\frac{t}{\varepsilon}, t)$ is fixed, then assertion of the lemma has already been established in proving the Bogoliubov lemma. Hence, assertion of the lemma holds true for the vector-function $f(\frac{t}{\varepsilon}, \Delta(t))$, where $\Delta(t)$ is a piecewise continuous function on an interval with a length T. Since $f(\frac{t}{\varepsilon}, t)$ is continuous in the second variable uniformly with respect to the first variable and is uniformly continuous in the second variable, we have that this vector-function can be approximated at any accuracy of the vector-function $f(\frac{t}{\varepsilon}, \Delta(t))$, where $\Delta(t)$ is a piecewise continuous function on any interval of the length T.

The lemma can be proved in another way. We approximate the vector-function $f(t,s)$ by the vector-function $\phi(t,s)$ that is a trigonometric polynomial in each variable. For the vector-function, $\phi(\frac{t}{\varepsilon}, t)$ is directly established by calculating the respective integral.

We now bring in analogs of the lemmas on regularity and stability for systems with fast and slow time. We formulate both assertions as one theorem.

Theorem 6.1. *Let the operator*

$$Lx = \frac{dx}{dt} - \bar{A}(t)x$$

be regular. Then for sufficiently small ε the operator

$$L_\varepsilon x = \frac{dx}{dt} - A(\frac{t}{\varepsilon}, t)x$$

is regular. If the zero solution of the system

$$\frac{dx}{dt} = \bar{A}(t)x \tag{6.7}$$

is asymptotically stable, then under sufficiently small ε the zero solution of system (6.1) is asymptotically stable. If the zero solution of system (6.7) is unstable, then the zero solution of system (6.1) for sufficiently small ε is unstable.

Proof. We determine the matrix-function

$$H(t,\varepsilon) = \int_{-\infty}^{\infty} G(t,s)[A(\frac{s}{\varepsilon}, s) - \bar{A}(s)]ds.$$

By virtue of Lemmas 6.1 and 6.2, as $\varepsilon \to 0$ the norm of the matrix $H(t,\varepsilon)$ tends to zero uniformly with respect to $t \in (-\infty, \infty)$. After the change

$$x = y + H(t,\varepsilon)y,$$

system (6.1) transforms into the system

$$\frac{dy}{dt} = \bar{A}(t)y + F(t,\varepsilon)y,$$

where

$$F(t,\varepsilon) = -(I + H(t,\varepsilon))^{-1}\left\{-H(t,\varepsilon)\bar{A}(t) + [A(\frac{t}{\varepsilon}, t) - \bar{A}(t)]H(t,\varepsilon)\right\}.$$

Evidently,

$$\lim_{\varepsilon \to 0} \sup_{-\infty < t < \infty} |F(t,\varepsilon)| = 0.$$

Further proof is completely similar to the proof of the lemmas on regularity and stability.

6.2 Example. Parametric Resonance in the Mathieu Equation with a Slowly Varying Coefficient

As an example to Theorem 6.1, we consider the problem of the parametric resonance in the Mathieu equation with slowly varying coefficient

$$\frac{d^2x}{dx^2} + \omega^2(1 + \varepsilon\varphi(\varepsilon t)\cos\lambda t)x = 0, \tag{6.8}$$

where $\varphi(t)$ is a periodic or almost periodic function with non-zero mean value, $\varepsilon > 0$ is a small parameter; ω, λ are real parameters. Transform equation (6.8) by making a change

$$x = a\cos\omega t + b\sin\omega t, \quad x' = -a\omega\sin\omega t + b\omega\cos\omega t$$

and solve the obtained system relative to the derivatives $\frac{da}{dt}$ and $\frac{db}{dt}$

$$\begin{aligned}\frac{da}{dt} &= (\tfrac{\varepsilon}{\omega}\varphi(\varepsilon t)\cos\lambda t(\tfrac{a}{2}\sin 2\omega t + b\sin^2\omega t)\\ \frac{db}{dt} &= -(\tfrac{\varepsilon}{\omega}\varphi(\varepsilon t)\cos\lambda t(a\cos^2\omega t + \tfrac{b}{2}\sin 2\omega t).\end{aligned}$$

Assume $\lambda = 2\omega$ and average it over the fast time t. We arrive at the system

$$\begin{aligned}\frac{d\bar{a}}{dt} &= -\varepsilon\frac{\varphi(\varepsilon t)}{4\omega}\bar{b},\\ \frac{d\bar{b}}{dt} &= -\varepsilon\frac{\varphi(\varepsilon t)}{4\omega}\bar{a}\end{aligned}. \tag{6.9}$$

Transform system (6.9) in terms of the slow time $\tau = \varepsilon t$ and obtain a system with almost periodic coefficients, small parameter being eliminated. This system is integrable. The general solution takes the form

$$\begin{aligned}\bar{a} &= a_0\cosh\tfrac{1}{4\omega}\int_{\tau_0}^{\tau}\varphi(s)ds - b_0\sinh\tfrac{1}{4\omega}\int_{\tau_0}^{\tau}\varphi(s)ds,\\ \bar{b} &= -a_0\sinh\tfrac{1}{4\omega}\int_{\tau_0}^{\tau}\varphi(s)ds + b_0\cosh\tfrac{1}{4\omega}\int_{\tau_0}^{\tau}\varphi(s)ds,\end{aligned} \tag{6.10}$$

where $\bar{a}(0) = a_0$, $\bar{b}(0) = b_0$. Since the mean value of the function $\varphi(\tau)$ is non-zero, then solution (6.10) is unbounded as $t \to \infty$. Therefore, at $\lambda = 2\omega$ there occurs a parametric resonance in equation (6.8).

6.3 Higher Approximations and the Problem of the Stability

Consider the problem of the stability of a zero solution for the following system of differential equations

$$\frac{dx}{dt} = \left(B_0 + \sum_{j=1}^{N}\varepsilon^j B_j(t,\tau)\right)x. \tag{6.11}$$

Here, $\varepsilon > 0$ is a small parameter, $\tau = \varepsilon t$ is a slow time, B_0 is a constant square matrix of order n. We say that the matrix-function $A(t,\tau)$ belongs to the class Σ_1 if its elements are trigonometric polynomials in the variable t and trigonometric polynomials in the variable τ. Namely, $A(t,\tau)$ takes the form

$$A(t,\tau) = \sum_{\lambda,\mu} a_{\lambda,\mu}e^{(\lambda t+\mu\tau)},$$

where $a_{\lambda,\mu}$ are the constant square matrices of order n, and λ, μ varies over a finite set of real values.

Further, we shall assume that the matrices $B_j(t,\tau)(j=1,2,\ldots,N)$ belong to the class Σ_1.

It is evident that if all eigenvalues of the matrix B_0 have negative real parts, or if the matrix B_0 has at least one eigenvalue with positive real part, then for sufficiently small ε, the zero solution of system (6.11) is stable or unstable, respectively. Therefore, we are going to consider only the critical case when the matrix B_0 has eigenvalues with non-positive real parts, some of the eigenvalues having zero real part.

Without loss of generality, we assume that all eigenvalues of the matrix B_0 are real (see Section 5.2). There exists a change such that stability remains unaffected and the matrix B_0 has real spectrum. As in section 5.2, we find a change of variables such that system (6.11) is transformed into the system

$$\frac{dy}{dt} = \left(B_0 + \sum_{j=1}^{N} \varepsilon^j A_j(\tau)\right) y + O(\varepsilon^{N+1}) \tag{6.12}$$

with no fast time t in the right-hand side terms up to order ε^N inclusive. This change is

$$x = y + \sum_{k=1}^{N} \varepsilon^k Y_k(t,\tau) y, \tag{6.13}$$

where the matrices $A_i(\tau), Y_i(t,\tau)\,(i = 1,2,\ldots,N)$ are to be determined. Moreover, the matrices $Y_i(t,\tau)$ must belong to the class Σ_1 and have zero mean value over t. Substituting (6.13) into (6.11) and with (6.12) taken into account yields

$$\left(B_0 + \sum_{k=1}^{N} \varepsilon^k A_k(\tau) + \sum_{k=1}^{N} \varepsilon^k Y_k(t,\tau) B_0 + \sum_{k,m=1}^{N} \varepsilon^{m+k} Y_k(t,\tau) A_m(\tau) + \right.$$

$$\left. + \sum_{k=1}^{N} \varepsilon^k \partial Y_k \partial t + \sum_{k=1}^{N} \varepsilon^{k+1} \partial Y_k \partial \tau \right) y = \left(B_0 + \sum_{k=1}^{N} \varepsilon^k B_0 Y_k(t,\tau) + \right.$$

$$\left. + \sum_{k=1}^{N} \varepsilon^k B_k(t,\tau) + \sum_{k,m=1}^{N} \varepsilon^{m+k} B_k(t,\tau) Y_m(t,\tau) \right) y.$$

Comparing the coefficients of ε in the right-hand and left-hand sides of the latter equality, we arrive at

$$\frac{\partial Y_1}{\partial t} - B_0 Y_1(t,\tau) + Y_1(t,\tau) = B_1(t,\tau) - A_1(\tau). \tag{6.14}$$

We seek the matrix $Y_1(t,\tau)$ of the class Σ_1 in the form

$$Y_1(t,\tau) = \sum_{\lambda} b_\lambda(\tau)e^{i\lambda t},$$

where λ varies over a finite set of values. As was already noted, the mean value of the matrix $Y_1(t,\tau)$ over t is assumed to be equal to the zero matrix. Suppose

$$B_1(t,\tau) = \sum_{\lambda} a_\lambda(\tau)e^{i\lambda t}$$

and choose the matrix $A_1(\tau)$ equal to the free term of this system, i.e.

$$A_1(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T B_1(t,\tau)dt.$$

Then, to find the matrix $b_\lambda(\tau)$, we obtain the equation

$$i\lambda b_\lambda(\tau) - B_0 b_\lambda(\tau) + b_\lambda(\tau)B_0 = a_\lambda(\tau).$$

Now we need the matrix $b_\lambda(\tau)$ in the form

$$b_\lambda(\tau) = \sum_{\mu} b_{\lambda\mu}e^{\mu_\lambda \tau},$$

where μ varies over a finite set of values, and $b_{\lambda\mu}$ are the constant square matrices of order n. The matrix $a_\lambda(\tau)$ is presented as

$$a_\lambda(\tau) = \sum_{\mu} a_{\lambda\mu}e^{i\mu t}.$$

To find the matrices $b_{\lambda\mu}$, we obtain the matrix equation

$$(i\lambda I - B_0)b_{\lambda\mu} + b_{\lambda\mu}B_0 = a_{\lambda\mu}.$$

This equation has a unique solution due to non-intersecting spectra of the matrices $i\lambda I - B_0$ and B_0.

Similarly, comparing the coefficients of ε^j ($j = 2,\dots,N$), we obtain the matrix equation

$$\frac{\partial Y_j}{\partial t} - B_0 Y_j(t,\tau) + Y_j(t,\tau)B_0 = F(t,\tau) - A_j(\tau), \tag{6.15}$$

where the matrix $F(t,\tau)$ is determined through the matrices $B_k(t,\tau), Y_k(t,\tau)$ ($j = 1,\dots,j-1$). Equation (6.15) has the same form as equation (6.14). Therefore, the problem of the choice of the matrix $A_j(\tau)$ and existence of the matrix $Y_j(t,\tau)$ of the class Σ_1 is solved in the same way.

Evidently, change (6.13) is a change with coefficients bounded for all t, τ . Hence, the problem of the stability of solutions of system (6.11) for sufficiently small ε is reduced to the problem of the stability of solutions of system (6.12) with slowly varying near-constant coefficients. If we introduce the differential operator

$$L(\varepsilon)y = \frac{dy}{dt} - B_0 y - \sum_{k=1}^{N} \varepsilon^k A_k(\tau) y$$

determined on the space B_n, then we can formulate a theorem of the stability of solutions of system (6.12). Recall that the regularity of the operator $L(\varepsilon)$ is equivalent to the exponential dichotomy of the solutions of the corresponding homogeneous equation. We call the operator $L(\varepsilon)$ stable if the solution of the corresponding homogeneous equation is stable and unstable if the solution of the corresponding homogeneous equation is unstable.

Theorem 6.2. *Let the operator $L(\varepsilon)$ be regular for $0 < \varepsilon < \varepsilon_0$ and the inequality*

$$||L^{-1}(\varepsilon)|| \leq \frac{C}{\varepsilon^N} \tag{6.16}$$

holds, where C is a constant independent of ε. Then, if the operator $L(\varepsilon)$ is stable for $0 < \varepsilon < \varepsilon_0$, the zero solution of system (6.12) is asymptotically stable for sufficiently small ε. If the operator $L(\varepsilon)$ is unstable for $0 < \varepsilon < \varepsilon_0$, then the zero solution of system (6.12) is unstable for sufficiently small ε.

The proof follows the same scheme as the proof of the lemma on stability. We only note that inequality (6.16) is equivalent to the exponential dichotomy of order k (see Section 5.7) of the solutions of the system

$$L(\varepsilon)y = 0.$$

Chapter 7

Asymptotic Integration

In this chapter we present results established by Burd and Karakulin [1998].

7.1 Statement of the Problem

The stability and asymptotic behavior of solutions of linear systems of differential equations

$$\frac{dx}{dt} = Ax + B(t)x,$$

where A is a constant matrix, and matrix $B(t)$ is small in a certain sense when $t \to \infty$ has been studied by many researchers (see papers by Levinson [1948], Hartman and Wintner [1955], Fedoryuk [1966], Harris and Lutz [1974], [1977], and books by Bellman [1953], Rapoport [1954], Coddington and Levinson [1955], Naimark [1968], Cesari [1971], and Eastham [1989]).

In Chapter 1.5 we described the results of Shtokalo on the investigation of the stability of solutions of the system of differential equations

$$\frac{dx}{dt} = Ax + \varepsilon B(t)x, \tag{7.1}$$

where $\varepsilon > 0$ is a small parameter, A is a constant square matrix, and $B(t)$ is a square matrix whose elements are trigonometric polynomials $b_{kl}(t)$ $(k, l = 1, ..., m)$ in the form

$$b_{kl}(t) = \sum_{j=1}^{m} b_j^{kl} e^{i\lambda_j t},$$

where λ_j $(j = 1, ..., m)$ are arbitrary real numbers. We shall denote by Σ the class of matrices whose elements are the trigonometric polynomials described above. The mean of a matrix from Σ is a constant matrix that consists of the constant terms $(\lambda_j = 0)$ of the elements of the matrix.

As we have seen before, using a change of variables similar to the ones used by Bogoliubov in the method of averaging, we can transform system (7.1) into a system with constant coefficients, which depend on a parameter ε, precisely up to the terms of any order in ε.

We also mention that the method of averaging in the first approximation was utilized in the papers by Samokhin and Fomin [1976, 1981], for studying the asymptotic behavior of solutions of a particular class of systems of equations with oscillatory decreasing coefficients.

In this chapter we adopt the method of Shtokalo for the problem of asymptotic integration of systems of linear differential equations with oscillatory decreasing coefficients.

We consider the following system of differential equations in n-dimensional space $\mathcal{R}^n$

$$\frac{dx}{dt} = \left\{ A_0 + \sum_{j=1}^{k} \frac{1}{t^{j\alpha}} A_j(t) \right\} x + \frac{1}{t^{(1+\delta)}} F(t)x. \tag{7.2}$$

Here, A_0 is a constant $n \times n$ matrix, and $A_1(t), A_2(t), \ldots, A_k(t)$ are $n \times n$ matrices that belong to Σ. We shall assume that matrix A_0 is in Jordan canonical form, a real number α and a positive integer k satisfy $0 < k\alpha \leq 1 < (k+1)\alpha$, $\delta > 0$, and $F(t)$ satisfies

$$||F(t)|| \leq C < \infty$$

for $t_0 \leq t < \infty$, where $||.||$ is some matrix norm in $\mathcal{R}^n$.

We are concerned with the behavior of solutions of system (7.2) when $t \to \infty$.

7.2 Transformation of the Basic System

We want to construct an invertible change of variables (for sufficiently large t,$t > t^* > t_0$) that would transform system (7.2) into

$$\frac{dy}{dt} = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y + \frac{1}{t^{(1+\varepsilon)}} G(t)y, \;\; \varepsilon > 0, \;\; t > t^*. \tag{7.3}$$

Here, $A_0, A_1, .., A_k$ are constants square matrices (moreover, A_0 is the same matrix as in (7.2)), and the matrix $G(t)$ has the same properties as the matrix $F(t)$ in system (7.2).

Without loss of generality, we can assume that all eigenvalues of the matrix A_0 are real. Indeed, if matrix A_0 has complex eigenvalues, then we can make a change of variables in (7.2)

$$y = e^{iRt} z,$$

where R is a diagonal matrix composed of the imaginary parts of eigenvalues of the matrix A_0. This change of variable with coefficients, which are bounded in t, $t \in (-\infty, \infty)$, transforms the matrix A_0 into the matrix $A_0 - iR$, which only has real eigenvalues.

We shall try to choose an invertible change of variables (for sufficiently large t), in the form

$$x = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\} y, \tag{7.4}$$

to transform system (7.2) into system (7.3), where $Y_0(t) = I$ is the identity matrix, and, $Y_1(t), Y_2(t), .., Y_k(t)$ are $n \times n$ matrices that belong to Σ and have zero mean value. By substituting (7.4) into (7.2), and replacing $\frac{dy}{dt}$ by the right-hand side of (7.3) we obtain

$$\begin{aligned} &\left\{\sum_{j=0}^{k} \tfrac{1}{t^{j\alpha}} Y_j(t)\right\} \left\{\sum_{j=0}^{k} \tfrac{1}{t^{j\alpha}} A_j\right\} y + \\ &+ \tfrac{1}{t^{(1+\varepsilon)}} \left\{\sum_{j=0}^{k} \tfrac{1}{t^{j\alpha}} Y_j(t)\right\} G(t) y + \tfrac{1}{t^{(1+\alpha)}} W(t) y + \left\{\sum_{j=1}^{k} \tfrac{1}{t^{j\alpha}} \tfrac{Y_j(t)}{dt}\right\} y = \\ &= \left\{A_0 + \sum_{j=1}^{k} \tfrac{1}{t^{j\alpha}} A_j(t)\right\} \left\{\sum_{j=0}^{k} \tfrac{1}{t^{j\alpha}} Y_j(t)\right\} y + \tfrac{1}{t^{(1+\delta)}} U(t) y, \end{aligned} \tag{7.5}$$

where

$$W(t) = \left\{ -\sum_{j=1}^{k} \frac{j\alpha}{t^{(j-1)\alpha}} Y_j(t) \right\}, \tag{7.6}$$

and

$$U(t) = F(t) \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\}. \tag{7.7}$$

Equating the terms that contain $t^{-j\alpha}$ $(j = 1, \ldots, k)$ in the left-hand and the right-hand sides of (7.5) yields a system of k linear matrix differential equation with constant coefficients

$$\begin{aligned} &\tfrac{dY_j(t)}{dt} - A_0 Y_j(t) + Y_j(t) A_0 = \\ &= \sum_{l=0}^{j-1} A_{j-l}(t) Y_l(t) - \sum_{l=0}^{j-1} Y_l(t) A_{j-l}, \quad (j = 1, \ldots, k). \end{aligned} \tag{7.8}$$

The solvability of system (7.8) was studied in Shtokalo [1961]. We represent $Y_j(t)$ as a finite sum

$$Y_j(t) = \sum_{\lambda \neq 0} y_\lambda^j e^{i\lambda t},$$

where y_λ^j are constants $n \times n$ matrices, and obtain matrix equations

$$i\lambda y_\lambda^j - A_0 y_\lambda^j + y_\lambda^j A_0 = b_\lambda^j.$$

Because all the eigenvalues of A are real, the matrix equations have unique solutions for $\lambda \neq 0$ (see, for instance, Gantmacher [1959], Daleckii and Krein [1974]). On each of the k steps of the solution process we determine the matrix

A_j from the condition that the right-hand side of (7.8) has a zero mean value. In particular, for $j = 1$

$$\frac{dY_1(t)}{dt} - A_0 Y_1(t) + Y_1(t) A_0 = A_1(t) - A_1,$$

where A_1 is the mean value of the matrix $A_1(t)$. The following results from (7.5).

Theorem 7.1. *System (7.2), for sufficiently large t, can be transformed using a change of variables (7.4) into a system*

$$\frac{dy}{dt} = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y + \frac{1}{t^{(1+\varepsilon)}} G(t) y,$$

where $\varepsilon > 0$, and $||G(t)|| \leq C_1 < \infty$.

Proof. Substituting (7.4) into (7.2) yields

$$\begin{aligned} &\left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t)\right\} \frac{dy}{dt} = \left\{A_0 + \textstyle\sum_{j=1}^{k} \frac{1}{t^{j\alpha}} A_j(t)\right\} \left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t)\right\} y - \\ &- \left\{\textstyle\sum_{j=1}^{k} \frac{1}{t^{j\alpha}} \frac{dY_j(t)}{dt}\right\} y - \frac{1}{t^{(1+\alpha)}} W(t) y + \frac{1}{t^{(1+\delta)}} U(t) y, \end{aligned}$$

where $W(t)$ and $U(t)$ are defined by (7.6) and (7.7), respectively. The last relation can be rewritten as

$$\begin{aligned} &\left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t)\right\} \left\{\frac{dy}{dt} - \left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j\right\} y\right\} = \\ &= \left\{A_0 + \textstyle\sum_{j=1}^{k} \frac{1}{t^{j\alpha}} A_j(t)\right\} \left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t)\right\} y - \left\{\textstyle\sum_{j=1}^{k} \frac{1}{t^{j\alpha}} \frac{dY_j(t)}{dt}\right\} y - \\ &- \left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t)\right\} \left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j\right\} y - \frac{1}{t^{(1+\alpha)}} W(t) y + \frac{1}{t^{(1+\delta)}} U(t) y. \end{aligned}$$

Due to (7.8) we get

$$\begin{aligned} &\left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t)\right\} \left\{\frac{dy}{dt} - \left\{\textstyle\sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j\right\} y\right\} = \\ &= \frac{1}{t^{(1+k)\alpha}} S(t) y - \frac{1}{t^{(1+\alpha)}} W(t) y + \frac{1}{t^{(1+\delta)}} U(t) y, \end{aligned} \tag{7.9}$$

where elements of the matrix $S(t)$ can be represented as $t^{-j\alpha} a_j(t)$ ($j = 0, \ldots, k$), and $a_j(t)$ are trigonometric polynomials. Therefore,

$$\frac{1}{t^{(1+k)\alpha}} S(t) - \frac{1}{t^{(1+\alpha)}} W(t) + \frac{1}{t^{(1+\delta)}} U(t) = \frac{1}{t^{(1+\varepsilon)}} R(t), \tag{7.10}$$

where $\varepsilon > 0$ and $R(t)$ satisfies

$$||R(t)|| \leq C_2 < \infty.$$

The identity (7.9), for sufficiently large t, can be rewritten as

$$\frac{dy}{dt} = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y + \frac{1}{t^{(1+\varepsilon)}} \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} Y_j(t) \right\}^{-1} R(t)y.$$

This fact along with (7.10) implies the theorem.

The main part of system (7.3)

$$\frac{dy}{dt} = \left\{ \sum_{j=0}^{k} \frac{1}{t^{j\alpha}} A_j \right\} y$$

does not have oscillating coefficients. This makes it simpler than the original system (7.2). In particular, the Fundamental Theorem of Levinson on asymptotic behavior of solutions of linear systems of differential equations (see Levinson [1948], Rapoport [1954], Coddington and Levinson [1955], Naimark [1968], Eastham [1989]) can be used for constructing the asymptotics of the fundamental matrix of the system (7.3). We state Levinson's Theorem in a form that is convenient for our consideration.

Theorem of Levinson. *Consider a system*

$$\frac{dx}{dt} = (A + V(t) + R(t))x, \tag{7.11}$$

where A is a constant matrix with distinct eigenvalues, the matrix $V(t)$ tends to zero matrix as $t \to \infty$, and

$$\int_{t_0}^{\infty} |V'(t)|dt < \infty \qquad \int_{t_0}^{\infty} |R(t)|dt < \infty.$$

Denote by $\lambda_j(t)$ the eigenvalues of the matrix $\Delta(t) = A + V(t)$. Assume that none of the differences $Re\lambda_k(t) - Re\lambda_j(t)$ changes its sign beginning with some sufficiently large t. Then the fundamental matrix of system (7.11) has the following form

$$X(t) = (P + o(1))exp \int_{t^*}^{t} \Delta(s)ds, \quad t > t^*, \quad t \to \infty,$$

where P is the matrix composed of the eigenvectors of the matrix A, and $\Delta(t)$ is a diagonal matrix whose elements are eigenvalues of the matrix $A + V(t)$.

This theorem implies the following theorem for system (7.3).

Theorem 7.2. *Assume that, among matrices $A_j (j = 0, \ldots, k)$, the first non-zero matrix is A_l and its eigenvalues are distinct. Then the fundamental matrix of system (7.3) has the following form*

$$X(t) = (P + o(1))exp \int_{t^*}^{t} \Lambda(s)ds, \quad t > t^*, \quad t \to \infty,$$

where P is a matrix composed of the eigenvectors of the matrix A_l, and $\Lambda(t)$ is a diagonal matrix whose elements are eigenvalues of the matrix $\sum_{j=l}^{k} t^{-j\alpha} A_j$.

To prove this theorem we just have to observe that the system of differential equations

$$\frac{dx}{dt} = \frac{1}{t^l} A_l x + \sum_{j=l+1} \frac{1}{t^j} A_j x$$

can be transformed into

$$\frac{dx}{d\tau} = \frac{1}{1-l} [A_l + \sum_{j=l+1} \frac{1}{t^{j-l}} A_j] x$$

using the change of variables $\tau = t^{1-l}$.

7.3 Asymptotic Integration of an Adiabatic Oscillator

As an example we consider an equation of an adiabatic oscillator

$$\frac{d^2 y}{dt^2} + (1 + \frac{1}{t^\alpha} \sin \lambda t) y = 0, \tag{7.12}$$

where λ, α are real numbers, and $0 < \alpha \leq 1$. The problem of asymptotic integration of the equation (7.12) has been studied in Harris and Lutz [1974, 1975, 1977], and Wintner [1946a, 1946b]. In particular, asymptotics of solutions for $\frac{1}{2} \leq \alpha \leq 1$ were obtained. The method that we proposed in this chapter can be used to obtain (in a simple manner) all known results on asymptotics of solutions of equation (7.12) as well as to establish new results.

We convert equation (7.12) into a system of equations ($x = (x_1, x_2)$) using a change of variables

$$y = x_1 \cos t + x_2 \sin t, \quad y' = -x_1 \sin t + x_2 \cos t. \tag{7.13}$$

which yields

$$\frac{dx}{dt} = \frac{1}{t^\alpha} A(t) x. \tag{7.14}$$

It will be convenient, as we will see below, to rewrite $A(t)$ in complex form as

$$A(t) = a_1 e^{i(\lambda+2)t} + \bar{a}_1 e^{-i(\lambda+2)t} + a_2 e^{i(\lambda-2)t} + \bar{a}_2 e^{-i(\lambda-2)t} + a_3 e^{i\lambda t} + \bar{a}_3 e^{-i\lambda t},$$

where

$$a_1 = \frac{1}{8}\begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}, \quad a_2 = \frac{1}{8}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad a_3 = \frac{1}{8}\begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix},$$

and the matrices $\bar{a}_1, \bar{a}_2, \bar{a}_3$ are complex conjugates to the matrices a_1, a_2, a_3, respectively.

The values of α and λ significantly affect the behavior of solutions of system (7.14). We denote by $R(t)$ a 2×2 matrix that satisfies

$$||R(t)|| \leq C_3 < \infty.$$

for all t.

First, assume $\frac{1}{2} < \alpha \leq 1$. For $\lambda \neq \pm 2$ system (7.3) becomes

$$\frac{dy}{dt} = \frac{1}{t^{1+\varepsilon}} R(t) y, \quad \varepsilon > 0.$$

Therefore, it is easy to see (taking into consideration change (7.13)), that the fundamental system of solutions of equation (7.12) for $\frac{1}{2} < \alpha \leq 1$, $\lambda \neq \pm 2$ as $t \to \infty$ has the form

$$x_1 = \cos t + o(1), \quad x_2 = \sin t + o(1),$$

$$x_1' = -\sin t + o(1), \quad x_2' = \cos t + o(1).$$

We shall represent the fundamental system of solutions of equation (7.12) as a matrix with rows x_1, x_2 and x_1', x_2'.

Now assume $\lambda = \pm 2$. More specifically let $\lambda = 2$. Then system (7.3) becomes

$$\frac{dy}{dt} = \frac{1}{t^\alpha} A_1 y + \frac{1}{t^{1+\varepsilon}} R(t) y, \quad \varepsilon > 0.$$

Here

$$a_2 + \bar{a}_2 = A_1 = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Theorem 7.2 implies that for $t \to \infty$ the fundamental matrix of system (7.14) has the following form

$$Y(t) = \begin{pmatrix} exp\left(\int_{t^*}^t \frac{1}{4s^\alpha} ds\right) & 0 \\ 0 & exp\left(-\int_{t^*}^t \frac{1}{4s^\alpha} ds\right) \end{pmatrix} [I + o(1)].$$

Therefore, for $\alpha = 1$, $\lambda = 2$ we obtain the fundamental system of solutions of equation (7.12) as

$$\begin{pmatrix} t^{\frac{1}{4}} \cos t & t^{-\frac{1}{4}} \sin t \\ -t^{\frac{1}{4}} \sin t & t^{-\frac{1}{4}} \cos t \end{pmatrix} [I + o(1)],$$

while for $1/2 < \alpha < 1$, $\lambda = 2$ for $t \to \infty$ we get

$$\begin{pmatrix} exp\left(\frac{t^{1-\alpha}}{4(1-\alpha)}\right)\cos t & exp\left(-\frac{t^{1-\alpha}}{4(1-\alpha)}\right)\sin t \\ -exp\left(\frac{t^{1-\alpha}}{4(1-\alpha)}\right)\sin t & exp\left(-\frac{t^{1-\alpha}}{4(1-\alpha)}\right)\cos t \end{pmatrix} [I + o(1)] \,.$$

We note that, for $\lambda = \pm 2$, and $\frac{1}{2} < \alpha \leq 1$, equation (7.12) has unbounded solutions. Moreover, for $\alpha = 1$ the solutions have a polynomial growth, while for $\alpha \neq 1$, they grow exponentially.

We now assume $\frac{1}{3} < \alpha \leq \frac{1}{2}$. In this case a change of variables (7.4) transforms (7.14) into

$$\frac{dy}{dt} = \frac{1}{t^\alpha} A_1 y + \frac{1}{t^{2\alpha}} A_2 y + \frac{1}{t^{1+\varepsilon}} R(t) y, \quad \varepsilon > 0.$$

If $\lambda \neq \pm 2, \pm 1$, then the matrix A_1 is zero, and matrix A_2 has the form

$$A_2 = i\left[\frac{1}{\lambda+2}(a_1\bar{a}_1 - \bar{a}_1 a_1) + \frac{1}{\lambda-2}(a_2\bar{a}_2 - \bar{a}_2 a_2) + \frac{1}{\lambda}(a_3\bar{a}_3 - \bar{a}_3 a_3)\right]. \tag{7.15}$$

Computing A_2 yields

$$A_2 = \frac{1}{4(\lambda^2 - 4)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The system

$$\frac{dy}{dt} = \frac{1}{t^{2\alpha}} A_2 y$$

can be integrated. We obtain that, for $t \to \infty$, the fundamental system of solutions of equation (7.12) with $\alpha = \frac{1}{2}$, $\lambda \neq \pm 2, \pm 1$, has the form

$$\begin{pmatrix} \cos(t + \gamma \ln t) & \sin(t + \gamma \ln t) \\ -\sin(t + \gamma \ln t) & \cos(t + \gamma \ln t) \end{pmatrix} [I + o(1)],$$

where $\gamma = \frac{1}{4(\lambda^2-4)}$. For $\frac{1}{3} < \alpha < \frac{1}{2}$, and $\lambda \neq \pm 2, \pm 1$, the fundamental system of solutions of equation (7.12) has the form

$$\begin{pmatrix} \cos\left(\frac{t^{1-2\alpha}}{4(1-2\alpha)(\lambda^2-4)}\right) & \sin\left(\frac{t^{1-2\alpha}}{4(1-2\alpha)(\lambda^2-4)}\right) \\ -\sin\left(\frac{t^{1-2\alpha}}{4(1-2\alpha)(\lambda^2-4)}\right) & \cos\left(\frac{t^{1-2\alpha}}{4(1-2\alpha)(\lambda^2-4)}\right) \end{pmatrix} [I + o(1)]$$

as $t \to \infty$.

We now assume $\alpha = \frac{1}{2}, \lambda = 1$. In this case A_1 is zero, and A_2 is determined by

$$iA_2 = -\frac{1}{3}a_1\bar{a}_1 + \frac{1}{3}\bar{a}_1 a_1 - \bar{a}_2 a_2 + a_2\bar{a}_2 - a_3\bar{a}_3 + \bar{a}_3 a_3 + a_2 a_3 + a_3 a_2 - \bar{a}_2\bar{a}_3 + \bar{a}_3\bar{a}_2.$$

A simple calculation yields

$$A_2 = \frac{1}{24}\begin{pmatrix} 0 & -5 \\ -1 & 0 \end{pmatrix}.$$

The corresponding system (7.3) has the following form

$$\frac{dy}{dt} = \frac{1}{t}A_2 y + \frac{1}{t^{1+\varepsilon}}R(t)y, \quad \varepsilon > 0.$$

By integrating the system

$$\frac{dy}{dt} = \frac{1}{t}A_2 y,$$

we obtain its fundamental matrix

$$Y(t) = \begin{pmatrix} -\sqrt{5}t^{\varrho} & \sqrt{5}t^{-\varrho} \\ t^{\varrho} & t^{-\varrho} \end{pmatrix},$$

where $\varrho = \frac{\sqrt{5}}{24}$. Then the fundamental system of solutions of equation (7.12) for $\alpha = \frac{1}{2}$, $\lambda = 1$, and $t \to \infty$, has the form

$$\begin{pmatrix} t^{\varrho}\sin(t-\beta) & t^{-\varrho}\sin(t+\beta) \\ t^{\varrho}\cos(t-\beta) & t^{-\varrho}\cos(t+\beta) \end{pmatrix}[I + o(1)],$$

where

$$\varrho = \frac{\sqrt{5}}{24}, \quad \beta = arctg\sqrt{5}, \quad 0 < \beta < \frac{\pi}{2}. \tag{7.16}$$

If $\frac{1}{3} < \alpha < \frac{1}{2}$ and $\lambda = 1$ we have the system

$$\frac{dy}{dt} = \frac{1}{t^{2\alpha}}A_2 y + \frac{1}{t^{1+\varepsilon}}R(t)y, \quad \varepsilon > 0.$$

Using Theorem 7.2 we obtain the asymptotics of the fundamental matrix of this system, and then, using the change (7.13), the asymptotics of the fundamental system of solutions of equation (7.12) for $\frac{1}{3} < \alpha < \frac{1}{2}$, $\lambda = 1$, and $t \to \infty$:

$$\begin{pmatrix} exp(\varrho\frac{t^{1-2\alpha}}{1-2\alpha})\sin(t-\beta) & exp(-\varrho\frac{t^{1-2\alpha}}{1-2\alpha})\sin(t+\beta) \\ exp(\varrho\frac{t^{1-2\alpha}}{1-2\alpha})\cos(t-\beta) & exp(-\varrho\frac{t^{1-2\alpha}}{1-2\alpha})\cos(t+\beta) \end{pmatrix}[I + o(1)],$$

where ϱ are β are defined by (7.16). Thus, for $\alpha = \frac{1}{2}$ and $\lambda = 1$ we observe a polynomial growth of solutions, while for $\frac{1}{3} < \alpha < \frac{1}{2}$ and $\lambda = 1$ the solutions grow exponentially.

Now let $\alpha = \frac{1}{2}$ and $\lambda = 2$. Simple calculations show that

$$A_1 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -\frac{1}{64} \\ \frac{1}{64} & 0 \end{pmatrix}.$$

Therefore, we get a system

$$\frac{dy}{dt} = \frac{1}{t^{\frac{1}{2}}}A_1 y + \frac{1}{t}A_2 y + \frac{1}{t^{1+\varepsilon}}R(t)y, \quad \varepsilon > 0. \tag{7.17}$$

We compute the eigenvalues of the matrix

$$\frac{1}{t^{\frac{1}{2}}}A_1 + \frac{1}{t}A_2,$$

integrate them, and, using Theorem 7.2, we obtain the asymptotics of the fundamental matrix of system (7.17). Next we find the fundamental system of solutions of equation (7.12) for $\alpha = \frac{1}{2}$, $\lambda = 2$ and $t \to \infty$:

$$\begin{pmatrix} exp(\phi(t))\cos t & exp-(\phi(t))\sin t \\ -exp(\phi(t))\sin t & exp-(\phi(t))\cos t \end{pmatrix} [I + o(1)],$$

where $\phi(t) = \frac{1}{32}[\sqrt{2^8 t - 1} - \arctan\sqrt{2^8 t - 1}]$. The last formulas give more precise asymptotics than the corresponding formulas in Harris and Lutz [1977].

For $\frac{1}{3} < \alpha < \frac{1}{2}$, $\lambda = 2$ instead of (7.17) we get a system

$$\frac{dy}{dt} = \frac{1}{t^{\alpha}}A_1 y + \frac{1}{t^{2\alpha}}A_2 y + \frac{1}{t^{1+\varepsilon}}R(t)y,$$

where $\varepsilon > 0$, with the same matrices A_1, A_2. Therefore, it is straightforward to write the asymptotics of the fundamental system of solutions of equation (7.12) for $\frac{1}{3} < \alpha < \frac{1}{2}$, $\lambda = 2$, and $t \to \infty$.

Finally, let $\alpha = \frac{1}{3}$, $\lambda \neq \pm 1$, and $\lambda \neq \pm 2$. Then, it turns out that A_1 is zero, and A_2 is defined by (7.15). Matrix A_3 differs from zero only if $\lambda = \pm\frac{2}{3}$. Assume $\lambda = \frac{2}{3}$. System (7.3) then becomes

$$\frac{dy}{dt} = \frac{1}{t^{\frac{2}{3}}}A_2 y + \frac{1}{t}A_3 y + \frac{1}{t^{1+\varepsilon}}R(t)y,$$

where $\varepsilon > 0$ and the matrices A_2 and A_3 are defined by

$$A_2 = \begin{pmatrix} 0 & -\frac{9}{128} \\ \frac{9}{128} & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -\frac{27}{1024} & 0 \\ 0 & \frac{27}{1024} \end{pmatrix}.$$

We compute the eigenvalues of the matrix

$$\frac{1}{t^{\frac{2}{3}}}A_2 + \frac{1}{t}A_3.$$

These eigenvalues have zero real parts, for sufficiently large t. Further, using the same scheme as before we find the asymptotics of the fundamental system of solutions of equation (7.12). We only note that the solutions of equation (7.12) are bounded for $\alpha = \frac{1}{3}$, $\lambda = \pm\frac{2}{3}$ as $t \to \infty$.

Exercise 7.1. Consider a system of equations

$$\frac{dx}{dt} + \omega_1^2 x = \frac{a\cos 2t}{t} y, \quad \frac{dy}{dt} + \omega_2^2 x = \frac{b\cos 2t}{t} x.$$

Construct the asymptotics of solutions of this system for $t \to \infty$, if
a) $\omega_1 + \omega_2 = 2$,
b) $\omega_1 - \omega_2 = 2$.

Chapter 8

Singularly Perturbed Equations

Consider a singularly perturbed differential operator

$$L(\varepsilon)x = \varepsilon \frac{dx}{dt} - Ax,$$

where $\varepsilon > 0$ is a small parameter, and A is a constant square matrix of order n. If all eigenvalues of the matrix A have non-zero real parts, then the above operator is regular in the space B_n, and for the solutions of a homogeneous system of equations

$$\varepsilon \frac{dx}{dt} - Ax = 0$$

take place an exponential dichotomy holds. This immediately implies that the space $U(\varepsilon)$ of this system can be represented as

$$U(\varepsilon) = U_+(\varepsilon) + U_-(\varepsilon).$$

For the solutions $x_+(t,\varepsilon) \in U_+(\varepsilon)$ the inequality

$$|x_+(t,\varepsilon)| \le M_+ e^{-\gamma_+ \frac{(t-s)}{\varepsilon}} |x_+(s,\varepsilon)|, \quad -\infty < s < t < \infty$$

holds true, and for the solutions $x_-(t,\varepsilon) \in U_-(\varepsilon)$ the inequality

$$|x_-(t,\varepsilon)| \le M_- e^{-\gamma_- \frac{(t-s)}{\varepsilon}} |x_+(s,\varepsilon)|, \quad -\infty < t < s < \infty$$

is fulfilled. Here, $M_+, M_-, \gamma_+, \gamma_-$ are positive constants.

Let us show that the same results are valid for singularly perturbed differential operators with periodic or almost periodic coefficients for sufficiently small values of parameter ε.

Consider a singularly perturbed differential operator

$$L(\varepsilon)x = \varepsilon \frac{dx}{dt} - A(t)x, \tag{8.1}$$

where $\varepsilon > 0$ is a small parameter, $A(t)$ is a square matrix of order n with its elements being almost periodic functions.

We study the problem of the regularity of the operator $L(\varepsilon)$. We will need a criterion derived by E. Muhamadiev for the regularity of operators with almost periodic coefficients (see Krasnosel'skii, Burd, and Kolesov [1973]).

Let h_j $j = 1, 2, \ldots$ be an arbitrary sequence of real numbers. If $f(t)$ is an almost periodic function, then from the sequence of almost periodic functions

$$f(t + h_j), \quad j = 1, 2, \ldots \tag{8.2}$$

we can choose a sub-sequence that uniformly converges on the real axis. We denote a set of almost periodic functions that involve all functions $f(t + h)$, $-\infty < h < \infty$ and the limits of sub-sequences (8.2) by $H[f(t)]$. A set of almost periodic matrices $H[A(t)]$ is determined similarly.

Theorem 8.1 (Muhamadiev). *The operator*

$$Lx = \frac{dx}{dt} + A(t)x$$

with an almost periodic matrix $A(t)$ is regular if and only if all homogeneous equations

$$\frac{dx}{dt} + A^*(t)x = 0 \quad (A^*(t) \in H[A(t)]),$$

whose only bounded solution on the entire real axis is zero.

We say that the spectrum of an almost periodic matrix $A(t)$ is separated from the imaginary axis if for all $t \in \mathcal{R}$ the matrix eigenvalues lie in a complex plane section defined by the inequality

$$|\Re\lambda| \geq \nu_0 > 0.$$

Theorem 8.2. *If the spectrum of a matrix $A(t)$ is separated from the imaginary axis, then operator (8.1) is uniformly regular for sufficiently small ε.*

Proof. Assume that the operator $L(\varepsilon)$ is not regular for sufficiently small ε. Then, by virtue of Theorem 8.1, there are sequences of the numbers $\varepsilon_j \to 0$ and almost periodic matrices $A_j(t) \in H[A(t)]$ such that each equation

$$\varepsilon_j \frac{dx}{dt} - A_j(t)x = 0 \tag{8.3}$$

has a solution that is normalized of 1 ($||x(t)||_{B_n} = 1$) and bounded for all $t \in \mathcal{R}$. We assume that at some point t_j the inequality $|x(t_j)| \geq 1/2$ holds true. By changing the time $t = \varepsilon_j\tau + t_j$, we transform equation (8.3) into

$$\frac{dy}{d\tau} - A_j(\varepsilon_j\tau + t_j)y = 0. \tag{8.4}$$

The solution of equation (8.4) is the function $y_j(\tau) = x_j(\varepsilon_j\tau + t_j)$, where the norm equals 1, and the derivative is bounded on entire real axis of a

j-independent constant. Therefore, due to the Arzell-Ascoli's theorem, the sequence $y_j(\tau)$ is compact on any finite interval. Without a loss of generality, we can assume that on each finite interval of change in τ, the sequence of the matrices $A_j(\varepsilon_j\tau + t_j)$ uniformly converge to a constant matrix A_0 from the set $H[A(t)]$, and the sequence $y_j(\tau)$ converges to some function $y_0(\tau)$. Evidently, $|y_0(\tau)| = 1$ and $|y_0(0)| \geq 1/2$. The vector-function $y_0(\tau)$ is the solution of the equation

$$\frac{dy}{d\tau} - A_0 y = 0$$

that has no trivial solutions bounded on the real axis, since the matrix A_0 has no eigenvalues lying on the imaginary axis.

The proof presented is due to V.F. Chaplygin [1973a]. The regularity of the operator $L(\varepsilon)$ implies the exponential dichotomy of the solutions of the homogeneous equation

$$\varepsilon\frac{dx}{dt} - A(t)x = 0. \tag{8.5}$$

Let us focus our attention on this problem and describe another proof of Theorem 8.2 . This proof is due to V.A. Coppel [1967, 1968] and K.W. Chang [1968] and is based on the following two Lemmas.

Lemma 8.1. *Let $A(t)$ be a continuously differentiable $n\times n$ matrix-function bounded in norm for all $t(|A(t)| \leq M)$, and let $A(t)$ have k eigenvalues with the real part $\Re\lambda \leq -\mu/2$ and $(n-k)$ eigenvalues with the real part $\Re\lambda \geq \mu/2$. Then there exists a positive constant $\beta = \beta(M,\mu)$ such that if the inequality*

$$|A'(t)| \leq \beta$$

holds for all t, then the system of differential equations

$$\frac{dx}{dt} = A(t)x \tag{8.6}$$

has a fundamental matrix $U(t)$ that obeys the inequalities

$$|U(t)PU^{-1}(s)| \leq Ke^{\frac{-\mu(t-s)}{4}}, \quad t \geq s,$$

$$|U(t)(I-P)U^{-1}(s)| \leq Ke^{\frac{-\mu(s-t)}{4}}, \quad s \geq t, \tag{8.7}$$

where K is a positive constant dependent only on M and μ, and

$$P = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Lemma 8.2. *Let system (8.6) have a fundamental matrix that obeys inequalities (8.7). Then there exists a positive constant $\gamma = \gamma(K,\mu)$ such that* *if $B(t)$ is a continuous matrix and $|B(t) - A(t)| \leq \gamma$ for all t, then the system of equations*

$$\frac{dy}{dt} = B(t)y$$

has a fundamental matrix $Y(t)$ that meets the exponential dichotomy conditions:

$$|Y(t)PY^{-1}(s)| \leq Le^{\frac{-\mu(t-s)}{8}}, \quad t \geq s,$$

$$|Y(t)(I-P)Y^{-1}(s)| \leq Le^{\frac{-\mu(s-t)}{8}}, \quad s \geq t,$$

where L is a positive constant dependent on only K and μ.

Using these two lemmas, it is easy to prove Theorem 8.2. In system (8.1), we change the time $\tau = \varepsilon t$ and obtain the system

$$\frac{dx}{d\tau} = A(\varepsilon\tau)x. \tag{8.8}$$

If the matrix $A(t)$ were differentiable, then the proof of Theorem 8.2 would immediately follow from Lemma 8.1. This matrix is not assumed to be differentiable and it thus is necessary to introduce an additional matrix

$$D_\varepsilon(\tau) = \int_\tau^{\tau+1} A(\varepsilon s)ds$$

and deduce Theorem 8.2 first for the system with the matrix $D_\varepsilon(\tau)$ and then for system (8.8). We emphasize again the importance of the fact that an exponential dichotomy takes place for the solutions of system (8.5) for sufficiently small ε. The space of the solutions $U(\varepsilon)$ can be represented as

$$U(\varepsilon) = U_+(\varepsilon) + U_-(\varepsilon).$$

For the solutions $x_+(t,\varepsilon) \in U_+(\varepsilon)$, the inequality

$$|x_+(t,\varepsilon)| \leq M_+ e^{-\gamma_+ \frac{(t-s)}{\varepsilon}} |x_+(s,\varepsilon)|, \quad -\infty < s < t < \infty$$

holds, and for the solutions $x_-(t,\varepsilon) \in U_-(\varepsilon)$ the inequality

$$|x_-(t,\varepsilon)| \leq M_- e^{-\gamma_- \frac{(t-s)}{\varepsilon}} |x_+(s,\varepsilon)|, \quad -\infty < t < s < \infty$$

is satisfied. Here, $M_+, M_-, \gamma_+, \gamma_-$ are positive constants.

Thus, if the spectrum of the matrix $A(t)$ is separated from zero, then, for sufficiently small ε, the zero solution of system (8.5) is asymptotically stable if all eigenvalues of the matrix $A(t)$ have negative real parts, and unstable if this matrix has eigenvalues with a positive real part.

Consider two differential expressions

$$\frac{dx}{dt} - A(t)y - B(t)y, \tag{8.9}$$

$$\varepsilon\frac{dy}{dt} - C(t)x - D(t)y, \tag{8.10}$$

and, where $A(t), D(t)$ are the square matrices, $B(t), C(t)$ are the rectangular matrices, $\varepsilon > 0$ is a small parameter. The elements of the matrices are the almost periodic functions. The differential expressions determine the operator $K(\varepsilon)$ in the space B_n. We introduce one more operator into consideration

$$K_0 x = \frac{dx}{dt} - \left[A(t) - B(t)D^{-1}(t)C(t)\right] x.$$

The operator $K(\varepsilon)$ is uniformly regular if the operator K_0 is regular and the spectrum of the matrix $D(t)$ is separated from zero. The proof of this assertion is similar to the proof of Theorem 8.2 (see Chaplygin [1973b]).

Consider a system of differential equations

$$\begin{array}{l} \frac{dx}{dt} = A(t)x + B(t)y \\ \varepsilon\frac{dy}{dt} = C(t)x + D(t)y\ . \end{array} \tag{8.11}$$

Theorem 8.3. *Let the zero solution of the system*

$$\frac{dx}{dt} - \left[A(t) - B(t)D^{-1}(t)C(t)\right] x = 0$$

be asymptotically stable. Let the spectrum of the matrix $D(t)$ lie in the left-hand half-plane. Then the zero solution of system (8.11) is asymptotically stable, for sufficiently small ε.

Part II

Averaging of Nonlinear Systems

Chapter 9

Systems in Standard Form. First Approximation

9.1 Problem Statement

Consider a system of differential equations

$$\frac{dx}{dt} = \varepsilon X(t, x), \tag{9.1}$$

where x is an n-dimensional vector, $\varepsilon > 0$ is a small parameter that varies on $(0, \varepsilon_0)$, the vector-function $X(t, x)$ is defined for $t \in \mathcal{R}$, $x \in D$, where D is a bounded set in n-dimensional space. The right-hand sides of system (9.1) are proportional to the small parameter. According to the terminology introduced by N.N. Bogoliubov such systems are called *systems in standard form.* Studies of many applied problems lead to the investigation of systems in standard form.

If there exists a mean value

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, \xi) dt = X_0(\xi), \quad \xi \in D,$$

then for system (9.1) we can write an averaged system

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi). \tag{9.2}$$

Let a system of algebraic equations

$$X_0(\xi) = 0$$

have a solution $\xi = \xi_0$ which, evidently, is a stationary solution of (9.2).

What are the conditions that guarantee that system (9.1) has a solution $x(t, \varepsilon)$ that is close to the solution $\xi = \xi_0$ of the averaged system (9.2)? The answer is given by the following theorem (Bogoliubov [1945]).

9.2 Theorem of Existence. Almost Periodic Case

To simplify the statement of the theorem we assume that ξ_0 is a zero vector.

Theorem 9.1. *Suppose*
1) $X(t,x)$ *is almost periodic in* t *uniformly with respect to* $x \in D$*;*
2) there exists a derivative $A(t) = X_x(t,0)$*, and, for* $t \in \mathcal{R}$, $|x_1|, |x_2| \le r \le a$

$$|X(t,x_1) - X(t,x_2) - A(t)(x_1 - x_2)| \le \omega(r)|x_1 - x_2|,$$

where $\omega(r) \to 0$ *as* $r \to 0$*;*
3) the matrix

$$A = \lim_{T\to\infty} \frac{1}{T} \int_0^T A(s)ds$$

does not have eigenvalues with a zero real part;
4) and

$$\lim_{T\to\infty} \left| \frac{1}{T} \int_0^T X(\sigma,0)d\sigma \right| = 0.$$

.

Then there exist a_0, $\varepsilon_1 > 0$ *such that for* $0 < \varepsilon < \varepsilon_1$ *system (9.1) has a unique almost periodic solution* $x(t,\varepsilon)$ *that lies in* $|x| \le a_0$*, for all* t*, and*

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |x(t,\varepsilon)| = 0.$$

We now make some remarks about the statement of the theorem. Condition 2) of Theorem 9.1 is certainly satisfied if the vector-function $X(t,x)$ has continuous partial derivatives $X_{x_i}(t,x)\,(i = 1,\ldots,n)$ in some neighborhood of $x = 0$, and these derivatives are continuous in x uniformly with respect to $t \in \mathcal{R}$. Condition 4) just means that the averaged system (9.2) has a stationary solution $\xi = 0$. The most significant is condition 3). The matrix of a linear system which appears, as a result of linearization of the averaged system on the stationary solution, does not have eigenvalues with zero real part. From this assumption and the lemma on regularity (see Section 3.2 above) it follows that, for sufficiently small ε, the operator

$$L_\varepsilon x = \frac{dx}{dt} - A(\frac{t}{\varepsilon})x$$

is regular.

We now prove the theorem.

Proof. First, we note that, due to the condition 1) of the theorem, the operator

$$Fx = X(t, x)$$

is defined and is continuous on the ball $||x|| \leq a$ of the space B_n of almost periodic vector-functions, and its image belongs to B_n. Due to conditions 1) and 2), $A(t)$ is an almost periodic matrix.

We write system (9.1) as

$$\frac{dx}{dt} = \varepsilon Ax + \varepsilon \left[A(t) - A\right] x + \varepsilon \left[X(t, x) - A(t)x\right],$$

and introduce a new time $\tau = \varepsilon t$. This yields

$$\frac{dx}{d\tau} = Ax + \left[A(\frac{\tau}{\varepsilon}) - a\right] x + \left[X(\frac{\tau}{\varepsilon}, x) - A(\frac{\tau}{\varepsilon})x\right]. \tag{9.3}$$

The problem of the existence of almost periodic solutions of system (9.3) is equivalent to the problem of the solvability in B_n of the operator equation

$$\begin{aligned} x(\tau) = &\int_{-\infty}^{\infty} G(\tau - \sigma) \left[A(\tfrac{\sigma}{\varepsilon}) - A\right] x(\sigma) d\sigma + \\ + &\int_{-\infty}^{\infty} G(\tau - \sigma) \left[X(\tfrac{\sigma}{\varepsilon}, x(\sigma)) - A(\tfrac{\sigma}{\varepsilon}) x(\sigma)\right] d\sigma \end{aligned} \tag{9.4}$$

where $G(\tau)$ is Green's function for the problem of bounded solutions of the system

$$\frac{dx}{d\tau} = Ax + f(\tau).$$

We now show that the linear operator

$$\Gamma(\varepsilon)h(\tau) = h(\tau) - \int_{-\infty}^{\infty} G(\tau - \sigma) \left[A(\frac{\sigma}{\varepsilon}) - A\right] h(\sigma) d\sigma$$

is continuously invertible in B_n, for sufficiently small ε. Indeed,

$$\Gamma(\varepsilon)h(\tau) = g(\tau),$$

where $g(\tau) \in B_n$, implies that $z(\tau) = h(\tau) - g(\tau)$ is an almost periodic solution of the system

$$\frac{dz}{d\tau} = A(\frac{\tau}{\varepsilon})z - \left[A(\frac{\tau}{\varepsilon}) - A\right] g(\tau).$$

As we have already observed, due to condition 3) and the lemma on regularity we get that, for sufficiently small ε, the operator

$$L_\varepsilon z = \frac{dz}{d\tau} - A(\frac{\tau}{\varepsilon})z$$

is regular. Therefore,

$$||z|| = ||h - g|| = ||L_\varepsilon^{-1}\left[A(\frac{\tau}{\varepsilon}) - A\right] g(\tau)|| \leq ||L_\varepsilon^{-1}|| \cdot K||g||, \tag{9.5}$$

where

$$\sup_{-\infty<\tau<\infty} |A(\frac{\tau}{\varepsilon}) - A| \leq K.$$

Inequality (9.5) implies that

$$||h|| = ||\Gamma^{-1}(\varepsilon)g|| \leq c||g||,$$

where c is some constant. Thus, for sufficiently small ε, all operators $\Gamma(\varepsilon)$ have uniformly bounded inverses $\Gamma^{-1}(\varepsilon)$ $(||\Gamma^{-1}(\varepsilon)|| \leq c)$. This allows us to consider an equivalent operator equation

$$x(\tau) = \Gamma^{-1}(\varepsilon) \int_{-\infty}^{\infty} G(\tau - \sigma) \left[X(\frac{\sigma}{\varepsilon}, x(\sigma)) - A(\frac{\sigma}{\varepsilon})x(\sigma)\right] d\sigma \tag{9.6}$$

instead of (9.4). Due to the condition 2), the operator

$$\Pi(x, \varepsilon) = \Gamma^{-1}(\varepsilon) \int_{-\infty}^{\infty} G(\tau - \sigma) \left[X(\frac{\sigma}{\varepsilon}, x(\sigma)) - A(\frac{\sigma}{\varepsilon})x(\sigma)\right] d\sigma \tag{9.7}$$

satisfies the Lipschitz condition

$$||\Pi(x, \varepsilon) - \Pi(y, \varepsilon)|| \leq cM\omega(r)||x - y|| \quad (||x||, ||y|| \leq r),$$

where M is some constant. This, in particular, implies that

$$||\Pi(x, \varepsilon)|| \leq cM\omega(r)||x|| + ||\Pi(0, \varepsilon)|| \quad (||x|| \leq r).$$

Let's estimate the norm of $\Pi(0, \varepsilon)$:

$$||\Pi(0, \varepsilon)|| = ||\Gamma^{-1}(\varepsilon) \int_{-\infty}^{\infty} G(\tau - \sigma)X(\frac{\sigma}{\varepsilon}, 0)d\sigma|| \leq cp(\varepsilon),$$

where

$$p(\varepsilon) = \sup_{-\infty<\tau<\infty} |\int_{-\infty}^{\infty} G(\tau - \sigma)X(\frac{\sigma}{\varepsilon}, 0)d\sigma|.$$

Condition 4) and Bogoliubov lemma (see Section 2.3) imply that

$$\lim_{\varepsilon \to 0} p(\varepsilon) = 0.$$

We choose ε_1 and a_0 such that, for $0 < \varepsilon < \varepsilon_1$,

$$cM\omega(a_0) = q < 1$$

and

$$||\Pi(0, \varepsilon)|| \le (1 - q)a_0.$$

Note that, for $\varepsilon_n \to 0$, a sequence $a_0(\varepsilon_n)$ can be selected so that $a_0(\varepsilon_n) \to 0$.

Operator (9.7) for $0 < \varepsilon < \varepsilon_1$ on the ball $||x|| \le a_0$ of B_n satisfies the conditions of the Contraction Mapping Theorem. Therefore, operator equation (9.6) has a unique solution $x(t, \varepsilon)$ in this ball.

Thus, the theorem is proved.

If $X(t, x)$ is a correct almost periodic function in t, then Bogoliubov lemma is not required for the proof (Remark 3.1).

We also note that the proof of Theorem 9 stated above is close to the proof outlined in the paper by Zabreiko, Kolesov, and Krasnosel'skii [1969].

9.3 Theorem of Existence. Periodic Case

For systems with periodic coefficients condition 3) of Theorem 9.1 can be relaxed. The regularity of the corresponding operator follows from the absence of a zero eigenvalue of the matrix A. We state, for convenience, an analog of Theorem 9.1 for the periodic case.

Theorem 9.2. *Assume that*
1) $X(t, x)$ is continuous and periodic in t with a period $T > 0$;
2) the vector-function $X(t, x)$ has continuous partial derivatives $X_{x_i}(t, x)\,(i = 1, \ldots, n)$ in some neighborhood of $x = 0$, and these derivatives are continuous in x uniformly with respect to t;
3) $A(t) = X_x(t, 0)$, and the matrix

$$A = \frac{1}{T}\int_0^T A(s)ds$$

does not have a zero eigenvalue;
4)

$$\left|\frac{1}{T}\int_0^T X(\sigma, 0)d\sigma\right| = 0.$$

Then there exist a_0 and $\varepsilon_1 > 0$ such that, for $0 < \varepsilon < \varepsilon_1$, system (9.1) has a unique T-periodic solution $x(t, \varepsilon)$ that lies in $|x| \le a_0$ for all t, and the norm of this solution tends to zero as $\varepsilon \to 0$ uniformly with respect to $t \in [0, T]$.

The proof of Theorem 9.2 essentially repeats the proof of Theorem 9.1. Instead of the lemma on regularity, one needs to use its analog for the periodic case (see Section 3.3). We also note that, in the periodic case, Bogoliubov lemma is not required (see Remark 3.1).

In the periodic case we can estimate the order in ε of the difference between the periodic solution of the exact system and the stationary solution of the averaged system.

We will present another method to prove Theorem 9.2 that employs a "usual" change of variables from the method of averaging. Such changes of variables are necessary for constructing the higher approximations in the method of averaging, and will be described in more detail later.

Consider a system in $\mathcal{R}^n$

$$\frac{dx}{dt} = \varepsilon X_1(t,x) + \varepsilon^2 X_2(t,x,\varepsilon), \tag{9.8}$$

where $\varepsilon > 0$ is a small parameter. Assume the following: vector-function $X_1(t,x)$ is defined for $t \in \mathcal{R}, x \in D \subset \mathcal{R}^n$, and is T-periodic in t; the vector-function $X_2(t,x,\varepsilon)$ is defined for $t \in \mathcal{R}$, $x \in D \subset \mathcal{R}^n$, and small ε, and is T-periodic in t; vector-functions X_1, X_2 are continuous in t and sufficiently smooth in $x \in D$.

In system (9.8) we make a change of variables

$$x = y + \varepsilon u(t,y), \tag{9.9}$$

where $u(t,y)$ is T-periodic in t and is defined as a periodic solution with zero mean value of the system

$$\frac{\partial u}{\partial t} = X_1(t,y) - \Sigma(y), \tag{9.10}$$

where

$$\Sigma(y) = \frac{1}{T}\int_0^T X_1(t,y)dt.$$

Clearly, equation (9.10) determines $u(t,y)$ uniquely.

Performing the change (9.9) yields

$$\frac{dy}{dt} + \varepsilon\frac{\partial u}{\partial t} + \varepsilon\frac{\partial u}{\partial y}\frac{dy}{dt} = \varepsilon X_1(t, y+\varepsilon u) + \varepsilon^2 X_2(t, y+\varepsilon u, \varepsilon).$$

Due to (9.10) we get

$$\left[I + \varepsilon\frac{\partial u}{\partial y}\right]\frac{dy}{dt} = \varepsilon[X_1(t,y+\varepsilon u) - X_1(t,y) + \Sigma(y)] + \varepsilon^2 X_2(t,y+\varepsilon u,\varepsilon).$$

This implies that, for sufficiently small ε,

$$\frac{dy}{dt} = \left[I + \varepsilon\frac{\partial u}{\partial y}\right]^{-1}\{\varepsilon[X_1(t,y+\varepsilon u) - X_1(t,y) + \Sigma(y)] + \varepsilon^2 X_2(t,y+\varepsilon u,\varepsilon)\}.$$

By representing the expression in square brackets as a power series in ε we get

$$\frac{dy}{dt} = \varepsilon\Sigma(y) + \varepsilon^2\left[X_{1x}(t, y)u(t.y) - \frac{\partial u}{\partial y}\Sigma(y)\right] + \left[I + \varepsilon\frac{\partial u}{\partial y}\right]^{-1}\varepsilon^2 X_2(t, y+\varepsilon u, \varepsilon)],$$

that we can write as

$$\frac{dy}{dt} = \varepsilon\Sigma(y) + \varepsilon^2 f(t, y, \varepsilon). \tag{9.11}$$

Theorem 9.2A. *Assume that the right hand sides of system (9.11) satisfy the conditions mentioned above, the averaged system*

$$\frac{d\bar{x}}{dt} = \varepsilon\Sigma(\bar{x}), \tag{9.12}$$

has a stationary solution $x_0 \in D$, i.e., $\Sigma(x_0) = 0$, and the matrix $A = \Sigma'(x_0)$ does not have zero eigenvalues. Then, for sufficiently small ε, there exists a unique solution $y^*(t, \varepsilon)$, which is T-periodic in t, of system (9.11), and

$$y^*(t, \varepsilon) - x_0 = O(\varepsilon).$$

Proof. In system (9.11) we make a change of variables

$$y = x_0 + z$$

to obtain

$$\begin{aligned} &\tfrac{dz}{dt} = \varepsilon\Sigma(x_0 + z) + \varepsilon^2 f(t, x_0 + z, \varepsilon) = \\ &\varepsilon Az + \varepsilon[\Sigma(x_0 + z) - Az] + \varepsilon^2 f(t, x_0 + z, \varepsilon). \end{aligned} \tag{9.13}$$

We replace (9.13) with an equivalent integral equation

$$z = \Pi(z, \varepsilon), \tag{9.14}$$

where

$$\Pi(z, \varepsilon) = \int_{-\infty}^{\infty} \varepsilon G_\varepsilon(t - s)[\Sigma(x_0 + z) - Az + \varepsilon f(s, x_0 + z, \varepsilon)],$$

and $G_\varepsilon(t)$ is Green's function for a problem of bounded solutions for the operator $L_z = \frac{dz}{dt} - \varepsilon Az$. The operator εL_z^{-1} is bounded in the space of periodic vector-functions P_T (see Section 3.3). The vector-function $\Sigma(x_0 + z) - Az$ satisfies the Lipschitz condition for some constant that depends on the norm of z, and tends to zero as this norm goes to zero. Therefore, the operator $\Pi(z.\varepsilon)$ satisfies the conditions of the Contraction Mapping Theorem in the space P_T, for sufficiently small ε, and, hence, there exists a unique periodic solution $x^*(t, \varepsilon)$. We use the method of successive approximations for computing the solutions z^* of the operator equation (9.14). Choosing $z_0 = 0$, yields $|z^*| = O(\varepsilon)$. This implies, under the conditions mentioned above, that system (9.1) has a unique T-periodic solution $x^*(t,\varepsilon)$, and $x^*(t,\varepsilon) = x_0 + O(\varepsilon)$.

Note that we could have assumed that the vector-functions $X_1(t,x)$ and $X_2(t,x,\varepsilon)$ have a finite number of simple discontinuities (jumps) in t on the period. After the first step of the method of successive approximations applied to the operator equation (9.14), we get a vector-function that is continuous in t.

9.4 Investigation of the Stability of an Almost Periodic Solution

We shall assume that the conditions of Theorem 9.1 are satisfied. We denote by $x_0(t,\varepsilon)$ the almost periodic solution of system (9.1) that exists due to the Theorem 9. We will study the connection between the stability of the solution $x_0(t,\varepsilon)$ and the stability of the trivial solution of system (9.2).

Theorem 9.3. *Assume that the conditions of Theorem 9.1 are satisfied, and the vector-function $X(t,x)$ has continuous partial derivatives $X_{x_i}(t,x)$ $(i = 1,\dots,n)$ in some neighborhood of $x = 0$, and that these are continuous in x uniformly with respect to $t \in \mathcal{R}$.*

Then

1) If all eigenvalues of the matrix A have negative real parts, then, for sufficiently small ε, the solution $x_0(t,\varepsilon)$ of system (9.1) is asymptotically stable.

2) If A has at least one eigenvalue with a positive real part, then, the solution $x_0(t,\varepsilon)$, for sufficiently small ε, is unstable.

3) If in the case 1) $\psi(t,t_0,x_0)\,(\psi(t_0,t_0,x_0))$ is a solution of the averaged system (9.2) that lies (along with its ρ-neighborhood) in the domain of attraction of the solution $x = 0$, then for any α, $(0 < \alpha < \rho)$ there exist numbers $\varepsilon_1(\alpha)\,(0 < \varepsilon_1 < \varepsilon_0)$ and $\beta(\alpha)$, such that for all $0 < \varepsilon < \varepsilon_1$ the solution $\varphi(t,t_0,\xi_0)$ of system (9.1) that lies in the domain of attraction of the solution $x_0(t,\varepsilon)$, for which $|x_0 - \xi_0| < \beta$, satisfies

$$|\varphi(t,t_0,\xi_0) - \psi(t,t_0,x_0)| < \alpha, \quad t \geq t_0.$$

Proof. In system (9.1) we make a change of variables

$$x = x_0(t,\varepsilon) + y$$

to get

$$\frac{dy}{dt} = \varepsilon X(t,x_0 + y) - \varepsilon X(t,x_0). \tag{9.15}$$

The problem of the stability of the solution $x_0(t,\varepsilon)$ of system (9.1) reduces to the problem of the stability of the trivial solution of system (9.15). We rewrite system (9.15) in the following form

$$\frac{dy}{dt} = \varepsilon A_1(t,\varepsilon)y + \varepsilon\omega(t,y), \tag{9.16}$$

where $A_1(t,\varepsilon) = X_x(t, x_0(t,\varepsilon))$ and

$$\omega(t,y) = X(x, x_0(t,\varepsilon)+y) - X(t, x_0(t,\varepsilon)) - A_1(t,\varepsilon)y.$$

Evidently, $\omega(t,0) \equiv 0$. The conditions of the theorem imply that the matrix $A_1(t,\varepsilon)$ is almost periodic in t uniformly with respect to ε,

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |A_1(t,\varepsilon) - A(t)| = 0, \tag{9.17}$$

and the vector-function $\omega(t,y)$ satisfies

$$|\omega(t,y_1) - \omega(t,y_2)| \le p(r)|y_1 - y_2|, \quad (|y_1|, |y_2| \le r), \tag{9.18}$$

where $\lim_{\varepsilon\to 0} p(r) = 0$. We now make a change of time $\tau = \varepsilon t$ in system (9.16). We obtain

$$\frac{dy}{d\tau} = A_1(\frac{\tau}{\varepsilon},\varepsilon)y + \omega(\frac{\tau}{\varepsilon},y),$$

that we rewrite as

$$\frac{dy}{d\tau} = A(\frac{\tau}{\varepsilon})y + [A_1(\frac{\tau}{\varepsilon},\varepsilon) - A(\frac{\tau}{\varepsilon})]y + \omega(\frac{\tau}{\varepsilon},y). \tag{9.19}$$

In system (9.19) we make a change of variables

$$y(\tau) = z(\tau) + H(\tau,\varepsilon)z(\tau),$$

where

$$H(\tau,\varepsilon) = \int_{-\infty}^{\infty} G_0(\tau - s)\left[A(\frac{s}{\varepsilon}) - A_0\right] ds,$$

and $G_0(\tau)$ is Green's function of the problem of bounded solutions of the system

$$\frac{dx}{d\tau} = Ax + f(\tau).$$

After the last change, system (9.19) takes the form

$$\frac{dz}{d\tau} = Az + D(\tau,\varepsilon)z + g(\frac{\tau}{\varepsilon},z), \tag{9.20}$$

where

$$D(\tau,\varepsilon) = \begin{array}{l}(I + H(\tau,\varepsilon))^{-1}[-H(\tau,\varepsilon)A - AH(\tau,\varepsilon)+ \\ A_1(\frac{\tau}{\varepsilon},\varepsilon)H(\tau,\varepsilon) + A_1(\frac{\tau}{\varepsilon},\varepsilon) - A(\frac{\tau}{\varepsilon})],\end{array}$$

$$g(\frac{\tau}{\varepsilon}, z) = (I + H(\tau, \varepsilon))^{-1} \omega(\frac{\tau}{\varepsilon}, (I + H(\tau, \varepsilon)) z).$$

Due to Bogoliubov lemma we have that

$$\lim_{\varepsilon \to 0} \sup_{-\infty < \tau < \infty} |H(\tau, \varepsilon)| = 0$$

which together with (9.17) implies that

$$\lim_{\varepsilon \to 0} \sup_{-\infty < \tau < \infty} |D(\tau, \varepsilon)| = 0.$$

Further, the vector-function $g(\frac{\tau}{\varepsilon}, z)$ satisfies the Lipschitz condition (9.18) with some function $q(r)$ for which

$$\lim_{r \to 0} q(r) = 0.$$

Clearly, the investigation of the problem of the stability, for sufficiently small ε, of the trivial solutions of systems (9.19) and (9.20) are equivalent. The rest of the proof is, in essence, the same as the proof of theorems on the stability in the first approximation (see Appendix B).

First, let all eigenvalues of the matrix A have negative real parts. Each solution $z(\tau)$ of system (9.20) is also a solution of the following system of integral equations

$$z(\tau) = e^{\tau A} z(0) + \int_0^\tau e^{(\tau - s)A} D(s, \varepsilon) z(s) ds + \int_0^\tau e^{(\tau - s)A} g(\frac{s}{\varepsilon}, z(s)) ds.$$

The matrix $e^{\tau A}$ satisfies

$$|e^{\tau A}| \le M_1 e^{-\gamma_1 \tau}, \quad \tau \ge 0,$$

where M_1, γ_1 are positive constants. Therefore, for $\tau \ge 0$

$$\begin{aligned} |z(\tau)| \le M_1 e^{-\gamma_1 \tau} |z(0)| + p(\varepsilon) \int_0^\tau M_1 e^{-\gamma_1 (\tau - s)} |z(s)| ds + \\ + \int_0^\tau M_1 e^{-\gamma_1 (\tau - s)} |g(\tfrac{s}{\varepsilon}, z(s))| ds, \end{aligned} \tag{9.21}$$

where

$$p(\varepsilon) = \sup_{-\infty < \tau < \infty} |D(s, \varepsilon)|, \quad \lim_{\varepsilon \to 0} p(\varepsilon) = 0.$$

Assume that we are given $\eta > 0$. We can assume that η is so small that

$$\beta = M_1 p(\eta) + M_1 q(\eta) \le \frac{\gamma_1}{2}.$$

We let $\delta = \eta/(1 + M_1)$ and show that $|z(0)| < \delta$ implies $|z(\tau)| < \eta$ for $\tau > 0$. Indeed, if τ_0 is the first instant for which $|z(\tau)| = \eta$, then (9.21) implies, taking into consideration

$$|g(\frac{s}{\varepsilon}, z(s))| \le q(\eta) |z(s)| \quad (|z(s)| \le \eta),$$

that, for $0 \le \tau \le \tau_0$,

$$|z(\tau)| \le M_1 e^{-\gamma_1 \tau}|z(0)| + \beta \int_0^\tau e^{-\gamma_1(\tau - s)}|z(s)|ds. \tag{9.22}$$

Due to the Gronwall-Bellman lemma,

$$\eta < M_1 e^{-(\gamma_1 - \beta)\tau_0}|z(0)|,$$

which, in turn, implies that

$$\eta < M_1 e^{-\frac{\gamma_1}{2}\tau_0} \frac{\eta}{1 + M_1} < \eta.$$

We obtained a contradiction.

Therefore, (9.22) is true for all $\tau > 0$. Also, the Gronwall-Bellman lemma implies that

$$|z(\tau)| \le M_1 e^{-\frac{\gamma_1}{2}\tau}|z(0)| \quad (\tau \ge 0,\, |z(0)| < \delta).$$

The first assertion of the theorem is proved.

We shall now assume that matrix A has eigenvalues with positive real part. Without a loss of generality we can assume that the matrix A can be represented in a block-diagonal form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where the matrix A_1 of order k has all eigenvalue with negative real parts, and the matrix A_2 of order $(n-k)$ has all eigenvalues with positive real parts. Matrices $e^{\tau A_1}$ and $e^{\tau A_2}$ yield the estimates

$$\left|e^{\tau A_1}\right| \le M_1 e^{-\gamma_1 \tau},\ \tau \ge 0, \quad \left|e^{\tau A_2}\right| \le M_2 e^{-\gamma_2 \tau},\ \tau \le 0,$$

where M_1, M_2, γ_1, and γ_2 are positive constants.

To prove the instability, it is sufficient to establish the existence of $r_0 > 0$, such that in any neighborhood of the origin there is an initial condition $z(0)$ of some solution $z(\tau)$, $(0 \le \tau < \infty)$ of system (9.13), that is not contained completely in the ball $|z| \le r_0$, for sufficiently small ε. The results described in Section 3.3 imply that solutions $z(\tau)$, which are bounded on $[0, \infty)$, of system (9.20) are the same as the solutions of the system of nonlinear integral equations

$$z(\tau) = Tz(\tau), \tag{9.23}$$

where

$$Tz(\tau) = \begin{pmatrix} e^{\tau A_1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} +$$

$$+\int_0^\tau \begin{pmatrix} e^{(\tau-s)A_1} & 0 \\ 0 & 0 \end{pmatrix} \left(D(s,\varepsilon)z(s) + g(\frac{s}{\varepsilon}, z(s)) \right) ds-$$

$$-\int_\tau^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{(\tau-s)A_2} \end{pmatrix} \left(D(s,\varepsilon)z(s) + g(\frac{s}{\varepsilon}, z(s)) \right) ds$$

Let $z^1(\tau)$ and $z^2(\tau)$ be two such solutions, which are bounded on $[0,\infty)$, of system (9.23), with $z^1_1(0) = z^2_1(0)$, i.e., the first k components of their vectors of initial conditions are the same, and $|z^1(\tau)|, |z^2(\tau)| \le r_0$ for $\tau \ge 0$. Then we get

$$|z^1(\tau) - z^2(\tau)| = |Tz^1(\tau) - Tz^2(\tau)| \le$$

$$\le M_1 \int_0^\tau [p(\varepsilon) + q(r_0)] e^{-\gamma_1(\tau-s)} |z^1(s) - z^2(s)| ds$$

$$+M_2[p(\varepsilon) + q(r_0)] \int_\tau^\infty e^{\gamma_2(\tau-s)} |z^1(s) - z^2(s)| ds.$$

Therefore,

$$|z^1(\tau) - z^2(\tau)| \le \frac{M_1}{\gamma_1}[p(\varepsilon) + q(r_0)] \left(1 - e^{-\gamma\tau}\right) \sup_{0\le s<\infty} |z^1(s) - z^2(s)| +$$

$$+\frac{M_2}{\gamma_2}[p(\varepsilon) + q(r_0)] \sup_{0\le s<\infty} |z^1(s) - z^2(s)| \le$$

$$\le \left(\frac{M_1}{\gamma_1} + \frac{M_2}{\gamma_2}\right) [p(\varepsilon) + q(r_0)] \sup_{0\le s<\infty} |z^1(s) - z^2(s)|.$$

We choose ε and r_0 so that

$$\left(\frac{M_1}{\gamma_1} + \frac{M_2}{\gamma_2}\right) [p(\varepsilon) + q(r_0)] < \frac{1}{2}.$$

Then

$$|z^1(\tau) - z^2(\tau)| < \frac{1}{2} \sup_{0\le s<\infty} |z^1(s) - z^2(s)|.$$

Therefore, $z^1(\tau) \equiv z^2(\tau)$.

Thus, the intersection of the ball $|z(0)| \le r_0$ with each hyperplane $z = z_1(0) + v$, where v is a vector whose first k components are zero and the last $(n-k)$ components are arbitrary, contains no more than one point from which the solutions that lie in this ball emanate. In other words, for "almost all" initial conditions from any neighborhood of the origin, there are corresponding solutions that leave the ball $|z| \le r_0$ for some $\tau > 0$. Therefore, the trivial solution of system (9.20), for sufficiently small ε, is unstable.

The assertion 3) follows from Theorem 13.5 (see Remark 13.6).

Thus, the theorem is proved.

9.5 More General Dependence on a Parameter

We consider a system that is more general than (9.1)

$$\frac{dx}{dt} = \varepsilon X(t, x.\varepsilon), \tag{9.24}$$

and the corresponding averaged system

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi), \tag{9.25}$$

where

$$X_0(\xi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(s, \xi, 0) ds.$$

If we would assume that the vector-function $X(t, x, \varepsilon)$ is continuous in ε uniformly with respect to $t \in \mathcal{R}$ and $x \in D$ (D is some bounded set in $\mathcal{R}^n$), and is almost periodic in t uniformly with respect to x and ε, then Theorems 9.1 and 9.3 are applicable to systems (9.24) and (9.25). In particular, one system that satisfies such an assumption is

$$\frac{dx}{dt} = \varepsilon X(t, x) + \varepsilon^2 X_1(t, x, \varepsilon), \tag{9.26}$$

where $X(t, x)$ satisfies the conditions of Theorems 9.1 and 9.3, $X_1(t, x, \varepsilon)$ is almost periodic in t uniformly with respect to x and ε, and is continuous in all of its variables.

