OXFORD MATHEMATICAL MONOGRAPHS

OXFORD MATHEMATICAL MONOGRAPHS

Hirschfeld: *Finite projective spaces of three dimensions*
Edmunds and Evans: *Spectral theory and differential operators*
Pressley and Segal: *Loop groups, paperback*
Evens: *Cohomology of groups*
Hoffman and Humphreys: *Projective representations of the symmetric groups:* Q-*Functions and Shifted Tableaux*
Amberg, Franciosi, and Giovanni: *Products of groups*
Gurtin: *Thermomechanics of evolving phase boundaries in the plane*
Faraut and Koranyi: *Analysis on symmetric cones*
Shawyer and Watson: *Borel's methods of summability*
Lancaster and Rodman: *Algebraic Riccati equations*
Thévenaz: *G-algebras and modular representation theory*
Baues: *Homotopy type and homology*
D'Eath: *Black holes: gravitational interactions*
Lowen: *Approach spaces: the missing link in the topology–uniformity–metric triad*
Cong: *Topological dynamics of random dynamical systems*
Donaldson and Kronheimer: *The geometry of four-manifolds, paperback*
Woodhouse: *Geometric quantization, second edition, paperback*
Hirschfeld: *Projective geometries over finite fields, second edition*
Evans and Kawahigashi: *Quantum symmetries of operator algebras*
Klingen: *Arithmetical similarities: Prime decomposition and finite group theory*
Matsuzaki and Taniguchi: *Hyperbolic manifolds and Kleinian groups*
Macdonald: *Symmetric functions and Hall polynomials, second edition, paperback*
Catto, Le Bris, and Lions: *Mathematical theory of thermodynamic limits: Thomas-Fermi type models*
McDuff and Salamon: *Introduction to symplectic topology, paperback*
Holschneider: *Wavelets: An analysis tool, paperback*
Goldman: *Complex hyperbolic geometry*
Colbourn and Rosa: *Triple systems*
Kozlov, Maz'ya and Movchan: *Asymptotic analysis of fields in multi-structures*
Maugin: *Nonlinear waves in elastic crystals*
Dassios and Kleinman: *Low frequency scattering*
Ambrosio, Fusco and Pallara: *Functions of bounded variation and free discontinuity problems*
Slavyanov and Lay: *Special functions: A unified theory based on singularities*
Joyce: *Compact manifolds with special holonomy*
Carbone and Semmes: *A graphic apology for symmetry and implicitness*
Boos: *Classical and modern methods in summability*
Higson and Roe: *Analytic K-homology*
Semmes: *Some novel types of fractal geometry*
Iwaniec and Martin: *Geometric function theory and nonlinear analysis*
Johnson and Lapidus: *The Feynman integral and Feynman's operational calculus, paperback*
Lyons and Qian: *System control and rough paths*
Ranicki: *Algebraic and geometric surgery*
Ehrenpreis: *The Radon transform*
Lennox and Robinson: *The theory of infinite soluble groups*
Ivanov: *The Fourth Janko Group*
Huybrechts: *Fourier-Mukai transforms in algebraic geometry*
Hida: *Hilbert modular forms and Iwasawa theory*
Boffi and Buchsbaum: *Threading Homology through algebra*
Vazquez: *The Porous Medium Equation*
Benzoni-Gavage and Serre: *Multi-dimensional hyperbolic partial differential equations*

Multidimensional Hyperbolic Partial Differential Equations

First-order Systems and Applications

Sylvie Benzoni-Gavage
Université Claude Bernard Lyon I
Lyon, France

Denis Serre
ENS de Lyon
Lyon, France

CLARENDON PRESS • OXFORD

2007

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Cape Town Dar es Salaam Hong Kong Karachi
Kuala Lumpur Madrid Melbourne Mexico City Nairobi
New Delhi Shanghai Taipei Toronto

With offices in

Argentina Austria Brazil Chile Czech Republic France Greece
Guatemala Hungary Italy Japan Poland Portugal Singapore
South Korea Switzerland Thailand Turkey Ukraine Vietnam

Oxford is a registered trade mark of Oxford University Press
in the UK and in certain other countries

Published in the United States
by Oxford University Press Inc., New York

First published 2007

British Library Cataloguing in Publication Data

Data available

Library of Congress Cataloging in Publication Data

Data available

Typeset by SPI Publisher Services, Pondicherry, India
Printed in Great Britain
on acid-free paper by
Biddles Ltd., King's Lynn, Norfolk

ISBN 0-19-921123-X 978-0-19-921123-4

1 3 5 7 9 10 8 6 4 2

PREFACE

Hyperbolic Partial Differential Equations (PDEs), and in particular first-order systems of conservation laws, have been a fashionable topic for over half a century. Many books have been written, but few of them deal with genuinely multidimensional hyperbolic problems: in this respect the most classical, though not so well-known, references are the books by Reiko Sakamoto, by Jacques Chazarain and Alain Piriou, and by Andrew Majda. Quoting Majda from his 1984 book, "*the rigorous theory of multi-D conservation laws is a field in its infancy*". We dare say it is still the case today. However, some advances have been made by various authors. To speak only of the stability of shock waves, we may think in particular of: Métivier and coworkers, who continued Majda's work in several interesting directions – weak shocks, lessening the regularity of the data, elucidation of the 'block structure' assumption in the case of characteristics with constant multiplicities (we shall speak here of *constantly hyperbolic* operators); Freistühler, who extended Majda's approach to undercompressive shocks, of which an important example is given by phase boundaries in van der Waals fluids, as treated by Benzoni-Gavage; Coulombel and Secchi, who dealt very recently with neutrally stable discontinuities (2D-vortex sheets), thanks to Nash–Moser techniques.

Even though it does not pretend to cover the most recent results, this book aims at presenting a comprehensive view of the state-of-the-art, with particular emphasis on problems in which modern tools of analysis have proved useful. A large part of the book is indeed devoted to initial boundary value problems (IBVP), which can only be dealt with by using symbolic symmetrizers, and thus necessitate pseudo-differential calculus (for smooth, non-constant coefficients) or even para-differential calculus (for rough coefficients and therefore also non-linear problems). In addition, the construction of symbolic symmetrizers conceals intriguing questions related to algebraic geometry, which were somewhat hidden in Kreiss' original paper and in the book by Chazarain and Piriou. In this respect we propose here new insight, in connection with constant coefficient IBVPs. Furthermore, the analysis of (linear) IBVPs, which are important in themselves, enables us to prepare the way for the (non-linear) stability analysis of shock waves. In the matter of complexity, stability of shocks is the culminating topic in this book, which we hope will contribute to make more accessible some of the finest results currently known on multi-D conservation laws. Finally, quoting Constantin Dafermos from his 2000 book, '*hyperbolic conservation laws and gas dynamics have been traveling hand-by-hand over the past one hundred and fifty years*'. Therefore it is not a surprise that we devote a significant part of this book to that specific and still important application. The idea of dealing with 'real' gases was inspired by the PhD thesis of Stéphane Jaouen after Sylvie Benzoni-Gavage was asked by his advisor, Pierre-Arnaud Raviart to act as a referee in the defense.

This volume contains enough material for several graduate courses – which were actually taught by either one of the authors in the past few years – depending on the topic one is willing to emphasize: hyperbolic Cauchy problem and IBVP, non-linear waves, or gas dynamics. It provides an extensive bibliography, including classical papers and very recent ones, both in PDE analysis and in applications (mainly to gas dynamics). From place to place, we have adopted an original approach compared to the existing literature, proposed new results, and filled gaps in proofs of important theorems. For some highly technical results, we have preferred to point out the main tools and ideas, together with precise references to original papers, rather than giving extended proofs.

We hope that this book will fulfill the expectations of researchers in both hyperbolic PDEs and compressible fluid dynamics, while being accessible to beginners in those fields. We have tried our best to make it self-contained and to proceed as gradually as possible (at the price of some repetition), so that the reader should not be discouraged by her/his first reading.

We warmly thank Jean-François Coulombel, whose PhD thesis (under the supervision of Benzoni-Gavage and with the kind help of Guy Métivier) provided the energy necessary to complete the writing of the most technical parts, for his careful reading of the manuscript and numerous useful suggestions. We also thank our respective families for their patience and support.

Lyon, April 2006

Sylvie Benzoni-Gavage
Denis Serre

CONTENTS

INTRODUCTION

Within the field of Partial Differential Equations (PDEs), the hyperbolic class is one of the most diversely applicable, mathematically interesting and technically difficult: these (certainly biased) qualifying terms may serve as milestones along an overview of the field, which we propose prior to entering the bulk of this book.

Applicability. Hyperbolic PDEs arise as basic models in many applications, and especially in various branches of physics in which finite-speed propagation and/or conservation laws are involved. To quote a few, and nonetheless fundamental examples, let us start with *linear* hyperbolic PDEs. The most ancient one is undoubtedly the wave equation – also known in one space dimension as the equation of vibrating strings – dating back to the work of d'Alembert in the eighteenth century, which is closely related to the transport equation. We also have in mind the Maxwell system of electromagnetism, as well as the equation associated with the Dirac operator. Theoretical physics is a source of several *semilinear* equations and systems – semilinearity being characterized by a linear principal part and non-linear terms in the subprincipal part – for example, the Klein/sine–Gordon equations, the Yang–Mills equations, the Maxwell system for polarized media, etc. The *non-linear* models – often quasilinear – are even more numerous. The most basic one is provided by the so-called Euler equations of gas dynamics, which opened the way (controversially) in the late nineteenth century to the shock waves theory (later revived, in the 1940s, by the atomic bomb research, and still of interest nowadays for more peaceful applications, in medicine for instance). Speaking of flows, a prototype of scalar, one-dimensional conservation law was introduced in the 1950s in traffic flow modelling (under some heuristic assumptions on the drivers' behaviour), which is nowadays referred to as the Lighthill–Whitham–Richards model. Other non-linear hyperbolic models include: the equations of elastodynamics (of which a linear version is widely used, in the modelling of earthquakes as well as in engineering problems with small deformations); the equations of chemical separation (chromatography, electrophoresis); the magnetohydrodynamics (MHD) equations – the coupling between fluid dynamics and electromagnetism being quite relevant for planets and other astrophysical systems – the Einstein equations of general relativity; non-linear versions of the Maxwell system for strong fields, for example the Born–Infeld model. Hyperbolic equations may also arise as a byproduct of an elaborate piece of analysis, as in the modulation theory of integrable Hamiltonian PDEs (like the Korteweg–de Vries equation and some non-linear Schrödinger equations), in which the envelopes of oscillating solutions are described by solutions of the (hopefully hyperbolic) Whitham equations.

This list of hyperbolic PDEs is by no means exhaustive. Of course most of them are to some extent approximate: more realistic models should also involve dissipation processes (for instance in continuum mechanics) or higher-order phenomena, and thus be (at least partially) parabolic or dispersive. However, large-scale phenomena are usually governed by the hyperbolic part: the relevance of hyperbolic PDEs in many applications is in no doubt.

Mathematical interest. For both mathematical reasons and physical relevance, hyperbolicity is associated with a space–time reference frame, in the sense that there exists a co-ordinate (most often the physical time) playing a special role compared to the other co-ordinates (usually spatial ones). Of course, changes of co-ordinates are always possible and we may speak of time-like co-ordinates and of space-like hypersurfaces: this terminology is familiar to people used to general relativity, and is also relevant in every situation where a hyperbolic operator is given. Except in one-dimensional frameworks, it is by no means possible to interchange the role of space and time variables: the distinction between time and space is a crucial feature of multidimensional hyperbolic PDEs, as we shall see in the analysis of Initial Boundary Value Problems.

Multidimensional hyperbolic PDEs constrast with one-dimensional ones from several points of view, in particular in connection with the important notion of *dispersion*. Indeed, recall that the most visible feature of hyperbolic PDEs is finite-speed propagation. In several space dimensions, when the information is propagated not merely by pure transport, it gets dispersed: this dispersion of signals is itself responsible for a damping phenomenon in all L^p norms with $p > 2$ (by contrast with what usually happens with the L^2 norm, independent of time by a conservation of energy principle), and is associated with special, space–time estimates called *Strichartz estimates*, obtained by fractional integration – Strichartz estimates have been proved much fruitful in particular in the analysis of semilinear hyperbolic Cauchy problems.

Another point worth mentioning is the diversity of mathematical tools that have been found useful to the theory of (linear) multidimensional hyperbolic PDEs, ranging from microlocal analysis to algebraic topology (not to mention those that still need to be invented, as we shall suggest below!). The former has been widely used to study the propagation of singularities in wave-like equations. In the same spirit, pseudo- (or even para-) differential calculus is of great help to study linear hyperbolic problems with variable coefficients, as we shall see in the third and fourth parts of this book. The link to algebraic topology might seem less obvious to unaware readers and deserves a little explanation. When studying constant-coefficients hyperbolic operators we are led to consider, in the frequency space, algebraic manifolds called *characteristic cones* – which are by definition zero sets of symbols, and are linked to finite-speed propagation. The fundamental solution, say E, of a constant-coefficients hyperbolic PDE is indeed known to be supported by the convex hull of Γ, the forward part of the dual of the characteristic cone. In some cases, it happens that E is supported by Γ only; the open set $\text{co}(\Gamma) \setminus \Gamma$, on which E vanishes, is then called a *lacuna*. For example, the wave equation in dimension $1+d$ with d odd and $d \geq 3$, has a lacuna: its fundamental solution is supported by the dual characteristic cone itself (this explains, for instance, the fact that light rays have no tail). The systematic study of *lacunæ* is related to the topology of real algebraic sets.

Compared to linear ones, non-linear problems display fascinating new features. In particular, several kinds of *non-linear waves* arise (shocks, rarefaction waves, as well as contact discontinuities). They are present already in one space dimension. The occurrence of shock waves is connected with a loss of regularity in the solutions in finite time, which can be roughly explained as follows: non-linearity implies that wave speeds depend on the state; therefore, a non-constant solution experiences a wave overtaking, which results in the creation of discontinuities in the derivatives of order $m-1$, if m is the order of the system; such discontinuities are called shock waves, or simply shocks. After blow-up, that is after creation of shock(s), solutions cannot be smooth any longer. This yields many questions: what is the meaning of the PDEs for non-smooth solutions; can we solve the system in terms of weak enough solutions, and if possible in a unique, physically relevant way? The answer to the first question has been given by the theory of distributions, which is somehow the mathematical counterpart of conservation principles in physics: conservation of mass, momentum and energy, for instance (or Ampère's and Faraday's laws in electromagnetism) make sense indeed as long as fields remain locally bounded. The drawback is – as has long been known – that weak solutions are by no means unique, and this seems to hurt the common belief that PDE models in physics describe deterministic processes. This apparent contradiction may be resolved by the use of a suitable *entropy condition*, most often reminiscent of the second principle of thermodynamics. In one space dimension, entropy conditions have been widely used in the last decades to prove global well-posedness results in the space of Bounded Variations (BV) functions – a space known to be inappropriate in several space dimensions, because of the obstruction on the L^p norms (see below for a few more details). Entropy conditions are expected to ensure also multidimensional well-posedness, even though we do not know yet what would be an appropriate space: one of the goals of this book is to present a starting point in this direction, namely (local in time) well-posedness within classes of piecewise smooth solutions.

Finally, the concept of time reversibility is quite intriguing in the framework of hyperbolic PDEs. On the one hand, as far as smooth solutions are concerned, many hyperbolic problems are time reversible, and this seems incompatible with the decay (already mentioned above) of L^p norms for $p > 2$ in several space dimensions. This paradox was actually resolved by Brenner [22, 23], who proved that multidimensional hyperbolic problems are ill-posed, in Hadamard's sense, in L^p for $p \neq 2$. Incidentally, Brenner's result shows that the space BV, which is built upon the space of bounded measures, itself close to L^1, cannot be appropriate for multidimensional problems. On the other hand, time reversibility is lost (as a mathematical counterpart of the second principle of thermodynamics) once shocks develop, whence a loss of information, the backward problem becoming ill-posed. As a matter of fact, shocks may be viewed as free boundaries and as such they can be sought as solutions of (non-standard) hyperbolic Initial Boundary Value Problems (IBVP): it turns out that most of the well-posed hyperbolic IBVPs are irreversible, as will be made clear in particular in this book – a large part of this volume is indeed dedicated to a systematic study of IBVPs, either for themselves, or in view of applications to well-posedness in the presence of shock waves.

Difficulty. Even when a functional framework is available, a rigorous analysis of hyperbolic problems often requires much more elaborate (or at least more technical) tools than for elliptic or parabolic problems, notably to cope with the lack of smoothing effects. The situation is even worse in the non-linear context, where functional analysis has been useless in the study of weak entropy solutions so far (except for first-order scalar equations). This is why our knowledge of global-in-time solutions is so poor, despite tremendous efforts by talented mathematicians. Speaking only about the Cauchy problem for quasilinear systems of first-order conservation laws, in space dimension d with n scalar unknowns, we know about well-posedness only in the following cases.

- Scalar problems ($n = 1$), thanks to Kruzkhov's theory [105].
- One space dimension ($d = 1$) and small data of bounded variation: existence results date back to Glimm's seminal work [70]; uniqueness and continuous dependence have been obtained by Bressan and coworkers (see, for instance, [25–27]).
- Small smooth data and large enough space dimension (for then dispersion can compete with non-linearity and prevent shock formation): most results from this point of view have been established by Klainerman and coworkers. See, for instance, Hörmander's book [88].

Amazingly enough, none of these results apply to such basic systems as the full gas dynamics equations in one space dimension ($n = 3$, $d = 1$) or the isentropic gas dynamics equations ($n = 2$) in dimension $d \geq 2$.

Other results solve only one part of the problem:

- Global existence for general data when $d = 1$ and $n = 2$ (under a *genuine non-linearity* assumption) by means of compensated compactness. This was achieved by DiPerna [49], following an idea by Tartar [202]. Solutions are then found in L^∞. Unfortunately, no uniqueness proof in such a large space has been given so far, except for weak–strong uniqueness (uniqueness in L^∞ of a classical solution).
- Local existence of smooth solutions for smooth data. This is quite a good result since it shows at least local well-posedness. It is attributed to several people (Friedrichs, Gårding, Kato, Leray, and possibly others),

depending on specific assumptions that were made. Unfortunately, its practical implications are limited by the smallness of the existence time – recall that shock formation precludes, in general, global existence results within smooth functions.

Having this (modest) state-of-the-art in mind, we can foresee a compromise regarding multidimensional weak solutions and non-linear problems: it will consist of the analysis of piecewise smooth solutions (involving a finite number of singularities like shock waves, rarefaction waves or contact discontinuities), tractable by 'classical' tools. This is the point of view we have adopted here, which defines the scope of this book: we shall consider either (possibly weak) solutions of linear problems with smooth coefficients or piecewise smooth solutions of non-linear problems – Cauchy problems and also of Initial Boundary Value Problems – to multidimensional hyperbolic PDEs. We now present a more detailed description of the contents.

We have chosen a presentation involving gradually increasing degrees of difficulty: this is the case for the ordering of the three main 'theoretical' parts of the book – the first one being devoted to linear Cauchy problems, the second one to linear Initial Boundary Value Problems, and the third one to non-linear problems; this is also the case inside those parts – the first two parts starting with constant coefficients before going to variable coefficients, and the third one starting from Cauchy problems, then going to IBVPs, and culminating with the shock waves stability analysis. As a consequence, readers should be able to find the information they need without having to enter overcomplicated frameworks: most chapters are indeed (almost) self-contained (and as a drawback, the book is not free from repetitions).

Another deliberate choice of ours has been to concentrate on first-order systems, even though we are very much aware that higher-order hyperbolic PDEs are also of great interest. This is mainly a matter of taste, because we come from the community of conservation laws. In addition, we think that the understanding of either one of those classes (first-order systems or higher-order scalar equations) basically provides the understanding of the other class (see, for instance, the book by Chazarain and Piriou [31], Chapter VII). Consistently with that choice, the main application we have considered is the first-order system of Euler equations in gas dynamics, to which the fourth part of the book is entirely devoted. We have tried to temperate this 'monomaniac' attitude by referring from place to place to higher-order equations, and in particular to the wave equation, which is the source of several examples throughout the theoretical chapters.

Finally, to keep the length of this book reasonable, we have decided not to speak of (nevertheless important) questions that are too far away from the shock waves theory. Thus the reader will not find anything about the propagation of singularities as developed by Egorov, Hörmander and Taylor. Likewise, non-local boundary operators as they appear, for instance, in absorbing or transparent boundary conditions will not be considered, and all numerical aspects of hyperbolic IBVPs will be omitted, despite their great theoretical and practical importance.

First part. The theory of linear Cauchy problems is most classical, even though some results are not that well-known. The chapter on constant-coefficient problems is the occasion of pointing out important definitions: Friedrichs symmetrizability; directions of hyperbolicity; strict hyperbolicity and more generally what we call *constant hyperbolicity* – the eigenvalues of the symbol of a so-called constantly hyperbolic operator are semisimple and of constant multiplicity, instead of being simple in the case of strict hyperbolicity. Throughout the book, all hyperbolic operators will be assumed either Friedrichs symmetrizable or constantly hyperbolic (or both), as is the case for most operators coming from physics. The chapter on variable-coefficients Cauchy problems presents, in more generality, the symmetrizers technique, and in particular introduces the notion of symbolic symmetrizers, thus illustrating the power of pseudo-differential calculus (for infinitely smooth coefficients) and even para-differential calculus (for coefficients of limited regularity).

Second part. The theory of Initial Boundary Value Problems (IBVP) is inspired from, but tremendously more complicated than, the theory of Cauchy problems. A kind of introductory chapter is devoted to the easier case of symmetric dissipative IBVPs. The second chapter addresses constant-coefficients IBVPs in a half-space, in which a central concept arises, namely the (uniform) Lopatinskiĭ condition. This stability condition dates back to the 1970s: simultaneously with a work by Lopatinskiĭ ([122], unnoticed in the West, Lopatinskiĭ being more famous for his older work on elliptic boundary value problems [121]), it was worked out by Kreiss [103], and independently by Sakamoto [174] for higher-order equations; in acknowledgement of Kreiss' work on first-order hyperbolic systems we shall rather call it the (uniform) Kreiss–Lopatinskiĭ condition, and we shall also speak of Kreiss' symmetrizers, which are symbolic symmetrizers adapted to IBVPs. The necessity of Kreiss' symmetrizers shows up indeed when a Laplace–Fourier transform is applied to the equations (Laplace in the time direction and Fourier in the spatial boundary direction): to obtain an a priori estimate without loss of derivatives we need to multiply the equations by a suitable matrix-valued function, depending homogeneously on space–time frequencies – thus being a symbol – in place of the energy tensor of the symmetric dissipative case; that matrix-valued symbol is what we call a Kreiss symmetrizer. The actual construction of Kreiss' symmetrizers is quite involved, and requires a good knowledge of linear algebra and real algebraic geometry. For this reason, a separate chapter is devoted to the construction of Kreiss' symmetrizers. The interplay with algebraic geometry (formerly developed by Petrovskiĭ, Oleinik and their school) is a deep reason why we need a structural assumption such as constant hyperbolicity: even with this, there remain tricky points to deal with, namely the so-called glancing points, where eigenvalues lack regularity. The chapter on variable-coefficient IBVPs focuses more on the calculus aspects of the theory: it shows how to extend well-posedness results to more general situations – variable coefficients with either infinite or limited regularity, non-planar boundaries – by means of pseudo- or para-differential calculus.

The remaining chapters of the second part are devoted to more peculiar topics: characteristic boundaries (which yield involved additional difficulties); homogeneous IBVPs (which turn out to require only a weakened version of the uniform Lopatinskiĭ condition); the so-called class WR, which consists of certain $\mathscr{C}^\infty$-well-posed problems and is generic in the sense that it is stable under small disturbances of the operators, but displays estimates with a loss of regularity. These topical chapters may be skipped by the reader insterested only in the applications to multidimensional shock stability.

Third part. We must admit that the current knowledge of non-linear multi-dimensional hyperbolic problems is very much limited: all well-posedness results presented in this part are *short-time* results; nevertheless, their proofs are not that easy. A first chapter reviews Cauchy problems: symmetric (or Friedrichs-symmetrizable) ones, but also those with symbolic symmetrizers (at is the case for constantly hyperbolic systems), for which well-posedness was not much known up to now (the only reference we are aware of is a proceedings paper by Métivier [132]). Well-posedness is to be understood in Sobolev spaces of sufficiently high index, or to be more precise, in $H^s(\mathbb{R}^d)$ with $s > d/2 + 1$ (the condition ensuring that $H^s(\mathbb{R}^d)$ is an algebra, whose elements are at least continuously differentiable, by Sobolev's theorem). In other words, we speak in that chapter only of smooth, or classical solutions, except in the very last section, where we recall the weak–strong uniqueness result of Dafermos and prepare the way for piecewise smooth solutions considered in the chapter on shock waves. Then 'standard' non-linear IBVPs are considered in a separate chapter, which is the occasion to see a simplified version of what is going on for shocks. The chapter on the persistence (or existence and stability) of single shock solutions was one of the main motivations to write this book. The idea was to give a comprehensive account of the work done by Majda in the 1980s [124–126], after it was revisited by Métivier and coworkers [56, 131, 133, 134, 136, 140]. Initially, we intended to cover also non-classical (multidimensional) shocks, as considered by Freistühler [58, 59] and Coulombel [40]. But for clarity we have preferred to concentrate on Lax shocks, while avoiding as much as possible to use their specific properties so that interested readers could either guess what happens for non-classical shocks or refer more easily to [40] for instance. We have also deliberately omitted the most recent developments on characteristic and/or non-constantly hyperbolic problems.

Fourth part. This concerns one of the most important applications of hyperbolic PDEs: gas dynamics. In fact, the theory of hyperbolic conservation laws was developed, in particular by Peter Lax in the 1950s, by *analogy with gas dynamics*: terms like 'entropy', 'compressive' (or 'undercompressive') shock are reminiscent of this analogy, and the so-called Rankine–Hugoniot jump conditions were initially derived (in the late nineteenth century) by these two engineers (Rankine and Hugoniot) in the framework of gas dynamics. There is a huge literature on gas dynamics, by engineers, by physicists and by mathematicians. In recent decades, the latter have had a marked preference for a familiar pressure law, usually referred to as the γ-law, for it simplifies, to some extent (depending on the explicit value of γ), the analysis of the Euler equations of gas dynamics. We have chosen here to consider more general pressure laws, which apply to so-called *real* – at least more realistic – fluids and not only perfect gases (as was the case in earlier mathematical papers, by Weyl [218], Gilbarg [69], etc.).

In a first chapter we address several basic questions, regarding hyperbolicity and symmetrizability. The second chapter is devoted to boundary conditions for real fluids, a very important topic for engineers, which has (surprisingly) not received much attention from mathematicians (see, however, the very nice review paper by Higdon [84]).

This applied part culminates with the shock-waves analysis for real fluids, in the last chapter. Even though it seems to belong to 'folklore' in the shock-waves community, the complete investigation of the Kreiss–Lopatinskiĭ condition for the Euler equations is hard to find in the literature: in particular, Majda gave the complete stability conditions in [126] but showed how to derive them only for isentropic gas dynamics; a complete, analytic proof was published only recently by Jenssen and Lyng [92]. By contrast, our approach is mostly algebraic, and works fine for full gas dynamics (of which the isentropic gas dynamics appear as a special, easier case). In addition, we give an explicit construction of Kreiss symmetrizers, which (to our knowledge) cannot be found elsewhere, and is fully elementary (compared to the sophisticated tools used for abstract systems).

Fifth part. This is only a (huge) appendix, collecting useful tools and techniques. The main topics are the Laplace transform – including Paley–Wiener theorems – pseudo-differential calculus, and its refinement called para-differential calculus. Less space demanding (or more classical) tools are merely introduced in the Notations section below.

NOTATIONS

The set of matrices with n rows and p columns, with entries in a field $\mathbb{K}$, is denoted by $\mathbf{M}_{n\times p}(\mathbb{K})$. If $p = n$, we simply write $\mathbf{M}_n(\mathbb{K})$. The latter is an algebra, whose neutral elements under addition and multiplication are denoted by 0_n and I_n, respectively. The space $\mathbf{M}_{n\times p}(\mathbb{K})$ may be identified to the set of linear maps from $\mathbb{K}^p$ to $\mathbb{K}^n$. The transpose matrix is written M^{T}. The group of invertible $n \times n$ matrices is $\mathbf{GL}_n(\mathbb{K})$. If $p = 1$, $\mathbf{M}_{n\times 1}(\mathbb{K})$ is identified with $\mathbb{K}^n$.

Given two matrices $M, N \in \mathbf{M}_n(\mathbb{C})$, their commutator $MN - NM$ is denoted by $[M, N]$.

If $\mathbb{K} = \mathbb{C}$, the adjoint matrix is written M^*. It is equal to $\overline{M}^{\mathrm{T}}$, where $\overline{M}$ denotes the conjugate of M. We equip $\mathbb{C}^m$ and $\mathbb{R}^m$ with the canonical Hermitian norm

$$\|x\| = \sqrt{\sum_j \mid x_j \mid^2} = (x^* x)^{1/2}.$$

This norm is associated to the scalar product

$$(x, y) = \sum_j x_j \overline{y}_j = y^* x.$$

The norm will sometimes be denoted $|x|$, especially when x is a space variable or a frequency vector (used in Fourier transform.)

A complex square matrix U is unitary if $U^*U = I_n$, or equivalently $UU^* = I_n$. The set $\mathbf{U}_n$ of unitary matrices is a compact subgroup of $\mathbf{GL}_n(\mathbb{C})$. Its intersection $\mathbf{O}_n$ with $\mathbf{M}_n(\mathbb{R})$ is the set of real orthogonal matrices. The special orthogonal group $\mathbf{SO}_n$ is the subgroup defined by the constraint $\det M = 1$.

As usual, $\mathbf{M}_{n\times p}(\mathbb{C})$ is equipped with the induced norm

$$\|M\| = \sup \frac{\|Mx\|}{\|x\|}.$$

When the product makes sense, one knows that $\|MN\| \le \|M\|\,\|N\|$. When $p = n$, $\mathbf{M}_n(\mathbb{C})$ is thus a normed algebra, and we have $\|M^k\| \le \|M\|^k$. If Q is a unitary (for instance real orthogonal) matrix, one has $\|Q\| = 1$. More generally, the norm is unitary invariant, which means that $\|M\| = \|PMQ\|$ whenever P and Q are unitary.

If $M \in \mathbf{M}_n(\mathbb{C})$, the set of eigenvalues of M, denoted by $\mathrm{Sp}M$, is called the *spectrum* of M. The largest modulus of eigenvalues of M is called the *spectral radius* of M, and denoted by $\rho(M)$. It is less than or equal to $\|M\|$, and such that

$$\rho(M) = \lim_{k\to+\infty} \|M^k\|^{1/k}.$$

The following formula holds,

$$\|M\|^2 = \rho(M^*M) = \rho(MM^*).$$

Several other norms on $\mathbf{M}_n(\mathbb{C})$ are of great interest, among which is the *Frobenius norm*, defined by

$$\|M\|_{\mathrm{F}} := \sqrt{\sum_{j,k} |\, m_{jk}|^2}.$$

Since $\|M\|_{\mathrm{F}}^2 = \mathrm{Tr}(M^*M) = \mathrm{Tr}(MM^*)$, we have $\|M\| \leq \|M\|_{\mathrm{F}}$.

A complex square matrix M is Hermitian if $M^* = M$. It is skew-Hermitian if $M^* = -M$. The Hermitian $n \times n$ matrices form an $\mathbb{R}$-vector space that we denote by $\mathbf{H}_n$. The cone of positive-definite matrices in $\mathbf{H}_n$ is denoted by $\mathbf{HPD}_n$. When M is Hermitian, we have $\|M\| = \rho(M)$. Every Hermitian matrix is diagonalizable with real eigenvalues, its normalized eigenvectors forming an orthonormal basis. The skew-Hermitian matrices with complex entries form an $\mathbb{R}$-vector space that we denote by $\mathbf{Skew}_n$. We remark that $\mathbf{M}_n(\mathbb{C}) = \mathbf{H}_n \oplus \mathbf{Skew}_n$ and $\mathbf{Skew}_n = i\mathbf{H}_n$. The intersections of $\mathbf{H}_n$, $\mathbf{HPD}_n$ and $\mathbf{Skew}_n$ with the subspace $\mathbf{M}_n(\mathbb{R})$ of matrices with real entries are denoted by $\mathbf{Sym}_n$, $\mathbf{SPD}_n$ and $\mathbf{Alt}_n$, respectively. We have $\mathbf{M}_n(\mathbb{R}) = \mathbf{Sym}_n \oplus \mathbf{Alt}_n$. Real symmetric matrices have real eigenvalues and are diagonalizable in an orthogonal basis.

Given an $n \times n$ matrix M, one defines its exponential by

$$\exp M = \mathrm{e}^M := \sum_{k=0}^{\infty} \frac{1}{k!} M^k,$$

which is a convergent series. The map $t \mapsto \exp(tM)$ is the unique solution of the differential equation

$$\frac{\mathrm{d}A}{\mathrm{d}t} = MA,$$

such that $A(0) = I_n$. It solves equivalently the ODE

$$\frac{\mathrm{d}A}{\mathrm{d}t} = AM.$$

The exponential behaves well with respect to conjugation, that is

$$\exp(PMP^{-1}) = P(\exp M)P^{-1}$$

for all invertible matrix P. The eigenvalues of $\exp A$ are the exponentials of those of A. In particular, $\rho(\exp A)$ is the exponential of the maximal real part $\mathrm{Re}\,\lambda$, as λ runs over $\mathrm{Sp}A$. The matrix $\exp(A+B)$ does not equal $(\exp A)(\exp B)$ in general, but it does when $AB = BA$. In particular, $\exp A$ is always invertible, with inverse $\exp(-A)$. Other useful formulæ are

$$\exp(M^{\mathrm{T}}) = (\exp M)^{\mathrm{T}}, \quad \exp \overline{M} = \overline{\exp M}, \quad \exp(M^*) = (\exp M)^*.$$

The exponential of a Hermitian matrix is Hermitian, positive-definite. The map

$$\exp : \mathbf{H}_n \to \mathbf{HPD}_n$$

is actually an homeomorphism. The exponential of a skew-Hermitian matrix is unitary.

Let $A \in \mathbf{M}_n(\mathbb{C})$ be given. The space $\mathbb{C}^n$ splits, in a unique way, as the direct sum of three invariant subspaces, namely the stable, unstable and central subspaces of A, denoted, respectively, $E_s(A)$, $E_u(A)$ and $E_c(A)$. Their invariance properties read

$$AE_s(A) = E_s(A), \quad AE_u(A) = E_u(A) \quad \text{and} \quad AE_c(A) \subset E_c(A).$$

The stable invariant subspace is formed of vectors x such that $(\exp tA)x$ tends to zero as $t \to +\infty$, and then the decay is exponentially fast. The unstable subspace is formed of vectors x such that $(\exp tA)x$ tends to zero (exponentially fast) as $t \to -\infty$. The central subspace consists of vectors such that $(\exp tA)x$ is polynomially bounded on $\mathbb{R}$. Since these spaces are invariant under A, this matrix operates on each one as an endomorphism, say A_s, A_u, A_c. The spectrum of A_s (respectively, A_u, A_c) has negative (respectively, positive, zero) real part. The union of these spectra is the whole spectrum of A, with the correct multiplicities. Hence the dimension of $E_s(A)$ is the number of eigenvalues of A of negative real part (these are called 'stable eigenvalues'), counted with multiplicities. When $E_c(A)$ is trivial, meaning that there is no pure imaginary eigenvalue, A is called *hyperbolic* (in the sense of Dynamical Systems).

Dunford–Taylor formula. Let γ be a Jordan curve, oriented in the trigonometric way, disjoint from SpA. Let σ be the part of SpA that γ enclose. Then the Cauchy integral

$$P_\sigma := \frac{1}{2i\pi} \int_\gamma (zI_n - A)^{-1} \mathrm{d}z$$

defines a projector (that is $P_\sigma^2 = P_\sigma$) whose range and kernel are invariant under A. (Moreover, A commutes with P_σ). The spectrum of the restriction of A to the range of P_σ is exactly the part of the spectrum of A that belongs to σ. In other words, $R(P_\sigma)$ is the direct sum of the generalized eigenspaces associated to those eigenvalues in σ.

More information about matrices and norms may be found in [187].

Functional spaces

Given an open subset Ω of $\mathbb{R}^n$, the set of infinitely differentiable functions (with values in $\mathbb{C}$) that are bounded as well as all their derivatives on $\overline{\Omega}$ is denoted by $\mathscr{C}_b^\infty(\overline{\Omega})$. The set of *compactly supported* infinitely differentiable functions (also called *test functions*) is denoted by $\mathscr{D}(\Omega)$. Its dual $\mathscr{D}'(\Omega)$ is the space of *distributions*. The derivation $\partial_j := \partial/\partial x_j$ is a bounded linear operator on $\mathscr{D}(\Omega)$. Its adjoint is therefore bounded on $\mathscr{D}'(\Omega)$. The distributional derivative, still denoted by ∂_j, is the adjoint of $-\partial_j$.

A multi-index α is a finite sequence $(\alpha_1,\dots,\alpha_n)$ of natural integers. Its length $|\alpha|$ is the sum $\sum_j \alpha_j$. The operator

$$\partial^\alpha := \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$$

is a derivation of order $|\alpha|$. We also use the notation

$$\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n},$$

when $\xi \in \mathbb{R}^n$.

Given a $\mathscr{C}^1$ function $f:\Omega\to\mathbb{C}$, the differential of f at point X is the linear form

$$\mathrm{d}f(X):\xi\mapsto \mathrm{d}f(X)\xi := \sum_{j=1}^n \xi_j\,\partial_j f(X).$$

The map $X\mapsto \mathrm{d}f(X)$ (that is the *differential* of f) is a differential form. The second differential, or *Hessian* of f at X is the bilinear form

$$\mathrm{D}^2 f(X):(\xi,\eta)\mapsto \sum_{i,j=1}^n \xi_i\,\eta_j\,\partial_i\partial_j f(X)\,.$$

We may define differentials of higher orders $\mathrm{D}^3 f,\dots$

Given a Banach space E, the Lebesgue space of measurable functions $u:\Omega\to E$ whose pth power is integrable, is denoted by $L^p(\Omega;E)$. When $E=\mathbb{R}$ or $E=\mathbb{C}$, we simply denote $L^p(\Omega)$ if there is no ambiguity. The norm in $L^p(\Omega;E)$ is

$$\|u\|_{L^p} := \left(\int_\Omega \|u(x)\|_E^p \mathrm{d}x\right)^{1/p}.$$

If $m\in\mathbb{N}$, the Sobolev space $W^{m,p}(\Omega;E)$ is the set of functions in $L^p(\Omega;E)$ whose distributional derivatives up to order m belong to L^p. Its norm is defined by

$$\|u\|_{W^{m,p}} := \left(\sum_{|\alpha|\le m} \|\partial^\alpha u\|_{L^p}^p\right)^{1/p}.$$

If $p=2$ and if E is a Hilbert space, $W^{m,2}(\Omega;E)$ is a Hilbert space and is denoted $H^m(\Omega;E)$, or simply $H^m(\Omega)$ if $E=\mathbb{C}$ or $E=\mathbb{R}$ or if there is no ambiguity.

Sobolev spaces of order s (instead of m) may be defined for every real number s. The simplest definition occurs when $p=2$, $\Omega=\mathbb{R}^n$ and $E=\mathbb{C}$, where $H^s(\mathbb{R}^n)$ is isomorphic to a weighted space $L^2((1+|\xi|^2)^s\mathrm{d}\xi)$ through the Fourier transform. For a crash course on $H^s(\Omega)$ (sometimes also denoted $H^s(\overline{\Omega})$), we refer the reader to Chapter II in [31]; for more details in more general situations, see for instance the classical monograph by Adams [1]. The notation H^s_w will stand for the Sobolev space H^s equipped with the weak topology instead of the (strong) Hilbert topology.

The Schwartz space of rapidly decreasing functions $\mathscr{S}(\mathbb{R}^n)$ will simply be denoted by $\mathscr{S}$ when no confusion can occur as concerns the space dimension. And similarly, its dual space, consisting of temperate distributions, will be denoted by $\mathscr{S}'$.

Other tools

We have collected in the appendix various additional tools, ranging from standard calculus and Fourier–Laplace analysis to pseudo-differential and para-differential calculus: we hope it will be helpful to the reader.

This page intentionally left blank

PART I

THE LINEAR CAUCHY PROBLEM

This page intentionally left blank

1

LINEAR CAUCHY PROBLEM WITH CONSTANT COEFFICIENTS

The general Cauchy problem

Let $d \geq 1$ be the space dimension and $x = (x_1, \dots, x_d)$ denote the space variable, t being the time variable. The Cauchy problem that we consider in this section is posed in the whole space $\mathbb{R}^d$, while t ranges on an interval, typically $(0, T)$, where $T \leq +\infty$.

A constant-coefficient first-order system is determined by $d + 1$ matrices $A^1, \dots, A^d, B$ given in $\mathbf{M}_n(\mathbb{R})$, where $n \geq 1$ is the *size* of the system. Then the Cauchy problem consists in finding solutions $u(x, t)$ of

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} A^\alpha \frac{\partial u}{\partial x_\alpha} = Bu + f, \tag{1.0.1}$$

where $f = f(x, t)$ and the initial datum $u(\cdot, t = 0) = a$ are given in suitable functional spaces. To shorten the notation, we shall rewrite equivalently

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u = Bu + f.$$

When $f \equiv 0$, the Cauchy problem is said to be *homogeneous*. A well-posedness property holds for the homogeneous problem when, given a in a functional space X, there exists one and only one solution u in $\mathscr{C}(0, T; Y)$, for some other functional space Y, the map

$$X \to \mathscr{C}(0, T; Y)$$
$$a \mapsto u$$

being continuous. 'Solution' is understood here in the distributional sense. Existence and continuity imply $X \subset Y$, since the map $a \mapsto u(0)$ must be continuous. We use the general notation

$$X \stackrel{S_t}{\to} Y$$
$$a \mapsto u(t).$$

Since a homogeneous system is, at a formal level, an autonomous differential equation with respect to time, we should like to have the semigroup property

$$S_{t+s} = S_t \circ S_s, \quad s, t \geq 0,$$

this of course requires that $Y = X$. We then say that the homogeneous Cauchy problem defines a continuous semigroup if for every initial data $a \in X$, there exists a unique distributional solution of class $\mathscr{C}(\mathbb{R}^+; X)$. Note that the word 'continuous' relies on the continuity with respect to time of the solution, but not on the continuity of $t \mapsto S_t$ in the operator norm. Semigroup theory actually tells us that, if X is a Banach space, the continuity in the operator norm corresponds to ordinary differential equations, a context that does not apply in PDEs.

When the homogeneous Cauchy problem defines a continuous semigroup on a functional space X, we expect to solve the non-homogeneous one using *Duhamel's formula*:

$$u(t) = S_t a + \int_0^t S_{t-s} f(s) \mathrm{d}s, \tag{1.0.2}$$

provided that at least $f \in L^1(0, T; X)$. For this reason, we focus on the homogeneous Cauchy problem and content ourselves in constructing the semigroup.

Before entering into the theory, let us remark that, since (1.0.1) writes

$$\frac{\partial u}{\partial t} = Pu + f,$$

where P is a differential operator of order less than or equal to one, the order with respect to time of this evolution equation, the Cauchy–Kowalevska theory applies. For instance, if $f = 0$, analytic initial data yield unique analytic solutions. However, these solutions exist only on a short time interval $(0, T^*(a))$. Since analytic data are unlikely in real life, and since local solutions are of little interest, we shall not concern ourselves with this result.

1.1 Very weak well-posedness

We first look at the necessary conditions for a very weak notion of well-posedness, where $X = \mathscr{S}(\mathbb{R}^d)$ (the Schwartz class) and $Y = \mathscr{S}'(\mathbb{R}^d)$, the set of tempered distributions. Surprisingly, this analysis will provide us with a rather strong necessary condition, sometimes called *weak hyperbolicity*[1].

Let us assume that the homogeneous Cauchy problem is well-posed in this context. Let a be a datum and u be the solution. From the equation

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} A^\alpha \frac{\partial u}{\partial x_\alpha} = Bu, \tag{1.1.3}$$

[1]Some authors call it simply *hyperbolicity*, and use the term *strong hyperbolicity* for the notion that we shall call *hyperbolicity*. Thus, depending on the authors, there is either the weak and normal hyperbolicities, or the normal and strong ones.

we obtain $u \in \mathscr{C}^\infty(0,T;Y)$. This allows us to Fourier transform (1.1.3) in the spatial directions. We obtain that (1.1.3) is equivalent to

$$\frac{\partial \hat{u}}{\partial t} + i\sum_{\alpha=1}^{d} \eta_\alpha A^\alpha \hat{u} = B\hat{u}.$$

Using the notation

$$A(\eta) := \sum_{\alpha=1}^{d} \eta_\alpha A^\alpha,$$

we rewrite this equation as an ODE in t, parametrized by η

$$\frac{\partial \hat{u}}{\partial t} = (B - iA(\eta))\hat{u}. \tag{1.1.4}$$

Since $\hat{u}(\cdot,0) = \hat{a}$, the solution of (1.1.4) is explicitly given by

$$\hat{u}(\eta,t) = \mathrm{e}^{t(B-iA(\eta))}\hat{a}(\eta). \tag{1.1.5}$$

By well-posedness (1.1.5) defines a tempered distribution for every choice of $\hat{a}$ in the Schwartz class, continuously in time. In other words, the bilinear map

$$(\phi,\psi) \mapsto \int_{\mathbb{R}^d} \psi(\eta)^* \mathrm{e}^{t(B-iA(\eta))}\phi(\eta)\,\mathrm{d}\eta, \tag{1.1.6}$$

which is well-defined for compactly supported smooth vector fields ϕ and ψ, is continuous in the Schwartz topology, uniformly for t in compact intervals.

Let λ be a simple eigenvalue of $A(\xi)$ for some $\xi \in \mathbb{R}^d$. Then, there is a $\mathscr{C}^\infty$ map $(t,\sigma) \mapsto (\mu, r)$, defined on a neighbourhood $\mathcal{W}$ of $(0,\xi)$, such that $\mu(0,\xi) = -i\lambda$ and

$$(t^2 B - iA(\sigma))r(t,\sigma) = \mu(t,\sigma)r(t,\sigma), \quad \|r\| \equiv 1.$$

Let us choose a non-zero compactly supported smooth function $\theta : \mathbb{R}^d \to \mathbb{C}$ with $\theta(0) \neq 0$. Then, for small enough $t > 0$, the condition $\eta - t^{-2}\xi \in \operatorname{Supp}\theta$ implies $(t, t^2\eta) \in \mathcal{W}$. For such a t, we may define two compactly supported smooth vector fields by

$$\phi^t(\eta) := \theta(\eta - t^{-2}\xi)r(t,t^2\eta), \quad \psi^t(\eta) := \theta(\eta - t^{-2}\xi)\ell(t,t^2\eta),$$

where ℓ is an eigenfield of the adjoint matrix $(t^2B - iA(\sigma))^*$, defined and normalized as above. We then apply (1.1.6) to (ϕ^t,ψ^t). The sequence $(\phi^t)_{t\to 0}$ is bounded in the Schwartz topology, and similarly is $(\psi^t)_{t\to 0}$. Therefore

$$\int_{\mathbb{R}^d} (\psi^t)^* \mathrm{e}^{t(B-\mathrm{i}A(\eta))}\phi^t\,\mathrm{d}\eta = \int_{\mathbb{R}^d} \mathrm{e}^{\mu(t,t^2\eta)/t}(\ell\cdot r)(t,t^2\eta)|\theta(\eta - \xi/t^2)|^2 \mathrm{d}\eta$$

is bounded as $t \to 0$. Since it behaves like $c\exp(-\mathrm{i}\lambda/t)$ for a non-zero constant c, we conclude that $\operatorname{Im}\lambda \leq 0$. Applying also this conclusion to the simple eigenvalue $\bar{\lambda}$, we find that λ is real.

The case of an eigenvalue of constant multiplicity in some open set of frequencies η can be treated along the same ideas; it must be real too. Finally, the points η at which the multiplicities are not locally constant form an algebraic submanifold, thus a set of void interior. By continuity, the reality must hold everywhere. We have thus proved

Proposition 1.1 *The* $(\mathscr{S},\mathscr{S}')$ *well-posedness requires that the spectrum of* $A(\xi)$ *be real for all* ξ *in* $\mathbb{R}^d$.

When $(\mathscr{S},\mathscr{S}')$ well-posedness does not hold, a Hadamard instability occurs: for most (in the Baire sense) data a in $\mathscr{S}$, and for all $T>0$, the Cauchy problem does not admit any solution of class $\mathscr{C}(0,T;\mathscr{S}')$. This is a consequence of the Principle of Uniform Boundedness.

Example The Cauchy–Riemann equations provide the simplest system for which this instability holds. One has $d=1$, $n=2$:

$$\partial_t u_1 + \partial_x u_2 = 0, \quad \partial_t u_2 - \partial_x u_1 = 0.$$

This example shows that a boundary value problem for a system of partial differential equations may be well-posed though the corresponding Cauchy problem is ill-posed.

The converse of Proposition 1.1 does not hold in general, mainly because of the interaction between non-semisimple eigenvalues of $A(\xi)$ with the mixing induced by B. Let us take again a simple example with $d=1$, $n=2$, and

$$A = A^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since the matrix

$$\exp(-i\xi A) = I_2 - i\xi A$$

has polynomial growth, the Cauchy problem for the operator $\partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is well-posed in the $(\mathscr{S},\mathscr{S}')$ sense, and even in the $(\mathscr{S},\mathscr{S})$ sense. Actually, its solution is explicitly given by

$$u_1(t) = a_1 - t a_2', \quad u_2(t) \equiv a_2.$$

(We see that there is an immediate loss of regularity.) However, with our non-zero B, the matrix $M := t(B - i\xi A)$ satisfies $M^2 = -it^2\xi I_2$, which implies that

$$\exp(t(B - i\xi A)) = \cos\omega\, I_2 + \frac{\sin\omega}{\omega} M,$$

where $\omega = t(i\xi)^{1/2}$. Since

$$\mathrm{Im}\ \omega = \pm t \left|\frac{\xi}{2}\right|^{1/2},$$

we see that offdiagonal coefficients of $\exp M$ grow like $\exp(c|\xi|^{1/2})$ as ξ tends to infinity, provided $t \neq 0$. Then a calculation similar to the one in the proof of Proposition 1.1 shows that this Cauchy problem is ill-posed in the $(\mathscr{S}, \mathscr{S}')$ sense.

1.2 Strong well-posedness

The previous example suggests that the notion of well-posedness in the (rather weak) $(\mathscr{S}, \mathscr{S}')$ sense might not be stable under small disturbance (the instability result would be the same with ϵB instead of B). For this reason, we shall merely consider the well-posedness when $Y = X$ and X is a Banach space. We then speak about strong well-posedness in X (or X-well-posedness). When this holds, the map $S_t : a \mapsto u(t)$ defines a continuous semigroup on X. It can be shown that if X is a Banach space, there exist two constants c, ω, such that

$$\|S_t\|_{\mathcal{L}(X)} \leq c\mathrm{e}^{\omega t}, \tag{1.2.7}$$

Proposition 1.2 *Let X be a Banach space. Then well-posedness (with $Y = X$) for some $B \in \mathbf{M}_n(\mathbb{R})$ implies the same property for all B.*

This amounts to saying that well-posedness is a property of $(A^1, \dots, A^d)$ alone.

Proof Assume strong well-posedness for a given matrix B_0. The problem

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} A^\alpha \frac{\partial u}{\partial x_\alpha} = B_0 u \tag{1.2.8}$$

defines a continuous semigroup $(S_t)_{t \geq 0}$. One has (1.2.7) with suitable constants c and ω. From Duhamel's formula, (1.1.3) with a matrix $B = B_0 + C$ instead of B_0, is equivalent to

$$u(t) = S_t a + \int_0^t S_{t-s} C u(s) \mathrm{d}s. \tag{1.2.9}$$

Then we can solve (1.2.9) by a Picard iteration. Let us denote by Ru the right-hand side of (1.2.9), and $I = (0, T)$ (with $T > 0$) a time interval where we look for a solution. Because of (1.2.7), there exists a large enough N so that R^N is contractant on $\mathscr{C}(I; X)$. Therefore, there exists a unique solution of (1.1.3) in $\mathscr{C}(I; X)$. Since T is arbitrary, the solution is global in time. □

1.2.1 *Hyperbolicity*

We first consider spaces X where the Fourier transform defines an isomorphism onto some other Banach space Z. Typically, X will be a Sobolev space $H^s(\mathbb{R}^d)^n$ and Z is a weighted L^2-space:

$$Z = L^2_s(\mathbb{R}^d)^n, \quad L^2_s(\mathbb{R}^d) := \{v \in L^2_{\mathrm{loc}}(\mathbb{R}^d)\,;\, (1 + |\xi|^2)^{s/2} v \in L^2(\mathbb{R}^d)\}.$$

Because of this example, we shall assume that multiplication by a measurable function g defines a continuous operator from Z to itself if and only if g is bounded.

Looking for a solution $u \in \mathscr{C}(I;X)$ of (1.1.3) is simply looking for a solution $v \in \mathscr{C}(I;Z)$ of

$$\frac{\partial v}{\partial t} = (B - iA(\eta))v, \quad v(\eta, 0) = \hat{a}(\eta). \tag{1.2.10}$$

Thanks to Proposition 1.2, we may restrict ourselves to the case where $B = 0_n$. Then v must obey the formula

$$v(\eta, t) = \mathrm{e}^{-\mathrm{i}tA(\eta)}\hat{a}(\eta),$$

where $\hat{a}$ is given in Z. In order that $v(t)$ belong to Z for all $\hat{a}$, it is necessary and sufficient that $\eta \mapsto \exp(-itA(\eta))$ be bounded. Since $tA(\eta) = A(t\eta)$, this is equivalent to writing

$$\sup_{\xi\in\mathbb{R}^d} \|\exp(iA(\xi))\| < +\infty. \tag{1.2.11}$$

Let us emphasize that this property does not depend on the time t, once $t \neq 0$.

Definition 1.1 *A first-order operator*

$$L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

is called hyperbolic *if the corresponding symbol $\xi \mapsto A(\xi)$ satisfies (1.2.11).*

More generally, a system (1.0.1) (whatever B is) that satisfies (1.2.11) is called a hyperbolic[2] *system of first-order PDEs.*

After having proven that hyperbolicity is a necessary condition, we show that it is sufficient for the H^s-well-posedness. It remains to prove the continuity of $t \mapsto v(t)$ with values in Z, when $\hat{a}$ is given in Z. For that, we write

$$\|v(\tau) - v(t)\|_Z^2 = \int_{\mathbb{R}^d} \left|\left(\mathrm{e}^{-\mathrm{i}\tau A(\eta)} - \mathrm{e}^{-\mathrm{i}tA(\eta)}\right)\hat{a}(\eta)\right|^2 (1 + |\eta|^2)^s \mathrm{d}\eta.$$

Thanks to (1.2.11), the integrand is bounded by $c|\hat{a}(\eta)|^2(1 + |\eta|^2)^s$, an integrable function, independent of τ. Likewise, it tends pointwisely to zero, as $\tau \to t$. Lebesgue's Theorem then implies that

$$\lim_{\tau\to t} \|v(\tau) - v(t)\|_Z = 0.$$

Let us summarize the results that we obtained:

[2]Some authors write *strongly hyperbolic* in this definition and keep the terminology *hyperbolic* for those systems that are well-posed in $\mathscr{C}^\infty$, that is whose a priori estimates may display a loss of derivatives.

Theorem 1.1

- *Let s be a real number. The Cauchy problem for*

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u = 0 \tag{1.2.12}$$

is H^s-well-posed if and only if this system is hyperbolic.

- *If the operator L (defined as above) is hyperbolic, then the Cauchy problem for (1.1.3) is H^s-well-posed for every real number s.*
- *In particular, the Cauchy problem is well-posed in H^s if and only if it is well-posed in L^2.*

Let us point out that hyperbolicity does not involve the matrix B.

Since the well-posedness in a Hilbertian Sobolev space holds or does not, independently of the regularity level s, we feel free to rename this property *strong well-posedness*.

Backward Cauchy problem We considered up to now the forward Cauchy problem, namely the determination of $u(t)$ for times t larger than the initial time. Its well-posedness within L^2 was shown to be equivalent to hyperbolicity. Reversing the time arrow amounts to making the change $\partial_t \mapsto -\partial_t$. This has the same effect as changing the matrices A^α into $-A^\alpha$. The L^2-well-posedness of the Cauchy problem is thus equivalent to the hyperbolicity of the new system

$$\partial_s u - \sum_\alpha A^\alpha \partial_\alpha u = -Bu.$$

This writes as

$$\sup_{\xi \in \mathbb{R}^d} \| \exp(-iA(\xi)) \| < +\infty,$$

which is the same as (1.2.11), *via* the change of dummy variable $\xi \mapsto -\xi$. Finally, the strong well-posedness of backward and forward Cauchy problems are equivalent to each other. For a hyperbolic system and a data $a \in H^s(\mathbb{R}^d)^n$, there exists a unique solution of (1.1.3) $u \in \mathscr{C}(\mathbb{R}; H^s(\mathbb{R}^d)^n)$ such that $u(0) = a$. Let us emphasize that here, t ranges on the whole line, not only on $\mathbb{R}^+$.

1.2.2 *Distributional solutions*

When (1.1.3) is hyperbolic, one can also solve the Cauchy problem for data in the set $\mathscr{S}'$ of tempered distribution. For that, we again use the Fourier transform since it is an automorphism of $\mathscr{S}'$. We again define $\hat{u}$ by the formula (1.1.5). We only have to show that this definition makes sense in $\mathscr{S}'$ for every t, and that u is continuous from $\mathbb{R}_t$ to $\mathscr{S}'$. For that, we have to show that $X(t) := \exp(t(B - iA(\eta)))$ is a $\mathscr{C}^\infty$ function of η, with slow growth at infinity, locally uniformly in time. We shall show that its derivatives are actually bounded with respect to η. The regularity is trivial, and we already know that $X(t)$ is bounded in η, locally in time. Denoting by X_α the derivative with respect to η_α, we have

$$\frac{\mathrm{d}X_\alpha}{\mathrm{d}t} = (B - iA(\eta))X_\alpha - iA^\alpha X,$$

and therefore

$$\frac{\mathrm{d}(X^{-1}X_\alpha)}{\mathrm{d}t} = -iX^{-1}A^\alpha X.$$

Using Duhamel's formula, as in the proof of Proposition 1.2, we see that

$$\|X(t)\| \leq c(1 + \|B\|\,|t|),$$

from which we deduce

$$\|X^{-1}X_\alpha\| \leq \frac{(1 + \|B\|\,|t|)^3 - 1}{3\|B\|} c^2 \|A^\alpha\|.$$

Finally, we obtain

$$\|X_\alpha\| \leq \frac{(1 + \|B\|\,|t|)^4}{3\|B\|} c^3 \|A^\alpha\|.$$

We leave the reader to estimate the higher derivatives and complete the proof of the following statement. The case of data in the Schwartz class is done in exactly the same way, since the Fourier transform is an automorphism of $\mathscr{S}$ and that $\mathscr{S}$ is stable under multplication by $\mathscr{C}^\infty$ functions with slow growth.

Proposition 1.3 *If L is hyperbolic, then the Cauchy problem for (1.1.3) is well-posed in both $\mathscr{S}$ and $\mathscr{S}'$.*

1.2.3 *The Kreiss' matrix Theorem*

Of course, since L^2-well-posedness implies $(\mathscr{S}, \mathscr{S}')$-well-posedness, hyperbolicity implies that the spectrum of $A(\xi)$ is real for all ξ in $\mathbb{R}^d$. It implies even more, that all $A(\xi)$ are diagonalizable. Though these two facts have a rather simple proof here, they do not characterize completely hyperbolic systems. We shall therefore describe the characterization obtained by Kreiss [102, 104]. This is an application of a deeper result that deals with strong well-posedness of general constant-coefficient evolution problems. However, since we focus only on first-order systems, we content ourselves with a statement with a simpler proof, due to Strang [199].

Theorem 1.2 *Let $\xi \mapsto A(\xi)$ be a linear map from $\mathbb{R}^d$ to $\mathbf{M}_n(\mathbb{C})$. Then the following properties are equivalent to each other:*

i) Every $A(\xi)$ is diagonalizable with pure imaginary eigenvalues, uniformly with respect to ξ:

$$A(\xi) = P(\xi)^{-1}\mathrm{diag}(i\rho_1, \ldots, i\rho_n)P(\xi), \quad (\rho_1(\xi), \ldots, \rho_n(\xi) \in \mathbb{R}),$$

with

$$\|P(\xi)^{-1}\| \cdot \|P(\xi)\| \leq C', \quad \forall \xi \in \mathbb{R}^d. \tag{1.2.13}$$

ii) There exists a constant $C > 0$*, such that*

$$\left\| \mathrm{e}^{tA(\xi)} \right\| \leq C, \quad \forall \xi \in \mathbb{R}^d, \forall t \geq 0. \tag{1.2.14}$$

iii) There exists a constant $C > 0$*, such that*

$$\left\|(zI_n - A(\xi))^{-1}\right\| \leq \frac{C}{\mathrm{Re}\ z}, \quad \forall \xi \in \mathbb{R}^d, \forall \mathrm{Re}\ z > 0. \tag{1.2.15}$$

Note that, replacing (z, ξ) by $(-z, -\xi)$, we also obtain (1.2.15) with Re $z \neq 0$. Applying Theorem 1.2, we readily obtain the following.

Theorem 1.3 *The Cauchy problem for a first-order system*

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u = 0, \quad x \in \mathbb{R}^d$$

is H^s*-well-posed if and only if the following two properties hold.*

- *The matrices* $A(\xi)$ *are diagonalizable with real eigenvalues,*

$$A(\xi) = P(\xi)^{-1} \mathrm{diag}(\rho_1(\xi), \ldots, \rho_n(\xi)) P(\xi), \quad (\rho_1, \ldots, \rho_n \in \mathbb{R}).$$

- *Their diagonalization is well-conditioned (one may also say that the matrices* $A(\xi)$ *are* uniformly *diagonalizable) :* $\sup_{\xi \in S^{d-1}} \|P(\xi)^{-1}\| \cdot \|P(\xi)\| < +\infty$*.*

Proof The fact that *i)* implies *ii)* is proved easily. Actually,

$$\|\mathrm{e}^{tA(\xi)}\| = \|P^{-1}\mathrm{e}^{tD}P\| \leq C'\|\mathrm{e}^{tD}\|.$$

When D is diagonal with pure imaginary entries, $\exp(tD)$ is unitary, and the right-hand side equals C'.

The fact that *ii)* implies *iii)* is easy too. The following equality holds provided the integral involved in it converges in norm

$$(A - zI_n) \int_0^\infty \mathrm{e}^{-zt} \mathrm{e}^{tA} \mathrm{d}t = -I_n. \tag{1.2.16}$$

Because of (1.2.14), the integral converges for every $z \in \mathbb{C}$ with positive real part. This gives a bound for the inverse of $zI_n - A$, of the form

$$\|(zI_n - A)^{-1}\| \leq \frac{C}{\mathrm{Re}\ z}, \quad \mathrm{Re}\ z > 0.$$

It remains to prove that *iii)* implies *i)*. Thus, let us assume (1.2.15). Replacing (z, ξ) by $(-z, -\xi)$, we see that the bound holds for Re $z \neq 0$, with $|\mathrm{Re}\ z|$ in the denominator. Thus the spectrum of $A(\xi)$ is purely imaginary.

Actually, $A(\xi)$ is diagonalizable, for if there were a non-trivial Jordan part, then $(zI_n - A(\xi))^{-1}$ would have a pole of order two or more, contradicting (1.2.15). Therefore, $A(\xi)$ admits a spectral decomposition

$$A(\xi) = i \sum_j \rho_j E_j,$$

where ρ_j is real and $E_j = E_j(\xi)$ is a projector ($E_j^2 = E_j$), with

$$E_j E_k = 0_n, \quad (k \neq j), \qquad \sum_j E_j = I_n.$$

Let us define

$$H = H(\xi) := \sum_j E_j E_j^*,$$

which is a positive-definite Hermitian matrix. Since $A(\xi)^* = -\sum_j \rho_j E_j^*$, it holds that

$$H(\xi)A(\xi) = -A(\xi)^* H(\xi),$$

from which it follows that $H(\xi)^{1/2} A(\xi) H(\xi)^{-1/2}$ is skew-Hermitian. As such, it is diagonalizable through a unitary transformation. Therefore $A(\xi) = P(\xi)^{-1} D(\xi) P(\xi)$, where $D(\xi)$ is diagonal with pure imaginary eigenvalues, and $P(\xi) = U(\xi) H^{1/2}$, where $U(\xi)$ is a unitary matrix.

We finish by proving that $P(\xi)$ is uniformly conditioned. Since $\|P^{\pm 1}\| = \|H^{\pm 1/2}\| = \|H\|^{\pm 1/2}$, this amounts to proving that $\|H\| \cdot \|H^{-1}\|$ is uniformly bounded. On the one hand, it holds that

$$|v|^2 = \left| \sum_j E_j v \right|^2 \leq n \sum_j |E_j v|^2 = n |H^{1/2} v|^2,$$

so that $\|H^{-1/2}\| \leq \sqrt{n}$. On the other hand, applying (1.2.15) to $\epsilon + i\rho_k$, we have

$$\| \sum_j (\epsilon + i\rho_k - i\rho_j)^{-1} E_j \| \leq \frac{C}{|\epsilon|}.$$

Letting $\epsilon \to 0$, we deduce that $\|E_j\| \leq C$, independently of ξ. It follows that $\|H\| \leq nC^2$. □

Remarks

i) A more explicit characterization of hyperbolic symbols has been established by Mencherini and Spagnolo when $n = 2$ or $n = 3$; see [129].

ii) The following example ($n = 3$ and $d = 2$), known as *Petrowski's example*, shows that the well-conditioning can fail for systems in which all matrices

$A(\xi)$ are diagonalizable with real eigenvalues. Let us take

$$A^1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One checks easily that the eigenvalues of $A(\xi)$ are real and distinct for $\xi_1 \neq 0$, while A^2 is already diagonal. Hence, $A(\xi)$ is always diagonalizable over $\mathbb{R}$. However, as ξ_1 tends to zero, one eigenvalue is identically zero, associated to the eigenvector $(\xi_2, \xi_1, -\xi_1)^T$, while another one is small, $\lambda \sim -\xi_1^2 \xi_2^{-1}$, associated to $(\xi_2, 0, \lambda\xi_2\xi_1^{-1})^T$. Both eigenvectors have the same limit $(\xi_2, 0, 0)^T$, which shows that $P(\xi)$ is unbounded as ξ_1 tends to zero. See a similar example in [108]. Oshime [155] has shown that Petrowsky's example is somehow canonical when $d = 3$. On the other hand, Strang [199] showed that when $n = 2$, the diagonalizability of every $A(\xi)$ is equivalent to hyperbolicity, and that such operators are actually Friedrichs symmetrizable in the sense of the next section.

iii) Uniform diagonalizability of $A(\xi)$ within real matrices has been shown by Kasahara and Yamaguti [93, 221] to be necessary and sufficient in order that the Cauchy problem for

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u = Bu$$

be $\mathcal{C}^\infty$-well-posed for every matrix $B \in \mathbf{M}_n(\mathbb{R})$. Of course, the sufficiency follows from Theorem 1.3 and Proposition 1.2. The necessity statement is even stronger than the one suggested by the example given in Section 1.1, since the diagonalizability within $\mathbb{R}$ is not sufficient. For instance, if $A(\xi)$ is given as in the Petrowski example, there are matrices B for which the Cauchy problem is ill-posed in the Hadamard sense.

1.2.4 *Two important classes of hyperbolic systems*

We now distinguish two important classes of hyperbolic systems.

Definition 1.2 *An operator*

$$L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

is said to be symmetric in Friedrichs' sense *[63], or simply* Friedrichs symmetric*, if all matrices A^α are symmetric; one may also say* symmetric hyperbolic*. More generally, it is* Friedrichs symmetrizable *if there exists a symmetric positive-definite matrix S_0 such that every $S_0 A^\alpha$ is symmetric.*

An operator M as above is said to be constantly[3] hyperbolic *if the matrices $A(\xi)$ are diagonalizable with real eigenvalues and, moreover, as ξ ranges along*

[3]We employ this shortcut in lieu of *hyperbolic with characteristic fields of constant multiplicities.*

S^{d-1}, *the multiplicities of eigenvalues remain constant. In the special case where all eigenvalues are real and simple for every* $\xi \in S^{d-1}$, *we say that the operator is* strictly hyperbolic.

Let us point out that in a constantly hyperbolic operator, the eigenvalues may have non-equal multiplicities, but the set of multiplicities remains constant as ξ varies. This implies in particular that the eigenspaces depend analytically on ξ for $\xi \neq 0$. This fact easily follows from the construction of eigenprojectors as Cauchy integrals (see the section 'Notations'.) To a large extent, the theory of constantly hyperbolic systems does not differ from the one of strictly hyperbolic systems. But the analysis is technically simpler in the latter case. This is why the theory of strictly hyperbolic operators was developed much further in the first few decades.

Theorem 1.4 *If an operator is Friedrichs symmetrizable, or if it is constantly hyperbolic, then it is hyperbolic.*

Proof Let the operator be Friedrichs symmetrizable by S_0. Then S_0^{-1} is positive-definite and admits a (unique) square root R symmetric positive-definite (see [187], page 78). Let us denote $S_0 A^\alpha$ by S^α, and $S(\xi) = \sum_\alpha \xi_\alpha S^\alpha$ as usual. Then

$$A(\xi) = S_0^{-1} S(\xi) = R(RS(\xi)R)R^{-1}.$$

The matrix $RS(\xi)R$ is real symmetric and thus may be written as $Q(\xi)^T D(\xi) Q(\xi)$, where Q is orthogonal. Then $A(\xi)$ is conjugated to $D(\xi)$, $A(\xi) = P(\xi)^{-1} D(\xi) P(\xi)$, with $P(\xi) = Q(\xi) R^{-1}$ and $P(\xi)^{-1} = RQ(\xi)^T$. Since our matrix norm is invariant under left or right multiplication by unitary matrices, we have

$$\|P(\xi)\| \, \|P(\xi)^{-1}\| = \|R\| \, \|R^{-1}\| = \sqrt{\rho(S_0)\rho(S_0^{-1})},$$

a number independent of ξ. The diagonalization is thus well-conditioned.

Let us instead assume that the system is constantly hyperbolic. The eigenspaces are continuous functions of ξ in S^{d-1}. Choosing continuously a basis of each eigenspace, we find locally an eigenbasis of $A(\xi)$, which depends continuously on ξ. This amounts to saying that, along every contractible subset of S^{d-1}, the matrices $A(\xi)$ may be diagonalized by a matrix $P(\xi)$, which depends continuously on ξ. If the set is, moreover, compact (for instance, a half-sphere), we obtain that $A(\xi)$ is diagonalizable with a uniformly bounded condition number. We now cover the sphere by two half-spheres and obtain a diagonalization of $A(\xi)$ that is well-conditioned on S^{d-1} (though possibly not continuously diagonalizable on the sphere). □

In the following example, though a symmetric as well as a strictly hyperbolic one, the diagonalization of the matrices $A(\xi)$ cannot be done continuously for all

$\xi \in S^1$:

$$\partial_t u + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 u + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2 u = 0. \tag{1.2.17}$$

Here, $\mathrm{Sp}(A(\xi)) = \{-|\xi|, |\xi|\}$. Each eigenvector, when followed continuously as ξ varies along S^1, rotates with a speed half of the speed of ξ. For $\xi = (\cos\theta, \sin\theta)^T$ and $\theta \in [0, 2\pi)$, the eigenvectors are

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \quad \begin{pmatrix} -\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.$$

The eigenbasis is reversed after one loop around the origin. This shows that the matrix $P(\xi)$ cannot be chosen continuously. In other words, the eigenbundle is non-trivial.

1.2.5 *The adjoint operator*

Let L be a hyperbolic operator as above. We define as usual the *adjoint operator* L^* by the identity

$$\int_{-\infty}^{+\infty} \int_{\mathbb{R}^d} (v \cdot (Lu) - u \cdot (L^* v)) \mathrm{d}x \, \mathrm{d}t = 0, \tag{1.2.18}$$

for every $u, v \in \mathscr{D}(\mathbb{R}^{d+1})^n$. Notice that the scalar product under consideration is the one in the L^2-space in (x,t)-variables.

With $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$, an integration by parts gives immediately the formula

$$L^* = -\partial_t - \sum^{\alpha} (A^\alpha)^T \partial_\alpha.$$

The matrix $A(\xi)^T$, being similar to $A(\xi)$, is diagonalizable. Since $A(\xi)^T$ is diagonalized by $P(\xi)^{-T}$ (with the notations of Theorem 1.3), and since the matrix norm is invariant under transposition, we see that $-L^*$ is hyperbolic too. If L is strictly, or constantly, hyperbolic, so is L^*. If L is Friedrichs symmetrizable, with $S^0 \in \mathbf{SDP}_n$ and $S^\alpha := S^0 A^\alpha \in \mathbf{Sym}_n$, then $(S^0)^{-1}$ symmetrizes $-L^*$ since it is positive-definite and $(S^0)^{-1} (A^\alpha)^T = (S^0)^{-1} S^\alpha (S^0)^{-1}$ is symmetric. Therefore, L^* is Friedrichs symmetrizable.

The adjoint operator will be used in the existence theory of the Cauchy problem (the duality method) or in the uniqueness theory (Holmgren's argument), the latter being useful even in the quasilinear case. Both aspects are displayed in Chapter 2.

1.2.6 *Classical solutions*

Let the system (1.1.3) be hyperbolic. According to Theorem 1.1, the Cauchy problem is well-posed in H^s. Using the system itself, we find that, whenever

$a \in H^s(\mathbb{R}^d)^n$,

$$u \in \mathscr{C}(\mathbb{R}; H^s(\mathbb{R}^d)^n) \cap \mathscr{C}^1(\mathbb{R}; H^{s-1}(\mathbb{R}^d)^n).$$

Let us assume that $s > 1 + d/2$. By Sobolev embedding, $H^s \subset \mathscr{C}^1$ and $H^{s-1} \subset \mathscr{C}$ hold. We conclude that all distributional first-order derivatives are actually continuous functions of space and time. Therefore, u belongs to $\mathscr{C}^1(\mathbb{R}^d \times \mathbb{R})^n$ and is a classical solution of (1.1.3).

More generally, $a \in H^s(\mathbb{R}^d)^n$ with $s > k + d/2$ implies that u is of class $\mathscr{C}^k$.

Let us consider the non-homogeneous Cauchy problem, with $a \in H^s(\mathbb{R}^d)^n$ and $f \in L^1(\mathbb{R}; H^s(\mathbb{R}^d)^n) \cap \mathscr{C}(\mathbb{R}; H^{s-1}(\mathbb{R}^d)^n)$ for $s > 1 + d/2$. Then Duhamel's formula immediately gives $u \in \mathscr{C}(\mathbb{R}; H^s(\mathbb{R}^d)^n)$, and the equation gives $\partial_t u \in \mathscr{C}(\mathbb{R}; H^{s-1}(\mathbb{R}^d)^n)$. We again conclude that u is $\mathscr{C}^1$ and is a classical solution of (1.0.1).

Since $H^s(\mathbb{R}^d)^n$ is dense in normal functional spaces, as L^2 or $\mathscr{S}'$, we see that classical solutions are dense in weaker solutions, like those in $\mathscr{C}(\mathbb{R}; L^2(\mathbb{R}^d)^n)$. We shall make use of this observation each time when some identity trivially holds for classical solutions.

The scalar case When $n = 1$, the unknown $u(x,t)$ is scalar-valued and all matrices are real numbers, denoted by $a^1, \ldots, a^n, b$. The supremum in (1.2.11) is equal to one, so that the equation is hyperbolic. It turns out that the Cauchy problem may be solved explicitly, thanks to the *method of characteristics*. Let $\vec{v}$ denote the vector with components a^α. Then a classical solution of (1.1.3) satisfies, for all $y \in \mathbb{R}^d$,

$$\frac{\mathrm{d}}{\mathrm{d}t} u(y + t\vec{v}, t) = bu(y + t\vec{v}, t),$$

which gives

$$u(y + t\vec{v}, t) = \mathrm{e}^{tb} a(y),$$

or

$$u(x,t) = \mathrm{e}^{tb} a(x - t\vec{v}). \tag{1.2.19}$$

This formula gives the distributional solution for $a \in \mathscr{S}'$ as well. The solution of the Cauchy problem for the non-homogeneous equation (1.0.1) is given by

$$u(x,t) = \mathrm{e}^{tb} a(x - t\vec{v}) + \int_0^t \mathrm{e}^{(t-s)b} f(x - (t-s)\vec{v}, s)\,\mathrm{d}s.$$

1.2.7 *Well-posedness in Lebesgue spaces*

The theory of the Cauchy problem is intimately related to Fourier analysis, which does not adapt correctly to Lebesgue spaces L^p other than L^2. The procedure followed above requires that $\mathcal{F}$ be an isomorphism from some space X to another one Z. It is known that $\mathcal{F}$ extends continuously from $L^p(\mathbb{R}^d)$ to its dual $L^{p'}(\mathbb{R}^d)$ when $1 \le p \le 2$, and only in these cases. Since $\mathcal{F}^{-1}$ is conjugated to $\mathcal{F}$ through complex conjugation, it satisfies the same property. Therefore, $\mathcal{F} : L^p(\mathbb{R}^d) \to L^{p'}(\mathbb{R}^d)$ is not an isomorphism for $p < 2$, since $p' > 2$. From this remark, we cannot find a well-posedness result in L^p for $p \neq 2$ by following the above strategy.

It has been proved actually by Brenner [22, 23] that, for hyperbolic systems, the Cauchy problem is ill-posed in L^p for every $p \neq 2$, except in the case where the matrices A^α commute to each other. In this particular case, system (1.2.12) actually decouples into a list of scalar equations, for which (1.2.19) shows the well-posedness in every L^p. To see the decoupling, we recall that commuting matrices that are diagonalizable may be diagonalized in a common basis $\mathcal{B} = \{r_1, \dots, r_n\}$: $A^\alpha r_j = \lambda_j^\alpha r_j$. Let us decompose the unknown on the eigenbasis:

$$u(x,t) = \sum_1^n w_j(x,t) r_j.$$

Then each w_j solves a scalar equation:

$$\partial_t w_j + \sum_\alpha \lambda_j^\alpha \partial_\alpha w_j = 0.$$

From the well-posedness of (1.2.12) and Duhamel's formula, we conclude that, for commuting matrices A^α, the hyperbolic Cauchy problem for (1.1.3) is also well-posed in every L^p. The matrices A^α do not need to commute with B.

See Section 1.5.2 for an interpretation of the ill-posedness in L^p $(p \neq 2)$, in terms of dispersion and so-called Stricharz estimates.

1.3 Friedrichs-symmetrizable systems

A system in Friedrichs-symmetric form

$$S_0 \partial_t u + \sum_\alpha S^\alpha \partial_\alpha u = 0$$

may always be transformed into a symmetric system with $S_0 = I_n$, using the new unknown $\tilde{u} := S_0^{1/2} u$. For the rest of this section, we shall only consider symmetric systems of the form (1.1.3).

A symmetric system admits an additional conservation law[4] in the form

$$\partial_t |u|^2 + \sum_\alpha \partial_\alpha (A^\alpha u, u) = 0, \tag{1.3.20}$$

where $(\cdot,\cdot)$ denotes the canonical scalar product and $|u|^2 := (u,u)$. Equation (1.3.20) is satisfied at least for $\mathscr{C}^1$ solutions of the system, when[5] $B = 0$. It can be viewed as an *energy identity*. Since classical solutions are dense in $\mathscr{C}(\mathbb{R}; L^2(\mathbb{R}^d)^n)$,

[4]By *conservation law*, we mean an equality of the form $\mathrm{Div}_{x,t} \vec{F} = 0$ that derives formally from the equation or system under consideration.

[5]Otherwise, the right-hand side of (1.3.20) should be $2(Bu, u)$. In the non-homogeneous case, we add also $2(f, u)$.

and since

$$u \mapsto \partial_t |u|^2 + \sum_\alpha \partial_\alpha (A^\alpha u, u)$$

is a continuous map from this class into $\mathscr{D}'(\mathbb{R}^{d+1})$, we conclude that (1.3.20) holds whenever $a \in L^2(\mathbb{R}^d)^n$.

With suitable decay at infinity, (1.3.20) implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |u(x,t)|^2 \mathrm{d}x = 0,$$

which readily gives

$$\|u(t)\|_{L^2} \equiv \|a\|_{L^2}. \tag{1.3.21}$$

Again, this identity is true for all data a given in $L^2(\mathbb{R}^d)^n$, since

- it is trivially true for $a \in \mathscr{S}$, where we know that $u(t) \in \mathscr{S}$, since such functions decay fast at infinity,
- $\mathscr{S}$ is a dense subset of L^2.

1.3.1 *Dependence and influence cone*

Actually, we can do a better job from (1.3.20). Let us first consider classical solutions, for some matrix B. The set $\mathcal{V}$ of pairs (λ, ν) such that the symmetric matrix $\lambda I_n + A(\nu)$ is non-negative is a closed convex cone. Given a point $(X, T) \in \mathbb{R}^d \times \mathbb{R}$, we define a set K by

$$K := \{(x,t)\,;\, \lambda(t-T) + (x-X)\cdot\nu \le 0, \forall(\lambda,\nu) \in \mathcal{V}\}.$$

As an intersection of half-spaces passing through (X, T), K is a convex cone with basis (X, T), and its boundary K has almost everywhere a tangent space, which is one of the hyperplanes $\lambda(t-T) + (x-X)\cdot\nu = 0$ for some (λ, ν) in the boundary of $\mathcal{V}$.

Given times $t_1 < t_2 < T$, we integrate the identity

$$\partial_t |u|^2 + \sum_\alpha \partial_\alpha (A^\alpha u, u) = 2(Bu, u)$$

on the truncated cone $K(t_1, t_2) := \{(x,t) \in K\,;\, t_1 < t < t_2\}$. Using Green's formula, we obtain

$$\int_{\partial K(t_1,t_2)} \left(n_0 |u|^2 + \sum_\alpha n_\alpha (A^\alpha u, u) \right) \mathrm{d}S = 2 \int_{K(t_1,t_2)} (Bu, u)\, \mathrm{d}x\, \mathrm{d}t, \tag{1.3.22}$$

where $\mathrm{d}S$ stands for the area element, while $\vec{n} = (n_1, \dots, n_d, n_0)$ is the outward unit normal. On the top $(t = t_2)$, $\vec{n} = (0, \dots, 0, 1)$, holds while on the bottom, $\vec{n} = (0, \dots, 0, -1)$. Denoting $\omega(t) := \{x\,;\, (x,t) \in K\}$, the corresponding contributions are thus

$$\int_{\omega(t_2)} |u(x,t_2)|^2 \mathrm{d}x - \int_{\omega(t_1)} |u(x,t_1)|^2 \mathrm{d}x.$$

On the lateral boundary, one has

$$\vec{n} = \frac{1}{\sqrt{\lambda^2 + |\nu|^2}}(\nu, \lambda)$$

for some (λ, ν) in $\mathcal{V}$, which depends on (x,t). The parenthesis in (1.3.22) becomes

$$\frac{1}{\sqrt{\lambda^2 + |\nu|^2}}((\lambda I_n + A(\nu))u, u).$$

Thus the corresponding integral is non-negative. Denoting by $y(t)$ the integral of $|u(t)|^2$ over $\omega(t)$, it follows that

$$y(t_2) - y(t_1) \leq 2 \int_{K(t_1,t_2)} (Bu, u)\, \mathrm{d}x\, \mathrm{d}t \leq 2\|B\| \int_{t_1}^{t_2} y(t)\, \mathrm{d}t.$$

Then, from the Gronwall inequality, we obtain that

$$y(t_2) \leq e^{2(t_2 - t_1)\|B\|} y(t_1).$$

In particular, for $0 < t < T$, we obtain

$$\int_{\omega(t)} |u(x,t)|^2 \mathrm{d}x \leq e^{2t\|B\|} \int_{\omega(0)} |a(x)|^2 \mathrm{d}x. \tag{1.3.23}$$

Because of the density of classical solutions in the set of L^2-solutions, and since its terms are L^2-continuous, we find that (1.3.23) is valid for every L^2-solutions.

Inequality (1.3.23) contains the following fact: If a vanishes identically on $\omega(0)$, then so does $u(t)$ on $\omega(t)$. Equivalently, the value of u at the point (X,T) (assuming that the solution is continuous) depends only on the restriction of the initial data a to the set $\omega(0)$.

Definition 1.3 *The set*

$$\omega(0) = \{x \in \mathbb{R}^d \,;\, (x - X)\cdot\nu \leq \lambda T,\ \forall(\lambda,\nu) \in \mathcal{V}\}$$

is the domain of dependence *of the point* (X,T).

Let us illustrate this notion with the system (1.2.17), to which we add a parameter c having the dimension of a velocity:

$$\partial_t u + c\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 u + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2 u = 0.$$

Since

$$\lambda I_2 + A(\nu) = \begin{pmatrix} \lambda + c\nu_1 & c\nu_2 \\ c\nu_2 & \lambda - c\nu_1 \end{pmatrix},$$

the cone $\mathcal{V}$ is given by the inequality $c|\nu| \le \lambda$. Thus the domain of dependence of (X,T) is the ball centred at X of radius cT.

We now fix a point x at initial time and look at those points (X,T) for which x belongs to their domains of dependence. Let us define a convex cone $\mathcal{C}^+$ by

$$\mathcal{C}^+ := \{y \in \mathbb{R}^d\,;\, \lambda + y\cdot\nu \ge 0,\, \forall(\lambda,\nu) \in \mathcal{V}\}.$$

Defining $y = (X-x)/T$, we have $1 + y\cdot\nu \ge 0$, that is $y \in \mathcal{C}^+$. Therefore $X = x + Ty \in x + T\mathcal{C}^+$. We deduce that u vanishes identically outside of $\operatorname{Supp} a + T\mathcal{C}^+$, where $a = u(\cdot,0)$. We have thus proved a propagation property:

Proposition 1.4 *Let the system (1.1.3) be symmetric. Given $a \in L^2(\mathbb{R}^d)^n$, let u be the solution of the Cauchy problem. Then, for $t_1 < t_2$,*

$$\operatorname{Supp} u(t_2) \subset \mathit{Supp}\, u(t_1) + (t_2 - t_1)\mathcal{C}^+. \tag{1.3.24}$$

Reversing the time arrow, we likewise have

$$\mathit{Supp}\, u(t_1) \subset \mathit{Supp}\, u(t_2) + (t_2 - t_1)\mathcal{C}^-, \tag{1.3.25}$$

where

$$\mathcal{C}^- := \{y \in \mathbb{R}^d\,;\, \lambda + y\cdot\nu \ge 0,\, \forall\nu \in -\mathcal{V}\}.$$

This result naturally yields the notion:

Definition 1.4 *Given a domain ω at initial time. The* influence domain *of ω at time $t > 0$ is the set $\omega + t\mathcal{C}^+$.*

Remark From Duhamel's formula, we extend the propagation property to the non-homogeneous problem. For instance, the solution for data $a \in L^2$ and $f \in L^1(0,T;L^2)$ satisfies

$$\operatorname{Supp} u(t) \subset (\operatorname{Supp} a + t\mathcal{C}^+) \cup \bigcup_{0<s<t} (\operatorname{Supp} f(s) + (t-s)\mathcal{C}^+). \tag{1.3.26}$$

1.3.2 *Non-decaying data*

Though the previous calculation applies only to solutions in $\mathscr{C}(\mathbb{R};L^2)$, where we already know the uniqueness of a solution, it can be used to construct solutions for much more general data than square-integrable ones.

First, the inequality (1.3.23) implies a propagation with finite speed: if $a \in L^2(\mathbb{R}^d)^n$ and $t > 0$, the support of $u(t)$ is contained in the sum $\operatorname{Supp} a + t\mathcal{C}^+$. We now use the following facts:

- L^2 is dense in $\mathscr{S}'$,
- for a in $\mathscr{S}'$, there exists a unique solution in $\mathscr{C}(\mathbb{R};\mathscr{S}')$ (see Proposition 1.3),
- the distributions that vanish on a given open subset of $\mathbb{R}^d$ form a closed subspace in $\mathscr{S}'$.

We conclude that (1.3.24) and (1.3.25) hold for a symmetric system when a is a tempered distribution.

We use this property to define a solution when the initial data is a (not necessarily tempered) distribution. Let a belong to $\mathscr{D}'(\mathbb{R}^d)^n$. Given a point $y \in \mathbb{R}^d$ and a positive number R, denote by $C(y;R)$ the set $y + R\mathcal{C}^-$. Choose a cut-off ϕ in $\mathscr{D}(\mathbb{R}^d)$, such that $\phi \equiv 1$ on $C(y,R)$. The product ϕa, being a compactly supported distribution, is a tempered one. Therefore, there exists a unique u^ϕ, solution of (1.1.3) in $\mathscr{C}(\mathbb{R};\mathscr{S}')$, with initial data ϕa. For two choices ϕ, ψ of cut-off functions, $(\phi - \psi)a$ vanishes on $C(y;R)$, so that $u^\phi(t)$ and $u^\psi(t)$ coincide on $C(y;R-t)$ for $0 < t < R$. This allows us to define a restriction of u^ϕ on the cone

$$K(y;R) := \bigcup_{0<t<R} \{t\} \times C(y;R-t).$$

As shown above, this restriction, denoted by $u_{y,R}$ does not depend on the choice of the cut-off. It actually depends only on the restriction of a on $C(y;R)$. Now, if a point (z,t) lies in the intersection of two such cones $K(y_1;R_1)$ and $K(y_2;R_2)$, it belongs to a third one $K(y_3;R_3)$, which is included in their intersection. The restrictions of u_{y_1,R_1} and u_{y_2,R_2} to $K(y_3;R_3)$ are equal, since they depend only on the restriction of a on $C(y_3;R_3)$. We obtain in this way a unique distribution $u \in \mathscr{C}(\mathbb{R}^+;\mathscr{D}')$, whose restriction on every cone $K(y;R)$ coincides with $u_{y,R}$. It solves (1.1.3) in the distributional sense, and takes the value a as $t = 0$. Reversing the time arrow, we solve the backward Cauchy problem as well.

This construction is relevant, for instance, when a is L^2_{loc} rather than square-integrable. It can be used also when a is in L^p_{loc} for $p \neq 2$, even though the corresponding solutions are not $\mathscr{C}(\mathbb{R};L^p)$ in general, because of Brenner's theorem.

1.3.3 *Uniqueness for non-decaying data*

The construction made above, though defining a unique distribution, does not tell us about the uniqueness in $\mathscr{C}(0,T;X)$ for $a \in X$, when $X = \mathscr{D}'(\mathbb{R}^d)^n$ or $X = L^2_{\mathrm{loc}}(\mathbb{R}^d)^n$ for instance. This is because we got uniqueness results through the use of Fourier transform, a tool that does not apply here. We describe below two relevant techniques.

Let us begin with $X = L^2_{\mathrm{loc}}$. We assume that $u \in \mathscr{C}(0,T;X)$ solves (1.1.3) with $a = 0$. We use the **localization** method. Let $K(y;R)$ be a cone as in the previous section, and $\phi \in \mathscr{D}(\mathbb{R}^d)$ be such that

$$\phi(x) = 1, \quad \forall x \in \bigcup_{0<t<R} C(y;R-t),$$

the latter set being the x-projection of $K(y;R)$. Multiplying (1.1.3) by ϕ, and denoting $v := \phi u$, we obtain

$$\partial_t v + \sum_\alpha A^\alpha \partial_\alpha v = Bv + f,$$

where $v \in \mathscr{C}(0,T;L^2(\mathbb{R}^d)^n)$ and

$$f := (\partial_t \phi + A(\nabla_x \phi))u \in \mathscr{C}(0,T;L^2(\mathbb{R}^d)^n).$$

At this point, we are allowed to write the energy estimate

$$\partial_t |v|^2 + \sum_\alpha \partial_\alpha (A^\alpha v, v) = 2\mathrm{Re}\ (Bv + f, v),$$

which gives for every $0 \le t_1 < t_2 < R$, after integration,

$$\int_{\omega(t_2)} |v(t_2)|^2 \mathrm{d}x \le \int_{\omega(t_1)} |v(t_1)|^2 \mathrm{d}x + \int_{t_1}^{t_2} \mathrm{d}t \int_{\omega(t)} 2\mathrm{Re}\ ((Bv,v) + (f,v))\mathrm{d}x, \tag{1.3.27}$$

where $\omega(t) := C(y; R-t)$. However, the equalities $v = u$, $f = 0$ hold in $K(y;R)$. Therefore (1.3.27) reduces to

$$\int_{\omega(t_2)} |u(t_2)|^2 \mathrm{d}x \le \int_{\omega(t_1)} |u(t_1)|^2 \mathrm{d}x + 2\int_{t_1}^{t_2} \mathrm{d}t \int_{\omega(t)} \mathrm{Re}\ (Bu,u)\,\mathrm{d}x.$$

This, with the Gronwall inequality, gives

$$\int_{\omega(t)} |u(x,t)|^2 \mathrm{d}x \le e^{2t\|B\|} \int_{\omega(0)} |u(x,0)|^2 \mathrm{d}x = 0.$$

Since y and R are arbitrary, we obtain $u \equiv 0$ almost everywhere, which is the uniqueness property.

We now turn to the case $X = \mathscr{D}'(\mathbb{R}^d)^n$, where the former argument does not work. Our main ingredient is the *Holmgren principle*, a tool that we shall develop more systematically in subsequent chapters. We assume that $u \in \mathscr{C}(0,T;X)$ solves (1.1.3) in the distributional sense. This means that, for every test function $\phi \in \mathscr{D}(\mathbb{R}^d \times (0,T))^n$, it holds that

$$\langle u, L^*\phi \rangle = 0, \quad L^* := -\partial_t - \sum_\alpha (A^\alpha)^T \partial_\alpha - B^T.$$

This may be rewritten as

$$\int_0^T \langle u(t), L^*\phi(t) \rangle \,\mathrm{d}t = 0. \tag{1.3.28}$$

Let ψ be a slightly more general test function: $\psi \in \mathscr{D}(\mathbb{R}^d \times (-\infty, T))^n$. Choosing $\theta \in \mathscr{C}^\infty(\mathbb{R})$ with $\theta(\tau) = 0$ for $\tau < 1$ and $\theta(\tau) = 1$ for $\tau > 2$, we define

$$\phi_\epsilon(x,t) = \theta(t/\epsilon)\psi(x,t).$$

We may apply (1.3.28) to ϕ_ϵ, which gives

$$\int_0^T \theta(t/\epsilon)\langle u(t), L^*\psi(t)\rangle \,\mathrm{d}t = \frac{1}{\epsilon}\int_0^T \theta'(t/\epsilon)\langle u(t), \psi(t)\rangle \,\mathrm{d}t.$$

Using the continuity in time, we may pass to the limit as $\epsilon \to 0^+$, and obtain

$$\int_0^T \langle u(t), L^*\psi(t)\rangle \,\mathrm{d}t = \langle u(0), \psi(0)\rangle.$$

Therefore, assuming $u(0) = 0$, we see that (1.3.28) is valid for ψ as well, that is to test functions in $\mathscr{D}(\mathbb{R}^d \times (-\infty, T))^n$.

We now choose an arbitrary test function $f \in \mathscr{D}(\mathbb{R}^d \times (0, T))^n$. Obviously, L^* is a hyperbolic operator and we can solve the backward Cauchy problem

$$L^*\chi = f, \quad \chi(T) = 0.$$

Extending f by zero for $t \leq 0$, we obtain a unique solution $\chi \in \mathscr{C}^\infty(-\infty, T; \mathscr{S})$. Applying (1.3.26) to this backward problem, we see that $\chi(t)$ has compact support for each time, with $\operatorname{Supp}\chi(t)$ included in a ball of the form $B_{\rho(T-t)}$, for a suitable constant ρ. Also, χ vanishes identically for t close enough to T (because f does). Truncating, we apply (1.3.28) to $\psi(x,t) = \theta(t+1)\chi(x,t)$. This gives $\langle u, f\rangle = 0$ for all test functions, that is $u = 0$. Therefore the Cauchy problem for a Friedrichs-symmetric operator has the uniqueness property in the class $\mathscr{C}(0, T; \mathscr{D}')$.

1.4 Directions of hyperbolicity

The situation for general (weakly) hyperbolic operators is not as neat as that for Friedrichs-symmetrizable ones. Non-symmetrizable operators do exist, as soon as $d = 2$ and $n = 3$, as shown by Lax [110]. The class of constantly hyperbolic operators provides a valuable and flexible alternative to Friedrichs-symmetrizable ones. Their analysis will lead us to several new and useful notions.

In this section, we shall not address the problem of propagation of the support (with finite velocity), which we solved in the symmetric case. This propagation holds true for constantly hyperbolic systems, but a rigorous proof needs a theory of the Cauchy problem for systems with variable coefficients. Such a theory will be done in Chapter 2, where we shall prove an accurate result.

1.4.1 *Properties of the eigenvalues*

The results in this section are essentially those of Lax [110], and the arguments follow Weinberger [217], though we give a more detailed proof of the claim below.

We begin by considering a subspace E in $\mathbf{M_n}(\mathbb{R})$, with the property that every matrix in E has a real spectrum. Without loss of generality, we may assume that I_n belongs to E. If $M \in E$, we denote by $\lambda_1(M) \leq \cdots \leq \lambda_n(M)$ the spectrum of M, counting with multiplicities. The functions λ_j are positively homogeneous of order one. They are continuous, but could be non-differentiable at crossing points. In the constantly hyperbolic case, however, they are analytic away from the origin.

Lemma 1.1 *Let A and B be matrices in E, with $\lambda_1(B) > 0$. Then the eigenvalues of $B^{-1}(\lambda I_n - A)$ are real.*

Proof From the assumption, we know that B is non-singular. Define a polynomial

$$P(X,Y) := \det(XI_n - A - YB),$$

which has degree n with respect to X as well as to Y. Define continuous functions $\phi_j(\mu) = \lambda_j(A + \mu B)$. From homogeneity and continuity, we have

$$\phi_j(\mu) \sim \begin{cases} \mu\lambda_j(B), & \text{as } \mu \to +\infty, \\ \mu\lambda_{n+1-j}(B) & \text{as } \mu \to -\infty. \end{cases}$$

Hence $\phi_j(\mu)$ tends to $\pm\infty$ with μ. By the Intermediate Value Theorem, it must take any prescribed real value λ at least once.

Thus, let λ^* be given and $\mu_j \in \mathbb{R}$ be a root of $\phi_j(\mu_j) = \lambda^*$ for each j. Given one of these roots, μ^*, let J be the number of indices such that $\mu_j = \mu^*$. Then λ^* is a root of $P(\cdot, \mu^*)$, of order J at least.

Claim 1.1 *The multiplicity of μ^* as a root of $P(\lambda^*, \cdot)$ is larger than or equal to J.*

This claim readily implies the lemma. Its proof is fairly simple when the ϕ_js are differentiable, for instance in the constantly hyperbolic case. But in the general case, one must use once more the assumption. To simplify the notations, we assume without loss of generality that $\lambda^* = \mu^* = 0$, by translating A to $A + \mu^* B - \lambda^* I_n$. Let N $(N \geq J)$ be the multiplicity of the null root of $P(\cdot, 0)$. The Newton's polygon of the polynomial P admits the vertices $(N, 0)$ and $(0, K)$.

Let δ be the edge of the Newton's polygon with vertex $(N, 0)$. We denote its other vertex by (j, k). Retaining only those monomials of P whose degrees (a, b) belong to δ, we obtain a polynomial $X^j Q$ with the following homogeneity:

$$Q(a^k X, a^{N-j} Y) = a^{k(N-j)} Q(X, Y).$$

It is a basic fact in algebraic geometry (see [35], Section 2.8) that, in the vicinity of the origin, the algebraic curve $P(x, y) = 0$ is described by simpler curves corresponding to the edges of the Newton polygon, up to analytic diffeomorphisms. In the present case, these diffeomorphisms have real coefficients (i.e. they preserve real vectors) since P has real coefficients. The 'simple' curve γ associated to δ is just that with equation $Q(x, y) = 0$. Hence, points (x, y) in γ with a real co-ordinate y must be real (because this is so in the curve $P = 0$.)

Let ω be a root of unity, of order $2(N - j)$, that is $\omega^{N-j} = -1$. Because of the homogeneity, the map $(x, y) \mapsto (\omega^k x, -y)$ preserves γ. If y is real, the map thus moves a real point into another one. Hence, ω^k is real, thus $\omega^{2k} = 1$. This implies that k is a multiple of $N - j$. In particular, $k \geq N - j$.

Since (j, k) is a vertex of the Newton polygon, lying between the vertices $(N, 0)$ and $(0, K)$, we have

$$\frac{j}{N} + \frac{k}{K} \leq 1.$$

Together with $k \geq N - j$, this implies $K \geq N$ and the claim. □

Suppose that in the proof of Lemma 1.1, one of the functions, say ϕ_l, is not strictly monotone. For a suitable real number λ, the equation $\phi_l(\mu) = \lambda$ will have at least three roots, and $P(\lambda, \cdot) = 0$ will have $n + 2$ roots at least, which is absurd. Therefore, the assumption $\lambda_1(B) > 0$ implies that $\mu \mapsto \lambda_j(A + \mu B)$ is monotone increasing. For a general B in E we may apply that to $B' := B - cI_n$ with $c < \lambda_1(B)$. Letting $c \to \lambda_1(B)$, we obtain that

$$\mu \to \lambda_j(A + \mu B) - \mu\lambda_1(B)$$

is non-decreasing. In particular,

$$\lambda_j(A + B) \geq \lambda_j(A) + \lambda_1(B), \quad \forall A, B \in E. \tag{1.4.29}$$

Reversing (A, B) into $(-A, -B)$, we also have

$$\lambda_j(A + B) \leq \lambda_j(A) + \lambda_n(B), \quad \forall A, B \in E. \tag{1.4.30}$$

In particular, with $j = 1$ in (1.4.29) and $j = n$ in (1.4.30), we obtain:

Proposition 1.5 *Let E be a vector space of real $n \times n$ matrices, whose every element has a real spectrum.*

The smallest eigenvalue is a concave function, while the largest is a convex one: For every A and B in E,

$$\lambda_1(A + B) \geq \lambda_1(A) + \lambda_1(B),$$
$$\lambda_n(A + B) \leq \lambda_n(A) + \lambda_n(B).$$

This applies to the space $E := \{A(\xi)\,;\, \xi \in \mathbb{R}^d\}$ when the operator $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is hyperbolic.

Remark When $E = \mathbf{Sym_n}(\mathbb{R})$, a space that obviously satisfies the assumption, (1.4.29) and (1.4.30) belong to the set of Weyl's inequalities. Given the spectra of A and B, but not A and B themselves, Horn's conjecture, proved recently by Klyachko [98] and Knutson and Tao [99], characterizes the set of possible spectra of $A + B$ as a convex polytope, defined through rather involved linear inequalities. It would be interesting to know[†] which of these inequalities remain

[†]This question has been solved recently, thanks to the efforts of J. Helton, V. Vinnikov and L. Gurvits. It turns out that every linear inequality that is valid for real symmetric matrices is valid for matrices in E. These inequalities actually apply to the roots of an arbitrary hyperbolic homogeneous polynomial.

true for a general subspace E treated in this section. The simplest ones in Horn's conjecture are Weyl's inequalities

$$\lambda_k(A+B) \leq \lambda_i(A) + \lambda_j(B), \qquad (k+n \leq i+j),$$

and

$$\lambda_k(A+B) \geq \lambda_i(A) + \lambda_j(B), \qquad (k+1 \geq i+j).$$

Next comes the Theorem of Lidskii, which tells us that, as a vector in $\mathbb{R}^n$, the spectrum of $A+B$ belongs to the convex hull of $(P^\sigma)_{\sigma\in\mathbf{S}_n}$ where P_σ has coordinates $\lambda_i(A) + \lambda_{\sigma(i)}(B)$ and σ runs over all permutations. See Exercises 11 of Chapter 3 and 19 of Chapter 5 in [187].

Lemma 1.1 can be improved in the following way. Given $\lambda^* \in \mathbb{R}$, let $\sigma_1, \dots, \sigma_s$ be the distinct eigenvalues of $M = B^{-1}(\lambda I_n - A)$. Let S_ℓ be the set of indices j such that $\mu_j(\lambda^*) = \sigma_\ell$ and J_ℓ its cardinality. Since each function ϕ_j is strictly monotone, we have $\lambda^* - \phi_j(\sigma_\ell) \neq 0$ for every j not in S_ℓ. Therefore, J_ℓ is precisely the multiplicity of the root λ^* of $P(\cdot, \sigma_\ell)$. From the claim, we know that J_ℓ is less than or equal to the multiplicity m_ℓ of σ_ℓ as a root of $P(\lambda^*, \cdot)$. Hence

$$n = J_1 + \cdots + J_s \leq m_1 + \cdots + m_s = n,$$

and we conclude that $m_\ell = J_\ell$ for each ℓ.

Lemma 1.2 *With the assumptions of Lemma 1.1, let a real pair satisfy* $P(\lambda, \mu) = 0$*, where* $P(X, Y) := \det(XI_n - A - YB)$*. Then the multiplicities of* λ *as a root of* $P(\cdot, \mu)$*, and of* μ *as a root of* $P(\lambda, \cdot)$*, coincide.*

Finally, we remark that $A \mapsto \max\{-\lambda_1(A), \lambda_n(A)\}$ is a semi-norm over such a space E as above.

1.4.2 *The characteristic and forward cones*

From now on, E is the set of matrices $\tau I_n + A(\xi)$ for $(\tau, \xi) \in \mathbb{R}^{1+d}$, where $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is a hyperbolic operator.

Definition 1.5 *The* characteristic cone *of the hyperbolic operator* $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ *is the set*

$$\mathrm{char}L := \{(\xi, \lambda) \in \mathbb{R}^d \times \mathbb{R}\,;\, \det(A(\xi) + \lambda I_n) = 0\}.$$

Its elements are the characteristic frequencies*. The connected component of* $(0, 1)$ *in* $(\mathbb{R}^d \times \mathbb{R}) \setminus \mathrm{char}L$ *is denoted by* Γ*; it is called the* forward cone.

Obviously, Γ is a kind of epigraph of λ_n:

$$\Gamma = \{(\xi, \lambda)\,;\, \lambda > \lambda_n(-\xi)\}.$$

According to Proposition 1.5, it is a convex cone in $\mathbb{R}^{d+1}$, a result originally due to Gårding [65, 66]. The terminology *forward cone* will be explained in the next section.

When L is constantly hyperbolic, the eigenvalues λ_j are analytic away from the origin. The function λ_n has therefore a non-negative Hessian $\mathbf{D}^2\lambda_n$. Because of homogeneity, this Hessian is indefinite,

$$\mathbf{D}^2\lambda_n(\xi)\xi = 0.$$

Therefore Γ is not strictly convex in the usual sense, and we shall say that λ_n is *transversally strictly convex* if the equality

$$\theta\lambda_n(\xi) + (1-\theta)\lambda_n(\xi') = \lambda_n(\theta\xi + (1-\theta)\xi'), \quad \theta \in (0,1)$$

implies $\xi' \in \mathbb{R}^+\xi$. We now prove that such a strict convexity holds for most systems.

Proposition 1.6 *Let the operator*

$$L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

be hyperbolic. Then the forward cone Γ *is convex.*

If L is constantly hyperbolic, then either the function λ_n is transversally strictly convex (and therefore λ_1 is transversally strictly concave), or the system is a vector-valued transport equation

$$\partial_t u + (\vec{V}\cdot)\nabla_x u = 0.$$

.

Proof If λ_n is not transversally strictly convex, there exists a segment $[\xi_1, \xi_2]$ on which λ_n is affine, and ξ_1, ξ_2 are not parallel. By homogeneity, λ_n is affine on the triangle with vertices $0, \xi_1, \xi_2$. Since $\lambda_n(0) = 0$, 'affine' actually means 'linear'. Let P be the plane spanned by ξ_1, ξ_2. Since λ_n is analytical away from the origin, and since $P \setminus \{0\}$ is a connected set, the restriction of λ_n to P is linear. It follows that $\lambda_n(\xi_1) = -\lambda_n(-\xi_1)$. In other words, $\lambda_n(\xi_1) = \lambda_1(\xi_1)$. This means that $A(\xi_1)$ has only one eigenvalue. Finally, the system being constantly hyperbolic, there must be only one eigenvalue for every ξ. Since $A(\xi)$ is diagonalizable, this gives $A(\xi) = \lambda_n(\xi)I_n$. Therefore, $\lambda_n(\xi) = \mathrm{Tr}\ A(\xi)/n$, which shows that λ_n is linear on the whole $\mathbb{R}^d$, thus there exists a vector $\vec{V}$ such that $\lambda_n(\xi) = \vec{V} \cdot \xi$. This ends the proof. □

1.4.3 *Change of variables*

The role of the cone Γ becomes clear when we consider changes of the space–time reference frame. Let us perform a linear change of independent variables

$$(x,t) \mapsto (y,s), \quad y = Rx + tV, \quad s = \lambda_0 t + \xi_0 \cdot x,$$

with $R \in M_d(\mathbb{R})$ and $V, \xi_0 \in \mathbb{R}^d$, chosen so that the whole matrix

$$\mathcal{R} := \begin{pmatrix} R & V \\ \xi_0^T & \lambda_0 \end{pmatrix}$$

is invertible. The system (1.1.3) is changed into

$$\frac{\partial u}{\partial s} + \sum_\alpha \widetilde{A}^\alpha \frac{\partial u}{\partial y_\alpha} = \widetilde{B}u,$$

where

$$\widetilde{A}^\alpha := (\lambda_0 I_n + A(\xi_0))^{-1} \left(\sum_\beta R_{\alpha\beta} A^\beta + V_\alpha I_n \right), \tag{1.4.31}$$

provided that (ξ_0, λ_0) is not characteristic. We consider the variable s as a new time variable and look at the Cauchy problem. Let us point out that it is not equivalent to the former Cauchy problem, since the data is now given on the hyperplane $\{s = 0\}$, instead of $\{t = 0\}$. Its strong well-posedness is equivalent to the hyperbolicity of the operator

$$\frac{\partial}{\partial s} + \sum_\alpha \widetilde{A}^\alpha \frac{\partial}{\partial y_\alpha}.$$

A change of variables that preserves t (that is with $\xi_0 = 0$, $\lambda_0 = 1$) is harmless, giving $\widetilde{A}(\eta) = A(\xi) + (\xi \cdot R^{-1}V)I_n$ with $\xi = R^T\eta$, so that hyperbolicity is preserved. Therefore, hyperbolicity is really a property of the pair (ξ_0, λ_0), which determines the direction of the hyperplane $\{s = 0\}$ where the Cauchy data is given. This leads us to the following.

Definition 1.6 *We say that the operator*

$$L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

is hyperbolic in the direction (ξ_0, λ_0), *if* (ξ_0, λ_0) *is not characteristic, and if, moreover, the operator*

$$\widetilde{L} := \frac{\partial}{\partial s} + \sum_\alpha \widetilde{A}^\alpha \frac{\partial}{\partial y_\alpha} \tag{1.4.32}$$

is hyperbolic, with $\widetilde{A}^\alpha$ *being defined in (1.4.31).*

Remarks

- In particular, $\partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is hyperbolic in the direction $(0, 1)$ if and only if it is hyperbolic in the sense that we considered so far.
- The hyperbolicity in directions (ξ_0, λ_0) and $(-\xi_0, -\lambda_0)$ are equivalent. Therefore, we may always restrict ourselves to $\lambda_0 \geq 0$.
- When $\partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is symmetric, and when $\lambda_0 I_n + A(\xi_0)$ is positive-definite, the new operator $\partial_s + \sum_\alpha \widetilde{A}^\alpha \partial_\alpha$ is Friedrichs symmetrizable, with $\lambda_0 I_n + A(\xi_0)$ as a symmetrizer. This statement is to be compared with

Theorem 1.5 below, with the remark that the positive-definiteness means that $\lambda_0 I_n + A(\xi_0)$ belongs to Γ in this case.

- If $(\xi_0, \lambda_0) \in \mathrm{char}L$, the matrices $\widetilde{A}^\alpha$ are not well-defined; the variable s cannot be taken as a time variable. In this case, we say that the hyperplane $\{s = 0\}$ is *characteristic*. We shall come back to this important notion later.