The proofs of the corresponding theorems for system (9.24) can be carried out using the same approaches as in the proofs of Theorems 9.1 and 9.3. It is convenient to state this result as a theorem.

Theorem 9.4. *Assume that*
1) $X(t, x, \varepsilon)$ is almost periodic in t uniformly with respect to x and ε, and $X(t, x, \varepsilon)$ is continuous in ε uniformly with respect to $t \in \mathcal{R}$, $x \in D$;
2) the vector-function $X(t, x, \varepsilon)$ has continuous partial derivatives $X_{x_i}(t, x, \varepsilon)$ $(i = 1, \ldots, n)$ in some neighborhood of $x = 0$, and these derivatives are continuous in x uniformly with respect to ε and $t \in \mathcal{R}$. We let $A(t) = X_x(t, 0, 0)$;
3) the matrix

$$A = \lim_{T\to\infty} \frac{1}{T} \int_0^T A(s) ds$$

has no eigenvalues with a zero real part;
4)

$$\lim_{T\to\infty} \left| \frac{1}{T} \int_0^T X(\sigma, 0, 0) d\sigma \right| = 0.$$

Then there exist a_0 and $\varepsilon_1 > 0$, such that for $0 < \varepsilon < \varepsilon_1$ system (9.24) has a unique almost periodic solution $x(t, \varepsilon)$ that lies in the ball $|x| \le a_0$ for all t, and

$$\lim_{\varepsilon \to 0} \sup_{-\infty < t < \infty} |x(t, \varepsilon)| = 0.$$

a) If all eigenvalues of A have negative real parts, then, for sufficiently small ε, and the solution $x(t, \varepsilon)$ of system (9.17) is asymptotically stable.
b) If matrix A has at least one eigenvalue with a positive real part, then the solution $x(t, \varepsilon)$, for sufficiently small ε, is unstable.
c) If in the case a) $\psi(t, t_0, x_0)$ $(\psi(t_0, t_0, x_0) = x_0)$ is a solution of the averaged system (9.25) that along with its ρ-neighborhood lies in the domain of attraction of the solution $x = 0$, then for any α, $(0 < \alpha < \rho)$ there exist $\varepsilon_1(\alpha)$, $(0 < \varepsilon_1 < \varepsilon_0)$, and $\beta(\alpha)$, such that for any $0 < \varepsilon < \varepsilon_1$ a solution $\varphi(t, t_0, \xi_0)$ of system (9.24), which lies in the domain of attraction of the solution $x(t, \varepsilon)$, and, for which $|x_0 - \xi_0| < \beta$, satisfies

$$|\varphi(t, t_0, \xi_0) - \psi(t, t_0, x_0)| < \alpha, \quad t \ge t_0.$$

9.6 Almost Periodic Solutions of Quasi-Linear Systems

Consider a system of differential equations

$$\frac{dx}{dt} = Ax + \varepsilon f(t, x, \varepsilon), \quad x \in \mathcal{R}^n, \tag{9.27}$$

where $\varepsilon > 0$ is a small parameter, A is a constant $n \times n$ matrix, the vector-function $f(t, x, \varepsilon)$ is almost periodic in t uniformly with respect to ε and $x \in D$, and is sufficiently smooth in x and ε. Here D is some bounded set in the space $\mathcal{R}^n$. System (9.27) is called a *quasi-linear* system. For $\varepsilon = 0$ it becomes a linear system with constant coefficients.

If the matrix A does not have eigenvalues with a zero real part, then system (9.27) (see Biryuk [1954a]) has a unique almost periodic solution, for sufficiently small ε. A proof of this fact can be easily carried out using the following scheme. We replace system (9.27) with a system of integral equations

$$x(t) = \varepsilon \int_{-\infty}^{\infty} G(t - s) f(s, x(s), \varepsilon) ds = \Pi(x, \varepsilon), \tag{9.28}$$

where $G(t)$ is Green's function of the almost periodic boundary value problem for the system

$$\frac{dx}{dt} = Ax. \tag{9.29}$$

The operator $\Pi(x, \varepsilon)$ acts in the space B_n of almost periodic vector-functions. If $f(t, x, \varepsilon)$ satisfies the Lipschitz condition in x, then, for sufficiently small ε, the operator $\Pi(x, \varepsilon)$ is a contraction and is invariant on some ball in B_n. Therefore, the Contraction Mapping Theorem implies the existence of a unique solution in this ball.

We now consider the case when all eigenvalues of the matrix A have a zero real part. We shall assume there exist n eigenvectors corresponding to the eigenvalues of A. In other words, we can select n linearly independent solutions of the system (9.29) so that they are periodic vector-functions. Then, the general solution of system (9.29) is an almost periodic vector-function in the following form

$$x(t) = e^{tA}c,$$

where c is a constant vector. The above assumption means that the Jordan canonical form of the matrix A is a diagonal matrix composed of its eigenvalues. We shall call such matrix A **semi-simple**. We make a change of variables in the system (9.27)

$$x(t) = e^{tA}y(t) \tag{9.30}$$

and obtain

$$\frac{dy}{dt} = \varepsilon e^{-tA} f(t, e^{tA}y, \varepsilon). \tag{9.31}$$

System (9.31) is a system in standard form with almost periodic coefficients, so it can be averaged. Theorem 9.4 can be applied to system (9.31). Due to Theorem 9.4 for a stationary solution of the averaged system there is, for sufficiently small ε, a corresponding almost periodic solution $y(t, \varepsilon)$ of system (9.31). Then an almost periodic solution of system (9.27) is defined by (9.30).

We now assume that the matrix A has $k < n$ eigenvalues with zero real parts, and $(n - k)$ eigenvalues with non-zero real parts. Again, we shall assume that for each eigenvalue with zero real part and multiplicity m_0 exist m_0 linearly independent corresponding eigenvectors.

In system (9.27) we make a change

$$x = Py, \tag{9.32}$$

where P is a constant matrix, such that matrix $P^{-1}AP$ has the block-diagonal form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Here A_1 is a $k \times k$ matrix whose eigenvalues have zero real parts, A_2 is a $(n-k) \times (n-k)$ matrix whose eigenvalues have non-zero real parts. We denote by y_1 the first k coordinates of the vector y, and by y_2 the last $(n - k)$ of its coordinates. Then after a change of variables (9.32) system (9.27) becomes

$$\begin{aligned} \tfrac{dy_1}{dt} &= A_1 y_1 + \varepsilon f_1(t, Py_1, Py_2, \varepsilon), \\ \tfrac{dy_2}{dt} &= A_2 y_2 + \varepsilon f_2(t, Py_1, Py_2, \varepsilon), \end{aligned} \tag{9.33}$$

where f_1 are the first k coordinates of the vector f, and f_2 are the last $(n-k)$ of its coordinates.

We now make a change

$$y_1 = e^{tA_1} z_1 \tag{9.34}$$

that yields

$$\begin{aligned} \tfrac{dz_1}{dt} &= \varepsilon g_1(t, z_1, y_2, \varepsilon), \\ \tfrac{dy_2}{dt} &= A_2 y_2 + \varepsilon g_2(t, z_1, y_2, \varepsilon), \end{aligned} \tag{9.35}$$

where

$$g_1(t, z_1, y_2, \varepsilon) = e^{-tA_1} f_1(t, Pe^{tA_1} z_1, Py_2, \varepsilon)$$

and

$$g_2(t, z_1, y_2, \varepsilon) = f_2(t, Pe^{tA_1} z_1, Py_2, \varepsilon).$$

For system (9.35) we write the averaged system

$$\frac{d\xi}{dt} = \varepsilon \Sigma(\xi), \tag{9.36}$$

where

$$\Sigma(\xi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T g_1(t, \xi, 0, 0) dt.$$

Evidently, system (9.36) is k-dimensional.

Suppose that system (9.36) has a stationary solution $\xi = \xi_0$, i.e., ξ_0 is a solution of the system of algebraic equations

$$\Sigma(\xi) = 0.$$

We would like to find out under what conditions an almost periodic solution of (9.35) that corresponds to the stationary solution of the averaged system exists? This problem has been considered by Malkin [1954]. Here, we state a more general result.

For the sake of simplicity we shall assume that ξ_0 is a zero-vector. Then, the vector-function $g_1(t, 0, 0, 0)$ has zero mean value, i.e.,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g_1(t, 0, 0, 0) dt = 0.$$

We introduce a matrix

$$A_0(t) = \frac{\partial g_1(t, u, 0, 0)}{\partial u}\Big|_{u=0},$$

and a matrix A_0 composed of its mean values

$$A_0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T A_0(s) ds = \Sigma_\xi(0).$$

Theorem 9.5. *Assume that the matrix A_1 is semi-simple and all of its eigenvalues have zero real parts, the eigenvalues of the matrix A_2 have non-zero real parts, and $g_1(t,0,0,0)$ is a correct almost periodic vector-function. We also assume that the eigenvalues of the matrix A_0 have non-zero real parts.*

Then, for sufficiently small ε, system (9.27) has an almost periodic solution $x_0(t,\varepsilon)$ that becomes an almost periodic solution of the linear system

$$\frac{dx}{dt} = Ax$$

when $\varepsilon = 0$. If all eigenvalues of matrices A_2 and A_0 have negative real parts, then the solution $x_0(t,\varepsilon)$ is asymptotically stable, for sufficiently small ε. If either of the matrices A_2 or A_0 has an eigenvalue with a positive real part, then the solution $x_0(t,\varepsilon)$ is unstable, for sufficiently small ε.

Proof. We make a change of variables in system (9.35)

$$z_1 = u + \varepsilon a(t), \quad z_2 = \varepsilon v, \tag{9.37}$$

where $a(t)$ is an almost periodic vector-function that has a zero mean value and satisfies

$$\frac{da}{dt} = g_1(t,0,0,0).$$

Almost-periodicity of $a(t)$ follows from the fact that $g_1(t,0,0,0)$ is a correct almost periodic vector-function. We note that $f(t,x,\varepsilon)$ is sufficiently smooth in x and ε. Making the change (9.37), and writing the right-hand sides of the system as a power series in ε yields

$$\begin{aligned} \tfrac{du}{dt} &= \varepsilon[g_1(t,u,0,0) - g_1(t,0,0,0) + \varepsilon U(t,u,v,\varepsilon)], \\ \tfrac{dv}{dt} &= A_2 v + [g_2(t,u,0,0) + \varepsilon V(t,u,v,\varepsilon)]. \end{aligned} \tag{9.38}$$

From system (9.38) we move to the system of operator equations in the space B_n. We rewrite system (9.38) in the following form

$$\begin{aligned} &\tfrac{du}{dt} = \varepsilon[A_0 u + (A_0(t) - A_0)u + (g_1(t,u,0,0) - g_1(t,0,0,0) - A(t)u) + \\ &+\varepsilon U(t,u,v,\varepsilon)], \\ &\tfrac{dv}{dt} = A_2 v + [g_2(t,0,0,0) + (g_2(t,u,0,0) - g_2(t,0,0,0) + \varepsilon V(t,u,v,\varepsilon)]. \end{aligned} \tag{9.39}$$

Due to the conditions of the theorem, the operator

$$L_1 u = \frac{du}{dt} - \varepsilon A_0 u$$

has a continuous inverse in the space B_k, and the operator

$$L_2 v = \frac{dv}{dt} - A_2 v$$

has a continuous inverse in the space B_{n-k}.

Let's estimate the norm of the operator L_1^{-1}. The equality

$$L_1^{-1}f = \int\limits_{-\infty}^{\infty} G_{1\varepsilon}(t-s)f(s)ds$$

implies

$$\begin{aligned}&||L_1^{-1}f|| = \sup\nolimits_{-\infty<t<\infty}|\int\limits_{-\infty}^{\infty} G_{1\varepsilon}(t-s)f(s)ds| \leq \\ &\leq \sup\nolimits_{-\infty<t<\infty}\int\limits_{-\infty}^{\infty}|G_{1\varepsilon}(t-s)|ds||f|| \leq \\ &\leq \int\limits_{-\infty}^{\infty} M_1 e^{-\gamma_1\varepsilon|t-s|}ds||f|| \leq \tfrac{M_1}{\varepsilon\gamma_1}||f||.\end{aligned}$$

Therefore,

$$||L_1^{-1}|| \leq \frac{M_1}{\varepsilon\gamma_1}, \quad M_1, \gamma_1 > 0.$$

The problem of the existence of almost periodic solutions of system (9.39) is equivalent to the problem of the existence of solutions of the system of operator equations

$$\begin{aligned}&u = \int\limits_{-\infty}^{\infty} G_{1\varepsilon}(t-s)\varepsilon[(A_0(s)-A_0)u + (g_1(s,u,0,0) - g_1(s,0,0,0) - A_0(s)u) + \\ &+\varepsilon U(s,u,v,\varepsilon)]ds, \\ &v = \int\limits_{-\infty}^{\infty} G_2(t-s)[g_2(s,0,0,0) + (g_2(s,u,0,0) - \\ &-g_2(s,0,0,0) + \varepsilon V(s,u,v,\varepsilon)]ds.\end{aligned} \tag{9.40}$$

where $G_{1\varepsilon}(t)$ and $G_2(t)$ are Green's functions of the problems of bounded solutions for the operators L_1 and L_2, respectively. We show that the linear operator

$$\Gamma(\varepsilon)h = h - \int\limits_{-\infty}^{\infty} G_1(\varepsilon(t-s))\varepsilon[(A_0(s)-A_0)]h(s)ds$$

is continuously invertible in B_k, for sufficiently small ε. Consider an operator equation in B_k

$$\Gamma(\varepsilon)h = g, \quad g \in B_k.$$

This implies that the vector-function $z(t) = h(t) - g(t)$ is an almost periodic solution of the system of differential equations

$$\frac{dz}{dt} = \varepsilon A_0 z + \varepsilon[(A_0(t) - A_0)]g(t).$$

The regularity of the operator L_1 implies

$$||z|| = ||h-g|| = ||L_1^{-1}\varepsilon[(A_0(t)-A_0)]g|| \leq \frac{M_1}{\gamma_1}K||g||,$$

where

$$\sup_{-\infty<t<\infty} |A_0(t) - A_0| = K.$$

From the last inequality follows

$$||h|| = |||\Gamma^{-1}(\varepsilon)g|| \leq c||g||,$$

where

$$c = 1 + \frac{M_1}{\gamma_1}K.$$

Thus, operators $\Gamma(\varepsilon)$, for sufficiently small ε, have uniformly bounded inverses

$$||\Gamma^{-1}(\varepsilon)|| \leq c.$$

We write (9.40) as

$$\begin{aligned} &u = \Gamma^{-1}(\varepsilon)\varepsilon \int_{-\infty}^{\infty} G_{1\varepsilon}(t-s))[(g_1(t,u,0,0) - g_1(t,0,0,0) - A_0(t)u)+ \\ &+\varepsilon U(t,u,v,\varepsilon)], \\ &v = \int_{-\infty}^{\infty} G_2(t-s)[g_2(s,0,0,0) + [g_2(t,0,0,0) + (g_2(t,u,0,0) \\ &-g_2(t,0,0,0) + \varepsilon V(t,u,v,\varepsilon)]. \end{aligned} \tag{9.41}$$

The smoothness of a vector-function $f(t,x,\varepsilon)$ implies the following. The vector-function $\omega(t,u) = g_1(t,u,0,0) - g_1(t,0,0,0) - A_0(t)u$ satisfies

$$|\omega(t,u_1) - \omega(t,u_2)| \leq p(r)|u_1 - u_2|, \quad |u_1|, |u_2| \leq r,$$

where $p(r) \to 0$ as $r \to 0$. The vector-functions $U(u,v,\varepsilon)$, $V(u,v,\varepsilon)$ are bounded in a norm in $\mathcal{R}^n$:

$$|U(u,v,\varepsilon)| \leq M_1, \quad |V(u,v,\varepsilon)| \leq M_2$$

and satisfy the Lipschitz condition in u,v:

$$\begin{aligned} |U(u_1,v_1,\varepsilon) - U(u_2,v_2,\varepsilon)| \leq K_1[|u_1 - u_2| + |v_1 - v_2|], \\ |V(u_1,v_1,\varepsilon) - V(u_2,v_2,\varepsilon)| \leq K_2[|u_1 - u_2| + |v_1 - v_2|]. \end{aligned}$$

The vector-function $\zeta(t,u) = g_2(t,u,0,0) - g_2(t,0,0)$ satisfies the Lipschitz condition

$$|\zeta(t,u_1) - \zeta(t,u_2)| \leq K_3|u_1 - u_2|.$$

Let's put $||L_2^{-1}|| = M_0$. Using the method of successive approximations we build sequences u_p and v_p that converge to a solution of the system (9.41). Namely, as the zero-th approximations we choose

$$u_0 = 0, \quad v_0 = L_2^{-1}g_2(t,0,0,0).$$

Further, for $p \geq 1$, we choose

$$u_p = \varepsilon\Gamma^{-1}(\varepsilon)[(g_1(t,u_p,0,0) - g_1(t,0,0,0) - A_0(t)u_p)+$$
$$+\varepsilon U(t,u_{p-1},v_{p-1},\varepsilon)],$$
$$v_p = L_2^{-1}[g_2(t,0,0,0) + (g_2(t,u_p,0,0) - g_2(t,0,0,0)+$$
$$+\varepsilon V(t,u_{p-1},v_{p-1},\varepsilon)].$$

Let's show that u_p and v_p remain for all p in $||u|| \leq a(\varepsilon)$, $||v - v_0|| \leq b(\varepsilon)$, where $a(\varepsilon)$, $b(\varepsilon)$ are some functions of ε which tend to zero as $\varepsilon \to 0$. We estimate the norms of u_p and v_p:

$$||u_p|| \leq c[p(r)||u_p|| + \varepsilon M_1],$$
$$||v_p - v_0|| \leq M_0[K_3||u_p|| + \varepsilon M_2].$$

We choose r so that $cp(r) < 1/2$. Then we get

$$||u_p|| \leq \frac{\varepsilon c M_1}{1 - cp(r)} < 2\varepsilon c M_1$$

and

$$||v_p - v_0|| < \varepsilon M_0[2cK_3M_1 + M_2].$$

These estimates imply the existence of $a(\varepsilon)$ and $b(\varepsilon)$. We now estimate the norms $||u_{p+1} - u_p||$ and $||v_{p+1} - v_p||$ to get

$$||u_{p+1} - u_p|| \leq c[p(r)||u_{p+1} - u_p|| + \varepsilon K_1(||u_p - u_{p-1}|| + ||v_p - v_{p-1}||)],$$
$$||v_{p+1} - v_p|| \leq M_0[K_3||u_{p+1} - u_p|| + \varepsilon K_2(||u_p - u_{p-1}|| + ||v_p - v_{p-1}||)].$$

If $||u_p - u_{p-1}|| \leq a_p$ and $||v_p - v_{p-1}|| \leq b_p$, then

$$||u_{p+1} - u_p|| \leq c\frac{\varepsilon K_1(a_p + b_p)}{1 - cp(r)} < 2c\varepsilon K_1(a_p + b_p)$$

and

$$||v_{p+1} - v_p|| < \varepsilon[(2M_0K_3cK_1 + K_2)(a_p + b_p)]||.$$

This implies that the sequences u_p and v_p converge in a norm of B_n to an almost periodic solution of system (9.41). Therefore, the existence of an almost periodic solution of system (9.27) is proved.

The proof of the second part of the theorem's assertion (about the stability of an almost periodic solution) is quite similar to the proof of stability in the first approximation (see Appendix B), so we omit it.

9.7 Systems with Fast and Slow Time

Consider a system

$$\frac{dx}{dt} = \varepsilon X(t, \tau, x), \tag{9.42}$$

where $x \in \mathcal{R}^n$, $\varepsilon > 0$ is a small parameter, and $\tau = \varepsilon t$ is a slow time. We shall assume that a vector-function $X(t, \tau, x)$ is almost periodic in the fast time t uniformly with respect to τ and x, is periodic in the slow time τ with a constant period T, and is continuous in x uniformly with respect to both t and τ. For system (9.42) we write the system, which is averaged in the fast time t,

$$\frac{dy}{dt} = \varepsilon Y(\tau, y), \tag{9.43}$$

where

$$Y(\tau, y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, \tau, y)dt.$$

After the change to the slow time, system (9.43) becomes a nonlinear system with periodic coefficients

$$\frac{dy}{d\tau} = Y(\tau, y). \tag{9.44}$$

Let system (9.44) have a periodic solution $y_0(\tau)$.

For system (9.42) one can state theorems that are similar to Theorems 9.1 and 9.3. The main condition is concerned with the linear system with periodic coefficients

$$\frac{dy}{d\tau} = A(\tau)y, \tag{9.45}$$

where $A(\tau) = Y'_y(\tau, y_0(\tau))$. If the operator

$$Ly = \frac{dy}{d\tau} - A(\tau)y$$

is regular in the space P_T, or, in other terms (see for example Roseau [1966]), the characteristic exponent of system (9.45) has non-zero real parts, then the system has a unique almost periodic solution $x(t, \varepsilon)$, for sufficiently small ε, and

$$\lim_{\varepsilon\to 0} ||x(t, \varepsilon) - y_0(\tau)|| = 0.$$

The stability properties of this solution are the same as the stability properties of the trivial solution of system (9.45).

We now give precise statements of these results.

The vector-function $X(t, \tau, x)$ is defined for $t, \tau \in \mathcal{R}$, $x \in D$, where D is a bounded set in n-dimensional space. In system (9.42) we make a change to the slow time τ to obtain

$$\frac{dx}{d\tau} = X(\frac{\tau}{\varepsilon}, \tau, x). \tag{9.46}$$

In system (9.46) we make a change $x(\tau) = z(\tau) + y_0(\tau)$ to get

$$\frac{dz}{d\tau} = X(\frac{\tau}{\varepsilon}, \tau, z + y_0(\tau)) - Y(\tau, y_0(\tau) = Z(\frac{\tau}{\varepsilon}, \tau, z).$$

Now, the periodic system averaged in the fast time has a trivial solution. It will be more convenient to state a theorem of the existence for an almost periodic solution that is close to the trivial solution of the averaged system. Therefore, we shall assume that system (9.43) has a trivial solution.

Theorem 9.6. *Assume that*
1) $X(t,\tau,x)$ is almost periodic in t uniformly with respect to $\tau \in \mathcal{R}$ and $x \in D$, and, is periodic in τ with a constant period T
2) System (9.43) has a trivial solution
3) There exists a derivative $A(t,\tau) = X_x(t,\tau,0)$
4) For $t,\tau \in \mathcal{R}$,$|x_1|$, $|x_2| \leq r \leq a$, the inequality

$$|X(t,\tau,x_1) - X(t,\tau,x_2) - A(t,\tau)(x_1 - x_2)| \leq \omega(r)|x_1 - x_2|$$

holds, where $\omega(r) \to 0$ as $r \to 0$;
5) The linear system with periodic coefficients

$$\frac{dy}{d\tau} = A(\tau)y,$$

where

$$A(\tau) = \lim_{T\to\infty} \frac{1}{T}\int\limits_0^T A(t,\tau)dt,$$

does not have characteristic exponents with zero real parts.

Then there exist a_0 and $\varepsilon_1 > 0$, such that for $0 < \varepsilon < \varepsilon_1$ system (9.42) has a unique solution $x(t,\varepsilon)$ that lies in the ball $|x| \leq a_0$ for all t, and

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |x(t,\varepsilon)| = 0.$$

Proof. We only present the main steps of the proof since it is similar to the proof of Theorem 9.1.

We note that, due to condition 1) of the theorem, the operator

$$Fx = X(x,\tau,x)$$

is defined and is continuous on the ball $||x|| \leq a$ in the space B_n of almost periodic vector-functions, and its values belong to B_n. Due to the conditions 1) and 2), the matrix $A(t,\tau)$ is almost periodic in t and periodic in τ with a period T.

We write system (9.42) as

$$\frac{dx}{d\tau} = A(\tau)x + \Big[A(\frac{\tau}{\varepsilon},\tau) - A(\tau)\Big]\,x + \Big[X(\frac{\tau}{\varepsilon},\tau,x) - A(\frac{\tau}{\varepsilon},\tau)x\Big].$$

The problem of the existence of almost periodic solutions of this system is equivalent to the solvability in the space B_n of the operator equation

$$\begin{aligned} x(\tau) = &\int_{-\infty}^{\infty} G(\tau,\sigma)\left[A(\tfrac{\sigma}{\varepsilon},\sigma) - A(\sigma)\right] x(\sigma)d\sigma + \\ &+ \int_{-\infty}^{\infty} G(\tau,\sigma)\left[X(\tfrac{\sigma}{\varepsilon},\sigma,x(\sigma)) - A(\tfrac{\sigma}{\varepsilon},\sigma)x(\sigma)\right] d\sigma \end{aligned} \tag{9.47}$$

where $G(\tau, s)$ is Green's function of the problem of the existence of T-periodic solutions of the system

$$\frac{dx}{d\tau} = A(\tau)x + f(\tau). \tag{9.48}$$

Here $f(\tau)$ is a T-periodic vector-function. Due to condition 4) of the theorem, system (9.48) has a unique periodic solution that can be represented as

$$x(\tau) = \int_{-\infty}^{\infty} G(\tau, s)f(s)ds.$$

Green's function $G(\tau, s)$ satisfies

$$|G(\tau, s)| \le Me^{-\gamma|\tau - s|},$$

where M,γ are positive constants.

Similarly to the proof of Theorem 9.1 we can show that the linear operator

$$\Gamma(\varepsilon)h(\tau) = h(\tau) - \int_{-\infty}^{\infty} G(\tau,\sigma)\left[A(\frac{\sigma}{\varepsilon},\sigma) - A(\sigma)\right] h(\sigma)d\sigma$$

is continuously invertible in B_n, for sufficiently small ε. To prove this statement instead of the lemma on regularity one has to utilize a generalized lemma on regularity (see Section 6.1).

Therefore, we can replace operator equation (9.47) with the following equivalent equation

$$x(\tau) = \Pi(x,\varepsilon) \equiv \Gamma^{-1}(\varepsilon) \int_{-\infty}^{\infty} G(\tau,\sigma)\left[X(\frac{\sigma}{\varepsilon},\sigma,z(\sigma)) - A(\frac{\sigma}{\varepsilon},\sigma)z(\sigma)\right] d\sigma. \tag{9.49}$$

The last step in the proof of the theorem deals with the following. We have to establish that operator $\Pi(x, \varepsilon)$ for $0 < \varepsilon < \varepsilon_1$ in the ball $||x|| \le a_0$ in B_n satisfies the Contraction Mapping Theorem. Condition 3) of the theorem implies that the operator $\Pi(x, \varepsilon)$ is a contraction. Due to the estimate on the norm $\Pi(0, \varepsilon)$ we get that the operator $\Pi(x, \varepsilon)$ maps a ball in B_n of sufficiently small radius onto itself.

We shall devote the remainder of this section to investigation of the stability of an almost periodic solution.

We suppose that the conditions of Theorem 9.6 are met. Denote by $x_0(t,\varepsilon)$ the almost periodic solution of system (9.42) that exists. We would like to study the stability of the solution $x_0(t,\varepsilon)$ depending on the stability of the trivial solution of system (9.47).

Theorem 9.7. *Assume that the conditions of Theorem 9.6 are met, the vector-function $X(t,\tau,x)$ has continuous partial derivatives $X_{x_i}(t,\tau.x)$, $(i = 1,\dots,n)$ in some neighborhood of $x=0$, and these derivatives are continuous in x uniformly with respect to $t,\tau \in \mathcal{R}$.*

Then, the following statements hold.

1) If all the characteristic exponents of system (9.45) have a negative real part, then, for sufficiently small ε, the solution $x_0(t,\varepsilon)$ of system (9.42) is asymptotically stable.

2) If, however, system (9.45) has at least one characteristic exponent with a positive real part, then the solution $x_0(t,\varepsilon)$ is unstable, for sufficiently small ε.

3) Assume that in case 1) $\psi(t,t_0,x_0,\varepsilon)$ is a solution of the averaged system (9.45) that lies along with its ρ-neighborhood ($\rho > 0$) in the domain of attraction of the solution $y = 0$. Then, for any $\alpha\,(0 < \alpha < \rho)$ there exist $\varepsilon_1(\alpha)$, which satisfies $0 < \varepsilon_1 < \varepsilon_0$, and $\beta(\alpha)$, such that, for all $0 < \varepsilon < \varepsilon_1$ a solution $\varphi(t,t_0,\xi_0,\varepsilon)$ of system (9.44) which lies in the domain of attraction of the solution $x_0(t,\varepsilon)$ for which $|x_0 - \xi_0| < \beta$, satisfies

$$|\varphi(t,t_0,\xi_0,\varepsilon) - \psi(t,t_0,x_0,\varepsilon)| < \alpha, \quad t \geq t_0.$$

Proof. In system (9.42) we make a change of variables

$$x = x_0(t,\varepsilon) + y$$

to obtain

$$\frac{dy}{dt} = \varepsilon X(t,\tau,x_0 + y) - \varepsilon X(t,\tau,x_0). \tag{9.50}$$

The problem of the stability of the solution $x_0(t,\varepsilon)$ of system (9.42) leads to the problem of the stability of the trivial solution of system (9.50). We write system (9.50) as

$$\frac{dy}{dt} = \varepsilon A_1(t,\tau,\varepsilon)y + \varepsilon\omega(t,\tau,y), \tag{9.51}$$

where

$$A_1(t,\tau,\varepsilon) = X_x(t,\tau,x_0(t,\tau,\varepsilon)),$$

and

$$\omega(t,\tau,y) = X(x,x_0(t,\tau,\varepsilon) + y) - X(t,\tau,x_0(t,\varepsilon)) - A_1(t,\tau,\varepsilon)y.$$

Evidently, $\omega(t,0) \equiv 0$. The conditions of the theorem imply that the matrix $A_1(t,\tau,\varepsilon)$ is almost periodic in t uniformly with respect to τ and ε, and

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t<\infty} |A_1(t,\tau,\varepsilon) - A(t,\tau)| = 0.$$

The rest of the proof of assertions 1) and 2) is the same as the proof of Theorem 9.3, and, in essence, is just a proof of the Theorem on the Stability in the First Approximation. This, however, utilizes a generalized Bogoliubov lemma and a generalization of the lemma on stability (see Section 6.1). Assertion 3) follows from Theorem 13.7 (see Remark 13.8).

We note that some less general results were obtained by Sethna [1967] and Roseau [1969, 1970] (see also Hale [1969]). These authors considered systems in the form

$$\frac{dx}{dt} = \varepsilon f(t,x,\varepsilon) + \varepsilon g(\varepsilon t, x, \varepsilon),$$

where $f(t,x,\tau,\varepsilon)$ is almost periodic in t and $g(\varepsilon t,x,\varepsilon)$ is periodic in εt with period T.

We now describe some generalizations of the obtained results.

Consider a more general system than (9.42)

$$\frac{dx}{dt} = \varepsilon X(t,\tau,x,\varepsilon), \quad \tau = \varepsilon t. \tag{9.52}$$

For system (9.52) we write the corresponding averaged system

$$\frac{d\xi}{dt} = \varepsilon X_0(\tau,\xi), \tag{9.53}$$

where

$$X_0(\tau,\xi) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(\tau,s,\xi,0)ds.$$

If we suppose that the vector-function $X(t,\tau,x,\varepsilon)$ is continuous in ε uniformly with respect to $t,\tau \in \mathcal{R}$, and $x \in D$, is almost periodic in t uniformly with respect to τ,x, and ε, and is periodic in τ with a constant period, then, the results of Theorems 9.6 and 9.7 apply to the system (9.52). The proofs can be carried out using the same ideas.

Finally, we note, that the approach considered above can be used for studying a system

$$\frac{dx}{dt} = \varepsilon X(t,\tau,x), \quad \tau = \varepsilon t,$$

where $X(t,\tau,x)$ is almost periodic in t uniformly with respect to τ and x and is almost periodic in τ uniformly with respect to t and x. In this case the system averaged in fast time,

$$\frac{dy}{d\tau} = Y(\tau, y),$$

is a system with almost periodic coefficients.

9.8 One Class of Singularly Perturbed Systems

Consider a system of differential equations

$$\frac{dx}{dt} = \varepsilon X(t,\tau,x,\varepsilon), \quad x \in \mathcal{R}^n, \tag{9.54}$$

where $\varepsilon > 0$ is a small parameter, and $\tau = \varepsilon^2 t$ is a slow time. System (9.54) with time τ is a singularly perturbed system. We shall assume that $X(t,\tau,x,\varepsilon)$ is almost periodic in t and τ uniformly with respect to the remaining variables, has continuous partial derivatives in x_i, $i = 1,2,\ldots,n$ in some bounded domain $D \subset \mathcal{R}^n$, and is continuous in ε uniformly with respect to the remaining variables. For system (9.54) we write a system, averaged in fast time t,

$$\frac{d\bar{x}}{dt} = \varepsilon \bar{X}(\tau, \bar{x}), \tag{9.55}$$

where

$$\bar{X}(\tau,\bar{x}) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X(t,\tau,\bar{x},0)dt.$$

The averaged system (9.55) is a singularly perturbed system with almost periodic coefficients and the time τ:

$$\varepsilon\frac{d\bar{x}}{dt} = \bar{X}(\tau,\bar{x}).$$

The following theorem was obtained by Ukhalov [1997].

Theorem 9.8. *Assume the following two conditions*
1) There exists a differentiable almost periodic function $x_0(\tau)$, such that

$$\bar{X}(\tau, x_0(\tau)) \equiv 0$$

for $\tau \in \mathcal{R}$.
2) The spectrum of the matrix

$$A(\tau) = \bar{X}_x(\tau, x_0(\tau))$$

is strictly separated from zero; i.e., all of its eigenvalues satisfy

$$|\Re\lambda(\tau)| \geq \gamma_0 > 0, \quad \tau \in \mathcal{R}.$$

Then we can find a_0 *and* ε_0*, such that, for* $0 < \varepsilon < \varepsilon_0$*, system (9.54) has a unique almost periodic solution* $x_0(t, \varepsilon)$*, which lies in the ball* $||x - x_0(\tau)|| \leq a_0$ *in* B_n*, and*

$$\lim_{\varepsilon \to 0} ||x_0(t, \varepsilon) - x_0(\tau)|| = 0.$$

The solution $x(t, \varepsilon)$ *is asymptotically stable if all eigenvalues of the matrix* $A(\tau)$ *have negative real parts. If* $A(\tau)$ *has at least one eigenvalue with a positive real part, the solution* $x(t, \varepsilon)$ *is unstable.*

Proof. The proof can be carried out using the same approach as in the proofs of Theorems 9.1 and 9.3. Here, we only mention the basic steps of the proof.

Clearly, it is sufficient to prove the existence and the stability of an almost periodic solution of system (9.54) for the case $x_0(\tau) \equiv 0$. We can always end up in this case after making a change of variables $x = x_0(\tau) + z$. Thus, we shall assume that $x_0(\tau) \equiv 0$. Then we get $\bar{X}(\tau, 0) \equiv 0$. Since $X(t, \tau, x, \varepsilon)$ is continuously differentiable in x in some ball, there exists a matrix

$$A(t, \tau, \varepsilon) = X_x(t, \tau, 0, \varepsilon),$$

such that for $t \in \mathcal{R}$, $\varepsilon \in (0, \varepsilon^*)$, $|x_1|, |x_2| \leq r$ the following inequality holds

$$|X(t, \tau, x_1, \varepsilon) - X(t, \tau, x_2, \varepsilon) - A(t, \tau, \varepsilon)(x_1 - x_2)| \leq \omega(r, \varepsilon)|x_1 - x_2|, \quad (9.56)$$

where $\omega(r, \varepsilon) \to 0$ as $r \to 0$ for any $\varepsilon \in (0, \varepsilon^*)$. We set $A(t, \tau) = A(t, \tau, 0)$ and

$$A(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T A(t, \tau) dt, \qquad (A(\tau) = \bar{X}_x(\tau, 0)).$$

First, we carry out the proof of the existence of an almost periodic solution. We rewrite system (9.54) in the following form (after we make a preliminary change of time $\tau = \varepsilon^2 t$)

$$\begin{aligned} &\varepsilon \tfrac{dx}{d\tau} = A(\tau)x + \left[A\left(\tfrac{\tau}{\varepsilon^2}, \tau\right) - A(\tau)\right] x + \left[X\left(\tfrac{\tau}{\varepsilon^2}, \tau, x, 0\right) - A\left(\tfrac{\tau}{\varepsilon^2}, \tau\right) x\right] + \\ &+ \left[X\left(\tfrac{\tau}{\varepsilon^2}, \tau, x, \varepsilon\right) - X\left(\tfrac{\tau}{\varepsilon^2}, \tau, x, 0\right)\right]. \end{aligned} \quad (9.57)$$

Due to the conditions of Theorems 9.8 and 8.2, the operator

$$L_\varepsilon x = \varepsilon \frac{dx}{d\tau} - A(\tau)x$$

is uniformly regular in B_n, for sufficiently small ε. Therefore, there exists an inverse operator L_ε^{-1}, $||L_\varepsilon^{-1}|| \leq N$ for $0 < \varepsilon < \varepsilon_1$. The problem of the existence of almost periodic solutions of system (9.57) is equivalent to the problem of the solvability in B_n of the operator equation

$$x = L_\varepsilon^{-1}\left\{\left[A\left(\tfrac{\tau}{\varepsilon^2},\tau\right) - A(\tau)\right]x + \left[X\left(\tfrac{\tau}{\varepsilon^2},\tau,x,0\right) - A\left(\tfrac{\tau}{\varepsilon^2},\tau\right)x\right] + \right.$$
$$\left. + \left[X\left(\tfrac{\tau}{\varepsilon^2},\tau,x,\varepsilon\right) - X\left(\tfrac{\tau}{\varepsilon^2},\tau,x,0\right)\right]\right\}.$$

Further, we show that the operator

$$\Gamma_\varepsilon h(\tau) = h(\tau) - L_\varepsilon^{-1}\left[A\left(\frac{\tau}{\varepsilon^2},\tau\right) - A(\tau)\right]h(\tau)$$

is continuously invertible in B_n, for sufficiently small ε. The problem leads to the problem of the existence and the uniqueness of the solution in B_n of the operator equation $\Gamma_\varepsilon h(\tau) = g(\tau)$, where $g(\tau) \in B_n$. This problem, in turn, leads to the problem of the regularity of the operator

$$K_\varepsilon y = \varepsilon\frac{dy}{d\tau} - A\left(\frac{\tau}{\varepsilon^2},\tau\right)y.$$

The proof of the regularity of the operator K_ε can be accomplished using the same approach as in the proof of the lemma on regularity (see Section 3.2). The proof uses the following analog of Bogoliubov lemma. For a system of equations

$$\varepsilon\frac{dy}{d\tau} = A(\tau)y + f(\left(\frac{\tau}{\varepsilon^2},\tau\right),$$

where $A(\tau)$ is an almost periodic matrix whose spectrum is strictly separated from the imaginary axis, the function $f(s,\tau)$ is almost periodic in s uniformly with respect to τ, is almost periodic in τ uniformly with respect to s, and,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(s,\tau)ds = 0,$$

there exists a unique almost periodic solution $y(t,\varepsilon)$ that satisfies

$$\lim_{\varepsilon\to 0}||y(\tau,\varepsilon)|| = 0.$$

Proof of this statement is quite similar to the proof of Bogoliubov lemma. After showing the continuous invertibility of the operator Γ_ε we come to the operator equation

$$x(\tau) = \Pi(x,\varepsilon), \tag{9.58}$$

where

$$\Pi(x,\varepsilon) = \Gamma_\varepsilon^{-1}L_\varepsilon^{-1}\left\{\left[X\left(\tfrac{\tau}{\varepsilon^2},\tau,x,0\right) - A\left(\tfrac{\tau}{\varepsilon^2},\tau\right)x\right] + \right.$$
$$\left. + \left[X\left(\tfrac{\tau}{\varepsilon^2},\tau,x,\varepsilon\right) - X\left(\tfrac{\tau}{\varepsilon^2},\tau,x,0\right)\right]\right\}.$$

Using the assumptions of the theorem (hence inequality (9.56)), similarly to the proof of Theorem 9.1, we establish that the operator $\Pi(x,\varepsilon)$ is, for sufficiently small ε, a contraction that maps the ball $||x|| \le a(\varepsilon)$ from B_n into itself. Therefore, the operator equation (9.58) has a unique solution in this ball. This solution is an almost periodic solution of system (9.54). The first assertion of the theorem is proved.

We now consider the problem of the stability of the solution $x_0(t,\varepsilon)$. Make the change of variables

$$x = x_0(t,\varepsilon) + y$$

in system (9.54) to get

$$\frac{dy}{dt} = \varepsilon[X(t,\tau,x_0+y,\varepsilon) - X(t,\tau,x_0,\varepsilon)]. \tag{9.59}$$

The problem of the stability of the almost periodic solution $x_0(t,\varepsilon)$ leads to the problem of the stability of the trivial solution of system (9.59). We write system (9.59) as

$$\frac{dy}{dt} = \varepsilon A_1(t,\tau,\varepsilon)y + \varepsilon P(t,\tau,y,\varepsilon), \tag{9.60}$$

where

$$A_1(t,\tau,\varepsilon) = X_x(t,\tau,x_0(t,\varepsilon),\varepsilon),$$
$$P(t,\tau,y,\varepsilon) = X(t,\tau,x_0+y,\varepsilon) - X(t,\tau,x_0,\varepsilon) - A_1(t,\tau,\varepsilon)y.$$

Clearly, $P(t,\tau,0,\varepsilon) \equiv 0$. The conditions of the theorem imply that the matrix $A_1(t,\tau,\varepsilon)$ is almost periodic in t uniformly with respect to τ, ε and is almost periodic in τ uniformly with respect to t and ε. Moreover,

$$\lim_{\varepsilon\to 0} \sup_{-\infty<t,\tau<\infty} |A_1(t,\tau,\varepsilon) - A(t,\tau)| = 0.$$

We rewrite (9.60) in the following form

$$\frac{dy}{dt} = \varepsilon A(t,\tau)y + \varepsilon[A_1(t,\tau,\varepsilon) - A(t,\tau)]y + \varepsilon P(t,\tau,y,\varepsilon), \tag{9.61}$$

and make a change of time $s = \varepsilon t$ in system (9.61) to get

$$\frac{dy}{ds} = A(\frac{s}{\varepsilon},\varepsilon s)y + [A_1(\frac{s}{\varepsilon},\varepsilon s,\varepsilon) - A(\frac{s}{\varepsilon},\varepsilon s)]y + P(\frac{s}{\varepsilon},\varepsilon s,y,\varepsilon). \tag{9.62}$$

In system (9.62) we make a change

$$y = [I + H(s,\varepsilon)]z,$$

where $H(s,\varepsilon)$ is an almost periodic matrix defined as an almost periodic solution of the matrix system

$$\frac{dH}{ds} = A(\varepsilon s)H + A(\frac{s}{\varepsilon},\varepsilon s) - A(\varepsilon s).$$

Thus, we obtain a system

$$\frac{dz}{ds} = A(\varepsilon s)z + D(s,\varepsilon)z + g(s,z,\varepsilon), \tag{9.63}$$

where matrix $D(s,\varepsilon)$ and function $g(s,z,\varepsilon)$ satisfy the following conditions

$$\lim_{\varepsilon\to 0} ||D(s,\varepsilon)|| = 0, \quad g(s,0,\varepsilon) \equiv 0.$$

Moreover, the function $g(s,z,\varepsilon)$ satisfies the Lipschitz condition in z with a constant $b(r)$ that tends to zero as $r \to 0$. The proof of the stability and the instability of the trivial solution of system (9.63) is almost exactly the same as the proof of the theorems on stability in the first approximation.

Chapter 10

Systems in the Standard Form. First Examples

10.1 Dynamics of Selection of Genetic Population in a Varying Environment

As an example, we consider dynamics of a selection of a Mendelian population with a genetic pool made up of only two alleles of a single gene. Denote these alleles by A and a, whereas the fitness of genotypes AA, Aa, aa at the moment t by

$$1-\varepsilon\alpha(t), \quad 1, \quad 1-\varepsilon\beta(t),$$

respectively. Here, $\varepsilon > 0$ is a small parameter, $\alpha(t)$, $\beta(t)$ are the periodic functions with the period ω and positive mean values α_0, β_0, respectively. Let $p_n(t)$, $q_n(t)$ be the frequencies of the alleles A and a in the generation n, respectively. The evolution equation has the form (see, for instance, Ewens [1979]).

$$\frac{dp_n}{dt} = \varepsilon p_n(1-p_n)\frac{\beta(t)-(\alpha(t)+\beta(t))p_n}{1-\varepsilon[(\alpha(t)+\beta(t))p_n^2+2\beta(t)p_n-\beta(t)]}. \tag{10.1}$$

Equation (10.1) is an equation in the standard form with periodic coefficients.

The averaged equation

$$\frac{d\bar{p}_n}{dt} = \varepsilon\bar{p}_n(1-\bar{p}_n)(\beta_0-(\alpha_0+\beta_0)\bar{p}_n) \tag{10.2}$$

has three equilibria

$$\bar{p_1}=0, \quad \bar{p_2}=1, \quad \bar{p_3}=\frac{\beta_0}{\alpha_0+\beta_0}.$$

The solutions $\bar{p_1}$, $\bar{p_2}$ are unstable. The solution $\bar{p_3}$ is asymptotically stable. Its domain of attraction is the interval $(0,1)$.

Theorem 9.4 implies that equation (10.1), for sufficiently small ε, has two unstable stationary solutions $p=0, p=1$, and an asymptotically stable solution with the period ω in the neighborhood of the solution $\bar{p_3}$ of the averaged equation.

Let $p_n(t,0,x_0)$ be the solution of (10.1) and $\bar{p}_n(t,0,\xi_0)$ be the solution of (10.2), where $\xi_0 > 0$, $\xi_0 \neq \frac{\beta_0}{\alpha_0+\beta_0}$. It follows from assertion c) of Theorem 9.4 that for any $\delta > 0$, there exists $\eta(\delta)$ such that for the solution $p_n(t,0,x_0)$ of equation (10.1) with the initial conditions meeting the inequality

$$|x_0 - \xi_0| < \eta(\delta),$$

the following inequality holds true

$$|p_n(t,0,x_0) - \bar{p}_n(t,0,\xi_0| < \delta, \quad t \geq 0.$$

Note that the averaged equation is integrable (as an equation with separable variables).

Now, let the mean values of α_0, β_0 be negative. Then the stationary solution $\bar{p_3}$ of the averaged equation is unstable. Equation (10.1), for sufficiently small ε, has a unique unstable periodic solution. The stationary solutions $\bar{p} = 0, \bar{p} = 1$ of equation (10.2) are asymptotically stable. Their domains of attraction lie within the intervals $(0, \frac{\beta_0}{\alpha_0+\beta_0})$, $(\frac{\beta_0}{\alpha_0+\beta_0}, 1)$, respectively.

For sufficiently small ε, the stationary solutions $p = 0, p = 1$ of equation (10.1) is asymptotically stable. If $\bar{p}_n(t,0,\xi_0)$ is the solution of (10.2) and $0 < \xi_0 < \frac{\beta_0}{\alpha_0+\beta_0}$, then for any $\delta > 0$ there exists $\eta(\delta)$ such that the inequality

$$|p_n(t,0,x_0) - \bar{p}_n(t,0,\xi_0| < \delta \quad t \geq 0$$

holds true. Here, $p_n(t,0,x_0)$ is the solution of (10.1) with the initial condition meeting the inequality

$$|x_0 - \xi_0| < \eta(\delta).$$

The same assertion holds when $\frac{\beta_0}{\alpha_0+\beta_0} < \xi_0 < 1$.

10.2 Periodic Oscillations of Quasi-Linear Autonomous Systems with One Degree of Freedom and the Van der Pol Oscillator

Consider a quasi-linear autonomous equation

$$\ddot{x} + k^2 x = \varepsilon f(x, \dot{x}, \varepsilon), \tag{10.3}$$

where $\varepsilon > 0$ is a small parameter, k is a real constant, $f(x,\dot{x},\varepsilon)$ is a function that is sufficiently smooth in x, $\dot{x}$, ε, and bounded in some limited domain of $x, \dot{x}, \varepsilon$. Transform equation (10.3) into the system of two equations

$$\dot{x} = y, \quad \dot{y} = -k^2 x + \varepsilon f(x, \dot{x}, \varepsilon) \tag{10.4}$$

and change over to the polar coordinates:

$$x = \rho \cos k\theta, \quad y = -k\rho \sin k\theta. \tag{10.5}$$

We obtain the system

$$\begin{aligned} \tfrac{d\rho}{dt} &= -\tfrac{\varepsilon}{k} f(\rho \cos k\theta, -k\rho \sin k\theta, \varepsilon) \sin k\theta = \varepsilon H_1(\rho, \theta, \varepsilon), \\ \tfrac{d\theta}{dt} &= 1 - \varepsilon \tfrac{f(\rho \cos k\theta, -k\rho \sin k\theta, \varepsilon) \cos k\theta}{k^2 \rho} = 1 + \varepsilon H_2(\rho, \theta, \varepsilon). \end{aligned} \tag{10.6}$$

We divide the first equation of system (10.6) by the second one and obtain the first-order differential equation

$$\frac{d\rho}{d\theta} = \varepsilon \frac{H_1(\rho, \theta, \varepsilon)}{1 + \varepsilon H_2(\rho, \theta, \varepsilon)} = H_1(\rho, \theta, 0) + O(\varepsilon) = \varepsilon H(\rho, \theta, \varepsilon). \tag{10.7}$$

Equation (10.7) is an equation in the standard form. Besides, the right-hand side of (10.7) is periodic in θ with the period $2\pi/k$. Let us compare equation (10.7) with the equation averaged over θ

$$\frac{d\bar{\rho}}{d\theta} = \varepsilon H_0(\bar{\rho}), \tag{10.8}$$

where

$$H_0(\bar{\rho}) = \frac{k}{2\pi} \int_0^{\frac{2\pi}{k}} H_1(\bar{\rho}, \theta, 0) d\theta = -\frac{k}{2\pi} \int_0^{\frac{2\pi}{k}} f(\bar{\rho} \cos k\theta, -k\bar{\rho} \sin k\theta, 0) \sin k\theta d\theta =$$

$$-\frac{1}{2\pi} \int_0^{2\pi} f(\bar{\rho} \cos u, -k\bar{\rho} \sin u, 0) \sin u du.$$

By virtue of Theorem 9.4, each stationary solution $\bar{\rho} = \rho_0$ of the equation

$$H_0(\bar{\rho}) = 0, \quad H_{0\bar{\rho}}(\rho_0) \neq 0 \tag{10.9}$$

corresponds to a $2\pi/k$-periodic solution of equation (10.7); this solution is asymptotically stable if

$$H_{0\bar{\rho}}(\rho_0) < 0, \tag{10.10}$$

and unstable if

$$H_{0\bar{\rho}}(\rho_0) > 0. \tag{10.11}$$

Let $\rho^*(\theta, \varepsilon)$ be a $2\pi/k$-periodic solution of equation (10.7) that corresponds to the stationary solution ρ_0 of the averaged equation. We pass on again to the variable t and consider the functions

$$x^*(t, \varepsilon) = \rho^*(\theta, \varepsilon) \cos k\theta, \quad \dot{x}^*(t, \varepsilon) = -k\rho^*(\theta, \varepsilon) \sin k\theta.$$

Let us show that the obtained functions will also be periodic with the period that depends on the parameter ε and initial conditions. We turn our attention to the second equation of system (10.6) that defines θ as a function of t. Assume that t and θ simultaneously vanish; in this case

$$t(\theta) = \int_0^\theta \frac{d\theta}{1 + \varepsilon H_2(\rho, \theta, \varepsilon)}.$$

Hence we obtain that

$$t(\theta + 2\pi/k) - t(\theta) = \int_\theta^{\theta+2\pi/k} \frac{d\theta}{1 + \varepsilon H_2(\rho, \theta, \varepsilon)}. \tag{10.12}$$

Since $\rho^*(\theta, \varepsilon)$ is a $2\pi/k$-periodic function, we see that the derivative of integral (10.12) equals zero. Therefore,

$$t(\theta + 2\pi/k) - t(\theta) = T(\varepsilon). \tag{10.13}$$

The period $T(\varepsilon)$ depends only on ε and the initial condition. Relation (10.13) shows that as t changes by $T(\varepsilon)$, the value of θ varies by $2\pi/k$, and, thus, the values $x^*(t, \varepsilon)$ and $\dot{x}^*(t, \varepsilon)$ are unaltered. Therefore, $x^*(t, \varepsilon)$ and $\dot{x}^*(t, \varepsilon)$ are the periodic functions with the period $T(\varepsilon)$. We have chosen $\theta(0) = 0$. Thus, $x^*(0, \varepsilon) = \rho^*(0, \varepsilon)$, $\dot{x}^*(0, \varepsilon) = 0$.

Consequently, given the satisfied condition (10.9), system (10.3), for sufficiently small ε, has the periodic solution $x^*(t, \varepsilon)$ with the period $T(\varepsilon)$. At $\varepsilon = 0$ this solution transforms into a periodic solution $\varphi(t) = \rho_0 \cos kt$ of the equation $\ddot{x} + k^2 x = 0$. The periodic solution $x^*(t, \varepsilon)$ can be found as series in terms of powers of ε; at that

$$x^*(t, \varepsilon) = \varphi(t) + O(\varepsilon). \tag{10.14}$$

We can also seek the period $T(\varepsilon)$ of the solution $x^*(t, \varepsilon)$ as the series

$$T(\varepsilon) = \frac{2\pi}{k}(1 + O(\varepsilon)).$$

Notice that the function $x^*(t + h, \varepsilon)$ at any real h is also a solution of equation (10.3). Hence, by the autonomy of equation (10.3), there exists a one-parameter family of periodic solutions.

We are now coming to the problem of the stability of the solution $x^*(t, \varepsilon)$. Linearizing equation (10.3) on the periodic solution $x^*(t, \varepsilon)$, we obtain a linear differential equation with the periodic coefficients

$$\ddot{y} + \left[k^2 - \varepsilon \frac{\partial f(x^*, \dot{x}^*, \varepsilon)}{\partial x^*}\right] y - \varepsilon \frac{\partial f(x^*, \dot{x}^*, \varepsilon)}{\partial \dot{x}^*} \dot{y} = 0. \tag{10.15}$$

Show that equation (10.15) has a periodic solution. Since $x^*(t,\varepsilon)$ is a periodic solution of equation (10.3), we have an identity

$$\ddot{x}^* + k^2 x^* \equiv \varepsilon f(x^*, \dot{x}^*, \varepsilon).$$

Differentiating this identity with respect to t yields

$$\frac{d^2}{dt^2}\left(\frac{dx^*}{dt}\right) + \left[k^2 - \varepsilon\frac{\partial f(x^*, \dot{x}^*, \varepsilon)}{\partial x^*}\right]\frac{dx^*}{dt} - \varepsilon\frac{\partial f(x^*, \dot{x}^*, \varepsilon)}{\partial \dot{x}^*}\frac{d}{dt}\left(\frac{dx^*}{dt}\right) \equiv 0.$$

Comparing the latter identity and equation (10.15) implies that equation (10.15) has the periodic solution $\dot{x}^*$.

To investigate the problem of the stability of a zero solution of equation (10.15), we employ the theory of second-order equations with periodic coefficients (see Coddington and Levinson [1955]).

Consider an equation

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0, \tag{10.16}$$

where $p(t)$, $q(t)$ are the periodic functions with the period ω. According to the Floquet Theory, stability of a zero solution of equation (10.16) is determined by eigenvalues of a monodromy matrix. Let us calculate the monodromy matrix. To do this, we take the linearly independent solutions $y_1(t)$, $y_2(t)$ of equation (10.16) with the initial conditions

$$y_1(0) = 1,\ \dot{y}_1(0) = 0,\ y_2(0) = 0,\ \dot{y}_2(0) = 1.$$

Along with the functions $y_1(t)$, $y_2(t)$, the functions $y_1(t+\omega)$, $y_2(t+\omega)$ are the solutions of equation (10.16). The latter functions are the linear combinations of the functions $y_1(t)$, $y_2(t)$:

$$\begin{aligned} y_1(t+\omega) &= a_1 y_1(t) + a_2 y_2(t),\\ y_2(t+\omega) &= b_1 y_1(t) + b_2 y_2(t). \end{aligned} \tag{10.17}$$

The matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

is the desired monodromy matrix. Differentiating equalities (10.17) and assuming $t = 0$ in equalities (10.17) and in equalities from the derivatives, we obtain

$$a_1 = y_1(\omega), \quad a_2 = \dot{y}_1(\omega),$$

$$b_1 = y_2(\omega), \quad b_2 = \dot{y}_2(\omega).$$

The monodromy matrix eigenvalues (called multipliers) are determined from the equation

$$\lambda^2 - 2A\lambda + B = 0,$$

where

$$A = \frac{1}{2}[y_1(\omega) + \dot{y}_2(\omega)], \quad B = y_1(\omega)\dot{y}_2(\omega) - y_2(\omega)\dot{y}_1(\omega).$$

The coefficient B equals the Wronski determinant of the solutions $y_1(t)$, $y_2(t)$ at $t = \omega$. From the Liouville formula for the Wronski determinant, it follows that

$$B = e^{-\int\limits_0^\omega p(t)dt}.$$

Let the eigenvalues of the monodromy matrix be distinct. If the eigenvalues of the monodromy matrix are less than 1 in absolute value, then the zero solution of equation (10.16) is asymptotically stable. If at least one eigenvalue is larger than 1 in absolute value, then the zero solution of equation (10.16) is unstable. We are interested in the case when equation (10.16) has an ω-periodic solution. Let $y_1(t)$ be a periodic solution. Then $a_1 = y_1(\omega) = 1$, $a_2 = \dot{y}_1(\omega) = 0$. It is easy to see that $\lambda_1 = 1$, $\lambda_2 = B$ in this case. Now it follows from the Floquet Theory that it is possible to take the functions $y_1(t)$ and $\eta(t) = e^{\frac{t}{\omega}\ln|B|}\psi(t)$, where $\psi(t)$ is $a\omega$-periodic function, as the linearly independent solutions. Therefore, if $|B| < 1$, then the zero solution of equation (10.16) is stable by Lyapunov. If $|B| > 1$, then the zero solution is unstable.

We now return to equation (10.15). We know that this equation has a periodic solution, and, hence, one multiplier equals 1. Comparing equation (10.15) and (10.16) implies that for equation (10.16)

$$B = e^{\int\limits_0^{\frac{2\pi}{k}} \varepsilon \frac{\partial f(x^*, \dot{x}^*, \varepsilon)}{\partial \dot{x}} dt}.$$

Therefore, the condition

$$\int\limits_0^{\frac{2\pi}{k}} \frac{\partial f(x^*, \dot{x}^*, \varepsilon)}{\partial \dot{x}} dt < 0 \tag{10.18}$$

quarantees the stability by Lyapunov for the zero solution of equation (10.15). As Andronov and Vitt showed [1933], condition (10.18) guarantees the stability by Lyapunov of the periodic solution $x^*(t, \varepsilon)$ of equation (10.3). Moreover, we can assert that each solution of (10.3), which is sufficiently close to the periodic solution, tends to one of the solutions of the family $x^*(t + h, \varepsilon)$ as $t \to \infty$. Coddington and Levinson [1955] showed that there exists $\delta > 0$ such that if the solution $z(t, \varepsilon)$ of equation (10.3) obeys the inequality

$$|z(t_1, \varepsilon) - x^*(t_0, \varepsilon)| < \delta$$

for some t_0 and t_1, then there exists a constant $c > 0$ such that

$$\lim_{t\to\infty} |z(t, \varepsilon) - x^*(t + c, \varepsilon)| = 0.$$

The number c is called an *asymptotic phase.*

Condition (10.18) and representation (10.14) produce the following sufficient condition for the stability of the periodic solution $x^*(t,\varepsilon)$ (at small ε):

$$\begin{gathered}\int_0^{\frac{2\pi}{k}} \frac{\partial f(\rho_0\cos kt, -k\rho_0\sin kt, 0)}{\partial \dot{x}} dt = \\ \frac{1}{k}\int_0^{2\pi} \frac{\partial f(\rho_0\cos u, -k\rho_0\sin u, 0)}{\partial \dot{x}} du = \Delta < 0.\end{gathered} \tag{10.19}$$

It is easier to solve the problem of the instability of the periodic solution $x^*(t,\varepsilon)$. If the inequality

$$\int_0^{\frac{2\pi}{k}} \frac{\partial f(x^*, \dot{x}^*, \varepsilon)}{\partial \dot{x}} dt > 0$$

holds, then the monodromy matrix of equation (10.16) has one multiplier larger than 1 in absolute value. The theorem of the instability in the first approximation implies that for sufficiently small ε, the solution $x^*(t,\varepsilon)$ is unstable. The sufficient condition of instability (for small ε) has the form

$$\begin{gathered}\int_0^{\frac{2\pi}{k}} \frac{\partial f(\rho_0\cos kt, -k\rho_0\sin kt, 0)}{\partial \dot{x}} dt = \\ = \frac{1}{k}\int_0^{2\pi} \frac{\partial f(\rho_0\cos u, -k\rho_0\sin u, 0)}{\partial \dot{x}} du = \Delta > 0.\end{gathered} \tag{10.20}$$

We can rewrite the obtained stability condition (10.19) (instability condition (10.20)). Transforming the left-hand side of the equation $H_0(\rho) = 0$ that determines ρ_0 and integrating it by parts, we obtain

$$H_0(\rho_0) = -\frac{1}{2\pi}\int_0^{2\pi} f(\rho_0\cos u, -k\rho_0\sin u, 0)\sin u du =$$

$$-\frac{1}{2\pi}\int_0^{2\pi} [\rho_0 \frac{\partial f(\rho_0\cos u, -k\rho_0\sin u, 0)}{\partial x} \sin u\cos u du +$$

$$+k\rho_0 \frac{\partial f(\rho_0\cos u, -k\rho_0\sin u, 0)}{\partial \dot{x}} \cos^2 u du] =$$

$$-\frac{1}{2\pi}\int_0^{2\pi} k\rho_0 \frac{\partial f(\rho_0\cos u, -k\rho_0\sin u, 0)}{\partial \dot{x}} du -$$

$$-\frac{1}{2\pi}\int_0^{2\pi} [\rho_0 \frac{\partial f(\rho_0\cos u, -k\rho_0\sin u, 0)}{\partial x} \cos u -$$

$$-k\rho_0\frac{\partial f(\rho_0\cos u, -k\rho_0\sin u, 0)}{\partial\dot{x}}\sin u]\sin u du = -k\Delta + H_{0\rho}(\rho_0) = 0.$$

Thus it follows that the condition of the stability for the periodic solution of equation (10.3) takes the form

$$H_{0\rho}(\rho_0) < 0,$$

i.e. coincides with condition (10.10). The condition of instability for this periodic solution coincides with condition (10.11).

10.3 Van der Pol Quasi-Linear Oscillator

As an example, we consider the well-known Van der Pol equation (see Van der Pol [1927]):

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}, \tag{10.21}$$

where $\varepsilon > 0$ is a small parameter. We transform equation (10.21) into the system by making change (10.5) and arrive at

$$\begin{aligned}\tfrac{d\rho}{dt} &= -\varepsilon\rho(\sin^2\theta - \rho^2\sin^2\theta\cos^2\theta),\\ \tfrac{d\theta}{dt} &= 1 - \tfrac{\varepsilon}{4}(\sin 2\theta + \rho^2\sin 4\theta).\end{aligned}$$

The averaged equation takes the form

$$\frac{d\bar{\rho}}{dt} = -\varepsilon\bar{\rho}(\frac{1}{2} - \frac{\bar{\rho}^2}{8}).$$

Hence, the stationary solutions are determined from the equation

$$H_0(\bar{\rho}) = \bar{\rho}(\frac{1}{2} - \frac{\bar{\rho}^2}{8}) = 0.$$

We have two stationary solutions $\rho_1 = 0$, $\rho_2 = 2$ with $H_{0\bar{\rho}}(\rho_1) > 0$ and $H_{0\bar{\rho}}(\rho_2) < 0$. Consequently, for sufficiently small ε, the zero solution of the Van der Pol equation (10.21) is unstable. Besides, equation (10.21) has a stable periodic solution with an amplitude close to 2 and a period close to 2π.

Exercise 10.1. Investigate the problem of the existence and stability of periodic solutions of the equation

$$\ddot{x} + x = \varepsilon(a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^5)\dot{x},$$

where $\varepsilon << 1$, $a_1 > 0$, $a_5 < 0$.

10.4 Resonant Periodic Oscillations of Quasi-Linear Systems with One Degree of Freedom

Consider a differential equation

$$\ddot{x} + k^2 x = \varepsilon f(t, x, \dot{x}, \varepsilon), \tag{10.22}$$

where $\varepsilon > 0$ is a small parameter, k is a number, $f(t, x, \dot{x})$ is a function periodic in t with the period 2π and sufficiently smooth in the variables x, $\dot{x}$, ε. Equation (10.22) is called **quasi-linear** or **weakly nonlinear**. It describes a system with one degree of freedom. This name is related to the fact that for $\varepsilon = 0$, equation (10.22) transforms into the linear differential equation

$$\ddot{x} + k^2 x = 0. \tag{10.23}$$

All solutions of equation (10.23) are periodic with the period $2\pi/k$. These solutions are called natural vibrations of the system described by equation (10.22).

A different approach to the theory stated below was described in the book by Malkin [1956]. This book influenced the contents of this and further sections of this chapter.

Assume that k either equals n, or k and n are close enough. We shall assume that the detuning $(n^2 - k^2)$ has an order of smallness ε, and

$$n^2 - k^2 = \varepsilon m,$$

where m is a finite quantity. Then equation (10.22) takes the form

$$\ddot{x} + n^2 x = \varepsilon F(t, x, \dot{x}, \varepsilon), \tag{10.24}$$

where

$$F(t, x, \dot{x}, \varepsilon) = mx + f(t, x, \dot{x}, \varepsilon). \tag{10.25}$$

For $\varepsilon = 0$, equation (10.24) turns into

$$\ddot{x} + n^2 x = 0. \tag{10.26}$$

The general solution of equation (10.26) assumes the form

$$x(t) = a \cos nt + b \sin nt$$

(a and b are arbitrary constants). This solution is periodic with the period $2\pi/n$ and hence is periodic with the period 2π.

Let us find the existence conditions for the 2π- periodic solution of equation (10.24).

Using a change of variables, we transform equation (10.24) in a system of two differential equations in the standard form. The corresponding change is called Van der Pol's change (Van der Pol [1927]), and the respective variables are called the Van der Pol variables. This change was already used for linear equations in chapter 4. We consider two methods of transforming equation (10.24) in a system in the standard form.

We use a change of variables

$$x(t) = a\cos nt + b\sin nt, \quad \dot{x} = -an\sin nt + bn\cos nt, \tag{10.27}$$

where $a(t)$, $b(t)$ are new variables. We will consider that the solution of equation (10.24) and its derivative have the same representation as the solution and the derivative of the solution of the linear equation

$$\ddot{x} + n^2 x = 0$$

but now $a(t), b(t)$ are the functions. Therefore, the Van der Pol change is essentially a method of variation of arbitrary constants. The change (10.27) leads to a system

$$\begin{aligned}&\dot{a}\cos nt + \dot{b}\sin nt = 0,\\ &-n\dot{a}\sin nt + n\dot{b}\cos nt = \varepsilon F(t, a\cos nt + b\sin nt, -an\sin nt + bn\cos nt, \varepsilon).\end{aligned}$$

We solve the latter system for $a(t)$, $b(t)$, using the Cramer's rule, and arrive at the system

$$\begin{aligned}&\dot{a} = -\varepsilon\tfrac{1}{n}F(t, a\cos nt + b\sin nt, -an\sin nt + bn\sin nt, \varepsilon)\sin nt,\\ &\dot{b} = \varepsilon\tfrac{1}{n}F(t, a\cos nt + b\sin nt, -an\sin nt + bn\sin nt, \varepsilon)\sin nt)\cos nt.\end{aligned} \tag{10.28}$$

System (10.28) is in the standard form. The right-hand side of system (10.28) is a 2π-periodic function of t. Therefore, we can apply Theorems 9.2A and 9.4 (see Sections 9.3 and 9.5) to system (10.28).

The averaged system takes the form

$$\begin{aligned}\tfrac{d\bar{a}}{dt} &= \varepsilon P(\bar{a}, \bar{b}),\\ \tfrac{d\bar{b}}{dt} &= \varepsilon Q(\bar{a}, \bar{b}),\end{aligned} \tag{10.29}$$

where

$$P(\bar{a}, \bar{b}) = \frac{1}{2\pi n}\int_0^{2\pi} F(s, a\cos ns + b\sin ns, -an\sin ns + bn\cos ns, 0)\sin nsds,$$

$$Q(\bar{a}, \bar{b}) = \frac{1}{2\pi n}\int_0^{2\pi} F(s, a\cos ns + b\sin ns, -an\sin ns + bn\cos ns, 0)\cos nsds.$$

The stationary solutions of system (10.29) are the solutions of the system of equations

$$P(\bar{a}, \bar{b}) = 0, \quad Q(\bar{a}, \bar{b}) = 0. \tag{10.30}$$

From Theorem 9.2A, which is applicable to the system (see Section 9.3), we arrive at the following result.

Theorem 10.1. *Let a_0, b_0 be the solution of system (10.30). Let the determinant of the matrix*

$$\Delta(a,b) = \begin{pmatrix} \frac{\partial P}{\partial \bar{a}} & \frac{\partial P}{\partial \bar{b}} \\ | & \\ \frac{\partial Q}{\partial \bar{a}} & \frac{\partial Q}{\partial \bar{b}} \end{pmatrix} \tag{10.31}$$

for $\bar{a} = a_0$, $\bar{b} = b_0$ be non-zero.

Then, for sufficiently small ε, system (10.28) has a unique 2π-periodic solution $a(t,\varepsilon)$, $b(t,\varepsilon)$ such that $a(t,0) = a_0$, $b(t,0) = b_0$.

Difference from zero of the determinant of matrix (10.31) means that the matrix of an averaged system linearized on the stationary solution (a_0, b_0) has no zero eigenvalue.

We say that the solution of system (10.30) is **simple** if the determinant of matrix (10.31) is non-zero.

Apparently, if the conditions of Theorem 10.1 hold, then equation (10.24), for sufficiently small ε, has the periodic solution

$$x(t,\varepsilon) = a(t,\varepsilon)\cos nt + b(t,\varepsilon)\sin nt \tag{10.32}$$

with the period 2π. This solution for $\varepsilon = 0$ turns into the periodic solution

$$a_0 \cos nt + b_0 \sin nt \tag{10.33}$$

of equation (10.26).

It is worth noting that the assertion of Theorem 10.1 can be interpreted as follows. General solution of equation (10.26) appears to be a family of periodic solutions that depends on two parameters. According to the assertion of Theorem 10.1, solution (10.33) corresponding to the periodic solution of equation (10.26) is singled out from this family and is called a generating solution.

The problem of the stability of a periodic solution of system (10.28) is solved with the help of Theorem 9.4. Namely, we obtain the following theorem.

Theorem 10.2. *Let the conditions of Theorem 10.1 hold, and the eigenvalues of matrix (10.31) for $a = a_0$, $b = b_0$ have negative real parts. Then the solution $a(t,\varepsilon)$, $b(t,\varepsilon)$, for sufficiently small ε, is asymptotically stable. If at least one eigenvalue of matrix (10.31) has a positive real part, then this solution is unstable.*

Evidently, solution (10.32) of equation (10.24) has the same properties.

Note that eigenvalues of the matrix $\Delta(a, b)$ have negative real parts if the trace of the matrix is negative and the determinant is positive.

Let us bring in another change of variables to transform equation (10.24) into a system in the standard form. This change is equivalent to the change (10.27) but is more convenient for use in applications.

The solution of equation (10.26) can be written as

$$x(t) = a\cos(nt + \psi),$$

where a is an amplitude of oscillations, and ψ is a phase of the oscillation. We assume that a and ψ are the functions of the variables t. Using a change

$$x = a(t)\cos(nt + \psi(t)), \quad \dot{x} = -na(t)\sin(nt + \psi(t)), \tag{10.34}$$

we transform equation (10.24) into the system of equations

$$\begin{aligned}&\dot{a}\cos(nt + \psi) - a\dot{\psi}\sin(nt + \psi) = 0,\\ &-n\dot{a}\sin(nt + \psi) - na\dot{\psi}\cos(nt + \psi) = \varepsilon F(t, a\cos(nt + \psi), -na\sin(nt + \psi, \varepsilon).\end{aligned}$$

We shall solve the resultant system for $\dot{a}$, $\dot{\psi}$. We arrive at the system

$$\begin{aligned}\dot{a} &= -\varepsilon\tfrac{1}{n}F(t, a\cos(nt + \psi), -an\sin(nt + \psi, \varepsilon)\sin(nt + \psi),\\ \dot{\psi} &= -\varepsilon\tfrac{1}{an}F(t, a\cos(nt + \psi), -an\sin(nt + \psi, \varepsilon)\cos(nt + \psi).\end{aligned} \tag{10.35}$$

For system (10.35), we can now state an assertion similar to that of Theorem 2.9. We introduce the functions

$$R(\bar{a}, \bar{\psi}) = \frac{1}{2\pi n}\int_0^{2\pi} F(s, a\cos(ns + \psi), -an\sin(ns + \psi, 0)\sin(ns + \psi)ds,$$

$$S(\bar{a}, \bar{\psi}) = \frac{1}{2\pi an}\int_0^{2\pi} F(s, a\cos(ns + \psi), -an\sin(ns + \psi, 0)\cos(ns + \psi)ds.$$

If $\bar{a} = a_0$, $\bar{\psi} = \psi_0$ is the solution of the system of equations

$$R(\bar{a}, \bar{\psi}) = 0, \quad S(\bar{a}, \bar{\psi}) = 0 \tag{10.36}$$

and the determinant of the matrix

$$\Delta(a, \psi) = \begin{pmatrix} \frac{\partial R}{\partial \bar{a}} & \frac{\partial R}{\partial \bar{\psi}} \\ \\ \frac{\partial S}{\partial \bar{a}} & \frac{\partial S}{\partial \bar{\psi}} \end{pmatrix} \tag{10.37}$$

is non-zero for $\bar{a} = a_0$, $\bar{\psi} = \psi_0$, then, for sufficiently small ε, system (10.35) has a unique 2π-periodic solution $a(t, \varepsilon)$, $\psi(t, \varepsilon)$ that for $\varepsilon = 0$ becomes the periodic function $a_0\cos(nt + \psi_0)$. This periodic solution is asymptotically stable if all eigenvalues of matrix (10.37) have negative real parts and unstable if at least one eigenvalue of the matrix has a positive real part.

10.5 Subharmonic Solutions

In nonlinear systems under the influence of a periodic perturbing force, periodic oscillations may arise not only when the perturbing force period T is close to the period of natural vibrations but also when it is close to nT, where n is an integer. In this case, the period of forced oscillations will equal nT.

Let us first consider a linear system described by the equation

$$\ddot{x} + 2\delta x + k^2 x = a\cos\omega t. \tag{10.38}$$

This equation has a unique periodic solution determined by the formula

$$x(t) = \frac{a}{\sqrt{(k^2-\omega^2)^2 + 4\delta^2\omega^2}}\cos(\omega t + \psi), \quad \tan\psi = -\frac{2\delta\omega}{k^2-\omega^2}.$$

This solution has the same period as the perturbing force. All other solutions of equation (10.38) approach the periodic solution as $t \to \infty$. Therefore, equation (10.38) cannot have solutions with the period that is a multiple to the period of perturbations. However if damping equals zero, then for $k = \omega/n$, the general solution of equation (10.38)

$$x(t) = Acos(\frac{\omega}{n}t + \psi) + \frac{n^2 a}{(1-n^2)\omega^2}\cos\omega t$$

is periodic with the period $2\pi n/\omega$. In real life systems described by linear differential equations of the form (10.38), by virtue of inevitable damping, there are only periodic solutions with the period of the perturbing force.

The situation is different for nonlinear systems. Here, we observe periodic oscillations with the period multiple to the period of perturbations if the period of natural vibrations equals $2\pi n/\omega$ or is close to it. This phenomenon is called a subharmonic resonance and the respective solution is called subharmonic. The exact theory of the subharmonic resonance for weakly nonlinear systems with one degree of freedom was developed by Mandelstam and Papaleksi [1932].

We consider an equation

$$\ddot{x} + k^2 x = f(t) + \varepsilon F(t, x, \dot{x}, \varepsilon), \tag{10.39}$$

where $\varepsilon > 0$ is a small parameter, $f(t)$ and $F(t, x, \dot{x}, \varepsilon)$ are periodic in t with the period 2π, $F(t, x, \dot{x}, \varepsilon)$ and sufficiently smooth in the variables $x, \dot{x}, \varepsilon$. We assume that k either equals $1/n$, where n is an integer, or is close to it. Assume that the detuning

$$\frac{1}{n^2} - k^2$$

has an order of smallness ε and

$$\frac{1}{n^2} - k^2 = \varepsilon m,$$

where m is a finite quantity. Then equation (10.39) takes the form

$$\ddot{x} + \frac{1}{n^2}x = f(t) + \varepsilon g(t, x, \dot{x}, \varepsilon), \tag{10.40}$$

where $g(t, x, \dot{x}, \varepsilon)$ is determined by formula (10.25). For $\varepsilon = 0$, equation (10.39) transforms into the equation

$$\ddot{x} + \frac{1}{n^2}x = f(t).$$

The general solution of this equation is

$$x(t) = a\cos\frac{1}{n}t + b\sin\frac{1}{n}t + \varphi(t),$$

where $\varphi(t)$ is the solution of an inhomogeneous equation corresponding to the perturbation $f(t)$. This solution is periodic with the period 2π, and, hence, with the period $2\pi n$.

We transform equation (10.40) using a change

$$x(t) = y(t) + \varphi(t)$$

into the equation

$$\ddot{y} + \frac{1}{n^2}y = \varepsilon h(t, y, \dot{y}, \varepsilon),$$

where

$$h(t, y, \dot{y}, \varepsilon) = g(t, y + \varphi(t), \dot{y} + \dot{\varphi}(t), \varepsilon).$$

Using a change of variables

$$x(t) = a\cos\frac{1}{n}t + b\sin\frac{1}{n}t, \quad \dot{x} = -a\frac{1}{n}\sin\frac{1}{n}t + b\frac{1}{n}\cos\frac{1}{n}t$$

or a change

$$x = a(t)\cos(\frac{1}{n}t + \psi(t)), \quad \dot{x} = -\frac{1}{n}a(t)\sin(\frac{1}{n}t + \psi(t)),$$

we obtain a system in the standard form such that its right-hand side is periodic with the period $2\pi n$. To study the problem of the existence of a $2\pi n$-periodic solution, we use Theorem 9.2A. Averaging the resultant system brings us to the problem of solvability of the systems of equations

$$P^*(\bar{a}, \bar{b}) = 0, \quad Q^*(\bar{a}, \bar{b}) = 0, \tag{10.41}$$

where

$$P^*(\bar{a},\bar{b}) = \int\limits_0^{2\pi n} h(s, a\cos\frac{1}{n}s + b\sin\frac{1}{n}s, -a\frac{1}{n}\sin\frac{1}{n}s + b\frac{1}{n}\cos\frac{1}{n}s, 0)\sin\frac{1}{n}sds,$$

$$Q^*(\bar{a},\bar{b}) = \int\limits_0^{2\pi} h(s, a\cos\frac{1}{n}s + b\sin\frac{1}{n}s, -a\frac{1}{n}\sin\frac{1}{n}s + b\frac{1}{n}\cos\frac{1}{n}s, 0)\cos\frac{1}{n}sds,$$

or

$$S^*(\bar{a},\bar{\psi}) = 0, \quad R^*(\bar{a},\bar{\psi}) = 0, \tag{10.42}$$

where

$$R^*(\bar{a},\bar{\psi}) = \int\limits_0^{2\pi n} h(s, a\cos(\frac{1}{n}s+\psi), -a\frac{1}{n}\sin(\frac{1}{n}s+\psi, 0)\sin(\frac{1}{n}s+\psi)ds,$$

$$S^*(\bar{a},\bar{\psi}) = \frac{1}{a}\int\limits_0^{2\pi n} h(s, a\cos(\frac{1}{n}s+\psi), -a\frac{1}{n}\sin(\frac{1}{n}s+\psi, 0)\cos(\frac{1}{n}s+\psi)ds.$$

If system (10.41) has a simple solution (a_0, b_0), then, for sufficiently small ε, equation (10.40) has a periodic solution with the period $2\pi n$. Such solution is called subharmonic. A similar assertion holds true when the problem is reduced to the consideration of system (10.42).

10.6 Duffing's Weakly Nonlinear Equation. Forced Oscillations

Duffing's equation (see Duffing [1918]) has the form

$$\ddot{x} + k^2x + \alpha x^3 = 0, \tag{10.43}$$

where k and α are real numbers, α being either positive or negative. Duffing's equation can be interpreted as an equation that describes oscillations of a nonlinear spring. If $\alpha > 0$, then the spring is called rigid, and if $\alpha < 0$, then the spring is called soft. Sometimes one refers to the rigid and soft elastic force, keeping in mind the nonlinear term αx^3 in Duffing's equation.

We consider Duffing's weakly nonlinear equation, assuming that $\alpha = \varepsilon\gamma$, where γ is a finite quantity. We study the problem of the resonance oscillations under the periodic perturbations with a small amplitude.

Consider an equation

$$\ddot{x} + k^2x = \varepsilon[A\cos t - \delta\dot{x} - \gamma x^3], \tag{10.44}$$

where A, $\delta > 0$ are real numbers. Along with the perturbing force, we introduce a small damping $\varepsilon\delta\dot{x}$. We assume that

$$1 - k^2 = \varepsilon m,$$

where m is a finite quantity. Then equation (10.44) can be be rewritten as

$$\ddot{x} + x = \varepsilon[mx + A\cos t - \delta\dot{x} - \gamma x^3]. \tag{10.45}$$

By a change

$$x = a\cos(t+\psi), \quad \dot{x} = -a\sin(t+\psi),$$

we transform equation (10.45) into the system

$$\begin{aligned}
&\dot{a} = -\varepsilon[\tfrac{1}{2}am\sin 2(t+\psi) + A\cos t\sin(t+\psi) + a\delta\sin^2(t+\psi) - \\
&-\gamma a^3\cos^3(t+\psi)\sin(t+\psi)], \\
&\dot{\psi} = -\varepsilon[m\cos^2(t+\psi) + \tfrac{A}{a}\cos t\cos(t+\psi) + \tfrac{1}{2}\delta\sin 2(t+\psi) - \\
&-\gamma a^2\cos^4(t+\psi)].
\end{aligned} \tag{10.46}$$

Averaging system (10.46) over the time t yields the averaged system

$$\begin{aligned}
&\dot{\bar{a}} = \varepsilon[-\tfrac{A}{2}\sin\bar{\psi} - \tfrac{\bar{a}\delta}{2}], \\
&\dot{\bar{\psi}} = \varepsilon[-\tfrac{m}{2} - \tfrac{A}{2a}\cos\bar{\psi} + \tfrac{3}{8}\gamma\bar{a}^2].
\end{aligned} \tag{10.47}$$

We first assume that $\delta = 0$. Then the stationary solutions of the averaged system are determined from the system of equations

$$\frac{A}{2}\sin\bar{\psi} = 0, \quad \frac{m}{2} + \frac{A}{2a}\cos\bar{\psi} - \frac{3}{8}\gamma a^2 = 0.$$

From the first equation we obtain $\bar{\psi} = 0$. Then the second equation turns into the equation

$$F(\bar{a}) = \frac{m\bar{a}}{2} + \frac{A}{2} - \frac{3}{8}\gamma\bar{a}^3 = 0$$

or

$$\bar{a}^3 - \frac{4m}{3\gamma}\bar{a} - \frac{4A}{3\gamma} = 0. \tag{10.48}$$

We shall assume that $A > 0$. Letting $\lambda = A/\varepsilon$ and returning to the previous notation, we arrive at the equation

$$\bar{a}^3 - \frac{4(1-k^2)}{3\alpha}\bar{a} - \frac{4\lambda}{3\alpha} = 0. \tag{10.49}$$

It is well known (see Birkhoff and MacLane [1977]) that the number of real roots of the cubic equation

$$x^3 + px + q = 0 \tag{10.50}$$

is determined by the sign of the discriminant of this equation

$$D = \frac{p^3}{27} + \frac{q^2}{4}.$$

If $D > 0$, then equation (10.50) has one real root, and if $D < 0$, equation (10.50) has three real roots. Returning to equation (10.49), we obtain

$$D = -\frac{16(1-k^2)^3}{9\alpha^3} + \frac{\lambda^2}{\alpha^2}.$$

Let $\alpha > 0$. If the detuning is negative, or, if it is positive but does not exceed some h defined by the inequality

$$\frac{16(1-k^2)^3}{9\alpha^3} < \frac{\lambda^2}{\alpha^2},$$

then equation (10.49) has one real root. But if $1 - k^2 > h$, then equation (10.49) has three real roots. Similarly we consider the case when $\alpha < 0$.

Let $\bar{a} = a_0$ be the solution of equation (10.49). The matrix $\Delta(\bar{a}, \bar{\psi})$ for $\bar{a} = a_0$, $\psi = 0$ takes the form

$$\Delta(a_0, 0) = \begin{pmatrix} 0 & -\varepsilon\frac{1}{2}A \\ -\varepsilon\frac{A}{2a_0^2} + \varepsilon\frac{3}{4}\gamma a_0 & 0 \end{pmatrix}. \tag{10.51}$$

The determinant of this matrix is non-zero if the root $\bar{a} = a_0$ of equation (10.48) is simple, since in this case

$$F'(a_0) = -\frac{A}{2a_0^2} + \frac{3}{4}\gamma a_0 \neq 0.$$

The latter expression can be written in the form

$$F'(a_0) = \frac{m}{2a_0} + \frac{3}{4}\gamma a_0.$$

From Theorem 10.1 it follows that, for sufficiently small ε, equation (10.45) has one or three periodic solutions with the period 2π. It is easy to see that if the inequality

$$\varepsilon\frac{m}{2a_0} + \varepsilon\frac{3}{4}\gamma a_0 > 0 \tag{10.52}$$

holds, the eigenvalues of matrix (10.51) are real and have opposite signs, and for

$$\varepsilon\frac{m}{2a_0} + \varepsilon\frac{3}{4}\gamma a_0 < 0, \tag{10.53}$$

the eigenvalues of matrix (10.51) are purely imaginary.

Hence, if inequality (10.52) holds, then, for sufficiently small ε, the periodic solutions of equation (10.46) are unstable. If inequality (10.53) holds, then Theorem 10.2 is not applicable.

Recall that $\varepsilon m = 1 - k^2$, $\varepsilon\gamma = \alpha$. Therefore, in the original variables, inequalities (10.52) and (10.53) can be written as

$$\frac{1-k^2}{2a_0} + \frac{3}{4}\alpha a_0 > 0, \qquad \frac{1-k^2}{2a_0} + \frac{3}{4}\alpha a_0 < 0.$$

If equation (10.48) has three real roots, then for the maximum and the minimum of the roots, inequality (10.53) holds, and for the root lying between them inequality (10.52) holds. Hence, in the case when three periodic solutions exist, one of them is unstable, and the stability of the other two is not determined by Theorem 10.2. If equation (10.48) has one real root, then the stability of the respective periodic solution is not determined by Theorem 10.2.

We now take damping into account, i.e., assume that $\delta > 0$. The stationary solutions of the averaged system are determined from the system of equations (we omit the bar above the variables)

$$R(a,\psi) = -\frac{A}{2}\sin\psi - \frac{a\delta}{2} = 0, \quad \Psi(a,\psi) = -\frac{m}{2} - \frac{A}{2a}\cos\psi + \frac{3}{8}\gamma a^2 = 0. \tag{10.54}$$

From the first equation, we obtain

$$\sin\psi = -\frac{a\delta}{A}. \tag{10.55}$$

This equation has a solution if the right-hand side is less than 1 in absolute value; at that, there are two solutions. Assume that this condition holds (it holds if, e.g. a and A are fixed and δ is small, or it holds for small a at fixed A and δ). Then, substituting (10.55) into the second equation of system (10.54) yields an equation for finding a

$$m - \frac{3}{4}\gamma a^2 \mp \sqrt{\frac{A^2}{a^2} - \delta^2} = 0. \tag{10.56}$$

Under certain conditions, equation (10.56) (e.g. for small δ) has one or three solutions. Assuming $a^2 = z$, we obtain a cubic equation with respect to z, which can be analyzed in the same way as it was done for equation (10.49).

The matrix of the averaged system linearized on the equilibrium a_0, ψ_0 takes the form

$$\Delta(a_0,\psi_0) = \begin{pmatrix} R_a(a_0,\psi_0) & R_\psi(a_0,\psi_0) \\ \Psi_a(a_0,\psi_0) & \Psi_\psi(a,\psi) \end{pmatrix}. \tag{10.57}$$

The trace of this matrix is negative, since

$$R_a(a,\psi) + \Psi_\psi(a,\psi) = -\delta.$$

Therefore, if the determinant

$$R_a(a,\psi)\Psi_\psi(a,\psi) - R_\psi(a,\psi)\Psi_a(a,\psi) \tag{10.58}$$

of matrix (10.57) is positive, then, for sufficiently small ε, equation (10.55) has an asymptotically stable periodic solution with the period 2π which conforms to the stationary solution of the averaged system. If the determinant of matrix (10.57) is negative, then the respective periodic solution of equation (10.55) is unstable.

We deduce the condition of the asymptotic stability (see Bogoliubov and Mitropolskiy [1961]) for equilibrium of an averaged system. Assume that the variables a and ψ are the functions of the detuning m. By differentiating (10.35) with respect to m we obtain

$$R_a \frac{da}{dm} + R_\psi \frac{d\psi}{dm} = 0, \quad \Psi_a \frac{da}{dm} + \Psi_\psi \frac{d\psi}{dm} = 0,$$

therefore,

$$(R_a \Psi\psi - \Psi_a R_\psi)\frac{da}{dm} = \Psi_m R_\psi - R_m \Psi\psi. \tag{10.59}$$

On the other hand,

$$R_\psi = -\frac{A}{2}\cos\psi, \quad R_m = 0, \quad \Psi_\psi = -\frac{\delta}{2}, \quad \Psi_m = -\frac{1}{2}.$$

Therefore, the right-hand side of equality (10.59) can be rewritten (with the second of equations in (10.54) taken into account):

$$-\frac{1}{2}(-\frac{A}{2}\cos\psi) = -\frac{ma}{4} + \frac{3}{16}\gamma a^3. \tag{10.60}$$

Thus, it follows from (10.59) and (10.60) that

$$(R_a \Psi\psi - \Psi_a R_\psi)\frac{da}{dm} = -\frac{ma}{4} + \frac{3}{16}\gamma a^3.$$

So, it is evident that the condition of asymptotic stability for the equilibrium of the averaged equation can be presented as

$$\begin{array}{lll} \frac{da}{dm} > 0, & if & -\frac{ma}{4} + \frac{3}{16}\gamma a^3 > 0, \\ \frac{da}{dm} < 0, & if & -\frac{ma}{4} + \frac{3}{16}\gamma a^3 < 0. \end{array} \tag{10.61}$$

The conditions obtained for the stability are convenient for graphic representation of the relationship between the amplitude a and the frequency k (the amplitude or resonance curve). Making use of (10.56), we construct curve (10.56) that in the original variables takes the form

$$1 - k^2 - \frac{3}{4}\alpha a^2 \mp \sqrt{\frac{\varepsilon^2 A^2}{a^2} - \varepsilon^2\delta^2} = 0 \tag{10.62}$$

as well as the so-called “skeleton curve”

$$-\frac{ma}{4} + \frac{3}{16}\gamma a^3 = 0$$

that in the original variables is

$$k^2 = 1 - \frac{3}{4}\alpha a^2. \tag{10.63}$$

Then on the branch of curve (10.62), which lies to the left of (10.63), the sections where a rises together with k are stable (i.e., corresponding to stable amplitudes); on the branch to the right of curve (10.63), on the contrary, the sections where a decreases as k rises are stable.

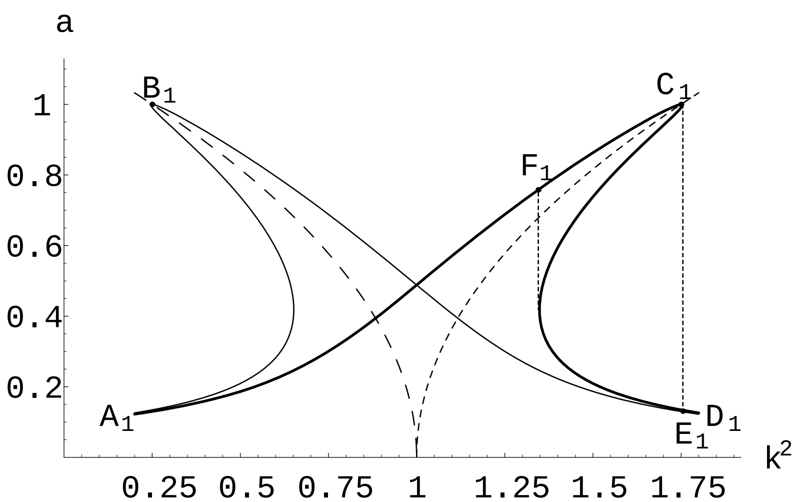

FIGURE 10.1: Amplitude Curves I for Harmonically Forced Damped Duffing's Equation.

In the case of three periodic solutions, two of them are asymptotically stable and one is unstable. If only one periodic solution exists, then it is asymptotically stable.

Figure 10.1 (for $a > 0$) shows the amplitude curves (the right-hand graph conforms to $\alpha < 0$, and the left-hand graph, to $\alpha > 0$) of the relationship between the oscillation amplitude a and k^2 at $\varepsilon = 0.1$, $A = \delta = 1$, $\alpha = \pm 1$. These curves allow analyzing of the character of oscillations in the system under study upon the change of frequency of natural vibrations. For example, let us consider the right-hand graph. As the frequencies of natural vibrations increase from small values, the amplitude of forced oscillations first rises along the curve A_1C_1. At the point C_1, there is a breakup; the value of the amplitude skips to the point E_1 on the curve C_1D_1 and changes along the curve C_1D_1 toward the point D_1 on the further increase of frequency. If we now decrease the frequency of natural vibrations, then the amplitude of forced oscillations will change along the curve D_1C_1 up to the point of inflection and then skip to the point F_1 of the curve C_1A_1, where upon it will change along the curve C_1A_1.

We note that while talking about the change of frequency of natural vibrations, we mean a very slow change such that at almost any instant the system can be regarded as stationary.

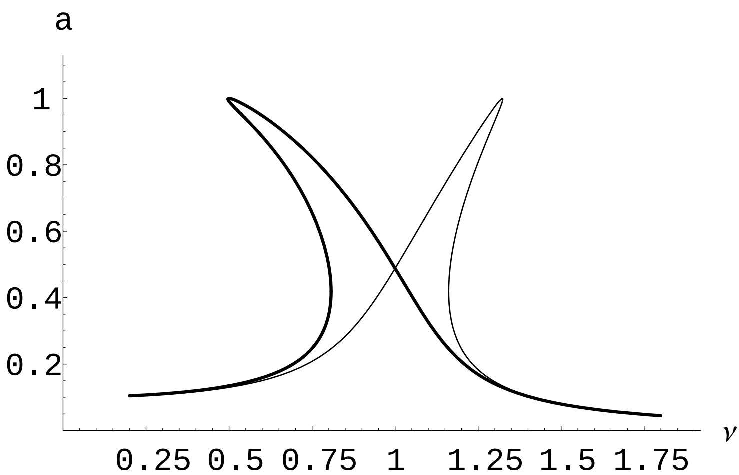

FIGURE 10.2: Amplitude Curves II for Harmonically Forced Damped Duffing's Equation.

We can write the initial problem as the following equation

$$\ddot{x} + x = \varepsilon[A\cos\nu t - \delta\dot{x} - \gamma x^3].$$

If we introduce a detuning

$$1 = \nu^2 + \varepsilon m$$

and pass to a system of equations using a change

$$x = a\cos(\nu t + \varphi), \quad \dot{x} = -a\nu\sin(\nu t + \varphi),$$

then, similarly to the previous case, we can obtain the amplitude curves of the relationship between the forced oscillations and the frequency of an external force. These curves are depicted in Figure 10.2, where the left-hand graph now conforms to the values $\alpha < 0$, and the right-hand graph, to $\alpha > 0$.

Exercise 10.2(a). Consider the problem of forced oscillations of Duffing's equation in the case when the induced force $f(t)$ is polyharmonic:

$$f(t) = a_1\cos t + a_2\cos 2t.$$

Does the relationship between the amplitudes a_1, a_2 play a role?

Exercise 10.2(b). Investigate the problem of the existence and stability of resonance forced oscillations of the equation

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x} + \varepsilon A \cos \omega t,$$

where $\varepsilon << 1$ and $1 - \omega^2 = \varepsilon k = O(\varepsilon)$.

10.7 Duffing's Equation. Forced Subharmonic Oscillations

We again consider Duffing's equation

$$\ddot{x} + \delta\dot{x} + k^2 x - \gamma x^3 = A \cos t. \tag{10.64}$$

We study the problem of the existence of subharmonic solutions of order 1/3, i.e., periodic solutions with the least period $\frac{2\pi}{3}$ in equation (10.64).

Assume that k is close to 1/3, and let

$$k^2 = \frac{1}{9} - \varepsilon\frac{1}{9}m.$$

It is convenient for us to change the time: $t = 3\tau$ and let $9\gamma = \varepsilon\gamma_1$, $9A = A_1$, $9\delta = \delta_1$. We obtain an equation in terms of the time τ:

$$\ddot{x} + x = A_1 \cos 3\tau + \varepsilon[mx - \delta_1\dot{x} + \gamma_1 x^3]. \tag{10.65}$$

The periodic solution of the equation

$$\ddot{x} + x = A_1 \cos 3\tau$$

has the form

$$\varphi(\tau) = -\frac{A_1}{8}\cos 3\tau.$$

In equation (10.65), we make a change

$$x(\tau) = y(\tau) + \varphi(\tau)$$

and obtain the equation

$$\ddot{y} + y = \varepsilon[m(y + \varphi) - \delta_1(\dot{y} + \dot{\varphi}) + \gamma_1(y + \varphi)^3].$$

Transform this equation into the system in the standard form by the change (10.34):

$$y = a\cos(\tau + \psi), \quad \dot{y} = -a\sin(\tau + \psi).$$

We arrive at the system

$$\begin{aligned}
&\dot{a} = -\varepsilon[m(a\cos(\tau+\psi) - \tfrac{A_1}{8}\cos 3\tau) - \\
&-\delta_1(-a\sin(\tau+\psi) + \tfrac{3A_1}{8}\sin 3\tau) + \\
&+\gamma_1(a\cos(\tau+\psi) - \tfrac{A_1}{8}\cos 3\tau)^3]\sin(\tau+\psi), \\
&\dot{\psi} = -\tfrac{1}{a}\varepsilon[m(a\cos(\tau+\psi) - \tfrac{A_1}{8}\cos 3\tau) - \\
&-\delta_1(-a\sin(\tau+\psi) + \tfrac{3A_1}{8}\sin 3\tau) + \\
&+\gamma_1(a\cos(\tau+\psi) - \tfrac{A_1}{8}\cos 3\tau)^3]\cos(\tau+\psi).
\end{aligned} \tag{10.66}$$

Averaging system (10.66) over the time τ and taking into account the identity

$$\cos^3\alpha = 3/4\cos\alpha + 1/4\cos 3\alpha$$

yields the averaged system

$$\begin{aligned}
&\dot{\bar{a}} = -\varepsilon\tfrac{3A_1\gamma_1\bar{a}^2}{64}\sin 3\bar{\psi} - \varepsilon\tfrac{a\delta_1}{2}, \\
&\dot{\bar{\psi}} = -\varepsilon\left[\tfrac{m}{2} + \tfrac{3\gamma_1\bar{a}^2}{8} - \gamma_1\tfrac{3A_1\bar{a}}{64}\cos 3\bar{\psi} + \tfrac{3A_1^2\gamma_1}{256}\right].
\end{aligned} \tag{10.67}$$

The stationary solutions of the averaged system are determined from the system of equations

$$\begin{aligned}
R(a,\psi) &= -\tfrac{3A_1\gamma_1\bar{a}^2}{64}\sin 3\bar{\psi} - \tfrac{a\delta_1}{2} = 0, \\
\Psi(a,\psi) &= -\left[\tfrac{m}{2} + \tfrac{3\gamma_1\bar{a}^2}{8} - \gamma_1\tfrac{3A_1\bar{a}}{64}\cos 3\bar{\psi} + \tfrac{3A_1^2\gamma_1}{256}\right] = 0.
\end{aligned} \tag{10.68}$$

First, we consider the case when $\delta_1 = 0$. Then the first equation of system (10.68) takes the form

$$\frac{3A_1\gamma_1\bar{a}^2}{64}\sin 3\bar{\psi} = 0.$$

Let $\bar{\psi}_1 = \pi/3$ be the solution of this equation. Then the second equation assumes the form

$$F(\bar{a}) = m + \frac{3\gamma_1\bar{a}^2}{4} + \frac{3A_1\gamma_1\bar{a}}{32} + \frac{3\gamma_1 A_1^2}{128} = 0. \tag{10.69}$$

Solutions of this equation are real under the following condition

$$-\frac{21\gamma_1^2 A_1^2}{1024} - m\gamma_1 \geq 0. \tag{10.70}$$

Hence, it is necessary that m and γ_1 have opposite signs. If this condition holds, then inequality (10.70) holds provided (in the original notation)

$$\left|k^2 - \frac{1}{9}\right| \geq \frac{21\cdot 81}{1024}A^2|\gamma|, \quad \left(k^2 - \frac{1}{9}\right)\gamma > 0. \tag{10.71}$$

This inequality was obtained by Malkin [1956] by the methods of perturbation theory. Stoker [1950] arrived at the inequality

$$\left|k^2 - \frac{1}{9}\right| \geq \frac{21}{1024k^2}A^2|\gamma|$$

using less strict methods.

If inequality (10.71) is fulfilled, then equation (10.69) has two solutions:

$$a_{1,2} = \frac{-\frac{3A_1\gamma_1}{16} \pm 2\sqrt{-\frac{63A_1^2\gamma_1^2}{1024} - 3m\gamma_1}}{3\gamma_1}.$$

The matrix of an averaged system linearized on the stationary solution takes the form

$$\Delta(a^*, \frac{\pi}{3}) = \begin{pmatrix} 0 & -\varepsilon\frac{9A_1\gamma_1(a^*)^2}{64} \\ \varepsilon\left(-\frac{3}{2}\gamma_1 a^* - \frac{3A_1\gamma_1}{32}\right) & 0 \end{pmatrix}, \tag{10.72}$$

where a^* is the solution of equation (10.69). It is easy to see that the determinant of this matrix is non-zero if inequality (10.71) is strict. In this case, equation (10.65), for sufficiently small ε, has two subharmonic solutions with the period $2\pi/3$, and, so does equation (10.64). On account of the formulas for the roots of $a_{1.2}$, we have the following result. The determinant of matrix (10.72) is negative if $a^* = a_1$ and is positive if $a^* = a_2$.

Thus, the stationary solution $(a_1, \pi/3)$ of the averaged system is unstable. Hence, for sufficiently small ε, one subharmonic solution is unstable. The problem of the stability of a subharmonic solution conforming to the stationary solution $(a_2, \pi/3)$ cannot be solved with the help of Theorem 10.2.

The stationary solution of the averaged system at $\psi_2 = 2\pi/3$ results in the same subharmonic solutions of equation (10.64) as the stationary solutions $(a_{1,2}, \pi/3)$.

We now return to the case of $\delta_1 \neq 0$. From the first equation of system (10.68), we obtain

$$\sin 3\bar{\psi} = -\frac{32\delta_1}{3A_1\gamma_1 a}.$$

This equation has solutions if the right-hand side is less than 1 in absolute value. Substituting this solution into the second equation yields the equation for determining $\bar{a}$:

$$m + \frac{3\gamma_1\bar{a}^2}{4} \mp \frac{3A_1\gamma_1}{32}\sqrt{1 - \frac{1024\delta_1^2}{9A_1^2\gamma_1^2\bar{a}^2}} + \frac{3A_1^2\gamma_1}{128} = 0.$$

Let us show the conditions for the stability of the stationary solution of the averaged system. Doing the same computations as in the previous clause, we obtain the following (with the arrows above the variables a, ψ omitted)

$$(R_a\Psi\psi - \Psi_a R_\psi)\frac{da}{dm} = \frac{3}{4}ma + \frac{9}{16}\gamma_1 a^3 + \frac{9}{512}A_1^2\gamma_1 a.$$

Therefore, the condition of the asymptotic stability of equilibrium of the averaged equation can be presented as

$$\begin{array}{lll} \frac{da}{dm} > 0, & if & \frac{3}{4}ma + \frac{9}{16}\gamma_1 a^3 + \frac{9}{512}A_1^2\gamma_1 a > 0, \\ \frac{da}{dm} < 0, & if & \frac{3}{4}ma + \frac{9}{16}\gamma_1 a^3 + \frac{9}{512}A_1^2\gamma_1 a < 0. \end{array}$$

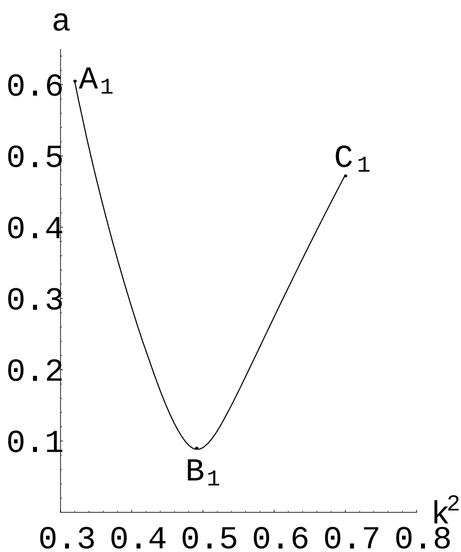

FIGURE 10.3: An Amplitude Curve for the Subharmonic of Order 1/3 of Damped Duffing's Equation.

Figure 10.3 demonstrates the amplitude curve of the relationship between the amplitude a and the square of eigenfrequency k^2 of system (10.64); the curve is determined by the equation

$$k^2 = \frac{1}{9} + \frac{3}{4}a^2\gamma + \sqrt{\frac{729A^2a^2}{1024} - \frac{1}{9}\delta^2} + \frac{243A\gamma}{128},$$

where $A = 1$, $\gamma = 0.2$, $\delta = 0.1$.

Exercise 10.3(a). Investigate the problem of the existence and the stability of a subharmonic of order 1/2 in the equation

$$\ddot{x} + \varepsilon\delta\dot{x} + \varepsilon\gamma x^2 = A\cos t,$$

where $\varepsilon > 0$ is a small parameter, $\delta > 0$, γ, and A are constants.

Exercise 10.3(b). Consider equation (10.46) with $A_1 = \varepsilon A_2 = O(\varepsilon)$. Do subharmonic solutions with the period $2\pi/3$ exist in equation (10.46)?

10.8 Almost Periodic Solutions of the Forced Undamped Duffing's Equation

Consider Duffing's equation

$$\ddot{x} + x - \varepsilon\nu x + \varepsilon^3 x^3 = f(t), \tag{10.73}$$

where $\varepsilon > 0$ is a small parameter, $\nu > 0$ is a constant, and $f(t)$ is an almost periodic function.

We shall be interested in the problem of the existence of an almost periodic solution of equation (10.73). Let $y = \varepsilon x$. Then equation (10.73) can be rewritten

$$\ddot{y} + y = \varepsilon[\nu x - x^3 + f(t)]. \tag{10.74}$$

Making a change (10.27):

$$y = a\cos t + b\sin t, \quad \dot{y} = -a\sin t + b\cos t$$

we transform equation (10.74) into the system in the standard form

$$\begin{aligned} \frac{da}{dt} &= -\varepsilon[\nu(a\cos t + b\sin t) - (a\cos t + b\sin t)^3 + f(t)]\sin t, \\ \frac{db}{dt} &= \varepsilon[\nu(a\cos t + b\sin t) - (a\cos t + b\sin t)^3 + f(t)]\cos t. \end{aligned} \tag{10.75}$$

Averaging the right-hand side of (10.75) yields the averaged system

$$\begin{aligned} \frac{d\bar{a}}{dt} &= -\varepsilon\left[\frac{\nu\bar{b}}{2} - \frac{3}{8}\bar{a}^2\bar{b} - \frac{3}{8}\bar{b}^3 - f_0\right], \\ \frac{d\bar{b}}{dt} &= \varepsilon\left[\frac{\nu\bar{a}}{2} - \frac{3}{8}\bar{a}^3 - \frac{3}{8}\bar{a}\bar{b}^2 + f_1\right], \end{aligned} \tag{10.76}$$

where

$$f_0 = \langle f(t)\sin t\rangle, \quad f_1 = \langle f(t)\cos t\rangle.$$

The stationary solutions of the averaged system are determined from the system of equations

$$\begin{aligned} \frac{\nu\bar{b}}{2} - \frac{3}{8}\bar{a}^2\bar{b} - \frac{3}{8}\bar{b}^3 - f_0 = 0, \\ \frac{\nu\bar{a}}{2} - \frac{3}{8}\bar{a}^3 - \frac{3}{8}\bar{a}\bar{b}^2 + f_1 = 0. \end{aligned} \tag{10.77}$$

Assume that $\bar{b} = 0, f_0 = 0$, and $f_1 > 0$. Then from system (10.77) we obtain the equation

$$3\bar{a}^3 - 4\nu\bar{a} - f_1 = 0. \tag{10.78}$$

Let us calculate the discriminant of cubic equation (10.78). We obtain that equation (10.78) has three real roots provided

$$\nu > \left(\frac{9f_1}{2}\right)^{\frac{2}{3}}. \tag{10.79}$$

Consider one of the roots, e.g. $\bar{a} = a_1 < 0$. By linearizing the right-hand side of averaged system (10.76) on the stationary solution $(a_1, 0)$, we obtain the matrix

$$A = \begin{pmatrix} 0 & -\frac{\nu}{2} + \frac{3}{8}a_1^2 \\ \frac{\nu}{2} - \frac{9}{8}a_1^2 & 0 \end{pmatrix}.$$

It is easy to see that eigenvalues of the matrix A are real and have opposite signs. Therefore, it follows from Theorems 9.1 and 9.3 that for sufficiently small ε, system (10.75) has an unstable almost periodic solution. It is clear that in the cases of $f_0 < 0$, $f_1 = 0$ and $f_0 = 0$, $|f_1| \neq 0$, we obtain the same results. Let us formulate these results as applied to Duffing's equation (10.73).

Theorem 10.3. *Let $f_0 f_1 = 0$, $|f_0 + f_1| = \mu > 0$, and ν be any fixed number that obeys inequality (10.79). Then there exists $\varepsilon_0 = \varepsilon_0(\nu) > 0$ such that for $0 < \varepsilon < \varepsilon_0$ equation (10.73) has the unstable almost periodic solution $x(t, \varepsilon)$, for which $\varepsilon x(t, \varepsilon) - (a \cos t + b \sin t) \to 0$ as $\varepsilon \to 0$. Here $a \neq 0$ and $b = 0$ if $f_1 \neq 0$, $f_0 = 0$, or $b \neq 0$, $a = 0$ if $f_0 \neq 0$, $f_1 = 0$.*

This theorem was deduced by Seifert (see Seifert [1971], [1972], and Fink [1974]). The problem of the existence of an almost periodic solution of Duffing's equation without damping was investigated by Moser [1965] using the methods of KAM theory (see Moser [1973]). He considered the equation

$$\ddot{x} + a^2(\mu)x + bx^3 = \mu f(t, x, \dot{x}), \tag{10.80}$$

where $f(t, x, \dot{x})$ is quasi-periodic in t with the basis frequencies $\omega_1, \ldots, \omega_m$ meeting the usual conditions of KAM theory. It is assumed that $f(t, x, \dot{x})$ is a real analytical function of $x, \dot{x}$ in some neighborhood of $x = \dot{x} = 0$ and $f(-t, x, -\dot{x}) = f(t, x, \dot{x})$. Under these conditions, there exists a real analytical function $a(\mu)$ and an almost periodic solution $x = \varphi(t, \mu)$ of equation (10.80) such that $a(0) = 1$, $\varphi(t, 0) \equiv 0$.

10.9 The Forced Van der Pol Equation. Almost Periodic Solutions in Non-Resonant Case

Consider a differential equation

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x} + A \sin \omega_1 t + B \sin \omega_2 t. \tag{10.81}$$

Here, $\varepsilon > 0$ is a small parameter, A and B are constants, the quotient ω_1/ω_2 is irrational. Assuming $A = B = 0$, we obtain the known Van der Pol equation (Van der Pol [1927]) that for all values of the parameter ε has a stable periodic solution (a limit cycle). For small ε, the period of this solution is close to the period of natural vibrations, i.e. to 2π.

Similarly to the problem of the periodic perturbance, we shall distinguish between the resonance and non-resonant cases. We say that a non-resonant case takes place when the value of the expression $m + m_1\omega_1 + m_2\omega_2$ is not of order ε while m, m_1, m_2 assume integers with values such that $|m| + |m_1| + |m_2| \leq 4$, and $m \neq 0$.

First we investigate the non-resonant case. For $\varepsilon = 0$, equation (10.81) turns into the linear inhomogeneous equation

$$\ddot{x} + x = A \sin \omega_1 t + B \sin \omega_2 t$$

with the general solution in the form

$$\begin{aligned} x(t) &= a \cos t + b \sin t + \tfrac{A}{1-\omega_1^2} \sin \omega_1 t + \tfrac{B}{1-\omega_2^2} \sin \omega_2 t, \\ \dot{x}(t) &= -a \sin t + b \cos t + \tfrac{A\omega_1}{1-\omega_1^2} \cos \omega_1 t + \tfrac{B\omega_2}{1-\omega_2^2} \cos \omega_2 t, \end{aligned} \tag{10.82}$$

where a, b are arbitrary constants. We shall treat formulas (10.82) as a change of variables. We take a, b as the new variables instead of x, $\dot{x}$ and arrive at the system of equations

$$\begin{aligned} &\tfrac{da}{dt} \cos t + \tfrac{db}{dt} \sin t = 0, \\ -&\tfrac{da}{dt} \sin t + \tfrac{db}{dt} \cos t = \varepsilon(1 - x^2) \tfrac{dx}{dt}. \end{aligned}$$

Thus we obtain the standard form system

$$\begin{aligned} \tfrac{da}{dt} &= -\varepsilon(1 - x^2) \tfrac{dx}{dt} \sin t \\ \tfrac{db}{dt} &= \varepsilon(1 - x^2) \tfrac{dx}{dt} \cos t \end{aligned} \tag{10.83}$$

where $x, \dot{x}$ should be replaced by their expressions from formulas (10.82). The mean value of the right-hand side of system (10.83) depends on whether the frequencies 1, ω_1, ω_2 are resonant or non-resonant. Assuming that the frequencies as non-resonant leads to the averaged system

$$\begin{aligned} \tfrac{d\bar{a}}{dt} &= \varepsilon P(\bar{a}, \bar{b}), \\ \tfrac{d\bar{b}}{dt} &= \varepsilon Q(\bar{a}, \bar{b}), \end{aligned} \tag{10.84}$$

where

$$\begin{aligned} P(\bar{a}, \bar{b}) &= a \left[\tfrac{1}{2} - \tfrac{1}{8}(\bar{a}^2 + \bar{b}^2) - \tfrac{A^2}{4(1-\omega_1^2)^2} - \tfrac{B^2}{4(1-\omega_2^2)^2} \right] \\ Q(\bar{a}, \bar{b}) &= b \left[\tfrac{1}{2} - \tfrac{1}{8}(\bar{a}^2 + \bar{b}^2) - \tfrac{A^2}{4(1-\omega_1^2)^2} - \tfrac{B^2}{4(1-\omega_2^2)^2} \right]. \end{aligned}$$

The stationary solutions of averaged system (10.84) are determined from the system of equations

$$P(\bar{a}, \bar{b}) = 0, \quad Q(\bar{a}, \bar{b}) = 0. \tag{10.85}$$

The system of equations (10.85) has the solution $\bar{a} = \bar{b} = 0$. If the non-zero stationary solutions exist, then they fit the equation

$$4 - (\bar{a}^2 + \bar{b}^2) - \frac{A^2}{(1 - \omega_1^2)^2} - \frac{B^2}{(1 - \omega_2^2)^2} = 0. \tag{10.86}$$

The matrix of the averaged system linearized on the stationary solution takes the form

$$\Delta(a_0, b_0) = \begin{pmatrix} C - \frac{1}{4}a_0^2 & -\frac{1}{4}a_0 b_0 \\ -\frac{1}{4}a_0 b_0 & C - \frac{1}{4}b_0^2 \end{pmatrix}, \tag{10.87}$$

where

$$C = \left[\frac{1}{2} - \frac{1}{8}(\bar{a}_0^2 + \bar{b}_0^2) - \frac{A^2}{4(1-\omega_1^2)^2} - \frac{B^2}{4(1-\omega_2^2)^2}\right].$$

For the stationary solution $a_0 = b_0 = 0$, the determinant of matrix (10.87) is non-zero, and this stationary solution is asymptotically stable if the inequality

$$\frac{A^2}{(1-\omega_1^2)^2} + \frac{B^2}{(1-\omega_2^2)^2} > 2 \tag{10.88}$$

holds and unstable provided

$$\frac{A^2}{(1-\omega_1^2)^2} + \frac{B^2}{(1-\omega_2^2)^2} < 2. \tag{10.89}$$

Hence, if inequality (10.88) is satisfied, then, for sufficiently small ε, equation (10.81) has an asymptotically stable almost periodic solution that for $\varepsilon = 0$ turns into the almost periodic function

$$\frac{A\omega_1}{1-\omega_1^2}\cos\omega_1 t + \frac{B\omega_2}{1-\omega_2^2}\cos\omega_2 t \tag{10.90}$$

containing only the frequencies ω_1 and ω_2. Inequality (10.88) will probably not hold if both frequencies ω_1 and ω_2 are sufficiently different from 1.

If inequality (10.89) holds, then equation (10.81) for sufficiently small ε has an unstable almost periodic solution that for $\varepsilon = 0$ turns into function (10.90).

If inequality (10.89) holds, then equations (10.87) has an infinite set of solutions lying on the circle

$$a^2 + b^2 = 4 - \frac{A^2}{(1-\omega_1^2)} - \frac{B^2}{(1-\omega_2^2)} = 0. \tag{10.91}$$

For the stationary solutions of the averaged system that satisfy equation (10.91), the determinant of matrix (10.87) equals zero and Theorem 10.1 is not applicable. In this case, in the neighborhood of the family of solutions (10.91), there exists a stable integral manifold of solutions of equation (10.81) (see Hale [1969]). We do not consider the problem of the existence of integral manifolds here and bring in some more elementary reasoning related to the family of solutions (10.91).

We rewrite system (10.83) as

$$\begin{aligned} \tfrac{da}{dt} &= \varepsilon A(t, a, b), \\ \tfrac{db}{dt} &= \varepsilon B(t, a, b). \end{aligned} \tag{10.92}$$

In this system, we make a standard change of the method of averaging

$$a = p + \varepsilon u(t, p, q), \quad b = q + \varepsilon v(t, p, q), \tag{10.93}$$

where the functions $u(t, p, q)$, $v(t, p, q)$ are determined from the equations

$$\frac{\partial u}{\partial t} = A(t, p, q) - P(p, q),$$
$$\frac{\partial v}{\partial t} = B(t, p, q) - Q(p, q),$$

as the functions with the zero mean value with respect to t. Here, $P(p, q)$, $Q(p, q)$ are determined by formulas (10.84), where $\bar{a}$, $\bar{b}$ are replaced by p, q, respectively. After the change (10.93), system (10.92) takes the form

$$\begin{aligned} \frac{dp}{dt} &= \varepsilon P(p, q) + \varepsilon^2 R_1(t, p, q, \varepsilon), \\ \frac{dq}{dt} &= \varepsilon Q(p, q) + \varepsilon^2 R_2(t, p, q, \varepsilon). \end{aligned} \tag{10.94}$$

Further, in system (10.94), we pass to the polar coordinates

$$p = M \cos \alpha, \quad q = M \sin \alpha$$

and let $\alpha = t - \theta$. As a result, we obtain the system

$$\begin{aligned} \frac{dM}{dt} &= \varepsilon R(M) + \varepsilon^2 Q_1(t, \theta, M, \varepsilon), \\ \frac{d\theta}{dt} &= 1 + \varepsilon^2 Q_2(t, \theta, M, \varepsilon), \end{aligned} \tag{10.95}$$

where

$$R(M) = \frac{M}{8}\left(4 - M^2 - \frac{2A^2}{(1-\omega_1^2)^2} - \frac{2B^2}{(1-\omega_2^2)^2}\right).$$

If inequality (10.89) holds, then the equation $R(M) = 0$ has the solution $M = M_0$, where

$$M_0^2 = 4 - \frac{2A^2}{(1-\omega_1^2)^2} - \frac{2B^2}{(1-\omega_2^2)^2}. \tag{10.96}$$

It is easy to see that $R'(M_0) = -1/4M_0^2 < 0$. Therefore, the stationary solution $M = M_0$ of the equation

$$\frac{dM}{dt} = \varepsilon R(M) \tag{10.97}$$

is asymptotically stable. The system of the first approximation

$$\frac{dM}{dt} = \varepsilon R(M), \quad \frac{d\theta}{dt} = 1$$

for system (10.95) has a periodic solution (a limit cycle) that is Lyapunov stable. Thus, in the first approximation, we have a family of the solutions $M = M_0$, $\theta = t + c$, where c is an arbitrary constant. This solution for equation (10.81) takes the form

$$x(t) = M_0 \cos(t + c) + \frac{A}{1-\omega_1^2} \sin \omega_1 t + \frac{B}{1-\omega_2^2} \sin \omega_2 t. \tag{10.98}$$

Hence, equation (10.81) in the first approximation is a family of almost periodic solutions, for which M_0 is defined by equality (10.96).

Return to system (10.95). The functions $Q_1(t,\theta,M,\varepsilon)$, $Q_2(t,\theta,M,\varepsilon)$ are almost periodic in t, periodic in θ and smooth in M. Therefore, these functions are bounded for $t \geq 0$ if the variable M changes in some bounded neighborhood of the point M_0. Since the solution $M = M_0$ of equation (10.97) is asymptotically stable, we see that it is uniformly asymptotically stable. The Malkin Theorem of the stability under constantly acting perturbations (see Appendix B) is applicable to the first equation in (10.95). Then, for any $\eta > 0$, it is possible to find $\delta > 0$ such that for sufficiently small ε the inequality

$$|M(t,\varepsilon) - M_0| < \eta, \quad t > 0$$

holds true if

$$|M(0,\varepsilon) - M_0| < \delta,$$

where $M(t,\varepsilon)$ is the solution of the first equation in system (10.95). From the second equation of system (10.95), we obtain that

$$\theta(t,\varepsilon) = \theta(0,\varepsilon) + (1 + \varepsilon^2\varphi(t,\varepsilon))t,$$

where the function $\varphi(t,\varepsilon)$ is bounded on $[0,\infty)$. The solution of equation (10.81) that conforms to the solution $M(t,\varepsilon), \theta(t,\varepsilon)$ of system (10.95) takes the form

$$\begin{aligned} x(t,\varepsilon) = M(t,\varepsilon)\cos(t(1+\varepsilon^2\varphi(t,\varepsilon)) + \theta(0,\varepsilon)) + \tfrac{A}{1-\omega_1^2}\sin\omega_1 t + \\ + \tfrac{B}{1-\omega_2^2}\sin\omega_2 t + \varepsilon x_1(t,\varepsilon), \end{aligned} \tag{10.99}$$

where the function $x_1(t,\varepsilon)$ is bounded on $[0,\infty)$. This follows from the formulas which define the functions $u(t,p,q)$, $v(t,p,q)$. Formula (10.99) shows that the first approximation (10.98) for sufficiently small ε gives satisfactory quantitative and qualitative characteristics of the exact solution.

10.10 The Forced Van der Pol Equation. A Slowly Varying Force

Consider a differential equation

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x} + A(\tau)\sin\omega_1 t + B(\tau)\sin\omega_2 t, \tag{10.100}$$

where $\tau = \varepsilon t$ is a slow time, $A(\tau)$, $B(\tau)$ are differentiable periodic functions of the variable τ with some period T. Therefore, we assume that the Van der Pol oscillator is affected by a sum of periodic forces with a slowly varying amplitude. The frequencies ω_1, ω_2 meet the conditions of absence of resonance described in 10.8.

Using a substitution

$$\begin{aligned} x &= a\cos t + b\sin t + \tfrac{A(\tau)}{1-\omega_1^2}\sin\omega_1 t + \tfrac{B(\tau)}{1-\omega_2^2}\sin\omega_2 t, \\ \dot{x} &= -a\sin t + b\cos t + \tfrac{A(\tau)\omega_1}{1-\omega_1^2}\cos\omega_1 t + \tfrac{B(\tau)\omega_2}{1-\omega_2^2}\cos\omega_2 t, \end{aligned} \tag{10.101}$$

we transform system (10.100) into

$$\begin{aligned} &\tfrac{da}{dt}\cos t + \tfrac{db}{dt}\sin t = -\tfrac{\varepsilon A'(\tau)}{1-\omega_1^2}\sin\omega_1 t + \tfrac{\varepsilon B'(\tau)}{1-\omega_2^2}\sin\omega_2 t, \\ &-\tfrac{da}{dt}\sin t + \tfrac{db}{dt}\cos t = \varepsilon(1-x^2)\tfrac{dx}{dt} - \tfrac{\varepsilon A'(\tau)\omega_1}{1-\omega_1^2}\cos\omega_1 t - \tfrac{\varepsilon B'(\tau)\omega_2}{1-\omega_2^2}\cos\omega_2 t. \end{aligned}$$

Solving the obtained system for $\frac{da}{dt}$, $\frac{db}{dt}$, we arrive at the system in the standard form

$$\begin{aligned} \tfrac{da}{dt} &= -\varepsilon(1-x^2)\tfrac{dx}{dt}\sin t - \left[\tfrac{\varepsilon A'(\tau)}{1-\omega_1^2}\sin\omega_1 t + \tfrac{\varepsilon B'(\tau)}{1-\omega_2^2}\sin\omega_2 t\right]\cos t + \\ &+ \left[\tfrac{\varepsilon A'(\tau)\omega_1}{1-\omega_1^2}\cos\omega_1 t + \tfrac{\varepsilon B'(\tau)\omega_2}{1-\omega_2^2}\cos\omega_2 t\right]\sin t, \\ \tfrac{db}{dt} &= \varepsilon(1-x^2)\tfrac{dx}{dt}\cos t - \left[\tfrac{\varepsilon A'(\tau)}{1-\omega_1^2}\sin\omega_1 t + \tfrac{\varepsilon B'(\tau)}{1-\omega_2^2}\sin\omega_2 t\right]\sin t - \\ &- \left[\tfrac{\varepsilon A'(\tau)\omega_1}{1-\omega_1^2}\cos\omega_1 t + \tfrac{\varepsilon B'(\tau)\omega_2}{1-\omega_2^2}\cos\omega_2 t\right]\cos t. \end{aligned} \tag{10.102}$$

The right-hand sides of the system depend on the fast time t and the slow time τ. We also notice that in the right-hand sides, x and $\frac{dx}{dt}$ are presented by formulas (10.101). We average the right-hand side of system (10.102) over the fast time t, assuming that the frequencies 1, ω_1, ω_2 are non-resonant. We obtain the averaged system

$$\begin{aligned} \tfrac{d\bar{a}}{dt} &= \varepsilon P(\tau, \bar{a}, \bar{b}), \\ \tfrac{d\bar{b}}{dt} &= \varepsilon Q(\tau, \bar{a}, \bar{b}), \end{aligned} \tag{10.103}$$

where

$$\begin{aligned} P(\bar{a}, \bar{b}) &= a\left[\tfrac{1}{2} - \tfrac{1}{8}(\bar{a}^2 + \bar{b}^2) - \tfrac{A^2(\tau)}{4(1-\omega_1^2)^2} - \tfrac{B^2(\tau)}{4(1-\omega_2^2)^2}\right] \\ Q(\bar{a}, \bar{b}) &= b\left[\tfrac{1}{2} - \tfrac{1}{8}(\bar{a}^2 + \bar{b}^2) - \tfrac{A^2(\tau)}{4(1-\omega_1^2)^2} - \tfrac{B^2(\tau)}{4(1-\omega_2^2)^2}\right]. \end{aligned}$$

Hence, the averaged system is a system with the T- periodic coefficients. The stationary (periodic) solution of the averaged system is $\bar{a} = \bar{b} = 0$. Linearizing the averaged equation on the solution $\bar{a} = \bar{b} = 0$, we obtain a system of decoupled equations that are written in terms of the time τ

$$\begin{aligned} \tfrac{d\bar{a}}{d\tau} &= \left(\tfrac{1}{2} - \tfrac{A^2(\tau)}{4(1-\omega_1^2)^2} - \tfrac{B^2(\tau)}{4(1-\omega_2^2)^2}\right)\bar{a}, \\ \tfrac{d\bar{b}}{d\tau} &= \left(\tfrac{1}{2} - \tfrac{A^2(\tau)}{4(1-\omega_1^2)^2} - \tfrac{B^2(\tau)}{4(1-\omega_2^2)^2}\right)\bar{b}. \end{aligned} \tag{10.104}$$

If inequality

$$\frac{\langle A^2(\tau)\rangle}{(1-\omega_1^2)^2} + \frac{\langle B^2(\tau)\rangle}{(1-\omega_2^2)^2} > 2 \tag{10.105}$$

is met, where $\langle A^2(\tau)\rangle$, $\langle B^2(\tau)\rangle$ are the mean values of the periodic functions $A^2(\tau)$, $B^2(\tau)$, respectively, then the zero solution of system (10.104) is asymptotically stable; if

$$\frac{\langle A^2(\tau)\rangle}{(1-\omega_1^2)^2} + \frac{\langle B^2(\tau)\rangle}{(1-\omega_2^2)^2} < 2, \tag{10.106}$$

then the zero solution of system (10.104) is unstable. It follows from Theorems 9.6 and 9.7 that, for sufficiently small ε, system (10.102), in a sufficiently small neighborhood of zero, has an almost periodic solution that is asymptotically stable if inequality (10.105) holds and unstable if inequality (10.106) holds. Thus, equation (10.100) has the almost periodic solution

$$x(t,\varepsilon) = a(t,\varepsilon)\cos t + b(t,\varepsilon)\sin t + \frac{A(\tau)}{1-\omega_1^2}\sin\omega_1 t + \frac{B(\tau)}{1-\omega_2^2}\sin\omega_2 t.$$

10.11 The Forced Van der Pol Equation. Resonant Oscillations

Consider only the resonance case when ω_1 differs from 1 by a quantity of order ε.

Let

$$1 = \omega_1^2 - \varepsilon m, \quad A = \varepsilon\lambda.$$

Then, instead of equation (10.81), we obtain the equation

$$\ddot{x} + \omega_1 x = \varepsilon(1-x^2)\dot{x} + \varepsilon m x + \varepsilon\lambda\sin\omega_1 t + B\sin\omega_2 t. \tag{10.107}$$

Now, by change (10.82), we pass to the system of variables a, b

$$\begin{aligned} \tfrac{da}{dt} &= -\tfrac{\varepsilon}{\omega_1}\left[(1-x_1^2)\tfrac{dx_1}{dt} + m x_1 + \lambda\sin\omega_1 t\right]\sin\omega_1 t, \\ \tfrac{db}{dt} &= \tfrac{\varepsilon}{\omega_1}\left[(1-x_1^2)\tfrac{dx_1}{dt} + m x_1 + \lambda\sin\omega_1 t\right]\cos\omega_1 t, \end{aligned} \tag{10.108}$$

where,

$$\begin{aligned} x_1 &= a\cos\omega_1 + b\sin\omega_1 t + \tfrac{B}{\omega_1^2-\omega_2^2}\sin\omega_2 t, \\ \tfrac{dx_1}{dt} &= -\omega_1 a\sin\omega_1 t + \omega_1 t\cos\omega_1 t\tfrac{B\omega_2}{\omega_1^2-\omega_2^2}\cos\omega_2 t. \end{aligned}$$

Averaging system (10.108) implies

$$\begin{aligned} \tfrac{d\bar{a}}{dt} &= \varepsilon\left[-\tfrac{\lambda}{2\omega_1} + \tfrac{m\bar{b}}{2\omega_1} + \tfrac{\bar{a}}{2}\left(1 - \tfrac{B^2}{2(\omega_1^2-\omega_2^2)} - \tfrac{\bar{a}^2+\bar{b}^2}{4}\right)\right], \\ \tfrac{d\bar{b}}{dt} &= \varepsilon\left[-\tfrac{m\bar{a}}{2\omega_1} + \tfrac{\bar{b}}{2}\left(1 - \tfrac{B^2}{2(\omega_1^2-\omega_2^2)} - \tfrac{\bar{a}^2+\bar{b}^2}{4}\right)\right]. \end{aligned} \tag{10.109}$$

For simplicity, we only consider the case of exact resonance, i.e., $m = 0$. In this case, to determine stationary solutions, we obtain the system of equations

$$\begin{aligned} -\tfrac{\lambda}{2\omega_1} + \tfrac{\bar{a}}{2}\left(1 - \tfrac{B^2}{2(\omega_1^2-\omega_2^2)} - \tfrac{\bar{a}^2+\bar{b}^2}{4}\right) &= 0, \\ \tfrac{\bar{b}}{2}\left(1 - \tfrac{B^2}{2(\omega_1^2-\omega_2^2)} - \tfrac{\bar{a}^2+\bar{b}^2}{4}\right) &= 0. \end{aligned} \tag{10.110}$$

System (10.110) has the solution $\bar{b} = 0$, $\bar{a} = a_0$, where a_0 is a root of the cubic equation

$$f(a_0) = -\frac{1}{8}a_0^3 + \left(\frac{1}{2} - \frac{B^2}{4(\omega_1^2 - \omega^2)^2}\right) a_0 - \frac{\lambda}{2\omega_1} = 0. \qquad (10.111)$$

The matrix of the averaged system linearized on the stationary solution $(a_0, 0)$ takes the form

$$\Delta(a_0, 0) = \begin{pmatrix} f'(a_0) & 0 \\ 0 & f'(a_0) + \frac{a_0^2}{4} \end{pmatrix}, \qquad (10.112)$$

where $f'(a_0)$ is the derivative of the left-hand side of equation (10.111) and $[f'(a_0) + \frac{a_0^2}{4} = \frac{\lambda}{2\omega_1} a_0]$. If equation (10.111) has one or three real solutions, then the determinant of matrix (10.112) is non-zero. In this case, equation (10.107), for sufficiently small ε will have one or three almost periodic solutions that at $\varepsilon = 0$ turn into the almost periodic function

$$a_0 \cos \omega_1 t + \frac{B}{\omega_1^2 - \omega_2^2} \sin \omega_2 t.$$

Because equation (10.111) cannot have a triple root, we see that there always exists at least one almost periodic solution. The conditions of the asymptotic stability of the obtained almost periodic solutions take the form

$$a_0 < 0, \quad f'(a_0) < 0.$$

Hence, those almost periodic solutions will be stable if they correspond to the negative roots of equation (10.111). Because the free term of equation (10.111) is negative ($\lambda > 0$), we have that equation (10.111) always has at least one negative root. If equation (10.111) has only one negative root, then $f'(a_0) < 0$ and the corresponding almost periodic solution is asymptotically stable for sufficiently small ε. If equation (10.111) has three negative roots, then almost periodic solutions conforming to the maximum and minimum roots are asymptotically stable, and the almost periodic solution that conforms to the respective mean root is unstable.

10.12 Two Weakly Coupled Van der Pol Oscillators

There have been a good number of publications devoted to the research into dynamics of two weakly coupled Van der Pol oscillators (see, e.g., Rand and Holmes [1980], Chakraborty and Rand [1988], Rand [2005], Camacho, Rand, and Howland [2004], and Qinsheng [2004]).

Here we consider two Van der Pol oscillators coupled with a weak gyroscopic link

$$\begin{aligned} \tfrac{d^2x_1}{dt^2} + \lambda_1^2 x_1 = \varepsilon\left[(1-x_1^2)\tfrac{dx_1}{dt} + n\tfrac{dx_2}{dt}\right], \\ \tfrac{d^2x_2}{dt^2} + \lambda_2^2 x_2 = \varepsilon\left[(1-x_2^2)\tfrac{dx_2}{dt} - n\tfrac{dx_1}{dt}\right] \end{aligned} \tag{10.113}$$

where $\varepsilon > 0$ is a small parameter, λ_1,λ_2, and n are positive constants. System (10.113) was also used in the paper by Mitropolskii and Samoilenko [1976b]. In this paper, as well as in the paper by Mitropolskii and Samoilenko [1976a], the method of averaging was used for a versatile analysis of weakly linear multidimensional systems of the first and second orders. It is assumed that the frequencies λ_1, λ_2 are non-resonant. It is shown that system (10.113) has a two-dimensional stable invariant torus and two unstable periodic solutions. Approximate formulas for the solutions were obtained.

Here, we confine ourselves to a more elementary analysis of the problem.

We assume that the numbers λ_1 and λ_2 are incommensurable. We transform system (10.113) by a change of variables

$$\begin{aligned} x_1 &= a_1\cos\lambda_1 t + b_1\sin\lambda_1 t, \quad \dot{x}_1 = -a_1\lambda_1 sin\lambda_1 t + b_1\lambda_1\cos\lambda_1 t, \\ x_2 &= a_2\cos\lambda_2 t + b_2\sin\lambda_2 t, \quad \dot{x}_2 = -a_2\lambda_2\sin\lambda_2 t + b_2\lambda_2\cos\lambda_2 t \end{aligned}$$

into the system in the standard form

$$\begin{aligned} \tfrac{da_1}{dt} &= -\tfrac{\varepsilon}{\lambda_1}F_1\sin\lambda_1 t, \\ \tfrac{db_1}{dt} &= \tfrac{\varepsilon}{\lambda_1}F_1\cos\lambda_1 t, \\ \tfrac{da_2}{dt} &= -\tfrac{\varepsilon}{\lambda_2}F_2\sin\lambda_2 t, \\ \tfrac{db_2}{dt} &= \tfrac{\varepsilon}{\lambda_2}F_2\cos\lambda_2 t, \end{aligned} \tag{10.114}$$

where

$$F_1 = (1-x_1^2)\frac{dx_1}{dt} + n\frac{dx_2}{dt}, \quad F_2 = (1-x_2^2)\frac{dx_2}{dt} - n\frac{dx_1}{dt}.$$

Averaging system (10.114) yields the averaged system

$$\begin{aligned} \tfrac{d\bar{a}_1}{dt} &= \varepsilon P(\bar{a}_1,\bar{b}_1), \quad \tfrac{d\bar{b}_1}{dt} = \varepsilon Q(\bar{a}_1,\bar{b}_1), \\ \tfrac{d\bar{a}_2}{dt} &= \varepsilon P(\bar{a}_2,\bar{b}_2), \quad \tfrac{d\bar{b}_2}{dt} = \varepsilon Q(\bar{a}_2,\bar{b}_2), \end{aligned} \tag{10.115}$$

where

$$P(\bar{a}_i,\bar{b}_i) = \tfrac{1}{2}\bar{a}_i\left[1-\tfrac{1}{4}(\bar{a}_i^2+\bar{b}_i^2)\right], \quad Q(\bar{a}_i,\bar{b}_i) = \tfrac{1}{2}\bar{b}_i\left[1-\tfrac{1}{4}(\bar{a}_i^2+\bar{b}_i^2)\right], \\ i = 1,2.$$

System (10.115) has the zero equilibrium

$$1)\ a_i = b_i = 0, \quad i = 1,2$$

and an infinite set of equilibria lying on the circles

$$2)\ a_1^2+b_1^2 = 4, \quad a_2^2+b_2^2 = 4, \quad 3)\ a_1 = b_1 = 0,\ a_2^2+b_2^2 = 4,$$

$$4)\ a_1^2 + b_1^2 = 4,\ a_2 = b_2 = 0.$$

In system (2.114), using the change of variables, we separate the averaged part. Rewrite system (10.114) as

$$\frac{da_i}{dt} = \varepsilon A_i(t, a_1, a_2, b_1, b_2), \quad \frac{db_i}{dt} = \varepsilon B_i(t, a_1, a_2, b_1, b_2), \quad i = 1, 2.$$

We now change over to new variables using the formulas

$$a_i = y_i + \varepsilon u_i(t, y_1, y_2, z_1, z_2), \quad b_i = z_i + \varepsilon v_i(t, y_1, y_2, z_1, z_2), \quad i = 1, 2,$$

where

$$\begin{aligned} &\tfrac{\partial u_1}{\partial t} = A_1(t, y_1, y_2, z_1, z_2) - P(y_1, y_2), \quad \tfrac{\partial u_2}{\partial t} = A_2(t, y_1, y_2, z_1, z_2) - \\ &-P(z_1, z_2), \\ &\tfrac{\partial v_1}{\partial t} = B_1(t, y_1, y_2, z_1, z_2) - Q(y_1, y_2), \quad \tfrac{\partial v_2}{\partial t} = B_2(t, y_1, y_2, z_1, z_2) - \\ &-Q(z_1, z_2). \end{aligned}$$

Then we obtain the system

$$\begin{aligned} \tfrac{dy_1}{dt} &= \varepsilon P(y_1, y_2) + \varepsilon^2 Y_1(t, y_1, y_2, z_1, z_2), \\ \tfrac{dy_2}{dt} &= \varepsilon P(z_1, z_2) + \varepsilon^2 Y_2(t, y_1, y_2, z_1, z_2), \\ \tfrac{dz_1}{dt} &= \varepsilon Q(y_1, y_2) + \varepsilon^2 Z_1(t, y_1, y_2, z_1, z_2), \\ \tfrac{dz_2}{dt} &= \varepsilon Q(z_1, z_2) + \varepsilon^2 Z_2(t, y_1, y_2, z_1, z_2). \end{aligned} \tag{10.116}$$

Again, we introduce new variables using the formulas

$$y_i = M_i \cos\alpha_i, \quad z_i = M_i \sin\alpha_1, \quad i = 1, 2$$

and let $\alpha_i = \lambda_i t - \theta_i$. As a result, we have the system

$$\begin{aligned} \tfrac{dM_i}{dt} &= \varepsilon R_i(M_i) + \varepsilon^2 Q_{1i}(t, \theta_1, \theta_2, M_1, M_2, \varepsilon), \quad i = 1, 2 \\ \tfrac{d\theta_i}{dt} &= \lambda_i + \varepsilon^2 Q_{2i}(t, \theta_1, \theta_2, M_1, M_2, \varepsilon), \quad i = 1, 2, \end{aligned} \tag{10.117}$$

where

$$R_i(M_i) = \frac{M_i}{8}(4 - M_i^2), \quad i = 1, 2.$$

It is easy to see that the stationary solution $M_1 = M_2 = 2$ of the system

$$\frac{dM_1}{dt} = R_1(M_1), \quad \frac{dM_2}{dt} = R_2(M_2) \tag{10.118}$$

is asymptotically stable and, therefore, uniformly asymptotically stable. The stationary solutions $M_1 = 0, M_2 = 2$, and $M_1 = 2, M_2 = 0$ are unstable. The system of the first approximation for (10.117) takes the form

$$\frac{dM_1}{dt} = R_1(M_1), \quad \frac{d\theta_1}{dt} = \lambda_1, \quad \frac{dM_2}{dt} = R_2(M_2), \quad \frac{d\theta_2}{dt} = \lambda_2.$$

The solution $M_1 = M_2 = 2$, $\theta_1 = \lambda_1 t + \theta_1^0$, $\theta_2 = \lambda_2 t + \theta_2^0$, where θ_1^0, θ_2^0 are arbitrary constants, in the initial variables takes the form

$$x_1(t) = 2\cos(\lambda_1 t + \theta_1^0), \quad x_2(t) = 2\cos(\lambda_2 t + \theta_2^0). \tag{10.119}$$

Hence, system (10.113) in the first approximation has a family of almost periodic solutions (10.119).

We return to system (10.117). The functions $Q_{1i}(t, \theta_1, \theta_2, M_1, M_2, \varepsilon)$, $i = 1, 2$ are almost periodic in t, periodic in θ_1, θ_2 and smooth in M_1, M_2. Therefore, these functions are bounded for $t \geq 0$ if the variables M_1, M_2 change in some bounded neighborhood of the point $(2, 2)$. Since the solution $M_1 = M_2 = 2$ of system (10.118) is uniformly asymptotically stable, we see that the Malkin Theorem on the stability under constantly acting perturbations (see Appendix B) is applicable to the first two equations of system (10.117). Thus, for any $\eta > 0$, it is possible to find $\delta > 0$, such that, for sufficiently small ε, for the solutions $M_i(t, \varepsilon)$, $i = 1, 2$ of system (10.117) the inequalities

$$|M_i(t, \varepsilon) - 2| < \eta, \quad i = 1, 2, \quad t > 0$$

hold provided

$$|M_i(0, \varepsilon) - 2| < \delta, \quad i = 1, 2.$$

From the third and fourth equations of system (10.117), we obtain

$$\theta_i(t, \varepsilon) = \theta_i(0, \varepsilon) + (1 + \varepsilon^2 \varphi_i(t, \varepsilon))t, \quad i = 1, 2,$$

where the functions $\varphi_i(t, \varepsilon)$, $i = 1, 2$ are bounded on $[0, \infty)$. The solution of system (2.113) that conforms to the solution $M_i(t, \varepsilon), \theta_i(t, \varepsilon), i = 1, 2$ of system (10.117) takes the form

$$x_i(t, \varepsilon) = M_i(t, \varepsilon)\cos(t(1 + \varepsilon^2 \varphi_i(t, \varepsilon)) + \theta_i(0, \varepsilon)) + \varepsilon x_{1i}(t, \varepsilon), \quad i = 1, 2, \tag{10.120}$$

where the functions $x_{1i}(t, \varepsilon)$, $i = 1, 2$ are bounded on $[0, \infty)$. This follows from formulas that define the functions $u_i(t, p, q)$, $v_i(t, p, q)$. Formula (10.120) shows that the first approximation (10.119) for sufficiently small ε provides a satisfactory characteristic of the exact solution.

10.13 Excitation of Parametric Oscillations by Impacts

As we saw in the first part of the book, a parametric resonance in a linear system often causes an unstable equilibrium state in a system. The situation is different in nonlinear systems, where the parametric resonance may generate stable stationary oscillations.

Consider a problem of parametric oscillations for the equation

$$\frac{d^2x}{dt^2} + \varepsilon\delta\frac{dx}{dt} + \omega^2[1+\varepsilon f(t)]x + \varepsilon\gamma x^3 = 0, \qquad (10.121)$$

where $\varepsilon > 0$ is a small parameter, $f(t)$ is a generalized derivative of the periodic function with a finite number of simple discontinuities (jumps) on the period, and ω, δ, γ are positive numbers. We introduce a variable $y(t)$ using the equation

$$\frac{dx}{dt} = y - \varepsilon\omega^2 g(t)x,$$

where $g(t)$ is a periodic function such that

$$g'(t) = f(t).$$

Then, instead of equation (10.121), we obtain an equivalent system of two first-order differential equations

$$\begin{array}{l} \frac{dx}{dt} = y - \varepsilon\omega^2 g(t)x, \\ \frac{dy}{dt} = -\omega^2 x + \varepsilon\omega^2 g(t)y - \varepsilon\delta y - \varepsilon\gamma x^3 + O(\varepsilon^2) \end{array}. \qquad (10.122)$$

We now move on to the new variables by means of a change

$$x = a\cos\nu t + b\sin\nu t, \quad y = -a\nu\sin\nu t + b\nu\cos\nu t, \qquad (10.123)$$

where the frequency ν will be chosen later, and obtain

$$\begin{array}{l} \frac{da}{dt}\cos\nu t + \frac{db}{dt}\sin\nu t = -\varepsilon\omega^2 g(t)x, \\ -\frac{da}{dt}\nu\sin\nu t + \frac{db}{dt}\nu\cos\nu t - \nu^2[a\cos\nu t + b\sin\nu t] = -\omega^2[a\cos\nu t + b\sin\nu t] + \\ (\varepsilon\omega^2 g(t) - \varepsilon\delta)[-a\nu\sin\nu t + b\nu\cos\nu t] - \varepsilon\gamma[a\cos\nu t + b\sin\nu t]^3 + O(\varepsilon^2). \end{array}$$

Solving the latter system for $\frac{da}{dt}$, $\frac{db}{dt}$, we arrive at the system

$$\begin{array}{l} \frac{da}{dt} = -\frac{\nu^2-\omega^2}{\nu}[\frac{a}{2}\sin 2\nu t + b\sin^2\nu t] - \\ -\varepsilon\omega^2 g(t)[a(\cos^2\nu t - \sin^2\nu t + b\sin 2\nu t] + \\ +\varepsilon\delta[-a\sin^2\nu t + \frac{b}{2}\sin 2\nu t] + \varepsilon\gamma\nu[a\cos\nu t + b\sin\nu t]^3\sin\nu t + O(\varepsilon^2), \\ \frac{db}{dt} = \frac{\nu^2-\omega^2}{\nu}[a\cos^2\nu t + \frac{b}{2}\sin 2\nu t] \\ -\varepsilon\omega^2 g(t)[-a\sin 2\nu t + b(\cos^2\nu t - \sin^2\nu t)] - \\ -\varepsilon\delta[-\frac{a}{2}\sin 2\nu t + b\cos^2\nu t] - \varepsilon\gamma\nu[a\cos\nu t + b\sin\nu t]^3\cos\nu t + O(\varepsilon^2). \end{array} \qquad (10.124)$$

We shall assume that $\nu^2 - \omega^2 = \varepsilon h$, where h is a constant. Then system (10.124) has a standard form, and we can use Theorems 9.2 and 9.3 to investigate the existence and stability of periodic solutions of system (10.124). As a function $f(t)$, we take a generalized periodic function corresponding to the Fourier series

$$f(t) \sim \sum_{k=1}^{\infty}\cos(2k-1)t.$$

The function $g(t)$ can be represented by the following series:

$$g(t) \sim \sum_{k=1}^{\infty} \frac{\sin(2k-1)t}{2k-1}.$$

Let $\nu = \frac{2k-1}{2}$, $(k = 1, 2\dots)$. We fix k and average it over t in the right-hand side of system (10.124). The averaged system then takes the form

$$\begin{aligned} \tfrac{da}{dt} &= \varepsilon[(-\tfrac{h}{2\nu} - \tfrac{\omega^2}{4\nu})b - \tfrac{\delta}{2}a + \tfrac{3\gamma}{8\nu}b(a^2+b^2)], \\ \tfrac{db}{dt} &= \varepsilon[(\tfrac{h}{2\nu} - \tfrac{\omega^2}{4\nu})a - \tfrac{\delta}{2}b - \tfrac{3\gamma}{8\nu}a(a^2+b^2)], \end{aligned} \tag{10.125}$$

where $\nu = \frac{2k-1}{2}$. We transform system (10.125) into the polar coordinates $a = \rho\cos\varphi$, $b = \rho\sin\varphi$, solve the transformed system for $\frac{d\rho}{dt}$, $\frac{d\varphi}{dt}$) and obtain

$$\begin{aligned} \tfrac{d\rho}{dt} &= \varepsilon[-\tfrac{\omega^2}{4\nu}\rho\sin 2\varphi - \tfrac{\delta}{2}\rho], \\ \tfrac{d\varphi}{dt} &= \varepsilon[\tfrac{h}{2\nu} - \tfrac{\omega^2}{4\nu}\cos 2\varphi - \tfrac{3\gamma}{8\nu}\rho^2]. \end{aligned} \tag{10.126}$$

The equilibria (ρ_0, φ_0) of system (10.126) are determined from the system of equations

$$\begin{aligned} -\tfrac{\omega^2}{4\nu}\rho\sin 2\varphi - \tfrac{\delta}{2}\rho = 0, \\ \tfrac{h}{2\nu} - \tfrac{\omega^2}{4\nu}\cos 2\varphi - \tfrac{3\gamma}{8\nu}\rho^2 = 0. \end{aligned} \tag{10.127}$$

Investigate the stability of the equilibria determined by system (10.127). Linearizing averaged system (10.126) on the equilibrium (ρ_0, φ_0), yields a linear system with the matrix

$$A = \begin{pmatrix} 0 & -\varepsilon\frac{\omega^2}{2\nu}\rho_0\cos 2\varphi_0 \\ -\varepsilon\frac{3\gamma}{4\nu}\rho_0 & \varepsilon\frac{\omega^2}{2\nu}\sin 2\varphi_0 \end{pmatrix}.$$

It is well known that eigenvalues of the second-order matrix have negative real parts if the trace of the matrix is negative while the determinant is positive. In our case, the trace of the matrix A is always negative (the inequality $\sin 2\varphi_0 < 0$ holds true, which follows from the first equality of (10.127)). The positiveness of the determinant of the matrix A, and thus the stability of the equilibrium, is determined by the inequality $\cos 2\varphi_0 < 0$. Therefore, in view of the second equation of system (10.126), we obtain the inequality

$$h - \frac{3\gamma}{4}\rho_0^2 < 0. \tag{10.128}$$

Therefore, inequality (10.128) implies the asymptotic stability of the respective equilibrium of an averaged system. By virtue of Theorems 9.2 and 9.3, the initial system of equations (10.122) and, therefore, equation (10.121), for sufficiently small ε, has an asymptotically stable periodic solution with the period $T = \frac{4\pi}{2k-1}$. Similarly, if the inequality

$$h - \frac{3\gamma}{4}\rho_0^2 > 0 \tag{10.129}$$

holds, then, for sufficiently small ε the equation (10.121) has an unstable periodic solution.

We eliminate the variable φ from system (10.127) and obtain the following relation between the amplitude ρ and the frequency of modulation ν

$$\rho^2 = \frac{4}{3\gamma\varepsilon}[\nu^2 - \omega^2 \mp \frac{1}{2}\varepsilon\sqrt{\omega^4 - 4\nu^2\delta^2}]. \tag{10.130}$$

First, we consider the case when $\gamma > 0$. The amplitude curve has two branches. The plus sign corresponds to the branch of asymptotically stable equilibrium, while the minus sign, to the branch of unstable equilibria. As the detuning $(\nu^2 - \omega^2)$ changes, due to negative values, from large negative values to positive ones, there are no oscillations until the detuning reaches a certain magnitude. After that, in the system the asymptotically stable periodic oscillations will increase in amplitude and then derail, i.e., turn into unstable oscillations. When the detuning reduces because of large positive values, the stable oscillations will be excited by a jump (a rigid excitation of oscillations). As the detuning reduces, they will decrease smoothly in amplitude. For $\gamma < 0$, we obtain a similar picture, but excitation of the stable oscillations by a jump occurs as the detuning increases.

The following graphs have been constructed for $\varepsilon = 0.1$, $\delta = 0.2$, $\gamma = 1$, $\nu = 0.5$, and under the values $\varepsilon = 0.1$, $\delta = 0.2$, $\gamma = -1$, $\nu = 0.5$, respectively. On the X-axis, we measure the detuning $(\nu^2 - \omega^2)$, and the amplitude of ρ is measured on the Y-axis.

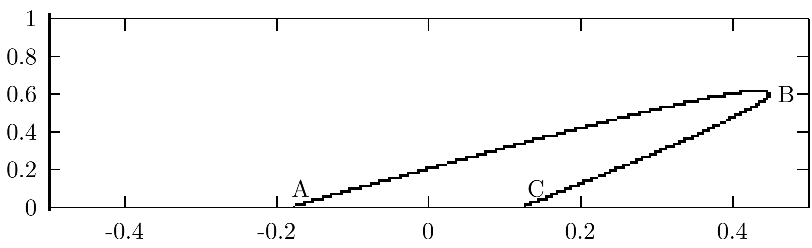

FIGURE 10.4: Parametric Oscillations in Duffing's Equation with $\gamma = 1$, $\nu = 0.5$.

In the first graph, AB is the branch of stable fixed points, and BC is the branch of unstable fixed points. In the second graph, BC is the branch of stable fixed points, and AB is the branch of unstable fixed points. The pattern is the same in the graphs, where only the value of the frequency was changed: we replaced $\nu = 0.5$ with $\nu = 1.5$.

To define boundaries of the synchronization zone, we need to make the right-hand side of equality (10.130) come to naught. The resonance zone in

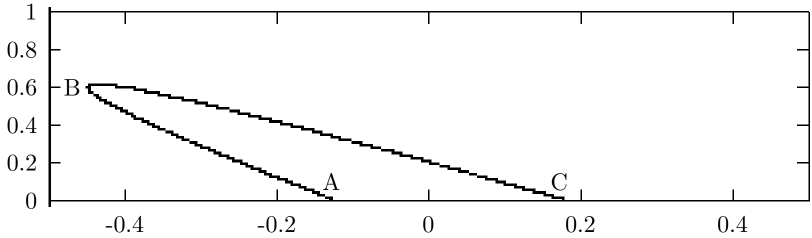

FIGURE 10.5: Parametric Oscillations in Duffing's Equation with $\gamma = -1$, $\nu = 0.5$.

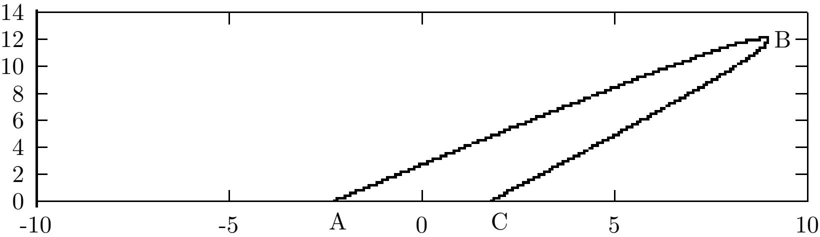

FIGURE 10.6: Parametric Oscillations in Duffing's Equation with $\gamma = 1$, $\nu = 1.5$.

the first approximation will be

$$\omega^2 - \frac{1}{2}\varepsilon\sqrt{\omega^4 - 4\omega^2\delta^2} < \nu^2 < \omega^2 - \frac{1}{2}\varepsilon\sqrt{\omega^4 - 4\omega^2\delta^2}.$$

Hence, the width of the resonance zone is

$$\Delta = \frac{1}{2}\varepsilon\omega\sqrt{\omega^2 - 4\delta^2}.$$

Damping reduces the interval of growing parametric resonance.