Hyperbolicity in the direction (ξ_0, λ_0) means that the matrices

$$\widetilde{A}(\eta) := (\lambda_0 I_n + A(\xi_0))^{-1} \left(A(R^T \eta) + (V \cdot \eta) I_n\right)$$

have a real spectrum for every $\eta \in \mathbb{R}^d$, and are uniformly diagonalizable. Lemma 1.1 tells us that this spectrum is real for every $\eta \in \mathbb{R}^d$, as soon as $\lambda_1(\lambda_0 I_n + A(\xi_0)) > 0$, which means $\lambda_0 + \lambda_1(A(\xi_0)) > 0$. On the other hand, one has, with the notations of Lemma 1.1,

$$\ker(B^{-1}(\lambda I_n - A) - \mu I_n) = \ker(\lambda I_n - A - \mu B).$$

Since now every element of E is diagonalizable, the dimension of the right-hand side equals the multiplicity of λ as a root of $P(\cdot, \mu)$. From Lemma 1.2, we deduce that the dimension of the left-hand side equals the multiplicity of μ as a root of $P(\lambda, \cdot)$. Hence the algebraic and geometric multiplicities coincide: $B^{-1}(\lambda - A)$ is diagonalizable. Applying this result to the above context, we conclude that $\widetilde{A}(\eta)$ is diagonalizable with a real spectrum, for every $\eta \in \mathbb{R}^d$. We leave the reader to verify that the diagonalization can be performed uniformly, using the assumption that it is true in E.

In two instances, the verification of this fact is rather easy. For, if L is symmetric and $(\lambda_0, \xi_0) \in \Gamma$, then $\lambda_0 I_n + A(\xi_0)$ is positive-definite and plays the role of a symmetrizer for $\tilde{L}$. On the other hand, assume that L is strictly hyperbolic (or more generally constantly hyperbolic). Looking back at the proof of Lemma 1.1, the functions ϕ_j and ϕ_k cannot coincide somewhere if $j \neq k$. Hence $B^{-1}(\lambda I_n - A)$ has distinct eigenvalues. It follows that $\tilde{L}$ is strictly (or constantly) hyperbolic too.

Therefore, we have the following result.

Theorem 1.5 *A hyperbolic operator L is hyperbolic in every direction of its forward cone. If L is either Friedrichs symmetrizable, or strictly, or constantly hyperbolic, then L has the same property in every direction of its forward cone.*

Comments It is known that when E is a subspace of $\mathbf{M_n}(\mathbb{R})$, consisting only on diagonalizable matrices with real eigenvalues, these eigenvalues may be labelled, at least locally, with the property that one-sided directional derivatives

$$\lim_{h \to 0+} \frac{\lambda(T + hT_1) - \lambda(T)}{h} =: \delta\lambda_T(T_1)$$

exist. However, $\delta\lambda_T(T_1)$ may be neither linear with respect to T_1, nor continuous in T. Although it is positively homogeneous in T_1, it may not satisfy

$$\delta\lambda_T(-T_1) = -\delta\lambda_T(T_1). \tag{1.4.33}$$

We illustrate these facts with a two-dimensional space, spanned by the matrices

$$A^1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A^2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix $\xi_1 A^1 + \xi_2 A^2$ has eigenvalues $\pm|\xi|$. We choose $\lambda_1 = -|\xi|$, $\lambda_2 = |\xi|$. These functions are not differentiable at the origin but have the one-sided directional derivatives mentioned above. There is no way to relabel the eigenvalues in order to satisfy (1.4.33). A more involved example is the set of symmetric $n \times n$ matrices.

A rather complete analysis of these facts may be found in Chapter 2 of Kato's book [95]. It also contains (Theorem 5.4 of [95]) the following amazing fact, which shows that the two-dimensional example above is optimal. When restricting to a curve $s \mapsto T(s)$ in E, where the parametrization is differentiable (respectively, analytic), one may label the eigenvalues in such a way that they are differentiable (respectively, analytic) with respect to s. In other words, one may satisfy (1.4.33) when there is only one scalar parameter, though it will be at the price of a loss of ordering.

1.4.4 *Homogeneous hyperbolic polynomials*

The theory of scalar equations of higher order involves the notion of *hyperbolic polynomials*. Let p be a homogeneous polynomial of degree n in $d+1$ variables $\xi_0, \dots, \xi_d$. We consider the equation

$$p\left(\frac{\partial}{\partial x_0}, \dots, \frac{\partial}{\partial x_d}\right) u = f.$$

According to Gårding [65], we say that p is hyperbolic in the direction of some real vector $a \in \mathbb{R}^{d+1}$ if for every vector $\xi \in \mathbb{R}^{d+1}$, the equation

$$p(\tau a + \xi) = 0$$

has n real roots, counting multiplicities. This implies $p(a) \neq 0$ and we may normalize p by $p(a) = 1$. Notice that the traditional case where $x_0 = t$ and p is hyperbolic in the direction of time corresponds to $a = (1, 0, \dots, 0)$. A typical example is $p(\xi) = \xi_0^2 - \xi_1^2 - \cdots - \xi_d^2$, which is associated to the wave operator $\partial_t^2 - \Delta_x^2$, and is hyperbolic in the 'direction of time' $a = (1, 0 \dots, 0)$.

The definition of hyperbolicity given above is equivalent to the $\mathcal{C}^\infty$ well-posedness of the Cauchy problem for the equation

$$p(\partial_0, \dots, \partial_d) u = f.$$

However, it does not imply L^2- or H^s-well-posedness (in a sense adapted to the order of the operator); it is merely the analogue of the weak hyperbolicty described in Section 1.1. We refer to [65] for the case where p is not homogeneous. Gårding's definition of hyperbolicity is the more general one, and extends, for instance, that of Petrowsky [158].

We shall not discuss here the Cauchy problem for general hyperbolic operators. This has given rise to an enormous literature. However, we do not resist to mention the remarkable convexity results obtained by Gårding in [66]. The first property is that the polynomial q, homogeneous of degree $n-1$, defined by

$$q(\xi) := \sum_{\alpha=0}^{d} a_\alpha \frac{\partial p}{\partial \xi_\alpha}$$

is hyperbolic in the direction of a too. This is the interlacing property of real zeroes of a univariate polynomial and its derivative. Let us give an immediate application. It is clear that a linear form is hyperbolic in every non-characteristic direction, and also that the product of polynomials that are hyperbolic in some direction a (the same for every one), is hyperbolic also in this direction. For instance, $\sigma_{d+1}(\xi) := \prod_\alpha \xi_\alpha$ is hyperbolic in the direction of $(1,\dots,1)$. Applying repeatedly the derivation in direction a, we deduce that every elementary symmetric polynomial $\sigma_k(\xi)$ is hyperbolic in the direction $(1,\dots,1)$. This is trivial if $k=1$ (pure transport), and this is well known if $k=2$, because σ_2 is a quadratic form of index $(1,d)$, positive on $(1,\dots,1)$.

The forward cone $C_p(a)$ is the connected component of a in the set defined by $p(\xi)>0$. As in the case of first-order systems, $C(a)$ is convex, and p is hyperbolic in the direction of b for every b in $C_p(a)$. If q is the a-derivative as above, then $C_p(a) \subset C_q(a)$, with obvious notation.

The nicest result is perhaps the following. Let P be the polarized form of p, meaning that

$$(\xi^1,\dots,\xi^n) \mapsto P(\xi^1,\dots,\xi^n)$$

is a symmetric multilinear form, such that $P(\xi,\dots,\xi) = p(\xi)$ for every $\xi \in \mathbb{R}^{1+d}$ (this is the generalization of the well-known polarization of a quadratic form). Then we have

$$\left(\xi^1 \in C_p(a),\dots,\xi^n \in C_p(a)\right) \Longrightarrow \left(p(\xi^1)\cdots p(\xi^n) \le P(\xi^1,\dots,\xi^n)^n\right). \quad (1.4.34)$$

We point out that when $n=2$, that is when p is a quadratic form of index $(1,d)$, this looks like the Cauchy–Schwarz inequality, except that (1.4.34) is in the opposite sense. An equivalent statement is that

$$\xi \mapsto p(\xi)^{1/n}$$

is a concave function over $C_p(a)$.

Gårding's results have had many consequences in various fields, including differential geometry, elliptic (!) PDEs (see, for instance, the article by Caffarelli *et al.* [29]) and interior point methods in optimization. A rather surprising byproduct is the concavity of the function[6]

$$H \mapsto (\det H)^{1/n}, \qquad H \in \mathbf{HPD}_n.$$

This property is reminiscent of the Alexandrov–Fenchel inequality

$$\mathrm{vol}(K_1)\mathrm{vol}(K_2) \leq V(K_1, K_2)^2$$

for convex bodies, where V denotes the *mixed volume*. The van der Waerden inequality for the permanent of a doubly stochastic matrix can be rewritten in terms of an inequality for hyperbolic polynomials, applied to σ_n in n indeterminates.

1.5 Miscellaneous

1.5.1 *Hyperbolicity of subsystems*

Let $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be a hyperbolic $n \times n$ operator. Given a linear subspace G of $\mathbb{R}^n$ of dimension m, with a projector π onto G, one may form a subsystem in m unknowns $v(x,t) \in G$ and m equations, governed by the operator $L' = \partial_t + \sum_\alpha \pi A^\alpha \partial_\alpha$. There is no reason, in general, why L' would be hyperbolic. The following result shows that a clever choice of π ensures this hyperbolicity.

Theorem 1.6 *Let $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be a hyperbolic $n \times n$ operator and ξ_0 belong to S^{d-1}. Given an eigenvalue λ_0 of the spectrum of $A(\xi_0)$, denote by π the eigenprojection onto the eigenspace $F(\lambda_0) := \ker(A(\xi_0) - \lambda_0 I_n)$.*

Then the operator

$$L' := \partial_t + \sum_{\alpha=1}^{d} \pi A^\alpha \partial_\alpha,$$

acting on functions valued in $F(\lambda_0)$ *(thus it is an $m \times m$ operator, m being the multiplicity of λ_0), is hyperbolic.*

This result is of low interest when L is symmetric hyperbolic (or more generally smmetrizable), for then π is an orthogonal projection, so that $\pi A(\xi) : F(\lambda_0) \to F(\lambda_0)$ is symmetric, thus L' is symmetric hyperbolic too.

Proof Using a linear change of unknowns, which amounts to conjugating the matrices A^α, we may assume that $A(\xi_0)$ is diagonal:

$$A(\xi_0) = \begin{pmatrix} \lambda_0 I_m & 0 \\ 0 & D_0 \end{pmatrix},$$

[6] This result is strictly better than the well-known concavity of $H \mapsto \log\det H$ for positive-definite Hermitian matrices. However, this latter statement has the advantage of having a form independent of n.

where $D_0 - \lambda_0 I_{n-m}$, of size $n - m$, is invertible. We decompose vectors and matrices accordingly:

$$X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A^\alpha = \begin{pmatrix} C^\alpha & F^\alpha \\ E^\alpha & D^\alpha \end{pmatrix}.$$

The theorem states that the $m \times m$ operator

$$L' := \partial_t + \sum_{\alpha=1}^{d} C^\alpha \partial_\alpha$$

is hyperbolic.

Since $\mathbb{C}^n$ is the direct sum $(\mathbb{R}^m \times \{0_{n-m}\}) \oplus (\{0_m\} \times \mathbb{R}^{n-m})$ of invariant subspaces of $A(\xi_0)$, corresponding to disjoint parts of the spectrum, standard perturbation theory (see Kato [95]) tells us that there exists a neighbourhood $\mathcal{V}$ of ξ_0 and an analytical map $\xi \mapsto K(\xi)$ from $\mathcal{V}$ to $M_{(n-m)\times m}(\mathbb{R})$, such that

i) $K(\xi_0) = 0$,

ii) the subspace

$$N(\xi) := \left\{ \begin{pmatrix} x \\ K(\xi)x \end{pmatrix} ; x \in \mathbb{R}^m \right\}$$

is invariant under $A(\xi)$.

Hence, $N(\xi)$ is invariant under the flow of $\dot{X} = A(\xi)X$. On this subspace, the flow is defined by $\dot{x} = Q(\xi)x$, $y = K(\xi)x$, where

$$Q(\xi) := C(\xi) + F(\xi)K(\xi).$$

Let us define

$$M := \sup_\xi \| \exp iA(\xi) \|,$$

which is finite by assumption. For every ξ in $\mathcal{V}$, $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^m$, it holds that

$$\| \exp(itQ(\xi))x_0 \| \le c_0 M(\|x_0\| + \|K(\xi)x_0\|),$$

where c_0 is accounted for the equivalence of the standard norm with $(x, K(\xi)x) \mapsto \|x\|$, on $N(\xi)$. In other words,

$$\| \exp(itQ(\xi)) \| \le c_0 M(1 + \|K(\xi)\|). \tag{1.5.35}$$

Let η be given in $\mathbb{R}^d$. One applies (1.5.35) to the vector $\xi = \xi_0 + s\eta$, for s small enough (in such a way that $\xi \in \mathcal{V}$) and $t = 1/s$. Since

$$Q(\xi) = \lambda_0 I_m + sC(\eta) + sF(\eta)K(\xi),$$

it holds that

$$\exp itQ(\xi) = \mathrm{e}^{it\lambda_0} \exp i(C(\eta) + F(\eta)K(\xi)).$$

Using (1.5.35) and passing to the limit as $s \to 0$, we obtain

$$\| \exp iC(\eta) \| \le c_1 M,$$

which proves the claim. □

We improve now Theorem 1.6 for constantly hyperbolic systems. Theorem 1.7 below is attributed to Lax. It turns out to be useful in geometrical optics in the presence of non-simple eigenvalues. It will also be valuable in the study of characteristic initial boundary value problems, see Section 6.1.3.

To begin with, we consider a constantly hyperbolic operator L and select an eigenvalue $\lambda(\xi)$, whose multiplicity, for $\xi \neq 0$, is denoted by m. Denote by π_ξ the eigenprojector onto $\ker(A(\xi) - \lambda(\xi) I_n)$. Obviously, λ and π are analytic functions on $\mathbb{R}^d \setminus \{0\}$.

Theorem 1.7 *Assume that L is constantly hyperbolic and adopt the above notations. Then, for every $\xi \neq 0$ and every $\eta \in \mathbb{R}^d$, it holds that*

$$\pi_\xi A(\eta) \pi_\xi = (\mathrm{d}\lambda(\xi) \cdot \eta) \pi_\xi.$$

Proof Differentiating the identity $(A(\xi) - \lambda(\xi))\pi_\xi = 0$, we obtain

$$(A(\xi) - \lambda(\xi))(\mathrm{d}\pi(\xi) \cdot \eta) + (A(\eta) - \mathrm{d}\lambda(\xi) \cdot \eta)\pi_\xi = 0.$$

We eliminate the factor $\mathrm{d}\pi(\xi) \cdot \eta$ by multiplying this equality by π_ξ on the left. □

In matrix terms, we may choose co-ordinates in $\mathbb{R}^n$ such that, for some vector $\xi \neq 0$,

$$A(\xi) = \begin{pmatrix} \lambda(\xi) I_m & 0 \\ 0 & A' \end{pmatrix}, \quad \det(A' - \lambda(\xi) I_{n-m}) \neq 0.$$

The theorem above tells us that if λ has a constant multiplicity, one has

$$A(\eta) = \begin{pmatrix} (\eta \cdot X) I_m & B(\eta) \\ C(\eta) & D(\eta) \end{pmatrix}, \quad \forall \eta \in \mathbb{R}^d,$$

for some vector $X \in \mathbb{R}^d$.

Corollary 1.1 *Let L be constantly hyperbolic, with an eigenvalue λ of multiplicity $m > n/2$. Then $\xi \mapsto \lambda(\xi)$ is linear.*

Proof From the assumption, there exists a non-zero vector x in the intersection of $\ker(A(\xi) - \lambda(\xi))$ and $\ker(A(\eta) - \lambda(\eta))$. On the one hand, $\pi_\xi x = x$. On the other hand, $A(\eta)x = \lambda(\eta)x$. Applying Theorem 1.7 gives $\lambda(\eta) = \mathrm{d}\lambda(\xi) \cdot \eta$. □

Remarks

- The example given in Section 1.2.3 shows that assuming only the diagonalizability on $\mathbb{R}^n$ of all matrices $A(\xi)$ does not ensure the hyperbolicity of the suboperator L', since one of the matrices $C(\eta)$ is a Jordan block $J(0;2)$ (take $d=2$, $n=3$, $\xi_0=\vec{e}^2$ and $\lambda_0=0$).

- The assumption of constant hyperbolicity in Theorem 1.7 may be relaxed by assuming only hyperbolicity with an eigenvalue $\sigma \mapsto \lambda(\sigma)$ of constant multiplicity in the neighbourhood of ξ.
- The conclusion in Theorem 1.7 may not be true when we drop the assumption of constant multiplicity. For instance, let us consider a symmetric hyperbolic operator L. We may assume that $\xi = \vec{e}^d$ and that $N(\xi)$ equals $\mathbb{R}^m \times \{0\}$. In other words, A^d is block-diagonal with the last block equal to λI_{n-m}. Then $\pi_\xi A(\eta)\pi_\xi$ is the first diagonal block of $A(\eta)$. It may be any linear map into the space of real symmetric $m \times m$ matrices. A refined analysis when λ_0 does not correspond to a locally constant multiplicity has been done by Lannes [107].
- The argument developed in the proof of Theorem 1.6 can be used in the context of parabolic-hyperbolic operators. We leave the reader to prove the following result (**Hint**: show that, for ξ large enough, the appropriate matrix has an invariant subspace $N(\xi)$, which tends to the subspace defined by $v=0$ as $\xi \to +\infty$).

Theorem 1.8 *Assume that the Cauchy problem for the system*

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u + \sum_\alpha B^\alpha \partial_\alpha v = 0,$$

$$\partial_t v + \sum_\alpha C^\alpha \partial_\alpha u + \sum_\alpha D^\alpha \partial_\alpha v = \sum_{\alpha,\beta} E^{\alpha\beta} \partial_\alpha \partial_\beta v$$

is well-posed in $L^2(\mathbb{R}^d \times \mathbb{R}_t^+)$. *Assume also that the diffusion matrix*

$$E(\xi) := \sum_{\alpha,\beta} \xi_\alpha \xi_\beta E^{\alpha\beta}$$

is non-singular for every $\xi \neq 0$. *Then the operator*

$$\partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

is hyperbolic.

- Likewise, one can consider first-order systems with damping (see [19, 147, 222, 223]). Again, we leave the reader to prove the following result (**Hint**: for $\xi = 0$, the subpace defined by $v = 0$ is invariant for the appropriate matrix. Extend it as an invariant subspace $N(\xi)$.)

Theorem 1.9 *Let $R \in \mathbf{M}_p(\mathbb{R})$ be given, with $1 \le p < n$. Assume that the Cauchy problem for the system*

$$\partial_t u + \sum_\alpha A^\alpha \partial_\alpha u + \sum_\alpha B^\alpha \partial_\alpha v = 0,$$

$$\partial_t v + \sum_\alpha C^\alpha \partial_\alpha u + \sum_\alpha D^\alpha \partial_\alpha v = Rv$$

is well-posed in $L^2(\mathbb{R}^d \times \mathbb{R}_t^+)$, uniformly in time, in the sense that there exists a constant M, independent of time, such that every solution satisfies

$$\|(u,v)(t)\|_{L^2} \le M\|(u,v)(0)\|_{L^2}.$$

Assume also that the damping matrix R is non-singular. Then the operator

$$\partial_t + \sum_\alpha A^\alpha \partial_\alpha$$

is hyperbolic.

This result is meaningful in the study of *relaxation models.*

1.5.2 *Strichartz estimates*

This section deals with norms of $L_t^p(L_x^q)$ type for functions $u(x,t)$, namely

$$\|u\|_{p,q} := \left(\int_{\mathbb{R}} \|u(\cdot,t)\|^p_{L^q(\mathbb{R}^d)} dt \right)^{1/p}.$$

Such norms define Banach spaces. Interpolation between the spaces associated to pairs (p_1, q_1) and (p_2, q_2) (say $p_1 \le p_2$) yields the spaces associated to (p,q), with

$$p_1 \le p \le p_2, \quad \left(\frac{1}{q_1} - \frac{1}{q_2}\right)\left(\frac{1}{p} - \frac{1}{p_2}\right) = \left(\frac{1}{p_1} - \frac{1}{p_2}\right)\left(\frac{1}{q} - \frac{1}{q_2}\right).$$

There are various types of Strichartz estimates. We shall neither list them all, nor give proofs, except in a single particular case (see below). Given a hyperbolic operator $L = \partial_t + A(\nabla_x)$, a Strichartz estimate is an inequality that typically bounds the $L_t^p(L_x^q)$-norm of the solution u of

$$Lu = f, \quad u(t=0) = u_0,$$

in terms of norms of f and u_0, taken in other functional spaces. By a duality argument, one deduces the general case from the simpler one $u_0 \equiv 0$. The latter follows from a dispersion inequality in the homogeneous case $f \equiv 0$, through the Fractional Integration Theorem (Hardy–Littlewood–Sobolev inequality).

As far as we know, the wave operator and its variant are the only ones that retained the attention of authors within hyperbolic problems. Therefore, we shall restrict ourselves to operators L that ‘divide’ the Dalembertian $\partial_t^2 - \Delta_x$, in the sense that $A(\xi)^2 \equiv |\xi|^2 I_n$. We then speak of *Dirac operators*, and $Lu = 0$ implies $\partial_t^2 u_j - \Delta_x u_j = 0$ for $j = 1, \dots, n$. The simplest example is

$$\partial_t u + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x u + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_y u = 0. \tag{1.5.36}$$

A more complicated one may be built from Pauli matrices:

$$A(\xi) = \begin{pmatrix} \xi_1 I_4 & \xi_2\sigma_2 + \xi_3\sigma_3 + \xi_4\sigma_4 \\ \xi_2\sigma_2^T + \xi_3\sigma_3^T + \xi_4\sigma_4^T & -\xi_1 I_4 \end{pmatrix},$$

with

$$\sigma_2 = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0_2 & J \\ -J & 0_2 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the seminal work by Strichartz [200], the homogeneous case is treated by noting that the Fourier transform of u is supported by the characteristic cone. It is important that, away from its singularity, this cone have non-zero curvature. In particular, we do not expect Strichartz estimates to hold when $\mathrm{char}(L)$ has a flat component, a fact that happens in linearized gas dynamics for instance, or in one-space dimension. In subsequent studies (see [96,203]), the estimate is obtained as a consequence of the conservation of energy, a dispersion inequality (an algebraic decay of $\|u(t)\|_{L^q}$ when $f \equiv 0$), Hardy–Littlewood–Sobolev inequalities and a so-called T^*T argument.

A typical Strichartz inequality for the wave equation $\partial_t^2\phi - \Delta_x\phi = 0$ is

$$\|\phi\|_{L_t^p(L_x^q)} \le c(p,q,d)\|\nabla_{x,t}\phi|_{t=0}\|_{L^2}, \tag{1.5.37}$$

which holds when

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - 1, \quad \frac{2}{p} + \frac{d-1}{q} \le \frac{d-1}{2}, \quad 2 \le p, q \le \infty, \quad d \ge 2, \tag{1.5.38}$$

with the exception of the triplet $(p,q,d) = (2,\infty,3)$. Translating in terms of u, we obtain an inequality

$$\|u\|_{L_t^p(L_x^q)} \le c(p,q,d)\|u_0\|_{\dot{H}^1}, \tag{1.5.39}$$

where $\dot{H}^1(\mathbb{R}^d)$ denotes the homogeneous Sobolev space of tempered distributions such that $\xi\hat{u}(\xi)$ is square-integrable. If $d \ge 3$, (1.5.39) contains the endpoint case

$$\|u\|_{L_t^\infty(L_x^{2^*})} \le c(d)\|u_0\|_{\dot{H}^1},$$

where

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}$$

is the Sobolev exponent:

$$\dot{H}^1(\mathbb{R}^d) \subset L^{2^*}(\mathbb{R}^d).$$

This particular inequality is an obvious consequence of the Sobolev embedding and the constancy of $\|u(t)\|_{\dot{H}^1}$.

For the same trivial reason, if $s \in (0, d/2)$, a 'Strichartz inequality' holds of the form

$$\|u\|_{L_t^\infty(L_x^{q(s)})} \leq c(s,d)\|u_0\|_{\dot{H}^s}, \tag{1.5.40}$$

where

$$\frac{1}{q(s)} = \frac{1}{2} - \frac{s}{d}.$$

Again, as in (1.5.39), this trivial result is simply the endpoint of a list of non-trivial ones. We also have

$$\|u\|_{L_t^p(L_x^q)} \leq c(p,q,s,d)\|u_0\|_{\dot{H}^s} \tag{1.5.41}$$

for

$$\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - s, \quad \frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}, \quad 2 \leq p, \quad d \geq 2, \tag{1.5.42}$$

with the exception of the triplets

$$(p,q,s) = \left(\frac{4}{d-1}, \infty, \frac{d+1}{4}\right).$$

We emphasize that (1.5.41) is scale invariant, in the sense that both sides have the same degree of homogeneity when u is replaced by u^λ, where $u^\lambda(x,t) := u(\lambda x, \lambda t)$, another solution of $Lv = 0$.

Strichartz estimates vs L^p-well-posedness From Brenner's theorem [22, 23], the Cauchy problem for a Dirac operator is ill-posed in every L^p-space but L^2. As a matter of fact, the matrices A^α of a Dirac operator satisfy

$$(A^\alpha)^2 = I_n, \quad A^\alpha A^\beta + A^\beta A^\alpha = 0_n \quad (\alpha \neq \beta),$$

which immediately imply $[A^\alpha, A^\beta] \neq 0_n$. We shall see that the ill-posedness may be viewed as a consequence of Strichartz estimates. This must be a rather general fact, as the lack of commutation of the matrices A^α of a hyperbolic operator L, is needed in order that $\mathrm{char}(L)$ have non-zero curvature.

Let P be the set of exponents p such that the Cauchy problem for L is well-posed in L^p. Obviously, 2 belongs to P. By standard interpolation theory (Riesz–Thorin theorem, see [15]), P is an interval. Next, the fact that L^* is also a Dirac operator, plus a duality argument, show that P is symmetric with respect to the involution $p \mapsto p'$ (as usual, $1/p + 1/p' = 1$).

Assume that P contains some element $q > 2$. Hence the solution operator S_t is uniformly bounded on L^q for $t \in (-1, 1)$. Let $s > 0$ be such that $q > q(s)$, and

q satisfies the inequalities in (1.5.42), so that (1.5.41) applies for some p. Writing

$$u(0) = \int_0^1 u(0)\,\mathrm{d}t = \int_0^1 S_{-t}u(t)\,\mathrm{d}t,$$

and using (1.5.41), we obtain an inequality

$$\|w\|_{L^q} \leq c\|w\|_{\dot{H}^s}, \quad \forall w \in \dot{H}^s(\mathbb{R}^d).$$

Such an inequality is obviously false, for instance because it is not scale invariant. We deduce that

$$P \subset [1, 2].$$

Since P is symmetric upon $p \mapsto p'$, we conclude that $P = \{2\}$, confirming Brenner's theorem for Dirac operators.

A proof for the 3-dimensional wave equation We give here the proof of (1.5.37) for the wave equation

$$\partial_t^2 \phi = \Delta_x \phi \tag{1.5.43}$$

in the special case $d = 3$. The constraints on (p, q) are therefore

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}, \quad 2 < p \leq \infty \quad (\text{that is } 6 \leq q < \infty).$$

To keep the presentation as short as possible, we limit ourselves to the proof of the inequality when $\phi|_{t=0} = 0$. We recall that, denoting ϕ_1 the time derivative of ϕ at initial time, the solution of (1.5.43) is given by

$$\phi(x, t) = \frac{1}{4\pi t} \int_{S(x;t)} \phi_1(y)\,\mathrm{d}s(y),$$

where $S(x;t)$ is the sphere of radius t, centred at x, and $ds(y)$ is the area element. Denote P_t the operator $\phi_1 \mapsto \phi(t)$. Fourier transforming the wave equation, we have easily

$$\widehat{P_t\chi}(\xi) = \frac{\sin t|\xi|}{|\xi|}\hat{\chi}(\xi),$$

which justifies the notation

$$P_t = \frac{\sin t|D|}{|D|}.$$

Since the symbol of P_t is real, it is a self-adjoint operator. The operator $P_s^* P_t = P_s P_t$ has symbol

$$\frac{(\sin t|\xi|)\,(\sin s|\xi|)}{|\xi|^2} = \frac{\cos(t-s)|\xi| - \cos(t+s)|\xi|}{2|\xi|^2}.$$

Therefore $P_s^* P_t$ is the convolution operator of kernel $(K(\cdot, t-s) - K(\cdot, t+s))/2$, where

$$K(\cdot, t) := \mathcal{F}^{-1} \frac{\cos t|\xi|}{|\xi|^2}.$$

It is not too difficult to compute, for $t > 0$,

$$K(x; t) = \frac{H(|x| - t)}{4\pi |x|},$$

where H is the Heaviside function. It follows immediately that

$$\|K(\cdot, t)\|_{L^r(\mathbb{R}^3)} = c_r t^{-1+3/r}, \quad r > 3.$$

From Young's inequality, we deduce (take $r = q/2$)

$$\|P_s^* P_t f\|_{L^q} \leq c(q) \left(|t-s|^{-1+6/q} + |t+s|^{-1+6/q} \right) \|f\|_{L^{q'}}, \tag{1.5.44}$$

provided that $q > 6$.

Given a function $f \in L^{p'}(0, +\infty; L_x^{q'})$, let us form

$$v := \int_0^\infty P_t f(t) \, \mathrm{d}t.$$

The T^*T argument consists in estimating v in $L^2(\mathbb{R}^3)$. First, we have

$$\begin{aligned} \|v\|_{L^2}^2 &= \left\langle \int_0^\infty P_t f(t) \, \mathrm{d}t, \int_0^\infty P_s f(s) \, \mathrm{d}s \right\rangle \\ &= \int_0^\infty \int_0^\infty \mathrm{d}t \, \mathrm{d}s \, \langle P_s^* P_t f(t), f(s) \rangle. \end{aligned}$$

Using (1.5.44) and the fact that $|t-s| \leq |t+s|$, we obtain

$$\begin{aligned} \|v\|_{L^2}^2 &\leq \int_0^\infty \int_0^\infty \mathrm{d}t \, \mathrm{d}s \, \|P_s^* P_t f(t)\|_{L^q} \|f(s)\|_{L^{q'}} \\ &\leq 2c(q) \int_0^\infty \int_0^\infty |t-s|^{-1+6/q} \|f(t)\|_{L^{q'}} \|f(s)\|_{L^{q'}} \mathrm{d}t \, \mathrm{d}s. \end{aligned}$$

Defining

$$G(t) := \int_0^\infty |t-s|^{-1+6/q} \|f(s)\|_{L^{q'}} \mathrm{d}s,$$

we infer from the Hölder inequality

$$\|v\|_{L^2}^2 \leq 2c(q) \, \|G\|_{L^p(0,+\infty)} \|f\|_{L_t^{p'}(L_x^{q'})}.$$

From the Hardy–Littlewood–Sobolev inequality, we also know

$$\|G\|_{L^p(0,+\infty)} \leq \gamma(q) \, \|f\|_{L_t^{p'}(L_x^{q'})},$$

provided that

$$\frac{1}{p} + \frac{3}{q} = \frac{1}{2}.$$

It follows that

$$\|v\|_{L^2}^2 \leq c'(q)\,\|f\|^2_{L_t^{p'}(L_x^{q'})}. \tag{1.5.45}$$

We conclude via a duality argument. First:

$$\begin{aligned}\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^3} f\phi \mathrm{d}x &= \int_0^{+\infty} \langle P_t\phi_1, f(t)\rangle\,\mathrm{d}t \\ &= \langle \phi_1, \int_0^{+\infty} P_t f(t)\,\mathrm{d}t\rangle.\end{aligned}$$

Using (1.5.45), we deduce

$$\left|\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^3} f\phi \mathrm{d}x\right| \leq \sqrt{c'(q)}\,\|\phi_1\|_{L^2}\|f\|_{L_t^{p'}(L_x^{q'})}.$$

Therefore,

$$\|\phi\|_{L_t^p(L_x^q)} = \sup_f \frac{\left|\int_0^{+\infty} \mathrm{d}t \int_{\mathbb{R}^3} f\phi \mathrm{d}x\right|}{\|f\|_{L_t^{p'}(L_x^{q'})}} \leq \sqrt{c'(q)}\,\|\phi_1\|_{L^2},$$

which is precisely the expected inequality in our case.

1.5.3 *Systems with differential constraints*

Several examples in natural sciences involve systems of a slightly more general form than (1.0.1), because of differential constraints that are satisfied by $u(t)$ at every time interval. Let us consider the homogeneous case, with $B = 0$. Then a typical system has the form

$$\partial_t u + \sum_{\alpha=1}^d A^\alpha \partial_\alpha u = 0, \quad \sum_{\beta=1}^d C^\beta \partial_\beta u = 0, \tag{1.5.46}$$

where $C^\beta \in M_{p\times n}(\mathbb{R})$, p being the number of constraints.

Such systems occur whenever one rewrites a higher-order system as a first-order one. Let us take, as an example, the wave equation

$$\partial_t^2 \phi = c^2 \Delta\phi, \tag{1.5.47}$$

where $c > 0$ is the wave velocity. Every solution of (1.5.47) yields a solution $u := (c\nabla_x\phi, -\partial_t\phi)$ of (1.2.12), where

$$A(\xi) = \begin{pmatrix} 0_d & c\xi \\ c\xi^T & 0 \end{pmatrix}.$$

Since $A(\xi)$ is symmetric, the corresponding system is hyperbolic. The spectrum of $A(\xi)$ is easily computed and consists of the simple eigenvalues $\pm c|\xi|$, and the multiple eigenvalue 0. The latter is actually spurious, only due to the fact that the mapping $\phi \mapsto u$ is not onto. Therefore, some solutions u do not correspond to solutions of the wave equation, whence the need of the constraint $\text{curl} v = 0$ on the d first components of $u = (v, w)$.

Another example is provided by Maxwell's system, which writes, in the absence of electric charges, as

$$\begin{aligned} \partial_t B + \text{curl} D &= 0, \\ \partial_t D - \text{curl} B &= 0, \\ \text{div} B &= 0, \\ \text{div} D &= 0. \end{aligned}$$

Again, the evolutionary part (the two first equations above), constitute a symmetric system, therefore a hyperbolic one. We compute easily the eigenvalues, $\pm|\xi|$ and 0, where zero has no physical significance, and must be ruled out with the help of the constraints.

Other examples come from field equations in relativity, where gauge invariance implies that natural variables are redundant.

The general philosophy is that the initial data is given satisfying the constraints, and the evolution must preserve them. Using a Fourier transform, it amounts to saying that $C(\xi)A(\xi)v$ must vanish when $C(\xi)v$ does. In other words, the kernel $N(\xi)$ of $C(\xi)$ is an invariant subspace of $A(\xi)$. As noticed by Dafermos [45], this property is fulfilled as soon as $C^\alpha A^\beta + C^\beta A^\alpha = 0$ holds for every pair (α, β). As a matter of fact, these identities, which hold frequently, imply $C(\xi)A(\xi) = 0$. In practice, all examples satisfy the following assumption **(CR)**:

for non-zero vectors $\xi \in \mathbb{R}^d$, the rank of $C(\xi)$ is constant.

This implies that the vector space $N(\xi)$ has a constant dimension and that it depends analytically on ξ.

We now characterize strong well-posedness of the Cauchy problem for (1.5.46). Standard functional spaces must be redefined according to the constraint. For instance, L^2-well-posedness is concerned with the following space

$$Z := \{u \in L^2(\mathbb{R}^d)^n \, ; \sum_\beta C^\beta \partial_\beta u = 0\}.$$

Equipped with the usual L^2-norm, Z is a Hilbert space. For an initial datum $a \in Z$, the solution is formally given by the formula

$$\hat{u}(\xi, t) = \exp(-itA(\xi))\hat{a}(\xi). \tag{1.5.48}$$

Since $a \in Z$, we know that $\hat{a}(\xi)$ belongs to $N(\xi)$ for almost every ξ. Then an estimate of the form $\|u(t)\|_Z \leq C\|a\|_Z$ holds if and only if

$$\sup_\xi \| \exp(-itA_N(\xi))\| < +\infty, \tag{1.5.49}$$

where $A_N(\xi)$ is the restriction of $A(\xi)$ to its invariant subspace $N(\xi)$. As before, (1.5.49) holds for some non-zero time t_0 if and only if it holds for every time $t \in \mathbb{R}$. For instance, the choice $t = -1$ gives the criterion for L^2-well-posedness:

$$\sup_\xi \| \exp(iA_N(\xi))\| < +\infty. \tag{1.5.50}$$

The L^2-well-posedness is again called *hyperbolicity*. As before, it requires that $A_N(\xi)$ (but not necessarily $A(\xi)$) be diagonalizable with real eigenvalues. It may be read in the light of the Kreiss–Strang Theorem 1.2, but for practical purposes, it is useful to consider two classes of well-posed system. The first one consists in the *constantly hyperbolic* systems, namely those for which $A_N(\xi)$ is diagonalizable with real eigenvalues of constant multiplicities, when $\xi \neq 0$. Strict or constant hyperbolicity still implies hyperbolicity.

The second important class consists in the *Friedrichs-symmetrizable* systems. Symmetrizability is the property that there exist a real symmetric definite-positive matrix S and a matrix $M \in M_{n\times p}(\mathbb{R})$, such that $S^\alpha := SA^\alpha + MC^\alpha$ is symmetric, for every $\alpha = 1, \dots, d$. Such systems obey the following energy identity

$$\partial_t(Su, u) + \sum_\alpha \partial_\alpha(S^\alpha u, u) = 0, \tag{1.5.51}$$

which yields the estimate

$$\int_{\mathbb{R}^d} (Su(x,t), u(x,t))\,\mathrm{d}x = \int_{\mathbb{R}^d} (Sa(x), a(x))\,\mathrm{d}x. \tag{1.5.52}$$

The positiveness of matrix S may actually be relaxed in a non-trivial way. For that, let us define a cone Λ in $\mathbb{R}^n$, by

$$\Lambda := \{\lambda \in \mathbb{R}^n\,;\, \exists \xi \neq 0,\, C(\xi)\lambda = 0\} = \bigcup_{\xi\neq 0} N(\xi).$$

The following statement is called *compensated compactness.*

Theorem 1.10 (Murat [145], Tartar [202]) *Let S be a symmetric $n \times n$ matrix. The quadratic form*

$$v \mapsto \int_{\mathbb{R}^d} (Sv, v)\,\mathrm{d}x$$

is positive-definite on Z if and only if $(S\lambda, \lambda) > 0$ for every non-zero $\lambda \in \Lambda$. In such a case, its square root defines a norm equivalent to $\|\cdot\|_Z$.

From this, we again obtain an L^2 estimate from (1.5.52), in some cases where there does not exist a positive-definite symmetrizer S.

Elastodynamics An important application of this calculus arises in elastodynamics. Non-linear elastodynamics obeys a second-order system in the unknown y called *displacement*. When written as a first-order system in terms of the first derivatives $u_{j\alpha} := \partial_\alpha y_j$ ($1 \le j \le d$, $0 \le \alpha \le d$ with $\partial_0 := \partial_t$), it must be supplemented with the compatibility relations

$$\partial_\alpha u_{j\beta} - \partial_\beta u_{j\alpha} = 0, \quad \alpha, \beta, j \ge 1.$$

We immediately compute that $v \in \Lambda$ if and only if the submatrix $(v_{j\alpha})_{1\le\alpha,j\le d}$ has rank at most one. For hyperelastic materials, this non-linear system is endowed with an energy density $W(\nabla y)$. A natural restriction is that the map $x \mapsto y$ preserves the orientation, so that $\det \nabla y > 0$ everywhere. In particular, the energy density $W(F)$ must become infinite as $\det F$ tends to zero. Besides, the frame indifference implies that $W(QF) = W(F)$ for every F, Q with $\det F > 0$ and $Q \in SO_d(\mathbb{R})$. It is shown in [36] (Theorem 4.8.1, page 170) that such a function cannot be convex[7]. Now, let us choose a matrix F in the vicinity of which W is not convex, locally. The constant state $\bar{u}$ defined by $\bar{u}_{j\alpha} := F_{j\alpha}$ if $\alpha \ne 0$ and zero otherwise is an equilibrium. Let us linearize the system about $\bar{u}$. The resulting system has constant coefficients and obeys the same differential constraints as the non-linear one. It is compatible with an energy identity (1.5.51), where (Su, u) encodes the second-order terms of the Taylor expansion of the full mechanical energy at $\bar{u}$. In particular, S is not positive. However, W can be *quasiconvex* at F, in the sense of Morrey [143], which means

$$\int_{\mathbb{R}^d} W(F + \nabla\psi)\,\mathrm{d}x \ge 0, \quad \forall \psi \in \mathscr{D}(\mathbb{R}^d). \tag{1.5.53}$$

Quasiconvexity implies the *Legendre–Hadamard inequality*

$$(S\lambda, \lambda) \ge 0, \quad \lambda \in \Lambda, \tag{1.5.54}$$

a weaker property than convexity. When (1.5.54) holds strictly for non-zero λ, the compensated-compactness Theorem tells us that (1.5.52) is a genuine estimate in Z. In such a case, the linearized problem is strongly L^2-well-posed.

We shall not consider in this chapter the local well-posedness of the non-linear system. The Cauchy problem for quasilinear systems of conservation laws is treated in Chapter 10. Let us mention only that a system of conservation laws endowed with a convex 'entropy' has a well-posedness property within smooth data and solutions (see Theorem 10.1). In elastodynamics, the system governs the evolution of $u = (v, F) = (\partial_t y, \nabla_x y)$. Since the entropy of our system is the energy $\frac{1}{2}|v|^2 + W(F)$, which is not convex, the above-mentioned theorem does not apply. However, Dafermos [46] has found a way to apply it, by rewriting the system of elastodynamics in terms of u and all minors of the matrix F. See also Demoulini *et al.* [48]. As a consequence, the local well-posedness is obtained whenever W is *polyconvex*, that is a convex function of F and its minors.

Electromagnetism Let us consider Maxwell's equations. The kernel $N(\xi)$ equals $\xi^\perp \times \xi^\perp$, where $\xi^\perp$ is the orthogonal of ξ in the Euclidean space $\mathbb{R}^3$.

[7] We warn the reader that the phase space $\mathbf{GL}_d^+(\mathbb{R})$, made of matrices F with $\det F > 0$, is not a convex set. Thus the convexity of a function is a meaningless notion.

Therefore Λ equals $\mathbb{R}^6$, and the symmetrizability has to be understood in the usual sense. In the vacuum, $H = B$ and $E = D$, for appropriate units. The system is already in symmetric form. In a 'linear' material medium, which may be anisotropic, H, E still are linear functions of B, D. For linear as well as non-linear media, there is a stored electromagnetic energy density $W(B, D)$, and E, H are given by the following formulæ (see [37])

$$E_j = \frac{\partial W}{\partial D_j}, \quad H_j = \frac{\partial W}{\partial B_j}.$$

In the linear case, W is a quadratic form. The Maxwell system is compatible, as long as we consider $\mathscr{C}^1$ solutions, with the *Poynting identity*, which expresses the conservation of energy

$$\partial_t W(B, D) + \mathrm{curl}(E \times H) = 0. \tag{1.5.55}$$

Let us consider the linearized system about some constant state $(\bar{B}, \bar{D})$. The former considerations show that if the matrix $S := \mathbf{D}^2 W(\bar{B}, \bar{D})$ is positive-definite, then the linear Cauchy problem is L^2-well-posed. We can actually relax the convexity condition, with the following observation. The Maxwell system is also compatible with the extra conservation law (herebelow, $u := (B, D)$)

$$\partial_t (B \times D) + \mathrm{div}\left(\frac{\partial W}{\partial B} \otimes B\right) + \mathrm{div}\left(\frac{\partial W}{\partial D} \otimes D\right) + \nabla(W - u \cdot \nabla_u W) = 0.$$

At the linearized level, we may consider a modified energy density ${}^t u S u + \det(X, B, D)$, where X is a given vector in $\mathbb{R}^3$. If there exists an X such that ${}^t u S u + \det(X, B, D)$ is positive-definite, then the linear system is Friedrichs symmetrizable and the Cauchy problem is L^2-well-posed. An obvious necessary condition for such an X to exist is that ${}^t u S u > 0$ whenever $B \times D = 0$ and $(B, D) \neq 0$. At the non-linear level, the same procedure as the one imagined by Dafermos in elastodynamics may be employed. The result is that the non-linear Maxwell's system is locally well-posed for smooth initial data and smooth solutions, whenever W can be written as a convex function of B, D and $B \times D$. (See [21, 188].)

1.5.4 *Splitting of the characteristic polynomial*

We give in this section a property of the characteristic polynomial $(X; \xi) \mapsto \det(X I_n + A(\xi))$, when the operator $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ is constantly hyperbolic.

Let us begin with an abstract result.

Lemma 1.3 *Let $P(X; \theta_1, \cdots, \theta_d)$ be a homogeneous polynomial of degree n in $1 + d$ variables, with real coefficients. Assume that the coefficient of X^n is non-zero. Assume also that for all θ in a non-void open subset $\mathcal{O}$ of $\mathbb{R}^d$, the polynomial $P_\theta := P(\cdot, \theta)$ has a root with multiplicity ≥ 2. Then P is reducible in $\mathbb{R}[X, \theta]$.*

Proof Let us denote by $R := \mathbb{R}[\theta_1, \cdots, \theta_d]$ the factorial ring of polynomials in d variables θ and by $k := \mathbb{R}(\theta_1, \cdots, \theta_d)$ the field of rational fractions in θ. We first consider P as an element of $k[X]$. Let us recall that $k[X]$ is a Euclidean ring, which has therefore a greatest common divisor (g.c.d.)

Let Q be the g.c.d of P and P' in $k[X]$, a monic polynomial of X. Its coefficients, belonging to k, are rational fractions of θ. We denote by Z the zero set of the product of denominators of these fractions; Z is a closed set with empty interior.

When $\theta \in \mathcal{O} \setminus Z$ (this is a non-void open set), $Q_\theta := Q(\cdot, \theta)$ has a non-trivial root, which means that either $Q_\theta \equiv 0$ or $d^\circ Q_\theta \geq 1$. However, the condition $Q_\theta \equiv 0$ defines a non-trivial algebraic manifold M (the intersection of the zero sets of the coefficients of Q), again a closed set with empty interior. Therefore, there exists a θ for which $d^\circ Q_\theta \geq 1$, and consequently $d_X^\circ Q \geq 1$.

Since Q divides P in $k[X]$, we write $P = QT$, with $T \in k[X]$. Multiplying by the l.c.m. of the denominators of all coefficients of Q and T (a least common multiple (l.c.m.) and a g.c.d. do exist in the factorial ring R), we have $g(\theta)P = Q_1 T_1$, where $g \in A$, $Q_1, T_1 \in R[X]$ and $0 < d_X^\circ Q_1 < n$. We recall that the *contents* of a polynomial $S \in R[X]$, denoted by $c(S)$, is the g.c.d. of all its coefficients. From Gauss' Lemma, $c(Q_1 T_1) = c(Q_1)c(T_1)$ and therefore $g = c(Q_1)c(T_1)$, since $c(P) = 1$ by assumption. We conclude that $P = Q_2 T_2$, where $Q_2 := c(Q_1)^{-1} Q_1 \in R[X]$ and similarly $R_2 \in R[X]$. Moreover, $0 < d_X^\circ Q_2 < n$, which shows that P is reducible in $R[X] = \mathbb{R}[X, \theta]$. $\square$

Corollary 1.2 *Let $P \in \mathbb{R}(X, \theta)$ be homogeneous with $d_X^\circ P = d_{(X,\theta)}^\circ P$. Let*

$$P = \prod_{l=1}^{L} P_l^{q_l}$$

its factorization into irreducible factors in $\mathbb{R}(X, \theta)$, the P_ls being pairwise distinct.

Then each P_l has the following property: for an open dense subset of values of θ in $\mathbb{R}^d$, the roots of $P_l(\cdot, \theta)$ are simple.

We now apply the corollary to the characteristic polynomial.

Proposition 1.7 *Let the operator $L := \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be constantly hyperbolic. Then the characteristic polynomial* $\det(XI_n + A(\xi))$ *splits as a product*

$$\prod_{l=1}^{L} P_l^{q_l}, \tag{1.5.56}$$

where the P_ls, normalized by $P_l(1, 0) = 1$, satisfy

- *Each P_l is a homogeneous polynomial of $(X; \xi)$,*
- *The P_ls are irreducible, pairwise distinct,*
- *For $\xi \in \mathbb{R}^d \setminus \{0\}$, the roots of $P_l(\cdot, \xi)$ are real and simple,*
- *For $\xi \in \mathbb{R}^d \setminus \{0\}$ and $l \neq k$, $P_l(\cdot, \xi)$ and $P_k(\cdot, \xi)$ do not have a common root.*

An example Let us consider Maxwell's equations, where $d = 3$, $n = 6$ and

$$A(\xi) = \begin{pmatrix} 0_3 & J(\xi) \\ -J(\xi) & 0_3 \end{pmatrix}, \quad J(\xi)V := \xi \times V.$$

An elementary computation gives:

$$\det(XI_6 + A(\xi)) = X^2(X^2 - |\xi|^2)^2.$$

Hence the splitting described in Proposition 1.7 corresponds to $P_1(X, \xi) = X$, $P_2(X, \xi) = X^2 - |\xi|^2$, $q_1 = q_2 = 2$.

1.5.5 *Dimensional restrictions for strictly hyperbolic systems*

We begin with a matrix theorem, due to Lax [111] in the case $n \equiv 2 \pmod 4$, and to Friedland *et al.*, [62] in the case $n \equiv 3, 4, 5 \pmod 8$:

Theorem 1.11 *Assume that* $n \equiv 2, 3, 4, 5, 6 \pmod 8$. *Let* V *be a subspace of* $\mathbf{M}_n(\mathbb{R})$ *with the property that every non-zero element in* V *has its eigenvalues real and pairwise distinct. Then* $\dim V \leq 2$.

In terms of hyperbolic operators, this tells us that strictly hyperbolic operators in space dimension $d \geq 3$ can exist only if $n \equiv 0, \pm 1 \pmod 8$ (assuming that the operator really involves all the space variables). This explains why constantly hyperbolic operators occur so frequently, as they exist in space dimension three at every size $n \geq 4$. The simplest examples are:

$n = 4$. Linearized isentropic gas dynamics.

$n = 5$. Linearized non-isentropic gas dynamics.

$n = 6$. Maxwell's equations. We know also of a non-equivalent example of this size.

Proof We prove only the Lax case $n \equiv 2 \pmod 4$. We argue by contradiction, assuming that $\dim V \geq 3$. We label the eigenvalues in the increasing order:

$$\lambda_1(M) < \cdots < \lambda_n(M), \quad M \neq 0_n.$$

Every non-zero element M in V, having real and distinct eigenvalues, is associated with finitely many (precisely 2^n) unitary bases of $\mathbb{R}^n$, which depend continuously on M. In other words, the set of (real) unitary eigenbases is a finite covering (with 2^n sheets) of $V \setminus \{0_n\}$. Since the base space is simply connected (because $d \geq 3$), the covering is trivial and a continuous map $M \mapsto \mathcal{B}(M)$ can actually be defined globally, where

$$\mathcal{B}(M) = \{r_1(M), \ldots, r_n(M)\}$$

is an eigenbasis. By continuity, all the bases $\mathcal{B}(M)$ have the same orientation.

On the one hand, it holds that

$$\lambda_j(-M) = -\lambda_{n-j+1}(M), \quad j = 1, \ldots, n.$$

It follows that

$$r_j(-M) = \pm r_{n-j+1}(M).$$

By continuity, the sign $\pm$ above is constant and depends only on j, but not on M. Denote it ρ_j. Exchanging j with $n-j+1$, we obtain

$$\rho_{n-j+1}\rho_j = 1. \tag{1.5.57}$$

On the other hand, we know that $\{r_1(M), \ldots, r_n(M)\}$ and $\{\rho_1 r_n(M), \ldots, \rho_n r_1(M)\} = \{r_1(-M), \ldots, r_n(-M)\}$ have the same orientation. Since $n \equiv 2 \pmod 4$, the order reversal[8]

$$\{r_1(M), \ldots, r_n(M)\} \mapsto \{r_n(M), \ldots, r_1(M)\}$$

reverses the orientation. Therefore, it must hold that

$$\prod_{j=1}^{n} \rho_j = -1.$$

This, however, is incompatible with (1.5.57) when n is even. □

Note that if $\dim V = 2$ (that is for strictly hyperbolic operators in two space dimensions), there does not need to exist a continuously defined eigenbasis on $V \setminus \{0_n\}$, as this set is not simply connected. For instance, the system (1.5.36) does not have this property: When following a loop around the origin in V, the eigenvectors are flipped.

1.5.6 *Realization of hyperbolic polynomial*

Let $p(X_0, \ldots, X_d)$, a homogeneous polynomial of degree n, be *hyperbolic* with respect to a vector $T \in \mathbb{R}^{d+1}$ in the sense of Gårding (see Section 1.4.4).

Given a hyperbolic polynomial p of degree n in $d+1$ variables, one may always assume that T is the first element $\vec{e}^0$ of the canonical basis. A natural question is whether p can be realized as p_L for some hyperbolic operator L. The case $d = 1$ is easy. Lax [110] conjectured that if $d = 2$, the answer is positive and one can choose a Friedrichs-symmetrizable operator. This has been proved recently by Lewis *et al.* [114], following a result by Helton and Vinnikov [82]. One easily sees that the hyperbolic polynomial $q(X) := X_0^2 - X_1^2 - \cdots - X_d^2$ cannot be realized if $d \geq 3$. However, it may happen that some power q^ℓ be realizable, as in the case of Maxwell's system, or Dirac systems. The fundamental question whether every hyperbolicity cone can be realized as a forward cone for some hyperbolic operator remains open so far.

Notice that two different hyperbolic operators L and L' can yield the same polynomial,

$$p_L = p_{L'}.$$

[8]This part of the proof would also work when $n \equiv 3 \pmod 4$.

This happens at least when L' is obtained from L by a linear change of variables. Then $(A^\alpha)' = P^{-1}A^\alpha P$ for some non-singular matrix. We can also make only a linear combination of the equations, which yields an operator $L'' = S_0\partial_t + \sum_\alpha S^\alpha\partial_\alpha$, where $S^\alpha := S_0A^\alpha$, which modifies p_L by a constant factor. Change of co-ordinates should also be allowed.

It is an important problem to classify the hyperbolic operators up to such a change. The characteristic cone is of course an invariant of this problem, but it is not the only one. There actually exist non-equivalent operators that have the same characteristic cone. A way to go forward, which has not been pushed so far, is to consider the characteristic bundle, whose basis is the characteristic variety (the projective set associated to the characteristic cone) and the fibres are the corresponding eigenfields. This bundle is modified by the change of variables and the combinations of equations, by its topology is not. Thus the Chern class of the bundle is a more accurate invariant. When the characteristics have variable multiplicities, the cone and the bundle are not smooth and the analysis becomes more difficult.

A similar problem, perhaps even more important is to classify within the set of symmetrizable operator, since the physics usually provides a Friedrichs symmetrizer, through an energy estimate or an entropy principle. Of course, the characteristic bundle remains a crucial tool. But some other invariants may appear, in particular in the case where L admits a linear velocity

$$\lambda(\xi) = V \cdot \xi,$$

which can be brought to the case $\lambda(\xi) \equiv 0$. See the discussion in Section 6.1.2.

2

LINEAR CAUCHY PROBLEM WITH VARIABLE COEFFICIENTS

The purpose of this chapter is to deal with variable-coefficient generalizations of the systems considered in Chapter 1. These are of the form

$$\frac{\partial u}{\partial t} + \sum_{\alpha=1}^{d} A^\alpha(x,t)\,\frac{\partial u}{\partial x_\alpha} \;=\; B(x,t)\,u \,+\, f(x,t), \tag{2.0.1}$$

where the $n \times n$ matrices A^α and B depend 'smoothly' on (x,t). Such systems may arise from constant-coefficient ones by change of variable, which is useful to show local uniqueness properties, see Section 2.2. Variable-coefficient systems also occur as the linearization of non-linear systems (see Chapter 10), in which case the matrices A^α and B may have a restricted regularity. In this chapter, unless otherwise stated, it will be implicitly assumed that B and A^α are $\mathscr{C}^\infty$ functions that are bounded as well as all their derivatives.

To be consistent with notations in Chapter 1 we may alternatively write (2.0.1) as

$$\partial_t u \;=\; P(t)\,u \,+\, f,$$

where $P(t)$ is the spatial differential operator

$$P(t) : u \mapsto P(t)\,u \;:=\; -\sum_{\alpha} A^\alpha(\cdot,t)\,\partial_\alpha u \,+\, B(\cdot,t)\,u. \tag{2.0.2}$$

Or, in short, (2.0.1) equivalently reads

$$L\,u \;=\; f,$$

where L denotes the evolution operator

$$L : u \mapsto L\,u \;:=\; \partial_t u \,-\, P(t)\,u. \tag{2.0.3}$$

These notations being fixed, we claim that Fourier analysis is not sufficient to deal with the Cauchy problem for the variable-coefficient operator L. We need pseudo-differential calculus (in the variable $x \in \mathbb{R}^d$, the time t playing the role of a parameter), and even para-differential calculus in the case of coefficients with a restricted regularity. For convenience, the results from this field we shall use are all collected in Appendix C.

2.1 Well-posedness in Sobolev spaces

The main issue regarding well-posedness is the derivation of a priori energy estimates of the form

$$\|u(t)\|_{H^s}^2 \leq C \left(\|u(0)\|_{H^s}^2 + \int_0^t \|Lu(\tau)\|_{H^s}^2 \, \mathrm{d}\tau \right).$$

The scalar case ($n = 1$) merely corresponds to a *transport operator* L, which is much easier to deal with than the case of genuinely matrix-valued operators. For clarity, we begin with this special case, which will serve as an introduction to the powerful technique of symmetrizers for systems.

2.1.1 *Energy estimates in the scalar case*

Proposition 2.1 *For a* scalar *operator*

$$L = \partial_t + a \cdot \nabla - b,$$

where $a(x,t) \in \mathbb{R}^d$ and $b(x,t) \in \mathbb{R}$ are smooth functions of (x,t), bounded as well as their derivatives, we have the following a priori estimates. For all $s \in \mathbb{R}$ and $T > 0$, there exists $C > 0$ so that for $u \in \mathscr{C}^1([0,T]; H^s) \cap \mathscr{C}([0,T]; H^{s+1})$ we have

$$\|u(t)\|_{H^s}^2 \leq C \left(\|u(0)\|_{H^s}^2 + \int_0^t \|Lu(\tau)\|_{H^s}^2 \, \mathrm{d}\tau \right).$$

Proof We start with the case $s = 0$. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2}^2 &= 2 \,\mathrm{Re}\, \langle Lu, u \rangle - 2 \,\mathrm{Re}\, \langle a \cdot \nabla u, u \rangle + 2 \,\mathrm{Re}\, \langle bu, u \rangle \\ &= 2 \,\mathrm{Re}\, \langle Lu, u \rangle + \int_{\mathbb{R}^d} (\, \mathrm{div} a + 2\, b) \, |u|^2 \end{aligned}$$

after integration by parts. Therefore, by the Cauchy–Schwarz inequality, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2}^2 \leq (\, 1 + \|\mathrm{div} a + 2\, b\|_{L^\infty} \,) \, \|u\|_{L^2}^2 + \|Lu\|_{L^2}^2,$$

or

$$\|u(t)\|_{L^2}^2 \leq \|u(0)\|_{L^2}^2 + \int_0^t \left(\|Lu(\tau)\|_{L^2}^2 + (1 + \|\mathrm{div} a + 2\, b\|_{L^\infty} \,) \, \|u(\tau)\|_{L^2}^2 \right) \mathrm{d}\tau.$$

By Gronwall's Lemma this implies

$$\|u(t)\|_{L^2}^2 \leq \mathrm{e}^{\gamma\, t} \, \|u(0)\|_{L^2}^2 + \int_0^t \mathrm{e}^{\gamma\,(t-\tau)} \, \|Lu(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \qquad \forall t \geq 0,$$

where $\gamma \geq 1 + \|\mathrm{div} a + 2\, b\|_{L^\infty}$. In particular, for all bounded interval $[0,T]$, there exists C_T (namely, $C_T = \exp(T\,(1 + \|\mathrm{div} a + 2\, b\|_{L^\infty}))$ so that

$$\|u(t)\|_{L^2}^2 \leq C_T \left(\|u(0)\|_{L^2}^2 + \int_0^t \|Lu(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \right) \qquad \forall t \in [0,T].$$

The general case $s \in \mathbb{R}$ actually follows from the special case $s = 0$ almost for free with the help of pseudo-differential calculus. Indeed, recall that the basic pseudo-differential operator Λ^s of symbol $\lambda^s(\xi) := (1 + \|\xi\|^2)^{s/2}$ is of order s (see Appendix C) and can be used to define the H^s norm

$$\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}.$$

Now, if $u \in \mathscr{C}^1([0,T];H^s) \cap \mathscr{C}([0,T];H^{s+1})$ then $\Lambda^s u \in \mathscr{C}^1([0,T];L^2) \cap \mathscr{C}([0,T];H^1)$ and therefore the inequality previously derived for $s = 0$ applies to $\Lambda^s u$. That is, we have

$$\|\Lambda^s u(t)\|_{L^2}^2 \leq C_T \left(\|\Lambda^s u(0)\|_{L^2}^2 + \int_0^t \|L \Lambda^s u(\tau)\|_{L^2}^2 \, d\tau \right).$$

Furthermore, the commutator

$$[L, \Lambda^s] = [a \cdot \nabla, \Lambda^s]$$

is of order $1 + s - 1 = s$ (see Appendix C). This implies the existence of $C_s(T)$ such that

$$\| [L, \Lambda^s] u(t) \|_{L^2} \leq C_s(T) \|u(t)\|_{H^s} \qquad \forall t \in [0,T].$$

Hence

$$\|\Lambda^s u(t)\|_{L^2}^2 \leq C_T \left(\|\Lambda^s u(0)\|_{L^2}^2 + 2 \int_0^t (\|\Lambda^s Lu(\tau)\|_{L^2}^2 + C_s(T)\|u(\tau)\|_{H^s}^2) \, d\tau \right).$$

Finally, by Gronwall's Lemma we obtain

$$\|u(t)\|_{H^s}^2 \leq C_T \, \mathrm{e}^{2C_s(T)\,T} \left(\|u(0)\|_{H^s}^2 + 2 \int_0^t \|L\, u(\tau)\|_{H^s}^2 \,) \, \mathrm{d}\tau \right).$$

□

As should be clear from this proof, the crucial point is the L^2 estimate, relying on the fact that the differential operator $\mathrm{Re}\,(a \cdot \nabla) = \frac{1}{2}\,(\,(a \cdot \nabla) + (a \cdot \nabla)^*\,)$ is bounded on L^2. The symmetrizer's technique described below aims at recovering a similar property for non-scalar operators.

2.1.2 *Symmetrizers and energy estimates*

There is a special class of systems for which energy estimates are almost as natural as for scalar equations. This is the class of Friedrichs-symmetrizable systems, which fulfill the following generalization of Definition 2.1.

Definition 2.1 *The system (2.0.1) is Friedrichs symmetrizable if there exists a* $\mathscr{C}^\infty$ *mapping* $S_0 : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbf{M}_n(\mathbb{R})$*, bounded as well as its derivatives, such that* $S_0(x,t)$ *is symmetric and uniformly positive-definite, and the matrices* $S_0(x,t)A^\alpha(x,t)$ *are symmetric for all* (x,t)*.*

Like scalar equations, Friedrichs-symmetrizable systems enjoy a priori estimates that keep track of coefficients. We give the L^2 estimate below, which is proved elementarily and will be extensively used in the non-linear analysis of Chapter 10. We postpone to the end of this section the more complete result in H^s, valid for any system admitting a symbolic symmetrizer (see Definition 2.3), whose proof takes advantage of Bony's para-differential calculus.

Proposition 2.2 *Assume that (2.0.1) is Friedrichs symmetrizable, with a symmetrizer S_0 satisfying*

$$\beta I_n \leq S_0 \leq \beta^{-1} I_n$$

in the sense of quadratic forms. We also assume that S_0, A^α, and their first derivatives are bounded, as well as B. Then, for all $T > 0$ and $u \in \mathscr{C}([0,T];H^1) \cap \mathscr{C}^1([0,T];L^2)$ we have

$$\beta^2 \|u(t)\|_{L^2}^2 \leq \mathrm{e}^{\gamma t} \|u(0)\|_{L^2}^2 + \int_0^t \mathrm{e}^{\gamma(t-\tau)} \|Lu(\tau)\|_{L^2}^2 \, \mathrm{d}\tau \qquad \forall t \in [0,T], \tag{2.1.4}$$

where L is defined in (2.0.3) and γ is chosen to be large enough, so that

$$\beta(\gamma - 1) \geq \left\| \partial_t S_0 + \sum_\alpha \partial_\alpha (S_0 A^\alpha) + S_0 B + B^{\mathrm{T}} S_0 \right\|_{L^\infty}. \tag{2.1.5}$$

Proof The proof is basically the same as the first part of the proof of Proposition 2.1. After integration by parts we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle S_0 u, u \rangle = 2 \,\mathrm{Re}\, \langle S_0 u, Lu \rangle + \langle Ru, u \rangle,$$

where

$$R = \partial_t S_0 + \sum_\alpha \partial_\alpha (S_0 A^\alpha) + S_0 B + B^{\mathrm{T}} S_0.$$

Of course, we have used here the symmetry of the matrices $S_0 A^\alpha$ and S_0. Integrating in time and using the Cauchy–Schwarz inequality (for the inner product $\langle S_0 \cdot, \cdot \rangle$) we arrive at

$$\langle S_0 u(t), u(t) \rangle \leq \langle S_0 u(0), u(0) \rangle + \int_0^t \langle S_0 Lu(\tau), Lu(\tau) \rangle \, \mathrm{d}\tau$$
$$+ \int_0^t (1 + \beta^{-1} \|R\|_{L^\infty}) \langle S_0 u(\tau), u(\tau) \rangle \, \mathrm{d}\tau.$$

By Gronwall's Lemma this implies

$$\langle S_0 u(t), u(t) \rangle \leq \mathrm{e}^{\gamma t} \langle S_0 u(0), u(0) \rangle + \int_0^t \mathrm{e}^{\gamma(t-\tau)} \langle S_0 Lu(\tau), Lu(\tau) \rangle \, \mathrm{d}\tau,$$

with $\gamma \geq (1 + \beta^{-1} \|R\|_{L^\infty})$. This yields the final estimate after multiplication by β. □

More generally, a priori estimates hold true for systems admitting a *functional symmetrizer*, defined as follows.

Definition 2.2 *Given a family of first-order (pseudo-)differential operators* $\{P(t)\}_{t\geq 0}$ *acting on* $\mathbb{R}^d$, *a* functional symmetrizer *is a* $\mathscr{C}^1$ *mapping*

$$\Sigma : \mathbb{R}^+ \to \mathscr{B}(L^2(\mathbb{R}^d; \mathbb{C}^n))$$

such that, for $0 \leq t \leq T$,

$$\Sigma(t) = \Sigma(t)^* \geq \alpha I, \tag{2.1.6}$$

for some positive α *depending only on* T, *and*

$$\mathrm{Re}\,(\Sigma(t)P(t)) := \frac{1}{2}(\Sigma(t)P(t) + P(t)^*\Sigma(t)^*) \in \mathscr{B}(L^2(\mathbb{R}^d)^n) \tag{2.1.7}$$

with a uniform bound on $[0, T]$.

Example For a Friedrichs-symmetrizable system of symmetrizer S_0, the simple multiplication operator $\Sigma(t) : u \mapsto \Sigma(t)u := S_0(\cdot, t)u$ is a functional symmetrizer. As a matter of fact, (2.1.6) merely follows from the analogous property of the matrices $S_0(x, t)$. And, because the matrices $S_0(x, t)A^\alpha(x, t)$ are symmetric, $2\,\mathrm{Re}\,(\Sigma(t)P(t))$ reduces to the multiplication operator associated with $(\sum \partial_\alpha(S_0 A^\alpha) + S_0 B + B^* S_0)(\cdot, t)$.

Theorem 2.1 *If a family of operators* $P(t)$ *admits a functional symmetrizer, then, for all* $s \in \mathbb{R}$ *and* $T > 0$, *there exists* $C > 0$ *so that for* $u \in \mathscr{C}^1([0, T]; H^s) \cap \mathscr{C}([0, T]; H^{s+1})$ *we have*

$$\|u(t)\|_{H^s}^2 \leq C \left(\|u(0)\|_{H^s}^2 + \int_0^t \|Lu(\tau)\|_{H^s}^2 \, \mathrm{d}\tau \right), \tag{2.1.8}$$

where L is defined in (2.0.3).

Proof The proof is very much like the one of Proposition 2.1. The first step is elementary. It consists in showing the estimate in (2.1.8) for $s = 0$. The second one infers the estimate for any s from the case $s = 0$ with the help of pseudo-differential calculus.

Case $s = 0$ From (2.1.6) we know that

$$\langle \Sigma(t)\, u(t)\,,\, u(t) \rangle_{L^2} \geq \alpha \|u(t)\|_{L^2}^2.$$

To bound the left-hand side we write

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \Sigma\, u\,,\, u \rangle = 2\,\mathrm{Re}\,\langle \Sigma\, L\, u\,,\, u \rangle + 2\,\mathrm{Re}\,\langle \Sigma\, P\, u\,,\, u \rangle + \langle \frac{d\Sigma}{dt} u\,,\, u \rangle.$$

Each term here above can be estimated by using the Cauchy–Schwarz inequality. For the first and last ones, we use uniform bounds in t of $\|\Sigma\|_{\mathscr{B}(L^2)}$ and $\|d\Sigma/dt\|_{\mathscr{B}(L^2)}$. For the middle term we use (2.1.7) and a uniform bound in t of Re $(\Sigma\, P)$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\langle\, \Sigma\, u\,,\, u\,\rangle \,\leq\, C_1\,(\,\|u\|_{L^2}^2 \,+\, \|L\,u\|_{L^2}^2\,).$$

Hence, we have

$$\alpha\,\|u(t)\|_{L^2}^2 \leq\, C_0\,\|u(0)\|_{L^2}^2 \,+\, C_1\,\int_0^t (\,\|u(\tau)\|_{L^2}^2 \,+\, \|L\,u(\tau)\|_{L^2}^2\,)\,\mathrm{d}\tau,$$

where $C_0 := \|\Sigma(0)\|_{\mathscr{B}(L^2)}$. We conclude by Gronwall's Lemma that (2.1.8) holds for $s = 0$ with $C = C'\exp(C'T)$, $C' := \max(C_1, C_0)/\alpha$.

General case Let s be an arbitrary real number. For $u \in \mathscr{C}^1([0,T]; H^s) \cap \mathscr{C}([0,T]; H^{s+1})$ the inequality previously derived for $s = 0$ applies to $\Lambda^s u$ and yields

$$\|\Lambda^s\, u(t)\|_{L^2}^2 \leq\, C\,\left(\,\|\Lambda^s\, u(0)\|_{L^2}^2 \,+\, \int_0^t \,\|L\,\Lambda^s\, u(\tau)\|_{L^2}^2\,\mathrm{d}\tau\,\right).$$

Writing

$$L\,\Lambda^s\, u \,=\, \Lambda^s\, L\, u \,+\, [\Lambda^s, P]\, u,$$

and observing that the commutator $[\Lambda^s, P]$ is of order $s + 1 - 1 = s$ (see Appendix C), we complete the proof exactly as in the scalar case (Proposition 2.1). □

Remark 2.1 By reversing time, that is, changing t to $T - t$ and $P(t)$ to $-P(T - t)$ in Theorem 2.1, we also obtain the estimate

$$\|u(t)\|_{H^s}^2 \,\leq\, C\,\left(\,\|u(T)\|_{H^s}^2 \,+\, \int_0^T \,\|Lu(\tau)\|_{H^s}^2\,\mathrm{d}\tau\,\right), \qquad (2.1.9)$$

for $u \in \mathscr{C}^1([0,T]; H^s) \cap \mathscr{C}([0,T]; H^{s+1})$.

The problem is now to construct symmetrizers. Except for Friedrichs-symmetrizable systems, this is not an easy task. We shall conveniently use symbolic calculus. Similarly as in Chapter 1, we denote

$$A(x,t,\xi) \,:=\, \sum_\alpha \,\xi_\alpha\, A^\alpha(x,t)\,, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}^+\,,\, \xi \in \mathbb{R}^d,$$

which can be viewed up to a $-i$ factor as the symbol of the principal part of the operator $P(t)$ defined in (2.0.2).

Definition 2.3 *A* symbolic symmetrizer *associated with* $A(x,t,\xi)$ *is a* $\mathscr{C}^\infty$ *mapping*

$$S : \mathbb{R}^d \times \mathbb{R}^+ \times (\mathbb{R}^d \backslash \{0\}) \to \mathbf{M}_n(\mathbb{C}),$$

homogeneous degree 0 *in its last variable* ξ*, bounded as well as all its derivatives with respect to* (x,t,ξ) *on* $\|\xi\| = 1$*, such that, for all* (x,t,ξ)

$$S(x,t,\xi) = S(x,t,\xi)^* \geq \beta I, \tag{2.1.10}$$

for some positive β*, uniformly on sets of the form* $\mathbb{R}^d \times [0,T] \times (\mathbb{R}^d \backslash \{0\})$ *(*$T > 0$*), and*

$$S(x,t,\xi)\, A(x,t,\xi) = A(x,t,\xi)^*\, S(x,t,\xi)^*. \tag{2.1.11}$$

Of course, a Friedrichs-symmetrizable system admits an obvious 'symbolic' symmetrizer independent of ξ

$$S(x,t,\xi) = S_0(x,t).$$

Note that, in general, a symbolic symmetrizer is not exactly a pseudo-differential symbol, due to the singularity allowed at $\xi = 0$. However, truncating about 0 does yield a pseudo-differential symbol in $\mathbf{S}^0$, which is unique modulo $\mathbf{S}^{-\infty}$ (see Appendix C). This enables us to associate S with a family of pseudo-differential operators $\widetilde{\Sigma}(t)$ of order 0 modulo infinitely smoothing operators. This in turn will enable us to construct a functional symmetrizer $\Sigma(t)$.

Remark 2.2 In the constant-coefficient case, neither $A(x,t,\xi)$ nor $S(x,t,\xi)$ depend on (x,t), and it is elementary to construct a functional symmetrizer based on S. This symmetrizer is of course independent of t and is just given by

$$\Sigma := \mathscr{F}^{-1}\, S\, \mathscr{F}$$

(where $\mathscr{F}$ denotes the usual Fourier transform). Then (2.1.6) holds with $\alpha = \beta$ since we have

$$\langle \Sigma\, u\,,\, v\rangle = \langle S\, \widehat{u}\,,\, \widehat{v}\rangle$$

for all u, $v \in L^2$. And (2.1.7) follows from (2.1.11), because of the relations

$$\langle \Sigma\, P\, u\,,\, v\rangle + \langle u\,,\, \Sigma\, P\, v\rangle = \langle S\, \mathscr{F}\, P\, u\,,\, \widehat{v}\rangle + \langle \widehat{u}\,,\, S\, \mathscr{F}\, P\, v\rangle$$
$$= \langle S\,(-\,i\,A + B)\,\widehat{u}\,,\, \widehat{v}\rangle + \langle \widehat{u}\,,\, S\,(-\,i\,A + B)\,\widehat{v}\rangle = \langle (S\,B + B^{\mathrm{T}}\,S)\,\widehat{u}\,,\, \widehat{v}\rangle.$$

Theorem 2.2 *Assuming that* $A(x,t,\xi)$ *admits a symbolic symmetrizer* $S(x,t,\xi)$ *(according to Definition 2.3), then the family* $P(t)$ *defined in* (2.0.2) *admits a functional symmetrizer* $\Sigma(t)$ *(as in Definition 2.2).*

Proof The proof consists of a pseudo-differential extension of Remark 2.2 above. As mentioned above, $S(\cdot,t,\cdot)$ can be associated with a pseudo-differential operator of order 0, $\widetilde{\Sigma}(t)$. We recall that the operator $\widetilde{\Sigma}(t)$ is not necessarily self-adjoint, even though the matrices $\widetilde{S}(x,t,\xi)$ are Hermitian. But $\widetilde{\Sigma}(t)^*$ differs from

$\widetilde{\Sigma}(t)$ by an operator of order -1 (since they are both of order 0), see Appendix C. As a first step, let us define

$$\Sigma(t) := \frac{1}{2}\,(\widetilde{\Sigma}(t) + \widetilde{\Sigma}(t)^*).$$

By Gårding's inequality, there exists $C_T > 0$ such that

$$\langle \Sigma(t)\, u\,,\, u\rangle \geq \frac{\beta}{2}\,\|u\|^2_{L^2} - C_T\,\|u\|^2_{H^{-1}}$$

for all $t \in [0,T]$ and $u \in L^2(\mathbb{R}^d)$. Now, noting that

$$\|u\|^2_{H^{-1}} = \langle \Lambda^{-2}\, u\,,\, u\rangle,$$

we can change $\Sigma(t)$ into $\Sigma(t) + C_T\,\Lambda^{-2}$ in order to have

$$\langle \Sigma(t)\, u\,,\, u\rangle \geq \frac{\beta}{2}\,\|u\|^2_{L^2}.$$

This modification does not alter the self-adjointness of $\Sigma(t)$ and gives (2.1.6) with $\alpha = \beta/2$. Furthermore, $\Sigma(t)\,P(t) + P(t)^*\,\Sigma(t)$ coincides with the operator of symbol

$$\widetilde{S}\,(-\,i\,A + B) + (-\,i\,A + B)^{\mathrm{T}}\,\widetilde{S} = \widetilde{S}\,B + B^{\mathrm{T}}\,\widetilde{S}$$

up to a remainder of order $0+1-1=0$, see Appendix C. (To simplify notations, we have omitted the dependence on the parameter t of the symbols.) Since $(\widetilde{S}\,B + B^{\mathrm{T}}\,\widetilde{S})(\cdot,t,\cdot)$ belongs to $\mathbf{S}^0$, $\Sigma(t)\,P(t) + P(t)^*\,\Sigma(t)$ is of order 0 and therefore is a bounded operator on L^2. □

As a consequence of Theorems 2.1 and 2.2, we have the following.

Corollary 2.1 *If $A(x,t,\xi) = \sum_\alpha \xi_\alpha\, A^\alpha(x,t)$ admits a symbolic symmetrizer then for all $s \in \mathbb{R}$ and $T > 0$, there exists $C > 0$ so that for $u \in \mathscr{C}^1([0,T];H^s) \cap \mathscr{C}([0,T];H^{s+1})$ we have*

$$\|u(t)\|^2_{H^s} \leq C\,\left(\|u(0)\|^2_{H^s} + \int_0^t \|Lu(\tau)\|^2_{H^s}\,d\tau \right),$$

where $L = \partial_t + \sum_\alpha A^\alpha\,\partial_\alpha - B$.

As already noted this applies in particular to Friedrichs-symmetrizable systems, but not only. Another important class of hyperbolic systems that do admit a symbolic symmetrizer is the one of constant multiplicity hyperbolic systems.

Theorem 2.3 *We assume that the system* (2.0.1) *is* constantly hyperbolic*, that is, the matrices $A(x,t,\xi)$ are diagonalizable with real eigenvalues $\lambda_1,\cdots,\lambda_p$ of constant multiplicities on $\mathbb{R}^d \times \mathbb{R}^+ \times (\mathbb{R}^d\backslash\{0\})$. We also assume that these matrices are independent of x for $\|x\| \geq R$. Then they admit a symbolic symmetrizer.*

Together with Theorem 2.2 this shows that constantly hyperbolic systems are symmetrizable and thus enjoy H^s estimates.

Proof The proof is based on spectral projections associated with $A(x,t,\xi)$, which are well-defined due to spectral separation. As a matter of fact, the assumptions imply that the spectral gap $|\lambda_j(x,t,\xi) - \lambda_k(x,t,\xi)|$ is bounded by below for $1 \leq j \neq k \leq p$, $(x,t) \in \mathbb{R}^d \times [0,T]$ and $\|\xi\| = 1$. Let us define

$$\rho := \frac{1}{2} \min\{|\lambda_j(x,t,\xi) - \lambda_k(x,t,\xi)|,\ 1 \leq j \neq k \leq p,\ (x,t) \in \mathbb{R}^d \times [0,T],\ \|\xi\| = 1\}$$

and the projectors

$$Q_j(x,t,\xi) := \frac{1}{2\,i\,\pi} \int_{|\lambda - \lambda_j(x,t,\xi)| = \rho\|\xi\|} (\lambda\, I_n - A(x,t,\xi))^{-1}\, \mathrm{d}\lambda$$

for $1 \leq j \leq p$. Since A and its eigenvalues λ_j are homogeneous degree 1 in ξ, we easily see by change of variables that Q_j is homogeneous degree 0. Furthermore, Q_j is independent of x for $\|x\| \geq R$. Then we introduce

$$S(x,t,\xi) := \sum_{j=1}^{p} Q_j(x,t,\xi)^*\, Q_j(x,t,\xi).$$

By construction, the matrix S is Hermitian. Moreover, we have for any vector $v \in \mathbb{C}^d$

$$v^* S v = \sum_{j=1}^{p} \|Q_j v\|^2 \geq \beta\, \|v\|^2,$$

where

$$\beta := \min\left\{ \sum_{j=1}^{p} \|Q_j(x,t,\xi) v\|^2;\ \|v\| = 1,\ \|x\| \leq R,\ 0 \leq t \leq T,\ \|\xi\| = 1 \right\} > 0$$

since $\sum_j Q_j = I_n$. This proves (2.1.10). And finally, since $Q_j A = A Q_j = \lambda_j Q_j$ because λ_j is semisimple, we have

$$w^* S A v = \sum_{j=1}^{p} \lambda_j\, w^*\, Q_j^*\, Q_j\, v$$

for all $v, w \in \mathbb{C}^n$, and thus SA is Hermitian. □

2.1.3 *Energy estimates for less-smooth coefficients*

The main purpose of this section is to obtain energy estimates for less-regular operators in which we keep track of coefficients (in the same spirit as in Proposition 2.2). This will be done in a framework preparing for later non-linear analysis. Namely, following Métivier [132], we assume the dependence of the matrices with respect to (x,t) occurs via a known but not necessarily very smooth function

$v(x,t)$. More precisely, we consider a family of operators

$$L_v = \partial_t + \sum_\alpha A^\alpha(v(x,t))\,\partial_\alpha$$

associated with functions $v \in L^\infty([0,T];H^s(\mathbb{R}^d)) \cap \mathscr{C}([0,T];H^s_w(\mathbb{R}^d))$ such that $\partial_t v \in L^\infty([0,T];H^{s-1}(\mathbb{R}^d)) \cap \mathscr{C}([0,T];H^{s-1}_w(\mathbb{R}^d))$. (We recall that H^s_w stands for the regular Sobolev space H^s equipped with the weak topology.) Observe that such functions belong to $\mathscr{C}^1(\mathbb{R}^d \times [0,T])$ as soon as $s > \frac{d}{2}+1$. As to the matrices A^α, they are assumed to be $\mathscr{C}^\infty$ functions of their argument $v \in \mathbb{R}^n$. We denote $A(v,\xi) = \sum_\alpha \xi_\alpha A^\alpha(v)$ and extend in a straightforward way the definition of a symbolic symmetrizer.

Definition 2.4 *A* symbolic symmetrizer *associated with $A(v,\xi)$ is a $\mathscr{C}^\infty$ mapping*

$$S : \mathbb{R}^n \times (\mathbb{R}^d \backslash \{0\}) \to \mathbf{M}_n(\mathbb{C}),$$

homogeneous degree 0 in ξ such that

$$S(v,\xi) = S(v,\xi)^* > 0 \ \ \textit{and}\ \ S(v,\xi)\,A(v,\xi) = A(v,\xi)^*\,S(v,\xi).$$

Theorem 2.4 *Assume that $A(v,\xi)$ admits a symbolic symmetrizer and take $T>0$, $s > \frac{d}{2}+1$. If*

$$\|v\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))} \leq \omega$$

and

$$\sup\left(\|v\|_{L^\infty([0,T];H^s(\mathbb{R}^d))}\,,\|\partial_t v\|_{L^\infty([0,T];H^{s-1}(\mathbb{R}^d))}\right) \leq \mu,$$

there exists $K = K(\omega) > 0$ and $C = C(\mu)$, $\gamma = \gamma(\mu)$ so that for all $u \in \mathscr{C}^1([0,T];H^m) \cap \mathscr{C}([0,T];H^{m+1})$, $\frac{d}{2}+1 < m \leq s$, we have

$$\|u(t)\|^2_{H^m} \leq K\,\mathrm{e}^{\gamma t}\,\|u(0)\|^2_{H^m} + C\int_0^t \mathrm{e}^{\gamma(t-\tau)}\,\|L_v u(\tau)\|^2_{H^m}\,\mathrm{d}\tau.$$

Proof As mentioned before, the proof takes advantage of Bony's para-differential calculus.

The idea is first to replace L_v by the para-differential operator

$$P_v = \partial_t + \sum_\alpha T_{A^\alpha(v)}\,\partial_\alpha.$$

To estimate the error, assume first that A^α vanishes at 0. Then by Proposition C.9 and Theorem C.12, we have

$$\|\,A^\alpha(v)\,u - T_{A^\alpha(v)}\,u\,\|_{H^m} \leq C\,\|u\|_{L^\infty}\,\|A^\alpha(v)\|_{H^m} \leq K(\|v\|_{L^\infty})\,\|v\|_{H^m}\,\|u\|_{L^\infty}$$

for all $u \in H^m$ ($\hookrightarrow L^\infty$). Hence

$$\|\,P_v u - L_v u\,\|_{H^m} \leq K(\|v\|_{L^\infty})\,\|v\|_{H^m}\,\|\nabla u\|_{L^\infty}$$

pointwisely in time for all $u \in \mathscr{C}^1([0,T];H^m) \cap \mathscr{C}([0,T];H^{m+1})$. By Sobolev embedding this estimate merely implies

$$\| P_v u \,-\, L_v u \|_{H^m} \,\leq\, C(\|v\|_{H^m})\, \|u\|_{H^m} \,.$$

In fact, the assumption $A^\alpha(0) = 0$ is superfluous. Indeed, denoting $\widetilde{A}^\alpha(v) = A^\alpha(v) - A^\alpha(0)$, we have

$$A^\alpha(v)\,\partial_\alpha \,-\, T_{A^\alpha(v)}\,\partial_\alpha \,=\, (A^\alpha(0)\,\partial_\alpha \,-\, T_{A^\alpha(0)}\,\partial_\alpha) \,+\, (\widetilde{A}^\alpha(v)\,\partial_\alpha \,-\, T_{\widetilde{A}^\alpha(v)}\,\partial_\alpha),$$

where the first term is a smoothing operator according to Theorem C.13[1] and the second operator can be estimated as before.

Once we have the estimate for $P_v - L_v$, it suffices to show the result for the operator P_v instead of L_v. Indeed, if

$$\|u(t)\|_{H^m}^2 \,\leq\, K\,\mathrm{e}^{\gamma t}\,\|u(0)\|_{H^m}^2 \,+\, C\int_0^t \mathrm{e}^{\gamma(t-\tau)}\,\|P_v u(\tau)\|_{H^m}^2\,\mathrm{d}\tau.$$

Then

$$\|u(t)\|_{H^m}^2 \leq K\,\mathrm{e}^{\gamma t}\,\|u(0)\|_{H^m}^2 \\ +2\,C\int_0^t \mathrm{e}^{\gamma(t-\tau)}\,\big(\,\|L_v u(\tau)\|_{H^m}^2 \,+\, C(\|v\|_{H^m})^2\,\|u(\tau)\|_{H^m}^2\,\big)\,\mathrm{d}\tau,$$

which implies by Gronwall's Lemma

$$\|u(t)\|_{H^m}^2 \,\leq\, K\,\mathrm{e}^{\widetilde{\gamma} t}\,\|u(0)\|_{H^m}^2 \,+\, C\int_0^t \mathrm{e}^{\widetilde{\gamma}(t-\tau)}\,\|L_v u(\tau)\|_{H^m}^2\,\mathrm{d}\tau,$$

with $\widetilde{\gamma} = \gamma + C\,C(\|v\|_{H^m})^2$.

The next step is to use the symbolic symmetrizer $S(v(x,t),\xi)$, which is Lipschitz in x, to construct a functional symmetrizer for P_v. The outline is a para-differential version of the proof of Theorem 2.2. Define the symbol $r(x,t,\xi) = \lambda^{2m}(\xi)\,S(v(x,t),\xi)$. Up to a small-frequency cut-off, $r(t) = r(\cdot,t,\cdot)$ belongs to Γ_1^{2m} (see Definition C.5) and thus is associated with a para-differential operator $T_{r(t)}$, simply denoted by T_r in what follows. The constants in the estimates (C.4.45) of r and its derivatives depend boundedly on $\|v\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))}$, that is on ω. Furthermore, since r is everywhere positive-definite Hermitian we know by Gårding's inequality (see Theorem C.18) there exist $\beta = \beta(\omega) > 0$ and $K_1 = K_1(\omega) > 0$ such that

$$\mathrm{Re}\,\langle T_r\,u\,,\,u\rangle \,\geq\, \beta\,\|u\|_{H^m}^2 \,-\, K_1\,\|u\|_{H^{m-1/2}}^2$$

for all $u \in H^m(\mathbb{R}^d)$. Noting that

$$\|u\|_{H^{m-1/2}}^2 \,=\, \langle \Lambda^{2m-1}\,u\,,\,u\rangle,$$

[1]In fact, we need here only a special, easy case of Theorem C.13 because $A^\alpha(0)$ is constant.

we may modify the symbol r into

$$\widetilde{r} = r + K_1 \lambda^{2m-1}$$

and get the estimate

$$\mathrm{Re}\ \langle T_{\widetilde{r}} u\,,\, u\rangle \;\geq\; \beta\, \|u\|_{H^m}^2$$

for the modified operator $T_{\widetilde{r}}$. We also have an upper bound (see Proposition C.17)

$$|\mathrm{Re}\ \langle T_{\widetilde{r}} u\,,\, u\rangle| \;\leq\; K_2(\omega)\, \|u\|_{H^m}^2$$

and $T_{\partial_t \widetilde{r}}$ enjoys a similar estimate

$$|\mathrm{Re}\ \langle T_{\partial_t \widetilde{r}} u\,,\, u\rangle| \;\leq\; C(\omega, \|\partial_t v\|_{L^\infty(\mathbb{R}^d \times [0,T])})\, \|u\|_{H^m}^2 .$$

Observing that $\|\partial_t v\|_{L^\infty(\mathbb{R}^d \times [0,T])}$ is controlled by $\|\partial_t v\|_{H^{s-1}}$ (by Sobolev embedding) and thus by μ, we shall merely write this new constant $C(\omega, \|\partial_t v\|_{L^\infty(\mathbb{R}^d \times [0,T])}) = C(\omega, \mu)$. Finally, from the symmetry of the symbol $r(x,t,\xi)\, a(v(x,t),\xi)$ we also have an estimate

$$|\mathrm{Re}\ \langle (T_{\widetilde{r}} + T_{\widetilde{r}}^*) \sum_\alpha T_{A^\alpha(v)} \partial_\alpha u\,,\, u\rangle| \;\leq\; K_3(\omega)\, \|u\|_{H^m}^2 .$$

The end of the proof is similar to the proof of Proposition 2.2. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Re}\ \langle T_{\widetilde{r}} u, u\rangle = \mathrm{Re}\ \langle (T_{\widetilde{r}} + T_{\widetilde{r}}^*)\, P_v u\,, u\rangle \;-\; \mathrm{Re}\ \langle (T_{\widetilde{r}} + T_{\widetilde{r}}^*) \sum_\alpha T_{A^\alpha(v)} \partial_\alpha u\,, u\rangle$$
$$+ \mathrm{Re}\ \langle T_{\partial_t \widetilde{r}} u\,,\, u\rangle.$$

Denoting $\Sigma = T_{\widetilde{r}} + T_{\widetilde{r}}^*$ and using the Cauchy–Schwarz inequality for the inner product $\langle \Sigma\cdot, \cdot\rangle$ we get from that identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \Sigma u, u\rangle \;\leq\; \langle \Sigma P_v u, P_v u\rangle \;+\; \langle \Sigma u, u\rangle \;+\; 2\, K_3(\omega)\, \|u\|_{H^m}^2 \;+\; 2\, C(\omega,\mu)\, \|u\|_{H^m}^2 .$$

Hence

$$\langle \Sigma u(t), u(t)\rangle \leq \langle \Sigma u(0), u(0)\rangle \;+\; \int_0^t \langle \Sigma P_v u(\tau), P_v u(\tau)\rangle\, \mathrm{d}\tau$$
$$+ \int_0^t (1 \;+\; 2\, (K_3(\omega) + C(\omega,\mu))/\beta(\omega))\, \langle \Sigma u(\tau), u(\tau)\rangle\, \mathrm{d}\tau.$$

Therefore by Gronwall's Lemma, we have

$$\langle \Sigma u(t), u(t)\rangle \;\leq\; \mathrm{e}^{\gamma\, t}\, \langle \Sigma u(0), u(0)\rangle \;+\; \int_0^t \mathrm{e}^{\gamma\,(t-\tau)}\, \langle \Sigma P_v u(\tau), P_v u(\tau)\rangle\, \mathrm{d}\tau$$

for $\gamma \geq (1 + 2(K_3(\omega) + C(\omega,\mu))/\beta(\omega))$. Coming back to the usual H^m norm, we get

$$\|u(t)\|_{H^m}^2 \leq \frac{K_2(\omega)}{\beta(\omega)} \left(e^{\gamma t} \|u(0)\|_{H^m}^2 + \int_0^t e^{\gamma(t-\tau)} \|P_v u(\tau)\|_{H^m}^2 \, d\tau \right).$$

The proof is complete once we note that any 'constant' depending on ω *a fortiori* depends boundedly on μ (by Sobolev embedding). □

Remark 2.3 The only point where we have used the assumption $m > \frac{d}{2} + 1$ is in the comparison between L_v and P_v. We shall see in the proof of Theorem 2.7 that we can bypass this assumption and still obtain energy estimates in the Sobolev space of negative index H^{-s}. Another possibility is to derive L^2 estimates, under the only assumption that v be $W^{1,\infty}$ in both time and space. This is the purpose of the next theorem.

Theorem 2.5 *Assume that $A(v,\xi)$ admits a symbolic symmetrizer and that*

$$\|v\|_{W^{1,\infty}(\mathbb{R}^d \times [0,T])} \leq \omega.$$

Then there exists $K = K(\omega) > 0$ and $\gamma = \gamma(\omega)$ so that for all $u \in \mathscr{C}^1([0,T];L^2) \cap \mathscr{C}([0,T];H^1)$

$$\|u(t)\|_{L^2}^2 \leq K \left(e^{\gamma t} \|u(0)\|_{L^2}^2 + \int_0^t e^{\gamma(t-\tau)} \|L_v u(\tau)\|_{L^2}^2 \, d\tau \right). \tag{2.1.12}$$

Proof We proceed exactly as in the proof of Theorem 2.4, just changing the way of estimating $P_v - L_v$. Indeed, examining the second step of that proof shows that it works for $m = 1$ as soon as we have bounds for $\|v\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))}$ and $\|\partial_t v\|_{L^\infty([0,T];L^\infty(\mathbb{R}^d))}$, which is the case by assumption. The estimate of $P_v - L_v$ is given by Corollary C.4, namely

$$\|P_v u - L_v u\|_{L^2} \leq C \max_\alpha \|A^\alpha(v)\|_{W^{1,\infty}} \|u\|_{L^2} \leq \widetilde{C} \|v\|_{W^{1,\infty}} \|u\|_{L^2}.$$

□

Remark 2.4 The estimate (2.1.12) here above applies in fact to any $u \in H^1(\mathbb{R}^d \times [0,T])$ by a density argument. For, $u \in H^1(\mathbb{R}^d \times [0,t])$ can be achieved as the limit of a sequence $u_k \in \mathscr{D}(\mathbb{R}^d \times [0,t])$, in such a way that u_k goes to u in $H^1([0,t];L^2(\mathbb{R}^d)) \hookrightarrow \mathscr{C}([0,t];L^2(\mathbb{R}^d))$ *and* $L_v u_k$ goes to $L_v u$ in $L^2(\mathbb{R}^d \times [0,t])$. Therefore, we can pass to the limit in the estimate (2.1.12) applied to u_k, written as

$$\max_{\tau \in [0,t]} \left(e^{-\gamma\tau} \|u_k(\tau)\|_{L^2}^2 \right) \leq K \left(\|u_k(0)\|_{L^2}^2 + \int_0^t e^{-\gamma\tau} \|L_v u_k(\tau)\|_{L^2}^2 \, d\tau \right).$$

Remark 2.5 If the operator L_v is Friedrichs symmetrizable, we also have a H^1 estimate

$$\|u(t)\|_{H^1}^2 \leq K \left(\mathrm{e}^{\gamma t} \|u(0)\|_{H^1}^2 + \int_0^t \mathrm{e}^{\gamma(t-\tau)} \|L_v u(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \right), \qquad (2.1.13)$$

for $u \in \mathscr{C}^1([0,T]; H^1(\mathbb{R}^d)) \cap \mathscr{C}([0,T]; H^2(\mathbb{R}^d))$ and $v \in W^{1,\infty}(\mathbb{R}^d \times (0,T))$, with K depending boundedly on $\|v\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d)}$ and $\gamma \geq \gamma_0$ depending boundedly on $\|v\|_{W^{1,\infty}(\mathbb{R}^d \times [0,T])}$.

The proof is most elementary. Denoting by $S_0(v)$ a Friedrichs symmetrizer, we see that for $v \in W^{1,\infty}$, the estimate in (2.1.13) is equivalent to

$$\|\widetilde{u}(t)\|_{H^1}^2 \leq \widetilde{K} \left(\mathrm{e}^{\gamma t} \|\widetilde{u}(0)\|_{H^1}^2 + \int_0^t \mathrm{e}^{\gamma(t-\tau)} \|\widetilde{L}_v \widetilde{u}(\tau)\|_{H^1}^2 \,\mathrm{d}\tau \right),$$

where $\widetilde{u}(x,t) = \sqrt{S_0(v(x,t))}\,u(x,t)$ and

$$\widetilde{L}_v = \partial_t + \sum_\alpha \widetilde{A}(v(x,t))^\alpha \,\partial_\alpha, \quad \text{with } \widetilde{A}(v) := \sqrt{S_0(v)}^{-1} S_0(v)\, A^\alpha(v) \sqrt{S_0(v)}^{-1}$$

being symmetric. Therefore, we can assume with no loss of generality (just dropping the tildas) that the matrices $A^\alpha(v)$ are symmetric. Thanks to this property we easily find that, for u smooth enough such that $L_v u = f$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{L^2(\mathbb{R}^d)}^2 = 2\,\mathrm{Re}\, \langle u\,,\, f\,\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_\beta \|\partial_\beta u\|_{L^2(\mathbb{R}^d)}^2 + 2\,\mathrm{Re} \sum_{\alpha,\beta} \langle \partial_\beta u\,,\, \partial_\beta(A^\alpha(v))\,\partial_\alpha u\,\rangle = 2\,\mathrm{Re} \sum_\beta \langle \partial_\beta u\,,\, \partial_\beta f\,\rangle,$$

hence by taking the sum and using the Cauchy–Schwarz inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|_{H^1(\mathbb{R}^d)}^2 \leq \gamma \|u\|_{H^1(\mathbb{R}^d)}^2 + \|f\|_{H^1(\mathbb{R}^d)}^2,$$

with $\gamma := 1 + \max_{\alpha,\beta} \|\partial_\beta(A^\alpha(v))\|_{L^\infty(\mathbb{R}^d \times (0,T))}$. After integration we get (2.1.13) with $K = 1$. □

2.1.4 *How energy estimates imply well-posedness*

The energy estimate in (2.1.8) easily implies uniqueness by linearity, for smooth enough solutions. The existence of a solution $u \in \mathscr{C}([0,T]; H^{s-1})$ for initial data $u(0) \in H^s$ can be obtained by a duality argument, applying the energy estimate in (2.1.8) to $-s$ and the adjoint operator L^*. The proof that u actually lies in $\mathscr{C}([0,T]; H^s)$ uses mollifiers, smooth solutions and their uniqueness. The whole result can be stated as follows, keeping the same compressed notations as in (2.0.2) and (2.0.3).

Theorem 2.6 *We assume the system* (2.0.1) *has* $\mathscr{C}_b^\infty$ *coefficients and is symmetrizable in the sense that the operator* $P(t)$ *admits a functional symmetrizer*

$\Sigma(t)$. We take $T > 0$, $s \in \mathbb{R}$ and $f \in L^2(0, T; H^s(\mathbb{R}^d))$, $g \in H^s(\mathbb{R}^d)$. Then there exists a unique $u \in \mathscr{C}([0, T]; H^s(\mathbb{R}^d))$ solution of the Cauchy problem

$$\partial_t u \,-\, P(t)\, u \,=\, f\,, \quad u(0) \,=\, g.$$

Furthermore, there exists $C > 0$ (independent of u!) such that for all $t \in [0, T]$

$$\|u(t)\|_{H^s}^2 \,\leq\, C\, \left(\|g\|_{H^s}^2 \,+\, \int_0^t \|f(\tau)\|_{H^s}^2 \, \mathrm{d}\tau \right). \qquad (2.1.14)$$

Finally, if $f \in \mathscr{C}^\infty([0, T]; H^{+\infty}(\mathbb{R}^d))$ and $g \in H^{+\infty}(\mathbb{R}^d)$ then $u \in \mathscr{C}^\infty([0, T]; H^{+\infty}(\mathbb{R}^d))$.

Remark 2.6 Of course the equation $\partial_t u \,-\, P(t)\, u \,=\, f$ is to be understood in the sense of distributions when s is low enough.

Remark 2.7 If the bounds associated with the symmetrizer $\Sigma(t)$ for $t \in [0, T]$ are independent of T, and if the source term f is given in $L^2(\mathbb{R}^+; H^s(\mathbb{R}^d))$ then the solution u is global and belongs to $L^2(\mathbb{R}^+; H^s(\mathbb{R}^d))$ too. If additionally, $s = k$ is a positive integer and if f belongs to $H^{k-1}(\mathbb{R}^d \times \mathbb{R}^+)$, then u belongs to $H^k(\mathbb{R}^d \times \mathbb{R}^+)$. This is due to the following simple observation – which will also be used in the context of Boundary Value Problems (Chapter 9).

Proposition 2.3 *As soon as the coefficients A^α and B of the operator*

$$P(t) \,=\, -\sum_\alpha A^\alpha(\cdot, t)\, \partial_\alpha \,+\, B(\cdot, t)$$

are $\mathscr{C}_b^\infty$ functions of (x, t) on $\mathbb{R}^d \times \mathbb{R}^+$, any $u \in L^2(\mathbb{R}^+; H^k(\mathbb{R}^d))$ such that $\partial_t u \,-\, P(t)\, u$ belongs to $H^{k-1}(\mathbb{R}^d \times \mathbb{R}^+)$, with k a positive integer, actually belongs to $H^k(\mathbb{R}^d \times \mathbb{R}^+)$.

Proof If $u \in L^2(\mathbb{R}^+; H^k(\mathbb{R}^d))$ we can show indeed that for all $m \in \mathbb{N}$ and all d-uple α such that $m + |\alpha| \leq k$, $\partial_t^m \partial^\alpha u$ belongs to $L^2(\mathbb{R}^d \times \mathbb{R}^+)$. This is trivial if $m = 0$, because of the identification $L^2(\mathbb{R}^+; L^2(\mathbb{R}^d)) \,=\, L^2(\mathbb{R}^d \times \mathbb{R}^+)$. The rest of the proof works by finite induction on m, using the decomposition

$$\partial_t^{m+1} \partial^\alpha u \,=\, \partial_t^m \partial^\alpha (\partial_t u \,-\, P(t)\, u) \,+\, \partial_t^m \partial^\alpha (P(t)\, u),$$

with $(\partial_t u \,-\, P(t)\, u) \,\in\, H^{k-1}(\mathbb{R}^d \times \mathbb{R}^+)$ and $P(t)\, u \,\in\, L^2(\mathbb{R}^+; H^{k-1}(\mathbb{R}^d))$. □

Proof of Theorem 2.6

Uniqueness As mentioned above, the uniqueness is easy to show. By linearity, it is sufficient to show that the only solution in $\mathscr{C}([0, T]; H^s(\mathbb{R}^d))$ for $f \equiv 0$ and $g \equiv 0$ is $u \equiv 0$. But if $u \in \mathscr{C}([0, T]; H^s(\mathbb{R}^d))$ satisfies (2.0.1) then necessarily u belongs to $\mathscr{C}^1([0, T]; H^{s-1}(\mathbb{R}^d))$. So we can apply Theorem 2.1 to the index $s - 1$. For $u(0) = g \equiv 0$ and $Lu = f \equiv 0$ this gives $u \equiv 0$.

Existence in $\mathscr{C}([0,T];H^{s-1})$ We note that the adjoint operator

$$L^* \,=\, -\,\partial_t \,-\, \sum_\alpha A^\alpha(x,t)^{\mathrm{T}}\partial_\alpha \,-\, \sum_\alpha \partial_\alpha A^\alpha(x,t)^{\mathrm{T}} \,-\, B^{\mathrm{T}}$$

(or its opposite, to fit with the standard time evolution) is also symmetrizable. As a matter of fact, $\Sigma(t)^{-1}$ is a functional symmetrizer for $P(t)^*$. In particular, the estimate (2.1.9) in Remark 2.1 can be applied to L^*. Now we introduce the space

$$\mathscr{E} \,:=\, \{\, \varphi \in \mathscr{C}^\infty([0,T];H^{+\infty}(\mathbb{R}^d))\,;\; \varphi(T) \,=\, 0\,\}.$$

Applying (2.1.9) to $-s$ (where s is the regularity index of the source term f) and L^* instead of L yields the estimate

$$\|\varphi(t)\|^2_{H^{-s}} \,\leq\, C \int_0^T \,\|L^*\varphi(\tau)\|^2_{H^{-s}}\,\mathrm{d}\tau \tag{2.1.15}$$

for all $\varphi \in \mathscr{E}$ and $t \in [0,T]$. Hence the operator L^* restricted to $\mathscr{E}$ is one-to-one. This enables us to define a unique linear form ℓ on $L^*\mathscr{E}$ by

$$\ell(L^*\varphi) \,=\, \int_0^T \langle\, f(t)\,,\, \varphi(t)\,\rangle_{H^s,H^{-s}}\,\mathrm{d}t \,+\, \langle\, g\,,\, \varphi(0)\,\rangle_{H^s,H^{-s}}. \tag{2.1.16}$$

By (2.1.15) and the Cauchy–Schwarz inequality, we see that

$$\ell(L^*\varphi)^2 \,\leq\, 2\,C\,(\,T\,\|f\|^2_{L^2(0,T;H^s)} \,+\, \|g\|^2_{H^s}\,)\;\; \|L^*\varphi\|^2_{L^2(0,T;H^{-s})}.$$

By the Hahn–Banach theorem, ℓ thus extends to a continuous form on $L^2(0,T;H^{-s})$. And by the Riesz theorem, we find $u \in L^2(0,T;H^s) = L^2(0,T;H^{-s})'$ such that

$$\ell(L^*\varphi) \,=\, \int_0^T \langle\, u(t)\,,\, L^*\varphi(t)\,\rangle_{H^s,H^{-s}}\,\mathrm{d}t \tag{2.1.17}$$

for all $\varphi \in \mathscr{E}$. In particular, for $\varphi \in \mathscr{D}(\mathbb{R}^d \times (0,T))$ we get that

$$\int_0^T \langle\, f(t)\,,\, \varphi(t)\,\rangle_{H^s,H^{-s}}\,\mathrm{d}t \,=\, \int_0^T \langle\, Lu(t)\,,\, \varphi(t)\,\rangle_{H^s,H^{-s}}\,\mathrm{d}t.$$

In other words, we have $Lu = f$ in the sense of distributions. This implies that $\partial_t u \,=\, P(t)u + f$ belongs to $L^2(0,T;H^{s-1})$, hence u belongs to $\mathscr{C}([0,T];H^{s-1})$. Finally, integrating by parts in (2.1.17), substituting $Lu = f$ and using (2.1.16), we obtain

$$\langle\, g\,,\, \varphi(0)\,\rangle_{H^s,H^{-s}} \,=\, \langle\, u(0)\,,\, \varphi(0)\,\rangle_{H^{s-1},H^{-s+1}}$$

for all $\varphi \in \mathscr{D}(\mathbb{R}^d \times [0,T))$. A standard argument then shows that $u(0) = g$.

Regularity If f belongs to $\mathscr{C}^\infty([0,T];H^{+\infty}(\mathbb{R}^d))$ and g belongs to $H^{+\infty}(\mathbb{R}^d)$ then u belongs to $\mathscr{C}([0,T];H^{+\infty}(\mathbb{R}^d))$ from the above construction applied to arbitrary large s. The equation $\partial_t u = P(t)u + f$ then implies that u belongs to $\mathscr{C}^k([0,T];H^{+\infty}(\mathbb{R}^d))$ for all $k \in \mathbb{N}$. If f and g are less regular, we can use mollifiers to construct sequences $f_k \in \mathscr{D}(\mathbb{R}^d \times [0,T])$ and $g_k \in \mathscr{D}(\mathbb{R}^d)$ such that

$$f_k \xrightarrow[k\to\infty]{L^2(0,T;H^s)} f\,,\ g_k \xrightarrow[k\to\infty]{H^s} g.$$

For all k there is a solution $u_k \in \mathscr{C}^\infty([0,T];H^{+\infty}(\mathbb{R}^d))$ corresponding to the source term f_k and the initial data g_k. Applying the estimate in (2.1.8) to $u_k - u_m$ yields the inequality

$$\|u_k(t) - u_m(t)\|^2_{H^s} \le C\left(\|g_k - g_m\|^2_{H^s} + \int_0^t \|f_k - f_m(\tau)\|^2_{H^s}\,\mathrm{d}\tau\right)$$

for all k, $m \in \mathbb{N}$. This implies that (u_k) is a Cauchy sequence in $\mathscr{C}([0,T];H^s)$ and thus converges, say towards $\widetilde{u} \in \mathscr{C}([0,T];H^s)$. In the limit we have $L\widetilde{u} = f$ and $\widetilde{u}(0) = g$. By uniqueness (in $\mathscr{C}([0,T];H^{s-1})$), we have $\widetilde{u} = u$, the solution constructed by a duality argument. Observe then that $\widetilde{u} = u$ satisfies the energy estimate in (2.1.14) by passing to the limit in (2.1.8) applied to u_k. This completes the proof. □

Remark 2.8 The end of this proof is sometimes referred to as a *weak=strong* argument. Indeed, if we say a *strong solution* is a solution that is the limit of infinitely smooth solutions of regularized problems, and a *weak solution* the one obtained by duality, it shows that any weak solution is necessarily a strong one.

Remark 2.9 This theorem is also valid backward, that is, prescribing $u(T)$ instead of $u(0)$. This fact will be used in the application of the Holmgren Principle below.

Remark 2.10 In the proof above, we have crucially used the infinite smoothness of coefficients. In fact, H^s-well-posedness is also true for H^s coefficients provided that s is large enough. We reproduce from [132] the corresponding precise result below, which is the continuation of Theorem 2.4 and is useful in non-linear analysis.

Theorem 2.7 *Assume $s > \frac{d}{2} + 1$ and take $T > 0$. For a function v belonging to $L^\infty([0,T];H^s(\mathbb{R}^d)) \cap \mathscr{C}([0,T];H^s_w(\mathbb{R}^d))$ and such that*

$$\partial_t v \in L^\infty([0,T];H^{s-1}(\mathbb{R}^d)) \cap \mathscr{C}([0,T];H^{s-1}_w(\mathbb{R}^d)),$$

we consider the differential operator

$$L_v = \partial_t + \sum_\alpha A^\alpha(v(x,t))\,\partial_\alpha,$$

where the $n \times n$ matrices A^α are $\mathscr{C}^\infty$ functions of their argument $v \in \mathbb{R}^n$.