If $f(t)$ is not a generalized but an ordinary periodic function, then the resonance zone, generally speaking, is reduced. For example, let the following Fourier series correspond to the function $f(t)$

$$\sum_{k=1}^{\infty} \frac{\cos(2k-1)t}{2k-1}.$$

Then the width of the resonance zone is

$$\Delta = \frac{1}{4}\varepsilon\omega^2\sqrt{1 - 16\delta^2}.$$

We now consider the problem of the rise of parametric oscillations under the action of a small periodic perturbation in a self-oscillatory system. The

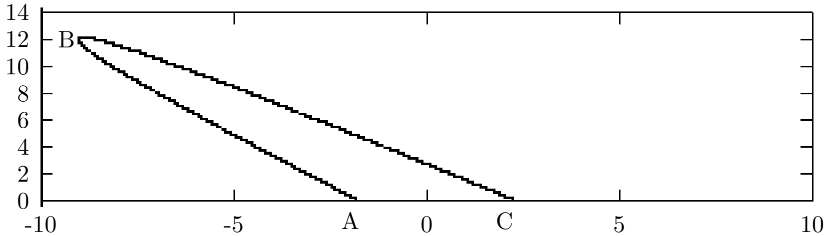

FIGURE 10.7: Parametric Oscillations in Duffing's Equation with $\gamma = -1$, $\nu = 1.5$.

respective equation takes the form

$$\frac{d^2x}{dt^2} + \varepsilon[(\delta + \gamma x^2)\frac{dx}{dt}] + \omega^2[1 + \varepsilon f(t)]x = 0, \tag{10.131}$$

where $\varepsilon > 0$ is a small parameter, $f(t)$ is a generalized periodic function that is a derivative of a periodic function with a finite number of simple discontinuities (jumps) on the period, and ω, δ, γ are positive numbers. The positiveness of δ means that in a self-oscillatory system, under the lack of parametric excitation, there is the asymptotically stable zero equilibrium and there is no oscillatory regime. We introduce a variable $y(t)$ using the equation

$$\frac{dx}{dt} = y - \varepsilon\omega^2 g(t)x,$$

where $g(t)$ is a periodic function obeying such that

$$g'(t) = f(t).$$

Then we move on to a system of differential equations with respect to the variables x, y similar to system (10.114). We introduce new variables using formulas (10.133) and assume $\nu^2 - \omega^2 = \varepsilon h$. As the function $g(t)$, we choose a function with the Fourier series

$$\sum_{k=1}^{\infty} \frac{\sin(2k-1)t}{2k-1}.$$

Let $\nu = \frac{2k-1}{2}$, $(k = 1, 2 \dots)$. We fix k and average the respective system over t in the standard form. By so doing, we arrive at the system

$$\begin{aligned} \tfrac{da}{dt} &= -\varepsilon[\tfrac{h}{2\nu}b + \tfrac{\omega^2}{4\nu}b + \tfrac{\delta}{2}a + \tfrac{\gamma}{8}a(a^2+b^2)], \\ \tfrac{db}{dt} &= \varepsilon[\tfrac{h}{2\nu}b - \tfrac{\omega^2}{4\nu}a + \tfrac{\delta}{2}b - \tfrac{\gamma}{8}b(a^2+b^2)]. \end{aligned}$$

Now pass to the polar coordinates. The average system takes the form

$$\begin{aligned} \tfrac{d\rho}{dt} &= -\varepsilon[\tfrac{\delta}{2}\rho + \tfrac{\omega^2}{4\nu}\rho\sin 2\varphi + \tfrac{\gamma\rho^3}{8}], \\ \tfrac{d\varphi}{dt} &= \varepsilon[\tfrac{h}{2\nu} - \tfrac{\omega^2}{4\nu}\cos 2\varphi]. \end{aligned}$$

We find the equilibrium of the system by equating the right-hand sides of the latter system to zero. It is easy to see that all equilibria of the averaged system are asymptotically stable. Therefore, for sufficiently small ε, equation (10.131) has asymptotically stable periodic solutions that increase as a result of parametric excitation. Discarding the variable φ from this system yields the relationship between the amplitude ρ and the frequency of modulation ν:

$$\rho^2 = \frac{2}{\varepsilon\gamma\nu}\sqrt{\varepsilon^2\omega^4 - 4(\nu^2 - \omega^2)^2} - 4\frac{\delta}{\gamma}.$$

The graph of the amplitude curve for the parameters $\varepsilon = 0.1$, $\delta = 0.2$, $\gamma = 1$, $\nu = 0.5$ takes the form (the X-axis depicts the detuning, whereas the Y-axis illustrates the amplitude ρ).

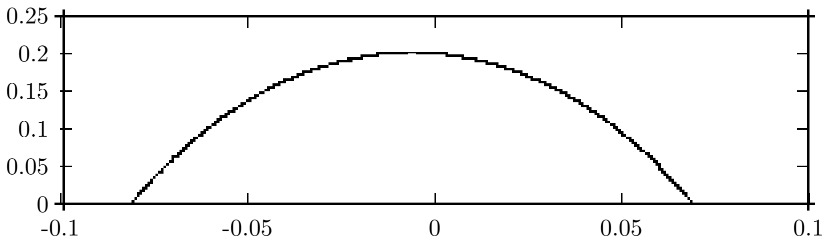

FIGURE 10.8: Parametric Oscillations in Van der Pol Equation.

Chapter 11

Pendulum Systems with an Oscillating Pivot

In this chapter we apply the theorems on the averaging on the infinite interval that were proved in Chapter 9 to the investigation of the stability of equilibria of pendulum systems with oscillating pivots. The approach in the application of Theorems 9.1, 9.3 and 9.4 consists of the following steps. The equations of motion of a system under investigation are written in Lagrange form. Next, we go to the Hamiltonian form of writing the equations of motion. We introduce a small parameter and make a change to a fast time. Thus, we obtain a system in standard form. The problem of the stability of equilibria is solved using the averaged equations in the first approximation.

11.1 History and Applications in Physics

The stabilizing effect of the vibration of a pivot of a pendulum became known as early as 1908. Stephenson [1908] showed that it is possible to stabilize the upper equilibrium of a pendulum with a vertically oscillating pivot. He investigated the case of a pendulum whose pivot receives a series of impulses that support its motion with a constant velocity along the line that forms a small angle with the rod of the pendulum. Stephenson defined a "mean" motion that is stable. He also obtained the conditions of the stability for the upper equilibrium of a pendulum whose pivot does fast vertical simple harmonic oscillations. Using similar methods Stephenson[1909] determined the conditions of the stability of two and of three rods that are linked together end to end and originally in a position of unstable equilibrium, if the pivot undergoes fast vertical oscillations.

The theory of the Mathieu equation for this problem was used in papers by Van der Pol [1925] and Strutt [1927]. Van der Pol and Strutt [1928] (see also Ince [1928]) considered the problem of the stability of solutions of the Mathieu equation and obtained the diagram of the stability of solutions in the plane of two parameters. They also discussed the conditions under which the oscillating influence can stabilize a system that originally was unstable. Hirsch [1930] considered the problem of the motion of a pendulum whose pivot undergoes small high frequency oscillations in the plane of the pendulum. Erdélyi [1934] conducted a complete investigation of small oscillations of a pendulum with a periodically oscillating pivot. He took damping into consideration, and utilized the Floquet theory and the theory of Hill's equation.

Lowenstern [1932] studied the effect of motions with high frequency and low amplitude which are applied to one class of dynamical systems and obtained, for the first time, results of some generality. He determined the equations of motion for general Lagrange systems that are exposed to fast oscillations, and the equations for small oscillations near an equilibrium. Lowenstern considered only periodic excitement.

Kapitsa [1951a, 1951b] investigated the problem of motion of a pendulum with an oscillating pivot in nonlinear settings. Kapitsa studied the stability of an inverted pendulum using a concept of an effective potential that he introduced (see also Landau and Lifchitz [1960, p. 93–95]). He also came up with an idea to apply a vibrating stabilization to other mechanical objects that differ from a pendulum, such as, for example, large molecules.

Bogoliubov [1950] obtained a rigorous mathematical proof of the stability of the upper equilibrium with a vertically oscillating pivot. He assumed that the amplitude of vibrations is small and their frequency is large. The proof is based on a very interesting transformation that allows one to obtain the answer in the first approximation of the method of averaging.

Bogdanoff [1962] generalized the results of Lowenstern on the case of small, fast quasi-periodic parametric excitements, but his analysis is limited to the case of linear equations. Bogdanoff and Citron [1965] experimentally demonstrated various effects in the behavior of a pendulum with an oscillating pivot.

Hemp and Sethna [1968] considered nonlinear dynamical systems with parametric excitements. They conducted an analysis of the influence of "fast" parametric excitements, and investigated the effect of the simultaneous influence of "fast" and "slow" parametric excitements. In particular, they considered the case when some of the frequencies of "fast" parametric excitements are close to each other.

Acheson [1993] investigated the problem of the stabilization of the upper equilibrium of an N-linked pendulum using small vertical oscillations of its pivot.

Burd, Zabreiko, Kolesov, and Krasnosel'skii [1969] (see Krasnosel'skii, Burd, and Kolesov [1973]) studied the problem of bifurcation of almost periodic oscillations from the upper equilibrium of a pendulum with a vertically oscillating pivot. Burd [1984] investigated the problem of bifurcation of almost periodic oscillations from the upper and the lower equilibria when the law of motion of the pivot is an almost periodic function with two frequencies that are close to each other.

Levi [1988] found a topological proof of the stabilization of the upper equilibrium of a pendulum with a vertically oscillating pivot. Levi [1998, 1999] also gave a very simple physical explanation of the stabilization of the upper equilibria by vibrations of its pivot. The main motif of his papers is an observation that behind the procedure of averaging lie some simple geometric facts. The discovery of stable π-kinks in the sine-Gordon equation under the influence of a fast oscillating force (see Zharnitsky, Mitkov, and Levi [1998]) is based on the same idea.

The concept of levitation of charged particles in an oscillating electrical field "Paul's trap" was introduced in 1958 (see Osberghaus, Paul, and Fischer [1958], and Paul [1990]). This work earned Paul a Nobel prize in 1989. The discovery of Paul's trap was preceded by the idea of a strong focusing in synchrotrons. (see Courant, Livingston, Snayder, and Blewett [1953]).

Recently Saito and Ueda [2003] suggested a new application of such mechanisms of stabilization to the production of bright solitons in two-dimensional Bose-Einstein condensate (here, a "bright" soliton means a stable solitary wave whose density is greater than the density of the background).

We also mention the works in which authors investigated the influence of high frequency vibrations on the presence of convection in fluids (Zenkovskaya and Simonenko [1966], Zenkovskaya and Shleikel [2002a, 2002b]).

Blekhman [1994] devoted his book to the description of amazing phenomena that take place under the action of vibrations in the nonlinear mechanical systems. These include the change of the state of a system under the influence of fast vibrations, the change of physical and mechanical properties and characteristics under the action of vibrations comparing to the slow actions, transformation of equilibria, particularly, their stabilization and destabilization under the action of vibrations, the change of frequencies of free oscillations in the system due to vibration, support of rotations by vibration, and self-synchronization of unbalanced rotors. Blekhman suggested a general approach to the studying of the aforementioned phenomena which he calls a "vibrational mechanics".

In recent years, interest has been renewed in the use of high frequency vibrations for control of low frequency properties of structures, i.e., their equilibria, stability, effective natural frequencies and amplitudes of vibrations (Champneys and Frazer [2000], Feeny and Moon [2000], Fidlin [2000], Fidlin and Thomsen [2001], Sudor and Bishop [1999], Thomsen and Tcherniak [1998], and Tcherniak [1999]).

We also mention the works of Yudovich [1997, 1998] on the vibro-dynamics of systems with constraints and a book by Strizhak [1981] that is devoted to the methods of investigation of pendulum systems.

11.2 Equation of Motion of a Simple Pendulum with a Vertically Oscillating Pivot

We consider one of the simplest oscillating systems: a material point of mass m, coupled via a massless rigid rod of length l (which is called the *length* of a pendulum) to the fixed pivot. Clearly, the trajectory of motion of the material point is an arc. We can also think of a pendulum in the form of a rigid body that can rotate in some vertical plane around its pivot.

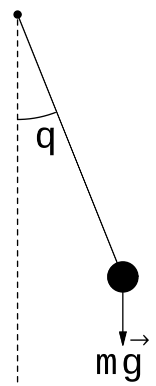

FIGURE 11.1: Simple Pendulum.

We shall assume that the environment in which the pendulum moves provides some damping that is proportional to the velocity of the motion and that a pivot is oscillating periodically or almost periodically.

First, we suppose that the pivot can move only along a vertical axis. It will be convenient to write the equation of motion of a pendulum in Hamiltonian form. We shall use the following notation: m is the mass of the pendulum, l is its length, c is a coefficient of damping, g is the gravitational constant, q is the angular displacement of the pendulum relative to the vertical axis, function $f(t)$ describes the law of motion of the pivot, and the Cartesian coordinates of the pendulum are $x = l \sin q$, $y = l \cos q + f(t)$. The kinetic energy of the pendulum is defined by

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m[l^2 \dot{q}^2 + 2l\dot{q}\dot{f} \sin q + \dot{f}^2],$$

where $\dot{x}$, $\dot{y}$, $\dot{q}$, $\dot{f}$ are derivatives of the functions $x(t)$, $y(t)$, $q(t)$, $f(t)$, respectively, while the potential energy is

$$V(q) = -mgl\cos q.$$

The dissipative function has the following form

$$R(\dot{q}) = cl^2\dot{q}^2.$$

The equation of motion in Lagrange form is written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0, \quad L = T - V.$$

Thus, the equation of motion of the pendulum is

$$\ddot{q} + \frac{2c}{m}\dot{q} + \left[\frac{g}{l} + \frac{\ddot{f}}{l}\right]\sin q = 0. \tag{11.1}$$

We transform this equation into Hamiltonian form. The generalized kinetic moment is defined by

$$p = \frac{\partial T}{\partial \dot{q}} = m(l^2\dot{q} + l\dot{f}\sin q),$$

which can be rewritten as

$$\dot{q} = \frac{1}{ml^2}p - \frac{\dot{f}}{l}\sin q. \tag{11.2}$$

Taking a derivative in (11.2) and utilizing (11.1) yields

$$\dot{p} = ml^2\left[-\frac{2c}{ml^2}p + \frac{2c}{ml}\dot{f}\sin q + \frac{\dot{f}}{ml^3}p\cos q - \frac{\dot{f}^2}{l^2}\sin q\cos q\right].$$

Finally, we obtain the desired system of the equation

$$\begin{aligned} \tfrac{dq}{dt} &= \tfrac{1}{ml^2}p - \tfrac{\dot{f}}{l}\sin q, \\ \tfrac{dp}{dt} &= -\tfrac{2c}{m}p + [2cl\sin q + \tfrac{p}{l}\cos q]\dot{f} - mgl\sin q - \tfrac{m}{2}\dot{f}^2\sin 2q. \end{aligned} \tag{11.3}$$

11.3 Introduction of a Small Parameter and Transformation into Standard Form

We shall consider forced movements $f(t)$ of a pivot which are described by three different laws.

First, we consider $f(t)$ that is defined as a trigonometric polynomial

$$f(t) = \sum_{k=1}^{N} \alpha_k \cos \nu_k t + \beta_k \sin \nu_k t, \tag{11.4}$$

where α_k, β_k, ν_k $(k = 1, 2, \ldots, N)$ are real numbers. We shall assume that amplitudes α_k, β_k are sufficiently small, and frequencies ν_k are sufficiently large in the following sense. There exists a small positive parameter ε, such that

$$\alpha_k = \varepsilon a_k,\ \beta_k = \varepsilon b_k,\ \nu_k = \frac{\omega_k}{\varepsilon},$$

where a_k, b_k, ω_k $(k = 1, 2, \ldots, N)$ are of order O(1) in ε. Then $f(t)$ can be written as

$$f(t) = \varepsilon \sum_{k=1}^{N} a_k \cos \omega_k \frac{t}{\varepsilon} + b_k \sin \omega_k \frac{t}{\varepsilon} = \varepsilon \phi(\frac{t}{\varepsilon}).$$

Second, we consider $f(t)$ that is defined by a periodic function

$$f(t) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \alpha \sin(2k-1)\nu t. \tag{11.5}$$

We introduce a small parameter ε by letting $\alpha = \varepsilon a$ and $\nu = \frac{\omega}{\varepsilon}$. Then $f(t)$ can be written as

$$f(t) = \varepsilon \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} a \sin(2k-1)\omega \frac{t}{\varepsilon} = \varepsilon \phi(\frac{t}{\varepsilon})$$

Third, $f(t)$ can be defined by

$$f(t) = \alpha \sin \nu t. \tag{11.6}$$

Letting $\alpha = \varepsilon a$ and $\nu = \frac{\omega}{\varepsilon}$, we get

$$f(t) = \varepsilon a \sin \frac{\omega}{\varepsilon} t = \varepsilon \phi(\frac{t}{\varepsilon}).$$

We note that in the case when $f(t)$ is defined by (11.5), the acceleration of the forced movement is a generalized periodic function $\ddot{f}(t)$ that corresponds to the Fourier series

$$\ddot{f}(t) = -\sum_{k=1}^{\infty} \sin(2k-1)t.$$

Therefore, equation (11.1) contains a generalized periodic function as a coefficient. However, system (11.3) contains only the function $\dot{f}(t)$ that is piecewise continuous and periodic.

We make a change to the fast time τ in (11.2) using $\varepsilon\tau = t$ and denote the operation of differentiating in τ by a dot. We get

$$\begin{aligned}\dot{q} &= \varepsilon\left(\tfrac{1}{ml^2}p - \tfrac{1}{l}\dot{\phi}(\tau)\sin q\right),\\ \dot{p} &= \varepsilon\left[-mgl\sin q - \tfrac{2c}{m}p + \left(2cl\sin q + \tfrac{p}{l}\cos q\right)\dot{\phi}(\tau) - m\dot{\phi}^2(\tau)\cos q\sin q\right].\end{aligned} \tag{11.7}$$

Thus, under the assumptions described above the system of equations of motion has the standard form.

11.4 Investigation of the Stability of Equilibria

The averaged system has the following form

$$\begin{aligned}\dot{\xi} &= \varepsilon\tfrac{1}{ml^2}\eta,\\ \dot{\eta} &= \varepsilon[-mgl\sin\xi - \tfrac{2c}{m}\eta - \tfrac{m}{2}\langle\dot{\phi}^2(\tau)\rangle\sin 2\xi],\end{aligned} \tag{11.8}$$

where

$$\langle\dot{\phi}^2(\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T \dot{\phi}^2(\tau)d\tau.$$

System (11.8) has three stationary solutions

$$I:(0,0),\quad II:(\pi,0),\quad III:(-\arccos\left(-\frac{gl}{\langle\dot{\phi}^2(\tau)\rangle}\right),0).$$

The latter stationary solution exists only if

$$gl \le \langle\dot{\phi}^2(\tau)\rangle. \tag{11.9}$$

System (11.8) also has stationary solutions I and II that correspond to the lower and the upper equilibria of a pendulum, respectively.

We investigate the stability of stationary solutions of system (11.8). For solution I the linearized system is

$$\begin{aligned}\dot{\psi} &= \varepsilon\tfrac{1}{ml^2}\varphi,\\ \dot{\varphi} &= \varepsilon[-mgl\psi - \tfrac{2c}{m}\varphi - m\langle\dot{\phi}^2(\tau)\rangle\psi],\end{aligned} \tag{11.10}$$

while for solution II (letting $\sigma = \xi - \pi$) the linearized system is

$$\begin{aligned}\dot{\sigma} &= \varepsilon\tfrac{1}{ml^2}\delta\\ \dot{\delta} &= \varepsilon[mgl\sigma - \tfrac{2c}{m}\delta - m\langle\dot{\phi}^2(\tau)\rangle\sigma].\end{aligned} \tag{11.11}$$

It is easy to see that the matrix of system (11.10) has a negative trace and a positive determinant. Therefore, the zero equilibrium of the averaged system is asymptotically stable. Theorem 9.3 implies that, for sufficiently small ε, the lower equilibrium of a pendulum is asymptotically stable.

The matrix of system (11.11) has a negative trace and a positive determinant if

$$gl < \langle \dot{\phi}^2(\tau) \rangle.$$

Therefore, under this condition the equilibrium $(0, \pi)$ of the averaged system is asymptotically stable. Theorem 9.3 implies that, for sufficiently small ε, the upper equilibrium of a pendulum is asymptotically stable.

We note that the inequality that determines the stability of the upper equilibrium has the following form in original time t

$$gl < \langle \dot{f}^2 \rangle.$$

For the almost periodic law of motion of a pivot (11.4) we get

$$\langle \dot{\phi}^2(\tau) \rangle = \frac{1}{2} \sum_{k=1}^{N} (a_k^2 + b_k^2)\omega_k^2 = \frac{1}{2} \sum_{k=1}^{N} (\alpha_k^2 + \beta_k^2)\nu_k^2.$$

For a periodic function (11.5) we have

$$\langle \dot{\phi}^2(\tau) \rangle = \frac{1}{2} \sum_{k=1}^{\infty} a^2\omega^2 \frac{1}{(2k-1)^2} = a^2\omega^2 \frac{\pi^2}{16},$$

while for function (11.6) we have

$$\langle \dot{\phi}^2(\tau) \rangle = \frac{a^2\omega^2}{2}.$$

Therefore, the upper equilibrium of a pendulum is asymptotically stable when

$$\sum_{k=1}^{N} (\alpha_k^2 + \beta_k^2)\nu_k^2 > 2gl \tag{11.12}$$

for function (11.4), when

$$a^2\omega^2 \frac{\pi^2}{8} > 2gl \tag{11.13}$$

for function (11.5), and when

$$a^2\omega^2 > 2gl \tag{11.14}$$

for function (11.6).

Comparison of inequalities (11.13) and (11.14) shows that the domain of the stability of the upper equilibrium of a pendulum when the law of vibration of pivot is defined by (11.5) is wider in comparison with the case when the law is defined by sine function. Inequality (11.14) in original variables has the form

$$\alpha^2\nu^2 > 2gl. \tag{11.15}$$

Inequality (11.15), as a condition of the stabilization of the upper equilibrium of a pendulum, was obtained by various methods by several authors (see Stephenson [1908], Erdélyi [1934], Kapitsa [1951a, 1951b], Bogoliubov [1950]). It is known that, in absence of vibrations of a pivot, the lower equilibrium is stable while the upper equilibrium is unstable. The preceding analysis leads us to the following result. If a pivot is vibrating according to the sine function law, the frequency of vibrations is sufficiently large, and the amplitude is sufficiently small, then the upper equilibrium of a pendulum can become stable. This result has been verified by numerous experiments.

Recall that this result was established by Stephenson as early as 1908, and its rigorous foundation using the method of averaging was laid by N.N. Bogoliubov. Here we considered more general laws of motion of a pivot and we obtained similar results.

For the stationary solution III of the averaged system that exists when inequality (11.9) holds, the linearized system has the form

$$\begin{aligned} &\dot\varphi = \varepsilon \tfrac{1}{ml^2}\psi, \\ &\dot\psi = \varepsilon[(-mgl\cos\xi_0 - m\langle\dot\phi^2(\tau)\rangle\cos 2\xi_0)\varphi - \tfrac{2c}{m}\psi], \end{aligned}$$

where

$$\cos\xi_0 = -\frac{gl}{\langle\dot\phi^2(\tau)\rangle}.$$

Clearly, the trivial solution of this system is unstable when inequality (11.9) holds. Theorems 9.1 and 9.3 imply that, for sufficiently small ε, an unstable almost periodic or periodic solution of system (11.7) corresponds to the stationary solution III of the averaged system (this depends on the selection of the law of motion of a pivot).

We now make some remarks about the problem considered above. The averaged system (11.8) can be written as a single differential equation of the second order. This equation in original time t has the following form (for the sine law of vibrations of a pivot)

$$\ddot\xi + \frac{2c}{m}\dot\xi + \left[\frac{g}{l} + \frac{\alpha^2\nu^2}{2l^2}\cos\xi\right]\sin\xi = 0. \tag{11.16}$$

Bogoliubov and Mitropolskiy [1961] observed that equation (11.6) describes an oscillating system that is similar to a pendulum with a fixed pivot, however the restoring force is proportional not to $\sin\xi$, but to $[\frac{g}{l} + \frac{\alpha^2\nu^2}{2l^2}\cos\xi]\sin\xi$. The frequency of small oscillations ignoring the damping is $\frac{g}{l} + \frac{\alpha^2\nu^2}{2l^2}$.

Exercise 11.1. Let the law of motion of a pivot be defined by

$$f(t) = \varepsilon a \sin\frac{\omega}{\varepsilon}t,$$

where $\varepsilon << 1$. In equation (11.1) of the motion of a pendulum, make a change of time $t = \tau/\varepsilon$ to get the equation (in time τ) :

$$\ddot{q} + \varepsilon\frac{2c}{m}\dot{q} + \left[\varepsilon^2\frac{g}{l} - \varepsilon\frac{a\omega^2}{l}\right]\sin q = 0.$$

Linearize this equation on the equilibrium $q = \pi$ and investigate the stability of the trivial solution of the linearized equation. (Hint: Use the method of Shtokalo (see Section 5.4)).

11.5 Stability of the Upper Equilibrium of a Rod with Distributed Mass

We now consider a pendulum whose mass is distributed along its rod. We denote the length of the rod by L, and its density by $\rho(y)$, where $0 \leq y \leq L$. The product of the mass m by the distance to the center of mass of the rod is

$$ml = \int_0^L \rho y dy,$$

and the inertial moment is

$$J = \int_0^L \rho y^2 dy.$$

The kinetic energy of the pendulum up to the terms that do not contain q and $\dot{q}$, is defined by

$$T = \frac{\dot{q}^2}{2}\int_0^L \rho y^2 dy + \dot{q}\dot{f}\sin q\int_0^L \rho y dy,$$

while the potential energy is

$$V = -g\cos q\int_0^L \rho y dy.$$

Taking into consideration a dissipation, we get an equation of a pendulum in the Lagrange form

$$\ddot{q} + c\dot{q} + \gamma[g + \ddot{f}]\sin q = 0,$$

where

$$\gamma = \frac{\int_0^L \rho y dy}{\int_0^L \rho y^2 dy}.$$

This equation differs from equation (11.1) in that the coefficient of $\sin q$ is multiplied by γ. Therefore, if the function $f(t)$ describes the law of forced movements of a pivot, then the condition that guarantees the stability of the upper equilibrium is

$$g\gamma < \langle \dot{f}^2 \rangle.$$

We shall assume that the rod of length L is solid and has uniform density. The linear density is $\rho = mL^{-1}$. In this case

$$\gamma = \frac{\int_0^L \rho y dy}{\int_0^L \rho y^2 dy} = \frac{3}{2}.$$

The condition on the stability of the upper equilibrium of the rod takes the form

$$\frac{2}{3} gl < \langle \dot{f}^2 \rangle.$$

This inequality implies that it is easier to stabilize a rod of length L with uniform density than a pendulum which has the mass concentrated at the endpoint.

Exercise 11.2. Assume that we have a conical rod. Consider two cases: a) the rod is fixed at the vertex of the cone; b) the pivot is located at the base of the cone. Find the conditions for stability of the upper equilibrium assuming that the law of vibration of a pivot is $f(t) = \varepsilon a \sin \frac{\omega}{\varepsilon} t$, $\varepsilon << 1$. Which of the cases is easier to stabilize?

11.6 Planar Vibrations of a Pivot

We now consider a pendulum with an oscillating pivot under more general assumptions about the motion of the pivot. Let the pivot oscillate simultaneously in horizontal and vertical directions according to

$$x = s(t),\ y = r(t).$$

Consider the case when a pivot makes sine-like harmonic oscillations, which have an amplitude α and a frequency ν, and which occur along the line that forms an angle θ with the y-axis. Then we have

$$s(t) = \alpha \sin \nu t \sin \theta,$$
$$r(t) = \alpha \sin \nu t \cos \theta.$$

Cartesian coordinates of a pendulum are $x = l \sin q + s(t)$, $y = -l \cos q + r(t)$. The kinetic energy is defined by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}[l^2\dot{q}^2 + 2l\dot{q}(\dot{s}\cos q + \dot{r}\sin q) + \dot{s}^2 + \dot{r}^2],$$

while the potential energy is $V(q) = -mgl \sin q$, and dissipative function is $R(\dot{q}) = cl^2\dot{q}^2$. The equations of motion in Lagrange form are

$$\ddot{q} + \frac{2c}{m}\dot{q} + \left[\frac{g}{l} + \frac{\ddot{r}}{l}\right]\sin q + \frac{\ddot{s}}{l}\cos q = 0.$$

As in section 11.2, we rewrite the system in Hamiltonian form

$$\begin{aligned} \tfrac{dq}{dt} &= \tfrac{1}{ml^2}p - \tfrac{\dot{s}}{l}\cos q - \tfrac{\dot{r}}{l}\sin q \\ \tfrac{dp}{dt} &= -\tfrac{2c}{m}p + (2cl\cos q - \tfrac{p}{l}\sin q)\dot{s} + (2cl\sin q + \tfrac{p}{l}\cos q)\dot{r} + \\ &+\tfrac{m}{2}(\dot{s}^2 - \dot{r}^2)\sin 2q - m\dot{s}\dot{r}\cos 2q - mgl\sin q. \end{aligned} \tag{11.17}$$

We assume that there exists a small positive parameter ε for which $\alpha = \varepsilon a$, $\nu = \frac{\omega}{\varepsilon}$. Then, we have

$$s(t) = \varepsilon a \sin \frac{\omega}{\varepsilon} t \sin \theta, \quad r(t) = \varepsilon a sin \frac{\omega}{\varepsilon} t \cos \theta.$$

By changing to the fast time $\varepsilon\tau = t$ and denoting the differentiation in τ as a dot above, we get

$$\begin{aligned} \dot{q} &= \varepsilon\left[\tfrac{1}{ml^2}p - \tfrac{\dot{s}(\tau)}{l}\cos q - \tfrac{\dot{r}(\tau)}{l}\sin q\right], \\ \dot{p} &= \varepsilon\left[-\tfrac{2c}{m}p + \dot{s}(\tau)(2cl\cos q - \tfrac{p}{l}\sin q) + \dot{r}(\tau)(2cl\sin q + \tfrac{p}{l}\cos q) + \right. \\ &\left. +\tfrac{m}{2}(\dot{s}^2(\tau) - \dot{r}^2(\tau))\sin 2q - m\dot{s}(\tau)\dot{r}(\tau)\cos 2q - mgl\sin q\right]. \end{aligned} \tag{11.18}$$

The averaged system takes the following form

$$\begin{aligned} \dot{\xi} &= \varepsilon\tfrac{1}{ml^2}\eta, \\ \dot{\eta} &= \varepsilon\left[-\tfrac{2c}{m}\eta - mgl\sin\xi + \tfrac{m}{2}(\langle\dot{s}^2(\tau)\rangle - \langle\dot{r}^2(\tau)\rangle)\sin 2\xi - \right. \\ &\left. -m\langle(\dot{s}(\tau)\dot{r}(\tau)\rangle\cos 2\xi\right]. \end{aligned} \tag{11.19}$$

Evidently,

$$\langle\dot{s}^2(\tau)\rangle = \frac{a^2\omega^2}{2}\sin^2\theta = \frac{\alpha^2\nu^2}{2}\sin^2\theta, \quad \langle\dot{r}^2(\tau)\rangle = \frac{\alpha^2\nu^2}{2}\cos^2\theta,$$

$$\langle \dot{s}(\tau)\dot{r}(\tau)\rangle = \frac{\alpha^2\nu^2}{4}\sin 2\theta.$$

System (11.19) can have multiple stationary solutions. For example, it has a stationary solution $(\frac{\pi}{2}, 0)$ if

$$\frac{\alpha^2\nu^2}{4}\sin 2\theta = gl. \tag{11.20}$$

The linearized system on this equilibrium is

$$\begin{aligned} \dot{\xi} &= \varepsilon \tfrac{1}{ml^2}\eta, \\ \dot{\eta} &= \varepsilon(m\xi\cos 2\theta - \tfrac{2c}{m}\eta). \end{aligned}$$

Therefore, equilibrium $(\frac{\pi}{2}, 0)$ is asymptotically stable when

$$\cos 2\theta < 0. \tag{11.21}$$

Theorems 9.1 and 9.3 imply that, for sufficiently small ε, and, under conditions (11.20) and (11.21), system (11.18) has an asymptotically stable periodic solution that lies in a small neighborhood of $(\frac{\pi}{2}, 0)$, i.e., the pendulum has stable periodic oscillations in the neighborhood of its horizontal state.

If the law of motion of the pivot is defined by

$$\begin{aligned} s(t) &= g(t)\sin\theta, \\ r(t) &= g(t)\cos\theta, \end{aligned}$$

where

$$g(t) = \varepsilon a \sum_{k=1}^{\infty} \frac{\sin(2k-1)\frac{\omega}{\varepsilon}t}{(2k+1)^2},$$

then the averaged equation has a stationary solution $(\frac{\pi}{2}, 0)$ when

$$a^2\omega^2\pi^2 \sin 2\theta = 32gl.$$

This solution is asymptotically stable when inequality (11.21) holds.

Exercise 11.3. Investigate the conditions for the existence and the stability of equilibria $(0, \pi/4)$ and $(0, \pi/3)$ of system (11.19) and corresponding periodic solutions of system (11.18).

11.7 Pendulum with a Pivot Whose Oscillations Vanish in Time

Recall the main result about the stability of the upper equilibrium with a vibrating pivot. We suppose that a pivot of a mathematical pendulum can move freely along a vertical axis and the law of motion of the pivot is defined by

$$f(t) = a \sin \omega t.$$

If the amplitude a of oscillations of the pivot is sufficiently small and their frequency ω is sufficiently large, then the upper equilibrium of the pendulum is stable, provided

$$a^2\omega^2 > 2gl, \tag{11.22}$$

where l is the length of the pendulum, and g is the gravitational constant.

In this section we consider the problem of the stability of the upper equilibrium of a pendulum in the case when the law of motion of a pivot is defined by

$$f(t) = \frac{a}{t^{\alpha-1}} \sin \omega t^{\alpha}, \tag{11.23}$$

where a, ω, α are constants, and $\alpha > 1$, i.e., the amplitude of motion of a pivot tends to zero as $t \to \infty$ (see Burd [1999], Ganina [1998], and Ganina, and Kolesov [2000]).

As we have already observed, the equation of motion of a pendulum with a pivot, which itself moves along a vertical axis, has the form

$$\ddot{q} + \frac{2c}{m}\dot{q} + \left(\frac{g}{l} + \frac{1}{l}\ddot{f}\right)\sin q = 0, \tag{11.24}$$

where m is the mass of a pendulum, c is the damping coefficient, $q(t)$ is the angular displacement of a pendulum relative to the vertical axis, and $f(t)$ is the law of motion of the pivot. It is, again, convenient to write the equation of motion in Hamiltonian form. The corresponding system of equations has the form

$$\begin{aligned} \dot{q} &= \tfrac{1}{ml^2}p - \tfrac{1}{l}\dot{f}\sin q, \\ \dot{p} &= -\tfrac{2c}{m}p + [2cl\sin q + \tfrac{p}{l}\cos q]\dot{f} - mgl\sin q - \tfrac{m}{2}\dot{f}^2\sin 2q. \end{aligned} \tag{11.25}$$

System (11.25) has two stationary solutions $(0,0)$ and $(\pi,0)$ which are the lower and the upper equilibria of a pendulum, respectively.

We investigate the stability of the upper equilibrium. By linearizing system (11.25) on the equilibrium $(\pi,0)$ we get the linear system

$$\begin{aligned} \dot{x}_1 &= \tfrac{1}{ml^2}x_2 + \tfrac{1}{l}\dot{f}x_1, \\ \dot{x}_2 &= -\tfrac{2c}{m}x_2 + [-2clx_1 + \tfrac{1}{l}x_2]\dot{f} + mglx_1 - \tfrac{m}{2}\dot{f}^2x_1. \end{aligned} \tag{11.26}$$

We would like to study the behavior of solutions of this system when $t \to \infty$. Introduce a new time τ using

$$\tau = t^{\alpha}$$

and take into consideration that, in time τ,

$$\frac{df}{dt} = \alpha a\omega\cos\omega\tau + \frac{(1-\alpha)a}{\tau}\sin\omega\tau.$$

As a result we get (with $z = (x_1, x_2)$)

$$\frac{dz}{d\tau} = \frac{1}{\tau^{1-\beta}} A_1(\tau) z + \frac{1}{\tau^{2-\beta}} F(\tau) z, \tag{11.27}$$

where $\beta = \frac{1}{\alpha} < 1$, and

$$A_1(\tau) = \begin{pmatrix} \frac{a\omega}{l}\cos\omega\tau & \frac{\beta}{ml^2} \\ -2a\omega cl\cos\omega\tau + mgl\beta - \frac{ma^2\cos^2\omega\tau}{\beta} & -\frac{2c\beta}{m} + \frac{a\omega\cos\omega\tau}{l} \end{pmatrix},$$

while the concrete form of the matrix $F(\tau)$ is insignificant for further consideration. We ought to emphasize that the elements of the matrix $F(\tau)$ are bounded for $0 < \tau_0 \le \tau < \infty$.

First, we suppose that $\alpha > 2$. Then, $1 - \beta > \frac{1}{2}$. In system (11.27) make a change of variables

$$z = y + \frac{1}{\tau^{1-\beta}} Y_1(\tau) y, \tag{11.28}$$

where the matrix $Y_1(\tau)$ is defined as a matrix whose elements are periodic functions with a zero mean value, and

$$\frac{dY_1}{d\tau} = A_1(\tau) - B_1, \tag{11.29}$$

where B_1 is a constant matrix to be defined later. Clearly, the matrix B_1 should be composed of mean values of elements of the matrix $A_1(\tau)$, i.e.,

$$B_1 = \begin{pmatrix} 0 & \frac{\beta}{ml^2} \\ mgl\beta - \frac{ma^2\omega^2}{2\beta} & -\frac{2c\beta}{m} \end{pmatrix}.$$

After the change (11.28) system (11.27) becomes, for sufficiently large τ,

$$\frac{dy}{d\tau} = \frac{1}{\tau^{1-\beta}} B_1 y + \frac{1}{\tau^{2-2\beta}} G(\tau) y, \tag{11.30}$$

where elements of the matrix $G(\tau)$ are bounded for $0 < \tau_0 \le \tau < \infty$. In the original time t system (11.30) has the form

$$\frac{dy}{dt} = \alpha B_1 y + \frac{\alpha}{t^{\alpha-1}} G(t) y. \tag{11.31}$$

Due to Levinson's Theorem (Levinson [1948], see also Section 7.2), linearly independent solutions of system (11.31) when $t \to \infty$ can be represented as

$$y_1(t) = e^{\lambda_1 t}(p_1 + o(1)), \quad y_2(t) = e^{\lambda_2 t}(p_2 + o(1)), \tag{11.32}$$

where λ_1 and λ_2 are the eigenvalues of the matrix αB_1, and p_1, p_2 are the corresponding eigenvectors. It is easy to see that if

$$2gl < \alpha^2 a^2 \omega^2, \tag{11.33}$$

the eigenvalues of the matrix αB_1 have negative real parts. Therefore, if inequality (11.33) holds, then the trivial solution of system (11.31) is asymptotically stable due to (11.32). Consequently, the trivial solution of system (11.27) is also asymptotically stable. This implies that if inequality (11.33) holds, then the trivial solution of system (11.26) is asymptotically stable. Due to the Theorem on Stability in the First Approximation, the upper equilibrium of the pendulum is asymptotically stable, provided that $\alpha > 2$ and that inequality (11.33) holds.

If $\frac{3}{2} < \alpha \leq 2$, then $\frac{1}{3} < 1 - \beta \leq \frac{1}{2}$. In this case, in order to transform the system into a form to which we can apply Levinson's Theorem, in place of change (11.28) we should make the change

$$z = y + \frac{1}{\tau^{1-\beta}} Y_1(\tau) y + \frac{1}{\tau^{2-2\beta}} Y_2(\tau) y, \tag{11.34}$$

where the matrix $Y_1(\tau)$ is defined by the same equation (11.29), the matrix $Y_2(\tau)$ and the constant matrix B_2 are determined from the equation

$$\frac{dY_2}{d\tau} = A_1(\tau) Y_1(\tau) - B_2.$$

Moreover, the elements of the matrix $Y_2(\tau)$ are periodic functions with a zero mean value. The change (11.34) transforms system (11.27) into the system

$$\frac{dy}{d\tau} = \frac{1}{\tau^{1-\beta}} B_1 y + \frac{1}{\tau^{2-2\beta}} B_2 y + \frac{1}{\tau^{3-3\beta}} L(\tau) y, \tag{11.35}$$

where elements of the matrix $L(\tau)$ are bounded functions for $\tau_0 \leq \tau < \infty$. A simple calculation shows that $B_2 = 0$ and, therefore, for linearly independent solutions of system (11.35) in time t asymptotic formulas (11.32) hold. Therefore, if $\frac{3}{2} < \alpha \leq 2$ and inequality (11.33) holds, then the trivial solution of system (11.26) is asymptotically stable. If $1 < \alpha \leq \frac{3}{2}$, the asymptotic formulas (11.32) might be different. However, the main asymptotic term would have the same form as in (11.32). The exponent α influences the speed of convergence of solutions to the equilibrium.

Thus, when a pivot of a mathematical pendulum moves along the vertical axis according to (11.23) the upper equilibrium is asymptotically stable provided inequality (11.33) holds.

Exercise 11.4(a). Linearize equation (11.24) on the equilibrium $q = \pi$. Investigate the stability of the trivial solution of the linearized system using the method described in section 7.

Exercise 11.4(b). Investigate the stability of the upper equilibrium of a pendulum if the law of motion of its pivot is given by

$$f(t) = ae^{-t} \sin e^t.$$

11.8 Multifrequent Oscillations of a Pivot of a Pendulum

In this section we suppose that the law of motion of a pivot of a pendulum is a trigonometric polynomial, and some frequencies of this polynomial are close to each other. Hemp and Sethna [1968] considered the case of two frequencies.

As before, it is convenient to write the equation of motion of a pendulum in Hamiltonian form. This way we get (see Section 11.3)

$$\begin{aligned} \tfrac{dq}{dt} &= \tfrac{2}{ml^2}p - \tfrac{\dot f}{l}\sin q, \\ \tfrac{dp}{dt} &= -\tfrac{2c}{m}p + \left[2cl\sin q + \tfrac{p}{l}\cos q\right]\dot f - mgl\sin q - \tfrac{m}{2}\dot f^2 \sin 2q. \end{aligned} \tag{11.36}$$

The forced motion of a pivot is defined by

$$f(t) = \sum_{k=1}^{N} \alpha_k \cos v_k t + \beta_k \sin \nu_k t,$$

where α_k, β_k, $\nu_k (k = 1, 2, \ldots, N)$ are real numbers. We shall assume that the amplitudes α_k,β_k are sufficiently small, while the frequencies ν_k are sufficiently large in the following sense. There exists a small parameter ε, such that

$$\alpha_k = \varepsilon a_k, \quad \beta_k = \varepsilon b_k, \quad \nu_k = \frac{\omega_k}{\varepsilon},$$

where a_k, b_k, ω_k, $(k = 1, \ldots, N)$ are of order O(1) in ε. Then $f(t)$ can be written as

$$f(t) = \varepsilon \sum_{k=1}^{N} a_k \cos \omega_k \frac{t}{\varepsilon} + b_k \sin \omega_k \frac{t}{\varepsilon} = \varepsilon \zeta(\frac{t}{\varepsilon}).$$

We shall say that the law of motion of a pivot is *monofrequent* if differences $(\omega_k - \omega_j)$ for all $j \neq k$ are of order O(1) in ε. If some of these differences are of order $O(\varepsilon)$ in ε, we shall say that the law of motion of a pivot is *multifrequent*.

In this section we consider the case when the law of motion of a pivot is multifrequent.

In the monofrequent case (see Section 11.3) the method of averaging allows us to obtain well-known results on the stability of equilibria $q = 0$ and $q = \pi$ of system (11.36), for sufficiently small ε. The lower equilibrium $q = 0$ is always stable, the upper equilibrium $q = \pi$ is stable if

$$M({\zeta'}^2(\tau)) = \lim_{T\to\infty} \frac{1}{T} \int_0^T {\zeta'}^2(\tau) d\tau > gl.$$

This follows from the analysis of averaged equations and the corresponding theorems on averaging.

The averaged equations have the following form

$$\dot{q} = \varepsilon \tfrac{1}{ml^2} p,$$
$$\dot{p} = \varepsilon \left[-mgl \sin q - \tfrac{2c}{m} p - \tfrac{m}{2} M(\dot{\zeta}^2(\tau)) \sin 2q \right].$$

In the multifrequent case the averaged equations are, in general, equations with almost periodic coefficients.

We shall limit our considerations to the case when two frequencies ω_1 and ω_2 are connected by

$$\omega_2 - \omega_1 = \varepsilon\Delta, \quad \Delta > 0,\ \Delta = O(1). \tag{11.37}$$

For simplicity we let

$$f(t) = \varepsilon \left[a_1 \cos \omega_1 \frac{t}{\varepsilon} + a_2 \cos \omega_2 \frac{t}{\varepsilon} \right] = \varepsilon \zeta \left(\frac{t}{\varepsilon} \right).$$

From (11.37) we obtain

$$\varepsilon \zeta (\frac{t}{\varepsilon}) = \varepsilon \left[a_1 \cos \omega_1 \frac{t}{\varepsilon} + a_2 \cos \left(\omega_1 + \varepsilon\Delta \right) \frac{t}{\varepsilon} \right]. \tag{11.38}$$

We introduce the fast time τ using $t = \varepsilon\tau$, and denote by the prime symbol differentiation in τ. Then from (11.36) we obtain the following system in standard form:

$$\begin{aligned} q' &= \varepsilon \left(\tfrac{1}{ml^2} p - \tfrac{1}{l} \zeta'(\tau) \sin q \right) \\ p' &= \varepsilon \left[-mgl \sin q - \tfrac{2c}{m} p + \left(\tfrac{p}{l} \cos q + 2cl \sin q \right) \zeta'(\tau) - \right. \\ & \left. -m\zeta'^2(\tau) \cos q \sin q \right]. \end{aligned} \tag{11.39}$$

Taking into consideration (11.38) we get in the right-hand side of (11.39) the terms that contain the fast time τ and the slow time $\varepsilon\tau$. We average system (11.39) in the fast time τ to obtain the averaged system

$$\begin{aligned} q' &= \varepsilon \tfrac{1}{ml^2} p, \\ p' &= \varepsilon \left[-mgl \sin q - \tfrac{2c}{m} p - \tfrac{m}{4} b(\tau) \sin 2q \right], \end{aligned} \tag{11.40}$$

where $b(\tau) = \omega_1^2 (a_1^2 + a_2^2 + 2a_1 a_2 \cos \Delta\tau)$. The averaged system (11.40) can be written as an equation of the second order in variable q. This equation in terms of the time t has the form

$$\ddot{q} + \frac{2c}{m} \dot{q} + \frac{g}{l} \sin q + \frac{b(t)}{4l^2} \sin 2q = 0, \tag{11.41}$$

where $b(t) = \nu_1^2 (\alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2 \cos \Delta t)$. Theorems 9.6 and 9.7 imply that the problem of the stability of the lower equilibrium of a pendulum, for sufficiently small ε, can be studied by investigating the stability of the trivial solution of the equation

$$\ddot{q} + \frac{2c}{m} \dot{q} + \frac{1}{2l^2} \left(2gl + b(t) \right) q = 0. \tag{11.42}$$

For the upper equilibrium of a pendulum we get

$$\ddot{q} + \frac{2c}{m}\dot{q} + \frac{1}{2l^2}\left(-2gl + b(t)\right) q = 0. \tag{11.43}$$

Analysis of equation (11.43) shows that the motion of a pivot of a pendulum according to the law defined by (11.38) can lead to destabilization of the upper equilibrium, which would be stable if the frequencies were not close. Bogdanoff and Citron [1965] demonstrated experimentally that such destabilizing effects of fast parametric excitements, whose frequencies are close to each other, actually take place.

We also note that the analysis of the equation (11.42) shows that the proximity of the two frequencies can destabilize the lower equilibrium.

Therefore, the oscillations of a pivot of a pendulum under the influence of two periodic forces with close frequencies can lead to new effects in the behavior of a pendulum.

We now assume that, instead of (11.37), the frequencies are related by

$$\omega_2 - 2\omega_1 = \varepsilon\Delta. \tag{11.44}$$

The relation (11.44) has no influence on the equations in the first approximation. Therefore, it can affect the stability properties of the equilibria if the problem of the stability cannot be solved using the first approximation alone.

Let the law of motion of a pivot in time τ be defined by

$$\zeta(\tau) = a_1 \cos\omega_1\tau + a_2\cos\omega_2\tau + a_3\cos\omega_3\tau$$

and

$$\omega_1 - \omega_2 = \varepsilon\Delta_1, \quad \omega_1 - \omega_3 = \varepsilon\Delta_2.$$

Then, the problem of the stability of the lower and the upper equilibria, for sufficiently small ε, reduced to the investigation of the stability of the trivial solution for equations (11.42) and (11.43), respectively. In this case, $b(t)$ is an almost periodic function.

We now assume that

$$\omega_1 - \omega_2 = \varepsilon^2\Delta \quad (\Delta = const,\ \Delta > 0) \tag{11.45}$$

and

$$f(t) = \varepsilon\left[a_1\cos\omega_1\frac{t}{\varepsilon} + a_2\cos(\omega_1 + \varepsilon^2\Delta)\frac{t}{\varepsilon}\right] = \varepsilon\zeta\left(\frac{t}{\varepsilon}\right). \tag{11.46}$$

Again, we make a change to the fast time $\tau = t/\varepsilon$ in (11.36). We obtain a system in standard form that in time $\tau_1 = \varepsilon^2\tau$ becomes a singularly perturbed system. Such systems were considered in Section 9.8. We average system (11.36) in the fast time τ to obtain a singularly perturbed system in the time τ_1:

$$\begin{aligned} \varepsilon q' &= \tfrac{1}{ml^2}p, \\ \varepsilon p' &= \left[-mgl\sin q - \tfrac{2c}{m}p - \tfrac{m}{4}b(\tau_1)\sin 2q\right], \end{aligned} \tag{11.47}$$

where
$$b(\tau_1) = \frac{1}{2}\omega_1^2(a_1^2 + a_2^2) + a_1 a_2 \omega_1^2 \cos \Delta\tau_1).$$
The degenerate system (we assume $\varepsilon = 0$ in (11.47))
$$\begin{array}{l} \frac{1}{ml^2}p = 0, \\ -mgl\sin q - \frac{2c}{m}p - \frac{m}{4}b(\tau_1)\sin 2q = 0 \end{array}$$
has four solutions
$$\begin{array}{l} 1)\, q_1 \equiv 0,\ p_1 \equiv 0, \quad 2)\, q_2 \equiv \pi,\ p_2 \equiv 0, \\ 3,4)\, q_{3,4} = \pm \arccos\left(\frac{-gl}{b(\tau_1)}\right),\ p_{3,4} \equiv 0. \end{array}$$
The two latter solutions exist only if the following inequality holds
$$\inf_{-\infty<\tau_1<\infty} b(\tau_1) > gl. \tag{11.48}$$
It is easy to see that
$$\begin{array}{l} \inf_{-\infty<\tau_1<\infty} b(\tau_1) = \begin{cases} \frac{1}{2}\omega_2^2(a_1 - a_2)^2,\ a_1 a_2 > 0, \\ \frac{1}{2}\omega_2^2(a_1 + a_2)^2,\ a_1 a_2 < 0, \end{cases} \\ \sup_{-\infty<\tau_1<\infty} b(\tau_1) = \begin{cases} \frac{1}{2}\omega_2^2(a_1 + a_2)^2,\ a_1 a_2 > 0, \\ \frac{1}{2}\omega_2^2(a_1 - a_2)^2,\ a_1 a_2 < 0, \end{cases} \end{array}$$
We linearize the averaged system on the equilibria 1), 2) of the degenerate system to obtain systems
$$\begin{array}{l} \varepsilon q_1' = \frac{1}{ml^2}p_1, \\ \varepsilon p_1' = \left[-mglq_1 - \frac{2c}{m}p_1 - \frac{m}{2}b(\tau_1)q_1\right] \end{array}$$
and
$$\begin{array}{l} \varepsilon q_2' = \frac{1}{ml^2}p_2, \\ \varepsilon p_2' = \left[-mglq_2 - \frac{2c}{m}p_2 + \frac{m}{2}b(\tau_1)q_2\right], \end{array}$$
respectively. Analysis of these systems along with Theorem 16.1 implies that, for sufficiently small ε, the lower equilibrium of a pendulum is asymptotically stable. The upper equilibrium is asymptotically stable provided
$$\inf_{-\infty<\tau_1<\infty} b(\tau_1) > gl$$
and is unstable provided
$$\sup_{-\infty<\tau_1<\infty} b(\tau_1) < gl.$$
This result was established by Ukhalov [1997].

Exercise 11.5. Show that, for sufficiently small ε, there exist two unstable almost periodic solutions of system (11.36) which tend to the solutions 3,4) of the degenerate system provided inequality (11.48) holds.

11.9 System Pendulum-Washer with a Vibrating Base (Chelomei's Pendulum)

Chelomei's pendulum (see Chelomei [1983]) is a system that consists of a rod that can rotate around a certain axis ("axis of support") and a rigid body ("washer") that can move along the rod.

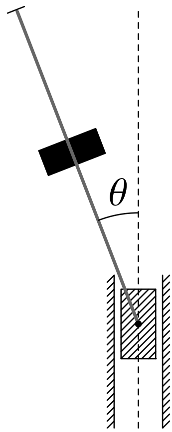

FIGURE 11.2: Chelomei's Pendulum.

V.N. Chelomei [1983] discovered experimentally that, due to a vertical vibration of the axis of support, the upper state of the rod can become stable under some conditions. In this case, the washer takes some fixed position on the rod. Chelomei's paper drew significant attention and several papers that dealt with this system have appeared. Menialov and Movchan [1984] studied the behavior of Chelomei's pendulum using the method of averaging. They assumed that the axis of support is vibrating in both horizontal and vertical directions. Blekhman and Malakhova [1986] using the same assumptions considered the behavior of the washer on a rigid rod, as well as an elastic rod that is vibrating in the regime of a standing wave. Ragulskis and Naginyavichus [1986] also studied the behavior of the washer on a vibrating elastic rod. Kirgetov [1986] obtained the conditions of the stability of the washer in the upper state of the rod using an alternative analytical method. In all these papers the authors showed that the stabilization of the upper equilibrium is achieved as a result of the horizontal component of vibrations of a support point. Thomsen and Tcherniak [2001] made a new attempt to explain the results of Chelomei's experiments. They showed that a small defect that violates the symmetry, such as a small deviation from a vertical excitement, can provide a mechanism for creation of stationary vibrations of an elastic rod.

To investigate Chelomei's pendulum we utilize the same method as we have used for studying a simple pendulum.

Consider a physical pendulum in the form of a non-uniform rod of mass M that is fixed at one end by a joint. Assume that the pendulum moves in a vertical plane around the support point. We also assume that a washer of mass m is put on the rod and can freely move along the rod. The inner diameter of the washer is the same as the thickness of the rod (see Figure 11.2). The support point of the pendulum makes high frequency periodic oscillations that have horizontal and vertical components. The equation of motion of the described system has the form (see Menialov and Movchan [1984]):

$$\begin{aligned}&(I_0 + I_1 + mx^2)\ddot{\varphi} + 2mx\dot{x}\dot{\varphi} + k_1\dot{\varphi} - (ML + mx)(g + \ddot{f}_1(t))\sin\varphi + \\ &(ML + mx)\ddot{f}_2(t)\cos\varphi = 0, \\ &\ddot{x} - x\dot{\varphi}^2 + k_2\dot{x} + (g + \ddot{f}_1(t))\cos\varphi + \ddot{f}_2(t)\sin\varphi = 0.\end{aligned} \tag{11.49}$$

Here, a dot denotes differentiation in time t, I_0 is the inertial moment of the rod without the washer relative to its axis of rotation, $I_1 + mx^2$ is the inertial moment of the washer, I_1 is the proper inertial moment of the washer, x is the current coordinate of the washer along the rod, φ is the current angle formed by the oscillating rod with a vertical axis, L is the distance from the center of mass of the rod to the support point, g is the gravitational constant, $k_1\dot{\varphi}$ is the frictional moment created by the motion of the entire system, $k_1\dot{x}$ is the friction force between the washer and the rod, $f_1(t)$ is the vertical component of oscillations of the support point, $f_2(t)$ is their horizontal component. We are not concerned with the exact form of functions $f_1(t)$, $f_2(t)$ for the moment. We let $p(x) = I_0 + I_1 + mx^2$, $r(x) = ML + mx$, and replace system (11.49) with a system of four equations of the first order. We make the change of variables

$$\begin{aligned}\dot{\varphi} &= \tfrac{\psi}{p(x)} + \tfrac{r(x)}{p(x)}[\dot{f}_1(t)\sin\varphi - \dot{f}_2(t)\cos\varphi], \\ \dot{x} &= \xi - \dot{f}_1(t)\cos\varphi - \dot{f}_2\sin\varphi.\end{aligned} \tag{11.50}$$

The change (11.50) introduces two new variables ψ and ξ, and is, in essence, a change that allows us to write system (11.49) in Hamiltonian form. Due to the first formula of (11.50) we get

$$\frac{d}{dt}(p(x)\dot{\varphi}) = p(x)\ddot{\varphi} + 2mx\dot{x}\dot{\varphi} = \frac{d}{dt}(\psi + r(x)\dot{f}_1(t)\sin\varphi - r(x)\dot{f}_2(t)\cos\varphi).$$

Substituting the last expression into the first equation of system (11.49) and taking into consideration the second formula of (11.50) yields

$$\begin{aligned}\dot{\psi} = &-\tfrac{k_1\psi}{p(x)} - \left[\left(m\xi + \tfrac{k_1 r(x)}{p(x)}\right)\sin\varphi + \tfrac{r(x)\psi}{p(x)}\cos\varphi\right]\dot{f}_1(t) - \\ &-\left[\left(-m\xi + \tfrac{k_1 r(x)}{p(x)}\right)\cos\varphi + \tfrac{r(x)\psi}{p(x)}\sin\varphi\right]\dot{f}_2(t) + \left(m - \tfrac{r^2(x)}{p(x)}\right)\cdot \\ &\cdot\left[\dot{f}_1^2(t)\sin\varphi\cos\varphi - \dot{f}_2^2(t)\sin\varphi\cos\varphi - \dot{f}_1(t)\dot{f}_2(t)\cos 2\varphi\right] + r(x)g\sin\varphi.\end{aligned}$$

By substituting the second formula of (11.50) into the second equation of system (11.49) we obtain

$$\begin{aligned}
&\dot{\xi} = -k_2\xi - \left[\tfrac{\psi}{p(x)}\left(1 - \tfrac{2xr(x)}{p(x)}\right)\sin\varphi - k_2\cos\varphi\right]\dot{f}_1(t) - \\
&-\left[-\tfrac{\psi}{p(x)}\left(1 - \tfrac{2xr(x)}{p(x)}\right)\cos\varphi - k_2\sin\varphi\right]\dot{f}_2(t) - \\
&-\tfrac{r(x)}{p(x)}\left(1 - \tfrac{xr(x)}{p(x)}\right)[\dot{f}_1^2(t)\sin^2\varphi + \\
&+\dot{f}_2^2(t)\cos^2\varphi - \dot{f}_1(t)\dot{f}_2(t)\sin 2\varphi] - g\cos\varphi + x\tfrac{\psi^2}{p^2(x)}.
\end{aligned}$$

Therefore, we get a system of four equations

$$\begin{aligned}
&\dot{\varphi} = \tfrac{\psi}{p(x)} + \tfrac{r(x)}{p(x)}[\dot{f}_1(t)\sin\varphi - \dot{f}_2(t)\cos\varphi], \\
&\dot{\psi} = -\tfrac{k_1\psi}{p(x)} - \left[\left(m\xi + \tfrac{k_1 r(x)}{p(x)}\right)\sin\varphi + \tfrac{r(x)\psi}{p(x)}\cos\varphi\right]\dot{f}_1(t) - \\
&-\left[\left(-m\xi + \tfrac{k_1 r(x)}{p(x)}\right)\cos\varphi + \tfrac{r(x)\psi}{p(x)}\sin\varphi\right]\dot{f}_2(t) + \left(m - \tfrac{r^2(x)}{p(x)}\right)\cdot \\
&\cdot\left[\dot{f}_1^2(t)\sin\varphi\cos\varphi - -\dot{f}_2^2(t)\sin\varphi\cos\varphi - \dot{f}_1(t)\dot{f}_2(t)\cos 2\varphi\right] + r(x)g\sin\varphi. \\
&\dot{x} = \xi - \dot{f}_1(t)\cos\varphi - \dot{f}_2\sin\varphi \\
&\dot{\xi} = -k_2\xi - \left[\tfrac{\psi}{p(x)}\left(1 - \tfrac{2xr(x)}{p(x)}\right)\sin\varphi - k_2\cos\varphi\right]\dot{f}_1(t) - \\
&-\left[-\tfrac{\psi}{p(x)}\left(1 - \tfrac{2xr(x)}{p(x)}\right)\cos\dot{\varphi} - k_2\sin\varphi\right]\dot{f}_2(t) - \\
&-\tfrac{r(x)}{p(x)}\left(1 - \tfrac{xr(x)}{p(x)}\right)[\dot{f}_1^2(t)\sin^2\varphi + \\
&+\dot{f}_2^2(t)\cos^2\varphi - \\
&-\dot{f}_1(t)\dot{f}_2(t)\sin 2\varphi] - g\cos\varphi + x\tfrac{\psi^2}{p^2(x)}.
\end{aligned} \tag{11.51}$$

We shall assume that the functions $f_1(t)$ and $f_2(t)$ are defined by

$$f_1(t) = \sum_{k=1}^{N} \alpha_k\cos\nu_k t + \beta_k\sin\nu_k t, \quad f_2(t) = \sum_{k=1}^{N} \gamma_k\cos\nu_k t + \delta_k\sin\nu_k t,$$

where α_k, β_k, γ_k, δ_k, and ν_k $(k = 1, \ldots, N)$ are real numbers. Further, we shall assume that the amplitudes α_k, β_k, γ_k, δ_k are sufficiently small while the frequencies ν_k are sufficiently large in the following sense. There exists a small positive parameter ε, such that

$$\alpha_k = \varepsilon a_k,\ \beta_k = \varepsilon b_k,\ \gamma_k = \varepsilon c_k,\ \delta_k = \varepsilon d_k,\ \nu_k = \frac{\omega_k}{\varepsilon},$$

where a_k, b_k, c_k, d_k, ω_k, $(k = 1, \ldots, N)$ are of order O(1) in ε. Then, the functions $f_1(t)$, $f_2(t)$ can be written as

$$\begin{aligned}
f_1(t) &= \varepsilon\textstyle\sum_{k=1}^{N} a_k\cos\omega_k\frac{t}{\varepsilon} + b_k\sin\omega\frac{t}{\varepsilon} = \varepsilon g_1(\frac{t}{\varepsilon}), \\
f_2(t) &= \varepsilon\textstyle\sum_{k=1}^{N} d_k\cos\omega_k\frac{t}{\varepsilon} + d_k\sin\omega\frac{t}{\varepsilon} = \varepsilon g_2(\frac{t}{\varepsilon}).
\end{aligned}$$

We make a change to the fast time $t = \varepsilon\tau$ in system (11.51) and denote differentiation in τ by the "prime" symbol. We obtain the system

$$\begin{aligned}
&\varphi' = \varepsilon\left[\tfrac{\psi}{p(x)} + \tfrac{r(x)}{p(x)}\left(g_1'(\tau)\sin\varphi - g_2'(\tau)\cos\varphi\right)\right],\\
&\psi' = \varepsilon\left\{-\tfrac{k_1\psi}{p(x)} - \left[\left(m\xi + \tfrac{k_1 r(x)}{p(x)}\right)\sin\varphi + \tfrac{r(x)\psi}{p(x)}\cos\varphi\right]g_1'(\tau)-\right.\\
&-\left[\left(-m\xi + \tfrac{k_1 r(x)}{p(x)}\right)\cos\varphi + \tfrac{r(x)\psi}{p(x)}\sin\varphi\right]g_2'(\tau)+\\
&+\left(m - \tfrac{r^2(x)}{p(x)}\right)\left[g_1'^2(\tau)\sin\varphi\cos\varphi - g_2'^2(\tau)\sin\varphi\cos\varphi - g_1'(\tau)g_2'(\tau)\cos 2\varphi\right]+\\
&\left.+r(x)g\sin\varphi\right\},\\
&x' = \varepsilon[\xi - g_1'(\tau)\cos\varphi - g_2'(\tau)\sin\varphi],\\
&\xi' = \varepsilon\left\{-k_2\xi - \left[\tfrac{\psi}{p(x)}\left(1 - \tfrac{2xr(x)}{p(x)}\right)\sin\varphi - k_2\cos\varphi\right]g_1'(\tau)-\right.\\
&-\left[-\tfrac{\psi}{p(x)}\left(1 - \tfrac{2xr(x)}{p(x)}\right)\cos\varphi - k_2\sin\varphi\right]g_2'(\tau)-\\
&-\tfrac{r(x)}{p(x)}\left(1 - \tfrac{xr(x)}{p(x)}\right)\left[g_1'^2(\tau)\sin^2\varphi + g_2'^2(\tau)\cos^2\varphi - g_1'(\tau)g_2'(\tau)\sin 2\varphi\right]-\\
&\left.-g\cos\varphi + x\tfrac{\psi^2}{p^2(x)}\right\}.
\end{aligned}\tag{11.52}$$

System (11.52) has a standard form. We average it and keep the same notation for its variables. This yields

$$\begin{aligned}
&\varphi' = \varepsilon\tfrac{\psi}{p(x)},\\
&\psi' = \varepsilon\left\{-\tfrac{k_1\psi}{p(x)} + P(x,\varphi)\right\},\\
&x' = \varepsilon\xi,\\
&\xi' = \varepsilon\left\{-k_2\xi - Q(x,\varphi)\right\},
\end{aligned}\tag{11.53}$$

where

$$\begin{aligned}
&P(x,\varphi) = \left(m - \tfrac{r^2(x)}{p(x)}\right)\left[(\langle g_1'^2(\tau)\rangle - \langle g_2'^2(\tau)\rangle)\sin\varphi\cos\varphi - \langle g_1'(\tau)g_2'(\tau)\rangle\cos 2\varphi\right]\\
&+r(x)g\sin\varphi,\\
&Q(x,\varphi) = \tfrac{r(x)}{p(x)}\left(1 - \tfrac{xr(x)}{p(x)}\right)\left[\langle g_1'^2(\tau)\rangle\sin^2\varphi + \langle g_2'^2(\tau)\rangle\cos^2\varphi-\right.\\
&\left.-\langle g_1'(\tau)g_2'(\tau)\rangle\sin 2\varphi\right] + g\cos\varphi,
\end{aligned}$$

and $\langle g(\tau)\rangle$ is the mean value of the function $g(\tau)$. We note that the averaged system (11.53) can be written as two equations of the second order:

$$\begin{aligned}
&p(x)\varphi'' + 2mxx'\varphi' + \varepsilon k_1\varphi' - \varepsilon^2 P(x,\varphi) = 0,\\
&x'' - x\varphi'^2 + \varepsilon k_2 x' + \varepsilon^2 Q(x,\varphi) = 0.
\end{aligned}\tag{11.54}$$

The equilibria of the averaged system (11.54) are defined from the system of equations

$$P(x,\,\varphi) = 0, \quad Q(x,\,\varphi) = 0. \tag{11.55}$$

System (11.55) has many solutions. We investigate the conditions that guarantee the existence and the stability of the solution $\varphi = 0$, i.e., the upper equilibrium of the pendulum. Due to the first equation of system (11.55) we obtain that for the existence of the solution $\varphi = 0$ the following inequality must hold

$$\langle g_1'(\tau) g_2'(\tau) \rangle = 0,$$

while using the second equation of the system (11.55) we get the condition

$$\langle g_2'^2(\tau) \rangle \neq 0.$$

The last inequality means that the averaged system has an equilibrium $\varphi = 0$, if the horizontal component of vibrations of the support point of a pendulum is non-zero.

We now find an equilibrium $x = x_0$ of the averaged system, such that $x = x_0$, $\varphi = 0$ is a stable equilibrium of system (11.55). We note that

$$P_\varphi(x_0, 0) = Q_x(x_0, 0) = 0.$$

The characteristic equation of the system, which is obtained by linearizing system (11.55) on the equilibrium $(x_0, 0)$, has the form

$$\left(\lambda^2 + \frac{k_1}{p(x)} \lambda - \frac{P_\varphi(x_0, 0)}{p(x)} \right) (\lambda^2 + k_2 \lambda - Q_x(x_0, 0)) = 0.$$

Hence, the conditions of the stability of the equilibrium $(x_0, 0)$ are

$$P_\varphi(x_0, 0) < 0, \quad Q_x(x_0, 0) < 0.$$

The equation for finding the equilibrium $x = x_0$ for $\varphi = 0$ is

$$-\frac{r(x)}{p(x)} \left(1 - \frac{x r(x)}{p(x)} \right) \langle g_2'^2(\tau) \rangle = g. \tag{11.56}$$

Equation (11.56) can be also written as

$$\frac{(ML + mx)(MLx - I)}{(I + mx^2)^2)} \langle g_2'^2(\tau) \rangle = g, \tag{11.57}$$

where $I = I_0 + I_1$. Finally, if we let

$$\zeta = \frac{MLx}{I}, \quad \mu = \frac{mI}{M^2 L^2}, \quad \sigma = \frac{ML \langle g_2'^2(\tau) \rangle}{gI},$$

then equation (11.57) can be written as

$$\Phi(\zeta) = \frac{(1 + \mu\zeta)(\zeta - 1)}{(1 + \mu\zeta^2)^2} = \frac{1}{\sigma}. \tag{11.58}$$

Depending on the value of σ (and, therefore, the value of $\langle g_2'^2(\tau) \rangle$), equation (11.58) may have one or two solutions, or have no solutions. We also observe that (11.58) implies that the following inequality holds

$$\zeta > 1. \tag{11.59}$$

The extremum $\zeta = \zeta^*$ of the function $\Phi(\zeta)$ for fixed μ is defined from equation

$$\mu^2\zeta^2(3-2\zeta) - \mu(3\zeta^2 - 6\zeta + 1) + 1 = 0; \tag{11.60}$$

moreover, we have to take (11.59) into consideration. Equation (11.60) in μ has no solutions if $\zeta < 3/2$, and has a unique positive solution if $\zeta > 3/2$. If

$$\frac{1}{\sigma} < \zeta^*,$$

then equation (11.58) has two solutions ζ_1, ζ_2, and

$$\Phi'(\zeta_1) > 0, \quad \Phi'(\zeta_2) < 0.$$

Since

$$Q(\zeta, 0) = \Phi(\zeta) - \frac{1}{\sigma},$$

we obtain

$$Q_\zeta(\zeta_1) > 0, \quad Q_\zeta(\zeta_2) < 0.$$

This implies that equilibrium $(x_{01}, 0)$, where $x_{01} = \frac{I\zeta_1}{ML}$, is unstable. Equilibrium $(x_{02}, 0))$, where $x_{02} = \frac{I\zeta_2}{ML}$, is stable if the inequality

$$P_\varphi(x_{02}, 0) = -\frac{M^2L^2 + 2MLx_{02} - mI}{I + mx_{02}^2}(\langle g_1'^2(\tau)\rangle - \langle g_2'^2(\tau)\rangle) +$$

$$+(ML + mx_{02})g < 0$$

holds. This, in turn, requires that the vertical component of the amplitude of oscillations of the support point be greater than its horizontal component.

Thus, under substantial additional conditions (we ought to say that for the rod of the finite length L we must have $0 \le x \le L$) the rod with the washer can be stabilized.

Chapter 12

Higher Approximations of the Method of Averaging

12.1 Formalism of the Method of Averaging for Systems in Standard Form

. Consider a system of differential equations in a standard form

$$\frac{dx}{dt} = \varepsilon X(t, x, \varepsilon), \tag{12.1}$$

where x is an n-dimensional vector, $\varepsilon > 0$ is a small parameter, the vector-function $X(t, x, \varepsilon)$ is defined for $t \in \mathcal{R}$, $x \in D$ and small ε, where D is a bounded set in $\mathcal{R}^n$. We assume that the right-hand sides $\varepsilon X(t, x, \varepsilon)$ in (12.1) can be expanded in powers of the small parameter ε:

$$\varepsilon X(t, x, \varepsilon) = \varepsilon X_1(t, x) + \varepsilon^2 X_2(t, x) + \varepsilon^3 X_3(t, x) + \dots .$$

Further, we suppose that the vector-functions $X_j(t, x)$, $j = 1, 2, \dots$ are almost periodic in t uniformly with respect to $x \in D$. However, **essential** limitations need to be imposed on the character of almost periodicity of these vector-functions. The thing is that to construct higher approximations of the method of averaging we use a close to identical, nonlinear transformation. Then we shall need to find almost periodic solutions of a system of differential equations with right-hand sides almost periodically dependent on t. It is possible to find these solutions if the right-hand sides of the system are the **correct** almost periodic functions. Recall that the almost periodic function $f(t)$ with the Fourier series

$$f(t) \sim \sum_{\nu} a_\nu e^{i\nu t}$$

is called **correct** if the following formula is valid

$$\int f(t)dt = a_0 t + g(t),$$

where a_0 is the mean value of the function $f(t)$, and $g(t)$ is an almost periodic function with the Fourier series

$$\sum_{\nu \neq 0} \frac{a_\nu}{i\nu} e^{i\nu t}.$$

Evidently, any periodic function, any trigonometric polynomial with an arbitrary set of frequencies, a sum of a periodic function and a trigonometric polynomial can serve as examples of the correct almost periodic functions. We note that in constructing higher approximations it is necessary to perform algebraic operations over the respective functions so as to remain within the function class under study. In what follows, we shall assume that we deal with one of the above classes of the correct almost periodic functions.

Hence, $X_j(t,x)$ allow for the expansion into the Fourier series

$$X_j(t,x) \sim X_{j0}(x) + \sum_{\nu\neq 0} X_{j\nu}(x)e^{i\nu t}, \quad j = 1,2,\dots.$$

Since only the real vector-functions $X_j(t,x)$ are under study, we see that the above expansion into the complex Fourier series is equivalent to the expansion into the real Fourier series

$$\frac{a_0}{2} + \sum_{\nu\neq 0}(a_{j\nu}\cos\nu t + b_{j\nu}\sin\nu t).$$

At the same time, the Fourier coefficients will be complex conjugate, i.e.,

$$X_{j(-\nu)} = \bar{X}_{j\nu}, \quad \frac{a_0}{2} = X_{j0}, \quad a_{j\nu} = X_{j\nu} + X_{j(-\nu)}, \quad b_{j\nu} = i(X_{j\nu} - X_{j(-\nu)}).$$

System (12.1) contains a slow variable $x\,(\frac{dx}{dt} \approx \varepsilon)$ and a fast variable t. N.N. Bogoliubov showed that we can change the variables so that to eliminate the fast variable t in the right-hand sides of system (12.1) at any order of accuracy with respect to ε. Let us describe this method.

We shall find a change of the variables

$$x = \xi + \varepsilon u_1(t,\xi) + \varepsilon^2 u_2(t,\xi) + \varepsilon^3 \dots, \tag{12.2}$$

where $\xi = \{\xi_1,\dots,\xi_n\}$ is a new unknown vector-function that satisfies the averaged system

$$\frac{d\xi}{dt} = \varepsilon\Sigma_1(\xi) + \varepsilon^2\Sigma_2(\xi) + \varepsilon^3 \dots, \tag{12.3}$$

which does not explicitly contain the time t, and $u_i(t,\xi)$ are some vector-functions that are almost periodic in t. We need to determine the unknown vector-functions $u_i(t,\xi)$, $i = 1,2,\dots$ that are the coefficients of transformation (12.2), and, also, the vector-functions $\Sigma_i(\xi)$, $i = 1,2,\dots$ in the right-hand sides of the averaged system (12.3). Using change (12.2), we transform system (12.1) into

$$\frac{d\xi}{dt} + \varepsilon\frac{\partial u_1(t,\xi)}{\partial\xi}\frac{d\xi}{dt} + \varepsilon\frac{\partial u_1(t,\xi)}{\partial t} + \varepsilon^2\frac{\partial u_2(t,\xi)}{\partial\xi}\frac{d\xi}{dt} + \varepsilon^2\frac{\partial u_2(t,\xi)}{\partial t} + \varepsilon^3\dots =$$

$$\varepsilon X_1(t,\xi + \varepsilon u_1(t,\xi) + \varepsilon^2 u_2(t,\xi) + \varepsilon^3\dots) + \varepsilon^2 X_2(t,\xi + \varepsilon u_1(t,\xi) +$$

$$+\varepsilon^2 u_2(t,\xi) + \varepsilon^3 \dots) + \varepsilon^3 \dots .$$

We emphasize that derivatives of the type $\frac{\partial u_i}{\partial \xi}$ are understood as matrices. In the latter equality, replacing the derivative $\frac{d\xi}{dt}$ by the right-hand side of system (12.3) yields

$$\varepsilon\Sigma_1(\xi) + \varepsilon\Sigma_2(\xi) + \varepsilon^2 \frac{\partial u_1(t,\xi)}{\partial \xi}\Sigma_1(\xi) + \varepsilon\frac{\partial u_1(t,\xi)}{\partial t} + \varepsilon^2\frac{\partial u_2(t,\xi)}{\partial t} + \varepsilon^3 \dots =$$

$$\varepsilon X_1(t,\xi) + \varepsilon^2 \frac{\partial X_1(t,\xi)}{\partial \xi} u_1(t,\xi) + \varepsilon X_2(t,\xi) + \varepsilon^3 \dots .$$

Equating the coefficients with the same power of ε in the left-hand and right-hand sides of the latter equality, we obtain an infinite system of equations. Let us write out the first two of them.

$$\Sigma_1(\xi) + \frac{\partial u_1(t,\xi)}{\partial t} = X_1(t,\xi), \tag{12.4}$$

$$\Sigma_2(\xi) + \frac{\partial u_1(t,\xi)}{\partial \xi}\Sigma_1(\xi) + \frac{\partial u_2(t,\xi)}{\partial t} = \frac{\partial X_1(t,\xi)}{\partial \xi} u_1(t,\xi) + X_2(t,\xi). \tag{12.5}$$

We successively determine the vector-functions $\Sigma_1(\xi)$,$\Sigma_2(\xi)$ and $u_1(t,\xi)$, $u_2(t,\xi)$ from identities (12.4) and (12.5). We write (12.4) as

$$\frac{\partial u_1(t,\xi)}{\partial t} = X_1(t,\xi) - \Sigma_1(\xi). \tag{12.6}$$

The vector-function $u_1(t,\xi)$ can be determined as almost periodic in t if the mean value of the almost periodic vector-function in the right-hand side of system (12.6) equals zero (of course, it is assumed that $X_1(t,\xi)$ is a correct almost periodic vector-function of the variable t). Hence, the vector-function $\Sigma_1(\xi)$ is **unambiguously** defined as the mean value of the almost periodic function $X_1(t,\xi)$:

$$\Sigma_1(\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X_1(t,\xi)dt.$$

Further, integrating the system of differential equations (12.6), we find the vector-function $u_1(t,\xi)$ at the accuracy up to an arbitrary vector-function of the variable ξ as follows

$$u_1(t,\xi) = \int_0^t \left[X_1(t,\xi) - \Sigma_1(\xi)\right] dt + u_{10}(\xi).$$

Note that the following Fourier series conforms to the vector-function $u_1(t,\xi)$

$$\sum_{\nu\neq 0} X_{1\nu}(\xi)\frac{e^{i\nu t}}{i\nu} + u_{10}(\xi).$$

By substituting the vector-functions $\Sigma_1(\xi)$, $u_1(t,\xi)$ into identity (12.5) we obtain

$$\frac{\partial u_2(t,\xi)}{\partial t} = \frac{\partial X_1(t,\xi)}{\partial \xi} u_1(t,\xi) + X_2(t,\xi) - \frac{\partial u_1(t,\xi)}{\partial \xi}\Sigma_1(\xi) - \Sigma_2(\xi). \quad (12.7)$$

Then, to define $u_2(t,\xi)$ as the almost periodic vector-function of t, we need to select $\Sigma_2(\xi)$ such that the mean value of the right-hand side of system (12.7) vanishes (it is again assumed that the right-hand side of system (12.7) is a correct almost periodic vector-function of t). It should be pointed out that $\Sigma_2(\xi)$ is no longer **defined unambiguously**. After that we find the vector-function $u_2(t,\xi)$. Continuing with the determination of the functions $\Sigma_i(\xi)$, $u_i(t,\xi)$ at the k-th step, we arrive at the system

$$\frac{\partial u_k(t,\xi)}{\partial t} = F(t,\xi,u_1(t,\xi),\ldots,u_{k-1}(t,\xi)) - \Sigma_k(\xi),$$

where $F(t,\xi,u_1(t,\xi),\ldots,u_{k-1}(t,\xi))$ is the already known correct almost periodic in t vector-function. The vector-function $\Sigma_k(\xi)$ is determined from the condition that the mean value of the right-hand side of the latter equality is equal to zero. Then we determine the vector-function $u_k(t,\xi)$ that is almost periodic in t.

Thus, we have shown that there exists a change of variables of the form (12.2) with the almost periodic coefficients that transforms system (12.1) into system (12.3). The truncated change

$$x = \xi + \varepsilon u_1(t,\xi) + \varepsilon^2 u_2(t,\xi) + \cdots + \varepsilon^k u_k(t,\xi) \quad (12.8)$$

transforms system (12.1) into the system

$$\frac{d\xi}{dt} = \varepsilon\Sigma_1(\xi) + \varepsilon\Sigma_2(\xi) + \cdots + \varepsilon^k\Sigma_k(\xi) + \varepsilon^{k+1}\Sigma_{k+1}(t,\xi,\varepsilon),$$

where the vector-function $\Sigma_{k+1}(t,\xi,\varepsilon)$ is almost periodic in t.

12.2 Theorem of Higher Approximations in the Periodic Case

Consider a system of differential equations in the space $\mathcal{R}^n$

$$\frac{dx}{dt} = \varepsilon f(t,x,\varepsilon), \quad (12.9)$$

where $\varepsilon > 0$ is a small parameter, the vector-function $f(t,x,\varepsilon)$ is defined for $t \in \mathcal{R}$, $x \in \mathcal{R}^n$ and small ε, is periodic in t with the period T independent of (x, ε), and is sufficiently smooth in x and ε. Assume that system (12.9) is representable as

$$\frac{dx}{dt} = \varepsilon f_1(t,x) + \varepsilon^2 f_2(t,x) + \cdots + \varepsilon^N f_N(t,x) + \varepsilon^{N+1} f_{N+1}(t,x,\varepsilon), \quad (12.10)$$

where each vector-function $f_i(t,x)$, $i = 1, \ldots, N$, $f_{N+1}(t,x,\varepsilon)$ is periodic in t with the period T.

Using a change of the form (12.8) with the T-periodic coefficients, we eliminate the variable t from the first N terms in the right-hand side of system (12.10) and obtain the system

$$\frac{dy}{dt} = \varepsilon \bar{f}_1(y) + \cdots + \varepsilon^N \bar{f}_N(y) + \varepsilon^{N+1} \tilde{f}(t,y,\varepsilon), \quad (12.11)$$

where $\tilde{f}(t,y,\varepsilon)$ has the same properties as $f_{N+1}(t,x,\varepsilon)$. If there exists a constant vector y_0 such that $\bar{f}(y_0) = 0$, and the matrix

$$A = \frac{\partial \bar{f}}{\partial y}(y_0) \quad (12.12)$$

has no eigenvalues with the zero real part, then, as it follows from Theorems 9.2 and 9.3, system (10.11), for sufficiently small ε, has a unique T-periodic solution $x^*(t,\varepsilon)$ with its properties of stability coinciding with the properties of stability of the stationary solution $y = y_0$ in the averaged system of the first approximation

$$\frac{dy}{dt} = \varepsilon \bar{f}_1(y).$$

We now consider a more general situation (see Murdock and Robinson [1980a], Hale and Pavlu [1983], Burd [1983], Murdock [1988]).

We assume that the matrix A has eigenvalues with the zero real part but has no zero eigenvalue. Then for sufficiently small ε, system (12.11) has a unique T-periodic solution but the problem on the stability of this solution cannot be solved in the first approximation. Consider the system

$$\frac{dy}{dt} = \varepsilon F(y,\varepsilon), \quad (12.13)$$

where $F(y,\varepsilon) = \bar{f}_1(y) + \cdots + \varepsilon^{N-1} \bar{f}_N(y)$, and write the initial system (12.11) in the form

$$\frac{dy}{dt} = \varepsilon F(y,\varepsilon) + \varepsilon^{N+1} \tilde{f}(t,y,\varepsilon). \quad (12.14)$$

Theorem 12.1. *Let there exist y_0 such that $F(y_0, 0) = 0$ and the matrix A defined by formula (12.12) has no zero eigenvalue. Then there exists $\varepsilon_1 > 0$ and the vector-functions $y_N(\varepsilon)$ and $y^*(t,\varepsilon)$ such that $y_N(\varepsilon)$ is the equilibrium of system (12.13), $y^*(t,\varepsilon)$ is the T-periodic solution of system*

(12.14), $y_N(0) = y^*(t,0) = y_0$ and $|y^*(t,\varepsilon) - y_N(\varepsilon)| = O(\varepsilon^N)$. Besides, the properties of stability of the solutions $y_N(\varepsilon)$ and $y^*(t,\varepsilon)$ are the same (the asymptotic stability of the equilibrium $y_N(\varepsilon)$ of system (12.13) implies, for sufficiently small ε, the asymptotic stability of the periodic solution $y^*(t,\varepsilon)$ of (12.14), and the instability of $y_N(\varepsilon)$ implies the instability of $y^*(t,\varepsilon)$ for sufficiently small ε). To put it more precisely, if the equations linearized on the equilibrium $y_N(\varepsilon)$ of (12.13) have an exponential dichotomy of order $k \le N$, then the equations linearized on the solution $y^*(t,\varepsilon)$ of system (12.14) have an exponential dichotomy of the same order.

Proof. If there exists y_0 such that $F(y_0,0) = 0$ and the matrix $A = (\partial F/\partial y)(y_0,0)$ is non-singular, then, as per the theorem of implicit function, for sufficiently small ε, there exists a unique vector-function $y_N(\varepsilon)$ such that $y_N(0) = y_0$ and $F(y_N(\varepsilon),\varepsilon) \equiv 0$. Hence, $y_N(\varepsilon)$ is the equilibrium of autonomous system (12.13). We transform system (12.14) by a change

$$y = y_N(\varepsilon) + z$$

into the system

$$\begin{aligned}\tfrac{dz}{dt} &= \varepsilon F(y_N(\varepsilon)+z,\varepsilon) + \varepsilon^{N+1}\tilde f(t, y_N(\varepsilon)+z,\varepsilon) = \\ &\varepsilon \tfrac{\partial F}{\partial y}(y_N(\varepsilon),\varepsilon)z + \varepsilon G(z,\varepsilon) + \varepsilon^{N+1}\tilde f(t, y_N(\varepsilon)+z,\varepsilon),\end{aligned} \tag{12.15}$$

where $G(0,\varepsilon) = 0$, $(\partial G/\partial z)(0,\varepsilon) = 0$. System (12.15) can be rewritten as

$$\frac{dz}{dt} = \varepsilon A(\varepsilon)z + \varepsilon[G(z,\varepsilon) + \varepsilon^N \tilde f(t, y_N(\varepsilon)+z,\varepsilon)], \tag{12.16}$$

where $A(\varepsilon) = \frac{\partial F}{\partial y}(y_N(\varepsilon),\varepsilon) = \frac{\partial F}{\partial y}(y_0,\varepsilon) + \varepsilon\tilde F(\varepsilon) = A + \varepsilon \tilde F(\varepsilon)$. The problem on the existence of a periodic solution of system (12.16) is reduced to the problem on the existence of the solution to the operator equation

$$z = \Pi(z,\varepsilon) =$$

$$\int_t^{t+T} \varepsilon\left[e^{-\varepsilon A(\varepsilon)T} - I\right]^{-1} e^{\varepsilon A(\varepsilon)(t-s)}[G(z,\varepsilon) + \varepsilon^N \tilde f(s, y_N(\varepsilon)+z,\varepsilon)]ds. \tag{12.17}$$

The operator in the right-hand side of system (12.17) acts in the space P_T of continuous T-periodic vector-functions. The linear operator

$$K(\varepsilon)f = \int_t^{t+T} \varepsilon\left[e^{-\varepsilon A(\varepsilon)T} - I\right]^{-1} e^{\varepsilon A(\varepsilon)(t-s)} f(s)ds$$

is bounded in the space P_T. Therefore, we show, almost word for word as in the proof of Theorem 9.1, that the operator $\Pi(z,\varepsilon)$ is a contraction for sufficiently small ε on some ball in the space P_T. Hence, for sufficiently small ε, there exists a unique periodic solution $z^*(t, \varepsilon)$ with the period T. Using the method of successive approximations with the initial approximation $z_0 = 0$, we obtain $|z^*(t, \varepsilon)| \leq \tilde{K}\varepsilon^N$, where $\tilde{K}$ is a constant. Taking into account the change formula, we obtain that there exists a T-periodic solution $y^*(t, \varepsilon)$ of system (12.14), $y^*(t, 0) = y_N(0) + z^*(t, 0) = y_0$ for which $||y^*(t, \varepsilon) - y_N(\varepsilon)|| = O(\varepsilon^N)$.

The equations linearized on the solution $y^*(t, \varepsilon)$ of system (12.14) take the form

$$\frac{du}{dt} = \varepsilon \frac{\partial F}{\partial y}(y^*(t, \varepsilon), \varepsilon)u + \varepsilon^{N+1} \frac{\partial \tilde{f}}{\partial y}(t, y^*(t, \varepsilon), \varepsilon)u =$$

$$\varepsilon \left(\frac{\partial F}{\partial y}(y_N(\varepsilon), \varepsilon) + O(\varepsilon^N) \right) u + \varepsilon^{N+1} \frac{\partial \tilde{f}}{\partial y}(t, y^*(t, \varepsilon), \varepsilon)u =$$

$$\varepsilon \frac{\partial F}{\partial y}(y_N(\varepsilon), \varepsilon)u + O(\varepsilon^{N+1}).$$

Thus, the equations linearized on the solution $y_N(\varepsilon)$ of system (12.13) and the equations linearized on the solution $y^*(t, \varepsilon)$ of system (12.14) coincide up to the order N. This implies the last assertion of the theorem.

Note that in the case when the matrix A has a zero eigenvalue, the situation with the problem on the existence and stability of periodic solutions is more complicated as can be seen from the example in Section 5.4 (see also Murdock and Robinson [1980a]). Here we do not consider this case.

12.3 Theorem of Higher Approximations in the Almost Periodic Case

Consider a system of differential equations

$$\frac{dx}{dt} = \varepsilon X(t, x, \varepsilon), \tag{12.18}$$

where $x \in R^n$, $\varepsilon > 0$ is a small parameter, $X(t, x, \varepsilon)$ is a vector-function determined for $t \in \mathcal{R}$, $x \in D$ (D is a bounded domain in $\mathcal{R}^n$) and sufficiently smooth in x and ε. The components $X(t, x, \varepsilon)$ are trigonometric polynomials in t with the frequencies independent of x, ε. We can assume an even more general t-dependence of the right-hand side of system (12.18). The only requirement is the possibility to construct equations of higher approximations.

We shall compare system (12.18) with the averaged system

$$\frac{dy}{dt} = \varepsilon Y_1(y), \tag{12.19}$$

where

$$Y_1(y) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t, y, 0)dt.$$

Let system (12.19) have the stationary solution $y = y_0$ and the matrix $A_0 = \frac{\partial Y_1}{\partial y}(y_0)$ have eigenvalues with the zero real part but no zero eigenvalue. We construct an averaged system of N-th approximations for system (12.18). We make a standard change of the method of averaging

$$x = y + \sum_{i=1}^{N} \varepsilon^i u_i(t, y) \tag{12.20}$$

that transforms (12.18) into

$$\frac{dy}{dt} = \sum_{i=1}^{N} \varepsilon^i Y_i(y) + \varepsilon^{N+1} G(t, y, \varepsilon). \tag{12.21}$$

A system of N-th approximation is called the shortened system

$$\frac{dy}{dt} = \sum_{i=1}^{N} \varepsilon^i Y_i(y).$$

Let

$$H(y, \varepsilon) = \sum_{i=1}^{N} \varepsilon^{i-1} Y_i(y).$$

By virtue of the assumed invertibility of the matrix A_0 and the theorem of implicit function, there exist $\varepsilon_0 > 0$ and a smooth vector-function $y_N(\varepsilon)$ defined for $0 < \varepsilon < \varepsilon_0$, $y_N(0) = y_0$ that is the equilibrium solution of a system of N-th approximation:

$$H(y_N(\varepsilon), \varepsilon) = 0.$$

We introduce the matrix $A(\varepsilon) = H_y(y_N(\varepsilon), \varepsilon)$ and the operator

$$L(\varepsilon) = \frac{dy}{dt} - A(\varepsilon)y$$

acting in the space B_n.

Theorem 12.2. *Let for $0 < \varepsilon < \varepsilon_0$ there exist an operator $L^{-1}(\varepsilon)$ that is inverse of the operator $L(\varepsilon)$ and the norm of the operator obeys the inequality*

$$||L^{-1}(\varepsilon)|| \le \frac{M}{\varepsilon^k}, \tag{12.22}$$

where k obeys the inequality $2k < N$.

Then, for sufficiently small ε, system (12.21) has a unique almost periodic solution $y_0(t,\varepsilon)$ ($y_0(t,0) = y_0$) in the ball $||y - y_N(\varepsilon)|| \leq r(\varepsilon)$ of the space B_n. This solution will be asymptotically stable, for sufficiently small ε, if all eigenvalues of the matrix $A(\varepsilon)$ have negative real parts, and unstable, for sufficiently small ε, if the matrix $A(\varepsilon)$ has at least one eigenvalue with a positive real part.

Proof. We make a change $y = y_N(\varepsilon) + z$ and transform system (12.21) into the system

$$\frac{dz}{dt} = \varepsilon H(y_N(\varepsilon) + z, \varepsilon) + \varepsilon^{N+1} G_1(t, z, \varepsilon), \tag{12.23}$$

where $G_1(t, z, \varepsilon) = G(t, y_N(\varepsilon) + z, \varepsilon)$. The initial problem is reduced to the problem of the existence of the almost periodic solution $z(t,\varepsilon)$ of system (12.23) such that

$$\lim_{\varepsilon \to 0} ||z(t,\varepsilon)|| = 0.$$

We write system (12.23) as

$$\frac{dz}{dt} = \varepsilon A(\varepsilon) z + \varepsilon F(z, \varepsilon) + \varepsilon^{N+1} G_1(t, z, \varepsilon), \tag{12.24}$$

where $F(z,\varepsilon) = H(y_N + z, \varepsilon) - A(\varepsilon)z$. In system (12.24), we pass to the slow time $\tau = \varepsilon t$ and obtain the system

$$\frac{dz}{d\tau} = A(\varepsilon) z + F(z, \varepsilon) + \varepsilon^N G_1(\frac{\tau}{\varepsilon}, z, \varepsilon). \tag{12.25}$$

Transform system (12.25) by using the change $z = \varepsilon^k u$, into the system

$$\frac{du}{d\tau} = A(\varepsilon) u + \varepsilon^{-k} F(\varepsilon^k u, \varepsilon) + \varepsilon^{N-k} G_1(\frac{\tau}{\varepsilon}, \varepsilon^k u, \varepsilon). \tag{12.26}$$

The problem of the existence of an almost periodic solution of system (12.26) is equivalent to the problem of solvability of the following operator equation in the space B_n

$$u = L^{-1}(\varepsilon)[\varepsilon^{-k} F(\varepsilon^k u, \varepsilon) + \varepsilon^{N-k} G_1(\frac{\tau}{\varepsilon}, \varepsilon^k u, \varepsilon)].$$

Since the expansion of the components of the vector-function $F(z,\varepsilon)$ starts with the terms of no lower than the second order over the variable z, we obtain that

$$\varepsilon^{-k} F(\varepsilon^k u, \varepsilon) = \varepsilon^k \Phi(u, \varepsilon),$$

wherein, $\Phi_i(0,\varepsilon) = \Phi_{iu}(0,\varepsilon) = 0$, $i = 1, \ldots, n$.

We show that the operator

$$\Pi(u,\varepsilon) = L^{-1}(\varepsilon)[\varepsilon^k \Phi(u,\varepsilon) + \varepsilon^{N-k} G_1(\frac{\tau}{\varepsilon}, \varepsilon^k u, \varepsilon)]$$

satisfies the conditions of the principle of contraction mappings on a ball with a sufficiently small radius in the space B_n. Evidently, the vector-function $\Phi(u,\varepsilon)$ obeys the inequality

$$|\Phi(u_1,\varepsilon)-\Phi(u_2,\varepsilon)|\leq\omega(r,\varepsilon)|u_1-u_2|,\quad |u_1|,|u_2|\leq r,$$

where $\omega(r,\varepsilon)\to 0$ as $r\to 0$. The vector-function $G_1(\tau/\varepsilon,u,\varepsilon)$ satisfies the Lipschitz condition in u with some constant K. In view of inequality (12.22), we obtain

$$||\Pi(u_1,\varepsilon)-\Pi(u_2,\varepsilon)||\leq M[\omega(r,\varepsilon)+K\varepsilon^{N-k}]||u_1-u_2||.$$

Then, we get

$$||\Pi(0,\varepsilon)||=||L^{-1}(\varepsilon)\varepsilon^{N-k}G_1(\frac{\tau}{\varepsilon},0,\varepsilon)||\leq MK_1\varepsilon^{N-2k},$$

where $K_1\geq||G_1(\frac{\tau}{\varepsilon},0,\varepsilon)||$. It is easy to infer from the latter two inequalities that the operator $\Pi(u,\varepsilon)$ meets the conditions of the principle of contraction mappings on a ball in B_n. Hence, in this ball, system (12.26), for sufficiently small ε, has a unique almost periodic solution $u_0(t,\varepsilon)$. Therefore, system (12.21), for sufficiently small ε, has a unique almost periodic solution $y_0(t,\varepsilon)=y_N(\varepsilon)+\varepsilon^k u_0(t,\varepsilon)$.

To prove the assertions of the stability for the solution $y_0(t,\varepsilon)$, it is sufficient to prove the similar assertions of the stability for the solution $z_0(\tau,\varepsilon)$ of system (12.25). By a change of the variable

$$y=z_0(\tau,\varepsilon)+z,$$

we transform system (12.25) into

$$\frac{dz}{d\tau}=A(\varepsilon)z+[H(z_0+z,\varepsilon)-H(z_0,\varepsilon)-A(\varepsilon)z]+\varepsilon^m[G(\tau/\varepsilon,z_0+z,\varepsilon)-$$

$$-G_1(\tau/\varepsilon,z_0,\varepsilon)].$$

Consider this linear system

$$\frac{dz}{d\tau}=A(\varepsilon)z. \tag{12.27}$$

For sufficiently small ε, the matrix $A(\varepsilon)$ has no eigenvalues with a zero real part. It follows from estimation of the operator $L^{-1}(\varepsilon)$ that system (12.27) has an exponential dichotomy of order k. We represent a space of the solutions $U(\varepsilon)$ of system (12.27) as

$$U(\varepsilon)=U_+(\varepsilon)+U_-(\varepsilon).$$

For the solutions $z_+(\tau,\varepsilon)\in U_+(\varepsilon)$, the inequality

$$|z_+(\tau,\varepsilon)|\leq|z_+(s,\varepsilon)|M_+\exp[-\gamma_+(\varepsilon)(\tau-s)],\quad -\infty<s<\tau<\infty$$

holds, and for the solutions $z_-(\tau, \varepsilon) \in U_-(\varepsilon)$, the inequality

$$|z_-(\tau, \varepsilon)| \leq |z_-(s, \varepsilon)| M_- \exp[-\gamma_-(\varepsilon)(\tau - s)], \quad -\infty < \tau < s < \infty$$

holds, where $\gamma_+(\varepsilon) = c_1 \varepsilon^k$, $\gamma_(\varepsilon) = c_2 \varepsilon^k$, $c_1, c_1 > 0$.

The final part of the proof of the theorem is almost word-for-word coincident with the proof of the theorems of the stability and instability of the first approximation (see Appendix B).

12.4 General Theorem of Higher Approximations in the Almost Periodic Case

If the matrix A_0 has a zero eigenvalue, then the problem of the existence of almost periodic solutions of system (12.18) is more complicated. The book by Krasnosel'skii, Burd, Kolesov [1973] presents the case when the matrix A_0 has a simple zero eigenvalue with all the other eigenvalues having non-zero real parts. It was additionally assumed that system (12.18) had the stationary solution $x(t) \equiv y_0$. They investigated the problem of the existence and stability of the non-stationary almost periodic solutions. In Ukhalov's thesis [1997], there was investigated the case when the matrix A_0 had a zero eigenvalue of arbitrary multiplicity. Under additional assumptions, the conditions for the existence and stability of the almost periodic solutions were established.

We again turn to system (12.18) and consider it under the same assumptions as in the previous clause.

We shall compare system (12.18) with averaged system (12.19). Let system (12.19) have the stationary solution $y = y_0$ and the matrix $A_0 = \frac{\partial Y_1}{\partial y}(y_0)$ have eigenvalues with a zero real part. The change (12.20) transforms (12.20) into system (12.18)

$$\frac{dy}{dt} = \sum_{i=1}^{m} \varepsilon^i Y_i(y) + \varepsilon^{m+1} G(t, y, \varepsilon). \tag{12.28}$$

We again assume

$$H(y, \varepsilon) = \sum_{i=1}^{m} \varepsilon^{i-1} Y_i(y).$$

Let the equation

$$H(y, \varepsilon) = 0 \tag{12.29}$$

for $0 < \varepsilon < \varepsilon_0$ have the bounded solution $y_0(\varepsilon)$, and the matrix $A(\varepsilon) = H'_y(y_0(\varepsilon), \varepsilon)$, for $\varepsilon = 0$, have eigenvalues with a zero real part. We introduce differential operator

$$L(\varepsilon)y = \frac{dy}{dt} - A(\varepsilon)y$$

that is defined on a set of continuously differentiable almost periodic vector-functions.

Theorem 12.3 (Ukhalov [1997]). *Let*
1) for $0 < \varepsilon < \varepsilon_1$ there exists an operator $L^{-1}(\varepsilon)$ that is inverse of the operator $L(\varepsilon)$ and the norm of this operator satisfies the inequality

$$||L^{-1}(\varepsilon)|| \leq \frac{M}{\varepsilon^{\alpha}}, \quad \alpha > 0;$$

2) for $0 < \varepsilon < \varepsilon_1$, $|y_1 - y_0(\varepsilon)| \leq r(\varepsilon)$, $|y_2 - y_0(\varepsilon)| \leq r(\varepsilon)$ the following inequality holds

$$|H(y_1, \varepsilon) - H(y_2, \varepsilon) - A(\varepsilon)(y_1 - y_2)| \leq K_0 r(\varepsilon)|y_1 - y_2|,$$

where $r(\varepsilon) = c_0\varepsilon^{\eta}$, $c_0 = const$, $K_0 = const$, $\eta > \alpha$;

$$|G(t, y_1, \varepsilon) - G(t, y_2, \varepsilon)| \leq K_1|y_1 - y_2|, \quad K_1 = const;$$

3) $m > \alpha + \eta$.
Then, for sufficiently small ε, system (12.21) has a unique almost periodic solution $y_0(t, \varepsilon)$ in the ball $U(y_0(\varepsilon), r(\varepsilon))$ ($||y - y_0(\varepsilon)|| \leq r(\varepsilon)$. This solution will be asymptotically stable, for sufficiently small, ε if all eigenvalues of the matrix $A(\varepsilon)$ have negative real parts, and, unstable, for sufficiently small ε, if the matrix $A(\varepsilon)$ has at least one eigenvalue with a positive real part.

Proof. In system (12.28) we change over to the slow time $\tau = \varepsilon t$ and obtain the system

$$\frac{dy}{d\tau} = H(y, \varepsilon) + \varepsilon^m G(\tau/\varepsilon, y, \varepsilon). \tag{12.30}$$

Rewrite (12.30) as

$$\frac{dy}{d\tau} = A(\varepsilon)y + [H(y, \varepsilon) - A(\varepsilon)y] + \varepsilon^m G(\tau/\varepsilon, y, \varepsilon). \tag{12.31}$$

The problem of the existence of an almost periodic solution of system (12.31) is equivalent to the problem of solvability of the following operator equation in B_n

$$y = \Pi(y, \varepsilon), \tag{12.32}$$

where

$$\Pi(y, \varepsilon) = L^{-1}(\varepsilon)[H(y, \varepsilon) - A(\varepsilon)y] + \varepsilon^m L^{-1}(\varepsilon)G(\tau/\varepsilon, y, \varepsilon).$$

Let us estimate the norm $||\Pi(y_1, \varepsilon) - \Pi(y_2, \varepsilon)||$, where y_1, $y_2 \in U(y_0(\varepsilon), r(\varepsilon))$. We obtain

$$||\Pi(y_1, \varepsilon) - \Pi(y_2, \varepsilon)|| \leq$$
$$\leq ||L^{-1}(\varepsilon)||||H(y_1, \varepsilon) - H(y_2, \varepsilon) - A(\varepsilon)(y_1 - y_2)||+$$

$$+\varepsilon^m||L^{-1}(\varepsilon)||||G(\tau/\varepsilon, y_1, \varepsilon) - G(\tau/\varepsilon, y_2, \varepsilon)|| \le$$
$$\le \frac{MK_0}{\varepsilon^\alpha} r(\varepsilon)||y_1 - y_2|| + \frac{MK_1}{\varepsilon^\alpha}\varepsilon^m||y_1 - y_2||.$$

Finally, we arrive at

$$||\Pi(y_1, \varepsilon) - \Pi(y_2, \varepsilon)|| \le (\Delta_1(\varepsilon)) + \Delta_2(\varepsilon))||y_1 - y_2)||, \tag{12.33}$$

where $\Delta_1(\varepsilon) = O(\varepsilon^{\eta-\alpha})$, $\Delta_2(\varepsilon) = O(\varepsilon^{m-\alpha})$, under $\varepsilon \to 0$, $\eta - \alpha > 0$, $m - \alpha > 0$. Let $y \in U(y_0(\varepsilon), r(\varepsilon))$. Then

$$\Pi(y, \varepsilon) = L^{-1}(\varepsilon)[H(y, \varepsilon) - A(\varepsilon)y] + \varepsilon^m L^{-1}(\varepsilon)G(\tau/\varepsilon, y) =$$

$$-L^{-1}(\varepsilon)A(\varepsilon)y_0(\varepsilon) + L^{-1}(\varepsilon)[H(y, \varepsilon) - A(\varepsilon)(y - y_0(\varepsilon)] + \varepsilon^m L^{-1}(\varepsilon)G(\tau/\varepsilon, y).$$

It follows from the definition of the operator $L^{-1}(\varepsilon)$ that

$$L^{-1}(\varepsilon)A(\varepsilon)y_0(\varepsilon) = -y_0(\varepsilon).$$

Hence,

$$\Pi(y, \varepsilon) - y_0(\varepsilon) =$$
$$= L^{-1}(\varepsilon)[H(y, \varepsilon) - H(y_0, \varepsilon) - A(\varepsilon)(y - y_0)] + \varepsilon^m L^{-1}(\varepsilon)G(\tau/\varepsilon, y).$$

Condition 2) of the theorem implies the validity of the estimate

$$||\Pi(y, \varepsilon) - y_0(\varepsilon)|| \le \Delta_3(\varepsilon) + \Delta_4(\varepsilon),$$

where $\Delta_3(\varepsilon) = O(\varepsilon^{2\eta-\alpha})$, $\Delta_4(\varepsilon) = O(\varepsilon^{m-\alpha})$ as $\varepsilon \to 0$; $2\eta - \alpha > \eta$, $m - \alpha > \eta$. Since we have that $r(\varepsilon) = c_0\varepsilon^\eta$, we have that for sufficiently small ε, it follows from $y \in U(y_0(\varepsilon), r(\varepsilon))$ that $\Pi(y, \varepsilon) \in U(y_0(\varepsilon), r(\varepsilon))$. It means that, for sufficiently small ε, the operator $\Pi(y, \varepsilon)$ maps the ball $U(y_0(\varepsilon), r(\varepsilon))$ onto itself. This and estimate (12.33) imply that the operator $\Pi(y, \varepsilon)$, for sufficiently small ε, is a contraction on the ball $U(y_0(\varepsilon), r(\varepsilon))$. Therefore, by virtue of the principle of contracting mappings, operator equation (12.32), for sufficiently small ε, in the ball $U(y_0(\varepsilon), r(\varepsilon))$ has a unique almost periodic solution $y = z_0(\tau, \varepsilon)$.

Thus, the existence of the almost periodic solution $y_0(t, \varepsilon)$ for system (12.21) is proved.

The proof of the assertions of the stability is similar to the respective proof in Theorem 12.2.

Remark 12.1.

It was assumed in the above proved theorems that the stationary solution $x(\varepsilon)$ of an averaged system is bounded on ε as $\varepsilon \to 0$. C. Holmes and P. Holmes [1981] studied the behavior of the solutions of Duffing's equation in the following form

$$\ddot{x} - x + x^3 = \gamma \cos \omega t - \delta \dot{x}$$

in the neighborhood of a periodic solution. Application of the method of perturbations resulted in the problem on the existence and stability of subharmonic solutions of the Mathieu nonlinear equation with quadratic and cubic nonlinearities. In the first approximation of the method of averaging, a nonlinear system of equations was obtained. In the second approximation, a nonlinear system that had the stationary solution $x(\varepsilon)$ with the asymptotic $x(\varepsilon) \sim 1/\sqrt{\varepsilon}$ as $\varepsilon \to 0$ was obtained. In this situation, the results of this chapter are not applicable.

12.5 Higher Approximations for Systems with Fast and Slow Time

We briefly describe the results gained in substantiating the validity of higher approximations for systems with fast and slow time. Consider a system of differential equations

$$\frac{dx}{dt} = \varepsilon X(t, \tau, x, \varepsilon), \quad \tau = \varepsilon t, \tag{12.34}$$

where $x \in \mathcal{R}^n$, $\varepsilon > 0$ is a small parameter, the vector-function $X(t, \tau, x, \varepsilon)$ is almost periodic in t uniformly with respect to τ, ε, periodic in τ with the period T, and sufficiently smooth in τ, x, ε.

We compare system (12.34) with the following system averaged over the fast time t

$$\frac{dy}{dt} = \varepsilon Y_1(\tau, y), \tag{12.35}$$

where

$$Y_1(\tau, y) = \lim_{T\to\infty} \int_0^T X(t, \tau, y, 0) dt.$$

Let system (12.35) have the T-periodic solution $y_0(\tau)$. The case when the linear periodic system

$$\frac{dy}{d\tau} = A(\tau) y, \tag{12.36}$$

where $A(\tau) = Y_{1y}(\tau, y_0(\tau))$, has no characteristic exponents with a zero real part was investigated in Section 9.7 (Theorem 9.6).

We now assume that system (12.36) has characteristic exponents with a zero real part. Using a change

$$x = z + \sum_{k=1}^{m} \varepsilon^k u_k(t, \tau, z),$$

where the vector-functions $u_k(t,\tau,z)$ are almost periodic in t and T-periodic in τ, system (12.34) can be transformed into

$$\frac{dz}{dt} = \sum_{k=1}^{m} \varepsilon^k Y_k(\tau, y) + \varepsilon^{m+1} G(t, \tau, z, \varepsilon); \tag{12.37}$$

i.e., we eliminate the fast variable t in the right-hand side of the system with the accuracy up to the terms of order ε^m. The system of k-th approximations is

$$\frac{dz}{dt} = \varepsilon \sum_{k=1}^{m} \varepsilon^k Y_k(\tau, y)$$

that at the time τ takes the form

$$\frac{dz}{d\tau} = \sum_{k=1}^{m} \varepsilon^{k-1} Y_k(\tau, y). \tag{12.38}$$

This is a system with the T-periodic coefficients. Let system (12.38) have the T-periodic solution $y^*(\tau,\varepsilon)$, and $y^*(\tau,0) = y_0(\tau)$. We introduce the linear system

$$\frac{dy}{d\tau} = A(\tau,\varepsilon)y,$$

where

$$A(\tau,\varepsilon) = \sum_{k=1}^{m} \varepsilon^{k-1} Y_{ky}(\tau, y^*(\tau,\varepsilon)).$$

If the operator

$$Ly = \frac{dy}{d\tau} - A(\tau,\varepsilon)y$$

is invertible in the space B_n of almost periodic vector-functions and the inverse operator satisfies the estimate

$$||L^{-1}|| \leq \frac{M}{\varepsilon^\alpha}, \quad M, \alpha > 0,$$

we have that under some additional assumptions, it is possible to prove an analog of Theorem 12.3 for system (12.37) (see Ukhalov [1997], [1998]).

12.6 Rotary Regimes of a Pendulum with an Oscillating Pivot

As an application of Theorem 12.2, we consider the problem of the stationary rotations of a physical pendulum with a vertically oscillating pivot.

It turns out that the pendulum may synchronously rotate at an angular velocity ω (ω is the frequency of harmonic vibrations of a pivot) and overcome resistances if they do not exceed a certain magnitude.

This problem was investigated by Bogoliubov [1950], Coughey [1960], Bogoliubov and Mitropolskiy [1961], Blekhman [1954, 1971].

We study this problem on the assumption that a pivot moves under the action of a quasi-periodic force.

Reduction of the System to Standard form. As is already known, in an environment with damping proportional to the velocity, the equation of motion of a pendulum with a pivot oscillating along a vertical axis takes the form

$$\frac{d^2\theta}{dt^2} + \lambda\frac{d\theta}{dt} + \omega_0^2 \sin\theta + \frac{\zeta(t)}{l}\sin\theta = 0, \tag{12.39}$$

where θ is the off-vertical deviation angle of a pendulum, λ is a damping factor, l is the length of a pendulum, $\omega_0^2 = \frac{g}{l}$, g is the acceleration of gravity, $\zeta(t) = \frac{d^2\xi}{dt^2}$, and $\xi(t)$ is the law of motion of a pivot. We investigate the case when the law of motion of a pivot is a sum of two harmonic functions

$$\xi(t) = \alpha_1 \cos\nu_1 t + \alpha_2 \cos\nu_2 t.$$

We introduce a small parameter into the equation of motion of a pendulum. We shall assume that the amplitudes α_1, α_2 are sufficiently small and the frequencies ν_1, ν_2 are sufficiently large in the following sense. There exists a small positive parameter ε such that

$$\nu_1 = \frac{\omega_1}{\varepsilon}, \quad \nu_1 = \frac{\omega_2}{\varepsilon}, \quad \alpha_1 = \varepsilon a_1, \quad \alpha_2 = \varepsilon a_2,$$

and ω_1, ω_2, a_1, a_2 are the quantities of order $O(1)$ with respect to ε.

We consider the problem of the existence of regimes that are close to the fast uniform rotation $\theta = \omega t + c$, where $\omega = \frac{1}{\varepsilon}$, $c = const$. In equation (12.39), we make a change $\theta = \omega t + \psi, \tau = \omega t$ and arrive at the equation

$$\frac{d^2\psi}{d\tau^2} + \frac{\lambda}{\omega}\frac{d\psi}{d\tau} + \frac{\omega_0^2}{\omega^2}\sin(\tau+\psi) + \frac{\zeta(\tau)}{l\omega^2}\sin(\tau+\psi) = 0.$$

In view of the formulas expressing α_1, α_2, ν_1, ν_2 via a_1, a_2, ω_1, ω_2, ε, we obtain

$$\frac{d^2\psi}{d\tau^2} + \varepsilon\lambda\frac{d\psi}{d\tau} + \varepsilon\lambda + \varepsilon^2\omega_0^2\sin(\tau+\psi) + \varepsilon g(\tau)\sin(\tau+\psi) = 0, \tag{12.40}$$

where

$$g(\tau) = -\frac{1}{l}[a_1\omega_1^2\cos\omega_1\tau + a_2\omega_2^2\cos\omega_2\tau].$$

From scalar equation (12.40), we go on to the system by introduction of a new variable φ using the formula

$$\frac{d\psi}{d\tau} = \sqrt{\varepsilon}\varphi.$$

For convenience, let $\mu = \sqrt{\varepsilon}$. Then we obtain the system

$$\begin{aligned} \tfrac{d\psi}{d\tau} &= \mu\varphi, \\ \tfrac{d\varphi}{d\tau} &= -\mu\lambda - \mu g(\tau)\sin(\tau+\psi) - \mu^2\lambda\varphi - \mu^3\omega_0^2\sin(\tau+\psi). \end{aligned} \tag{12.41}$$

Study of System (12.41).

According to the Bogoliubov method, we make a change of variables

$$\begin{aligned} \psi &= \xi + \mu u_1(\tau,\xi,\eta) + \mu^2 u_2(\tau,\xi,\eta), \\ \varphi &= \eta + \mu v_1(\tau,\xi,\eta) + \mu^2 v_2(\tau,\xi,\eta) \end{aligned}$$

and transform system (12.41) into the system

$$\begin{aligned} \tfrac{d\xi}{d\tau} &= \mu A_1(\xi,\eta) + \mu^2 A_2(\xi,\eta) + \mu^3 A_3(\tau,\xi,\eta,\varepsilon) \\ \tfrac{d\eta}{d\tau} &= \mu B_1(\xi,\eta) + \mu^2 B_2(\xi,\eta) + \mu^3 B_3(\tau,\xi,\eta,\varepsilon). \end{aligned}$$

To find the functions $A_i(\xi,\eta)$, $B_i(\xi,\eta)$, $u_i(\tau,\xi,\eta)$, $u_i(\tau,\xi,\eta)$ $(i = 1, 2)$, we obtain the following system of equations

$$\begin{aligned} &A_1 + \tfrac{\partial u_1}{\partial\tau} = \eta, \quad B_1 + \tfrac{\partial v_1}{\partial\tau} = -g(\tau)\sin(\tau+\xi), \quad A_2 + \tfrac{\partial u_2}{\partial\tau} = v_1, \\ &B_2 + \tfrac{\partial v_1}{\partial\xi}A_1 + \tfrac{\partial v_1}{\partial\eta}B_1 + \tfrac{\partial v_2}{\partial\tau} = \lambda\eta - g(\tau)u_1(\tau,\xi,\eta)\cos(\tau+\xi). \end{aligned} \tag{12.42}$$

From system (12.42), we arrive at

$$\begin{aligned} &A_1 = \eta, \quad u_1 = 0, \quad A_2 = 0, \\ &B_1 = -\lambda - \langle g(\tau)\sin(\tau+\xi)\rangle, \quad B_2 = -\lambda\eta. \end{aligned} \tag{12.43}$$

Recall that $\langle g(\tau)\sin(\tau+\xi)\rangle$ is the mean value of the almost periodic function $g(\tau)\sin(\tau+\xi)$. For $\omega_1 = 1$, the averaged equations of the first approximations take the form

$$\frac{d\xi}{d\tau} = \mu\eta, \quad \frac{d\eta}{d\tau} = -\mu\lambda + \mu\frac{a_1}{2l}\sin\xi. \tag{12.44}$$

For $\omega_2 = 1$, they are

$$\frac{d\xi}{d\tau} = \mu\eta, \quad \frac{d\eta}{d\tau} = -\mu\lambda + \mu\frac{a_2}{2l}\sin\xi. \tag{12.45}$$

The stationary solutions are determined from the equations

$$\eta_{0i} = 0, \quad \sin\xi_{0i} = \frac{2\lambda l}{a_i}, \quad i = 1, 2. \tag{12.46}$$

Solution of the second of the equations exists if the inequality

$$\frac{2\lambda l}{a_i} < 1, \quad i = 1, 2 \tag{12.47}$$

holds. Under condition (12.47), we obtain two solutions

$$0 < \xi_{0i}^1 < \frac{\pi}{2}, \quad \frac{\pi}{2} < \xi_{0i}^2 < \pi.$$

The matrix linearized on the stationary solution (ξ^1_{0i}), $(i = 1, 2)$ takes the form

$$A_0 = \begin{pmatrix} 0 & 1 \\ \frac{a_i}{2l}\cos\xi^1_{0i} & 0 \end{pmatrix}.$$

For ξ^1_{0i}, one eigenvalue of the matrix A_0 is positive and the other is negative. It follows from Theorems 9.1 and 9.3 that, for sufficiently small ε, equation (12.40) has the solution

$$\psi_i(\tau) = \xi^1_{01} + \varepsilon f_i(t, \varepsilon), \tag{12.48}$$

where $f_i(t, \varepsilon)$ is the almost periodic function of t, and this solution is unstable. For the stationary solution ξ^2_{0i}, the eigenvalues of A_0 are purely imaginary. Thus, in this case, we need to investigate the equations of higher approximations. It follows from formulas (12.43) that the equations of the second approximation for $\omega_i = 1\,(i = 1, 2)$ take the form

$$\frac{d\xi}{d\tau} = \mu\eta, \quad \frac{d\eta}{d\tau} = -\mu\lambda + \mu\frac{a_i}{2l}\sin\xi - \mu^2\lambda\eta, \quad i = 1, 2. \tag{12.49}$$

Equation (12.49) has the same stationary solutions as (12.44) and (12.45). The matrix

$$A_0 + \mu A_1 = \begin{pmatrix} 0 & 1 \\ \frac{a_i}{2l}\cos\xi^2_{0i} & -\mu\lambda \end{pmatrix}$$

linearized on ξ^2_{0i} has both eigenvalues with negative real parts. It is easy to see that the conditions of Theorem 12.2 hold. To formulate the obtained result, we note that inequality (12.47) can be written as

$$\lambda = \frac{\alpha_i\nu_i}{2l} < 1, \quad i = 1, 2. \tag{12.50}$$

If inequality (12.50) holds, the averaged equations have the stationary solutions ξ^1_{0i}, ξ^2_{0i}, whereas the regime ξ^1_{0i} is unstable and the regime ξ^2_{0i} is asymptotically stable.

Theorem 12.4. *Let inequalities (12.50) hold. Then equation (12.41), for sufficiently small ε, has four solutions of the form*

$$\psi^j_i = \xi^j_{0i} + \varepsilon f^j_i(t, \varepsilon), \quad i = 1, 2;\ j = 1, 2,$$

where $f^j_i(t, \varepsilon)$ are the almost periodic functions. The solutions ψ^1_i, $(i = 1, 2)$ are unstable and ψ^2_i $(i = 1, 2)$ are asymptotically stable.

Hence, equation (12.41) has four solutions

$$\theta^j_i = \nu_i t + \xi^j_{0i} + f^j_i(t, \varepsilon) \quad i = 1, 2;\ j = 1, 2. \tag{12.51}$$

We shall call the solutions of the form (12.51) quasi-stationary rotary regimes. We say that the quasi-stationary rotary regime is stable (unstable) if the solution ψ^j_i has the respective property.

Similarly to the previous case, we can show that there exist four quasi-stationary rotary regimes at the rotation of a pendulum to the opposite side (in formulas (12.51) ν_i should be replaced by $-\nu_i$).

Recall (see Bogoliubov and Mitropolskiy [1961]) the mechanical interpretation of inequality (12.51), that is the power spent to overcome forces resisting pendulum rotation at the angular velocity ν_i, must not attain the limit value $I\frac{a_i}{2l}\nu_i^2$, where I is the moment of inertia of a pendulum.

Study the problem on the existence of sub-rotary quasi-stationary rotary regimes, i.e. regimes with a uniform rotation frequency $\omega = \frac{m\nu_1+s\nu_2}{r}$, where m, s, r are integers. Additionally, we shall assume that the damping factor λ is of order $O(\varepsilon)$, namely

$$\lambda = \varepsilon\gamma, \quad \gamma = O(1). \tag{12.52}$$

To find the sub-rotary regimes, we need to construct averaged equations of the fourth approximation. We make a change

$$\begin{aligned}\psi &= \xi + \mu u_1(\tau,\xi,\eta) + \mu^2 u_2(\tau,\xi,\eta) + \mu^3 u_3(\tau,\xi,\eta) + \mu^4 u_4(\tau,\xi,\eta),\\ \varphi &= \eta + \mu v_1(\tau,\xi,\eta) + \mu^2 v_2(\tau,\xi,\eta) + \mu^3 v_3(\tau,\xi,\eta) + \mu^4 v_4 2(\tau,\xi,\eta)\end{aligned}$$

and obtain the averaged equations

$$\begin{aligned}\tfrac{d\xi}{d\tau} &= \mu A_1(\xi,\eta) + \mu^2 A_2(\xi,\eta) + \mu^3 A_3(\xi,\eta) + \mu^4 A_4(\xi,\eta),\\ \tfrac{d\eta}{d\tau} &= \mu B_1(\xi,\eta) + \mu^2 B_2(\xi,\eta) + \mu^3 B_3(\xi,\eta) + \mu^4 B_4(\xi,\eta).\end{aligned}$$

For this case, the formulas, which define the coefficients A_i, B_i $(i = 1, 2, 3, 4)$, take the form

$$\begin{aligned}&A_1 = \eta,\ A_i = 0\,(i = 2,3,4),\ B_1 = -\langle g(\tau)\sin(\tau+\xi)\rangle,\ B_2 = 0,\\ &B_3 = -\langle g(\tau)u_2(\tau,\xi,\eta)\cos(\tau+\xi)\rangle - \omega_0^2\langle\sin(\tau+\xi)\rangle - \gamma,\\ &B_4 = -\gamma\eta - \langle g(\tau)u_3(\tau,\xi,\eta)\cos(\tau+\xi)\rangle.\end{aligned}$$

First, we note that equality (12.52) implies that inequality (12.47) holds for small ε. Therefore, there exist four quasi-stationary rotary regimes with the frequencies ν_1, ν_2; at that, all regimes are unstable, since the respective stationary regimes obey the inequalities

$$0 < \xi_{0i}^j < \frac{\pi}{2} \quad (i,j = 1,2).$$

Now let $\omega_i \neq 1\,(i = 1,2)$. Then $B_1 = 0$. If one of the relations $\omega_1 - \omega_2 = 2$ or $\omega_1 + \omega_2 = 2$ hold, then

$$B_3 = -\gamma + \frac{a_1 a_2 \omega_1^2 \omega_2^2}{4l^2(1-\omega_1)^2}\sin 2\xi, \quad B_4 = -\gamma\eta.$$

The averaged equations of the fourth approximation take the form

$$\xi' = \mu\eta, \quad \eta' = -\mu^3\left(\gamma - \frac{a_1 a_2 \omega_1^2 \omega_2^2}{4l^2(1-\omega_1)^2}\sin 2\xi\right) - \mu^4\gamma\eta. \tag{12.53}$$

Let $\eta = \mu z$. Then system (12.53) can be written as

$$\xi' = \varepsilon z, \quad z' = \varepsilon \left(\gamma - \frac{a_1 a_2 \omega_1^2 \omega_2^2}{4l^2(1-\omega_1)^2} \sin 2\xi \right) - \varepsilon^2 \gamma z \tag{12.54}$$

so that it takes the same form as system (12.49). Hence, the complete system will differ from system (12.49) by the terms of order $o(\varepsilon)$. We can make use of Theorem 12.2. We obtain the conditions for the existence of stationary regimes with the frequency $\frac{\nu_1-\nu_2}{2}$ (in view of the formulas that define a_1, a_2, ω_1, ω_2, λ via α_1, α_2, ν_1, ν_2, γ, $\omega = \varepsilon^{-1}$) as the inequality

$$\frac{\lambda l^2(\nu_1-\nu_2)(\nu_1+\nu_2)^2}{2\alpha_1\alpha_2\nu_1^2\nu_2^2} < 1 \tag{12.55}$$

and with the frequency $\frac{\nu_1+\nu_2}{2}$ as the inequality

$$\frac{\lambda l^2(\nu_1-\nu_2)^2(\nu_1+\nu_2)}{2\alpha_1\alpha_2\nu_1^2\nu_2^2} < 1. \tag{12.56}$$

If inequalities (12.55) and (12.56) hold, the averaged equations have four stationary solutions

$$0 < \xi_1^j < \frac{\pi}{4}, \ \frac{\pi}{4} < \xi_2^j < \frac{\pi}{2}, \ \pi < \xi_3^j < \frac{5\pi}{4}, \ \frac{5\pi}{4} < \xi_4^j < \frac{3\pi}{2}, \quad j = 1, 2,$$

where the first and the third solutions are unstable, whereas the second and the fourth are asymptotically stable.

Using Theorem 12.2, we obtain that for equation (12.40), provided inequality (12.55) holds (it is easy to see that inequality (12.56) follows from inequality (12.55)), for sufficiently small ε, there exist four quasi-stationary rotary regimes with the frequency $\frac{\nu_1-\nu_2}{2}$ and four quasi-stationary rotary regimes with the frequency $\frac{\nu_1+\nu_2}{2}$. Among these eight regimes, four are asymptotically stable and four are unstable. If a pendulum rotates in the opposite direction, there exist eight more regimes of the same type.

The existence of quasi-stationary rotary regimes of other types requires additional assumptions on the smallness of damping factor with respect to ε. Let us briefly describe the case when

$$\lambda = \varepsilon^2 \gamma, \quad \gamma = O(1).$$

Under this condition, we construct averaged equations of the sixth approximation and obtain the conditions for the existence of quasi-stationary rotary regimes with the frequencies $\frac{\nu_1}{2}$ and $\frac{\nu_2}{2}$ due to the force of gravity. Namely, for $\omega_i = 2\,(i = 1, 2)$ the averaged equations take the form

$$\dot{\xi} = \mu\eta, \quad \dot{\eta} = \mu^5 \left(-\gamma + \frac{g}{4l^2} a_i \omega_i^2 \sin 2\xi \right) - \mu^6 \gamma \eta.$$

The problem of the existence of sub-rotary regimes in the case of harmonic vibrations of a pivot was investigated in the work of Bogoliubov (Jr.) and Sadovnikov [1961], where, however, a computational error was made.

The problem under study was investigated in detail in the articles by the author [1983, 1996a].

Exercise 12.1. Study the problem of the existence of quasi-rotary regimes with the frequencies $\frac{\nu_1}{2}$ and $\frac{\nu_2}{2}$ in equation (12.39).

12.7 Critical Case Stability of a Pair of Purely Imaginary Roots for a Two-Dimensional Autonomous System

We consider a two-dimensional autonomous system

$$\begin{aligned} \tfrac{dx}{dt} &= P(x,y) = -by + \varphi(x,y), \\ \tfrac{dy}{dt} &= Q(x,y) = bx + \psi(x,y). \end{aligned} \tag{12.57}$$

Here, b is a real number, $\varphi(x,y)$ and $\psi(x,y)$ are the series in x and y that converge in some neighborhood of the origin start with the terms no less than to the second power. Therefore, we can write

$$\varphi(x,y) = P_2(x,y) + P_3(x,y) + \dots,$$

$$\psi(x,y) = Q_2(x,y) + Q_3(x,y) + \dots,$$

where $P_i(x,y)$, $Q_i(x,y)\,(i = 2,3,\dots)$ are the homogeneous polynomials in x, y of the power i. System (12.57) has a zero solution. To study the problem on the stability of this solution, we cannot use theorems of the stability of the first approximation, because the matrix of the linear system

$$\begin{aligned} \tfrac{dx}{dt} &= -by, \\ \tfrac{dy}{dt} &= bx \end{aligned}$$

has purely imaginary eigenvalues $\pm ib$.

This case is called a critical case of stability. The problem under study was solved by Lyapunov (reprinted in [1992]).

To study this problem, we use the method of averaging.

We transform system (12.57), by making a change $x = \rho\cos\theta$, $y = \rho\sin\theta$, (i.e. introducing polar coordinates) and arrive at the system

$$\begin{aligned} \tfrac{d\rho}{dt} &= \varphi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta, \\ \tfrac{d\theta}{dt} &= b - \tfrac{\psi(\rho\cos\theta,\rho\sin\theta)\rho\sin\theta - \varphi(\cos\theta,\rho\sin\theta)\rho\cos\theta}{\rho^2}. \end{aligned} \tag{12.58}$$

We reduce the right-hand sides of the resultant system by ρ for $\rho \neq 0$ and then complete definition by continuity for $\rho = 0$. The right-hand side of the first equation will take the form

$$\varphi(\rho\cos\theta, \rho\sin\theta)\cos\theta + \psi(\rho\cos\theta, \rho\sin\theta)\sin\theta = \rho^2 R_2(\theta) + \rho^3 R_3(\theta) + \cdots + \\ +\rho^k R_k(\theta) + \ldots,$$

where

$$R_k(\theta) = [P_k(\cos\theta, \sin\theta)\cos\theta + Q_k(\cos\theta, \sin\theta)\sin\theta], \quad k = 1, 2, \ldots.$$

It should be recorded that $R_k(\theta)$ is a polynomial in $\sin\theta$, $\cos\theta$. Each term of this polynomial takes the form

$$c_{pq}\sin^p(\theta)\cos^q(\theta), \tag{12.59}$$

where c_{pq} is a real number, $p + q = k + 1$.

The right-hand side of the second equation becomes

$$b - \tfrac{\psi(\rho\cos\theta, \rho\sin\theta)\rho\sin\theta - \varphi(\cos\theta, \rho\sin\theta)\rho\cos\theta}{\rho^2} = b - \rho S_2(\theta) - \rho^2 S_3(\theta) - \cdots - \\ -\rho^k S_k(\theta) - \ldots,$$

where

$$S_k(\theta) = [P_k(\cos\theta, \sin\theta)\sin\theta - Q_k(\cos\theta, \sin\theta)\cos\theta], \quad k = 1, 2, \ldots.$$

It is evident that $S_k(\theta)$ is a polynomial in $\sin\theta$, $\cos\theta$ with its terms taking the form of (12.59).

We are interested in the behavior of the solutions of system (12.57) in the neighborhood of a zero equilibrium, thus, we introduce a small parameter $\varepsilon > 0$ (scaling parameter), assuming that $\rho = \varepsilon r$.

Then we obtain the system

$$\begin{aligned} \tfrac{dr}{dt} &= \varepsilon r^2 R_2(\theta) + \varepsilon^2 r^3 R_3(\theta) + \cdots + \varepsilon^{k-1} r^k R_k(\theta) + \ldots, \\ \tfrac{d\theta}{dt} &= b - \varepsilon r S_2(\theta) - \varepsilon^2 r^2 S_3(\theta) - \cdots - \varepsilon^{k-1} r^{k-1} S_k(\theta) - \ldots. \end{aligned} \tag{12.60}$$

Dividing the first equation of system (12.60) by the second one and expanding the right-hand side in terms of powers of ε yields the first-order equation in the standard form

$$\frac{dr}{d\theta} = \varepsilon r^2 A_1(\theta) + \varepsilon^2 r^3 A_2(\theta) + O(\varepsilon^3), \tag{12.61}$$

where

$$A_1(\theta) = \frac{1}{b} R_2(\theta), \quad A_2(\theta) = \frac{1}{b} R_3(\theta) + \frac{1}{b^2} R_2(\theta) S_2(\theta). \tag{12.62}$$

The right-hand side of system (12.61) are the 2π-periodic functions of θ.

Prior to applying the method of averaging we state the following.

Lemma. *The mean value of the periodic function* $\sin^p\theta\cos^q\theta$ *equals zero if* $p+q$ *is an odd number.*

We find averaged equations of the first approximation for (12.61). To do this, we average $A_1(\theta)$. By virtue of the formula that defines $A_1(\theta)$ and the lemma, it follows that the mean value of $A_1(\theta)$ equals zero. Hence, we need to construct an averaged equation of second approximation.

By L_1 we denote the mean value of the function $A_2(\theta)$, i.e.

$$L_1 = \frac{1}{2\pi}\int_0^{2\pi} A_2(\theta)d\theta.$$

The number L_1 is called **the first Lyapunov quantity**.

In equation (12.61) we make a standard change of the method of averaging

$$r = y + \varepsilon u_1(\theta)y^2 + \varepsilon^2 u_2(\theta)y^3.$$

Then we arrive at the equation

$$(1+2\varepsilon u_1(\theta)y + 3\varepsilon^2 u_2(\theta)y^2)\tfrac{dy}{d\theta} + \varepsilon\tfrac{du_1}{d\theta}y^2 + \varepsilon^2\tfrac{du_2}{d\theta}y^3 = \\ \varepsilon A_1(\theta)y^2 + \varepsilon^2(2u_1(\theta)A_1(\theta) + A_2(\theta))y^3 + O(\varepsilon^3).$$

Let

$$\frac{du_1}{d\theta} = A_1(\theta), \quad \frac{du_2}{d\theta} = A_2(\theta) + 2A_1(\theta)u_1(\theta) - L_1.$$

After this change, equation (12.61) assumes the following form (we should take into account that the mean value of the function $A_1(\theta)u_1(\theta)$ equals zero, since this function is a derivative of the periodic function)

$$\frac{dy}{d\theta} = \varepsilon^2 L_1 y^3 + O(\varepsilon^3). \tag{12.63}$$

Hence, the averaged equation of the second approximation is

$$\frac{d\bar{r}}{d\theta} = \varepsilon^2 L_1\bar{r}^3. \tag{12.64}$$

It is possible to examine stability of zero solution (12.64) based directly on the fact that the equation is integrable. It is convenient to come to this result, using the Lyapunov function (see Appendix B or Krasovskii [1963], and Malkin [1966]). As the Lyapunov function, we take the function $V(\bar{r}) = \frac{1}{2}L_1\bar{r}^2$. In view of equation (12.64), the derivative $V(\bar{r})$ takes the form

$$\dot{V} = \frac{dV}{d\theta} = \varepsilon^2 L_1^2\bar{r}^4.$$

Both functions V and $\dot{V}$ are of positive definite. If $L_1 > 0$, the functions V and $\dot{V}$ have the same sign and, therefore, by virtue of the Chetaev Theorem, the solution $r = 0$ is unstable. For $L_1 < 0$, the functions V and $\dot{V}$ have opposite signs and, therefore, by virtue of the Lyapunov Theorem, the zero solution of equation (12.59) is asymptotically stable.

Turning to equation (12.63), we obtain that if we take the function $V(r) = \frac{1}{2}L_1 r^2$ as the Lyapunov function for this equation, then, for sufficiently small ε, the functions V and $\dot{V}$ take the same signs at $L_1 > 0$ and different signs at $L_1 < 0$. Hence, the zero solution of equation (12.63), for sufficiently small ε, is asymptotically stable if $L_1 < 0$ and unstable if $L_1 > 0$. This assertion will hold true for equation (12.61) as well and, hence, for system (12.57). We formulate this result as a theorem.

Theorem 12.5. *If the first Lyapunov quantity $L_1 < 0$, then the zero solution of system (12.57) is asymptotically stable. If $L_1 < 0$, then the zero solution of system (12.57) is unstable.*

If $L_1 = 0$, we need to calculate the second Lyapunov quantity L_2. We shall not dwell on the calculation of L_2 and only note that the averaged equation in this case has the form

$$\frac{d\bar{r}}{d\theta} = \varepsilon^4 L_2 \bar{r}^5. \tag{12.65}$$

As in the case with $L_1 \neq 0$, we obtain that the solution of equation (12.65) is asymptotically stable for $L_2 < 0$ and unstable for $L_2 > 0$. The same assertion also holds true for system (12.57), since after the averaging change, the initial system takes the form

$$\frac{dy}{d\theta} = \varepsilon^4 L_2 y^5 + O(\varepsilon^5).$$

If $L_2 = 0$, it is necessary to calculate the third Lyapunov quantity L_3. Generally, if

$$L_1 = L_2 = \cdots = L_{k-1} = 0,$$

the averaged equation

$$\frac{d\bar{r}}{d\theta} = \varepsilon^{2k} L_k \bar{r}^{2k+1}.$$

This fact in a somewhat different form was noted by Lyapunov (reprinted in [1992]). As an example, we consider the equation

$$\ddot{x} + x = \beta x^2 + \gamma \dot{x}^3 + \delta x \dot{x}^2, \tag{12.66}$$

where β, γ, δ are some constants. Making a change

$$x = r\cos\theta, \quad \dot{x} = -r\sin\theta,$$

we arrive at the system

$$\begin{aligned} \tfrac{dr}{dt} &= -F(r,\theta)\sin\theta, \\ \tfrac{d\theta}{dt} &= 1 - r^{-1}F(r,\theta)\cos\theta, \end{aligned}$$

where

$$F(r,\theta) = \beta r^2 \cos^2\theta - \gamma r^3 \sin^3\theta + \delta r^3 cos\theta \sin^2\theta.$$

Let us find $A_2(\theta)$. By virtue of formula (12.62), the function $A_2(\theta)$ takes the form

$$A_2(\theta) = \gamma \sin^4\theta - \delta\cos\theta\sin^3\theta - \beta^2\cos^5\theta\sin\theta + \beta\delta\cos^4\theta\sin^2\theta -$$
$$-\beta\delta\cos^4\theta\sin^2\theta - \delta^2\sin^3\theta\cos^3\theta.$$

Calculating the mean value of $A_2(\theta)$ yields $L_1 = \frac{3}{8}\gamma$. Hence, if $\gamma < 0$, then the zero solution of equation (12.66) is asymptotically stable, and if $\gamma > 0$, then the zero solution of equation (12.66) is unstable.

12.8 Bifurcation of Cycle (the Andronov-Hopf Bifurcation)

It might be well to mention how the names of Andronov and Hopf are related to bifurcation of cycle. In the two-dimensional case, this bifurcation was discovered by Andronov in 1931. Hopf investigated the multidimensional case in 1942 (see Marsden and McCracken [1976] concerning the history of the problem).

Consider a two-dimensional autonomous system

$$\frac{dz}{dt} = A(\alpha)z + F(z,\alpha), \quad z \in R^2, \tag{12.67}$$

where $\alpha \in (-\alpha_0, \alpha_0)$ is a real parameter, $A(\alpha)$ is a square matrix of order 2; $z = (x, y)$, $F(0,\alpha) = 0$ and the components of the vector-function $F(z,\alpha)$ are the power series of x, y that converge in some neighborhood of the origin and start with the terms no less than the second power. We shall assume that the right-hand sides of system (12.67) depend smoothly on the parameter α.

The eigenvalues of the matrix $A(\alpha)$ have the form

$$\lambda_{1,2}(\alpha) = \alpha \pm i\omega(\alpha), \quad \omega(0) \neq 0. \tag{12.68}$$

For $\alpha = 0$, the eigenvalues of the matrix $A(\alpha)$ are purely imaginary. Suppose that the matrix $A(\alpha)$ has a canonical form, i.e.,

$$A(\alpha) = \begin{pmatrix} \alpha & -\omega(\alpha) \\ \omega(\alpha) & \alpha \end{pmatrix}.$$

Then system (12.67) is transformed into

$$\begin{aligned} \dot{x} &= \alpha x - \omega(\alpha)y + \textstyle\sum_{j=2}^{\infty} A_j(x,y,\alpha), \\ \dot{y} &= \omega(\alpha)x + \alpha y + \textstyle\sum_{j=2}^{\infty} B_j(x,y,\alpha). \end{aligned} \tag{12.69}$$

Here, $A_j(x, y, \alpha)$, $B_j(x, y, \alpha)$ are the homogeneous polynomials in the variables x, y of order j.

For $\alpha = 0$, system (12.69) turns into the system

$$\begin{aligned} \dot{x} &= -\omega(0)y + \textstyle\sum_{j=2}^{\infty} A_j(x, y, 0), \\ \dot{y} &= \omega(0)x + \textstyle\sum_{j=2}^{\infty} B_j(x, y, 0). \end{aligned} \tag{12.70}$$

In Section 12.7, we investigated the problem of the stability of a zero solution for system (12.70). Here, the governing part belongs to the Lyapunov quantities L_i, $i = 1, 2, \ldots$. In particular, if $L_1 < 0$, then the zero solution of system (12.70) is asymptotically stable. If $L_1 > 0$, then the zero solution of system (12.70) is unstable.

It turns out that the Lyapunov quantities also play a decisive role in studying the Andronov-Hopf bifurcation.

By making a change of the variables $x = r\cos\theta$, $y = r\sin\theta$, we convert system (12.69) into the polar coordinates

$$\begin{aligned} \dot{r} &= \alpha r + r^2 C_3(\theta, \alpha) + r^3 C_4(\theta, \alpha) + \ldots, \\ \dot{\theta} &= \omega(\alpha) + r D_3(\theta, \alpha) + r^2 D_4(\theta, \alpha), \end{aligned} \tag{12.71}$$

where

$$\begin{aligned} C_j(\theta, \alpha) &= (\cos\theta) A_{j-1}(\cos\theta, \sin\theta, \alpha) + (\sin\theta) B_{j-1}(\cos\theta, \sin\theta, \alpha), \\ D_j(\theta, \alpha) &= (\cos\theta) B_{j-1}(\cos\theta, \sin\theta, \alpha) - (\sin\theta) A_{j-1}(\cos\theta, \sin\theta, \alpha). \end{aligned}$$

Note that C_j and D_j are the homogeneous polynomials to the power j in $(\cos\theta, \sin\theta)$.

Our task is to find periodic solutions of system (12.71) for $\alpha \to 0$ such that $r \to 0$. We introduce a small parameter $\varepsilon > 0$ and assume

$$r = \varepsilon r_1, \quad \alpha = \varepsilon \alpha_1,$$

where a new variable r_1 is considered in the neighborhood of some number r_0, which will be chosen later together with α_1 to be the function of ε. System (12.71) will be written as

$$\begin{aligned} \dot{r}_1 &= \varepsilon[\alpha_1 r_1 + r_1^2 C_3(\theta)] + \varepsilon^2[r_1^3 C_4(\theta + \alpha_1 r_1^2 C_3^1(\theta)] + O(\varepsilon^3), \\ \dot{\theta} &= \omega(0) + \varepsilon[\alpha_1 \omega'(0) + r_1 D_3(\theta)] + O(\varepsilon^2), \end{aligned} \tag{12.72}$$

where $C_i(\theta) = C_i(\theta, 0)$, $i = 3, 4$, $C_3^1(\theta) = (\partial/\partial\alpha) C_3(\theta, 0)$, $D_3(\theta) = D_3(\theta, 0)$. We divide the first equation of system (12.72) by the second equation and obtain the first-order equations in the standard form with 2π-periodic coefficients:

$$\frac{dr_1}{d\theta} = \frac{\varepsilon[\alpha_1 r_1 + r_1^2 C_3(\theta)] + \varepsilon^2[r_1^3 C_4(\theta) + \alpha_1 r_1^2 C_3^1(\theta)] + O(\varepsilon^3)}{\omega(0) + \varepsilon[\alpha_1 \omega'(0) + r_1 D_3(\theta)] + O(\varepsilon^2)}. \tag{12.73}$$

To find the periodic solution of equation (12.73), assume that $\alpha_1 = \varepsilon$. Then equation (12.73) takes the form

$$\frac{dr_1}{d\theta} = \frac{\varepsilon r_1^2 C_3(\theta) + \varepsilon^2[r_1 + r_1^3 C_4(\theta)] + O(\varepsilon^3)}{\omega(0) + O(\varepsilon)}. \tag{12.74}$$

The numerator in the right-hand side of equation (12.74) differs from that in the right-hand side of equation (12.61) in Section 12.7 in only the additional summand $\varepsilon^2 r_1$. Therefore, calculating similarly, after a change

$$r_1 = y + \varepsilon u_1(\theta) y^2 + \varepsilon^2 u_2(\theta) y^3$$

yields the equation

$$\frac{dy}{d\theta} = \varepsilon^2 \omega(0)^{-1}[y + y^3 L_1] + O(\varepsilon^3),$$

where L_1 is the first Lyapunov quantity determined by the formula

$$L_1 = \frac{1}{2\pi} \int_0^{2\pi} [C_4(\theta, 0) - \omega^{-1}(0) C_3(\theta, 0) D_3(\theta, 0)] d\theta. \tag{12.75}$$

So, the averaged equation of the second approximation has the form

$$\frac{d\bar{y}}{d\theta} = \varepsilon^2 \omega(0)^{-1}[\bar{y} + \bar{y}^3 L_1]. \tag{12.76}$$

If $L_1 < 0$, then equation (12.76) has two non-negative stationary solutions $\bar{y}_1 = 0$ and $\bar{y}_2 = (-L_1)^{-1/2}$. The solution $\bar{y}_1$ is unstable, and the solution $\bar{y}_2$ is asymptotically stable. Hence, on account of Theorem 12.1, equilibrium of $\bar{y}_2$ corresponds to the asymptotically stable 2π-periodic solution $r(\theta, \varepsilon)$ of equation (12.75). Passing to the variable t, we find $x(t, \varepsilon)$ and $y(t, \varepsilon)$ that conform to the respective 2π-periodic function $r(\theta, \varepsilon)$. The functions thus produced will also be periodic but the period will depend on the parameter ε and the initial conditions. To show this, we turn to the second equation of system (12.72), which defines θ as the function of t. We assume that t and θ simultaneously vanish. In this case

$$\omega(0)[t(\theta + 2\pi) - t(\theta)] = \int_\theta^{\theta+2\pi} \frac{d\theta}{1 + O(\varepsilon)}.$$

Since $r(\theta, \varepsilon)$ is a 2π-periodic function, we see that the derivative of the integral equals zero. Therefore

$$\omega(0)[t(\theta + 2\pi) - t(\theta)] = T(\varepsilon). \tag{12.77}$$

The value $T(\varepsilon)$ depends only on ε and the initial condition. Relation (12.77) shows that if t changes by $T(\varepsilon)$, the value of θ changes by 2π, and, consequently, the functions $x(t,\varepsilon)$ and $y(t,\varepsilon)$ are unaltered, and there are the periodic functions with the period $T(\varepsilon)\,(T(0) = 2\pi/\omega(0))$. We chose $\theta(0) = 0$. Hence, $x(0,\varepsilon) = r(0,\varepsilon), y(0) = 0$.

In that way, we have proved the existence of a limit cycle in a small neighborhood of zero equilibrium for system (12.67) if $L_1 < 0$.

When $L_1 > 0$, we obtain the existence of the cycle on the assumption $\alpha = -\varepsilon$. In this case, the averaged equation of the second approximation takes the form

$$\frac{d\bar{y}}{d\theta} = \varepsilon^2\omega(0)^{-1}[-\bar{y} + \bar{y}^3 L_1].$$

The stationary solution $\bar{y}_1 = 0$ is asymptotically stable, and the stationary solution $\bar{y}_2 = L^{-1/2}$ is unstable.

It remains to show that in a sufficiently small neighborhood of zero equilibrium, there are no other periodic solutions, except those obtained by the above method. The proof is presented in the article of Chow and Mallet-Paret [1977], where they also considered the multidimensional case. It is possible to study the stability of the limit cycle using various methods (see Hassard, Kazarinoff, and Wan [1981]), in particular, the method considered in Section 10.2. We only note that the cycle is stable when $L_1 < 0$ and unstable when $L_1 > 0$.

A cycle bifurcation is said to take place in system (12.67) when the parameter α changes. If the first Lyapunov quantity L_1 is negative, then the zero equilibrium loses stability, and a stable limit cycle forms when α changes from negative to positive values. If the first Lyapunov quantity L_1 is positive, then the zero equilibrium becomes stable, and an unstable limit cycle forms when the parameter α changes from positive to negative values.

Let us focus our attention on calculating the amplitude of the limit cycle. Let $L_1 < 0$. Owing to the choice of parameters, $\varepsilon = \sqrt{\alpha}$. Averaged equation (12.76) is rewritten as

$$\frac{d\bar{y}}{d\theta} = \alpha\omega(0)^{-1}[\bar{y} + \bar{y}^3 L_1].$$

Then it follows from Theorem 9.2A that

$$y = (-L_1)^{-1/2} + O(\alpha).$$

Hence,

$$r = \left(\frac{\alpha}{-L_1}\right)^{\frac{1}{2}} + O(\alpha^{\frac{3}{2}}).$$

The constant

$$A = \left(\frac{\alpha}{-L_1}\right)^{\frac{1}{2}}$$

is called the amplitude of the limit cycle. The period of oscillations is determined by the formula

$$T(\alpha) = \frac{2\pi}{\omega(0)}[1 + O(\alpha^{\frac{1}{2}})].$$

For simplicity, we assumed the eigenvalues of the matrix $A(\alpha)$ to be presented in the form (12.68). In the general case, the eigenvalues of the matrix $A(\alpha)$ take the form

$$\lambda_{1,2}(\alpha) = f(\alpha) \pm i\omega(\alpha),$$

where $f(0) = 0$ and $f'(0) = \nu \neq 0$ which transforms averaged equation (12.75) into

$$\frac{d\bar{y}}{d\theta} = f(\varepsilon\alpha)y + \varepsilon^2 y^3 L_1 =$$

$$\varepsilon\alpha\nu y + \varepsilon^2 y^3 L_1 = [\pm\nu\bar{y} + \bar{y}^3 L_1], \quad \pm = -sgn\ (\nu L_1),$$

where $\alpha = -sgn\ (\nu L_1)\varepsilon$. The amplitude of the cycle is defined by the formula

$$A = \left(-\frac{\nu\alpha}{L_1}\right)^{\frac{1}{2}}.$$

As an example, we consider the equation

$$\ddot{x} - 2\alpha\dot{x} + x = (1 - \alpha)[\beta x^2 + \gamma\dot{x}^3 + \delta x\dot{x}^2], \tag{12.78}$$

where α is a real parameter, and β, γ, δ are some constants. For $\alpha = 0$, we obtain equation (12.66) from the previous section. As was shown there, the first Lyapunov quantity L_1 is negative if $\gamma < 0$ and positive if $\gamma > 0$. Therefore, if the parameter α changes from negative to positive values at $\gamma < 0$ in the equation under study, the bifurcation of the stable limit cycle occurs. When the parameter α changes from positive to negative values at $\gamma > 0$, the unstable limit cycle is formed.

As the second example, we consider the Josephson autonomous equation (see Sanders [1983])

$$\beta\ddot{\psi} + (1 + \gamma\cos\psi)\dot{\psi} + \sin\psi = \alpha, \tag{12.79}$$

where $\beta > 0$, γ, α are real parameters. We shall assume that $0 < \alpha < 1$. Equation (12.79) has two stationary solutions $\sin\psi_{1,2} = \alpha$, where $0 < \psi_1 < \pi/2$, $\pi/2 < \psi_2 < \pi$. Linearizing equation (12.79) on equilibria produces the linear equation

$$\beta\ddot{w} + (1 + \gamma\cos\psi_{1,2})\dot{w} + w\cos\psi_{1,2} = 0.$$

It is evident that the stationary solution ψ_2 is always unstable. The stability of the stationary solution ψ_1 depends on the sign in the expression $(1+\gamma\cos\psi_1)$. For

$$\gamma^* = -\frac{1}{\cos\psi_1} \tag{12.80}$$

we obtain the critical case of stability (a pair of purely imaginary roots).

Consider the problem on bifurcation of a periodic solution when the parameter γ passes the critical value (12.80).

Calculate the first Lyapunov quantity. Transform equation (12.79) into the system of equations

$$\dot{\psi} = -y, \quad \dot{y} = \frac{1}{\beta}[\sin\psi - \alpha + (1 + \gamma\cos\psi)y]. \tag{12.81}$$

Assume $\psi = \psi_1 + x$ and expand the right-hand side of system (12.81) in terms of powers of x until the third-order terms. We shall discard the terms of higher order and let $\gamma = \gamma^*$. Thereby, we arrive at the system

$$\begin{aligned} &\dot{x} = -y, \\ &\dot{y} = \tfrac{1}{\beta}[x\cos\psi_1 - \tfrac{1}{2}x^2\sin\psi_1 - \tfrac{1}{6}x^3\cos\psi_1 - \tfrac{x^2 y}{2} - \gamma^* xy\sin\psi_1. \end{aligned} \tag{12.82}$$

We introduce the notation

$$\omega^2 = \frac{\cos\psi_1}{\beta}$$

and make the change $y = \omega z$. In this case, instead of system (12.82), we obtain the system

$$\dot{x} = -\omega z, \quad \dot{z} = \omega x - \frac{x^2\sin\psi_1}{2\beta\omega} - \frac{x^3\omega}{6} - \frac{\gamma^* xz\sin\psi_1}{\beta} + \frac{x^2 z}{2\beta}. \tag{12.83}$$

By going to the polar coordinates and supposing $x = r\cos\theta$, $z = r\sin\theta$ in system (12.83), we get the system

$$\begin{aligned} \frac{dr}{dt} &= -\frac{r^2\cos^2\theta\sin\theta\sin\psi_1}{2\beta\omega} - \frac{\gamma^* r^2\cos\theta\sin^2\theta\sin\psi_1}{\beta} - \frac{r^3\omega\cos^3\theta}{6} - \frac{r^3\cos^2\theta\sin^2\theta}{2\beta} \\ \frac{d\theta}{dt} &= \omega - \frac{r\cos^3\theta\sin\psi_1}{2\beta\omega} - \frac{\gamma^* r\cos^2\theta\sin\theta\sin\psi_1}{\beta} + O(r^2). \end{aligned}$$

Calculating L_1 from formula (12.75) yields

$$L_1 = -\frac{(\gamma^*)^2}{16\beta} < 0.$$

Hence, as the parameter γ passes the critical value γ^*, the equilibrium loses stability, and the stable periodic solution is formed.

Because $f'(\gamma *) = \nu = \cos\psi_1/2\beta$, we have that the amplitude of the cycle is

$$A = 2\sqrt{2(1 + \gamma\cos\psi_1)}\cos^{3/2}\psi_1.$$

Using the method of averaging, it is possible to study a wide range of bifurcation problems. For example, we can consider a system of equations

$$\frac{dz}{dt} = A(\alpha)z + g(t, z, \alpha),$$

where the matrix $A(\alpha)$ is of the same form as that in system (12.67), $g(t, z, \alpha)$ is almost periodic in t uniformly with respect to z, α, and $g = (|z|^2)$.

Chapter 13

Averaging and Stability

In this chapter, we investigate the problem of closeness of non-stationary solutions of the exact and averaged equations on an infinite time interval. We believe, the first theorems of this type were proved in the works of Banfi [1967], and Sethna [1970]. Here, we infer the corresponding assertions based on special theorems of stability under constantly acting perturbations (obtained by the author [1977, 1979b, 1986]).

13.1 Basic Notation and Auxiliary Assertions

We shall use the following notation: $|x|$ is the norm of the element $x \in \mathcal{R}^n$, I is the interval $[0, \infty)$, $B_x(K) = \{x : x \in \mathcal{R}^n, |x| \le K\}$, $G = I \times B_x(K)$. Consider a vector-function $f(t, x)$ that is defined on G with values in $\mathcal{R}^n$, bounded in norm, continuous in x uniformly with respect to t and has no more than a finite number of jump discontinuities in t on each finite interval. We denote this function as

$$S_x(f) = \sup_{|t_2 - t_1| \le 1} \left| \int_{t_1}^{t_2} f(s, x) ds \right|, \quad x \in B_x(K).$$

Lemma 13.1. *For $f(t, x)$ defined on G the following inequality holds:*

$$\left| \int_{t_0}^{t} f(s, x(s)) ds \right| \le (T + 1) \sup_{|t_2 - t_1| \le 1} \left| \int_{t_1}^{t_2} f(s, x(s)) ds \right|, \quad t,\ t_1,\ t_2 \in [t_0, t_0 + T],$$

where $x(t)$ is the function defined on $[t_0, t_0 + T]$ with values in $B_x(K)$, and the vector-function $f(t, x(t))$ is integrable on $[t_0, t_0 + T]$.

Proof. The lemma follows from the evident inequality

$$\left| \int_{t_0}^{t} f(s, x(s)) ds \right| \le \left| \int_{t_0}^{t_0+1} f(s, x(s)) ds \right| + \left| \int_{t_0+1}^{t_0+2} f(s, x(s)) ds \right| + \cdots +$$

$$+\left|\int\limits_{t_0+[t]}^{t} f(s,x(s))ds\right| \le ([t]+1)\sup_{|t_2-t_1|\le 1}\left|\int\limits_{t_1}^{t_2} f(s,x(s))ds\right|,$$

where ($[t]$ is an integer part of t).

Lemma 13.2. *Let the vector-function $f(t,x)$ be defined on G and continuous in x uniformly with respect to $t \in I$. Let the vector-function $x(t)$ be continuous and its values belong to $B_x(K)$.*

Then for any $\eta > 0$ it is possible to specify an ε such that

$$\sup_{|t_2-t_1|\le 1}\left|\int\limits_{t_1}^{t_2} f(s,x(s))ds\right| < \eta, \quad (t_1,t_2)\in[0,T], \quad 0<t<\infty$$

if $S_x(f) < \varepsilon$.

Proof. By the lemma conditions, for any $\eta > 0$ it is possible to specify $\delta > 0$ such that $|f(t,x_1) - f(t,x_2)| < \eta/2$ at $|x_1 - x_2| < \delta$. Denote by $x^0(t)$ a step vector-function with values in $B_x(K)$ such that $|x(t) - x^0(t)| < \delta$, $t \in [0,T]$. At that, in each interval no longer than a unity, the function $x^0(t)$ possesses no more than k different values, the number k depending only upon δ. Let x_j $(j = 1,\dots,k)$ be the values of $x^0(t)$ in the interval $|t_1 - t_2| \le 1$. Assume $\varepsilon = \eta/2k$. Then we have

$$\left|\int\limits_{t_1}^{t_2} f(s,x(s))ds\right| \le \left|\int\limits_{t_1}^{t_2}[f(s,x(s)) - f(s,x^0(s))]ds\right| +$$

$$+\left|\int\limits_{t_1}^{t_2} f(s,x^0(s))ds\right| \le \frac{\eta}{2} + \sum_{j=1}^{k}\left|\int\limits_{t_1}^{t_2} f(s,x_j)ds\right| \le \frac{\eta}{2} + \frac{\eta}{2k}k = \eta.$$

The latter inequality is valid for any t_1, t_2 satisfying the inequality $|t_2-t_1| \le 1$, which proves the lemma.