If $A(v,\xi) = \sum_\alpha A^\alpha(v)\xi_\alpha$ *admits a symbolic symmetrizer, then for all*

$$f \in L^\infty([0,T];H^s(\mathbb{R}^d)) \cap \mathscr{C}([0,T];H^s_w(\mathbb{R}^d)) \quad \textit{and} \quad g \in H^s(\mathbb{R}^d),$$

the Cauchy problem

$$L_v u = f\,, \quad u(0) = g$$

has a unique solution $u \in L^2([0,T];H^s(\mathbb{R}^d))$. *Furthermore,* u *belongs to* $\mathscr{C}([0,T];H^s(\mathbb{R}^d))$ *and enjoys an estimate*

$$\|u(t)\|^2_{H^s} \le K\,\mathrm{e}^{\gamma t}\,\|u(0)\|^2_{H^s} + C\int_0^t \mathrm{e}^{\gamma(t-\tau)}\,\|L_v u(\tau)\|^2_{H^s}\,\mathrm{d}\tau\,,$$

where $K > 0$ *depends boundedly on* $\|v\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))}$ *and* C, γ *depend boundedly on*

$$\sup\left(\|v\|_{L^\infty([0,T];H^s(\mathbb{R}^d))}\,,\,\|\partial_t v\|_{L^\infty([0,T];H^{s-1}(\mathbb{R}^d))}\right).$$

Sketch of proof As for Theorem 2.6, the existence part of the proof uses the adjoint operator

$$L_v^* = -\partial_t - \sum_\alpha \partial_\alpha\,(A^\alpha(v))^{\mathrm{T}}$$

of L_v and energy estimates in the space of negative index H^{-s}. The existence of a symmetrizer for L_v^*, namely S^{-1}, is of course crucial to derive those estimates. By Theorem 2.4 it does imply an energy estimate, though a priori only in H^m with $m > \frac{d}{2} + 1$. However, as already observed in Remark 2.3, the only place where the proof of Theorem 2.4 makes use of the restriction on m is in the comparison between L_v^* and

$$P_v^* = -\partial_t - \sum_\alpha \partial_\alpha\,T_{(A^\alpha(v))^*}.$$

If we invoke Proposition C.15, which shows that

$$\|\,(L_v^* - P_v^*)\,\phi\|_{H^{-s}} \le C(\|v\|_{H^s})\,\|\,\phi\,\|_{H^{-s}}$$

for all $\phi \in \mathscr{C}_0^\infty(\mathbb{R}^d \times [0,T])$, and perform the estimate of $\|\phi\|_{H^{-s}}$ in terms of $\|P_v^*\,\phi\|_{H^{-s}}$ as in the proof of Theorem 2.4, we eventually obtain

$$\|\phi(t)\|^2_{H^{-s}} \le K\,\mathrm{e}^{\gamma t}\,\|\phi(0)\|^2_{H^{-s}} + C\int_0^t \mathrm{e}^{\gamma(t-\tau)}\,\|L_v\phi(\tau)\|^2_{H^{-s}}\,\mathrm{d}\tau$$

for all $\phi \in \mathscr{D}(\mathbb{R}^d \times [0,T])$. Once we have this estimate, the same arguments as in the proof of Theorem 2.6 show the existence of a solution $u \in L^2([0,T],H^s)$. Using the equation $L_v u = f$, we see that additionally $u \in H^1([0,T];H^{s-1}(\mathbb{R}^d)) \hookrightarrow \mathscr{C}([0,T];H^{s-1}(\mathbb{R}^d))$.

The rest of the proof consists in showing more regularity on u. This can be done thanks to the estimate in Theorem 2.4 for $m = s$ and a *weak-strong* argument, the latter being also used to prove u satisfies the H^s energy estimate (hence its uniqueness). See [132] for more details. (Also see the proof of Theorem 2.8 hereafter for a similar method in L^2.) □

Theorem 2.7 deals with smooth solutions. There is also a L^2-well-posedness result for Lipschitz coefficients, relying on the energy estimate (2.1.12) in Theorem 2.5, which can be stated as follows.

Theorem 2.8 *Let v belong to $W^{1,\infty}(\mathbb{R}^d \times (0,T))$, and consider the operator*

$$L_v = \partial_t + \sum_\alpha A^\alpha(v(x,t))\,\partial_\alpha,$$

where A^α are $\mathscr{C}^\infty$ functions of their argument $v \in \mathbb{R}^n$ and $A(v,\xi) = \sum_\alpha \xi_\alpha A^\alpha(v)$ admits a symbolic symmetrizer. Then for all $f \in L^2(\mathbb{R}^d \times (0,T))$ and $g \in L^2(\mathbb{R}^d)$, the Cauchy problem

$$L_v u = f\,, \quad u(0) = g$$

has a unique solution $u \in \mathscr{C}([0,T];L^2(\mathbb{R}^d))$, which enjoys the estimate

$$\|u(t)\|^2_{L^2} \leq K \left(\mathrm{e}^{\gamma t}\, \|u(0)\|^2_{L^2} + \int_0^t \mathrm{e}^{\gamma(t-\tau)}\, \|f(\tau)\|^2_{L^2}\, \mathrm{d}\tau \right),$$

where $K > 0$ and $\gamma > 0$ depend boundedly on $\|v\|_{W^{1,\infty}(\mathbb{R}^d)}$.

Proof Using the energy estimate (2.1.12) in Theorem 2.5, we can find a weak solution $u \in L^2([0,T];L^2(\mathbb{R}^d))$ by the same duality argument as in the proof of Theorem 2.7, with $H^{\pm s}$ replaced by L^2. It remains to show u in fact belongs to $\mathscr{C}([0,T];L^2(\mathbb{R}^d))$, and does satisfy the L^2 energy estimate in (2.1.12).

We take a mollifying kernel $\rho \in \mathscr{D}^\infty(\mathbb{R}^d;\mathbb{R}^+)$, define $\rho_\varepsilon(x) = \varepsilon^{-d}\,\rho(x/\varepsilon)$, and consider the operator R_ε associated with the convolution by ρ_ε. Then $u_\varepsilon := R_\varepsilon u$ belongs to $L^2([0,T];H^{+\infty}(\mathbb{R}^d))$ and goes to u in $L^2([0,T];L^2(\mathbb{R}^d))$, and similarly for $f_\varepsilon := R_\varepsilon f$, while $g_\varepsilon := R_\varepsilon g$ belongs to $H^{+\infty}(\mathbb{R}^d)$ and goes to g in $L^2(\mathbb{R}^d)$ when ε goes to zero. Furthermore, by Theorem C.14 we have

$$\lim_{\varepsilon\to 0} \|[L_v, R_\varepsilon]u(t)\|_{L^2(\mathbb{R}^d)} = 0.$$

Therefore,

$$\partial_t u_\varepsilon = f_\varepsilon + L_v\, u_\varepsilon - [L_v, R_\varepsilon]u$$

belongs to $L^2([0,T];L^2(\mathbb{R}^d))$, and consequently u_ε belongs to $H^1(\mathbb{R}^d \times [0,T])$. Thanks to Remark 2.4 this additional regularity of u_ε allows us to apply the energy estimate in (2.1.12) (Theorem 2.5) to u_ε, and of course also to $u_\varepsilon - u_{\varepsilon'}$.

Hence

$$\|u_\varepsilon(t) - u_{\varepsilon'}(t)\|^2_{L^2} \leq K \left(\mathrm{e}^{\gamma t} \, \|u_\varepsilon(0) - u_{\varepsilon'}(0)\|^2_{L^2} \right.$$
$$\left. + \int_0^t \mathrm{e}^{\gamma(t-\tau)} \|L_v u_\varepsilon(\tau) - L_v u_{\varepsilon'}(\tau)\|^2_{L^2} \mathrm{d}\tau \right)$$

for all ε, $\varepsilon' \in (0,1)$. Since $u_\varepsilon(0) = g_\varepsilon$ goes to $u(0) = g$ in $L^2(\mathbb{R}^d)$, and (using the Lebesgue theorem in the time direction),

$$L_v u_\varepsilon \; = \; f_\varepsilon \, + \, [L_v, R_\varepsilon] u$$

goes to $L_v u = f$ in $L^2(\mathbb{R}^d \times (0,T))$, the inequality here above implies that (u_ε) is a Cauchy sequence in $\mathscr{C}([0,T]; L^2(\mathbb{R}^d))$. By uniqueness of limits in $L^2([0,T]; L^2(\mathbb{R}^d))$ (and the Lebesgue theorem again), u is necessarily the limit of (u_ε) in $\mathscr{C}([0,T]; L^2(\mathbb{R}^d))$. Finally, by passing to the limit in

$$\|u_\varepsilon(t)\|^2_{L^2} \, \leq \, K \left(\mathrm{e}^{\gamma t} \, \|u_\varepsilon(0)\|^2_{L^2} \, + \, \int_0^t \, \mathrm{e}^{\gamma(t-\tau)} \, \|L_v u_\varepsilon(\tau)\|^2_{L^2} \, \mathrm{d}\tau \right)$$

we obtain

$$\|u(t)\|^2_{L^2} \, \leq \, K \left(\mathrm{e}^{\gamma t} \, \|u(0)\|^2_{L^2} \, + \, \int_0^t \, \mathrm{e}^{\gamma(t-\tau)} \, \|L_v u(\tau)\|^2_{L^2} \, \mathrm{d}\tau \right) .$$

Uniqueness readily follows from this estimate. □

Finally, for Friedrichs-symmetrizable systems with Lipschitz coefficients, H^1-well-posedness also holds true, as stated in the following result, which will be used in the Initial Boundary Value Problem theory.

Theorem 2.9 *Assume $v \in W^{1,\infty}(\mathbb{R}^d \times (0,T))$ and the operator*

$$\partial_t \, + \, \sum_\alpha \, A^\alpha(w) \, \partial_\alpha$$

is Friedrichs symmetrizable for w in a domain containing the range of v. (As usual, A^α are supposed to be $\mathscr{C}^\infty$ functions of w, and so the Friedrichs symmetrizer.) Then for all $g \in H^1(\mathbb{R}^d)$, the Cauchy problem

$$L_v u \, = \, 0 \, , \quad u(0) \, = \, g$$

has a unique solution $u \in \mathscr{C}([0,T]; H^1(\mathbb{R}^d)) \cap \mathscr{C}^1([0,T]; L^2(\mathbb{R}^d))$.

Proof Unsurprisingly, the proof relies on a regularization of the coefficients and on the H^1 estimate in (2.1.13). Consider a mollifier ρ_ε and $v_\varepsilon = \rho_\varepsilon * v$. Then v_ε is bounded independently of ε in $W^{1,\infty}(\mathbb{R}^d \times (0,T))$ and goes to v in $\mathscr{C}([0,T]; L^\infty(\mathbb{R}^d))$. By Theorem 2.6, the Cauchy problem

$$L_{v_\varepsilon} \, u_\varepsilon \, = \, 0 \, , \, u_\varepsilon(0) \, = \, \rho_\varepsilon * g$$

admits a unique solution $u_\varepsilon \in \mathscr{C}^\infty([0,T];H^\infty(\mathbb{R}^d))$. Applying (2.1.13) to u_ε we see (u_ε) is uniformly bounded in $\mathscr{C}([0,T];H^1(\mathbb{R}^d))$. Furthermore, (u_ε) is a Cauchy sequence in $\mathscr{C}([0,T];L^2(\mathbb{R}^d))$. Indeed, by the L^2 energy estimate (2.1.12), we have

$$\|u_\varepsilon(t) - u_{\varepsilon'}(t)\|_{L^2(\mathbb{R}^d))} \leq K \left(e^{\gamma t}\, \|u_\varepsilon(0) - u_{\varepsilon'}(0)\|^2_{L^2} + \int_0^t e^{\gamma(t-\tau)}\, \|L_v(u_\varepsilon - u_{\varepsilon'})(\tau)\|^2_{L^2}\, d\tau \right)$$

and

$$\|L_v(u_\varepsilon - u_{\varepsilon'})(\tau)\|_{L^2} \leq (\, \|u_\varepsilon\|_{\mathscr{C}([0,T];H^1(\mathbb{R}^d))} + \|u_{\varepsilon'}\|_{\mathscr{C}([0,T];H^1(\mathbb{R}^d))}\,) \times \max_\alpha \|A^\alpha(v_\varepsilon) - A^\alpha(v_{\varepsilon'})\|_{\mathscr{C}([0,T];L^\infty(\mathbb{R}^d))}.$$

Therefore, u_ε converges to some $u \in \mathscr{C}([0,T];L^2(\mathbb{R}^d))$. Furthermore, for all $t \in [0,T]$, $u_\varepsilon(t)$ belongs to $H^1(\mathbb{R}^d)$ by weak compactness of bounded balls in $H^1(\mathbb{R}^d)$, and by L^2–H^1 interpolation, u_ε converges to u in $\mathscr{C}([0,T];H^s(\mathbb{R}^d))$ for all $s \in [0,1)$. It is not obvious that u belongs to $\mathscr{C}([0,T];H^1(\mathbb{R}^d))$ though. If we can prove that u is continuous at $t=0$ in the space $H^1(\mathbb{R}^d)$, then for the same reason, by translating time it will also be right continuous at any $t_0 \in [0,T]$, as well as left continuous by reversing the time. Now by an $\varepsilon/3$ argument together with the continuity of u_ε at $t=0$, the convergence of $\rho_\varepsilon * g$ to g in H^s and the convergence of u_ε to u in $\mathscr{C}([0,T];H^s(\mathbb{R}^d))$, we already see that

$$\lim_{t \searrow 0} \|u(t) - g\|_{H^s(\mathbb{R}^d)} = 0$$

for all $s \in [0,1)$, and we want to prove that

$$\lim_{t \searrow 0} \|u(t) - g\|_{H^1(\mathbb{R}^d)} = 0.$$

We can first show that $u(t)$ converges to g as t goes to $0+$ in $H^1_w(\mathbb{R}^d)$, the Sobolev space $H^1(\mathbb{R}^d)$ equipped with the weak topology. Indeed, for all ϕ in H^{-1} and $\psi \in H^{-s}$ (for $s<1$), we have

$$|\,\langle\, \phi\,,\, u(t) - g\,\rangle_{H^{-1},H^1}\,| \;\leq\; \|u(t)-g\|_{H^1}\, \|\phi - \psi\|_{H^{-1}} + \|\psi\|_{H^{-s}}\, \|u(t)-g\|_{H^s}\,,$$

in which $\|\phi - \psi\|_{H^{-1}}$ can be made arbitrarily small, $\|u(t)-g\|_{H^1}$ is bounded independently of t and $\|u(t)-g\|_{H^s}$ is already known to go to zero with t. So by a standard result on weak topology, the strong convergence of $u(t)$ to g in H^1 will be proved if we can show that

$$\limsup_{t \searrow 0} \|u(t)\|_{H^1} \;\leq\; \|g\|_{H^1}.$$

And this inequality is a (tricky) consequence of (2.1.13) applied to u_ε, or more precisely of the refined version

$$\|u_\varepsilon(t)\|_{1,v_\varepsilon(t)} \;\leq\; e^{\gamma t}\, \|u_\varepsilon(0)\|_{1,v_\varepsilon}, \tag{2.1.18}$$

where we have defined a modified, equivalent norm on H^1 by

$$\|u\|_{1,v}^2 = \|\sqrt{S_0(v)}\,u\|_{L^2} + \sum_\alpha \|\sqrt{S_0(v)}\,\partial_\alpha u\|_{L^2}^2,$$

with S_0 a Friedrichs symmetrizer of the operator. We postpone the proof of (2.1.18) for a moment, and complete the proof of Theorem 2.9. Observe that $(u_\varepsilon(t))$ also converges to $u(t)$ uniformly on $[0,T]$ in H^1_w. Indeed, for all ϕ in H^{-1} and $\psi \in H^{-s}$ with $s \in (0,1)$, we have

$$\sup_{t\in[0,T]} |\langle \phi, (u_\varepsilon - u)(t)\rangle_{H^{-1},H^1}| \le \|u_\varepsilon - u\|_{\mathscr{C}([0,T];H^1(\mathbb{R}^d))} \|\phi - \psi\|_{H^{-1}} + \|\psi\|_{H^{-s}} \|u_\varepsilon - u\|_{\mathscr{C}([0,T];H^s(\mathbb{R}^d))},$$

where the first term in the right-hand side can be made arbitrarily small, and the second term is already known to tend to 0. Therefore, using the convergence of v_ε in $\mathscr{C}([0,T];L^\infty(\mathbb{R}^d))$, we get by passing to the limit in (2.1.18),

$$\sup_{\tau\in[0,t]} \|u(t)\|_{1,v(t)} \le \limsup_{\varepsilon\searrow 0} \sup_{\tau\in[0,t]} \|u_\varepsilon(t)\|_{1,v_\varepsilon} \le \mathrm{e}^{\gamma t}\, \|g\|_{1,v(0)},$$

hence

$$\limsup_{t\searrow 0} \|u(t)\|_{1,v(t)} \le \|g\|_{1,v(0)}.$$

This shows that $u(t)$ does go to g strongly in H^1 when t goes to zero. In conclusion, up to manipulating the time as explained above, this implies u belongs to $\mathscr{C}([0,T];H^1(\mathbb{R}^d))$, and by passing to the limit in $L_{v_\varepsilon} u_\varepsilon = 0$ in the sense of distributions, $L_v u = 0$, which implies u is also in $\mathscr{C}^1([0,T];L^2(\mathbb{R}^d))$. □

Proof of the inequality in (2.1.18) We omit the subscript ε. Revisiting the proof of (2.1.13) (in the case $L_v u = 0$) and differentiating the equation

$$\partial_t u + \sum_\alpha A^\alpha(v)\,\partial_\alpha u = 0$$

before multiplying by $S_0(v)$, we get

$$\partial_t\partial_\beta u + \sum_\alpha A^\alpha(v)\,\partial_\alpha\partial_\beta u = \sum_\alpha [\partial_\beta\,,\, A^\alpha(v)\partial_\alpha]\,u,$$

where

$$\|[\partial_\beta\,,\, A^\alpha(v)\partial_\alpha]\,u(t)\|_{L^2(\mathbb{R}^d)} \le \|A^\alpha(v(t))\|_{W^{1,\infty}(\mathbb{R}^d)}\, \|u(t)\|_{H^1(\mathbb{R}^d)}.$$

Now we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|^2_{1,v(t)} = \langle R_v u, u\rangle \;+\; \sum_\beta \langle R_v\partial_\beta u, \partial_\beta u\rangle$$
$$+\,2\,\mathrm{Re}\,\sum_{\alpha,\beta}\langle S_0(v)\partial_\beta u\,,\,[\partial_\beta\,,\,A^\alpha(v)\partial_\alpha]\,u\,\rangle,$$

where

$$R_v \;:=\; \partial_t S_0(v) \;+\; \sum_\alpha \partial_\alpha(S_0(v)\,A^\alpha(v)),$$

and infer by the Cauchy–Schwarz inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|u(t)\|_{1,v(t)} \;\leq\; \gamma\,\|u(t)\|_{1,v(t)},$$

with

$$\gamma \;:=\; \frac{1}{2}\,\|S_0(v)^{-1}\|_{L^\infty(\mathbb{R}^d\times[0,T])}\,\big(\|R_v\|_{L^\infty(\mathbb{R}^d\times[0,T])} \;+\; \sum_\alpha \|A^\alpha(v(t))\|_{W^{1,\infty}(\mathbb{R}^d)}\,\big).$$

□

2.2 Local uniqueness and finite-speed propagation

Most of the definitions introduced in Chapter 1 extend to the variable-coefficient systems by 'freezing' the coefficients. We already used the notions of Friedrichs-symmetrizable systems (Definition 2.1) and constantly hyperbolic systems (in Theorem 2.3) as a generalization of Definition 1.2. In view of Definition 1.5, we can also associate to system (2.0.1) a *characteristic cone*

$$\mathrm{char}(x,t) \;:=\; \{\,(\xi,\lambda)\in\mathbb{R}^d\times\mathbb{R}\,;\,\det(A(x,t,\xi)+\lambda I_n)=0\,\}$$

at each point $(x,t)\in\mathbb{R}^d\times\mathbb{R}^+$, and the corresponding *forward cone* $\Gamma(x,t)$, which is the connected component of $(0,1)$ in $(\mathbb{R}^d\times\mathbb{R})\setminus\mathrm{char}(x,t)$. Additionally, for a symmetrizable system with symbolic symmetrizer $S(x,t,\xi)$, we can also define

$$\Upsilon(x,t) \;=\; \{\,(\xi,\tau)\in\mathbb{R}^d\times\mathbb{R}\,;S(x,t,\xi)\,(\,\tau\,I_n \;+\; A(x,t,\xi)\,)\;>\;0\,\}, \qquad (2.2.19)$$

where positivity is to be understood in the sense of Hermitian matrices. Observe that the set $\Upsilon(x,t)$ is still an open cone due to the homogeneity degree 0 of S in ξ. If the operator L is constantly hyperbolic, then $\Gamma(x,t)$ is the set of (ξ,λ) such that all the roots τ of the equation

$$\det\,(\,A(x,t,\xi)+(\lambda+\tau)I_n\,)\;=\;0$$

are strictly negative, that is,

$$\Gamma(x,t)\;=\;\{\,(\xi,\lambda)\,;\;\mathrm{Sp}(\,\lambda\,I_n\;+\;A(x,t,\xi))\subset(0,+\infty)\,\}.$$

We thus have, for a system both Friedrichs symmetrizable and constantly hyperbolic,

$$\Gamma(x,t) = \{ (\xi,\lambda) ; \; S(x,t,\xi) (\lambda I_n + A(x,t,\xi)) > 0 \} = \Upsilon(x,t)$$

for any symmetrizer $S(x,t,\xi)$.

Definition 2.5 *Let $\mathcal{H}$ be a smooth hypersurface in $\mathbb{R}^d \times \mathbb{R}^+$. Denoting by $\vec{n}$ the normal vector to $\mathcal{H}$, we say that $\mathcal{H}$ is*

i) characteristic *at point (x,t) if $\vec{n}(x,t) \in \mathrm{char}(x,t)$,*
ii) space-like *at point (x,t) if $\vec{n}(x,t) \in -\Gamma(x,t) \cup \Gamma(x,t)$.*

If i), or respectively ii), holds for all (x,t), $\mathcal{H}$ is simply said to be characteristic*, or* space-like*, respectively.*

By definition of Γ, a space-like surface is, of course, not characteristic. And the most natural example of a space-like surface is $\{ t = 0 \}$! The interest of space-like surfaces is that they are associated with local uniqueness results.

Theorem 2.10

i) Assuming that the system (2.0.1) *is* constantly hyperbolic *and $f \equiv 0$, let $\mathcal{H}$ be a space-like hypersurface at (x_0,t_0). Then there exists a neighbourhood $\mathscr{N}$ of (x_0,t_0) such that, if u is a $\mathscr{C}^1$ solution of* (2.0.1) *in $\mathscr{N}$ and $u|_{\mathcal{H}\cap\mathscr{N}} \equiv 0$ then $u|_{\mathscr{N}} \equiv 0$. (The reader may refer to Figure 2.1.)*

ii) Assuming that the system (2.0.1) *is Friedrichs symmetrizable and $f \equiv 0$, let $\mathcal{L}$ be a* lens *made of two space-like surfaces, $\mathcal{H}$ and $\mathscr{K}$, sharing the*

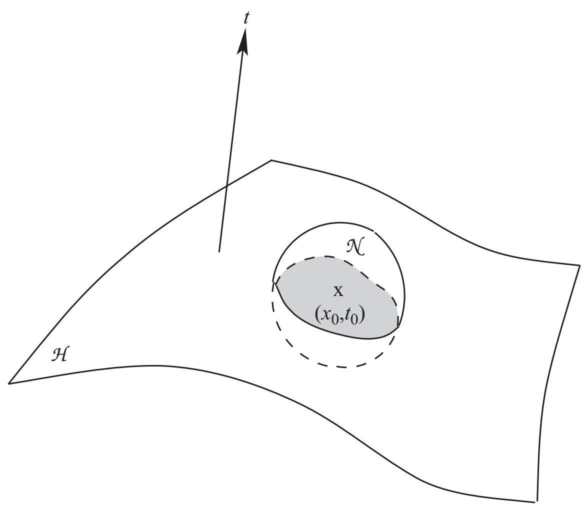

Figure 2.1: Illustration of local uniqueness for constantly hyperbolic systems

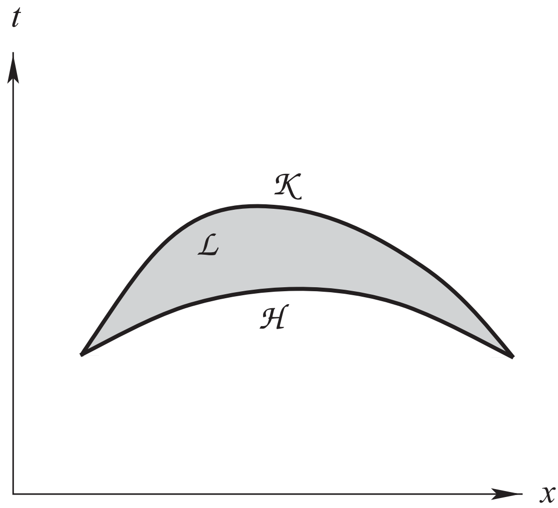

Figure 2.2: Illustration of local uniqueness for Friedrichs-symmetrizable systems

same boundary. If u is a $\mathscr{C}^1$ solution of (2.0.1) *in $\mathcal{L}$ and if $u \equiv 0$ on $\mathcal{H}$, for instance, then $u \equiv 0$ also on $\mathscr{K}$. (The reader may refer to Figure 2.2.)*

Proof We begin with *ii)*, the proof of which is more elementary. It is analogous to the localized computations performed in Sections 1.3.1 and 1.3.3. We denote by S_0 a Friedrichs symmetrizer of (2.0.1). Multiplying (2.0.1) by $u^* S_0$ we get

$$\int_{\mathcal{L}} \partial_t (S_0 u, u) \, + \, \int_{\mathcal{L}} \textstyle\sum_\alpha \partial_\alpha (S_0 A^\alpha u, u) \, = \, \int_{\mathcal{L}} (Ru, u),$$

where $R := \partial_t S_0 + \sum_\alpha \partial_\alpha (S_0 A^\alpha) + S_0 B + B^{\mathrm{T}} S_0$ (like in Proposition 2.2). Integrating the left-hand side yields the equality

$$\int_{\mathscr{K}} n_0 \, (S_0 u, u) \, + \, \int_{\mathscr{K}} \textstyle\sum_\alpha n_\alpha (S_0 A^\alpha u, u)$$

$$= \int_{\mathcal{H}} n_0 \, (S_0 u, u) \, + \, \int_{\mathcal{H}} \textstyle\sum_\alpha n_\alpha (S_0 A^\alpha u, u) \, + \, \int_{\mathcal{L}} (Ru, u) \, ,$$

where n_0 denotes the t-component and n_α the x_α-component of $\vec{n}$. Since $\vec{n}$ belongs to $-\Upsilon \cup \Upsilon$, the matrix $S_0 \, (\, n_0 \, I_n \, + \, \sum_\alpha n_\alpha A^\alpha \,)$ is definite, with the same sign at all points (x, t) of $\partial \mathcal{L}$ by continuity. Assume, for instance, that it is positive. Then there exists $\gamma > 0$ such that

$$S_0 \, (\, n_0 \, I_n \, + \, \textstyle\sum_\alpha n_\alpha A^\alpha \,) \, \geq \, \gamma \, I_n$$

in $\partial\mathcal{L}$. Hence we have

$$\gamma \int_{\mathscr{K}} \|u\|^2 \leq C \left(\int_{\mathcal{H}} \|u\|^2 + \int_{\mathcal{L}} \|u\|^2 \right),$$

where $C = \max_{\mathcal{L}}(\|S_0\| + \sum_\alpha \|S_0 A^\alpha\|, \|R\|)$. By the multidimensional Gronwall's Lemma (see Lemma A.3 in Appendix A) we conclude that there exists C' (independent of u) so that

$$\int_{\mathscr{K}} \|u\|^2 \leq C' \int_{\mathcal{H}} \|u\|^2.$$

If $u|_{\mathcal{H}} \equiv 0$ then clearly $u|_{\mathscr{K}} \equiv 0$. □

Proof of *i)* The proof relies on a change of variables, transforming $\mathcal{H}$ into the hyperplane $\{t = 0\}$ and preserving constant hyperbolicity, and on the Holmgrem principle applied to the new (Cauchy) problem. We consider a local diffeomorphism χ such that $\chi(x_0, t_0) = (0, 0)$ and

$$\widetilde{\mathcal{H}} := \chi(\mathcal{H}) = \{ (\widetilde{x}, \widetilde{t}) \, ; \, \widetilde{t} = 0 \}.$$

In particular, $\nabla_{(x,t)}\widetilde{t}$ is parallel to $\vec{n}$. We can even choose χ so that

$$\partial_t \widetilde{t} = n_0 , \quad \partial_\alpha \widetilde{t} = n_\alpha .$$

The transformed operator $\widetilde{L}$ under χ is defined by

$$(\widetilde{L}v)(\widetilde{x}, \widetilde{t}) = L(v \circ \chi)(x, t).$$

More specifically, it reads

$$\widetilde{L} = \widetilde{A}^0 \partial_{\widetilde{t}} + \sum_\alpha \widetilde{A}^\alpha \partial_{\widetilde{x}_\alpha} - \widetilde{B},$$

where

$$\widetilde{A}^0(\widetilde{x}, \widetilde{t}) = n_0 \, I_n + \sum_\beta n_\beta A^\beta(x, t) , \quad \widetilde{A}^\alpha(\widetilde{x}, \widetilde{t}) = \partial_t \widetilde{x}_\alpha \, I_n + \sum_\beta (\partial_{x_\beta} \widetilde{x}_\alpha) A^\beta(x, t)$$

and $\widetilde{B}(\widetilde{x}, \widetilde{t}) = B(x, t)$. Note that $\widetilde{A}^0(0, 0)$ is an invertible matrix since $\vec{n}(x_0, t_0) \notin \mathrm{char}(x_0, t_0)$. Furthermore, we have

$$\widetilde{\tau} \, \widetilde{A}^0(\widetilde{x}, \widetilde{t}) + \widetilde{A}(\widetilde{x}, \widetilde{t}, \widetilde{\xi}) = (\, \widetilde{\tau} \, n_0 + \lambda \,) \, I_n + A(x, t, \xi + \widetilde{\tau} \, \nu), \tag{2.2.20}$$

where $\nu = (n_1, \cdots, n_d)$ and

$$\lambda = \widetilde{\xi} \cdot \partial_t \widetilde{x} , \quad \xi = \sum_\beta \widetilde{\xi}_\beta \, \nabla_x \widetilde{x}_\beta .$$

Observing that

$$(\xi, \lambda) = (\widetilde{\xi}, 0) \, \mathrm{d}\chi(x, t) \quad \text{and} \quad \vec{n} = (\nu, n_0) = (0, 1) \, \mathrm{d}\chi(x, t),$$

we see that (ξ, λ) and (ν, n_0) are not parallel, unless $\widetilde{\xi} = 0$. By assumption, either $\vec{n}(x_0, t_0)$ or $-\vec{n}(x_0, t_0)$ belongs to $\Gamma(x_0, t_0)$. Then we know from Theorem 1.5 of Chapter 1 that the operator is constantly hyperbolic in the direction $\vec{n}$, which means that for all (ξ, λ) not parallel to $\vec{n} = (\nu, n_0)$, the roots σ of the polynomial

$$\det(A(x_0, t_0, \xi + \sigma\nu) + (\lambda + \sigma n_0) I_n)$$

are real with constant multiplicities. Because of (2.2.20) this shows that the roots $\widetilde{\tau}$ of

$$\det(\,\widetilde{\tau}\,\widetilde{A}^0(0, 0) \;+\; \widetilde{A}(0, 0, \widetilde{\xi}, \widetilde{\tau})\,)$$

are real with constant multiplicities for $\widetilde{\xi} \neq 0$. This means that the transformed operator $\widetilde{L}$ is constantly hyperbolic at point $(0, 0)$, and thus also in the neighbourhood of $(0, 0)$. Up to replacing $\widetilde{A}^\alpha(x, t)$ by $\widetilde{A}^\alpha(\,\theta(\|(x, t)\|)\,(x, t)\,)$ for all $\alpha \in \{0, \cdots, d\}$, where θ is a smooth cut-off function, we can assume that $\widetilde{L}$ is globally defined, constantly hyperbolic, and has constant coefficients outside some bounded ball. So we are led to show the result for $\widetilde{L}$ instead of L, and $\widetilde{\mathcal{H}} := \{\,(\widetilde{x}, \widetilde{t})\,;\; \widetilde{t} = 0\,\}$ instead of $\mathcal{H}$.

From now on we drop the tildas. The (hyperbolic system associated with) operator L^* meets the assumptions of Theorem 2.3. Thus Theorem 2.6 applies to L^*. This will enable us to apply the Holmgren principle to L.

We must show the existence of a neighbourhood $\mathscr{N}$ of $(0, 0)$ such that if u is a $\mathscr{C}^1$ solution of $Lu = 0$ in $\mathscr{N}$ and $u(x, 0) = 0$ for $(x, 0) \in \mathscr{N}$ then $u(x, t) = 0$ for all $(x, t) \in \mathscr{N}$. Without loss of generality, we can consider a conical neighbourhood $\mathscr{N}$, foliated by the hypersurfaces with boundary

$$\begin{aligned}\mathcal{H}_\theta \;:=\; \{\,(x, t)\,;\;& \theta^3\,\|x\|^2 \;-\; V^2\,(\,t \;-\; \theta\,T\,)^2 \;+\; \theta^2\,V^2\,T^2\,(1 - \theta) \;=\; 0\,,\\ & 0 \leq t \leq \theta\,T\,,\; \|x\| \leq V\,T\,\}\,, \quad \theta \in [0, 1)\,.\end{aligned}$$

(The reader may refer to Fig. 2.3.) Choosing

$$V \;=\; \max\,\{\,|\lambda_j(x, t, \xi)|\,;\; \|\xi\| = 1\,\},$$

where $\lambda_j(x, t, \xi)$ denote, as usual, the eigenvalues of $A(x, t, \xi)$ (recall that $A(x, t, \xi)$ has been modified to be independent of (x, t) outside a bounded ball), it is not difficult to show that all the hypersurfaces $\mathcal{H}_\theta$ are space-like, as $\mathcal{H}_0 \subset \mathcal{H}$. To be precise, one can compute that a unit normal vector to $\mathcal{H}_\theta$ at point (x, t) reads $\vec{n} = (\nu, n_0) = \vec{N}/\|\vec{N}\|$ where $\vec{N} \;:=\; (\,\theta^3\,x\,,\; V^2(\theta T - t)\,)$. And thus the matrix $(A(x, t, \nu) + n_0 I_n)$ has eigenvalues

$$\mu_j \;=\; \frac{1}{\|\vec{N}\|}\;\left(\,V^2(\theta T - t) + \lambda_j(x, t, \theta^3 x)\,\right).$$

By definition of V, we have

$$|\lambda_j(x, t, \theta^3 x)| \;\leq\; V\,\theta^3\,\|x\|,$$

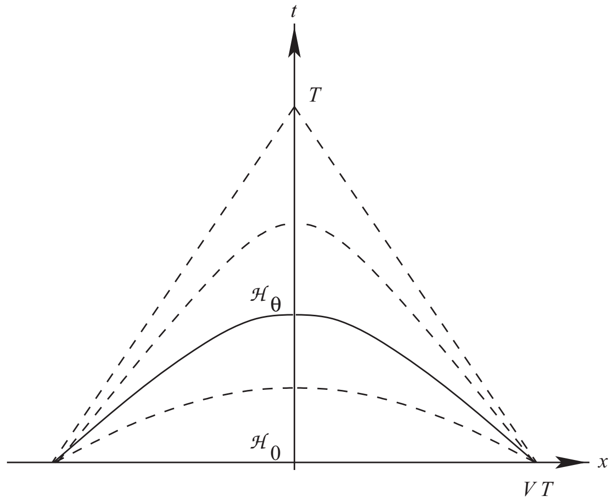

Figure 2.3: Foliation of a conical neighbourhood $\mathscr{N}$ of $(0,0)$

and thus

$$\left|\frac{\lambda_j(x,t,\theta^3 x)}{V^2(\theta T - t)}\right| \leq \frac{\theta^3\,\|x\|}{\sqrt{\theta^3\,\|x\|^2 + \theta^2\,V^2\,T^2\,(1-\theta)}} < 1 \text{ on } \mathcal{H}_\theta.$$

This proves that $\mu_j > 0$ and thus $\vec{n}$ belongs to $-\Gamma$. Therefore, $\mathcal{H}_\theta$ is space-like.

Now, similarly as in the first part of the proof, we can transform by change of variable the problem

$$\begin{cases} L^*\varphi = 0\,, \\ \varphi|_{\mathcal{H}_\theta} = g \end{cases} \tag{2.2.21}$$

for $g \in \mathscr{D}(\mathcal{H}_\theta)$ into a standard Cauchy problem

$$\begin{cases} \widetilde{L^*}\widetilde{\varphi} = 0\,, \\ \widetilde{\varphi}|_{\widetilde{t}=0} = \widetilde{g}\,, \end{cases}$$

with $\widetilde{L^*}$ being constantly hyperbolic and having constant coefficients outside a bounded ball. So by Theorems 2.3 and 2.6 the problem (2.2.21) admits a unique smooth solution φ. Now, denoting by $\mathcal{L}_\theta$ the lens lying in between $\mathcal{H}_\theta$ and $\mathcal{H}_0$, we have for all $\mathscr{C}^1$ solution u of $Lu = 0$ in $\mathscr{N}$

$$0 = \int_{\mathcal{L}_\theta} (\varphi, Lu) - \int_{\mathcal{L}_\theta} (L^*\varphi, u) = \int_{\mathcal{H}_\theta} (g, (n_0 + \textstyle\sum_\alpha n_\alpha\, A^\alpha\,)\, u) - \displaystyle\int_{\mathcal{H}_0} (\varphi, u),$$

where $\vec{n} = (n_1, \cdots, n_d, n_0)$ denotes the normal vector to $\mathcal{H}_\theta$. Thus if $u(x,0) = 0$ for all $(x,0) \in \mathscr{N}$ we see that

$$\int_{\mathcal{H}_\theta} (\, g, (n_0 + \textstyle\sum_\alpha n_\alpha\, A^\alpha\,)\, u\,) \;=\; 0.$$

Since this holds for any g this implies that

$$(n_0 + \textstyle\sum_\alpha n_\alpha\, A^\alpha\,)\, u \;\equiv\; 0$$

on $\mathcal{H}_\theta$. But the matrix $(n_0 + \sum_\alpha n_\alpha\, A^\alpha\,)$ is known to be invertible since $\vec{n}$ does not belong to the characteristic cone. Finally, this shows that $u \equiv 0$ on $\mathcal{H}_\theta$ for all θ. □

Theorem 2.10 has a counterpart/consequence in terms of *finite-speed propagation*, which is as follows.

Theorem 2.11

i) We assume that the system (2.0.1) *is* constantly hyperbolic *and has constant coefficients outside a compact set. We also take $f \equiv 0$, and set*

$$V \;=\; \max\,\{\, |\lambda_j(x,t,\xi)|\,;\; \|\xi\| = 1\,\},$$

with $\lambda_j(x,t,\xi)$ the eigenvalues of $A(x,t,\xi)$. To any point $(X,T) \in \mathbb{R}^d \times (0,+\infty)$ we associate the conical set

$$C \;:=\; \bigcup_{0<t<T} \Omega(t) \times \{t\} \qquad \textit{with } \Omega(t) \;:=\; \{\, x\,;\; \|x - X\| < V(T-t)\,\},$$
$$0 \le t < T.$$

If $u \in \mathscr{C}^1([0,T]; H^s)$ is a (weak) solution of (2.0.1) *(with $s \in \mathbb{R}$) such that $u|_{\Omega(0)} = 0$, then $u|_C = 0$.*

ii) Assuming that the system (2.0.1) *is Friedrichs symmetrizable (and still $f \equiv 0$), the same result holds on changing the balls $\Omega(t)$ to the convex sets*

$$\Omega(t) \;:=\; \bigcap_{\|\xi\|=1} \{\, x\,; v(\xi)(t-T) + \xi\cdot(x - X) \;\le\; 0\,\},$$

where

$$v(\xi) \;:=\; \max_{(x,t)} \max\,\{\, v \in \mathbb{R}\,;\; (\xi, v) \in \mathrm{char}(x,t)\,\}.$$

Proof *i)* We first show the result for $u \in \mathscr{C}^1(\mathbb{R}^d \times [0,T])$ a smooth solution. The proof is based on a connectedness argument using Theorem 2.10 *i)*. We take $\varepsilon \in (0,T)$ and define $T_\varepsilon = T - \varepsilon$,

$$C_\varepsilon \;:=\; \{\, x\,;\; 0 < t < T_\varepsilon\,,\; \|x - X\| < V(T_\varepsilon - t)\,\}.$$

Similarly as in the proof of Theorem 2.10 *i)*, we can construct space-like surfaces $\mathcal{H}_\theta^\varepsilon$, depending smoothly on θ, such that

$$C_\varepsilon \,=\, \bigcup_{\theta\in(0,1)} \mathcal{H}_\theta^\varepsilon \quad \text{and} \quad \mathcal{H}_0^\varepsilon \subset \Omega(0).$$

Then, if $u \in \mathscr{C}^1(\mathbb{R}^d \times [0,T])$ is the solution of (2.0.1) and $u|_{\Omega(0)} = 0$, the set

$$\Theta \,:=\, \{\, \theta \in [0,1)\,;\; u|_{\mathcal{H}_\theta} \equiv 0 \,\}$$

of course contains 0, and is a closed subset of $[0,1)$ by a continuity argument. But, covering the compact sets $\mathcal{H}_\theta$ by a finite number of neighbourhoods $\mathscr{N}$, Theorem 2.10 *i)* also shows that Θ is open. Therefore, we have $\Theta \,=\, [0,1)$, which means that $u \equiv 0$ on $C_\varepsilon \,=\, \bigcup_{0<t<T_\varepsilon} \Omega(t) \times \{t\}$.

In general, we can proceed by regularization. Setting $f \,:=\, Lu$ and $g \,:=\, u|_{t=0}$, we have by assumption that $f = 0$ in $\bigcup_{0<t<T} \Omega(t)$ and $g = 0$ in $\Omega(0)$. Introducing a mollifier ρ_k in $\mathbb{R}^d$, the Cauchy problem

$$Lv \,=\, f * \rho_k \quad , \quad v|_{t=0} \,=\, g * \rho_k,$$

has a unique solution $v = u_k$, which is in $\mathscr{C}^1(\mathbb{R}^d \times [0,T])$ by Theorem 2.6. For k large enough, we have $f * \rho_k = 0$ in C_ε and $g * \rho_k = 0$ in $\mathcal{H}_0^\varepsilon$ (with the same notations as before). We thus infer that, for k large enough, $u_k = 0$ in C_ε by the first part of the proof. But the energy estimate in (2.1.14) shows that

$$u_k \xrightarrow[n\to\infty]{\mathscr{C}([0,T];H^s)} u,$$

since

$$Lu_k \,=\, f * \rho_k \xrightarrow[k\to\infty]{\mathscr{C}([0,T];H^s)} f \,=\, Lu \quad \text{and } u_k|_{t=0} \,=\, g * \rho_k \xrightarrow[n\to\infty]{H^s} g \,=\, u|_{t=0}.$$

Passing to the limit, we conclude that $u = 0$ in C_ε.

ii) The proof is a natural generalization of Section 1.3.1. It is roughly the same as in Theorem 2.10 *ii)*, except that we are going to consider lenses $\mathcal{L}$ that are only *weakly space-like*. Let us consider

$$\mathcal{L}_\varepsilon \,:=\, \bigcup_{0\le s\le T-\varepsilon} \Omega(s) \times \{s\}.$$

The boundary of $\mathcal{L}_\varepsilon$ is made of three parts, the bottom $\mathcal{T} \,=\, \Omega(0) \times \{0\}$, the top $\mathcal{T}_\varepsilon \,=\, \Omega(T-\varepsilon) \times \{T-\varepsilon\}$, and the side $\mathcal{S}$. The top and bottom are obviously space-like. As regards the side, for all $(y,s) \in \mathcal{S}$, there exists ξ_0 such that

$$\{\,(x,t)\,; v(\xi_0)(t-T) + \xi_0 \cdot (x - X) \,=\, 0\,\}$$

is a hyperplane of support for $\mathcal{L}_\varepsilon$ at point (y,s). A normal vector to $\mathcal{L}_\varepsilon$ at point (y,s) is necessarily of the form $\vec{n} = (\xi_0, v_0; = v(\xi_0))$. By definition of $v(\xi)$, we know that v_0 is not smaller than any v such that the matrix $vI_n + A(y,s,\xi_0)$ is singular. Therefore, all the eigenvalues of $v_0 I_n + A(y,s,\xi_0)$ are non-negative.

This implies in particular that the symmetric matrix $S_0(y,s)\,(v_0 I_n + A(y,s,\xi_0))$ is non-negative. Therefore, integrating by parts the identity

$$\int_{\mathcal{L}_\varepsilon} \partial_t(S_0 u, u) + \int_{\mathcal{L}_\varepsilon} \textstyle\sum_\alpha \partial_\alpha(S_0 A^\alpha u, u) = \int_{\mathcal{L}_\varepsilon} (Ru, u),$$

(with the same notations as in the proof of Theorem 2.10*ii)*), we get

$$0 \geq \int_{\mathcal{T}_\varepsilon} (S_0 u, u) + \int_{\mathcal{L}_\varepsilon} (Ru, u),$$

if u vanishes on the bottom. Then by the Cauchy–Schwarz inequality and the definiteness of S_0, there exists $\alpha > 0$ such that

$$\int_{\mathcal{T}_\varepsilon} \|u\|^2 \leq \alpha \int_{\mathcal{L}_\varepsilon} \|u\|^2.$$

Since this holds for all $\varepsilon \in (0, T]$, we can conclude by the Gronwall Lemma. □

2.3 Non-decaying infinitely smooth data

As a consequence of Theorem 2.11, we have that, under either one of the assumptions, the Cauchy problem can be uniquely solved for any smooth data, without any assumption on their far-field behaviour.

Theorem 2.12 *We assume that the system* (2.0.1) *is either* constantly hyperbolic *with constant coefficients outside a compact set, or Friedrichs symmetrizable. Then, for $f \in \mathscr{C}^\infty(\mathbb{R}^d \times [0,T])$ and $g \in \mathscr{C}^\infty(\mathbb{R}^d)$, there exists a unique $u \in \mathscr{C}^\infty(\mathbb{R}^d \times [0,T])$ such that*

$$Lu = f \quad \text{and} \quad u|_{t=0} = g.$$

Proof Uniqueness directly follows from Theorem 2.11. Existence is based on truncation of f and g. We take a smooth cut-off function θ and define

$$f_k(x,t) = \theta\left(\frac{\|x\|}{k}\right) f(x,t)\,, \quad g_k(x,t) = \theta\left(\frac{\|x\|}{k}\right) g(x,t)$$

for all positive integers k. To fix the ideas, we assume that $\theta \equiv 1$ on the unit ball. By Theorem 2.6, there exists a unique solution $u_k \in \mathscr{C}^\infty([0,T]; H^{+\infty})$ of the Cauchy problem

$$Lu_k = f_k, \quad u_k|_{t=0} = g_k.$$

Furthermore, Theorem 2.11 shows that u_k is compactly supported, and that u_k coincides with u_m for all $m \geq k > VT$ on the cylinder $\{\, \|x\| \leq k - VT \,\} \times [0,T]$. Therefore, the sequence (u_k) is convergent in $\mathscr{C}^\infty(\mathbb{R}^d \times [0,T])$ and the limit solves the Cauchy problem

$$Lu = f, \quad u|_{t=0} = g.$$

□

2.4 Weighted in time estimates

We conclude this chapter by an estimate that will be useful in Chapter 9. It does not actually deal with the Cauchy problem. It gives global in time exponentially weighted estimates. The important feature of these estimates is that the weight depends on a parameter, denoted by γ, which has to be chosen large enough for the estimate to hold. This gives the flavour of the machinery that will be used in Chapter 9 for the resolution of Initial Boundary Value Problems.

Theorem 2.13 *We assume that the system* (2.0.1) *is symmetrizable with uniform bounds on* $\Sigma(t)$ *(including the lower bound* $\alpha > 0$ *in* (2.1.6)*) and on* $\mathrm{d}\Sigma/\mathrm{d}t$ *for* $t \in \mathbb{R}$*. We also assume that the matrix* B *is uniformly bounded on* $\mathbb{R}^{d+1}$*. Then there exist* $C > 0$ *and* $\gamma_0 > 0$ *so that for* $u \in \mathscr{D}(\mathbb{R}^{d+1})$ *and* $\gamma \geq \gamma_0$ *we have*

$$\gamma \left(\int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|u(t)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}t \right)^{1/2} \; \leq \; C \; \left(\int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|Lu(t)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}t \right)^{1/2} . \tag{2.4.22}$$

Remark 2.11 According to Theorem 2.2, this result applies in particular to systems admitting a symbolic symmetrizer (see Definition 2.3) with bounds valid for all $t \in \mathbb{R}$.

Proof Elementary calculus yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[\mathrm{e}^{-2\gamma t} \, \langle \, \Sigma \, u \, , \, u \, \rangle \right] \; = \; \mathrm{e}^{-2\gamma t} \, (\, 2 \, \mathrm{Re} \; \langle \, \Sigma \, L \, u \, , \, u \, \rangle \; + \; 2 \, \mathrm{Re} \; \langle \, \Sigma \, P \, u \, , \, u \, \rangle$$
$$+ \, \langle \, \frac{\mathrm{d}\Sigma}{\mathrm{d}t} \, u \, , \, u \, \rangle \; - \; 2\gamma \, \langle \, \Sigma \, u \, , \, u \, \rangle \,)$$

for all γ. Then, integrating on $\mathbb{R}$ and estimating all terms by the Cauchy–Schwarz inequality, we obtain a constant C_1 (depending on α and bounds for Σ, $\mathrm{d}\Sigma/\mathrm{d}t$ and $\mathrm{Re}\;(\Sigma P)$) such that

$$\gamma \int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|u(t)\|^2_{L^2(\mathbb{R}^d))} \, \mathrm{d}t \; \leq \; C_1 \int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|u(t)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}t$$
$$+ \, C_1 \left(\int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|u(t)\|^2_{L^2(\mathbb{R}^d))} \, \mathrm{d}t \right)^{1/2} \left(\int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|Lu(t)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}t \right)^{1/2}$$

for all $\gamma > 0$. This implies that

$$\gamma \int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|u(t)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}t \leq C_1 \left((1 + \varepsilon\gamma) \int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|u(t)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}t \right.$$
$$\left. + \, \frac{1}{4\varepsilon\gamma} \int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \, \|Lu(t)\|^2_{L^2(\mathbb{R}^d)} \, \mathrm{d}t \right)$$

for any $\varepsilon > 0$. Choosing for instance $\varepsilon = 1/(3C_1)$ yields the desired estimate with $\gamma_0 = 3C_1$ and $C \; = \; 3C_1/2$. □

This page intentionally left blank

PART II

THE LINEAR INITIAL BOUNDARY VALUE PROBLEM

This page intentionally left blank

3

FRIEDRICHS-SYMMETRIC DISSIPATIVE IBVPs

We begin the analysis of the Initial Boundary Value Problem (IBVP). Although this book is devoted to general hyperbolic operators, the study of Friedrichs-symmetric ones with dissipative boundary conditions allows us to uncover crucial concepts and methods. In particular, we shall see in Chapter 4 that a suitable tool for proving strong well-posedness is a symbolic symmetrizer for which the boundary condition is strongly dissipative. This motivates us to devote a full chapter to the IBVP for a Friedrichs-symmetrizable operator, when the boundary condition is dissipative in a classical sense. Then the symmetrizer is classical, instead of symbolic. Since it is given with the system, we do not have to build it.

Of course, we could develop a full theory of dissipative IBVPs in the Friedrichs sense, with variable coefficients and general domains. But since a full account of the theory of IBVPs will be given in the next chapters, we may restrict ourselves to the simplest possible situation. Namely, our operators have constant coefficients, and the physical domain is the half-space

$$\Omega := \{x \in \mathbb{R}^d\,;\, x_d > 0\}.$$

We shall frequently use the notation $x = (y, x_d)$, where $y \in \mathbb{R}^{d-1}$ are the coordinates along the boundary. Frequency vectors are also split into $\xi = (\eta, \xi_d)$, with $\eta \in \mathbb{R}^{d-1}$.

3.1 The weakly dissipative case

Let $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be a Friedrichs-symmetric operator, meaning that the matrices A^α are symmetric.

A general IBVP takes the form

$$Lu = f, \quad x \in \Omega,\, t > 0, \tag{3.1.1}$$

$$Bu = g \quad x_d = 0,\, t > 0, \tag{3.1.2}$$

$$u = u_0, \quad x \in \Omega,\, t = 0, \tag{3.1.3}$$

where $B \in \mathbf{M}_{p\times n}(\mathbb{R})$ and A^α have constant entries. We shall see in a moment that the number p of scalar boundary conditions must equal that of the positive eigenvalues of A^d. It thus usually differs from n. If $g = 0$, then we speak of the *homogeneous* IBVP. If instead g is an arbitrary vector field of given regularity, the IBVP is *non-homogeneous*.

To begin with, we consider the homogeneous IBVP. Thus let u be a classical solution of (3.1.1)–(3.1.3) with $g \equiv 0$. Let us integrate the energy conservation law (3.1.1) on Ω. Assuming that $u(t), \nabla_x u$ are square-integrable, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u(x,t)|^2 \mathrm{d}x = \int_{\partial\Omega} (A^d u, u)\,\mathrm{d}y + 2\int_\Omega (f,u)\,\mathrm{d}x. \tag{3.1.4}$$

To obtain an a priori estimate from (3.1.4), we need an upper bound of $(A^d v, v)$, knowing the value of Bv. Of course, we cannot use the Gronwall Lemma in order to control the boundary integral, since there is no trace in $L^2(\Omega)$. The existence of such a bound amounts to assuming the (strict) *dissipativeness* of the boundary condition.

Definition 3.1 *We say that the boundary condition (3.1.2) is* dissipative *for the symmetric operator L in the domain Ω defined by $x_d > 0$, if A^d is non-positive on* $\ker B$*:*

$$(v \in \mathbb{R}^n,\ Bv = 0) \Longrightarrow (A^d v, v) \le 0.$$

For a more general domain with smooth boundary, we say that the boundary condition (3.1.2) is dissipative *for the Friedrichs-symmetric operator L, if $A(\nu)$ is non-negative on* $\ker B$ *at every boundary point $x \in \partial\Omega$, ν being the outward unit normal to $\partial\Omega$ at x (in that case, B often depends on $x \in \partial\Omega$ itself, through ν):*

$$Bv = 0 \Longrightarrow (A(\nu)v, v) \ge 0.$$

We say that $A(\nu)$ is the normal matrix.

For a reason that will become clear in Chapter 4, we shall only consider maximal dissipative boundary conditions:

Definition 3.2 *We say that the boundary condition (3.1.2) is maximal* dissipative *if it is dissipative, and if* $\ker B$ *is not a proper subspace of some linear subspace on which A^d is non-positive.*

This definition generalizes in an obvious way to general domains.

Lemma 3.1 *Assume that B is maximal dissipative for L. Then* $\ker A^d \subset \ker B$.

Proof Given $u \in \ker A^d$, let us define $N := \mathbb{R}u + \ker B$. For $v \in \ker B$, we have

$$(A^d(u+v), u+v) = (A^d v, u+v) = (v, A^d(u+v)) = (v, A^d v),$$

which is non-positive by assumption. Since $\ker B \subset N$, we must have $N = \ker B$ by maximality. In other words, $u \in \ker B$. □

Assuming that the boundary condition is maximal dissipative, we shall solve the *homogeneous* IBVP. This kind of well-posedness, which is qualified as *weak*, or *homogeneous*, in the literature, is expected because of the following consequence of (3.1.4) when $g \equiv 0$

$$\|u(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)} + \int_0^t \|f(s)\|_{L^2(\Omega)} \mathrm{d}s.$$

To prove the weak well-posedness, we remark that the homogeneous IBVP has the abstract form of a differential equation (3.1.5) below, and thus can be treated within the theory of continuous semigroups in a Hilbert space. The space that we consider is $X = L^2(\Omega)$. We shall use a restricted version of the Hille–Yosida theorem (see, for instance, [28, 51]):

Theorem 3.1 *Let X be a Hilbert space, $D(A)$ a linear subspace and $A : D(A) \to X$ be a maximal monotone operator. Then, for every $u_0 \in D(A)$, there exists one and only one $u \in \mathscr{C}([0, +\infty); D(A)) \cap \mathscr{C}^1([0, +\infty); X)$, such that*

$$\begin{cases} \frac{\mathrm{d}u}{\mathrm{d}t} + Au = 0 \text{ on } [0, +\infty), \\ u(0) = u_0. \end{cases} \tag{3.1.5}$$

Moreover, one has

$$\|u(t)\|_X \leq \|u_0\|_X, \quad \forall t \geq 0.$$

We recall that the linear operator A is called *monotone* if $(Au, u) \geq 0$ for all $u \in D(A)$, and *maximal monotone* if, moreover, $I_X + A$ is onto, that is

$$\forall f \in X, \exists u \in D(A) \text{ such that } u + Au = f. \tag{3.1.6}$$

Since A is maximal monotone, $D(A)$ is dense in X and A is closed. The fact that the map $u_0 \mapsto u(t)$, defined on $D(A)$, is non-expanding for the X-norm, allows us to extend it continuously as a bounded operator $S_t \in \mathscr{L}(X)$. The family $(S_t)_{t \geq 0}$ is a continuous semigroup:

$$S_{t+s} = S_t \circ S_s, \quad \lim_{t \to s} S_t u_0 = S_s u_0.$$

Since A will be a differential operator, it will not be bounded in X. Thus we do not expect that the semigroup is continuous in the operator norm (‘uniform continuity’).

The use of the semigroup gives a sense to the well-posedness of the Cauchy problem (3.1.5) in X. We call $u(t) := S_t u_0$ the *unique solution* in $\mathscr{C}(\mathbb{R}^+; X)$ with initial data $u_0 \in X$. Since it is the limit of strong solutions, it is a solution in the distributional sense.

We first consider the case where not only $g \equiv 0$ but also $f \equiv 0$. Then, the IBVP takes the form (3.1.5), where

$$A := -\mathcal{L} = \sum_\alpha A^\alpha \partial_\alpha,$$

and

$$D(A) := \{u \in L^2(\Omega)\,;\, Au \in L^2(\Omega) \text{ and } Bu = 0 \text{ on } \partial\Omega\}.$$

3.1.1 *Traces*

To make the definition of $D(A)$ mathematically correct, we need to explain the meaning of Bu on $\partial\Omega$. We recall that if a vector field $\vec{q} : \Omega \to \mathbb{R}^d$ and its divergence are square-integrable, then $\vec{q}$ admits a uniquely defined normal trace $\gamma_\nu \vec{q} \in H^{-1/2}(\partial\Omega)$, such that the following Green's formula holds

$$\int_\Omega (\vec{q} \cdot \nabla\phi + \phi\, \mathrm{div}\vec{q})\, \mathrm{d}x = \langle \gamma_\nu \vec{q}, \gamma_0 \phi \rangle_{H^{-1/2}, H^{1/2}}, \quad \forall \phi \in H^1(\Omega),$$

$\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ being the well-known trace operator. The trace γ_ν extends continuously (see [208], Theorem 1.2) the map ('normal trace')

$$\vec{q} \mapsto \nu \cdot \vec{q}|_{\partial\Omega}\,,$$

a priori defined for smooth vector fields on Ω, where ν is still the outward unit normal.

Given $u \in L^2(\Omega)^n$, such that $Au \in L^2(\Omega)^n$, and applying γ_ν to each vector field $\vec{q^i}$, defined by

$$q^i_\alpha := (A^\alpha u)_i\,,$$

we obtain a trace $A^d u\big|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$ (recall that $\nu \equiv -\vec{e^d}$ here). Then, because of Lemma 3.1, there exists a matrix $M \in M_{p\times n}$ such that $B = MA^d$. Then the trace of Bu is simply M times the trace of $A^d u$. Therefore $D(A)$ is well-defined.

Several calculations will be made using the Fourier transform in the y-variables, $\mathscr{F}_y$. A vector field $\vec{q}$ as above has the following properties: $(\eta, x_d) \mapsto \mathscr{F}_y q_\alpha$ is square-integrable for every α, as well as $(\eta, x_d) \mapsto \partial_d \mathscr{F}_y q_d + i\sum_1^{d-1} \eta_\alpha \mathscr{F}_y q_\alpha$. It follows that

$$\mathscr{F}_y q_d \in L^2_{loc}(\mathbb{R}^{d-1}; H^1(0, +\infty)).$$

Because of Sobolev embedding, we conclude that

$$\mathscr{F}_y q_d \in L^2_{loc}(\mathbb{R}^{d-1}; \mathscr{C}([0, +\infty)).$$

Then, testing against smooth functions, we find easily that $\mathscr{F}_y \gamma_\nu \vec{q}$, which belongs to $L^2((1+|\eta|^2)^{-1/2} d\eta)$, coincides almost everywhere with $\gamma_0[(\mathscr{F}_y q_d)(\eta, \cdot)]$, where γ_0 is the trace operator on $H^1(\mathbb{R}^+)$.

From this remark, we conclude that, in order to verify that u belongs to $D(A)$ when u and Au are already in X, it is sufficient to prove that $\gamma_0 B \mathscr{F}_y u$ vanishes almost everywhere. Also, proving that $A^d u$ has a square-integrable trace (instead of being in $H^{-1/2}$) amounts to proving that $\gamma_0 A^d \mathscr{F}_y u$ is square-integrable.

3.1.2 *Monotonicity of A*

Since A commutes with translations along the variable y, we consider the conjugated operator A_F, obtained after a Fourier transform in the y variable:

$$A_F v = A^d \frac{\partial v}{\partial x_d} + iA(\eta)v,$$
$$D(A_F) = \mathscr{F}_y D(A) = \{v \in L^2(\Omega)\,;\, A_F v \in L^2(\Omega) \text{ and } Bv = 0 \text{ on } \partial\Omega\}.$$

Let us point out that in this definition, the functions are complex-valued. The trace of Bv has been defined in the previous section.

The monotonicity of A is the property that

$$\langle Au, u\rangle \geq 0, \quad \forall u \in D(A). \tag{3.1.7}$$

By Plancherel's formula, it is equivalent to

$$\text{Re } \langle A_F v, v\rangle \geq 0, \quad \forall v \in D(A_F). \tag{3.1.8}$$

From the previous section, we know that, for almost every $\eta \in \mathbb{R}^{d-1}$, the field $A^d v(\eta,\cdot)$ is in $H^1(\mathbb{R}^+)$ and its trace is well-defined. In particular, we know that $\gamma_0 Bv(\eta,\cdot) = 0$ for almost every η.

Let us define a matrix S as follows. Since A^d is symmetric, we have $\mathbb{R}^d = R(A^d) \oplus^\perp \ker A^d$. If $w \in \ker A^d$, we set $Sw = 0$, while if $w \in R(A^d)$, there is a unique $w' \in R(A^d)$ such that $A^d w' = w$, and then we set $Sw = w'$. We easily check that S is symmetric and that SA^d is the orthogonal projector[1] onto $R(A^d)$. In particular, the bilinear form $w \mapsto (A^d w, w)$ can be rewritten as $w \mapsto (SA^d w, A^d w)$. Dissipativeness means Re $(SA^d w, A^d \bar{w}) \leq 0$ on $\ker B$.

Let $w \in L^2(\mathbb{R}^+)$ be such that $A^d \mathrm{d}w/\mathrm{d}x_d \in L^2(\mathbb{R}^+)$. Then $z := A^d w \in H^1(\mathbb{R}^+)$. Given $\eta \in \mathbb{R}^{d-1}$, let us compute:

$$\begin{aligned}\int_0^{+\infty} \text{Re } \left(A^d \frac{\mathrm{d}w}{\mathrm{d}x_d} + iA(\eta)w, \bar{w}\right) \mathrm{d}x_d &= \int_0^{+\infty} \text{Re } \left(A^d \frac{dw}{dx_d}, \bar{w}\right) \mathrm{d}x_d \\ &= \int_0^{+\infty} \text{Re } \left(\frac{dz}{dx_d}, S\bar{z}\right) \mathrm{d}x_d \\ &= -\frac{1}{2}\text{Re } (Sz(0), \bar{z}(0)) \geq 0.\end{aligned}$$

Now, if $v \in D(A_F)$, we have $v(\eta,\cdot) \in L^2(\mathbb{R}^+)$ and $A^d dv(\eta,\cdot)/dx_d \in L^2(\mathbb{R}^+)$ for almost every η. Hence we deduce, for every non-negative test function $\phi \in \mathscr{D}_+(\mathbb{R}^{d-1})$,

$$\int_\Omega \phi(\eta) \text{Re } (A_F v, \bar{v})\, \mathrm{d}x_d \mathrm{d}\eta \geq 0.$$

[1] The matrix S is called the Moore–Penrose generalized inverse of A^d. It coincides with the inverse when A^d is non-singular. See [187], Section 8.4.

Finally, we let ϕ tend monotonically to one everywhere. The right-hand side tends to Re $\langle A_F v, v\rangle$ and we obtain the inequality (3.1.8). This proves that A is a monotone operator.

3.1.3 *Maximality of A*

We now solve the equation

$$u + Au = f, \tag{3.1.9}$$

where f is given in $L^2(\Omega)$ and u is searched in $D(A)$. Thanks to a Fourier transform, it is enough to solve $v + A_F v = g$, where $g := \mathscr{F}_y f$ and $v := \mathscr{F}_y u$. The latter equation decouples as a set of ODEs with boundary condition, parametrized by $\eta \in \mathbb{R}^{d-1}$:

$$v + iA(\eta)v + A^d v' = g(\eta,\cdot), \quad Bv(\eta,0) = 0. \tag{3.1.10}$$

In order to simplify the presentation, we shall make the unnecessary assumption that A^d is non-singular, the so-called *non-characteristic* case. The general case is treated similarly, after a projection of the ODE onto $\ker A^d$ and $R(A^d)$.

To solve the differential equation in (3.1.10), we split v and g into their stable and unstable components, with respect to the matrix

$$\mathcal{A}(\eta) := -\left(A^d\right)^{-1}(I_n + iA(\eta)).$$

We shall denote $E_\pm(\eta)$ the corresponding stable and unstable subspaces[2] in $\mathbb{C}^n$ (see the introductory section 'Notations'). With obvious notations, we split

$$v = v_s + v_u, \quad (A^d)^{-1} g = g_s + g_u, \quad v_s, g_s \in E_-(\eta), \quad v_u, g_u \in E_+(\eta).$$

Let $(S(z))_{z\in\mathbb{R}}$ be the group generated by $\mathcal{A}(\eta)$, that is $S(z) = \exp z\mathcal{A}(\eta)$. We look for a solution v of the form

$$v_s(\eta, x_d) = S(x_d)v_0 + \int_0^{x_d} S(x_d - z) g_s(\eta, z)\,\mathrm{d}z, \tag{3.1.11}$$

$$v_u(\eta, x_d) = -\int_{x_d}^{+\infty} S(x_d - z) g_u(\eta, z)\,\mathrm{d}z. \tag{3.1.12}$$

Since $g \in L^2(\Omega)$, the partial function $g(\eta,\cdot)$ is square-integrable for almost every η. For such ηs, the integrals in (3.1.11) and (3.1.12) are convolution products of the components $g_s(\eta,\cdot)$, $g_u(\eta,\cdot)$, with integrable kernels. Actually, denoting by S_s and S_u the restrictions of S to the invariant subspaces $E_\pm(\eta)$, we know that $S_s(z)$ and $S_u(-z)$ decay exponentially fast as $z \to +\infty$. Then

$$\int_0^{x_d} S(x_d - z) g_s(\eta, z)\mathrm{d}z = \widetilde{S}_s * \widetilde{g}_s(x_d), \quad x_d > 0,$$

[2]It is shown in Lemma 4.1 that the real part of the eigenvalues of $\mathcal{A}(\eta)$ does not vanish. Hence $\mathbb{C}^n = E_-(\eta) \oplus E_+(\eta)$.

where $h \mapsto \widetilde{h}$ is the extension from $\mathbb{R}^+$ to $\mathbb{R}$ by zero. Another formula resembling the one above holds for the integral in (3.1.12). By Young's inequality, the convolution products belong to L^2.

Hence, Equations (3.1.11) and (3.1.12) define an L^2-function v, provided one chooses v_0 in the stable subspace $E_-(\eta)$. Obviously, $v(\eta, \cdot)$ solves the ODE in (3.1.10). In order to satisfy the homogeneous boundary condition, it remains to solve

$$Bv_0 = B \int_0^{+\infty} S(-z) g_u(\eta, z) \, \mathrm{d}z, \quad v_0 \in E_-(\eta). \tag{3.1.13}$$

Lemma 3.2 *Under the above assumptions (L Friedrichs symmetric, B maximal dissipative), it holds that*

$$E_-(\eta) \oplus \ker B = \mathbb{C}^n. \tag{3.1.14}$$

Equation (3.1.13) *admits a unique solution* v_0.

Proof We first show (3.1.14), admitting a result of Hersh (Lemma 4.1) proved in the next chapter. Let $U_0 \in E_-(\eta)$ be given, and define $U(x_d) := S(x_d)U_0$. It satisfies the differential equation

$$(I_n + iA(\eta))U + A^d \frac{\mathrm{d}U}{\mathrm{d}x_d} = 0$$

and decays exponentially fast at $+\infty$. Multiplying on the left of this equation by U^*, and taking the real part, we obtain

$$\|U\|^2 + \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x_d} U^* A^d U = 0.$$

Integrating from 0 to $+\infty$, we derive

$$2 \int_0^{+\infty} \|U(x_d)\|^2 \mathrm{d}x_d = \mathrm{Re}\ (A^d U_0, U_0).$$

If, moreover, $U_0 \in \ker B$, the right-hand side is non-positive, by dissipative assumption. It follows that

$$\int_0^{+\infty} \|U(x_d)\|^2 \mathrm{d}x_d \leq 0,$$

that is $U \equiv 0$ and $U_0 = 0$. This proves that $E_-(\eta) \cap \ker B = \{0\}$.

The conclusion is obtained by proving that $\dim E_-(\eta) + \dim \ker B = n$. Since B is maximal dissipative, the dimension of its kernel is the number of non-positive eigenvalues of A^d. The fact that $\dim E_-(\eta)$ equals the number of positive eigenvalues of A^d will be proved in a much more general context in Lemma 4.1.

Thanks to (3.1.14), B is an isomorphism from $E_-(\eta)$ onto $R(B)$. This ensures the unique solvability of (3.1.13). □

At this stage, we have built, in a unique way, a solution of (3.1.10). It is defined for almost every $\eta \in \mathbb{R}^{d-1}$ and is square-integrable with respect to x_d. Clearly, it is also measurable in (η, x_d). Given η, we may apply the energy estimate, which reads

$$2\int_0^{+\infty} \|v(\eta,x_d)\|^2 \mathrm{d}x_d = \mathrm{Re}\ (A^d v(\eta,0), v(\eta,0)) + 2\mathrm{Re}\ \int_0^{+\infty} (v,g)(\eta,x_d)\,\mathrm{d}x_d. \tag{3.1.15}$$

Because of $Bv(\eta,0)=0$ and the dissipativeness, we derive

$$\int_0^{+\infty} \|v(\eta,x_d)\|^2 \mathrm{d}x_d \le \mathrm{Re}\ \int_0^{+\infty} (v,g)(\eta,x_d)\,\mathrm{d}x_d,$$

which gives, thanks to the Cauchy–Schwarz inequality,

$$\int_0^{+\infty} \|v(\eta,x_d)\|^2 \mathrm{d}x_d \le \int_0^{+\infty} \|g(\eta,x_d)\|^2 \mathrm{d}x_d.$$

Integrating with respect to η, we obtain that $v \in L^2(\Omega)$. Using Plancherel's formula, we conclude that $u \in L^2(\Omega)$ as well. By Fourier inversion, $u + Au = f$ holds in the distributional sense. Therefore, $Au = f - u$ is square-integrable too. At last, the trace of Bu is zero because that of Bv is so. Therefore u belongs to $D(A)$ and A is maximal monotone.

Applying Theorem 3.1, we have

Theorem 3.2 *Let $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be a symmetric hyperbolic operator, and let the boundary matrix $B \in M_{p\times n}(\mathbb{R})$ be maximal dissipative. Finally, let $D(A)$ be the functional space*

$$D(A) := \left\{ u \in L^2(\Omega)^n\,;\, \sum_\alpha A^\alpha \partial_\alpha u \in L^2(\Omega)^n \text{ and } Bu = 0 \text{ on } \partial\Omega \right\}.$$

Then the homogeneous IBVP in $\Omega \times \mathbb{R}_t^+$,

$$Lu(x,t) = 0, \quad Bu(y,0,t) = 0, \quad u(x,0) = u_0(x) \tag{3.1.16}$$

is L^2-well-posed in the following sense. For every $u_0 \in D(A)$, there exists a unique $u \in \mathscr{C}([0,+\infty); D(A)) \cap \mathscr{C}^1([0,+\infty); L^2)$ that solves $Lu = 0$ as an ODE in $X = L^2(\Omega)^n$, such that $u(0) = u_0$. Furthermore,

$$t \mapsto \|u(t)\|_{L^2}$$

is non-increasing.

The map $u_0 \mapsto u(t)$ therefore extends uniquely as a continuous semigroup of contractions in $L^2(\Omega)^n$ (see the book by Pazy [156] for the semigroup theory). This allows us to define a solution in the weaker case of a square-integrable datum. Such weak (or *mild*) solutions are distributional solutions of $Lu = 0$, since they are limits of stronger solutions. The same density argument shows that they satisfy the boundary condition $Bu = 0$ in the trace sense, as explained in Section 3.1.1.

Data with $f \not\equiv 0$ We may use the semigroup defined in the Theorem 3.2, together with Duhamel's Formula

$$u(t) = S_t u_0 + \int_0^t S_{t-s} f(s)\,\mathrm{d}s,$$

provided f is integrable from $(0,T)$ to $X = L^2(\Omega)^n$. This defines a mild solution of $Lu = f$, $Bu(y,0,t) = 0$ and $u(t=0) = u_0$. This mild solution is a distributional one. For instance, if $f \in L^2((0,T)\times\Omega)$, we easily obtain the following estimate (see also Section 3.2.1):

$$\mathrm{e}^{-2\gamma T}\|u(T)\|_{L^2}^2 + \gamma\int_0^T \mathrm{e}^{-2\gamma t}\|u(t)\|_{L^2}^2\,\mathrm{d}t \le \|u_0\|_{L^2}^2 + \frac{1}{\gamma}\int_0^T \mathrm{e}^{-2\gamma t}\|f(t)\|_{L^2}^2\,\mathrm{d}t. \tag{3.1.17}$$

Variational IBVPs. In some physically relevant cases, a second-order IBVP is formed by the Euler–Lagrange equations of some Lagrangian

$$\iint \left(\frac{1}{2}|u_t|^2 - W(\nabla_x u)\right)\mathrm{d}x\,\mathrm{d}t,$$

where W is a quadratic form on $\mathbf{M}_{d\times n}(\mathbb{R})$. We also mean that the boundary conditions are of Neumann type and figure the Euler–Lagrange equations along the boundary. These special boundary conditions are *conservative*, in the sense that $\langle A(\nu)u, u\rangle \equiv 0$, when rewriting the IBVP to the first order. If W is positive-definite, we can apply the above procedure, Friedrichs symmetrizing the problem thanks to the energy

$$\mathcal{E}[u] = \int \left(\frac{1}{2}|u_t|^2 + W(\nabla_x u)\right)\mathrm{d}x =: \mathcal{E}_c[u] + \mathcal{E}_p[u].$$

More generally, the Hille–Yosida Theorem applies whenever $\mathcal{E}_p$ is convex and coercive over $H^1(\Omega)^n$, $\Omega := \mathbb{R}^{d-1}\times(0,+\infty)$. A natural question is whether the strong well-posedness of a variational IBVP implies that $\mathcal{E}_p$ is convex and coercive over $H^1(\Omega)^n$ (a property that is not equivalent to the positive-definiteness of W). The answer turns out to be positive, see [189]. In particular, there is no need to construct a symbolic symmetrizer (as we do in Chapter 7 in a more general context). See also [190] for the case where $n = d$ and $(H^1)^d$ is replaced by the subspace with the incompressibility constraint $\operatorname{div} u = 0$.

3.2 Strictly dissipative symmetric IBVPs

The symmetric dissipative IBVP occurs in several problems of physical importance, in elasticity and electromagnetism, when energy must be conserved along an evolution process. However, the theory suffers from a major weakness when one wishes to treat free-boundary value problems by Picard iterations, for one is unable to control norms of the trace $\gamma_0 u$ on $\partial\Omega$ (or that of $\gamma_0 A^d u$ in the characteristic case), in terms of the same norms of $\gamma_0 Bu$.

To improve the theory, we need a notion stronger than dissipativeness. We present it first in the non-characteristic case (recall that this means $\det A^d \neq 0$).

Definition 3.3 *Let L be Friedrichs symmetric. If $\Omega := \{x_d > 0\}$ is non-characteristic (that is, A^d is non-singular), we say that the boundary condition (4.1.2) is* strictly dissipative *if the three properties below hold:*

i) A^d is negative-definite on $\ker B$*:*

$$(v \neq 0,\, Bv = 0) \Longrightarrow (A^d v, v) < 0,$$

ii) $\ker B$ *is maximal for the above property,*

iii) B is onto.

For a more general domain with smooth boundary, the dissipation property must be with respect to $-A(\nu)$, at every boundary point $x \in \Omega$, ν being the outward unit normal to $\partial\Omega$ at x (in that case, B often depends on x itself):

$$(v \neq 0,\, Bv = 0) \Longrightarrow (A(\nu)v, v) > 0.$$

The fact that B is onto is natural in the non-homogeneous boundary value problem, since the boundary condition itself must be solvable at the algebraic level; otherwise there would be a trivial obstacle to the well-posedness of the IBVP. As above, the dimension of $\ker B$ equals the number of negative eigenvalues of A^d. Since B is onto, this means that p equals the number of positive eigenvalues of A^d.

We are going to show that under strict dissipativeness, an a priori estimate holds for the full IBVP (3.1.1)–(3.1.3), which is much better than the one encountered before in the sense that it controls the L^2-norm of $\gamma_0 A^d u$ (therefore that of $\gamma_0 u$ in the non-characteristic case) in terms of that of $\gamma_0 Bu$.

An equivalent formulation of strict dissipativeness is given by the following

Lemma 3.3 *Assume that the boundary condition is strictly dissipative. Then there exist two positive constants ε, C such that the quadratic form $w \mapsto \varepsilon|w|^2 + (A^d w, w) - C|Bw|^2$ is negative-definite.*

Proof We argue by contradiction. If the lemma was false, there would be a sequence $(w_m)_{m\in\mathbb{N}}$ with the properties that $|w_m| \equiv 1$ and

$$\frac{1}{m}|w_m|^2 + (A^d w_m, w_m) - m|Bw_m|^2 \geq 0.$$

By compactness, we may assume that $(w_m)_m$ converges towards some w. Since $|Bw_m| = \mathcal{O}(1/\sqrt{m})$, we find $w \in \ker B$. On an other hand, $\frac{1}{m}|w_m|^2 + (A^d w_m, w_m) \geq 0$ gives in the limit the inequality $(A^d w, w) \geq 0$. Finally, $w \neq 0$ since $|w| = 1$. This contradicts the assumption. □

3.2.1 *The a priori estimate*

We assume that u is a smooth solution of the full IBVP, with smooth decay as $|x|$ goes to infinity. Multiplying (3.1.1) by u^* and taking the real part, we obtain

$$\partial_t |u|^2 + \sum_{\alpha=1}^{d} \partial_\alpha (u^* A^\alpha u) = 2\mathrm{Re}\ (f, u).$$

Integrating on Ω, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u(t)|^2 \mathrm{d}x = 2\mathrm{Re}\ \int_\Omega (f(t), u(t))\,\mathrm{d}x + \int_{\partial\Omega} u^* A^d u\,\mathrm{d}y. \tag{3.2.18}$$

Using Lemma 3.3, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |u(t)|^2 \mathrm{d}x + \varepsilon \int_{\partial\Omega} |u(t)|^2 \mathrm{d}y \le 2\mathrm{Re}\ \int_\Omega (f(t), u(t))\,\mathrm{d}x + C \int_{\partial\Omega} |g(t)|^2\,\mathrm{d}y.$$

Let γ be a positive number. We apply the latter estimates to $v := \exp(-\gamma t) u$, for which $Lv = -\gamma v + \exp(-\gamma t) f =: F$, $v(t = 0) = u_0$, and $Bv = \exp(-\gamma t) g$ on the boundary. By Young's inequality, we have

$$2\mathrm{Re}\ (F, v) \le \exp(-2\gamma t) \left(\frac{1}{\gamma} |f|^2 - \gamma |u|^2 \right).$$

We thus obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} e^{-2\gamma t} \|u(t)\|_{L^2}^2 + \gamma \mathrm{e}^{-2\gamma t} \|u(t)\|_{L^2}^2 + \varepsilon \mathrm{e}^{-2\gamma t} \|\gamma_0 u(t)\|_{L^2}^2 \\ &\quad \le \mathrm{e}^{-2\gamma t} \left(\frac{1}{\gamma} \|f(t)\|_{L^2}^2 + C \|g(t)\|_{L^2}^2 \right), \end{aligned}$$

where $\|\cdot\|_{L^2}$ denotes the norm in either $L^2(\Omega)$ or $L^2(\partial\Omega)$. We now integrate from 0 to $T > 0$ and obtain

$$\begin{aligned} &\mathrm{e}^{-2\gamma T} \|u(T)\|_{L^2}^2 + \int_0^T \mathrm{e}^{-2\gamma t} \left(\gamma \|u(t)\|_{L^2}^2 + \varepsilon \|\gamma_0 u(t)\|_{L^2}^2 \right) \mathrm{d}t \\ &\le \|u_0\|_{L^2}^2 + \int_0^T \mathrm{e}^{-2\gamma t} \left(\frac{1}{\gamma} \|f(t)\|_{L^2}^2 + C \|g(t)\|_{L^2}^2 \right) \mathrm{d}t. \end{aligned}$$

Notation For positive γ, T, we define a 'norm' $\|\cdot\|_{\gamma,T}$ by

$$\|u\|_{\gamma,T}^2 := \int_0^T \int_{\partial\Omega} \mathrm{e}^{-2\gamma t} |\gamma_0 u(y,t)|^2 \mathrm{d}y\,\mathrm{d}t + \gamma \int_0^T \int_\Omega \mathrm{e}^{-2\gamma t} |u(x,t)|^2 \mathrm{d}x\,\mathrm{d}t.$$

We warn the reader that this expression does not define a functional space, and therefore cannot be considered as a genuine norm, since the corresponding closure of $\mathscr{D}(\bar{\Omega})$ is isomorphic to $L^2(\Omega) \times L^2(\partial\Omega)$, so that the function on the boundary is no longer a trace of the interior function.

We summarize this weighted estimate in the following

Proposition 3.1 *For the symmetric, strictly dissipative IBVP in $\Omega = \{x_d > 0\}$, there holds the following a priori estimate for every positive numbers γ, T,*

$$\begin{aligned} & e^{-2\gamma T}\|u(T)\|_{L^2}^2 + \|u\|_{\gamma,T}^2 \\ & \leq C\left(\|u(0)\|_{L^2}^2 + \int_0^T e^{-2\gamma t}\left(\frac{1}{\gamma}\|(Lu)(t)\|_{L^2}^2 + \|\gamma_0 Bu(t)\|_{L^2}^2\right) dt\right), \end{aligned} \tag{3.2.19}$$

where the constant $C > 0$ depends neither upon f, g, u_0, u, nor on γ, T.

The characteristic case The definition of strict dissipation given in Definition 3.3 was appropriate when the boundary is not characteristic. Of course, one says that the boundary is *characteristic* if the normal matrix $A(\nu)$ is singular. In the general case, we ask that the quadratic form $v \mapsto (A^d v, v)$, defined on $\ker B$, be non-positive, and vanishes only on $\ker A^d$ (in particular, $\ker A^d \subset \ker B$, as quoted before). This assumption yields the fact that the quadratic form $w \mapsto \varepsilon|A^d w|^2 + (A^d w, w) - C|Bw|^2$ is non-positive and vanishes only on $\ker A^d$ (compare with Lemma 3.3). The norm $\|\cdot\|_{\gamma,T}$ in the a priori estimate (3.2.19) is then weakened into

$$\|u\|_{\gamma,T}^2 := \int_0^T \int_{\partial\Omega} e^{-2\gamma t}|\gamma_0 A^d u(y,t)|^2 dy\, dt + \gamma \int_0^T \int_\Omega e^{-2\gamma t}|u(x,t)|^2 dx\, dt.$$

3.2.2 *Construction of $\hat{u}$ and u*

The existence of the solution of the homogeneous boundary value problem (that is, with $g \equiv 0$) was obtained in Section 3.1. Since strict dissipativeness implies dissipativeness, we may use this construction to solve our problem when $g \equiv 0$. Following the details of this construction, we verify that the estimate (3.2.19) holds true (without the g term of course). Since our problem is linear, it remains to treat the pure boundary value problem. Hence we shall assume that $u_0 \equiv 0$ and $f \equiv 0$. Since $u_0 \equiv 0$, we may extend the solution and the data g by zero to negative times. We therefore have to solve a problem in the domain defined by

$$x \in \Omega = \mathbb{R}^{d-1} \times (0, \infty), \quad t \in \mathbb{R},$$

and we may assume that g is supported in a slab $\partial\Omega \times [0, T]$. This problem is the occasion to describe for the first time, but in a simple context, the use of the Fourier–Laplace transform (see Appendix B.3). We warn the reader that we apply the Fourier transformation to the variable y only, since x_d does not vary on the whole line but only[3] on $(0, +\infty)$.

We now face the pure boundary value problem. Taking the Fourier–Laplace transform (in variables (y, t)) of the PDEs and of the boundary condition, this

[3]One might apply the Laplace transform to the variable x_d, but this would not help anyway.

problem is formally equivalent to the following:

$$(\tau I_n + iA(\eta))\hat{u} + A^d \frac{\partial \hat{u}}{\partial x_d} = 0, \qquad x_d > 0, \tag{3.2.20}$$

$$B\hat{u}(\eta, 0, \tau) = \hat{g}(\eta, \tau). \tag{3.2.21}$$

The problem (3.2.20) and (3.2.21) is clearly a set of uncoupled differential-algebraic problems, parametrized by (η, τ). We may solve separately each of them (or merely almost all of them). Doing so, the partial derivative will of course be seen as an ordinary derivative.

The solutions of (3.2.20) are smooth functions (see their description in Section 4.2.3, when the boundary is characteristic). Since $\hat{u}(\cdot, \cdot, \gamma + i\cdot)$ is expected to be square-integrable with respect to $\mathrm{d}\eta\,\mathrm{d}x_d\,\mathrm{d}\sigma$, $\hat{u}(\eta, \cdot, \tau)$ must be square-integrable on $\mathbb{R}^+$. This means here that it decays exponentially fast as $x_d \to +\infty$. In other words, the integrability property translates into

$$\hat{u}(\eta, 0, \tau) \in E_-(\eta, \tau). \tag{3.2.22}$$

This, together with the boundary condition (3.2.21), determines in a unique way $\hat{u}(\eta, 0, \tau)$ (adapt Lemma 3.2), and hence $\hat{u}(\eta, x_d, \tau)$.

Once we have determined the function $\hat{u}$, it remains to show that it is the Laplace–Fourier transform of some function u, which satisfies the IBVP. We begin by noticing that, since the coefficients of (3.2.20) are holomorphic in τ, the map $\tau \mapsto E_-(\eta, \tau)$ is holomorphic too. And since g itself is τ-holomorphic, the trace $\hat{u}(\eta, 0, \tau)$ is holomorphic, as the solution of a Cramer system with holomorphic coefficients. Hence, $\hat{u}$ itself is τ-holomorphic.

We now turn to the estimate of $\hat{u}$. Since it decays exponentially fast as $x_d \to +\infty$, we may multiply (3.2.20) by $\hat{u}^*$, take the real part, and integrate with respect to x_d on $\mathbb{R}^+$. Because of the symmetry of $A(\eta)$ and A^d, we obtain

$$(\mathrm{Re}\ \tau) \int_0^{+\infty} |\hat{u}|^2 \mathrm{d}x_d = \frac{1}{2} \hat{u}(0)^* A^d \hat{u}(0).$$

Let now $\varepsilon > 0$ and $C > 0$ be such that the quadratic form $w \mapsto \varepsilon |A^d w|^2 + (A^d w, w) - C|Bw|^2$ is non-positive. We deduce

$$(\mathrm{Re}\ \tau) \int_0^{+\infty} |\hat{u}|^2 dx_d + \varepsilon |A^d \hat{u}(0)|^2 \leq C|\hat{g}|^2.$$

This inequality implies that if $\hat{g}$ is square-integrable in (η, σ) for some value of γ, then the left-hand side enjoys the same property. An integration yields the inequality

$$\gamma \int_{\mathbb{R}} \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} |\hat{u}|^2 \mathrm{d}\eta\,\mathrm{d}x_d\,\mathrm{d}\sigma + \varepsilon \iint_{x_d=0} |A^d \hat{u}|^2 \mathrm{d}\eta\,\mathrm{d}\sigma \leq C \iint_{x_d=0} |\hat{g}|^2 \mathrm{d}\eta\,\mathrm{d}\sigma. \tag{3.2.23}$$

In particular, the trace at $x_d = 0$ of $A^d\hat{u}$ is square-integrable. Assuming, for instance, that g belongs to $L^2(\partial\Omega \times \mathbb{R}^+)$ (actually, some growth is allowed as t increases), the right-hand side of (3.2.23) is bounded. Thus, $\|\hat{u}(\cdot,\cdot,\gamma+i\cdot)\|_{L^2}$ is an $\mathcal{O}(1/\gamma)$. Since $\hat{u}$ is τ-holomorphic, the theorem of Paley–Wiener (see Rudin [169], chapter 19) implies that there exists a function $u : \Omega \times (0,+\infty)$ with the following properties:

- For every $\gamma > 0$, the function $e^{-\gamma t}u$ is square-integrable,
- The function $\hat{u}$ is the Fourier–Laplace transform of u, with respect to the variables (y,t).

The inverse Fourier–Laplace transform of (3.2.20) shows that u satisfies the system $Lu = 0$. In particular, Lu is square-integrable and $A^d u$ must admit a trace on $x_d = 0$, of class $H^{-1/2}$, which we denote abusively $\gamma_0 A^d u$. Since $\ker B$ contains $\ker A^d$, the trace of Bu makes sense as well.

The following estimate follows from (3.2.23) and the Parseval formula:

$$\gamma \int_0^{+\infty} \mathrm{e}^{-2\gamma t}\|u(t)\|^2_{L^2(\Omega)}\mathrm{d}t \le C\int_0^{+\infty} \mathrm{e}^{-2\gamma t}\|g(t)\|^2_{L^2(\partial\Omega)}\mathrm{d}t. \tag{3.2.24}$$

At last, arguing as in Section 3.1.1, we see that $e^{-\gamma t}\gamma_0 A^d u$ is square-integrable (rather than of class $H^{-1/2}$), and that the following estimate holds

$$\gamma \int_0^{+\infty} \mathrm{e}^{-2\gamma t}\|u(t)\|^2_{L^2(\Omega)}\mathrm{d}t + \varepsilon\int_0^{+\infty} \mathrm{e}^{-2\gamma t}\|\gamma_0 A^d u(t)\|^2_{L^2(\partial\Omega)}\mathrm{d}t$$
$$\le C\int_0^{+\infty} \mathrm{e}^{-2\gamma t}\|g(t)\|^2_{L^2(\partial\Omega)}\mathrm{d}t. \tag{3.2.25}$$

We summarize our result in the following statement.

Theorem 3.3 *Let L be a Friedrichs-symmetric (hence hyperbolic) operator. Let $\Omega = \mathbb{R}^{d-1} \times (0,+\infty)$ be the spatial domain. Finally, let B be a strictly dissipative boundary matrix.*

Then, for every data $u_0 \in L^2(\Omega)$, $g \in L^2((0,T)\times\partial\Omega)$ and $f \in L^2((0,T)\times\Omega)$, there exists a unique solution of the Initial Boundary Value Problem (3.1.1)–(3.1.3) in the class $u \in L^2((0,T)\times\Omega)$. In addition, $A^d u$ admits a trace on the boundary, of class $L^2((0,T)\times\partial\Omega)$. Finally, we have $u \in \mathcal{C}([0,T];L^2(\Omega))$, and the estimate (3.2.19).

4

INITIAL BOUNDARY VALUE PROBLEM IN A HALF-SPACE WITH CONSTANT COEFFICIENTS

In this chapter, we drop the assumption of Friedrichs-symmetry and therefore dissipativeness, while keeping the other features of Chapter 3: L is a hyperbolic operator with constant coefficients, and the spatial domain is a half-space. We impose linear boundary conditions. Of course, most physical problems yield a Friedrichs-symmetric operator, but we wish to consider boundary conditions that are not dissipative in the sense of Definitions 3.1, 3.2 and 3.3.

Our main purpose is the well-posedness of such an Initial Boundary Value Problem (IBVP). We postpone the study of more natural linear problems (general smooth domain, variable coefficients) to Chapter 9, which uses the results displayed in the present one, together with those of Chapter 2.

4.1 Position of the problem

Let

$$L := \partial_t + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha$$

be a hyperbolic operator (with $A^\alpha \in M_n(\mathbb{R})$), and let B be a constant real-valued $q \times n$ matrix. Let Ω be the half-space in $\mathbb{R}^d$, defined by the inequality $x_d > 0$. Denote the tangential variables $(x_1, \ldots, x_{d-1})$ by y. We have

$$\Omega = \{(y, x_d)\,;\, y \in \mathbb{R}^{d-1},\, x_d > 0\}.$$

The general problem that we have in mind is still

$$(Lu)(x,t) = f(x,t), \quad x_d, t > 0,\, y \in \mathbb{R}^{d-1}, \tag{4.1.1}$$

$$Bu(y,0,t) = g(y,t), \quad t > 0,\, y \in \mathbb{R}^{d-1}, \tag{4.1.2}$$

$$u(x,0) = u_0(x), \quad x_d > 0,\, y \in \mathbb{R}^{d-1}. \tag{4.1.3}$$

Here above, the source term $f(x,t)$, the boundary data $g(y,t)$ and the initial data $u_0(x)$ are given in suitable functional spaces.

Since practical applications involve systems with variable coefficients, the present chapter investigates *robust* existence and stability results. As for the Cauchy problem (see Chapter 1), or the symmetric strictly dissipative case (Chapter 3), the terminology ‘robust results’ refers to strong well-posedness theorems, where the solution not only exists and is unique, but is estimated in the same norms as the data.

4.1.1 *The number of scalar boundary conditions*

Before going further, we observe that we may always choose the matrix B with full rank q, since otherwise g would have to satisfy compatibility conditions, and we could reduce the set of boundary conditions by extracting r independent lines, with $r = \operatorname{rank} \mathrm{B}$. Therefore, we shall always assume that $q = \operatorname{rank} \mathrm{B}$. In particular, we have $q \leq n$. The most significant object is $\ker B$, rather than B itself, since a multiplication on the left by a regular $q \times q$ matrix G transforms our boundary condition into an equivalent one, whose matrix is $B' := GB$.

As we shall see in Sections 4.2 and 4.3, the matrix B in the boundary condition (4.1.2) must satisfy several requirements of algebraic type, for the IBVP to be well-posed. In this section, we concentrate on its rank q, that is the number of scalar boundary data that we need.

With this purpose in mind, let us consider data that do not depend on the tangential variable y: we have $f = f(x_d, t)$, $g = g(t)$ and $u_0 = u_0(x_d, t)$. Assuming that the IBVP is well-posed (which means at least existence and uniqueness) in some appropriate functional space, the corresponding solution must have the same translational invariance : $u = u(x_d, t)$. This means that the reduced system

$$(\partial_t + A^d \partial_d)u = f, \quad Bu(0,t) = g(t), \quad u(x_d, 0) = u_0(x_d) \tag{4.1.4}$$

is well-posed in the quarter-plane $\{x_d > 0, t > 0\}$. By assumption (hyperbolicity) A^d is diagonalizable. Up to a linear tranformation of dependent variables, we may assume that A^d is already diagonal, $A^d = \operatorname{diag}(a_1, \dots, a_n)$, with $a_1 \geq \cdots \geq a_n$. Each co-ordinate u_j must obey the transport equation

$$(\partial_t + a_j \partial_d)u_j = f_j. \tag{4.1.5}$$

Denote by p the number of positive eigenvalues: $a_p > 0 \geq a_{p+1}$. We say that p is the *number of incoming characteristics*[1]. Split the unknown into an incoming and an outgoing part,

$$u_+ := \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix}, \quad u_- := \begin{pmatrix} u_{p+1} \\ \vdots \\ u_n \end{pmatrix}.$$

Integrating (4.1.5), we observe that u_- is uniquely determined by its value at initial time:

$$u_j(x_d, t) = u_{0j}(x_d - a_j t) + \int_0^t f_j(x_d - a_j(t-s), s)\mathrm{d}s, \quad j \geq p+1.$$

[1] In a half-space defined by the reverse inequality $x_d < 0$, the incoming characteristics correspond to $a_j < 0$.

Let ℓ be a linear form vanishing on the vector space $B(\mathbb{R}^p \times \{0_{n-p}\})$, and let L denote ℓB, so that $L_1 = \cdots = L_p = 0$ (ℓ and L are row vectors with q and n components, respectively). From (4.1.2), we obtain

$$\sum_{j=p+1}^{n} L_j \left(u_{0j}(-a_j t) + \int_0^t f_j(a_j(s-t), s)\mathrm{d}s \right) = \ell g(t). \tag{4.1.6}$$

When $\ell \neq 0$, (4.1.6) is a non-trivial compatibility condition for the data (u_0, g, f). Such a constraint prevents a general existence result from being obtained. Therefore, well-posedness requires

$$B(\mathbb{R}^p \times \{0_{n-p}\}) = \mathbb{R}^q,$$

which implies $p \geq q$. In other words,

Existence for every reasonnable data requires that the number of boundary conditions be less than or equal to the number of incoming characteristics.

We now turn to uniqueness, by considering the homogeneous IBVP ($f \equiv 0$, $u_0 \equiv 0$, $g \equiv 0$). From above, we already know that $u_- \equiv 0$. Let $R \in \mathbb{R}^p$ be such that $(R, 0)^T \in \ker B$. Let us choose a smooth function v of one variable, with $v \equiv 0$ on $[0, +\infty)$, and define u_+ by its co-ordinates

$$u_j(x_d, t) := v\left(\frac{x_d}{a_j} - t\right) R_j, \quad j = 1, \ldots, p.$$

We verify immediately that u is a solution of the IBVP. By uniqueness, we must have $u \equiv 0$. This means $R = 0$. Therefore, well-posedness requires $(\mathbb{R}^p \times \{0_{n-p}\}) \cap \ker B = \{0\}$. Since the dimension of $\ker B$ is $n - q$, this means in particular $p \leq q$. In other words,

Uniqueness requires that the number of boundary conditions be larger than or equal to the number of incoming characteristics.

Gathering these results, we obtain that for the IBVP (4.1.1)–(4.1.3) to be well-posed, it is necessary that

$$\mathbb{R}^n = (\mathbb{R}^p \times \{0\}) \oplus \ker B.$$

In particular, *the number of independent scalar boundary conditions* (the rank of B) *must be equal to the number of incoming characteristics*, that is

$$q = p.$$

Going back to a general matrix A^d, not necessarily diagonal, this reads

Proposition 4.1 *For the IBVP (4.1.1)–(4.1.3) to be well-posed, it is necessary that*

$$\mathbb{R}^n = U(A^d) \oplus \ker B, \tag{4.1.7}$$

where we recall that $U(A^d)$ is the unstable subspace of A^d.

4.1.2 *Normal IBVP*

The previous analysis concerned only the well-posedness of the one-dimensional IBVP, where the differential operator is $L_0 := \partial_t + A^d \partial_d$. Going back to the general case where A^d is diagonalizable (but not necessarily diagonal), we conclude that in order that this reduced IBVP be well-posed in reasonable spaces like L^2, it is necessary and sufficient to have

$$\mathbb{R}^n = (\ker B) \oplus E^u(A^d), \tag{4.1.8}$$

where $E^u(A^d)$ denotes the unstable invariant subspace of A^d.

As we already saw in Section 3.1.1, the need for a correct definition of the trace of Bu when u and Lu belong to a weak class, like $L^2(\Omega \times (0,T))$, requires that $\ker A^d \subset \ker B$. This leads us to the following.

Definition 4.1 *We say that the IBVP (4.1.1)–(4.1.3) is* normal *if*

- $B \in M_{p\times n}(\mathbb{R})$*, where* p *is the number of positive eigenvalues of* A^d*,*
- $\ker A^d \subset \ker B$*,*
- *property (4.1.8) holds true.*

Of course, dimensionality ensures that $p = \operatorname{rank} B$ for a normal IBVP. Using the characteristics, one easily shows that, in one space dimension, a normal hyperbolic IBVP is well-posed in L^2, in the sense that, for every data

$$u_0 \in L^2(\mathbb{R}^+), \quad g \in L^2(0,T), \quad f \in L^2(\mathbb{R}^+ \times (0,T)),$$

there exists a unique solution $u \in L^2(\mathbb{R}^+ \times (0,T))$, which satisfies, moreover, $u(0,\cdot) \in L^2(0,T)$, with obvious norm estimates.

4.2 The Kreiss–Lopatinskiĭ condition

We derive in this section a condition that turns out to be necessary for any kind of very weak well-posedness. The idea is very similar to the one followed in Section 1.1. In particular, we examine only the semigroup aspects of the IBVP, that is we shall only consider a homogeneous boundary condition ($g \equiv 0$). Of course, as seen in Chapter 3, we do not expect a simultaneous characterization of the well-posedness of non-homogeneous IBVP.

A Fourier transform with respect to the full space variable x being impossible, we content ourselves with $\mathscr{F}_y$, the Fourier transform with respect to the tangential variables. As we saw in Chapter 3, it is worth treating the time variable by a Laplace transform. Hence, we begin by looking for particular solutions of the form (*normal mode analysis*)

$$u(x,t) := e^{\tau t + i\eta\cdot y} U(x_d),$$

where $\eta \in \mathbb{R}^{d-1}$ and $\tau \in \mathbb{C}$. Since we aim to find a necessary condition, we are only interested in those solutions that could contradict well-posedness, that is those that grow rapidly as time increases, while being temperate in space. Thus we restrict ourselves to complex numbers τ of positive real part.

A field u defined as above solves $Lu = 0$ if and only if

$$(\tau I_n + iA(\eta))U + A^d \frac{\mathrm{d}U}{\mathrm{d}x_d} = 0. \tag{4.2.9}$$

Up to now, we did not assume anything about A^d but hyperbolicity. Since this allows A^d to be singular, Equation (4.2.9) need not be an ODE. We recall and generalize a notion introduced in Chapter 3:

Definition 4.2 *We consider the hyperbolic IBVP (4.1.1)–(4.1.3). The boundary* $\{x_d = 0\}$ *is said to be* characteristic *if the matrix* A^d *is singular (that is* $\det A^d = 0$*). When the IBVP is posed in a more general spatial domain* Ω *with a smooth boundary,* $\partial\Omega$ *is said to be* characteristic *if the matrix* $A(\nu)$ *is singular,* ν *being the outward normal vector field. It is* non-characteristic *otherwise.*

This notion is local in nature. The boundary can be characteristic on a part of the boundary only, this part being either a set of full dimension $d-1$ in $\partial\Omega$, or a submanifold. However, the theory of the IBVP is essentially open when the rank of $A(\nu)$ varies along a connected component of $\partial\Omega$.

4.2.1 *The non-characteristic case*

In order to make the exposition as clear as possible, we first suppose that the boundary is non-characteristic. Then (4.2.9) may be recast as an ODE in $\mathbb{C}^n$,

$$\frac{\mathrm{d}U}{\mathrm{d}x_d} = \mathcal{A}(\tau,\eta)U, \quad \mathcal{A}(\tau,\eta) := -(A^d)^{-1}(\tau I_n + iA(\eta)). \tag{4.2.10}$$

The following lemma is fundamental in the theory.

Lemma 4.1 (Hersh) *Under the hyperbolicity assumption, and for* $\eta \in \mathbb{R}^{d-1}$ *and* $\operatorname{Re}\tau > 0$*, the matrix* $\mathcal{A}(\tau,\eta)$ *does not have any pure imaginary eigenvalue. The number of stable eigenvalues (see the introductory paragraph 'Notations'), counted with multiplicities, equals p, the number of positive eigenvalues of* A^d*.*

Proof Let ω be a pure imaginary root of the characteristic polynomial of $\mathcal{A}(\tau,\eta)$,

$$P(X;\tau,\eta) := \det(XI_n - \mathcal{A}(\tau,\eta)).$$

Thus ω satisfies

$$\det(\tau I_n + iA(\eta) + \omega A^d) = 0.$$

Then, hyperbolicity implies $\tau \in i\mathbb{R}$, which contradicts the assumption. This proves that $\mathcal{A}(\tau,\eta)$ may not have a pure imaginary eigenvalue. Since P depends continuously on (τ,η) and has a constant degree, we infer that the number of roots with positive real part (counted with multiplicities) may not vary locally.

Since $\{\mathrm{Re}\,\tau > 0\} \times \mathbb{R}^{d-1}$ is a connected set, this number must be constant. We evaluate it by choosing $\eta = 0$, $\tau = 1$: We have

$$\mathcal{A}(1,0) = -(A^d)^{-1},$$

whose eigenvalues are the $-1/a_j$s. □

Using this lemma, we decompose the space $\mathbb{C}^n$ as the direct sum of the stable and unstable spaces of $\mathcal{A}(\tau,\eta)$. Recall that the *stable* (respectively, *unstable*) subspace is the sum of generalized eigenspaces of $\mathcal{A}(\tau,\eta)$ corresponding to eigenvalues of negative (respectively, positive) real parts. We denote $E_-(\tau,\eta) = E^s(\mathcal{A}(\tau,\eta))$ (respectively, $E_+(\tau,\eta) = E^u(\mathcal{A}(\tau,\eta))$.) These are the spaces of *incoming* (respectively, *outgoing*) modes. As mentioned in the section 'Notations', these spaces can be characterized by means of contour integrals. Choosing a large enough loop γ in the half-space $\{\mathrm{Re}\,\omega > 0\}$, enclosing the unstable eigenvalues (namely the ones with positive real parts) of $\mathcal{A}(\tau,\eta)$, the projector onto $E_+(\tau,\eta)$, along $E_-(\tau,\eta)$, is given by the formula

$$\pi_+(\tau,\eta) = \frac{1}{2i\pi}\int_\gamma (zI_n - \mathcal{A}(\tau,\eta))^{-1}\mathrm{d}z. \tag{4.2.11}$$

A similar formula holds for the projector $\pi_- = I_n - \pi_+$. Since we may vary slightly the arguments (τ,η) without changing the contour (because of the continuity of the roots of a polynomial), we infer from (4.2.11) that the maps $(\tau,\eta) \mapsto \pi_\pm(\tau,\eta)$ are holomorphic in τ, analytic in η, which amounts to saying:

Lemma 4.2 *The stable and unstable subspaces $E_\pm(\tau,\eta)$ depend holomorphically on τ, analytically on η. In particular, their dimensions do not depend on (τ,η) as long as $\eta \in \mathbb{R}^{d-1}$ and* $\mathrm{Re}\,\tau > 0$.

Fix now $\eta \in \mathbb{R}^{d-1}$ and $\mathrm{Re}\,\tau > 0$. Given an initial datum $U(0)$, the Cauchy problem for (4.2.10) admits a unique solution U. Decomposing $U(0)$ into a stable part $U_{0-} := \pi_- U(0)$ and an unstable one U_{0+}, the solution reads

$$U(x_d) = U_-(x_d) + U_+(x_d), \quad U_\pm(x_d) := \exp(x_d\mathcal{A}(\tau,\eta))U_{0\pm}.$$

Because of Lemma 4.1, the matrix

$$\exp\left(x_d\mathcal{A}(\tau,\eta)|_{E_-}\right)$$

decays exponentially fast as x_d tends to $+\infty$. Similarly, the inverse of

$$\exp\left(x_d\mathcal{A}(\tau,\eta)|_{E_+}\right)$$

decays exponentially fast. This shows that U_- decays exponentially fast, while U_+ is not polynomially bounded, except in the case $U_{0+} = 0$ (that is $U(0) \in E_-(\tau,\eta)$). Therefore, in order that U be a tempered distribution on $\mathbb{R}^+$, it is necessary and sufficient that $U(0) \in E_-(\tau,\eta)$. In that case, U actually decays exponentially fast, and is therefore square-integrable.

For this reason, we admit only those solutions of (4.2.10) for which $U(0) \in E_-(\tau,\eta)$. They take their values in $E_-(\tau,\eta)$ and form a vector space of dimension p. Let U be such a solution and u be the corresponding solution of $Lu = 0$. If, moreover, $BU(0) = 0$, then u satisfies the homogeneous boundary condition $Bu(y, 0, t) \equiv 0$. At initial time,

$$u(y, x_d, 0) = e^{i\eta\cdot y} U(x_d)$$

belongs to every Hölder space $\mathscr{C}^{k,\alpha}(\Omega)$, while the norm of $u(\cdot, t)$ grows exponentially fast (like $\exp(t\mathrm{Re}\ \tau)$) as t increases, provided $U(0) \neq 0$. Now, rescaling both space and time variables yields the parametrized solution to the homogeneous IBVP:

$$u^\lambda(x, t) := u(\lambda x, \lambda t), \quad \lambda \in (0, +\infty).$$

As $\lambda \to +\infty$, the sequence $\left(u^\lambda(\cdot, 0)\right)_\lambda$ grows at most polynomially in Hölder spaces, with respect to λ, while the sequence $\left(u^\lambda(t)\right)_\lambda$ grows always exponentially fast for given positive time. This shows that the mapping

$$u(\cdot, 0) \mapsto u(\cdot, t), \quad (t > 0),$$

if ever defined, may not be continuous between Hölder spaces, even at the price of a loss of derivatives. From the Principle of Uniform Boundedness, we conclude that this map cannot actually be defined.

This analysis shows that a necessary condition for well-posedness in Hölder spaces is that $E_-(\tau,\eta) \cap \ker B = \{0\}$ for every $\eta \in \mathbb{R}^{d-1}$ and $\mathrm{Re}\ \tau > 0$.

Definition 4.3 *We say that the hyperbolic IBVP (4.1.1)–(4.1.3) satisfies the* Kreiss–Lopatinskiĭ condition *(or briefly the* Lopatinskiĭ condition*) if*

$$E_-(\tau,\eta) \cap \ker B = \{0\}$$

for every $\eta \in \mathbb{R}^{d-1}$ *and* $\mathrm{Re}\ \tau > 0$.

Making $\eta = 0$, we see that this condition implies that the IBVP is normal in the sense of Definition 4.1. Because of Lemma 4.1, the Lopatinskiĭ condition is equivalent to saying that

$$\mathbb{C}^n = E_-(\tau,\eta) \oplus \ker B, \quad \forall \eta \in \mathbb{R}^{d-1}, \forall \mathrm{Re}\ \tau > 0.$$

Note that, contrary to the hyperbolicity assumption, this condition is not invariant by time reversing. If one wishes to solve the backward IBVP, one needs to consider the Laplace variable τ of negative real part, instead of positive ones. Also, the number of boundary conditions must be equal to the number of negative (instead of positive) eigenvalues of A^d.

The names of Lopatinskiĭ and Kreiss have been associated with the stability condition described above since their seminal works [121–123] and [103]. Hersh's article [83] is anterior, but the merit of Kreiss, as well as of Sakamoto [174] in the context of higher-order scalar operators, was to understand the role of the uniform version of the Lopatinskiĭ condition in the well-posedness theory. Lopatinskiĭ is much better known for his analysis of the *elliptic* boundary value problems, but did contribute to the hyperbolic problem [122]. The algebraic conditions that ensure the well-posedness in the elliptic theory resemble actually very much those of the hyperbolic case. The fundamental paper by Agmon *et al.* [2], for instance, manipulates the same kind of stable subspaces as we do here.

The Lopatinskiĭ condition is by nature the fact that the following linear differential problem is well-posed in $L^2(\mathbb{R}^+)$ when $x_0 \in \mathbb{C}^m$ and $F \in L^2(\mathbb{R}^+)$ are given

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} X \\ Y \end{pmatrix} = M\begin{pmatrix} X \\ Y \end{pmatrix} + F(s), \qquad X(0) = x_0.$$

An important instance of such a problem occurs in control theory, X being the state and Y the adjoint state. See [72] for a discussion. In the context of initial boundary value problems, the matrix of the ODE depends on parameters, say the Laplace–Fourier frequencies, and we shall need some kind of uniformity of this well-posedness. This important aspect will be discussed in Section 4.3.

4.2.2 *Well-posedness in Sobolev spaces*

The solutions considered in Section 4.2.1 are not square-integrable with respect to y and therefore cannot be used directly in the study of the well-posedness in Sobolev spaces. In order to prove that well-posedness in this context still requires that the Kreiss–Lopatinskiĭ condition be fulfilled, we improve our construction.

Let us assume that the Kreiss–Lopatinskiĭ condition fails at some point (τ_0, η_0) with $\eta_0 \in \mathbb{R}^{d-1}$ and $\operatorname{Re} \tau_0 > 0$. We shall use an analytical tool described in Section 4.6, called the *Lopatinskiĭ determinant*. This is a function $\Delta(\tau, \eta)$, which is analytic in η and holomorphic in τ, with the property that the Kreiss–Lopatinskiĭ condition fails precisely at zeroes of Δ. Its construction will be explained in Section 4.6.1.

As η moves around η_0, the holomorphic function $\Delta(\cdot, \eta)$ keeps as many zeroes close to η_0 as the multiplicity of the root τ_0 of $\Delta(\cdot, \eta_0)$ (Rouché's theorem[2]). Since the multiplicities of zeroes are upper semicontinuous functions of η, we may choose a zero (τ_1, η_1) such that, when η moves in the neighbourhood $\mathcal{V}$ of η_1, $\Delta(\cdot, \eta)$ has a unique root close to τ_1, obviously with a constant multiplicity m. This root, denoted by $T(\eta)$, is analytic, because it is a simple root of

$$\frac{\partial^{m-1}}{\partial \tau^{m-1}} \Delta(\cdot, \eta).$$

For each $\eta \in \mathcal{V}$, the space $F(\eta) := \ker B \cap E_-(T(\eta), \eta)$ is non-trivial and, as above, we may assume that it has a constant dimension and is analytic in η.

[2] A correct use of Rouché's theorem requires that $\Delta(\cdot, \eta_0)$ do not vanish identically. This is proved in Lemma 8.1.

Then we may choose a non-zero analytic vector field $X(\eta)$ in it. Taking $X(\eta)$ as an initial datum, (4.2.9) defines a function $x_d \mapsto U(x_d;\eta)$.

Now, each of the functions

$$(x,t) \mapsto \exp(T(\eta)t + i\eta \cdot y)\, U(x_d;\eta)$$

solves $Lu = 0$, as well as $Bu(y,0,t) = 0$. To build an L^2 function with the same property, we choose a non-zero scalar function $\phi \in \mathscr{D}(\mathcal{V})$ and define

$$u(x,t) := \int_{\mathcal{V}} \phi(\eta) \exp(T(\eta)t + i\eta \cdot y)\, U(x_d;\eta)\, \mathrm{d}\eta.$$

Plancherel's Formula yields

$$\int_\Omega |u(t)|^2 \mathrm{d}x = (2\pi)^{1-d} \int_\Omega |\phi(\eta)|^2 \exp(2t\mathrm{Re}\ T(\eta))|U(x_d;\eta)|^2 \mathrm{d}\eta \mathrm{d}x_d.$$

We see that the L^2-norm of $u(t)$ increases exponentially fast as t grows. Performing a rescaling as in Section 4.2.1, we find a sequence of solutions $(u^\lambda)_\lambda$, such that the L^2-norm of $u^\lambda(t)$ increases exponentially fast as $\lambda \to +\infty$ for any given $t > 0$, while the L^2-norm of $u^\lambda(0)$ is a constant times $\lambda^{-d/2}$. Even the H^s-norm of $u^\lambda(0)$ is polynomially bounded in λ for every s. We conclude that the IBVP cannot be well-posed in Sobolev spaces, even at the price of a loss of derivatives.

Proposition 4.2 *The Kreiss–Lopatinskiĭ condition is necessary for the well-posedness of the IBVP in either Hölder or Sobolev spaces. When it fails, no estimate can hold in such norms, even at the price of a loss of derivatives.*

In other words, the failure of the Kreiss–Lopatinskiĭ condition implies a Hadamard instability.

4.2.3 *The characteristic case*

When A^d is singular, (4.2.9) is not an ODE any longer. To mimic the analysis of Section 4.2.1, we need to extract from (4.2.9) an ODE. To do that, we first observe that, since A^d is diagonalizable, $\mathbb{C}^n$ is the direct sum of $R(A^d)$ and $\ker A^d$,

$$\mathbb{C}^n = R(A^d) \oplus \ker A^d.$$

Denoting by π the projector onto $\ker A^d$, along $R(A^d)$, we decompose $U = r + k$, with $k := \pi U$, $r := (I_n - \pi)U$. Then (4.2.9) is equivalent to

$$A^d \frac{\mathrm{d}r}{\mathrm{d}x_d} + (I_n - \pi)(\tau I_n + iA(\eta))(r+k) = 0, \tag{4.2.12}$$

$$\pi(\tau I_n + iA(\eta))(r+k) = 0. \tag{4.2.13}$$

From Theorem 1.6, we know that the endomorphism $\pi A(\eta)$ of $\ker A^d$ has a real spectrum. Hence $\pi(\tau I_n + iA(\eta))$ is non-singular on $\ker A^d$. Therefore, we may invert (4.2.13), as $k = M(\tau,\eta)r$, with $M(\tau,\eta) \in \mathscr{L}(R(A^d);\ker A^d)$. Then (4.2.12) becomes an ODE,

$$\frac{\mathrm{d}r}{\mathrm{d}x_d} = \mathcal{B}(\tau, \eta) r.$$

Given an initial datum $r(0)$, it admits a unique solution $r(x_d)$, and k is determined by $k(x_d) = M(\tau, \eta) r(x_d)$. Our next result is

Lemma 4.3 *For* $\mathrm{Re}\, \tau > 0$ *and* $\eta \in \mathbb{R}^{d-1}$*, the matrix* $\mathcal{B}(\tau, \eta)$ *does not have pure imaginary eigenvalues. Consequently, the number of eigenvalues of positive (respectively negative) real part does not depend on* (τ, η)*. It equals the number of negative (respectively positive) eigenvalues of* A^d*.*

Proof For λ to be an eigenvalue of $\mathcal{B}(\tau, \eta)$, it is necessary and sufficient that there exists an $r \in R(A^d)$, non-zero, and a $k \in \ker A^d$, such that

$$\lambda A^d r + (\tau I_n + iA(\eta))(r + k) = 0.$$

This amounts to saying that

$$(\tau I_n + iA(\eta) + \lambda A^d)(r + k) = 0.$$

Therefore one must have $\det(\tau I_n + iA(\eta) + \lambda A^d) = 0$. If λ is pure imaginary, then τ is so, by hyperbolicity assumption. The rest of the proof is similar to that of Lemma 4.1. □

We conclude from Lemma 4.3 that bounded solutions of (4.2.9) on $\mathbb{R}^+$ actually decay exponentially fast at $+\infty$, and form a vector space of dimension p. They take values in a p-dimensional vector space $E_-(\tau, \eta)$, again called the *stable subspace* of (4.2.9). The space $E_-(\tau, \eta)$ is made of sums $r + M(\tau, \eta) r$, with r in the stable subspace $E^s(\mathcal{B}(\tau, \eta))$.

Mimicking Sections 4.2.1 and 4.2.2, we see that a necessary condition for well-posedness in either Hölder or Sobolev spaces is again the *Kreiss–Lopatinskiĭ condition*, which reads

$$\mathbb{C}^n = \ker B \oplus E_-(\tau, \eta), \quad \forall \eta \in \mathbb{R}^{d-1}, \forall \mathrm{Re}\, \tau > 0.$$

To finish this section, we notice the following property, whose meaning is that the failure of the Kreiss–Lopatinskiĭ condition cannot come from the characteristic nature of the boundary.

Proposition 4.3 *For every* $\eta \in \mathbb{R}^{d-1}$ *and* $\mathrm{Re}\, \tau > 0$*, it holds that*

$$E_-(\tau, \eta) \cap \ker A^d = \{0\}.$$

Proof Let $u = r + k$ belong to $E_-(\tau, \eta)$. From (4.2.13), $\pi(\tau I_n + iA(\eta))(r + k) = 0$ holds, that is $u = r + M(\tau, \eta) r$. If, moreover, $u \in \ker A^d$, then $r = 0$ and therefore $u = 0$. □

4.3 The uniform Kreiss–Lopatinskiĭ condition

The symmetric, strictly dissipative case is very suggestive because its estimate (3.2.19) is the kind that we need for iterative purposes: It measures the solution u, its value at a given time T and its trace on the boundary, in the same norms as the corresponding data, respectively f, u_0 and g.

Another nice feature of (3.2.19) is that it is invariant under the rescaling $(x, t, u) \mapsto (\lambda x, \lambda t, u)$. This transforms (f, g, u_0) into $(\lambda f, g, u_0)$ and (γ, T) into $(\gamma/\lambda, \lambda T)$. All five terms in (3.2.19) are multiplied by the same power of λ. This immediately gives the following

Lemma 4.4 *Assume that the (not necessarily symmetric) hyperbolic IBVP (4.1.1)–(4.1.3) satisfies the a priori estimate (3.2.19) for every compactly supported, smooth u, for a given value of the parameter $\gamma > 0$ and every time $T > 0$. Then the estimate holds for every parameter γ with the same constant C. Similarly, if the estimate holds for a given time $T > 0$ and every $\gamma > 0$, then it holds for every $T, \gamma > 0$ with the same constant C.*

This suggests to introduce a stronger notion of well-posedness, which turns out to be suitable for a generalization to variable-coefficients IBVPs. It turns out to be well-suited for non-homogeneous IBVPs too.

Definition 4.4 *Let us consider a non-characteristic hyperbolic IBVP (4.1.1)–(4.1.3) in the domain $x_d > 0, t > 0$. We say that this IBVP is* strongly *well-posed in L^2 if the a priori estimate (3.2.19) holds for every smooth, rapidly decaying (in x) u, and every value of $\gamma, T > 0$, with a fixed constant C.*

The presence of the parameter γ in (3.2.19) (see (4.3.14) below) provides some flexibility in a non-linear iteration. It can be adjusted to ensure contraction in local-in-time problems.

4.3.1 *A necessary condition for strong well-posedness*

Let v be the partial Fourier transform of u, with respect to y, the estimate (3.2.19) is equivalent to

$$
\begin{aligned}
&\mathrm{e}^{-2\gamma T}\|v(T)\|_{L^2}^2 + \|v\|_{\gamma,T}^2 \qquad\qquad (4.3.14)\\
&\quad \le C\left(\|v(0)\|_{L^2}^2 + \int_0^T \mathrm{e}^{-2\gamma t}\left(\frac{1}{\gamma}\|(\hat{L}v)(t)\|_{L^2}^2 + \|\gamma_0 Bv(t)\|_{L^2}^2\right)\mathrm{d}t\right),
\end{aligned}
$$

where $\hat{L} = \partial_t + iA(\eta) + A^d\partial_d$. Since (4.3.14) reads

$$\int_{\mathbb{R}^d} \Phi(\eta)\mathrm{d}\eta \le 0,$$

where $\Phi(\eta)$ depends only on the restriction $v(\eta, \cdot, \cdot)$ (but does not depend on η-derivatives), it decouples as parametrized inequalities $\mathrm{Out}[w] \le C\,\mathrm{In}_\eta[w]$ between measurements of the output and the input, for every $\eta \in \mathbb{R}^{d-1}$ and every smooth $w(x_d, t)$ with fast decay as $x_d \to +\infty$, where

$$\mathrm{Out}[w] := \mathrm{e}^{-2\gamma T} \|w(T)\|_{L^2}^2 + \int_0^T \mathrm{e}^{-2\gamma t} \mathrm{d}t \left(\gamma \int_0^{+\infty} |w|^2 \mathrm{d}x_d + |w(0,t)|^2 \right)$$

$$\mathrm{In}_\eta[w] := \|w(0)\|_{L^2}^2 + \int_0^T \mathrm{e}^{-2\gamma t} \mathrm{d}t \left(\frac{1}{\gamma} \int_0^{+\infty} |L_\eta w|^2 \mathrm{d}x_d + |Bw(0,t)|^2 \right).$$

Hereabove, L_η is obtained from $\hat{L}$ by freezing η. We emphasize that the constant C is the same as that in (4.3.14). In particular, it does not depend on η. Since $\mathrm{Out}[w]$ is a sum of positive terms, the estimate (3.2.19) implies the inequality

$$\int_0^T \mathrm{e}^{-2\gamma t} |w(0,t)|^2 \mathrm{d}t \leq C \, \mathrm{In}_\eta[w], \tag{4.3.15}$$

for every η and smooth, fast decaying w. Let us now choose a complex number τ with $\mathrm{Re}\,\tau > 0$ and a vector $V \in E_-(\tau, \eta)$. We apply (4.3.15) to the space-decaying function

$$w := \mathrm{e}^{t\tau} \exp(x_d \mathcal{A}(\tau, \eta)) V,$$

for which we have $L_\eta w \equiv 0$. We then obtain

$$|V|^2 \leq C \left(|BV|^2 + \left(\int_0^T \exp(2(\mathrm{Re}\,\tau - \gamma)t) \mathrm{d}t \right)^{-1} \int_0^{+\infty} |w|^2 \mathrm{d}x_d \right).$$

Let us choose γ in the interval $(0, \mathrm{Re}\,\tau)$. As $T \to +\infty$, the integral

$$\int_0^T \exp(2(\mathrm{Re}\,\tau - \gamma)t) \mathrm{d}t$$

tends to infinity. Passing to the limit, we obtain

$$|V|^2 \leq C|BV|^2.$$

In conclusion, we have found a necessary condition for strong well-posedness, in the form

$$\forall \mathrm{Re}\,\tau > 0,\ \forall \eta \in \mathbb{R}^{d-1},\ \forall V \in E_-(\tau, \eta), \quad |V|^2 \leq C|BV|^2, \tag{4.3.16}$$

for some finite constant C, independent of (τ, η, v).

We point out that (4.3.16) implies the Kreiss–Lopatinskiĭ condition (that is, $BV = 0$ and $V \in E_-(\tau, \eta)$ imply $V = 0$). However, it is a stronger property, since the Kreiss–Lopatinskiĭ condition is only equivalent to an estimate of the form

$$|V| \leq C(\tau, \eta)|BV|, \quad \forall V \in E_-(\tau, \eta),$$

where the number C might not be uniformly bounded (though being an homogeneous function of (τ, η), of degree zero). Hence, (4.3.16) exactly means that the Kreiss–Lopatinskiĭ condition holds, with a uniform constant. This justifies the following

Definition 4.5 *Let L be hyperbolic, A^d be invertible. Given $B \in M_{p\times n}(\mathbb{R})$, we say that the IBVP (4.1.1)–(4.1.3) satisfies the* uniform Kreiss–Lopatinskiĭ condition *(UKL) in the domain $x_d > 0, t > 0$, if*

- *p equals the number of positive eigenvalues of A^d,*
- *there exists a number $C > 0$, such that*

$$|V| \leq C|BV|, \quad \forall \eta \in \mathbb{R}^{d-1}, \forall \mathrm{Re}\, \tau > 0, \forall V \in E_-(\tau, \eta).$$

We point out that the inequality $|V| \leq C|BV|$ for a single pair (τ, η) already implies that p is larger than or equal to the number of positive eigenvalues of A^d.

Remark We have shown above that (UKL) is a necessary condition for the L^2-well-posedness of the pure (namely $f \equiv 0$, $u_0 \equiv 0$) boundary value problem. If in addition, the operator L is Friedrichs symmetric (or symmetrizable as well), it is rather easy (see [185], page 199–200) to show that (UKL) is also a sufficient condition for the pure L^2-well-posedness. Thus, it seems at first glance that the (UKL) condition concerns *only* the pure boundary value problem, though the general IBVP decouples into this one, plus a pure Cauchy problem, via the Duhamel formula. It is therefore fascinating that the (UKL) condition actually enables us to prove the L^2 estimates for the complete IBVP, at least when L is constantly hyperbolic and the boundary is not characteristic, as we shall see in Chapter 5.

4.3.2 *The characteristic IBVP*

Recall that we assume $\ker A^d \subset \ker B$ in the characteristic case (A^d is singular.) The best control of boundary terms that we expect is that of $A^d u$. Control of the components of u in the kernel of A^d will at least involve higher-order norms (norms of derivatives) of the data. Consequently, we must adapt our definition as follows. We recall that γ_0 denotes the trace operator on the boundary (while γ is some positive real number).

Definition 4.6 *Consider a (possibly characteristic) hyperbolic IBVP (4.1.1)–(4.1.3) in the domain $x_d > 0, t > 0$. We say that this IBVP is* strongly L^2*-well-posed if* $\ker A^d \subset \ker B$ *and if, moreover, the quantity*

$$\mathrm{e}^{-2\gamma T}\|u(T)\|^2_{L^2} + \int_0^T \mathrm{e}^{-2\gamma t} dt \left(\int_{\partial\Omega} |\gamma_0 A^d u(y,t)|^2 \mathrm{d}y + \gamma \int_\Omega |u(x,t)|^2 \mathrm{d}x \right)$$

is bounded by above by

$$C\left(\|u(0)\|^2_{L^2} + \int_0^T \mathrm{e}^{-2\gamma t} \left(\frac{1}{\gamma}\|(Lu)(t)\|^2_{L^2} \mathrm{d}t + \|\gamma_0 Bu(t)\|^2_{L^2} \right) \mathrm{d}t \right),$$

for every smooth, rapidly decaying (in x) u, and every value of $\gamma, T > 0$, with a fixed constant C.

As before, we obtain a necessary condition for strong well-posedness, in the form of a (UKL) condition

$$\exists C > 0, \quad \left(\eta \in \mathbb{R}^{d-1}, \mathrm{Re}\,\tau > 0, V \in E_-(\tau,\eta)\right) \Longrightarrow (|A^d V| \leq C|BV|). \quad (4.3.17)$$

Let us note that (4.3.17) not only implies

$$E_-(\tau,\eta) \cap \ker B \subset \ker A^d.$$

In view of Proposition 4.3, this actually ensures the Kreiss–Lopatinskiĭ condition

$$E_-(\tau,\eta) \cap \ker B = \{0\},$$

though uniformity holds only[3] 'modulo $\ker A^d$', according to (4.3.17).

4.3.3 *An equivalent formulation of (UKL)*

We shall prove later that the (UKL) condition is actually a necessary and sufficient condition for well-posedness, at least for the important class of constantly hyperbolic operators, in the non-characteristic case[4]. However, this condition, as defined above, does not give a practical tool when one faces a particular IBVP, because the computation of the constant $C(\tau,\eta)$ is intricate, and it is not easy to see whether it is upper bounded as (τ,η) varies. It turns out that there is a much more explicit way to check (UKL) condition. To explain what is going on, we begin with the following observation, which will be proved in Chapter 5.

> We recall that the set $\mathbf{G}(n,p)$ of vector subspaces of dimension p in $\mathbb{C}^n$ is a compact differentiable manifold, called the *Grassmannian manifold.* This object is isomorphic to the homogeneous space (set of left cosets) $\mathbf{GL}_p(\mathbb{R})\backslash\mathbf{M}^0_{n\times p}(\mathbb{R})$, where $\mathbf{M}^0_{n\times p}(\mathbb{R})$ denotes the dense open set of $\mathbf{M}_{n\times p}(\mathbb{R})$ consisting in matrices of full rank p.

Lemma 4.5 *Assume that the operator L is constantly hyperbolic and the boundary is non-characteristic. Then the map $(\tau,\eta) \mapsto E_-(\tau,\eta)$ (already defined for* $\mathrm{Re}\,\tau > 0$*, valued in $\mathbf{G}(n,p)$) admits a unique limit at every boundary point $(i\rho,\eta)$ (with $\rho \in \mathbb{R}$, $\eta \in \mathbb{R}^{d-1}$), with the exception of the origin.*

It is then natural to call $E_-(i\rho,\eta)$ this limit. We emphasize that, in general, $E_-(i\rho,\eta)$ only contains, but need not be equal to, the stable subspace of the

[3] In view of Proposition 4.3, we also have an estimate $|V| \leq C'|BV|$ on $E_-(\tau,\eta)$, at least when $\mathrm{Re}\,\tau > 0$. However, we do not know whether $E_-(\tau,\eta) \cap \ker A^d$ is trivial at boundary points $\mathrm{Re}\,\tau = 0$. This left the possibility that $C' = C'(\tau,\eta)$ and that the estimate of V in terms of BV be non-uniform.

[4] We should temper this sentence. By well-posedness, we mean strong L^2-well-posedness of the non-homogeneous BVP. We shall see in Chapter 8 that strong well-posedness may hold true in some complicated space, when the Kreiss–Lopatinskiĭ condition is satisfied but not uniformly. In Chapter 7, we show that the IBVP with an homogeneous boundary condition needs a property weaker than (UKL).

differential equation $A^d U' + i(\rho I_n + A(\eta))U = 0$, since the latter has dimension less than or equal to p.

Let us assume that the IBVP defined by the pair (L, B) satisfies (UKL) condition. Then, by continuity, (4.3.16) still holds when $\mathrm{Re}\,\tau = 0$, which means that $E_-(\tau,\eta) \cap \ker B = \{0\}$ for these parameters too. Conversely, assume that this intersection is trivial for every non-zero pair (τ,η) with $\mathrm{Re}\,\tau \geq 0$. Then, for every such pair, the number

$$c(\tau,\eta) := \sup\left\{\frac{|V|}{|BV|}\,;\, V \in E_-(\tau,\eta),\, V \neq 0\right\}$$

is finite. The function $(\tau,\eta) \mapsto c$ is continuous and homogeneous of degree zero. Since the hemisphere defined by $\mathrm{Re}\,\tau \geq 0$, $\eta \in \mathbb{R}^{d-1}$ and $|\tau|^2 + |\eta|^2 = 1$ is compact, we infer that this function is upper bounded. Hence, the IBVP satisfies (UKL) condition. Finally,

Corollary 4.1 *Let L be constantly hyperbolic and the boundary be non-characteristic. Then the IBVP (4.1.1)–(4.1.3) satisfies the uniform Kreiss–Lopatinskiĭ condition if, and only if,* $E_-(\tau,\eta) \cap \ker B = \{0\}$ *for every non-zero pair* (τ,η) *with* $\mathrm{Re}\,\tau \geq 0$ *and* $\eta \in \mathbb{R}^{d-1}$.

This corollary gives a practical tool for checking the (UKL) condition. The main difficulty during calculations being the computation of $E_-(\tau,\eta)$ when $\mathrm{Re}\,\tau$ vanishes.

Remark We warn the reader that the continuous extension of $E_-(\tau,\eta)$ to boundary points ($\mathrm{Re}\,\tau = 0$) may not exist for non-constantly hyperbolic operators (operators for which eigenvalues do cross each other). This happens even within the class of Friedrichs-symmetric systems. See, however, the deep analysis [135] by Métivier and Zumbrun of Friedrichs-symmetric IBVPs with characteristic fields of variables multiplicities, which works out in the case of MHD.

4.3.4 *Example: The dissipative symmetric case*

To show the relevance of the notion of the uniform Kreiss–Lopatinskiĭ condition, we prove:

Proposition 4.4 *Let L be Friedrichs symmetric. If B is dissipative, then the IBVP satisfies the Kreiss–Lopatinskiĭ condition. If B is strictly dissipative, the IBVP satisfies (UKL).*

Proof Let $\eta \in \mathbb{R}^{d-1}$ and $\mathrm{Re}\,\tau > 0$ be given. Let u be an element of $E_-(\tau,\eta)$ and U be the solution of the differential-algebraic Cauchy problem

$$(\tau I_n + iA(\eta))U + A^d \frac{\mathrm{d}U}{\mathrm{d}x_d} = 0, \quad U(0) = u.$$

Then U decays exponentially fast at $+\infty$. Since L is Friedrichs symmetric, the standard energy estimates gives

$$2(\mathrm{Re}\,\tau)\int_0^{+\infty}|U|^2\mathrm{d}x_d=(A^d u,u).$$

If B is dissipative and if $u\in\ker B$, we deduce that

$$\int_0^{+\infty}|U|^2\mathrm{d}x_d\le 0,$$

hence $U\equiv 0$ and $u=0$. Therefore $E_-(\tau,\eta)\cap\ker B=\{0\}$. This is the Kreiss–Lopatinskiĭ condition.

If B is strictly dissipative, we know that there exist positive constants ε and C such that the Hermitian form

$$w\mapsto\varepsilon|A^d w|^2+(A^d w,w)-C|Bw|^2$$

is non-positive. With u as above, we immediately obtain

$$\varepsilon|A^d u|^2\le C|Bu|^2.$$

Since this inequality does not depend on (τ,η), the Kreiss–Lopatinskiĭ condition is satisfied uniformly. □

It is not true in general that every IBVP satisfying (UKL) can be put in a symmetric, strictly dissipative form. A crude reason is that if $n\ge 3$ and $d\ge 2$, most hyperbolic operators are not symmetrizable. However, when $Lu=(\partial_t^2-\Delta_x)u$ with $u:\Omega\to\mathbb{R}^n$ (hence L is diagonal) and the IBVP is coupled through first-order boundary conditions, Godunov *et al.* proved the converse of Proposition 4.4; see, for instance, [71].

4.4 The adjoint IBVP

The existence of a solution of a general IBVP will be proved by a duality argument[5]. Hence, we need to construct an adjoint IBVP. To a pair (L,B), where L is a hyperbolic operator and B a boundary matrix, we shall associate a pair (L',C), such that whenever

$$Lu=f,\quad L'v=F\quad\text{in } x_d>0$$

and

$$Bu=g,\quad Cv=h\quad\text{on } x_d=0,$$

[5]In the present context of constant coefficients and a half-space domain, the solution could be constructed directly, as we did in the symmetric dissipative case. But we have in mind the generalization to variable coefficients.

a duality formula holds. Of course, L' will be the adjoint operator L^* found in Chapter 1,

$$L^* = -\partial_t - \sum_\alpha (A^\alpha)^T \partial_\alpha.$$

Assume that $u, v \in \mathscr{D}(\Omega \times \mathbb{R})^n$ (for the sake of simplicity, the time variable runs through the whole line) and let us compute

$$\begin{aligned} \int_{\mathbb{R}} \int_\Omega ((v, f) - (u, F)) \mathrm{d}x \, \mathrm{d}t &= \int_{\mathbb{R}} \int_\Omega ((v, Lu) - (u, L^* v)) \mathrm{d}x \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \int_\Omega (\partial_t (u, v) + \sum_\alpha \partial_\alpha (A^\alpha u, v)) \mathrm{d}x \, \mathrm{d}t \qquad (4.4.18) \\ &= - \int_{\mathbb{R}} \int_{\partial\Omega} (A^d u, v) \, \mathrm{d}y \, \mathrm{d}t, \end{aligned}$$

from Green's formula. The left-hand side is a bilinear form in the variables (u, f) and (v, F), respectively. Similarly, we search a decomposition of A^d in such a way that the right-hand side be bilinear in (u, g) and (v, h). It will be sufficient that A^d reads $C^T N + M^T B$, for suitable matrices M and N, for then

$$(A^d u, v) = (Nu, Cv) + (Bu, Mv) = (Nu, h) + (g, Mv).$$

Moreover, in order that the (backward) adjoint IBVP be well-posed, we ask in particular that it be normal, which at least requires that C is $q \times n$, where q is the number of negative eigenvalues of $(A^d)^T$, that is of A^d. In the non-characteristic case, this means $q = n - p$, while in general it only implies $q \leq n - p$.

Let us begin with the non-characteristic case. We choose any $(n - p) \times n$ matrix X that is onto, such that

$$\begin{pmatrix} B \\ X \end{pmatrix} \in \mathbf{GL}_n(\mathbb{R}).$$

Such a matrix exists since B is onto. Let us write the inverse blockwise as (Y, D), where D is $n \times (n - p)$. From the identity $YB + DX = I_n$, we may choose $C := (A^d D)^T$, $M = (A^d Y)^T$ and $N = X$.

We notice that $R(D) \subset \ker B$, since $BD = 0_{p \times (n-p)}$. However, both spaces have the same dimension $n - p$, since B is onto and D is one-to-one. Hence $R(D) = \ker B$ and

$$\ker C = (R(A^d D))^\perp = (A^d R(D))^\perp = (A^d \ker B)^\perp. \qquad (4.4.19)$$

Identity (4.4.19) shows that $\ker C$, which is the meaningful object in a boundary condition, does not depend on the choice of the complement X. It actually does not even depend on the procedure; we leave the reader to verify that the duality identity (4.4.18) only needs $\ker C = (A^d \ker B)^\perp$.

The characteristic case can be treated in the same spirit. However, the matrix $(A^d D)^T$ built above is $(n - p) \times n$ instead of being $q \times n$. Hence, we must first reduce the matrix A^d to the block-diagonal form

$$\begin{pmatrix} 0_m & 0 \\ 0 & a^d \end{pmatrix}, \quad a^d \in \mathbf{GL}_{n-m}(\mathbb{R}).$$

Because of $\ker A^d \subset \ker B$, we then have $B = (0, B_1)$, where B_1 is $p \times (n-m)$ and is onto. Then we proceed as before, replacing B by B_1 and n by $n-m$. We thus find a matrix C_1, such that $a^d = M_1^T B_1 + C_1^T N_1$ and $\ker C_1 = (a^d \ker B_1)^\perp$. Then the following matrices work:

$$C = \begin{pmatrix} 0 \\ C_1 \end{pmatrix}, \quad M = (0, M_1), \quad N = (0, N_1).$$

Finally, we obtain the result:

Proposition 4.5 *Let a pair (L, B) be given, where L is hyperbolic, $B \in \mathbf{M}_{p\times n}(\mathbb{R})$, where p is the number of incoming characteristics, and* $\ker A^d \subset \ker B$. *Let L^* be the adjoint of L, with q the number of its incoming characteristics, that is the number of negative eigenvalues of A^d. Then there exists a matrix $C \in \mathbf{M}_{q\times n}(\mathbb{R})$ and matrices M, N such that $A^d = C^T N + M^T B$. The matrix C is characterized uniquely, up to left multiplication, by the identity* $\ker C = (A^d(\ker B))^\perp$.

When u, v are smooth fields on $\Omega \times \mathbb{R}$, decaying fast enough at infinity, the following identity holds

$$\int_{\mathbb{R}}\int_{\Omega}((Lu, v) - (L^*v, u))\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}}\int_{\partial\Omega}((Nu, Cv) + (Bu, Mv))\mathrm{d}y\,\mathrm{d}t = 0. \tag{4.4.20}$$

As a corollary, we obtain

Theorem 4.1 *Given a normal hyperbolic IBVP, defined by (L, B), there exists an adjoint IBVP, defined by the pair (L^*, C), which is normal hyperbolic (backward in time) and satisfies the identity (4.4.20). One has*

$$L^* = -\partial_t - \sum_\alpha (A^\alpha)^T \partial_\alpha,$$

$$A^d = C^T N + M^T B,$$

where $C \in \mathbf{M}_{q\times n}(\mathbb{R})$ satisfies $\ker C = (A^d(\ker B))^\perp$. *Moreover,*

$$\ker A^d = \ker N \cap \ker B. \tag{4.4.21}$$

Finally, the former IBVP is the adjoint of the latter.

Proof It remains to prove that the adjoint problem is backward normal and to check the validity of (4.4.21).

By backward normal, we mean that the IBVP obtained from (L^*, C) by the time reversion $t \to -t$ is normal. This amounts to saying that $C \in \mathbf{M}_{q\times n}(\mathbb{R})$, where q is the number of negative eigenvalues of $(A^d)^T$ (or of A^d as well), and that $\mathbb{R}^n = \ker C \oplus E^u(-(A^d)^T)$. In other words, it remains to check that

$$\mathbb{R}^n = (\ker C)^\perp + (E^u(A^d) \oplus \ker A^d). \tag{4.4.22}$$

However, our construction satisfies $(\ker C)^\perp = R(C^T) = A^d(\ker B)$. Therefore, (4.4.22) follows from the sequence

$$\mathbb{R}^n = R(A^d) + \ker A^d = A^d(\ker B \oplus E^u(A^d)) + \ker A^d \subset A^d(\ker B) \\ + A^d(E^u(A^d)) + \ker A^d.$$

Last, $\ker N \cap \ker B \subset \ker A^d$ is trivial, and the converse follows from the facts that $\ker A^d \subset \ker B$ and that C is onto. □

The method of duality will need the following important fact.

Theorem 4.2 *Let (L, B) define a normal hyperbolic IBVP on $\Omega = \{x_d > 0\}$, and let (L^*, C) define its dual IBVP. Let $(\tau, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}$ be given, such that* Re $\tau > 0$. *It holds that*

$$(\mathbb{C}^n = E_-(\tau, \eta) \oplus \ker B) \iff (\mathbb{C}^n = E_-^*(-\bar{\tau}, -\eta) \oplus \ker C),$$

where $E_-^*(z, \sigma)$ denotes the stable invariant subspace for the differential-algebraic equation

$$(A^d)^T V' + (zI_n + iA(\sigma)^T)V = 0.$$

Consequently, (L, B) satisfies the Kreiss–Lopatinskiĭ property if and only if (L^*, C) satisfies the Kreiss–Lopatinskiĭ property, backward in time. And similarly, (L, B) satisfies the uniform Kreiss–Lopatinskiĭ property if and only if (L^*, C) satisfies the uniform Kreiss–Lopatinskiĭ property, backward in time.

Proof We first note that, whenever V solves $(A^d)^T V' - (\bar{\tau} I_n + iA(\eta)^T)V = 0$ and U solves $A^d U' + (\tau I_n + iA(\eta))U = 0$, then $(V^* A^d U)' = 0$ and therefore $V^T A^d U$ is constant. If, moreover, U and V decay as $x \to +\infty$, this constant is zero. In other words, we have

Lemma 4.6 *Assume that* Re $\tau > 0$ *and $\eta \in \mathbb{R}^{d-1}$. If $u \in E_-(\tau, \eta)$ and $v \in E_-^*(-\bar{\tau}, -\eta)$, then $v^* A^d u = 0$.*

By continuity, the equality also holds true if Re $\tau \geq 0$.

Let us assume that $\mathbb{C}^n = E_-^*(-\bar{\tau}, -\eta) \oplus \ker C$, that is $C(E_-^*) = C(\mathbb{C}^n) = \mathbb{C}^q$. Then let u belong to $\ker B \cap E_-(\tau, \eta)$. From Lemma 4.6 and the decomposition of A^d, we have, for all v in $E_-^*(-\bar{\tau}, -\eta)$,

$$0 = v^* A^d u = (Cv)^* N u,$$

which amounts to saying that $Nu = 0$. From (4.4.21), we obtain $u \in \ker A^d$. Proposition 4.3 then gives $u = 0$, proving that $\mathbb{C}^n = E_-(\tau, \eta) \oplus \ker B$.

The converse follows after the exchange of (L, B) and (L^*, C).

We now prove the uniform part of the theorem. Let us assume that the dual IBVP satisfies (UKL) condition, meaning that there exists a positive constant c such that, for every complex number τ with $\mathrm{Re}\,\tau > 0$, for every $\eta \in \mathbb{R}^{d-1}$ and every vector $v \in E_-^*(-\bar{\tau}, -\eta)$, it holds that

$$|(A^d)^T v| \leq c_0\,|Cv|.$$

From Theorem 4.1, there exists a positive number c_1 such that the following inequality holds

$$|A^d u| \leq c_1(|Nu| + |Bu|).$$

Using the Kreiss–Lopatinskiĭ condition for the dual problem, and then its uniformity, we may write

$$\begin{aligned} |Nu| &= \sup\left\{ \frac{(Cv, Nu)}{|Cv|}\,;\, v \in E_-^*(-\bar{\tau}, -\eta) \right\} \\ &\leq c_0 \sup\left\{ \frac{(Cv, Nu)}{|(A^d)^T v|}\,;\, v \in E_-^*(-\bar{\tau}, -\eta) \right\}. \end{aligned}$$

Since the kernel of M contains that of $(A^d)^T$, there exists another constant c_2 such that $|Mv| \leq c_2|(A^d)^T v|$. Using now the decomposition of A^d, we conclude that

$$|Nu| \leq c_0 c_2 |Bu| + c_0 \sup\left\{ \frac{(v, A^d u)}{|(A^d)^T v|}\,;\, v \in E_-^*(-\bar{\tau}, -\eta) \right\}.$$

The use of Lemma 4.6 now gives the expected result: if $u \in E_-(\tau, \eta)$, then

$$|A^d u| \leq c_1(1 + c_0 c_2)|Bu|,$$

where the constants do not depend on the pair (τ, η). □

4.5 Main results in the non-characteristic case

The main statement of this chapter is the following strong well-posedness result in L^2. It completes the study of the strongly dissipative symmetric case (see Chapter 3).

Theorem 4.3 *Let Ω be the half-plane $\{x \in \mathbb{R}^d\,;\, x_d > 0\}$. Let*

$$L = \partial_t + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha + R$$

be a constantly hyperbolic first-order operator, with constant coefficients. Assume that the boundary is non-characteristic ($\det A^d \neq 0$*), and let p be the number of positive eigenvalues of A^d (the number of incoming characteristics). Let $B \in M_{p\times n}(\mathbb{R})$ have rank p.*

Then the IBVP

$$\begin{aligned} Lu &= f, \quad \text{in } \Omega \times (0,T), \\ Bu &= g, \quad \text{on } \partial\Omega \times (0,T), \\ u &= u_0, \quad \text{in } \Omega \times \{0\}, \end{aligned}$$

is strongly well-posed in L^2 if, and only if, the uniform Kreiss–Lopatinskiĭ condition holds. In other words:

- *On the one hand, if* (3.2.19) *holds for smooth solutions, then the IBVP satisfies (UKL).*
- *On the other hand, assuming that (UKL) holds, we have the following existence and uniqueness property: for all data $f \in L^2(\Omega \times (0,T))$, $g \in L^2(\partial\Omega \times (0,T))$ and $u_0 \in L^2(\Omega)$, there exists a unique $u \in L^2(\Omega \times (0,T))$ with the following properties:*
  - *It satisfies $Lu = f$ in $\Omega \times (0,T)$,*
  - *Its trace on $\partial\Omega \times (0,T)$ (which is known to belong to $H^{-1/2}$ because of $Lu \in L^2(\Omega \times (0,T))$) is square-integrable, and satisfies $Bu = g$,*
  - *It belongs to $\mathscr{C}([0,T]; L^2(\Omega))$, and satisfies $u(t=0) = u_0$,*
  - *Finally,* (3.2.19) *holds for every $\gamma > \gamma_1$. Here, γ_1 and the constant C depend only on A^α, B, R, but not on T, f, g or u_0; when $R = 0$, one may take $\gamma_1 = 0$.*

A related result holds when the boundary is characteristic. However, we shall establish it for a smaller class of admissible operators. We postpone the corresponding analysis to Chapter 6.

4.5.1 *Kreiss' symmetrizers*

The proof that (UKL) implies well-posedness follows the lines of Chapter 2, but displays new ideas, coming partly from Chapter 3.

We begin by the analysis of the BVP, that is a boundary value problem, posed for all time $t \in \mathbb{R}$, thus without initial condition. We first prove the analogue of (3.2.19), but without final and initial states:

$$\int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} (\gamma \|u(t)\|_{L^2}^2 + \|\gamma_0 u(t)\|_{L^2}^2) \mathrm{d}t \\ \leq C \int_{\mathbb{R}} \mathrm{e}^{-2\gamma t} \left(\frac{1}{\gamma} \|(Lu)(t)\|_{L^2}^2 + \|\gamma_0 B u(t)\|_{L^2}^2 \right) \mathrm{d}t. \tag{4.5.23}$$

It readily implies that the solution is unique. Since the adjoint BVP satisfies the same assumptions as the direct one, it enjoys the same estimate. Through a duality argument, using Hahn–Banach and Riesz theorems, we obtain the existence of the solution of the BVP. The resolution stands of course in the space associated to the norms present in (4.5.23), say in L^2_γ.

The passage from the well-posedness of the BVP to that of the IBVP is made in three steps, and is due to Rauch. The first one is causality; we prove that if the source f and the boundary data vanish for negative times, then so does the solution. This allows us to solve the IBVP when the initial data vanishes identically. The next step consists in a new estimate, namely that of $u(\cdot, T)$ in L^2_γ, when $u(\cdot, 0) \equiv 0$. The last one is again a duality argument.

The proof of the estimate (4.5.23) mimics that of the Friedrichs-symmetric case with strict dissipation, considered in Chapter 3. However, a dissipative symmetrizer has not been given a priori and we have to build it. A main technical difficulty is that this symmetrizer, called a Kreiss symmetrizer, is symbolic, thus it depends on the frequencies (τ, η). Its construction is lengthy and is postponed to Chapter 5. Theorem 5.1 tells us that there exists a map $(\tau, \eta) \mapsto K(\tau, \eta)$ (the *Kreiss symmetrizer*), defined for $\eta \in \mathbb{R}^d$ and Re $\tau > 0$, with the following properties:

i) $(\tau, \eta) \mapsto K$ is bounded and homogeneous of degree zero,

ii) $\Sigma(\tau, \eta) := K(\tau, \eta)A^d$ is Hermitian,

iii) There exists a positive constant c_0, independent of (τ, η), such that

$$w^*\Sigma(\tau, \eta)w \leq -c_0\|w\|^2, \quad \forall w \in \ker B \tag{4.5.24}$$

iv) There exists a positive constant, say again c_0, independent of (τ, η), such that

$$\text{Re } (v^*M(\tau, \eta)v) \geq c_0(\text{Re } \tau)\|v\|^2, \quad \forall v \in \mathbb{C}^n, \tag{4.5.25}$$

where

$$M(\tau, \eta) := K(\tau, \eta)(\tau I_n + iA(\eta)).$$

Note that in the Friedrichs-symmetric, strictly dissipative case, one can simply choose $K \equiv I_n$, which is classical instead of symbolic. Point *ii)* is the symmetry property, while point *iii)* is the strict dissipation.

Remark Estimate (4.5.23) can be used the same way as (3.2.19) to show the necessity of (UKL). We leave the reader to adapt the calculations of Section 4.3.1.

4.5.2 *Fundamental estimates*

Recall that the construction of the Kreiss symmetrizer is postponed to the next chapter, under appropriate assumptions. We thus suppose that L is constantly hyperbolic, that the boundary is non-characteristic and that the boundary condition satisfies (UKL), and we admit in the remainder of the present chapter that these properties ensure the existence of a dissipative symmetrizer K (Theorem 5.1).

Let u be given in $\mathscr{D}(\overline{\Omega} \times \mathbb{R}_t)$, meaning that u is extendable to $\mathbb{R}^d \times \mathbb{R}$ as a $\mathscr{C}^\infty$ function with compact support. For the sake of clarity, we define $f = Lu$

and $g = \gamma_0 Bu$. Let us define the Laplace transform in time, Fourier transform in y

$$h \mapsto \mathcal{L}h(\tau,\eta) := \iint_{\mathbb{R}^{d-1}\times\mathbb{R}} \mathrm{e}^{-\tau t - i\eta\cdot y} h(y,t)\,\mathrm{d}y\,\mathrm{d}t \quad \eta\in\mathbb{R}^{d-1},\ \mathrm{Re}\ \tau > 0.$$

Since we shall deal with smooth and compactly supported functions, we shall never discuss the convergence of the integral. We need the following auxiliary functions:

$$U(\cdot,\cdot,x_d) := \mathcal{L}[u(\cdot,x_d,\cdot)], \quad F(\cdot,\cdot,x_d) := \mathcal{L}[f(\cdot,x_d,\cdot)], \quad G := \mathcal{L}g.$$

Integration by parts yields the identities

$$\mathcal{L}[\partial_\alpha h] = i\eta_\alpha \mathcal{L}h, \quad \alpha = 1,\dots,d-1, \qquad \mathcal{L}[\partial_d h] = \partial_d \mathcal{L}h, \qquad \mathcal{L}[\partial_t h] = \tau\mathcal{L}h.$$

Therefore, we have

$$(\tau I_n + iA(\eta) + R)U + A^d U' = F,$$

where the prime denotes the x_d-derivative. Multiplying on the left by $U^* K(\tau,\eta)$, we have

$$U^*(M(\tau,\eta) + K(\tau,\eta)R)U + U^*\Sigma(\tau,\eta)U' = U^* K(\tau,\eta)F.$$

Let us take the real part in this identity. Since Σ is Hermitian, we have

$$\mathrm{Re}\ (U^*\Sigma(\tau,\eta)U') = \frac{1}{2}(U^*\Sigma(\tau,\eta)U)'.$$

Using points *i)* and *iv)* above, and integrating in x_d over $\mathbb{R}^+$, we derive an inequality

$$(c_0\mathrm{Re}\ \tau - C_1)\int_0^\infty \|U\|^2 \mathrm{d}x_d \le \frac{1}{2}U(0)^*\Sigma(\tau,\eta)U(0) + C\int_0^\infty \|U\|\,\|F\|\,\mathrm{d}x_d.$$

We now appeal to Lemma 3.3: There exist positive constants ε and C, such that the Hermitian form $w \mapsto \varepsilon\|w\|^2 + w^*\Sigma(\tau,\eta)w - C\|Bw\|^2$ is non-positive. Checking the proof of the lemma, we easily see that the constants may be chosen independently of (τ,η). We may write c_0 instead of ε. We deduce therefore the bound

$$(c_0\mathrm{Re}\ \tau - C_1)\int_0^\infty \|U\|^2 \mathrm{d}x_d + \frac{c_0}{2}\|U(0)\|^2 \le C\int_0^\infty \|U\|\,\|F\|\,\mathrm{d}x_d + C\|BU(0)\|^2,$$

where the argument 0 stands for the x_d-variable. If $C_1 > 0$, we take a threshold γ_1 larger than C_1/c_0, to obtain

$$\gamma\int_0^\infty \|U\|^2 \mathrm{d}x_d + \frac{1}{2}\|U(0)\|^2 \le C\int_0^\infty \|U\|\,\|F\|\,\mathrm{d}x_d + C\|BU(0)\|^2,$$

for $\gamma = \operatorname{Re}\tau > \gamma_1$. Using now the Cauchy–Schwarz inequality, we obtain

$$\gamma\int_0^\infty \|U\|^2 \mathrm{d}x_d + \|U(0)\|^2 \leq C\left(\frac{1}{\gamma}\int_0^\infty \|F\|^2 \mathrm{d}x_d + \|BU(0)\|^2\right).$$

Integrating in η, we obtain

$$\gamma\|U(\tau)\|_{L^2}^2 + \|U(\tau,0)\|_{L^2}^2 \leq C\left(\frac{1}{\gamma}\|F(\tau)\|_{L^2}^2 + \|BU(\tau,0)\|_{L^2}^2\right),$$

where the L^2-norm is taken in terms of η and, for U and F, in terms of x_d too. Integrating then with respect to the imaginary part of τ and using Plancherel formula, we obtain the weighted estimate

$$\gamma\iint_{\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|u\|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|\gamma_0 u\|^2 \mathrm{d}y\,\mathrm{d}t \tag{4.5.26}$$
$$\leq C\left(\frac{1}{\gamma}\iint_{\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|Lu\|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|\gamma_0 Bu\|^2 \mathrm{d}y\,\mathrm{d}t\right).$$

We consider now less-regular functions. For this purpose, we define the weighted spaces L^2_γ of measurable functions u such that $(x,t)\mapsto \mathrm{e}^{-\gamma t}u(x,t)$ is square-integrable. These are Hilbert spaces with the obvious norms

$$\|u\|_\gamma := \|\mathrm{e}^{-\gamma t}u\|_{L^2}.$$

Such spaces may concern functions defined either on $\Omega\times\mathbb{R}_t$, or on $\partial\Omega\times\mathbb{R}_t$. We define in a similar way weighted Sobolev spaces H^k_γ. We note that, given u in L^2_γ such that $Lu \in L^2_\gamma$, the trace $\gamma_0 u$ is well-defined in a class $H^{-1/2}$, weighted by $\mathrm{e}^{\gamma t}$, which we denote $H_\gamma^{-1/2}$.

Lemma 4.7 *Assume that u, Lu and $\gamma_0 Bu$ are in the class L^2_γ. Then*

i) The trace $\gamma_0 u$ belongs to the class L^2_γ.
ii) The function u satisfies (4.5.26).

Proof Using a cut-off function, we may restrict ourselves to the case of a compactly supported functions.

We begin with the easy case where u belongs to $H^1_\gamma(\Omega\times\mathbb{R})$ and has a compact support. Then point *i*) is obvious. On the other hand, u is the limit in H^1_γ of functions $u^\epsilon \in \mathscr{D}(\bar{\Omega}\times\mathbb{R})$. Hence Lu and $\gamma_0 Bu$ are the limits in L^2_γ of the sequences Lu^ϵ and $\gamma_0 Bu^\epsilon$. The latter satisfy (4.5.26), which remains valid in the limit.

We turn to the general case of an L^2-function with compact support. Let ρ^ϵ be a mollifier in the variables (y,t) (the tangential variables.) A convolution by ρ^ϵ yields a function u^ϵ, compactly supported and $\mathscr{C}^\infty$ with respect to y and t. Besides, $Lu^\epsilon = \rho^\epsilon * (Lu)$ and $\gamma_0 Bu^\epsilon = \rho^\epsilon * (\gamma_0 Bu)$ are smooth in (y,t). The sequences $(u^\epsilon)_\epsilon$, $(Lu^\epsilon)_\epsilon$ and $(\gamma_0 Bu^\epsilon)_\epsilon$ converge, respectively, to u, Lu and $\gamma_0 Bu$ in L^2_γ. In particular, they are Cauchy in L^2_γ.

When $0 \le \alpha < d$, we have $\partial_\alpha u^\epsilon = (\partial_\alpha \rho^\epsilon) * u$, with $\partial_0 := \partial_t$. Hence $\partial_\alpha u^\epsilon$ is square-integrable. Next, the identity

$$\partial_d u^\epsilon = (A^d)^{-1}(Lu^\epsilon - \partial_t u^\epsilon - \sum_{\alpha=0}^{d-1} A^\alpha \partial_\alpha u^\epsilon)$$

shows that $\partial_d u^\epsilon$ is square-integrable[6] too. Hence u^ϵ belongs to $H^1_\gamma(\Omega \times \mathbb{R})$, and we are allowed to use (4.5.26).

This estimate shows that the sequences $(u^\epsilon)_\epsilon$ and $(\gamma_0 u^\epsilon)_\epsilon$ are Cauchy, hence converge, in L^2_γ. This proves that $\gamma_0 u$ is actually an L^2_γ-function. Similarly, u^ϵ is Cauchy in $\mathscr{C}(I; L^2(\Omega))$, for every bounded interval I. Hence, it converges in this space and that proves the continuity of u with respect to time, with L^2-values. Finally, the estimate passes to the limit. □

Lemma 4.7 immediately gives the following results:

Corollary 4.2 *Let $f(x,t)$ and $g(y,t)$ be given in the classes L^2_γ. Then the solution of the boundary value problem $Lu = f$ $(x \in \Omega,\ t \in \mathbb{R})$, $\gamma_0 Bu = g$ $(y \in \partial\Omega,$ $t \in \mathbb{R})$, if it exists, must be unique in the class L^2_γ.*

Corollary 4.3 *Assume that u, Lu and $\gamma_0 Bu$ are of class L^2_γ for every γ larger than some finite threshold. Assume also that Lu and Bu vanish identically for $t < T$. Then u vanishes for $t < T$.*

Proof Because of translational invariance, we may take $T = 0$, meaning that $Lu \equiv 0$ and $\gamma_0 Bu \equiv 0$ for $t < 0$. Then the right-hand side of (4.5.26) is an $o(\mathrm{e}^{\epsilon\gamma})$ as $\gamma \to +\infty$, for every positive ϵ. Hence the left-hand side has the same property, which implies that $u \equiv 0$ for $t < 0$. □

Corollary 4.3 is a principle of *causality.*

4.5.3 *Existence and uniqueness for the boundary value problem in L^2_γ*

We proceed by duality, using the fact that the dual space to L^2_γ is $L^2_{-\gamma}$, when using the standard product

$$(u, v) := \iint_{\Omega \times \mathbb{R}} u(x,t) \cdot v(x,t)\, \mathrm{d}x\, \mathrm{d}t.$$

In particular, we have

$$\|u\|_\gamma = \sup_v \frac{|(u,v)|}{\|v\|_{-\gamma}}.$$

Assume that u and Lu belong to L^2_γ. Then the well-defined boundary trace $\gamma_0 u$ belongs to $H^{-1/2}_\gamma$, and we have a Green's formula: For every $v \in H^1_\gamma$, it holds

[6] We point out the importance that the boundary be non-characteristic in this argument.

that

$$(Lu, v) - (u, L^* v) + \langle \gamma_0 Bu, \gamma_0 Mv \rangle + \langle \gamma_0 Nu, \gamma_0 Cv \rangle = 0, \tag{4.5.27}$$

where the adjoint operator L^* is defined by

$$L^* := -\partial_t - \sum_\alpha (A^\alpha)^T \partial_\alpha,$$

and we recall that $A^d = M^T B + C^T N$. The boundary terms in (4.5.27) are duality products between $H_\gamma^{-1/2}$ and $H_{-\gamma}^{1/2}$.

Since $A(\xi)$ and $A(\xi)^T$ are similar, L^* is constantly hyperbolic too. We have seen (Theorem 4.2) that the backward adjoint IBVP satisfies (UKL). Therefore, the latter admits a dissipative symmetrizer[7] and we may use an estimate similar to (4.5.26): If v, L^*v and $\gamma_0 Cv$ are of class $L^2_{-\gamma}$, then $\gamma_0 v$ is of class $L^2_{-\gamma}$, and v satisfies

$$\begin{aligned}\gamma \iint_{\Omega\times\mathbb{R}} & \mathrm{e}^{2\gamma t}\|v\|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{2\gamma t}\|\gamma_0 v\|^2 \mathrm{d}y\,\mathrm{d}t \\ &\leq C\left(\frac{1}{\gamma}\iint_{\Omega\times\mathbb{R}} \mathrm{e}^{2\gamma t}\|L^* v\|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{2\gamma t}\|\gamma_0 Cv\|^2 \mathrm{d}y\,\mathrm{d}t\right), \end{aligned}\tag{4.5.28}$$

at least for $\gamma > \gamma_1$.

Define the subspace X_γ of $L^2_{-\gamma}$, whose elements are the functions of the form L^*v, where v and L^*v are in $L^2_{-\gamma}$, such that $\gamma_0 Cv \equiv 0$. From (4.5.28), the map $L^*v \mapsto v$ is well-defined and continuous from X_γ into $L^2_{-\gamma}$.

Given functions $f(x,t)$ and $g(y,t)$ of class L^2_γ, we may define a linear form ℓ on X_γ by

$$L^* v \mapsto \ell(L^* v) := \iint_{\Omega\times\mathbb{R}} (v, f)\,\mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times\mathbb{R}} (g, Mv)\,\mathrm{d}y\,\mathrm{d}t,$$

where we recall that $A^d = M^T B + C^T N$. Estimate (4.5.28) and the Cauchy–Schwarz inequality show that ℓ is continuous, with

$$\begin{aligned}|\ell(L^* v)| &\leq \|v\|_{-\gamma}\|f\|_\gamma + C\|\gamma_0 v\|_{-\gamma}\|g\|_\gamma \\ &\leq C\left(\gamma^{-1}\|f\|_\gamma + \gamma^{-1/2}\|g\|_\gamma\right)\|L^* v\|_{-\gamma}.\end{aligned}$$

Thanks to the Hahn–Banach and Riesz Theorems, there exists a function $u(x,t)$, of class L^2_γ, satisfying

$$\ell(L^* v) = (u, L^* v), \tag{4.5.29}$$

and

$$\gamma\|u\|^2_\gamma \leq C\left(\gamma^{-1}\|f\|^2_\gamma + \|g\|^2_\gamma\right). \tag{4.5.30}$$

[7] A dissipative symmetrizer for a backward IBVP obeys the same requirements as for the direct IBVP, except that an inequality has to be reversed. We leave the reader to write the details.

Testing (4.5.27) against functions L^*v where $v \in \mathscr{D}(\Omega \times \mathbb{R})$, we obtain $Lu = f$ in the distributional sense and $\langle \gamma_0 Bu - g, Mv\rangle = 0$. However, the equality $\ker C = (A^d \ker B)^\perp$ and the fact that B is onto imply that $M : \ker C \to \mathbb{R}^p$ is onto, and therefore we may replace Mz by any test function. It follows that $\gamma_0 Bu = g$.

With Lemma 4.7 and Corollary 4.2, we conclude with the well-posedness of the Boundary Value Problem:

Lemma 4.8 *Given $f(x,t)$ and $g(y,t)$ in the classes L^2_γ, there exists a unique u in L^2_γ, solution of the Boundary Value Problem $Lu = f$ and $\gamma_0 Bu = g$, relative to the half-space $\Omega \times \mathbb{R}$.*

Moreover, this solution satisfies (4.5.26)*.*

Applying this result to the adjoint problem, we obtain

Corollary 4.4 *The space X_γ defined above coincides with $L^2_{-\gamma}$.*

4.5.4 *Improved estimates*

In order to pass from the well-posedness of the BVP to that of the IBVP, we need to improve (4.5.26) in the following way.

Lemma 4.9 *With the above assumptions, every smooth and compactly supported function u satisfies for every $T \in \mathbb{R}$:*

$$\begin{aligned}
&\mathrm{e}^{-2\gamma T}\int_\Omega \|u(T)\|^2 \mathrm{d}x + \gamma \int_{-\infty}^T \int_{\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|u\|^2 \mathrm{d}x\, \mathrm{d}t \\
&\qquad + \int_{-\infty}^T \int_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|\gamma_0 u\|^2 \mathrm{d}y\, \mathrm{d}t \\
&\qquad \leq C\left(\frac{1}{\gamma}\int_{-\infty}^T \int_{\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|Lu\|^2 \mathrm{d}x\, \mathrm{d}t + \int_{-\infty}^T \int_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|\gamma_0 Bu\|^2 \mathrm{d}y\, \mathrm{d}t\right),
\end{aligned} \tag{4.5.31}$$

where the constant C does not depend on γ, T or u.

Proof The first step is to replace the integrals over $\mathbb{R}$ by integrals over $(-\infty, T)$. This immediately follows from Corollary 4.3 and from the existence part. Let $\tilde f$ be defined by $\tilde f = Lu$ if $t > T$ and $\tilde f = 0$ if $t < T$. Define in a similar way $\tilde g$. Then let $\tilde u \in L^2_\gamma$ be the solution associated to $\tilde f$ and $\tilde g$. Then $\tilde u$ vanishes for $t < T$ (Corollary 4.3). The estimate (4.5.26) for $v := u - \tilde u$ precisely reads

$$\begin{aligned}
&\gamma \int_{-\infty}^T \int_{\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|u\|^2 \mathrm{d}x\, \mathrm{d}t + \int_{-\infty}^T \int_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|\gamma_0 u\|^2 \mathrm{d}y\, \mathrm{d}t \\
&\qquad \leq C\left(\frac{1}{\gamma}\int_{-\infty}^T \int_{\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|Lu\|^2 \mathrm{d}x\, \mathrm{d}t + \int_{-\infty}^T \int_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{-2\gamma t}\|\gamma_0 Bu\|^2 \mathrm{d}y\, \mathrm{d}t\right),
\end{aligned}$$

We turn to the estimate of the L^2-norm of $u(T)$. The proof below, which we borrow from Rauch's PhD thesis [161], covers the physically significant case of a Friedrichs-symmetric operator. For a proof in full generality, we refer to [162].

Thus let us assume that L is Friedrichs symmetric. Integrating the identity

$$\partial_t \left(\mathrm{e}^{-2\gamma t}|u|^2\right) + \sum_\alpha \partial_\alpha \left(\mathrm{e}^{-2\gamma t} u^* A^\alpha u\right) = 2\mathrm{e}^{-2\gamma t}\left(\langle Lu, u\rangle - \gamma|u|^2\right)$$

on $\Omega \times (-\infty, T)$, we obtain

$$\begin{aligned}&\mathrm{e}^{-2\gamma T}\|u(T)\|_{L^2}^2 + 2\gamma \int_{-\infty}^T \mathrm{e}^{-2\gamma t}\|u(t)\|_{L^2}^2 \mathrm{d}t \\ &\quad = 2\int_{-\infty}^T \mathrm{e}^{-2\gamma t}(u(t), Lu(t))_{L^2}\mathrm{d}t + \int_{-\infty}^T \mathrm{e}^{-2\gamma t}(A^d\gamma_0 u(t), \gamma_0 u(t))_{L^2}\mathrm{d}t.\end{aligned}$$

The Cauchy–Schwarz inequality yields

$$\mathrm{e}^{-2\gamma T}\|u(T)\|_{L^2}^2 \le \frac{1}{2\gamma}\int_{-\infty}^T \mathrm{e}^{-2\gamma t}\|Lu(t)\|_{L^2}^2 \mathrm{d}t + C\int_{-\infty}^T \mathrm{e}^{-2\gamma t}\|\gamma_0 u(t)\|_{L^2}^2 \mathrm{d}t.$$

We conclude with the help of (4.5.26). □

Working now as in Lemma 4.7, we obtain

Proposition 4.6 *Let u, Lu and $\gamma_0 Bu$ be of class L^2_γ. Then u is continuous in time, with values in $L^2(\Omega)$, and satisfies* 4.5.31.

4.5.5 *Existence for the initial boundary value problem*

Existence Given three functions $f \in L^2_\gamma(\Omega \times \mathbb{R}^+)$, $g \in L^2_\gamma(\partial\Omega \times \mathbb{R}^+)$ and $u_0 \in L^2(\Omega)$, we define a linear form on $L^2_{-\gamma}$ (see Corollary 4.4 above) by

$$\ell_0(L^* v) := \int_0^{+\infty}\int_\Omega (v, f)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega u_0 \cdot v(0)\,\mathrm{d}x + \int_0^{+\infty}\int_{\partial\Omega}(g, \gamma_0 M v)\,\mathrm{d}y\,\mathrm{d}t.$$

We may think that the time integrals run over $\mathbb{R}_t$, and that we have extended f and g by zero to negative times.

Once more, ℓ_0 is well-defined on some subspace of $L^2_{-\rho}$ and continuous, for every ρ larger than γ. This gives the existence of a u^ρ in L^2_ρ, with the property that $\ell_0(F) = (u^\rho, F)$ every F in $L^2_{-\rho}$. Since $L^2_{-\rho} \cap L^2_{-\gamma}$ is dense in $L^2_{-\gamma}$, we see that $u^\rho = u^\gamma$ almost everywhere. In other words, the common value u belongs to L^2_ρ for every ρ larger than γ.

Testing ℓ_0 against elements of X_γ given by the vs in $\mathscr{D}(\Omega \times \mathbb{R}^*)$, we obtain the differential equation

$$Lu = f, \quad (t \neq 0)$$

in the distributional sense. This allows us to apply Green's identity

$$\iint_{\Omega\times(S,T)}((Lu,v)-(u,L^*v))\mathrm{d}x\,\mathrm{d}t+\langle\gamma_0 Bu,\gamma_0 Mv\rangle_{\partial\Omega\times(S,T)}$$
$$+\langle\gamma_0 Nu,\gamma_0 Cv\rangle_{\partial\Omega\times(S,T)}=\langle u(T^-),v(T)\rangle-\langle u(S^+),v(S)\rangle, \qquad (4.5.32)$$

both in $(-\infty,0)$ and $(0,+\infty)$. This gives

$$\iint_{\Omega\times\mathbb{R}^*}(v\cdot Lu-u\cdot L^*v)\mathrm{d}x\,\mathrm{d}t=\langle\gamma_0 Bu,\gamma_0 Mv\rangle_{\partial\Omega\times\mathbb{R}}+\langle\gamma_0 Nu,\gamma_0 Cv\rangle_{\partial\Omega\times\mathbb{R}}$$
$$-\langle[u]_{t=0},v(0)\rangle_\Omega,$$

where $[u]_{t=0}$ denotes the difference of the traces of u at $t=0^+$ and $t=0^-$. We thus obtain

$$\langle\gamma_0 Bu,Mv\rangle_{\partial\Omega\times\mathbb{R}}=\int_0^{+\infty}\int_{\partial\Omega}(g,Mv)\,\mathrm{d}y\,\mathrm{d}t, \qquad (4.5.33)$$

$$\langle[u]_{t=0},v(0)\rangle_\Omega=\int_\Omega u_0\cdot v(0)\,\mathrm{d}x. \qquad (4.5.34)$$

The argument that we developed in the previous section applies to (4.5.33) and shows that $\gamma_0 Bu=g\chi_{t>0}$. In particular, Lu and $\gamma_0 Bu$ vanish for $t<0$. Given $\epsilon>0$, let $\phi\in\mathscr{C}^\infty$ be a function of time, satisfying $\phi\equiv 1$ for $t<-\epsilon$, and $\phi\equiv 0$ for $t>-\epsilon/2$. Then $u^\epsilon:=\phi u$ is in L^2_γ, as well as Lu^ϵ and[8] $\gamma_0 Bu^\epsilon$. Since Lu^ϵ and $\gamma_0 Bu^\epsilon$ vanish for $t<-\epsilon$ (where they coincide with Lu and $\gamma_0 Bu$), and since u^ϵ, Lu^ϵ, $\gamma_0 Bu^\epsilon$ belong to L^2_ρ for every $\rho>\gamma$, Corollary 4.3 tells that u^ϵ vanishes for $t<-\epsilon$. Since ϵ is arbitrary, we deduce that u vanishes on $t<0$.

In particular, the trace of u at $t=0^-$ is zero, and (4.5.34) amounts to saying that $u(t=0^+)=u_0$. Hence there exists a solution of the full IBVP, which lies in $L^2_\gamma(\Omega\times\mathbb{R}^+)$. We shall see in a moment that this is unique. Because of the bound

$$|\ell_0(L^*v)|\le C\left(\gamma^{-1}\|f\|_\gamma+\gamma^{-1/2}\|u_0\|_{L^2}+\gamma^{-1/2}\|g\|_\gamma\right)\|L^*v\|_{-\gamma}$$

given by (4.5.28), this solution satisfies the estimate

$$\gamma\|u\|_\gamma^2\le C\left(\frac{1}{\gamma}\|f\|_\gamma^2+\|u_0\|_{L^2}^2+\|g\|_\gamma^2\right), \qquad (4.5.35)$$

for every $\gamma>\gamma_1$, where C does not depend on γ.

Uniqueness Let $u\in L^2_\gamma(\Omega\times\mathbb{R}^+_t)$ be such that $Lu=0$. Hence $\gamma_0 u$ and $u(0)$ make sense. Assume that $u(0)\equiv 0$ and $\gamma_0 Bu\equiv 0$. Green's Formula yields

$$\iint_{\Omega\times\mathbb{R}^+}(u,L^*v)\,\mathrm{d}x\,\mathrm{d}t=\langle\gamma_0 Nu,\gamma_0 Cv\rangle_{\partial\Omega\times\mathbb{R}^+},$$

[8] Note that Lu itself is not in $L^2_\gamma(\Omega\times\mathbb{R})$.

for every $v \in \mathscr{D}(\bar\Omega \times \mathbb{R})$. Extending u to negative times by zero, we obtain

$$\iint_{\Omega\times\mathbb{R}} (u, L^* v)\,\mathrm{d}x\,\mathrm{d}t = \langle \gamma_0 N u, \gamma_0 C v\rangle_{\partial\Omega\times\mathbb{R}^+}.$$

In particular, $Lu = 0$ on $\Omega \times \mathbb{R}$ and $\gamma_0 B u = 0$ on $\partial\Omega \times \mathbb{R}$. Using Corollary 4.2, we deduce $u \equiv 0$.

Improved estimates Apply (4.5.32) with $(S, T) = (0, +\infty)$:

$$\iint_{\Omega\times(0,+\infty)} (u, L^* v)\mathrm{d}x\,\mathrm{d}t - \langle \gamma_0 N u, \gamma_0 C v\rangle = \iint_{\Omega\times(0,+\infty)} (f, v)\mathrm{d}x\,\mathrm{d}t + \langle g, \gamma_0 M v\rangle + \langle u_0, v(0)\rangle.$$

According to (4.5.28), the left-hand side is bounded above by

$$C\left(\gamma^{-1}\|f\|_\gamma + \gamma^{-1/2}\|u_0\|_{L^2} + \gamma^{-1/2}\|g\|_\gamma\right)\left(\|L^* v\|_{-\gamma} + \|\gamma_0 C v\|_{-\gamma}\right).$$

We thus obtain the estimate

$$\gamma\|u\|_\gamma^2 + \|\gamma_0 N u\|_\gamma^2 \le C\left(\frac{1}{\gamma}\|f\|_\gamma^2 + \|u_0\|_{L^2}^2 + \|g\|_\gamma^2\right).$$

However, since $\ker N \cap \ker B = \ker A^d = \{0\}$, this really means

$$\gamma\|u\|_\gamma^2 + \|\gamma_0 u\|_\gamma^2 \le C\left(\frac{1}{\gamma}\|f\|_\gamma^2 + \|u_0\|_{L^2}^2 + \|g\|_\gamma^2\right). \tag{4.5.36}$$

If u_0 vanished, the argument employed for uniqueness would show that, after extension by zero to negative times, Lu still belongs to L^2_γ. Then (4.5.32) would be valid. We are now going to prove that it remains valid for general data u_0 in L^2. For that purpose, it is enough to assume $f \equiv 0$ and $g \equiv 0$. By density, we may also assume that u_0 belongs to $\mathscr{D}(\Omega)$.

From the uniqueness result above, tangential derivatives $\partial_\alpha u$ ($\alpha = 1, \ldots, d-1$), being the solutions of the IBVP corresponding to $f^\alpha = 0$, $g^\alpha = 0$ and $u_{0\alpha} := \partial_\alpha u_0$, belong to L^2_γ. Similarly, $\partial_t u$ is the solution corresponding to $f^t = 0$, $g^t = 0$ and the initial data $-\sum_\alpha A^\alpha \partial_\alpha u_0$, and thus belongs to L^2_γ. Hence $\partial_d u = -\partial_t u - \sum_\alpha A^\alpha \partial_\alpha u$ is L^2_γ. Hence u is $H^1(\gamma)$ and we may integrate the energy identity on the slab $\Omega \times (0, T)$, as in the proof of Lemma 4.7.

4.5.6 *Proof of Theorem 4.3*

It remains to treat the case of a time interval $(0, T)$. Let $f \in L^2(\Omega \times (0, T))$, $g \in L^2(\partial\Omega \times (0, T))$ and $u_0 \in L^2(\Omega)$ be given. The extensions of f and g by zero, for times $t > T$, belong to L^2_γ for every γ. We thus obtain a unique solution u of the IBVP in $\Omega \times \mathbb{R}^+$. Its restriction to times $t \in (0, T)$ furnishes a solution of the IBVP in the slab, with the required estimate.

We now prove uniqueness. Assume that f, g and u_0 vanish identically. and $u \in L^2(\Omega \times (0, T))$ satisfies the IBVP. If $\epsilon \in (0, T)$, choose ϕ in $\mathscr{D}(\mathbb{R})$ such that

$\phi(t) = 1$ if $t < T - \epsilon$ and $\phi(t) = 0$ if $t > T - \epsilon/2$. Extending u by zero to $t \notin (0, T)$, we obtain that ϕu and $L[\phi u]$ are in L^2_γ for every γ. Since $L[\phi u]$ and $\gamma_0 B(\phi u)$ vanish for $t < T - \epsilon$, Corollary 4.3 tells that u vanish for $T - \epsilon$. Since ϵ is arbitrary, we conclude that $u = 0$.

4.5.7 *Summary*

We summarize below the strategy that we followed for proving the existence and uniqueness, and establishing estimates for the IBVP.

- With the help of the Kreiss' symmetrizer, we establish an a priori estimate in L^2_γ, when $u \in \mathscr{D}(\Omega \times \mathbb{R}_t)$.
- By truncation and convolution, we extend this estimate to the case where u, Lu and $\gamma_0 Bu$ are of class L^2_γ.
- This implies uniqueness for the BVP.
- This implies also a causality property: If $Lu \equiv 0$ and $\gamma_0 Bu \equiv 0$ in the past (say for $t < T$), then $u \equiv 0$ in the past.
- Since (UKL) and constant hyperbolicity pass to the adjoint BVP, we also have an estimate for the latter when its data and solution are of class $L^2_{-\gamma}$.
- By a duality argument, which uses Hahn–Banach and Riesz Theorems, the BVP is solvable in the class L^2_γ, for data in L^2_γ.
- Thanks to the existence result and to the causality, one may replace $\mathbb{R}_t$ by $(-\infty, T)$ in the estimates. Also, the IBVP with a zero initial data admits a solution.
- Thanks to the energy estimate (when the operator is Friedrichs symmetric), we also have an estimate of $u(T)$ in $L^2(\Omega)$.
- We deduce that, for data in L^2_γ, the solution is continuous with values in $L^2(\Omega)$.
- Thanks to the causality property and to the time-pointwise estimate, and using a duality argument, the IBVP is solvable in the class L^2_γ.
- As above, the IBVP has a causality property and estimates on $\Omega \times (0, T)$ instead of $\Omega \times (0, +\infty)$.

4.5.8 *Comments*

BVP *vs* IBVP We emphasize that the existence and the estimates for the Boundary Value Problem do not automatically imply the corresponding results for the Initial Boundary Value Problem. Let us consider an abstract differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + f, \tag{4.5.37}$$

which is a model for the homogeneous BVP ($Bu(0,t) \equiv 0$). Let us assume that (4.5.37) has an existence property with an estimate

$$\int_{\mathbb{R}} \mathrm{e}^{-2\gamma t}\|u(t)\|^2 \mathrm{d}t \le \frac{C^2}{\gamma^2}\int_{\mathbb{R}} \mathrm{e}^{-2\gamma t}\|f(t)\|^2 \mathrm{d}t, \quad \forall \gamma > 0, \tag{4.5.38}$$

where the norm is taken in some Hilbert space. Letting $\gamma \to +\infty$, we find as usual that if f vanished in the past, then u does too.

It is not hard, using the Parseval Identity, to prove that $\tau I - A$ has a bounded inverse for every τ of positive real part, with the estimate

$$\|(\tau I - A)^{-1}\| \le \frac{C'}{\mathrm{Re}\ \tau}. \tag{4.5.39}$$

However, (4.5.38) is essentially equivalent to (4.5.39), and it is not possible from there to derive a pointwise estimate for the semigroup generated by A (assuming that it exists). This prevents us from proving anything about the Cauchy problem for (4.5.37) without some additional information about A.

Historically, this difficulty was encountered by Kreiss [103], who solved only the (non-homogeneous) BVP. The extension of his results to the full IBVP was obtained later by Rauch [161, 162]. We shall face this difficulty in the homogeneous IBVP (that is with a zero boundary condition, see Chapter 7) and in the so-called $\mathcal{WR}$ case, see Chapter 8.

4.6 A practical tool

4.6.1 *The Lopatinskiĭ determinant*

We are going to define in this section a (not very) practical tool called the *Lopatinskiĭ determinant.* This is a function $(\tau, \eta) \mapsto \Delta(\tau, \eta)$, with the following properties

i) It is well-defined for $\eta \in \mathbb{R}^{d-1}$ and Re $\tau > 0$,

ii) It is jointly analytic in (τ, η), and therefore holomorphic in τ,

iii) It vanishes precisely at points violating the Lopatinskiĭ condition.

To fill these three properties, it is enough to construct a basis

$$\beta(\tau, \eta) = \{X_1(\tau, \eta), \ldots, X_p(\tau, \eta)\}$$

of $E_-(\tau, \eta)$, which satisfies the first two ones. Then we define the *Lopatinskiĭ determinant* as

$$\Delta(\tau, \eta) := \det(BX_1(\tau, \eta), \ldots, BX_p(\tau, \eta)), \tag{4.6.40}$$

since the vanishing of the determinant is equivalent to the existence of a non-trivial linear combination

$$X := \sum_1^p c_j X_j(\tau, \eta)$$

such that $BX = 0$, which amounts to $X \in \ker B \cap E_-(\tau, \eta)$.

To construct β, we use the procedure described by Kato in [95], Section 4.2, which goes as follows. Let $z \mapsto P(z)$ be an analytic function with values in projectors, where z ranges on a simply connected domain. From identity $P^2 = P$, one easily finds $P' = [Q, P]$, with $Q := [P', P]$. Then the linear Cauchy problem $M' = QM$, $M(z_0) = I_n$ is globally solvable and yields the formula

$$M(z)^{-1}P(z)M(z) \equiv P(z_0).$$

Therefore, given a basis β_0 of the range of $P(z_0)$, the set $\beta(z) := M(z)\beta_0$ is a basis of the range of $P(z)$, and is analytic in z.

When P depends on several variables, this procedure cannot be done simultaneously in general. If $Q_j := [\partial P/\partial z_j, P]$, simultaneity requires the compatibility condition $\partial Q/\partial z_k - \partial Q/\partial z_j = [Q_j, Q_k]$, though in practice we have oddly $\partial Q/\partial z_k - \partial Q/\partial z_j = 2[Q_j, Q_k]$! However, we may apply Kato's procedure successively to each of the arguments, provided that at each step, a Cauchy problem is posed in a simply connected region. Because ODEs with analytic coefficients propagate analyticity, the resulting matrix M is jointly analytic in its arguments. The inelegant fact is that the result depends on the order in which we solve ODEs, because of the lack of compatibility.

Applying these ideas to the projectors $\pi_-(\tau, \eta)$, which are jointly analytic, we now have

Lemma 4.10 *For $\eta \in \mathbb{R}^{d-1}$ and $\mathrm{Re}\,\tau > 0$, the space $E_-(\tau, \eta)$ admits a basis $\beta(\tau, \eta)$, which is jointly analytic in (τ, η), and thus holomorphic in τ.*

The symmetric case We restrict ourselves here to the non-characteristic case. When the operator L is symmetric, that is $A(\xi)^T = A(\xi)$ for every ξ in $\mathbb{R}^d$, an alternative construction can be done, with the help of the following

Lemma 4.11 *In the symmetric case with a non-characteristic boundary, one has for every $\eta \in \mathbb{R}^{d-1}$ and $\mathrm{Re}\,\tau > 0$*

$$E^u(A^d) \cap E_+(\tau, \eta) = \{0\}, \tag{4.6.41}$$

where $E^u(A^d)$ stands for the unstable invariant subspace of A^d.

To our knowledge, the validity of property (4.6.41) under the assumption of hyperbolicity, instead of symmetry, remains an open question.

Proof Let u_0 belong to $E_+(\tau, \eta)$. Then the unique solution

$$A^d u' + (\tau + iA(\eta))u = 0, \quad u(0) = u_0$$

decays exponentially fast at $-\infty$. Multiplying the ODE by u^* and integrating, we obtain

$$u_0^* A^d u_0 = -2(\mathrm{Re}\,\tau) \int_{-\infty}^0 |u|^2 \mathrm{d}x_d \leq 0.$$

If, moreover, $u_0 \in E^u(A^d)$, the unique solution of

$$v' = A^d v, \quad v(0) = u_0$$

decays exponentially fast at $-\infty$. Multiplying the ODE by $v^* A^d$ and integrating, we obtain

$$u_0^* A^d u_0 = 2 \int_{-\infty}^0 |A^d v|^2 \mathrm{d}x_d \geq 0.$$

We conclude that $u_0^* A^d u_0 = 0$, which readily implies that $u \equiv 0$. Thus $u_0 = 0$. □

Thanks to the lemma, the map $\pi_-(\tau, \eta) : E^u(A^d) \to E_-(\tau, \eta)$ is injective, thus bijective since both spaces have dimension p. Now, given a basis γ_0 of $E^u(A^d)$, we obtain a basis $\gamma(\tau, \eta) := \pi_-(\tau, \eta)\gamma_0$ of $E_-(\tau, \eta)$, which is jointly analytic.

An abstract definition of Δ We use here the exterior algebra $\Lambda(\mathbb{C}^n)$. For a construction of this object, we refer, for instance, to Harris' book [81]. The (non-commutative) algebra $\Lambda(\mathbb{C}^n)$ is spanned by $\mathbb{C}^n$ under the associative product (*exterior product*) $\wedge$. The exterior product basically satisfies $X \wedge Y = -Y \wedge X$ for every X, Y in $\mathbb{C}^n$. This rule defines a *graded algebra*,

$$\Lambda(\mathbb{C}^n) = \Lambda_0(\mathbb{C}^n) \oplus \cdots \oplus \Lambda_n(\mathbb{C}^n),$$

with

$$\Lambda_0(\mathbb{C}^n) = \mathbb{C}, \quad \Lambda_1(\mathbb{C}^n) = \mathbb{C}^n$$

and

$$\dim \Lambda_k(\mathbb{C}^n) = \begin{pmatrix} n \\ k \end{pmatrix}.$$

Elements of $\Lambda_k(\mathbb{C}^n)$ are called k-vectors. Given a basis $\{e^1, \ldots, e^n\}$ of $\mathbb{C}^n$, they are linear combinations of $e^{j_1} \wedge \cdots \wedge e^{j_k}$, where $j_1 < \cdots < j_k$. When F is a k-dimensional subspace of $\mathbb{C}^n$, we may define a k-vector X_F by $X_F := X^1 \wedge \cdots \wedge X^k$, where $\{X^1, \ldots, X^k\}$ is a given basis of F. One verifies that different bases of F give the same X_F, up to a non-zero scalar factor. This procedure defines a unique one-dimensional subspace in $\Lambda_k(\mathbb{C}^n)$. The map $F \mapsto \mathbb{C}X_F$ is one-to-one, but not onto, because not all k-vectors are simple exterior products.

Let $M \in M_{m \times n}(\mathbb{C})$ be a matrix. For $k \in \mathbb{N}$, define a linear map $M^{(k)} : \Lambda_k(\mathbb{C}^n) \to \Lambda_k(\mathbb{C}^m)$ by

$$e^{j_1} \wedge \cdots \wedge e^{j_k} \mapsto Me^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_k} + \cdots + e^{j_1} \wedge \cdots \wedge e^{j_{k-1}} \wedge Me^{j_k}$$

and linearity. It satisfies the identity

$$M^{(k)}(X^1 \wedge \cdots \wedge X^k) = MX^1 \wedge X^2 \wedge \cdots \wedge X^k + \cdots + X^1 \wedge \cdots \wedge X^{k-1} \wedge MX^k,$$

for all $X^1, \ldots, X^k$ in $\mathbb{C}^n$. Let us assume $m = n$ and let $(\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of M, counted with multiplicity. One easily verifies that the eigenvalues of $M^{(k)}$, counted with multiplicities, are the sums

$$\lambda_{j_1} + \cdots + \lambda_{j_k}, \quad j_1 < \cdots < j_k.$$

When applying this observation to the matrix $\mathcal{A}(\tau, \eta)$, we find that its 'pth sum' admits a unique eigenvalue $\mu(\tau, \eta)$ of minimal real part, namely the sum of eigenvalues of $\mathcal{A}(\tau, \eta)$ with negative real part. Moreover, $\mu(\tau, \eta)$ is a simple eigenvalue, whose eigenvector is X_F, for $F := E_-(\tau, \eta)$. Using Kato's argument, we may construct a jointly analytic choice $X(\tau, \eta)$ of X_F. For instance

$$X(\tau, \eta) = X_1(\tau, \eta) \wedge \cdots \wedge X_p(\tau, \eta)$$

works. Then we may define

$$\vec{\Delta}(\tau, \eta) := B^{(p)} X(\tau, \eta).$$

This expression belongs to $\Lambda^{(p)}(\mathbb{C}^p)$, a one-dimensional vector space. The link between both definitions is

$$\vec{\Delta} = \Delta \; e^1 \wedge \cdots \wedge e^p,$$

where $\{e^1, \ldots, e^p\}$ is any basis of $\mathbb{C}^p$ with determinant one. We shall not distinguish Δ from $\vec{\Delta}$ in the following.

4.6.2 *'Algebraicity' of the Lopatinskiĭ determinant*

We show in this section that the Lopatinskiĭ locus, that is the set of zeroes of the Lopatinskiĭ determinant Δ, is a subset of an algebraic manifold of codimension one. In general, this subset is strict, although it has the same codimension. In other words, there exists a single polynomial $\mathbf{Lop}(X, \eta)$ such that $\Delta(\tau, \eta) = 0$ implies $\mathbf{Lop}(i\tau, \eta) = 0$. Obviously, $\mathbf{Lop}$ is a homogeneous polynomial, so that its zero set may be viewed as a projective variety. More importantly, it has real coefficients, so that the zeroes $(i\rho, \eta)$ of Δ on the boundary $\mathrm{Re}\, \tau = 0$ belong to a *real* algebraic variety.

Recall first that, in the non-characteristic case, Δ is defined as the determinant in $\mathbb{C}^p$ of vectors $Br_1(\tau, \eta), \ldots, Br_p(\tau, \eta)$, where the r_js span the stable subspace of $\mathcal{A}(\tau, \eta)$. When $\mathcal{A}$ is diagonalizable, a generic property, r_j may be taken as an eigenvector associated to $\mu_j(\tau, \eta)$, one of the stable eigenvalues. Using the polynomial $P(X, \xi) := \det(XI_n + A(\xi))$, the eigenvalues are constrained by $P(\tau, i\eta, \mu_j) = 0$, or equivalently $P(-i\tau, \eta, -i\mu_j) = 0$.

Since r_j solves $(\tau I_n + A(i\eta, \mu_j))r_j = 0$, a choice of r_j can be made polynomially in (τ, η, μ_j). For instance, we may choose the first column of $M(\tau, \eta, \mu_j)$, where $M(\tau, \eta, \mu)$ is the transpose of the matrix of cofactors of $\tau I_n + A(i\eta, \mu)$, since

$$(\tau I_n + A(i\eta, \mu))M(\tau, \eta, \mu) = (\det(\tau I_n + A(i\eta, \mu)))I_n,$$

and the right-hand side vanishes whenever μ is an eigenvalue of $\mathcal{A}$. Note that the columns of M are non-trivial; for instance, they do depend on τ. From now on, let us denote by $R(\tau, \eta, \mu)$ this choice.

The zeroes of the Lopatinskiĭ determinant therefore satisfy the following list of polynomial equations:

$$\begin{aligned} \det(BR(\tau,\eta,\mu_1),\dots,BR(\tau,\eta,\mu_p)) &= 0,\\ P(-i\tau,\eta,-i\mu_1) &= 0,\\ &\vdots\\ P(-i\tau,\eta,-i\mu_p) &= 0. \end{aligned}$$

Using the resultant, we may eliminate μ_1, considered as a dummy variable, between the first two equations. These are thus replaced by a polynomial equation in $\tau, \eta, \mu_2, \dots, \mu_p$. Using again the resultant, we eliminate successively $\mu_2, \dots, \mu_p$ and end with a single polynomial equation

$$\mathbf{Lop}(i\tau, \eta) = 0.$$

Practical aspects

- The procedure described above may be the simplest one, in the sense that it gives the simplest result **Lop**. This seems to be true when P is irreducible. However, in practical situations, symmetry properties are responsible for the presence of multiple eigenvalues of $A(\xi)$, which yield a splitting of P (see Proposition 1.7, for instance). When P does split, our procedure must be reconsidered and gives rise to a polynomial of lower degree. See Chapter 13 for convincing examples within gas dynamics.
- A flaw of **Lop** is that it has been built without any consideration about the sign of the real part of the μ_js. Therefore, its zero set also contains the zeroes of fake Lopatinskiĭ determinants, where the μ_js are chosen arbitrarily in the spectrum of $\mathcal{A}$. The manifold defined by $\mathbf{Lop}(i\tau, \eta) = 0$ thus contains irrelevant parts, which have to be removed on a case-by-case analysis. This difficulty is always encountered when one wishes to check the Lopatinskiĭ condition, or (UKL).
- Another difficulty arises when $\mathcal{A}$ displays a Jordan block. In this case, the columns of M do not span the corresponding generalized eigenspace of $\mathcal{A}$, since it only spans the classical eigenspace. Therefore, it may happen that **Lop** admits spurious zeroes, which do not correspond to zeroes of Δ. This has been the cause of a misunderstanding in [166], where some Lax shocks in an ideal gas flow were unduly claimed to be unstable. See [42] for a detailed explanation.
- Similarly, there are points (τ, η) where the first column of M vanishes, therefore it does not span an eigenspace. At such points, **Lop** vanishes

automatically. Since the set of such points is algebraic, its equation must divide **Lop** or merely a power of it, according to Hilbert's *Nullstellensatz*. Since the corresponding factor of **Lop** does not involve the coefficients of the boundary operator B at all, it is easily identified and can be removed immediately. An alternative method consists in the replacement of the first column of M by another one. The choice of the jth column of M yields a polynomial $\mathbf{Lop}_j$. These polynomials differ only by these spurious factors. Taking their g.c.d., one obtains a simpler polynomial $\mathbf{Lop}_0$ that vanishes everywhere Δ does.

- Points where $\mathcal{A}$ displays a Jordan block also form an algebraic variety, whose equation enters automatically as a factor in **Lop**. This factor may be identified since it does not involve B, and is then removed. However, it is not removed by taking the g.c.d. $\mathbf{Lop}_0$, since it is present in every $\mathbf{Lop}_j$.

Example Consider the system $\partial_t u + A(\nabla_x)u = 0$, with $d = n = 2$ and

$$A(\xi) := \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & -\xi_1 \end{pmatrix}.$$

The spectrum of $A(\xi)$ consists in $\pm|\xi|$. Hence the boundary condition at $x_2 = 0$ must be scalar (one incoming characteristics): $Bu = b_1 u_1 + b_2 u_2$. Given an eigenvalue μ of $\mathcal{A}(\tau, \eta)$, that is a root of $\tau^2 + \eta^2 = \mu^2$, a typical eigenvector is $R(\tau, \eta) := (i\eta - \tau, \mu)^T$. The only exception is the point given by $\tau = i\eta$ (a boundary point) and $\mu = 0$; near such a point, a convenient choice of an eigenvector would be $R'(\tau, \eta) := (-\mu, \tau + i\eta)^T$.

The Lopatinskiĭ determinant is $\Delta(\tau, \eta) = b_1(i\eta - \tau) + b_2\mu$. Eliminating μ between $\Delta = 0$ and $\tau^2 + \eta^2 = \mu^2$, we obtain the equation

$$b_2^2(\tau^2 + \eta^2) = b_1^2(i\eta - \tau)^2.$$

Therefore,

$$\mathbf{Lop}(z, \eta) = b_1^2(\eta + z)^2 + b_2^2(\eta^2 - z^2) = (\eta + z)(b_1^2(\eta + z) + b_2^2(\eta - z)).$$

The fact that **Lop** vanishes at the point $z = -\eta$, regardless of the value of $\vec{b}$, reflects the fact that R does not span an eigenspace at this point. A computation with the choice $R'(\tau, \eta)$ would have given a similar result with the factor $z - \eta$ instead of $z + \eta$. This factor is thus irrelevant, and the vanishing of the Lopatinskiĭ determinant must imply that of the simpler polynomial

$$\mathbf{Lop}_0(z, \eta) = b_1^2(\eta + z) + b_2^2(\eta - z).$$

This formula shows that the IBVP satisfies the Lopatinskiĭ condition for every $\vec{b}$ such that $b_1 \neq \pm b_2$. What the formula hides is that the IBVP still satisfies the Lopatinskiĭ condition when $b_1 = b_2$, while it does not if $b_1 = -b_2$. The replacement of the half-space $x_2 > 0$ by the domain $x_2 < 0$, or the reversal of the time arrow, exchanges the roles. Hence, it is important to keep in mind that the vanishing set of $\mathbf{Lop}_0$ encodes not only the vanishing of the Lopatinskiĭ determinant, but also a number of other properties.

4.6.3 *A geometrical view of (UKL) condition*

Denote by $W(k,n)$ the Grassmannian manifold, consisting of the linear subspaces of dimension k in $\mathbb{C}^n$. For instance, $W(1,n)$ is just the projective space $P_{n-1}(\mathbb{C})$. A correct definition is the following one. Recall that $\Lambda_k(\mathbb{C}^n)$ is the set of elements of degree k in the exterior algebra $\Lambda(\mathbb{C}^n)$. Consider the subset $\Lambda_k^S(\mathbb{C}^n)$ of non-zero simple k-vectors, that is of the form X_F, where F is some linear subspace of dimension k, in the notations of Section 4.6.1. This set is a cone, and $W(k,n)$ is its quotient by the relation $X \sim Y$ iff X and Y are parallel. In other words, $W(k,n)$ is the projective space associated to $\Lambda_k^S(\mathbb{C}^n)$. The Grassmannian manifolds are compact and endowed with an analytic structure. For a theory of Grassmannian manifolds, we refer to Harris' book [81].

Let M be a linear subspace in $\mathbb{C}^n$, of dimension $n-p$. One easily sees that the subset M° in $W(p,n)$ consisting of p-dimensional subspaces that meet M non-trivially, is closed and therefore compact.

Assume now that the IBVP (4.1.1)–(4.1.3) satisfies the uniform Kreiss–Lopatinskiĭ condition. Let $\mathcal{W}$ be the subset of $W(p,n)$, consisting of the spaces $E_-(\tau,\eta)$ for $\eta \in \mathbb{R}^{d-1}$ and Re $\tau > 0$. The (non-uniform) Lopatinskiĭ condition tells us that $\mathcal{W}$ does not meet $(\ker B)^\circ$. Uniformity obviously tells us more. If F belongs to the closure of $\mathcal{W}$, then there exists a sequence (τ_m,η_m), such that $E_-(\tau_m,\eta_m)$ converges towards F. Then the inequality $|V| \le C|BV|$ passes to the limit and therefore holds on F. This proves that $F \notin (\ker B)^\circ$.

Conversely, assume that $(\ker B)^\circ$ does not meet the closure of $\mathcal{W}$. When $F \in \overline{\mathcal{W}}$, let C_F be the best constant in the inequality $|V| \le C|BV|$ for $V \in F$. Obviously, $F \mapsto C_F$ is continuous. Since $\overline{\mathcal{W}}$ is compact, $F \mapsto C_F$ is bounded. In other words, the IBVP satisfies (UKL) condition. We can summarize as follows

Lemma 4.12 *If the IBVP (4.1.1)–(4.1.3) is hyperbolic and has the correct number of boundary conditions, then the uniform Kreiss–Lopatinskiĭ condition is equivalent to*

$$(\ker B)^\circ \bigcap \overline{\mathcal{W}} = \emptyset$$

in the Grassmann manifold $W(p,n)$.

Remark Because of homogeneity, we may also define the set $\mathcal{W}$ by

$$\mathcal{W} = \{E_-(\tau,\eta)\,;\, |\tau|^2 + \|\eta\|^2 = 1\}.$$

The closure of $\mathcal{W}$ differs from $\mathcal{W}$ itself only by limits of sequences $E_-(\tau_m,\eta_m)$ when Re $\tau_m \to 0^+$ and η_m converges.

4.6.4 *The Lopatinskiĭ determinant of the adjoint IBVP*

Recall (Theorem 4.2) that the Kreiss–Lopatinskiĭ condition is satisfied by the adjoint IBVP if and only if it is satisfied by the original IBVP, and similarly for the uniform KL condition. This suggests that a relation holds between the respective Lopatinskiĭ determinants. We state it now. For the sake of simplicity, we restrict ourselves to the non-characteristic case.

Theorem 4.4 *Assume that (L, B) is normal and the boundary is non-characteristic. Let (τ_0, η_0) (with $\mathrm{Re}\,\tau_0 \geq 0$, $\eta_0 \in \mathbb{R}^{d-1}$) be a point in the neighbourhood of which a Lopatinskiĭ determinant Δ is well-defined, meaning that $E_-(\tau, \eta)$ is locally continuous.*

Then E_-^* is well-defined in a neighbourhood of $(-\bar{\tau}_0, -\eta_0)$, and one may take the function

$$(\theta, \sigma) \mapsto \overline{\Delta(-\bar{\theta}, -\sigma)}$$

as a Lopatinskiĭ determinant for the dual IBVP.

Remark Taking the complex conjugate in the formula above is useful only in that it preserves holomorphy in θ as $\mathrm{Re}\,\theta < 0$.

Proof The first statement is a consequence of the fact that

$$E_-^*(-\bar{\tau}, -\eta) = (A^d E_-(\tau, \eta))^\perp.$$

We now turn to the construction of the adjoint Lopatinskiĭ determinant $\Delta^*(\theta, \sigma)$. Recall the definition (4.6.40), where $\{X_1(\tau, \eta), \ldots, X_p(\tau, \eta)\}$ is a regular basis of $E_-(\tau, \eta)$. Let us choose constant vectors $X_{p+1}, \ldots, X_n$ such that

$$\{X_1(\tau_0, \eta_0), \ldots, X_p(\tau_0, \eta_0), X_{p+1}, \ldots, X_n\}$$

is a basis of $\mathbb{C}^n$. Then, in a neighbourhood of (τ_0, η_0), the matrix

$$X(\tau, \eta) := (X_1(\tau, \eta), \ldots, X_p(\tau, \eta), X_{p+1}, \ldots, X_n)$$

is non-singular. Denote by $Y_1(\tau, \eta), \ldots, Y_n(\tau, \eta)$ the column vectors of $Y := (A^d X(\tau, \eta))^{-*}$. Then $Y_{p+1}, \ldots, Y_n$ forms a regular basis of $E_-^*(-\bar{\tau}, -\eta)$.

Let us write these matrices blockwise:

$$X = (X_-, X_+), \quad Y = (Y_-, Y_+),$$

where the minus blocks are $n \times p$ and the plus blocks are $n \times (n-p)$. Our Lopatinskiĭ determinants are defined by the formula

$$\Delta(\tau, \eta) = \det(BX_-(\tau, \eta)), \quad \Delta^*(-\bar{\tau}, -\eta) = \det(CY_+(\tau, \eta))$$

(note that both sides of the last equality are anti-holomorphic in τ).

Finally, we recall the equality

$$A^d = C^T N + M^T B.$$

Defining the $n \times n$ matrices

$$P := \begin{pmatrix} B \\ N \end{pmatrix}, \quad Q := \begin{pmatrix} M \\ C \end{pmatrix},$$

we have $Q^T P = A^d$. It follows that

$$(QY)^*(PX) = Y^* Q^T P X = Y^* A^d X = I_n,$$

thus $QY = (PX)^{-*}$. However, we have by definition

$$PX = \begin{pmatrix} BX_- & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad QY = \begin{pmatrix} \cdot & \cdot \\ \cdot & CY_+ \end{pmatrix}.$$

Using Schur's formula (see Proposition 8.1.2 and Corollary 8.1.1 in [187]), one sees that

$$\overline{\Delta^*(-\bar{\tau}, -\eta)} = \frac{\Delta(\tau, \eta)}{\det(PX)}.$$

Since P and X are locally non-singular, the function $(\tau, \eta) \mapsto \det(PX)$ and its inverse are smooth . Therefore, the formula above, together with a renormalization of, say, the vector Y_n, allow us to take simply

$$\overline{\Delta^*(-\bar{\tau}, -\eta)} := \Delta(\tau, \eta).$$

□

5

CONSTRUCTION OF A SYMMETRIZER UNDER (UKL)

This chapter is devoted to the proof of the existence of a Kreiss' symmetrizer in the non-characteristic case. A technical, though important ingredient in the proof is to establish the so-called 'block structure' property for the matrix $\mathcal{A}$ at points (τ, η) where Re $\tau = 0$ (boundary points), of *glancing* type. Although we focus on problems with constant coefficients in a half-space, our construction is flexible enough to handle the case where the data (normal direction to the boundary, entries of the symbol, boundary matrix) are parametrized. This fact is crucial in the applications to variable-coefficient problems, especially non-linear ones, and/or in general domains.

5.1 The block structure at boundary points

5.1.1 *Proof of Lemma 4.5*

Let us recall the terms of Lemma 4.5:

Assume that the operator L is constantly hyperbolic and the boundary $\{x_d = 0\}$ is non-characteristic. Then the map $(\tau, \eta) \mapsto E_-(\tau, \eta)$ (already defined for Re $\tau > 0$) admits a unique limit in the Grassmannian $G(n, p)$ at every boundary point $(i\rho, \eta)$ (meaning that $\rho \in \mathbb{R}$, $\eta \in \mathbb{R}^{d-1}$), with the exception of the origin.

Proof We first prove that the stable spectrum of $\mathcal{A}(\tau, \eta)$ admits a continuous extension up to the boundary. Let $(i\rho_0, \eta_0)$ be a non-zero boundary point, and let $\omega(\tau, \eta)$ be an eigenvalue of $\mathcal{A}(\tau, \eta)$ for Re $\tau > 0$, depending continuously on (τ, η). As (τ, η) tends to $(i\rho_0, \eta_0)$, the omega-limit set of $\omega(\tau, \eta)$ is connected, by continuity and boundedness of ω, and by connectedness of the domain. However, this omega-limit set is contained in the spectrum of $\mathcal{A}(i\rho_0, \eta_0)$, a discrete set. It is therefore a singleton. This shows that $\omega(\tau, \eta)$ has a limit as (τ, η) tends to $(i\rho_0, \eta_0)$.

Denote by $\hat{\omega}_1, \dots, \hat{\omega}_r$ those distinct eigenvalues of $\mathcal{A}(i\rho_0, \eta_0)$ that are limits, as $\tau \to i\rho_0$ and Re $\tau > 0$, of stable eigenvalues of $\mathcal{A}(\tau, \eta)$. Obviously, Re $\hat{\omega}_j \leq 0$, although eigenvalues with non-positive real part need not belong to $\{\hat{\omega}_1, \dots, \hat{\omega}_r\}$ in general.

When (τ, η) is close to $(i\rho_0, \eta_0)$ and Re $\tau > 0$, $E_-(\tau, \eta)$ may be split as a direct sum of invariant subspaces $F_1(\tau, \eta), \dots, F_r(\tau, \eta)$, so that the eigenvalues of $\mathcal{A}(\tau, \eta)$ on F_j have the single limit $\hat{\omega}_j$ as (τ, η) tends to $(i\rho_0, \eta_0)$. Each F_j inherits the analyticity of E_-.

It will be sufficient to prove that each of these spaces F_j has a limit as (τ,η) tends to $(i\rho_0,\eta_0)$. We select an index j and denote $F(\tau,\eta)=F_j(\tau,\eta)$, $\hat\omega=\hat\omega_j$ for the sake of simplicity. Let $\hat F$ be a cluster point of $F(\tau,\eta)$ as (τ,η) tends to $(i\rho_0,\eta_0)$. It will be enough to prove the uniqueness of $\hat F$. By continuity, $\hat F$ is invariant under $\mathcal{A}(i\rho_0,\eta_0)$, and is associated to the sole eigenvalue $\hat\omega$. Therefore, we have $\hat F\subset\hat G$, where $\hat G$ denotes the generalized eigenspace of $\mathcal{A}(i\rho_0,\eta_0)$, associated to the eigenvalue $\hat\omega$.

Let us begin with the easy case, when $\mathrm{Re}\,\hat\omega<0$. Classically, $\hat G$ locally extends analytically as an invariant subspace $G(\tau,\eta)$ of $\mathcal{A}(\tau,\eta)$. But since the corresponding eigenvalues will keep a negative real part, we find that $G(\tau,\eta)\subset E_-(\tau,\eta)$, so that $F(\tau,\eta)$ must be equal to $G(\tau,\eta)$ for (τ,η) close to $(i\rho_0,\eta_0)$, and therefore $\hat F=\hat G$. Notice that this argument can be used to prove that every eigenvalue ω of $\mathcal{A}(i\rho_0,\eta_0)$ with negative real part must belong to the list $\hat\omega_1,\dots,\hat\omega_r$, with its full multiplicity.

There remains the case $\hat\omega=i\mu_0$, with $\mu_0\in\mathbb{R}$. By Proposition 1.7, we decompose the characteristic polynomial $P_{\tau,\eta}$ of $\mathcal{A}(\tau,\eta)$ as $P_{\tau,\eta}(X)=P_0(\tau,i\eta,X)Q(\tau,i\eta,X)^q$, where P_0 and Q are homogeneous polynomials with real entries and

- ρ_0 is a simple root of $Q(\cdot,\eta_0,\mu_0)$,
- $P_0(\rho_0,\eta_0,\mu_0)\neq 0$.

Let $N\geq 1$ be the multiplicity of μ_0 as a root of $Q(\rho_0,\eta_0,\cdot)$. The eigenvalues of $\mathcal{A}(\tau,\eta)$ that are close to $i\mu_0$ are roots of $Q(-i\tau,\eta,-i\cdot)$ and their multiplicities are multiples of q. Since $F(\tau,\eta)$ is the sum of some of the corresponding generalized eigenspaces, q divides its dimension. We shall denote $l:=(\dim F(\tau,\eta))/q$. Then $\dim\hat F=lq$.

Let $\mathcal{O}$ denote the open set of pairs $(\tau,\eta)\in\mathbb{C}\times\mathbb{R}^{d-1}$ for which the factors $P_k(\tau,i\eta,\cdot)$ in (1.5.56) have simple roots, distinct for distinct indices k. Likewise, we denote by $\mathcal{O}_{\mathbb{C}}$ when allowing complex values for both τ and η. For instance, $(i,0)\in\mathcal{O}$ holds. The complement of $\mathcal{O}_{\mathbb{C}}$, being the zero set of the discriminant Δ of $\Pi_1^L P_k(\tau,\eta,\cdot)$, is an algebraic variety of complex codimension one. Therefore, $\mathcal{O}_{\mathbb{C}}$ is dense and arcwise connected. Likewise, $\mathcal{O}$ is dense in $\mathbb{C}\times\mathbb{R}^{d-1}$, for otherwise the polynomial Δ, vanishing on a non-void open set, would vanish identically, contradicting the fact that $(i,0)\in\mathcal{O}$. We shall admit for a moment the following

Lemma 5.1 *For every pair (τ,η) in $\mathcal{O}_{\mathbb{C}}$, the matrix $\mathcal{A}(\tau,\eta)$ is diagonalizable.*

In particular, there is a neighbourhood $\mathcal{V}$ of $(i\rho_0,\eta_0)$, such that if $\mathrm{Re}\,\tau>0$ and $(\tau,\eta)\in\mathcal{V}\cap\mathcal{O}$, then $F(\tau,\eta)$ is the sum of l eigenspaces, all of them being of dimension q. Therefore, the minimal polynomial of $\mathcal{A}$ over F has the form $\Pi_1^l(X-\omega_j(\tau,\eta))$. Since $\mathcal{O}$ is dense, we obtain by continuity that a polynomial of degree l annihilates the restriction of $\mathcal{A}$ on F, for every $(\tau,\eta)\in\mathcal{V}$ with

Re $\tau > 0$. Letting (τ, η) tend towards $(i\rho_0, \eta_0)$, we see that the polynomial $(X - i\mu_0)^l$ annihilates the restriction of $\mathcal{A}(i\rho_0, \eta_0)$ on $\hat{F}$.

Let us now define the Jordan tower of $\mathcal{A}(i\rho_0, \eta_0)$, associated to the eigenvalue $i\mu_0$,

$$G_k := \ker(\mathcal{A}(i\rho_0, \eta_0) - i\mu_0)^k, \quad g_k := \dim G_k - \dim G_{k-1}.$$

Classical linear algebra tells us that $g_1 \geq \cdots \geq g_l$. Also, G_1 is simply the kernel of $\rho_0 I_n + A(\eta_0, \mu_0)$, which has dimension q by assumption, just because ρ_0 has multiplicity q and $A(\eta_0, \mu_0)$ is diagonalizable. Thus $g_1 = q$. Since $\hat{F} \subset G_l$, we have $ql = \dim \hat{F} \leq \dim G_l = g_1 + \cdots + g_l \leq lg_1 = lq$. Therefore $\dim G_l = \dim \hat{F}$, and we conclude that $\hat{F} = G_l$, which is the uniqueness property. □

Proof of Lemma 5.1 From Theorem 1.5, the matrix

$$\mathcal{A}(i\rho, \eta) = -i(A^d)^{-1}(\rho I_n + A(\eta))$$

is diagonalizable for real ρ and η, provided $\rho >> |\eta|$.

In a small neighbourhood of $(i, 0)$, let $\omega_j(\tau, \eta)$ be the distinct eigenvalues of $\mathcal{A}(\tau, \eta)$, each of them being of constant multiplicity m_j, and therefore holomorphic. The generalized eigenspaces are holomorphic too and one can choose, using Kato's procedure, holomorphic bases $\{X_j^1, \dots, X_j^{m_j}\}$. Given two indices j and $k \leq m_j$, the holomorphic map

$$V(\tau, \eta) := (\mathcal{A}(\tau, \eta) - \omega_j(\tau, \eta))X_j^k(\tau, \eta)$$

satisfies

$$(\mathrm{Re}\ \tau = 0, \mathrm{Im}\ \eta = 0) \Longrightarrow (V(\tau, \eta) = 0),$$

according to the beginning of the proof. Hence, V vanishes identically. This shows that $\mathcal{A}(\tau, \eta)$ is diagonalizable in a neighbourhood of $(i, 0)$.

Given a point (τ, η) in $\mathcal{O}_{\mathbb{C}}$, there exists a path Γ in $\mathcal{O}_{\mathbb{C}}$, connecting $(i, 0)$ to (τ, η). From the definition of $\mathcal{O}_{\mathbb{C}}$, the generalized eigenspaces of $\mathcal{A}$ can be followed holomorphically along Γ. Using the same argument of holomorphic continuation as above, we see that $\mathcal{A}(\tau, \eta)$ is diagonalizable. □

5.1.2 *The block structure*

The proof of Lemma 4.5 tells us much more than actually stated. First, $E_-(i\rho_0, \eta_0)$ is the direct sum of subspaces that we denote by E_j (possibly many) and E_s (only one), each one being invariant for the matrix $\mathcal{A}(i\rho_0, \eta_0) = -i\mathcal{B}(\rho_0, \eta_0)$, with

$$\mathcal{B}(\rho_0, \eta_0) := (A^d)^{-1}(\rho_0 I_n + A(\eta_0)) \in M_n(\mathbb{R}).$$

We notice that the spectrum of $-i\mathcal{B}$ remains unchanged under the symmetry with respect to the imaginary axis. The component E_s is exactly the stable subspace of $\mathcal{A}(i\rho_0, \eta_0)$. It may be trivial, or equal to E_-, or something else in between. Obviously, Lemma 4.1 does not apply up to $\tau = i\rho_0$. The component E_j is an invariant subspace on which $\mathcal{A}(i\rho_0,\eta_0)$ has a unique eigenvalue $i\mu_j$, the μ_js being pairwise distinct real numbers. Thus the direct sum of the E_js is the 'central' part of E_-. We warn the reader that E_j does not coincide, in general, with the generalized eigenspace $\ker(\mathcal{A}-i\mu_j I_n)^n$ (see Proposition 5.1 below), so that the central part of $E_-(i\rho_0,\eta_0)$ is only a subspace of the central invariant subspace of $\mathcal{A}$. What the proof above tells us is that, for each index j, the restriction of $\mathcal{A}$ to E_j is similar to the very regular Jordan block

$$J(i\mu_j;q,l) := \begin{pmatrix} i\mu_j I_q & I_q & 0_q & \dots & \\ 0_q & \ddots & \ddots & O & \vdots \\ \vdots & & \ddots & \ddots & 0_q \\ & O & & \ddots & I_q \\ & & \dots & 0_q & i\mu_j I_q \end{pmatrix},$$

where l stands for the number of diagonal blocks and q stands for the size of each one (the multiplicity of $-\rho_0$ as an eigenvalue of $A(\eta,\mu_j)$.)

What is even more interesting is that the proof can be adapted to the study of the generalized eigenspace $\ker(\mathcal{A}-i\mu_j I_n)^n$. As a matter of fact, this subspace extends analytically to nearby values of (τ,η), as an invariant subspace $H(\tau,\eta)$ of $\mathcal{A}(\tau,\eta)$. The latter is the sum of generalized eigenspaces associated to the eigenvalues of $\mathcal{A}$ that are close to $i\mu_j$. These are precisely the roots of $Q(-i\tau,\eta,-i\cdot)$ that are close to $i\mu_j$, and their multiplicities as eigenvalues are q times their multiplicities as roots. Thus, the dimension of H equals qN, with the notation of the previous section. Again, $\mathcal{A}$ being diagonalizable for (τ,η) in $\mathcal{O}$, we see that the minimal polynomial of the restriction of $\mathcal{A}$ to H is a polynomial of degree at most N for (τ,η) close to $(i\rho_0,\eta_0)$. By continuity, this still holds at point $(i\rho_0,\eta_0)$. Hence, E_j is included in $\ker(\mathcal{A}(i\rho_0,\eta_0)-i\mu_0)^N$. The same convexity argument about the Jordan tower shows that the spaces G_k have dimensions kq for every k up to N. Therefore, the Jordan block of $\mathcal{A}(i\rho_0,\eta_0)$, corresponding to its eigenvalue $i\mu_j$, is exactly the very regular $J(i\mu_j;q,N)$.

Let us now investigate the link between the numbers l and N. The roots of $Q(-i\tau,\eta,\cdot)$ behave, as (ρ,η) varies near (ρ_0,η_0), like Nth roots of unity of the discriminant $D(-i\tau,\eta)$ of the polynomial (Puiseux's theory), modulo higher-order terms. Therefore, they form an approximately regular N-agon, centred on the real axis. Among the N vertices, exactly l have a negative imaginary part. In order to evaluate l, we may fix η to the value η_0 and let $\tau = i\rho_0+\gamma$ vary along $i\rho_0+\mathbb{R}^+$. Using Newton's polygon, plus the fact that $\partial Q/\partial\rho(\rho_0,\eta_0,\mu_0)\neq 0$, we obtain that the N roots of $Q(-i\tau,\eta_0,\cdot)$ near μ_0 obey

$$(\mu-\mu_0)^N \sim i\gamma\left(\frac{\partial Q}{\partial\rho}\bigg/ N!\frac{\partial^N Q}{\partial\mu^N}\right)(\rho_0,\eta_0,\mu_0) =: ci\gamma.$$

One immediately concludes that $|2l-N|\leq 1$. In other words, $l=N/2$ in the even case, while $l=(N\pm 1)/2$ in the odd case.

The above results, which we summarize in the following proposition, express the fact that a constantly hyperbolic operator, associated to a non-characteristic boundary, satisfies the so-called *block structure condition*, introduced by Kreiss [103] as an assumption, and proved by Métivier [134] in our context.