We introduce one more vector-function

$$w(t,x) = -\int\limits_{t}^{\infty} e^{(t-s)} f(s,x)ds,$$

where $f(t,x) \in G$ and $|f(t,x)| < M < \infty$.

Lemma 13.3. *For any $\eta > 0$ it is possible to specify an ε such that*

$$|w(t,x)| < \eta$$

if $S_x(f) < \varepsilon$.

Proof. In the expression

$$w(t,x) = -\int_t^\infty e^{(t-s)} f(s,x)ds,$$

we make a change $s = t + \tau$ and obtain

$$w(t,x) = -\int_0^\infty e^{-\tau} f(t+\tau,x)d\tau.$$

The latter equality can be written as

$$w(t,x) = -\int_0^\infty e^{-\tau} \frac{d}{d\tau}\left[\int_t^{t+\tau} f(\sigma,x)d\sigma\right].$$

Integrating it by parts yields

$$w(t,x) = \int_0^\infty e^{-\tau}\left[\int_t^{t+\tau} f(\sigma,x)d\sigma\right] d\tau.$$

Now, assertion of the lemma results from the equality

$$w(t,x) = \sum_{k=0}^{\infty}\int_k^{k+1} e^{-\tau}\left[\int_t^{t+\tau} f(\sigma,x)d\sigma\right] d\tau.$$

13.2 Stability under Constantly Acting Perturbations

We consider a system of differential equations in $\mathcal{R}^n$

$$\frac{dx}{dt} = X(t,x) + R(t,x), \tag{13.1}$$

where the vector-functions $X(t,x)$ and $R(t,x)$ are defined on G and continuous in t.

Along with equation (13.1), we study an unperturbed system of differential equations

$$\frac{dy}{dt} = X(t,y). \tag{13.2}$$

Assume that system (13.2) has the solution $\psi(t,t_0,\xi_0)$ $((\psi(t_0,t_0,\xi_0) = \xi_0)$ such that it is determined for all $t \geq t_0 \geq 0$ and, together with its ρ-neighborhood ($\rho > 0$), belongs to set G.

We shall now use such concepts as the uniform asymptotic stability and uniform asymptotic stability with respect to a part of the variables (see Appendix B, Definitions B.4 and B.6). We only make the following remark.

Remark 13.1.
Note that later on, we shall use a non classical definition of asymptotic stability with respect to a part of the variables. Namely, in the definition of asymptotic stability with respect to a part of the variables, it is sufficient to assume that the initial conditions are close on some coordinates $i = 1, \dots, k < n$ rather than on all coordinates.

Theorem 13.1. *Let a vector-function $X(t,x)$ be bounded on the set G and satisfies the Lipschitz condition with some constant L:*

$$|X(t,x_1) - X(t,x_2)| \le L|x_1 - x_2|, \quad x_1, x_2 \in B_x(K). \tag{13.3}$$

Let $R(t,x)$ be continuous in x uniformly with respect to $t \in I$ and bounded on the set G. Let the solution $\psi(t,t_0,\xi_0)$ of system (13.2) be uniformly asymptotically stable.

Then for any $\varepsilon > 0\,(0 < \varepsilon < \rho)$ it is possible to specify $\eta_1(\varepsilon)$, $\eta_2(\varepsilon)$ such that for all solutions $x(t,t_0,x_0) \in B_x(K)\,(x(t_0,t_0,x_0) = x_0)$ of (13.1), which are defined for $t \ge t_0$ and have initial conditions obeying the inequality

$$|x_0 - \xi_0| < \eta_1(\varepsilon),$$

and for all $R(t,x)$ meeting the inequality

$$S(R) < \eta_2(\varepsilon),$$

for all $t > t_0$, the following inequality is valid

$$|x(t,t_0,x_0) - \psi(t,t_0,\xi_0)| < \varepsilon. \tag{13.4}$$

Proof. Let $y(t,t_0,x_0)$ be the solution of system (13.2) with initial condition identical to that of the solution $x(t,t_0,x_0)$ of system (13.1). These solutions satisfy the integral equations

$$y(t,t_0,x_0) = x_0 + \int_{t_0}^{t} X(s, y(s,t_0,x_0))ds,$$

$$x(t,t_0,x_0) = x_0 + \int_{t_0}^{t} [X(s,x(s,t_0,x_0)) + R(s,x(s,t_0,x_0))]ds.$$

This implies

$$\begin{array}{l} |x(t,t_0,x_0)-y(t,t_0,x_0)|\le \int\limits_{t_0}^{t} |X(s,x(s,t_0,x_0))-X(s,y(s,t_0,x_0))|\,ds+ \\ +\left|\int\limits_{t_0}^{t} R(s,x(s,t_0,x_0))ds\right| \end{array}.$$

Using condition (13.3) of the theorem, we arrive at the inequality

$$|x(t,t_0,x_0)-y(t,t_0,x_0)|\le L\int\limits_{t_0}^{t} |x(s,t_0,x_0)-y(s,t_0,x_0)|ds+f(t),$$

where

$$f(t)=\left|\int\limits_{t_0}^{t} R(s,x(s,t_0,x_0))ds\right|.$$

From the known integral inequality (see, e.g., Barbashin [1967]) we obtain

$$|x(t,t_0,x_0)-y(t,t_0,x_0)|\le f(t)+L\int\limits_{t_0}^{t} e^{L(t-s)}f(s)ds.$$

It follows from Lemma 13.1 that for $t_0\le t\le t_0+T$

$$\begin{array}{l} |x(t,t_0,x_0)-y(t,t_0,x_0)|\le \\ \le (T+1)\left(1+LTe^{LT}\right)\sup_{|t_2-t_1|\le 1}\left|\int\limits_{t_1}^{t_2} R(s,x(s,t_0,x_0))ds\right|. \end{array} \tag{13.5}$$

By virtue of the uniform asymptotic stability of the solution $\psi(t,t_0,\xi_0)$ of equation (13.2), there exist the numbers $\delta<\varepsilon$ and $T_0>0$ such that the inequality $|x_0-\xi_0|<\delta$ implies

$$\begin{array}{l} |y(t,t_0,x_0)-\psi(t,t_0,\xi_0)|<\frac{\varepsilon}{2},\quad t\ge t_0, \\ |y(t_0+T_0,t_0,x_0)-\psi(t_0+T_0,t_0,\xi_0)|<\frac{\delta}{2}. \end{array} \tag{13.6}$$

It follows from Lemma 13.2 that it is possible to select the number $\eta_2(\varepsilon)$ so that to satisfy the inequality

$$|x(t,t_0,x_0)-y(t,t_0,x_0)|<\frac{\delta}{2},\quad t_0\le t\le t_0+T. \tag{13.7}$$

Then we have

$$|x(t,t_0,x_0)-\psi(t,t_0,\xi_0)|<\frac{\varepsilon}{2}+\frac{\delta}{2}<\varepsilon,\quad t_0\le t\le t_0+T.$$

Further, from (13.6) and (13.7) (for $T \le T_0$) we have

$$|x(t_0 + T, t_0, x_0) - \psi(t_0 + T, t_0, \xi_0)| < \delta.$$

Thus, within the time interval $[t_0, t_0 + T]$, the solution $x(t, t_0, x_0)$ will not go beyond the limits of the ε-neighborhood of the solution $\psi(t, t_0, \xi_0)$ and will lie in the δ-neighborhood of $\psi(t, t_0, \xi_0)$ at the time instant $t = t_0 + T$. Let us take the time instant $t = t_0 + T$ as the initial time and perform a similar reasoning. We check that the solution $x(t, t_0, x_0)$ does not go beyond the limits of ε-neighborhood of the solution $\psi(t, t_0, \xi_0)$ for $t_0 + T \le t \le t_0 + 2T$ and, besides, $x(t_0 + 2T, t_0, x_0)$ lies in δ-neighborhood of the solution $\psi(t, t_0, \xi_0)$. Proceeding with the reasoning, we obtain $|x(t, t_0, x_0) - \psi(t, t_0, \xi_0)| < \varepsilon$ for $t_0 + (n-1)T \le t \le t_0 + nT$ and, besides, $|x(t_0 + nT, t_0, x_0) - \psi(t_0 + nT, t_0, \xi_0)| < \delta$, which is the proof of the theorem.

Note that the last part of the proof follows the reasoning for Lemma 6.3 from the book by Barbashin [1967].

Remark 13.2. In proving Theorem 13.1, we assumed that the solution $x(t, t_0, \xi_0)$ is defined for all $t \ge t_0$. However, if the conditions of local theorem for the existence of solutions of equation (13.1) are fulfilled and $S_x(R)$ is sufficiently small for $x \in B_x(K)$, then the solution $x(t, t_0, x_0)$ with the initial condition close enough in norm to the initial condition of the solution $\psi(t, t_0, x_{i_0})$ of equation (13.2) is defined for all $t \ge t_0$, which is easily concluded from inequality (13.5).

Remark 13.3. Theorem 13.1 remains valid on the assumption that the vector-functions $X(t, x)$ and $R(t, x)$ have no more than the finite number of simple discontinuities (jumps) in t on each finite interval.

Remark 13.4. If the solution $\psi(t, t_0, \xi_0)$ of system (13.2) is uniformly asymptotically stable only with respect to a part of the variables $\psi_1, \ldots, \psi_k$, $k < n$, then, the reasoning similar to that in Theorem 13.1, implies a theorem asserted in the following way:
for any $\varepsilon > 0\,(0 < \varepsilon < \rho)$ it is possible to specify the numbers $\eta_1(\varepsilon)$, $\eta_2(\varepsilon)$ such that for all solutions $x(t, t_0, x_0) \in B_x(K)\,(x(t_0, t_0, x_0) = x_0)$ of equation (13.1), which are defined for $t \ge t_0$ and have initial data obeying the inequality

$$|x_{i0} - \xi_{0i}| < \eta_1(\varepsilon), \quad i = 1, 2, \ldots, k < n,$$

and for all $R(t, x)$ meeting the inequality

$$S(R) < \eta_2(\varepsilon),$$

the following inequality is valid for all $t > t_0$

$$|x_i(t, t_0, x_0) - \psi_i(t, t_0, \xi_0)| < \varepsilon, \quad i = 1, \ldots, k < n.$$

Theorem 13.1 is the theorem of stability under constantly acting perturbations. It differs from the theorems by Malkin and Krasovskii-Germaidze (see Appendix B, Definitions B.4, B.5, and Theorem B.6) in a more general assumption on the "smallness" of the perturbation $R(t,x)$. It is precisely these conditions of smallness of the perturbation $R(t,x)$ in terms of the theorems by Malkin or Krasovskii-Germaidze that imply the smallness of $S(R)$. The function $\sin t/\varepsilon$ presents an example of a perturbation such that $S(\sin(t/\varepsilon))$ is small for small ε but the quantities

$$|\sin \frac{t}{\varepsilon}|, \qquad \int |\sin \frac{t}{\varepsilon}| dt$$

are not small for this function. Theorem 13.1 allows quickly oscillating functions to be included into the constantly acting perturbations.

It is assumed in Theorem 13.1 that the solution of system (13.1) belongs to the set $B_x(K)$.

We bring in another theorem on stability under constantly acting perturbations that is free from this assumption.

For the vector-function $f(t,x)$ defined on G and bounded in norm by the constant:

$$|f(t,x)| \le M$$

we write $f(t,x) \in M(G)$.

Theorem 13.2. *Let $X(t,x) \in M(G)$ satisfy the Lipschitz condition with some constant in the spatial variable x . Let $R(t,x) \in M_1(G)$ and $R_{x_i}(t,x) \in M_2(G)$, $i = 1,2,\dots,n$. Assume that system (13.2) has the solution $\psi(t,t_0,\xi_0)$ $((\psi(t_0,t_0,$ $\xi_0)$ that is defined for all $t \ge t_0 \ge 0$ and, together with its ρ-neighborhood ($\rho > 0$), belongs to the set G. Let this solution be uniformly asymptotically stable.*

Then for any $\varepsilon > 0\,(0 < \varepsilon < \rho)$ it is possible to specify the numbers $\eta_1(\varepsilon)$, $\eta_2(\varepsilon)$, $\eta_3(\varepsilon)$ such that the solution $x(t,t_0,x_0)$ $(x(t_0,t_0,x_0) = x_0)$ of system (13.1) with the initial conditions obeying the inequality

$$|x_0 - \xi_0| \le \eta_1(\varepsilon)$$

and for $R(t,x)$ meeting the inequalities

$$S(R) < \eta_2(\varepsilon), \quad S(R_{x_i}) < \eta_3(\varepsilon),\ i = 1,\dots,n$$

at $|x| < \varepsilon$, for all $t \ge t_0$, fulfills the inequality

$$|x(t,t_0,x_0) - \psi(t,t_0,\xi_0)| < \varepsilon. \tag{13.8}$$

Proof. By a change

$$x = z + w(t,z),$$

where

$$w(t,z) = -\int_t^\infty e^{(t-s)} R(s,z) ds,$$

we transform system (13.1) into

$$\left(I + \frac{\partial w}{\partial z}\right)\frac{dz}{dt} + w = X(t, z + w(t,z)) + R(t, z + w(t,z)). \tag{13.9}$$

Select $\eta_3(\varepsilon)$ such that the matrix $I + \frac{\partial w}{\partial z}$ is invertible (by virtue of Lemma 13.3 $|\frac{\partial w}{\partial z_i}| < \eta_3(\varepsilon)$, $i = 1, \dots, n$). Then (13.9) can be rewritten as (on account of $\frac{\partial w}{\partial t} = w + R(t,z)$)

$$\frac{dz}{dt} = \left[I + \frac{\partial w}{\partial z}\right]^{-1}\left[-w + \left[I + \frac{\partial w}{\partial z}\right] X(t,z) - \right.$$
$$\left. -\frac{\partial w}{\partial z} X(t,z) + (X(t, z+w) - X(t,z)) + (R(t, z+w) - R(t,z))\right],$$

or,

$$\frac{dz}{dt} = X(t,z) + \left[I + \frac{\partial w}{\partial z}\right]^{-1}\left[H(t,z,w) - \frac{\partial w}{\partial z} X(t,z)\right], \tag{13.10}$$

where

$$H(t,z,w) = -w + (X(t,z+w) - X(t,z)) + (R(t,z+w) - R(t,z)).$$

To complete the proof of the theorem, it is sufficient to apply the Malkin Theorem of stability under constantly acting perturbations (see Appendix B, Theorem B.6) to system (13.10), since $H(t,z,w)$ can be made arbitrarily small together with w.

Remark 13.5. When the solution of equation (13.2) is uniformly asymptotically stable with respect to a part of the variables, it is possible to deduce an analog of Theorem 13.2, where inequality (13.8) will be replaced with the inequality

$$|x_i(t, t_0, x_0) - \psi_i(t, t_0, \xi_0)| < \varepsilon, \quad i = 1, \dots, k < n.$$

13.3 Integral Convergence and Closeness of Solutions on an Infinite Interval

We apply Theorems 13.1 and 13.2 to the problem of averaging on an infinite interval. Provisionally, we introduce the concept of integral convergence of the right-hand sides of differential equations and determine the relationship between the convergence and closeness of solutions on an infinite interval.

Consider a system of equations

$$\frac{dx}{dt} = X(t, x, \varepsilon), \tag{13.11}$$

where $\varepsilon > 0$ is a small parameter, and $(t, x) \in G$. Let $G_\varepsilon = (0, \varepsilon_0] \times G$. We say that $X(t, x, \varepsilon)$ **converges integrally** to $X(t, x)$ if

$$\lim_{\varepsilon \to 0} \sup_{|t_2 - t_1| \le 1} \left| \int_{t_1}^{t_2} [X(s, x, \varepsilon) - X(s, x)]\, ds \right| = 0$$

for each $x \in B_x(K)$. Theorem 13.1 immediately has the following implication.

Theorem 13.3. *Let*
1) the vector-function $X(t, x, \varepsilon)$ be defined for $(t, x, \varepsilon) \in G_\varepsilon$, have no more than a finite number of simple discontinuities (jumps) in t on each finite interval and be continuous in x uniformly with respect to t, ε;
2) $|X(t, x, \varepsilon)| \le M_1, \quad (t, x, \varepsilon) \in G_\varepsilon$;
3) the vector-function $X(t, x, \varepsilon)$ integrally converge to the vector-function $X(t, x)$ that is continuous in all variables and bounded in norm by some constant M on the set G;
4) $X(t, x)$ satisfy the Lipschitz condition with some constant L:

$$|X(t, x_1) - X(t, x_2)| \le L|x_1 - x_2|, \quad x_1, x_2 \in B_x(K),\ t \in I;$$

5) the system

$$\frac{dy}{dt} = X(t, y) \tag{13.12}$$

have the uniformly asymptotically stable solution $y = \psi(t, t_0, \xi_0)$ (uniformly asymptotically stable with respect to a part of the variables $y_1, \dots, y_k$, $k < n$) that together with its ρ-neighborhood ($\rho > 0$) belongs to the set G.

Then for any α $(0 < \alpha < \rho)$ there exist $\varepsilon_1(\alpha)$ $(0 < \varepsilon_1 < \varepsilon_0)$ and $\beta(\alpha)$ such that for all $0 < \varepsilon < \varepsilon_1$, the solution $x(t, t_0, x_0) \in B_x(K)$ of system (5.2.11), which is defined for $t \ge t_0$ and $|x_0 - \xi_0| < \beta(\alpha)$ $(|x_{0i} - \xi_{0i}| < \beta(\alpha)$, $i = 1, \dots, k)$, obeys the inequality

$$|\psi(t, t_0, \xi_0) - x(t, t_0, x_0)| < \alpha, \quad t \ge t_0$$

$$(|\psi_i(t, t_0, \xi_0) - x_i(t, t_0, x_0)| < \alpha, \quad i = 1, \dots, k < n, \quad t \ge t_0).$$

.

Theorem 13.3 directly ensues from Theorem 13.1, Remarks 13.3 and 13.4 if system (13.11) is rewritten as

$$\frac{dx}{dt} = X(t, x) + R(t, x, \varepsilon),$$

where $R(t, x, \varepsilon) = X(t, x, \varepsilon) - X(t, x)$.

Theorem 13.3 implies the following result.

Theorem 13.4. *Let*
1) for each ε *the vector-functions* $X(t, x, \varepsilon)$, $X_{x_i}(t, x, \varepsilon)$, $i = 1, \dots, n$ *be defined on* G *and continuous in all variables, and* $X_{x_i}(t, x, \varepsilon)$ *be continuous in* x *uniformly with respect to* t, ε;
2) $|X(t, x, \varepsilon)| \le M_1$, $|X_{x_i}(t, x, \varepsilon)| \le M_2$, $i = 1, \dots, n$ *on* G_ε;
3) the vector-functions $X(t, x, \varepsilon)$, $X_{x_i}(t, x, \varepsilon)$ *integrally converge to* $X(t, x)$, $X_{x_i}(t, x)$, *respectively;*
4) the system

$$\frac{dx}{dt} = X(t, x)$$

have the uniformly asymptotically stable solution $y = \psi(t, t_0, \xi_0)$ *(uniformly asymptotically stable with respect to a part of the variables* $y_1, \dots, y_k$, $k < n$*) that together with its* ρ*-neighborhood (*$\rho > 0$*) belongs to the set* G.

Then for any $\alpha\,(0 < \alpha < \rho)$ *there exist* $\varepsilon_1(\alpha)\,(0 < \varepsilon_1 < \varepsilon_0)$ *and* $\beta(\alpha)$ *such that for all* $0 < \varepsilon < \varepsilon_1$ *the solution* $x(t, t_0, x_0) \in B_x(K)$ *of system (13.11) defined for* $t \ge t_0$, *for which* $|x_0 - \xi_0| < \beta(\alpha)$ $(|x_{0i} - \xi_{0i}| < \beta(\alpha)$, $i = 1, \dots, k)$, *satisfies the inequality*

$$|\psi(t, t_0, \xi_0) - x(t, t_0, x_0)| < \alpha, \quad t \ge t_0$$

$$(|\psi_i(t, t_0, \xi_0) - x_i(t, t_0, x_0)| < \alpha, \quad i = 1, \dots, k < n, \quad t \ge t_0).$$

.

13.4 Theorems of Averaging

Theorems 13.3 and 13.4 contain some results on averaging on an infinite interval for differential equations in standard form. Consider a system of differential equations

$$\frac{dx}{dt} = \varepsilon X(t, x, \varepsilon), \quad x \in \mathcal{R}^n, \tag{13.13}$$

where $X(t, x, \varepsilon)$ is defined for $(t, x, \varepsilon) \in G_\varepsilon$.

Theorem 13.5. *Let*
1) in each of the variables x, ε, *the vector-function* $X(t, , x, \varepsilon)$ *be continuous uniformly with respect to the other variables;*

2) $|X(t,x,\varepsilon)| \le M_1, \quad (t,x,\varepsilon) \in G_\varepsilon$*;*
3) there exist the limit uniformly with respect to $t \in I$

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_t^{t+T} X(t,x,0)dt = \bar{X}(x)$$

for each $(t,x,\varepsilon) \in G_\varepsilon$ *and* $\bar{X}(x)$ *be bounded in norm by some constant* M_2 *on* $B_x(K)$*;*
4) $\bar{X}(x)$ *meet the Lipschitz condition with some constant* L*:*

$$\left|\bar{X}(x_1) - \bar{X}(x_2)\right| \le L|x_1 - x_2|, \quad x_1, x_2 \in B_x(K)$$

5) the system

$$\frac{dx}{dt} = \bar{X}(x), \tag{13.14}$$

have the uniformly asymptotically stable solution $\psi(t,t_0,\xi_0)$ *(uniformly asymptotically stable with respect to a part of the variables* $x_1,\dots,x_k$*,* $k<n$*) that together with its* ρ*-neighborhood* $(\rho>0)$ *belongs to the set* G*.*

Then for any $\alpha\,(0<\alpha<\rho)$ *there exist* $\varepsilon_1(\alpha)\,(0<\varepsilon_1<\varepsilon_0)$ *and* $\beta(\alpha)$ *such that for all* $0<\varepsilon<\varepsilon_1$ *the solution* $\varphi(t,t_0,x_0) \in B_x(K)$ *of system (13.13), for which* $|x_0-\xi_0| < \beta(\alpha)\,(|x_{0i}-\xi_{0i}| < \beta(\alpha),\ i=1,\dots,k)$ *obeys the inequality*

$$|\psi(t,t_0,\xi_0) - \varphi(t,t_0,x_0)| < \alpha, \quad t \ge t_0$$

$$(|\psi_i(t,t_0,\xi_0) - \varphi_i(t,t_0,x_0)| < \alpha, \quad i=1,\dots,k<n, \quad t\ge t_0).$$

In order to show that Theorem 13.5 ensues from Theorem 13.3, we need to check that condition 3) of Theorem 13.5 implies the fulfillment of condition 3) of Theorem 13.3. The integral convergence here means (on transition to the slow time $\tau = \varepsilon t$ in system (13.13)) that

$$\lim_{\varepsilon\to 0} \sup_{|t_2-t_1|\le 1} \left| \int\limits_{t_1}^{t_2} \left[X(\frac{\tau}{\varepsilon},x,\varepsilon) - \bar{X}(x)\right] d\tau \right| = \lim_{\varepsilon\to 0} \Pi(\varepsilon) = 0, \quad x \in B_x(K).$$

We show that for any $\delta > 0$ at sufficiently small ε

$$\Pi(\varepsilon) < \delta. \tag{13.15}$$

Evidently, due to the continuity of $X(\frac{\tau}{\varepsilon},x,\varepsilon)$ in the third variable uniformly with respect to the other variables for sufficiently small ε

$$\Pi(\varepsilon) \le \sup_{|t_2-t_1|\le 1} \left| \int\limits_{t_1}^{t_2} \left[X(\frac{\tau}{\varepsilon},x,0) - \bar{X}(x)\right] d\tau \right| + \frac{\delta}{2}.$$

Let us take arbitrary numbers t_1, t_2 that satisfy the condition $|t_1 - t_2| \le 1$ and prove that, for sufficiently small ε

$$\left| \int\limits_{t_1}^{t_2} \left[X(\frac{\tau}{\varepsilon}, x, 0) - \bar{X}(x) \right] d\tau \right| < \frac{\delta}{2}.$$

Thus, we get follows (13.15). We have

$$\left| \int\limits_{t_1}^{t_2} \left[X(\frac{\tau}{\varepsilon}, x, 0) - \bar{X}(x) \right] d\tau \right| = \left| \varepsilon \int\limits_{\frac{t_1}{\varepsilon}}^{\frac{t_2}{\varepsilon}} \left[X(u, x, 0) - \bar{X}(x) \right] du \right| =$$

$$= \left| \frac{1}{\frac{t_2 - t_1}{\varepsilon}} \int\limits_{\frac{t_1}{\varepsilon}}^{\frac{t_2}{\varepsilon}} \left[X(u, x, 0) - \bar{X}(x) \right] du \right| = \left| \frac{1}{T} \int\limits_{\frac{t_1}{\varepsilon}}^{\frac{t_2}{\varepsilon} + T} \left[X(u, , x, 0) - \bar{X}(x) \right] du \right|,$$

where $T = \frac{t_2 - t_1}{\varepsilon} \to \infty$ as $\varepsilon \to 0$.

From this and condition 3) follows the theorem.

Remark 13.6. Theorem 13.5 implies assertion of 3) in Theorem 9.3. Indeed, let $x_0(t, \varepsilon)$ be an almost periodic solution of the system

$$\frac{dx}{dt} = \varepsilon X(t, x)$$

that is close to the stationary solution $y = y_0$ of the averaged system

$$\frac{dy}{dt} = \varepsilon Y(y).$$

The stationary solution y_0 is asymptotically stable and, consequently, uniformly asymptotically stable. The solutions of the averaged system that lie within the domain of attraction of the stationary solution y_0 are also uniformly asymptotically stable. Therefore, solutions of the original system with the initial conditions that are sufficiently close to the initial condition of solution of the averaged system that lies in the domain of attraction of the solution y_0 will be close on an infinite interval as is stated by Theorem 13.5.

Theorem 13.4 implies the following theorem of averaging for the system

$$\frac{dx}{dt} = \varepsilon X(t, x, \varepsilon). \tag{13.16}$$

Theorem 13.6. *Let*

1) $X(t, x, \varepsilon)$, $X_{x_i}(t, x, \varepsilon)$ *be defined on* G*, continuous in all variables and*

bounded;
2) the functions $X(t,x,\varepsilon)$, $X_{x_i}(t,x,\varepsilon)$ *be continuous in each of the variables* x, ε *uniformly with respect to the other variables;*
3) there exist the limit uniformly with respect to t

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x,0)dt = \bar{X}(x)$$

in the considered domain of variables;
4) the system

$$\frac{dx}{dt} = \bar{X}(x,0)$$

have the uniformly asymptotically stable solution $\psi(t,t_0,\xi_0)$ *(uniformly asymptotically stable with respect to a part of the variables* $x_1,\ldots,x_k$, $k<n$*) that together with its* ρ*-neighborhood belongs to the set* G.

Then for any $\alpha\,(0<\alpha<\rho)$ *there exist* $\varepsilon_1(\alpha)\,(0<\varepsilon_1<\varepsilon_0)$ *and* $\beta(\alpha)$ *such that for all* $0<\varepsilon<\varepsilon_1$ *the solution* $\varphi(t,t_0,x_0)\in B_x(K)$ *of system (13.16), for which* $|x_0-\xi_0|<\beta(\alpha)\,(|x_{0i}-\xi_{0i}|<\beta(\alpha)$, $i=1,\ldots,k)$, *obeys the inequality*

$$|\psi(t,t_0,\xi_0)-\varphi(t,t_0,x_0)|<\alpha, \quad t\geq t_0$$

$$(|\psi_i(t,t_0,\xi_0)-\varphi_i(t,t_0,x_0)|<\alpha, \quad i=1,\ldots,k<n, \quad t\geq t_0).$$

Proof of Theorem 13.6 is similar to the proof of Theorem 13.5.

An analogous theorem to Theorems 13.5 and 13.6 was proved in the paper by Banfi [1967] (see also Filatov [1971], and Mitropolskii and Homa [1983]).

Remark 13.7. If we replace the requirement for uniform asymptotic stability of the solution of an averaged system with the requirement of asymptotic stability, then Theorem 13.6, generally speaking, is not correct. The following example illustrates this fact. Let us consider a scalar differential equation

$$\frac{dx}{dt} = \varepsilon x^2 + \varepsilon f(t), \tag{13.17}$$

where $\varepsilon>0$ is a small parameter, and $f(t)$ is a periodic function with zero mean value. The averaged equation

$$\frac{dx}{dt} = \varepsilon x^2$$

has the asymptotically stable solution $x(t)$ with the initial condition $x(0)=-1$ that is not uniformly asymptotically stable. The solution $x(t)$ tends to zero as $t\to\infty$ together with all solutions with negative initial conditions, whereas the solutions with positive initial conditions tend to infinity as $t\to\infty$. If

Theorem 13.6 was applied in this case, then equation (13.17) would have, for sufficiently small ε, a bounded solution with the initial condition close to the initial condition of the solution $x(t)$. It follows from the Massera Theorem (see Pliss [1966]) that equation (13.17) has a periodic solution. But equation (13.17) cannot have periodic solutions. Substituting the periodic solution into equation (13.17) yields a zero mean value of the left-hand side and a positive mean value of the right-hand side. Hence, equation (13.17) has no bounded solutions although the averaged equation has an asymptotically stable bounded solution.

13.5 Systems with Fast and Slow Time

Theorems 13.5 and 13.6 are generalized over the system of differential equations with fast and slow time. We briefly describe the respective results.

Consider a system with fast and slow time

$$\frac{dx}{dt} = \varepsilon X(t,\tau,x,\varepsilon), \quad \tau = \varepsilon t, \quad x \in \mathcal{R}^n, \tag{13.18}$$

where $X(t,\tau,x,\varepsilon)$ is defined for $(t,x,\varepsilon) \in G_\varepsilon$.

Theorem 13.7. *Let*
1) the function $X(t,\tau,x,\varepsilon)$ be continuous in each of the variables τ, x, ε uniformly with respect to the other variables;
2) $|X(t,\tau,x,\varepsilon)| \le M_1, \quad (t,x,\varepsilon) \in G_\varepsilon$;
3) there exist the limit uniformly with respect to $t \in I$

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_t^{t+T} X(t,\tau,x,0)dt = \bar{X}(\tau,x)$$

for each $(t,x) \in G$ and $\bar{X}(\tau,x)$ be bounded in norm by some constant M_2 on G;
4) the function $\bar{X}(\tau,x)$ meet the Lipschitz condition with some constant L:

$$\left|\bar{X}(\tau,x_1) - \bar{X}(\tau,x_2)\right| \le L|x_1 - x_2|, \quad x_1, x_2 \in B_x(K),\ \tau \in I$$

and be continuous in τ uniformly with respect to x;
5) the system

$$\frac{dx}{d\tau} = \bar{X}(\tau,x)$$

have the uniformly asymptotically stable solution $\psi(t,t_0,\xi_0)$ (uniformly asymptotically stable with respect to a part of the variables $x_1,\dots,x_k$, $k<n$) that together with its ρ-neighborhood ($\rho > 0$) belongs to the set G.

Then for any $\alpha\,(0 < \alpha < \rho)$ *there exist* $\varepsilon_1(\alpha)\,(0 < \varepsilon_1 < \varepsilon_0)$ *and* $\beta(\alpha)$ *such that for all* $0 < \varepsilon < \varepsilon_1$ *the solution* $\varphi(t, t_0, x_0) \in B_x(K)$ *of system (13.18), for which* $|x_0 - \xi_0| < \beta(\alpha)(|x_{0i} - \xi_{0i}| < \beta(\alpha),\ i = 1, \dots, k)$, *obeys the inequality*

$$|\psi(t, t_0, \xi_0) - \varphi(t, t_0, x_0)| < \alpha, \quad t \geq t_0$$

$$(|\psi_i(t, t_0, \xi_0) - \varphi_i(t, t_0, x_0)| < \alpha, \quad i = 1, \dots, k < n, \quad t \geq t_0).$$

As in the proof of Theorem 13.5, we only need to check whether condition 3) of Theorem 13.7 entails the fulfillment of condition 3) in Theorem 13.3. The integral convergence here means (on transition to the slow time $\tau = \varepsilon t$ in equation (13.18)) that

$$\lim_{\varepsilon \to 0} \sup_{|t_2 - t_1| \leq 1} \left| \int_{t_1}^{t_2} \left[X(\frac{\tau}{\varepsilon}, \tau, x, \varepsilon) - \bar{X}(\tau, x) \right] d\tau \right| = \lim_{\varepsilon \to 0} \Pi(\varepsilon) = 0, \quad x \in B_x(K).$$

We show that $\Pi(\varepsilon)$ is small for small ε. Evidently, by virtue of the continuity of $X(\frac{\tau}{\varepsilon}, \tau, x, \varepsilon)$ in the fourth variable, uniformly with respect to the other variables, it is sufficient to show that

$$\Pi_1(\varepsilon) = \sup_{|t_2 - t_1| \leq 1} \left| \int_{t_1}^{t_2} \left[X(\frac{\tau}{\varepsilon}, \tau, x, 0) - \bar{X}(\tau, x) \right] d\tau \right|$$

tends to zero as $\varepsilon \to 0$. This assertion takes place if we fix the second variable τ in the vector-function $X(\frac{\tau}{\varepsilon}, \tau, x, 0)$ and, respectively, the variable τ in the vector-function $\bar{X}(\tau, x)$, i.e., on the assumption that $\tau = \tau_0 = const$. Hence, $\Pi_1(\varepsilon) \to 0$ as $\varepsilon \to 0$ if, with respect to this variable, the vector-functions $X(\frac{\tau}{\varepsilon}, \tau, x, 0)$ and $X(\tau, x)$ are the step functions in the interval $[t_1, t_2]$. The continuity of the vector-functions $X(\frac{\tau}{\varepsilon}, \tau, x, 0)$ and $\bar{X}(\tau, x)$ in the variable τ uniformly with respect to the other variables implies the limit equality

$$\lim_{\varepsilon \to 0} \Pi_1(\varepsilon) = 0.$$

Theorem 13.7 generalizes the result established in the work of Sethna [1970].

Remark 13.8. Theorem 13.7 implies the assertion 3) of Theorem 9.7. Indeed, let $x_0(t, \varepsilon)$ be an almost periodic solution of the system

$$\frac{dx}{dt} = \varepsilon X(t, \tau, x)$$

and be close to the periodic solution $y_0(t, \varepsilon)$ of the averaged system

$$\frac{dy}{dt} = \varepsilon Y(\tau, y).$$

The periodic solution $y_0(t,\varepsilon)$ is asymptotically stable and, therefore, uniformly asymptotically stable. The solutions of the averaged system that lie in the domain of attraction of the periodic solution $y_0(t,\varepsilon)$ are also uniformly asymptotically stable. Thus, solutions of the original system with the initial conditions close enough to the initial condition of the solution of the averaged system lying in the domain of attraction of the solution $y_0(t,\varepsilon)$ will be close on an infinite interval.

Also, we can generalize Theorem 13.6 for the case (13.18).

Theorem 13.8. *Let*
1) $X(t,\tau,x,\varepsilon)$, $X_{x_i}(t,\tau,x,\varepsilon)$ be defined on G_ε, continuous in all variables and bounded;
2) in each of variables τ,x,ε the functions $X(t,\tau,x,\varepsilon)$, $X_{x_i}(t,\tau,x,\varepsilon)$ be continuous in each of the variables uniformly with respect to the other variables;
3) there exist the limit uniformly with respect to t

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X(t,\tau,x,0)dt = \bar{X}(\tau,x)$$

in the studied domain of variables;
4) the system

$$\frac{dx}{dt} = \bar{X}(\tau,x)$$

have the uniformly asymptotically stable solution $\psi(t,t_0,\xi_0)$ (uniformly asymptotically stable with respect to a part of the variables $x_1,\dots,x_k$, $k<n$) that which together with its ρ-neighborhood belongs to the set G.

Then for any $\alpha\,(0<\alpha<\rho)$ there exist $\varepsilon_1(\alpha)\,(0<\varepsilon_1<\varepsilon_0)$ and $\beta(\alpha)$ such that for all $0<\varepsilon<\varepsilon_1$ the solution $\varphi(t,t_0,x_0)\in B_x(K)$ of system (13.18), for which the $|x_0-\xi_0|<\beta(\alpha)\,(|x_{0i}-\xi_{0i}|<\beta(\alpha)$, $i=1,\dots,k)$, obeys the inequality

$$|\psi(t,t_0,\xi_0)-\varphi(t,t_0,x_0)|<\alpha,\quad t\ge t_0$$
$$(|\psi_i(t,t_0,\xi_0)-\varphi_i(t,t_0,x_0)|<\alpha,\quad i=1,\dots,k<n,\quad t\ge t_0).$$

Proof of Theorem 13.8 is similar to the proof of Theorem 13.7.

13.6 Closeness of Slow Variables on an Infinite Interval in Systems with a Rapidly Rotating Phase

The presented method allows study of the problem on closeness of the exact and averaged equations on an infinite interval for equations with the so-called rapidly rotating phase. Such equations will be investigated in detail in the following chapters.

Consider a system of differential equations

$$\begin{aligned} \tfrac{dx}{dt} &= \varepsilon X(x, y, \varepsilon), \\ \tfrac{dy}{dt} &= \omega(x) + \varepsilon Y(x, y, \varepsilon), \end{aligned} \tag{13.19}$$

where x is the n-dimensional vector, y is a scalar variable, $\varepsilon > 0$ is a small parameter. The system contains slow variables $x_1, x_2, \ldots, x_n$ and a fast variable y.

We assume that the functions $X(x, y, \varepsilon)$ and $Y(x, y, \varepsilon)$ are periodic in the fast variable y with the period 2π. We bring in a theorem that ensues from Theorem 13.5.

Theorem 13.9. *Let*
1) the functions $X(x, y, \varepsilon)$, $Y(x, y, \varepsilon)$ be defined for $x \in B_x(K)$, $y \in (-\infty, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, continuous in the variables x, ε uniformly with respect to y and have no more than a finite number of simple discontinuities (jumps) in y on every finite interval;
2) the function $\omega(x)$ meet the Lipschitz condition with some variable L for $x \in B_x(K)$ and the inequality

$$\omega(x) > c > 0,$$

hold, where c is a constant;
3) there exist a constant M such that

$$|X(x, y, \varepsilon)| \leq M, \quad |Y(x, y, \varepsilon)| \leq M, \quad x \in B_x(K),\ y \in (-\infty, \infty),\ \varepsilon \in [0, \varepsilon_0];$$

4) the vector-function

$$\bar{X}(x) = \frac{1}{2\pi} \int_0^{2\pi} X(x, y, 0)dy$$

meet the Lipschitz condition with some constant L_1;
5) the system

$$\frac{dx}{d\tau} = \bar{X}(x) \tag{13.20}$$

have the uniformly asymptotically stable solution $\psi(\tau, \tau_0, \xi_0)$ that together with its ρ-neighborhood ($\rho > 0$) belongs to the domain $B_x(K)$.

Then for the slow variables x of the solution of system (13.19) the assertion of Theorem 13.5 holds true; i.e. for any α ($0 < \alpha < \rho$) there exist $\varepsilon_1(\alpha)$ ($0 < \varepsilon_1 < \varepsilon_0$) and $\beta(\alpha)$ such that for all $0 < \varepsilon < \varepsilon_1$ the solution $x(t, t_0, x_0) \in B_x(K)$ of system (13.19), for which the $|x_0 - \xi_0| < \beta(\alpha)$, obeys the inequality

$$|\psi(t, t_0, \xi_0) - x(t, t_0, x_0)| < \alpha, \quad t \geq t_0.$$

Proof. We make a change $\tau = \varepsilon t$, $\alpha = \varepsilon y$ and transform system (13.19) into the system

$$\begin{aligned} \frac{dx}{d\tau} &= X(x, \tfrac{\alpha}{\varepsilon}, \varepsilon), \\ \frac{d\alpha}{d\tau} &= \omega(x) + \varepsilon Y(x, \tfrac{\alpha}{\varepsilon}, \varepsilon) \end{aligned} \tag{13.21}$$

Conditions 2) and 3) of the theorem imply that, for sufficiently small ε, the function $\alpha(\tau)$ is monotonic and, therefore, α can be taken as an independent variable instead of τ. System (13.21) is rewritten as

$$\begin{aligned} \frac{dx}{d\alpha} &= \frac{1}{\omega(x)} X(x, \tfrac{\alpha}{\varepsilon}, \varepsilon) + \varepsilon X_1(x, \alpha, \varepsilon), \\ \frac{d\tau}{d\alpha} &= \frac{1}{\omega(x)} + \varepsilon Y_1(x, \alpha, \varepsilon), \end{aligned} \tag{13.22}$$

where $X_1(x, \alpha, \varepsilon), Y_1(x, \alpha, \varepsilon)$ have the same properties as $X(x, \frac{\alpha}{\varepsilon}, \varepsilon)$, $Y(x, \frac{\alpha}{\varepsilon}, \varepsilon)$, respectively. It is easy to see that the right-hand sides of system (13.22), for $\varepsilon \to 0$, integrally converge to the right-hand sides of the system

$$\begin{aligned} \frac{dx}{d\alpha} &= \frac{1}{\omega(x)} \bar{X}(x) \\ \frac{d\tau}{d\alpha} &= \frac{1}{\omega(x)}, \end{aligned} \tag{13.23}$$

that at the initial time τ takes the form

$$\begin{aligned} \frac{dx}{d\tau} &= \bar{X}(x) \\ \frac{d\alpha}{d\tau} &= \omega(x). \end{aligned} \tag{13.24}$$

The solution of system (13.23) is in correspondence to the solution $\psi(\tau, \tau_0, \xi_0)$ of system (13.24), uniformly asymptotically stable in the variable x, and the respective solution of system (13.23) has the same property. Applying Theorem 13.5 to system (13.22) and taking into account the fact that transition from the time τ to the time α is in one-to-one to ways differentiable mapping of some neighborhood of the solution of system (13.23) into some neighborhood of the solution of system (13.24), for sufficiently small ε, yields the assertion of the theorem.

Now consider the following system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X(\tau, x, y, \varepsilon), \\ \frac{dy}{dt} &= \omega(x) + \varepsilon Y(\tau, x, y, \varepsilon), \end{aligned} \tag{13.25}$$

where x is the n-dimensional vector, y is a scalar variable, ε is a small parameter varying within the interval $(0, \varepsilon_0], \tau = \varepsilon t$ is the slow time. We assume that the functions $X(\tau, x, y, \varepsilon)$ and $Y(\tau, x, y, \varepsilon)$ are periodic in the variable y with the period 2π. Then, the following theorem is proved in exactly the same way as Theorem 13.9.

Theorem 13.10. *Let*
1) the vector-functions $X(\tau, x, y, \varepsilon)$, $Y(\tau, x, y, \varepsilon)$ *be defined for* $x \in B_x(K)$,

$y \in (-\infty, \infty)$, $\varepsilon \in (0, \varepsilon_0]$, $\tau \in I$, continuous in the variables τ, x, ε uniformly with respect to y, and have no more than a finite number of simple discontinuities (jumps) in y on each finite interval;
2) the function $\omega(x)$ meet the Lipschitz condition with some constant L for $x \in B_x(K)$ and the inequality

$$\omega(x) > c > 0,$$

holds, where c is a constant;
3) there exist a constant M such that

$$|X(\tau, x, y, \varepsilon)| \leq M, \quad |Y(\tau, x, y, \varepsilon)| \leq M, \quad x \in B_x(K),\ y \in (-\infty, \infty),$$
$$\varepsilon \in [0, \varepsilon_0],\ \tau \in I;$$

4) the vector-function

$$\bar{X}(\tau, x) = \frac{1}{2\pi} \int_0^{2\pi} X(\tau, x, y, 0)dy$$

meet the Lipschitz condition with some constant L_1 in the variable x and be continuous in τ uniformly with respect to x;
5) the system

$$\frac{dx}{d\tau} = \bar{X}(\tau, x)$$

have the uniformly asymptotically stable solution $\psi(\tau, \tau_0, \xi_0)$ that together with its ρ-neighborhood ($\rho > 0$) belongs to the domain $B_x(K)$.

Then for the slow variables x of the solution of system (13.25), the assertion of Theorem 13.9 holds true.

Chapter 14

Systems with a Rapidly Rotating Phase

When a system of ordinary differential equations involves a small parameter, the variables can be subdivided into fast and slow ones. The principle of averaging allows fast variables to be eliminated and equations to be written with only slow variables. The major part in all the problems related to the principle of averaging belongs to the changes of variables. The changes make it possible to eliminate fast variables from equations of motion within the given accuracy and thus separate slow motion from the fast one.

We shall consider equations called the equations with a rapidly rotating phase. Such equations arise, for example, when describing the motion of conservative systems with one degree of freedom and subject to small perturbations.

14.1 Near Conservative Systems with One Degree of Freedom

Consider a near conservative differential equation

$$\ddot{z} + f(z) = \varepsilon G(z, \dot{z}), \tag{14.1}$$

where ε is a small parameter. We shall assume that for an unperturbed equation

$$\ddot{z} + f(z) = 0, \tag{14.2}$$

the general solution (also called general integral) is known and can be written as

$$z = q_0(x, t + t_0), \tag{14.3}$$

where x and t_0 are arbitrary constants. Besides, we shall suppose that $f(0) = 0$ and consider a domain in the phase plane of the variables $z, \dot{z}$ such that all solutions of unperturbed equation (14.2) are the periodic functions of time. Hence, the function $q_0(x, t + t_0)$ will be the periodic function of t with the period T dependent upon x (in the general case). Now, instead of the variable

t, we introduce a new variable in order to obtain a periodic solution with a constant period. Let

$$y = \omega(x)(t + t_0).$$

Here, the factor $\omega(x)$ is selected so that the function

$$q(x, y) = q_0(x, t + t_0)$$

is periodic with the period 2π. This condition defines unambiguously the quantity $\omega(x)$ that is a normalizing factor. By analogy with the linear case, we take x to denote amplitude, $\omega(x)$ to denote frequency and the variable y to denote phase.

The function $q(x, y)$ identically satisfies equation (14.2), i.e.

$$\ddot{z} + f(z) = \omega^2(x)q_{yy}(x, y) + f(q(x, y)) \equiv 0. \tag{14.4}$$

Now, from equation (14.1), we go on to a system of differential equations using the method of variation of arbitrary constants. This transition to the system is similar to the transition to the system in standard form from the equation of second-order. In the theory of oscillations, this method is referred to as the Van der Pol method.

We introduce new variables I and y, where the variable I is the function of the variable x. We shall show this dependence later. Now, we make a change

$$z = q(I, y), \tag{14.5}$$

$$\dot{z} = \omega(x)q_y(I, y), \tag{14.6}$$

where I, y are new variables, differentiate (14.5) and equate the obtained expression with (14.6). Thus, the first relation derived is

$$q_I\dot{x} + q_y\dot{y} = \omega q_y. \tag{14.7}$$

This equation is the condition for consistency formulas of the change (14.5), (14.6). Substituting (14.6) in equation (14.1) yields the second relation

$$(\omega'(x)q_y + \omega(x)q_{Iy})\dot{I} + \omega(x)q_{yy}\dot{y} = -f(q) + \varepsilon G(q, \omega(x)q_y), \tag{14.8}$$

where $\omega'(x) = \frac{d\omega}{dx}$. It is convenient to write equation (14.8) as

$$-(\omega'(x)q_y + \omega(x)q_{Iy})\dot{I} - \omega(x)q_{yy}\dot{y} = f(q) - \varepsilon G(q, \omega(x)q_y). \tag{14.9}$$

System (14.7), (14.9) is the system of two differential equations in I, y. At the same time, with respect to the derivatives $\dot{I}$, $\dot{y}$, this system is the linear algebraic system of equations. The determinant of this system is

$$\Delta(I) = \begin{vmatrix} q_I & q_y \\ -(\omega'(x)q_y + \omega(x)q_{Iy}) & -\omega(x)q_{yy} \end{vmatrix}.$$

We select I based on the condition

$$\Delta(I) = 1.$$

Computing the determinant results in the equation

$$-\omega(x)q_I q_{yy} + (\omega'(x)q_y^2 + \omega(x)q_{Iy})q_y = 1. \tag{14.10}$$

The left-hand side of (14.10) is the periodic function of the variable y with the period 2π. We calculate the mean value of both sides of (14.10) and obtain

$$\frac{1}{2\pi}\int_0^{2\pi}[-\omega(x)q_I q_{yy} + \omega'(x)q_y^2 + \omega(x)q_{Iy}q_y]dy = 1 \tag{14.11}$$

We write an integral to be corresponding to the first term of the left-hand side

$$-\int_0^{2\pi}\omega(x)q_I q_{yy}dy.$$

Having integrated the latter integral by parts, we obtain

$$-\omega(x)q_I q_y\,\Big|_0^{2\pi} + \int_0^{2\pi}\omega(x)q_{Iy}q_y dy.$$

The non-integral term equals zero, since $q_I q_y$ - 2π is a periodic function. Equation (14.11) will be written as

$$\frac{1}{2\pi}\int_0^{2\pi}[2\omega(x)q_{Iy}q_y + \omega'(x)q_y^2]dy = 1. \tag{14.12}$$

The left-hand side of (14.12) can be represented as

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d}{dI}(\omega(x)q_y^2)dy = \frac{d}{dI}\left[\frac{1}{2\pi}\int_0^{2\pi}\omega(x)q_y^2 dy\right].$$

Therefore, (14.12) will hold true on the assumption

$$I = \frac{1}{2\pi}\int_0^{2\pi}\omega(x)q_y^2 dy. \tag{14.13}$$

The variable I is called the action variable.

Note that the action variable can be calculated from the formula

$$I = \frac{1}{2\pi} \int_0^{2\pi} \dot{z} dz. \tag{14.14}$$

We solve system (14.7) - (14.9) relative to $\dot{I}$, $\dot{y}$ using Cramer's rule. We calculate two determinants. For the first one, we obtain:

$$\Delta_1 = \begin{vmatrix} \omega q_y & q_y \\ f(q) - \varepsilon G & -\omega(x) q_{yy} \end{vmatrix} = -\omega^2 q_y q_{yy} - f(q) q_y + \varepsilon G q_y = \varepsilon G q_y,$$

since

$$f(q) = -\omega^2 q_{yy}.$$

For the second one, we obtain:

$$\Delta_2 = \begin{vmatrix} q_I & \omega q_y \\ -(\omega'(x) q_y + \omega(x) q_{Iy}) & f(q) - \varepsilon G \end{vmatrix} = q_I f(q) - \varepsilon G q_y + \omega\omega' q_y^2 + \omega^2 q_{Iy} q_y =$$

$$\omega[-\omega q_I q_{yy} + \omega' q_y^2 + \omega q_{Iy} q_y] - \varepsilon G q_I = \omega \Delta(I) - \varepsilon G q_I = \omega - \varepsilon G q_I.$$

Thus, we have the required system of equations

$$\begin{aligned} \tfrac{dI}{dt} &= \varepsilon G(q, \omega q_y) q_y, \\ \tfrac{dy}{dt} &= \omega(I) - \varepsilon G(q, \omega q_y) q_I, \end{aligned} \tag{14.15}$$

where $\omega(I) = \omega(x(I))$. The initial system in the neighborhood of the periodic solution is said to be written in the action-angle variables. System (14.15) is called a system with a fast phase, or a system with a rapidly rotating phase. The action variable is a slowly varying variable, since $\frac{dI}{dt} \approx \varepsilon$, and the angle (phase) variable changes rapidly, ($\frac{dy}{dt} \approx \omega(I) = \omega(x(I))$).

The periodic solution of unperturbed equation (14.2) in the action-angle variables takes the form

$$\begin{aligned} \tfrac{dI}{dt} &= 0, \\ \tfrac{dy}{dt} &= \omega(I). \end{aligned}$$

Hence, on the periodic orbit $I = I_0 = const$, $q = \omega(I_0)t$.

14.2 Action-Angle Variables for a Hamiltonian System with One Degree of Freedom

Unperturbed equation (14.2) can be written as a Hamiltonian system with one degree of freedom

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \tag{14.16}$$

with the Hamiltonian

$$H(q,p) = \frac{p^2}{2} - \int^{q} f(z)dz,$$

where $z = q$, $\dot{z} = p$.

The action-angle variables can be introduced for the Hamiltonian system with one degree of freedom. Our statement of this problem follows the book by Markeev [1999]. The phase space of this system is a plane of the variables q, p. There exist periodic solutions of two types. In motions of the first type, the functions $p(t)$ and $q(t)$ are periodic with the same period. Such motions are called oscillatory. In motions of the second type, $q(t)$ is not periodic but when it increases or decreases by some q_0, the configuration of the system does not change. Such motions are called rotary.

The action-angle variables are introduced in the following way. From equation $H(q,p) = h$ we find the function $p = p(q,h)$. Then we calculate the action variable as a function of h by the formula

$$I = \frac{1}{2\pi} \oint p(q,h)dq,$$

where the integral is taken over the complete cycle of q variation (a cycle of oscillations or rotations, depending on the type of motion defined by the equation $H(q,p) = h$). Transformation of the function $I = I(h)$ gives $h = h(I)$. The generating function that assigns a canonical change of the variables $q,p \to I,y$ has the form

$$V(q,I) = \int p(q,h(I))dq.$$

Implicitly, the change of $q,p \to I,y$ is given by the formulas

$$p = \frac{\partial V}{\partial q}, \quad y = \frac{\partial V}{\partial I}.$$

The new Hamiltonian function is

$$\mathcal{H} = h(I).$$

In terms of the action-angle variables, equations of motion take the form

$$\frac{dI}{dt} = 0, \quad \frac{dy}{dt} = \omega(I).$$

Note that when the variable q fulfills the complete cycle of variation, the variable of angle y grows by 2π. Denoting the increment of the angle variable during the complete cycle of q variation as Δy, we obtain

$$\Delta y = \oint \frac{\partial y}{\partial q}dq = \oint \frac{\partial^2 V}{\partial I q}dq = \frac{\partial}{\partial I}\oint \frac{\partial V}{\partial q}dq = \frac{\partial V}{\partial I}(2\pi I) = 2\pi.$$

14.3 Autonomous Perturbations of a Hamiltonian System with One Degree of Freedom

Consider a perturbed Hamiltonian system

$$\begin{aligned} \tfrac{dx}{dt} &= \tfrac{\partial H}{\partial y} + \varepsilon f(x,y), \\ \tfrac{dy}{dt} &= -\tfrac{\partial H}{\partial x} + \varepsilon g(x,y), \end{aligned} \tag{14.17}$$

where the Hamiltonian function H and the functions $f(x,y)$, $g(x,y)$ are sufficiently smooth in the variables x, y in some domain $G \subset R^2$. We assume that the unperturbed system has a domain $D_0 \subset G$ filled with oscillatory or rotary motions.

In system (14.17), we pass from the variables x, y to the variables action-angle I, θ using the canonical transformation

$$x = U(I,\theta), \quad y = V(I,\theta),$$

where $U(I,\theta)$, $V(I,\theta)$ are the periodic functions in θ with the period 2π. After the transformation we obtain the system

$$\begin{aligned} \tfrac{dI}{dt} &= \varepsilon[f(U,V)\tfrac{\partial V}{\partial \theta} + g(U,V)\tfrac{\partial U}{\partial \theta}], \\ \tfrac{d\theta}{dt} &= \omega(I) + \varepsilon[f(U,V)\tfrac{\partial V}{\partial I} - g(U,V)\tfrac{\partial U}{\partial I}]. \end{aligned} \tag{14.18}$$

Let us introduce the notation

$$X(I,\theta) = [f(U,V)\frac{\partial V}{\partial \theta} + g(U,V)\frac{\partial U}{\partial \theta}], \quad Y(I,\theta) = [f(U,V)\frac{\partial V}{\partial I} - g(U,V)\frac{\partial U}{\partial I}].$$

Then system (14.18) will be written as

$$\frac{dI}{dt} = \varepsilon X(I,\theta), \quad \frac{d\theta}{dt} = \omega(I) + \varepsilon Y(I,\theta). \tag{14.19}$$

The right-hand sides of system (14.19) are the periodic functions of the phase θ. We shall assume that the functions $X(I,\theta)$, $Y(I,\theta)$ are bounded in magnitude in some bounded domain of the plane $\mathcal{R}^2$. Then, for sufficiently small ε, the sign of the right-hand side of the second equation in system (14.19) coincides with the sign of the function $\omega(I)$ that is non-zero as the frequency of the periodic solution of the unperturbed system. Hence, the variable θ, for sufficiently small ε, is a monotonic function of the variable t and can be taken as a new independent variable. Dividing the first equation of system (14.19) by the second one yields the first-order differential equation

$$\frac{dI}{d\theta} = \varepsilon \frac{X(I,\theta)}{\omega(I) + \varepsilon Y(I,\theta)}. \tag{14.20}$$

We average the right-hand side of (14.20) over θ and obtain the averaged equation of the first approximation

$$\frac{d\bar{I}}{dy} = \varepsilon \frac{\bar{X}(\bar{I})}{\omega(\bar{I})}, \tag{14.21}$$

where

$$\bar{X}(\bar{I}) = \frac{1}{2\pi} \int_0^{2\pi} X(I, \theta) d\theta. \tag{14.22}$$

If the algebraic equation

$$\bar{X}(I) = 0 \tag{14.23}$$

has the solution $I = I_0$, then this solution is a stationary solution of the averaged equation. Besides, if the inequality

$$\bar{X}_I(I_0) \neq 0 \tag{14.24}$$

holds (i.e. the root $I = I_0$ of equation (14.22) is simple), then, by virtue of Theorem 9.2, equation (14.21), for sufficiently small ε, has the periodic solution $I(\theta, \varepsilon)$ with the period 2π and $I(\theta, 0) = I_0$. It follows from Theorem 9.3 that the obtained periodic solution is asymptotically stable if

$$\bar{X}_I(I_0) < 0 \tag{14.25}$$

and unstable if

$$\bar{X}_I(I_0) > 0. \tag{14.26}$$

The solution $I(\theta, \varepsilon)$ corresponds to the limit cycle (isolated closed phase curve) $x(t, \varepsilon), y(t, \varepsilon)$ of system (14.17). Then we proceed with the reasoning as it is in Section 10.2. In order to obtain the solution $I(t, \varepsilon), \theta(t, \varepsilon)$ of system (14.19) from $I(\theta, \varepsilon)$, we need to know θ as the function of t. Therefore, we have to solve the equation

$$\dot{\theta} = \omega(I(\theta, \varepsilon)) + \varepsilon Y(I(\theta, \varepsilon), \theta). \tag{14.27}$$

The period of the function $I(t, \varepsilon)$ is also determined from this solution. It is necessary to find the solution $\theta(t, \varepsilon)$ of equation (14.27) with the initial condition $\theta(0, \varepsilon) = 0$ and choose $T = T(\varepsilon)$ such that $\theta(T, \varepsilon) = 2\pi$. Then $T(\varepsilon)$ is just the period in t of the function $\theta(t, \varepsilon)$ and the corresponding solution of system (14.17), since

$$\theta(t + T(\varepsilon)), \varepsilon) = \theta(t, \varepsilon) + 2\pi.$$

The limit cycle $x(t, \varepsilon), y(t, \varepsilon)$ is stable if inequality (14.25) holds and unstable if inequality (14.26) holds.

It is convenient to formulate this result as a theorem as applied to system (14.17).

Theorem 14.1. *Let a perturbed conservative system in terms of action-angle variables have the form (14.19). Let the function $X(\bar{I})$ be defined by formula (14.22) and equation (14.23) have the solution I_0 obeying inequality (14.24).*

Then system (14.17), for sufficiently small ε, in some neighborhood of $U(L_0)$ of the periodic or rotary solution L_0 of an unperturbed system has a unique limit cycle (a closed orbit) L_ε, and $L_\varepsilon \to L_0$ as $\varepsilon \to 0$. The limit cycle L_ε is stable if inequality (14.25) holds and unstable if inequality (14.26) holds.

Theorem 14.1 in different terms was obtained by Pontrjagin [1934] (see also Andronov, Leontovich, Gordon, and Maier [1971]). We bring in the respective formulation.

Theorem 14.2. *Let L_0 be a closed orbit of unperturbed Hamiltonian system (14.16) and $q = \varphi(t)$, $p = \psi(t)$ be the respective motion. Let τ be the period of the functions $\varphi(t)$ and $\psi(t)$. If*

$$\int_0^\tau [g(\varphi(s), \psi(s))\varphi'(s) - f(\varphi(s), \psi(s))\psi'(s)]ds = 0,$$

$$l = \int_0^\tau [g'_y(\varphi(s), \psi(s)) + f'_x(\varphi(s), \psi(s))]ds \neq 0,$$

then there exist the numbers $\mu > 0$ and $\delta > 0$ such that
a) for any ε, $|\varepsilon| < \delta$, system (14.17) in the neighborhood of L_0 has one and only one closed orbit L_ε, and $L_\varepsilon \to L_0$ as $\varepsilon \to 0$;
b) this orbit is stable if $\varepsilon l < 0$ and unstable if $\varepsilon l > 0$.

The phase space of system (14.19) appears to be a direct product of the interval $\Delta = (I_1, I_2) \in \mathcal{R}^1$ and the circle S^1. Assume that in this domain the system has no equilibria. Then, the following theorem holds.

Theorem 14.3. *Let the equation*

$$\bar{X}(I) = 0$$

on the interval Δ have only simple roots. Then, for sufficiently small ε, each of such roots corresponds to the limit cycle of system (14.17). The stable and unstable cycles alternate.

The detailed research into the autonomously perturbed Hamiltonian systems is presented in the book by Morozov [1998].

14.4 Action-Angle Variables for a Simple Pendulum

As an example, we find action-angle variables for the equation of a pendulum

$$\ddot{x} + \Omega^2 \sin x = 0, \tag{14.28}$$

where $\Omega^2 = \frac{g}{l}$ (g is the gravitational acceleration, l is the pendulum length). The solution of equation (14.28) is expressed by the elliptic Jacobian functions. Then we follow Appel [1909]). The required knowledge of the elliptic Jacobian functions is given in the reference guide by Gradstein and Ryzhik [1965]). We present some standard notation and formulas that will be of further use. By $k\,(0 \le k \le 1)$ we denote the modulus of elliptical function, $K(k)$, $E(k)$ are the complete elliptic integrals of the first and second kinds, respectively:

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$
$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\,d\varphi = \int_0^1 \frac{\sqrt{1-k^2}}{\sqrt{1-x^2}}dx,$$

$k' = \sqrt{1-k^2}$ is an additional modulus, $K'(k)$ is an additional elliptic first-kind integral defined by the formula $K'(k) = K(k')$. For three elliptic Jacobian functions, we shall use the standard notation $sn\ u$, $cn\ u$, $dn\ u$. Recall some known relations

$$sn^2\ u + cn^2\ u = 1, \quad k^2 sn^2\ u + dn^2\ u = 1, \quad \frac{d}{du} sn\ u = cn\ u dn\ u,$$

$$\frac{d}{du} cn\ u = -sn\ u dn\ u, \quad \frac{d}{du} dn\ u = -k^2 sn\ u dn\ u.$$

Return to equation (14.28). The integral of energy for equation (14.28) can be written as

$$\frac{\dot{x}^2}{4\Omega^2} - \cos x = h, \quad h = const,$$

or

$$\frac{\dot{x}^2}{4\Omega^2} + \sin^2\frac{x}{2} = h^* = \frac{\dot{x}^2(0)}{4\Omega^2} + \sin^2\frac{x(0)}{2}.$$

Depending on the value of h^*, i.e. choice of initial conditions, we obtain the oscillatory or rotary motions of a pendulum. Consider the case of the oscillatory motions first. This case corresponds to h^* that satisfies the inequality $0 < h^* < 1$. If $x(0) = 0$, then this inequality means that $\dot{x}^2(0) < 4\Omega^2$. Taking into account that the velocity of the pendulum motion $v = l\dot{x}$, we obtain the inequality $v^2(0) < 4l^2\Omega^2$, or $v(0) < 4lg$. Let $h^* = \sin^2\frac{\alpha}{2}$. Then α is the maximum angle of the off-vertical deviation of a pendulum: $|x| \le \alpha$. We have

$$\dot{x}^2 = 4\Omega^2(\sin^2\frac{\alpha}{2} - \sin^2\frac{x}{2}).$$

If the pendulum rises, then

$$\frac{dx}{2\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{x}{2}}}=\Omega dt.$$

Integrating yields

$$\int\limits_0^x \frac{du}{2\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{u}{2}}}=\Omega(t+t_0).$$

Hence, the general solution of equation (14.28) is

$$x(t)=2\arcsin ksn\ \Omega(t+t_0).$$

On the assumption that $x(0)=0$, the solution is

$$x(t)=2\arcsin ksn\ \Omega t. \tag{14.29}$$

Formula (14.29) presents an expression for the oscillatory motions of a pendulum, at that $k=\sin\frac{\alpha}{2}$. The period and frequency of oscillations of a pendulum are determined by the formulas

$$T=\frac{4K(k)}{\Omega},\qquad \omega=\frac{\pi\Omega}{2K(k)}.$$

In equation (14.28) we pass to the action-angle variables (I,θ) using the formulas

$$x=2\arcsin ksn\ \left[\frac{2K(k)}{\pi}\theta\right]=X_1(I,\theta), \tag{14.30}$$

$$\dot{x}=2k\Omega cn\ \left[\frac{2K(k)}{\pi}\theta\right]=Y_1(I,\theta), \tag{14.31}$$

where $\theta=\frac{\pi}{2K(k)}\Omega t$. The action variable is determined by the formula

$$I=\frac{1}{2\pi}\int\limits_0^{2\pi}\dot{x}dx=\frac{1}{2\pi}\int\limits_0^{2\pi}\sqrt{4\Omega^2h^*-4\Omega^2\sin^2\frac{x}{2}}dx=\frac{\Omega}{\pi}\int\limits_0^{2\pi}\sqrt{k^2-\sin^2\frac{x}{2}}dx.$$

During complete oscillation, x runs twice the interval $[-\alpha,\alpha]$, therefore

$$I=\frac{2\Omega}{\pi}\int\limits_{-\alpha}^{\alpha}\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{x}{2}}dx=\frac{4\Omega}{\pi}\int\limits_0^{\alpha}\sqrt{\sin^2\frac{\alpha}{2}-\sin^2\frac{x}{2}}dx.$$

Making a change $\sin\frac{x}{2}=u\sin\frac{\alpha}{2}$, we obtain

$$I=\frac{8\Omega k^2}{\pi}\int\limits_0^1\frac{\sqrt{1-u^2}}{\sqrt{1-k2u^2}}du.$$

The latter integral is readily represented as

$$I = \frac{8\Omega}{\pi}[E(k) - k'^2 K(k)].$$

Let us calculate the derivative I with respect to k. With allowance for the formulas of differentiation of the elliptic integrals with respect to modulus

$$\frac{dE}{dk} = \frac{E(k) - K(k)}{k}, \quad \frac{dK}{k} = \frac{E(k) - k'^2 K(k)}{k},$$

we obtain

$$\frac{dI}{dk} = \frac{8\Omega}{\pi} k K(k).$$

Since I is a monotonic function of k, then $I(k)$ has the inverse function $k(I)$ and

$$\frac{dk}{dI} = \frac{\pi}{8\Omega k K(k)}.$$

Manipulation over (14.30) and (14.31) yields the system

$$\frac{dI}{dt} = 0, \quad \frac{d\theta}{dt} = \frac{\pi\Omega}{2K(k)}.$$

Let us consider the rotary motions of a pendulum. We assume that $h^* > 1$ in the energy integral. If $x(0) = 0$, then the inequality $\dot{x}^2(0) > 4\Omega^2$ or $v(0) > 4lg$ holds. We write the energy integral as

$$\dot{x}^2 = 4\Omega^2 h^* - 4\Omega^2 \sin^2\frac{x}{2} = 4\Omega^2 h^* \left(1 - \frac{1}{h^*}\sin^2\frac{x}{2}\right)$$

and suppose $k^2 = 1/h^*$. We get

$$\left(\frac{dx}{dt}\right)^2 = 4\Omega^2 h^* (1 - k^2 \sin^2\frac{x}{2}).$$

Thus,

$$\frac{d\frac{x}{2}}{\sqrt{1 - k^2 \sin^2\frac{x}{2}}} = \frac{\Omega}{k} dt.$$

The general solution of equation (14.28) takes the form

$$x(t) = \pm 2 \arcsin sn\, \frac{\Omega}{k}(t + t_0),$$

or

$$x(t) = \pm 2 \arcsin sn\, \frac{\Omega}{k} t$$

if $x(0) = 0$. Evidently,

$$\dot{x}(t) = \pm 2\frac{\Omega}{k} dn\, \frac{\Omega}{k} t.$$

The plus sign stands for the anticlockwise rotation of a pendulum, while the minus sign denotes the clockwise rotation. The action variable is determined by the formula

$$I = \frac{1}{2\pi}\int_0^{2\pi} \dot{x}dx = \frac{\Omega}{k\pi}\int_0^{2\pi} \sqrt{1 - k^2 \sin^2 \frac{x}{2}}dx = \frac{4\Omega}{k\pi}E(k).$$

Then, we get

$$\frac{dI}{dk} = -\frac{4K(k)\Omega}{k^2\pi}.$$

Consequently, there exists the inverse function

$$\frac{dk}{dI} = -\frac{k^2\pi}{4K(k)\Omega}.$$

Changing over the action-angle variables (I, θ) by the formulas

$$x(t) = \pm 2 \arcsin sn\ (\frac{K(k)}{\pi}\theta) = X_2(I, \theta),$$

$$\dot{x}(t) = \frac{2\Omega}{k} dn\ (\frac{K(k)}{\pi}\theta) = Y_2(I, \theta),$$

we obtain the system

$$\frac{dI}{dt} = 0, \quad \frac{d\theta}{dt} = \frac{\pi\Omega}{kK(k)}.$$

14.5 Quasi-Conservative Vibro-Impact Oscillator

Here, we follow the work by Babitskii, Kovaleva, and Krupenin [1982] and the book by Babitskii and Krupenin [2001] (see also Babitskii [1998]).

Consider a linear oscillation system with an equation of motion

$$\ddot{x} + \Omega^2 x = 0$$

that describes harmonic oscillations of a unit mass body fixed on a spring with a rigidity Ω^2. The phase portrait of this system are the ellipses

$$H(x, \dot{x}) = \frac{1}{2}(\dot{x}^2 + \Omega^2 x^2) = E = const.$$

At the point $x = \Delta$, we arrange an immovable limiter and assume that once the coordinate x reaches the value Δ, an instant elastic impact occurs in the system so that if $x = \Delta$ at the time instant t_α, then the relation

$$\dot{x}(t_\alpha - 0) = -\dot{x}(t_\alpha + 0) \tag{14.32}$$

holds.

On the presence of a gap, when $\Delta > 0$, the ellipses, which conform to the linear system are "cut" by a vertical line $x = \Delta$ and their left-hand parts correspond to their true trajectories. If the energy level in the linear system is insufficient to attain the level $x = \Delta$, then linear oscillations with the frequency Ω take place. Under collisions, the oscillation frequency $\omega > \Omega$ and rises with energy but no greater than the value 2Ω so that

$$\Omega < \omega < 2\Omega, \quad \Delta > 0. \tag{14.33}$$

Under pull $\Delta < 0$, the oscillation frequency ω obeys the inequality

$$2\Omega < \omega < \infty, \quad \Delta < 0. \tag{14.34}$$

At $\Delta = 0$ we have an ellipse cut in half. Therefore, for all values of energy, the image point passes any phase trajectory for the same time with the doubled velocity 2Ω so

$$\omega = 2\Omega, \quad \Delta = 0. \tag{14.35}$$

Similarly, we study the case when there are two symmetric limiters in the system (see Babitskii and Krupenin [2001]).

Condition (14.33) suggests that variation of the impulse Φ_0 in the neighborhood of the impact instant t_α takes the form

$$J = \dot{x}_- - \dot{x}_+ = 2\dot{x}_-, \quad \dot{x}_- > 0,$$

where $\dot{x}_\alpha = \dot{x}(t_\alpha \mp 0)$.

The resultant force becomes local at $t = t_\alpha$. Hence

$$\Phi_0 \,|_{t=t_\alpha} = J\delta(t - t_\alpha) \tag{14.36}$$

and

$$\int\limits_{t_\alpha - 0}^{t_\alpha + 0} \Phi_0 dt = J.$$

Impacts occur periodically when $t_\alpha = t_0 + \alpha T$, where α is an integer, and T is the period between impacts calculated by the equality $T = 2\pi\omega^{-1}$ and (14.33)-(14.34). Thus, for $\infty < t < \infty$, we obtain a T-periodic continuation of (14.36)

$$\Phi_0 = J\delta_T(t - t_0),$$

where $\delta_T(t)$ - T is the periodic δ-function.

Solution of the equation

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = 0 \tag{14.37}$$

is understood as the T-periodic function $x(t)$ such that its substitution into this equation transforms it into a correct equality (from the viewpoint of the theory of distributions) of the form

$$\ddot{x} + \Omega^2 x + J\delta_T(t - t_0) = 0,$$

where t_0 is an arbitrary constant, and for all $\alpha = 0, \pm 1, \ldots$

$$x(t_0 + \alpha T) = \Delta, \quad J = 2\dot{x}_-(t_0 + \alpha T).$$

At the same time, the restrictions

$$x(t) \leq \Delta, \quad \dot{x}_- > 0 \tag{14.38}$$

are fulfilled, and the periods of oscillations, depending on the sign of Δ, fit the frequency ranges of (14.33)-(14.35).

To describe the solution analytically, we assume $t_0 = 0$. In this case, for $0 \leq t < T_0$, the solution of equation (14.37) has the form

$$\begin{aligned} &x(t) = -J\kappa[\omega_0(J)(t - t_0), \omega_0(J)], \\ &\kappa(t, \omega_0) = \tfrac{1}{2\Omega}\tfrac{\cos[\Omega(t - T_0/2)]}{\sin(\Omega T_0/2)}, \quad J(\omega_0) = -2\Omega\Delta\tan\tfrac{\Omega T_0}{2}, J \geq 0, \end{aligned}$$

and the third relation here determines the smooth dependence $\omega_0(J)$ at $\Delta \neq 0$, whereas $\omega_0 = 2\Omega$ at $\Delta = 0$.

The representation found should be continued with respect to periodicity. We arrive at

$$\kappa(t, \omega_0) = \frac{\omega_0}{2\pi\Omega^2} + \frac{\omega_0}{\pi}\sum_{k=1}^{\infty}\frac{\cos k\omega_0 t}{\Omega^2 - k^2\omega_0^2}.$$

Geometric conditions of an impact often result in frequency intervals (14.33)-(14.35). Note that when $\Delta = 0$, the solution $x(t)$ for $0 \leq t < \pi/\Omega$ takes the form

$$x(t) = -\frac{J}{2\Omega}\sin\Omega t,$$

where J is a frequency-independent arbitrary constant.

Now consider a perturbed vibro-impact oscillator

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = \varepsilon g(t, x, \dot{x}), \tag{14.39}$$

where ε is a small parameter. We call such oscillators quasi-conservative. We assume $\psi = \omega_0 t$ and transform equation (14.39) into the system in terms of the variables J, ψ (impulse-phase), by making a change

$$\begin{aligned} x &= -J\kappa[\psi, \omega_0 J], \\ \dot{x} &= -J\omega_0(J)\kappa_\psi[\psi, \omega_0 J], \end{aligned} \tag{14.40}$$

where

$$\kappa[\psi, \omega_0 J] = \omega_0^{-1}\left[\frac{1}{2\pi\Omega_0^2} + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\cos k\psi}{\Omega_0^2 - k^2}\right], \quad \Omega_0 = \Omega[\omega_0(J)]^{-1}.$$

Change (14.40) is not smooth; at $\psi = 2l\pi$, where l is an integer, the function κ_ψ has finite discontinuities, thus, in new variables, impacts occur when $\psi = 2l\pi$. Making the substitution (14.40), we arrive at the system (see Babitskii and Krupenin [2001])

$$\begin{aligned} \frac{dJ}{dt} &= -4\varepsilon\omega_0 g(t, -J\kappa, -J\omega_0\kappa_\psi)\kappa_\psi, \\ \frac{d\psi}{dt} &= \omega_0(J) - 4\varepsilon\omega_0 J^{-1} g(t, -J\kappa, -J\omega_)\kappa_\psi)(-J\kappa)_J. \end{aligned} \tag{14.41}$$

This is a system with a rapidly rotating phase, where the right-hand sides are periodic in ψ and have finite discontinuities (κ_ψ) at the points $\psi = 2l\pi$. The dependencies $\omega_0(J)$ have the form

$$\begin{aligned} &\omega_0(J) = \frac{\pi\Omega}{\pi - \arctan[J/(2\Omega\Delta)]}, \quad \Delta > 0, \quad \Omega < \omega_0 < 2\Omega, \\ &\omega_0(J) = -\frac{\pi\Omega}{\arctan[J/(2\Omega\Delta)]}, \quad \Delta < 0, \quad 2\Omega < \omega_0 < \infty, \\ &\omega_0 = 2\Omega = const, \quad \Delta = 0. \end{aligned}$$

Now let the perturbation be independent of t, i.e., be autonomous. Assume that

$$g(x, \dot{x}) = (\alpha - \beta x^2)\dot{x}.$$

The oscillatory system

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = \varepsilon(\alpha - \beta x^2)\dot{x}$$

is called autoresonance. Changing over to the impulse-phase variables and averaging the first equation over the fast variable yields

$$\bar{X}(J) = \frac{J}{2\sin^2(1/2\omega T)}\left[\alpha\left(1 - \frac{\sin\Omega T}{\Omega T}\right) - \frac{\beta J^2(1 - \sin 2\Omega T/2\Omega T)}{16\Omega^2\sin^2(1/2\omega T}\right].$$

Here, we should take into account that $J = -2\Omega\Delta\tan(1/2\Omega T)$ and $\pi/\Omega < T < 2\pi/\Omega$ for $\Delta > 0$. To find stationary solutions of the averaged equation, we obtain the transcendental equation

$$\bar{X}(J) = 0.$$

It follows from Theorem 14.1 that each simple root J_0 of this equation corresponds to the periodic solution of the autoresonance. Stability of this periodic solution is determined by the sign of the number $X_J(J_0)$.

14.6 Formal Scheme of Averaging for the Systems with a Rapidly Rotating Phase

Consider a system

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X(x, \psi, \varepsilon), \\ \frac{d\psi}{dt} &= \omega(x) + \varepsilon\Psi(x, \psi, \varepsilon). \end{aligned} \tag{14.42}$$

Here, x is the n-dimensional vector; ψ is a scalar, $\varepsilon > 0$ is a small parameter, the scalar function $\omega(x)$ in the range of variables x_i obeys the inequality

$$|\omega(x)| > c > 0.$$

Assume that the right-hand sides of system (14.42) are expandable in terms of the parameter ε

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X_1(x,\psi) + \varepsilon^2 X_2(x,\psi) + O(\varepsilon^3), \\ \frac{d\psi}{dt} &= \omega(x) + \varepsilon \Psi_1(x,\psi) + \varepsilon^2 \Psi_2(x,\psi) + O(\varepsilon^3), \end{aligned} \tag{14.43}$$

where $O(\varepsilon^3)$ are the terms of order ε^3 as $\varepsilon \to 0$. Then, we suppose that all functions $X_i(x,\psi)$, $\Psi_i(x,\psi)$ are periodic in ψ with the period 2π. In system (14.43), the variables x_i change slowly (their rate of change is proportional to the small parameter ε), and the phase ψ changes relatively fast, as, $\dot{\psi} \sim 1$. Let us describe the formal scheme of averaging of system (14.43). We shall analyze the solution of system (14.43) that meets the initial conditions $x(t_0) = x_0$, $\psi(t_0) = \psi_0$ on an asymptotically large period of time t of order $1/\varepsilon$. The slow variables x_i, within $\Delta t \sim 1/\varepsilon$, will acquire some limited increments, and the fast phase ψ may acquire a great increment during this time. The problem in averaging of system (14.43) consists in getting a simpler averaged system, where the slow variables x_i and the fast phase ψ will be separated. Besides, the fast phase ψ must be eliminated from the right-hand sides of the averaged system.

To obtain an averaged system as well as to separate the fast and slow variables, we make a change of the variables

$$\begin{aligned} x &= \xi + \varepsilon u_1(\xi,\eta) + \varepsilon^2 u_2(\xi,\eta) + O(\varepsilon^3), \\ \psi &= \eta + \varepsilon v_1(\xi,\eta) + \varepsilon^2 v_2(\xi,\eta) + O(\varepsilon^3). \end{aligned} \tag{14.44}$$

It is naturally assumed that at $\varepsilon = 0$ the variables x, ψ and the new variables ξ, η coincide, respectively. The variables ξ and η are to be separated in the averaged system, thus, we find the averaged system in the form

$$\begin{aligned} \frac{d\xi}{dt} &= \varepsilon \Sigma_1(\xi) + \varepsilon^2 \Sigma_2(\xi) + O(\varepsilon^3), \\ \frac{d\eta}{dt} &= \omega(\xi) + \varepsilon \Phi_1(\xi) + \varepsilon^2 \Phi_2(\xi) + O(\varepsilon^3), \end{aligned} \tag{14.45}$$

where $\Sigma_i(\xi)$, $\Phi_i(\xi)$ are to be defined.

Averaged system (14.45) is substantially simpler than original system (14.42), since in (14.45) the system of the slow variables of motion is integrated independently of the fast variable η, and after determination of the slow variables ξ_i, the fast variable η can be found by squaring.

We differentiate the formulas of the change of variables (14.44) in terms of system (14.45) and obtain

$$\begin{aligned}
&\tfrac{d\xi}{dt} + \varepsilon \tfrac{\partial u_1(\xi,\eta)}{\partial d\xi}\tfrac{d\xi}{dt} + \varepsilon \tfrac{\partial u_1(\xi,\eta)}{\partial \eta}\tfrac{d\eta}{dt} + \varepsilon^2 \tfrac{\partial u_2(\xi,\eta)}{\partial \xi}\tfrac{d\xi}{dt} + \varepsilon^2 \tfrac{\partial u_2(\xi,\eta)}{\partial \eta}\tfrac{d\eta}{dt} + O(\varepsilon^3) = \\
&\varepsilon X_1(\xi + \varepsilon u_1(\xi,\eta) + \varepsilon^2 u_2(\xi,\eta) + O(\varepsilon^3),\ \eta + \varepsilon v_1(\xi,\eta) + \\
&+\varepsilon^2 v_2(\xi,\eta) + O(\varepsilon^3)) + +\varepsilon^2 X_2(\xi + \varepsilon u_1(\xi,\eta) + \varepsilon^2 u_2(\xi,\eta) + O(\varepsilon^3),\ \eta + \\
&+\varepsilon v_1(\xi,\eta) + \varepsilon^2 v_2(\xi,\eta) + O(\varepsilon^3)) + O(\varepsilon^3), \\
&\tfrac{d\eta}{dt} + \varepsilon \tfrac{\partial v_1(\xi,\eta)}{\partial d\xi}\tfrac{d\xi}{dt} + \varepsilon \tfrac{\partial v_1(\xi,\eta)}{\partial \eta}\tfrac{d\eta}{dt} + \varepsilon^2 \tfrac{\partial v_2(\xi,\eta)}{\partial \xi}\tfrac{d\xi}{dt} + \varepsilon^2 \tfrac{\partial v_2(\xi,\eta)}{\partial \eta}\tfrac{d\eta}{dt} + O(\varepsilon^3) = \\
&\omega(\xi + \varepsilon u_1(\xi,\eta) + \varepsilon^2 u_2(\xi,\eta) + O(\varepsilon^3)) + \varepsilon \Psi_1(\xi + \varepsilon u_1(\xi,\eta) + \\
&+\varepsilon^2 u_2(\xi,\eta) + +O(\varepsilon^3)),\ \eta + \varepsilon v_1(\xi,\eta) + \\
&+\varepsilon^2 v_2(\xi,\eta) + O(\varepsilon^3)) + \varepsilon^2 \Psi_2(\xi + \varepsilon u_1(\xi,\eta) + \varepsilon^2 u_2(\xi,\eta) + O(\varepsilon^3)),\ \eta + \\
&+\varepsilon v_1(\xi,\eta) + \varepsilon^2 v_2(\xi,\eta) + O(\varepsilon^3)) + O(\varepsilon^3),
\end{aligned} \tag{14.46}$$

where the expressions of the form $\frac{\partial u_1(\xi,\eta)}{\partial \xi} = \{\frac{\partial u_1^i(\xi,\eta)}{\partial \xi_j}\}$ and the analogous ones (containing the first derivatives) should be understood as matrices. In (14.46), we replace the derivatives $\dot{\xi}$, $\dot{\eta}$ with the right-hand sides of averaged system (14.45) and expand all functions as power series of ε. Equating the coefficients at ε to the same power in the left-hand and right-hand sides of the derived equality results in a recurrent system of equations for determining the unknown functions $u_i(\xi,\eta)$, $v_i(\xi,\eta)$, $\Sigma_i(\xi)$, $\Phi_i(\xi)$, $i = 1, 2, \ldots$. We write the first two equations obtained by equating the coefficients at ε to the first power:

$$\Sigma_1(\xi) + \frac{\partial u_1(\xi,\eta)}{\partial \eta}\omega(\xi) = X_1(\xi,\eta), \tag{14.47}$$

$$\Phi_1(\xi) + \frac{\partial v_1(\xi,\eta)}{\partial \eta}\omega(\xi) = \frac{\partial \omega(\xi)}{d\xi}u_1(\xi,\eta) + \Psi_1(\xi,\eta). \tag{14.48}$$

Note that (14.47) is an equation in the n-dimensional space, and (14.48) is a scalar equation. We determine the vector-functions $\Sigma_1(\xi)$ and $u_1(\xi,\eta)$. Recall that the vector-function $X_1(\xi,\eta)$ is periodic in η with the period 2π. For the 2π-periodic function $f(\xi)$, Fourier series expansion is written as

$$f(\eta) \sim \sum_{n=-\infty}^{\infty} f_n e^{in\eta}.$$

Consider a differential equation

$$\frac{dx}{d\eta} = f(\eta).$$

Evidently, this equation has periodic solutions only when the mean value of the periodic function $f(\eta)$

$$f_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\eta)d\eta$$

equals zero; at that, the periodic solution is determined accurately up to an arbitrary constant and has the form

$$x(\eta) \sim \sum_{n\neq 0} \frac{f_n}{in} e^{in\eta} + const.$$

We turn to equation (14.47). Let us try to determine the vector-function $u_1(\xi, \eta)$ as a 2π-periodic vector-function of the variable η. We rewrite (14.47)

$$\frac{\partial u_1(\xi,\eta)}{\partial \eta}\omega(\xi) = X_1(\xi,\eta) - \Sigma_1(\xi). \tag{14.49}$$

On the assumption

$$\Sigma_1(\xi) = \frac{1}{2\pi}\int_0^{2\pi} X_1(\xi,\eta)d\eta, \tag{14.50}$$

the mean value of the periodic vector-function in the right-hand side of system (14.49) equals zero. Therefore, if equality (14.50) is valid, then the vector-function $u_1(\xi, \eta)$ is determined as the 2π-periodic vector-function of the variable η. Note that it is determined accurately up to an arbitrary vector-function of the variable ξ:

$$u_1(\xi,\eta) = \sum_{n\neq 0} \frac{f_n(\xi)}{in}\frac{e^{in\eta}}{\omega(\xi)} + u_{10}(\xi). \tag{14.51}$$

Thus, formula (14.50) unambiguously determines the vector-function $\Sigma_1(\xi)$, and formula (14.51) ambiguously determines the vector-function $u_1(\xi, \eta)$.

Now, recast (14.48) as

$$\frac{\partial v_1(\xi,\eta)}{\partial \eta}\omega(\xi) = \frac{\partial \omega(\xi)}{d\xi}u_1(\xi,\eta) + \Psi_1(\xi,\eta) - \Phi_1(\xi). \tag{14.52}$$

Equation (14.52) has the same form as (14.49). The right-hand side of the system is the known 2π-periodic function of the variable η, since the vector-function $u_1(\xi, \eta)$ is already determined. Therefore, on the assumption

$$\Phi_1(\xi) = \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{\partial \omega(\xi)}{d\xi}u_1(\xi,\eta) + \Psi_1(\xi,\eta)\right] d\eta,$$

the function $v_1(\xi, \eta)$ will be determined as the 2π-periodic function η. Note that the functions $\Phi_1(\xi)$ and $v_1(\xi, \eta)$ are determined ambiguously. To calculate $\Sigma_i(\xi)$, $\Phi_i(\xi)$, $u_i(\xi, \eta)$, $v_i(\xi, \eta), i \geq 2$ we obtain the equations similar to systems (14.49) and (14.52):

$$\frac{\partial u_i(\xi,\eta)}{\partial \eta}\omega(\xi) = F_i(\xi,\eta) - \Sigma_i(\xi), \quad \frac{\partial v_i(\xi,\eta)}{\partial \eta}\omega(\xi) = G_i(\xi,\eta) - \Phi_i(\xi),$$

where the vector-functions $F_i(\xi, \eta)$ and the functions $G_i(\xi, \eta)$ depend on the functions $\Sigma_k(\xi)$, $\Phi_k(\xi)$ and $u_k(\xi, \eta)$, $v_k(\xi, \eta)$, where $k = 1, 2, \ldots, i-1$. Thus, the vector-functions $\Sigma_i(\xi)$ and the functions $\Phi_i(\xi)$ are determined as the mean values of $F_i(\xi, \eta)$ and $G_i(\xi, \eta)$, respectively. Thereby, we sequentially determine the coefficients in the formula of the change of variables (14.44) and the coefficients of the averaged system at any accuracy with respect to the small parameter ε.

Note that in practice, it is normally possible to calculate only the coefficients at ε to the first and second power.

Consider averaged system (14.45). The system of a k-th approximation will be called the following system for the slow variables

$$\frac{d\xi_k}{dt} = \sum_{l=1}^{k} \varepsilon^l \Sigma_l(\xi_k), \tag{14.53}$$

which was obtained by casting out the terms of order ε^{k+1} and higher in the averaged system for the slow variables. Since we integrate system (14.53) on the time interval $t \sim 1/\varepsilon$, then, generally speaking, $(\xi - \xi_k) \sim \varepsilon^k$. Hence, the solution $\xi_k(t, \varepsilon)$ of system (14.53) gives the k-th approximation for the solution ξ of averaged system (14.45). Finding approximate values of the slow variables $\xi_k(t, \varepsilon)$ provides the possibility of determining approximately the fast variable η from the second equation of the averaged system. We arrived at

$$\eta = \eta_0 + \int_{t_0}^{t} [\omega(\xi_k) + \varepsilon\Phi_1(\xi_k) + \varepsilon^2\Phi_2(\xi_k) + \varepsilon^3 \ldots]dt. \tag{14.54}$$

Since $\xi - \xi_k \sim \varepsilon^k$, then $\omega(\xi) - \omega(\xi_k) \sim \varepsilon^k$. Integration is on the interval $t \sim 1/\varepsilon$. Therefore, the fast phase is determined from equation (14.54) with an error $\sim \varepsilon^{k-1}$, whereas the slow variables $\xi_k(t, \varepsilon)$ are determined with an error ε^k. The computation accuracy for the fast variable is an order lower than that for the slow variables. However, in particular cases, for example, $\omega = const$, the fast and slow variables are calculated at the same accuracy.

Thus, it is natural to regard an averaged system of the k-th approximation as the system comprising system (14.53) and the equation

$$\frac{d\eta_{k-1}}{dt} = \omega(\xi_k) + \sum_{l=1}^{k-1} \varepsilon^l \Phi_l(\xi_k). \tag{14.55}$$

In view of their importance, let us write the system of the first approximation

$$\frac{d\xi_1}{dt} = \varepsilon\Sigma_1(\xi_1) \tag{14.56}$$

and the system of the second approximation

$$\begin{aligned} \tfrac{d\xi_2}{dt} &= \varepsilon\Sigma_1(\xi_2) + \varepsilon^2\Sigma_2(\xi_2), \\ \tfrac{d\eta_1}{dt} &= \omega(\xi_2) + \varepsilon\Phi_1(\xi_2). \end{aligned} \tag{14.57}$$

Knowing the averaged values ξ_k and η_{k-1}, we can find the respective approximations for the original variables x_k and ψ_{k-1}

$$x_k = \xi_k + \sum_{l=1}^{k-1} \varepsilon^l u_l(\xi_k, \eta_{k-1}),$$
$$\psi_{k-1} = \eta_{k-1} + \sum_{l=1}^{k-2} \varepsilon^l v_l(\xi_k, \eta_{k-1}).$$

Keeping the required accuracy, these equations can be written as

$$x_k = \xi_k + \sum_{l=1}^{k-1} \varepsilon^l u_l(\xi_{k-l}, \eta_{k-l}),$$
$$\psi_{k-1} = \eta_{k-1} + \sum_{l=1}^{k-2} \varepsilon^l v_l(\xi_{k-l-1}, \eta_{k-l-1}).$$

Consider the problem on setting initial conditions for the solutions of the averaged equations. Assume that the initial conditions are expandable in power series of the parameter ε, i.e.

$$\begin{aligned} x(t_0, \varepsilon) &= x_0^0 + \varepsilon x_o^1 + \varepsilon^2 x_0^2 + O(\varepsilon^3), \\ \psi(t_0, \varepsilon) &= \psi_0^0 + \varepsilon \psi_0^1 + \varepsilon^2 \psi_0^2 + O(\varepsilon^3). \end{aligned} \tag{14.58}$$

Using expansions (14.58), it is possible to find the equations for determining initial values for the k-th approximation averaged system through substituting the formula of the change of variables (14.44) at $t = t_0$ into the left-hand sides of expression (14.56) and equating the coefficients at ε to the same power. In particular, equations of the first approximation should be solved at the initial conditions $\xi_1(t_0, \varepsilon) = x_0^0$.

Introducing the slow time $\tau = \varepsilon t$ transforms the equation of first approximation (14.56) into

$$\frac{d\xi_1}{d\tau} = \Sigma_1(\xi_1). \tag{14.59}$$

This equation should be solved on the interval $\Delta\tau \sim 1$. The parameter ε is thus eliminated from the first-approximation equations, and integration of (14.59) on the finite interval $\Delta\tau \sim 1$ is simpler than integration of equations (14.56) on the interval $\Delta t \sim 1/\varepsilon$. We can find the solution of the k-th approximation equations in the form

$$\xi_k = \xi_1 + \sum_{l=1}^{k-1} \varepsilon^l \delta\xi_l.$$

In this case, to determine the corrections $\delta\xi_l$,we obtain linear inhomogeneous systems of equations.

The formal scheme of averaging is more comprehensively presented in the book by Volosov and Morgunov [1971].

Chapter 15

Systems with a Fast Phase. Resonant Periodic Oscillations

Consider a system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon X(t, x, \psi, \varepsilon), \\ \frac{d\psi}{dt} &= \omega(x) + \varepsilon \Psi(t, x, \psi, \varepsilon), \end{aligned} \tag{15.1}$$

where x is a scalar variable, ψ is a rapidly rotating phase, $\varepsilon > 0$ is a small parameter, $t \in \mathcal{R}$.

We shall assume that the functions $X(t, x, \psi, \varepsilon)$ and $\Psi(t, x, \psi, \varepsilon)$ are periodic in the phase ψ with the period 2π and periodic in the variable t with the period $T = 2\pi/\nu$. System (15.1) contains one slow variable x and two fast variables ψ and t. The presence of the two fast variables substantially complicates the study of system (15.1) (see Morozov [1998]), since it is possible that the frequencies ω and ν are commensurable.

Definition of Resonance. We say that a resonance takes place in system (15.1) if

$$\omega(x) = \frac{q}{p}\nu, \tag{15.2}$$

where p, q are prime integers.

For system (15.1), the condition of resonance (15.2) entails commensurability of the natural vibration period τ (the period of unperturbed motion) and the period of perturbation:

$$\tau = \frac{p}{q}\frac{2\pi}{\nu}.$$

Condition (15.2) with the fixed p and q can be considered as an equation with respect to x. We denote its solution by x_{pq}. The solution $x = x_{pq}$ is called a resonance level. In this case, condition (15.2) singles out the resonance curves among the closed phase curves of an unperturbed system. We shall call the resonance level $x = x_{pq}$ non-degenerate if $\frac{d\omega}{dx}(x_{pq}) \neq 0$.

The behavior of the solutions of system (15.1) in the neighborhood of the resonance levels is described in Morozov [1998], where non-degenerate passable, partly passable and non-passable resonance levels, as well as degenerate resonance levels are under consideration. Also, the topology of resonance zones, existence of periodic solutions, nontrivial hyperbolic sets, and transition from the exact resonance to a non-resonance level are studied. The main emphasis is placed on the global behavior of the solutions.

In addition, we want to note the paper by Murdock [1975]. Here, the following differential equation was considered

$$\ddot{\theta} = \varepsilon f(t, \theta, \dot{\theta}),$$

where $\varepsilon << 1$ and $f(t, \theta, \dot{\theta})$ is periodic in t and θ with the period 2π. This equation can be written as the system

$$\dot{\theta} = \omega, \quad \dot{\omega} = \varepsilon f(t, \theta, \omega).$$

Classification of the numbers ω is presented with respect to various types of resonance levels, and the local and global behavior of the solutions.

In this book, we only investigate the problem of the existence and stability of periodic solutions in the neighborhood of a non-degenerate resonance level. Essential is the fact that analysis of the problem requires averaged second-approximation equations to be constructed.

15.1 Transformation of the Main System

To examine the qualitative behavior of the solutions of system (15.1) in the $\mu = \sqrt{\varepsilon}$-neighborhood of an individual resonance level
$x = x_{pq}$

$$U_\mu = \{(x, \varphi) : x_{pq} - c\mu < x < x_{pq} + c\mu, \quad 0 \leq \varphi < 2\pi, \quad c = const > 0 \ ,\}$$

transform equation (15.1) into a more convenient form. We make a change

$$\psi = \varphi + \frac{q}{p}\nu t$$

and arrive at the system

$$\begin{aligned} \tfrac{dx}{dt} &= \varepsilon X(t, x, \varphi + \tfrac{q}{p}\nu t, \varepsilon), \\ \tfrac{d\varphi}{dt} &= \omega(x) - \tfrac{q}{p}\nu + \varepsilon\Psi(t, x, \varphi + \tfrac{q}{p}\nu t, \varepsilon). \end{aligned}$$

Again make a change

$$x = x_{pq} + \mu z$$

and expand the right-hand side of the transformed system into a series in terms of the powers of the parameter μ:

$$\begin{aligned} \tfrac{dz}{dt} &= \mu X(t, x_{pq}, \varphi + \tfrac{q}{p}\nu t, 0) + \mu^2 X_x(t, x_{pq}, \varphi + \tfrac{q}{p}\nu t, 0)z + O(\mu^3), \\ \tfrac{d\varphi}{dt} &= \mu\omega_x(x_{pq})z + \tfrac{1}{2}\mu^2\omega_{xx}(x_{pq})z^2 + \mu^2\Psi(t, x_{pq}, \varphi + \tfrac{q}{p}\nu t, 0) + O(\mu^3). \end{aligned} \tag{15.3}$$

System (15.3) is the standard form of the system that contains only one fast variable t. The right-hand sides of system (15.3) are periodic in t with the least period $\frac{2\pi p}{\nu}$. We now make a standard transformation of the method of averaging in order to eliminate the fast variable t from the right-hand sides of system (15.3) with the accuracy up the terms of order μ^2. We shall find this transformation in the following form

$$z = \xi + \mu u_1(t,\eta) + \mu^2 u_2(t,\eta)\xi, \quad \varphi = \eta + \mu^2 v_2(t,\eta), \tag{15.4}$$

where $u_1(t,\eta)$, $u_2(t,\eta)$, $v_2(t,\eta)$ are defined as the periodic functions of the variable t with the period $\frac{2\pi p}{\nu}$ and a zero mean value from the equations

$$\begin{aligned} \tfrac{\partial u_1}{\partial t} &= X(t, x_{pq}, \eta + \tfrac{q}{p}\nu t, 0) - X_0(\eta), \\ \tfrac{\partial u_2}{\partial t} &= X_x(t, x_{pq}, \eta + \tfrac{q}{p}\nu t, 0) - \tfrac{\partial u_1}{\partial \eta}\omega_x(x_{pq}) - X_1(\eta), \\ \tfrac{\partial v_2}{\partial t} &= \Psi(t, x_{pq}, \eta + \tfrac{q}{p}\nu t, 0) - \tfrac{\partial u_1}{\partial \eta}\omega_x(x_{pq}) - \Psi_0(\eta). \end{aligned}$$

The functions $X_0(\eta)$, $X_1(\eta)$ and $\Psi_0(\eta)$ are determined from the formulas

$$X_0(\eta) = \frac{\nu}{2\pi p}\int_0^{\frac{2\pi p}{\nu}} X(t, x_{pq}, \eta + \frac{q}{p}\nu t, 0)dt, \tag{15.5}$$

$$X_1(\eta) = \frac{\nu}{2\pi p}\int_0^{\frac{2\pi p}{\nu}} X_x(t, x_{pq}, \eta + \frac{q}{p}\nu t, 0)dt,$$

$$\Psi_0(\eta) = \frac{\nu}{2\pi p}\int_0^{\frac{2\pi p}{\nu}} \Psi(t, x_{pq}, \eta + \frac{q}{p}\nu t, 0)dt.$$

Note that the functions $X_0(\eta)$, $X_1(\eta)$ and $\Psi_0(\eta)$ are periodic in η with the least period $\frac{2\pi}{p}$. For definiteness, we show that by the example of $X_0(\eta)$. Calculation of the integral in formula (15.5) reduces to calculation of the integrals in the following form

$$\alpha_{mk} = \int_0^{\frac{2\pi p}{\nu}} e^{im(\eta + \frac{q}{p}\nu t)} e^{ik\nu t} dt = e^{im\eta}\int_0^{\frac{2\pi p}{\nu}} e^{i(m\frac{q}{p}+k)\nu t} dt.$$

Evidently, $\alpha_{mk} = 0$ if $m\frac{q}{p} + k = 0$. Since m and k are integers, we see that the latter equality takes place only when $m = rp$, where r is an integer. Therefore, $e^{im\eta} = e^{irp\eta}$ and, consequently, the least period of the function $X_0(\eta)$ in η equals $\frac{2\pi}{p}$. The change (15.4) results in the system

$$\begin{aligned} \tfrac{d\xi}{dt} &= \mu X_0(\eta) + \mu^2 X_1(\eta)\xi + O(\mu^3), \\ \tfrac{d\eta}{dt} &= \mu\omega_x(x_{pq})\xi + \mu^2\Psi_0(\eta) + \tfrac{1}{2}\mu^2\omega_{xx}(x_{pq})\xi^2 + O(\mu^3). \end{aligned} \tag{15.6}$$

15.2 Behavior of Solutions in the Neighborhood of a Non-Degenerate Resonance Level

Assume that there exists a number η_0 $(0 < \eta_0 < \frac{2\pi}{p})$ such that

$$X_0(\eta_0) = 0, \tag{15.7}$$

and η_0 is a simple root of equation (15.7), i.e.,

$$X_{0\eta}(\eta_0) \neq 0.$$

In this case, the averaged system of the first approximation

$$\frac{d\xi}{dt} = \mu X_0(\eta), \quad \frac{d\eta}{dt} = \mu\omega_x(x_{pq})\xi \tag{15.8}$$

has the solution

$$\xi = 0, \quad \eta = \eta_0. \tag{15.9}$$

Linearizing the right-hand side of system (15.8) on solution (15.9) yields the matrix

$$A_0(\mu) = \begin{pmatrix} 0 & \mu X_{0\eta}(\eta_0) \\ \mu\omega_x(x_{pq}) & 0 \end{pmatrix}.$$

Given the satisfied inequality

$$X_{0\eta}(\eta_0) \cdot \omega_x(x_{pq}) > 0, \tag{15.10}$$

the matrix $A_0(\mu)$ has real eigenvalues of opposite signs. Then Theorem 9.4 has the following implications.

Theorem 15.1. *Let there exist a number η_0 such that it obeys equality (15.7) and inequality (15.10). Then in the $\sqrt{\varepsilon}$-neighborhood of a resonance point x_{pq}, for sufficiently small ε, there exists a unique unstable periodic solution of system (15.1) with the period $\frac{2\pi p}{\nu}$.*

We now assume that instead of inequality (15.10) the opposite inequality

$$X_{0\eta}(\eta_0) \cdot \omega_x(x_{pq}) < 0 \tag{15.11}$$

holds. In this case, the eigenvalues of the matrix $A_0(\mu)$ are purely imaginary. To investigate the problem of the existence and stability of the periodic solutions of system (15.1) in the neighborhood of the resonance point x_{pq}, we need to have the second-approximation equations in the following form

$$\begin{aligned} \tfrac{d\xi}{dt} &= \mu X_0(\eta) + \mu^2 X_1(\eta)\xi, \\ \tfrac{d\eta}{dt} &= \mu\omega_x(x_{pq})\xi + \mu^2\Psi_0(\eta) + \tfrac{1}{2}\mu^2\omega_{xx}(x_{pq})\xi^2. \end{aligned} \tag{15.12}$$

It follows from conditions (15.7), and (15.11) and the theorem of implicit function that, for sufficiently small μ, there exists a unique function $w(\mu) = (\xi(\mu), \eta_0)$ such that $\xi(0) = 0$ and $w(\mu)$ is the equilibrium state of system (15.12). The system linearized on this equilibrium will have the following matrix

$$A_1(\mu) = \begin{pmatrix} \mu^2 X_1(\eta_0) & \mu X_{0\eta}(\eta_0) \\ \mu\omega_x(x_{pq}) & \mu^2 \Psi_{0\eta}(\eta_0) \end{pmatrix}.$$

It is easy to see that the eigenvalues of the matrix $A_1(\mu)$, for sufficiently small μ, have negative real parts if the inequality

$$X_1(\eta_0) + \Psi_{0\eta}(\eta_0) < 0 \qquad (15.13)$$

holds, and positive real parts if the inequality

$$X_1(\eta_0) + \Psi_{0\eta}(\eta_0) > 0 \qquad (15.14)$$

holds. It follows from Theorem 12.1 that, for sufficiently small μ, there exists a unique periodic solution $w(t,\mu) = (\xi(t,\mu), \eta(t,\mu))$ of system (15.6) with the period $\frac{2\pi p}{\nu}$; this solution is asymptotically stable if inequality (15.13) holds and unstable if inequality (15.14) holds, at that, $||w(t,\mu) - w(\mu)|| = O(\mu^3)$. We formulate the obtained result as applied to system (15.1).

Theorem 15.2. *Let η_0 meet equality (15.7) and inequality (15.11). Let the inequality*

$$X_1(\eta_0) + \Psi_{0\eta}(\eta_0) \neq 0$$

hold true. Then, for sufficiently small ε, system (15.1) has a unique periodic solution with the period $\frac{2\pi p}{\nu}$ in the ε-neighborhood of the resonance point x_{pq}. This solution is asymptotically stable if inequality (15.13) holds and unstable if inequality (15.14) holds.

15.3 Forced Oscillations and Rotations of a Simple Pendulum

Forced resonance oscillations and rotations of a simple (mathematical) pendulum are described by the equation

$$\ddot{x} + \varepsilon\gamma\dot{x} + \Omega^2 \sin x = \varepsilon a \cos \nu t, \qquad (15.15)$$

where $\varepsilon > 0$ is a small parameter. Here, $\Omega^2 = g/l$ (see Section 14.4), $\gamma > 0$ is a damping coefficient, a, ν are real numbers. Recall (see Section 14.4) that the unperturbed equation of a pendulum

$$\ddot{x} + \Omega^2 \sin x = 0$$

has the solution

$$x(t) = 2\arcsin ksn\ \Omega t, \quad x(0) = 0$$

and in terms of the action-angle variables (I, θ) takes the form

$$\frac{dI}{dt} = 0, \quad \frac{d\theta}{dt} = \frac{\pi\Omega}{2K(k)}.$$

We now transform perturbed equation (15.15) into a system of equations using the action-angle variables

$$x = 2\arcsin ksn\ \left[\frac{2K(k)}{\pi}\theta\right] = X(I, \theta), \quad \dot{x} = 2k\Omega cn\left[\frac{2K(k)}{\pi}\theta\right] = Y(I, \theta),$$

where $\theta = \frac{\pi}{2K(k)}\Omega t$. We obtain the system

$$\begin{aligned} \tfrac{dI}{dt} &= \varepsilon\left\{a\sin\nu t - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}\theta\right]\right\}\tfrac{\partial X}{\partial\theta}, \\ \tfrac{d\theta}{dt} &= \tfrac{\pi\Omega}{2K(k)} - \varepsilon\left\{a\sin\nu t - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}\theta\right]\right\}\tfrac{\partial X}{\partial I}. \end{aligned} \tag{15.16}$$

We say that a resonance takes place in system (15.16) provided

$$\frac{\pi\Omega}{2K(k(I))} = \frac{r}{s}\nu, \tag{15.17}$$

where r, s are coprime integers. Denote the value of I at which equality (15.17) holds by I_{rs}. By making a change

$$\theta = \varphi + \frac{r}{s}\nu t,$$

we transform system (15.16) into

$$\begin{aligned} \tfrac{dI}{dt} &= \varepsilon\left\{a\sin\nu t - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}(\varphi + \tfrac{r}{s}\nu t)\right]\right\}X_\theta(I, \varphi + \tfrac{r}{s}\nu t), \\ \tfrac{d\varphi}{dt} &= \omega(I) - \varepsilon\left\{a\sin\nu t - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}(\varphi + \tfrac{r}{s}\nu t)\right]\right\}X_I(I, \varphi + \tfrac{r}{s}\nu t), \end{aligned} \tag{15.18}$$

where

$$\omega(I) = \frac{\pi\Omega}{2K(k)} - \frac{r}{s}\nu.$$

Then we make a change

$$I = I_{rs} + \mu z$$

and expand the right-hand side of the transformed system into a series in terms of the powers of the parameter μ:

$$\begin{aligned} \tfrac{dz}{dt} &= \mu\left\{a\sin\nu t - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}(\varphi + \tfrac{r}{s}\nu t)\right]\right\}[X_\theta(I_{rs}, \varphi + \tfrac{r}{s}\nu t) + \\ &+ \mu X_{\theta I}(I_{rs}, \varphi + \tfrac{r}{s}\nu t)z] + O(\mu^3), \\ \tfrac{d\varphi}{dt} &= \mu\omega_I(I_{rs})z + \tfrac{1}{2}\mu^2\omega_{II}(I_{rs})z^2 + \\ &+ \mu^2\left\{a\sin\nu t - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}(\varphi + \tfrac{r}{s}\nu t)\right]\right\}X_I(I_{rs}, \varphi + \tfrac{r}{s}\nu t) + O(\mu^3). \end{aligned} \tag{15.19}$$

The right-hand side of system (15.19) is periodic in t with the period $T = \frac{2\pi s}{r\nu}$. The averaged second-approximation system has the form

$$\begin{aligned} \tfrac{d\xi}{dt} &= \mu X_0(\eta) + \mu^2 X_1(\eta)\xi + O(\mu^3), \\ \tfrac{d\eta}{dt} &= \mu\omega_x(x_{pq})\xi + \mu^2\Psi_0(\eta) + \tfrac{1}{2}\mu^2\omega_{xx}(x_{pq})\xi^2 + O(\mu^3), \end{aligned} \tag{15.20}$$

where $X_0(\eta)$ is the mean value over t of the first summand of the first equation in the right-hand side of system (15.19), $X_1(\eta)\xi$ is the mean value over t of the second summand of the first equation in the right-hand side of system (15.19), $\Psi_0(\eta)$ is the mean value over t of the third summand of the second equation in the right-hand side of system (15.19).

Now, to investigate system (15.19) in the neighborhood of the resonance point I_{rs}, we make use of Theorems 15.1 and 15.2.

Calculate the derivative of the function $\omega(I)$ at the resonance point I_{rs}

$$d = \omega'(I_{rs}) = \frac{d}{dI}\left(\frac{\pi\Omega}{2K(k(I))}\right)|_{I=I_{rs}} = -\frac{\pi^2}{16k^2k'^2K^3(k)}\left[E(k) - k'^2K(k)\right].$$

Hence, $d < 0$, since $E(k) - k'^2K(k) > 0$ for $0 < k < 1$. Now calculate the mean value over t for the right-hand side of the first summand in the first equation of system (15.19). Expansion of the function $X(I,\theta)$ into the Fourier series for the case of oscillatory motion of a pendulum takes the form (see Gradstein and Ryzhik [1965])

$$X(I,\theta) = 8\sum_{n=0}^{\infty}\frac{a_n(q)}{2n+1}\sin(2n+1)\theta,$$

where

$$q = \exp\left(-\pi\frac{K'(k)}{K(k)}\right), \quad a_n(q) = \frac{q^{n+1/2}}{1+q^{2n+1}}.$$

The function $\frac{\partial X}{\partial\theta}$ is expanded into the Fourier series as follows

$$\frac{\partial X}{\partial\theta} = 8\sum_{n=0}^{\infty}a_n(q)\cos(2n+1)\theta = \frac{4kK(k)}{\pi}cn\left[\frac{2K(k)}{\pi}\theta\right],$$

whereas the function $\frac{\partial X}{\partial I}$ has the following expansion

$$\frac{\partial X}{\partial I} = \frac{\partial X}{\partial k}\frac{dk}{dI} = \frac{\pi^3}{4k^3k'^2K^3(k)\Omega}\sum_{n=0}^{\infty}\frac{q^{n+1/2}(1-q^{2n+1})}{(1+q^{2n+1})^2}\sin(2n+1)\theta.$$

Note that we have utilized the formula

$$\frac{d}{dk}\frac{K'(k)}{K(k)} = -\frac{\pi}{2k^2k'^2K^2(k)}.$$

Further on, we get

$$-2\gamma k\Omega cn\left[\frac{2K(k)}{\pi}\theta\right]X_\theta(I,\theta) = -\frac{\gamma}{\pi}8k^2K(k)\Omega cn^2\left[\frac{2K(k)}{\pi}\theta\right]$$

and

$$\frac{1}{2\pi}\int\limits_0^{2\pi} cn^2\left[\frac{2K(k)}{\pi}\theta\right]d\theta = \frac{1}{4K(k)}\int\limits_0^{4K(k)} cn^2 u du = \frac{1}{k^2K(k)}[E(k) - k'^2K(k)].$$

We calculate the mean value for the right-hand side of the first equation of system (15.19). It is apparent that this mean value can only be non-zero when $r = 1$, $s = 2n+1$ $(n = 0, 1, \dots)$. If $r = 1$, $s = 2n+1$, the mean value equals

$$f(\varphi) = -\frac{1}{2}aa_n(q)\sin[(2n+1)\varphi] - \gamma I_{rs}.$$

In calculating the mean value, we took into account the following formula

$$I = \frac{8\Omega k^2}{\pi}[E(k) - k'^2K(k)].$$

Hence, equation (15.7) takes the form

$$\sin[(2n+1)\varphi] = -\frac{2\gamma I_{rs}}{aa_n(q)} = A.$$

Thus we obtain $(4n+2)$ different values of the number φ_0 (η_0 in (15.7))

$$\varphi_{0l} = \frac{l}{2n+1}\pi - \frac{(-1)^l \arccos A}{2n+1}, \quad l = 1, \dots, 4n+2. \tag{15.21}$$

The solution of equation (15.21) exists if

$$|A| < 1. \tag{15.22}$$

At large n, inequality (15.22) is invalid, since

$$\lim_{n\to\infty} a_n(q) = 0.$$

Whether (15.22) holds also depends on the value γ. The derivative of $f(\varphi)$ at the point φ_{0l} equals

$$b = f_\varphi(\varphi_{0l}) = -(-1)^l\frac{2n+1}{2}a_n(q)a\sqrt{1-A^2}.$$

If l is even, then

$$bd = \omega_I(I_{rs})f_\varphi(\varphi_{0l}) > 0.$$

The conditions of Theorem 15.1 are satisfied. Therefore, in the $\sqrt{\varepsilon}$-neighborhood of the resonance point $I_{1,2n+1}$ in system (15.15), for sufficiently small ε, there exists an unstable solution periodic in t with the period $\frac{2\pi}{\nu}$. If l is odd, then

$$bd < 0.$$

In this case, we should consider the second approximation equations. Calculate the functions $X_1(\eta)$ and $\Psi_0(\eta)$. It is easy to reveal that the mean value is

$$\langle 2k\gamma cn\left[\frac{2K(k)}{\pi}(\varphi + \frac{1}{2n+1}\nu t)\right] X_I(I_{rs}, \varphi + \frac{1}{2n+1}\nu t)\rangle = 0.$$

Therefore,

$$X_1(\eta) = \langle a \sin \nu t X_{\theta I}(I_{rs}, \eta + \frac{1}{2n+1}\nu t)\rangle - \gamma,$$

$$\Psi_0(\eta) = -\langle a \sin \nu t X_{I\theta}(I_{rs}, \eta + \frac{1}{2n+1}\nu t)\rangle.$$

Hence,

$$X_1(\varphi_{0l}) + \Psi_0(\varphi_{0l}) = -\gamma.$$

It is convenient to formulate the implications of Theorems 15.1 and 15.2 as a theorem.

Theorem 15.3. *Let $I_{1,2n+1}$ be a resonance point, i.e.,*

$$\frac{\pi}{2K(k(I_{1,2n+1}))} = \frac{1}{2n+1}.$$

Let inequality (15.22) hold. Then, for sufficiently small ε, equation (15.15) has $(2n+1)$ unstable resonance periodic solutions in the $\sqrt{\varepsilon}$-neighborhood of the resonance point and $(2n+1)$ asymptotically stable resonance periodic solutions in the ε-neighborhood of the resonance point. For $\varepsilon = 0$, these periodic solutions become periodic solutions of the unperturbed equation.

We also note that in view of the inequality

$$\frac{\pi\Omega}{2K(k)} < \Omega \quad (K(k) > \frac{\pi}{2},\, k > 0),$$

not all resonance points exist. For example, the main resonance point $I_{1,1}$ exists if $\nu < \Omega$ and does not exist if $\nu > \Omega$.

Consider the case of rotary motions of an unperturbed pendulum. We choose the same perturbation

$$f(t) = a \sin \nu t$$

and, using the action-angle variables, obtain the system of equations

$$\begin{aligned} \tfrac{dI}{dt} &= \varepsilon[f(t) - \gamma Y(I,\theta)]X_\theta(I,\theta), \\ \tfrac{d\theta}{dt} &= \tfrac{\pi\Omega}{kK(k)} - \varepsilon[f(t) - \gamma Y(I,\theta)]X_I(I,\theta), \end{aligned}$$

where

$$x = X(I,\theta) = 2\arcsin sn\left[\frac{K(k)}{\pi}\theta\right], \quad \dot{x} = Y(I,\theta) = \frac{2\Omega}{k}dn\left[\frac{K(k)}{\pi}\theta\right].$$

The resonance points are determined from the equation

$$\frac{\pi\Omega}{kK(k)} = \frac{r}{s}\nu.$$

After the change $\theta = \varphi + \frac{r}{s}\nu t$, we obtain the system

$$\begin{aligned} \tfrac{dI}{dt} &= \varepsilon[f(t) - \gamma Y(I, \varphi + \tfrac{r}{s}\nu t)]X_\theta(I, \varphi + \tfrac{r}{s}\nu t), \\ \tfrac{d\varphi}{dt} &= \omega(I)\tfrac{\pi\Omega}{kK(k)} - \varepsilon[f(t) - \gamma Y(I, \varphi + \tfrac{r}{s}\nu t)]X_I(I, \varphi + \tfrac{r}{s}\nu t), \end{aligned} \tag{15.23}$$

where $\omega(I) = \frac{\pi\Omega}{kK(k)} - \frac{r}{s}\nu$. The derivative $\omega(I)$ at the resonance point equals

$$\omega'(I_{rs}) = \frac{\pi^2 E(k)}{4k'^2 K^3(k)} > 0.$$

For further calculations note that the following expansions into the Fourier series take place

$$X(I,\theta) = \theta + 4\sum_{l=1}^{\infty}\frac{q^l}{l(1+q^{2l}}\sin l\theta,$$

$$X_\theta(I,\theta) = 1 + 4\sum_{l=1}^{\infty}\frac{q^l}{(1+q^{2l}}\cos l\theta = \frac{2K(k)}{\pi}dn\left[\frac{K(k)}{\pi}\theta\right],$$

$$X_I(I,\theta) = -\frac{4\pi^3}{k'^2K^3(k)\Omega}\sum_{l=1}^{\infty}\frac{q^l(1-q^{2l})}{(1+q^{2l})^2}\sin l\theta.$$

The mean value of the first summand in the first equation of system (15.23) is non-zero only at $r = 1$, $s = n$. The equation for determining the number φ_0 takes the form

$$\sin n\varphi = -\frac{\gamma I_{1,n}}{2ab_n(q)} = B, \tag{15.24}$$

where $b_n(q) = \frac{q^n}{1+q^{2n}}$. For $|B| < 1$, equation (15.24) has $2n$ solutions

$$\varphi_{0l} = \frac{l\pi}{n} - \frac{(-1)^l \arcsin B}{n}, \quad l = 0, \dots, 2n-1.$$

The derivative $X_{0\eta}(\eta_0)$ (in notation of Theorem 15.1) equals

$$X_{0\eta}(\eta_0) = -(-1)^l 2anb_n(q)\sqrt{1-B^2}$$

and, therefore, the number $X_{0\eta}(\eta_0)$ is positive when l is odd and negative when l is even. The further calculations are similar to those we performed in the case of oscillatory motions of an unperturbed pendulum. We arrive at the following theorem.

Theorem 15.4. *Let $I_{1,n}$ be a resonance point, i.e.,*

$$\frac{\pi}{k(I_{1,n})K(k(I_{1,n}))} = \frac{1}{n}.$$

Let inequality (15.22) hold. Then, for sufficiently small ε, equation (15.15) has n unstable resonance periodic solutions in the $\sqrt{\varepsilon}$-neighborhood of the resonance point and n asymptotically stable resonance periodic solutions in the ε-neighborhood of the resonance point. At $\varepsilon = 0$, these periodic solutions become rotary motions of the unperturbed equation.

We introduce an additional summand (instance) in the right-hand side of the equation of a pendulum

$$\ddot{x} + \varepsilon\gamma\dot{x} + \Omega^2 \sin x = \varepsilon a \sin \nu t + \varepsilon M,$$

where M is a constant. During oscillatory motions of an unperturbed pendulum, the constant M has no influence on the existence of resonance periodic solutions. When an unperturbed pendulum rotates, then a new summand arises in calculating the number φ_0 . The number φ_0 is determined from the equation

$$\sin n\varphi = -\frac{\gamma I_{1,n} - M}{2ab_n(q)}. \tag{15.25}$$

It follows from equation (15.25) that when $M > 0$, the small M contributes to the rise of the resonance solutions. If M is large, then there are no resonant solutions. This feature of the rotary motions was pointed out by Chernousko [1963].

The forced oscillations and rotations of a pendulum were investigated from a different standpoint in the work of Markeev and Churkina [1985].

15.4 Resonance Oscillations in Systems with Impacts

The results we gained are applicable to a quasi-conservative system with impacts

$$x'' + \Omega^2 x + \Phi(x, x') = \varepsilon g(t, x, x'),$$

where $\Phi(x, x')$ is the operator of impact interaction, the function $g(t, x, x')$ is periodic in t with the period $\frac{2\pi}{\nu}$.

Introduce the impulse-phase variables J, ψ using the formulas

$$x = -J\chi(\psi), \quad x' = -J\omega_0(J)\chi_\psi(\psi), \tag{15.26}$$

where the functions $\chi(\psi)$, $\omega_0(J)$ are determined earlier in Section 14.3. After this change, we obtain the system

$$\begin{aligned}\tfrac{dJ}{dt} &= -4\varepsilon\omega_0(J)g(t,-J\chi(\psi),-J\omega_0(J)\chi_\psi)(\psi)\chi_\psi(\psi),\\ \tfrac{d\psi}{dt} &= \omega_0(J)-4\varepsilon\omega_0(J)J^{-1}g(t,-J\chi(\psi),-J\omega_0(J)\chi_\psi)(\psi)(-J\chi(\psi))_J\end{aligned}\tag{15.27}$$

that has the form of system (15.1). Therefore, we can use Theorems 15.1 and 15.2 to investigate resonant solutions . The respective calculations were performed in the work of Burd and Krupenin [1999].

As an example, consider an equation

$$x''+\Omega^2x+\Phi(x,x')=\varepsilon[a\sin(\nu t+\delta)-\gamma x'],\tag{15.28}$$

where $\gamma>0$, Ω, a,ν, δ are real constants. Transformation of (15.26) results in the system

$$\begin{aligned}\tfrac{dJ}{dt} &= -4\varepsilon\omega_0(J)[a\sin(\nu t+\delta)+\gamma J\omega_0(J)\chi_\psi(\psi)]\chi_\psi(\psi),\\ \tfrac{d\psi}{dt} &= \omega_0(J)-4\varepsilon\omega_0(J)J^{-1}[a\sin(\nu t+\delta)+\gamma J\omega_0(J)\chi_\psi(\psi)](-J\chi(\psi))_J.\end{aligned}\tag{15.29}$$

Let J_{pq} be the solution of the equation

$$\omega_0(J_{pq})=\frac{q}{p}\nu.$$

By making a change $\psi=\varphi+\frac{q}{p}\nu t$, we transform system (15.26) into

$$\begin{aligned}\tfrac{dJ}{dt} &= -4\varepsilon\omega_0(J)[a\sin(\nu t+\delta)+\gamma J\omega_0(J)\chi_\psi(\varphi+\tfrac{q}{p}\nu t)]\chi_\psi(\varphi+\tfrac{q}{p}\nu t)+\tfrac{q}{p}\nu t),\\ \tfrac{d\varphi}{dt} &= \omega_0(J)-\tfrac{q}{p}\nu-4\varepsilon\omega_0(J)J^{-1}[a\sin(\nu t+\delta)+\\ &+\gamma J\omega_0(J)\chi_\psi(\varphi+\tfrac{q}{p}\nu t)](-J\chi(\varphi+\tfrac{q}{p}\nu t))_J.\end{aligned}\tag{15.30}$$

Again, make a change

$$J=J_{pq}+\mu z$$

and expand the right-hand side of the transformed system into a series power of μ. Then we find the averaged equations. To calculate $X_0(\eta)$, which is the first term of the first equation in the averaged system (15.6), we need to average the sum of two terms. The first term is

$$-4\omega_0(J_{pq})[a\sin(\nu t+\delta)]\chi_\psi(\varphi+\frac{q}{p}\nu t).\tag{15.31}$$

Since

$$\chi_\psi(\varphi+\frac{q}{p}\nu t)=-\omega_0(J_{pq})^{-1}\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{k\sin k(\varphi+\frac{q}{p}\nu t)}{\Omega_0^2-k^2},\quad \Omega_0=\Omega[\omega_0(J_{pq}]^{-1},$$

we have to average summands in the form

$$\sin(\nu t+\delta)\sin k(\varphi+\frac{q}{p}\nu t),\quad k=1,2,\dots.$$

It is easy to see that the mean value of function (15.31) will be non-zero if and only if $q = 1$, $p = n\,(n = 1, 2, \dots)$. For $q = 1$, $p = n$ it equals

$$\frac{2a\nu^2}{\pi n(\Omega^2 - \nu^2)} \cos(n\eta - \delta).$$

The mean value of the second summand

$$-4\gamma J_{pq}\omega_0^2(J_{pq})\chi_\psi(\varphi + \frac{q}{p}\nu t)\chi_\psi(\varphi + \frac{q}{p}\nu t)$$

equals

$$-\frac{\gamma J_{pq}}{2}\left(1 + \frac{4\Omega^2\Delta^2}{J_{pq}^2}\right).$$

In calculating the mean value, we used the following equality

$$\sin^{-2}\pi\Omega_0 = 1 + 4J^{-2}\Omega^2\Delta^2.$$

Consequently, the number η_0 is determined as the solution of the equation

$$\cos(n\eta - \delta) = \frac{\gamma J_{pq}\pi n}{4a\nu^2}(\Omega^2 - \nu^2)\left(1 + \frac{4\Omega^2\Delta^2}{J_{pq}^2}\right) = A_n. \tag{15.32}$$

Since $A_n \to \infty$ as $n \to \infty$, we see that equation (15.32) can have solutions only for a finite number of the values of n. If equation (15.32) has solutions for the given value n, then these solutions are determined by the following formulas

$$\eta_{0l} = \frac{\delta}{n} \pm \frac{\arccos A_n}{n} + \frac{2l\pi}{n}, \quad l = 0, \dots, n-1.$$

Calculating the derivative of the function $X_0(\eta)$ at the points η_{0l} yields

$$X_{0\eta}(\eta_{0l}) = \pm\frac{2a\nu^2}{\pi(\Omega^2 - \nu^2)}\sqrt{1 - A_n^2}, \tag{15.33}$$

and, therefore, (15.33) has a positive sign at n points and a negative sign at n points.

Simple computations show that $X_1(\eta_{0l}) + Y_{0\eta}(\eta_{0l}) < 0$. Theorems 15.1 and 15.2 have the following implications.

If the resonance point J_{n1} is the solution of the equation

$$\omega(J_{n1}) = \frac{\nu}{n},$$

then, for sufficiently small ε, equation (15.28) has n unstable resonance periodic solutions with the period $\frac{2\pi n}{\nu}$ in the $\sqrt{\varepsilon}$-neighborhood of the resonance point J_{n1} and n asymptotically stable resonance periodic solutions with the period $\frac{2\pi n}{\nu}$ in the ε-neighborhood of the resonance point J_{n1}.

Chapter 16

Systems with Slowly Varying Parameters

16.1 Problem Statement. Transformation of the Main System

The formalism of the method of averaging for investigating resonant solutions in systems with slowly varying coefficients was evolved by Mitropolskii [1964]. Periodic perturbations of two-dimensional systems with a rapidly rotating phase and slowly varying coefficients were considered by Morrison [1968]. He deduced equations of the second approximation and obtained the conditions for the closeness of solutions of the exact and averaged equations on a finite asymptotically large time interval.

In this chapter (see Burd [1996b, 1997]), we consider almost periodic perturbed two-dimensional systems with a rapidly rotating phase and slowly varying coefficients. We investigate the conditions of the existence and stability for stationary resonant almost periodic solutions.

Consider a system of differential equations

$$\frac{dx}{dt} = \varepsilon f(x, \varphi, \psi, \varepsilon), \quad \frac{d\varphi}{dt} = \omega(x, \tau) + \varepsilon g(x, \varphi, \psi, \tau, \varepsilon). \tag{16.1}$$

Here, $\varepsilon > 0$ is a small parameter, $\tau = \varepsilon t$ is a slow time, $x(t)$, $\varphi(t)$, $\psi(t)$ are scalar functions, and

$$\frac{d\psi}{dt} = \Omega(\tau).$$

We shall assume that the functions $f(x, \varphi, \psi, \varepsilon)$, $g(x, \varphi, \psi, \tau, \varepsilon)$ are sufficiently smooth in x, φ, ε, and the function $\omega(x, \tau)$ is sufficiently smooth in x. Besides, the functions

$$f(x, \varphi, \psi, \varepsilon), \quad g(x, \varphi, \psi, \tau, \varepsilon), \quad \omega(x, \tau)$$

are almost periodic in the variables ψ, τ uniformly with respect to the other variables. The function $\Omega(\tau)$ is the correct almost periodic function of τ and is separated from zero for all τ

$$\inf_{-\infty<\tau<\infty} |\Omega(\tau)| \neq 0.$$

(Recall that the almost periodic function $f(t)$ is called correct if the integral of this function is representable in the form

$$\int_0^t f(s)ds = \langle f\rangle t + f_1(t),$$

where $f_1(t)$ is the almost periodic function.)

System (16.1) is a system with two slow variables x, τ and two fast variables φ, ψ. We will investigate the case of resonance; there exists an almost periodic function $x_0(\tau)$ such that

$$\omega(x_0(\tau), \tau) \equiv 0.$$

Assume that the resonance is non-degenerate, i.e.,

$$\inf_{-\infty<\tau<\infty} |\omega_x(x_0(\tau), \tau)| \neq 0.$$

We shall study the behavior of solutions of system (16.1) in the $\mu = \sqrt{\varepsilon}$-neighborhood of the resonance point $x_0(\tau)$. We make a change

$$x = x_0(\tau) + \mu z$$

and expand the right-hand side of system (16.1) in terms of the powers of μ. As a result, we obtain the system

$$\begin{aligned} &\tfrac{dz}{dt} = \mu\left[f(x_0(\tau), \varphi, \psi, \tau, 0) - \tfrac{dx_0}{d\tau}\right] + \mu^2 f_x(x_0(\tau), \varphi, \psi, \tau, 0)z+ \\ &+\mu^3 F(z, \varphi, \psi, \tau, \mu), \\ &\tfrac{d\varphi}{dt} = \mu\omega_x(x_0(\tau), \tau)z + \tfrac{1}{2}\mu^2\omega_{xx}(x_0(\tau), \tau)z^2 + \mu^2 g(x_0(\tau), \varphi, \psi, \tau, 0)+ \\ &+\mu^3 G(z, \varphi, \psi, \tau, \mu). \end{aligned} \tag{16.2}$$

System (16.2) contains only one fast variable ψ. We now make the standard change of the method of averaging in order to eliminate the fast variable in the right-hand side of system (16.2) to the accuracy of the terms of order μ^2. We find the change in the form

$$z = \xi + \mu u_1(\eta, \psi, \tau) + \mu^2 u_2(\eta, \psi, \tau)\xi, \quad \varphi = \eta + \mu^2 v_2(\eta, \psi, \tau),$$

where the functions $u_i(\eta, \psi, \tau)$, $(i = 1, 2)$, $v_2(\eta, \psi, \tau)$ are almost periodic in ψ, τ. The change results in the system

$$\begin{aligned} &\tfrac{d\xi}{dt} = \mu\left[f_0(\eta, \tau) - \tfrac{dx_0}{d\tau}\right] + \mu^2 f_1(\eta, \tau)\xi + \mu^3 F_1(\xi, \eta, \psi, \tau, \mu), \\ &\tfrac{d\eta}{dt} = \mu\omega_x(x_0, \tau)\xi + \tfrac{1}{2}\mu^2\omega_{xx}(x_0, \tau)\xi^2 + \mu^2 g_0(\eta, \tau) + \mu^3 G_1(\xi, \eta, \psi, \tau, \mu), \end{aligned} \tag{16.3}$$

where f_0, f_1, g_0 are determined as the mean values over ψ:

$$\begin{aligned} &f_0(\eta, \tau) = \langle f(x_0(\tau), \varphi, \psi, \tau, 0)\rangle, \quad f_2(\eta, \tau) = \langle f_x(x_0(\tau), \varphi, \psi, \tau, 0)\rangle, \\ &g_0(\eta, \tau) = g(x_0(\tau), \varphi, \psi, \tau, 0)\rangle. \end{aligned}$$

The functions $u_i(\eta, \psi, \tau)$, $(i = 1, 2)$, $v_2(\eta, \psi, \tau)$ are determined as almost periodic solutions in ψ with the zero mean value from the equations

$$\begin{aligned}
\Omega(\tau)\tfrac{\partial u_1}{\partial \psi} &= f(x_0(\tau), \varphi, \psi, \tau, 0) - f_0(\eta, \tau),\\
\Omega(\tau)\tfrac{\partial u_2}{\partial \psi} &= f_x(x_0(\tau), \varphi, \psi, \tau, 0) - u_{1\eta}(\eta, \psi, \tau)\omega_x(x_0, \tau) - f_1(\eta, \tau),\\
\Omega(\tau)\tfrac{\partial v_2}{\partial \psi} &= g(x_0(\tau), \varphi, \psi, \tau, 0) - \omega_x(x_0, \tau)u_1(\eta, \psi, \tau) - g_0(\eta, \tau).
\end{aligned}$$

In so doing, it is assumed that the functions

$$f(x_0(\tau), \varphi, \psi, \tau, 0),\ f_x(x_0(\tau), \varphi, \psi, \tau, 0),\ g(x_0(\tau), \varphi, \psi, \tau, 0),\ u_1(\eta, \psi, \tau)$$

are the correct almost periodic functions of ψ. System (16.3) at the time τ is a singularly perturbed system of the following form

$$\begin{aligned}
\mu\tfrac{d\xi}{d\tau} &= \left[f_0(\eta, \tau) - \tfrac{dx_0}{d\tau}\right] + \mu f_1(\eta, \tau)\xi + \mu^2 F_1(\xi, \eta, \psi, \tau, \mu),\\
\mu\tfrac{d\eta}{dt} &= \omega_x(x_0, \tau)\xi + \tfrac{1}{2}\mu\omega_{xx}(x_0, \tau)\xi^2 + \mu g_0(\eta, \tau) + \mu^2 G_1(\xi, \eta, \psi, \tau, \mu).
\end{aligned} \tag{16.4}$$

16.2 Existence and Stability of Almost Periodic Solutions

Let there exist an almost periodic function $\eta_0(\tau)$ such that

$$f_0(\eta, \tau) = \frac{dx_0}{d\tau}. \tag{16.5}$$

In this case, the degenerate system derived from (16.4) at $\mu = 0$ has the solution

$$\xi = 0, \quad \eta = \eta_0(\tau). \tag{16.6}$$

Linearizing the right-hand side of system (16.4) at $\mu = 0$ on solution (16.6) yields the matrix

$$A_0(\tau) = \begin{pmatrix} 0 & f_{0\eta}(\eta_0, \tau) \\ \omega_x(x_0, \tau) & 0 \end{pmatrix}.$$

If

$$\omega_x(x_0, \tau) f_{0\eta}(\eta_0, \tau) > \sigma_0 > 0, \quad \tau \in (-\infty, \infty), \tag{16.7}$$

where σ_0 is a constant, then the eigenvalues of the matrix $A_0(\tau)$ are real and have different signs. Then it follows from the results of Chapter 8 that the operator

$$L(\mu)z = \frac{dz}{d\tau} - \frac{1}{\mu}A_0(\tau)z \quad (z = (\xi, \eta)),$$

for sufficiently small μ, is uniformly regular in the space B_2 of two-dimensional almost periodic functions. Hence, an inhomogeneous system

$$\mu\frac{dz}{d\tau} = A_0(\tau)z + f(\tau),$$

where $f(\tau) \in B_2$, has a unique almost periodic solution $z(\tau, \mu)$ determined by the formula

$$z(\tau, \mu) = L^{-1}(\mu) f = \frac{1}{\mu} \int_{-\infty}^{\infty} K(\tau, s, \mu) f(s) ds.$$

Here,

$$|K(\tau, s, \mu)| \leq M \exp\left(-\frac{\gamma}{\mu}|\tau - s|\right) \quad (-\infty < \tau, s < \infty), \tag{16.8}$$

and M, γ are positive constants.

We also note that the zero solution of the system

$$\mu \frac{dz}{d\tau} = A_0(\tau) z \tag{16.9}$$

is unstable, since the eigenvalues of the matrix $A_0(\tau)$ have opposite signs.

We transform system (16.4) using a change

$$u = \eta - \eta_0(\tau)$$

and write the obtained system in the vector form ($z = (\xi, u)$) with separating the matrix $A_0(\tau)$:

$$\mu \frac{dz}{d\tau} = A_0(\tau) z + H(z, \psi, \tau, \mu). \tag{16.10}$$

The components $H(z, \psi, \tau, \mu)$ have the form

$$f_0(u + \eta_0, \tau) - \tfrac{dx_0}{d\tau} - f_{0\eta}(\eta_0, \tau) u + \mu f_1(u + \eta_0, \tau)\xi + \mu^2 F_1(\xi, u + \eta_0, \psi, \tau, \mu),$$
$$-\mu \tfrac{d\eta_0}{d\tau} + \tfrac{1}{2}\mu \omega_{xx}(x_0, \tau)\xi^2 + \mu g_0(u + \eta_0, \tau) + \mu^2 G_1(\xi, u + \eta_0, \psi, \tau, \mu).$$

Evidently, the following inequality is valid

$$|H(0, \psi, \tau, \mu)| \leq p(\mu), \tag{16.11}$$

where $p(\mu) \to 0$ as $\mu \to 0$. Then, by virtue of the smoothness of $H(z, \psi, \tau, \mu)$ in z for $|z_1|$, $|z_2| \leq r$, the following inequality holds

$$|H(z_1, \psi, \tau, \mu) - H(z_2, \psi, \tau, \mu)| \leq p_1(r, \mu)|z_1 - z_2|, \tag{16.12}$$

where $p_1(r, \mu)$ 0 as $r \to 0$.

The problem of almost periodic solutions of system (16.10) is equivalent to the problem of solvability of the operator equation

$$z(\tau, \mu) = \frac{1}{\mu} \int_{-\infty}^{\infty} K(\tau, s, \mu) H(z, \psi, s, \mu) ds = \Pi(z, \mu) \tag{16.13}$$

in the space B_2. Let us show that the operator $\Pi(z,\mu)$ satisfies the conditions of the principle of contraction mappings in some ball $||z|| \le a_0(\mu)$ in the space B_2. According to inequalities (16.8) and (16.12)

$$||\Pi(z_1,\mu) - \Pi(z_2,\mu)|| \le \frac{2M}{\gamma} p_1(r,\mu)||z_1 - z_2||, \quad ||z_1||, ||z_2|| \le r.$$

Therefore, in particular, inequality (16.11) implies that

$$||\Pi(z,\mu)|| \le \frac{2M}{\gamma} p_1(r,\mu) + ||\Pi(0,\mu)|| \le \frac{2M}{\gamma} p_1(r,\mu) + p(\mu)\frac{2M}{\gamma}.$$

We now choose the numbers $a_0(\mu)$ and μ_1 so that for $0 < \mu < \mu_1$ the inequalities

$$\frac{2M}{\gamma} p_1(a_0,\mu) = q < 1, \quad ||\Pi(0,\mu)|| < (1-q)a_0$$

hold. Thus, the operator $\Pi(z,\mu)$ for $0 < \mu < \mu_1$ on the ball $||z|| \le a_0$ in the space B_2 satisfies the conditions of the principle of contraction mappings. Note that as $\mu \to 0$, it is possible to choose the sequence $a_0(\mu)$ so that $a_0(\mu) \to 0$.

Thus, the operator equation (16.13) in the ball $||z|| \le a_0(\mu)$ has a unique solution $z_*(\tau,\mu)$ such that as $\mu \to 0$ it tends to $(0,0)$ uniformly with respect to τ. Hence, system (16.10), for sufficiently small μ, has a unique almost periodic solution $z_*(\tau,\mu)$. In its turn, system (16.4), for sufficiently small μ, has a unique almost periodic solution such that as $\mu \to 0$ tends to $(0, \eta_0(\tau))$ uniformly with respect to τ. Therefore, system (16.3), for sufficiently small μ, has a unique almost periodic solution.

To investigate the stability of the almost periodic solution $z_*(\tau,\mu)$ of system (16.10), we make a change $z = z_*(\tau,\mu) + y(\tau,\mu)$ and obtain the system

$$\mu\frac{dy}{d\tau} = A_0(\tau)y + H_1(y,\psi,\tau,\mu), \tag{16.14}$$

where

$$H_1(y,\psi,\tau,\mu) = H(z_*(\tau,\mu) + y(\tau,\mu),\psi,\tau,\mu) - H(z_*(\tau,\mu),\psi,\tau,\mu).$$

The problem of the stability of the almost periodic solution $z_*(\tau,\mu)$ is reduced to the problem of the stability of the zero solution of system (16.14). Taking into account the exponential estimates on the solutions of system (16.9) (see Chapter 8), based on the theorem of the stability in first approximation, we obtain that, for sufficiently small μ, the zero solution of system (16.14) is unstable. Hence, the almost periodic solution $z_*(\tau,\mu)$ of system (16.10) is unstable.

We shall state this result as a theorem as applied to system (16.2).

Theorem 16.1. *Let $x_0(\tau)$ be an almost periodic function such that*

$$\omega(x_0(\tau),\tau) \equiv 0, \qquad \inf_{-\infty<\tau<\infty} |\omega_x(x_0(\tau),\tau)| \ne 0.$$

Let $f(x_0(\tau), \varphi, \psi, \tau, 0)$ be a correct almost periodic function ψ, and $\Omega(\tau)$ be a correct almost periodic function τ and

$$\inf_{-\infty<\tau<\infty} |\Omega(\tau)| \neq 0.$$

Let there exist an almost periodic function $\eta_0(\tau)$ such that

$$f_0(\eta(\tau), \tau) = \frac{dx_0}{d\tau}$$

and inequality (16.7) holds:

$$\omega_x(x_0, \tau) f_{0\eta}(\eta_0, \tau) > \sigma_0 > 0, \quad \tau \in (-\infty, \infty), \quad \sigma_0 = const.$$

In this case, system (16.2), for sufficiently small μ, has a unique unstable almost periodic solution.

Hence, given the satisfied conditions of Theorem 16.1 in the μ-neighborhood of the resonance point $x_0(\tau)$, there exists a unique unstable almost periodic solution.

Now let, instead of inequality (16.7), the contrary inequality hold

$$\omega_x(x_0, \tau) f_{0\eta}(\eta_0, \tau) < \sigma_1 < 0, \quad -\infty < \tau < \infty, \quad \sigma_1 = const. \qquad (16.15)$$

If inequality (16.15) holds, the eigenvalues of the matrix $A_0(\tau)$ for all τ are purely imaginary. Now we need to consider averaged equations of higher approximations (we used only the first approximation in satisfying inequality (16.7)). First, we write system (16.2) in more detail (with the accuracy up to the terms of order μ^3)

$$\begin{aligned}
&\tfrac{dz}{dt} = \mu\left[f(x_0(\tau), \varphi, \psi, \tau, 0) - \tfrac{dx_0}{d\tau}\right] + \mu^2 f_x(x_0(\tau), \varphi, \psi, \tau, 0) z + \\
&+\mu^3 f_\varepsilon(x_0(\tau), \varphi, \psi, \tau, 0) + \tfrac{1}{2}\mu^3 f_{xx}(x_0, \varphi, \psi, \tau, 0) z^2 + O(\mu^4), \\
&\tfrac{d\varphi}{dt} = \mu\omega_x(x_0(\tau), \tau) z + \tfrac{1}{2}\mu^2 \omega_{xx}(x_0(\tau), \tau) z^2 + \mu^2 g(x_0(\tau), \varphi, \psi, \tau, 0) + \\
&\mu^3 g_x(x_0(\tau), \varphi, \psi, \tau, 0) + \tfrac{1}{6}\mu^3 \omega_{xxx}(x_0(\tau), \tau) z^3 + O(\mu^4).
\end{aligned} \qquad (16.16).$$

Now, we consider a narrower neighborhood of the resonance point $x_0(\tau)$. Let

$$z = \mu y.$$

System (16.16) by a change

$$\varphi = \eta_0(\tau) + \mu\beta$$

is transformed into

$$\begin{aligned}
&\tfrac{dy}{dt} = f(x_0, \eta_0, \psi, \tau, 0) - \tfrac{dx_0}{d\tau} + \mu f_\varphi(x_0, \eta_0, \psi, \tau, 0)\beta + \\
&+\tfrac{1}{2}\mu^2 f_{\varphi\varphi}(x_0, \eta_0, \psi, \tau, 0)\beta^2 + +\mu^2 f_\varepsilon(x_0, \eta_0, \psi, \tau, 0) + \mu^2 f_x(x_0, \eta_0, \psi, \tau, 0) + \\
&+O(\mu^3), \\
&\tfrac{d\beta}{dt} = \mu\omega_x(x_0, \tau) + \mu\left[g(x_0, \eta_0, \psi, \tau, 0) - \tfrac{d\eta_0}{d\tau}\right] + \mu^2 f_\varphi(x_0, \eta_0, \psi, \tau, 0) + O(\mu^3).
\end{aligned} \qquad (16.17)$$

We make the change of the method of averaging in the following form

$$\begin{aligned} &y = \xi + \xi_0(\tau) + z_0(\psi,\tau) + \mu u_1(\psi,\tau)\eta + \mu^2 u_2(\psi,\tau,\xi,\eta),\\ &\beta = \eta + \mu v_1(\psi,\tau) + \mu^2 v_2(\psi,\tau)\eta, \end{aligned} \tag{16.18}$$

where the functions $\xi_0(\tau)$, $z_0(\psi,\tau)$ will be chosen later. Substituting (16.18) into (16.17) yields

$$\begin{aligned}
&\tfrac{d\xi}{dt} + \mu^2 \tfrac{d\xi_0}{d\tau} + \tfrac{\partial z_0}{\partial \psi}\Omega(\tau) + \mu^2 \tfrac{\partial z_0}{\partial \tau} + \mu \tfrac{\partial u_1}{\partial \psi}\Omega(\tau)\eta + \mu u_1(\psi,\tau)\tfrac{d\eta}{dt} + \\
&\mu^2 \tfrac{\partial u_2}{\partial \psi}\Omega(\tau) + \mu^2 \tfrac{\partial u_2}{\partial \xi}\tfrac{d\xi}{dt} + \mu^2 \tfrac{\partial u_2}{\partial \eta}\tfrac{d\eta}{dt} + O(\mu^3) = \\
&f(x_0,\eta_0,\psi,\tau,0) - \tfrac{dx_0}{d\tau} + \mu f_\varphi(x_0,\eta_0,\psi,\tau,0)\eta + \tfrac{1}{2}\mu^2 f_{\varphi\varphi}(x_0,\eta_0,\psi,\tau,0)\eta^2 + \\
&+\mu^2 f_\varphi(x_0,\eta_0,\psi,\tau,0)v_1(\psi,\tau) + \mu^2 f_x(x_0,\eta_0,\psi,\tau,0)(\xi + \xi_0(\tau) + z_0(\psi,\tau)) + \\
&+\mu^2 f_\varepsilon(x_0,\eta_0,\psi,\tau,0) + O(\mu^3),\\
&\tfrac{d\eta}{dt} + \mu \tfrac{\partial v_1}{\partial \psi}\Omega(\tau) + \mu^2 \tfrac{\partial v_2}{\partial \psi}\Omega(\tau)\eta + \mu v_2(\psi,\tau)\tfrac{d\eta}{dt} = \\
&\mu\omega_x(x_0,\tau)(\xi + \xi_0(\tau) + z_0(\psi,\tau)) + \mu g(x_0,\eta_0,\psi,\tau,0) + \\
&+\mu^2 g_\varphi(x_0,\eta_0,\psi,\tau,0)\eta - \mu \tfrac{d\eta_0}{d\tau} + \mu^2 \omega_x(x_0,\tau)u_1(\psi,\tau)\eta + O(\mu^3).
\end{aligned}$$

Thus, we obtain equations for determining the functions $z_0(\psi,\tau)$, $u_1(\psi,\tau)$, $v_1(\psi,\tau)$:

$$\begin{aligned}
&\tfrac{\partial z_0}{\partial \psi}\Omega(\tau) = f(x_0,\eta_0,\psi,\tau,0) - \tfrac{dx_0}{d\tau},\\
&\tfrac{\partial u_1}{\partial \psi}\Omega(\tau) = f_\varphi(x_0,\eta_0,\psi,\tau,0) - \langle f_\varphi(x_0,\eta_0,\psi,\tau,0)\rangle,\\
&\tfrac{\partial v_1}{\partial \psi}\Omega(\tau) = \omega_x(x_0,\tau)z_0(\psi,\tau) + g(x_0,\eta_0,\psi,\tau,0) + \omega_x(x_0,\tau)\xi_0 - \tfrac{d\eta_0}{d\tau}.
\end{aligned}$$

Here, $z_0(\psi,\tau)$ is defined as the almost periodic function ψ with the zero mean value ($\langle f(x_0,\eta_0,\psi,\tau,0)\rangle = \frac{dx_0}{d\tau}$). Recall that $f(x_0,\eta_0,\psi,\tau,0)$ is the correct almost periodic function ψ. The function $u_1(\psi,\tau)$ is also defined as the almost periodic function in ψ with the zero mean value ($f_\varphi(x_0,\eta_0,\psi,\tau,0)$ is the correct almost periodic function ψ). To calculate $v_1(\psi,\tau)$, we first define the almost periodic function $\xi_0(\tau)$ by the equality

$$\omega_x(x_0,\tau)\xi_0 - \frac{d\eta_0}{d\tau} = -\langle g(x_0,\eta_0,\psi,\tau,0)\rangle.$$

Then for $v_1(\psi,\tau)$ we obtain the equation

$$\frac{\partial v_1}{\partial \psi}\Omega(\tau) = \omega_x(x_0,\tau)z_0(\psi,\tau) + g(x_0,\eta_0,\psi,\tau,0) - \langle g(x_0,\eta_0,\psi,\tau,0)\rangle.$$

It is convenient to determine $v_1(\psi,\tau)$ with the formula

$$v_1(\psi,\tau) = v(\psi,\tau) + w(\tau),$$

where $v(\psi,\tau)$ is the almost periodic function ψ with the zero mean value, and $w(\tau)$ is the almost periodic function τ that we shall choose later. To find the functions $u_2(\xi,\eta,\psi,\tau)$, we obtain the equation

$$\begin{aligned}
&\tfrac{\partial v_2}{\partial \psi}\Omega(\tau) + \tfrac{d\xi_0}{d\tau} + \tfrac{\partial z_0}{\partial \psi} + u_1(\psi,\tau)\omega_x(x_0,\tau)\xi_0(\tau) = f_\varphi(x_0,\eta_0,\psi,\tau,0)v(\psi,\tau) + \\
&f_\varphi(x_0,\eta_0,\psi,\tau,0)w(\tau) + \tfrac{1}{2}f_{\varphi\varphi}(x_0,\eta_0,\psi,\tau,0)\eta^2 + \\
&+f_x(x_0,\eta_0,\psi,\tau,0)(\xi + \xi_0(\tau) + z_0(\psi)) + f_\varepsilon(x_0,\eta_0,\psi,\tau,0).
\end{aligned}$$

We select $w(\tau)$ from the equality

$$w(\tau)\langle f_\varphi(x_0,\eta_0,\psi,\tau,0)\rangle = \tfrac{d\xi_0}{d\tau} - \langle f_\varphi(x_0,\eta_0,\psi,\tau,0)v(\psi,\tau)\rangle - \\ -\langle f_x(x_0,\eta_0,\psi,\tau,0)\rangle(\xi_0 + z_0(\psi,\tau)) + \langle f_\varepsilon(x_0,\eta_0,\psi,\tau,0)\rangle.$$

This can be done, since, as follows from condition (16.15), $\langle f_\varphi(x_0,\eta_0,\psi,\tau,0)\rangle = f_{0\eta}(\eta_0,\tau) \neq 0$. If we represent $u_2(\xi,\eta,\psi,\tau)$ as

$$u_2(\xi,\eta,\psi,\tau) = p_2(\psi,\tau)\xi + q_2(\psi,\tau)\eta^2 + r_2(\psi,\tau),$$

then to determine the functions $p_2(\psi,\tau)$, $q_2(\psi,\tau)$, $r_2(\psi,\tau)$ we obtain the equations

$$\tfrac{\partial p_2}{\partial\psi}\Omega(\tau) = -\omega_x(x_0,\tau)v(\psi,\tau) + f_x(x_0,\eta_0,\psi,\tau,0) - \langle f_x(x_0,\eta_0,\psi,\tau,0)\rangle,$$
$$\tfrac{\partial q_2}{\partial\psi}\Omega(\tau) = \tfrac{1}{2}f_{\varphi\varphi}(x_0,\eta_0,\psi,\tau,0) - \tfrac{1}{2}\langle f_{\varphi\varphi}(x_0,\eta_0,\psi,\tau,0)\rangle,$$
$$\tfrac{\partial p_2}{\partial\psi}\Omega(\tau) = [\xi_0 + z_0(\tau,\tau)]f_x(x_0,\eta_0,\psi,\tau,0) - \tfrac{d\xi_0}{d\tau} - \tfrac{\partial z_0}{\partial\tau} + \\ +[v(\psi,\tau) + w(\tau)]f_\varphi(x_0,\eta_0,\psi,\tau,0) + f_\varepsilon(x_0,\eta_0,\psi,\tau,0).$$

By virtue of the choice of the function of $w(\tau)$, the mean value over ψ in the right-hand side of the latter equation equals zero. Therefore, $r_2(\psi,\tau)$ is determined as the almost periodic function ψ with the zero mean value (it is naturally assumed that the functions

$$f_x(x_0,\eta_0,\psi,\tau,0)z_0(\psi,\tau),\ f_\varphi(x_0,\eta_0,\psi,\tau,0)v(\psi,\tau),\ f_\varepsilon(x_0,\eta_0,\psi,\tau,0)$$

are the correct almost periodic functions of ψ). From the first and second equations, the functions $p_2(\psi,\tau)$, $q_2(\psi,\tau)$ are determined as the almost periodic functions of ψ with the zero mean value ($v(\psi,\tau)$, $f_{\varphi\varphi}(x_0,\eta_0,\psi,\tau,0)$ are the correct almost periodic functions of ψ). To find the functions of $v_2(\psi,\tau)$, we obtain the equation

$$\frac{\partial v_2}{\partial\psi}\Omega(\tau) = -\omega_x(x_0,\tau)v(\psi,\tau) + g_\varphi(x_0,\eta_0,\psi,\tau,0) - \langle g_\varphi(x_0,\eta_0,\ \tau,0)\rangle.$$

If $g_\varphi(x_0,\eta_0,\psi,\tau,0)$ is a correct almost periodic function ψ, then $v_2(\psi,\tau)$ is determined as an almost periodic function of ψ with the zero mean value. Therefore, all the functions involved in the change formula are found.