Proposition 5.1 *We assume that L is constantly hyperbolic and that A^d is invertible. Let $(i\rho, \eta)$ be a boundary point ((ρ, η) is real and non-zero). Then*

i) Given a purely imaginary eigenvalue $i\mu$ of $\mathcal{A}(i\rho, \eta)$, the corresponding Jordan factor is a 'regular' Jordan block $J(i\mu; q, N)$, where q is precisely the multiplicity of $-\rho$ as an eigenvalue of $A(\eta, \mu)$.

ii) The space $E_-(i\rho, \eta)$ is the direct sum of

- *the stable subspace of $\mathcal{A}(i\rho, \eta)$,*
- *the subspaces $E_j(\rho, \eta) := \ker(\mathcal{A}(i\rho, \eta) - i\mu_j)^{l_j}$ for some index $l_j \in [(N_j - 1)/2, (N_j + 1)/2]$, where N_j is as in point i)).*

We complete this information with the following.

Lemma 5.2 *Let the operator $\partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be constantly hyperbolic. Let $(\eta_0, \mu_0) \in \mathbb{R}^d$ be a non-zero point, and let $-\rho_0$ be an eigenvalue of $A(\eta_0, \mu_0)$. Obviously, $-\rho_0$ is real, and we denote its multiplicity by q. We denote by R an $n \times q$ matrix, whose columns span the eigenspace $\ker(\rho_0 I_n + A(\eta_0, \mu_0))$, and similarly by L a $q \times n$ matrix, whose rows span the left-eigenspace of $\rho_0 I_n + A(\eta_0, \mu_0)$:*

$$L(\rho_0 I_n + A(\eta_0, \mu_0)) = 0, \quad (\rho_0 I_n + A(\eta_0, \mu_0))R = 0, \quad rk\, R = rk\, L = q.$$

Then it holds that

$$LA^d R = -\frac{\mathrm{d}\rho}{\mathrm{d}\mu}(\mu_0) LR \tag{5.1.1}$$

in $M_n(\mathbb{R})$ (LR is non-singular, since $A(\eta_0, \mu_0)$ is diagonalizable), where $\rho(\mu)$ is the root of $P(\cdot, \eta_0, \mu)$ such that $\rho(\mu_0) = \rho_0$.

Finally, $\mathrm{d}\rho/\mathrm{d}\mu(\rho_0)$ is non-zero if and only if the multiplicity of $i\mu_0$, as an eigenvalue of $\mathcal{A}(i\rho_0, \eta_0)$ (from Proposition 1.7, it is a multiple of q), equals q, that is if $N = 1$. In that case, $\mathrm{d}\rho/\mathrm{d}\mu(\rho_0)$ is negative if $l = 1$, positive if $l = 0$.

Proof As above, we denote by $P = P_0 Q^q$ a factorization in which ρ_0 is a simple root of $Q(\cdot, \eta_0, \mu_0)$. Classically, there is a unique locally defined analytic function $\mu \mapsto \rho$ that solves $Q(\cdot, \eta_0, \mu) = 0$, with $\rho(\mu_0) = \rho_0$. Also, the right- and left-eigenspaces depend analytically upon μ. This means that the bases of these eigenspaces can be chosen analytically. In other words, R and L can be extended analytically with the properties

$$L(\mu)(\rho(\mu) I_n + A(\eta_0, \mu)) = 0, \quad (\rho(\mu) I_n + A(\eta_0, \mu))R(\mu) = 0,$$
$$\mathrm{rk}\, R(\mu) = \mathrm{rk}\, L(\mu) = q.$$

Differentiating with respect to μ, we obtain

$$(\rho_0 I_n + A(\eta_0, \mu_0))\frac{\mathrm{d}R}{\mathrm{d}\mu} + \left(\frac{\mathrm{d}\rho}{\mathrm{d}\mu} I_n + A^d\right) R = 0.$$

Multiplying on the left by L, we obtain relation (5.1.1).

Obviously, the multiplicity of $i\mu_0$ as an eigenvalue of $\mathcal{A}(i\rho_0, \eta_0)$ equals q times the multiplicity of μ_0 as a root of $Q(\rho_0, \eta_0, \cdot)$. It equals q if and only if

$$\frac{\partial Q}{\partial \mu}(\rho_0, \eta_0, \mu_0) \neq 0,$$

which amounts to saying that $\mathrm{d}\rho/\mathrm{d}\mu(\mu_0) \neq 0$.

We finally consider the case $N = 1$. For (τ, η) close to $(i\rho_0, \mu_0)$, $\mathcal{A}(\tau, \eta)$ admits a unique eigenvalue ω close to $i\mu_0$. It is an analytic function of (τ, η), and

$$\frac{\partial \omega}{\partial \tau} = \left(\frac{\mathrm{d}\rho}{\mathrm{d}\mu}\right)^{-1},$$

the latter being a non-zero real number. Since $\mathrm{Re}\,\omega \neq 0$ when $\mathrm{Re}\,\tau > 0$, we see that ω is a stable eigenvalue for $\mathrm{Re}\,\tau > 0$, if and only if $\mathrm{d}\rho/\mathrm{d}\mu(\mu_0) < 0$. □

Important remark All the results in Sections 4.3.3 and 5.1 do not really use the fact that $(\tau, \eta) \mapsto \mathcal{A}(\tau, \eta)$ is a polynomial function. The properties are valid also when its coefficients are rational fractions, provided that the following assumptions hold true:

i) When $\tau = i\rho$ with $\rho \in \mathbb{R}$, the matrix $\mathcal{B}(\rho, \eta) := i\mathcal{A}(i\rho, \eta)$ has real entries,

ii) The rational fraction $P(\tau, \eta, \omega) := \det(\omega I_n - \mathcal{A}(\eta, \tau))$ is homogeneous,

iii) When $\omega = i\xi$ with $\xi \in \mathbb{R}$, the roots of $P(\cdot, \eta, i\xi)$ are purely imaginary, and their multiplicities do not vary with $(\eta, \xi) \neq 0$ (note that, because of Property *i*), the polynomial $(\rho, \eta, \xi) \mapsto P(i\rho, \eta, i\xi)$ has real coefficients),

iv) Given such an eigenvalue $i\lambda_j$, with multiplicity m_j, the kernel of $\xi I_n + \mathcal{B}(\lambda_j, \eta)$ has dimension m_j.

v) The boundary points $(i\rho, \eta)$ under consideration are not poles of $\mathcal{A}$.

In practice, Properties *i*) and *ii*) will be ensured by the construction of $\mathcal{A}$, while Properties *iii*) and *iv*) will express a form of constant hyperbolicity.

5.2 Construction of a Kreiss symmetrizer under (UKL)

We prove in this section the following important result.

Theorem 5.1 *Let a normal hyperbolic IBVP be defined by the domain*

$$\Omega = \{x \in \mathbb{R}^d\,;\, x_d > 0\},$$

with a linear first-order operator L and a boundary matrix B, both with constant coefficients. Assume that L is constantly hyperbolic and that the boundary is *non-characteristic (*$\det A^d \neq 0$*). Assume finally that the IBVP satisfies the uniform Kreiss–Lopatinskiĭ condition.*

Then there exists a matrix-valued $\mathscr{C}^\infty$*-map* $(\tau, \eta) \mapsto K(\tau, \eta)$*, on* $\mathrm{Re}\,\tau \geq 0$, $\eta \in \mathbb{R}^{d-1}$, $|\tau| + |\eta| \neq 0$*, such that*

i) the matrix $\Sigma(\tau, \eta) := K(\tau, \eta)A^d$ *is Hermitian,*

ii) there exists a number $c > 0$ *such that, for every* (τ, η) *and every* $x \in \ker B$*, the inequality* $x^*\Sigma(\tau, \eta)x \leq -c\|x\|^2$ *holds,*

iii) there exists a number $c_0 > 0$ *such that, for every* (τ, η)*, the inequality* $\mathrm{Re}\, M \geq c_0(\mathrm{Re}\,\tau)I_n$ *holds in the sense of symmetric matrices, where*

$$M = M(\tau, \eta) := -(\Sigma\mathcal{A})(\tau, \eta) = K(\tau, \eta)(\tau I_n + iA(\eta))$$

and $\mathrm{Re}\, M$ *denotes the Hermitian matrix* $\frac{1}{2}(M + M^*)$*.*

If instead, the matrices A^α *and* B *are parametrized (for instance, if* L *has variable coefficients) with regularity* $\mathscr{C}^k$ *with respect to the parameters* z*, then such a symmetrizer* Σ *can be chosen with the same regularity: derivatives* $\partial^m_{\tau,\eta}\partial^l_z\Sigma$ *are continuous whenever* $l \leq k$*.*

Comments

- Since A^d is invertible, it is equivalent to search for a K or for a Σ. In the following, we shall always work in terms of Σ. The situation with a characteristic boundary ($\det A^d = 0$) raises significant new difficulties. It will be treated in Chapter 6.
- The matrix K is called a Kreiss symmetrizer or a *dissipative symmetrizer*, or simply a *symmetrizer*. It plays, in the present framework, the role that the identity I_n played for symmetric operators with a strictly dissipative boundary condition. As a matter of fact, if L is symmetric, then $K \equiv I_n$ satisfies trivially point *i)*, while $\mathrm{Re}\, M = (\mathrm{Re}\,\tau)I_n$. Finally, *ii)* is simply the dissipation assumption.
- When L is symmetric, it may happen that the IBVP satisfies the (UKL) condition though the boundary condition is not dissipative. In such a case, the symmetrizer K provided by the theorem differs from I_n.
- All the arguments hold true when the data depend on parameters, exactly as developed in the proof. Hence we shall present them for a single data, depending only on (τ, η).

The proof of Theorem 5.1 is long and technical, though interesting in its own way. We shall split it into several steps. From Step 1 to Step 18, we detail the construction in the case of a strictly hyperbolic operator. We examine afterwards which steps need a further study for a constantly hyperbolic operator and how to adapt the proof to this more general framework.

Step 1 We shall actually build a K that is homogeneous with degree zero, with respect to (τ,η). Therefore, it will be enough to build K or Σ for (τ,η) in the unit hemisphere, Re $\tau \geq 0$, $\eta \in \mathbb{R}^{d-1}$, $|\tau|^2 + |\eta|^2 = 1$.

Step 2 Since the hemisphere is compact, and since the properties to fill define convex cones, it will be enough to *localize* the construction, namely to build a $\Sigma_{\rm loc}$ in the vicinity of every point, then to cover the hemisphere by a finite set of such neighbourhoods, and at last to define a Σ from the corresponding $\Sigma_{\rm loc}$s, using a partition of unity with real non-negative coefficients. From now on, we thus give ourselves a point (τ_0,η_0) and we look for a solution Σ near this point.

Step 3 The case of interior points, namely Re $\tau_0 > 0$ is easier, since, choosing Σ independently of (τ,η), it is enough to find a Σ in H_n satisfying $\Sigma|_{{\rm ker} B} < 0$ and Re $M > 0$ at the sole point (τ_0,η_0). Hence, the construction needs only to be done pointwisely instead of locally, at interior points.

Step 4 Given an interior point (τ,η), we build a symmetizer. We shall use two lemmas.

Lemma 5.3 *Let $\mathcal{A}$ be a hyperbolic matrix, and let E_-, E_+ be its stable and unstable invariant subspaces. Let us define*

$$X_- := \{H \in \mathbf{H}_n\,;\, E_- \subset {\rm ker} H\}.$$

Then the map $T : H \mapsto$ Re $(H\mathcal{A})$ is an automorphism of X_-.

Moreover, if $T(H) \geq 0$ with $E_- = {\rm ker} T(H)$, then $H \geq 0$ with $E_- = {\rm ker} H$.

Proof We first prove that T has a right-inverse. For that, let $X \in X_-$ be given. If $x = x_- + x_+$ with $x_\pm \in E_\pm$, we define

$$x^* H x := \int_{-\infty}^0 y(s)^* X y(s)\,{\rm d}s,$$

where y is the solution of ${\rm d}y/{\rm d}s = \mathcal{A}y$, $y(0) = x_+$. The integral converges since y decays exponentially fast at $-\infty$. The above equality defines a unique Hermitian matrix $H \in X_-$.

Given $h \in \mathbb{R}$ and $x(h) := x_- + y(h)$, we have

$$x(h)^* H x(h) := \int_{-\infty}^h y(s)^* X y(s)\,{\rm d}s.$$

Differentiating at the origin, we obtain

$$x^*(\mathcal{A}^* H + H\mathcal{A})x = x^* X x,$$

which means $T(H) = X$ (note that only the component of $x'(0)$ along E_+ is important in this calculation.)

The integral thus defines a right-inverse, which proves that T is an automorphism. If, moreover, $X = T(H)$ is non-negative, the integral formula shows that

H is non-negative. Assume additionally that $E_- = \ker X$. If $x \in \ker H$, we have

$$\int_{-\infty}^{0} y(s)^* X y(s)\, \mathrm{d}s = 0,$$

which implies that $y \in E_-$, that is $y \equiv 0$ and hence $x \in E_-$. □

We apply Lemma 5.3 to $\mathcal{A}(\tau, \eta)$. Let us choose a non-negative element K_+ in X_-, whose kernel is $E_-(\tau, \eta)$, and let us define $H_+ = T^{-1}(K_+)$. From Lemma 5.3, it is non-negative and $E_-(\tau, \eta) = \ker H_+$. Similarly, choosing a non-positive Hermitian matrix K_- such that $E_+(\tau, \eta) = \ker K_-$, the equation Re $(H\mathcal{A}) = K_-$ admits a unique non-negative Hermitian solution H_- whose kernel is $E_+(\tau, \eta)$.

From the Kreiss–Lopatinskiĭ condition $\mathbb{C}^n = E_-(\tau, \eta) \oplus \ker B$, there exists a matrix P such that, for $x_\pm \in E_\pm$,

$$(x_- + x_+ \in \ker B) \Longleftrightarrow (x_- = P x_+).$$

Since the restriction of H_+ to $E_+(\tau, \eta)$ is positive-definite, the restriction of $bH_+ - P^* H_- P$ to $E_+(\tau, \eta)$ is positive-definite for a large enough positive number b. We now define $\Sigma := -bH_+ + H_-$. Because of the choice of b, it satisfies point *ii)*. Moreover, Re $M = bK_+ - K_-$. As a sum of two non-negative Hermitian matrices, it is non-negative and its kernel is contained in the intersection of both kernels, which is $E_-(\tau, \eta) \cap E_+(\tau, \eta) = \{0\}$. This matrix is therefore positive-definite.

Remark When $\mathcal{A}(\tau, \eta)$ is not hyperbolic, a fact that happens at some boundary points, at least when $|\tau| >> |\eta|$, there cannot exist an Hermitian Σ such that Re $(\Sigma\mathcal{A})$ is negative-definite. Actually, if x is an eigenvector associated with a pure imaginary eigenvalue μ of $\mathcal{A}$, then Re $x^* \Sigma \mathcal{A} x = (\text{Re } \mu) x^* \Sigma x = 0$.

Summary After Step 4, it remains to construct a Σ in a neighbourhood of any given point $(i\rho_0, \eta_0)$ of the boundary.

Step 5 We recall our notation $\tau = \gamma + i\rho$. Let $Q \in \mathbf{GL}_n(\mathbb{R})$ be given. Making the change of unknowns $v = Qu$ amounts to replacing the matrices A^α, $\mathcal{B}(\rho, \eta)$ and B by

$$a^\alpha := Q A^\alpha Q^{-1}, \quad \beta(\rho, \eta) := Q \mathcal{B}(\rho, \eta) Q^{-1}, \quad b := BQ^{-1}.$$

Then, a local solution $\Sigma(\tau, \eta)$ of our problem yields a local solution $\sigma(\tau, \eta)$ of the problem associated to the matrices a^α, β and b, through the correspondence $\Sigma =: {}^t\!Q \sigma Q$. Similarly, M is replaced by m, defined by $M =: {}^t\!Q m Q$. One may even allow Q to depend smoothly on (ρ, η).

Step 6 Let $(i\rho_0, \eta_0)$ be a non-zero boundary point. Using a change of basis as above, we may assume that

$$Q_0 \mathcal{B}(\rho_0, \eta_0) Q_0^{-1} =: \beta_0 = \operatorname{diag}(\beta_{\mathrm{c}0}, \beta_{\mathrm{h}0}),$$

where 'c' and 'h' stand for the central and hyperbolic parts of $\mathcal{A}(i\rho_0, \eta_0)$. Namely, $\beta_{\mathrm{c}0}$ has only real eigenvalues, while $\beta_{\mathrm{h}0}$ has only non-real eigenvalues. One may even assume that β_{c0} has a canonical Jordan form. From the strict hyperbolicity of L, we know that eigenspaces of $\mathcal{B}(\rho_0, \eta_0)$, when associated to real eigenvalues, are lines because

$$\ker(\mathcal{B}(\rho, \eta) - \mu I_n) = \ker(\rho I_n + A(\eta, -\mu)).$$

Therefore, distinct Jordan blocks of β_{c0} have distinct eigenvalues. We shall denote by m the size of β_{c0} and by β_j $(1 \le j \le J)$ its Jordan blocks. The size and the eigenvalue of β_j are denoted by m_j and μ_j.

Next, it is well-known[1] that an analytical function $Q(\rho, \eta)$ may be found in a neighbourhood of (ρ_0, η_0), so that $Q(\rho_0, \eta_0) = Q_0$ and

$$Q(\rho, \eta)\mathcal{B}(\rho, \eta)Q(\rho, \eta)^{-1} =: \beta(\rho, \eta) = \operatorname{diag}(\beta_{\rm c}(\rho, \eta), \beta_{\rm h}(\rho, \eta)).$$

We point out that, by a continuity argument, $\beta_{\rm h}$ still has non-real eigenvalues, though the eigenvalues of $\beta_{\rm c}$ need not remain real. From upper semicontinuity of geometric multiplicities, the eigenspaces of $\beta_{\rm c}$ are lines.

Step 7 From the last two steps, we are led to the construction of a local solution $\sigma(\tau, \eta)$ in a neighbourhood $\mathcal{V}$ of $(i\rho_0, \eta_0)$, associated to the matrices a^d, β and b. Here, all three matrices depend smoothly on (ρ, η), but do not depend on γ.

We shall specialize σ as follows:

$$\sigma(\gamma + i\rho, \eta) = \operatorname{diag}(S + i\gamma T, \sigma_{\rm h}),$$

where $S = S(\rho, \eta)$, $T = T(\rho, \eta)$, $\sigma_{\rm h}$ are such that

$$S \in \mathbf{Sym}_m, \quad T \in \mathbf{Alt}_m, \quad \sigma_{\rm h} \in \mathbf{H}_{n-m}.$$

In particular, S and T have real entries[2] and this ensures that σ is Hermitian. From $m := \sigma(\gamma(a^d)^{-1} + i\beta)$, we obtain

$$m = \begin{pmatrix} iS\beta_{\rm c} + \gamma(S(a^d)^{-1}_{cc} - T\beta_{\rm c}) + O(\gamma^2) & O(\gamma) \\ O(\gamma) & i\sigma_{\rm h}\beta_{\rm h} + O(\gamma) \end{pmatrix}.$$

From the Remark in Step 4, there is no hope that the term $S\beta_{\rm c}$ would help in the positivity of Re m. Therefore, we shall ask that

$$S\beta_{\rm c} \in \mathbf{Sym}_m, \tag{5.2.2}$$

for every (ρ, η) in a neighbourhood $\mathcal{W}$ of (ρ_0, η_0). Denoting

$$\begin{aligned} Y(\rho, \eta) :&= S(a^d)^{-1}_{cc} - T\beta_{\rm c} + {}^t(S(a^d)^{-1}_{cc} - T\beta_{\rm c}) \\ &= S(a^d)^{-1}_{cc} + (a^d)^{-t}_{cc}S - T\beta_{\rm c} + {}^t\beta_{\rm c}T, \end{aligned}$$

[1]See, for instance, the procedure used in the proof of Theorem 2.3.

[2]We recall that, unless another ground field is specified, elements of $\mathbf{Sym}_n$ and $\mathbf{Alt}_n$ have real entries.

we obtain

$$\text{Re } m = \begin{pmatrix} \frac{\gamma}{2}Y(\rho,\eta) + O(\gamma^2) & O(\gamma) \\ O(\gamma) & \text{Re } (i\sigma_{\rm h}\beta_{\rm h}) + O(\gamma) \end{pmatrix}. \tag{5.2.3}$$

Step 8 We wish now to replace the local construction by a pointwise one. Let us first assume that $Y(\rho,\eta)$ and Re $(i\sigma_{\rm h}\beta_{\rm h})$ be positive-definite at (ρ_0,η_0). By continuity, they remain so, uniformly in some neighbourhood of (ρ_0,η_0) that we still call $\mathcal{W}$. Then, decomposing vectors of $\mathbb{C}^n$ into their 'c' and 'h' components, we have for every (ρ,η) in $\mathcal{W}$

$$\text{Re } (X^*mX) \geq c\gamma|X_{\rm c}|^2 + c|X_{\rm h}|^2 + O(\gamma)|X_{\rm c}|\,|X_{\rm h}|,$$

where c is some positive constant. Using the Cauchy–Schwarz inequality, we see that m satisfies point *iii)* near $(i\rho_0,\eta_0)$.

In other words, the point *iii)* will be fulfilled locally whenever the following properties hold

$$Y(\rho_0,\eta_0) \in \mathbf{SPD}_m(\mathbb{R}), \tag{5.2.4}$$

$$\text{Re } (i\sigma_{\rm h}\beta_{\rm h0}) \in \mathbf{HPD}_{n-m}. \tag{5.2.5}$$

Step 9 We finish the work begun in Step 8. To do so, we study condition (5.2.2).

Lemma 5.4 *The equation $S\beta_{\rm c}(\rho,\eta) = {}^t\beta_{\rm c}(\rho,\eta)S$ defines a subspace of dimension m in $\mathbf{Sym}_m$, for every (ρ,η) in a small neighbourhood of (ρ_0,η_0). This subspace varies smoothly with (ρ,η).*

Proof We only have to prove that the map Λ

$$S \mapsto S\beta_{\rm c} - {}^t\beta_{\rm c}S$$

$$\mathbf{Sym}_m \to \mathbf{Alt}_m$$

is onto. Then the analytic dependence of $\ker\Lambda$ follows from that of Λ and from the constancy of the dimension.

Recall that the eigenspaces of $\beta_{\rm c}$ are (not necessarily real) lines for every value of (ρ,η) in $\mathcal{W}$. Since the property to prove is equivalent to the injectivity of the transpose of Λ:

$$s \mapsto \beta_{\rm c}s - s\,{}^t\beta_{\rm c}$$

$$\mathbf{Alt}_m \to \mathbf{Sym}_m,$$

the Lemma will be a direct consequence of the following one, together with the fact that ${}^t\beta_{\rm c}$ is similar to its transpose. □

Lemma 5.5 *Let two matrices $s, D \in \mathbf{M}_n(\mathbb{C})$ be given. We assume that the eigenspaces of D are lines, that s is alternate, and that ${}^tDs - sD = 0$. Then $s = 0$.*

Proof The fact that the eigenspaces of D are lines means that the minimal polynomial of D equals its characteristic one. This amounts to saying that there exists a vector x such that $\mathbb{C}^n$ equals the Krylov subspace $\mathrm{Span}\{x, Dx, D^2x, \dots\}$ (see Exercise 16, Chapter 2 in [187]). In other words, as a $\mathbb{C}[D]$-module, $\mathbb{C}^n$ has dimension one.

Denoting by a the alternate form defined by s, we have assumed that $a(Dz, y) = a(z, Dy)$ for every $z, y \in \mathbb{C}^n$. Applying recursively this equality, we obtain for every $k, l \in \mathbb{N}$

$$a(D^k x, D^l x) = a(D^l x, D^k x) = -a(D^k x, D^l x),$$

so that $a(D^k x, D^l x) = 0$. Since the vectors $D^k x$ span $\mathbb{C}^n$, we conclude that $a = 0$, that is $s = 0$. □

Assume now that a matrix S_0 in $\mathbf{Sym}_m$ satisfies $S_0\beta_{c0} \in \mathbf{Sym}_m$. Then, from Lemma 5.4, it is possible to find an analytic function $(\rho, \eta) \mapsto S$ from $\mathcal{W}$ to $\mathbf{Sym}_m$ that satisfies (5.2.2) in $\mathcal{W}$ and $S(\rho_0, \eta_0) = S_0$.

Hence, in order that σ satisfy all requirements near a point (ρ_0, η_0), it is enough that (5.2.4) and (5.2.5) hold and

$$S_0\beta_{c0} \in \mathbf{Sym}_m. \tag{5.2.6}$$

In other words, we have replaced the local construction of σ by a pointwise one. Thanks to this simplification, we shall drop the index '0' from now on.

Summary After Step 9, it remains to construct three matrices $S \in \mathbf{Sym}_m$, $T \in \mathbf{Alt}_m$ and $\sigma_\mathrm{h} \in \mathbf{H}_{n-m}$ with the properties that at a given point (ρ, η) in the sphere S^{d-1}, (5.2.2), (5.2.4) and (5.2.5) hold together with

$$\sigma|_{\ker b} < 0, \tag{5.2.7}$$

where $\sigma := \mathrm{diag}(S, \sigma_\mathrm{h})$.

Step 10 Since T occurs only in (5.2.4), we analyse first this property, assuming that S is known. Then $Y = Y_S - T\beta_\mathrm{c} + {}^t\beta_\mathrm{c} T$, where $Y_S \in \mathbf{Sym}_m$ is given in terms of S. Let us remember that $\beta = \mathrm{diag}(\beta_1, \dots)$, where the β_js are Jordan blocks with distinct real eigenvalues μ_j. We decompose $T =: (T_{jk})_{j,k}$ block-wise accordingly. We note that the block T_{jk} occurs only in the (j, k)-block of Y, through ${}^t\beta_j T_{jk} - T_{jk}\beta_k$. Therefore, using the well-known fact that $X \mapsto NX - XN'$ is an automorphism of $M_{rs}(K)$ whenever N and N' have disjoint spectra, we may choose uniquely the offdiagonal blocks T_{jk} in such a way that Y be block-diagonal. We emphasize that, Y_S being symmetric, this choice is consistent with the skew-symmetric form of T that we ask for: it holds that $T_{kj} = -\,{}^tT_{jk}$.

Step 11 We continue Step 10 by determining the (skew-symmetric) diagonal blocks T_{jj}. For that purpose, we use the following lemma.

Lemma 5.6 *Let y belong to $\mathbf{Sym}_m$, and let β be an upper Jordan block of size m. Then the following properties are equivalent,*

- *there exists a skew-symmetric t, such that ${}^t\beta t - t\beta + y$ is positive-definite.*
- $y_{11} > 0$.

Proof The direct implication is trivial, because one always has $(t\beta)_{11} = 0$. To prove the converse, we proceed by induction. There is nothing to prove if $m = 1$. If $m \geq 2$, we assume that the lemma is true at order $m - 1$. We use hats for $(m-1) \times (m-1)$ upper-left blocks. Since $y_{11} > 0$, the induction hypothesis ensures the existence of an $(m-1) \times (m-1)$ skew-symmetric matrix $\hat{t}$, such that $ss_{m-1} := {}^t\hat{\beta}\hat{t} - \hat{t}\hat{\beta} + \hat{y} > 0$. We now define

$$t := \begin{pmatrix} \hat{t} & X \\ -{}^tX & 0 \end{pmatrix}, \quad X := \begin{pmatrix} 0_{m-2} \\ u \end{pmatrix}.$$

A straightforward computation gives

$${}^t\beta t - t\beta + y = \begin{pmatrix} ss_{m-1} & \cdot \\ \cdot & 2u + y_{mm} \end{pmatrix},$$

where the offdiagonal terms do not depend on u. Choosing $u > 0$ large enough, the resulting matrix is positive-definite. □

Summary Thanks to Steps 10 and 11, we have reduced our task to the construction of $S \in \mathbf{Sym}_m$ and $\sigma_{\mathrm{h}} \in \mathbf{H}_{n-m}$ with the properties that at a given point (ρ, η) in the sphere S^{d-1}, (5.2.2), (5.2.5) and (5.2.7) hold together with

$$y_j > 0, \quad 1 \leq j \leq J, \tag{5.2.8}$$

where y_j stands for the upper-left coefficient of the jth diagonal block Y_{jj} of Y_S.

Step 12 We decompose as above S blockwise. Then (5.2.2), together with $S \in \mathbf{Sym}_m$, give ${}^t\beta_j S_{jk} = S_{jk}\beta_k$. Using the fact mentioned in Step 10, we find that $S_{jk} = 0$ when $j \neq k$. Hence, S has to be block-diagonal. We shall denote by S_j its diagonal blocks.

Step 13 Hence, $S\beta_{\mathrm{c}} = \mathrm{diag}(S_1\beta_1, \dots)$ and (5.2.2) reduces to $S_j\beta_j \in \mathbf{Sym}_{m_j}$. Since $S_j \in \mathbf{Sym}_{m_j}$ as well, we find by inspection that the S_js have the general form

$$\begin{pmatrix} & & & s_1 \\ & O & \iddots & s_2 \\ & \iddots & \iddots & \\ \iddots & \iddots & \iddots & \\ s_1 & s_2 & & s_{m_j} \end{pmatrix}, \tag{5.2.9}$$

with arbitrary numbers $s_1, \dots, s_{m_j}$.

Step 14 Since the block Y_{jj} equals $S_j a_j + {}^ta_j S_j$, where a_j is the jth diagonal block of $(a^d)^{-1}$, we obtain $y_j = 2s_1^j a_1^j$, where s_1^j stands for the s_1-coefficient of S_j and a_1^j stands for the last coefficient of the first column of a_j. In other words,

$a_1^j = (a_j)_{m_j 1}$. At this point, it is essential to know the sign of a_1^j. For this, we note that a_1^j equals $\ell'(a^d)^{-1}r'$, where

$$r' = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \quad \text{and} \quad \ell' = (\dots, 0, 1, 0, \dots)$$

are the right and left eigenvectors of β, corresponding to the block β_j. Defining $\ell := \ell' Q$ and $r = Q^{-1}r'$, we obtain $a_1^j = \ell(A^d)^{-1}r$, where r, ℓ are right and left eigenvectors of $\mathcal{B}$. With the notations of Lemma 5.2, $r = R$ and $\ell = LA^d$ (this again shows (5.1.1) when $m_j \geq 2$), hence $a_1^j = LR$. In particular, a_1^j is non-zero.

We notice that, when $m_j = 1$, $\ell' = {}^t r'$ holds and therefore $\ell' r' = 1 > 0$, which translates into $LA^d R > 0$. This fixes the respective orientations of L and R. In the opposite case, one has $\ell' r' = 0$, that is $LA^d R = 0$, which leaves the orientations of L, R independent of each other. In other words, the sign of a_1^j is well-defined if $m_j = 1$, as the one of $(LR)(LA^d R)$, while it is free (depending on our choice of the Jordan basis) when $m_j > 1$.

Recalling Proposition 5.1, we know that $E_-(i\rho, \eta)$ is a direct sum of invariant subspaces E_j $(1 \leq j \leq J)$ and $E_{\rm h}$, corresponding either to the block β_j or to $\beta_{\rm h}$. The dimension of $E_{\rm h}$ is half the size $n - m$ of $\beta_{\rm h}$, while that of E_j belongs to $[m_j - 1/2, m_j + 1/2]$.

If $m_j = 1$, then Proposition 5.1 and Lemma 5.2 show that a_1^j is positive if and only if E_j is non-trivial. Therefore $a_1^j > 0$ if and only if $R \in E_-(i\rho, \eta)$.

If $m_j > 1$, then E_j is non-trivial. The sign of a_1^j may be choosen at our convenience.

Step 15 Let us denote $e_- := QE_-(\tau, \eta)$, so that, by the (UKL) condition, it holds that $e_- \cap \ker b = \{0\}$. From the description given in Proposition 5.1, we have $e_- = e_{\rm c} \oplus e_s$, where $e_{\rm c} \subset \mathbb{C}^m \times \{0\}$ and $e_s \subset \{0\} \times \mathbb{C}^{n-m}$. The component e_s, which corresponds to the stable subspace of $\mathcal{A}(i\rho, \eta)$, is the invariant subspace of β associated to the eigenvalues (of $\beta_{\rm h}$) of negative imaginary parts. The 'central' component $e_{\rm c}$ is spanned by some of the m vectors of the Jordan basis for $\beta_{\rm c}$. Within the Jordan basis associated to the jth block β_j, the l_j first vectors are in $e_{\rm c}$, while the other ones are not, where $|l_j - m_j/2| \leq 1/2$. This allows us to write $e_{\rm c} = \oplus_j e_j$. Denoting also by f_j the subspace spanned by the $m_j - l_j$ last vectors of this basis, by f_c their direct sum and by f_u the 'unstable' component, namely the invariant subspace associated to the eigenvalues (of $\beta_{\rm h}$) of positive imaginary parts, we obtain $\mathbb{C}^n = e_{\rm c} \oplus e_s \oplus f_{\rm c} \oplus f_u =: e_- \oplus f_+$. Note that, in spite of the notation, f_+ is not the limit, as (τ, η) tends to $(i\rho, \eta)$, of $QE_+(\tau, \eta)$.

Given a vector x in $\mathbb{C}^n$, we write its block-decomposition

$$x = \begin{pmatrix} \vdots \\ x_j \\ \vdots \\ x_{\rm h} \end{pmatrix},$$

with obvious notations. Then, in each block x_j $(1 \le j \le J)$, we split into e_j and f_j components:

$$x_j = \begin{pmatrix} x_{j-} \\ x_{j+} \end{pmatrix}.$$

Finally, we denote by $x_{\rm h} = x_s + x_u$ the decomposition in $\{0\} \times \mathbb{C}^{n-m}$. Hence, the decomposition of x into e_- and f_+ components reads

$$x = x_- + x_+ =: \begin{pmatrix} \vdots \\ \begin{pmatrix} x_{j-} \\ 0 \end{pmatrix} \\ \vdots \\ x_s \end{pmatrix} + \begin{pmatrix} \vdots \\ \begin{pmatrix} 0 \\ x_{j+} \end{pmatrix} \\ \vdots \\ x_u \end{pmatrix}.$$

In the following, we shall identify x_-, x_+ with the vectors

$$\begin{pmatrix} \vdots \\ x_{j-} \\ \vdots \\ x_s \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ x_{j+} \\ \vdots \\ x_u \end{pmatrix},$$

respectively.

Step 16 The assumption that the Kreiss–Lopatinskiĭ property holds at the boundary point $(i\rho, \eta)$ means exactly that the subspace ker$b$ has an equation of the form $x_- = Px_+$, where P is an appropriate linear map. From the description of $x_\pm$ above, we write P blockwise

$$P = \begin{pmatrix} P_{jk} & P_{ju} \\ P_{sk} & P_{su} \end{pmatrix}_{1 \le j,k \le J}.$$

Let us summarize what we are looking for. We wish to find matrices $S_j \in \mathbf{Sym}_{m_j}$ of the form (5.2.9), and a Hermitian matrix $\sigma_{\rm h}$, with the following properties:

- for every j, $s_1^j \neq 0$,
- when $m_j = 1$, then s_1^j is negative if e_j is trivial and positive otherwise,
- Re $(i\sigma_{\rm h}\beta_{\rm h})$ is positive-definite,
- the restriction of $\sigma = \mathrm{diag}(\dots, S_j, \dots, \sigma_{\rm h})$ to ker$b$ is negative-definite.

An important point is that we do not require that β_j be a standard Jordan block. Therefore we are free to make a change of variables $x \mapsto x' = Rx$, provided it alters neither the block-diagonal form of σ, nor the form (5.2.9) of each diagonal block. We choose a diagonal change of variable

$$R = R(\epsilon) = \operatorname{diag}(\dots, R_j, \dots, I_{n-m}),$$

where

$$R_j = \operatorname{diag}(\dots, \epsilon^3, \epsilon, \epsilon^{-1}, \epsilon^{-3}, \dots) =: \operatorname{diag}(U_j, V_j).$$

In other words,

$$x'_{j-} = U_j x_{j-}, \quad x'_{j+} = V_j x_{j+}, \quad x'_s = x_s, \quad x'_u = x_u.$$

It is important to note that this change of variables does not modify the overall structure (5.2.9). Therefore, we have to solve the same problem as described above, with the only change that the equation of ker$b$ is replaced by $x'_- = P(\epsilon)x'_+$ for some real number $\epsilon \neq 0$, where

$$P(\epsilon) = \begin{pmatrix} U_j P_{jk} V_k^{-1} & U_j P_{ju} \\ P_{sk} V_k^{-1} & P_{su} \end{pmatrix}_{1 \le j,k \le J}.$$

Step 17 When $\epsilon \to 0$, $P(\epsilon)$ tends to

$$P_0 = \begin{pmatrix} 0 & 0 \\ 0 & P_{su} \end{pmatrix}_{1 \le j,k \le J}.$$

Let us assume that a solution σ'_0 of the above problem has been found, when ker$b$ is replaced by the subspace defined by $x'_- = P_0 x'_+$. Then, by continuity, σ'_0 is still a solution of the problem for some small non-zero value of ϵ. Hence, going back to $\epsilon = 1$ through the change of variable, we obtain our matrix σ. This shows that we need only to solve the problem in Step 16, when ker$b$ is the subspace defined by

$$x_{j-} = 0 \quad (1 \le j \le J), \quad x_s = P_{su} x_u.$$

Step 18 We study the latter problem. When $x_{j-} = 0$ and $x_s = P_{su} x_u$, then

$$x^* \sigma x = \sum_{j=1}^{J} x_{j+}^* \hat{S}_j x_{j+} + x_{\mathrm{h}}^* \sigma_{\mathrm{h}} x_{\mathrm{h}},$$

where $\hat{S}_j$ is one of the following three matrices

$$\begin{pmatrix} s_1^j & s_2^j & & \\ s_2^j & \iddots & \iddots & \\ & \iddots & \iddots & \\ & & & s_{m_j}^j \end{pmatrix}, \quad \begin{pmatrix} s_2^j & s_3^j & & \\ s_3^j & \iddots & \iddots & \\ & \iddots & \iddots & \\ & & & s_{m_j}^j \end{pmatrix}, \quad \begin{pmatrix} s_3^j & s_4^j & & \\ s_4^j & \iddots & \iddots & \\ & \iddots & \iddots & \\ & & & s_{m_j}^j \end{pmatrix}.$$

The problem clearly decouples. On the one hand, we wish to find, for each j, a negative-definite matrix $\hat{S}_j$ of the form above. This can be found easily when $m_j > 1$, since then there is no constraint on the entries $s_1^j, \dots$. When $m_j = 1$, there are two cases. Either e_j is trivial, and then $\hat{S}_j = (s_1^j)$, where the constraint is $s_1^j < 0$, obviously compatible with our task. Or e_j is non-trivial, then $\hat{S}_j$ is void and there is nothing to prove. Hence the problem concerning the $\hat{S}_j$s can always be solved.

On the other hand, we have to find a Hermitian matrix $\sigma_{\rm h}$, such that $\mathrm{Re}\,(i\sigma_{\rm h}\beta_{\rm h})$ is positive-definite, and the restriction of $\sigma_{\rm h}$ to the space defined by $x_s = P_{su}x_u$ is negative-definite. Noting that, from the Kreiss–Lopatinskiĭ condition, this subspace is transverse to the stable subspace of $-i\beta_{\rm h}$, this is exactly the same problem as the one solved in Step 4.

Step 19 We now turn to the case of a constantly hyperbolic operator. Only Steps 6, 9, 11, 13, 14, 16 and 18 need some adaptation.

In Step 6, the description of $\beta_{\rm c0}$ is given by Proposition 5.1. The blocks β_j have distinct eigenvalues. Each one is a regular Jordan block $J(\mu; q, N)$.

Step 20 The adaptation of Lemma 5.4 in Step 9 is subtle. Actually, the dimension of the space of symmetric matrices S such that $S\beta_{\rm c}(\rho,\eta) = {}^t\beta_{\rm c}(\rho,\eta)S$, though constant, will not be equal to m, but to another number, see below. Let us first note that we can consider instead the complex dimension of the set of complex symmetric matrices with this property[3]. Now, any conjugation $\beta \mapsto P^{-1}\beta P$ induces the transformation $S \mapsto {}^tPSP$ on the solutions, thus preserving the dimension. We use this argument in two ways. First, we may assume that

$$\beta_{\rm c}(\rho,\eta) = \mathrm{diag}(\dots, \beta_j(\rho,\eta), \dots),$$

where β_j are smooth functions of their arguments, and the $\beta_j(\rho_0,\eta_0)$s are the regular Jordan blocks $J(\mu_j; q_j, N_j)$, with distinct real eigenvalues, described in Step 19. Then the fact recalled in Step 10 tells us that solutions S must be block-diagonal too, say $\mathrm{diag}(\dots, S_j, \dots)$, since the β_js keep disjoint spectra. Next, we may assume that each $\beta_j(\rho,\eta)$ has a Jordan form (here, P need not depend smoothly on (ρ,η)). Let us note that the arguments in Section 5.1 adapt to every eigenvalue of $\beta(\rho,\eta)$ (not only the pure imaginary ones): to each eigenvalue, there corresponds a unique Jordan block, a regular one. More precisely, $\beta_j(\rho,\eta)$ is a collection of blocks $J(\omega; q_j, N_\omega)$, with ω in some finite set Ω_j. The geometric multiplicity q_j is the same for all elements ω, because geometric multiplicity is upper semicontinuous, and because all ωs are roots of the same factor $P_k^{q_j}$ in the characteristic polynomial (see Proposition 1.7).

[3]We prefer considering solutions in $\mathbf{Sym}_m(\mathbb{C})$ rather than in H_m, because the latter is not a complex vector space.

Again, the diagonal block S_j must be of the form $\mathrm{diag}(\dots, S_j^\omega, \dots)$, where

$$S_j^\omega J(\omega; q_j, N_\omega) = {}^tJ(\omega; q_j, N_\omega) S_j^\omega.$$

As in step 13, we find that S_j^ω must have the form (5.2.9), where now the terms $s_1, \dots$ must be $q_j \times q_j$ symmetric matrices. Such matrices from a vector space of dimension $N_\omega q_j(q_j+1)/2$. Summing over Ω_j, we see that the set of solutions S_j has dimension $N_j q_j(q_j+1)/2$. Finally, the set of solutions S has dimension

$$\sum_j \frac{1}{2} N_j q_j (q_j + 1),$$

obviously a constant number in a neighbourhood of (ρ_0, η_0).

Step 21 In Step 11, Lemma 5.6 adapts straightforwardly to the context of regular Jordan block $J = J(\mu; q; N)$. The existence of a skew-symmetric real matrix t such that ${}^tJt - tJ + y$ is positive-definite is equivalent to the positivity of the upper-left diagonal $q \times q$ block y_{11}. The proof goes by induction on N. Therefore, condition (5.2.8) is unchanged up to the fact that y_{11} is now a $q \times q$ block, instead of a scalar.

Then, $y_j = s_1^j a_1^j + {}^ta_1^j s_1^j$, where $s_1^j \in \mathbf{Sym}_q$ is to be chosen. The lower-left block a_1^j may be written, as in Step 14, as LR, where $L = \ell' Q (A^d)^{-1}$, $R = Q^{-1} r'$ and

$$r' = \begin{pmatrix} \vdots \\ 0_q \\ I_q \\ 0_q \\ \vdots \end{pmatrix}, \quad \text{and} \quad \ell' = (\dots, 0_q, I_q, 0_q, \dots).$$

Rows of L and columns of R span the left and right kernels of $\rho I_n + A(\eta, \mu)$ (notations of Lemma 5.2). In particular, $a_1^j \in \mathbf{GL}_q(\mathbb{R})$. When N (which plays the rôle of m_j) is larger than 1, $\ell' r' = 0_q$ holds, hence $LA^dR = 0_q$, which is consistent with identity (5.1.1) and the fact that $\mathrm{d}\rho/\mathrm{d}\mu$ vanishes at this point. Then, we are free to compose L and R with two independent invertible $q \times q$ matrices (this is reflected in a modification of the change of basis Q). This allows us to prescribe any arbitrary invertible value to a_1^j, for instance $\pm I_q$. Consequently, $y_j = \pm 2 s_1^j$ and we may choose any positive- or negative-definite s_1^j.

On the other hand, when $N = 1$, $LA^dR = I_q$ holds, hence from (5.1.1)

$$a_1^j = -\left(\frac{\mathrm{d}\rho}{\mathrm{d}\mu}\right)^{-1} I_q, \quad y_j = -2\left(\frac{\mathrm{d}\rho}{\mathrm{d}\mu}\right)^{-1} s_1^j.$$

In conclusion, s_1^j must be chosen positive-definite if E_j is non-trivial, but negative-definite if E_j is trivial.

Step 22 In the requirements of Step 16, the first item must be read $\pm s_1^j \in$ $\mathbf{SPD}_{q_j}$ for every j (instead of $s_1^j \neq 0$). Next, each diagonal block R_j must be chosen in the block-diagonal form

$$R_j = \mathrm{diag}(\ldots, \epsilon^3 I_{q_j}, \epsilon I_{q_j}, \epsilon^{-1} I_{q_j}, \epsilon^{-3} I_{q_j}, \ldots).$$

The rest of the step works in exactly the same way.

Finally, there is even more room for the choice of $\hat{S}_j$ in Step 18. It may again be chosen so as to be negative-definite.

The proof is complete. □

Comment The construction of the dissipative symmetrizer with some uniform estimates reduces quite easily (Steps 1 and 2) to a local problem. At interior points, the construction can even be done pointwise (Step 3), with a Lyapunov stability argument (Step 4). Thus, most of the proof is devoted to the construction in the neighbourhood of boundary points $(i\rho, \eta)$. At such points, we separate the hyperbolic and the central part of $\mathcal{B}(\rho, \eta)$ (Steps 5 and 6). The symmetrizer is block-diagonal in terms of the invariants subspaces of $\mathcal{B}(\rho, \eta)$ (Step 7). The treatment of the hyperbolic part is similar to the case of interior points (Step 18). Thus there remains the central part. What is hidden in the proof above is that the construction is rather simple when the central part of $\mathcal{B}(\rho, \eta)$ is semisimple, that is diagonalizable. Suppose that its eigenvalues are simple, or that their multiplicities are locally constant. Then the symmetrizer is block-diagonal and certain parts of the analysis (Steps 13 to 18) are essentially trivial. Thus the deeper part of the proof concerns the so-called *glancing points*, which are boundary points $(i\rho, \eta)$ at which two (or more) real eigenvalues of $\mathcal{B}(\rho, \eta)$ cross each other. This is the only place where we make use of the assumption of strict or constant hyperbolicity. Finally, Steps 19 to 22 only adapt the construction from the strictly to the constantly hyperbolic case.

6

THE CHARACTERISTIC IBVP

6.1 Facts about the characteristic case

In this section, we place ourselves in the characteristic case: A^d is singular. We denote by m the dimension of its kernel. Using a linear transformation, we may assume that

$$A^d = \begin{pmatrix} 0_m & 0 \\ 0 & a^d \end{pmatrix},$$

with $a^d \in \mathbf{GL}_{n-m}(\mathbb{R})$. Since we may not hope for a better result than in the non-characteristic case, we also assume that the operator L is constantly hyperbolic. From Theorem 1.7, the upper-left block in the block decomposition of $A(\eta)$ is of the form $l(\eta)I_m$, where l is a linear form on the space of frequencies[1]. In other words, l is a vector in the physical space, whose last component vanishes. Without loss of generality, the change of variable $(x,t) \mapsto (x - tl, t)$ preserves the physical space $\{x_d > 0\}$ and leads to the situation where

$$A(\eta) = \begin{pmatrix} 0_m & a_{12}(\eta) \\ a_{21}(\eta) & a_2(\eta) \end{pmatrix}.$$

We recall that the boundary matrix B satisfies[2] $\ker A^d \subset \ker B$, which means here that B has the form $B = (0_{p\times m}, B_2)$.

In the following, we often use block decomposition. For instance, a generic vector $u \in \mathbb{C}^n$ will split as $(v,w)^T$, with $v \in \mathbb{C}^m$. Thus the kernel of A^d is given by the equation $w = 0$. From Proposition 4.3, we have $E_-(\tau,\eta) \cap \ker A^d = \{0\}$, hence $E_-(\tau,\eta)$ is isomorphic to its projection $e_-(\tau,\eta)$ on the w-component. The subspace $e_-(\tau,\eta)$ can be defined equivalently as the stable subspace of the Schur complement

$$\mathcal{A}_2(\tau,\eta) := -(a^d)^{-1}\left(\tau I_{n-m} + ia_2(\eta) + \frac{1}{\tau} a_{21}(\eta)a_{12}(\eta)\right).$$

The reciprocal map from $e_-(\tau,\eta)$ to $E_-(\tau,\eta)$ is obviously

$$w \mapsto \left(-\frac{i}{\tau} a_{12}(\eta)w, w\right).$$

The matrix $\mathcal{A}_2$ has been shown to be hyperbolic (Lemma 4.3).

[1] Whenever $m \geq 2$, this is in contradiction with Assumption 1.1 of [127], where the upper-left block was required to have simple eigenvalues.

[2] This property is called *reflexivity* in [150, 151].

As mentioned in Section 5.1, $(\tau,\eta)\mapsto e_-(\tau,\eta)$, and $(\tau,\eta)\mapsto E_-(\tau,\eta)$ also, admit continuous extensions at boundary points, except perhaps at $\tau=0$. It is unclear whether the property $E_-(\tau,\eta)\cap\ker A^d=\{0\}$ extends to such points, though it will be a necessary condition for the uniform Kreiss–Lopatinskiĭ condition, because of $\ker A^d\subset\ker B$.

6.1.1 *A necessary condition for strong well-posedness*

We now present a new restriction that the strong well-posedness imposes to the IBVP. Amazingly enough, this restriction concerns the operator L only, but not the boundary matrix B. It is thus of a very different nature from the uniform Lopatinskiĭ condition. Of course, this condition is trivially satisfied when the boundary is non-characteristic.

We start from the estimate in Definition 4.6. Considering a trivial initial data $u_0\equiv 0$, we may extend u by zero to negative times and then take the Laplace–Fourier transform in (t,y). The estimate implies

$$\iiint_0^{+\infty}\gamma|\hat u(\gamma+i\sigma,\eta,x_d)|^2\mathrm{d}\sigma\mathrm{d}\eta\mathrm{d}x_d+\iint|A^d\hat u(\gamma+i\sigma,\eta,0)|^2\mathrm{d}\sigma\mathrm{d}\eta$$
$$\leq C\left(\iiint_0^{+\infty}\frac{1}{\gamma}|\widehat{Lu}(\gamma+i\sigma,\eta,x_d)|^2\mathrm{d}\sigma\mathrm{d}\eta\mathrm{d}x_d\ +\ \iint|B\hat u(\gamma+i\sigma,\eta,0)|^2\mathrm{d}\sigma\mathrm{d}\eta\right).$$

Since $\widehat{Lu}(\tau,\eta,x_d)=(\tau+iA(\eta))\hat u+A^d\hat u'$ decouples with respect to (τ,η), this estimate is equivalent to

$$(\mathrm{Re}\ \tau)\int_0^{+\infty}|\hat u(\tau,\eta,x_d)|^2\mathrm{d}x_d+|A^d\hat u(\tau,\eta,0)|^2$$
$$\leq\ C\left(\frac{1}{\mathrm{Re}\ \tau}\int_0^{+\infty}|\widehat{Lu}(\tau,\eta,x_d)|^2\mathrm{d}x_d+|B\hat u(\tau,\eta,0)|^2\right),$$

for every pair (τ,η) with $\mathrm{Re}\ \tau>0$. The constant C, being the same as above, does not depend on (τ,η).

We now specialize to the solutions of the differential-algebraic system $(\tau I_n+iA(\eta))\hat u+iA^d\hat u'=0$ that decay at $+\infty$. These solutions take values in $E_-(\tau,\eta)$. Using $B\hat u=B_2w$ with $\hat u=:(v,w)^T$, the above estimate amounts to

$$(\mathrm{Re}\ \tau)\left(\|v\|_{L^2}^2+\|w\|_{L^2}^2\right)+|w(0)|^2\leq C|B_2w|^2. \tag{6.1.1}$$

Since $\tau v+ia_{12}(\eta)w=0$, (6.1.1) reduces to the inequality

$$(\mathrm{Re}\ \tau)\left(|\tau|^{-2}\|a_{12}(\eta)w\|_{L^2}^2+\|w\|_{L^2}^2\right)+|w(0)|^2\leq C|B_2w(0)|^2. \tag{6.1.2}$$

Estimate (6.1.2) splits into three inequalities, among which two are familiar to us. For instance, one of them is the uniform Kreiss–Lopatinskiĭ condition[3]

$$\big(\mathrm{Re}\ \tau > 0,\, \eta \in \mathbb{R}^{d-1},\, w(0) \in e_-(\tau,\eta)\big) \Longrightarrow (|w(0)| \le C|B_2 w(0)|).$$

The new fact is the estimate

$$\|a_{12}(\eta)w\|_{L^2} \le \frac{C|\tau|}{\sqrt{\mathrm{Re}\ \tau}}|B_2 w(0)|.$$

However, since we shall require the (UKL) condition, the only new information is the coarser estimate

$$\|a_{12}(\eta)w\|_{L^2} \le \frac{C|\tau|}{\sqrt{\mathrm{Re}\ \tau}}|w(0)|, \quad \forall w(0) \in e_-(\tau,\eta), \tag{6.1.3}$$

where $w(x_d) := \exp(x_d \mathcal{A}_2(\tau,\eta))w(0)$.

We emphasize that (6.1.3) does not depend on the particular choice of a boundary condition. It is a property of $e_-(\tau,\eta)$ itself, which is necessary in order that *some* boundary condition exists for which the IBVP is strongly well-posed. For this reason, we use the following notion.

Definition 6.1 *Let L be a hyperbolic operator and $\lambda(\xi)$ be an eigenvalue of $A(\xi)$, of constant multiplicity m in a neighbourhood of $\mathbf{e}^d$, with $\lambda(\mathbf{e}^d) = 0$. Without loss of generality, we may assume also that $\mathrm{d}\lambda(\mathbf{e}^d) = 0$; hence A^d and $A(\eta)$ have the structure described above. We then say that L is* stabilizable[4] *if (6.1.3) holds true, with a constant C independent of (η,τ) in the unit hemisphere.*

The previous analysis shows that:

Proposition 6.1 *With the notations above, assume that there exists a boundary matrix β with $\beta u := \beta_2 w$, such that the IBVP associated to (L,β) in the half-plane $\{x_d > 0\}$ is strongly well-posed. Then L is stabilizable.*

We shall discuss two examples, taken from [127]. In the first one, the property does not hold, while in the second, we show that it does for every Friedrichs-symmetrizable system. The latter might be seen as a consequence of Proposition 6.1, since it is rather easy to find a strictly dissipative boundary condition for a symmetric operator, and such a boundary condition automatically satisfies (UKL).

First example We consider the case where $a_{21} \equiv 0$. Then the operator L decouples, since w obeys a differential system $L_2 w = f_2$, which does not involve

[3]Remember that in a characteristic IBVP, we expect a boundary estimate of only $A^d u$, that is of the w component.

[4]One should say *stabilizable in direction* $\mathbf{e}^d$.

the component v at all. Since the boundary condition involves w only, the IBVP itself decouples between a sub-IBVP

$$L_2 w = f_2, \quad w|_{t=0} = w_0, \quad B_2 w|_{x_d=0} = g_2,$$

and an ODE for v, where w enters as a source term:

$$v_t = f_1 - a_{12}(\nabla_y)w.$$

Let us assume that L_2 is constantly hyperbolic and that the sub-IBVP satisfies the (UKL) condition. Then, for L^2 data, the solution w is uniquely defined (see Theorem 4.3) and is L^2. The fact that the source term $a_{12}(\nabla_y)w$ is only of class H^{-1} is not necessarily a cause of trouble; after all, the pure Cauchy problem is well-posed in L^2 since L is constantly hyperbolic[5], though it decouples as well.

In the present case, $e_-(\tau, \eta)$ is precisely the stable subspace at frequency (τ, η) for the operator L_2. From Lemma 4.5, it extends continuously at boundary points, and in particular at $(0, \eta)$, provided η is non-zero. Since $\mathcal{A}_2(\tau, \eta)$ does not display any singularity, we may pass to the limit in (6.1.3) if the full IBVP is well-posed, obtaining $a_{12}(\eta)w = 0$ almost everywhere when $w(0) \in e_-(0, \eta)$. Since w is continuous (as the solution of the differential equation $w' = \mathcal{A}_2(0, \eta)w$), this implies $a_{12}(\eta)w(0) = 0$. In other words, the well-posedness of the full IBVP requires that

$$e_-(0, \eta) \subset \ker a_{12}(\eta), \quad \forall \eta \in \mathbb{R}^{d-1}. \tag{6.1.4}$$

In most cases, (6.1.4) implies that $a_{12} \equiv 0$. For instance, if $n - m = 2$ and L_2 has one positive velocity in each direction, then $e_-(0, \eta)$ is a line, spanned by some eigenvector r of $\mathcal{A}_2(0, \eta)$. Let λ be the corresponding eigenvalue: $(\lambda a^d + i a_2(\eta))r = 0$. This vector cannot be real, since otherwise λ would be purely imaginary, $\lambda = i\mu$, and L_2 would have a zero velocity in the direction (η, μ), contradicting the assumption of constant hyperbolicity for L. Since a_{12} is real valued, (6.1.4) implies that both Re r and Im r belong to its kernel. Hence the kernel has dimension at least two, which means $a_{12} \equiv 0$.

In conclusion, if $a_{21} \equiv 0$ and if L_2 is constantly hyperbolic with non-vanishing velocities, so that L is itself constantly hyperbolic, then most of the choices of the non-zero matrix $a_{12}(\eta)$ violate the stability condition (6.1.3).

We finish this section by giving an example of such an operator L_2 ($n - m = 2$, strict hyperbolicity, with a negative and a positive velocity in each direction):

$$L_2 = \partial_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_1 + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_2.$$

We point out that the above analysis works under the weaker assumption

$$a_{21}(\eta) a_{12}(\eta) \equiv 0.$$

[5]Here, the velocities of L_2 may not vanish, in order that L be constantly hyperbolic.

Friedrichs-symmetrizable operators We have the following result:

Theorem 6.1 *Let L be Friedrichs symmetrizable. Then* (6.1.3) *holds true.*

Proof Multiplying by the Friedrichs symmetrizer, we may assume that L has the form

$$L = S_0 \partial_t + \sum_\alpha S^\alpha \partial_\alpha,$$

with symmetric matrices S^α and S_0, the latter being positive-definite. Let u be a decaying solution of

$$(\tau S_0 + iS(\eta))u + S^d u' = 0.$$

Multiplying on the left by u^*, taking the real part and integrating, we obtain

$$(\text{Re } \tau)\|u\|_{L^2}^2 \leq \frac{1}{2} u^*(0) S^d u(0) = \frac{1}{2} w^*(0) s^d w(0).$$

This contains in particular Inequality (6.1.3), since the v component of u is

$$-i\tau^{-1} a_{12}(\eta) w.$$

□

6.1.2 *The case of a linear eigenvalue*

The purpose of this section is to identify a natural class of operators L for which a symbolic dissipative symmetrizer could be constructed. Our motivations are twofold. First, we wish to admit some of the operators that we encounter frequently in physics. Second, the L^2-well-posedness necessitates a few additional properties, one of them being stabilizability.

Let $\xi \mapsto \lambda(\xi)$ be the eigenvalue of $A(\xi)$, responsible for the characteristicity of the boundary. Thus $\lambda(\mathbf{e}^d) = 0$. Because of the constant hyperbolicity, λ is differentiable for $\xi \neq 0$. Since it is homogeneous of degree one, we deduce $\mathrm{d}\lambda(\mathbf{e}^d)\mathbf{e}^d = 0$. On the other hand, we do not alter the nature of the IBVP by chosing a moving frame that travels at a constant speed, parallel to the boundary. This amounts to changing the variable as $(x, t) \mapsto (x', t)$, with $x' = x - tl$ and $l_d = 0$ (see above). Choosing $l = \mathrm{d}\lambda(\mathbf{e}^d)$, we are led to the case where $\mathrm{d}\lambda(\mathbf{e}^d) = 0$.

There are realistic cases where this reduction yields the property $\lambda \equiv 0$. For instance, λ could have been linear in the initial setting, a fact that happens, for instance, when its multiplicity m is strictly larger than $n/2$ (Corollary 1.1). This is the case in hydrodynamics when the fluid velocity is tangential along the boundary (no mass transfer across the boundary). A second possibility is that rotational invariance makes the spectrum depend only on $|\xi|$. Then $\lambda \equiv 0$ follows directly from $\lambda(\mathbf{e}^d) = 0$. This happens for Maxwell's equations in electromagnetism.

We summarize in the following proposition the properties displayed by the matrices $A(\xi)$ when $\lambda \equiv 0$ is an eigenvalue of constant multiplicity $m \geq 1$. In particular, the eigenvalues of $\mathcal{A}_2$ remain bounded as τ goes to zero. This fundamental property is crucial when estimating the derivatives of the solution of the IBVP with compatible data. It has been shown in [127], through counterexamples, that the failure of this property is responsible for a loss of order in the estimates of derivatives: L^2-norms of derivatives of order $r \geq 1$ require the L^2-norms of derivatives of the boundary data, at some order (typically $2r$) strictly larger than r. See also a discussion at the end of the section.

Proposition 6.2 *Assume that $d \geq 2$ and L is constantly hyperbolic, with $\lambda = 0$ being an eigenvalue of $A(\xi)$, of multiplicity $m \geq 1$ (for $\xi \neq 0$). Without loss of generality, assume that A^d has the form*

$$\begin{pmatrix} 0_m & 0 \\ 0 & a^d \end{pmatrix}.$$

Then:

- *Given p, the number of positive eigenvalues of A^d, the matrix $A(\xi)$ has precisely p positive eigenvalues and p negative eigenvalues for $\xi \neq 0$. In particular, $n - m = 2p$ is even.*
- *One has*

$$A(\eta, \rho) = \begin{pmatrix} 0_m & a_{12}(\eta) \\ a_{21}(\eta) & a_2(\eta) + \rho a^d \end{pmatrix}, \quad \eta \in \mathbb{R}^{d-1}, \rho \in \mathbb{R}.$$

- *It holds that*

$$a_{12}(\eta)(a_2(\eta) + \rho a^d)^{-1} a_{21}(\eta) = 0_m, \tag{6.1.5}$$

  whenever $\xi = (\eta, \rho) \in \mathbb{C}^d$ satisfies $\det(a_2(\eta) + \rho a^d) \neq 0$. In particular, it holds that

$$a_{12}(\eta)(a^d)^{-1} a_{21}(\eta) = 0_m, \quad \eta \in \mathbb{C}^{d-1}. \tag{6.1.6}$$

  More generally, we have

$$a_{12}(\eta)(a^d)^{-1} \left(a_2(\eta)(a^d)^{-1}\right)^k a_{21}(\eta) = 0_m, \quad \eta \in \mathbb{C}^{d-1}, k \in \mathbb{N}. \tag{6.1.7}$$

- *If $\xi = (\eta, \xi_d) \in \mathbb{C}^d$ satisfies $\det(a_2(\eta) + \xi_d a^d) \neq 0$, then $\dim(\ker A(\xi)) = m$.*
- *If $\xi = (\eta, \rho) \in \mathbb{R}^d$ satisfies $\det(a(\eta) + \rho a^d) \neq 0$, then*

$$(-\infty, 0) \cap Sp\left(a_{12}(\eta)(a_2(\eta) + \rho a^d)^{-2} a_{21}(\eta)\right) = \emptyset. \tag{6.1.8}$$

- *The eigenvalues of $\mathcal{A}_2(\tau, \eta)$ admit finite limits as $\tau \to 0$ with $\eta \neq 0$. The real part of these limits does not vanish. These limits are the roots μ of*

$$\det(a_2(\eta) + i\mu a^d) \det(I_m + a_{12}(\eta)(a_2(\eta) + i\mu a^d)^{-2} a_{21}(\eta)) = 0,$$

  which is a polynomial equation.

Proof Since $\lambda = 0$ has a constant multiplicity and $A(\xi)$ has only real eigenvalues, and since the unit sphere of $\mathbb{R}^d$ is connected, the number of positive eigenvalues of $A(\xi)$ is constant, thus equal to p by taking $\xi = \mathbf{e}^d$. Since the spectrum of $A(\xi)$ is the opposite of that of $A(-\xi)$, the number of negative eigenvalues is p also. The total number of eigenvalues is n on the one hand, $m+2p$ on the other hand.

The second point is a direct consequence of Theorem 1.7, since $\pi_\xi A(\eta)\pi_\xi$ is the upper-left block of $A(\eta)$ when $\xi = \mathbf{e}^d$.

Assume $\xi = (\eta,\rho) \in \mathbb{R}^d$, with $\det(a_2(\eta)+\rho a^d) \neq 0$. By assumption, $\ker A(\eta,\rho)$ has dimension m. One easily finds

$$\ker A(\eta,\rho) = \{(v, -(a_2(\eta)+\rho a^d)^{-1}a_{21}(\eta)v) \,|\, v \in \ker a_{12}(\eta)(a_2(\eta) + \rho a^d)^{-1}a_{21}(\eta)\}.$$

This implies $\dim \ker a_{12}(\eta)(a_2(\eta)+\rho a^d)^{-1}a_{21}(\eta) \geq m$. Since this kernel is a subspace of $\mathbb{R}^m$, it must therefore equal $\mathbb{R}^m$. Hence the matrix vanishes. In particular, we find

$$\ker A(\eta,\rho) = \{(v, -(a_2(\eta)+\rho a^d)^{-1}a_{21}(\eta)v) \,|\, v \in \mathbb{R}^m\}. \tag{6.1.9}$$

Analyticity ensures that (6.1.5) holds true even for complex values of (η,ρ). Hence, formula (6.1.9) remains valid (replacing $\mathbb{R}^m$ by $\mathbb{C}^m$). This shows that $\dim\ker A(\xi) = m$ whenever $\det(a_2(\eta)+\xi_d a^d) \neq 0$. Expanding (6.1.5) in terms of $1/\rho$ as ρ tends to infinity, we obtain (6.1.6) and (6.1.7).

When $\xi \neq 0$ is real, diagonalizability tells us that $\ker A(\xi)^2 = \ker A(\xi)$. This amounts to

$$-1 \notin \mathrm{Sp}\left(a_{12}(\eta)(a_2(\eta)+\rho a^d)^{-2}a_{21}(\eta)\right).$$

By homogeneity, this is equivalent to (6.1.8) (notice that (6.1.8) is obvious if L is symmetric).

By assumption, the characteristic polynomial $P(X;\xi) := \det(XI_n - A(\xi))$ factorizes as $X^m Q(X;\xi)$, where Q is itself a polynomial in X. Since X^m is unitary, the quotient Q is itself polynomial in ξ. As a polynomial in (X,ξ), Q is homogeneous of degree m. Since the multiplicity of the null root of $P(\cdot;\xi)$ is exactly m when $\xi \neq 0$ is real, we have

$$Q(0;\xi) \neq 0, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \tag{6.1.10}$$

Since P contains the monomial $X^m \xi_d^{n-m} \det a^d$ and since $\det a^d \neq 0$, the degree of Q in ξ_d is exactly $n-m$. Using Schur's Formula, we have

$$Q(X;\xi) = \det(XI_{n-m} - a_2(\eta) - \xi_d a^d - X^{-1}a_{21}(\eta)a_{12}(\eta)).$$

Substituting $X = i\tau$ and $\xi_d = i\mu$, we obtain

$$Q(i\tau;\eta,i\mu) = (\det(-ia^d))\det(\mathcal{A}_2(\eta,\tau) - \mu I_{n-m}).$$

Hence the eigenvalues of $\mathcal{A}_2(\eta,\tau)$ are the roots of $Q(i\tau;\eta,i\cdot)$. Since the degree of Q with respect to its last argument equals its total degree, these roots are continuous functions of (η,τ) everywhere, and especially at the origin.

When $\tau \to 0$, these roots tend to those of $Q(0;\eta, i\cdot)$. Because of (6.1.10), these limits have a non-zero real part (notice that $\eta \neq 0$ here.) To obtain an expression of $Q(0,\eta,i\mu)$, we use again Schur's formula, obtaining

$$P(X;\xi) = \det(XI_{n-m} - a_2(\eta) - \xi_d a^d)\, \det(XI_m - a_{12}(\eta) \times (XI_{n-m} - a_2(\eta) - \xi_d a^d))^{-1} a_{21}(\eta)).$$

Expanding

$$(XI_{n-m} - a_2(\eta) - \xi_d a^d)^{-1} = -\sum_{k\geq 0} X^k (a_2(\eta) + \xi_d a^d)^{-k-1},$$

and using (6.1.5), we find

$$\det(XI_m - a_{12}(XI_{n-m} - a_2 - \xi_d a^d))^{-1} a_{21}) = X^m \det(I_m + a_{12}(a_2 + \xi_d a^d)^{-2} a_{21}) + O(X^{m+1}).$$

Hence

$$Q(X;\xi) = \det(XI_{n-m} - a_2(\eta) - \xi_d a^d)\, \det(I_m + a_{12}(\eta)(a_2(\eta) + \xi_d a^d)^{-2} a_{21}(\eta)) + O(X).$$

Setting $X = 0$ and $\xi_d = i\mu$, we find the limit equation. □

Comments

i) It happens frequently that $a_{21}(\eta)a_{12}(\eta) \neq 0_{n-m}$ (we have seen that this is a necessary condition for stabilizability.) For instance, if L is Friedrichs symmetric, it happens whenever $a_{12}(\eta) \neq 0$, a natural fact when the IBVP does not decouple between a trivial ODE $v_t = f_1$ and an IBVP of smaller size. If $a_{21}(\eta)a_{12}(\eta) \neq 0_{n-m}$, then the matrix $\mathcal{A}_2(\eta,\tau)$ is unbounded as $\tau \to 0$, though its eigenvalues have finite limits. Hence it does not remain uniformly diagonalizable: Eigenvectors associated to distinct eigenvalues tend to become parallel. However, eigenvalues do not merge in general (a counterintuitive fact, but $\mathcal{A}_2$ does not have a limit as $\tau \to 0$); for instance, if $m = n-2$, then $p = 1$ and $Q(\tau, i\eta, \cdot)$ has exactly one root of positive real part and one of negative real part, this dichotomy persists as $\tau \to 0$, since the roots of $Q(0; i\eta, \cdot)$ may not belong to $i\mathbb{R}$. Hence the roots remain distinct.

ii) In the Friedrichs-symmetric case, (6.1.6) tells us exactly that the range $R(a_{21}(\eta))$ is an isotropic subspace for the quadratic form q_d defined on $\mathbb{R}^{n-m} = \mathbb{R}^{2p}$ by $(a^d)^{-1}$. Since q_d is non-degenerate with p positive and p negative eigenvalues, its maximal isotropic subspaces have dimension p. Hence $\operatorname{rk} a_{21}(\eta) \leq p$. Let us examine two examples. The first one is Maxwell's system of electromagnetism, for which $(d,n,m,p) = (3,6,2,2)$. Easy calculations show that $\operatorname{rk} a_{21}(\eta) = 2 = p$ for every $\eta \neq 0$. Thus our inequality is sharp. Its optimality is reinforced by the following observation.

Since $p = (n - m)/2 = 2$, the set $I_2(q_d)$ of q_d-isotropic planes is a manifold of dimension one[6]. This dimension fits with that of the projective space on which the map $\eta \mapsto R(a_{21}(\eta))$ is defined. One checks easily that the range of this map is precisely a connected component of $I_2(q_d)$.

The second example is the linearized isentropic gas dynamics (or acoustics). With the normalization $\lambda \equiv 0$, the ground velocity is zero. In suitable units, the operator reads

$$L \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} \rho_t + \operatorname{div} u \\ u_t + \nabla \rho \end{pmatrix}.$$

Here, $(n, m, p) = (d + 1, d - 1, 1)$. Again, the equality $\operatorname{rk} a_{21}(\eta) = 1 = p$ holds. However, the set of isotropic lines is discrete (it consists in two elements). Thus the map $\eta \mapsto R(a_{21}(\eta))$ is constant, although it is defined on a projective space of dimension $d - 2$. This constancy could be checked by direct calculation.

Why should $\lambda \equiv 0$ hold?

Our first motivation is an observation made by Majda and Osher in [127]. As mentioned in Proposition 6.2, the eigenvalues of $\mathcal{A}_2(\tau, \eta)$ have finite limits as τ tends to zero, the singularity of $\mathcal{A}_2$. It turns out that when the eigenvalue, responsible for the characteristic nature of the boundary, is not a linear function of the frequency, then $\mathcal{A}_2(\tau, \eta)$ admit eigenvalues that tend to infinity as $\tau \to 0$. Majda and Osher showed that such a behaviour, although compatible with strong L^2-estimates, is responsible for a loss in the estimates of the derivatives. Typically, boundary data of class H^2 are needed in order that the trace of the solution be of class H^1.

The following analysis, due to Ohkubo [150], is related to the former comment and gives a partial explanation for the choice of a constant (or a linear) eigenvalue. Let us assume that the IBVP is well-posed in L^2, in the sense of Definition 4.6. We wish to show that this well-posedness extends to the class H^1. Thus we give ourselves data $f = Lu$, $g = \gamma_0 Bw$ and $a \equiv 0$ that belong to H^1 in their respective domains. By assumption, we know that there exists a unique solution $u = (v, w)$ that is at least square-integrable on $(0, T) \times \Omega$, and such that w has a square-integrable trace along the boundary. Thus we seek for the integrability of the first-order derivatives. First of all, we differentiate the equation $Lu = f$ in directions parallel to the boundary (tangential derivatives). We see that $\partial_\alpha u$ solves the IBVP

$$LX_\alpha = \partial_\alpha f, \quad \gamma_0 BX_\alpha = \partial_\alpha g, \quad X_\alpha(0) = 0.$$

Since the data are square-integrable, we obtain that $\partial_\alpha u \in L^2$ and that $\partial_\alpha w$ has a well-defined trace that is square-integrable too. The same procedure may be

[6] It may be identified to the set of straight lines contained in a one-sheeted hyperboloid. In particular, it has two connected components.

applied to the time derivative X_0:

$$LX_0 = \partial_t f, \quad \gamma_0 B X_0 = \partial_0 g, \quad X_0(0) = f(0).$$

Hence $\partial_t u$ and $\partial_t w$ have the required properties.

We now turn towards the normal derivative $\partial_d u$. Using the second part of the system, we have

$$\partial_d w = (a^d)^{-1} \left(f_2 - \partial_t w - a_2(\nabla_y) w - a_{21}(\nabla_y) v \right). \tag{6.1.11}$$

Equation (6.1.11) immediately tells us that $\partial_d w$ is square-integrable and admits a square-integrable trace[7]. However, it remains to study $\partial_d v$. To handle this term, we differentiate the first part of the system and we use (6.1.11). We obtain an ODE

$$\partial_t \left(\partial_d v - a_{12}(\nabla_y)(a^d)^{-1} w \right) = \partial_d f_1 - a_{12}(\nabla_y)(a^d)^{-1} f_2 + a_{12}(\nabla_y)(a^d)^{-1} a_2(\nabla_y) w$$
$$+ a_{12}(\nabla_y)(a^d)^{-1} a_{21}(\nabla_y) v.$$

Assuming Property (6.1.6) and also

$$a_{12}(\eta)(a^d)^{-1} a_2(\eta) \equiv 0, \tag{6.1.12}$$

which together imply (6.1.5), the right-hand side reduces to $\partial_d f_1 - a_{12}(\nabla_y)(a^d)^{-1} f_2$, a square-integrable data. Thus $\partial_d v - a_{12}(\nabla_y)(a^d)^{-1} w$, hence $\partial_d v$, is square-integrable (notice that we do not need a boundary condition for $\partial_d v$.)[8]

Ohkubo's conclusion is that a sufficient condition for the IBVP be well-posed in H^1 is that both (6.1.6) and (6.1.12) hold true. Though it is not clear that these conditions are necessary, it looks reasonable to restrict ourselves to the class of such operators. This is the choice that we shall adopt in Section 6.2. Using a series expansion, we see that these conditions imply (6.1.5), which means that $\ker A(\xi)$ has dimension m whenever $\det(a_2(\eta) + \rho a^d) \neq 0$. By a continuity argument, we see that zero is an eigenvalue of $A(\xi)$ of multiplicity larger than or equal to m, for every direction ξ, whence $\lambda \equiv 0$.

6.1.3 *Facts in two space dimensions*

From now on, we assume that our operator L is symmetric, with a zero eigenvalue of constant multiplicity $m \geq 1$. We denote by $p = (n - m)/2$ the number of positive (negative) eigenspeeds in every direction $\xi \neq 0$.

We shall need to consider the isotropic cone of $A(\xi)$:

$$\Gamma(\xi) := \{u \in \mathbb{R}^n \,;\, u^T A(\xi) u = 0\}.$$

[7]Notice that the analysis would be complete at this stage if the boundary was not characteristic. This explains why only one condition (UKL) governs the well-posedness of non-characteristic IVBPs in every Sobolev space, while we need an extra condition in the characteristic case, to pass from the well-posedness in L^2 to that in H^1.

[8]The way we used (6.1.11) does not seem optimal. We expect that a cleverer analysis could yield the weaker condition (6.1.5), instead of (6.1.6) and (6.1.12).

The intersection of these cones as ξ runs through $\mathbb{R}^d$ is denoted Γ. Obviously, $\ker A(\xi)$ is a subset of $\Gamma(\xi)$. Actually, we proved in Theorem 1.7 that $\ker A(\xi') \subset \Gamma(\xi)$ for every ξ, ξ' with $\xi' \neq 0$. This implies not only

$$\ker A(\xi') \subset \Gamma, \tag{6.1.13}$$

but also

$$\ker A(\xi) + \ker A(\xi') \subset \Gamma(\xi) \cap \Gamma(\xi'), \quad \forall \xi \neq 0, \, \xi' \neq 0. \tag{6.1.14}$$

We now assume that $d = 2$ and denote by H the sum of the kernels $\ker A(\xi)$ as ξ runs the unit circle. Then $\Gamma = \Gamma(\xi) \cap \Gamma(\xi')$ whenever $\xi \wedge \xi' \neq 0$. Hence (6.1.13), (6.1.14) and the fact that each $\Gamma(\xi)$ is a quadratic cone give us:

Proposition 6.3 *Assume that $d = 2$, the operator L is symmetric and the zero eigenvalue has constant multiplicity $m \geq 1$. Then it holds that*

$$\sum_{\xi \neq 0} \ker A(\xi) =: H \, \subset \, \Gamma := \bigcap_{\xi} \Gamma(\xi). \tag{6.1.15}$$

Comments

- It is worth pointing out that the sum (of vector spaces of dimension m) H can be a rather big subspace. However, Propositions 6.2 and 6.3 tell us that, as any isotropic subspace $\Gamma(\xi)$, its dimension is less than or equal to $m + p$. What is even more striking is that in the examples of physical interest[9], the dimension of this sum is actually *equal* to $m + p$. Hence all the quadratic forms $A(\xi)$ have a maximal isotropic subspace in common! As we shall see in the following, this property plays a crucial role, so that we shall take it as an assumption for the planar restrictions of L.
- Property (6.1.15) would be false in higher space dimension, for there is no reason why $u^T A(\xi'') u'$ would vanish for $u \in \ker A(\xi)$ and $u' \in \ker A(\xi')$ when ξ, ξ', ξ'' are linearly independent.

In the next result, we assume that H is of maximal dimension (namely $m + p$) and we use the canonical form for the symbol $A(\xi)$:

$$A(\xi) = \begin{pmatrix} 0_m & a_{12}(\xi) \\ a_{21}(\xi) & a_2(\xi) \end{pmatrix} \in \mathbf{Sym}_n, \quad a^d := a_2(\mathbf{e}^d).$$

Since H contains $\ker A^d = \mathbb{R}^m \times \{0\}$, it splits as $\mathbb{R}^m \times H_1$ where $\dim H_1 = p$.

Lemma 6.1 *Assume that $d = 2$ and $\dim H = m + p$ (maximal dimension). Then H_1 is its own a^d-orthogonal space. If $\xi \neq 0$, then H is its own $A(\xi)$-orthogonal subspace.*

Proof Since a^d is non-degenerate, $\dim H_1^{\perp} = 2p - \dim H_1 = p$. Since $H_1^{\perp}$ contains H_1, it thus equals H_1. Similarly, H is self-orthogonal with respect

[9] For example, acoustics, linearized gas dynamics, Maxwell's equations, linear elasticity.

to $A(\xi)$. Since it contains $\ker A(\xi)$, its $A(\xi)$-orthogonal must be of dimension $m+n-\dim H = \dim H$, whence the result. □

Corollary 6.1 *Assume that $d=2$, the operator L is Friedrichs symmetric and that the subspace $H := \sum_{\xi\neq 0}\ker A(\xi)$ is of maximal dimension, namely $m+p$. Then it is possible to choose the u-co-ordinates in such a way that $A(\xi)$ reads*

$$A(\xi) = \begin{pmatrix} 0_{m+p} & \beta(\xi) \\ \beta(\xi)^T & \delta(\xi) \end{pmatrix}, \tag{6.1.16}$$

where $\delta(\xi)\in \mathbf{Sym}_p$, while $\beta(\xi)\in \mathbf{M}_{(m+p)\times p}$ is injective *for $\xi\neq 0$.*

Conversely, if $A(\xi)$ reads as above and $\beta(\xi)$ is injective for $\xi\neq 0$, then the operator $L=\partial_t + A(\nabla_x)$ is Friedrichs symmetric, and zero is an eigenspeed of multiplicity m in every direction of the plane.

Proof Assume that H is of maximal dimension $m+p$. One chooses orthogonal co-ordinates in which $H=\mathbb{R}^{m+p}\times\{0\}$. Since the co-ordinates are orthogonal, the symbol remains symmetric. Because H is isotropic for every $A(\xi)$, the symbol has the form described in (6.1.16), with $\delta(\xi)$ symmetric. Let q belong to $\ker\beta(\xi)$ and $\xi\neq 0$. Then

$$A(\xi)\begin{pmatrix} 0 \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ \delta(\xi)q \end{pmatrix} \in H^\perp.$$

From Lemma 6.1, we deduce that $(0,q)^T\in H$, which means $q=0$. Hence $\beta(\xi)$ is injective.

Conversely, assuming that $\beta(\xi)$ is injective, we see that $\ker A(\xi) = \ker\beta(\xi)^T \times \{0\}$. Since the rank of β^T equals that of β, namely p, we find that $\ker\beta(\xi)^T$ is of dimension $m+p-p=m$. □

In the following, we shall consider symmetric operators in any space dimension $d\geq 2$. Given a vector $\eta\in\mathbb{R}^{d-1}=\mathbb{R}^{d-1}\times\{0\}$, we shall denote $H(\eta)$ the sum of the kernels $\ker A(\xi)$ as $\xi\neq 0$ runs over the plane spanned by η and $\mathbf{e}^d$. From Proposition 6.3, $H(\eta)$ is isotropic for both A^d and $A(\eta)$. Since $H(\eta)$ contains $\ker A^d = \mathbb{R}^m\times\{0\}$, it splits as $\mathbb{R}^m\times H_1(\eta)$, where $H_1(\eta)$ is isotropic for both a^d and $a_2(\eta)$. As mentioned above, we shall assume that $H(\eta)$ is of maximal dimension, namely $m+p$, meaning that $H_1(\eta)$ has dimension p. In particular, we have the following identity:

$$(a^d)^{-1}H_1(\eta) = H_1(\eta)^\perp. \tag{6.1.17}$$

6.1.4 *The space $E_-(0,\eta)$*

We now characterize the limit $E_-(0,\eta)$ of the stable subpace $E_-(\tau,\eta)$ when $\tau\to 0$ with $\mathrm{Re}\,\tau>0$. Recall that the latter is associated to the differential-algebraic system

$$(\tau I_n + iA(\eta))u + A^d u' = 0.$$

Following the analysis of the previous section, we assume that L is Friedrichs symmetric, that zero is an eigenvalue of multiplicity $m \geq 1$, that $d \geq 2$ and that $\dim H_1(\eta) = p$ for every $\eta \in \mathbb{R}^{d-1} \setminus \{0\}$. As above, we use the block decomposition $u = (v, w)^T$ with $v \in \mathbb{C}^m$, $w \in \mathbb{C}^{n-m} = \mathbb{C}^{2p}$ and

$$A(\eta + \xi_d \mathbf{e}^d) = \begin{pmatrix} 0_m & a_{12}(\eta) \\ a_{21}(\eta) & a_2(\eta) + \xi_d a^d \end{pmatrix}.$$

If $w \in H_1(\eta)$ and $v \in \mathbb{R}^m$, then $(v, w)^T \in H(\eta)$, hence $(v, w)^T \in \Gamma(\eta)$, meaning that $2v^T a_{12}(\eta) w + w^T a_2(\eta) w = 0$. Since this is valid for every v, we conclude that $a_{12}(\eta) w = 0$, whence

$$H_1(\eta) \subset \ker a_{12}(\eta). \tag{6.1.18}$$

Introducing $H_2(\eta) := H_1(\eta)^\perp$ (with respect to the standard Euclidian product), we derive that

$$R(a_{21}(\eta)) \subset H_2(\eta). \tag{6.1.19}$$

The differential system rewrites

$$\tau v + i a_{12}(\eta) w = 0, \quad i a_{21}(\eta) v + (\tau + i a_2(\eta)) w + a^d w' = 0,$$

from which we may eliminate v:

$$a^d w' + (\tau + i a_2(\eta)) w + \frac{1}{\tau} a_{21}(\eta) a_{12}(\eta) w = 0. \tag{6.1.20}$$

We shall denote $e_-(\tau, \eta)$ the w-projection of $E_-(\tau, \eta)$, that is the stable subspace of the system (6.1.20).

In the decomposition $\mathbb{C}^{2p} = H_1(\eta) \oplus^\perp H_2(\eta)$, we denote by $\pi = \pi(\eta)$ the orthogonal projection onto $H_1(\eta)$. With $q_1 = \pi w$ and $q_2 = \tau^{-1}(1 - \pi) w \in H_2(\eta)$, we have $w = q_1 + \tau q_2$ and the system reads

$$a^d q_1' + i a_2(\eta) q_1 + a_{21}(\eta) a_{12}(\eta) q_2 + \tau \{a^d q_2' + q_1 + i a_2(\eta) q_2 + \tau q_2\} = 0. \tag{6.1.21}$$

We write an equivalent system by projecting onto $H_1(\eta)$ and $H_2(\eta)$ separately. The important point is that the first three terms in (6.1.21) belong to $H_2(\eta)$ (use the fact that $H_1(\eta)$ is isotropic simultaneously for a^d and for $a_2(\eta)$, and use (6.1.19)). Hence the system becomes

$$\begin{aligned} a^d q_1' + i a_2(\eta) q_1 + a_{21}(\eta) a_{12}(\eta) q_2 &= -\tau (1 - \pi) \{a^d q_2' + i a_2(\eta) q_2 + \tau q_2\}, \\ \pi a^d q_2' + q_1 + i \pi a_2(\eta) q_2 &= 0. \end{aligned}$$

This can be written in a compact form as

$$\frac{\mathrm{d}}{\mathrm{d}x} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = Z(\tau, \eta) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}.$$

Since we proceeded by changes of variables, $Z(\tau, \eta)$ is conjugated to $\mathcal{A}_2(\tau, \eta)$, hence it is of hyperbolic type. Its stable subspace is of dimension p. What is even more important is that $Z(\tau,\eta)$ has a limit $Z(0,\eta)$ that is hyperbolic too. As a matter of fact, the formal limit of the above system, as $\tau \to 0$, writes as

$$a^d q_1' + ia_2(\eta)q_1 + a_{21}(\eta)a_{12}(\eta)q_2 = 0, \tag{6.1.22}$$

$$\pi a^d q_2' + q_1 + i\pi a_2(\eta)q_2 = 0. \tag{6.1.23}$$

Since a^d is invertible, the first line determines q_1' in terms of q_1, q_2. On the other hand, if $\pi a^d q = 0$, then $a^d q \in H_2(\eta)$. From (6.1.17), we conclude that $q \in H_1(\eta)$. Hence the second line determines q_2', showing that the system can be written in the form $q' = Z(0,\eta)q$ and that $Z(0,\eta)$ is the limit of $Z(\tau,\eta)$.

Proposition 6.4 *The matrix $Z(0,\eta)$ is hyperbolic: It does not have pure imaginary eigenvalues.*

Proof Let q satisfy $Z(0,\eta)q = i\alpha q$ with $\alpha \in \mathbb{R}$. Denoting by $ir_1 + r_2$ the decomposition of q, we have

$$-(\alpha a^d + a_2(\eta))r_1 + a_{21}(\eta)a_{12}(\eta)r_2 = 0,$$

$$\pi(\alpha a^d + a_2(\eta))r_2 + r_1 = 0.$$

One may assume that r_1 and r_2 are real. Multiplying the first line by r_2^T, the second one by r_1^T and summing, we obtain the identity

$$|r_1|^2 + |a_{12}(\eta)r_2|^2 = 0.$$

It follows that $q \in H_2(\eta) \cap \ker a_{12}(\eta)$. Then the equations reduce to $(\alpha a^d + a_2(\eta))q \in H_2(\eta)$. All this means

$$A(\xi)\begin{pmatrix} 0 \\ q \end{pmatrix} \in \{0\} \times H_2(\eta) = H(\eta)^\perp, \quad \xi := \eta + \alpha \mathbf{e}^d.$$

Using Lemma 6.1, we conclude that $(0,q)^T \in H(\eta)$, namely $q \in H_1(\eta)$. Since $q \in H_2(\eta)$, this gives $q = 0$. □

Lemma 6.2 *There exists a linear map $N(0,\eta)$ such that the equation of the stable subspace of $Z(0,\eta)$ is*

$$q_1 = N(0,\eta)q_2.$$

Moreover, one has $\ker N(0,\eta) \subset \ker a_{12}(\eta)$.

Proof Since the dimension of the stable subspace is p, we have to prove that its intersection with $H_1(\eta)$ is trivial. Thus, let $q(x)$ be a solution of the limit differential system, vanishing at $+\infty$. Multiply (6.1.22) by q_2^* and (6.1.23) by q_1^*, sum both equations, then take the real part. One obtains

$$|q_1|^2 + |a_{12}(\eta)q_2|^2 + \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Re}\,(q_2^* a^d q_1) = 0.$$

Integrate now from 0 to $+\infty$, we obtain

$$\int_0^{+\infty}\left(|q_1|^2+|a_{12}(\eta)q_2|^2\right)\,\mathrm{d}x=\mathrm{Re}\,(q_2(0)^*a^dq_1(0)).$$

If $q_2(0)$ vanishes, we deduce that $q_1\equiv 0$, hence $q_1(0)=0$. This proves that the stable subspace is parametrized by q_2.

Assume now that $N(0,\eta)r=0$ with $r\in H_2(\eta)$. The solution of the differential system with initial data $q_1(0)=0=N(0,\eta)r$, $q_2(0)=r$, vanishes at $+\infty$. Hence $q_1\equiv 0$ and $a_{12}(\eta)q_2\equiv 0$. Whence $a_{12}(\eta)r=0$. □

A standard perturbation argument tells us that the stable subspace of $Z(0,\eta)$ is the limit of that of $Z(\tau,\eta)$. Thus there exists a linear map $N(\tau,\eta)$ that depends smoothly on (τ,η) when τ is small, such that the latter has an equation $q_1=N(\tau,\eta)q_2$. In other words, $E_-(\tau,\eta)$ is given by

$$E_-(\tau,\eta)=\{(-ia_{12}(\eta)r,N(\tau,\eta)r+\tau r)\,;\,r\in H_2(\eta)\}.$$

Passing to the limit as $\tau\to 0$, we obtain

$$\{(-ia_{12}(\eta)r,N(0,\eta)r)\,;\,r\in H_2(\eta)\}\subset E_-(0,\eta). \tag{6.1.24}$$

When the right-hand side in (6.1.24) has dimension p, the embedding is an equality that gives a complete description of $E_-(0,\eta)$. However, one does not see any good reason why this would hold true for every $\eta\in\mathbb{R}^{d-1}\setminus\{0\}$.

From Lemma 6.2, equality occurs in (6.1.24) unless $N(0,\eta)$ is singular. We examine the latter property. If $r\in\ker N(0,\eta)$, then there exists (see the proof of the lemma) a non-trivial $q(x)$ that takes values in $H_2(\eta)\cap\ker a_{12}(\eta)$, and such that (see (6.1.23))

$$\pi(\xi_d a^d q'+ia_{12}(\eta)q)=0.$$

Let $x^k\exp(\sigma x)r_0$ be a leading term in q as $x\to+\infty$, then $\mathrm{Re}\,\sigma<0$ and $\pi(\sigma a^d+ia_{12}(\eta))r_0=0$. In other words, the vector $U:=(0,r_0)^T$ satisfies

$$U\in H(\eta)^\perp,\quad A(\eta+\xi_d\mathbf{e}^d)U\in H(\eta)^\perp, \tag{6.1.25}$$

for a $\xi_d=-i\sigma$ that is non-real. Conversely, let $U\neq 0$ satisfy (6.1.25) with a non-real ξ_d. We have $U=(0,r)^T$ with $r\in H_2(\eta)$. Up to a complex conjugacy, we may assume that the imaginary part of ξ_d is positive. Then $q(x):=\exp(i\xi_d)r$ is a decaying solution of our differential system that satisfies $q_1(0)=0$ and $q_2(0)=r\neq 0$. Hence $N(0,\eta)$ is singular. We summarize our results in the following statement.

Proposition 6.5 *The linear map $N(0,\eta)$ is singular if and only if there exists a non-zero vector $U\in\mathbb{C}^n$ that solves the generalized eigenvalue problem* (6.1.25).

When $N(0,\eta)$ is regular, then

$$E_-(0,\eta)=\{(-ia_{12}(\eta)M(0,\eta)z,z)\,;\,z\in H_1(\eta)\}, \tag{6.1.26}$$

where $M(0,\eta) := N(0,\eta)^{-1}$*. In particular, the* w*-projection of* $E_-(0,\eta)$*, as well as the limit of* $e_-(\tau,\eta)$ *as* $\tau \to 0$*, equal* $H_1(\eta)$*.*

Note that the eigenvalue problem (6.1.25) could not have a non-trivial solution when ξ_d is real, as shown in Proposition 6.4.

Examples of physical interest

In practice, we are interested in examples that come from physics, where conservation laws are likely to hold. We have in mind fluid dynamics, electromagnetism and elasticity. The systems are known as acoustics, linearized gas dynamics, Maxwell's equations and linear isotropic elasticity. All of them have a zero eigenspeed of constant multiplicity; in the case of linearized gas dynamics, one must assume a solid boundary, and choose a frame moving with the boundary velocity.

Except for the elasticity, one has the remarkable feature

$$a_2(\eta) \equiv 0_{2p}. \tag{6.1.27}$$

We emphasize that (6.1.27) prevents the eigenvalue problem (6.1.25) from having a non-trivial solution, since $H(\eta)$ is its own A^d-orthogonal. Hence the w-projection of $E_-(0,\eta)$ equals $H_1(\eta)$ in every direction.

Let us examine linear elastodynamics. There is no loss of generality in assuming that $d = 2$, since all the above analysis deals within planes spanned by $\mathbf{e}^d$ and a tangent vector η. The system reads as

$$\partial_t F + \nabla z = 0, \quad \partial_t z + \mathrm{div} T = 0,$$

with $F(x,t) \in \mathbf{M}_2(\mathbb{R})$, $z(x,t) \in \mathbb{R}^2$ and

$$T = \lambda(F + F^T) + \mu(\mathrm{Tr}F)I_2.$$

The vector field z represents the opposite of the material velocity, while the stress tensor T is an isotropic function of the infinitesimal deformation tensor. Actually, the skew-symmetric part $F_{12} - F_{21}$ decouples from the rest and we may restrict ourselves to the system that governs the evolution of z and the symmetric part of F. Since the system admits a quadratic energy

$$\frac{1}{2}|z|^2 + \frac{\lambda}{4}|F + F^T|^2 + \frac{\mu}{2}(\mathrm{Tr}F)^2,$$

it is Friedrichs symmetrizable. Given $c_P := \sqrt{2\lambda + \mu}$ (the velocity of 'pressure waves'), the choice of variables

$$u := (2\sqrt{\lambda(\lambda+\mu)}\, F_{11},\ c_P\sqrt{\lambda}\,(F_{12} + F_{21}),\ c_P^2 F_{22} + \mu F_{11},\ c_P z_1,\ c_P z_2)^T$$

puts the system in the symmetric form $Lu = 0$, where $A(\xi)$ has the canonical form

$$A(\xi) = \begin{pmatrix} 0_m & a_{12}(\eta) \\ a_{21}(\eta) & a_2(\eta) + \xi_d a^d \end{pmatrix}, \quad \xi = (\eta, \xi_d).$$

We have

$$a_2(\eta) + \xi_d a^d = \begin{pmatrix} 0_2 & b(\xi) \\ b(\xi)^T & 0_2 \end{pmatrix}, \quad b(\xi) = \begin{pmatrix} \xi_d\sqrt{\lambda} & \eta\sqrt{\lambda} \\ \eta\mu/c_P & \xi_d c_P \end{pmatrix} \tag{6.1.28}$$

and $a_{12}(\eta) = (0, 0, 2\eta\sqrt{\lambda(\lambda+\mu)}/c_P, 0)^T$.

In these variables, one checks that $H(\mathbf{e}^1) = \mathbb{R}^3 \times \{0\}$. Since $n = 5$, $m = 1$ and $p = 2$, $H(\mathbf{e}^1)$ is of dimension $m + p$ and the previous analysis applies. Let us consider now the eigenvalue problem (6.1.25). If $(0, r)^T$ is a solution, then r has the form $(0, q)^T$ with $q \in \mathbb{C}^2$ and $q \in \ker b(\xi) \cap \ker a_{12}(\eta)$. Here, $\xi = (\eta, \xi_d)$ with a non-real ξ_d. Since

$$\det b(\xi) = (\xi_d^2 c_P - \mu^2/c_P)\sqrt{\lambda}$$

cannot vanish (otherwise ξ_d would be real), we must have $q = 0$. Thus, there is no non-trivial solution. We conclude that $N(0, \mathbf{e}^1)$ is non-singular.

More generally, in any space dimension and for every tangent vector $\eta \neq 0$, the space $H(\eta)$ is of dimension $m + p$ and the limit $E_-(0, \eta)$ is parametrized as in (6.1.26).

6.1.5 *Conclusion*

Let us summarize the results obtained in this section. It is difficult to characterize hyperbolic operators L admitting an eigenvalue λ of constant multiplicity $m \geq 1$, such that $\lambda(\mathbf{e}^d) = 0$. Hence, we have simplified the class of admissible operators by using as natural as possible arguments.

First, we explained why a linear λ is likely to occur. Without loss of generality, this allows us to restrict ourselves to $\lambda \equiv 0$. A nice consequence of this assumption is the boundedness of the spectrum of $\mathcal{A}_2(\tau, \eta)$ as $\tau \to 0$, a property needed for an acceptable estimate of derivatives of the solution of the IBVP. Another one is that $n - m =: 2p$ is even whenever $d \geq 2$.

Next, the concept of stabilization led us to a slightly more restricted class. We showed that symmetric operators are stabilizable, but that far-from-symmetric ones (say $a_{21}a_{12} \equiv 0_{n-m}$) are not. Since the examples encountered in the natural sciences are endowed with a quadratic 'entropy' (being often an energy) and thus are Friedrichs symmetrizable, we restrict ourselves to symmetric symbols $A(\xi)$. Without loss of generality, one may write

$$A(\xi) = \begin{pmatrix} 0_m & a_{12}(\eta) \\ a_{12}(\eta)^T & a_2(\eta) + \xi_d a^d \end{pmatrix}, \quad \xi = \eta + \xi_d \mathbf{e}^d.$$

For every non-zero tangent vector, we introduce the sum $H(\eta)$ of the kernels $\ker A(\xi)$ as ξ runs over $\mathbb{R}\eta \oplus \mathbb{R}\mathbf{e}^d$. We showed that it is an isotropic subspace for all these $A(\xi)$ simultaneously. In particular, its dimension is not larger than $m + p$. Examples from physics suggest we can assume that

i) the dimension of $H(\eta)$ is *maximal*, namely it equals $m + p$ for every tangent $\eta \neq 0$, *ii)* the generalized eigenvalue problem (6.1.25) does not have a non-trivial solution.

Under these assumptions, we know that the stable subspace $E_-(\tau,\eta)$ admits a unique limit $E_-(0,\eta)$ as τ tends to zero, keeping a positive real part, and this limit is described in (6.1.26). And since the boundary matrix has the form $B=(0,B_2)$, we have:

Proposition 6.6 *Assume properties i) and ii) above. Then the uniform Kreiss–Lopatinskii condition is satisfied in the neighbourhood of* $(0,\eta)$ *if, and only if,*

$$\mathbb{C}^{2p} = \ker B_2 \oplus H_1(\eta). \tag{6.1.29}$$

6.1.6 *Ohkubo's case*

This is the case where we shall be able to construct the dissipative symmetrizer in the next section. Recall that Ohkubo considers Friedrichs-symmetric operators with $a_2 \equiv 0_{n-m}$. Thus we have

$$A(\xi) = \begin{pmatrix} 0_m & a_{12}(\eta) \\ a_{21}(\eta) & \xi_d a^d \end{pmatrix}, \quad a_{21}(\eta)^T = a_{12}(\eta), \quad (a^d)^T = a^d.$$

The important point is the following.

Proposition 6.7 *In Ohkubo's case, it holds for* $\eta \in \mathbb{R}^{d-1} \setminus \{0\}$ *that*

$$\ker a_{12}(\eta) = R((a^d)^{-1} a_{21}(\eta)),$$

or equivalently

$$\ker(a_{12}(\eta)(a^d)^{-1}) = R(a_{21}(\eta)).$$

In particular, the subspaces $\ker a_{12}(\eta)$ *and* $R(a_{21}(\eta))$ *have dimension* p.

Proof We already have the inclusion

$$R((a^d)^{-1} a_{21}(\eta)) \subset \ker a_{12}(\eta), \tag{6.1.30}$$

from (6.1.6).

Let us make $\xi = (\eta, 0)$ with $\eta \neq 0$. Then $\ker A(\xi) = \ker a_{21}(\eta) \times \ker a_{12}(\eta)$. Since 0 is an eigenvalue of multiplicity m whenever $\xi \neq 0$, we infer that

$$m = \dim \ker a_{12}(\eta) + \dim \ker a_{21}(\eta).$$

However, we also have

$$m = \operatorname{rk} a_{21}(\eta) + \dim \ker a_{21}(\eta).$$

We obtain therefore

$$\operatorname{rk}((a^d)^{-1} a_{21}(\eta)) = \operatorname{rk} a_{21}(\eta) = \dim \ker a_{12}(\eta),$$

implying the equality in (6.1.30). □

Corollary 6.2 *In Ohkubo's case, the conclusion (6.1.29) of Proposition 6.6 applies.*

Proof We have to check properties *i)* and *ii)*.

When $\xi_d \neq 0$, we have

$$\ker A(\xi) = \{(\xi_d X, -(a^d)^{-1} a_{21}(\eta) X \,|\, X \in \mathbb{R}^m\}.$$

Also, when $\xi_d = 0$, it holds that

$$\ker A(\xi) = \ker a_{21}(\eta) \times \ker a_{12}(\eta).$$

It follows easily that $H(\eta) = \mathbb{R}^m \times \ker a_{12}(\eta)$. This space is of dimension $m+p$, hence maximal.

On the other hand, let U be a solution of (6.1.25). Since $H(\eta)^\perp = \{0\} \times R(a_{21}(\eta))$, we have $U = (0, a_{21}(\eta)z)$. Then

$$A(\eta + \xi_d \mathbf{e}^d)U = \begin{pmatrix} a_{12}(\eta)a_{21}(\eta)z \\ \xi_d a^d a_{21}(\eta)z \end{pmatrix}.$$

We deduce that $a_{12}(\eta)a_{21}(\eta)z = 0$, whence $U = 0$. □

Remarks We warn the reader that our assumption on the symbol does depend on the choice of space co-ordinates in which $\{x_d = 0\}$ is the boundary of the domain. A consistent assumption is that there is a tangent vector W, such that $a_2(\eta) = (\eta \cdot W)a^d$ for every η in $\mathbb{R}^{d-1}$. This allows us to choose co-ordinates in which $a_2 \equiv 0_{n-m}$, while the boundary is still $\{x_d = 0\}$. In other words, the lower-right block in the symbol needs to move along a one-dimensional space (notice that a^d is a generator of this line since it is a non-zero matrix.) It is clear from (6.1.28) that the system of isotropic elasticity does not satisfy this assumption. We leave the reader to check that the Maxwell system of electromagnetism, as well as the system of acoustics, do. Since isotropic elasticity does not belong to the Ohkubo's class, the construction of the forthcoming section does not apply to this system. Since the choice of Ohkubo's assumption is sufficient but not necessary, this does not simply anything about the solvability of the general IBVP under (UKL) in elasticity. A more elaborate construction, specific to the system of isotropic elasticity, is given in [141, 142], which proves the solvability of IBVPs in space dimension two and three.

6.2 Construction of the symmetrizer; characteristic case

The solvability of the IBVP under (UKL) will be shown, as in the non-characteristic case, with the help of a symmetrizer $K(\tau, \eta)$. We recall that we restrict to Friedrichs symmetric operators $L = \partial_t + A(\nabla_x)$ that have a null eigenvalue of constant multiplicity m, and such that $A(\xi)$ has the form[10]

$$A(\eta,\xi_d) = \begin{pmatrix} 0_m & a_{12}(\eta) \\ a_{21}(\eta) & \xi_d a^d \end{pmatrix}, \quad \eta \in \mathbb{R}^{d-1},\, \xi_d \in \mathbb{R}.$$

The symmetrizer will satisfy again the two requirements that $\Sigma := KA^d$ is Hermitian and $\text{Re}\; M(\tau,\eta) \geq c_0(\text{Re}\;\tau)I_n$, with the notations of Section 5.2. Also, it will be homogeneous of degree zero and bounded.

As mentioned before, it is not possible to obtain an estimate of the full trace of u on the boundary, in the same norm as the boundary data. What we may reasonably expect is an estimate of the trace of $A^d u$. For this reason, the restriction of Σ to the kernel of B cannot be negative-definite since it vanishes on $\ker A^d$. Hence, we ask only that this restriction be non-positive, while vanishing only on $\ker A^d$. In other words, this restriction will be bounded from above by $-c_0(A^d)^T A^d$, with $c_0 > 0$. Since $\ker A^d \subset \ker B$, this condition will involve only the vectors $w \in \mathbb{C}^{n-m}$ such that $(0,w)^T \in \ker B$. The space of these vectors is simply $\ker B_2$.

The special form of A^d and the fact that Σ is Hermitian immediately imply that K has the form

$$K(\tau,\eta) = \begin{pmatrix} K_1(\tau,\eta) & 0 \\ K_{21}(\tau,\eta) & K_2(\tau,\eta) \end{pmatrix},$$

where $K_2 a^d =: \Sigma_2$ is Hermitian. We now list the objects we are looking for:

i) Find parametrized matrices $K_1(\tau,\eta)$, $K_{21}(\tau,\eta)$ and $K_2(\tau,\eta)$, homogeneous of degree zero and uniformly bounded on $\text{Re}\;\tau > 0$, $\tau \in \mathbb{R}^{d-1}$,

ii) Such that $\Sigma_2(\tau,\eta) := K_2(\tau,\eta)a^d$ is Hermitian,

iii) The restriction of Σ_2 to $\ker B_2$ must be less than $-c_0 I_{n-m}$, with $c_0 > 0$ independent of (τ,η). In other words, we need a positive c_1, independent of (τ,η), such that

$$\Sigma_2 \leq -c_0 I_{n-m} + c_1 B_2^T B_2.$$

That inequality may also be written

$$\Sigma \leq -c_2 (A^d)^T A^d + c_1 B^T B,$$

iv) And finally,

$$\text{Re}\; M(\tau,\eta) \geq c_0(\text{Re}\;\tau)I_n, \quad \forall(\tau,\eta), \tag{6.2.31}$$

where

$$M(\tau,\eta) := K(\tau,\eta)(\tau I_n + iA(\eta,0)) = \begin{pmatrix} \tau K_1 & iK_1 a_{12} \\ \tau K_{21} + iK_2 a_{21} & iK_{21}a_{12} + \tau K_2 \end{pmatrix}.$$

[10]Notice that we work only in the simpler and natural case $a_2 \equiv 0_{n-m}$. To our knowledge, the systematic construction of the symmetrizer in a more general context is an open question.

We note that, since L is symmetric, the choice $K \equiv I_n$ fills three of the four requirements. The remaining one, that Σ_2 restricted to $\ker B_2$ is negative-definite, would then be a dissipation assumption.

Our assumption, besides constant hyperbolicity, is the uniform Kreiss–Lopatinskiĭ condition that $E_-(\tau,\eta) \cap \ker B$ is trivial for every pair with $\mathrm{Re}\,\tau \geq 0$, $\eta \in \mathbb{R}^{d-1}$, $(\tau,\eta) \neq 0$. The definition of $E_-(\tau,\eta)$ for $\mathrm{Re}\,\tau = 0$, $\tau \neq 0$, is as usual by continuous extension. Such an extension to points with $\mathrm{Re}\,\tau = 0$, $\tau \neq 0$ exists and follows from the same arguments as in the non-characteristic case. The continuous extension to points $(0,\eta)$ was described in Proposition 6.6.

In the following, we shall use the following property, equivalent to (UKL), that $e_-(\tau,\eta) \cap \ker B_2$ is trivial for every pair with $\mathrm{Re}\,\tau \geq 0$, $\eta \in \mathbb{R}^{d-1}$, $(\tau,\eta) \neq 0$.

Steps 1, 2 and 3 of the proof of Theorem 5.1 work as well. Therefore, we are led to construct the symmetrizer pointwise at interior points ($\mathrm{Re}\,\tau > 0$) and locally at boundary points ($\mathrm{Re}\,\tau = 0$). We shall split the latter case into two subcases, according to whether $\tau \neq 0$ or $\tau = 0$.

Interior points When (τ,η) is a fixed interior point ($\mathrm{Re}\,\tau > 0$), we only have to build $K(\tau,\eta)$ satisfying *ii)*, $\Sigma_2 < 0$ on $\ker B_2$ and $\mathrm{Re}\,M > 0$ at the sole point (τ,η).

Following Step 4 of the non-characteristic case (see Chapter 4), we first choose a Hermitian matrix Σ_2 that satisfies $\Sigma_2 < 0$ on $\ker B_2$ and $\mathrm{Re}\,\Sigma_2\mathcal{A}_2(\tau,\eta) < 0$. Such a Σ_2 does exist, because of (UKL): $(\ker B_2) \cap e_-(\tau,\eta) = \{0\}$.

Next, we choose

$$K_2 = \Sigma_2(a^d)^{-1}, \quad K_{21} = -\frac{i}{\tau}K_2 a_{21},$$

so that

$$M(\tau,\eta) = \begin{pmatrix} \tau K_1 & iK_1 a_{12} \\ 0 & -\Sigma_2\mathcal{A}_2 \end{pmatrix}.$$

It remains to choose $K_1 = \epsilon k_1$, with $k_1 \in HDP_m$ and $\epsilon > 0$ small enough. This ensures $\mathrm{Re}\,M > 0$ and ends the construction in the case of an interior point.

Ordinary boundary points ($\mathrm{Re}\,\tau = 0$**, but** $\tau \neq 0$**)** The above computation suggests to rewrite M in the general form

$$M(\tau,\eta) = \begin{pmatrix} \tau K_1 & iK_1 a_{12} \\ K' & -\Sigma_2\mathcal{A}_2 + \frac{i}{\tau}K' a_{12} \end{pmatrix}, \quad K' := \tau K_{21} + iK_2 a_{21}.$$

With this notation, we are searching a triplet (K_1, K', Σ_2) instead of (K_1, K_{21}, K_2). The matrix Σ_2 is Hermitian. Matrices K_1 and Σ_2 are homogeneous of degree zero, while K' has degree one.

As long as τ does not approach zero, we may choose $K_1 = \epsilon I_m$ and $K' = i\epsilon a_{12}^T$, with $\epsilon > 0$ small enough. This would have worked well in the case of interior points yet. Such a choice yields

$$\text{Re } M(\tau,\eta) = \begin{pmatrix} \epsilon(\text{Re } \tau)I_m & 0 \\ 0 & \text{Re } (-\Sigma_2\mathcal{A}_2) - \epsilon(\text{Re } \tau)|\tau|^{-2}a_{12}^T a_{12} \end{pmatrix}.$$

Let (τ_0,η_0) be given, with Re $\tau_0 \geq 0$ and $\tau_0 \neq 0$. Following the non-characteristic case, the uniform Kreiss–Lopatinskiĭ property ensures that there exists a Hermitian matrix $\Sigma_2(\tau,\eta)$ defined in a neighbourhood $\mathcal{V}$ of (τ_0,η_0), homogeneous of degree zero and bounded, whose restriction to F^d is negative-definite, and such that Re $(-\Sigma_2\mathcal{A}_2) \geq 2c_0(\text{Re } \tau)I_{n-m}$. Property *iv)* will then be satisfied provided $\epsilon a_{12}^T a_{12} \leq c_0|\tau|^2 I_{n-m}$ in $\mathcal{V}$, which is true when ϵ is small enough.

We emphasize that the above construction of Σ_2 needs the block-structure property for the matrix $\mathcal{A}_2(\tau,\eta)$ near glancing points (see [134]). This property is ensured by the important remark in Section 5.1 and the following identity, coming from Schur's complement formula (see [187]):

$$\det(\omega I_{n-m} - \mathcal{A}_2(\tau,\eta)) = \frac{\det(\tau I_n + A(i\eta,\omega))}{\tau^m \det a^d}, \tag{6.2.32}$$

where the right-hand side is analytic near (τ_0,η_0,ω) for every ω.

Central points ($\tau = 0$) From now on, we shall focus our attention on the vicinity of points of the form $(0,\eta_0)$ with $\eta_0 \neq 0$, where the previous analysis does not apply.

In such a neighbourhood, the symmetrizer (actually an incomplete one, see below) will be chosen as a linear combination of three simpler matrices:

$$K(\tau,\eta) = K^{II}(\eta) - \lambda K^I(\eta) + \mu K^{III}(\eta), \quad \lambda > 0,\ \mu > 0.$$

The real parameters λ,μ will be chosen later. The role of λ is to ensure the dissipativeness of the boundary condition, while the role of μ is to ensure that Re $M \geq c_0\gamma I_n$ with a positive c_0. It turns out that it is possible to choose λ first, and then to adapt μ.

The three pieces are given by the formulæ

$$K^I := \text{diag}(0_m, a_{21}a_{12}(a^d)^{-1}),$$
$$K^{II} := \text{diag}(a_{12}(a^d)^{-2}a_{21}, a_{21}a_{12}(a^d)^{-2} + (a^d)^{-1}a_{21}a_{12}(a^d)^{-1}),$$

where we drop the argument η for simplicity, and (recalling our notation $\tau = \gamma + i\rho$)

$$K^{III} := \begin{pmatrix} 0_m & 0 \\ -k_2 a_{21} & \rho k_2 \end{pmatrix}, \quad k_2 := \sigma(a^d)^{-1},$$
$$\sigma = i(a_{21}a_{12}(a^d)^{-1} - (a^d)^{-1}a_{21}a_{12}).$$

Note that $\Sigma := KA^d$ has the form diag$(0_m,\Sigma_2)$. We denote by $\Sigma_2^I,\ldots$ the blocks that correspond to $K^I,\ldots$. Then

$$\Sigma_2^I = a_{21}a_{12}, \quad \Sigma_2^{II} = a_{21}a_{12}(a^d)^{-1} + (a^d)^{-1}a_{21}a_{12}, \quad \Sigma_2^{III} = \rho\sigma.$$

These matrices are clearly Hermitian, thus $\Sigma_2 = \Sigma_2^{II} - \lambda\Sigma_2^I + \mu\Sigma_2^{III}$ is too.

We shall often work with variables (p,q) in $\mathbb{C}^{n-m}$, where $w = p+q$ and $p \in \ker a_{12}$, $q \in R(a_{21})$. Because of the Kreiss–Lopatinskiĭ condition, written at $\tau = 0$, we have $\mathbb{C}^{n-m} = \ker B_2 \oplus \ker a_{12}$. In other words, $\ker B_2$ has an equation of the form $p = Dq$, where $D(\eta)$ is some linear operator. Obviously, D depends smoothly, thus boundedly, on $\eta/|\eta|$. On $\ker B_2$, the norms $|w|$ and $|q|$ are equivalent, uniformly in η.

As the Hermitian form $w \mapsto |a_{12}q|^2$ is positive-definite on $\ker B_2$, there exists a (large enough) λ such that the form

$$w \mapsto w^*\Sigma_2^{II}w - \lambda w^*\Sigma_2^I w$$

is negative-definite on $\ker B_2$, uniformly in η. Regardless of the value of μ, the restriction of Σ_2 to $\ker B_2$ will be uniformly negative-definite for ρ, hence τ, small enough.

We now turn to the study of the Hermitian form $Q(u) := \mathrm{Re}\ (u^*Mu)$, in which we use repeatedly (6.1.6). It decomposes naturally into $Q^{II} - \lambda Q^I + \mu Q^{III}$, where λ has been fixed yet and μ is still at our disposal. The formulæ for each piece are:

$$\begin{aligned} Q^I(u) &= \mathrm{Re}\ (\tau q^* a_{21}a_{12}(a^d)^{-1}p), \\ Q^{II}(u) &= \gamma\left(|(a^d)^{-1}a_{21}v|^2 + |a_{12}(a^d)^{-1}p|^2\right) + \mathrm{Re}\ (\tau q^* a_{21}a_{12}(a^d)^{-2}w), \\ Q^{III}(u) &= \mathrm{Re}\ \left(w^*k_2(-\gamma a_{21}v - ia_{21}a_{12}q + \rho\tau w)\right), \end{aligned}$$

where we notice that

$$w^*k_2 = i(q^*a_{21}a_{12}(a^d)^{-1} - p^*(a^d)^{-1}a_{21}a_{12})(a^d)^{-1}. \tag{6.2.33}$$

Let us establish some bounds, where the numbers $c_0, c_1 > 0$ may be taken independently of η. On the one hand, we have

$$|\lambda Q^I(u)| \le c_1|\tau|\,|p|\,|q|, \quad Q^{II}(u) \ge c_0\gamma\left(|a_{21}v|^2 + |p|^2\right) - c_1|\tau|\,|w|\,|q|. \tag{6.2.34}$$

On the other hand, (6.2.33) yields

$$\begin{aligned} w^*k_2(-\gamma a_{21}v - ia_{21}a_{12}q + \rho\tau w) &= iq^*a_{21}a_{12}(a^d)^{-2}(-\gamma a_{21}v - ia_{21}a_{12}q + \rho\tau w) \\ &\quad - i\rho\tau p^*(a^d)^{-1}a_{21}a_{12}(a^d)^{-1}p \\ &= |(a^d)^{-1}a_{21}a_{12}q|^2 + \rho(\rho - i\gamma)|a_{12}(a^d)^{-1}p|^2 \\ &\quad + \mathcal{O}(\gamma|q|\,|a_{21}v|) + \mathcal{O}(|\rho\tau|\,|q|\,|w|), \end{aligned}$$

whence

$$Q^{III}(u) \ge c_0\left(|q|^2 + \rho^2|p|^2\right) - c_1|q|(\gamma|a_{21}v| + |\rho\tau|\,|w|). \tag{6.2.35}$$

Let us choose $\mu > 0$ large enough so that the quadratic form $\mu c_0(X^2+Y^2) - 2c_1XY$ is positive-definite. This allows us to absorb the bad terms $-c_1|\rho|\,|p|\,|q|$

of Q^{II} and $-\lambda Q^{I}$ into the positive terms of μQ^{III}. This results, with a slight change of the positive numbers c_0, c_1, in the following lower bound

$$Q(u) \geq c_0\gamma\left(|a_{21}v|^2 + |p|^2\right) + c_0\left(|q|^2 + \rho^2|p|^2\right) \\ - c_1|q|\left(|\rho|\,|q| + \gamma|w| + \gamma|a_{21}v| + \rho^2|w|\right).$$

Using Young's inequality in the last three terms, we obtain

$$Q(u) \geq c_0\left(\gamma(|a_{21}v|^2 + |p|^2) + \rho^2|p|^2\right) + \left(\frac{c_0}{2} - c_1\rho\right)|q|^2 \\ - \frac{3c_1^2}{2c_0}\left(\gamma^2|w|^2 + \gamma^2|a_{21}v|^2 + \rho^4|w|^2\right).$$

It is now clear that there is a neighbourhood $\mathcal{V}$ of the origin, such that $\tau \in \mathcal{V}$ implies

$$Q(u) := \mathrm{Re}\ (u^*Mu) \geq c_0\gamma\left(|a_{21}v|^2 + |p|^2\right) + c_0\left(\frac{1}{3}|q|^2 + \rho^2|p|^2\right). \tag{6.2.36}$$

Looking back at our results, we find only one flaw, which lies in (6.2.36): The form Q dominates only $a_{21}v$ but not v itself. If a_{21} is one-to-one, that is $n = 3m$, then our construction gives us the symmetrizer that we were looking for, otherwise it does not. In the general case, let us denote by K^0 the symmetrizer that we have defined above, and Σ_2^0, Q^0 the corresponding Hermitian forms. Our final symmetrizer will be of the form $K = K^0 + \epsilon I_n$. We shall have

$$\Sigma_2 = \Sigma_2^0 + \epsilon a^d, \quad Q(u) = Q^0(u) + \epsilon\gamma|u|^2.$$

For every choice of $\epsilon > 0$, the form Q satisfies our requirement

$$Q(u) \geq c_0\gamma|u|^2$$

for an appropriate $c_0 > 0$. At last, a small enough ϵ does not hurt the negativeness of the restriction of Σ_2^0 on $\ker B_2$. This completes the construction of a dissipative symmetrizer. We leave the readers to convince themselves that the inequalities satisfied by Σ_2 and Q are uniform in η.

7

THE HOMOGENEOUS IBVP

We continue the analysis of strong L^2-well-posedness that we treated in the previous chapters. However, we weaken our requirements by considering only the *homogeneous* IBVP:

$$(Lu)(x,t) = f(x,t), \quad x_d, t > 0,\ y \in \mathbb{R}^{d-1}, \tag{7.0.1}$$

$$Bu(y,0,t) = 0, \qquad t > 0,\ y \in \mathbb{R}^{d-1}, \tag{7.0.2}$$

$$u(x,0) = u_0(x), \quad x_d > 0,\ y \in \mathbb{R}^{d-1}. \tag{7.0.3}$$

The boundary condition is therefore $Bu(t,y,0) = 0$, instead of $Bu = g$ with a general data g. An important consequence is that (7.0.1)–(7.0.3) is a semigroup problem: At a formal level, the solution should read

$$u(t) = S_t u_0 + \int_0^t S_{t-s} f(s)\mathrm{d}s,$$

where $(S_t)_{t\geq 0}$ is a semigroup on some Banach space. What we expect is that this is a continuous semigroup on each Sobolev space $H^\sigma(\Omega)$. However, it is not so simple to follow the standard strategy of semigroup theory, and we shall merely use the same tools as in the non-homogeneous IBVP: Fourier–Laplace transformation, dissipative symmetrizer, Lopatinskiĭ condition, Paley–Wiener Theorem and duality. The main difference with the non-homogeneous theory lies in the estimate we ask for: since the input consists only in two terms, say $u(\cdot,0)$ and Lu, we do not require an estimate for the boundary value of the solution. This has two important consequences. On the one hand, the Kreiss–Lopatinskiĭ condition is weakened. On the other hand, strongly well-posed IBVPs in the above sense may admit *surface waves* of finite energy.

When dealing with second-order IBVPs, an important case, which we do not treat here, is when the pair (L,B) expresses the Euler–Lagrange equations of some Lagrangian

$$\mathscr{L}[u] := \int\int \left(\frac{1}{2}|u_t|^2 - W(\nabla_x u)\right) \mathrm{d}x\,\mathrm{d}t$$

over $H^1(\Omega \times \mathbb{R}_t)$. This special situation encompasses the examples of the wave equation with the Neumann condition, and of linearized elastodynamics with zero normal stress at the boundary. The theory of such problems is done in [189,190], where it is shown that the Hille–Yosida Theorem can always be applied.

The present chapter is thus devoted to the more general situation of first-order problems.

As usual, strong well-posedness means that we have control of the solution exactly in the same terms as for the data. Since the boundary condition does not really involve a data (it is trivial), we do not require any control of the trace of the solution along the boundary. Therefore, we say that the homogeneous IBVP (7.0.1)–(7.0.3) is *strongly well-posed* in L^2 if for every $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$, there exists a unique solution $u \in \mathscr{C}([0,T];L^2(\Omega))$ with the estimate

$$\mathrm{e}^{-2\gamma T}\|u(T)\|_{L^2}^2 + \gamma \int_0^T \mathrm{e}^{-2\gamma t}\|u(t)\|_{L^2}^2 \mathrm{d}t \leq C\left(\|u_0\|_{L^2}^2 + \frac{1}{\gamma}\int_0^T \mathrm{e}^{-2\gamma t}\|f(t)\|_{L^2}^2 \mathrm{d}t\right). \tag{7.0.4}$$

Hereabove, all norms are taken in $L^2(\Omega)$.

BVP *vs* IBVP When the initial data u_0 vanishes identically, we may extend u and f to negative times by zero, so that the extension $\tilde{u}$ is a solution of the BVP for $L\tilde{u} = \tilde{f}$ on $\mathbb{R} \times \Omega$, instead of $(0,+\infty) \times \Omega$. We say that the homogeneous BVP is strongly well-posed in L^2 if for every $\gamma > 0$ and every $f \in L^2_\gamma(0,+\infty;L^2(\Omega))$, the IBVP with $u_0 \equiv 0$ has a unique solution in $L^2_\gamma(0,+\infty;L^2(\Omega))$, with the estimate

$$\int_0^T \mathrm{e}^{-2\gamma t}\|u(t)\|_{L^2}^2 \mathrm{d}t \leq \frac{C}{\gamma^2}\int_0^T \mathrm{e}^{-2\gamma t}\|f(t)\|_{L^2}^2 \mathrm{d}t, \tag{7.0.5}$$

where C does not depend either on f or on γ.

We therefore say that our BVP is *strongly well-posed* if, for every u smooth with compact support in space and time, the property $Bu(y,0,t) \equiv 0$ implies an estimate

$$\int_0^T \mathrm{e}^{-2\gamma t}\|u(t)\|_{L^2}^2 \mathrm{d}t \leq \frac{C}{\gamma^2}\int_0^T \mathrm{e}^{-2\gamma t}\|Lu(t)\|_{L^2}^2 \mathrm{d}t. \tag{7.0.6}$$

We have seen in Section 4.5.5 that the well-posedness of the BVP is a step towards that of the IBVP, the passage from the former to the latter needing an extra argument, due to Rauch. Because we do not estimate the trace of the solution, Rauch's argument does not work out in the homogeneous case. For this reason, the well-posedness of the homogeneous IBVP remains an open question. So far, the only cases where the IBVP is fully understood is the symmetric dissipative one (see Section 3.1), and more generally the 'variational' case treated in [189, 190].

Need for a modified Lopatinskiĭ condition When considering Friedrichs-symmetric operators with classical dissipative symmetrizers (see Chapter 3), we observed that the amount of dissipation needed to treat the homogeneous IBVP was weaker than that required for the non-homogeneous problem. Since

Chapter 3 stated sufficient but not necessary criteria for well-posedness, we cannot draw a definitive conclusion from it. However, we anticipate that some condition weaker than (UKL) would ensure the homogeneous well-posedness, at least for constantly hyperbolic operators.

Before entering into the details, we make the following observations. On the one hand, since our preliminary study of Section 4.2 dealt with the homogeneous problem, we already know that the Kreiss–Lopatinskiĭ condition is necessary for well-posedness. On the other hand, the well-posedness of the non-homogeneous IBVP clearly implies that of the homogeneous one. Therefore, at least for constantly hyperbolic operators, the truth for the homogeneous case must lie somewhere between the Kreiss–Lopatinskiĭ condition and its uniform version.

7.1 Necessary conditions for strong well-posedness

In this section, we derive a necessary condition that we expect also to be sufficient when the differential operator is constantly hyperbolic. The sufficiency would be ensured, as in the non-homogeneous case, by the existence of a Kreiss symmetrizer, though a weakly dissipative one, instead of strongly dissipative. We postpone its construction to Section 7.2.

For the sake of simplicity, we assume that the boundary is non-characteristic. Also, it will be comfortable to consider only constantly hyperbolic operators. This allows us to extend by continuity the stable subspace $E_-(\tau,\eta)$ and the Lopatinskiĭ determinant to boundary points ($\mathrm{Re}\ \tau = 0$, $(\tau,\eta)\neq(0,0)$).

Our necessary condition will be derived from the stability of the homogeneous BVP (instead of the corresponding IBVP). As a matter of fact, it is not hard to see that the well-posedness with $u_0 \equiv 0$ implies the well-posedness of the BVP.

At the level of the Laplace–Fourier transform, the estimate (7.0.6) amounts to saying that

$$\|v\|_{L^2} \leq \frac{C}{\mathrm{Re}\ \tau}\|F\|_{L^2}, \tag{7.1.7}$$

for every $v \in L^2(\mathbb{R}^+)$ such that

$$F := v' - \mathcal{A}(\tau,\eta)v \in L^2(\mathbb{R}^+), \quad \mathrm{Re}\ \tau > 0,\ \eta \in \mathbb{R}^{d-1}$$

and

$$Bv(0) = 0.$$

Let us introduce the decomposition $v = v_s + v_u$ into a stable part $v_s \in E_-(\tau,\eta)$ and an unstable one $v_u \in E_+(\tau,\eta)$. Decomposing $F := (A^d)^{-1}\mathcal{L}f$, we have

$$\frac{\mathrm{d}v_u}{\mathrm{d}x_d} = \mathcal{A}_u(\tau,\eta)v_u + F_u, \tag{7.1.8}$$

$$\frac{\mathrm{d}v_s}{\mathrm{d}x_d} = \mathcal{A}_s(\tau,\eta)v_u + F_s, \tag{7.1.9}$$

where $\mathcal{A}_{u,s}$ represent the restrictions of $\mathcal{A}$ to $E_{\pm}$. Let us fix a pair (τ, η), with $\operatorname{Re}\tau > 0$. The spectra of $\mathcal{A}_{u,s}$ have, respectively, positive/negative real parts. Since $v_{u,s}$ are square-integrable on $\mathbb{R}^+$, and $F_{u,s}$ are square-integrable, we have

$$v_u(x_d) = -\int_{x_d}^{+\infty} \mathrm{e}^{(x_d - z)\mathcal{A}_u} F_u(z)\mathrm{d}z, \tag{7.1.10}$$

$$v_s(x_d) = \mathrm{e}^{x_d \mathcal{A}_s} v_s(0) + \int_0^{x_d} \mathrm{e}^{(x_d - z)\mathcal{A}_s} F_s(z)\mathrm{d}z, \tag{7.1.11}$$

where $v_s(0)$ is to be determined in $E_-(\tau, \eta)$.

Denoting

$$k_{u,r}(\tau, \eta) := \|z \mapsto \exp(-z\mathcal{A}_u)\|_{L^r(\mathbb{R}^+)}, \quad k_{s,r}(\tau, \eta) := \|z \mapsto \exp(z\mathcal{A}_s)\|_{L^r(\mathbb{R}^+)},$$

Young's inequality gives

$$\|v_u\|_{L^2} \leq k_{u,1}(\tau, \eta)\|F_u\|_{L^2} \tag{7.1.12}$$

and similarly

$$\|v_s\|_{L^2} \leq k_{s,1}(\tau, \eta)\|F_s\|_{L^2} + k_{s,2}(\tau, \eta)|v_s(0)|. \tag{7.1.13}$$

The boundary condition writes as

$$Bv_s(0) = B\int_0^{+\infty} \mathrm{e}^{-z\mathcal{A}_u(\tau,\eta)} F_u(z)\,\mathrm{d}z.$$

Let us assume the Lopatinskiĭ condition, since it is necessary for the stability. Then $B : E_-(\tau, \eta) \to \mathbb{C}^p$ is an isomorphism for every pair (τ, η) with $\operatorname{Re}\tau > 0$ and $\eta \in \mathbb{R}^{d-1}$. We denote by $\beta(\tau, \eta)$ the norm of its inverse:

$$\beta(\tau, \eta) = \sup\left\{ \frac{|w|}{|Bw|}\,;\, w \in E_-(\tau, \eta) \right\}.$$

Then we have

$$|v_s(0)| \leq \|B\|\beta(\tau, \eta)k_{u,2}(\tau, \eta)\|F_u\|_{L^2},$$

which yields the estimate

$$\|v\|_{L^2} \leq (\|B\|\beta(\tau, \eta)k_{u,2}(\tau, \eta)k_{s,2}(\tau, \eta) + k_{u,1}(\tau, \eta))\,\|F_u\|_{L^2} + k_{s,1}(\tau, \eta)\|F_s\|_{L^2}. \tag{7.1.14}$$

Finally, introducing the norm $N(\tau, \eta)$ of the splitting $z \mapsto (z_s, z_u)$, we end with

$$\|v\|_{L^2} \leq N(\tau, \eta)\,(\|B\|\beta(\tau, \eta)k_{u,2}(\tau, \eta)k_{s,2}(\tau, \eta) + k_{u,1}(\tau, \eta) + k_{s,1}(\tau, \eta))\,\|F\|_{L^2}. \tag{7.1.15}$$

We now discuss whether (7.1.14) may result in (7.1.7) or not (this is essentially generalizing the study of 2-D wave-like equations by Miyatake [139]). Up to unlikely cancellations, (7.1.14) implies (7.1.7) if, and only if,

$$N(\tau,\eta)\left(\beta(\tau,\eta)k_{u,2}(\tau,\eta)k_{s,2}(\tau,\eta)+k_{u,1}(\tau,\eta)+k_{s,1}(\tau,\eta)\right)$$

is bounded by a constant times $(\text{Re }\tau)^{-1}$. Thus we evaluate the magnitude of each term N, β, $k_{(u,s),r}$ as (τ,η) approaches P_0. Since we know that the stability holds at interior points ($\text{Re }\tau > 0$), we may focus on the neighbourhood $\mathcal{V}$ of boundary points $P_0 = (i\rho_0, \eta_0)$. Likewise, the stability holds true in $\mathcal{V}$ as soon as the Kreiss–Lopatinskiĭ condition is satisfied at P_0, since we were able to build a dissipative Kreiss symmetrizer in this case. Therefore, we may consider only those boundary points where the Lopatinskiĭ determinant Δ vanishes. We thus assume $\Delta(P_0) = 0$.

Let us begin with non-glancing points. Since then $E_-(i\rho_0,\eta_0)$ and $E_+(i\rho_0,\eta_0)$ are supplementary, $N(\tau,\eta)$ remains bounded in $\mathcal{V}$. Generically, $\beta(\tau,\eta)$ is of the order of $\Delta(\tau,\eta)^{-1}$. Since Δ is analytic in $\mathcal{V}$, the only bound in terms of Re τ is $\beta = \mathcal{O}((\text{Re }\tau)^{-m})$ where m is the order at which Δ vanishes at P_0. Since $m \geq 1$, we see that the required stability holds only if on the one hand $m = 1$ and on the other hand $k_{(u,s),2}$ are bounded. However, since the eigenvalues of $\mathcal{A}(\tau,\eta)$ behave smoothly in $\mathcal{V}$, we see that $k_{s,2}$ is of the order of

$$(\min|\text{Re }\mu_j(\tau,\eta)|)^{-1/2}, \quad (\mu_1,\dots,\mu_p) = \text{Sp}\mathcal{A}_s(\tau,\eta).$$

A similar formula holds for the unstable part. Therefore $k_{u,2}$ and $k_{s,2}$ remain bounded if and only if $\mathcal{A}(P_0)$ does not have any pure imaginary eigenvalue. The boundary points where this situation holds true form the *elliptic* part of the frequency boundary. In conclusion, the vanishing of Δ at non-glancing non-elliptic (in particular in the hyperbolic part $\mathcal{H}$) boundary frequencies yields instability in the homogeneous BVP, while its vanishing at non-glancing elliptic points is harmless, provided Δ vanishes at first order only.

Remark Since $\mathcal{A}(P_0) = i\mathcal{B}(P_0)$, where the matrix $\mathcal{B}(P_0)$ has real entries, the ellipticity of P_0 implies that the eigenvalues of $\mathcal{B}(P_0)$ come by complex conjugated pairs. Exactly half of them have a positive imaginary part. We deduce that an elliptic zone may occur in the frequency boundary, only if $n = 2p$.

We now turn to glancing points. Then at least one pair of eigenvalues coalesce into a pure imaginary one, with (generically) the formation of a Jordan block in $\mathcal{A}(P_0)$. In particular, the splitting $\mathbb{C}^n = E_-(\tau,\eta) \oplus E_+(\tau,\eta)$ does not hold at P_0. This implies that $N(\tau,\eta)$ is unbounded in $\mathcal{V}$. Typically, the coalescing eigenvalues experience an algebraic square root singularity at P_0, and therefore every relevant quantity displays such a singularity. Thus, generically, bounds of the form

$$N(\tau,\eta) = \mathcal{O}(\sqrt{\text{Re }\tau}), \quad k_{(u,s),r}(\tau,\eta) = \mathcal{O}\left((\text{Re }\tau)^{-1/2r}\right)$$

holds. If these bounds are optimal, as we expect, then the stability needs that $\beta(\tau,\eta)$ be bounded in $\mathcal{V}$, meaning that the Kreiss–Lopatinskiĭ condition holds true at P_0.

Conclusion Let L be a given constantly hyperbolic operator. An homogeneous BVP for which the Kreiss–Lopatinskiĭ condition holds true at every frequency point but elliptic boundary ones, at which Δ vanishes at most to first order, is likely to be L^2-well-posed.

On the contrary, if Δ vanishes at some point outside the elliptic part of the frequency boundary, the BVP is likely to be ill-posed in $L^2(\Omega)$ (an exception to this rule occurs if $(n,p)=(2,1)$; see below). Similarly, if Δ vanishes at some elliptic point, but to the order two or more, the BVP is likely to be ill-posed in $L^2(\Omega)$.

Of course, the proof of the well-posedness of an homogeneous BVP, for which Δ vanishes to first order at some elliptic point and nowhere else, needs the construction of some kind of dissipative symmetrizer. The symmetrizer $K(\tau,\eta)$ that we need here should display the following properties (we keep the notations of Chapter 5):

i) The map $(\tau,\eta)\mapsto K(\tau,\eta)$ is $\mathscr{C}^\infty$ on Re $\tau\geq 0$, $\eta\in\mathbb{R}^{d-1}$, $|\tau|+|\eta|\neq 0$, and is homogeneous of degree zero,

ii) the matrix $\Sigma(\tau,\eta):=K(\tau,\eta)A^d$ is Hermitian,

iii) the Hermitian form $x^*\Sigma(\tau,\eta)x$ is non-positive on $\ker B$,

iv) there exists a number $c_0>0$ such that, for every (τ,η), the inequality Re $M\geq c_0(\text{Re }\tau)I_n$ holds in the sense of Hermitian matrices, where

$$M=M(\tau,\eta):=-(\Sigma\mathcal{A})(\tau,\eta)=K(\tau,\eta)(\tau I_n+iA(\eta)).$$

The only change with respect to the non-homogeneous BVP, is point *iii)*, where the restriction of Σ to $\ker B$ needs only to be non-positive, instead of negative-definite. We therefore speak of a *weakly* dissipative symmetrizer. Of course, the Kreiss symmetrizer that we constructed in Chapter 5 is convenient at every point where the Kreiss–Lopatinskiĭ is satisfied. Thus it remains only to build a symmetrizer in the neighbourhood of those (elliptic boundary) points where Δ vanishes. We postpone this task to Section 7.2. For the moment, we content ourselves with the following calculus, which shows that the existence of such a weakly dissipative symmetrizer implies the L^2-stability of the homogeneous BVP. If u is smooth and compactly supported, and satisfies $Bu=0$ on the boundary, then its Fourier–Laplace transform $\hat{u}$ satisfies

$$(\tau I_n+iA(\eta))\hat{u}+A^d\frac{\mathrm{d}\hat{u}}{\mathrm{d}x_d}=\widehat{Lu}$$

together with $B\hat{u}=0$ for $x_d=0$. Multiplying on the left by $K(\tau,\eta)$ yields

$$M(\tau,\eta)\hat{u}+\Sigma(\tau,\eta)\frac{\mathrm{d}\hat{u}}{\mathrm{d}x_d}=K(\tau,\eta)\widehat{Lu}.$$

Multiplying again by $\hat{u}^*$, taking the real part and using the fact that Σ is Hermitian, we obtain

$$2\hat{u}^*\text{Re }(M(\tau,\eta))\hat{u}+\frac{\mathrm{d}}{\mathrm{d}x_d}(\hat{u}^*\Sigma(\tau,\eta)\hat{u})=2\text{Re }(\hat{u}^*K(\tau,\eta)\widehat{Lu}).$$

Integrating in x_d, using the Cauchy–Schwarz inequality, the dissipativeness and the positivity of Re M, we end with

$$c_0^2(\mathrm{Re}\ \tau)^2 \int_0^{+\infty} |\hat{u}|^2 \mathrm{d}x_d \le \int_0^{+\infty} |K\widehat{Lu}|^2 \mathrm{d}x_d.$$

We observe that K is uniformly bounded. Therefore, after an integration in (η, ρ) where as usual $\tau = \gamma + i\rho$, we have

$$\iiint_{\mathbb{R}\times\Omega} |\hat{u}(\gamma + i\cdot, \cdot, \cdot)|^2 \mathrm{d}\rho\, \mathrm{d}\eta\, \mathrm{d}x_d \le \frac{C}{\gamma^2} \iiint_{\mathbb{R}\times\Omega} |\widehat{Lu}(\gamma + i\cdot, \cdot, \cdot)|^2 \mathrm{d}\rho\, \mathrm{d}\eta\, \mathrm{d}x_d.$$

The Plancherel formula gives the stability estimate (7.0.5).

Remark We emphasize that we did not use the regularity of K in this computation. The regularity becomes useful when dealing with variable coefficients.

The case $(n, p) = (2, 1)$ A more careful analysis would employ the norm

$$\tilde{\beta}(\tau, \eta) := \beta(\tau, \eta) \| B : E_+(\tau, \eta) \to \mathbb{C}^p \|$$

instead of $\beta(\tau, \eta)$ itself. In other words, one may define equivalently

$$\hat{\beta}(\tau, \eta) := \sup \left\{ \frac{|w|}{|z|} \,;\, w \in E_-(\tau, \eta),\ z \in E_+(\tau, \eta),\ Bw = Bz \right\}.$$

We have

$$|v_s(0)| \le \hat{\beta}(\tau, \eta) k_{u,2} \|F_u\|_2,$$

yielding (7.1.15) with $\hat{\beta}$ instead of β. In general, the size of $\hat{\beta}$ is not different from that of β, because the norm of $B : E_+(\tau, \eta) \to \mathbb{C}^p$ is not small. However, there is one situation where $\hat{\beta}$ remains bounded when β blows up. Such a cancellation happens when the point P_0 where Δ vanishes[1] satisfies $E_+(P_0) \subset E_-(P_0)$; in particular, P_0 is a glancing point. If, moreover, $n = 2$ and $p = 1$, then $E_+(P_0) = E_-(P_0)$, and the vanishing of Δ actually means

$$E_+(P_0) = E_-(P_0) = \ker B.$$

Generically, if $R_+(\tau, \eta)$ is an analytic generator of $E_+(\tau, \eta)$, then $(\tau, \eta) \mapsto BR_+$ vanishes at the same order[2] as $\Delta := BR_-$, and therefore $\hat{\beta}$ remains bounded.

Conclusion When $n = 2$, $p = 1$ and L is constantly hyperbolic, the homogeneous BVP is likely to be well-posed if and only if the Lopatinskiĭ determinant vanishes at most either at boundary frequencies of elliptic type, or at glancing points.

[1]Necessarily a boundary point since we require the Lopatinskiĭ condition at interior points.
[2]Near glancing points, R_+ and R_- are *conjugate* quantities, in the algebraic sense.

7.1.1 *An illustration: the wave equation*

We illustrate the previous, rather formal, analysis at the level of the wave equation

$$\partial_t^2 u = c^2 \Delta u, \quad (x_d > 0)$$

with a boundary condition of the form

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x_d} + \vec{b}\cdot\nabla_y u = 0. \tag{7.1.16}$$

The status of the Kreiss–Lopatinskiĭ condition for this BVP is well-known. See, for instance, [13], Section 4, or just make your own computations. The (UKL) is satisfied if and only if

$$a < 0, \quad |\vec{b}| < c. \tag{7.1.17}$$

The wave equation displays an elliptic zone in the boundary of the frequencies, which is made of pairs $(\tau = i\rho, \eta)$ such that $|\rho| < c|\eta|$. The glancing points are the pairs $(\pm ic|\eta|, \eta)$. The Lopatinskiĭ determinant vanishes in the elliptic zone and nowhere else if, and only if,

$$a = 0, \quad |\vec{b}| < c. \tag{7.1.18}$$

Finally, Δ vanishes at a glancing point if, and only if,

$$a \le 0, \quad |\vec{b}| = c. \tag{7.1.19}$$

We show below that every uniformly stable non-homogeneous BVP (meaning that (7.1.17) holds) belongs to the class of strictly dissipative Friedrichs-symmetrizable systems. By this, we mean that there exists a vector $\vec{p}$ such that the modified quadratic energy

$$E_p[u] := \frac{u_t^2 + c^2|\nabla u|^2}{2} + u_t(\vec{p}\cdot\nabla u),$$

is strictly convex, and the boundary flux $Q_p \cdot \nu = -(Q_p)_d$ is negative-definite on the subspace defined by the boundary condition (7.1.16). We next show that under (7.1.18), the BVP is weakly dissipative: Q_p is non-positive subjected to the constraint (7.1.16). This proves that under (7.1.18), the homogeneous BVP is well-posed in $L^2(\Omega)$ (see Section 3.1), although this is not true for the non-homogeneous BVP. The same situation occurs under (7.1.19) (except for the borderline case $a = 0$, $|\vec{b}| = c$), confirming our analysis of the situation $(n, p) = (2, 1)$.

The energy E_p is convex (respectively, strictly convex) if and only if $|\vec{p}| \le c$ (respectively, $|\vec{p}| < c$). It satifies the identity

$$\partial_t E_p = \mathrm{div} Q_p, \quad Q_p = c^2(\partial_t u + \vec{p}\cdot\nabla u)\nabla u + \frac{1}{2}((\partial_t u)^2 - c^2|\nabla u|^2)\vec{p}.$$

From this identity, we extract

$$Q_p \cdot \nu = -c^2(\partial_t u + \vec{p} \cdot \nabla u)\frac{\partial u}{\partial x_d} + \frac{1}{2}p_d\left(c^2|\nabla u|^2 - (\partial_t u)^2\right).$$

Let us introduce the notations:

$$z = \frac{\partial u}{\partial x_d}, \quad w = \nabla_y u, \quad \vec{p} = (\vec{q}, p_d)^T.$$

After elimination of $\partial_t u$ (thanks to (7.1.16)), there remains a quadratic form

$$\mathcal{Q}_p(w,z) = c^2(az + (\vec{b} - \vec{q}) \cdot w)z + \frac{1}{2}p_d(c^2|w|^2 - c^2z^2 - (az + \vec{b} \cdot w)^2).$$

The general problem of dissipativeness in the classical Friedrichs sense is to find a $\vec{p}$ with $|\vec{p}| < c$, such that the quadratic form $\mathcal{Q}_p$ is non-positive (negative-definite yields strict dissipation).

Let us begin with the transitional case (7.1.18). We have

$$\mathcal{Q}_p(w,z) = c^2z(\vec{b} - \vec{q}) \cdot w + \frac{1}{2}p_d(c^2|w|^2 - c^2z^2 - (\vec{b} \cdot w)^2).$$

Since

$$\mathcal{Q}_p(w,0) = \frac{1}{2}p_d(c^2|w|^2 - (\vec{b} \cdot w)^2), \quad \mathcal{Q}_p(0,z) = -\frac{1}{2}p_d c^2 z^2,$$

where the two quadratic forms are of opposite signs, the only way that $\mathcal{Q}_p$ may be non-positive is that $p_d = 0$. Then $\mathcal{Q}_p$ reduces to $c^2z(\vec{b} - \vec{q}) \cdot w$ and therefore $\vec{q}$ must equal $\vec{b}$. Hence $\vec{p} = (\vec{b}, 0)^T$, which satisfies $|\vec{p}| < c$ indeed. Thus the IBVP is weakly (because $\mathcal{Q}_p \equiv 0$) Friedrichs dissipative[3].

Let us consider now the transitional case (7.1.19). The choice

$$\vec{q} = \frac{c^2}{c^2 + a^2}\vec{b}, \quad p_d = \frac{ac^2}{c^2 + a^2}$$

satisfies $|\vec{p}| \le c$, with the strict inequality when $a < 0$. Then the boundary flux is

$$\mathcal{Q}_p(w,z) = \frac{ac^2}{2(c^2 + a^2)}\left((c^2 + a^2)z^2 + c^2|w|^2 - (\vec{b} \cdot w)^2\right), \tag{7.1.20}$$

which is non-positive (though not negative-definite, of course). Thus the corresponding IBVP is weakly dissipative. In particular, the homogeneous IBVP is well-posed in $L^2(\Omega)$. The only exception is when $a = 0$, where the condition (7.1.19) meets (7.1.18). Then, specialization to w parallel to $\vec{b}$ shows that for $\mathcal{Q}_p$ to be non-positive, we need

$$p_d = 0, \quad \vec{b} \cdot \vec{q} = c^2.$$

[3]Notice that this IBVP is conservative, rather than dissipative, as $\mathcal{Q}_p \equiv 0$ implies that the integral of E_p is preserved. In this situation, both the forward and the backward IBVPs are well-posed.

Because of $|\vec{p}| \leq c$, we deduce that necessarily $\vec{q} = \vec{b}$. Since then $|\vec{p}| = c$, the energy E_p is convex but **not strictly**. Presumably, the corresponding homogeneous IBVP is not strongly well-posed in $L^2(\Omega)$.

Finally, we consider the (UKL) case (7.1.17). The same choice as above yields the same formula (7.1.20), where the quadratic form is now negative-definite. Hence the IBVP is strictly dissipative in the Friedrichs sense. Notice that in the simple case $a < 0$, $\vec{b} = 0$, the naive choice $\vec{p} = 0$ yields weak but not strong dissipativeness; the corrector $p_d \partial_t u \partial_d u$ in the energy thus plays a rather important role.

7.2 Weakly dissipative symmetrizer

We turn to the construction of a weakly dissipative symbolic symmetrizer. We recall the properties to fulfil:

i) The map $(\tau, \eta) \mapsto K(\tau, \eta)$ is $\mathscr{C}^\infty$ on Re $\tau \geq 0$, $\eta \in \mathbb{R}^{d-1}$, $|\tau| + |\eta| \neq 0$, and is homogeneous of degree zero,

ii) the matrix $\Sigma(\tau, \eta) := K(\tau, \eta) A^d$ is Hermitian,

iii) the Hermitian form $x^* \Sigma(\tau, \eta) x$ is non-positive on $\ker B$,

iv) there exists a number $c_0 > 0$ such that, for every (τ, η), the inequality Re $M \geq c_0(\text{Re }\tau) I_n$ holds in the sense of Hermitian matrices, where

$$M = M(\tau, \eta) := -(\Sigma \mathcal{A})(\tau, \eta) = K(\tau, \eta)(\tau I_n + i A(\eta)).$$

We assume throughout this section that the operator L is constantly hyperbolic, and that the Lopatinskiĭ condition holds true everywhere in the closed hemisphere Re $\tau \geq 0$, $|\tau|^2 + |\eta|^2 = 1$, but at some elliptic points of the boundary Re $\tau = 0$. In particular, $n = 2p$. As explained in the previous section, we ask that the Lopatinskiĭ determinant Δ vanish in a non-degenerate way: its differential does not vanish simultaneously. Note that we do not allow[4] Δ to vanish at some glancing point.

We may anticipate that a dissipative symmetrizer will degenerate somehow at boundary points P_0 where the Lopatinskiĭ condition fails. For we already now that $\Sigma(\tau, \eta)$, restricted to $E_-(\tau, \eta)$, is positive when Re $\tau > 0$. By continuity, the restriction of $\Sigma(P_0)$ to $E_-(P_0)$ must be non-negative. Since its restriction to $\ker B$ is non-positive, we see that $\Sigma(P_0)$ vanishes on $E_-(P_0) \cap \ker B$. This immediately implies that $E_-(P_0) \cap \ker B$ is contained in the kernel of the restrictions of $\Sigma(P_0)$ to both $E_-(P_0)$ and $\ker B$. In other words, it holds that

$$w^* \Sigma(P_0) v = 0, \quad \forall v \in E_-(P_0) \cap \ker B, \forall w \in E_-(P_0) + \ker B. \tag{7.2.21}$$

Actually, noticing that $x_d \mapsto v^* \Sigma v$ is non-increasing along the flow of $\mathcal{A}$ (because of Re $(\Sigma \mathcal{A}) \leq 0_n$), we deduce the more remarkable constraint that $v^* \Sigma(P_0) v$ vanishes identically over the Krylov space of $E_-(P_0) \cap \ker B$ for $\mathcal{A}(P_0)$. Recall

[4] The case $(n, p) = (2, 1)$ with a constantly hyperbolic operator is isomorphic to the system (1.2.17), for which the analysis is explicit and similar to that of Section 7.1.1.

that the Krylov space is the smallest invariant subspace for $\mathcal{A}(P_0)$, containing $E_-(P_0) \cap \ker B$. We denote it by $\mathrm{Kry}[E_-(P_0) \cap \ker B\,;\,\mathcal{A}(P_0)]$. As above, we infer the necessary condition that

$$w^*\Sigma(P_0)v = 0, \quad \forall v \in \mathrm{Kry}[E_-(P_0) \cap \ker B\,;\,\mathcal{A}(P_0)], \forall w \in E_-(P_0). \qquad (7.2.22)$$

Notice that this orthogonality property does not hold for $w \in \ker B$ in general, because $\mathrm{Kry}[E_-(P_0) \cap \ker B\,;\,\mathcal{A}(P_0)]$ is not included in $\ker B$.

For some reason that we do not fully understand, we shall be able to construct a weakly dissipative symmetrizer when

$$\mathrm{Kry}[E_-(P_0) \cap \ker B\,;\,\mathcal{A}(P_0)] = E_-(P_0), \qquad (7.2.23)$$

and only in this case. Notice that this is a generic situation. It is an open question whether the homogeneous BVP remains well-posed when (7.2.23) happens to fail.

Since the construction of dissipative symmetrizers is a local task (Steps 1 and 2 of the proof of Theorem 5.1), and since in that proof we performed this local construction near every point where the Lopatinskiĭ condition holds true, we need only construct K in a neighbourhood of an elliptic point $P_0 = (i\rho_0, \eta_0)$ of the boundary, assuming that $\Delta(P_0) = 0$. Hence we are looking for a smooth function $(\tau, \eta) \mapsto \Sigma(\tau, \eta)$, defined in a neighbourhood of P_0 with values in Hermitian matrices, such that the restriction of Σ to $\ker B$ is non-positive and $\mathrm{Re}\,(\Sigma\mathcal{A}) \le -(\mathrm{Re}\,\tau)I_n$. Notice that since P_0 is a non-glancing point, we may extend $E_\pm(\tau, \eta)$ analytically in a neighbourhood of P_0. Since P_0 is elliptic, $E_-(\tau, \eta)$ equals the stable subspace of $\mathcal{A}(\tau, \eta)$ for every (τ, η) in a neighbourhood of P_0. This is in sharp contrast with the case of hyperbolic boundary points, discussed in Chapter 8. Note that Δ extends analytically too, and that it vanishes if and only if $E_-(\tau, \eta)$ intersects $\ker B$ non-trivially.

Choice of co-ordinates Since P_0 is an elliptic point, $\mathcal{A}$ is smoothly diagonalizable in some neighbourhood $\mathcal{P}$ of P_0:

$$\mathcal{A}(P) = Q(P)^{-1}a(P)Q(P), \quad a = \mathrm{diag}(\mu_1(P), \dots, \mu_n(P)),$$

with Q and μ_j analytic in P. The ordering of the μ_j is such that $\mu_1, \dots, \mu_p$ have negative real parts while $\mu_{p+1}, \dots, \mu_n$ (recall that $n = 2p$) have positive real parts. We denote

$$a_-(P) = \mathrm{diag}(\mu_1, \dots, \mu_p), \quad a_+(P) = \mathrm{diag}(\mu_{p+1}, \dots, \mu_n).$$

Finally, noticing that $\mathcal{A} = i\mathcal{B}$, where $\mathcal{B} \in \mathbf{M}_n(\mathbb{R})$ whenever $\mathrm{Re}\,\tau = 0$, we have $a_+^* = -a_-$ along the boundary, and therefore

$$a_+(P)^* + a_-(P) = \mathcal{O}(\mathrm{Re}\,\tau). \qquad (7.2.24)$$

Thanks to this change of co-ordinates, we look for a Hermitian matrix $\sigma(P)$, which serves to define $\Sigma(P)$ through $\Sigma := Q^T\sigma Q$. Denoting $b(P) := BQ^{-1}$, it will have to satisfy

i) $\sigma|_{\ker b} \leq 0$,
ii) Re $(\sigma a) \leq -(\text{Re }\tau)I_n$ for Re $\tau \geq 0$.

The stable subspace $e_-(P)$ of $a(P)$ is constant, equal to $\mathbb{C}^p \times \{0\}$. Thus the Lopatinskiĭ condition holds whenever

$$\left(b(P)\begin{pmatrix} v_- \\ 0 \end{pmatrix} = 0\right) \Longrightarrow (v_- = 0).$$

Therefore we may define the Lopatinskiĭ determinant by $\Delta(P) = \det b_-$, where b_- is the $p \times p$ block at left: $b = (b_-, b_+)$. We recall that $b(P) \in \mathbf{M}_{p\times 2p}(\mathbb{C})$.

We assume that

$$\dim E_-(P_0) \cap \ker B = 1 \tag{7.2.25}$$

and that

$$\partial\Delta/\partial\tau(P_0) \neq 0. \tag{7.2.26}$$

Then the set $\{\Delta = 0\}$ is locally a submanifold with an equation of the form $\tau = i\rho_0 + f(\eta - \eta_0)$, where f is analytic. Assuming that the IBVP satisfies the Lopatinskiĭ condition at interior points (obviously a necessary restriction), we have Re $f \leq 0$. Up to a renormalization of eigenvectors of $\mathcal{A}$, we may assume that actually

$$\Delta = \tau - i\rho_0 - f(\eta - \eta_0).$$

In particular, we have

$$\text{Re }\Delta \geq \text{Re }\tau. \tag{7.2.27}$$

Because of the Lopatinskiĭ condition, we know that for $\Delta(P) \neq 0$, $\ker b(P)$ has an equation of the form

$$v_- = k(P)v_+,$$

where $P \mapsto k(P) \in \mathbf{M}_p(\mathbb{C})$ is analytic, with a singularity along $\Delta = 0$ that we describe below.

Because of (7.2.25), $\dim\ker b_-(P_0) = 1$ holds and therefore $\operatorname{rk} b_-(P_0) = p - 1$. This implies that the cofactor matrix $\widehat{b_-(P_0)}$ does not vanish. Since $\widehat{b_-(P_0)}^T b_-(P_0) = 0_p$, $\widehat{b_-(P_0)}$ has a kernel of dimension $p-1$ at least. Being nonzero, $\widehat{b_-(P_0)}$ must be of rank one. Let us write $\widehat{b_-(P_0)} = \delta_0\beta_0^T$, where $\beta_0, \delta_0 \in \mathbb{C}^p$. Since $b(P_0)$ has rank p, and since $\delta_0^T b_-(P_0) = 0$, we see that $\delta_0^T b_+(P_0) \neq 0$, from which it follows that $\widehat{b_-(P_0)}^T b_+(P_0)$ is of rank one exactly. The above analysis is true at every vanishing point of Δ. Therefore, using (7.2.26), we may find analytical maps $P \mapsto \beta(P), \delta(P), h(P)$ such that

$$\widehat{b_-(P)}^T b_+(P) = \beta(P)\delta(P)^T - \Delta(P)h(P).$$

Noting now that k is given by $-b_-^{-1}b_+$, we deduce the formula

$$k(P) = -\frac{1}{\Delta}\beta\delta^T + h, \tag{7.2.28}$$

where β, δ are non-vanishing analytic vectors and h is an analytic matrix. Observe that $\begin{pmatrix} \beta_0 \\ 0 \end{pmatrix}$ spans $e_- \cap \ker b(P_0)$. Hence (7.2.23) amounts to saying that none of the co-ordinates of β_0 vanish.

We look for a symmetrizer in block form

$$\sigma(P) = \begin{pmatrix} (\text{Re } \tau)I_p & m^* \\ m & -\kappa I_p \end{pmatrix}, \tag{7.2.29}$$

where the constant $\kappa > 0$ and $m(P) \in \mathbf{M}_p(\mathbb{C})$ are to be chosen below, the latter in a smooth way. We begin to fulfilling point *i)*. When Re $\tau \geq 0$, every vector in $\ker b(P)$ has the form $(k(P)w, w)$ where $w \in \mathbb{C}^p$ is an arbitrary vector. Hence the non-positivity of σ over $\ker b$ is equivalent to $\tilde{k} \leq 0_p$ where

$$\tilde{k} := \begin{pmatrix} k^* & I_p \end{pmatrix} \sigma \begin{pmatrix} k \\ I_p \end{pmatrix} = (\text{Re } \tau)k^*k + 2\text{Re } (mk) - \kappa I_p.$$

We extract from $\tilde{k}$ a singular part k_s, so that $\tilde{k} = k_s + k_r$, where k_r is locally bounded upon Re $\tau > 0$. Denoting $d := \bar{\delta}$, we have

$$k_s(P) = \frac{\text{Re } \tau}{|\Delta|^2}|\beta|^2 dd^* - 2\text{Re } \left(\frac{1}{\Delta}m\beta d^*\right),$$

$$k_r(P) = (\text{Re } \tau)\text{Re } \left(\frac{1}{\Delta}h^*\beta d^*\right) + (\text{Re } \tau)h^*h - 2\text{Re } (mh) - \kappa I_p.$$

We stress the fact that $|\text{Re } \tau/\Delta|$ is bounded by 1 for Re $\tau > 0$ because of (7.2.27), which justifies the presence of the first term in k_r.

Since we shall need to do so in order to fulfill point *ii)*, we limit ourselves to *diagonal* matrices m. To begin with, because of assumption (7.2.23), namely $\beta_j(P_0) \neq 0$ for $j = 1, \dots, p$, we may choose locally a diagonal $m(P)$ in such a way that $m\beta = d$; just take

$$m = \text{diag}(d_1/\beta_1, \dots, d_p/\beta_p). \tag{7.2.30}$$

Then

$$|\Delta|^2 k_s = (\text{Re } \tau - 2\text{Re } \Delta)dd^*$$

is non-positive because of (7.2.27). Therefore $\tilde{k} \leq k_r$. Then the three first terms in k_r are completely determined and bounded in a neighbourhood $\mathcal{V}$ of P_0 under the constraint Re $\tau > 0$. Therefore, taking $\kappa > 0$ large enough, we see that k_r, and consequently $\tilde{k}$, is non-negative within $\mathcal{V}_+ := \mathcal{V} \cap \{\text{Re } \tau > 0\}$. In conclusion, under the choice (7.2.30) (essentially mandatory), the restriction of σ to $\ker b$ is non-positive for every $P \in \mathcal{V}_+$ provided κ is a large enough positive constant.

Let us turn now to point *ii)*. We have

$$\text{Re }(\sigma a) = \begin{pmatrix} (\text{Re }\tau)\text{Re }a_- & \frac{1}{2}(a_-^* m^* + m^* a_+) \\ \frac{1}{2}(m a_- + a_+^* m) & -\kappa \text{Re }a_+ \end{pmatrix}.$$

Since $a_\pm$ are diagonal, the subscript indicating the sign of the real parts of the diagonal entries, we have $\pm\text{Re }a_\pm > 0_p$. Therefore, $\text{Re }(\sigma a) \leq -c_0(\text{Re }\tau)I_n$ will hold true locally with $c_0 > 0$, provided $\kappa > 0$ and the following property holds true[5]:

$$m a_- + a_+^* m = \mathcal{O}(\text{Re }\tau). \tag{7.2.31}$$

Since m and $a_\pm$ are smooth, this amounts to saying that $m a_- + a_+^* m$ vanishes along the boundary $\{\text{Re }\tau = 0\}$. Recalling that in this instance, a_+^* equals $-a_-$, we deduce the following equivalent form to (7.2.31):

$$[m, a_-] = 0, \quad \text{whenever Re }\tau = 0. \tag{7.2.32}$$

In the (generic) case where a_- has simple eigenvalues, this just tells us that m is diagonal for $\text{Re }\tau = 0$.

Summarizing the above analysis, we conclude that if (7.2.23) holds true, then there exists a weakly dissipative symmetrizer σ in a domain $\mathcal{V} \cap \{\text{Re }\tau > 0\}$ where $\mathcal{V}$ is a neighbourhood of P_0, σ being given by (7.2.29) and (7.2.30) with $\kappa > 0$ large enough. This allows us to state

Theorem 7.1 *Let $L = \partial_t + \sum_\alpha A^\alpha \partial_\alpha$ be a constantly hyperbolic operator, such that $A^d \in \mathbf{GL}_n(\mathbb{R})$. Assume that $n = 2p$ where p is the number of characteristics incoming in $\Omega = \{x_d > 0\}$ (so that there may exist elliptic frequency boundary points).*

Let $B \in \mathbf{M}_{p\times n}(\mathbb{R})$ have rank p, such that the Kreiss–Lopatinskiĭ condition is fulfilled at every point $(\tau, \eta) \neq (0,0)$ with $\text{Re }\tau \geq 0$*, except possibly at some* elliptic *frequency boundary points.*

Assume finally that, at boundary points P where it vanishes, the Lopatinskiĭ determinant Δ satisfies (7.2.27)*, and that $E_-(P) \cap \ker b$ is of dimension* 1*, its Krylov space under $\mathcal{A}(P)$ being equal to $E_-(P)$.*

Then there exists a smooth dissipative symmetrizer $\Sigma(\tau, \eta)$ for the homogeneous *IBVP*

$$Lu = f \ \textit{in}\ \Omega \times (0,T), \quad Bu = 0 \ \textit{on}\ \partial\Omega \times (0,T).$$

Note that the smoothness of the construction and the robustness of the assumptions allow us to treat operators L and B with variable coefficients, as well as homogeneous IBVPs in general domains with smooth boundaries.

Corollary 7.1 *Let Ω be either a half-space, or a bounded open domain of $\mathbb{R}^d$ with a smooth boundary. Let L be a constantly hyperbolic operator with smooth*

[5] Rigorously speaking, we only need a bound by $|\text{Re }\tau|^{1/2}$. But since everything is smooth, this amounts to having a bound by $\text{Re }\tau$.

coefficients, $\partial\Omega$ being non-characteristic. Let $B(x) \in \mathbf{M}_{p\times n}(\mathbb{R})$ be a boundary matrix of rank p everywhere on $\partial\Omega$. If Ω is unbounded, we assume that L and B have constant coefficients outside of some ball.

Assume that the Kreiss–Lopatinskiĭ condition is fulfilled at every point $(\tau, \eta; x)$ ($x \in \partial\Omega$, $\eta \in T_x\partial\Omega$, $\mathrm{Re}\ \tau \geq 0$ and $(\tau, \eta) \neq (0, 0)$), except possibly at some elliptic *frequency boundary points.*

Assume at last that at these vanishing points, the Lopatinskiĭ determinant vanishes at first order only (say (7.2.27)), that the (non-trivial) intersection of ker$B$ *with $E_-(\tau, \eta; x)$ is a line, whose Krylov subspace under $\mathcal{A}(\tau, \eta; x)$ is $E_-(\tau, \eta; x)$ itself.*

Then the homogeneous *BVP*

$$Lu = f \ \text{in}\ \Omega \times (0, T), \quad Bu = 0 \ \text{on}\ \partial\Omega \times (0, T), \quad u(\cdot, 0) = 0 \ \text{in}\ \Omega$$

is well-posed in L^2, in the sense that for every $f \in L^2(\Omega \times (0, T))$, there exists a unique solution $u \in L^2(\Omega \times (0, T)$, which satisfies, furthermore

$$\int_0^T \mathrm{e}^{-2\eta t} \|u(t)\|_{L^2}^2 \mathrm{d}t \leq \frac{C_T}{\eta^2} \int_0^T \mathrm{e}^{-2\eta t} \|f(t)\|_{L^2}^2 \mathrm{d}t$$

for every $\eta > 0$.

7.3 Surface waves of finite energy

When the Lopatinskiĭ condition fails at an elliptic point $(i\rho, \eta)$ of the boundary, we know that there exists a non-trivial solution of the homogeneous BVP, of the form

$$U(x, t) = \mathrm{e}^{i(\rho t + \eta\cdot y)} V(x_d),$$

where $V \in L^2(0, +\infty)$. This solution displays important properties. On the one hand, it is a travelling wave in the direction η parallel to the boundary, with velocity $\sigma = -\rho/|\eta|$. Contrary to waves associated with a zero (τ, η) of the Lopatinskiĭ determinant Δ with $\mathrm{Re}\ \tau > 0$, this wave is not responsible for a Hadamard instability. On the other hand, its energy density per unit surface area, namely

$$\int_0^\infty |V(x_d)|^2 \, \mathrm{d}x_d,$$

is finite, and thus such waves can be used directly to construct exact solutions of the homogeneous BVP. This makes a contrast with the case studied in the next chapter, when Δ vanishes on the boundary at non-elliptic points.

Because of these two properties, such special solutions are called (elementary) *surface waves*. By extension, we also call any linear combination of elementary ones a surface wave. At first glance, one may think that since the vanishing of Δ at some boundary point but not at interior points ($\mathrm{Re}\ \tau > 0$) is highly non-generic, the elementary surface waves are likely to form a single line, and there is no non-trivial linear combination. We may object to this observation with three facts.

- To begin with, the scale invariance of the BVP implies that $U^\lambda(x,t) := U(\lambda x, \lambda t)$ is again a solution. We can therefore construct (composite) surface waves by the formula

  $$u(x,t) := \int_0^\infty U^\lambda(x,t)\,\mathrm{d}\mu(\lambda),$$

  where μ is a finite measure. If $\mathrm{d}\mu = \phi(\lambda)\mathrm{d}\lambda$ with $\phi \in \mathscr{D}(0,+\infty)$, such a solution belongs to the Schwartz class in the η direction. Thus we can *localize* the wave in the direction of propagation. However, it remains a travelling wave in that direction.
- Next, many physically relevant systems are rotationally invariant, in the sense that $d = 3$ and the orthogonal group $\mathbf{O}_3$ (or perhaps only the subgroup $\mathbf{SO}_3$ of rotation) acts on both the independent variable x (in a natural way), and on the dependent one u, leaving the PDEs unchanged. Of course, the half-plane Ω is left invariant only by the subgroup $\mathbf{SO}_2$, and invariance means that the action of $\mathbf{SO}_2$ preserves the homogeneous boundary condition. In such a case, $\mathbf{SO}_2$ also acts on the set of elementary surface waves, which is thus rather large. Let us denote by $U^{R,\lambda}$ the image of U^λ by the action of the rotation R. This wave propagates in the direction $R\eta$. If $(R,\lambda) \mapsto \phi(R,\lambda)$ is a test function, we can define a surface wave by the formula

  $$u^\phi(x,t) := \int_0^\infty \int_{\mathbf{SO}_2} U^{R,\lambda}(x,t)\phi(R,\lambda)\,\mathrm{d}R\,\mathrm{d}\lambda. \tag{7.3.33}$$

  Such a solution is now localized in every spatial direction, in the sense that for each time t, it is square-integrable over the physical space Ω. We warn the reader that this construction does not furnish all the finite energy solutions of the homogeneous BVP, however. Thus it cannot be used to solve the IBVP in a closed form.
- Finally, it has been shown in [189] that for second-order BVPs that come from a Lagrangian through the Euler–Lagrange equations, there exists at least one elementary surface wave in every direction. This situation contrasts with the general case, where the vanishing of Δ at some elliptic point but at no interior point does not persist under small variations of L and/or B. Together with the scale invariance, this yields a set of elementary waves U^η parametrized by non-zero vectors $\eta \in \mathbb{R}^2$ (more generally by $\eta \in \mathbb{R}^{d-1}$). Thus a construction similar to (7.3.33) works, where the integral is taken over $\mathbb{R}^2$.

A remarkable fact is that the decay of surface waves is slower than the decay we observe in the Cauchy problem: The dispersion of the energy is affected by the boundary condition and by the possibility for waves to travel along the boundary.

Assume, for instance, that L is symmetrizable, and that the boundary condition is conservative for the associated quadratic energy. Thus the total energy is conserved. If we just have a Cauchy problem in $\mathbb{R}^d \times (0, +\infty)$, the energy is expected to propagate in the direction of the characteristic cone of L (a kind of Huyghens principle, see, for instance, [184]). After a large time t, and if the initial data is compactly supported, the solution concentrates in a corona $C_t := B + t\Lambda$, where Λ denotes the group velocities in every direction of $\mathbb{R}^d$, a $(d-1)$-dimensional manifold. Since the area of C_t is of order t^{d-1}, the energy density typically decays like t^{1-d}, meaning that the amplitude of the solution decays as $t^{(1-d)/2}$. If we have a conservative BVP instead, with surface waves, then the total energy asymptotically splits into two parts. One part is associated with the bulk waves, which are waves propagating away from the boundary. These waves obey more or less the same description as in the Cauchy problem. The other part of the energy is carried by surface waves and thus remains localized in a strip along the boundary. Since we anticipate the same kind of dispersion as above, though along the boundary instead of in Ω, we expect that the surface energy concentrates in a domain $B + t\Lambda^S$, where Λ^S denotes the group velocities of surface waves in every direction of the boundary, a $(d-2)$-dimensional manifold. Therefore the amplitude of surface waves typically decays as $t^{1-d/2}$, instead of $t^{(1-d)/2}$. Thus the decay of surface waves is weaker than that of bulk waves: the former are roughly $\sqrt{t}$ times larger than the latter for large time. We point out that this effect can be reinforced by inhomogeneities of the boundary. For instance, if $d = 3$ and $\partial\Omega = G \times \mathbb{R}$, where G is a non-flat curve, one frequently observes *guided waves* in the direction of x_2. A wave guided along an m-dimensional subspace of the boundary typically decays as $t^{(1-m)/2}$. For instance, a guided wave in our three-dimensional space (thus $m = (d-1) - 1 = 1$) does not decay at all!

Example: Rayleigh waves in elastodynamics The best known example of surface waves arises in elastodynamics, where they are called *Rayleigh waves*. They are responsible for the damage in earthquakes: the Earth is a half-space at a local scale, and can be considered as an elastic medium. The vibrations of the Earth obey exactly the Euler–Lagrange equation of the Lagrangian equal to the difference of the kinetic and bulk energy. In particular, the boundary condition is zero normal stress. As explained above, a significant part of the elastic energy is concentrated along the surface. This energy is formed of kinetic and deformation energies, the latter being observable once the earthquake has gone away. When an earthquake happens in a mountain range, a guided wave can form, which almost does not decay, reinforcing the damages in a narrow strip. This phenomenon was evoked à propos of Kobé's earthquake in 1995.

It is worth noting that the Rayleigh waves travel much slower than bulk waves. The latter split into two families: compression waves, also called P-*waves*, for which the medium vibrates in the direction of propagation, and shear waves, called S-*waves* with perpendicular vibration. When the bulk energy is convex and coercive over $H^1(\Omega)$ (a more demanding property than coercivity over $H^1(\mathbb{R}^3)$), their velocities c_{P} and c_{S} satisfy $0 < c_{\mathrm{S}} < c_{\mathrm{P}}$; see [189]. The normal stress of these waves does not vanish at the boundary and thus they can propagate only away from the surface. It is a linear combination of P- and S-waves that satisfy the boundary condition, and thus forms the Rayleigh waves. Amazingly, this combination has the effect of lowering the wave velocity. The square of the Rayleigh velocity c_{R} is the unique positive solution of the quartic equation (see [185], Section 14.2 for details)

$$\left(\frac{X}{2c_{\mathrm{S}}^2} - 1\right)^4 = \left(\frac{X}{c_{\mathrm{S}}^2} - 1\right)\left(\frac{X}{c_{\mathrm{P}}^2} - 1\right). \tag{7.3.34}$$

Strangely enough, Equation (7.3.34) has *two* positive solutions if $c_{\mathrm{P}} < c_{\mathrm{S}}$. However, this fact is meaningless because then we can establish that the IBVP is Hadamard ill-posed: the Lopatinskiĭ condition fails at some interior point (see [189]).

Other examples The simplest Lagrangian to which the results of [189] apply is

$$u \mapsto \int\!\!\int (u_t^2 - c^2|\nabla_x u|^2)\,\mathrm{d}x\,\mathrm{d}t$$

with a scalar field $u(x,t)$. The corresponding IBVP is the wave equation with the Neumann boundary condition. This is a borderline case, where the surface wave has infinite energy. It corresponds to the vanishing of the Lopatinskiĭ determinant at the boundary of the elliptic region of boundary points. This is the exceptional situation with $n = 2$ and $p = 1$ that we discussed above.

A more appealing example arises in the modelling of liquid–vapour phase transitions, which can be roughly described as follows. We consider a fluid governed by the isothermal Euler equations, which express the conservation of mass and momentum. (See Chapter 13 for the complete system.) We assume an equation of state $p = P(\rho)$ (ρ the density, p the pressure), where P is van der Waals-like, in particular it is not monotone. Each interval where P is increasing corresponds to a phase, typically either vapour or liquid. A phase transition may be viewed as a sharp discontinuity between a liquid state and a vapour state. Physically interesting phase transitions are subsonic, and therefore correspond to undercompressive discontinuities (see Chapter 12 for this notion). In this case, Rankine–Hugoniot conditions are not sufficient to select physically relevant patterns: an additional algebraic relation is needed, which is called a kinetic relation in the mathematical theory of undercompressive shocks (see [113]). One way to derive this relation for phase boundaries was introduced by Slemrod [196] and Truskinovsky [213], who pointed out that the apparent jump should in fact correspond to a microscopical internal structure called a viscous-capillary profile. (See also the survey paper [54].) In [9], Benzoni-Gavage considered the zero-viscosity limit of the viscosity-capillarity criterion of Slemrod and Truskinovsky, and found surface waves of finite energy at the linearized level (see Chapter 12 for the stability analysis of free boundary problems). In the light of the more recent work [189], that is not surprising. Indeed, the capillarity criterion gives as a kinetic relation the dynamical analogue of the equality of chemical potentials between phases at equilibrium; alternatively, it means that the total energy (= kinetic energy T plus free energy F) is conserved across the discontinuity. In particular, the corresponding free boundary problem is time reversible and can be viewed as the Euler–Lagrange system of the Lagrangian

$$\int (T - F)\,\mathrm{d}x\,\mathrm{d}t.$$

8

A CLASSIFICATION OF LINEAR IBVPs

We continue the analysis of the linear IBVP with constant coefficients, in the half-space

$$\Omega = \{x \in \mathbb{R}^d\,;\, x_d > 0\}.$$

Assume for the sake of simplicity that the boundary $\{x_d = 0\}$ is non-characteristic:

$$\det A^d \neq 0.$$

We are interested in this chapter in classifying IBVPs. Because practical applications involve pairs (L, B) with variable coefficients, we are interested in notions that are stable upon a small variation of either L, B or both. Thus our parameter space $\mathcal{IB}$ will be that of pairs (L, B) where L is hyperbolic and the IBVP is normal, namely

$$E_-(1, 0) \cap \ker B = \{0\}.$$

The structure of $\mathcal{IB}$ may be rough at some points, where the operator is not strictly or constantly hyperbolic. We note, however, that a small perturbation preserves strict hyperbolicity, because the unit sphere is compact. Since normality is an open condition, we deduce that normal pairs (L, B) with L strictly hyperbolic are interior points of $\mathcal{IB}$. We denote the corresponding class $\mathcal{IB}_S$. Likewise, the set of pairs (L, B), where L is a symmetric operator, is a linear space in which the set $\mathcal{IB}_F$ (F for 'Friedrichs symmetric') of normal pairs is a dense open subset.

We say that a property (**P**) is 'robust' if it defines an open subset in either $\mathcal{IB}_S$ or $\mathcal{IB}_F$, and then we specify the context. Our goal is to identify robust classes, in terms of the estimates available for the corresponding IBVPs. We essentially follow [13, 186]. The reader interested in the transitions between robust classes should go to [13].

Because the description of $E_-(\tau, \eta)$ is not completely understood along the boundary $\{\text{Re}\,\tau = 0\,,\, \eta \in \mathbb{R}^{d-1}\}$ for general hyperbolic operators, we may wish sometimes to restrict our study within either the class $\mathcal{IB}_C$ (L is constantly hyperbolic) or $\mathcal{IB}_F$ (L is Friedrichs symmetrizable).

8.1 Some obvious robust classes

- The first robust class that we encountered in this book is, within $\mathcal{IB}_F$, that of systems with a strictly dissipative boundary condition. See Section 3.2.
- Within the class $\mathcal{IB}_C$, we may think of systems that satisfy the (UKL) condition. As a matter of fact, the Lopatinskiĭ determinant Δ is a continuous function over the hemisphere

$$|\tau|^2 + |\eta|^2 = 1, \quad \text{Re}\,\tau \geq 0, \quad \eta \in \mathbb{R}^{d-1}.$$

It is not hard to extend Lemma 4.5 so that if L and B depend on parameters ϵ, then Δ can be defined continuously in terms of (τ, η, ϵ). Assume now that (UKL) holds true for $\epsilon = 0$. This means that the continuous function $|\Delta(\cdot, \cdot, 0)|$ is bounded below by a positive number. Since the half-sphere is compact, this remains true for small values of ϵ. Hence, nearby IBVPs of which the operator is constantly hyperbolic satisfy (UKL).
- Finally, the set of strongly unstable IBVPs is also a robust class. Let (L_ϵ, B_ϵ) be a continuous family in $\mathcal{IB}$, with (L_0, B_0) strongly ill-posed. This means that the Lopatinskiĭ determinant Δ_0 of (L_0, B_0) vanishes at some point (τ_0, η_0) with $\text{Re}\,\tau_0 > 0$. In a neighbourhood $\mathcal{V}$ of (τ_0, η_0), E_- depends continuously on L. Therefore, we may define in $\mathcal{V} \times (-\alpha, \alpha)$ a Lopatinskiĭ determinant $\Delta(\cdot, \cdot, \epsilon)$ that is continuous in ϵ, analytic in η and holomorphic in τ. We now fix $\eta = \eta_0$ and let vary ϵ. For $\epsilon = 0$, the holomorphic function $f_0(\tau) = \Delta(\tau, \eta_0, 0)$ vanishes at τ_0. Note that $\eta_0 \neq 0$ since the IBVP is normal.

Lemma 8.1 *f_0 is a non-trivial holomorphic function.*

Proof. The analytic function $F := \Delta(\cdot, \cdot, 0)$ is positively homogeneous. If f_0 vanished identically, then $F(\tau, \eta)$ would vanish provided $\text{Re}\,\tau > 0$ and $\eta \in \mathbb{R}^{+*}\eta_0$. By continuity, it would also vanish for $\eta = 0$. But this contradicts the normality of the IBVP. □

It follows that τ_0 is an isolated zero of f_0. Now, Rouché's Theorem tells us that holomorphic functions nearby f_0 do have roots near τ_0. In particular, for small enough values of ϵ, there exist a root $\tau(\epsilon)$ of $\Delta(\tau, \eta_0, \epsilon) = 0$, with $\tau(0) = \tau_0$ and hence $\text{Re}\,\tau(\epsilon) > 0$. Therefore, IBVPs with (L, B) nearby (L_0, B_0) are strongly unstable.

8.2 Frequency boundary points

In this section, we study in more detail the structure of the 'stable' subspace $E_-(\tau, \eta)$ when $\tau = i\rho$ is pure imaginary. As we shall see below, this structure depends highly on the region of the boundary set $\{\text{Re}\,\tau = 0\,, \eta \in \mathbb{R}^{d-1}\}$. We recall that in general, although $E_-(i\rho_0, \eta_0)$ is invariant under $\mathcal{A}(i\rho_0, \eta_0)$, it does not equal the stable subspace of this matrix. By definition it is only the limit of the stable subspace $E_-(\tau,\eta)$ when $(\tau,\eta)\to(i\rho_0,\eta_0)$ with $\mathrm{Re}\,\tau>0$. In particular, it contains the stable subspace:

$$E_s\left(\mathcal{A}(i\rho_0,\eta_0)\right)\subset E_-(i\rho_0,\eta_0). \tag{8.2.1}$$

Likewise, we have

$$E_u\left(\mathcal{A}(i\rho_0,\eta_0)\right)\cap E_-(i\rho_0,\eta_0)=\{0\}. \tag{8.2.2}$$

8.2.1 *Hyperbolic boundary points*

We say that a subspace of $\mathbb{C}^n$ is of *real type* if it admits a basis formed of vectors in $\mathbb{R}^n$, or equivalently if it is the complexification of some subspace of $\mathbb{R}^n$.

We recall that for a hyperbolic operator $L=\partial_t+A(\nabla_x)$, the characteristic cone char(L) is the set defined by the equation

$$(\rho,\xi)\in\mathbb{R}\times\mathbb{R}^n,\quad \det(\rho I_n+A(\xi))=0.$$

In the complement of char(L), the connected component of $(1,0)$ has been denoted Γ. It is a convex open cone, whose elements are the time-like vectors. See Section 1.4 for details.

Theorem 8.1 *Let $(\rho,\xi)\in\Gamma$ be given, with $\xi=:(\eta,\xi_d)$. Then the limit $E_-(i\rho,\eta)$ of $E_-(\tau,\eta')$ as $(\tau,\eta')\to(i\rho,\eta)$ with* $\mathrm{Re}\,\tau>0$, *exists and is given by the formula*

$$E_-(i\rho,\eta)=E_u\left((A^d)^{-1}(\rho I_n+A(\xi))\right). \tag{8.2.3}$$

In particular, $E_-(i\rho,\eta)$ is of real type.

This result is by no means obvious because of:

Lemma 8.2 *Under the assumption of Theorem 8.1, the matrix $\mathcal{A}(i\rho,\eta)$ is diagonalizable with pure imaginary eigenvalues.*

In particular, $E_-(i\rho,\eta)$ is *not* the stable space of $\mathcal{A}(i\rho,\eta)$, the latter reducing to $\{0\}$. We begin with the proof of the lemma.

Proof It suffices to prove that $\mathcal{B}(\rho,\eta):=(\rho I_n+A(\eta))(A^d)^{-1}$ is $\mathbb{R}$-diagonalizable. This is obvious if $\rho=0$ and $\xi\parallel\vec{e}_d$.

Otherwise, we apply Theorem 1.5 to the pair (ρ,ξ): the matrix

$$(\rho I_n+A(\xi))^{-1}\left(\sum_{\alpha,\beta}m_\beta r_{\alpha\beta}A^\alpha+(V\cdot m)I_n\right)$$

is diagonalizable over the reals whenever

$$M(R,V):=\begin{pmatrix}R & \xi\\ {}^tV & \rho\end{pmatrix},\qquad R\in\mathbf{M}_d(\mathbb{R}),\ V\in\mathbb{R}^d$$

is non-singular and $m \in \mathbb{R}^d$. We observe that, since (ρ, ξ) is not parallel to $(0, \vec{e}_d)$, we can choose a pair $(R, V) \in \mathbf{GL}_d(\mathbb{R}) \times \mathbb{R}^d$, such that

$$V^T R^{-1} \vec{e}_d = 0 \quad \text{and} \quad V^T R^{-1} \xi \neq \rho.$$

The inequality tells us that $M(R, V)$ is non-singular, because of Schur's complement formula

$$\det M(R, V) = (\det R)(\rho - V^T R^{-1} \xi).$$

Defining $m := R^{-1} \vec{e}_d$, we have $V \cdot m = 0$, and therefore

$$M(R, V) \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} \vec{e}_d \\ 0 \end{pmatrix}.$$

With this choice, the matrix $\sum_{\alpha,\beta} m_\beta r_{\alpha\beta} A^\alpha + (V \cdot m) I_n$ equals A^d, and Theorem 1.5 tells us that $(\rho I_n + A(\xi))^{-1} A^d$ is diagonalizable over the reals. Shifting by $\xi_d I_n$ and conjugating by A^d, the same is true for $\mathcal{B}(\rho, \eta)$. □

We prove now Theorem 8.1.

Proof Let δ be the segment joining the point $(1, 0)$ to (ρ, ξ) in $\mathbb{R} \times \mathbb{R}^n$. From Proposition 1.6, every point of δ is in Γ. Therefore, Lemma 8.2 applies along δ. Parametrizing δ by $s \in [0, 1]$, we may decompose

$$\mathbb{C}^n = E_s \left((A^d)^{-1} (\rho_s I_n + A(\xi_s)) \right) \oplus E_u \left((A^d)^{-1} (\rho_s I_n + A(\xi_s)) \right),$$

since the matrix under consideration has non-zero real eigenvalues. Hence the stable and unstable subspaces extend analytically as invariant subspaces to a small neighbourhood $\mathcal{V}$ of δ. The above splitting still holds in $\mathcal{V}$. In particular, we have

$$\mathbb{C}^n = E_s \left((A^d)^{-1} (-i\tau I_n + A(\xi)) \right) \oplus E_u \left((A^d)^{-1} (-i\tau I_n + A(\xi)) \right),$$

whenever $\gamma = \text{Re } \tau$ is small and $(\rho = \text{Im } \tau, \xi) \in \delta$.

Since the space $E_u \left((A^d)^{-1} ((-i\gamma + \rho_s) I_n + A(\xi_s)) \right)$ is invariant under $\mathcal{A}(\gamma + i\rho_s, \eta_s)$, we may consider the restriction of the latter matrix to that subspace. When $\gamma > 0$, its spectrum $\sigma(\gamma, \rho', \xi')$ consists of complex numbers whose real part is non-zero. By continuity in the connected set $\{\gamma > 0\} \cap \mathcal{V}$, we deduce that the number of eigenvalues with positive (resp., negative) real part remains constant. Therefore, it can be computed at points close to $(0, 1, 0)$, with $\gamma > 0$; for instance at the point $(\gamma, 1, 0)$ with $\gamma > 0$. We then have $\mathcal{A}(\gamma + i, 0) = -(\gamma + i)(A^d)^{-1}$ and

$$E_u \left((A^d)^{-1} (-i\tau I_n + A(\xi)) \right) = E_u \left((1 - i\gamma)(A^d)^{-1} \right) = E_u((A^d)^{-1}).$$

Therefore $\sigma(\gamma, 1, 0)$ consists only in numbers of negative real part. Thus, $\gamma > 0$ implies that the elements of $\sigma(\gamma, \rho, \xi)$ have negative real parts, whence

$$E_u \left((A^d)^{-1} (-i\tau I_n + A(\xi)) \right) \subset E_-(\tau, \eta).$$

Finally, the equality of dimensions implies the equality of these subspaces. Letting then $\gamma \to 0^+$, we complete the proof of the theorem. □

By symmetry, $E_-(i\rho, \eta)$ is of real type when $(\rho, \xi) \in -\Gamma$. In this case, we have the formula

$$E_-(i\rho, \eta) = E_s\left((A^d)^{-1}(\rho I_n + A(\xi))\right).$$

In particular, we have

$$E_-(-i\rho, -\eta) = E_-(i\rho, \eta), \quad (\rho, \xi) \in \Gamma.$$

This can be seen also as the consequence of the more general formula

$$E_-(\bar{\tau}, -\eta) = E_-(\tau, \eta) \quad \eta \in \mathbb{R}^d, \, \mathrm{Re}\, \tau > 0. \tag{8.2.4}$$

This shows that the formula above stands without any restriction on the frequencies:

$$E_-(-i\rho, -\eta) = E_-(i\rho, \eta), \quad (\rho, \xi) \in \mathbb{R} \times \mathbb{R}^d, \quad (\rho, \xi) \neq (0, 0). \tag{8.2.5}$$

Definition 8.1 *The set of pairs $(\rho, \xi) \in (\mathbb{R} \times \mathbb{R}^d) \setminus \{(0,0)\}$ such that $\mathcal{A}(i\rho, \eta)$ is diagonalizable with a pure imaginary spectrum, is called the* hyperbolic *set of the boundary of the frequency domain. We denote the hyperbolic set by $\mathcal{H}$. It is a cone:*

$$(y \in \mathbb{R}^*, \, X \in \mathcal{H}) \Longrightarrow yX \in \mathcal{H}.$$

There is a slight abuse of words in this definition, as boundary points are of the form $(i\rho, \eta)$, rather than (ρ, η), but this terminology is much easier to handle. Hyperbolic boundary points are those for which the real matrix

$$\mathcal{B}(\rho, \eta) := (A^d)^{-1}(\rho I_n + A(\eta))$$

is $\mathbb{R}$-diagonalizable. Theorem 8.1 tells us that

$$\pi\Gamma \subset \mathcal{H}, \tag{8.2.6}$$

where Γ is the forward cone and $\pi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}^{d-1}$ is the projection where the last entry is deleted:

$$\pi(\rho, \eta, \omega) = (\rho, \eta).$$

We shall justify below the use of the word ‘hyperbolic’ for these boundary frequencies; they are a candidate for the propagation phenomena for weakly well-posed IBVPs.

8.2.2 *On the continuation of $E_-(\tau, \eta)$*

We digress here with a strange result, saying that the uniform continuity of $(\tau, \eta) \mapsto E_-(\tau, \eta)$, proved in the constantly hyperbolic case (Lemma 4.5), fails in general. More importantly, it even fails in the nice class of Friedrichs-symmetric operators!

Theorem 8.2 *Let $n \geq 3$ and $1 \leq p \leq n-1$ be given. Then there exist $d \geq 2$ and a symmetric operator*

$$L = \partial_t + \sum_{\alpha=1}^{d} A^\alpha \partial_\alpha, \quad A^\alpha \in \mathbf{Sym}_n(\mathbb{R}),$$

with A^d of signature $(p, n-p)$ (p positive and $n-p$ negative eigenvalues), such that the map $(\tau, \eta) \mapsto E_-(\tau, \eta)$ does not admit a continuous extension to $\mathrm{Re}\,\tau \geq 0$, $|\tau|^2 + |\eta|^2 = 1$.

Proof We choose

$$d = \frac{n(n+1)}{2} - 1,$$

and $\{A^1, \dots, A^d\}$ a basis of

$$\{S \in \mathbf{Sym}_n(\mathbb{R})\,;\, \mathrm{Tr}\, S = 0\}.$$

Since $\max(p, n-p) \geq 2$, we may specify $\epsilon = \pm 1$ and

$$(A^d)^{-1} = \begin{pmatrix} \epsilon & {}^t0 \\ 0 & J \end{pmatrix},$$

so that J is indefinite.

From Theorem 8.1, we know that $E_-(\tau, \eta)$ has a limit $E_-(i\rho, \eta)$ at points $(\rho, \eta) \in \pi\Gamma$. Our purpose will be to show that the map $(\rho, \eta) \mapsto E_-(i\rho, \eta)$ does not extend continuously to the boundary of $\pi\Gamma$. From Theorem 8.1, this amounts to proving that the map

$$(\rho, \xi) \mapsto E_u\left((A^d)^{-1}(\rho I_n + A(\xi))\right)$$

does not extend continuously to the boundary of Γ.

We proceed *ad absurdum*. So let us assume that $E_u\left((A^d)^{-1}(\rho I_n + A(\xi))\right)$ is continuous up to the boundary of Γ, but the origin of course. Identifying $\mathbb{R} \times \mathbb{R}^d$ to $\mathbf{Sym}_n(\mathbb{R})$ through

$$(\rho, \xi) \mapsto \rho I_n + A(\xi),$$

it holds that

$$\Gamma = \mathbf{SPD}_n.$$

Thus we have supposed that $S \mapsto E_u((A^d)^{-1}S)$ is continuous up to $\partial\Gamma$ but the origin. Taking

$$S = \begin{pmatrix} 1 & {}^t0 \\ 0 & \Sigma \end{pmatrix},$$

this implies that the map $\Sigma \mapsto E_u(J\Sigma)$, which is well-defined on $\mathbf{SPD}_{n-1}$, extends continuously up to the boundary, *including the origin*. Since the closure of $\mathbf{SPD}_{n-1}$ is a closed cone, and since $\Sigma \mapsto E_u(J\Sigma)$ is positively homogeneous of degree zero, this map would have to be constant. Since J is indefinite, this is obviously false. □

8.2.3 *Glancing points*

Roughly speaking, glancing points are boundary points (ρ, η) in a neighbourhood of which $E_-(\tau, \eta)$ does not behave analytically, or at least at which the Implicit Function Theorem does not apply. Since $E_-(\tau, \eta)$ is the sum of generalized eigenspaces of $\mathcal{A}(\tau, \eta)$ associated to stable eigenvalues ($\operatorname{Re} \mu < 0$), this means that $\mathcal{B}(\rho, \eta)$ admits a real eigenvalue $-\omega$, whose multiplicity does not persist locally (crossing eigenvalues). In other words, an acceptable definition of *glancing points* $(\rho, \eta) \in (\mathbb{R} \times \mathbb{R}^{d-1}) \setminus \{(0,0)\}$ is that there exists an $\omega \in \mathbb{R}$ such that

$$P(\rho, \eta, \omega) = 0, \quad \frac{\partial P}{\partial \xi_d}(\rho, \eta, \omega) = 0, \tag{8.2.7}$$

where P is the characteristic polynomial

$$P(X, \xi) := \det(X I_n + A(\xi)).$$

When some irreducible factor occurs twice or more in P (see Proposition 1.7), one should replace P by the product of its distinct irreducible factors, in this definition.

We denote by $\mathcal{G}$ the set of glancing points. Elimination of ω in (8.2.7) yields the result that $\mathcal{G}$ is contained in a real algebraic variety.

From the definition, $\mathcal{G}$ contains the apparent boundary of $\operatorname{char}(L)$ for an observer sitting at infinity in the ξ_d-direction. It also contains the projection of self-intersections of Γ. Self-intersections occur when L is not constantly hyperbolic. It might happen that such a self-intersection projects within $\mathcal{H}$, showing that the construction of dissipative symmetrizers can be a difficult task even at hyperbolic points, in spite of the nice description given in Theorem 8.1.

Let (ρ, η) be a typical point of the apparent boundary of Γ from the ξ_d-direction. By *typical*, we mean that $P(\rho, \xi) = 0$, and the multiplicity m of ρ as a root of $P(\cdot, \xi)$ is strictly less than the multiplicity M of ω as a root of $P(\rho, \eta, \cdot)$. Since $A(\xi)$ is diagonalizable, the kernel of $\mathcal{A}(i\rho, \eta) - i\omega I_n$, which equals that of $\rho I_n + A(\xi)$, has dimension m. Since $m < M$, the eigenvalue $i\omega$ of $\mathcal{A}(i\rho, \eta)$ is not semisimple; this matrix is not diagonalizable.

Recall that the maximal eigenvalue $\lambda_+(\xi)$ of $A(\xi)$ is a convex function that is analytic in the constantly hyperbolic case. We have shown a kind of strict convexity in Proposition 1.6. When this convexity is slightly stronger, say when $\ker \mathrm{D}^2 \lambda_+(\xi) = \mathbb{R}\xi$ for every $\xi \in \mathcal{S}^{d-1}$, then the above analysis applies to the points $(\rho = \lambda_+(-\xi), \xi)$, and we deduce that on the boundary of $\mathcal{H}$, the matrix $\mathcal{A}(i\rho, \eta)$ is not diagonalizable. Therefore, $\mathcal{H}$ is a connected component of the set of pairs $(\rho, \eta) \in (\mathbb{R} \times \mathbb{R}^{d-1}) \setminus \{(0,0)\}$ such that $\mathcal{A}(i\rho, \eta)$ is diagonalizable.

The same argument as above shows that if (ρ, η) is not a glancing point, then every pure imaginary eigenvalue of $\mathcal{A}(i\rho, \eta)$ is semisimple and locally analytic.

In particular, $E_-(\cdot,\cdot)$ admits an analytical extension in a small neighbourhood of non-glancing points.

8.2.4 *The Lopatinskiĭ determinant along the boundary*

The Lopatinskiĭ determinant $\Delta(\tau,\eta)$ may be defined everywhere we have an $E_-(\tau,\eta)$ defined by continuity, and it is analytic whenever $(\tau,\eta)\mapsto E_-(\tau,\eta)$ has this property. In particular, Δ has an analytic extension to non-glancing points.

The noticable property of the restriction of Δ to non-glancing points is that, chosing an analytical basis of $E_-(i\rho,\eta)$ that is made of real vectors when $(\rho,\eta)\in\mathcal{H}\setminus\mathcal{G}$, Δ becomes a *real analytic* function on this open subset, which we denote $D(\rho,\eta)$. In particular, its zero set in $\mathcal{H}\setminus\mathcal{G}$ is an analytic submanifold. Section 4.6.2 shows that

$$D^{-1}(0)\cap(\mathcal{H}\setminus\mathcal{G})$$

is actually contained in a real *algebraic* variety. This will be of great importance below.

Since D is homogeneous on the boundary, it will be appropriate to work on the projective space $\mathbb{P}(\mathbb{R}\times\mathbb{R}^{d-1})$. We shall denote by $\mathbb{H}$ and $\mathbb{G}$ the projectivized objects obtained from $\mathcal{H}$ and $\mathcal{G}$.

8.3 Weakly well-posed IBVPs of real type

Let $(L_0,B_0)\in\mathcal{IB}_C$ be given. In particular, E_- and Δ are continuable up to the boundary $\{\text{Re}\,\tau=0\,;\,\eta\in\mathbb{R}^{d-1}\}$ but at the origin $(0,0)$. Assume that the corresponding IBVP satisfies the Kreiss–Lopatinskiĭ condition, but not uniformly. Specifically, we require that $D^{-1}(0)$ is contained in $\mathbb{H}\setminus\mathbb{G}$. In particular, since $D^{-1}(0)$ is compact and $\mathbb{H}\setminus\mathbb{G}$ is open, $D^{-1}(0)$ does not meet the boundary of $\mathbb{H}\setminus\mathbb{G}$.

Examples If $d=2$, the boundary points form a plane without its origin. The projective object is a line, in which $\mathbb{H}\setminus\mathbb{G}$ is a finite collection of open segments. Our requirement is that the zeros of D belong to $\mathbb{H}\setminus\mathbb{G}$. Since D is analytic, these zeros are isolated.

If $d\geq 3$, the zero set $\beta:=D^{-1}(0)\cap(\mathbb{H}\setminus\mathbb{G})$ is generically a smooth real analytic hypersurface in a real $(d-1)$-projective space. For instance, if $d=3$, it is a finite collection of loops. Note that the presence of such hypersurfaces is compatible with the (non-uniform) Lopatinskiĭ condition: Given a point (ρ_0,η_0) in β, where $\mathrm{d}D\neq 0$, the complex zeros form a complex analytic, locally smooth, manifold. More precisely, assume $(\partial D/\partial\rho)(\rho_0,\eta_0)\neq 0$. Since $\Delta(\tau,\eta)=D(-i\tau,\eta)$, zeros of Δ locally satisfy

$$i\tau+\rho_0\sim\frac{(\eta-\eta_0)\cdot\nabla_\eta D}{\partial D/\partial\rho}.$$

Together with the Implicit Function Theorem, this shows that the set of zeros (τ,η) of Δ, with a real component η, is locally a manifold of real (projective) dimension $d-2$. Thus it coincides with the set of pairs $(i\rho, \eta)$ such that $(\rho, \eta) \in \beta$. In other words, Δ does not vanish for $\mathrm{Re}\,\tau > 0$ when (τ, η) is close to $(i\rho_0, \eta_0)$ and η is real. We point out that the situation is very different when D vanishes at some boundary point outside of $\mathbb{H}$.

In [13], we have denoted $\mathcal{WR}$ the class described above. We summarize its definition below.

Definition 8.2 *We say that the IBVP associated to the pair (L, B) is of class $\mathcal{WR}$ if the following properties are satisfied:*

- *The operator L is constantly hyperbolic,*
- *The boundary is non-characteristic ($\det A^d \neq 0$),*
- *The non-uniform Kreiss–Lopatinskiĭ condition is satisfied (Δ does not vanish for $\mathrm{Re}\,\tau > 0$ and $\eta \in \mathbb{R}^{d-1}$),*
- *The Kreiss–Lopatinskiĭ condition is locally uniform at boundary points out of $\mathbb{H} \setminus \mathbb{G}$,*
- *The real analytic set $D^{-1}(0) \cap (\mathbb{H} \setminus \mathbb{G})$ is non-void, and it holds that*

$$(D = 0) \Longrightarrow \left(\frac{\partial D}{\partial \rho} \neq 0\right). \tag{8.3.8}$$

In particular, this analytic set is smooth.

Because $E_-(\tau, \eta)$ is always continuable at points of $\pi\Gamma$, provided L is hyperbolic, we might weaken the first assumption above, at the price of replacing $\mathbb{H} \setminus \mathbb{G}$ by $\mathbb{H}_0$, the projectivization of $\pi\Gamma$. In order to keep our discussion clear, we call $\mathcal{WR}_C$ the class defined by the list of conditions above, and by $\mathcal{WR}_0$ the weakened class. Likewise, $\mathcal{WR}_{0F}$, $\mathcal{WR}_{0C}$ denote the subclasses of $\mathcal{WR}_0$ in which L is either Friedrichs symmetric or constantly hyperbolic.

The main point is that classes of the type $\mathcal{WR}$ are *open*, thus robust, in their natural environment. For instance, $\mathcal{IB}_F$ is naturally an open set in an $\mathbb{R}$-vector space. The fact that $\mathcal{WR}_{0F}$ is open is due to the following facts:

- The uniform Kreiss–Lopatinskiĭ condition is robust, as seen in Section 8.1. From compactness, if (L_0, B_0) belongs to $\mathcal{WR}_{0F}$, a small perturbation (L, B) still satisfies the Lopatinskiĭ condition out of $\mathbb{H} \setminus \mathbb{G}$.
- In $\mathbb{H} \setminus \mathbb{G}$, the Kreiss–Lopatinskiĭ condition fails only at zeros of D. When $D^{-1}(0)$ is a smooth manifold, a small analytic perturbation of D yields a small smooth perturbation of the zero set. There do not appear any new connected components, neither along $\mathbb{H} \setminus \mathbb{G}$, nor in the interior $\mathrm{Re}\,\tau > 0$. Likewise, components of $D^{-1}(0)$ may not move towards the interior.
- Therefore, pairs (L, B) in $\mathcal{IB}_F$, which are close to (L_0, B_0), still belong to $\mathcal{WR}_{0F}$.

8.3.1 *The adjoint problem of a BVP of class $\mathcal{WR}$*

Since existence theorems for linear BVPs rely upon the duality method, through the Hahn–Banach and Riesz Theorems, we have to consider the adjoint BVP. We recall that it is a *backward* BVP, meaning that we work in weighted spaces L^2_θ with $\theta < 0$, in contrast to spaces L^2_γ with $\gamma > 0$ for the forward BVP. At the level of the Laplace–Fourier analysis, this means that the relevant time frequencies θ are those with $\mathrm{Re}\,\theta < 0$. We recall (see Sections 4.4 and 4.6.4) that the adjoint operator L^* and the dual boundary matrix $C \in \mathbf{M}_{n-p,n}(\mathbb{R})$ are given by

$$L^* = -\partial_t - \sum_\alpha (A^\alpha)^T \partial_\alpha, \quad A^d = C^T N + M^T B,$$

for some matrices M, N of full rank.

Since A^d is non-singular, because of Lemma 4.6 and the equality

$$\dim E_-(\tau,\eta) + \dim E_-^*(-\bar{\tau},-\eta) = n,$$

we have

$$E_-^*(-\bar{\tau},-\eta) = \left(A^d E_-(\tau,\eta)\right)^\perp .$$

In particular, $E_-(i\rho,\eta)$ is of real type if and only if $E_-^*(i\rho,-\eta)$ is of real type. More precisely, it holds that

$$\mathcal{H}^* = \{(\rho,-\eta)\,;\,(\rho,\eta) \in \mathcal{H}\},$$

where $\mathcal{H}^*$ denotes the hyperbolic set associated to the adjoint BVP. We check easily that the same relation holds true for the glancing sets

$$\mathcal{G}^* = \{(\rho,-\eta)\,;\,(\rho,\eta) \in \mathcal{G}\}.$$

Recall (Theorem 4.2) that the adjoint BVP satisfies the Kreiss–Lopatinskiĭ condition at point $(-\bar{\tau},-\eta)$ if and only if the original one does at point (τ,η). As a matter of fact, we have seen (Theorem 4.4) that, if $\Delta(\tau,\eta)$ is a Lopatinskiĭ determinant for the direct BVP, then

$$(\theta,\sigma) \mapsto \overline{\Delta(-\bar{\theta},-\sigma)}, \quad \mathrm{Re}\,\theta < 0,\ \sigma \in \mathbb{R}^{d-1}$$

is a Lopatinskiĭ determinant for the dual BVP. This implies the following relation between D and D^*, the latter being associated to the adjoint BVP:

$$D^*(\rho,-\eta) = D(\rho,\eta).$$

It immediately follows that the various classes $\mathcal{WR},\dots$ are preserved when we pass to the dual problem. In particular, we have

$$(D^*)^{-1}(0) = \{(\rho,\sigma\,;\,(\rho,-\sigma) \in D^{-1}(0)\}.$$

8.4 Well-posedness of unsual type for BVPs of class $\mathcal{WR}$

We give ourself a BVP of class $\mathcal{WR}_{0C}$. The operator L is therefore constantly hyperbolic, with A^d invertible. We denote

$$\Lambda := \{(i\rho, \eta)\,;\,(\rho,\eta) \in D^{-1}(0)\}.$$

This is the set of boundary points at which the Kreiss–Lopatinskiĭ condition fails. We first establish optimal a priori estimates for the boundary value problem. Then we prove an existence result by duality.

8.4.1 *A priori estimates (I)*

As usual, we work at the level of the Laplace–Fourier transform v. We begin with the case of a homogeneous initial data $u(0,\cdot) \equiv 0$ and extend u by zero to negative times.

We must estimate v and its trace along the boundary, using the equations

$$\tau v + iA(\eta)v + A^d \frac{\partial v}{\partial x_d} = \mathscr{L}f, \quad x_d > 0, \tag{8.4.9}$$

$$Bv(\tau,\eta,0) = G(\tau,\eta) := \mathscr{L}g. \tag{8.4.10}$$

As long as (τ,η) does not approach a point of Λ, the construction done in Chapter 5 applies, and we have a dissipative symbolic symmetrizer $K(\tau,\eta)$, which depends smoothly on (τ,η), as well as of possibly additional parameters. This implies an estimate of the form

$$\mathrm{Re}\;\tau \int_0^\infty |v|^2 \mathrm{d}x_d + |v(0)|^2 \le C(\tau,\eta)\left(\frac{1}{\mathrm{Re}\;\tau}\int_0^\infty |\mathscr{L}f|^2 \mathrm{d}x_d + |G|^2\right). \tag{8.4.11}$$

The number $C(\tau,\eta)$ above is bounded away from Λ. However, it does blow up as (τ,η) tends to a point of Λ, since its boundedness is equivalent to the non-vanishing of Δ.

In order to obtain an estimate up to Λ, we use the splitting $v = v_s + v_u$ introduced in Chapter 7. We recall below formulæ (7.1.10) and (7.1.11) for a given pair (τ,η) with $\mathrm{Re}\;\tau > 0$:

$$v_u(x_d) = -\int_{x_d}^{+\infty} \mathrm{e}^{(x_d - z)\mathcal{A}_u} F_u(z)\mathrm{d}z,$$

$$v_s(x_d) = \mathrm{e}^{x_d \mathcal{A}_s} v_s(0) + \int_0^{x_d} \mathrm{e}^{(x_d - z)\mathcal{A}_s} F_s(z)\mathrm{d}z,$$

where $F := (A^d)^{-1}\mathscr{L}f$.

Since we know that strong estimates hold away from Λ, we concentrate on points (τ,η) in a neighbourhood $\mathcal{V}$ of some $(\tau_0 = i\rho_0, \eta_0) \in \Lambda$. We make use of the following statement.

Lemma 8.3 *Assume that L is a constantly hyperbolic operator. Then, given a point (ρ_0,η_0) in $\pi\Gamma$, there exists a neighbourhood $\mathcal{V}$ of $(\tau_0 = i\rho_0, \eta_0)$, in which* *the following bounds hold uniformly for* $z > 0$*:*

$$\| \exp(-z\mathcal{A}_u(\tau,\eta))\| \leq Ce^{-\omega z \mathrm{Re}\ \tau}, \tag{8.4.12}$$

$$\| \exp(z\mathcal{A}_s(\tau,\eta))\| \leq Ce^{-\omega z \mathrm{Re}\ \tau}. \tag{8.4.13}$$

Here c *and* ω *are positive constants.*

These bounds follow from three important facts:

- The eigenvalues of $\mathcal{A}(\tau,\eta)$ remain of constant multiplicities in $\mathcal{V}$, and therefore are smooth functions.
- They are real along Re $\tau = 0$,
- The imaginary parts of their derivatives $\partial\mu/\partial\tau(\tau_0,\eta_0)$ do not vanish, because this point is non-glancing.

Actually, the same statement holds true near every non-glancing point, since non-real eigenvalues at (τ_0,η_0) are harmless.

Corollary 8.1 *Under the assumptions of Lemma 8.3, it holds that*

$$\|z \mapsto \exp(-z\mathcal{A}_u)\|_{L^r(\mathbb{R}^+)} \leq C(\mathrm{Re}\ \tau)^{-1/r}, \tag{8.4.14}$$

$$\|z \mapsto \exp(z\mathcal{A}_s)\|_{L^r(\mathbb{R}^+)} \leq C(\mathrm{Re}\ \tau)^{-1/r}, \tag{8.4.15}$$

uniformly in $\mathcal{V}$*.*

Applying (8.4.12) to (7.1.10), with the help of a Young inequality for the convolution $L^1 * L^2 \subset L^2$, we obtain the first estimate, which is a strong one:

$$\|v_u\|_{L^2(R^+)} \leq \frac{c}{\mathrm{Re}\ \tau}\|F_u\|_{L^2(\mathbb{R}^+)}. \tag{8.4.16}$$

In the following, we abreviate $\|\cdot\|_{L^2(\mathbb{R}^+)} =: \|\cdot\|_2$ when no confusion is possible.

The constant of integration $v_s(0)$ must be determined from the boundary condition:

$$Bv_s(0) = B\int_0^{+\infty} \mathrm{e}^{-z\mathcal{A}_u}F_u(z)\,\mathrm{d}z + G. \tag{8.4.17}$$

Since we assume the Kreiss–Lopatinskiĭ condition at the interior points, (8.4.17) together with $v_s(0) \in E_-(\tau,\eta)$ determine uniquely $v_s(0)$. We note, however, that $v_s(0)$ does not remain bounded as (τ,η) tends to (τ_0,η_0), for general data F and G. Introducing the eigenprojectors $\pi_\pm(\tau,\eta)$ onto $E_\pm(\tau,\eta)$, we have $v_s = \pi_- v$, $v_u = \pi_+ v$, and so on. The linear map $v \mapsto (\pi_- v, \pi_+ v)$ is bounded, uniformly in $\mathcal{V}$ since E_- and E_+ are transverse at (τ_0,η_0). Because of the Kreiss–Lopatinskiĭ condition, the linear map

$$v \mapsto \begin{pmatrix} Bv \\ \pi_+ v \end{pmatrix}$$

is non-singular. We emphasize that its inverse is *not* uniformly bounded as $(\tau,\eta) \to (\tau_0,\eta_0)$, since

$$v \mapsto \begin{pmatrix} Bv \\ \pi_+(\tau_0,\eta_0)v \end{pmatrix}$$

is singular. As a matter of fact, the Lopatinskiĭ determinant $\Delta(\tau,\eta)$ equals, up to a smooth and non-vanishing factor, the determinant of $(B, \pi_+(\tau,\eta))$. Therefore, the matrix

$$q(\tau,\eta) := \Delta(\tau,\eta) \begin{pmatrix} B \\ \pi_+(\tau,\eta) \end{pmatrix}^{-1}$$

remains bounded as $(\tau,\eta) \to (\tau_0,\eta_0)$.

Since

$$\Delta(\tau,\eta)v_s(0) = q(\tau,\eta) \begin{pmatrix} B \int_0^{+\infty} \mathrm{e}^{-z\mathcal{A}_u} F_u(z)\,dz + G \\ 0 \end{pmatrix},$$

we obtain the estimate

$$|\Delta(\tau,\eta)v_s(0)| \leq C \left(\frac{1}{\sqrt{\mathrm{Re}\ \tau}} \|F_u\|_2 + |G| \right), \tag{8.4.18}$$

where C is uniform in $\mathcal{V}$.

Define now

$$\tilde{p}(\tau,\eta)v := v_u + \Delta(\tau,\eta)v_s,$$

that is

$$\tilde{p}(\tau,\eta) = \pi_+(\tau,\eta) + \Delta(\tau,\eta)\pi_-(\tau,\eta). \tag{8.4.19}$$

This $n \times n$ matrix-valued symbol depends smoothly on (τ,η) in $\mathcal{V}$ and is homogeneous of degree zero. Definition 8.4.19 works also outside of $\mathcal{V}$. The linear operator $\tilde{p}(\tau,\eta)$ is non-singular, and its inverse is bounded except in a neighbourhood of Λ. The symbol $\tilde{p}$ is continuous up to the boundary $\mathrm{Re}\ \tau = 0$, except for glancing points, where it is even not bounded. Hence we smooth it out in a neighbourhood of the glancing points, in such a way that the new symbol $p(\tau,\eta)$ fulfils the following properties:

- p coincides with $\tilde{p}$, except in a small neighbourhood of glancing points; in particular, they coincide in a neighbourhood of Λ,
- $(\tau,\eta) \mapsto p(\tau,\eta)$ is smooth and homogeneous of degree zero,
- $p(\tau,\eta)$ is non-singular everywhere on $\mathrm{Re}\ \tau \geq 0$ but along Λ.

We warn the reader that, when smoothing $\tilde{p}$, we lose the holomorphy in τ. Therefore, p is not the symbol of a unique pseudo-differential operator. Instead, we have a collection of ψDOs

$$P_\gamma\left(\frac{1}{i}\partial_t,\frac{1}{i}\nabla_y\right)$$

of respective symbols

$$(\rho,\eta)\mapsto p_\gamma(\rho,\eta):=p(\gamma+i\rho,\eta).$$

From homogeneity, the principal symbol of P_γ is p_0, for every $\gamma\geq 0$. In particular, P_γ is of order zero and is microlocally elliptic at every pair $((\rho,\eta);v)$ but those for which $(\tau,\eta)\in\Lambda$ and $v\in E_-(\tau,\eta)\cap\ker B$. Its characteristic cone is precisely Λ.

8.4.2 *A priori estimates (II)*

Improvement of (8.4.18)

Let us focus on the neighbourhood of $(\tau_0,\eta_0)\in\Lambda$, since elsewhere the estimates are those of the uniformly stable case (UKL). We therefore have $p=\pi_++\Delta\pi_-$. Going back to (8.4.17), we use the fact that $(A^d)^{-1}M^T$ is a right inverse of B (see Section 4.4), and rewrite

$$Bv_s(0)=B\left(\int_0^{+\infty}\mathrm{e}^{-z\mathcal{A}_u}F_u(z)\,\mathrm{d}z+(A^d)^{-1}M^TG\right).$$

Next, we decompose $(A^d)^{-1}M^TG$, using $1=p+(1-p)=p+(1-\Delta)\pi_-$:

$$Bv_s(0)=B\left(\int_0^{+\infty}\mathrm{e}^{-z\mathcal{A}_u}F_u(z)\,\mathrm{d}z+p(A^d)^{-1}M^TG+(1-\Delta)\pi_-(A^d)^{-1}M^TG\right).$$

Since $v_s(0)$ is in E_-, this gives

$$v_s(0)=v_{s0}+(1-\Delta)\pi_-(A^d)^{-1}M^TG,$$

where $v_{s0}\in E_-$ is the solution of

$$Bv_{s0}=B\left(\int_0^{+\infty}\mathrm{e}^{-z\mathcal{A}_u}F_u(z)\,\mathrm{d}z+p(A^d)^{-1}M^TG\right).$$

Following the same argument as above, v_{s0} satisfies an estimate of the form (8.4.18), with G replaced by $p(A^d)^{-1}M^TG$.

Since $\Delta p^{-1}=\Delta\pi_++\pi_-$ is uniformly bounded and $(A^d)^{-1}M^T$ is one-to-one, we have a bound

$$|\Delta(\tau,\eta)G|\leq|p(A^d)^{-1}M^TG|.$$

Gathering all these inequalities, we obtain the more accurate estimate

$$|\Delta(\tau,\eta)v_s(0)|\leq C\left(\frac{1}{\sqrt{\mathrm{Re}\,\tau}}\|F_u\|_2+|p(A^d)^{-1}M^TG|\right),\qquad(8.4.20)$$

where C is uniform in $\mathcal{V}$.

Main estimates

We now estimate $\Delta(\tau,\eta)v_s$ when $(\tau,\eta)\in\mathcal{V}$. From (7.1.11), (8.4.15) and the Young inequality, we have

$$\|\Delta(\tau,\eta)v_s\|_2 \le C|\Delta(\tau,\eta)|\left(\frac{1}{\sqrt{\mathrm{Re}\ \tau}}|v_s(0)|+\frac{1}{\mathrm{Re}\ \tau}\|F_s\|_2\right).$$

Thanks to (8.4.18), this yields

$$\|\Delta(\tau,\eta)v_s\|_2 \le C\left(\frac{1}{\mathrm{Re}\ \tau}\|p(\tau,\eta)F\|_2+\frac{1}{\sqrt{\mathrm{Re}\ \tau}}|p(A^d)^{-1}M^TG|\right). \quad (8.4.21)$$

Merging (8.4.16) and (8.4.21), and using the *uniformly* dissipative Kreiss symmetrizer outside a neighbourhood of Λ, we conclude that the following estimate holds *uniformly* in (τ,η):

$$\|p(\tau,\eta)v\|_2 \le C\left(\frac{1}{\mathrm{Re}\ \tau}\|p(\tau,\eta)F\|_2+\frac{1}{\sqrt{\mathrm{Re}\ \tau}}|p(A^d)^{-1}M^TG|\right). \quad (8.4.22)$$

Likewise, (8.4.18) yields the uniform estimate

$$\|p(\tau,\eta)v(0)\|_2 \le C\left(\frac{1}{\sqrt{\mathrm{Re}\ \tau}}\|p(\tau,\eta)F\|_2+|p(A^d)^{-1}M^TG|\right). \quad (8.4.23)$$

Using, finally, Plancherel's Formula, we derive our fundamental estimate

$$\begin{aligned}\gamma\iint_{\Omega\times\mathbb{R}} &\mathrm{e}^{-2\gamma t}\|P_\gamma u\|^2\mathrm{d}x\,\mathrm{d}t+\iint_{\partial\Omega\times\mathbb{R}}\mathrm{e}^{-2\gamma t}\|P_\gamma\gamma_0 u\|^2\mathrm{d}x\,\mathrm{d}t\\ &\le C\left(\frac{1}{\gamma}\iint_{\Omega\times\mathbb{R}}\mathrm{e}^{-2\gamma t}\|P_\gamma(A^d)^{-1}Lu\|^2\mathrm{d}x\,\mathrm{d}t\right.\\ &\qquad\left.+\iint_{\partial\Omega\times\mathbb{R}}\mathrm{e}^{-2\gamma t}\|P_\gamma\gamma_0(A^d)^{-1}M^TBu\|^2\mathrm{d}x\,\mathrm{d}t\right),\end{aligned} \quad (8.4.24)$$

whenever γ is positive and u is smooth, compactly supported. This is clearly weaker than (4.5.26), as expected. However, it has the nice feature that only $P_\gamma(A^d)^{-1}Lu$ and $P_\gamma\gamma_0(A^d)^{-1}M^TBu$, instead of Lu and $\gamma_0 BU$, are required to be square-integrable. Therefore, there is some hope that such an estimate could be useful in nonlinear problems when using a fixed-point argument.

Remarks

- Note that $(A^d)^{-1}M^TB$ is a projector, whose kernel is $\ker B$. Therefore, (8.4.24) is the same as (4.5.26), up to the presence of the operator P_γ everywhere. In other words, one passes from one estimate to the other by changing the norm both in data and in output. We arrive at the strange conclusion that a BVP of class $\mathcal{WR}_{0C}$ displays *strong* well-posedness in some Hilbert space, though not in L^2. This space is inhomogeneous in frequency, and in space-time variables as well, since x_d and (y,t) play different roles.

- When Δ vanishes at the first order only along Λ, the norms $\|P_\gamma w\|$ are intermediate between the L^2-norm and the H^{-1}-norm. Thus (8.4.24) implies, in the case of constant coefficients in a non-characteristic half-space, the estimates 'with loss of one derivative' obtained by Coulombel [38,39,41,43]. Coulombel's estimates also hold true with variable coefficients, as well as in some characteristic cases. We leave for a future study an extension of our estimates in such contexts.
- In the case of a BVP of class $\mathcal{WR}$ for the wave operator, a nicer analysis can be made, see [191]. In a suitable range of parameters, the BVP can be decomposed in a sequence of two hyperbolic BVP, each one satisfying (UKL). The sequence of a priori estimates yields an estimate similar to (8.4.24). This decomposition is robust to variation of coefficients, and thus can be employed in BVP with variable coefficients.