The change transforms the system into

$$\tfrac{d\xi}{dt} = \mu a(\tau)\eta + \mu^2[b(\tau)\xi + c(\tau)\eta^2] + \mu^3F_2(\xi,\eta,\psi,\tau,\mu),$$
$$\tfrac{d\eta}{dt} = \mu d(\tau)\xi + \mu^2 e(\tau)\eta + \mu^3G_2(\xi,\eta,\psi,\tau,\mu), \tag{16.19}$$

where

$$a(\tau) = \langle f_\varphi(x_0,\eta_0,\ \tau,0)\rangle, \quad b(\tau) = \langle f_x(x_0,\eta_0,\ \tau,0)\rangle,$$
$$c(\tau) = \tfrac{1}{2}\langle f_{\varphi\varphi}(x_0,\eta_0,\ \tau,0)\rangle, \quad d(\tau) = \omega_x(x_0,\tau), \quad e(\tau) = \langle f_\varphi(x_0,\eta_0,\ \tau,0)\rangle. \tag{16.20}$$

Condition (16.16) in new notation takes the form

$$a(\tau)d(\tau) < \sigma_1 < 0.$$

As was noted, it follows from this condition that the eigenvalues of the first-approximation matrix are purely imaginary for all τ. We reduce system (16.19) to "standard form", i.e., to the form where the first-approximation matrix is zero. Assume

$$\delta(\tau) = \left[-\frac{d(\tau)}{a(\tau)}\right]^{1/2}$$

and introduce the function

$$\chi(\tau) = \frac{1}{\mu}\int\limits_0^\tau (-a(s)d(s))^{1/2}ds.$$

Let $a(\tau)\delta(\tau)$ be a correct almost periodic function. We now go on to the new variables using the formulas

$$\begin{aligned}\xi &= (A\cos\chi + B\sin\chi) + \mu R(A,B,\tau,\chi)\\ \eta &= \delta(\tau)(B\cos\chi - A\sin\chi) + \mu R(A,B,\tau,\chi),\end{aligned} \tag{16.21}$$

where the functions $R(A,B,\tau,\chi)$, $S(A,B,\tau,\chi)$ will be found as periodic in χ with the period 2π, and equations for the variables A, B will take the form

$$\frac{dA}{dt} = \mu^2 P(A,B,\tau) + O(\mu^3), \quad \frac{dB}{dt} = \mu^2 Q(A,B,\tau) + O(\mu^3). \tag{16.22}$$

Substituting (16.21) into (16.19) and taking (16.22) into account yields the equations

$$\begin{aligned}a(\tau)\left[\delta(\tau)\tfrac{\partial R}{\partial\chi} - S\right] + (P\cos\chi + Q\sin\chi) &= b(\tau)(A\cos\chi + B\sin\chi),\\ a(\tau)\left[\tfrac{\partial S}{\partial\chi} + \delta(\tau)R\right] &= \left[e(\tau) - \tfrac{\delta'(\tau)}{\delta(\tau)}\right](B\cos\chi - A\sin\chi) + P\cos\chi - Q\sin\chi.\end{aligned} \tag{16.23}$$

Elimination of $R(A,B,\tau,\chi)$ from equations (16.23) gives

$$\begin{aligned}&a(\tau)\left[\tfrac{\partial^2 S}{\partial\chi^2} + S\right] - 2(P\cos\chi + Q\sin\chi) =\\ &-\left[b(\tau) + e(\tau) - \tfrac{\delta'(\tau)}{\delta(\tau)}\right](A\cos\chi + B\sin\chi).\end{aligned} \tag{16.24}$$

In order for $S(A,B,\tau,\chi)$ to be periodic, it is necessary that equation (16.24) have no items with $\cos\chi$ and $\sin\chi$. Hence we obtain

$$\begin{aligned}P(A,B,\tau,\chi) &= \tfrac{1}{2}A\left[b(\tau) + e(\tau) - \tfrac{\delta'(\tau)}{\delta(\tau)}\right],\\ Q(A,B,\tau,\chi) &= \tfrac{1}{2}B\left[b(\tau) + e(\tau) - \tfrac{\delta'(\tau)}{\delta(\tau)}\right].\end{aligned}$$

In this case, the solution of the equation (16.24) takes the form

$$S(A,B,\tau,\chi)=C_1(A,B,\tau)\cos\chi+C_2(A,B,\tau)\sin\chi.$$

Now, from the first equation of system (16.23), we derive

$$R(A,B,\tau,\chi)=\tfrac{\delta(\tau)}{2d(\tau)}\left[b(\tau)+e(\tau)-\tfrac{\delta'(\tau)}{\delta(\tau)}\right](B\cos\chi-A\sin\chi)+$$
$$\tfrac{1}{8}(C_1(A,B,\tau)\sin\chi-C_2(A,B,\tau)\sin\chi).$$

Evidently, we can assume $C_1(A,B,\tau)=C_2(A,B,\tau)\equiv 0$. Therefore, the change

$$\xi=(A\cos\chi+B\sin\chi)+\mu\tfrac{\delta(\tau)}{2d(\tau)}\left[[b(\tau)+e(\tau)-\tfrac{\delta'(\tau)}{\delta(\tau)}\right](B\cos\chi-A\sin\chi),$$
$$\eta=\delta(\tau)(B\cos\chi-A\sin\chi)$$

transforms system (16.19) into the system

$$\begin{aligned}\tfrac{dA}{dt}&=\mu^2\tfrac{A}{2}\left[b(\tau)+e(\tau)-\tfrac{\delta'(\tau)}{\delta(\tau)}\right]+\mu^2 f_1(A,B,\chi,\tau)+O(\mu^3),\\ \tfrac{dB}{dt}&=\mu^2\tfrac{B}{2}\left[b(\tau)+e(\tau)-\tfrac{\delta'(\tau)}{\delta(\tau)}\right]+\mu^2 f_2(A,B,\chi,\tau)+O(\mu^3),\end{aligned}\tag{16.25}$$

where the functions $f_1(A,B,\chi,\tau)$, $f_2(A,B,\chi,\tau)$ contain the terms with respect to A, B no lower than square. Transition to the time τ transforms system (16.25) into the system

$$\begin{aligned}\tfrac{dA}{d\tau}&=\tfrac{1}{2}\left[b(\tau)+e(\tau)-\tfrac{\delta'(\tau)}{\delta(\tau)}\right]A+f_1(A,B,\chi,\tau)+O(\mu),\\ \tfrac{dB}{dt}&=\tfrac{1}{2}\left[b(\tau)+e(\tau)-\tfrac{\delta'(\tau)}{\delta(\tau)}\right]B+f_2(A,B,\chi,\tau)+O(\mu),\end{aligned}\tag{16.26}$$

By $M(\tau)$ we denote the diagonal matrix

$$M(\tau)=\left[b(\tau)+e(\tau)-\frac{\delta'(\tau)}{\delta(\tau)}\right]I,$$

where I is an identity matrix.

The system

$$\frac{dz}{d\tau}=M(\tau)z,\quad (z=(A,B))$$

determines a continuously invertible operator $Lz=\frac{dz}{d\tau}-M(\tau)z$ in the space B_2 if the mean value of the almost periodic function $b(\tau)+e(\tau)-\frac{\delta'(\tau)}{\delta(\tau)}$ and, consequently, the mean value of the almost periodic function $b(\tau)+e(\tau)$ is non-zero (the mean value of the almost periodic function $\delta'(\tau)/\delta(\tau)$ equals zero). So, if $\langle b(\tau)+e(\tau)\rangle\neq 0$, then

$$L^{-1}f=\int\limits_{-\infty}^{\infty}G(\tau,s)f(s)ds,$$

where

$$|G(\tau, s)| \le M \exp(-\gamma|\tau - s|), \quad -\infty < \tau, s < \infty, \quad M, \gamma > 0.$$

We write system (16.26) in the vector form

$$\frac{dz}{d\tau} = M(\tau)z + F_1(z, \chi, \tau) + \mu F_2(z, \chi, \tau). \tag{16.27}$$

The problem of the almost periodic solutions of system (16.27) is equivalent to the problem of the solvability in the space B_2 for the operator equation

$$z(\tau) = \Pi(z, \mu) = \int_{-\infty}^{\infty} G(\tau, s)[F_1(z, \chi, s) + \mu F_2(z, \chi, s, \mu)]ds. \tag{16.28}$$

Evidently, the following inequalities hold

$$|F_1(0, \chi, \tau) + \mu F_2(0, \chi, \tau, \mu)| \le \omega_1(\mu),$$
$$|F_1(z_1, \chi, \tau) - F_1(z_2, \chi, \tau)| \le \omega_2(r)|z_1 - z_2|, \quad |z_1|, |z_2| \le r,$$

where $\omega_1(\mu) \to 0$ as $\mu \to 0$, and $\omega_2(r) \to 0$ as $r \to 0$. These inequalities, similarly to the proof of Theorem 16.1, imply that the operator $\Pi(z, \mu)$, for sufficiently small μ, satisfies the conditions of the principle of contraction mappings in a ball of the radius $a(\mu)$ in the space B_2. Therefore, operator equation (16.28) in this ball has a unique almost periodic solution $z_*(\tau, \mu)$. The problem on the stability of this solution is investigated with the help of theorems of the stability in the first approximation. If $\langle b(\tau) + e(\tau)\rangle < 0$, then the solution $z_*(\tau, \mu)$, for sufficiently small μ, is asymptotically stable, and if $\langle b(\tau) + e(\tau)\rangle > 0$, then the solution $z_*(\tau, \mu)$ is unstable. We formulate the result obtained as a theorem.

Theorem 16.2. *Let a resonance function $x_0(\tau)$ meet the conditions of Theorem 16.1. Let the inequality*

$$a(\tau)d(\tau) < \sigma_1 < 0, \quad \in (-\infty, \infty).$$

hold. Suppose

$$f(x_0, \eta_0, \psi, \tau, 0), f_x(x_0, \eta_0, \psi, \tau, 0), f_\varepsilon(x_0, \eta_0, \psi, \tau, 0), g(x_0, \eta_0, \psi, \tau, 0)$$

are correct almost periodic functions of ψ. Besides, some other functions arising in changing (16.18) are correct almost periodic functions of ψ. Let the function $\Omega(\tau)$ meet the conditions of Theorem 16.1. Let $a(\tau)\delta(\tau)$ be correct almost periodic function and, finally, the inequality

$$\langle b(\tau) + e(\tau)\rangle \ne 0$$

hold. Then system (16.2) in the ε-neighborhood of the resonance point, for sufficiently small ε, has a unique almost periodic solution such that is asymptotically stable if

$$\langle b(\tau) + e(\tau)\rangle < 0$$

and unstable if

$$\langle b(\tau) + e(\tau)\rangle > 0.$$

16.3 Forced Oscillations and Rotations of a Simple Pendulum. The Action of a Double-Frequency Perturbation

As an example, consider a problem of forced oscillations and rotations of a mathematical pendulum under the action of a double-frequency perturbation, with the frequencies differing between one another by terms of order ε. The corresponding equation has the form

$$\ddot{x} + \gamma\dot{x} + \Omega^2 x = a_1 \sin\omega t + a_2 \sin(\omega t + \varepsilon\Delta t), \tag{16.29}$$

where Ω, γ, a_1, a_2, ω, Δ are real positive numbers. The perturbation function $f(t) = a_1 \sin\omega t + a_2 \sin(\omega t + \varepsilon\Delta t)$ can be presented as

$$f(t,\tau) = E(\tau)\sin(\omega t + \delta(\tau)), \quad \tau = \varepsilon t, \tag{16.30}$$

where

$$E(\tau) = \sqrt{a_1^2 - 2a_1a_2\cos\Delta\tau + a_2^2}, \quad \cos\delta(\tau) = \tfrac{a_1 + a_2\cos\Delta\tau}{E(\tau)},$$
$$\sin\delta(\tau) = \tfrac{a_2\sin\Delta\tau}{E(\tau)}.$$

Function (16.30) is periodic in t with the period $2\pi/\omega$ and in τ with the period $2\pi/\Delta$, at that, $E(\tau)$ is strictly positive if $a_1 \neq a_2$, which will be assumed.

As is shown in Section 15.3, in the case of oscillatory motions of an unperturbed pendulum, by making a change

$$x = 2\arcsin k sn\left[\frac{2K(k)}{\pi}\theta\right] = X(I,\theta), \quad \dot{x} = 2k\Omega cn\left[\frac{2K(k)}{\pi}\theta\right] = Y(I,\theta), \tag{16.31}$$

where $\theta = \frac{\pi}{2K(k)}\Omega t$, we can transform equation (16.29) into a system with a fast phase

$$\begin{aligned} \tfrac{dI}{dt} &= \varepsilon\left\{f(t,\tau) - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}\theta\right]\right\}\tfrac{\partial X}{\partial\theta}, \\ \tfrac{d\theta}{dt} &= \tfrac{\pi\Omega}{2K(k)} - \varepsilon\left\{f(t,\tau) - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}\theta\right]\right\}\tfrac{\partial X}{\partial I}. \end{aligned} \tag{16.32}$$

Here, the function $X(I,\theta)$ is determined in Section 15.3, and $K(k)$ is a complete elliptic integral of the first kind.

We say that a resonance takes place in system (16.32) under condition of the satisfied equality

$$\frac{\pi\Omega}{2K(k(I))} = \frac{r}{s}\nu, \tag{16.33}$$

where r, s are coprime integers. We denote the value of I, at which equality (16.33) holds, by I_{rs}. By a change

$$\theta = \varphi + \frac{r}{s}\nu t,$$

we transform system (16.32) into the system

$$\begin{aligned}\tfrac{dI}{dt} &= \varepsilon \left\{ f(t,\tau) - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}(\varphi + \tfrac{r}{s}\nu t)\right]\right\} X_\theta(I, \varphi + \tfrac{r}{s}\nu t),\\ \tfrac{d\varphi}{dt} &= \omega(I) - \varepsilon \left\{ f(t,\tau) - 2\gamma k\Omega cn\left[\tfrac{2K(k)}{\pi}(\varphi + \tfrac{r}{s}\nu t)\right]\right\} X_I(I, \varphi + \tfrac{r}{s}\nu t),\end{aligned} \tag{16.34}$$

where

$$\omega(I) = \frac{\pi\Omega}{2K(k)} - \frac{r}{s}\nu.$$

Note that I_{rs} is independent of τ. Therefore, $d(\tau) = d$ is independent of τ and $d < 0$, similarly to 15.3. We calculate the mean value over t for the right-hand side of the first equation in system (16.34). This mean value can only be non-zero when $r = 1$, $s = 2n+1$ $(n = 0, 1, \dots)$. If $r = 1$, $s = 2n+1$, then the mean value equals

$$f(\varphi,\tau) = \frac{1}{2}E(\tau)a_n(q)\sin[\delta(\tau) - (2n+1)\varphi] - \gamma I_{rs}.$$

Recall (see Section 15.3) that

$$q = \exp\left(-\pi\frac{K'(k)}{K(k)}\right), \quad a_n(q) = \frac{q^{n+1/2}}{1+q^{2n+1}}.$$

The function $\varphi_0(\tau)$ is determined from the equation

$$\sin[\delta(\tau) - (2n+1)\varphi] = \frac{2\gamma I_{rs}}{E(\tau)a_n(q)} = A_n(\tau).$$

The solution exists if

$$|A_n(\tau)| < 1. \tag{16.35}$$

At large n the inequality does not hold, since

$$\lim_{n\to\infty} a_n(q) = 0.$$

The possibility of inequality (16.35) holding also depends on the value of γ. If inequality (14.35) is satisfied, then we obtain $(4n+2)$ various values of the function $\varphi_0(\tau)$ (the function $\eta_0(\tau)$ in notation (16.5))

$$\varphi_{0l}(\tau) = \frac{1}{2n+1}(\delta(\tau) - l\pi - (-1)^l \arcsin A_n(\tau), \quad l = 0, \dots, 4n+1.$$

The derivative of the function $f(\varphi, \tau)$ with respect to φ at the point $\varphi_{0l}(\tau)$ equals

$$a(\tau) = f_\varphi(\varphi_{0l}(\tau), \tau) = -(-1)^l \frac{2n+1}{2} a_n(q) E(\tau) \sqrt{1 - A_n^2(\tau)}.$$

If l is even, then

$$a(\tau)d(\tau) > 0, \quad \tau \in (-\infty, \infty).$$

The conditions of Theorem 16.1 are satisfied. Therefore, in the $\sqrt{\varepsilon}$-neighborhood of the resonance point $I_{1,2n+1}$, system (16.33), for sufficiently small ε, has an unstable almost periodic solution.

If l is odd, then

$$a(\tau)d(\tau) < 0, \quad \tau \in (-\infty, \infty).$$

Now we need to calculate the mean value of the function $b(\tau) + e(\tau)$. It is easy to calculate that

$$\langle 2k\gamma cn \left[\frac{2K(k)}{\pi}\theta\right] X_I(I, \theta)\rangle = 0 \quad (\theta = \varphi + \frac{1}{2n+1}\omega t).$$

Therefore,

$$\begin{aligned} b(\tau) &= \langle f(t, \tau) X_{\theta I}(I, \theta)\rangle - \gamma, \\ e(\tau) &= -\langle f(t, \tau) X_{I\theta}(I, \theta)\rangle. \end{aligned}$$

Hence,

$$b(\tau) + e(\tau) = -\gamma.$$

The conditions of Theorem 16.2 are satisfied. Therefore, in the ε-neighborhood of the resonance point $I_{1,2n+1}$, system (16.34), for sufficiently small ε, has an asymptotically stable almost periodic solution.

We could also assume that γ is not a constant but an almost periodic function $\gamma(\tau)$ with a positive mean value. The functions $E(\tau)$ and $\delta(\tau)$ can be supposed to be periodic as well.

The rotary motions of an unperturbed pendulum is studied similarly. In this case, $2n$ resonance points can exist.

Analogously, a more general pendulum equation can be investigated

$$\ddot{x} + \Omega^2(\tau)\sin x = \varepsilon[\gamma(\tau)\dot{x} + E(\tau)\sin(\nu + \delta(\tau))]. \tag{16.36}$$

Here, $\Omega(\tau)$ is a correct almost periodic function such that it satisfies the conditions described in the beginning of the chapter, $\frac{d\nu}{dt} = \omega(\tau)$, $\omega(\tau)$ is a correct almost periodic function separated from zero, $\gamma(\tau)$ is a correct almost periodic function with a positive mean value.

Consider solutions of the equation

$$\ddot{x} + \Omega^2(\tau)\sin x = 0$$

within a sub-domain of the domain of oscillatory motions for all τ, the boundary of this sub-domain being independent of τ. Transform equation (16.36) into a system using change (16.31). The resonance points I_{rs} are determined from the equation

$$\frac{\pi\Omega(\tau)}{2K(k(I_{rs}))} = \frac{r}{s}\omega(\tau),$$

where r, s are coprime integers. Making the change $\theta = \varphi + (r/s)\nu$ in the respective system and calculating the mean values over ν in the right-hand sides, we obtain that the function $f_0(\varphi, \tau)$ can only be non-zero at $r = 1$, $s = 2n + 1$. The equation for determining $\varphi_0(\tau)$ assumes the form

$$\sin(\delta(\tau) - (2n+1)\varphi) = \frac{2\frac{dI_{rs}}{d\tau} + 2\gamma(\tau)I_{rs}(\tau)}{E(\tau)a_n(q)}.$$

Calculation of the coefficients $a(\tau)$, $b(\tau)$, $c(\tau)$ and $d(\tau)$ yields the same results as in the previous case. Thus, for equation (16.36), the assertions are the same as those for equation (16.34).

The above scheme is also applicable to investigation of the resonance solutions of a pendulum with a vertically or horizontally oscillating pivot, according to the law

$$\xi = \varepsilon E(\tau)\sin(\nu + \delta(\tau)), \quad \frac{d\nu}{dt} = \omega(\tau),$$

where $E(\tau)$, $\delta(\tau), \omega(\tau)$ are periodic or almost periodic functions.

Part III

Appendices

Appendix A

Almost Periodic Functions

In this section we describe the basic properties of almost periodic functions used throughout this book in more detail than in Section 1.2. The theory of almost periodic functions is comprehensively covered in many books (e.g., Levitan [1953], Fink [1974], Corduneanu [1989]).

The functions of this class are defined for all $t \in (-\infty, \infty)$ (we shall write $t \in \mathcal{R}$).

Let the following expression be the *trigonometric polynomial*

$$T_n(t) = \sum_{k=1}^{n} a_k \cos \omega_k t + b_k \sin \omega_k t, \tag{A.1}$$

where a_k, b_k, ω_k are real numbers. It is convenient to write expression (A.1) in the complex form

$$T_n(t) = \sum_{k=1}^{n} c_k e^{i\lambda t},$$

where λ_k are real numbers.

There exist the trigonometric polynomials that differ from the periodic function. Indeed, let us take the polynomial $f(t) = e^{it} + e^{i\pi t}$. Assume that $f(t)$ is the periodic function with some period ω. The identity $f(t+\omega) = f(t)$ takes the form

$$(e^{i\omega} - 1)e^{it} + (e^{i\pi\omega} - 1)e^{i\pi t} \equiv 0.$$

Since the functions e^{it} and $e^{i\pi t}$ are linearly independent, we have that

$$e^{i\omega} - 1 = 0,\ e^{i\pi\omega} - 1 = 0.$$

Hence, $\omega = 2k\pi$ and $\pi\omega = 2h\pi$, where k and h are integers. These equalities cannot hold simultaneously.

Definition A.1. The function $f(t)$ defined for $t \in \mathcal{R}$ will be called *almost periodic* if it can be represented as a limit of the uniform convergence on the entire real axis of the sequence $T_n(t)$ of the trigonometric polynomials in the form (A.1). That is, for any $\varepsilon > 0$ there can be found a natural number N such that when $n > N$

$$\sup_{-\infty < t < \infty} |f(t) - T_n(t)| < \varepsilon.$$

Evidently, any continuous periodic function will be almost periodic in terms of Definition A.1. Now let us describe the properties of almost periodic functions.

Theorem A.1. *Every almost periodic function is continuous and bounded for $t \in \mathcal{R}$.*

Proof. Since the polynomials $T_n(t)$, $n = 1, 2, \dots$ are continuous, we see that their uniform limit on $\mathcal{R}$ is a continuous function. Let us show that the almost periodic function $f(t)$ is bounded. Assume that $T_1(t)$ is a trigonometric polynomial such that

$$\sup_{-\infty<t<\infty} |f(t) - T_1(t)| < 1.$$

Let $\sup_{-\infty<t<\infty} |T_1(t)| = M$, then

$$|f(t)| \le |f(t) - T_1(t)| + |T_1(t)| \le M + 1$$

and, therefore,

$$\sup_{-\infty<t<\infty} |f(t)| \le M + 1.$$

Theorem A.2. *Every almost periodic function $f(t)$ is uniformly continuous.*

Proof. Evidently, for any t_1, $t_2 \in \mathcal{R}$ and the trigonometric polynomial $T_n(t)$, the following inequality holds true

$$|f(t_1) - f(t_2)| \le |f(t_1) - T_n(t_1)| + |T_n(t_1) - T_n(t_2)| + |T_n(t_2) - f(t_2)|. \quad (A.2)$$

Let $\varepsilon > 0$. Choose n such that the inequalities

$$|f(t_1) - T_n(t_1)| < \frac{\varepsilon}{3}, \quad |f(t_2) - T_n(t_2)| < \frac{\varepsilon}{3}$$

hold. Since $T_n(t)$ is uniformly continuous, we have that there exists $\delta > 0$ such that for $|t_1 - t_2| < \delta$ the inequality

$$|T_n(t_1) - T_n(t_2)| < \frac{\varepsilon}{3}$$

is satisfied. It follows from (A.2) that for $|t_1 - t_2| < \delta$ the following inequality is valid

$$|f(t_1) - f(t_2)| < \varepsilon$$

Theorem A.3. *If $f(t)$ is an almost periodic function and c is a constant, then $cf(t)$, $f(t + c)$, $f(ct)$ are the almost periodic functions.*

The proof is evident.

Theorem A.4. *If $f(t)$ and $g(t)$ are the almost periodic functions, then $f(t) \pm g(t)$ and $f(t) \cdot g(t)$ are the almost periodic functions.*

Proof. We shall show that $f(t) + g(t)$ is an almost periodic function. Let $\varepsilon > 0$ and $T_\varepsilon(t)$, $R_\varepsilon(t)$ be the trigonometric polynomials such that

$$|f(t) - T_\varepsilon(t)| < \frac{\varepsilon}{2}, \quad |g(t) - R_\varepsilon(t)| < \frac{\varepsilon}{2}.$$

In this case,

$$|f(t) + g(t) - [T_\varepsilon(t) + R_\varepsilon(t)| \le |f(t) - T_\varepsilon(t)| + |g(t) - R_\varepsilon(t)|. \tag{A.3}$$

The required assertion follows from (A.3). Now let $T_\varepsilon(t)$ and $R_\varepsilon(t)$ be the trigonometric polynomials such that

$$|f(t) - T_\varepsilon(t)| < \frac{\varepsilon}{2(M+1)}, \quad |g(t) - R_\varepsilon(t)| < \frac{\varepsilon}{2(M+1)},$$

where M is a constant, for which the following inequalities hold

$$|f(t)| \le M, \quad |g(t)| \le M.$$

Then

$$|T_\varepsilon(t)| \le |T_\varepsilon(t) - f(t)| + |f(t)| < \varepsilon.$$

We obtain

$$|f(t)g(t) - T_\varepsilon(t)R_\varepsilon(t)| \le |g(t)||f(t) - T_\varepsilon(t)| + |T_\varepsilon(t)||g(t) - R_\varepsilon(t)| < \varepsilon.$$

The latter inequality implies the almost periodicity of $f(t), g(t)$.

Corollary A.1. *If $P(z_1, z_2, \ldots, z_k)$ is a polynomial of variables $z_1, z_2, \ldots, z_k$ and $f_1(t), f_2(t), \ldots, f_k(t)$ are the almost periodic functions, then $F(t) = P(f_1, f_2, \ldots, f_k)$ is an almost periodic function.*

Theorem A.5. *The limit of a uniformly convergent sequence of almost periodic functions is an almost periodic function.*

Proof. Let $f_k(t)$, $k = 1, 2, \ldots$ be a sequence of the almost periodic functions that uniformly converge to the function $g(t)$. Show that $g(t)$ is an almost periodic function. Using defined $\varepsilon > 0$, we can specify K such that for $k > K$ (by virtue of the uniform convergence)

$$|g(t) - f_k(t)| < \frac{\varepsilon}{2}, \quad t \in \mathcal{R}.$$

For the almost periodic function $f_k(t)$, it is possible to define the trigonometric polynomial $T_k(t)$ such that

$$|f_k(t) - T_k(t)| < \frac{\varepsilon}{2}, \quad t \in \mathcal{R}.$$

Then for all t the inequality

$$|g(t) - T_k(t)| \leq |g(t) - f_k(t)| + |f_k(t) - T_k(t)| < \varepsilon$$

holds true. Therefore, the function $g(t)$ can be uniformly approximated by the trigonometric polynomials.

Theorem A.6. *Let the function $\Phi(z_1, z_2, \ldots, z_k)$ be uniformly continuous on a closed bounded set M in a k-dimensional space. Let $f_1(t)$,$f_2(t)$,$\ldots$,$f_k(t)$ be the almost periodic functions and* $(f_1(t),\, f_2(t), \ldots, f_k(t)) \in M$ *for all* $t \in \mathcal{R}$. *Then* $F(t) = \Phi(f_1(t),\, f_2(t), \ldots, f_k(t))$ *is an almost periodic function.*

Proof. By virtue of the Weierstrass Theorem on the approximation of continuous functions with polynomials, for any $\varepsilon > 0$ there exists a polynomial $P_\varepsilon(z_1,\, z_2, \ldots,\, z_k)$ of variables $z_1,\, z_2, \ldots,\, z_k$ such that

$$|\Phi(z_1,\, z_2, \ldots . z_k) - P_\varepsilon(z_1,\, z_2, \ldots,\, z_k)| < \varepsilon. \tag{A.4}$$

The function $P_\varepsilon(f_1(t), \ldots, f_k(t))$ is almost periodic by virtue of the corollary to Theorem A.4. Assertion of the theorem results from inequality (A.4) and Theorem A.5.

Corollary A.2. *Let $f(t)$, $g(t)$ be the almost periodic functions and*

$$\inf_{-\infty<t<\infty} |g(t)| > 0$$

then $f(t)/g(t)$ is an almost periodic function.

Proof. It is sufficient to show that $1/g(t)$ is an almost periodic function. The conditions of the corollary and boundedness of the function $g(t)$ imply that $0 < m \leq g(t) \leq M$. It is evident that the function $\Phi(z) = 1/z$ is uniformly continuous on the set $m \leq |z| \leq M$. Therefore, Theorem A.6 implies that $1/g(t)$ is the almost periodic function.

We consider the function $f(t, x_1,\, x_2, \ldots, x_k)$ that depends on the vector parameter $x = (x_1, \ldots, x_k)$ varying in the bounded set $M \in \mathcal{R}^k$.

Definition A.2. The function $f(t, x)$ is called almost periodic in t and uniform with respect to $x \in M$ if for any $\varepsilon > 0$ it is possible to specify a natural number N such that for $n > N$

$$\sup_{-\infty<t<\infty} |f(t, x) - \sum_{l=1}^{n} c_l(x) e^{i\lambda_l t}| < \varepsilon,$$

with the functions $c_l(x)$ being continuous on M.

It is possible to show that the function $f(t, x)$ is almost periodic in t uniformly with respect to x if it is almost periodic in t for every fixed x and continuous in $x \in M$ uniformly with respect to $t \in \mathcal{R}$.

Theorem A.8. *If the function $f(t, x)$ is almost periodic in t uniformly with respect to $x \in M$ and $x_i(t)$, $(i = 1, \ldots, k)$ are the almost periodic functions with*

$$(x_1(t),\, x_2(t), \ldots,\, x_k(t)) \in M$$

for $t \in \mathcal{R}$, then $f(t,\, x_1(t), \ldots,\, x_k(t))$ is an almost periodic function.

Proof. Without loss of generality, we assume that the set M is bounded. Then the functions $c_i(x)$, $i = 1, \ldots, k$ are uniformly continuous on M. By virtue of Theorems A.6 and A.4, the function

$$\sum_{l=1}^{n} c_l(x_1(t), \ldots, x_k(t)) e^{i\lambda_l t}$$

is almost periodic. Therefore, Theorem A.5 implies that
$f(t,\, x_1(t), \ldots, x_k(t))$ is the almost periodic function.

The continuous function $f(t)$ is called the *normal function* if the family of the functions $\{f(t + h)\}(-\infty < h < \infty)$ has the following property: on any infinite sequence of the functions

$$f(t + h_1),\ f(t + h_2), \ldots$$

it is possible to choose a sub-sequence that uniformly converges at all t.

Theorem A.9. *The almost periodic function $f(t)$ is a normal function.*

This theorem is proved in the following manner (see Corduneanu [1989]). It is easy to see that the function $e^{i\lambda t}$ is normal. It is then proved that a trigonometric polynomial is a normal function, then, using Definition A.1, the property of the normal arbitrary almost periodic function is determined.
S. Bochner showed that the property of the normality of a function is a necessary and sufficient condition of the almost periodicity of the function.

Theorem A.10. *Let $f(t)$ be an almost periodic function and its derivative $f'(t)$ be uniformly continuous on the entire real axis. Then $f'(t)$ is an almost periodic function.*

Proof. Consider the functions

$$\varphi_k(t) = k[f(t + \frac{1}{k}) - f(t)], \quad k = 1, 2, \ldots.$$

Using the mean-value theorem, we get

$$\varphi_k(t) = f'(t + \frac{\theta_k}{k}), \quad 0 < \theta_k < 1.$$

For $k \to \infty$, the sequence of the almost periodic functions $\varphi_k(t)$ uniformly converges for $t \in \mathcal{R}$ to the function $f'(t)$. Thus, $f'(t)$ is the almost periodic function.

Consider the problem of the integrability of an almost periodic function later. The following property of an almost periodic function is especially important to us.

Theorem A.11. *For an almost periodic function f(t) there exists a limit*

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_a^{a+T} f(t)dt = \langle f(t) \rangle$$

that is uniform with respect to a. The number $\langle f(t)\rangle$ is independent of the choice of a and is called the mean value of the almost periodic function $f(t)$.

Proof. Let $f(t) = T_n(t)$ be a trigonometric polynomial:

$$T_n(t) = c_0 + \sum_{l=1}^{n} c_l e^{i\lambda_l t},$$

where λ_l are non-zero real numbers. In this case,

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_a^{a+T} T_n(t)dt = c_0 + \sum_{l=1}^{n} c_l \frac{e^{i\lambda_l(a+T)} - e^{i\lambda_l a}}{i\lambda_l T}.$$

This implies the inequality

$$\left| \frac{1}{T} \int\limits_a^{a+T} T_n(t)dt - c_0 \right| \leq \frac{2}{T} \sum_{l=1}^{n} \left| \frac{c_l}{\lambda_l} \right|,$$

which, in turns, it follows that

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_a^{a+T} T_n(t)dt = c_0.$$

For an arbitrary almost periodic function $f(t)$, we take a trigonometric polynomial $T_n(t)$ such that

$$|f(t) - T_n(t)| < \frac{\varepsilon}{3}, \quad t \in \mathcal{R},$$

where $\varepsilon > 0$. Since for $T_n(t)$ there exists a mean value, we see that it is possible to specify $T(\varepsilon)$ such that for $T_1, T_2 \geq T(\varepsilon)$

$$\left| \frac{1}{T_1} \int\limits_a^{a+T_1} T_n(t)dt - \frac{1}{T_2} \int\limits_a^{a+T_2} T_n(t)dt \right| < \frac{\varepsilon}{3}.$$

Consequently,

$$\left| \frac{1}{T_1} \int\limits_a^{a+T_1} f(t)dt - \frac{1}{T_2} \int\limits_a^{a+T_2} f(t)dt \right| \leq \frac{1}{T_1} \int\limits_a^{a+T_1} |f(t) - T_n(t)|dt +$$

$$+ \left| \frac{1}{T_1} \int\limits_a^{a+T_1} T_n(t)dt - \frac{1}{T_2} \int\limits_a^{a+T_2} T_n(t)dt \right| < \frac{\varepsilon}{3} + \frac{1}{T_2} \int\limits_a^{a+T_2} |f(t) - T_n(t)|dt < \varepsilon$$

for $T_1, T_2 \geq T(\varepsilon)$. The latter inequality shows that, uniformly with respect to a, there exists the limit

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_a^{a+T} f(t)dt = \langle f(t) \rangle.$$

We show that this limit is independent of a:

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_a^{a+T} f(t)dt = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^{T} f(t)dt.$$

Since $|f(t)| \leq M < \infty$, we have that for $a > 0$, $T > 0$

$$\frac{1}{T} \left| \int\limits_a^{a+T} f(t)dt - \int\limits_0^{T} f(t)dt \right| = \frac{1}{T} \left| \int\limits_T^{a+T} f(t)dt - \int\limits_0^{a} f(t)dt \right| \leq \frac{2aM}{T}.$$

This implies the desired equality.

Let $f(t)$ be a periodic function with the period ω. We present the real number T in the form $T = n\omega + \alpha_n$, where n is an integer and α_n meets the inequality

$$0 \leq \alpha_n \leq \omega.$$

If $T \to \infty$, then $n \to \infty$. We calculate the mean value of the function f(t):

$$\langle f(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)dt = \lim_{n\to\infty} \frac{1}{n\omega + \alpha_n} \int_0^{n\omega+\alpha_n} f(t)dt =$$

$$\lim_{n\to\infty} \frac{1}{n\omega+\alpha_n}\left\{\sum_{i=0}^{n-1}\int_{i\omega}^{(i+1)\omega} f(t)dt + \int_{n\omega}^{n\omega+\alpha_n} f(t)dt\right\} =$$

$$= \lim_{n\to\infty} \frac{1}{n\omega+\alpha_n}\left\{n\int_0^{\omega} f(t)dt + \int_0^{\alpha_n} f(t)dt\right\} = \frac{1}{\omega}\int_0^{\omega} f(t)dt.$$

Thus, the mean value introduced for the periodic functions coincides with the ordinary mean value of a periodic function.

The existence of the mean value allows a Fourier series to be constructed for an almost periodic function. Let f(t) be an almost periodic function. Since the function $e^{i\lambda t}$ is periodic for any real λ, we obtain that the product $f(t)e^{i\lambda t}$ is an almost periodic function. Hence, there exists the mean value

$$a(\lambda) = \langle f(t)e^{i\lambda t}\rangle.$$

Of fundamental importance is the fact that the function $a(\lambda)$ may be non-zero for only countable set of λ, at most. The numbers $\lambda_1, \dots, \lambda_n, \dots$ are called the Fourier exponents, and the numbers $a_1, \dots, a_n, \dots$ are the Fourier coefficients of the function f(t).

Thus, for any almost periodic f(t), we can associate a Fourier series:

$$f(t) \sim \sum_n a_n e^{i\lambda_n t}.$$

The Fourier series allow for formal operations. Let f(t) and g(t) be the almost periodic functions and

$$f(t) \sim \sum_n a_n e^{i\lambda_n t} = \sum_\lambda a(\lambda)e^{i\lambda t},$$

$$g(t) \sim \sum_n b_n e^{i\mu_n t} = \sum_\lambda b(\lambda)e^{i\lambda t}.$$

Then:
1) $kf(t) \sim \sum_n ka_n e^{i\lambda_n t} \quad (k = constant)$,
2) $e^{i\lambda t}f(t) \sim \sum_n a_n e^{i(\lambda_n+\lambda)t}$,
3) $f(t+\alpha) \sim \sum_n a_n e^{i\lambda\alpha}e^{i\lambda_n t} \quad (\alpha \in \mathcal{R})$,
4) $\bar{f}(t) \sim \sum_n \bar{a}_n e^{-i\lambda_n t}$,
5) $f(t)+g(t) \sim \sum_\lambda (a(\lambda)+b(\lambda))e^{i\lambda t}$,
6) $f(t)\cdot g(t) \sim \sum_n c_n e^{i\nu_n t}$, where

$$c_n = \sum_{\lambda_p+\mu_q=\nu_n} a_p b_q.$$

If the derivative of the almost periodic function f(t) is an almost periodic function, then its Fourier series ensues from the Fourier series f(t) using a termwise differentiation. If an indefinite integral of the almost periodic function $f(t)$ is an almost periodic function, then

$$F(t) = \int_0^t f(t)dt \sim c + \sum_n \frac{a_n}{i\lambda_n} e^{i\lambda_n t} \quad (\lambda_n \neq 0).$$

Consider the integration of the almost periodic functions. If f(t) is a periodic function with a non-zero mean value, then the following equality is valid

$$\int_0^t f(t)dt = \langle f \rangle t + g(t),$$

where g(t) is a periodic function. Generally speaking, the latter equality does not hold for the almost periodic functions. There exist the almost periodic functions with a non-zero mean value such that their integral is unbounded and is, thus, not an almost periodic function. An example of such a function is the function

$$f(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} e^{i \frac{1}{k^2} t}.$$

We shall call the almost periodic function $f(t)$ **correct** if the following equality is satisfied

$$\int_0^t f(t)dt = \langle f \rangle t + g(t)$$

where $g(t)$ is an almost periodic function. The function $f(t)$ is correct if it is a trigonometric polynomial. If the Fourier exponents of a periodic function are separated from zero $\lambda_n \geq \delta > 0$, then this function is also correct.

If there exists a finite set of the numbers $\omega_1, \omega_2, \ldots, \omega_m$ such that each Fourier exponent of the almost periodic function is a linear combination of these numbers

$$\lambda_n = n_1\omega_1 + \cdots + n_m\omega_m,$$

where $n_1, \ldots, n_m$ are integers, then this almost periodic function is called **quasi-periodic**. The quasi-periodic functions can be obtained from the functions of several variables. For instance, let $F(x, y)$ be a function periodic in each variable with the period 2π. In this case, $F(\omega_1 t, \omega_2 t)$ is a quasi-periodic function if the numbers ω_1, ω_2 are incommensurable.

It follows from the above properties of the almost periodic functions that they generate a linear space. If we introduce the norm

$$||f(t)|| = \sup_{-\infty<t<\infty} |f(t)|,$$

then this space forms a Banach space (a complete normalized linear space). This space is denoted as B. It is easy to see that the mean value is a linear functional on this space, i.e. the mean value has the following properties:

1) $\langle cf(t) \rangle = c\langle f(t) \rangle$ (c=constant),

2) $\langle (f(t) + g(t)) \rangle = \langle f(t) \rangle + \langle g(t) \rangle$,

3) If the sequence of the almost periodic functions $f_1(t), \ldots, f_n(t), \ldots$ uniformly converges for $t \in \mathcal{R}$ to the almost periodic function f(t), then

$$\lim_{n \to \infty} \langle f_n(t) \rangle = \langle f(t) \rangle.$$

Appendix B

Stability of the Solutions of Differential Equations

In this book we place high emphasis on the method of averaging on an infinite interval as applied to the research into stability of the solutions of differential equations. For this reason we provide some definitions and results of the Lyapunov Stability Theory, that we use in this book. A more detailed account of the theory of stability is presented in Krasovskii [1963], Malkin [1966], Hahn [1963], and Rumiantsev, and Oziraner [1987].

B.1 Basic Definitions

Consider a system of differential equations

$$\frac{dx}{dt} = X(t, x), \tag{B.1}$$

where the vector-function $X(t, x)$ is defined for $t \in [0, \infty)$, $x \in D$, and the D is a domain of the n-dimensional Euclidean space $\mathcal{R}^n$. We assume that for system (B.1) the conditions of the local theorem of the existence and uniqueness solutions of the initial problem are fulfilled. Let $\varphi(t, t_0, x_0)$ ($\varphi(t_0, t_0, x_0) = x_0$) be the solutions of system (B.1) defined for $t \geq t_0 \geq 0$.

Definition B.1. *The solution $\varphi(t, t_0, x_0)$ is stable in the sense of Lyapunov (hereinafter, Lyapunov stable) if for any $\varepsilon > 0$ it is possible to specify $\delta > 0$ such that for the solution $\psi(t, t_0, \xi)$ of system (B.1) the following inequality holds*

$$|\psi(t, t_0, \xi) - \varphi(t, t_0, x_0)| < \varepsilon$$

for $t \geq t_0$ only if

$$|\xi_0 - x_0| < \delta.$$

Unless the conditions of the definition hold, the solution $\varphi(t, t_0, x_0)$ is **unstable**.

Definition B.2. *The solution $\varphi(t, t_0, x_0)$ of system (B.1) is asymptotically stable if it is Lyapunov stable and if there exists $\sigma > 0$ such that*

$$\lim_{t\to\infty} |\psi(t, t_0, \xi) - \varphi(t, t_0, x_0)| = 0, \qquad (B.2)$$

if

$$|\xi - x_0| < \sigma.$$

The domain $G_\sigma \subset \mathcal{R}^n$ in the space of initial conditions is called the domain of attraction of the solution $\varphi(t, t_0, x_0)$ if for the solutions of system (B.1), which originate in the G_σ, the limit equality (B.2) holds.

Definition B.3. *The solution $\varphi(t, t_0, x_0)$ of system (B.1) is uniformly asymptotically stable if it is asymptotically stable and if for any $\eta > 0$ it is possible to specify $T(\eta)$ such that the inequality*

$$|\psi(t, t_0, \xi) - \varphi(t, t_0, x_0)| < \eta$$

holds for $t \geq t_0 + T(\eta)$ and for any initial time t_0 and coordinate of initial perturbations ξ in the domain of attraction of the solution $\varphi(t, t_0, x_0)$.

We present an example of the asymptotically stable but not a uniformly asymptotically stable solution. Consider an equation

$$\dot{x} = x^2. \qquad (B.3)$$

The solution $x \equiv 0$ of this equation is unstable, since solutions with positive initial data increase monotonically. Solutions with negative initial condition also increase and tend to zero as $t \to \infty$. Therefore, every solution with a negative initial condition is asymptotically stable. But each such solution will not be uniformly asymptotically stable. At sufficiently large t, in the small neighborhood of this solution there will be the points with positive x-coordinates.

If the equilibrium state $x = a$ of an autonomous system is asymptotically stable, then it is uniformly asymptotically stable. If the periodic solution of a system of differential equations with periodic coefficients is asymptotically stable, then this periodic solution is uniformly asymptotically stable.

Along with system (B.1), we consider the perturbed system

$$\frac{dx}{dt} = X(t, x) + R(t, x), \qquad (B.4)$$

where the vector-function $R(t, x)$ describes constantly acting perturbations. We shall assume that system (B.4) satisfies the conditions of local theorem of the existence and uniqueness.

Definition B.4. *The solution $\varphi(t, t_0, x_0)$ of unperturbed system (B.1) is called stable under constantly acting perturbations if for any $\varepsilon > 0$ there exist two other numbers $\eta_1(\varepsilon) > 0$ and $\eta_2(\varepsilon) > 0$ such that every solution $x(t, t_0, \xi)$ of system (B.4) with the initial condition ξ meeting the inequality*

$$|\xi - x_0| < \eta_1(\varepsilon)$$

under an arbitrary perturbation $R(t, x)$, which in the domain $t \geq t_0$, $|x| < \varepsilon$ meets the inequality

$$|R(t, x)| < \eta_2(\varepsilon)$$

under all $t > t_0$, meets the inequality

$$|x(t, t_0, \xi) - \varphi(t, t_0, x_0| < \varepsilon.$$

This definition belongs to Malkin (see Malkin [1966]). It is worth noting that the stability under constantly acting perturbations is also referred to as a total stability (see Hahn [1963]).

The above definition assumes that constantly acting perturbations are small for all time values t. Krasovskii and Germaidze (see Krasovskii [1963]) considered the case when perturbations may be large at the particular time moments but small on average. They introduced the following definition.

Definition B.5. *The solution $\varphi(t, t_0, x_0)$ of unperturbed system (B.1) is called* **stable under constantly acting perturbations** *bounded on average if for any pair of the numbers $\varepsilon > 0$ and $T > 0$ it is possible to specify two numbers $\delta > 0$ and $\eta > 0$ such that provided the satisfied inequality*

$$\int_t^{t+T} \psi(s) ds < \eta,$$

where $\psi(t)$ is a continuous function meeting the condition

$$|R(t, x)| \leq \psi(t)$$

for $|x| < \varepsilon$, every solution $x(t, t_0, \xi)$ of system (B.4) with the initial condition meeting the inequality

$$|\xi - x_0| < \delta$$

for all $t > t_0$, obeys the inequality

$$|x(t, t_0, \xi) - \varphi(t, t_0, x_0| < \varepsilon.$$

We state a few more definitions related to the concept of the stability with respect to a part of the variables (see Rumiantsev, and Oziraner [1987], and Vorotnikov [1997]).

Definition B.6. *The solution $\varphi(t, t_0, x_0)$ of system (B.1) is referred to as:*
a) stable with respect to a part of the variables $x_1, x_2, \ldots, x_k$, $k < n$ if for any $\varepsilon > 0$ it is possible to specify $\delta > 0$ such that for the solution $\psi(t, t_0, \xi)$ of system (B.1) the inequality

$$|\psi_i(t, t_0, \xi) - \varphi_i(t, t_0, x_0)| < \varepsilon, \quad i = 1, 2, \ldots, k$$

holds for $t \geq t_0$ only if

$$|\xi_0 - x_0| < \delta;$$

b) asymptotically stable with respect to a part of the variables $x_1, x_2, \ldots, x_k$, $k < n$ if it is stable with respect to a part of the variables $x_1, x_2, \ldots, x_k$, $k < n$ and if there exists $\sigma > 0$ such that for $|\xi - x_0| < \sigma$ the limit equality

$$\lim_{t \to \infty} |\psi_i(t, t_0, \xi) - \varphi_i(t, t_0, x_0)| = 0, \quad i = 1, 2, \ldots, k < n$$

holds true;
c) uniformly asymptotically stable with respect to a part of the variables $x_1, x_2, \ldots, x_k$, $k < n$ if it is asymptotically stable with respect to a part of the variables and if for any $\eta > 0$ it is possible to specify $T(\eta)$ such that the inequality

$$|\psi_i(t, t_0, \xi) - \varphi_i(t, t_0, x_0)| < \eta, \quad i = 1, 2, \ldots, k < n$$

holds for $t \geq t_0 + T(\eta)$ and for any the initial time instant t_0 and coordinate of initial perturbations ξ in the domain $|\xi - x_0| < \sigma$.

B.2 Theorems of the Stability in the First Approximation

We shall assume that $f(t, x)$ is continuously differentiable with respect to the spatial variable x in the neighborhood of the solution $\varphi(t)$ whose stability is under study. The problem of the stability of the solution $\varphi(t)$ of system (B.1) can be reduced to the problem of the stability of the zero solution if we make a change

$$x = y + \varphi(t).$$

After the change, we arrive at the system

$$\frac{dy}{dt} = f(t, y + \varphi) - f(t, \varphi)$$

that can be written as

$$\frac{dy}{dt} = A(t)y + \omega(t, y), \tag{B.5}$$

where $\omega(t, y) = f(t, y + \varphi) - f(t, \varphi) - A(t)y$ and $A(t) = f_x(t, \varphi)$. Evidently, $\omega(t, 0) \equiv 0$. Hence, the problem is reduced to the problem of the stability of the zero solution to system (B.5). Evidently, $\omega(t, y)$ contains only members of order of smallness higher than the first order over the spatial variable y in the neighborhood of zero. This condition can be written as

$$|\omega(t, x)| \leq q(r)|x| \quad (|x| \leq r), \tag{B.6}$$

where $q(r)$ is monotonic and

$$\lim_{r \to 0} q(r) = 0.$$

The system of linear equations

$$\frac{dx}{dt} = A(t)x \tag{B.7}$$

is called the system of the first approximation for system (B.5). It is natural to find the conditions such that the stability of the zero solution of system (B.7) implies the stability of the zero solution of system (B.5). The respective theorems are called the theorems of the stability in the first approximation. These theorems hold true under rather general assumptions on elements of the matrix $A(t)$ and the **exponential dichotomy** of the solutions of system (B.6).

Assume that the matrix $A(t) = A$ is constant and prove two theorems of the stability in the first approximation. The scheme of proving these theorems is used repeatedly in this book for various situations.

Consider the following system of equations

$$\frac{dx}{dt} = Ax + \omega(t, x), \tag{B.8}$$

where A is a constant matrix, the vector-function $\omega(t, x)$, $\omega(t, 0) \equiv 0$ is defined and continuous in a set of variables for $0 \leq t < \infty$ and x from some ball $|x| < r_0$. Besides, we assume that $\omega(t, x)$ satisfies inequality (B.6).

Theorem B.1. *Let all eigenvalues of the matrix A have negative real parts. Then the zero solution of system (B.8) is asymptotically stable.*

Proof. Every solution $x(t)$ of system (B.8) is simultaneously a solution of the system of integral equations

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A}\omega(s, x(s))ds. \tag{B.9}$$

The conditions of the theorem result in an estimation

$$|e^{tA}| \leq Me^{-\gamma t} \quad (t \geq 0),$$

where M, γ are positive constants. Therefore, for $t \geq 0$ the inequality

$$|x(t)| \leq Me^{-\gamma t}|x(0)| + \int_0^t Me^{-\gamma(t-s)}|\omega(s, x(s))|ds \qquad (B.10)$$

holds true. Let ε be a sufficiently small number such that $\beta = Mq(\varepsilon) \leq \frac{\gamma}{2}$. Assume $\delta = \varepsilon/(1+M)$. First we shall show that the inequality $|x(0)| < \delta$ implies the inequality $|x(t)| < \varepsilon$ for $t > 0$. Indeed, if t_0 is the first time instant when $|x(t_0)| = \varepsilon$, then (B.10) implies that for $0 \leq t \leq t_0$

$$|x(t)| \leq Me^{-\gamma t}|x(0)| + \beta \int_0^t Me^{-\gamma(t-s)}|x(s)|ds. \qquad (B.11)$$

By virtue of the Gronwall-Bellman lemma (see Section 3.4)

$$|x(t_0)| = \varepsilon \leq Me^{-(\gamma-\beta)t}|x(0)|,$$

whence it follows that

$$\varepsilon < Me^{-\frac{\gamma t_0}{2}} \cdot \frac{\varepsilon}{M} < \varepsilon.$$

We have arrived at a contradiction.

Hence, inequality (B.11) holds true under all $t \geq 0$. It follows from the Gronwall-Bellmann lemma that

$$|x(t)| \leq Me^{-\frac{\gamma t_0}{2}}|x(0)|, \quad t \geq 0, \quad |x(0)| \leq \delta.$$

The theorem is proved.

Consider now the theorem of the instability in the first approximation. We shall assume that $\omega(t, x)$ satisfies the Lipschitz condition in the following form

$$|\omega(t, x) - \omega(t, y)| \leq p(r)|x - y|, \quad |x|, |y| \leq r,$$

where $p(r) \to 0$ as $r \to 0$.

Theorem B.2. *Let the matrix A have at least one eigenvalue with positive real part.*

Then the zero solution of system (B.8) is unstable.

Proof. To prove the theorem, it is sufficient to determine the existence of $r_0 > 0$ such that in any neighborhood of the origin of coordinates there is the initial value $x(0)$ of some solution $x(t)$ $(0 \le t < \infty)$ of system (B.8), which is not fully contained within the ball $|x| \le r_0$.

Without loss of generality, we can assume that the matrix A has no eigenvalues with the zero real part. If there are such eigenvalues, then the change $x = e^{\lambda t}y$, where $\lambda > 0$ is selected small so that the matrix $A - \lambda I$ has eigenvalues with positive real parts, results in the system with the linear part matrix having no eigenvalues that lie on the imaginary axis.

For definiteness, we suppose that the matrix A has k eigenvalues with negative real parts and $(n-k)$ eigenvalues with positive real parts. Without loss of generality, we assume that the matrix A has a block-diagonal form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 is a matrix of order k such that its eigenvalues have negative real parts, and A_2 is a matrix of order $(n-k)$ such that its eigenvalues have positive real parts. For system (B.8) we consider the problem of the existence of solutions bounded on the half axis $[0, \infty)$. As was shown in section 3.4, this problem is equivalent to the problem of the existence of solutions for the system of integral equations

$$x(t) = \begin{pmatrix} e^{tA_1} & 0 \\ 0 & 0 \end{pmatrix} y(0) + \int_0^t \begin{pmatrix} e^{(t-s)A_1} & 0 \\ 0 & 0 \end{pmatrix} \omega(s, x(s))ds +$$

$$- \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{(t-s)A_2} \end{pmatrix} \omega(s, x(s))ds.$$

Let $x^1(t)$ and $x^2(t)$ be two solutions of system (B.8) such that $x_1^1(0) = x_1^2(0)$, $|x^1(t)|, |x^2(t)| \le r_0$ for $t \ge 0$. Here, $x_1^1(0)$, $x_1^2(0)$ are used to denote the first k coordinates of the vector of the initial conditions. Then

$$x^1(t) - x^2(t) = \int_0^t \begin{pmatrix} e^{(t-s)A_1} & 0 \\ 0 & 0 \end{pmatrix} [\omega(s, x^1(s)) - \omega(s, x^2(s))]ds -$$

$$- \int_t^\infty \begin{pmatrix} 0 & 0 \\ 0 & e^{(t-s)A_2} \end{pmatrix} [\omega(s, x^1(s)) - \omega(s, x^2(s))]ds$$

Hence,

$$|x^1(t) - x^2(t)| \le Mp(r_0) \int_0^t e^{-\gamma(t-s)} |x^1(s) - x^2(s)|ds +$$

$$+ Mp(r_0) \int_t^\infty e^{-\gamma(t-s)} |x^1(s) - x^2(s)|ds$$

and, consequently,

$$|x^1(t) - x^2(t)| \leq Mp(r_0)\frac{1}{\gamma}(2 - e^{-\gamma t}) \sup_{0 \leq s < \infty} |x^1(s) - x^2(s)|. \qquad (B.12)$$

We choose r_0 such that the inequality $2Mp(r_0) < \gamma$ holds. Then it follows from inequality (B.12) that $x^1(t) \equiv x^2(t)$.

Thus, among the solutions of system (B.8) with the fixed $x_1(0)$ (i.e. fixed first k components of the vector of the initial conditions), there is no more than one solution bounded on $[0, \infty)$, with the norm not higher than r_0. Therefore, there exist infinitely many initial values in any neighborhood of the origin of coordinates, that correspond to the solutions of system (B.8) such that for some $t > 0$ go beyond the bounds of the ball $|x| \leq r_0$. Therefore, the zero solution of system (B.8) is unstable.

The theorem is proved.

B.3 The Lyapunov Functions

For the cases when the theorems of the stability in the first approximation do not work, Lyapunov suggested the method called the Lyapunov Function Method.

Consider a scalar function $v(x_1, x_2, \ldots, x_n)$ defined in a domain $D \subset \mathcal{R}^n$ that involves the origin of coordinates $0 = (0, 0, \ldots, 0)$. We shall assume that in the domain D the function $v(x)$ is continuous and has continuous partial derivatives $\frac{\partial v}{\partial x_i}$ $(i = 1, 2, \ldots, n)$, and $v(0) = 0$. Later on, the functions $v(x)$ meeting the above conditions shall be referred to as the **Lyapunov Functions**.

The function $v(x)$ will be called **positive** in the domain D if $v(x) \geq 0$ at all points of this domain, negative if $v(x) \leq 0$ for $x \in D$. If one of these inequalities hold, then the function $v(x)$ is the function of constant sign. The function $v(x)$ will be called **positive definite** in the domain D if $v(x) > 0$ for $x \neq 0$ and **negative definite** if $v(x) < 0$ for $x \neq 0$. In both cases, the function $v(x)$ is called **definite**. Finally, the function $v(x)$ is called alternating in the domain D if it possesses values of different signs in this domain.

Let us bring in some examples. The function $v(x) = x_1^2 + x_2^2 + x_3^2$ is positive definite in $\mathcal{R}^3$, the function $v(x) = x_1^2 + x_2^2$ is positive in $\mathcal{R}^3$, and the function $v(x) = x_1^2 + x_2^2 - x_3^2$ is alternating in $\mathcal{R}^3$.

Up until now, we have considered the Lyapunov functions without relating them differential equations. Now let us define an autonomous system of differential equations

$$\frac{dx}{dt} = f(x), \quad x \in \mathcal{R}^n. \qquad (B.13)$$

According to system (B.13), the total derivative of the function $v(x)$ wiil be called the derivative $\frac{d}{dt}v(\varphi(t))$, where $x = \varphi(t)$ is the solution of system (B.13). It is evident that

$$\frac{d}{dt}v(\varphi(t)) = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}\frac{d\varphi_i}{dt} = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f(\varphi(t)). \tag{B.14}$$

We denote this derivative by $\dot{v}$. We introduce a vector

$$grad\ v = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right).$$

Then $\dot{v}$ is a scalar product of the vectors *grad v* and f

$$\dot{v}(\varphi(t)) = \langle gradv, f(\varphi(t)\rangle.$$

It follows from (B.14) that in order to calculate the total derivative $\dot{v}$ at the point $t = t_0$, it is not necessary to know the solution $\varphi(t)$ of system (B.13), and only its value $\varphi(t_0) = a$ is required. In this case

$$\dot{v}(t_0) = \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}|_{x=a} \cdot f_i(a) = \langle gradv(a), f(\varphi(a)\rangle.$$

Thus, the sign in the right-hand side of the latter equality at all the points of some domain guarantees the sign of $\dot{v}(t)$ on all the solutions of system (B.13) in this domain. This fact is the basis of theorems that allow studying the properties of the stability of the solutions to system (B.13) by the properties of the Lyapunov functions and their derivatives by virtue of the system.

We shall assume that system (B.13) has the zero solution.

Theorem B.3 (The Lyapunov Theorem of Stability). *If in the domain D, for system (B.13), there exists a definite function $v(x)$ such that its total derivative, by virtue of system (B.13), is a function of a constant sign and the sign of this function is opposite to the sign of the function $v(x)$, then the solution $x = 0$ is stable in the Lyapunov sense.*

Theorem B.4 (The Lyapunov Theorem of Asymptotic Stability). *If in the domain D, there exists a definite function $v(x)$ such that its derivative, by virtue of system (B.13), is also definite and opposite to the sign of $v(x)$, then the solution $x = 0$ is asymptotically stable.*

Consider the problem of the conditions for instability of the zero solution of system (B.13). To find the instability of a solution, it is sufficient to find in the arbitrarily small neighborhood of the point $x = 0$ at least one solution that goes beyond the boundaries of the ball T_ε with the radius ε for some $\varepsilon > 0$. We present an assertion of instability by Chetaev.

Let the Lyapunov function $v(x)$ be defined in the ball T_μ, $\mu > 0$. The domain of positivity of the function $v(x)$ is called the set of the points $x \in T_\mu$ such that

$$v(x) > 0.$$

The surface $V(x) = 0$ is the boundary of the domain of positivity of ($V(0) = 0$). For example, for the function $v(x_1, x_2) = x_1 - x_2^2$, the boundary of the domain of positivity will be the parabola $x_1 = x_2^2$. If the function $v(x)$ is positive, then its domain of positivity coincides with the full neighborhood of the point $x = 0$.

Theorem B.5. *If for system (B.13) it is possible to find a function $v(x)$ such that in the arbitrarily small neighborhood of the point $x = 0$, there exists a domain of positivity and the derivative $\dot{v}(x)$ of this function, by virtue of system (B.13), is positive at all points of the domain of positivity of the function $V(x)$, then the solution $x = 0$ is unstable.*

As an example, we consider a scalar differential equation

$$\dot{x} = F(x) = gx^m + g_{m+1}x^{m+1} + \dots, \tag{B.15}$$

where $m \geq 2$, g, $g_{m+1}, \dots$ are some constants and the series $F(x)$ converges for sufficiently small x. We investigate the stability of the zero solution of equation (B.15) using the Lyapunov functions.

If m is an odd number, then we let

$$v(x) = \frac{1}{2}gx^2.$$

For $\dot{v}(x)$, we have

$$\dot{v} = g^2x^{m+1} + g \cdot g_{m+1}x^{m+2} + \dots.$$

Both functions $v(x)$ and $\dot{v}(x)$ are definite for sufficiently small x. If $g > 0$, then the functions $v(x)$ and $\dot{v}(x)$ are of the same sign and, therefore, by virtue of the Chetaev Theorem, the solution $x = 0$ is unstable. If $g < 0$, then the functions $v(x)$ and $\dot{v}(x)$ have opposite signs for sufficiently small x and, therefore, by virtue of the Lyapunov Theorem, the zero solution is asymptotically stable.

For even m, we assume

$$v(x) = x.$$

In this case, the function $\dot{v}(x)$ is definite, and the function $v(x)$ itself, regardless of the sign of g, can have the values of the same sign as $\dot{v}(x)$. Hence, both for $g > 0$ and for $g < 0$, the conditions of the Chetaev Theorem hold, thus, the solution $x = 0$ is unstable.

The Lyapunov functions are also used to prove the theorems of the stability under constantly acting perturbations. We formulate the following theorem.

Theorem B.6. *If the solution $x(t, t_0, x_0)$ of system (B.1) is uniformly asymptotically stable, then*
1) $x(t, t_0, x_0)$ is stable under constantly acting perturbations;
2) $x(t, t_0, x_0)$ is stable under constantly acting perturbations bounded on average.

Appendix C

Some Elementary Facts from the Functional Analysis

C.1 Banach Spaces

A linear space is defined as a nonempty set L such that each pair of its elements f, g is associated with an element $f + g \in L$ called a sum of the elements f, g. The summation has the following properties:

(i) $f + g = g + f$,

(ii) $f + (g + h) = (f + g) + h, \quad f, g, h \in L$

(iii) there exists a unique element 0 (called zero in L) such that $f + 0 = f$ for all $f \in L$,

(iv) for each $f \in L$ there exists a unique element $(-f) \in L$ such that $f + (-f) = 0$.

Elements of a linear space can be multiplied by scalars. As scalars, real or complex numbers are used. Each scalar α and each element $f \in L$ make up a new element $\alpha f \in L$, and for any two scalars α, β the following relations hold true:

(v) $\alpha(f + g) = \alpha f + \alpha g$,

(vi) $(\alpha + \beta)f = \alpha f + \beta f$,

(vii) $(\alpha\beta)f = \alpha(\beta)f$,

(viii) $1 \cdot f = f$.

Hence, L is called a real linear space if the scalars are real numbers and a complex linear space if the scalars are complex numbers. The elements f, g, $h, \ldots$ of the space L are called points or vectors. Of the most interest are the linear spaces, where elements are functions. Normally, some restrictions are imposed upon the class of functions that generate a linear space. For example, a linear space can be generated by continuous periodic functions with some fixed period $T > 0$. Continuous periodic vector-functions with the period T also constitute a linear space.

A subset L_1 of the space L is a linear space, which has the same rules of procedure as in L, and is called a subspace. For example, a family of all continuous periodic functions with the period T and the zero mean value, is called a subspace of the space of all continuous T-periodic functions.

Linear space is a purely algebraic concept. To perform an analysis in this space, we should introduce the concept of distance between the elements. The distance can be introduced for the elements of an arbitrary set X as follows. For any two elements f, $g \in X$, we introduce a number $d(f, g)$ with the following properties:

(i) $d(f, g) = 0 \to f = g$ and, vice versa, if $f = g$, then $d(f, g) = 0$,
(ii) $d(f, g) = d(g, f)$,
(iii) $d(f, g) = d(f, h) + d(h, g)$ (inequality of triangle).

The function $d(f, g)$ is called a metrics, and the set X with this metrics is called a metric space.

We are only interested in the metric spaces that are the linear spaces at the same time. We can introduce a metrics in a linear space using the concept of the norm.

The norm in the linear space L is introduced as follows. Each element $f \in L$ corresponds to the number $||f||$ (norm of the element) that has the following properties:

(i) $||f|| \geq 0$ and $||f|| = 0$ if and only if $f = 0$,
(ii) $||\alpha f|| = |\alpha|||f||$ for any number α,
(iii) $||f + g|| \leq ||f|| + ||g||$ (inequality of triangle).

The linear space with the norm is called the normalized linear space. This space is a metric space. It is easy to see that the metrics can be determined by the formula

$$d(f, g) = ||f - g||.$$

In the normalized linear space L, we introduce a convergence. Let f_n $(n = 1, 2, \dots)$ be a sequence of elements of L. This sequence converges to the element $f \in L$ if $||f_n - f|| \to 0$ as $n \to \infty$. It follows from the inequality of triangle that the sequence has a unique limit.

The sequence f_j $(j = 1, 2, \dots)$ is called **fundamental** if for any $\varepsilon > 0$ it is possible to specify a positive integer $N(\varepsilon)$ such that for $m, n > N(\varepsilon)$ the following inequality holds

$$||f_n - f_m|| < \varepsilon.$$

The Cauchy principle asserts that a fundamental sequence of real numbers converges to a real number. In an arbitrary normalized linear space, the Cauchy principle does not hold.

Let us bring in some examples. In a set of rational numbers, we take a modulus of a rational number as the norm. Then the set of the rational numbers with this norm generates a normalized linear space r. The fundamental sequence of the rational numbers converges but its limit can also be a real number, which is not an element of the space r. As the second example, we take a family of polynomials defined on the interval $[0, 1]$. We introduce the norm:

$$||P_k(t)|| = \max_{t \in [01]} |P_k(t)|,$$

where $|P_k(t)|$ is a modulus of the polynomial $P_k(t)$. We obtain a normalized linear space. The properties of the norm result from the properties of the modulus. Convergence on this space coincides with the uniform convergence of the sequence of polynomials on the interval $[0, 1]$ though the sequence of polynomials on $[0, 1]$ can uniformly converge to an arbitrary continuous function rather than to a polynomial.

A normalized linear space, where every fundamental sequence converges to an element of this space, is called a complete normalized linear space or a **Banach space**.

C.2 Linear Operators

Let E_x and E_y be the normalized linear spaces. If each element $x \in E_x$ corresponds to an element $y \in E_y$ by a certain rule, then we say that an operator A is defined. The space E_x is called the domain of definition of the operator A, and E_y is the range of values of the operator A. We write $Ax = y$. The operator can also act in one space. Then its domain of definition and the range of values lie within the same space and may not coincide with the whole space. The operator A is called linear if the following two conditions hold for any x,y

(i) $A(x + y) = Ax + Ay$,

(ii) $A(\alpha x) = \alpha Ax$ for any scalar α from the collection of scalars of the given space E_x.

We bring in an example of a linear operator. Consider one of the most frequently used Banach spaces. By $C[0, 1]$ we denote a linear space of continuous function determined on the interval $[0, 1]$ with the norm

$$||x(t)|| = \max_{t\in[01]} |x(t)|.$$

We show that this space is a Banach space. Let $x_n(t) \in C[0, 1]$, $n = 1, 2, \ldots$ be a sequence such that

$$||x_n(t) - x_m(t)|| \to 0$$

for $m, n \to \infty$. It means that for the sequence $x_n(t)$, the Cauchy condition of the uniform convergence on $[0, 1]$ holds. Let $x_0(t)$ be the limit of the sequence $x_n(t)$. As a limit of the uniformly convergent sequence, the function $x_0(t)$ is continuous. Hence, $x_0(t) \in C[0, 1]$ and $||x_n(t) - x_0(t)|| \to 0$. Therefore, $C[0, 1]$ is a Banach space. We determine the operator in the space $C[0, 1]$ by the formula

$$y = Ax = \int_0^1 K(t, s)x(s)ds, \tag{C.1}$$

where the function $K(t, s)$ is continuous in the family of variables in the square $0 \le t \le 1$, $0 \le s \le 1$. Evidently, if $x(t) \in C[0,1]$, then $y(t) \in C[0,1]$. The operator A maps the space $C[0,1]$ onto itself. It is easy to verify that the operator A is linear.

The second example is the operator

$$y = Bx = \int_0^t x(s)ds \qquad (C.2)$$

in the space $C[0,1]$. The domain of definition of the operator B is the whole space. The range of values consists of continuously differentiable functions $y(t)$, which meet the condition $y(0) = 0$, since the continuous function integral with a varying upper limit is continuously differentiable. Linearity of the operator B is directly testable.

The linear operator A is continuous if $||Ax_n - Ax|| \to 0$ as $||x_n - x|| \to 0$. It is easy to verify that operators (C.1), (C.2) are continuous.

The operator A is called bounded if there exists a constant $M > 0$ such that $||Ax|| \le M||x||$ for $x \in E_x$.

It turns out that the operator A is continuous if and only if it is bounded.

Norm of an Operator Let A be a linear bounded operator. The least constant M that obeys the inequality

$$||Ax|| \le M||x||$$

is called the norm of the operator A and is denoted by $||A||$.

Hence, by definition, the number $||A||$ has the following properties:

a) for any $x \in E_x$ the following inequality holds true

$$||Ax|| \le ||A||||x||,$$

b) for any $\varepsilon > 0$ there exists an element x_ε such that

$$||Ax_\varepsilon|| > (||A|| - \varepsilon)||x_\varepsilon||.$$

The norm of the operator A is determined by the formula

$$||A|| = \sup_{||x|| \le 1} ||Ax||. \qquad (C.3)$$

Let us show that this formula holds true. If $||x|| \le 1$, then

$$||Ax|| \le ||A||||x|| \le ||A||.$$

Hence,

$$\sup_{||x|| \le 1} ||Ax|| \le ||A||. \qquad (C.4)$$

We take the element x_ε from property b) of the operator norm. Let

$$z_\varepsilon = \frac{x_\varepsilon}{||x_\varepsilon||}.$$

Then

$$||Az_\varepsilon|| = \frac{1}{||x_\varepsilon||}||Ax_\varepsilon|| > \frac{1}{||x_\varepsilon||}(||A|| - \varepsilon)||x_\varepsilon|| = ||A|| - \varepsilon.$$

Since $||z_\varepsilon||$=1, we have

$$\sup_{||x||\le 1} ||Ax|| \ge ||Az_\varepsilon|| > ||A|| - \varepsilon.$$

Thus,

$$\sup_{||x||\le 1} ||Ax|| \ge ||A||. \tag{C.5}$$

Inequalities (C.4) and (C.5) imply that (C.3) is valid.

For the operator A determined by (C.1), it is easy to give an upper estimate for the norm. Indeed,

$$||Ax|| = \max\nolimits_{t\in[0,1]} |\int_0^1 K(t,s)x(s)ds| \le$$
$$\le \max\nolimits_{t\in[0,1]} \int_0^1 |K(t,s)|ds \max\nolimits_{t\in[0,1]} |x(t)| = \max\nolimits_{t\in[0,1]} \int_0^1 |K(t,s)|ds||x(t)||.$$

Hence,

$$||A|| \le \int_0^1 |K(t,s)|ds. \tag{C.6}$$

A more sophisticated reasoning shows that inequality (C.6) can be an equality.

In applications, it is often sufficient to give an upper estimate of the norm of an operator.

C.3 Inverse Operators

For the linear operator A in the space E, an inverse operator is an operator B such that

$$AB = BA = I,$$

where I is a unit operator. We write $B = A^{-1}$. The concept of the inverse operator is related to the problems of the existence and uniqueness of solutions to the operator equation

$$Ax = y, \tag{C.7}$$

where y is a defined element of the space E, and x is the desired element of the space E. The linear algebraic equations, linear differential equations, linear integral equations etc. relate to the equations of the form (C.7) If there exists an operator A^{-1}, then the operator equation (C.7) has the solution

$$x = A^{-1}y$$

obtained by a direct substitution of the latter equation into (C.7). Let x_1 be another solution of equation (C.7), i.e.

$$Ax_1 = y.$$

Then, applying the operator A^{-1} to both sides of the latter equation, we arrive at

$$A^{-1}Ax_1 = x_1 = A^{-1}y = x.$$

Consequently, the solution $x = A^{-1}y$ is unique.

It is easy to verify that the operator A^{-1} is linear if A is a linear operator. At the same time, the continuity of A does not imply the continuity of the inverse operator in the general case. It may turn out that the operator A is unbounded, while the inverse operator is bounded.

Let us bring in a few theorems that assign the existence conditions for an inverse linear bounded operator. The operator A maps its domain of definition onto the range of values. If this mapping is one-to-one, then there exists an inverse linear operator A^{-1}.

Theorem C.1. *Let the operator A determined on the space E for any $x \in E$ meet the condition*

$$||Ax|| \geq m||x||, \quad m > 0, \tag{C.8}$$

where m is a constant. Then there exists the linear bounded operator A^{-1}.

Proof. It follows from condition (C.8) that the operator A is one-to-one. If $Ax_1 = y$ and $Ax_2 = y$, then $A(x_2 - x_1) = 0$ and, as per (C.8)

$$m||x_1 - x_2|| \leq ||A(x_1 - x_1)|| = 0,$$

whence it follows that $x_1 = x_2$. Therefore, there exists the inverse linear operator A^{-1}. Boundedness of the operator A^{-1} follows from inequality (C.8):

$$||A^{-1}y|| \leq \frac{1}{m}||AA^{-1}y|| = \frac{1}{m}||y||.$$

The latter inequality also implies that $||A^{-1}|| \leq \frac{1}{m}$.

Theorem C.2. *Let A be a linear bounded operator in the Banach space E. If*

$$||A|| \leq q < 1,$$

then the inverse of operator $(I - A)$ is inverse bounded and

$$||(I - A)^{-1}|| \leq \frac{1}{1-q}.$$

Proof. Consider an operator series

$$I + A + A^2 + \cdots + A^n + \ldots.$$

This series is dominated by the convergent numerical series (we keep in mind that $||I|| = 1$, $||A^n|| \leq ||A||^n$)

$$1 + ||A|| + ||A||^2 + \cdots + ||A||^n + \cdots \leq 1 + q + q^2 + \cdots + q^n + \cdots = \frac{1}{1-q}.$$

Therefore, the series converges to some operator denoted by B. We have

$$\begin{aligned} &B(I - A) = (I + A + A^2 + \cdots + A^n + \ldots)(I - A) = \\ &= (I + A + A^2 + \cdots + A^n + \ldots) - (A + A^2 + \cdots + A^{n+1} + \ldots) = I \end{aligned}$$

and, similarly, $(I - A)B = I$. Hence, $B = (I - A)^{-1}$ and $||(I - A)^{-1}|| \leq \frac{1}{1-q}$.

The following theorem is the Banach Theorem on Inverse Operator, which is not presented here in its full generality. The unbounded linear operator A is not continuous. Generally speaking, the fact that the sequence $x_n \to x_0$ by the norm does not imply that the sequence Ax_n tends to some limit. However, some unbounded linear operators have a weaker property that partly substitutes for the continuity.

Let A be a linear operator with a domain of definition $D(A)$. If the conditions $x_n \in D(A)$, $x_n \to x_0$, $Ax_n \to y_0$ imply that $x_0 \in D(A)$ and $Ax_0 = y_0$, then A is called a closed operator.

Theorem C.3. *Let A be a closed linear operator such that its range of values coincides with the whole space. If it is one-to-one, then the existing inverse operator A^{-1} is bounded.*

Thus, if for the closed linear operator A in the Banach space E the operator equation

$$Ax = y$$

for any $y \in E$ has a unique solution $x \in D(A)$, then there exists the continuous inverse operator A^{-1}.

As an example to the Banach Theorem, we consider the differential operator $Ax = \frac{dx}{dt}$ in the space $C[0, 1]$ the operator being determined on the functions that meet the condition

$$x(0) = 0. \qquad (C.9)$$

Hence, the domain $D(A)$ of definition of the operator A in the space $C[0,1]$ consists of continuously differentiable functions satisfying condition (C.9).

The operator A is unbounded. To prove this, we consider a sequence of functions

$$x_n(t) = \frac{1}{\sqrt{n}} \sin nt, \quad n = 1, 2, \ldots .$$

All functions of this sequence lie within the domain of definition of the operator A, and the sequence by the norm uniformly converges to $x_0(t) \equiv 0$, whereas the sequence $Ax_n = \sqrt{n} \cos nt$ is unbounded in the norm.

The operator A is closed. If $x_n \to x_0$ and $Ax_n \to y$, then, in terms of analysis, it means that the sequence of the continuous functions $x_n(t)$ uniformly converges to the function $x_0(t)$, and the sequence of the derivatives $x'_n(t)$ uniformly converges to the function $y(t)$. By virtue of the known theorem of analysis, $x'_n(t)$ converges to the function $x'_0(t)$, which implies the closeness of the operator.

The range of values of the operator coincides with the whole space. This results from the fact that the problem

$$\frac{dx}{dt} = z(t) \quad x(0) = 0$$

for any $z \in C[0,1]$ has a unique solution

$$x(t) = \int_0^t z(s)ds. \tag{C.10}$$

Hence, Theorem C.3 implies that the operator A has a continuous inverse operator determined by formula (C.10).

C.4 Principle of Contraction Mappings

The method of successive approximations is well known and is widely used to prove the existence of the solutions to algebraic, differential, integral equations and to construct approximate solutions. Within the functional analysis, this method follows a general scheme and results in the principle of contraction mappings.

Theorem C.4. *Let the Banach space E enclose the operator A that maps the points from E back onto the points of this space and for all $x, y \in E$*

$$||A(x) - A(y)|| \leq q||x - y||, \tag{C.11}$$

where $q < 1$ is independent of x and y. Then exists there one and only one point $x_0 \in E$ such that $A(x_0) = x_0$.

The point x_0 is called a fixed point of the operator A. If the operator A satisfies inequality (C.11), then it is called a contracting operator in the space E.

Proof. Take an arbitrary element $x \in E$ and construct a sequence

$$x_1 = A(x),\, x_2 = A(x_1), \ldots, x_n = A(x_{n-1}), \ldots.$$

Let us show that x_n is the Cauchy sequence. To do this, note that

$$\begin{array}{l} ||x_1 - x_2|| = ||A(x) - A(x_1)|| \le q||x - x_1|| = q||x - A(x)||, \\ ||x_2 - x_3|| = ||A(x_1) - A(x_2)|| \le q||x_1 - x_2|| \le q^2||x - A(x)||, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ ||x_n - x_{n+1}|| \le q^n||x - A(x)||, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

Then

$$\begin{array}{l} ||x_n - x_{n+m}|| \le ||x_n - x_{n+1}|| + \\ ||x_{n+1} - x_{n+2}|| + \cdots + ||x_{n+m-1} - x_{n+m}|| \le \\ \le (q^n + q^{n+1} + \cdots + q^{n+m-1})||x - A(x)|| = \\ = \frac{q^n - q^{n+m}}{1-q}||x - A(x)||. \end{array} \qquad (C.12)$$

Since $q < 1$, we obtain

$$||x_n - x_{n+m}|| \le \frac{q^n}{1-q}||x - A(x)||.$$

Hence, $||x_n - x_{n+m}|| \to 0$ as $n \to \infty$, $m > 0$. Consequently, the sequence x_n converges to some element $x_0 \in E$. We shall prove that $A(x_0) = x_0$. We have the inequality

$$\begin{array}{l} ||x_0 - A(x_0)|| \le ||x_0 - x_n|| + ||x_n - A(x_0)|| = \\ = ||x_0 - x_n|| + ||A(x_{n-1} - A(x_0)|| \le \\ \le ||x_0 - x_n|| + q||x_{n-1} - x_0||. \end{array}$$

For any $\varepsilon > 0$, and sufficiently large n, the following inequalities hold

$$||x_0 - x_n|| < \frac{\varepsilon}{2}, \quad ||x_{n-1} - x_0|| < \frac{\varepsilon}{2}.$$

Hence,

$$||x_0 - A(x_0)|| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we see that $||x_0 - A(x_0)|| = 0$, i.e. $A(x_0) = x_0$. It remains to show that the operator $A(x)$ has a unique fixed point. Assume there are two such points:

$$A(x_0) = x_0, \quad A(y_0) = y_0,$$

then

$$||x_0 - y_0|| = ||A(x_0) - A(y_0)|| \leq q||x_0 - y_0||.$$

Since $q < 1$, we have $x_0 = y_0$.

Remark C.1. Changing over to a limit in formula (C.12) as $m \to \infty$ allows estimation of the n-th approximation

$$||x_n - x_0|| \leq \frac{q^n}{(1-q)}||x - A(x)||.$$

Remark C.2. Successive approximations of x_n can be constructed for any element $x \in E$. The choice of the element will only affect the speed of the convergence of x_n to the fixed point x_0.

Remark C.3. Suppose in the Banach space E there exists a closed ball with the center at the point $a \in E$ and radius $r > 0$. This ball is called the set $S(a, r)$ composed of the points $x \in E$ satisfying the inequality $||x - a|| \leq r$. We often have to consider an operator A such that it is contracting on the ball $S(a, r)$. Then the principle of contraction mappings can be applied under one additional condition that the operator A maps this ball onto itself, and the successive approximations do not go outside this ball. For example, let, additionally to inequality (C.11), the inequality

$$||a - A(a)| \leq (1 - q)r$$

hold. Then, if $x \in S(a, r)$, so $A(x) \in S(a, r)$. Indeed,

$$||a - A(x)|| \leq ||A(x) - A(a)|| + ||A(a) - a|| \leq q||x - a|| + (1 - q)r \leq$$

$$\leq qr + (1 - q)r = r.$$

The operator A will have the single fixed point in the ball $S(a, r)$.

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