8.4.3 *The estimate for the adjoint BVP*

We have seen that the adjoint BVP is of class $\mathcal{WR}$ too, with the Lopatinskiĭ determinant

$$\Delta_*(\theta,\sigma) = \overline{\Delta(-\bar{\theta},-\sigma)}.$$

The standard objects, when associated with the adjoint BVP, are indexed with a subscript $*$, for instance $E_{-*}, E_{+*}, \dots$ The superscript $*$ is kept for denoting adjoint operators. For instance, the adjoint π_-^* of the projection onto E_-, of kernel E_+, is the projection onto $E_+^\perp$, with kernel $E_-^\perp$. Since we have

$$E_{\pm *} = \left(A^d E_\pm\right)^\perp,$$

we find the useful identities

$$\pi_-^*(A^d)^T = (A^d)^T\pi_{+*}, \quad \pi_+^*(A^d)^T = (A^d)^T\pi_{-*}. \tag{8.4.25}$$

Hereabove, and everywhere in the following, we always assume the relations $\theta = -\bar{\tau}$ and $\sigma = -\eta$. Using (8.4.25), we deduce

$$\tilde{p}^*(A^d)^T = (A^d)^T(\pi_{-*} + \bar{\Delta}\pi_{+*}),$$

from which we infer

$$\tilde{p}^*(A^d)^T\tilde{p}_* = \bar{\Delta}(A^d)^T.$$

In the smoothing process, it is possible to keep track of this identity, in the following way

$$p^*(A^d)^T p_* = \bar{\delta}(A^d)^T, \tag{8.4.26}$$

where $\delta(\tau,\eta)$ is smooth, homogeneous of degree zero, and coincides with Δ outside of a small neighbourhood of glancing points. Also, δ vanishes only along Λ.

The analogue of (8.4.24) for the adjoint problem is obviously

$$\gamma \iint_{\Omega\times\mathbb{R}} \mathrm{e}^{2\gamma t}\|P_{-\gamma *}w\|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{2\gamma t}\|P_{-\gamma *}\gamma_0 w\|^2 \mathrm{d}x\,\mathrm{d}t \leq C\left(\frac{1}{\gamma}\iint_{\Omega\times\mathbb{R}} \mathrm{e}^{2\gamma t}\|P_{-\gamma *}(A^d)^{-T}L^* w\|^2 \mathrm{d}x\,\mathrm{d}t + \iint_{\partial\Omega\times\mathbb{R}} \mathrm{e}^{2\gamma t}\|P_{-\gamma *}\gamma_0 (A^d)^{-T}N^T C w\|^2 \mathrm{d}x\,\mathrm{d}t\right), \tag{8.4.27}$$

At the level of the Laplace–Fourier transform, this amounts to saying that

$$|\mathrm{Re}\ \theta|\,\|p_*(\theta,\sigma)\hat{w}\|_2^2 + |p_*(\theta,\sigma)\hat{w}(0)|^2 \leq C\left\{\frac{1}{|\mathrm{Re}\ \theta|}\|p_*(\theta,\sigma)(A^d)^{-T}\widehat{L^*w}\|_2^2 + |p_*(\theta,\sigma)(A^d)^{-T}N^T C\hat{w}(0)|^2\right\}. \tag{8.4.28}$$

8.4.4 *Existence result for the BVP*

As in Section 4.5.5, we employ a duality argument. We do not give full details, which essentially mimic those of Section 4.5. We content ourselves with explaining what role our modified estimates play.

Let us fix a $\gamma > 0$. We give ourself a pair (f,g) with

$$P_\gamma (A^d)^{-1} f \in L^2_\gamma, \quad P_\gamma (A^d)^{-1} M^T g \in L^2_\gamma. \tag{8.4.29}$$

We denote by Y_γ the set of distributions of the form $L^* w$ with the properties that

$$P_{-\gamma *}(A^d)^{-T} L^* w \in L^2_{-\gamma}, \quad P_{-\gamma} w \in L^2_{-\gamma}, \quad \gamma_0 C w = 0.$$

Admitting that (8.4.27) holds true on Y_γ, we deduce that the linear map $L^* w \mapsto w$ is well-defined (a uniqueness property), with some continuity properties. We then define a linear functional

$$\ell(L^* w) := \iint_{\Omega\times\mathbb{R}} (w,f)\,\mathrm{d}x\mathrm{d}t + \iint_{\partial\Omega\times\mathbb{R}} (Mw,g)\,\mathrm{d}y\mathrm{d}t.$$

To begin with, we majorize the first integral. Using the Plancherel formula, it amounts to the same to deal with the integral of $(\hat{w},\hat{f})$. This has the effect of decoupling frequencies, so that we need an estimate of the integral with respect to x_d only. To this end, we write

$$\int_0^\infty (\hat{w}, \hat{f})\,\mathrm{d}x_d = \left((A^d)^T\hat{w}, (A^d)^{-1}\hat{f}\right)_2 = \left((A^d)^T p_* \frac{\hat{w}}{\bar{\delta}}, p(A^d)^{-1}\hat{f}\right)_2,$$

where the last equality makes use of (8.4.26). Using Cauchy–Schwarz, we infer

$$|(\hat{w}, \hat{f})_2| \leq C \left\| p_* \frac{\hat{w}}{\bar{\delta}} \right\|_2 \|p(A^d)^{-1}\hat{f}\|_2.$$

Applying now (8.4.28), we obtain

$$|(\hat{w}, \hat{f})_2| \leq \frac{C}{\gamma|\delta|} \left\| p_*(A^d)^{-T}\widehat{L^*w} \right\|_2 \|p(A^d)^{-1}\hat{f}\|_2.$$

Playing the same game with the second integral, and using the fact that p is uniformly bounded, we obtain

$$\begin{aligned} &|(\hat{w}, \hat{f})_2 + (M\hat{w}(0), \hat{g})| \\ &\quad \leq \frac{C}{|\delta|}\left(\frac{1}{\gamma}\|p(A^d)^{-1}\hat{f}\|_2 + \frac{1}{\sqrt{\gamma}}|p(A^d)^{-1}M^T\hat{g}|\right)\left\|p_*(A^d)^{-T}\widehat{L^*w}\right\|_2. \end{aligned}$$

This shows that ℓ extends continuously to the space of functions W such that

$$\frac{1}{\bar{\delta}} p_*(A^d)^{-T}\hat{W} \in L^2(\mathrm{Re}\ \theta = -\gamma).$$

The dual of this space is, from (8.4.26), the space of functions U such that

$$A^d p\hat{U} \in L^2(\mathrm{Re}\ \tau = \gamma).$$

In other words, it is the set of Us such that

$$P_\gamma U \in L^2_\gamma. \tag{8.4.30}$$

Therefore, there exists a u with property (8.4.30), such that

$$\ell(L^*w) = \iint_{\Omega\times\mathbb{R}} (L^*w, u)\,\mathrm{d}x\mathrm{d}t.$$

Additionally, we have

$$\|P_\gamma u\|_\gamma \leq C\left(\frac{1}{\gamma}\left\|P_\gamma(A^d)^{-1}f\right\|_\gamma + \frac{1}{\sqrt{\gamma}}\|P_\gamma(A^d)^{-1}M^T g\|_\gamma\right).$$

This u is the solution of the boundary value problem for $t \in \mathbb{R}$. Note that the duality method gives directly the dependency upon $P_\gamma(A^d)^{-1}M^T g$, instead of g, confirming the analysis done in Section 8.4.2.

8.4.5 *Propagation property*

There is an important difference between the classes $\mathcal{WR}$ and (UKL), as far as the propagation (of support or of singularities) is concerned. For a uniformly stable IBVP, it can be proved that the signals propagate not faster than in the pure Cauchy problem. A simple calculus, using energy inequality, shows that in the strictly dissipative Friedrichs-symmetric case, the support of the solution does not propagate faster than expected.

This is no longer true for an IBVP of class $\mathcal{WR}_{0C}$, because of the fact that the 'boundary symbol' $p(i\rho,\eta)$ vanishes along the set Λ, which is strictly included in the forward characteristic cone in general.

Let us consider as an example the wave equation

$$\partial_t^2 u = c^2 \Delta_x u,$$

with the boundary condition

$$\frac{\partial u}{\partial \nu} = \gamma \frac{\partial u}{\partial t} + g,$$

with $\gamma \in (0, 1/c)$ a constant, and $\partial u/\partial \nu$ the normal derivative. This problem can be recast at a first-order system. We leave the reader to check that this BVP is of class $\mathcal{WR}_{0C}$, and to compute that Λ consists in the pairs $(i\rho, \eta)$ such that

$$|\eta|^2 = \rho^2 \left(\frac{1}{c^2} - \gamma^2 \right).$$

This reveals that signals propagate along the boundary at the velocity

$$c' := \frac{c}{1 - c^2\gamma^2},$$

which is larger than c. In particular, c' tends to $+\infty$ as γ tends to $1/c$; recall that the IBVP is not normal (and therefore ill-posed) for $\gamma = 1/c$, and that it does not satisfy at all the Kreiss–Lopatinskiĭ condition for $\gamma > 1/c$.

9

VARIABLE-COEFFICIENTS INITIAL BOUNDARY VALUE PROBLEMS

This chapter is the logical continuation of Chapter 4 on Initial Boundary Value problems. We are concerned here with *variable-coefficient* operators

$$L := \partial_t + \sum_{j=1}^{d} A^j(x,t)\,\partial_j,$$

where x lies in a domain Ω strictly smaller than $\mathbb{R}^d$ having a smooth boundary $\partial\Omega$: we will not consider domains with corners or edges despite their physical interest (e.g. in fluid dynamics with the entrance of pipes), because the analysis of the corresponding Initial Boundary Value Problems is not well understood up to now (see [100, 153, 154, 175]).

Unless otherwise specified, the matrices $A^j(x,t)$ will be $\mathscr{C}^\infty$ functions of (x,t), independent of (x,t) outside a compact subset of $\mathbb{R}^d \times \mathbb{R}^+$. We assume throughout this chapter that the operator L is hyperbolic in the direction of t, which means in particular that the *characteristic matrix*

$$A(x,t,\xi) = \sum_{j=1}^{d} \xi_j\, A^j(x,t)$$

is diagonalizable in $\mathbb{R}$ for all $\xi \in \mathbb{R}^d \backslash \{0\}$. In fact, we will mostly concentrate on the more restricted class of *constantly hyperbolic* operators, for which by definition the eigenvalues of $A(x,t,\xi)$ are of constant multiplicities.

Boundary conditions are supposed to be encoded by a smooth matrix-valued function $B : (x,t) \in \partial\Omega \times (0,+\infty) \mapsto B(x,t)$, the rank of $B(x,t)$ being prescribed by a frozen-coefficients analysis: accordingly with Proposition 4.1, the rank of $B(x,t)$ must be equal to the number of incoming characteristics, that is, the number of negative eigenvalues (counted with multiplicity) of the characteristic matrix computed at $(x,t,\xi = \nu(x))$, where $\nu(x)$ is the exterior unit normal vector to $\partial\Omega$ at point x.

Independently of the specific boundary conditions, we will thus require that the number of negative eigenvalues of $A(x,t,\nu(x))$ be constant, which is definitely not innocent; unless A is independent of (x,t) and ν is independent of x, i.e. L has constant coefficients and Ω is a half-space, as in Chapter 4. This requirement on the number of negative eigenvalues of $A(x,t,\nu(x))$ is (more generally) fulfilled by non-characteristic problems in domains with connected boundaries. (We warn the reader though, that it might be difficult to ensure the whole boundary is non-characteristic: see Section 11.1.1 for a more detailed discussion in a quasilinear framework.) As a matter of fact, if the boundary $\partial\Omega$ is everywhere non-characteristic, that is, if the matrix $A(x,t,\nu(x))$ is non-singular along $\partial\Omega \times (0,+\infty)$, and if additionally $\partial\Omega$ is connected, the hyperbolicity of the operator L implies that the eigenvalues of $A(x,t,\nu(x))$ are split into a constant number of negative ones and a constant number of positive ones: indeed, if $A(x,t,\nu(x))$ is non-singular and only has real eigenvalues then of course it has no eigenvalue on the imaginary axis; in the language of ODEs, this means $A(x,t,\nu(x))$ is a hyperbolic matrix for all $(x,t) \in \partial\Omega \times (0,+\infty)$ and a connectedness argument then shows the dimension of its stable subspace must be constant along $\partial\Omega \times (0,+\infty)$.

Denoting by p the number (assumed constant, no matter the restriction) of incoming characteristics, we may suppose without loss of generality that B is everywhere of maximal rank p, that is, $B(x,t) \in \mathbf{M}_{p\times n}(\mathbb{R})$ for all $(x,t) \in \partial\Omega \times (0,+\infty)$, the p rows of $B(x,t)$ being independent.

Additional requirements are to be imported from Chapter 4. Not only must the rank of B coincide with the dimension of the stable subspace $E^s(A)$ of A but we should have the *normality condition*:

$$\mathbb{R}^n \;=\; \ker B(x,t) \oplus E^s(A(x,t,\nu(x))) \quad \text{for all } (x,t) \in \partial\Omega \times (0,+\infty)\,. \qquad (9.0.1)$$

If we were to consider possibly characteristic boundaries, we should also require that

$$\ker A(x,t,\nu(x)) \subset \ker B(x,t) \quad \text{for all } (x,t) \in \partial\Omega \times (0,+\infty)\,.$$

But we will concentrate on non-characteristic problems.

Finally, we will need an assumption specific to variable coefficients: we ask that the kernel of B admit a *smooth basis*, that is, a family of $\mathscr{C}^\infty$ vector-valued functions $(e_{p+1},\ldots,e_n)$ such that

$$\text{Span }(e_{p+1}(x,t),\ldots,e_n(x,t)) \;=\; \ker B(x,t)$$

for all $(x,t) \in \partial\Omega \times (0,+\infty)$. A standard result in differential topology [85] (p. 97), saying that any vector bundle over a contractible manifold is trivial, implies the existence of such a smooth basis for some particular boundaries $\partial\Omega$: for instance, a hyperplane is contractible. For non-contractible boundaries $\partial\Omega$, the existence of a smooth basis is a non-trivial assumption.

Our aim is to solve Initial Boundary Value Problems (IBVP) of the form

$$(Lu)(x,t) = f(x,t), \quad x \in \Omega\,,\ t > 0\,, \qquad (9.0.2)$$

$$(Bu)(x,t) = g(x,t), \quad x \in \partial\Omega\,,\ t > 0\,, \qquad (9.0.3)$$

$$u(x,0) = u_0(x), \quad x \in \Omega\,, \qquad (9.0.4)$$

where the source term f, the boundary data g and the initial condition u_0 are given in a Sobolev space H^s, $s \geq 0$. We may also add a zeroth-order operator to L.

As for the Cauchy problem, well-posedness crucially relies on energy estimates. As for the Cauchy problem, the basic tools for deriving energy estimates are *symmetrizers*. However, symmetrizers for IBVPs are much more complicated than for the Cauchy problem: they were first introduced by Kreiss in his celebrated paper [103] (see also [160]); up to now, the main reference is the (unfortunately depleted) book by Chazarain and Piriou [31], where Kreiss' symmetrizers are constructed in detail and used to show the well-posedness of IBVPs. For constant-coefficient problems, Chapter 5 of this book gives a new construction of Kreiss' symmetrizers, and Chapter 4 shows how to use them for the well-posedness theory.

The purpose of this chapter is to answer the two main questions: 1) how do Kreiss' symmetrizers associated with frozen coefficients systems yield energy estimates for variable coefficients? and 2) how do energy estimates imply the well-posedness of Initial Boundary Value Problems?

As for the Cauchy problem, we shall at first deal with infinitely smooth coefficients. In this case, the answers to questions 1) and 2) make use of pseudo-differential calculus with parameter. They are contained in Chapter VII of [31]. Our presentation is different but makes use of the same arguments.

Coefficients with poorer regularity (arising, for instance, in the resolution of quasilinear problems) are trickier to deal with, and will be considered separately. The main reference on this topic is the (unpublished) PhD thesis of Mokrane [140], which takes advantage of para-differential calculus. We shall give a presentation here that parallels the smooth coefficients theory, and point out the special features related to poor regularity.

In any case, we will only consider non-characteristic problems, for the analysis of variable-coefficient characteristic problems is still in its infancy: apart from the seminal paper by Majda and Osher [127], most results known concern dissipative boundary conditions for Friedrichs-symmetrizable systems; as far as *linear* problems are concerned, the main references are the papers by Rauch [163, 164] (see Chapter 11 for further references concerning quasilinear problems).

Another deliberate choice by us is to consider first the simplest case of a planar boundary and explain afterwards how to deal with arbitrary domains through co-ordinate charts.

9.1 Energy estimates

As announced above, we proceed gradually and first consider a half-space Ω, which we assume to be $\Omega = \{ x \in \mathbb{R}^d \,;\, x_d > 0 \}$ without loss of generality. Furthermore, we assume $\partial\Omega$ is *non-characteristic*, which means that $A^d(y, 0, t)$ is invertible for all $(y, t) = (x_1, \dots, x_{d-1}, t) \in \mathbb{R}^{d-1} \times (0, +\infty)$ (with an obvious convention if $d = 1$).

The main purpose here is to derive energy estimates for the Boundary Value Problem

$$(Lu)(y,x_d,t) = f(y,x_d,t), \quad y \in \mathbb{R}^{d-1}\,,\ t \in \mathbb{R}\,,\ x_d > 0\,, \tag{9.1.5}$$

$$(Bu)(y,0,t) = g(y,t), \quad y \in \mathbb{R}^{d-1}\,,\ t \in \mathbb{R}\,, \tag{9.1.6}$$

in which the coefficients of L and B are supposed to be well-defined and smooth for all $t \in \mathbb{R}$. We have in mind energy estimates that are weighted in time, as in Theorem 2.13. This is why we introduce the new unknown $\widetilde{u}_\gamma(x,t) := e^{-\gamma t}\, u(x,t)$ and new data $\widetilde{f}_\gamma$, $\widetilde{g}_\gamma$ in the same way. We observe that (9.1.5)(9.1.6) is equivalent to

$$L_\gamma \widetilde{u}_\gamma = \widetilde{f}_\gamma, \quad \Omega \times \mathbb{R}\,, \tag{9.1.7}$$

$$B\widetilde{u}_\gamma = \widetilde{g}_\gamma, \quad \partial\Omega \times \mathbb{R} \tag{9.1.8}$$

with

$$L_\gamma \ :=\ \partial_t + \gamma + \sum_{j=1}^{d} A^j \partial_j\,.$$

The whole difficulty lies in the fact that the principal part $L_0 \ =\ L$ of the operator L_γ is in general *not hyperbolic* in the direction of x_d. This is not specific to variable coefficients. For, the hyperbolicity of L in the direction of x_d requires that, for any 'frequency'[1] $\xi\ =\ (\eta,\delta)\ =\ (\eta_1,\ldots,\eta_{d-1},\delta)\ \in\ \mathbb{R}^d\backslash\{0\}$ the polynomial

$$\det(\,\delta\, I_n\ +\ \sum_{j=1}^{d-1}\ \eta_j\ A^j a\ +\ \zeta\, A^d\,)$$

only has real roots ζ. This holds true in the frequency region called *hyperbolic* for obvious reasons, which is in general not the whole frequency space.

Example In gas dynamics (see Chapter 13), the polynomial above can be explicitly factorized as

$$(\,\delta\ +\ (\eta,\zeta)\cdot \mathbf{u}\,)^{d-1}\,\Big(\,(\,\delta\ +\ (\eta,\zeta)\cdot \mathbf{u}\,)^2\ -\ c^2\,(\|\eta\|^2\ +\ \zeta^2)\Big)\,,$$

where $\mathbf{u}$ denotes the gas velocity and c the sound speed. If the first factor only has real roots, in fact the multiple one

$$\zeta\ =\ -\,(\delta\ +\ \eta\cdot \check{\mathbf{u}})/u_d\,,$$

where $\check{\mathbf{u}}\ :=\ (u_1,\ldots,u_{d-1})$ denotes the tangential part of the velocity $\mathbf{u} = (\check{\mathbf{u}}, u_d)$, the second factor has real roots if and only if

$$(\delta\ +\ \eta\cdot\check{\mathbf{u}})^2\ \geq\ (c^2\ -\ u_d^2)\ \|\eta\|^2\,.$$

[1]The term frequency here is used in a wide mathematical sense, and does not refer specifically to time oscillations.

This inequality holds true everywhere for supersonic flows ($u_d^2 \geq c^2$). But for a subsonic flow ($u_d^2 < c^2$), the *hyperbolic region* of L in the direction x_d is restricted to the cone

$$\{\, \xi = (\delta, \eta)\, ;\; (\delta + \eta \cdot \check{\mathbf{u}})^2 \geq (c^2 - u_d^2)\, \|\eta\|^2 \,\}.$$

It is notable that, for points $\xi = (\delta, \eta)$ on the boundary of the cone, i.e. such that

$$(\delta + \eta \cdot \check{\mathbf{u}})^2 = (c^2 - u_d^2)\, \|\eta\|^2 \,,$$

called *glancing modes*, the matrix

$$(A^d)^{-1} \,(\, \delta\, I_n + \sum_{j=1}^{d-1} \eta_j\, A^j \,)\,,$$

which will turn out to play a crucial role in the analysis, is *not* diagonalizable.

Coming back to the abstract problem (9.1.5)(9.1.6) and considering the special role played by the variable x_d, we rewrite (9.1.7) in the equivalent form

$$\partial_d \widetilde{u}_\gamma - P^\gamma(x_d)\, \widetilde{u}_\gamma = (A^d)^{-1}\, \widetilde{f}_\gamma \,, \tag{9.1.9}$$

$$P^\gamma(x_d) := -\,(A^d)^{-1}\,(\, \partial_t + \gamma + \sum_{j=1}^{d-1} A^j \partial_j \,)\,.$$

This notation emphasizes the dependence of P^γ on the parameter x_d, but it should be clear to the reader that $P^\gamma(x_d)$ is a variable-coefficient differential operator in (y, t). More precisely, in the terminology recalled in the appendix (section C.2), for all $x_d \geq 0$, $\{P^\gamma(x_d)\}_{\gamma \geq 1}$ is a family of differential operators of order 1 on $\{(y, t) \in \mathbb{R}^{d-1} \times \mathbb{R}\}$, their symbols being

$$a(y, t, \eta, \delta, \gamma; x_d) := -\,(A^d(y, x_d, t))^{-1}\,(\, (i\delta + \gamma)\, I_n + i\, A(y, x_d, t, \eta)\,)\,,$$

where

$$A(y, x_d, t, \eta) = \sum_{j=1}^{d-1} \eta_j\, A^j(y, x_d, t)\,.$$

Consistently with Chapter 4, we will also use the shorter notation

$$\mathcal{A}(x, t, \eta, \tau) := a(y, t, \eta, \delta, \gamma; x_d),$$

where $x = (y, x_d)$ and $\tau = \gamma + i\delta$.

Dropping for a while the tildas and the γ subscript in (9.1.7)(9.1.8), we are facing a problem of the form

$$\begin{array}{ll} \partial_{x_d} u - P^\gamma(x_d)\, u = f\,, & x_d > 0\,, \\ Bu = g\,, & x_d = 0\,. \end{array}$$

If one compares with the Cauchy problem studied in Chapter 2

$$\begin{aligned} &\partial_t u - P(t)\, u = f\,, \quad && t > 0\,,\\ &u = g\,, && t = 0\,, \end{aligned}$$

there are two intertwined differences: 1) the whole vector u is prescribed on the 'boundary' ($t = 0$) in the latter case, whereas in the former, only part of u is prescribed on the 'boundary' ($x_d = 0$) and 2) unlike $\partial_t - P(t)$, the operator $\partial_{x_d} u - P^\gamma(x_d)$ is *not hyperbolic* in general, because L is not hyperbolic in the direction x_d. The gap between hyperbolic Initial Value Problems and Initial *Boundary* Value Problems should now be clear to the reader.

9.1.1 *Functional boundary symmetrizers*

In what follows, the space $\mathbb{R}^d$ is to be understood as the product of the boundary of Ω by the real line in the time direction, i.e. $\mathbb{R}^d = \{\,(y,t)\,;\; y \in \mathbb{R}^{d-1}\,,\, t \in \mathbb{R}\,\}$. The delimiters $\langle\,,\,\rangle$ stand for the inner product on $L^2(\mathbb{R}^d, \mathrm{d}y\,\mathrm{d}t)$.

Definition 9.1 *A (functional)* boundary symmetrizer *for* (9.1.5)(9.1.6) *is a family of $\mathscr{C}^1$ mappings*

$$\begin{aligned} \mathrm{R}^\gamma : \mathbb{R}^+ &\to \mathscr{B}(L^2(\mathbb{R}^d;\mathbb{R}^n))\\ x_d &\mapsto \quad \mathrm{R}^\gamma(x_d) \end{aligned}$$

for $\gamma \geq \gamma_0 \geq 1$, with bounds for $\mathrm{R}^\gamma(x_d)$ and $\mathrm{d}\mathrm{R}^\gamma/\mathrm{d}x_d$ that are uniform in both $\gamma \geq \gamma_0$ and $x_d \geq 0$, such that

- *the operator $\mathrm{R}^\gamma(x_d)$ is self-adjoint,*
- *the operator*

$$\mathrm{Re}\,(\,\mathrm{R}^\gamma(x_d)P^\gamma(x_d)\,) := \frac{1}{2}\,(\,\mathrm{R}^\gamma(x_d)P^\gamma(x_d) + (\,\mathrm{R}^\gamma(x_d)P^\gamma(x_d)\,)^*)$$

  belongs to $\mathscr{B}(L^2(\mathbb{R}^d;\mathbb{R}^n))$ and as such satisfies the lower bound

$$\mathrm{Re}\,(\,\mathrm{R}^\gamma(x_d)P^\gamma(x_d)\,) \geq C\,\gamma\,I\,, \tag{9.1.10}$$

  with $C > 0$ independent of both $x_d \geq 0$ and $\gamma \geq \gamma_0$,
- *there exist positive constants α and β so that*

$$\langle \mathrm{R}^\gamma(0)u\,,\,u\rangle \;\geq\; \alpha\,\|u\|^2_{L^2} - \beta\,\|Bu\|^2_{L^2} \tag{9.1.11}$$

  for all $u \in L^2(\mathbb{R}^d;\mathbb{R}^n)$ and all $\gamma \geq \gamma_0$.

To some extent, this definition resembles the one given in Chapter 2 (Definition 2.2) for functional symmetrizers concerning the Cauchy problem. If one compares the two definitions, the main novelties here are, besides the parameter γ, the inequality (9.1.11), which is meant to deal with boundary terms in the energy estimates, and also that a *non-negative lower bound* for $\mathrm{Re}\,(\,\mathrm{R}^\gamma(x_d)P^\gamma(x_d)\,)$ is requested: this is because the symmetrizer is designed to find a lower bound for the derivative of $\langle \mathrm{R}^\gamma(x_d)\,u\,,\; u\rangle$ with respect to x_d, eventually leading to a control of the trace of u at $x_d = 0$. (Observe the counterpart of this trace for the Cauchy problem is merely the initial data, so that a rough bound $\mathrm{Re}\,(\,\Sigma(t)P(t)\,) \geq -C\,I$ suffices to show well-posedness in that case.)

In the case of Friedrichs symmetrizability, which is a property of the sole operator L, we easily get boundary symmetrizers, provided that the boundary matrix B is *strictly dissipative*: the definition of strict dissipativity given in Chapter 3 for Initial Boundary Value Problems with constant coefficients extends in a straightforward way to variable coefficients; we give it below for a general domain Ω with outward unit normal vector ν.

Definition 9.2 *Assume $L = \partial_t + \sum_{j=1}^{d} A^j \partial_j$ is a Friedrichs-symmetrizable operator, with Friedrichs symmetrizer S_0 (see Definition 2.1). The boundary matrix B is called* strictly dissipative *if there exist $\alpha > 0$ and $\beta > 0$ so that for all $(x,t) \in \partial\Omega \times \mathbb{R}$ and all $v \in \mathbb{R}^n$;*

$$v^{\mathrm{T}}\, S_0(x,t)\, A(x,t,\nu(x))\, v \;\geq\; \alpha\,\|v\|^2 \;-\; \beta\,\|B(x,t)v\|^2\,, \tag{9.1.12}$$

where ν denotes the outward normal to $\partial\Omega$.

In the case $\Omega = \{x_d > 0\}$, the strict dissipativity of B means

$$v^*\; S_0(y,0,t)\, A^d(y,0,t)v \;\leq\; -\,\alpha\,\|v\|^2 \;+\; \beta\,\|B(y,t)v\|^2 \tag{9.1.13}$$

for all $(y,t) \in \mathbb{R}^d$ and all $v \in \mathbb{C}^n$. If the inequality (9.1.13) is satisfied then the operator

$$R^\gamma(x_d)\;:\;u\;\mapsto\;R^\gamma(x_d)\,u\;:=\;-\,S_0(\cdot,x_d,\cdot)\,A^d(\cdot,x_d,\cdot)\,u$$

defines a boundary symmetrizer according to Definition 9.1. As a matter of fact, the inequality in (9.1.11) directly follows from (9.1.13) applied to $v = u(y,t)$, after integration in (y,t). And we easily compute

$$\begin{aligned}\langle\,\mathrm{Re}\,(\,\mathrm{R}^\gamma(x_d)P^\gamma(x_d)\,)\,u\,,\,u\,\rangle \;=\;& \gamma\,\langle\,S_0(\cdot,x_d,\cdot)\,u\,,\,u\,\rangle\\ &-\frac{1}{2}\,\langle\,(\,\partial_t S_0 + \sum_{j=1}^{d-1}\partial_j(S_0A^j)\,)\,u\,,\,u\,\rangle\,.\end{aligned}$$

Taking a lower bound for S_0, say $\sigma > 0$, and an upper bound for $\partial_t S_0 + \sum \partial_j(S_0A^j)$, say K, we clearly have for $\gamma \geq 2K/\sigma$,

$$\langle\,\mathrm{Re}\,(\,\mathrm{R}^\gamma(x_d)P^\gamma(x_d)\,)\,u\,,\,u\,\rangle\;\geq\;\frac{\gamma\,\sigma}{2}\,\|u\|^2_{L^2}\,,$$

which is (9.1.10) with $C = \sigma/2$. This shows that Friedrichs-symmetrizable operators together with strictly dissipative boundary matrices are endowed with boundary symmetrizers. In what follows, we will consider boundary matrices that are not necessarily dissipative, as in Chapter 4.

Proposition 9.1 *If there is a boundary symmetrizer for* (9.1.5)(9.1.6)*, then there exists* $\gamma_0 \geq 1$ *and* $c > 0$ *so that for* $\gamma \geq \gamma_0$ *and* $u \in \mathscr{D}(\overline{\Omega} \times \mathbb{R})$ *we have*

$$\gamma \int_{\mathbb{R}} \int_{x_d > 0} \mathrm{e}^{-2\gamma t} \, \|u(x,t)\|^2 \, \mathrm{d}x \, \mathrm{d}t \, + \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{-2\gamma t} \, \|u(y,0,t)\|^2 \, \mathrm{d}y \, \mathrm{d}t$$

$$\leq c \left(\frac{1}{\gamma} \int_{\mathbb{R}} \int_{x_d > 0} \mathrm{e}^{-2\gamma t} \, \|(Lu)(x,t)\|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{-2\gamma t} \, \|(Bu)(y,0,t)\|^2 \, \mathrm{d}y \, \mathrm{d}t \right), \tag{9.1.14}$$

and more generally,

$$\gamma \, \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+; H^s_\gamma(\mathbb{R}^d))} \, + \, \|\widetilde{u}_\gamma(0)\|^2_{H^s_\gamma(\mathbb{R}^d)}$$

$$\leq c \left(\frac{1}{\gamma} \, \|L_\gamma \widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+; H^s_\gamma(\mathbb{R}^d))} \, + \, \|B \widetilde{u}_\gamma(0)\|^2_{H^s_\gamma(\mathbb{R}^d)} \right), \tag{9.1.15}$$

where $\widetilde{u}_\gamma(x,t) \, = \, \mathrm{e}^{-\gamma t} \, u(x,t)$*,* $L_\gamma \, = \, L \, + \, \gamma$*, and* $H^s_\gamma(\mathbb{R}^d)$ *denotes the usual Sobolev space of index* s *equipped with the* γ*-depending norm:*

$$\|v\|_{H^s_\gamma(\mathbb{R}^d)} \, = \, \left(\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} (\, \gamma^2 \, + \, \|\eta\|^2 \, + \, \delta^2 \,)^s \, |(\mathscr{F}_{(y,t)} v)(\eta, \delta)|^2 \, \mathrm{d}\delta \, \mathrm{d}\eta \right)^{1/2}$$

$$= \, \|\Lambda^{s,\gamma} \, v\|_{L^2(\mathbb{R}^d)} \, .$$

(Note that by the definition of L_γ and $\widetilde{u}_\gamma$, $L_\gamma \widetilde{u}_\gamma \, = \, \mathrm{e}^{-\gamma t} \, L \, u$.)

Proof The proof is very similar to that of Theorem 2.1 in Chapter 2 (also see the proof of Theorem 2.13).

We begin with the L^2 estimate (9.1.14). We readily have

$$\frac{d}{dx_d} \langle \, \mathrm{R}^\gamma \, \widetilde{u}_\gamma \, , \, \widetilde{u}_\gamma \, \rangle \, = \, 2 \, \mathrm{Re} \, \langle \, \mathrm{R}^\gamma \, (A^d)^{-1} L_\gamma \, \widetilde{u}_\gamma \, , \, \widetilde{u}_\gamma \, \rangle \, + \, 2 \, \mathrm{Re} \, \langle \, \mathrm{R}^\gamma P^\gamma \, \widetilde{u}_\gamma \, , \, \widetilde{u}_\gamma \, \rangle$$

$$+ \, \langle \, \frac{d\mathrm{R}^\gamma}{dx_d} \, \widetilde{u}_\gamma \, , \, \widetilde{u}_\gamma \, \rangle \, .$$

The first and last terms can be bounded by below using uniform bounds for $\|(A^d)^{-1}\|$, $\|\mathrm{R}^\gamma\|_{\mathscr{B}(L^2)}$ and $\|\mathrm{dR}^\gamma / \mathrm{d}x_d\|_{\mathscr{B}(L^2)}$. The middle term is bounded by below in view of (9.1.10). Hence there exist $C_1 > 0$ and C_2 so that for all $\varepsilon > 0$ and $\gamma \geq \gamma_0$

$$\frac{d}{dx_d} \langle \, \mathrm{R}^\gamma \, \widetilde{u}_\gamma \, , \, \widetilde{u}_\gamma \, \rangle \, \geq \, (\, (2C - C_1 \varepsilon) \gamma \, - \, C_2 \,) \, \|\widetilde{u}_\gamma(x_d)\|^2_{L^2(\mathbb{R}^d)}$$

$$- \frac{C_1}{4 \varepsilon \gamma} \, \|L_\gamma \, \widetilde{u}_\gamma(x_d)\|^2_{L^2(\mathbb{R}^d)} \, .$$

Integrating in x_d and using (9.1.11) we get

$$\alpha \, \|\widetilde{u}_\gamma(0)\|^2_{L^2(\mathbb{R}^d)} \, - \, \beta \, \|B\widetilde{u}_\gamma(0)\|^2_{L^2(\mathbb{R}^d)} \, \leq \, (\,(C_1\varepsilon - 2C)\gamma \, + \, C_2\,) \, \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} + \frac{C_1}{4\varepsilon\gamma} \, \|L_\gamma \, \widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \, .$$

Choosing, for instance, $\varepsilon \, = \, C/C_1$, we find that for $\gamma \geq 2C_2/C$

$$\gamma \, \frac{C}{2} \, \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \, + \, \alpha \, \|\widetilde{u}_\gamma(0)\|^2_{L^2(\mathbb{R}^d)} \leq \frac{C_1^2}{4C\gamma} \, \|L_\gamma \, \widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} + \beta \, \|B\widetilde{u}_\gamma(0)\|^2_{L^2(\mathbb{R}^d)} \, .$$

This implies the L^2 estimate in (9.1.14) with $c = \max(C_1^2/2C, \beta)/\min(C/4, \alpha)$

$$\text{for} \quad \gamma \geq \max(2C_2/C, 2C_1/C)\,.$$

The proof of (9.1.15) makes use of pseudo-differential calculus with parameter (see Section C.2). Indeed, applying the L^2 estimate obtained above

$$\begin{aligned} &\gamma \, \|v\|^2_{L^2(\mathbb{R}^+;L^2(\mathbb{R}^d))} \, + \, \|v(0)\|^2_{L^2(\mathbb{R}^d)} \\ &\leq \, c \, \left(\frac{1}{\gamma} \, \|L_\gamma v\|^2_{L^2(\mathbb{R}^+;L^2(\mathbb{R}^d))} \, + \, \|Bv(0)\|^2_{L^2(\mathbb{R}^d)} \right) \end{aligned}$$

to $v \, = \, \Lambda^{s,\gamma} \, \widetilde{u}_\gamma$, where the operator $\Lambda^{s,\gamma}$ acts on the variables (y, t), we readily get

$$\begin{aligned} &\gamma \, \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+;H^s_\gamma(\mathbb{R}^d))} \, + \, \|\widetilde{u}_\gamma(0)\|^2_{H^s_\gamma(\mathbb{R}^d)} \\ &\leq \, c \, \left(\frac{1}{\gamma} \, \|L_\gamma \, \Lambda^{s,\gamma} \, \widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+;L^2(\mathbb{R}^d))} \, + \, \|B \, \Lambda^{s,\gamma} \, \widetilde{u}_\gamma(0)\|^2_{L^2(\mathbb{R}^d)} \right). \end{aligned}$$

Now, using that $\{\,[\,L_\gamma \, , \, \Lambda^{s,\gamma}\,]\}_{\gamma\geq 1}$ is a family of pseudo-differential operators of order $1 + s - 1 = s$, that $\{\,[\,B \, , \, \Lambda^{s,\gamma}\,]\}_{\gamma\geq 1}$ is a family of pseudo-differential operators of order $s - 1$ and $\|v\|_{H^{s-1}_\gamma} \lesssim \gamma^{-1} \, \|v\|_{H^s_\gamma}$ for all v we obtain a constant $c_s \geq 2c$ so that

$$\begin{aligned} &\gamma \, \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+;H^s_\gamma(\mathbb{R}^d))} \, + \, \|\widetilde{u}_\gamma(0)\|^2_{H^s_\gamma(\mathbb{R}^d)} \\ &\leq \, c_s \Big(\tfrac{1}{\gamma} \, \|L_\gamma \, \widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+;H^s_\gamma(\mathbb{R}^d))} \, + \, \tfrac{1}{\gamma} \, \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+;H^s_\gamma(\mathbb{R}^d))} \\ &\qquad + \|B \, \widetilde{u}_\gamma(0)\|^2_{H^s_\gamma(\mathbb{R}^d)} \, + \, \tfrac{1}{\gamma^2} \, \|\widetilde{u}_\gamma(0)\|^2_{H^s_\gamma(\mathbb{R}^d)} \Big) \, . \end{aligned}$$

For $\gamma \, \geq \, \sqrt{2c_s}$, the left-hand side absorbs the extra terms in the right-hand side, and we obtain (9.1.15) with $c = 2c_s$. □

Remark 9.1 By density of $\mathscr{D}$ in H^1, the energy estimate (9.1.14) extends to any $u \in \mathrm{e}^{-\gamma t} H^1(\bar{\Omega} \times \mathbb{R})$ (which is sufficient to pass to the limit in the right-hand side of (9.1.14) applied to $\mathscr{D}$-approximations of u).

9.1.2 *Local/global Kreiss' symmetrizers*

Kreiss' work [103] has shown that strictly dissipative problems are not the only ones to enjoy estimates of the kind (9.1.14). In other words, there are boundary value problems that do admit boundary symmetrizers, even though they are not strictly dissipative (in the sense of Definition 9.2). Such problems are those that satisfy a stability condition based on a *normal modes* analysis: in fact, the stability condition derived by Kreiss involves a *refined* normal modes analysis, in that it pays attention to the so-called *neutral* modes. (These terms will be explained below.) This is in contrast with prior work – dating back to the 1940s in gas dynamics – in which the role of neutral modes was not well understood. For examples of problems that satisfy Kreiss' stability condition without being strictly dissipative, see for instance Chapter 13 on gas dynamics.

Kreiss introduced a tool known nowadays as a Kreiss symmetrizer, which turned out to imply energy estimates without loss of derivatives. He performed the proof of these estimates for constant coefficients and claimed, advisedly, that they were satisfied also in the case of variable coefficients [103]. The proof was later completed by several authors. The actual construction of Kreiss symmetrizers, assuming Kreiss' stability condition, is the bulk part of the original paper [103]. This *tour de force* is explained in Chapter 5 – and also in Chapter 13 for the (linearized) Euler equations in the constant-coefficients case. Its extension to variable coefficients does not contain major difficulties but deserves some explanation.

Before that, let us pause and come back to the analysis of the Boundary Value Problem (9.1.5)(9.1.6). We have decomposed the derivation of energy estimates into the following steps:

i) the derivation of energy estimates from functional symmetrizers,

ii) the construction of functional symmetrizers from symbolic ones.

iii) the construction of symbolic symmetrizers.

The splitting between *i)* and *ii)* parallels Chapter 2 (on the Cauchy problem). Step *i)* has been done in the previous section. Step *ii)* is the main purpose of this section. If step *iii)* is rather easy for the Cauchy problem (assuming only constant multiplicities of eigenvalues, see Theorem 2.3), it is a tough piece for the Boundary Value Problem. However, once the work is done for constant coefficients (see [103] or Chapter 5 in this book), its extension to variable coefficients is not difficult. Details will be given in Section 9.1.3.

Historical note Kreiss' stability condition is often referred to as the (uniform) Lopatinskiĭ–Kreiss–Sakamato condition, as both the Russian Lopatinskiĭ and the Japanese Sakamato have contributed to the theory, independently of Kreiss:

even though Lopatinskiĭ is more famous for his work on elliptic boundary value problems [123], he did contribute to the early developments of the hyperbolic theory [122] and his name here is not misplaced (as seen in Chapter 4, it is more specifically attached to the search for unstable modes by means of the so-called Lopatinskiĭ determinant); and Sakamoto performed simultaneaously with Kreiss a similar work on higher-order hyperbolic equations (see [171–174]).

We adopt here a presentation of Kreiss' symmetrizers slightly different from Chapter 5, which is adapted to variable coefficients and insists on *local* Kreiss' symmetrizers.

Notations In the whole chapter, we denote by $\mathbb{C}^+ = \{ \tau \in \mathbb{C} \,;\ \mathrm{Re}\,\tau \geq 0 \}$ the closed right-half complex plane. Furthermore, in this section and in the next one, we use the following shortcuts:

- the physical space–time set is
$$\mathbb{Y} := \overline{\Omega} \times \mathbb{R} = \{ (y, x_d, t) \,;\ y \in \mathbb{R}^{d-1} \,,\ x_d \in \mathbb{R}^+ \,,\ t \in \mathbb{R} \} \,,$$
and its boundary is $\partial\mathbb{Y} = \partial\Omega \times \mathbb{R} = \{ (y, 0, t) \,;\ y \in \mathbb{R}^{d-1} \,,\ t \in \mathbb{R} \}$;
- the 'frequency' set is
$$\mathbb{P} := (\mathbb{C}^+ \times \mathbb{R}^{d-1}) \backslash (0,0) = \{ (\tau, \eta) \neq (0,0) \,;\ \tau \in \mathbb{C}^+ \,,\ \eta \in \mathbb{R}^{d-1} \} \,,$$
and its intersection with the unit sphere is $\mathbb{P}_1 = \{ (\tau,\eta) \in \mathbb{P}; |\tau|^2 + \|\eta\|^2 = 1 \}$;
- the whole space–time–frequency set is
$$\mathbb{X} := \{ X = (y, x_d, t, \eta, \tau) \,; (y, x_d, t) \in \mathbb{Y} \,,\ (\tau, \eta) \in \mathbb{P} \}$$
and $\mathbb{X}_1^0 = \mathbb{X}^0 \cap \mathbb{X}_1$ with
$$\mathbb{X}^0 := \{ X = (y, 0, t, \eta, \tau) \,; (y, 0, t) \in \partial\mathbb{Y} \,,\ (\tau, \eta) \in \mathbb{P} \} \,,$$
$$\mathbb{X}_1 := \{ X = (y, x_d, t, \eta, \tau) \,; (y, x_d, t) \in \mathbb{Y} \,,\ (\tau, \eta) \in \mathbb{P}_1 \} \,.$$

Finally, for all $X \in \mathbb{X}$, we denote
$$\mathcal{A}(X) = -\,(A^d(x,t))^{-1} \,(\, \tau\, I_n + i\, A(x,t,\eta)\,) \,.$$

Definition 9.3 *A* Kreiss' symmetrizer *for* (9.1.5)(9.1.6) *at some point* $\underline{X} \in \mathbb{X}$ *is a* $\mathscr{C}^\infty$ *matrix-valued function* r *in some neighbourhood* $\mathscr{V}$ *of* $\underline{X}$ *in* $\mathbb{X}$*, which is associated with another* $\mathscr{C}^\infty$ *matrix-valued function* T*, and such that*

i) the matrix $r(X)$ *is Hermitian and* $T(X)$ *is invertible for all* $X \in \mathscr{V}$*,*

ii) there exists $C > 0$ *independent of* $X \in \mathscr{V}$ *so that*
$$\mathrm{Re}\ (\, r(X)\, T(X)^{-1} \mathcal{A}(X) T(X)\,) \geq C\,\gamma\, I_n \,, \tag{9.1.16}$$

iii) and additionally, if $\underline{X} \in \mathbb{X}^0$*, there exist* $\alpha > 0$ *and* $\beta > 0$ *independent of* $X \in \mathscr{V}$ *so that*
$$r(X) \geq \alpha\, I_n - \beta\, (B(y,t) T(X))^* B(y,t) T(X) \,. \tag{9.1.17}$$

Remark 9.2 The existence of a Kreiss symmetrizer at points in $\mathbb{X}\backslash\mathbb{X}^0$ is obviously independent of the boundary conditions!

We focus here on how Kreiss' symmetrizers generate boundary functional symmetrizers (and thus the estimates in (9.1.14)). This part of the analysis was not performed by Kreiss in [103]. It appeared later, in particular in [31], even though not really stated in this way.

Theorem 9.1 *Assume the matrices A^j and B are $\mathscr{C}^\infty$ functions of $(y, x_d, t) \in \mathbb{Y}$ and $(y, t) \in \partial\mathbb{Y}$, respectively, and that they are constant outside a compact set. If* (9.1.5)(9.1.6) *admits a Kreiss' symmetrizer at any point of $\mathbb{X}_1$ then it admits a (functional) boundary symmetrizer.*

Proof Our first aim is to construct a global symmetrizer $\mathcal{R}(X)$ defined for all $X \in \mathbb{X}$ and such that for some positive constant $\widetilde{C}$

$$\mathrm{Re}\,(\,\mathcal{R}(X)\,\mathcal{A}(X)\,) \,\geq\, \widetilde{C}\,\gamma\, I_n\,,\ X \in \mathbb{X}\,, \tag{9.1.18}$$

and for other positive constants $\widetilde{\alpha}$ and $\widetilde{\beta}$

$$\mathcal{R}(X) \,\geq\, \widetilde{\alpha}\, I_n \,-\, \widetilde{\beta}\, B(y,t)^{\mathrm{T}} B(y,t)\,,\ X \in \mathbb{X}^0\,. \tag{9.1.19}$$

Note that these inequalities look very much like microlocal versions of (9.1.10) and (9.1.11), respectively, with $\mathrm{R}^\gamma(x_d)$ of symbol $\mathcal{R}(\cdot, x_d, \cdot, \cdot, \cdot, \gamma)$. So a little pseudo-differential calculus with parameter γ will enable us to conclude.

Suppose the matrices A^j and B are constant outside the open ball $B(0; M)$. By homogeneity in $(\tau, \eta) \in \mathbb{P}$ and in $(x, t) \in \mathbb{Y}\backslash B(0; M)$ of the inequalities in (9.1.18) and (9.1.19), it is sufficient to construct $\mathcal{R}$ in $\mathbb{K} := \mathbb{X} \cap (\overline{B(0; M)} \times \mathbb{P}_1)$ and then extend it by

$$\mathcal{R}(x, t, \eta, \tau) \,=\, \mathcal{R}\Big(\, M\, \frac{(x,t)}{\|(x,t)\|}\,,\, \frac{(\eta,\tau)}{\|(\eta,\tau)\|}\,\Big)$$

for all $(x, t, \eta, \tau) \in (\mathbb{Y}\backslash B(0; M)) \times \mathbb{P}$.

Now, the compact set $\mathbb{K}$ can be covered by a finite number of neighbourhoods $\mathscr{V}_i$, of points in $\mathbb{X}\backslash\mathbb{X}_1^0$ together with a finite number of neighbourhoods $\mathscr{V}_j^0$, of points in $\mathbb{X}_1^0$, on which we have well-defined Hermitian matrices r_i, r_j^0, and invertible matrices T_i, T_j^0 satisfying the requirements of Definition 9.3. Let us introduce a partition of unity associated with the covering $\{\mathscr{V}_i, \mathscr{V}_j^0\}$, that is functions $\varphi_i \in \mathscr{D}(\mathscr{V}_i; [0,1])$ and $\varphi_j^0 \in \mathscr{D}(\mathscr{V}_j^0; [0,1])$ satisfying the identity

$$\sum_i \varphi_i + \sum_j \varphi_j^0 \equiv 1\,.$$

Then we define a global Kreiss' symmetrizer on $\mathbb{K}$ by

$$\mathcal{R}(X) := \sum_i \varphi_i(X)\,(T_i(X)^{-1})^*\,r_i(X)\,T_i(X)^{-1} + \sum_j \varphi_j^0(X)\,(T_j^0(X)^{-1})^*\,r_j^0(X)\,T_j^0(X)^{-1}\,.$$

Note that $\mathcal{R}(X)$ reduces to

$$\sum_j \varphi_j^0(X)\,(T_j^0(X)^{-1})^*\,r_j^0(X)\,T_j^0(X)^{-1}$$

for $X \in \mathbb{K} \cap \mathbb{X}^0$. By construction, $\mathcal{R}(X)$ is always Hermitian.

Defining C as the minimum of the constants C_i and C_j^0 occurring in (9.1.16) for r_i and r_j^0, we easily see that

$$\mathrm{Re}\,(\,\mathcal{R}(X)\,\mathcal{A}(X)\,) \;\geq\; C\,\gamma\,\mathcal{S}(X)\,,$$

where the Hermitian matrix

$$\mathcal{S}(X) := \sum_i \varphi_i(X)\,(T_i(X)^{-1})^*\,T_i(X)^{-1} + \sum_j \varphi_j^0(X)\,(T_j^0(X)^{-1})^*\,T_j^0(X)^{-1}$$

is uniformly bounded by below, say by $\sigma > 0$ on the compact set $\mathbb{K}$. Hence (9.1.18) holds true with $\widetilde{C} := \sigma\,C$.

Similarly, taking α, respectively β, the minimum of the α_j^0, respectively, the maximum of the β_j^0 in the estimates (9.1.17) for r_j^0, we obtain that for all $v \in \mathbb{C}^n$ and all $X \in \mathbb{K} \cap \mathbb{X}^0$

$$v^*\,\mathcal{R}(X)\,v \;\geq\; \alpha\,v^{\mathrm{T}}\,\mathcal{S}(X)\,v \;-\; \beta\,\|B(y,t)\,v\|^2\,.$$

This shows (9.1.19) with $\widetilde{\alpha} := \sigma\,\alpha$.

Once we have on hand the Hermitian matrices $\mathcal{R}(X)$ satisfying (9.1.18) and (9.1.19), it is not difficult to construct a boundary symmetrizer in the sense of Definition 9.1. By a slight abuse of notation, let us simply write $\mathcal{R}(x_d)$ for the function

$$(y,t,\eta,\delta,\gamma) \;\mapsto\; \mathcal{R}(y,x_d,t,\eta,\tau = \gamma + i\delta)\,.$$

This is a symbol in the variables (y,t) with parameter γ and of degree 0. Therefore, by Theorem C.6*i)*, $\{\mathrm{Op}^\gamma(\mathcal{R}(x_d))\}_{\gamma \geq 1}$ is a family of operators of order 0. We claim that

$$\mathrm{R}^\gamma(x_d) \;:=\; \frac{1}{2}\Big(\,\mathrm{Op}^\gamma(\mathcal{R}(x_d)) \;+\; \mathrm{Op}^\gamma(\mathcal{R}(x_d))^*\,\Big)$$

defines a functional boundary symmetrizer. Indeed, by Theorem C.6*ii)*, $\{\mathrm{R}^\gamma(x_d)\}$ differs from $\{\mathrm{Op}^\gamma(\mathcal{R}(x_d))\}$ by a family of order -1. By Remark C.2, this error can be absorbed by the zero-order terms for γ large enough. Now, by the parameter version of Gårding's inequality (Theorem C.7), we deduce from (9.1.19) the inequality

$$\langle \mathrm{R}^\gamma(0)u\,,\,u\rangle \;+\; \beta\,\|Bu\|_{L^2}^2 \;\geq\; \frac{\widetilde{\alpha}}{4}\,\|u\|_{L^2}^2$$

for γ large enough. The case of (9.1.18) is a little trickier, and requires the *sharp* Gårding's inequality (Theorem C.8, in which the smoothness of coefficients is crucial). Applying Theorem C.8 to the degree 1 symbol $\mathcal{R}(X)\,\mathcal{A}(X) \;-\; \widetilde{C}\,\gamma\,I_n$, we infer from (9.1.18) that

$$\mathrm{Re}\;\langle \mathrm{R}^\gamma(x_d)P^\gamma(x_d)u\,,\,u\rangle \;\geq\; \frac{\widetilde{C}}{4}\;\gamma\;\|u\|_{L^2}^2$$

for $\gamma \geq \gamma_0$ large enough. □

Remark 9.3 It is somewhat surprising that, on the one hand, a functional boundary symmetrizer R^γ need only be defined for γ large enough and, on the other hand, (local) Kreiss symmetrizers are considered up to points where $\gamma \,=\, \mathrm{Re}\;\tau$ is zero. The reason is twofold, in relation to both the homogeneity and compactness arguments invoked in the proof of Theorem 9.1. Indeed, the construction of R^γ for $\gamma \geq \gamma_0$ requires Kreiss symmetrizers at points $(x, t, \eta, \tau = \gamma + i\delta)$ with $\gamma \geq \gamma_0$ but not only: in fact, Kreiss symmetrizers are needed on a *compact* subset $\underline{\mathbb{P}}$ of the frequency set $\mathbb{P}$ containing all points of the form $(\tau_1, \eta_1) \;=\; (\tau, \eta)/\|(\tau, \eta)\|$ with $(\tau, \eta) \in \mathbb{P}$ and $\mathrm{Re}\;\tau \geq \gamma_0$; letting $\|\eta\|$ go to infinity, we see that $\underline{\mathbb{P}}$ must contain $\mathbb{P}_1$ and in particular points (τ_1, η_1) such that $\mathrm{Re}\;\tau_1 \;=\; 0$.

9.1.3 *Construction of local Kreiss' symmetrizers*

We keep the notations of the previous section, and discuss the construction of (local) Kreiss' symmetrizers at points of $\mathbb{X}_1$. As already noted, Kreiss' symmetrizers at points $X \;=\; (y, x_d, \eta, \tau)$ with $x_d > 0$ do not depend on the boundary conditions: their construction will not necessitate any tricky assumption other than the constant hyperbolicity of the operator L. Otherwise, for points in $\mathbb{X}_1^0$, we may distinguish between the case $\mathrm{Re}\;\tau > 0$ (i.e. $(\tau, \eta) \in \overset{\circ}{\mathbb{P}}$) and the case $\mathrm{Re}\;\tau = 0$, the latter being much trickier than the former: at points of

$$\breve{\mathbb{X}}^0 \;:=\; \{X = (y, 0, t, \eta, \tau) \in \mathbb{X}_1^0\,;\;\mathrm{Re}\;\tau > 0\,\}\,,$$

the existence of Kreiss' symmetrizers relies on the Lopatinskiĭ condition; at points of $\mathbb{X}_1^0 \backslash \breve{\mathbb{X}}^0$ it requires the (uniform) Kreiss–Lopatinskiĭ condition.

Before going into detail, let us introduce some additional material. In what follows, we denote

$$\breve{\mathbb{X}} \;:=\; \{X = (y, x_d, t, \eta, \tau) \in \mathbb{X}_1\,;\;\mathrm{Re}\;\tau > 0\,\}\,.$$

The hyperbolicity of the operator L ensures that the matrix $\mathcal{A}(X)$ is hyperbolic – in the sense of ODEs – for all $X \in \breve{\mathbb{X}}$: this observation dates back to Hersh [83] and has already been used in Chapter 4 for constant-coefficient problems. Therefore, for all $X \in \check{\mathbb{X}}$, we may consider the *stable subspace* of $\mathcal{A}(X)$, which we denote by $E_-(X)$. By Dunford–Taylor's formula, E_- is locally smooth. More precisely, if $\mathscr{V}$ is a small enough open subset of $\check{\mathbb{X}}$, there exists a closed contour Γ enclosing all eigenvalues of $\mathcal{A}(X)$ of negative real part for $X \in \mathscr{V}$, and the formula

$$P_-(X) := \frac{1}{2\,i\,\pi} \int_\Gamma (\,z\,I_n \;-\; \mathcal{A}(X)\,)^{-1}\,\mathrm{d}z$$

defines a projector onto the stable subspace $E_-(X)$ such that Ker $P_-(X)$ is the unstable subspace of $\mathcal{A}(X)$. Clearly, P_- inherits the regularity of $\mathcal{A}$: namely, it is analytic in (η, τ) and $\mathscr{C}^\infty$ in (x,t). The projector P_- will be our basic tool to construct Kreiss' symmetrizers at points of $\check{\mathbb{X}}$.

Remark 9.4 The above representation of $E_-(X)$ shows its dimension is locally constant and thus independent of X in the connected set $\check{\mathbb{X}}$. Observing that, at $X = (y,0,t,0,1)$, the matrix $\mathcal{A}(X)$ reduces to

$$\mathcal{A}(y,0,t,0,1) \;=\; -\,(A^d(y,0,t))^{-1} \;=\; (A(y,0,t,\nu(y,0)))^{-1}\,,$$

we see the stable subspace $E_-(y,0,t,0,1)$ of $\mathcal{A}(y,0,t,0,1)$ coincides with the stable subspace $E^s(A(y,0,t,\nu(y,0)))$. Consequently, if the rank p of $B(y,t)$ is known to be the dimension of $E^s(A(y,0,t,\nu(y,0))$, the dimension of $E_-(X)$ also equals p for all $X \in \check{\mathbb{X}}$.

The Lopatinskiĭ condition

We call the Lopatinskiĭ condition at some point $X \in \check{\mathbb{X}}^0$ the requirement that the mapping $B(y,t)_{|E_-(X)} : E_-(X) \to \mathbb{R}^p$ be an isomorphism. This algebraic condition is equivalent to an analytical condition on the homogeneous constant-coefficients problem, say $(\Pi_{(y,t)})$, obtained by *freezing* the coefficients at $(y,0,t)$ in (9.1.5)(9.1.6) and by taking $f = g = 0$ as source terms. The mapping $B(y,t)_{|E_-(X)}$ turns out to be an isomorphism if and only if the problem $(\Pi_{(y,t)})$ does not have any non-trivial solution with the following features: square integrability in the direction orthogonal to the boundary; oscillations with wave vector η in the direction of the boundary; exponential-type behaviour $\mathrm{e}^{\tau\,\cdot}$ in time. Would they exist, such solutions would be called *normal modes*. The equivalence between the algebraic condition and the analytical one is a straightforward consequence of the definition of $E_-(X)$, the stable subspace of the hyperbolic matrix $\mathcal{A}(X)$.

For $B(y,t)_{|E_-(X)} : E_-(X) \to \mathbb{R}^p$ to be an isomorphism, an obvious necessary condition is $\dim E_-(X) = p$, which is true as soon as $p = \dim E^s(A(y,0,t,\nu(y,0))$ (see Remark 9.4 above). Once $\dim E_-(X) = p$ is known, it suffices to check that $B(y,t)_{|E_-(X)}$ is one-to-one, a quantitative version of this condition being

(L_X) There exists $C > 0$ so that for all $V \in E_-(X)$, $\quad \|V\| \leq C\,\|\,B(y,t)\,V\,\|\,.$

Clearly, (L_X) is an open condition. In other words, the constant C is locally uniform in $\breve{\mathbb{X}}^0$. Furthermore, by homogeneity degree 0 of E_-, the condition (L_X) is equivalent (with the same constant C) to (L_{X_1}), where $X_1 \in \breve{\mathbb{X}}^0_1$ is defined by $X_1 = (y, 0, t, \eta_1, \tau_1)$, $(\eta_1, \tau_1) = (\tau, \eta)/\|(\tau, \eta)\|$ if $X = (y, 0, t, \eta, \tau)$.

Remark 9.5 As already observed in Remark 9.4 above, the stable subspace $E_-(y, 0, t, 0, 1)$ of $\mathcal{A}(y, 0, t, 0, 1)$ coincides with the stable subspace $E^s(A(y, 0, t, \nu(y, 0)))$. Therefore, the condition (L_X) at $X = (y, 0, t, 0, 1)$ requires in particular that the intersection of $\ker B(y, t)$ and $E^s(A(y, 0, t, \nu(y, 0))$ be zero. If we also know that the rank of $B(y, t)$ equals the dimension of $E^s(A(y, 0, t, \nu(y, 0))$, the normality condition in (9.0.1) is just a reformulation of $(\mathrm{L}_{(y,0,t,0,1)})$: this is why there will be no need to mention (9.0.1) in the assumptions of the main theorems ((9.0.1) will be a consequence of those assumptions). One may observe additionally that $(\mathrm{L}_{(y,0,t,0,1)})$ amounts to a *one-dimensional stability condition*, in which no transversal modes (in $\mathrm{e}^{i\eta y}$) are considered.

The uniform Kreiss–Lopatinskiĭ condition

The extension of (L_X) to points X with $\mathrm{Re}\,\tau = 0$ looks the same but with a careful definition of $E_-(X)$, no longer the stable subspace of $\mathcal{A}(X)$. For, the matrix $\mathcal{A}(y, 0, t, \eta, \tau)$ with $\mathrm{Re}\,\tau = 0$ is no longer hyperbolic in general. This is where the so-called *neutral* modes come into play: by definition, the time behaviour of neutral modes is $\mathrm{e}^{\tau\, t}$ with $\mathrm{Re}\,\tau = 0$; but in fact only those modes with amplitude in $E_-(X)$ are to be considered, with $E_-(X)$ defined by *continuous extension* of the projector $P_-(X)$ as $E_-(X) = \mathrm{Im}(P_-(X))$.

Lemma 9.1 *Assume the operator L is constantly hyperbolic, for all* $\underline{X} \in \mathbb{X}^0 \backslash \breve{\mathbb{X}}^0$. *Then there exists a projector* $P_-(\underline{X})$ *of rank* p *such that*

$$P_-(\underline{X}) = \lim_{X \overset{\mathbb{X}}{\to} \underline{X}} P_-(X)\,.$$

This innocent-looking result is highly non-trivial: observe indeed that in general some eigenvalues of $\mathcal{A}(y, x_d, t, \eta, \tau)$ cross the contour Γ as $\mathrm{Re}\,\tau$ approaches zero. The proof for a constant-coefficients operator L is given in Chapter 5. A careful look at the arguments shows that they remain valid for variable coefficients. The details are left to the reader.

Remark 9.6 In general, for $X = (y, 0, t, \eta, \tau) \in \mathbb{X}^0$ with $\mathrm{Re}\,\tau = 0$, the actual stable subspace of $\mathcal{A}(X)$ is strictly embedded in $E_-(X)$, which also contains a *part of the center subspace* of $\mathcal{A}(X)$.

Once the subspace $E_-(X)$ is properly defined at all points $X \in \mathbb{X}^0$, the Lopatinskiĭ condition at those points still reads as (L_X).

Now, what we call the uniform Kreiss–Lopatinskiĭ condition is merely the following.

(UKL) The condition (L_X) is satisfied for all $X \in \mathbb{X}^0$.

Remark 9.7 The term *uniform* attached to Kreiss–Lopatinskiĭ refers to the fact that the constant C in (L_X) may be chosen to be uniform in $\mathbb{X}_1^0$. Indeed, for $X = (y, 0, t, \eta, \tau) \in \mathbb{X}_1^0$, the constant C in (L_X) depends continuously on (y, t, η, τ), is independent of (y, t) outside a compact subset of $\mathbb{R}^d$ (provided that L and B have constant coefficients outside a compact set) and (τ, η) lies in the compact set $\mathbb{P}_1$.

Practical verification of (UKL)

The usual way to check whether (UKL) is satisfied consists in finding a basis of $E_-(X)$, say $(\mathrm{e}_1(X), \ldots, \mathrm{e}_p(X))$ (a classical argument of Kato [95], p. 99–101 implies that this basis can be chosen to depend smoothly on X) and in looking for the zeroes of the so-called Lopatinskiĭ determinant

$$\Delta(X) := \det(B(y,t)\,\mathrm{e}_1(X), \ldots, B(y,t)\,\mathrm{e}_p(X))\,.$$

As a matter of fact, the mapping $B(y,t)_{|E_-(X)}$ is an isomorphism if and only if $\Delta(X) \neq 0$ and therefore (UKL) is clearly equivalent to

$$\Delta(X) \neq 0 \quad \forall X \in \mathbb{X}^0\,.$$

In practice, the Lopatinskiĭ determinant $\Delta(X)$ is usually derived for $X \in \breve{\mathbb{X}}^0$ first, and then extended by continuity to points $X = (y, 0, t, \eta, \tau)$ such that $\mathrm{Re}\ \tau = 0$. However, we warn the reader that this limiting procedure requires some care to avoid introducing fake zeroes. In any case, we claim the search for zeroes of Δ is mostly algebraic. See Section 4.6 for more details (and also Chapter 13.2.18 for an actual example).

Failure of (UKL)

If a zero of Δ is found in $\breve{\mathbb{X}}^0$, the well-posedness of the BVP (9.1.5)(9.1.6) is hopeless: as was shown in Chapter 4 the existence of non-trivial normal modes with $\mathrm{Re}\ \tau > 0$ is responsible for a Hadamard instability.

If Δ does not vanish on $\breve{\mathbb{X}}^0$ but has a zero $(y, 0, t, \eta, \tau)$ with $\mathrm{Re}\ \tau = 0$, we say the BVP is weakly stable. The non-trivial kernel of $B(y,t)_{|E_-(y,0,t,\eta,\tau)}$ implies the existence of non-trivial neutral modes, which oscillate as $\mathrm{e}^{\tau\, t}$ and are bounded but not necessarily square-integrable transversally to the boundary: in this case (UKL) fails but it is still possible to construct (weaker) Kreiss' symmetrizers and obtain energy estimates with (a limited) loss of derivatives; this goes beyond the scope of this book and we refer to [41, 170] for more details.

Construction of Kreiss' symmetrizers

The hard part of the job is the construction of Kreiss' symmetrizers for constant-coefficients problems: In general, it involves an intricate piece of matrix analysis and algebraic geometry, see Chapter 5; It is easier in the specific case of gas dynamics, see Chapter 13. Using the constant-coefficients construction we can indeed construct, for *fixed* $(y, x_d, t) \in \mathbb{Y}$, Hermitian matrices $r(y, x_d, t, \eta, \tau)$

and invertible matrices $T(y, x_d, t, \eta, \tau)$ satisfying the inequalities in (9.1.16) and (9.1.17) for all $(\tau, \eta) \in \mathbb{P}_1$.

The main thing to do here is to allow variations of (y, x_d, t) and check the smoothness of r and T with respect to (x, t). In fact, the careful construction made in Chapter 5 is robust under perturbation of coefficients. We will not go into details in all cases. For clarity, we just give the explicit construction in the easiest cases, namely the first and second one in the following list.

i) At points of $\breve{\mathbb{X}}_1$, Kreiss' symmetrizers are easily found by a block reduction of the matrix $\mathcal{A}(X)$ and the Lyapunov matrix theorem.

ii) At points $\underline{X} \in \mathbb{X}_1 \backslash \breve{\mathbb{X}}_1$ where the matrix $\mathcal{A}$ is smoothly diagonalizable – which means there exist invertible matrices $T(X)$ depending smoothly on X in some neighbourhood of $\underline{X}$ such that $T(X)^{-1}\mathcal{A}(X)T(X)$ is diagonal – Kreiss' symmetrizers exist in diagonal form.

iii) At 'generic' points of $\mathbb{X}_1 \backslash \breve{\mathbb{X}}_1$, the construction of Kreiss' symmetrizers is undoubtedly cumbersome.

On the one hand, the smooth diagonalizability condition in *ii)* implies in particular that the distinct eigenvalues of $\mathcal{A}(X)$ are smooth functions of X, analytic in (η, τ), in some neighbourhood of $\underline{X}$. On the other hand, *iii)* comprises the so-called case of *glancing* points, where two (or more) eigenvalues of $\mathcal{A}(X)$ collide, and is much more involved. It requires a specific assumption, named by Majda the *block-structure condition*, which is now known to hold true for any constantly hyperbolic system [134]. For the case of systems with variable multiplicities, we refer to the recent work of Métivier and Zumbrun [135].

Remark 9.8 In gas dynamics, $\mathcal{A}(X)$ is smoothly diagonalizable at all points except the glancing points, in such a way that case *iii)* reduces to those special, glancing points, where the construction of a Kreiss' symmetrizer is still rather easy (see Chapter 13).

Construction of Kreiss symmetrizers in case *i)* Let us fix $\underline{X} \in \breve{\mathbb{X}}_1$ and a neighbourhood $\mathscr{V}$ of $\underline{X}$ in $\breve{X}_1$ where the projector $P_-(X)$ onto $E_-(X)$ is well-defined, while $P_+(X) := I_n - P_-(X)$ is a projector onto the unstable subspace $E_+(X)$ of $\mathcal{A}(X)$. Using Kato's argument [95], we can find smooth bases of the ranges of $P_-(X)$ and $P_+(X)$. (Here and in what follows, *smooth* always means $\mathscr{C}^\infty$ in (y, x_d, t) and analytic in (τ, η).) Then the matrix $T(X)$ composed of the corresponding column vectors is invertible, depends smoothly on $X \in \mathscr{V}$, and reduces $\mathcal{A}(X)$ to a block-diagonal form:

$$a(X) := T(X)^{-1}\,\mathcal{A}(X)\,T(X) = \left(\begin{array}{c|c} \mathcal{A}^-(X) & 0 \\ \hline 0 & \mathcal{A}^+(X) \end{array}\right),$$

where $\mathcal{A}^-$ only has a spectrum of negative real part and $\mathcal{A}^+$ only has a spectrum of positive real part. We now invoke the following result, which is a variant of the Lyapunov matrix theorem.

Lemma 9.2 *If A is a matrix whose spectrum entirely lies in the open half-plane*

$$\{ z\, ;\ \mathrm{Re}\ z\ >\ 0 \}$$

there exists a unique positive-definite Hermitian matrix H such that

$$\mathrm{Re}\ (H\,A)\ =\ I\,.$$

Furthermore, H admits the explicit representation

$$H\ =\ \int_{-\infty}^{0} \mathrm{e}^{t\,A^*}\ \mathrm{e}^{t\,A}\ \mathrm{d}t\,.$$

As a consequence, if A depends smoothly on a parameter, so does H.

Applying Lemma 9.2 to $\mathcal{A}^+(X)$ and $-\mathcal{A}^-(X)$ we obtain two positive-definite Hermitian matrices $H^+(X)$ and $H^-(X)$, depending smoothly on $X \in \mathscr{V}$ such that

$$\mathrm{Re}\ (H^{\pm}(X)\,\mathcal{A}^{\pm}(X))\ =\ \pm I\,.$$

Let us now consider a matrix $r(X)$ of the form

$$r(X)\ =\ \left(\begin{array}{c|c} -\,H^-(X) & 0 \\ \hline 0 & \mu\,H^+(X) \end{array}\right)\,,$$

with $\mu > 0$ to be determined. By construction of $H^{\pm}$,

$$\mathrm{Re}\ (r(X)\,a(X))\ =\ \left(\begin{array}{c|c} I & 0 \\ \hline 0 & \mu\,I \end{array}\right)\,.$$

So (9.1.16) is satisfied as soon as $\mu \geq 1$ (recall that for $X \in \mathbb{X}_1$, the last component has a real part $\gamma \leq 1$). This is all that we have to do if $\underline{X}$ is not in $\breve{\mathbb{X}}^0$. If $\underline{X} \in \breve{\mathbb{X}}^0_1$, the actual choice of μ will come from the fulfillment of (9.1.17). By (UKL), up to diminishing $\mathscr{V}$ to a compact neighbourhood of $\underline{X}$, we find a constant C such that

$$\|\,P_-(X)\,V\,\|^2\ \leq\ C\left(\,\|P_+(X)\,V\|^2 + \|\,B(y,t)\,V\,\|^2\,\right) \qquad \text{for all } V \in \mathbb{C}^n\,,$$
$$\text{for all } X \in \mathscr{V}\,.$$

Because of the block structure of $r(X)$, we have for all $v = \begin{pmatrix} v_- \\ v_+ \end{pmatrix} \in \mathbb{C}^n$

$$v^*\,r(X)\,v\ =\ -\,v_-^*\,H^-(X)\,v_-\ +\ \mu\,v_+^*\,H^+(X)\,v_+\ \geq\ \|v_-\|^2\ +\ b\,\mu\,\|v_+\|^2$$
$$-\,(1+c)\,\|v_-\|^2$$

with $b > 0$ such that $H^+\ \geq\ b\,I$ and $c > 0$ such that $H^-\ \leq\ c\,I$. Now, denoting $V\ =\ T(X)\,v$ we observe that

$$P_-(X)\,V\ =\ T(X)\begin{pmatrix} v_- \\ 0 \end{pmatrix}\,, \quad P_+(X)\,V\ =\ T(X)\begin{pmatrix} 0 \\ v_+ \end{pmatrix}$$

and thus

$$\|T(X)\|^{-1}\,\|P_\pm(X)\,V\| \;\leq\; \|v_\pm\| \;\leq\; \|T(X)^{-1}\|\,\|P_\pm(X)\,V\|\,.$$

Therefore

$$\begin{aligned} v^*\,r(X)\,v \geq\; & \|T(X)\|^{-2}\,(\,\|P_-(X)\,V\|^2 \,+\, b\,\mu\,\|P_+(X)\,V\|^2\,) \\ & -(1+c)\,\|T(X)^{-1}\|^2\,\|P_-(X)\,V\|^2 \\ \geq\; & \|T(X)\|^{-2}\Big(\|P_-(X)V\|^2 + (b\mu - C(1+c)\|T(X)^{-1}\|^2)\|P_+(X)V\|^2\Big) \\ & -C\,(1+c)\,\|T(X)^{-1}\|^2\,\|\,B(y,t)\,V\,\|^2\,. \end{aligned}$$

So if μ is chosen greater than

$$\mu_0 \;:=\; 2\,C\,(1+c)\,\max_{X\in\mathscr{V}}\|T(X)^{-1}\|^2/b\,,$$

we have

$$v^*\,r(X)\,v \;\geq\; \alpha\,\|v\|^2 \;-\; \beta\,\|\,B(y,t)\,T(X)\,v\,\|^2$$

for

$$\alpha \;=\; \frac{1}{2}\,\min_{X\in\mathscr{V}}\|T(X)\|^{-2}\,\min\Big(1,\,\frac{1}{2}\,b\,\mu_0\Big) \quad \text{and} \quad \beta \;=\; \frac{1}{2}\,b\,\mu_0\,.$$

This proves the estimate in (9.1.17).

Construction of Kreiss symmetrizers in case *ii)* It basically works as in the previous case, by passing to the limit in the projection operators. Thus we still have an estimate of the form

$$\|\,P_-(X)\,V\,\|^2 \leq C\,\Big(\,\|P_+(X)\,V\|^2 + \|\,B(y,t)\,V\,\|^2\Big) \text{ for all } V\in\mathbb{C}^n \text{ and } X\in\mathscr{V}\,,$$

where $\mathscr{V}$ is a neighbourhood of $\underline{X}$ in $\mathbb{X}$. Reducing $\mathcal{A}(X)$ to diagonal form

$$a(X) \;=\; \left(\begin{array}{c|c} \mathcal{A}^-(X) & 0 \\ \hline 0 & \mathcal{A}^+(X) \end{array}\right),$$

with $\mathcal{A}^\pm(X)$ diagonal and

$$\pm\,\mathrm{Re}\;\mathcal{A}^\pm(X) \;\geq\; C\,\gamma\,I\,,$$

we get (9.1.16), and also (9.1.17) for μ large enough, merely by setting

$$r(X) \;=\; \left(\begin{array}{c|c} -I & 0 \\ \hline 0 & \mu\,I \end{array}\right).$$

Construction of Kreiss symmetrizers in case *iii)* We refer to Chapter 5. See also [31].

In conclusion, we have the following.

Theorem 9.2 *Under the following assumptions:*

- *The operator L is* constantly hyperbolic, *that is, the matrices $A(x,t,\xi)$ are diagonalizable with real eigenvalues of constant multiplicities on $\Omega \times \mathbb{R} \times (\mathbb{R}^d \backslash \{0\})$;*
- *The boundary $\partial\Omega$ of $\Omega = \{ x \in \mathbb{R}^d ;\ x_d > 0\}$ is* non-characteristic, *in that the matrix $A^d(x,t)$ is non-singular along $\partial\Omega \times \mathbb{R}$,*
- *The boundary matrix B is of constant, maximal rank equal to the number of incoming characteristics along $\partial\Omega \times \mathbb{R}$, which are defined as the positive eigenvalues of A^d (counted with multiplicity),*
- *The uniform Kreiss–Lopatinskiĭ condition* (UKL) *is satisfied;*

there exists a Kreiss' symmetrizer (according to Definition 9.3) at any point of $\mathbb{X}_1^0$.

As a consequence of this theorem together with Proposition 9.1 and Theorem 9.1 we have the following.

Theorem 9.3 *Under the assumptions of Theorems 9.1 and 9.2, for all $s \in \mathbb{R}$, there exists $\gamma_s \geq 1$ and $C_s > 0$ such that for all $\gamma \geq \gamma_s$ and $u \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$,*

$$\gamma \, \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+;H^s_\gamma(\mathbb{R}^d))} \, + \, \|\widetilde{u}_{\gamma \,|\, x_d=0}\|^2_{H^s_\gamma(\mathbb{R}^d)}$$

$$\leq \, C_s \, \left(\frac{1}{\gamma} \, \|L_\gamma \widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^+;H^s_\gamma(\mathbb{R}^d))} \, + \, \|B\widetilde{u}_{\gamma \,|\, x_d=0}\|^2_{H^s_\gamma(\mathbb{R}^d)} \right),$$

where $\widetilde{u}_\gamma(x,t) \, = \, \mathrm{e}^{-\gamma t} \, u(x,t)$ and $\|v\|_{H^s_\gamma(\mathbb{R}^d)} \, = \, \|\Lambda^{s,\gamma} v\|_{L^2(\mathbb{R}^d)}$.

Remark 9.9 It is also possible to derive estimates in weighted Sobolev spaces

$$\mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}) \, := \, \{ \, u \, = \, \mathrm{e}^{\gamma \, t} \, \widetilde{u}_\gamma \, ; \, \widetilde{u}_\gamma \, \in \, H^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}) \, \}$$

for $m \in \mathbb{N}$. (Sometimes, we will use the (abuse of) notation $\mathrm{e}^{\gamma t} \, H^m_\gamma$ instead of $\mathscr{H}^m_\gamma$.) Recalling that for functions of $(y,t) \in \mathbb{R}^d$,

$$\|v\|^2_{H^m_\gamma(\mathbb{R}^d)} \, \simeq \, \sum_{|\alpha| \leq m} \gamma^{2(m-|\alpha|)} \, \|\partial^\alpha v\|^2_{L^2(\mathbb{R}^d)} \, ,$$

(the sign $\simeq$ standing for two-sided inequalities with constants independent of γ and v, see Remark C.1), hence also

$$\|\mathrm{e}^{-\gamma \, t} \, u \, \|^2_{H^m_\gamma(\mathbb{R}^d)} \, \simeq \, \sum_{|\alpha| \leq m} \gamma^{2(m-|\alpha|)} \, \|\mathrm{e}^{-\gamma \, t} \, \partial^\alpha u\|^2_{L^2(\mathbb{R}^d)} \, ,$$

it is natural to equip $\mathscr{H}^m_\gamma(\bar{\mathcal{O}})$, for any open domain $\mathcal{O}$ of $\mathbb{R}^n$, with the norm defined by

$$\|u\|^2_{\mathscr{H}^m_\gamma(\bar{\mathcal{O}})} := \sum_{|\alpha|\leq m} \gamma^{2(m-|\alpha|)} \, \|\mathrm{e}^{-\gamma\, t}\, \partial^\alpha u\|^2_{L^2(\mathcal{O})} \, .$$

The result of Theorem 9.3 may be viewed as a $L^2(\mathbb{R}^+; \mathscr{H}^m_\gamma(\mathbb{R}^d))$ estimate:

$$\gamma\,\|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\mathbb{R}^d))} + \|u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} \leq C\left(\frac{1}{\gamma}\,\|Lu\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\mathbb{R}^d))} + \|Bu_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}\right) .$$

In fact, this also implies a $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ estimate, as shown in the following.

Corollary 9.1 *Under the assumptions of Theorems 9.1 and 9.2, for all $m \in \mathbb{N}$, there exists $\gamma_m \geq 1$ and $C_m > 0$ such that for all $\gamma \geq \gamma_m$ and $u \in \mathscr{D}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$,*

$$\begin{aligned}&\gamma\,\|u\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}\\ &\leq C_m\left(\frac{1}{\gamma}\,\|Lu\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|Bu_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}\right).\end{aligned}$$

Proof Since

$$\|u\|_{\mathscr{H}^0_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} = \|\mathrm{e}^{-\gamma t}u\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \, ,$$

the case $m = 0$ is just a reformulation of the estimate in Theorem 9.3 (with $s = 0$).

For $m \geq 1$, recall that

$$\|u\|_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} = \sum_{|\alpha|\leq m} \gamma^{2(m-|\alpha|)} \, \|\mathrm{e}^{-\gamma\, t}\, \partial^\alpha u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \, .$$

We already have an estimate for the terms of this sum that do not involve derivatives with respect to x_d, for which $\alpha_d = 0$. It remains to bound the terms for which $\alpha_d \geq 1$. This is done by using the equality

$$\mathrm{e}^{-\gamma t}\,\partial_d\, u = (A^d)^{-1}\,(\,L_\gamma\,\widetilde{u}_\gamma - \gamma\,\widetilde{u}_\gamma - \partial_t\widetilde{u}_\gamma - \sum_{j=1}^{d-1} A^j\,\partial_j\,\widetilde{u}_\gamma\,)\,, \qquad (9.1.20)$$

which already implies the $\mathscr{H}^1_\gamma$ estimate. Indeed, we have

$$\|\mathrm{e}^{-\gamma t}\,\partial_d\, u\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \leq C'\,\|L_\gamma\,\widetilde{u}_\gamma\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + C'\,\|\widetilde{u}_\gamma\|_{L^2(\mathbb{R}^+;H^1_\gamma(\mathbb{R}^d))}\,,$$

where $C' > 0$ depends only on $\|(A^d)^{-1}\|_{L^\infty}$ and $\|A^j\|_{L^\infty}$. Since

$$\|u\|^2_{\mathscr{H}^1_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} = \|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^1_\gamma(\mathbb{R}^d))} + \|\mathrm{e}^{-\gamma t}\,\partial_d\, u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})}\,,$$

we get by using the $L^2(\mathbb{R}^+;\mathscr{H}^1_\gamma(\mathbb{R}^d))$ estimate,

$$\begin{aligned}\gamma\,\|u\|^2_{\mathscr{H}^1_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} &+ \|u_{|x_d=0}\|^2_{\mathscr{H}^1_\gamma(\mathbb{R}^d)}\\ &\leq \frac{C}{\gamma}\,\|Lu\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^1_\gamma(\mathbb{R}^d))} + C\,\|Bu_{|x_d=0}\|^2_{\mathscr{H}^1_\gamma(\mathbb{R}^d)}\\ &\quad+\frac{2C'}{\gamma}\,\gamma^2\,\|\mathrm{e}^{-\gamma t}\,Lu\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + 2C'\gamma\,\|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^1_\gamma(\mathbb{R}^d))},\end{aligned}$$

hence, using again the $L^2(\mathbb{R}^+;\mathscr{H}^1_\gamma(\mathbb{R}^d))$ estimate,

$$\begin{aligned}\gamma\,\|u\|^2_{\mathscr{H}^1_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} &+ \|u_{|x_d=0}\|^2_{\mathscr{H}^1_\gamma(\mathbb{R}^d)}\\ &\leq \max\left(C(1+2C'),2C'\right)\frac{1}{\gamma}\,\|Lu\|^2_{\mathscr{H}^1_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})}\\ &\quad+C(1+2C')\,\|Bu_{|x_d=0}\|^2_{\mathscr{H}^1_\gamma(\mathbb{R}^d)}\,.\end{aligned}$$

More generally, the $\mathscr{H}^k_\gamma$ estimates for $k\leq m$ can be proved by induction, differentiating the equality (9.1.20). Details are omitted (see the proof of Theorem 9.7, which even shows $\mathscr{H}^m_\gamma$ estimates for less-smooth coefficients). □

9.1.4 *Non-planar boundaries*

Theorem 9.3/Corollary 9.1 have natural extensions to more general domains Ω. At first, we may consider a domain Ω diffeomorphic to a half-space. Since a change of variables preserves the constant hyperbolicity of the operator (see, for instance, the proof of Theorem 2.10 in Chapter 2) and the non-characteristicness of the boundary, we get an extended version of Theorem 9.3 (for $s=0$) by changing the other assumptions accordingly. In particular, the uniform Kreiss–Lopatinskiĭ condition has to do with points in the set

$$\mathbb{X}^0_1 := \{\,X = (x,t,\xi,\tau)\,;\ (x,\xi)\in T^*\partial\Omega\,,\ \tau\in\mathbb{C}^+\,,\ |\tau|^2 + \|\xi\|^2 = 1\,\}\,,$$

where $T^*\partial\Omega$ is the cotangent bundle of $\partial\Omega$. (We recall that $\tau\in\mathbb{C}^+$ means Re $\tau\geq 0$.) As done before, we denote by ν the exterior unit normal vector field on $\partial\Omega$. Then, for all $X\in\mathbb{X}^0_1$ we denote by $E_-(X)$ the stable subspace of

$$\mathcal{A}(X) := A(x,t,\nu(x))^{-1}\,(\,\tau\,I_n + i\,A(x,t,\xi)\,)\,.$$

In this generalized framework, the condition (L_X) reads as before:

(L_X) There exists $C>0$ so that for all $V\in E_-(X)$, $\quad\|V\|\leq C\,\|\,B(y,t)\,V\|\,.$

Theorem 9.4 *We assume that $\overline{\Omega}$ is diffeomorphic to a half-space. Other assumptions are, the coefficients (in the operator L and in the boundary matrix B) are constant outside a compact subset of $\overline{\Omega}\times\mathbb{R}$ and furthermore:*

- **(CH)** *The operator L is* constantly hyperbolic*;*
- **(NC)** *The boundary $\partial\Omega$ is* non-characteristic*;*

(N) *The boundary matrix* $B(x,t)$ *is of constant, maximal rank* $p = \dim E^s(A(x,t,\nu(x)))$ *for* $(x,t) \in \partial\Omega \times \mathbb{R}$;

(UKL) (L_X) *holds for all* $X \in \mathbb{X}_1^0$.

Then there exists $C > 0$ *such that for all* $u \in \mathscr{D}(\overline{\Omega} \times \mathbb{R})$ *and all* $\gamma \geq 1$,

$$\gamma \, \|\widetilde{u}_\gamma\|^2_{L^2(\overline{\Omega}\times\mathbb{R})} \, + \, \|\widetilde{u}_\gamma\|^2_{L^2(\partial\Omega\times\mathbb{R})} \, \leq \, C \, \left(\frac{1}{\gamma} \, \|L_\gamma \widetilde{u}_\gamma\|^2_{L^2(\overline{\Omega}\times\mathbb{R})} \, + \, \|B\widetilde{u}_\gamma\|^2_{L^2(\partial\Omega\times\mathbb{R})} \right), \tag{9.1.21}$$

where $\widetilde{u}_\gamma(x,t) \, = \, \mathrm{e}^{-\gamma t} \, u(x,t)$.

Proof If

$$\begin{aligned} \Phi \, : \, \bar{\Omega} &\to D \, = \, \{\, \widetilde{x} \, ; \, \widetilde{x}_d \, \geq \, 0 \,\} \\ x &\mapsto \widetilde{x} \, = \, \Phi(x) \end{aligned}$$

is a diffeomorphism then under the space–time change of variables $\Psi : (x,t) \mapsto (\Phi(x), t)$, the operator L becomes $\widetilde{L}$ such that

$$(\widetilde{L}\widetilde{v})(\widetilde{x}, t) \, = \, (Lv)(x,t)$$

for all (smooth enough) $v = \widetilde{v} \circ \Psi$. By the chain rule we easily see that

$$\widetilde{L} \, = \, \partial_t \, + \, \sum_{j=1}^{d} \widetilde{A}^j \partial_{\widetilde{x}_j} \, ,$$

where

$$\widetilde{A}^j(\widetilde{x}, t) \, := \, \sum_{k=1}^{d} \partial_{x_k} \widetilde{x}_j(x) \, A^k(x,t) \quad \text{for all } j \in \{1, \ldots, d\} \, .$$

Observing that

$$\widetilde{A}(\, \widetilde{x}, t, \widetilde{\xi} \,) \, := \, \sum_j \, \widetilde{\xi}_j \, \widetilde{A}^j(\widetilde{x}, t) \, = \, A(x,t,\xi)$$

with $\xi \, = \, \widetilde{\xi} \cdot \mathrm{d}\Phi(x)$, *i.e.* $\xi_k \, = \, \sum_{j=1}^d \widetilde{\xi}_j \, \partial_{x_k} \widetilde{x}_j(x)$ for all $k \in \{1, \ldots, d\}$ (so that $\widetilde{\xi} \in T^*_{\widetilde{x}} \partial D$), we claim that the stable subspace of

$$\widetilde{\mathcal{A}}(y, 0, t, \eta, \tau) \, := \, - \, (\, \widetilde{A}^d(y, 0, t) \,)^{-1} \, \left(\tau \, I_n \, + \, i \, \sum_{j=1}^{d-1} \eta_j \, \widetilde{A}^j(y, 0, t) \right)$$

is the same as the stable subspace of

$$\mathcal{A}(x, t, \xi, \tau) \, = \, A(x, t, \nu(x))^{-1} \, (\, \tau \, I_n \, + \, i \, A(x, t, \xi) \,)$$

for $x = \Phi^{-1}(y,0) \in \partial\Omega$ and $\xi = (\eta, 0) \cdot \mathrm{d}\Phi(x)$. Indeed, we have

$$\widetilde{\mathcal{A}}(y,0,t,\eta,\tau) \;=\; -\left(\sum_{k=1}^{d} \partial_{x_k}\widetilde{x}_d(x)\; A^k(x,t)\right)^{-1} \left(\tau\, I_n \;+\; i\, A(x,t,\xi)\right),$$

and $\partial_{x_1}\widetilde{x}_d(x), \ldots, \partial_{x_d}\widetilde{x}_d(x)$ are, up to a positive factor, the components of $-\nu(x)$. Therefore, the uniform Kreiss–Lopatinskiĭ condition for the problem

$$\widetilde{L}\widetilde{u} \;=\; \widetilde{f}\,, \quad \widetilde{B}\widetilde{u} \;=\; \widetilde{g}$$

in the half-space $\{\,(y,z)\,;\; z \;>\; 0\,\}$ is equivalent to the uniform Kreiss–Lopatinskiĭ condition stated above for the original problem

$$Lu \;=\; f\,, \quad Bu \;=\; g$$

in Ω.

So a Boundary Value Problem in the domain Ω is fully equivalent to a Boundary Value Problem in the half-space D, the latter satisfying the assumptions of Theorem 9.3 if the former satisfies the assumptions of Theorem 9.4. And the L^2 estimates for either one of the BVP are clearly equivalent. □

If Ω is a smooth relatively compact domain instead of being globally diffeomorphic to a half-space, the same result is true.

Theorem 9.5 *Assume Ω is a relatively compact domain with $\mathscr{C}^\infty$ boundary. Then the energy estimate* (9.1.21) *in Theorem 9.4 is valid under the other assumptions* **(CH)***,* **(NC)***,* **(N)** *and* **(UKL)***.*

Proof The ideas are the same as in the proof of Theorem 9.4, except that we use co-ordinate charts instead of a global diffeomorphism. Let $(U_j)_{j\in\{0,\ldots,J\}}$ be a covering of $\overline{\Omega}$ by chart open subsets, with $U_0 \subset \Omega$, and consider $(\varphi_j)_{j\in\{0,\ldots,J\}}$ an associated partition of unity, that is, $\varphi_j \in \mathscr{D}(U_j)$ for all j and $\sum_{j=0}^{J} \varphi_j \;\equiv\; 1$. In what follows, we use the convenient notation $\lesssim$, which means 'less than or equal to a (harmless) constant times...'

On the one hand, by Theorem 2.13, which applies to the present case because of our constant hyperbolicity assumption (see Theorems 2.3 and 2.2), there is a $\gamma_0 > 0$ so that for $u \in \mathscr{D}(\overline{\Omega})$ and $\gamma \geq \gamma_0$

$$\gamma\, \|\mathrm{e}^{-\gamma t}\varphi_0\, u\,\|^2_{L^2(\overline{\Omega}\times\mathbb{R})} \;\lesssim\; \frac{1}{\gamma}\, \|\mathrm{e}^{-\gamma t}L(\varphi_0\, u)\|^2_{L^2(\overline{\Omega}\times\mathbb{R})}\,.$$

Then, since the commutator $[\,L\,,\,\varphi_0\,]$ is of order 0, this implies

$$\gamma\, \|\mathrm{e}^{-\gamma t}\varphi_0\, u\,\|^2_{L^2(\overline{\Omega}\times\mathbb{R})} \;\lesssim\; \frac{1}{\gamma}\, \|\mathrm{e}^{-\gamma t}Lu\|^2_{L^2(\overline{\Omega}\times\mathbb{R})} \;+\; \frac{1}{\gamma}\, \|\mathrm{e}^{-\gamma t}u\|^2_{L^2(\overline{\Omega}\times\mathbb{R})}\,.$$

On the other hand, since each U_j, $j \geq 1$ is diffeomorphic to a half-space, Theorem 9.4 shows that

$$\gamma \, \|\mathrm{e}^{-\gamma t} \varphi_j \, u \, \|^2_{L^2(\overline{\Omega} \times \mathbb{R})} \, + \, \|\mathrm{e}^{-\gamma t} \varphi_j \, u \, \|^2_{L^2(\partial\Omega \times \mathbb{R})}$$

$$\lesssim \frac{1}{\gamma} \|\mathrm{e}^{-\gamma t} L(\varphi_j \, u)\|^2_{L^2(\overline{\Omega} \times \mathbb{R})} \, + \, \|\mathrm{e}^{-\gamma t} B \varphi_j \, u\|^2_{L^2(\partial\Omega \times \mathbb{R})}$$

$$\lesssim \; \frac{1}{\gamma} \, \|\mathrm{e}^{-\gamma t} L u\|^2_{L^2(\overline{\Omega} \times \mathbb{R})} \, + \, \frac{1}{\gamma} \, \|\mathrm{e}^{-\gamma t} \, u\|^2_{L^2(\overline{\Omega} \times \mathbb{R})} \, + \, \|\mathrm{e}^{-\gamma t} B \, u\|^2_{L^2(\partial\Omega \times \mathbb{R})} .$$

Summing all the inequalities obtained for $j \in \{0, \ldots, J\}$, we get

$$\gamma \, \|\mathrm{e}^{-\gamma t} \, u \, \|^2_{L^2(\overline{\Omega} \times \mathbb{R})} \, + \, \|\mathrm{e}^{-\gamma t} u \, \|^2_{L^2(\partial\Omega \times \mathbb{R})}$$

$$\lesssim \; \frac{1}{\gamma} \, \|\mathrm{e}^{-\gamma t} L \, u\|^2_{L^2(\overline{\Omega} \times \mathbb{R})} \, + \, \frac{1}{\gamma} \, \|\mathrm{e}^{-\gamma t} \, u\|^2_{L^2(\overline{\Omega} \times \mathbb{R})} + \, \|\mathrm{e}^{-\gamma t} \, B \, u\|^2_{L^2(\partial\Omega \times \mathbb{R})} \, ,$$

hence the announced estimate for γ large enough. □

9.1.5 *Less-smooth coefficients*

Preparing the way for Chapter 11 on non-linear problems, we consider here initial boundary value problems in which, similarly as in Theorem 2.4 for the Cauchy problem, the matrices A^j and B depend not directly on (x, t) but on $v(x, t)$, with v Lipschitz continuous, and A^j and B are $\mathscr{C}^\infty$ of $w \in \mathbb{R}^n$ (or w in an open subset of $\mathbb{R}^n$). For this purpose we are going to modify the rest of our notations accordingly.

To simplify the presentation, we go back to the basic domain $\Omega = \{x; x_d > 0\}$.

Notations Introducing contractible open subsets of $\mathbb{R}^n$, say $\mathbb{W}$ and $\mathbb{W}^0 \subset \mathbb{W}$, intended to contain, respectively, $v(\bar{\Omega})$ and $v(\partial\Omega)$, we redefine in this section

$$\mathbb{X}_1 \, := \, \{ \, X \, = \, (w, \eta, \tau) \, \in \, \mathbb{W} \times \mathbb{R}^{d-1} \times \mathbb{C}^+ \, , \; |\tau|^2 \, + \, \|\eta\|^2 \, = \, 1 \, \} \, ,$$

$$\mathbb{X}^0_1 \, := \, \{ \, X \, = \, (w, \eta, \tau) \, \in \, \mathbb{W}^0 \times \mathbb{R}^{d-1} \times \mathbb{C}^+ \, , \; |\tau|^2 \, + \, \|\eta\|^2 \, = \, 1 \, \} \, .$$

For all $X \; = \; (w, \eta, \tau) \in \mathbb{X}_1$ such that $\mathrm{Re}\, \tau > 0$, $E_-(X)$ denotes the stable subspace of

$$\mathcal{A}(X) \; := \; - \, (A^d(w))^{-1} \, (\, \tau \, I_n \, + \, i \, A(w, \eta, 0) \,) \, ,$$

where $A(w, \xi) \; := \; \sum_{j=1}^d \, \xi_j \, A^j(w)$ for all $(w, \xi) \in \mathbb{W} \times \mathbb{R}^d$, and E_- is extended by continuity to all points of $\mathbb{X}_1$. For a given Lipschitz continuous v, L_v stands for the variable-coefficients differential operator

$$L_v \; := \; \partial_t \, + \, \sum_j \, A^j_v \, \partial_j \, , \quad \text{where} \;\; A^j_v(x, t) \; := \; A^j(v(x, t)) \, ,$$

and B_v stands for the variable-coefficients boundary matrix defined by

$$B_v(y, t) \; = \; B(v(y, 0, t))$$

for all $(y,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Finally, for any $\omega > 0$ we denote by $\mathbb{V}_\omega$ the set of Lipschitz continuous

$$v : \mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{W}$$

such that v is constant outside a compact subset, $v_{|x_d=0}$ takes its values in $\mathbb{W}^0$ and

$$\|v\|_{W^{1,\infty}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}))} \leq \omega .$$

Theorem 9.6 *Given $\mathscr{C}^\infty$ matrix-valued mappings A^j $(j = 1, \dots, d)$ and B on $\mathbb{W}$ and $\mathbb{W}^0$, respectively, our main assumptions are that*

- *the operator*

$$\partial_t + \sum_j A^j(w)\,\partial_j$$

  is constantly hyperbolic, *that is, the matrices $A(w,\xi)$ are diagonalizable with real eigenvalues of constant multiplicities on $\mathbb{W} \times (\mathbb{R}^d \backslash \{0\})$;*
- *the matrix $A^d(w)$ is non-singular for all $w \in \mathbb{W}^0$;*
- *the matrix $B(w)$ is for all $w \in \mathbb{W}^0$ of maximal rank equal to the number of positive eigenvalues of $A^d(w)$ (counted with multiplicity),*
- *the uniform Kreiss–Lopatinskiĭ condition holds:*

  (UKL) *for all $X = (w, \eta, \tau) \in \mathbb{X}_1^0$, there exists $C > 0$ so that*

$$\|V\| \leq C\,\|\,B(w)\,V\| \qquad \text{for all } V \in E_-(X)\,.$$

Then there exists $c = c(\omega) > 0$ and $\gamma_0 = \gamma_0(\omega) > 0$ such that for all $\gamma \geq \gamma_0$, for all $v \in \mathbb{V}_\omega$, for all $u \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$, $\widetilde{u}_\gamma(x,t) := \mathrm{e}^{-\gamma t}\,u(x,t)$ satisfies

$$\gamma\,\|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|(\widetilde{u}_\gamma)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}$$
$$\leq c\Big(\frac{1}{\gamma}\,\|(\gamma + L_v)\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|(B_v\widetilde{u}_\gamma)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}\Big).$$

Proof We first rewrite the equation $L_v\,u = f$ as

$$\partial_d\widetilde{u}_\gamma - P_v^\gamma\,\widetilde{u}_\gamma = (A_v^d)^{-1}\,\widetilde{f}_\gamma\,, \quad P_v^\gamma := -\,(A_v^d)^{-1}\,(\,\partial_t + \gamma + \sum_{j=1}^{d-1} A_v^j\partial_j\,)\,.$$

Here the notation $(A_v^d)^{-1}$ stands for

$$(x,t) \mapsto (A_v^d(x,t))^{-1} = (A^d(v(x,t))^{-1}\,.$$

Since the matrix $(A_v^d(x,t))^{-1}$ is uniformly bounded by a constant depending only on ω for $v \in \mathbb{V}_\omega$, the problem amounts to proving that for all $v \in \mathbb{V}_\omega$ and all $u \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$

$$\gamma \, \|u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \lesssim \frac{1}{\gamma} \, \|f\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \quad (9.1.22)$$

with $f = \partial_d u - P_v^\gamma u$ and $g = B_v \, u_{|x_d=0}$. In fact, it suffices to prove (9.1.22) for the paralinearized expressions $f = \partial_d u - T^\gamma_{\mathcal{A}_v} u$, where $\mathcal{A}_v(x,t,\eta,\tau) := \mathcal{A}(v(x,t),\eta,\tau)$, and $g = T^\gamma_{B_v} u_{|x_d=0}$. Indeed, the error estimates in Theorem C.20 show that

$$\| P_v^\gamma u - T^\gamma_{\mathcal{A}_v} u \|_{L^2} \leq C(\omega) \, \|u\|_{L^2} \, , \quad \gamma \, \| B_v \, u - T^\gamma_{B_v} u \|_{L^2} \leq C(\omega) \, \|u\|_{L^2} \, .$$

Consequently, if we have the estimate

$$\gamma \, \|u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \lesssim \left(\frac{1}{\gamma} \, \|\partial_d u - T^\gamma_{\mathcal{A}_v} u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|T^\gamma_{B_v} u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \right)$$

we also have, for γ large enough

$$\gamma \, \|u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \lesssim \left(\frac{1}{\gamma} \, \|\partial_d u - P_v^\gamma u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|B_v \, u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \right).$$

The proof of the estimate for the para-linearized problem

$$\partial_d u - T^\gamma_{\mathcal{A}_v} u = f \, , \quad T^\gamma_{B_v} u_{|x_d=0} = g$$

of course relies on Kreiss symmetrizers. But we must overcome the fact that the sharp Gårding inequality is not true for Lipschitz coefficients. This is why we introduce after Chazarain and Piriou [31] a refined version of Kreiss symmetrizers, which will allow us to use the standard Gårding inequality.

Definition 9.4 *Given $\mathscr{C}^\infty$ functions*

$$\begin{array}{llll} \mathcal{A} : \mathbb{X} & \to \mathbf{M}_n(\mathbb{C}) & \textit{and} \quad B : \mathbb{W}^0 & \to \mathbf{M}_{p\times n}(\mathbb{R}) \\ \quad X = (w,\tau,\eta) & \mapsto \mathcal{A}(X) & \quad w & \mapsto B(w) \, , \end{array}$$

a refined Kreiss symmetrizer for $\mathcal{A}$ and B at some point $\underline{X} \in \mathbb{X}_1$ is a matrix-valued function r in some neighbourhood $\mathscr{V}$ of $\underline{X}$ in $\mathbb{X}_1$, which is associated with another matrix-valued function T, both being $\mathscr{C}^\infty$, and such that

i) the matrix $r(X)$ is Hermitian and $T(X)$ is invertible for all $X \in \mathscr{V}$,

ii) the matrix $\mathrm{Re}\,(\, r(X)\, T(X)^{-1} \mathcal{A}(X) T(X)\,)$ is block-diagonal, with blocks $h_0(X)$ and $h_1(X)$ such that $h_0(X)/\gamma$ is $\mathscr{C}^\infty$ and

$$\mathrm{Re}\,(\, h_0(X)\,) \geq C\,\gamma\, I_p \, , \quad \mathrm{Re}\,(\, h_1(X)\,) \geq C \;\; I_{n-p} \, , \quad (9.1.23)$$

for some $C > 0$ independent of $X \in \mathscr{V}$

iii) and additionally, if $\underline{X} \in \mathbb{X}^0$*, there exist* $\alpha > 0$ *and* $\beta > 0$ *independent of* $X \in \mathscr{V}$ *so that*

$$r(X) \geq \alpha I_n - \beta (B(w)T(X))^* B(w)T(X). \tag{9.1.24}$$

End of the proof of Theorem 9.6. Our assumptions allow us to construct local Kreiss symmetrizers r, which, moreover, satisfy the refined property in *ii)* (see Section 9.1.3 for some hints and [31], pp. 381–390 for details): we shall point out where this property is so important; besides this 'technical' point, the proof of Theorem 9.6 follows a standard strategy, which we describe now.

For any $v \in \mathbb{V}_\omega$, we may consider the mapping r_v defined by $r_v(y, x_d, t, \eta, \tau) = r(v(y, x_d, t), \eta, \tau)$ and extend it to a global symmetrizer $\mathcal{R}_v$, homogeneous degree 0 in (η, τ) as in the proof of Theorem 9.1. The resulting matrix-valued function

$$\mathcal{R}_v(x_d) : (y, t, \eta, \delta, \gamma) \mapsto \mathcal{R}_v(y, x_d, t, \eta, \tau = \gamma + i\delta)$$

may be viewed, for all x_d, as a symbol in the variables (y, t) (with associated frequencies (η, δ)) and parameter γ, which belongs to Γ^0_1, the order 0 coming from the homogeneity degree 0 in (η, τ) and the regularity index is 1 coming from the fact that v, hence also $\mathcal{A}_v$ and B_v, are Lipschitz in (y, t). By construction, $\mathcal{R}_v$ satisfies inequalities

$$\mathcal{R}_v(y, 0, t, \eta, \tau) \geq \alpha I_n - \beta B_v(y, t)^{\mathrm{T}} B_v(y, t), \tag{9.1.25}$$

$$\mathrm{Re}\,(\mathcal{R}_v(y, x_d, t, \eta, \tau) \mathcal{A}_v(y, x_d, t, \eta, \tau)) \geq C \gamma I_n, \tag{9.1.26}$$

for some constants $\alpha > 0$, $\beta > 0$ and $C > 0$ depending only on the Lipschitz bound ω for v. (We have not used the refined property in *ii)* yet.) Note also that $\mathcal{R}_v$ is Lipschitz continuous in x_d. We attempt a para-differential version of the proof of Theorem 9.1/Proposition 9.1 by considering the family of operators

$$\mathrm{R}^\gamma_v(x_d) := \frac{1}{2} (T^\gamma_{\mathcal{R}_v(x_d)} + (T^\gamma_{\mathcal{R}_v(x_d)})^*).$$

By construction, $\mathrm{R}^\gamma_v(x_d)$ is a self-adjoint, bounded operator on $L^2(\mathbb{R}^d, \mathrm{d}y\, \mathrm{d}t)$, whose norm is bounded uniformly in x_d *and* γ. This is true also for $\mathrm{d}\mathrm{R}^\gamma_v/\mathrm{d}x_d$. Additionally, by Theorem C.21 and Remark C.2,

$$\|\mathrm{R}^\gamma_v(x_d) - T^\gamma_{\mathcal{R}_v(x_d)}\|_{\mathcal{B}(L^2)} \lesssim \frac{1}{\gamma}.$$

Hence, the inequality in (9.1.25) together with the error estimates in Theorems C.20 and C.22 and the Gårding inequality in Theorem C.23 imply

$$\langle \mathrm{R}^\gamma_v(0)\, u(0), u(0) \rangle + \beta\, \mathrm{Re}\, \langle T^\gamma_{B_v{}^{\mathrm{T}} B_v} u(0), u(0) \rangle \geq \frac{\alpha}{2} \|u\|^2_{L^2(\mathbb{R}^d, \mathrm{d}y\, \mathrm{d}t)}$$

for γ large enough. (Here $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\mathbb{R}^d, \mathrm{d}y\, \mathrm{d}t)$.) Assume for now that we also have

$$\mathrm{Re} \int \langle \mathrm{R}^\gamma_v T^\gamma_{\mathcal{A}_v} u, u \rangle\, \mathrm{d}x_d \geq \gamma \frac{C}{2} \|u\|^2_{L^2(\mathbb{R}^d \times \mathbb{R}^+, \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}x_d)} \tag{9.1.27}$$

(which can *not* be deduced from (9.1.26), as for the Gårding inequality in Theorem C.23 to apply to the degree 1 symbol $\mathrm{R}_v^\gamma \mathcal{A}_v$ it would require $\lambda^{\gamma,1}$ instead of γ in the right-hand side of (9.1.26)): then it is easy to complete the proof of Theorem 9.6. Indeed, integrating in x_d the equality

$$\frac{\mathrm{d}}{\mathrm{d}x_d}\langle\, \mathrm{R}_v^\gamma\, u\,,\, u\,\rangle \;=\; \langle\, \frac{\mathrm{d}\mathrm{R}_v^\gamma}{\mathrm{d}x_d}\, u\,,\, u\,\rangle \;+\; 2\,\mathrm{Re}\,\langle\, \mathrm{R}_v^\gamma(\partial_d u \;-\; T^\gamma_{\mathcal{A}_v}\, u)\,,u\,\rangle$$
$$+\,2\,\mathrm{Re}\,\langle\, \mathrm{R}_v^\gamma\, T^\gamma_{\mathcal{A}_v}\, u\,,u\,\rangle\,,$$

we get

$$\frac{\alpha}{2}\,\|u(0)\|^2_{L^2(\mathbb{R}^d)} \;-\; \beta\,\mathrm{Re}\;\langle\, T^\gamma_{B_v{}^{\mathrm{T}}B_v}\, u(0)\,,\, u(0)\,\rangle$$

$$\leq\; (\,C_2 \;+\; \gamma\,(\varepsilon\, C_1 \;-\; C)\,)\;\|u\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \;+\; \frac{C_1}{4\varepsilon\gamma}\;\|f\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}\,.$$

In this inequality, the constants C_1 and C_2 come from bounds for $\mathrm{R}_v^\gamma(x_d)$ and $\mathrm{d}\mathrm{R}_v^\gamma/\mathrm{d}x_d$, $\varepsilon > 0$ is arbitrary, and we have set $f \;=\; \partial_d u \;-\; T^\gamma_{\mathcal{A}_v}\, u$. Now, choosing $\varepsilon = C/(2C_1)$, we obtain

$$\frac{\alpha}{2}\,\|u(0)\|^2_{L^2(\mathbb{R}^d)} \;+\; \gamma\,\frac{C}{4}\;\|u\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \;\leq\; \beta\,\mathrm{Re}\;\langle\, T^\gamma_{B_v{}^{\mathrm{T}}B_v}\, u(0)\,,\, u(0)\,\rangle$$
$$+\;\frac{C_1^2}{2C\gamma}\;\|f\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}$$

for all $\gamma \geq 4C_2/C$. Finally, using again Theorems C.21, C.22 and Remark C.2, we arrive at

$$\frac{\alpha}{4}\,\|u(0)\|^2_{L^2(\mathbb{R}^d)} \;+\; \gamma\,\frac{C}{4}\;\|u\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \;\leq\; \beta\;\|T^\gamma_{B_v}\, u(0)\|^2_{L^2(\mathbb{R}^d)}$$
$$+\;\frac{C_1^2}{2C\gamma}\;\|f\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}$$

for γ large enough. □

Proof of (9.1.27) This is where we are to make use of *ii)*. Indeed, going back to the construction of the global symmetrizer $\mathcal{R}_v$ (see the proof of Theorem 9.1), we see it is of the form

$$\mathcal{R}_v(x,t,\eta,\tau) \;=\; \sum_j\; P_j(X)^*\, r_j(X)\, P_j(X)\,,$$

where the sum is finite, $X \;=\; (v(x,t),\eta,\tau = \gamma + i\delta)$, and $P_j(X)$ ($= \varphi(X)^{1/2}\, T_j(X)^{-1}$ with the notations of Theorem 9.1), $r_j(X)$ are homogeneous degree 0 in (η,τ), with $\sum_j P_j^* P_j$ uniformly bounded by below. Additionally, the refined property in *ii)* for the r_j shows that

$$\mathcal{R}_v(x,t,\eta,\tau)\,\mathcal{A}(X) \;=\; \sum_j P_j(X)^* \left(\begin{array}{c|c} \gamma\, h_{0,j}(X) & 0 \\ \hline 0 & h_{1,j}(X) \end{array}\right) P_j(X)\,,$$

with $h_{0,j}$ homogeneous degree 0, $h_{1,j}$ homogeneous degree 1 in (η,τ), satisfying lower bounds (see (9.1.23))

$$\mathrm{Re}\;(h_{0,j}(X)) \;\geq\; C_j\, I_{p_j}\,, \quad \mathrm{Re}\;(h_{1,j}(X)) \;\geq\; C_j\, \lambda^{\gamma,1}(\eta,\delta)\, I_{n-p_j}\,.$$

Both these bounds are elligible for the Gårding inequality (Theorem C.23), hence

$$\mathrm{Re}\;\langle T^\gamma_{h_{0,j}}\, u_{0,j}\,,\, u_{0,j}\,\rangle \geq \frac{C_j}{4}\,\|u_{0,j}\|^2_{L^2}\,,$$

$$\textit{i.e.}\;\; \mathrm{Re}\;\langle \gamma\, T^\gamma_{h_{0,j}}\, u_{0,j}\,,\, u_{0,j}\,\rangle \geq \gamma\,\frac{C_j}{4}\,\|u_{0,j}\|^2_{L^2}$$

for all $u_{0,j}$ with values in $\mathbb{C}^{p_j}$ (upper block) and

$$\mathrm{Re}\;\langle T^\gamma_{h_{1,j}}\, u_{1,j}\,,\, u_{1,j}\,\rangle \geq \frac{C_j}{4}\,\|u_{1,j}\|^2_{H^{1/2}} \;\geq\; \gamma\,\frac{C_j}{4}\,\|u_{1,j}\|^2_{L^2}$$

for all $u_{1,j}$ with values in $\mathbb{C}^{n-p_j}$ (lower block). Now the conclusion will follow from standard error estimates. Indeed,

$$\mathrm{Re}\;\langle \mathrm{R}^\gamma_v\, T^\gamma_{\mathcal{A}_v}\, u\,,\, u\,\rangle \;\geq\; \sum_j \left\langle \left(\begin{array}{c|c} \gamma\, T^\gamma_{h_{0,j}} & 0 \\ \hline 0 & T^\gamma_{h_{1,j}} \end{array}\right) T^\gamma_{P_j}\, u\,,\, T^\gamma_{P_j}\, u \right\rangle \;-\; C'\,\|u\|^2_{L^2}$$

$$\geq\; \sum_j \gamma\,\frac{C_j}{4}\,\|T^\gamma_{P_j}\, u\|^2_{L^2} \;-\; C'\,\|u\|^2_{L^2}$$

by Theorems C.21, C.22 first and the inequalities obtained above, hence

$$\mathrm{Re}\;\langle \mathrm{R}^\gamma_v\, T^\gamma_{\mathcal{A}_v}\, u\,,\, u\,\rangle \;\geq\; (\,C\,\gamma \;-\; C''\,)\,\|u\|^2_{L^2}$$

once more by Theorems C.21, C.22, Remark C.2 and the Gårding inequality (Theorem C.23, applied to the degree 0 symbol $\sum_j P_j^* P_j$), which proves (9.1.27) for large enough $\gamma \geq 2C''/C$. □

It is possible to go further than the L^2 estimates of Theorem 9.6 and derive Sobolev estimates, provided that v enjoys some additional, though limited regularity.

Theorem 9.7 *In the framework of Theorem 9.6, assume, moreover, that* $\mathbb{W}$ *and* $\mathbb{W}^0$ *contain zero and* v *is compactly supported, with* $v \in H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ *and* $v_{|x_d=0} \in H^m(\mathbb{R}^{d-1}\times\mathbb{R})$ *for some integer* $m > (d+1)/2+1$ *and*

$$\|v\|_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}))} \;\leq\; \mu\,, \quad \|v_{|x_d=0}\|_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}))} \;\leq\; \mu\,.$$

Then there exists $\gamma_m = \gamma_m(\omega, \mu) \geq 1$ *and* $C_m = C_m(\omega, \mu) > 0$ *such that, for all* $\gamma \geq \gamma_m$*, for all* $u \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$

$$\gamma \, \|u\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \;+\; \|u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}$$

$$\leq\; C_m \, \left(\frac{1}{\gamma} \, \|L_v u\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \;+\; \|B_v u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} \right) .$$

Proof The idea is to prove first an estimate in $L^2(\mathbb{R}^+; \mathscr{H}^m_\gamma(\mathbb{R}^d))$ and then infer the estimate in $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ by differentiation with respect to x_d.

1) Tangential derivatives The $L^2(\mathbb{R}^+; \mathscr{H}^m_\gamma(\mathbb{R}^d))$ estimate will be deduced from the result of Theorem 9.6 (regarded as a $L^2(\mathbb{R}^+; \mathscr{H}^0_\gamma(\mathbb{R}^d))$ estimate) applied to derivatives of u in the (y, t) directions. In order to estimate the 'error terms' we will also need the non-linear estimate in Theorem C.12, as well as the product and commutator estimates in Theorem C.10 and Corollary C.2, or more precisely their 'parameter version' (the proof of which is left as an exercise) stated in the following.

Lemma 9.3

- *For all* q*,* r*,* s *with* $r + s > 0$ *and* $q \leq \min(r, s)$*,* $q < r + s - d/2$*, there exists* $C > 0$ *so that for all* $a \in H^r$ *and all* $u \in \mathscr{H}^s_\gamma$*,*

$$\|a\,u\|_{\mathscr{H}^q_\gamma} \;\leq\; C \, \|a\|_{H^r} \, \|u\|_{\mathscr{H}^s_\gamma} .$$

- *If* m *is an integer greater than* $d/2 + 1$ *and* α *is a* d*-uple of length* $|\alpha| \in [1, m]$*, there exists* $C > 0$ *such for all* $\gamma \geq 1$*, that for all* a *in* H^m *and all* $u \in \mathscr{H}^{|\alpha|-1}_\gamma$*,*

$$\| \, \mathrm{e}^{-\gamma\, t} \; [\, \partial^\alpha \, , \, a] \, u \, \|_{L^2} \;\leq\; C \, \|a\|_{H^m} \, \|u\|_{\mathscr{H}^{|\alpha|-1}_\gamma} .$$

So, let us take a d-uple $\alpha \;=\; (\alpha_0, \alpha')$ of length $|\alpha| \leq m$ (with $m > (d+1)/2 + 1$), and denote

$$u_\alpha \;:=\; \partial^\alpha u \;=\; \partial_t^{\alpha_0} \, \partial_y^{\alpha'} u$$

for $u \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$. The $L^2(\mathbb{R}^+; \mathscr{H}^0_\gamma(\mathbb{R}^d))$ estimate (Theorem 9.6) applied to u_α reads

$$\gamma \, \|\mathrm{e}^{-\gamma t} \, u_\alpha\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \;+\; \|\mathrm{e}^{-\gamma t} \, (u_\alpha)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}$$

$$\leq\; c \left(\frac{1}{\gamma} \, \|\mathrm{e}^{-\gamma t} \, L_v u_\alpha\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \;+\; \|\mathrm{e}^{-\gamma t} \, (B_v u_\alpha)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \right) . \tag{9.1.28}$$

Our main task is to estimate the right-hand side in terms of $f := L_v u$, $g := B_v u_{|x_d=0}$, and 'error terms' depending on u, and check the error terms can be absorbed in the left-hand side. Let us denote $f_\alpha := L_v u_\alpha$ and $g_\alpha := (B_v u_\alpha)_{|x_d=0}$ and begin with the easier case of g_α. By definition we have

$$g_\alpha = \partial^\alpha g + [B_v, \partial^\alpha] u_{|x_d=0} \,.$$

The estimate of the first part is trivial, and the estimate of the commutator will obviously rely on Lemma 9.3. A slight difficulty arises here because, unlike v, B_v is not H^m in general (recall that $B(w)$ is of constant, generally non-zero rank for $w \in \mathbb{W}^0$!). However, since v is H^m *and* compactly supported,

$$B_v = B_0 + C_v \,, \quad \text{with } C_v := B_v - B_0 \in H^m$$

by Theorem C.12, and

$$\|C_v\|_{H^m(\mathbb{R}^d)} \leq c_m(\|v\|_{L^\infty}) \, \|v_{|x_d=0}\|_{H^m(\mathbb{R}^d)} \,.$$

Therefore

$$\begin{aligned} \|\mathrm{e}^{-\gamma t} [B_v, \partial^\alpha] u_{|x_d=0}\|_{L^2(\mathbb{R}^d)} &= \|\mathrm{e}^{-\gamma t} [C_v, \partial^\alpha] u_{|x_d=0}\|_{L^2(\mathbb{R}^d)} \\ &\leq c_{m,0}(\omega,\mu) \, \|u_{|x_d=0}\|_{\mathscr{H}_\gamma^{|\alpha|-1}(\mathbb{R}^d)} \end{aligned}$$

by Lemma 9.3, which implies

$$\|\mathrm{e}^{-\gamma t} [B_v, \partial^\alpha] u_{|x_d=0}\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{\gamma} \, c_{m,0}(\omega,\mu) \, \|u_{|x_d=0}\|_{\mathscr{H}_\gamma^{|\alpha|}(\mathbb{R}^d)} \,.$$

(It is the factor $1/\gamma$ that will enable us to absorb this contribution in the left-hand side). This yields a constant $\widetilde{c}_{m,0} = \widetilde{c}_{m,0}(\omega,\mu)$ such that

$$\begin{aligned} &\sum_{|\alpha| \leq m} \gamma^{2(m-|\alpha|)} \, \|\mathrm{e}^{-\gamma t} \, g_\alpha\|^2_{L^2(\mathbb{R}^d)} \\ &\leq \widetilde{c}_{m,0}(\omega,\mu) \, \left(\|g\|^2_{\mathscr{H}_\gamma^m(\mathbb{R}^d)} + \frac{1}{\gamma^2} \, \|u_{|x_d=0}\|^2_{\mathscr{H}_\gamma^m(\mathbb{R}^d)} \right) . \end{aligned}$$

The estimate of f_α is a little trickier, because the coefficient of ∂_d in L_v is the nonconstant matrix-valued function A_d^{-1}. For this reason we write f_α in the following twisted way:

$$f_\alpha = A_v^d \, \partial^\alpha (\, (A_v^d)^{-1} f \,) + A_v^d \, [(A_v^d)^{-1} L_v, \partial^\alpha] u \,.$$

On the one hand, we have by definition of the $\mathscr{H}_\gamma^m$ norm,

$$\begin{aligned} &\gamma^{m-|\alpha|} \, \|\mathrm{e}^{-\gamma t} \, A_v^d \, \partial^\alpha (\, (A_v^d)^{-1} f \,)\|_{L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})} \\ &\leq \|A_v^d\|_{L^\infty} \, \|(A_v^d)^{-1} f\|_{L^2(\mathbb{R}^+; \mathscr{H}_\gamma^m(\mathbb{R}^d))} \,, \end{aligned}$$

and by Lemma 9.3 and Theorem C.12,

$$\begin{aligned} &\|(A_v^d)^{-1} f\|_{L^2(\mathbb{R}^+; \mathscr{H}_\gamma^m(\mathbb{R}^d))} \\ &\leq \left(\|(A_0^d)^{-1}\| + a_m \, \|v\|_{L^2(\mathbb{R}^+; H^m(\mathbb{R}^d))} \right) \|f\|_{L^2(\mathbb{R}^+; \mathscr{H}_\gamma^m(\mathbb{R}^d))} \end{aligned}$$

with a_m depending continuously on $\|v\|_{L^\infty}$. On the other hand, the commutator reads

$$[(A_v^d)^{-1}\,L_v,\partial^\alpha]u \;=\; [\,(A_v^d)^{-1}\,,\,\partial^\alpha]\,\partial_t u \,+\, \sum_{j=1}^{d-1}[\,(A_v^d)^{-1}\,A_v^j\,,\,\partial^\alpha]\,\partial_j u\,.$$

Therefore, applying again Lemma 9.3 and Theorem C.12 (using the same trick as for B to deal with the fact that the matrices $(A^d(0))^{-1}\,A^j(0)$ are non-zero), we find a constant $c_{m,1}$ depending only on $\|v\|_{L^\infty}$ and $\|v\|_{L^2(\mathbb{R}^+;H^m(\mathbb{R}^d))}$ such that

$$\begin{aligned}&\|\mathrm{e}^{-\gamma t}\,[(A_v^d)^{-1}\,L_v,\partial^\alpha]u\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})}\\&\quad\le\; c_{m,1}\;\|(\partial_t u,\partial_1 u,\ldots,\partial_{d-1}u)\|_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^{|\alpha|-1}(\mathbb{R}^d))}\,,\end{aligned}$$

hence

$$\|\mathrm{e}^{-\gamma t}\,[(A_v^d)^{-1}\,L_v,\partial^\alpha]u\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})}\;\le\;c_{m,1}\;\|u\|_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^{|\alpha|}(\mathbb{R}^d))}\,.$$

So, finally, we find a constant $\widetilde{c}_{1,m}=\widetilde{c}_{1,m}(\omega,\mu)$ such that

$$\begin{aligned}&\sum_{|\alpha|\le m}\gamma^{2(m-|\alpha|)}\;\|\mathrm{e}^{-\gamma t}\,f_\alpha\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})}\\&\quad\le\;\widetilde{c}_{1,m}\,\Big(\;\|f\|^2_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^m(\mathbb{R}^d))}\;+\;\|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^m(\mathbb{R}^d))}\;\Big)\,.\end{aligned}$$

Consequently, summing on α the inequality (9.1.28) and using the estimates of the right-hand side obtained here above we find that

$$\begin{aligned}&\gamma\,\|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^m(\mathbb{R}^d))}\;+\;\|u_{|x_d=0}\|^2_{\mathscr{H}_\gamma^m(\mathbb{R}^d)}\\&\quad\le\;\widetilde{c}\,\Big(\,\tfrac{1}{\gamma}\,\|f\|^2_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^m(\mathbb{R}^d))}\;+\;\|g\|^2_{\mathscr{H}_\gamma^m(\mathbb{R}^d)}\;+\;\frac{1}{\gamma}\,\|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^m(\mathbb{R}^d))}\\&\qquad\qquad+\;\frac{1}{\gamma^2}\,\|u_{|x_d=0}\|^2_{\mathscr{H}_\gamma^m(\mathbb{R}^d)}\Big),\end{aligned}$$

with $\widetilde{c}=c\max(\widetilde{c}_{0,m},\widetilde{c}_{1,m})$, hence

$$\begin{aligned}&\gamma\,\|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^m(\mathbb{R}^d))}\;+\;\|u_{|x_d=0}\|^2_{\mathscr{H}_\gamma^m(\mathbb{R}^d)}\\&\quad\le\;2\widetilde{c}\,\Big(\,\tfrac{1}{\gamma}\,\|L_v u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}_\gamma^m(\mathbb{R}^d))}\;+\;\|B_v u_{|x_d=0}\|^2_{\mathscr{H}_\gamma^m(\mathbb{R}^d)}\,\Big)\end{aligned}\tag{9.1.29}$$

for $\gamma\ge\sqrt{2\widetilde{c}}$.

2) Normal derivatives Once we have (9.1.29), we readily infer a bound for the L^2 norm of $\mathrm{e}^{-\gamma t}\partial_d u$ by writing

$$\partial_d u\;=\;(A_v^d)^{-1}\Big(f\;-\;\partial_t u\,+\,\sum_{j=1}^{d-1}A_v^j\,\partial_j u\Big)\,.\tag{9.1.30}$$

More generally, for any differentiation operator ∂^α in the (y,t) direction only, the d-uple α being of length $|\alpha| \leq m-1$, writing

$$\partial_d \partial^\alpha u \,=\, (A_v^d)^{-1} \left(f_\alpha \,-\, \partial_t \partial^\alpha u \,+\, \sum_{j=1}^{d-1} A_v^j \,\partial_j \partial^\alpha u \right),$$

and using the estimate of $f_\alpha := L_v\, \partial^\alpha u$ obtained in the first part, namely

$$\gamma^{2(m-|\alpha|)}\, \|\mathrm{e}^{-\gamma t} f_\alpha\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \lesssim \|f\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\mathbb{R}^d))} + \|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^{|\alpha|}_\gamma(\mathbb{R}^d))}, \tag{9.1.31}$$

together with the $L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\mathbb{R}^d))$ estimate (9.1.29) we find that

$$\gamma\, \|\mathrm{e}^{-\gamma t}\partial_d u_\alpha\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \lesssim \tfrac{1}{\gamma}\, \|L_v u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\mathbb{R}^d))} + \|B_v u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}\,.$$

Finally, we can show by induction on k, differentiation of (9.1.30) and repeated use of Lemma 9.3 (in dimension $d+1$ instead of d), the estimate

$$\gamma\, \|\mathrm{e}^{-\gamma t}\partial_d^k u_\alpha\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\mathbb{R})} \lesssim \tfrac{1}{\gamma}\, \|L_v u\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}))} + \|B_v u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}\,.$$

for all integer k and all d-uple such that $k+|\alpha| \leq m$. (The details are left to the reader.) Summing all these inequalities with (9.1.29) we obtain the $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ estimate announced. □

Remark 9.10 In the proof of Theorem 9.7 here above, we have used, in a crucial way, the inequality

$$\|u\|_{\mathscr{H}^{k-1}_\gamma(\mathbb{R}^d)} \lesssim \frac{1}{\gamma}\, \|u\|_{\mathscr{H}^k_\gamma(\mathbb{R}^d)}\,,$$

which either can be viewed as a straightforward consequence of the equivalence $\|u\|_{\mathscr{H}^k_\gamma(\mathbb{R}^d)} \simeq \|\mathrm{e}^{-\gamma t}\, u\|_{H^k_\gamma(\mathbb{R}^d)}$ (independently of γ) and the inequality

$$\|v\|_{H^{m-1}_\gamma(\mathbb{R}^d)} \leq \frac{1}{\gamma}\, \|v\|_{H^m_\gamma(\mathbb{R}^d)}\,,$$

or can be proved directly by using the definition

$$\|u\|_{\mathscr{H}^{m-1}_\gamma(\mathbb{R}^d)} = \sum_{|\alpha|\leq m-1} \gamma^{2(m-1-|\alpha|)}\, \|\mathrm{e}^{-\gamma t}\partial^\alpha u\|_{L^2(\mathbb{R}^d)}$$

and the inequality

$$\|\mathrm{e}^{-\gamma t} w\|_{L^2(\mathbb{R}^d)} \leq \frac{1}{\gamma}\, \|\mathrm{e}^{-\gamma t}\partial_t w\|_{L^2(\mathbb{R}^d)} \tag{9.1.32}$$

for $w \in \mathscr{H}^1_\gamma(\mathbb{R}^d) = \mathrm{e}^{\gamma t} L^2(\mathbb{R}^d)$. The latter can be proved in a completely elementary way. Indeed, by integration by parts we have $\gamma\, \|\mathrm{e}^{-\gamma t} w\|^2_{L^2} = \mathrm{Re}\, \langle \mathrm{e}^{-\gamma t} w\,,\, \mathrm{e}^{-\gamma t}\partial_t w\rangle$, which implies (9.1.32) merely by the Cauchy–Schwarz inequality. We point out this here because, when $\mathbb{R}^d$ is replaced by $\mathbb{R}^d \times [0,T]$ (as we shall do later), the inequality (9.1.32) is no longer valid.

9.2 How energy estimates imply well-posedness

As for the Cauchy problem, energy estimates can be used in a duality argument to show the well-posedness of Initial Boundary Value Problems (IBVP). However, this is far from being straightforward, as already seen in Chapter 4 in the constant-coefficients case. The first step is to show the well-posedness of the Boundary Value Problem, posed for $t \in \mathbb{R}$ in weighted spaces, with boundary data for $x \in \partial\Omega$. This is basically where the energy estimates/duality argument are used. The second step deals with the well-posedness of the special, IBVP with zero initial data, which we also call *homogeneous IBVP* (the term homogeneous referring by convention to the initial data and not to the boundary data, contrary to the terminology of Chapter 7). The final step concerns the general IBVP, with compatibility conditions needed for the regularity of solutions.

Sections 9.2.1, 9.2.2 and 9.2.3 describe these steps successively, for smooth coefficients. Section 9.2.4 will be devoted to coefficients with poorer regularity.

9.2.1 *The Boundary Value Problem*

The resolution of the Boundary Value Problem (BVP) relies on a duality argument, which requires the definition of an adjoint BVP. We proceed as in Section 4.4. We first observe that for smooth enough functions u and v,

$$\int_{\Omega\times\mathbb{R}} (\,v^{\mathrm{T}}\,L\,u\,-\,u^{\mathrm{T}}\,L^*\,v) = \int_{\Omega\times\mathbb{R}} \partial_t(v^{\mathrm{T}}\,u)\,+\,\sum_j \partial_j(v^{\mathrm{T}}\,A^j\,u)$$

$$= \int_{\partial\Omega}\int_{\mathbb{R}} v^{\mathrm{T}}(x,t)\,A(x,t,\nu(x))\,u(x,t)\,\mathrm{d}\mu(x)\,\mathrm{d}t,$$

(where μ denotes the measure on $\partial\Omega$) after integration by parts. We thus need to decompose the matrix $A(x,t,\nu(x))$ according to the boundary matrix $B(x,t)$ in order to formulate an adjoint BVP. This is the purpose of the following abstract result, which can be applied to $\mathscr{W} := \partial\Omega\times\mathbb{R}$ and, with a slight abuse of notation, $A(x,t) = A(x,t,\nu(x))$ (invertible if and only if our BVP is non-characteristic).

Lemma 9.4 *Given a smooth manifold* $\mathscr{W}$, *assume that* $A \in \mathscr{C}^\infty(\mathscr{W};\mathbf{GL}_n(\mathbb{R}))$ *and* $B \in \mathscr{C}^\infty(\mathscr{W};\mathbf{M}_{p\times n}(\mathbb{R}))$. *If* B *is everywhere of maximal rank* p *and if* $\ker B$ *admits a smooth basis, there exists* $N \in \mathscr{C}^\infty(\mathscr{W};\mathbf{M}_{(n-p)\times n}(\mathbb{R}))$ *such that*

$$\mathbb{R}^n \,=\, \ker B \,\oplus\, \ker N$$

everywhere on $\mathscr{W}$. *Furthermore, there exist*

$$C \in \mathscr{C}^\infty(\mathscr{W};\mathbf{M}_{(n-p)\times n}(\mathbb{R})) \quad \textit{and} \quad M \in \mathscr{C}^\infty(\mathscr{W};\mathbf{M}_{p\times n}(\mathbb{R}))$$

such that

$$\mathbb{R}^n \,=\, \ker C \,\oplus\, \ker M\,, \quad A \,=\, M^{\mathrm{T}}\,B \,+\, C^{\mathrm{T}}\,N\,, \quad \ker C \,=\, (A\ker B)^\perp$$

everywhere on $\mathscr{W}$.

Proof of (9.1.27) The row vectors (of length n) of B, $b_1, \dots, b_p$ say are, by assumption, $\mathscr{C}^\infty$ functions on $\mathscr{W}$. By assumption, there exists also a family of independent vector-valued smooth functions $(e_{p+1}, \dots, e_n)$ spanning $\ker B$ everywhere on $\mathscr{W}$. Hence we have

$$\ker B \;=\; \mathrm{Span}(e_{p+1}, \dots, e_n) \;=\; (\,\mathrm{Span}(b_1^*, \dots, b_p^*)\,)^\perp\,,$$

and $(b_1^*, \dots, b_p^*, e_{p+1}, \dots, e_n)$ is a basis of $\mathbb{R}^n$. The linear mapping

$$\mathbb{R}^n \;\to\; \mathbb{R}^{n-p},$$

which associates to any vector $u \in \mathbb{R}^n$ its $(n-p)$ last components in that basis is obviously one-to-one on $\ker B$, and its matrix (in the 'canonical' bases of $\mathbb{R}^n$ and $\mathbb{R}^{n-p}$) is

$$N \;=\; \begin{pmatrix} 0 \; I_{n-p} \end{pmatrix} P^{-1} \quad \text{with} \quad P \;:=\; \begin{pmatrix} b_1^* \dots b_p^* \; e_{p+1} \dots e_n \end{pmatrix}.$$

By construction, the mapping $w \in \mathscr{W} \mapsto N(w) \in \mathbf{M}_{(n-p)\times n}(\mathbb{R})$ is of class $\mathscr{C}^\infty$. Therefore, the square matrix

$$\mathbb{B} := \begin{pmatrix} B \\ N \end{pmatrix}$$

is invertible everywhere on $\mathscr{W}$, and its inverse $\mathbb{B}^{-1}$ is a $\mathscr{C}^\infty$ function on $\mathscr{W}$. Defining D and Y as the $(n \times p)$ and $(n \times (n-p))$ blocks in $\mathbb{B}^{-1}$, in such a way that

$$\begin{pmatrix} Y \; D \end{pmatrix} \begin{pmatrix} B \\ N \end{pmatrix} = \begin{pmatrix} B \\ N \end{pmatrix} \begin{pmatrix} Y \; D \end{pmatrix} \;=\; I_n\,,$$

we obtain M and C as

$$M \;=\; (A\,Y)^{\mathrm{T}} \quad \text{and} \quad C \;=\; (A\,D)^{\mathrm{T}}.$$

By construction, M and C are $\mathscr{C}^\infty$ on $\mathscr{W}$. The remaining algebraic details are left to the reader. □

Thanks to this lemma we have the identity

$$\int_{\Omega\times\mathbb{R}} (\,v^{\mathrm{T}}\,L\,u \;-\; u^{\mathrm{T}}\,L^*\,v) \;+\; \int_{\partial\Omega\times\mathbb{R}} (\,(Mv)^{\mathrm{T}}\,(Bu) \;+\; (Nu)^{\mathrm{T}}\,(Cv)\,)$$
$$= 0 \quad \text{for all } \; u\,,\, v\,. \tag{9.2.33}$$

Furthermore, the (uniform) Lopatinski condition for the BVP (9.1.5) is equivalent to the backward (uniform) Lopatinski condition for the adjoint BVP

$$(L^*u)(x,t) = f(x,t), \quad x \in \Omega\,,\; t \in \mathbb{R}\,, \tag{9.2.34}$$

$$(Cu)(x,t) = g(x,t), \quad x \in \partial\Omega\,,\; t \in \mathbb{R}\,. \tag{9.2.35}$$

Indeed, Theorem 4.2 in Chapter 4, is valid pointwisely in (x,t).

We use below the following slight abuse of notation. For any (Hilbert) space H of functions u of (x,t), $\mathrm{e}^{\gamma t}\,H$ stands for the space of functions u such that $(x,t) \mapsto \widetilde{u}_\gamma(x,t) = \mathrm{e}^{-\gamma t}\,u(x,t)$ belongs to H. This will be used for $H = H^s_\gamma(\mathbb{R}^d)$, $H = L^2(\mathbb{R}^+; H^s_\gamma(\mathbb{R}^d))$ and $H = H^k(\mathbb{R}^d \times \mathbb{R}^+)$.

Our aim is to solve (9.1.5)(9.1.6) in $\mathrm{e}^{\gamma t}\,L^2(\mathbb{R}^+; H^s_\gamma(\mathbb{R}^d))$ for $s \geq 0$. For 'small' values of s (i. e. $s \leq d/2+1$) the boundary condition (9.1.6) is to be understood in the sense of traces, thanks to the following result attributed to Friedrichs.

Theorem 9.8 *Assuming $\partial\Omega$ is non-characteristic for the operator L (whose coefficients are supposed to be $\mathscr{C}^\infty$, or even Lipschitz, functions of (x,t) that are constant outside a compact subset of $\Omega \times \mathbb{R}$), we consider the subspace*

$$E = \{\, u \in L^2(\Omega \times \mathbb{R})\,;\; L\,u \in L^2(\Omega \times \mathbb{R})\,\}$$

equipped with the graph norm. Then $\mathscr{D}(\overline{\Omega} \times \mathbb{R})$ is dense in E and the map

$$\begin{array}{ccc} \mathscr{D}(\overline{\Omega} \times \mathbb{R}) & \to & \mathcal{D}(\partial\Omega \times \mathbb{R}) \\ \varphi & \mapsto & \varphi_{|\partial\Omega\times\mathbb{R}} \end{array}$$

admits a unique continuous extension from E to $H^{-1/2}(\partial\Omega \times \mathbb{R})$.

Proof For general domains Ω, the question reduces, through co-ordinate charts, to the case of a hyperplane, see [31] for more details. For simplicity, we directly assume that

$$\Omega = \{\, x\,;\; x_d > 0\}\,.$$

Furthermore, multiplying L by $(A^d)^{-1}$, we may assume without loss of generality that the coefficient of ∂_d in L is the identity matrix – by assumption $(A^d)^{-1}$ is smooth and uniformly bounded.

We first show the existence of a constant C so that for any $\varphi \in \mathscr{D}(\overline{\Omega} \times \mathbb{R})$,

$$\|\varphi_{|\partial\Omega\times\mathbb{R}}\|_{H^{-1/2}(\partial\Omega\times\mathbb{R})} \leq C\,(\,\|\varphi\|_{L^2(\Omega\times\mathbb{R})} + \|L\varphi\|_{L^2(\Omega\times\mathbb{R})}\,)\,.$$

Indeed, if $\mathbf{1}$ denotes the characteristic function of $\mathbb{R}^+$, we may associate to any function u the function u^d defined by $u^d(x,t) = u(x,t)\,\mathbf{1}(x_d)$. Then, by the so-called jump formula

$$L(\varphi^d) = (L\varphi)^d + \varphi_{|\{x_d=0\}} \otimes \delta_{\{x_d=0\}}\,.$$

Now, an easy calculation (using Fourier transform) shows that the $H^{-1/2}$ norm of $\varphi_{|\{x_d=0\}}$ is proportional to the H^{-1} norm of $\varphi_{|\{x_d=0\}} \otimes \delta_{\{x_d=0\}}$. Therefore, there exists $C > 0$ so that

$$\begin{aligned} \|\varphi_{|\{x_d=0\}}\|_{H^{-1/2}} &\leq C\,(\,\|L(\varphi^d)\|_{H^{-1}} + \|(L\varphi)^d\|_{H^{-1}}\,) \\ &\leq C\,(\,C'\,\|\varphi^d\|_{L^2} + \|(L\varphi)^d\|_{L^2}\,) = C\,(\,C'\,\|\varphi\|_{L^2} + \|L\varphi\|_{L^2}\,) \end{aligned}$$

where C' is the norm of L as an operator $L^2 \to H^{-1}$.

To show the density of $\mathscr{D}(\overline{\Omega} \times \mathbb{R})$ in E, we can restrict ourselves – by standard cut-off – to the approximations of functions $u \in E$ that are compactly supported. Take a smooth kernel $\rho \in \mathscr{C}^\infty(\mathbb{R}^{d+1}; \mathbb{R}^+)$ compactly supported in $\{x_d \leq 0\}$ such that $\int_{\mathbb{R}^{d+1}} \rho = 1$ and consider the associated mollifier $\rho_\varepsilon(x) = \varepsilon^{-d-1}\, \rho(x/\varepsilon, t/\varepsilon)$. We know that the convolution operator by ρ_ε, say R_ε, tends to the identity in L^2 when ε goes to 0. In particular, for any $u \in E$, $R_\varepsilon(u^d)$ tends to u^d in $L^2(\mathbb{R}^{d+1})$ and thus $\varphi_\varepsilon := R_\varepsilon(u^d)_{|\{x_d>0\}}$ tends to u in $L^2(\Omega \times \mathbb{R})$. Since ρ_ε is supported in $\{x_d \leq 0\}$, the smooth function φ_ε is also compactly supported if u is so. It remains to show that $L\varphi_\varepsilon$ tends to Lu. But

$$L\varphi_\varepsilon = (R_\varepsilon \circ L)(u^d)_{|\{x_d>0\}} + [\, L \,,\, R_\varepsilon \,](u^d)_{|\{x_d>0\}} \,.$$

By Theorem C.14 (which extends a classical lemma of Friedrichs [63] to Lipschitz coefficients), the last term is known to tend to 0 in L^2. Furthermore, since $L(u^d) - (Lu)^d$ is supported in $\{x_d = 0\}$ and ρ_ε is supported in $\{x_d \leq 0\}$, the first term equivalently reads

$$(R_\varepsilon \circ L)(u^d)_{|\{x_d>0\}} = R_\varepsilon((Lu)^d)_{|\{x_d>0\}} \,,$$

which does tend to Lu in $L^2(\Omega \times \mathbb{R})$. □

We are now in a position to prove the following, which covers both the frameworks of Theorem 9.4 and Theorem 9.5.

Theorem 9.9 *Assume that Ω is either (globally) diffeomorphic to a half-space or a relatively compact domain with $\mathscr{C}^\infty$ boundary. We also make the usual assumptions* **(CH)** *(constant hyperbolicity of the operator L),* **(NC)** *(non-characteristicity of the boundary $\partial\Omega$ with respect to L),* **(N)** *(normality of the boundary data B) with the additional fact that* $\ker B$ *admits a smooth basis on $\partial\Omega \times \mathbb{R}$ and* **(UKL)** *(uniform Kreiss–Lopatinskiĭ condition for (L, B) in Ω).*

Then there exists $\gamma_0 \geq 1$ such that for all $\gamma \geq \gamma_0$, for all $f \in \mathrm{e}^{\gamma t}\, L^2(\Omega \times \mathbb{R})$ and all $g \in \mathrm{e}^{\gamma t}\, L^2(\partial\Omega \times \mathbb{R})$, there is one and only one solution $u \in \mathrm{e}^{\gamma t}\, L^2(\Omega \times \mathbb{R})$ of the Boundary Value Problem (9.1.5)(9.1.6)*. Furthermore, the trace of u on $\partial\Omega \times \mathbb{R}$ belongs to $\mathrm{e}^{\gamma t}\, L^2(\partial\Omega \times \mathbb{R})$, and $\widetilde{u}_\gamma = \mathrm{e}^{-\gamma t} u$ enjoys an estimate*

$$\gamma\, \|\widetilde{u}_\gamma\|^2_{L^2(\Omega\times\mathbb{R})} + \|(\widetilde{u}_\gamma)_{|\partial\Omega\times\mathbb{R}}\|^2_{L^2(\partial\Omega\times\mathbb{R})} \leq c \left(\frac{1}{\gamma}\, \|\widetilde{f}_\gamma\|^2_{L^2(\Omega\times\mathbb{R})} + \|\widetilde{g}_\gamma\|^2_{L^2(\partial\Omega\times\mathbb{R})} \right) \tag{9.2.36}$$

for some constant $c > 0$ depending only on γ_0. Furthermore, for all $k \in \mathbb{N}$, there exists $\gamma_k \geq \gamma_0$ such that, for all $\gamma \geq \gamma_k$, for all $f \in \mathrm{e}^{\gamma t}\, H^k(\Omega \times \mathbb{R})$ and all $g \in \mathrm{e}^{\gamma t}\, H^k(\partial\Omega \times \mathbb{R})$, the solution u of (9.1.5)(9.1.6) *belongs to $\mathrm{e}^{\gamma t}\, H^k(\Omega \times \mathbb{R})$ and its trace belongs to $\mathrm{e}^{\gamma t}\, H^k(\partial\Omega \times \mathbb{R})$.*

Note: The first part of the theorem is the extension to variable coefficients of Lemma 4.8 in Chapter 4. The last part specifies the degree of regularity of the solution for more regular data.

Proof As for the Cauchy problem, a technical difficulty lies in the fact that the energy estimate is known a priori for solutions smoother than L^2. More precisely, either Theorem 9.4 or Theorem 9.5 gives (9.1.21) for $u \in \mathscr{D}(\overline{\Omega} \times \mathbb{R})$. This implies (9.1.21) *a priori* only for $u \in \mathrm{e}^{\gamma t} H^1(\overline{\Omega} \times \mathbb{R})$: indeed, we can approach $u \in \mathrm{e}^{\gamma t} H^1(\overline{\Omega} \times \mathbb{R})$ by some $u_\varepsilon \in \mathscr{D}(\overline{\Omega} \times \mathbb{R})$ in such a way that u_ε goes to u in $\mathrm{e}^{\gamma t} L^2(\overline{\Omega} \times \mathbb{R})$, $(u_\varepsilon)_{|\partial\Omega\times\mathbb{R}}$ tends to the trace of u in $\mathrm{e}^{\gamma t} L^2(\partial\Omega \times \mathbb{R})$, but also Lu_ε tends to Lu in $\mathrm{e}^{\gamma t} L^2(\overline{\Omega} \times \mathbb{R})$. In fact, it will turn out that (9.1.21) is met as soon as u belongs to $L^2(\overline{\Omega} \times \mathbb{R})$: this fact may be viewed as a consequence of the regularity part of the theorem, which we admit for the moment. □

Uniqueness It is straightforward if we use the regularity part. For, by linearity it suffices to prove that the only $\mathrm{e}^{\gamma t} L^2$ solution of the homogeneous problem

$$Lu = 0 \text{ on } \Omega \times \mathbb{R}, \quad Bu = 0 \text{ on } \partial\Omega \times \mathbb{R} \tag{9.2.37}$$

is 0. But if $u \in \mathrm{e}^{\gamma t} L^2(\partial\Omega \times \mathbb{R})$ solves (9.2.37), then u belongs to $\mathrm{e}^{\gamma_1 t} H^1(\partial\Omega \times \mathbb{R})$ (because 0 belongs to H^1!) and therefore satisfies the energy estimate (9.2.36) with $\gamma \geq \gamma_1$ and $f \equiv 0$, $g \equiv 0$, which of course implies $u = 0$ almost everywhere.

Existence It is shown by duality as in the constant-coefficients case (also see Theorem 2.6 in Chapter 2 for the Cauchy problem). The details are slightly different here because we have not proved yet the a priori estimate for L^2 data. Introduce the space

$$\mathscr{E} := \{ v \in \mathrm{e}^{-\gamma t} H^1(\overline{\Omega} \times \mathbb{R});\ C\, v_{|\partial\Omega\times\mathbb{R}} = 0 \}$$

equipped with the norm

$$\|v\|_\gamma := \|\mathrm{e}^{\gamma t}\, v\|_{L^2(\Omega\times\mathbb{R})} .$$

Similarly, to simplify the writing we denote here

$$|v|_\gamma := |\mathrm{e}^{\gamma t}\, v|_{L^2(\partial\Omega\times\mathbb{R})} .$$

The energy estimate for the *backward* and *homogeneous* boundary value problem associated with L^* reads

$$\gamma\, \|v\|_\gamma^2 + |v|_\gamma^2 \leq \frac{c}{\gamma}\, \|L^*\, v\|_\gamma^2 ,$$

for some $c > 0$ independent of both v and γ. In particular, it shows that L^* restricted to $\mathscr{E}$ is one-to-one and thus enables us to define a linear form ℓ on $L^*\mathscr{E}$ by

$$\ell(L^* v) = \int_{\Omega\times\mathbb{R}} v^{\mathrm{T}} f + \int_{\partial\Omega\times\mathbb{R}} (M\, v)^{\mathrm{T}}\, g ,$$

satisfying the estimate

$$\begin{aligned}|\ell(L^*v)| &\leq \|f\|_{-\gamma}\,\|v\|_\gamma \;+\; \|M\|_{L^\infty}\,|g|_{-\gamma}\,|v|_\gamma \\ &\lesssim \Big(\frac{1}{\gamma}\,\|f\|_{-\gamma} \;+\; \frac{1}{\gamma^{1/2}}\,|g|_{-\gamma}\Big)\;\|L^*\,v\|_\gamma .\end{aligned}$$

Therefore, by the Hahn–Banach theorem ℓ extends to a continuous form on the weighted space $\mathrm{e}^{-\gamma\,t}L^2(\overline{\Omega}\times\mathbb{R})$, and by the Riesz theorem, there exists u in the dual space $\mathrm{e}^{\gamma\,t}L^2(\overline{\Omega}\times\mathbb{R})$ such that

$$\ell(L^*v) \;=\; \int_{\Omega\times\mathbb{R}} u^{\mathrm{T}}L^*v\,.$$

Thanks to (9.2.33), this yields

$$\int_{\Omega\times\mathbb{R}} v^{\mathrm{T}}Lu \;+\; \int_{\partial\Omega\times\mathbb{R}} (M\,v)^{\mathrm{T}}\,Bu \;-\; \int_{\Omega\times\mathbb{R}} v^{\mathrm{T}}f \;-\; \int_{\partial\Omega\times\mathbb{R}} (M\,v)^{\mathrm{T}}\,g \;=\; 0$$

for all $v\in\mathscr{E}$. In particular, for $v\in\mathscr{D}(\Omega\times\mathbb{R})$, the boundary terms vanish and we infer that $Lu\;=\;f$ in the sense of distribution. Consequently, we have

$$\int_{\partial\Omega\times\mathbb{R}} (M\,\varphi)^{\mathrm{T}}\,Bu \;-\; \int_{\partial\Omega\times\mathbb{R}} (M\,\varphi)^{\mathrm{T}}\,g \;=\; 0$$

for all compactly supported $\mathscr{C}^\infty$ function φ on the boundary $\partial\Omega\times\mathbb{R}$ such that $C\,\varphi\;=\;0$. Since at each point, $M_{|\mathrm{Ker}C}:\mathrm{Ker\ C}\to\mathbb{C}^p$ is onto, and M has $\mathscr{C}^\infty$ coefficients, this implies

$$\int_{\partial\Omega\times\mathbb{R}} \psi^{\mathrm{T}}\,Bu \;-\; \int_{\partial\Omega\times\mathbb{R}} \psi^{\mathrm{T}}\,g \;=\; 0$$

for all $\psi\in\mathscr{D}(\partial\Omega\times\mathbb{R})$, hence the boundary condition $Bu\;=\;g$ in the sense of distributions.

Regularity A key ingredient is the following tricky result.

Theorem 9.10 (Chazarain–Piriou) *Assume that the operator L and the boundary operator B have $\mathcal{C}^\infty$ coefficients, constant outside a compact subset of $\Omega\times\mathbb{R}$, that the boundary $\partial\Omega$ is non-characteristic for the operator L, and that the pair (L,B) is endowed with the a priori estimate*

$$\gamma\,\|\varphi\|^2_{L^2(\overline{\Omega}\times\mathbb{R})} \;+\; \|\varphi_{|\partial\Omega}\|^2_{L^2(\partial\Omega\times\mathbb{R})} \;\leq\; c\,\left(\frac{1}{\gamma}\,\|L_\gamma\varphi\|^2_{L^2(\overline{\Omega}\times\mathbb{R})} \;+\; \|B\varphi\|^2_{L^2(\partial\Omega\times\mathbb{R})}\right) \tag{9.2.38}$$

for some $c>0$ independent of $\gamma\geq\gamma_0$ and $\varphi\in H^1(\overline{\Omega}\times\mathbb{R})$. Then for all integer $m\geq-1$, the four conditions

$$\begin{aligned}&v\in H^m(\overline{\Omega}\times\mathbb{R})\cap L^2(\overline{\Omega}\times\mathbb{R})\,,\; L_\gamma\,v\in H^{m+1}(\overline{\Omega}\times\mathbb{R}) \quad \textit{for all}\;\gamma\geq\gamma_m,\\ &v_{|\partial\Omega\times\mathbb{R}}\in H^m(\partial\Omega\times\mathbb{R})\,, \qquad B\,v_{|\partial\Omega\times\mathbb{R}}\in H^{m+1}(\partial\Omega\times\mathbb{R})\,,\end{aligned}$$

imply

$$v \in H^{m+1}(\overline{\Omega} \times \mathbb{R}) \quad \textit{and} \quad v_{|\partial\Omega\times\mathbb{R}} \in H^{m+1}(\partial\Omega \times \mathbb{R})\,.$$

Note: in this statement, $v_{|\partial\Omega\times\mathbb{R}}$ is to be understood as the trace of v on $\partial\Omega \times \mathbb{R}$, as defined in Theorem 9.8, and belongs at least to $H^{-1/2}(\partial\Omega \times \mathbb{R})$.

Admitting temporarily Theorem 9.10, we can easily complete the proof of the regularity part in Theorem 9.9. For, if $u \in \mathrm{e}^{\gamma t} L^2(\overline{\Omega} \times \mathbb{R})$ is such that $Lu = f$ belongs to $\mathrm{e}^{\gamma t} L^2(\overline{\Omega} \times \mathbb{R})$ and $Bu_{|\partial\Omega\times\mathbb{R}} = g$ belongs to $\mathrm{e}^{\gamma t} L^2(\partial\Omega \times \mathbb{R})$, Theorem 9.10 applied to $m = -1$ and $v = \widetilde{u}_\gamma$ readily implies $u_{|\partial\Omega\times\mathbb{R}}$ belongs to $\mathrm{e}^{\gamma t} L^2(\partial\Omega \times \mathbb{R})$, as claimed. Now, if additionally $f \in \mathrm{e}^{\gamma t} H^1(\Omega \times \mathbb{R})$ and $g \in \mathrm{e}^{\gamma t} H^1(\partial\Omega \times \mathbb{R})$, we can apply Theorem 9.10 to $m = 0$ and $v = \widetilde{u}_\gamma$, and thus obtain that u belongs to $\mathrm{e}^{\gamma t} H^1(\Omega \times \mathbb{R})$ and its trace to $\mathrm{e}^{\gamma t} H^1(\partial\Omega \times \mathbb{R})$. By induction, we thus show that u and its trace inherit exactly the Sobolev index of f and g.

Energy estimate To complete the proof of Theorem 9.9 it remains to show that the energy estimate in (9.2.36) is valid as soon as u belongs to $\mathrm{e}^{\gamma t}L^2$. To prove this fact we can regularize the data f and g and use the regularity part of Theorem 9.9 together with the uniqueness of solutions. For any $f \in \mathrm{e}^{\gamma t}L^2(\overline{\Omega} \times \mathbb{R})$ and $g \in \mathrm{e}^{\gamma t}L^2(\partial\Omega \times \mathbb{R})$, there exist $f_\varepsilon \in \mathscr{D}(\overline{\Omega} \times \mathbb{R})$ and $g_\varepsilon \in \mathscr{D}(\partial\Omega \times \mathbb{R})$ such that

$$\mathrm{e}^{-\gamma t} f_\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(\overline{\Omega}\times\mathbb{R})} \mathrm{e}^{-\gamma t} f\,, \quad \mathrm{e}^{-\gamma t} g_\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(\partial\Omega\times\mathbb{R})} \mathrm{e}^{-\gamma t} g\,.$$

For all ε, there exists $u_\varepsilon \in \mathrm{e}^{\gamma t}H^1(\overline{\Omega} \times \mathbb{R})$ with $\gamma \geq \gamma_1$ solving the regularized problem

$$Lu_\varepsilon \;=\; f_\varepsilon \;\text{ on } \Omega \times \mathbb{R}\,, \quad Bu_\varepsilon \;=\; g_\varepsilon \;\text{ on } \partial\Omega \times \mathbb{R}\,.$$

The energy estimate (9.2.36) is valid for the (smooth) difference $(u_\varepsilon - u_{\varepsilon'})$ and thus shows that both $(u_\varepsilon)_{\varepsilon>0}$ and $((u_\varepsilon)_{|\partial\Omega\times\mathbb{R}})_{\varepsilon>0}$ are Cauchy sequences, in $\mathrm{e}^{\gamma t}L^2(\overline{\Omega} \times \mathbb{R})$ and $\mathrm{e}^{\gamma t}L^2(\partial\Omega \times \mathbb{R})$ respectively. Therefore, there exist $\underline{u} \in \mathrm{e}^{\gamma t}L^2(\overline{\Omega} \times \mathbb{R})$ and $\underline{u}_0 \in \mathrm{e}^{\gamma t}L^2(\partial\Omega \times \mathbb{R})$ such that

$$\|\mathrm{e}^{-\gamma t}\,(u_\varepsilon - \underline{u})\|_{L^2(\overline{\Omega}\times\mathbb{R})} \to 0 \quad \text{and} \quad \|\mathrm{e}^{-\gamma t}\,((u_\varepsilon)_{|\partial\Omega\times\mathbb{R}} - \underline{u}_0)\|_{L^2(\overline{\Omega}\times\mathbb{R})} \to 0$$

as ε goes to 0. By construction, $Lu_\varepsilon = f_\varepsilon$ converges to $f = Lu$ in $\mathrm{e}^{\gamma t} L^2(\overline{\Omega} \times \mathbb{R})$. Consequently, by Theorem 9.8 the trace $u_{\varepsilon|\partial\Omega\times\mathbb{R}}$ converges to $\underline{u}_{|\partial\Omega\times\mathbb{R}}$ in $\mathrm{e}^{\gamma t} H^{-1/2}(\partial\Omega \times \mathbb{R})$, hence by uniqueness of limits in the sense of distributions, $\underline{u}_{|\partial\Omega\times\mathbb{R}} = \underline{u}_0$. In the limit, we thus get $B\underline{u}_{|\partial\Omega\times\mathbb{R}} = g$, and $\underline{u}$ solves the same BVP as u. Therefore $\underline{u} = u$ and by passing to the limit in the energy estimate (9.2.36) for u_ε, we obtain (9.2.36) for u. □

Remark 9.11 The previous regularization method and Corollary 9.1 applied to u_ε, show (at least when Ω is a half-space) that there exists $C_k > 0$ so that for $f \in \mathscr{H}^k_\gamma(\Omega \times \mathbb{R})$ and $g \in \mathscr{H}^k_\gamma(\partial\Omega \times \mathbb{R})$ with $\gamma \geq \gamma_k$, the solution u of the

Boundary Value Problem (9.1.5)(9.1.6) satisfies

$$\gamma \, \|u\|^2_{\mathscr{H}^k_\gamma(\Omega\times\mathbb{R})} \,+\, \|u_{|x_d=0}\|^2_{\mathscr{H}^k_\gamma(\partial\Omega\times\mathbb{R})} \leq C_k \,\left(\frac{1}{\gamma}\, \|f\|^2_{\mathscr{H}^k_\gamma(\Omega\times\mathbb{R})} \,+\, \|g\|^2_{\mathscr{H}^k_\gamma(\partial\Omega\times\mathbb{R})} \right). \tag{9.2.39}$$

Proof of Theorem 9.10 (Note: In the case $m = -1$, the only thing to show is that the trace of v belongs to $L^2(\partial\Omega \times \mathbb{R})$.)

For general domains Ω, the proof should naturally involve co-ordinate charts, and to some extent this would hide the heart of the matter. We prefer to give the proof in the simpler case $\Omega = \{\, x\,;\; x_d > 0\,\}$, $\overline{\Omega} \times \mathbb{R}$ being identified with $\mathbb{R}^d \times \mathbb{R}^+ = \{\,(y,t,x_d)\,;\; x_d \geq 0\,\}$, and $\partial\Omega \times \mathbb{R}$ with $\mathbb{R}^d = \{\,(y,t)\,\}$. The outline follows the proof given by Chazarain and Piriou [31] in the case of smooth compact Ω, save for co-ordinate charts.

A first useful remark is that it suffices to show v actually belongs to $L^2(\mathbb{R}^+; H^{m+1}(\mathbb{R}^d))$. Indeed, Proposition 2.3 applied with x_d instead of t shows that $v \in L^2(\mathbb{R}^+; H^{m+1}(\mathbb{R}^d))$ and $L_\gamma v \in H^m(\mathbb{R}^d \times \mathbb{R}^+))$ imply $v \in H^{m+1}(\mathbb{R}^d \times \mathbb{R}^+)$.

Next, we are to use the following observation on Sobolev spaces.

Proposition 9.2 *Take $s \in \mathbb{R}$. If $v \in H^s(\mathbb{R}^d)$ and if there exists $C > 0$ so that*

$$\|v\|_{s,\theta} \,:=\, \int_{\mathbb{R}^d} \frac{(1+\,\|\xi\|^2)^{s+1}}{1+\,\|\theta\,\xi\|^2}\; |\widehat{v}(\xi)|^2 \;\mathrm{d}\xi \;\leq\; C$$

for all $\theta \in (0,1]$, then v belongs to $H^{s+1}(\mathbb{R}^d)$.

(This is a consequence of Fatou's Lemma: take the lim inf of the inequality when θ goes to 0.) So, showing additional regularity for $v \in H^s$ amounts to finding a uniform bound of $\|v\|_{s,\theta}$. In this respect, one may try to use the left part of the two-sided estimate given by the following technical result.

Proposition 9.3 *Take $s \in \mathbb{R}$, $r > s+1$ and $\rho \in \mathscr{D}(\mathbb{R}^d)$ such that*

$$\widehat{\rho}(\xi) \,=\, \mathcal{O}(\|\xi\|^r)$$

in the neighbourhood of 0 and that $\widehat{\rho}$ does not vanish identically on any ray $\{\, t\,\xi\,;\; t \in \mathbb{R}^+\}$, $\xi \neq 0$. For $\varepsilon > 0$, we define

$$\rho_\varepsilon \,:\, x \,\mapsto\, \varepsilon^{-d}\,\rho(x/\varepsilon)\,,$$

and consider R_ε the convolution operator by ρ_ε. Then there exist C and $C' > 0$ such that

$$C\,\|v\|^2_{s,\theta} \,\leq\, \|v\|^2_{H^s(\mathbb{R}^d)} \,+\, n_{s,\theta}(v) \,\leq\, C'\,\|v\|^2_{s,\theta},$$

$$n_{s,\theta}(v) := \int_0^1 \, \|R_\varepsilon\, v\|^2_{L^2(\mathbb{R}^d)}\varepsilon^{-2(s+1)} \,\left(1 \,+\, \frac{\theta^2}{\varepsilon^2}\right)^{-1} \frac{\mathrm{d}\varepsilon}{\varepsilon},$$

for all $\theta \in (0,1]$ and all $v \in H^s(\mathbb{R}^d)$.

(The proof is technical but elementary: it can be found in [31], Chapter 2. There do exist kernels ρ satisfying all the requirements: take say $\sigma \in \mathscr{D}(\mathbb{R}^d)$ such that $\int \sigma \neq 0$ and define, for instance, $\rho = \Delta^{r/2}\sigma$.) Another important preliminary result is the following.

Theorem 9.11 *If R_ε is a smoothing operator as in Proposition 9.3, if P (resp., B) is a differential operator of order 1 (resp., 0), with $\mathscr{C}^\infty$ coefficients that are constant outside a compact subset of $\mathbb{R}^d$, there exists $C'' > 0$ such that*

$$\int_0^1 \|[\,P\,,\,R_\varepsilon]\,v\|^2_{L^2(\mathbb{R}^d)}\,\varepsilon^{-2(s+1)}\,\left(1+\frac{\theta^2}{\varepsilon^2}\right)^{-1}\,\frac{\mathrm{d}\varepsilon}{\varepsilon}\,\leq\,C''\,\|v\|^2_{s,\theta}\,,$$

$$\int_0^1 \|[\,B\,,\,R_\varepsilon]\,v\|^2_{L^2(\mathbb{R}^d)}\,\varepsilon^{-2(s+2)}\,\left(1+\frac{\theta^2}{\varepsilon^2}\right)^{-1}\,\frac{\mathrm{d}\varepsilon}{\varepsilon}\,\leq\,C''\,\|v\|^2_{s,\theta}\,,$$

for all $\theta \in (0,1]$, all $v \in H^s(\mathbb{R}^d)$.

(These are specials cases of a general result on pseudo-differential operators, the proof of which involves commutator decompositions and repeated use of Proposition 9.3; see [31], Chapter 4.)

In particular, if $(A^d)^{-1}$ is smooth and uniformly bounded, we may apply Theorem 9.11 to

$$P(x_d)\,=\,-\,(A^d)^{-1}\left(\partial_t+\sum_{j=1}^{d-1}A^j\partial_j\right),$$

which depends continuously on x_d (viewed here as a parameter): this will give an estimate for the operator $L^d \,:=\, (A^d)^{-1}\,L$ since $[\,L^d\,,\,R_\varepsilon] \,=\, [\,-\,P\,,\,R_\varepsilon]$. And applying Theorem 9.11 to $B \,=\, (A^d)^{-1}$ will eventually give an estimate for the operator

$$L^d_\gamma \,:=\, (A^d)^{-1}\,L_\gamma \,=\, \partial_z \,-\, P(x_d) \,-\, \gamma\,(A^d)^{-1}\,,$$

uniformly in γ.

Now, the characterization of H^s functions that are actually in H^{s+1}, given by Proposition 9.2, the two-sided inequality provided by Proposition 9.3, the commutator's estimates given by Theorem 9.11, and the energy estimate (9.2.38), altogether enable us to complete the proof of Theorem 9.10. Consider $v \in H^m(\mathbb{R}^d \times \mathbb{R}^+) \cap L^2(\mathbb{R}^d \times \mathbb{R}^+)$ such that $L_\gamma\,v \in H^{m+1}(\mathbb{R}^d \times \mathbb{R}^+)$, $v_{|x_d=0} \in H^m(\mathbb{R}^d)$ and $B\,v_{|x_d=0} \in H^{m+1}(\mathbb{R}^d)$ for $m \geq -1$, and define $\varphi_\varepsilon \,:=\, R_\varepsilon(v)$, where R_ε is a smoothing operator as in Proposition 9.3, in the variables $(y,t) \in \mathbb{R}^d$. By construction, the function φ_ε belongs to $L^2(\mathbb{R}^+; H^{+\infty}(\mathbb{R}^d))$, and

$$L^d_\gamma\,\varphi_\varepsilon \,=\, R_\varepsilon\,L^d_\gamma\,\varphi \,+\, [L^d_\gamma, R_\varepsilon]\,\varphi$$

belongs to $L^2(\mathbb{R}^d \times \mathbb{R}^+)$: for the first term we use the smoothness of $(A^d)^{-1}$, that $L_\gamma\,\varphi$ is at least L^2 and that R_ε is a bounded operator on L^2; for the commutator, we use that there are no derivatives with respect to x_d and apply Friedrichs Lemma ([63], or Theorem C.14) in the (y,t) variables. Therefore, applying again Proposition 2.3, with x_d playing the role of t, we infer that φ_ε belongs to $H^1(\mathbb{R}^d \times \mathbb{R}^+)$, and we can apply the L^2 estimate (9.2.38) to φ_ε, which reads

$$\gamma \, \|\varphi_\varepsilon\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \, + \, \|(\varphi_\varepsilon)_{|x_d=0}\|^2_{L^2(\mathbb{R}^d)} \\ \leq \, c \, \left(\frac{1}{\gamma} \, \|L_\gamma \varphi_\varepsilon\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \, + \, \|B(\varphi_\varepsilon)_{|x_d=0}\|^2_{L^2(\mathbb{R}^d)} \right) .$$

Multiplying by $\varepsilon^{-2m-3}\left(1 \, + \, \frac{\theta^2}{\varepsilon^2}\right)^{-1}$, integrating in $\varepsilon \in (0,1]$, and using Theorem 9.11 (with $s=m$ and $s=m-1$) we get a constant $C>0$ such that

$$\gamma \, \int_0^{+\infty} \, \|v\|^2_{m,\theta} \, \mathrm{d}x_d \, + \, \|v_{|x_d=0}\|^2_{m,\theta} \\ \leq \, C \left(\tfrac{1}{\gamma} \, \int_0^{+\infty} \, \|v\|^2_{m,\theta} \, \mathrm{d}x_d \, + \, \gamma \, \int_0^{+\infty} \, \|v\|^2_{m-1,\theta} \, \mathrm{d}x_d \right. \\ \left. + \, \tfrac{1}{\gamma} \, \int_0^{+\infty} \, n_{m,\theta}(L_\gamma v(x_d)) \, \mathrm{d}x_d \, + \, n_{m,\theta}(Bv_{|x_d=0}) \, + \, \|v_{|x_d=0}\|^2_{m-1,\theta} \right),$$

For γ large enough the first term in the right-hand side can be absorbed in the left-hand side. Therefore, applying Propositions 9.2 and 9.3, we get

$$\gamma \, \int_0^{+\infty} \, \|v\|^2_{m,\theta} \, \mathrm{d}x_d \, + \, \|v_{|x_d=0}\|^2_{m,\theta} \\ \leq \, C \left(\frac{1}{\gamma} \, \|L_\gamma v\|^2_{L^2(\mathbb{R}^+;H^{m+1}(\mathbb{R}^d))} \, + \, \|Bv_{|x_d=0}\|^2_{H^{m+1}(\mathbb{R}^d)} \, + \, \gamma \, \|v\|^2_{L^2(\mathbb{R}^+;H^m(\mathbb{R}^d))} \right. \\ \left. + \, \|v_{|x_d=0}\|^2_{H^m(\mathbb{R}^d)} \right).$$

The right-hand side being independent of θ, Proposition 9.2 thus shows that v belongs to $L^2(\mathbb{R}^+; H^{m+1}(\mathbb{R}^d))$ and $v_{|x_d=0}$ belongs to $H^{m+1}(\mathbb{R}^d)$. This completes the proof of Theorem 9.10. □

9.2.2 *The homogeneous IBVP*

Theorem 9.12 *In the framework of Theorem 9.9, if $f \in L^2(\bar{\Omega} \times [0,T])$ and $g \in L^2(\partial\Omega \times [0,T])$ the problem*

$$Lu = f \;\; on \;\; \Omega \times (0,T) \, , \tag{9.2.40}$$

$$Bu = g \;\; on \;\; \partial\Omega \times (0,T) \, , \tag{9.2.41}$$

$$u_{|t=0} \, = 0 \;\; on \;\; \Omega \, , \tag{9.2.42}$$

admits a unique solution $u \in L^2(\bar{\Omega} \times [0,T])$. *Furthermore, the trace* $u_{\partial\Omega \times [0,T]}$ *belongs to* $L^2(\partial\Omega \times [0,T])$, *and there exist* γ_0 *and* $c > 0$ *depending only on* ω *so that for all* $\gamma \geq \gamma_0$,

$$\gamma \, \|e^{-\gamma t} \, u\|^2_{L^2(\Omega \times (0,T))} + \|e^{-\gamma t} \, u_{|\partial\Omega \times [0,T]}\|^2_{L^2(\partial\Omega \times (0,T))} \tag{9.2.43}$$

$$\leq c \left(\tfrac{1}{\gamma} \, \|e^{-\gamma t} \, f\|^2_{L^2(\Omega \times (0,T))} + \|e^{-\gamma t} \, g\|^2_{L^2(\partial\Omega \times (0,T))} \right).$$

Furthermore, if $f \in H^k(\bar{\Omega} \times [0,T])$ *and* $g \in H^k(\partial\Omega \times [0,T])$ *with* $k \geq 1$ *and* $\partial_t^j f = 0$, $\partial_t^j g = 0$ *at* $t = 0$ *for all* $j \in \{0, \ldots, k-1\}$, *then* u *belongs to* $H^k(\bar{\Omega} \times [0,T])$ *and satisfies* $\partial_t^j u = 0$ *at* $t = 0$ *for all* $j \in \{0, \ldots, k-1\}$.

Remark 9.12

i) According to Theorem 9.8, the equality in (9.2.42) is meaningful at least in $H^{-1/2}(\Omega)$ for any square-integrable u satisfying (9.2.40) with f also square-integrable.

ii) It will appear in the proof of Theorem 9.12 that solving the homogeneous IBVP (9.2.40)–(9.2.42) for $f \in L^2(\bar{\Omega} \times [0,T])$ and $g \in L^2(\partial\Omega \times [0,T])$ is equivalent to solving

$$\begin{cases} L\widetilde{u} = \widetilde{f} \text{ on } \Omega \times (-\infty, T)\,, \\ B\widetilde{u} = \widetilde{g} \text{ on } \partial\Omega \times (-\infty, T)\,, \\ \widetilde{u} = 0 \quad \text{on } \Omega \times (-\infty, 0)\,, \end{cases}$$

for $\widetilde{f} \in L^2(\bar{\Omega} \times (-\infty, T])$ and $\widetilde{g} \in L^2(\partial\Omega \times (-\infty, T])$ vanishing on $(-\infty, 0)$.

Proof To prove the existence of a solution u, we first extend f and g by zero for $t < 0$ and $t > T$. The resulting functions, say $\breve{f}$ and $\breve{g}$, obviously belong to $e^{\gamma t} \, L^2(\bar{\Omega} \times \mathbb{R})$ and $e^{\gamma t} \, L^2(\partial\Omega \times \mathbb{R})$, respectively, for any real number γ. Therefore, we may apply Theorem 9.9 to $\breve{f}$ and $\breve{g}$ as source term and boundary term, respectively. This yields a solution $\breve{u} \in e^{\gamma_0 t} \, L^2(\bar{\Omega} \times \mathbb{R})$ of the boundary value problem

$$L\breve{u} = \breve{f} \text{ on } \Omega \times \mathbb{R}\,, \quad B\breve{u} = \breve{g} \text{ on } \partial\Omega \times \mathbb{R}\,. \tag{9.2.44}$$

By construction $u := \breve{u}_{|t \in [0,T]}$ clearly satisfies (9.2.40) and (9.2.41). However, the fact that u vanishes at $t = 0$ needs some verification.

Similarly as in the proof of Theorem 9.8, we denote by $\mathbf{1}$ the characteristic function of $\mathbb{R}^+$, and for any function v, $v^0(x,t) = v(x,t)\,\mathbf{1}(t)$. By that theorem, where $\Omega \times \mathbb{R}$ is replaced by $\{\,(x,t)\,;\, t > 0\,\}$ (whose boundary is non-characteristic because of the hyperbolicity of L in the direction of t), we know that $\breve{u}$ admits a trace $\breve{u}_{|t=0} \in H^{-1/2}(\mathbb{R}^d)$ and we have the jump formula

$$L(\breve{u}^0) = (L\breve{u})^0 + \breve{u}_{|t=0} \otimes \delta_{\{t=0\}}\,.$$

So we will be able to conclude that $\breve{u}_{|t=0} = 0$ if we can show that $L(\breve{u}^0) = (L\breve{u})^0$, which will be the case if $\breve{u}^0 = \breve{u}$. The latter equality equivalently means that $\breve{u}_{t<0} = 0$, which follows from the following support theorem.

Theorem 9.13 *In the framework of Theorem 9.9, if $f \in L^2(\bar{\Omega} \times \mathbb{R})$ and $g \in L^2(\partial\Omega \times \mathbb{R})$ both vanish for $t < t_0$, then the solution u of the boundary value problem*

$$Lu = f \quad on \;\; \Omega \times \mathbb{R}\,, \qquad Bu = g \quad on \;\; \partial\Omega \times \mathbb{R}$$

also vanishes for $t < t_0$.

We postpone the proof of this result, and complete the proof of Theorem 9.12.

For the uniqueness of the solution u, we must show that the only solution in L^2 of the homogeneous problem

$$\begin{aligned} Lu \quad &= 0 \text{ on } \;\Omega \times (0,T)\,, \\ Bu \quad &= 0 \text{ on } \;\partial\Omega \times (0,T)\,, \\ u_{|t=0} &= 0 \text{ on } \;\Omega\,, \end{aligned}$$

is the trivial solution 0. So assume u is a solution. Using the same notation as before and again the jump formula we see u^0 solves the problem

$$\begin{aligned} Lu &= 0 \text{ on } \;\Omega \times (-\infty,T)\,, \\ Bu &= 0 \text{ on } \;\partial\Omega \times (-\infty,T)\,. \end{aligned}$$

Now, we introduce a smooth cut-off function θ such that $\theta \equiv 1$ on $(-\infty,\tau]$, $\tau < T$, and $\theta \equiv 0$ on $[T,+\infty)$. Then both $\theta\, u^0$ and $L(\theta\, u^0)$ belong to $L^2(\bar{\Omega} \times \mathbb{R})$, and the trace of $B\,\theta u^0$ belongs to $L^2(\partial\Omega \times \mathbb{R})$. Furthermore, $L(\theta\, u^0) \equiv 0$ and $B\,\theta\, u^0 \equiv 0$ for $t < \tau$. Hence by Theorem 9.13 again, $\theta\, u^0 = 0$ for $t < \tau$. Since this is true for all $\tau < T$ and $\theta\, u^0 = u$ for $t \in [0,\tau]$, we infer that $u \equiv 0$ (a.e) in $[0,T)$, as expected.

So the unique solution of (9.2.40)–(9.2.42) is necessarily the one constructed in the first step, i.e. $u = \breve{u}_{|t\in[0,T]}$. Therefore, the (weighted) localized L^2 estimate (9.2.43) for the homogeneous IBVP is a consequence of the weighted L^2 estimate (9.2.36) for the BVP, applied to $\breve{u}$.

Finally, the assumptions on the partial derivatives of f and g if $f \in H^k(\bar{\Omega} \times [0,T])$ and $g \in H^k(\partial\Omega \times [0,T])$ show that they admit extensions as functions in $H^k(\bar{\Omega} \times \mathbb{R})$ and $H^k(\partial\Omega \times \mathbb{R})$, respectively, which vanish for $t < 0$. By Theorem 9.9, the solution of the corresponding BVP belongs to $H^k(\bar{\Omega} \times \mathbb{R})$, and by Theorem 9.13, its restriction to $(-\infty,T])$ depends only on f and g (and not on their extensions for $t > T$), so it coincides with the restriction of $\breve{u}$ (solution of (9.2.44) above) to $(-\infty,T])$: this implies $\breve{u}_{|t<T} \in H^k(\bar{\Omega} \times (-\infty,T])$, hence $u = \breve{u}_{|t\in[0,T]} \in H^k(\bar{\Omega} \times [0,T])$, and the fact that $\breve{u}_{|t<0} = 0$ does imply that $\partial_t^j u = 0$ at $t = 0$ for all $j \in \{0,\dots,k-1\}$. □

Proof of Theorem 9.13 Without loss of generality, we may assume $t_0 = 0$. By assumption, f belongs to $\mathrm{e}^{\gamma t} L^2(\bar{\Omega} \times \mathbb{R})$ and g belongs to $\mathrm{e}^{\gamma t} L^2(\partial\Omega \times \mathbb{R})$ for all $\gamma > 0$. So, by Theorem 9.9, for all $n \in \mathbb{N}$, there exists a unique $u_n \in \mathrm{e}^{\gamma_n t} L^2(\Omega \times \mathbb{R})$ with $\gamma_n = \gamma_0 + n$ such that

$$Lu_n = f \text{ on } \Omega \times \mathbb{R}, \quad Bu_n = g \text{ on } \partial\Omega \times \mathbb{R},$$

and by the energy estimate (9.2.36),

$$\|\mathrm{e}^{-\gamma_n t} u_n\|_{L^2}^2 \lesssim \frac{1}{\gamma_0} \|\mathrm{e}^{-\gamma_0 t} f\|_{L^2}^2 + \|\mathrm{e}^{-\gamma_0 t} g\|_{L^2}^2 =: C^2 .$$

Furthermore, we claim u_n is independent of n for large enough n. Indeed, consider a $\mathscr{C}^\infty$ function

$$\begin{aligned} \theta : \quad \mathbb{R} &\to [0,1] \\ t \leq 0 &\mapsto \ 1, \\ t > 1 &\mapsto \mathrm{e}^{-t} . \end{aligned}$$

Then, for all n, $v_n(x,t) := \theta(t)\,(u_{n+1} - u_n)(x,t)$ defines a function in $\mathrm{e}^{\gamma_n t} L^2(\partial\Omega \times \mathbb{R})$ such that

$$(L - \theta'(t)/\theta(t))\, v_n = 0 \text{ on } \Omega \times \mathbb{R}, \quad Bv_n = 0 \text{ on } \partial\Omega \times \mathbb{R}.$$

Therefore, by the uniqueness part of Theorem 9.9 applied to the operator $(L - \theta'(t)/\theta(t))$ (the additional smooth, zero-order term being harmless), we find that $v_n = 0$ (for n large enough so that γ_n is larger than the γ_0 corresponding to the modified operator).

Then the fact that $u_n = u$ vanishes for $t < 0$, is an easy consequence of the uniform bound

$$\|\mathrm{e}^{-\gamma_n t} u\|_{L^2} \leq C .$$

Indeed, take $\varepsilon > 0$ and $\varphi \in \mathscr{D}(\Omega \times (-\infty, -\varepsilon))$. Then

$$|\langle u\,,\,\varphi\rangle| \leq \|\mathrm{e}^{-\gamma_n t} u\|_{L^2}\, \|\mathrm{e}^{\gamma_n t} \varphi\|_{L^2} \leq C\,\mathrm{e}^{-\gamma_n \varepsilon}\, \|\varphi\|_{L^2} ,$$

which goes to zero when $n \to +\infty$. Therefore, $u_{|t \in (-\infty, -\varepsilon)} \equiv 0$ (a.e.) for all $\varepsilon > 0$. □

9.2.3 *The general IBVP (smooth coefficients)*

As already pointed out in Chapter 4 for constant-coefficients problems, the resolution of the Initial Boundary Value Problem (9.0.2)–(9.0.4) with non-zero initial data u_0 is not a direct consequence of previous results.

- A natural (or naive) approach to the general IBVP is to consider and solve the Cauchy problem

$$Lv = 0, \quad v_{|t=0} = u_0$$

(with u_0 extended by zero outside Ω) and then look for $u = w + v$ with

$$Lw = f \text{ in } \Omega\,, \quad Bw = g - Bv \text{ on } \partial\Omega\,,\ w_{|t=0} = 0\,. \tag{9.2.45}$$

Unfortunately, for L^2 data f and u_0, v is square-integrable in Ω but its trace on $\partial\Omega$ has no reason to be also square-integrable (recall that by Theorem 9.8 it is a priori $H^{-1/2}$), which obviously hinders the resolution of (9.2.45).

- An alternative is to consider as a preliminary problem not a Cauchy problem but an Initial Boundary Value Problem of the form

$$Lv = 0 \text{ in } \Omega\,, \quad B_0 v = 0 \text{ on } \partial\Omega\,,\ v_{|t=0} = u_0\,, \tag{9.2.46}$$

no matter what B_0 is, provided it yields a solution v with L^2 trace on $\partial\Omega$: if we are able to solve (9.2.46) in such a way that $v_{|\partial\Omega}$ is in L^2, then we will be able to solve (9.2.45) in L^2 (thanks to Theorem 9.12), hence also the general IBVP (by taking $u = w + v$). In other words, the resolution of the general IBVP relies on the resolution of a particular IBVP, with a possibly different boundary matrix, with non-zero initial data u_0 but with zero source term f and zero boundary data g.

For the resolution of (9.2.46), a natural idea is to proceed as for the Cauchy problem, by duality. In this respect, we need a pointwise estimate of the form

$$\|v(T)\|_{L^2(\Omega)} \leq \mathrm{e}^{\gamma T}\, \|v(0)\|_{L^2(\Omega)}\,, \tag{9.2.47}$$

for γ large enough, independent of $v \in \mathscr{C}^1([0,T]; L^2(\Omega)) \cap \mathscr{C}([0,T]; H^1(\Omega))$ such that

$$Lv = 0 \text{ in } \Omega\,, \quad B_0 v = 0 \text{ on } \partial\Omega\,.$$

Deriving such an estimate was the main purpose of the paper by Rauch [162], following his PhD thesis [161]. We shall not go into (thankless) details regarding the derivation of (9.2.47) in general, for which we refer to [162]. But in the case of a Friedrichs-symmetrizable operator L (as in [161]), the task is much easier: proving (9.2.47) is a matter of integrations by parts and of suitable choice of the boundary matrix B_0, as we explain now.

Let us assume that L admits a Friedrichs symmetrizer S_0, with

$$\sigma\, I_n \leq S_0 \leq \sigma^{-1}\, I_n$$

for some positive σ. We take $v \in \mathscr{C}^1([0,T]; L^2(\Omega)) \cap \mathscr{C}([0,T]; H^1(\Omega))$ to ensure that our computations are valid. If $Lv = 0$ then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^* S_0 v &= -2\int_\Omega \sum_j v^* S_0 A^j\, \partial_j v + \int_\Omega v^* (\partial_t S_0)\, v \\ &= -\int_{\partial\Omega} \sum_j \nu_j\, v^* S_0 A^j\, v + \int_\Omega v^* (\partial_t S_0 + \sum_j \partial_j (S_0 A^j))\, v.\end{aligned}$$

If $\partial\Omega$ is non-characteristic, the matrix

$$S_0(x,t)\, A(x,t,\nu(x))$$

is hyperbolic along $\partial\Omega$, which allows us to define $B_0(x,t)$ as the spectral projection onto its stable subspace. If $B_0 v \,=\, 0$ on $\partial\Omega$ then

$$v(x,t)^*\, S_0(x,t)\, A(x,t,\nu(x))\, v(x,t) \,\geq\, 0$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^* S_0 v \,\leq\, \int_\Omega v^*(\partial_t S_0 \,+\, \sum_j \partial_j(S_0 A^j))\, v\,.$$

This implies, after integration,

$$\|v(t)\|_{L^2} \,\leq\, \mathrm{e}^{\gamma\, t}\, \|v(0)\|_{L^2}$$

with

$$\gamma \,=\, \frac{1}{2\sigma}\, \|\partial_t S_0 + \sum_j \partial_j(S_0 A^j)\|_{L^\infty(\Omega\times(0,t))}\,.$$

Theorem 9.14 (Rauch) *Under the assumptions of Theorem 9.2, there exists* $\gamma_0 \geq 1$ *and* $C_0 > 0$ *such that for all* $\gamma \geq \gamma_0$*, for all* $T > 0$ *and all* $u \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$*,*

$$\mathrm{e}^{-2\gamma T}\, \|u_{|t=T}\|^2_{L^2(\Omega)} \,+\, \gamma\, \|\mathrm{e}^{-\gamma t}\, u\|^2_{L^2(\Omega\times(0,T))} \,+\, \|\mathrm{e}^{-\gamma t}\, u_{|\partial\Omega}\|^2_{L^2(\partial\Omega\times(0,T))}$$

$$\leq\, C_0\, \left(\|u_{|t=0}\|^2_{L^2(\Omega)} \,+\, \frac{1}{\gamma}\, \|\mathrm{e}^{-\gamma t}\, Lu\|^2_{L^2(\Omega\times(0,T))} + \|\mathrm{e}^{-\gamma t}\, Bu_{|x_d=0}\|^2_{L^2(\partial\Omega\times(0,T))} \right)\,.$$

Proof hints If (additionally) L is Friedrichs symmetrizable, a duality argument and the counterpart of the energy estimate (9.2.43) for the adjoint homogeneous IBVP easily yield the refined energy estimate as stated in Theorem 9.14 (for details, see the proof of Theorem 9.19 below, which holds true for Lipschitz coefficients). Without Friedrichs symmetrizability, the proof of Theorem 9.14 is much more technical (and valid only for smooth coefficients). Rauch [162] first derives a $\mathscr{H}^{d-1}_\gamma$ estimate, by using a method due to Gårding and Leray, then an estimate of negative index for the adjoint problem, and finally shows how to 'raise' this estimate up to an L^2 one.

Using Theorem 9.14 we can prove the L^2-well-posedness of the general IBVP, as stated below.

Theorem 9.15 *In the framework of Theorem 9.9, for all* $f \in L^2(\bar\Omega \times [0,T])$*,* $g \in L^2(\partial\Omega \times [0,T])$*,* $u_0 \in L^2(\bar\Omega)$*, the problem*

$$\begin{cases} Lu \;\; = f \;\; \textit{on} \;\; \Omega \times (0,T)\,, \\ Bu \;\; = g \;\; \textit{on} \;\; \partial\Omega \times (0,T)\,, \\ u_{|t=0} = u_0 \;\; \textit{on} \;\; \Omega\,, \end{cases}$$

admits a unique solution $u \in L^2(\bar\Omega \times [0,T])$*, which is such that* $u_{\partial\Omega\times[0,T]} \in L^2(\partial\Omega \times [0,T])$*. Furthermore,* u *belongs to* $\mathscr{C}([0,T]; L^2(\bar\Omega))$ *and satisfies an estimate of the form*

$$\|u(T)\|^2_{L^2(\Omega)} + \tfrac{1}{T}\,\|u\|^2_{L^2(\Omega\times(0,T))} + \|u_{|\partial\Omega\times[0,T]}\|^2_{L^2(\partial\Omega\times(0,T))}$$

$$\leq c\,\Big(\,\|u_0\|^2_{L^2(\Omega)} + T\,\|f\|^2_{L^2(\Omega\times(0,T))} + \|g\|^2_{L^2(\partial\Omega\times(0,T))}\Big)\,,$$

with c *independent of* u_0*,* f*,* g*,* u *and* T*.*

Proof Once we have the energy estimate of Theorem 9.14, the proof of L^2-well-posedness follows a standard procedure, and works even for rough (Lipschitz) coefficients. To avoid too much repetition, we refer to the proof of Theorem 9.19 below. □

Additional regularity and compatibility conditions

We recall that for the IBVP with zero initial data, Theorem 9.12 says solutions are H^k for H^k source term f and H^k boundary data g, provided that

$$\partial_t^\ell f = 0 \quad\text{and}\quad \partial_t^\ell g = 0 \quad\text{at}\;\; t=0\,,\qquad \text{for all}\;\; \ell \in \{0,\dots,k-1\}\,.$$

In fact, these conditions are not optimal: weaker conditions are sufficient to ensure the H^k regularity of solutions, even for the general IBVP. These are *compabilitity conditions* between the source term f, the boundary data g and the initial data u_0, looking rather complicated but easy to understand. For, if $Lu = f$ with u of class $\mathscr{C}^p$ with respect to time and f of class $\mathscr{C}^{p-1}$, a straightforward proof by induction shows that

$$\partial_t^q u = \sum_{\ell=0}^{q-1} \binom{q-1}{\ell} P_\ell\, \partial_t^{q-1-\ell} u + \partial_t^{q-1} f\,,$$

where $P_0 := \partial_t + L = -\sum_j A^j(x,t)\,\partial_j$ and $P_p := -\sum_j (\partial_t^p A^j)(x,t)\,\partial_j$ for all integer $p \geq 1$; now if $Bu = g$ on $\partial\Omega$, Leibniz' rule shows that

$$\partial_t^\ell g = \sum_{q=0}^{\ell} \binom{\ell}{q} (\partial_t^{\ell-q} B)\, \partial_t^q u\,.$$

Consequently, the compatibility conditions (necessary for such a u to exist) are

(CC$_p$) the functions $u_q : x \in \bar\Omega \mapsto u_q(x)$ defined inductively by

$$u_q(x) = \sum_{\ell=0}^{q-1} \binom{q-1}{\ell} P_\ell^0 u_{q-1-\ell}(x) + \partial_t^{q-1} f(x,0) \quad \text{for all}\;\; q \in \{1,\dots,p\}$$

are such that

$$\partial_t^\ell g(x,0) = \sum_{q=0}^{\ell} \binom{\ell}{q} (\partial_t^{\ell-q} B)(x,0)\, u_q(x) \quad \text{for all } x \in \partial\Omega, \text{and all } \ell \in \{1,\dots,p\}.$$

Here above we have used the notation

$$P_\ell^0 := -\sum_j (\partial_t^\ell A^j)(x,0)\,\partial_j\,.$$

Theorem 9.16 (Rauch–Massey) *In the framework of Theorem 9.15, if, moreover, the initial data belongs to $H^k(\bar\Omega)$, the source term f belongs to $H^k(\bar\Omega \times [0,T])$, and the boundary data belongs to $H^{k+1/2}(\partial\Omega \times [0,T])$ for some integer $k \geq 1$, and if u_0, f, g satisfy the compatibility conditions in* $(\mathbf{CC}_p)$ *for all $p \in \{0,\dots,k-1\}$, then the solution u belongs to $\mathscr{C}^r([0,T]; H^{k-r}(\bar\Omega))$ for all $r \in \{0,\dots,k\}$.*

We omit the proof here; see [165].

Remark 9.13 If $u_0 \equiv 0$ and *if* $\partial_t^\ell f(x,0) = 0$ for all $\ell \in \{1,\dots,k-1\}$, the set of compatibility conditions $(\mathbf{CC}_p)$ for $p \in \{0,\dots,k-1\}$ reduce to

$$\partial_t^\ell g(x,0) = 0 \quad \text{for all } \ell \in \{1,\dots,k-1\}\,,$$

as expected.

9.2.4 *Rough coefficients*

The L^2-well-posedness of the BVP is true as soon as the coefficients of both (the principal part of) the operator and the boundary conditions are Lipschitz. More precisely, we have the following result.

Theorem 9.17 *In the framework and with the assumptions of Theorem 9.6, with the additional fact that $\mathbb{W}^0$ is contractible, there exists $\gamma_0 = \gamma_0(\omega) \geq 1$ such that for all $\gamma \geq \gamma_0$, for all $v \in \mathbb{V}_\omega$, for all $f \in \mathrm{e}^{\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and all $g \in \mathrm{e}^{\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R})$, there is one and only one solution $u \in \mathrm{e}^{\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R})$ of the Boundary Value Problem*

$$L_v\, u = f \ \text{ on } \{x_d > 0\}\,, \quad B_v\, u = g \ \text{ on } \{x_d = 0\}\,. \tag{9.2.48}$$

Furthermore, the trace of u at $x_d = 0$ belongs to $\mathrm{e}^{\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R})$, and $\widetilde{u}_\gamma = \mathrm{e}^{-\gamma t} u$ enjoys an estimate

$$\gamma \, \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|\widetilde{u}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \leq c \left(\frac{1}{\gamma} \|\widetilde{f}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|\widetilde{g}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \right) \tag{9.2.49}$$

for some constant $c > 0$ depending only on ω.

Proof As in Theorem 9.9 the existence part relies on a duality argument using the adjoint BVP, which can still be defined thanks to Lemma 9.4: indeed, substituting $\mathbb{W}^0$ to Ω we get $\mathscr{C}^\infty$ mappings C, N, and M on $\mathbb{W}^0$ with values in $\mathbf{M}_{(n-p)\times n}(\mathbb{R})$ and $\mathbf{M}_{p\times n}(\mathbb{R})$, respectively, such that $\mathbb{R}^n = \ker C(w) \oplus \ker M(w)$,

$$A^d(w) = M(w)^{\mathrm{T}} B(w) + C(w)^{\mathrm{T}} N(w) \quad \text{and} \quad \ker C(w) = (A^d(w) \ker B(w))^\perp$$

for all $w \in \mathbb{W}^0$; hence the following identity

$$\int_{x_d>0} (z^{\mathrm{T}} L_v u - u^{\mathrm{T}} (L_v)^* z) + \int_{x_d=0} ((M_v z)^{\mathrm{T}} B_v u + (N_v u)^{\mathrm{T}} C_v z) = 0 \tag{9.2.50}$$

for all Lipschitz-continuous v such that $v_{|x_d=0}$ maps $\mathbb{R}^{d-1} \times \mathbb{R}$ into $\mathbb{W}^0$ (and consistently with the notation B_v, $C_v := C \circ v$, $M_v := M \circ v$ and $N_v := N \circ v$) and for all $u \in \mathrm{e}^{\gamma t} H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and $z \in \mathrm{e}^{-\gamma t} H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$.

Note: Thanks to Theorem 9.8, the identity (9.2.50) still holds true when the regularity of u is relaxed to

$$u \in \mathrm{e}^{\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}) \text{ and } L_v u \in \mathrm{e}^{\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$$

(or symmetrically, if $z \in \mathrm{e}^{-\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and $(L_v)^* z \in \mathrm{e}^{\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$, but $u \in \mathrm{e}^{\gamma t} H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$). For completeness, we now explain the duality argument (which merely parallels what has been done in the proof of Theorem 9.9). Introduce the subspace of $\mathrm{e}^{-\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$

$$\mathscr{E} := \{ z \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}) ; \ C_v z_{|x_d=0} = 0 \}$$

and denote for simplicity

$$\|z\|_\gamma := \|\mathrm{e}^{\gamma t} z\|_{L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})},$$

$$|z|_\gamma := |\mathrm{e}^{\gamma t} z|_{L^2(\mathbb{R}^{d-1} \times \mathbb{R})}.$$

The energy estimate

$$\gamma \|z\|_\gamma^2 + |z|_\gamma^2 \leq \frac{c}{\gamma} \|(L_v)^* z\|_\gamma^2$$

(a consequence of Theorem 9.6 for the *backward* and *homogeneous* BVP associated with $(L_v)^*$) enables us to define a linear form ℓ on $(L_v)^* \mathscr{E}$ by

$$\ell((L_v)^* z) = \int_{x_d>0} z^{\mathrm{T}} f + \int_{x_d=0} (M_v z)^{\mathrm{T}} g,$$

such that

$$|\ell((L_v)^* z)| \leq \|f\|_{-\gamma} \|z\|_\gamma + \|M_v\|_{L^\infty} |g|_{-\gamma} |z|_\gamma$$

$$\lesssim \Big(\frac{1}{\gamma} \|f\|_{-\gamma} + \frac{1}{\gamma^{1/2}} |g|_{-\gamma} \Big) \|(L_v)^* z\|_\gamma .$$

By the Hahn–Banach Theorem ℓ thus extends to a continuous form on the weighted space $\mathrm{e}^{-\gamma t}L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$, which shows by the Riesz Theorem the existence of u in the dual space $\mathrm{e}^{\gamma t}L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ such that

$$\ell((L_v)^*z) \;=\; \int_{x_d>0} u^{\mathrm{T}}(L_v)^*z$$

for all $z\in\mathscr{E}$. In particular, this implies by definition of ℓ on $(L_v)^*\mathscr{E}$ that for all $z\in\mathscr{D}(\mathbb{R}^{d-1}\times(0,+\infty)\times\mathbb{R})$,

$$\int_{x_d>0} (\,f^{\mathrm{T}}\,z \;-\; u^{\mathrm{T}}\,(L_v)^*z\,) \;=\; 0\,,$$

hence $L_v\,u \;=\; f$ in the sense of distributions. Therefore, $L_v u\;\in\;\mathrm{e}^{\gamma t}L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ and we may apply the identity (9.2.50). This yields

$$\int_{x_d=0} (M_v z)^{\mathrm{T}}\,(\,B_v u \;-\; g\,) \;=\; 0$$

for all $z\in\mathscr{E}$, hence also

$$\int_{x_d=0} (M_v\,\varphi)^{\mathrm{T}}\,(\,B_v u \;-\; g\,) \;=\; 0$$

for all compactly supported $\mathscr{C}^\infty$ function φ on the boundary $\mathbb{R}^{d-1}\times\mathbb{R}$ such that $C_v\,\varphi\;\equiv\;0$. Since at each point $w\in\mathbb{W}^0$, the linear mapping $M(w)_{|\mathrm{Ker}C(w)}$: Ker $C(w)\;\to\mathbb{C}^p$ is onto, and $v_{|x_d=0}\;:\;\mathbb{R}^{d-1}\times\mathbb{R}\to\mathbb{W}^0$ is Lipschitz-continuous, this implies

$$\langle\psi\,,\,B_v u\;-\;g\,\rangle \;=\; 0$$

for all $\psi\in\mathrm{e}^{-\gamma t}\;H^{1/2}(\mathbb{R}^{d-1}\times\mathbb{R})$, hence the boundary condition $B_v u\;=\;g$ holds true in $\mathrm{e}^{\gamma t}H^{-1/2}(\mathbb{R}^{d-1}\times\mathbb{R})$.

In this way we have obtained a weak solution $u\in\mathrm{e}^{\gamma t}L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ of our BVP. The next step is a so-called *weak=strong* argument, showing that all $\mathrm{e}^{\gamma t}L^2$ solutions are in fact limits of smooth solutions of regularized problems. This will imply in particular the validity of the estimate of Theorem 9.6 for any $\mathrm{e}^{\gamma t}L^2$ solution, hence the uniqueness.

We take a standard mollifier in the (y,t) variables ρ_ε, and R_ε the associated convolution operator, and define

$$u_\varepsilon^\gamma \;=\; R_\varepsilon\mathrm{e}^{-\gamma t}u\,,\quad F_\varepsilon^\gamma = R_\varepsilon(A_v^d)^{-1}\mathrm{e}^{-\gamma t}f\,,\quad g_\varepsilon^\gamma = R_\varepsilon\mathrm{e}^{-\gamma t}g\,.$$

For all $\varepsilon>0$, g_ε^γ belongs to $H^{+\infty}(\mathbb{R}^d\times\mathbb{R})$ and goes to $\widetilde{g}_\gamma\;=\;\mathrm{e}^{-\gamma t}g$ in $L^2(\mathbb{R}^d\times\mathbb{R})$ as ε goes to zero, and F_ε^γ belongs to $L^2(\mathbb{R}^+;H^{+\infty}(\mathbb{R}^d\times\mathbb{R}))$ and goes to $\widetilde{F}_\gamma\;=\;(A_v^d)^{-1}\mathrm{e}^{-\gamma t}f$ in $L^2(\mathbb{R}^+;L^2(\mathbb{R}^d\times\mathbb{R}))$. Similarly, u_ε^γ belongs to $L^2(\mathbb{R}^+;H^{+\infty}(\mathbb{R}^d\times\mathbb{R}))$ and goes to $\widetilde{u}_\gamma\;=\;\mathrm{e}^{-\gamma t}u$ in $L^2(\mathbb{R}^+;L^2(\mathbb{R}^d\times\mathbb{R}))$, while $(u_\varepsilon^\gamma)_{|x_d=0}$ belongs to $H^{+\infty}(\mathbb{R}^d\times\mathbb{R})$ and goes to $(\widetilde{u}_\gamma)_{|x_d=0}\;=\;\mathrm{e}^{-\gamma t}u_{|x_d=0}$ in $H^{-1/2}(\mathbb{R}^d\times\mathbb{R})$.

If we can show additional regularity of u_ε^γ in the x_d variable, we will be allowed to apply the energy estimate of Theorem 9.6 to u_ε^γ and pass to the limit to obtain the estimate for $\widetilde{u}_\gamma$. This will be possible thanks to the following result on commutators, which is a straightforward consequence of Theorem C.14 and Remark C.6, stated here as a lemma only for convenience.

Lemma 9.5 *With the notations defined above and, as in the proof of Theorem 9.6,*

$$P_v^\gamma \,:=\, -\,(A_v^d)^{-1}\left(\partial_t \,+\, \gamma \,+\, \sum_{j=1}^{d-1} A^j(v(x,t))\,\partial_j\,\right)$$

we have

$$\lim_{\varepsilon\to 0}\|\,[P_v^\gamma,R_\varepsilon]\,\mathrm{e}^{-\gamma\,t}u\,\|_{L^2} \,=\, 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0}\|\,[B_v,R_\varepsilon]\,\mathrm{e}^{-\gamma\,t}u_{|x_d=0}\,\|_{L^2} \,=\, 0\,.$$

Going on with the proof of Theorem 9.17, we claim that u_ε^γ belongs to $H^1(\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R})$. Indeed, we have

$$\partial_d u_\varepsilon^\gamma \,=\, F_\varepsilon^\gamma \,+\, P_v^\gamma\, u_\varepsilon^\gamma \,-\, [P_v^\gamma,R_\varepsilon]\,\mathrm{e}^{-\gamma\,t}u\,,$$

where the first two terms belong to $L^2(\mathbb{R}^+;H^{+\infty}(\mathbb{R}^d\times\mathbb{R}))$ and the third one is bounded in $L^2(\mathbb{R}^+;L^2(\mathbb{R}^d\times\mathbb{R}))$ (as a byproduct of the first limit in Lemma 9.5), so that $\partial_d u_\varepsilon^\gamma$ is in L^2, as well as the other derivatives $\partial_j u_\varepsilon^\gamma$. Consequently, by density of $\mathscr{D}(\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R})$ in $H^1(\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R})$, this allows us to apply the inequality in (9.1.22) to $u_\varepsilon^\gamma - u_{\varepsilon'}^\gamma$, hence

$$\begin{aligned}
&\gamma\,\|u_\varepsilon^\gamma - u_{\varepsilon'}^\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|(u_\varepsilon^\gamma - u_{\varepsilon'}^\gamma)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}\\
&\lesssim \frac{1}{\gamma}\,\|(\partial_d \,-\, P_v^\gamma)(u_\varepsilon^\gamma - u_{\varepsilon'}^\gamma)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|B_v\,(u_\varepsilon^\gamma - u_{\varepsilon'}^\gamma)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})},\\
&\lesssim \frac{1}{\gamma}\,\|F_\varepsilon^\gamma - F_{\varepsilon'}^\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \frac{1}{\gamma}\,\|[P_v^\gamma,R_\varepsilon - R_{\varepsilon'}]\,\mathrm{e}^{-\gamma\,t}u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})}\\
&\quad +\|g_\varepsilon^\gamma - g_{\varepsilon'}^\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})} \,+\, \|[B_v,R_\varepsilon - R_{\varepsilon'}]\,\mathrm{e}^{-\gamma\,t}u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}\,.
\end{aligned}$$

By Lemma 9.5 here above, the commutator terms tend to zero as ε and ε' go to zero, and the other terms tend to zero as well, as noticed at the beginning. This shows that $(u_\varepsilon^\gamma)_{\varepsilon>0}$ and $((u_\varepsilon^\gamma)_{|x_d=0})_{\varepsilon>0}$ are Cauchy sequences in $L^2(\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R})$ and $L^2(\mathbb{R}^d\times\mathbb{R})$, respectively, which we already knew for the former but not for the latter: the convergence of traces was a priori known in $H^{-1/2}(\mathbb{R}^d\times\mathbb{R})$; by uniqueness of limits in the sense of distributions, this shows that $(\widetilde{u}_\gamma)_{|x_d=0}$ is the limit of $((u_\varepsilon^\gamma)_{|x_d=0})_{\varepsilon>0}$ also in $L^2(\mathbb{R}^d\times\mathbb{R})$. So by passing to the limit in the inequality in (9.1.22) for u_ε^γ, which reads

$$\begin{aligned}
&\gamma\,\|u_\varepsilon^\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|(u_\varepsilon^\gamma)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}\\
&\lesssim\, \frac{1}{\gamma}\,\|(\partial_d \,-\, P_v^\gamma)u_\varepsilon^\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|B_v\,(u_\varepsilon^\gamma)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R})}\,,
\end{aligned}$$

we obtain the same inequality for $\widetilde{u}_\gamma$, hence (9.2.49) thanks to the boundedness of $(A_v^d)^{-1}$. □

Once we have Theorem 9.17 we can prove the L^2-well-posedness of the IBVP with *zero initial data* exactly as in the case of smooth coefficients (Theorem 9.12): as a matter of fact, the smoothness of coefficients does not play any role in the proof of the support theorem (Theorem 9.13), and thus the proof of Theorem 9.12 still works for Lipschitz coefficients, hence the following well-posedness result for the homogeneous IBVP.

Theorem 9.18 *With the assumptions of Theorem 9.6, for all $f \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$ and $g \in L^2(\mathbb{R}^{d-1} \times [0,T])$, for all Lipschitz-continuous*

$$v : \mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T] \to \mathbb{W},$$

constant outside a compact subset and such that $v_{|x_d=0}$ takes its values in $\mathbb{W}^0$, there exists a unique $u \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$ solution of the IBVP

$$L_v u = f \ \text{ for } x_d > 0,\ t \in (0,T) \quad \text{and} \quad B_v u_{|x_d=0} = g,\ u_{|t=0} = 0.$$

Furthermore, $u_{|x_d=0}$ belongs to $L^2(\mathbb{R}^{d-1} \times [0,T])$ and there exist $\gamma_0 > 0$ and $c > 0$, depending only on $\|v\|_{W^{1,\infty}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T]))}$ such that for all $\gamma \geq \gamma_0$,

$$\gamma\, \|\mathrm{e}^{-\gamma t}\, u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|\mathrm{e}^{-\gamma t}\, u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,T])} \tag{9.2.51}$$

$$\leq c \left(\tfrac{1}{\gamma}\, \|\mathrm{e}^{-\gamma t}\, f\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|\mathrm{e}^{-\gamma t}\, g\|^2_{L^2(\mathbb{R}^{d-1}\times[0,T])} \right).$$

For the general IBVP we need timewise bounds, which are not easy to obtain (see Section 9.2.3 regarding smooth coefficients). When coefficients are just Lipschitz-continuous, the only known result is due to Métivier [136] and assumes Friedrichs symmetrizability.

Theorem 9.19 (Métivier) *With the assumptions of Theorem 9.6 and the additional one that the operator L is Friedrichs symmetrizable, for all $f \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$, $g \in L^2(\mathbb{R}^{d-1} \times [0,T])$, $u_0 \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+)$, for all Lipschitz-continuous*

$$v : \mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T] \to \mathbb{W},$$

constant outside a compact subset and such that $v_{|x_d=0}$ takes its values in $\mathbb{W}^0$, the problem

$$\begin{cases} L_v u & = f \ \text{ for } x_d > 0,\ t \in (0,T), \\ B_v u_{|x_d=0} & = g \ \text{ for } t \in (0,T), \\ u_{|t=0} & = u_0 \text{ for } x_d > 0, \end{cases} \tag{9.2.52}$$

admits a unique solution $u \in L^2(\mathbb{R}^{d-1} \times [0,T])$, which is such that $u_{\partial\Omega\times[0,T]} \in L^2(\mathbb{R}^{d-1} \times [0,T])$. Furthermore, u belongs to $\mathscr{C}([0,T]; L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+))$ and *satisfies an estimate of the form*

$$\|u(T)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} + \tfrac{1}{T}\,\|u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,T])}$$

$$\leq c\,\left(\|u_0\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} + T\,\|f\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|g\|^2_{L^2(\mathbb{R}^{d-1}\times[0,T])}\right), \tag{9.2.53}$$

with c *depending only on* $\|v\|_{W^{1,\infty}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])}$.

Proof There are basically three (unequal) steps: 1) improve Theorem 9.18 by showing the solution of the homogeneous IBVP is *continuous in time* and satisfies a refined version of (9.2.51), namely

$$\mathrm{e}^{-2\gamma T}\,\|u_{|t=T}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} + \gamma\,\|\mathrm{e}^{-\gamma t}\,u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])}$$
$$+\,\|\mathrm{e}^{-\gamma t}\,u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,T])}$$
$$\leq c\left(\frac{1}{\gamma}\,\|\mathrm{e}^{-\gamma t}\,f\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|\mathrm{e}^{-\gamma t}\,g\|^2_{L^2(\mathbb{R}^{d-1}\times[0,T])}\right);$$

2) handle the IBVP with 'regularized' initial data $u_0 \in H^1(\mathbb{R}^d)$;
3) conclude by a density argument for initial data $u_0 \in L^2(\mathbb{R}^d)$.

Step 1) Recall that the solution given by Theorem 9.18 is $u = \breve{u}_{|[0,T]}$, where $\breve{u}$ is the solution of the BVP

$$L_v\breve{u} = \breve{f}\,, \quad B_v\breve{u}_{|x_d=0} = \breve{g}\,,$$

with $\breve{f}$ and $\breve{g}$ being extensions by zero for $t<0$ or $t>T$, of f and g, respectively. Furthermore, the 'weak=strong argument' in the proof of Theorem 9.17 shows if R_ε is a regularization operator in the (y,t) directions,

$$\begin{cases} u^\gamma_\varepsilon := R_\varepsilon\mathrm{e}^{-\gamma t}\breve{u} & \to\ \mathrm{e}^{-\gamma t}\breve{u} & \text{in } L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}),\\ (u^\gamma_\varepsilon)_{|x_d=0} & \to\ \mathrm{e}^{-\gamma t}\breve{u}_{|x_d=0} & \text{in } L^2(\mathbb{R}^{d-1}\times\mathbb{R}),\\ F^\gamma_\varepsilon := R_\varepsilon\mathrm{e}^{-\gamma t}(A^d_v)^{-1}\breve{f} & \to\ \mathrm{e}^{-\gamma t}(A^d_v)^{-1}\breve{f} & \text{in } L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}),\\ g^\gamma_\varepsilon := R_\varepsilon\mathrm{e}^{-\gamma t}\breve{g} & \to\ \mathrm{e}^{-\gamma t}\breve{g} & \text{in } L^2(\mathbb{R}^d), \end{cases}$$

with $u^\gamma_\varepsilon \in L^2(\mathbb{R}^+;H^{+\infty}(\mathbb{R}^d))$, $F^\gamma_\varepsilon \in L^2(\mathbb{R}^+;H^{+\infty}(\mathbb{R}^d))$, $g^\gamma_\varepsilon \in H^{+\infty}(\mathbb{R}^d)$, and additionally $u^\gamma_\varepsilon \in H^1(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$. Assume for a moment the refined energy estimate for the homogeneous IBVP, i.e. for all γ large enough, for all $\tau\in\mathbb{R}$ and all $w \in H^1(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ such that $w_{|t<0} \equiv 0$,

$$\mathrm{e}^{-2\gamma\tau}\|w(\tau)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \tag{9.2.54}$$
$$+\,\gamma\,\|\mathrm{e}^{-\gamma t}w\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,\tau])} + \|\mathrm{e}^{-\gamma t}w_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,\tau])}$$
$$\leq c'\left(\tfrac{1}{\gamma}\|\mathrm{e}^{-\gamma t}L_vw\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,\tau])} + \|\mathrm{e}^{-\gamma t}B_vw\|^2_{L^2(\mathbb{R}^{d-1}\times[0,\tau])}\right).$$

Then, applying it to $w = \mathrm{e}^{\gamma t}\,(u^\gamma_\varepsilon - u^\gamma_{\varepsilon'})$ and using Lemma 9.5 to deal with commutators, we easily show $(u^\gamma_\varepsilon)_{|t\in[0,T]}$ is a Cauchy sequence in $\mathscr{C}([0,T]; L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+))$ when ε goes to zero, hence has a limit in that Banach space, and that this limit must be u by uniqueness of solutions. Additionally, by passing to the limit in the refined estimate (9.2.54) applied to $\mathrm{e}^{\gamma t}\,u^\gamma_\varepsilon$, we see it is also satisfied by u.

Therefore, step 1) will be complete when (9.2.54) is proved. By density of $\mathscr{D}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ in $H^1(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}) \hookrightarrow H^1(\mathbb{R}^+; L^2(\mathbb{R}^{d-1}\times\mathbb{R})) \hookrightarrow \mathscr{C}(\mathbb{R}^+; L^2(\mathbb{R}^{d-1}\times\mathbb{R}))$, it suffices to prove (9.2.54) for $w\in\mathscr{D}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ such that $w_{|t<0}\equiv 0$. (Any element of $H^1(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ vanishing for $t<0$ can be approached by such a w: it suffices to choose a mollifier supported in $\{t>0\}$.) Now, introducing a Friedrichs symmetrizer S_v for L_v, with

$$\sigma\, I_n \,\leq\, S_v \,\leq\, \sigma^{-1}\, I_n$$

for $\sigma>0$ (independent of v, which is bounded by assumption), denoting $L^\gamma_v = (\gamma + L_v)$ and (as usual) $\widetilde{w}_\gamma = \mathrm{e}^{-\gamma t}\,w$ (in such a way that $\mathrm{e}^{-\gamma t} L_v w = L^\gamma_v\,\widetilde{w}_\gamma$), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{x_d>0}\widetilde{w}^*_\gamma S_v\widetilde{w}_\gamma \;&=\; 2\,\mathrm{Re}\,\iint_{x_d>0}\widetilde{w}^*_\gamma S_v\,L^\gamma_v\widetilde{w}_\gamma \;+\;\iint_{x_d>0}\widetilde{w}^*_\gamma(\partial_t S_v)\,\widetilde{w}_\gamma\\ &\qquad -2\,\iint_{x_d>0}\sum_j\,\widetilde{w}^*_\gamma S_v A^j_v\,\partial_j\widetilde{w}_\gamma\\ &=\;\iint_{x_d>0}\widetilde{w}^*_\gamma(\partial_t S_v + \sum_j\partial_j(S_v A^j_v))\,\widetilde{w}_\gamma\\ &\qquad +2\,\mathrm{Re}\,\iint_{x_d>0}\widetilde{w}^*_\gamma S_v\,L^\gamma_v\widetilde{w}_\gamma \;+\;2\,\int_{x_d=0}\widetilde{w}^*_\gamma S_v\,A^d_v\widetilde{w}_\gamma\,.\end{aligned}$$

This implies after integration (using Cauchy–Schwarz and Young's inequalities),

$$\begin{aligned}\|\widetilde{w}_\gamma(t)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \;\leq\; &(\gamma_0+\gamma)\,\|\widetilde{w}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,t])}\\ &+\tfrac{1}{\gamma\,\sigma^4}\,\|L^\gamma_v\widetilde{w}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,t])}\\ &+\delta\,\|(\widetilde{w}_\gamma)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,t])},\end{aligned}$$

with $\gamma_0 = \frac{1}{\sigma}\|\partial_t S_v + \sum_j \partial_j(S_v A^j_v)\|_{L^\infty(\mathbb{R}^{d-1}\times\mathbb{R}^+\times(0,t))}$ and $\delta := \frac{2}{\sigma}\,\|S_v A^d_v\|_{L^\infty(\mathbb{R}^{d-1}\times\mathbb{R}^+\times(0,t))}$. And the L^2 norms of $\widetilde{w}_\gamma$ and its trace at $x_d=0$ are controlled by the inequality in (9.2.51). To be precise, we thus get that for $\gamma\geq\gamma_0$,

$$\begin{aligned}&\|\widetilde{w}_\gamma(t)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)}\\ &\quad\leq\;(a+\sigma^{-4})\,\frac{1}{\gamma}\;\|L^\gamma_v\widetilde{w}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,t])}\;+\;a\,\|B_v(\widetilde{w}_\gamma)_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,t])}\end{aligned}$$

with $a = c\max(2,\delta)$. Going back to w itself, we see this inequality together with (9.2.51) gives the refined inequality in (9.2.54) with $c' = c + a + \sigma^{-4}$.

Step 2) Assuming the initial data is better than square-integrable, say $u_0 \in H^1(\mathbb{R}^{d-1}\times\mathbb{R}^+)$, we may solve the corresponding IBVP by the 'naive' approach. Namely, we can extend v and u_0 to the whole space $\mathbb{R}^d$ (in such a way that the extended functions are still Lipschitz-continuous and in H^1, respectively, and that the extended operator L_v is still Friedrichs symmetrizable) and we consider the Cauchy problem

$$L_v\, u^c \,=\, 0\,,\; u^c_{|t=0} \,=\, u_0\,.$$

By Theorem 2.8 we know this problem admits a (unique) solution $u^c \in \mathscr{C}([0,T];L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+))$, and thanks to Friedrichs symmetrizability, by Theorem 2.9), u^c even belongs to $\mathscr{C}([0,T];H^1(\mathbb{R}^d))$. This additional regularity shows u does have a L^2 trace on the hyperplane $\{x_d = 0\}$. Therefore, by Theorem 9.18, the homogeneous IBVP

$$L_v\, u^h \,=\, f\,,\; Bu^h_{|x_d=0} \,=\, g \,-\, Bu^c_{|x_d=0}\,,\; u^h_{|t=0} \,=\, 0\,,$$

admits a (unique) square-integrable solution u^h, having a square-integrable trace on $\{x_d = 0\}$. And of course, the sum $u = u^c + u^h$ solves the IBVP

$$L_v\, u \,=\, f\,,\; Bu_{|x_d=0} \,=\, g\,,\; u_{|t=0} \,=\, u_0\,.$$

But obviously this construction of a solution to the general IBVP does not show its uniqueness. Uniqueness will follow from the energy estimate, which requires a little more work.

We first observe the solution obtained here above is a 'strong' one (as the sum of the smooth u^c and the 'strong' solution u^h of a homogeneous IBVP). Therefore (thanks to Lemma 9.5 about commutators), it is sufficient to prove the energy estimate

$$\begin{aligned}\|u_{|t=T}\|^2_{L^2} \,+\, \frac{1}{T}\,\|u\|^2_{L^2} \,+\, \|u_{|x_d=0}\|^2_{L^2} \,\leq\, c\,\Big(\,\|u_{|t=0}\|^2_{L^2} \,+\, T\,\|L_v u\|^2_{L^2}\\ +\|B_v u_{|x_d=0}\|^2_{L^2}\,\Big)\,,\end{aligned}$$

(where these L^2 norms are, from left to right, on $\mathbb{R}^{d-1}\times\mathbb{R}^+$, $\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T]$, or $\mathbb{R}^{d-1}\times[0,T]$) for u smooth enough, in H^1 say. This can be done by a duality argument and step 1) applied *backward* to the adjoint problem

$$(L_v)^* z \,=\, f'\,,\quad C_v z_{|x_d=0} \,=\, g'\,,\quad z_{|t=T} \,=\, 0\,, \tag{9.2.55}$$

where C_v is defined as in the proof of Theorem 9.17. Indeed, revisiting the duality identity (9.2.50) when t lies in the bounded interval $[0,T]$ we get

$$\int_{x_d>0} u^{\mathrm{T}} f' - \int_{x_d=0} (N_v u)^{\mathrm{T}} g' = \int_{x_d>0} z^{\mathrm{T}} L_v u + \int_{x_d=0} (M_v z)^{\mathrm{T}} B_v u + \int_{t=0} z^{\mathrm{T}} u$$

for all z satisfying (9.2.55) with $f' \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$ and $g' \in L^2(\mathbb{R}^{d-1} \times [0,T])$ and all $u \in H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$. Denoting for convenience (similarly as in the proof of Theorem 9.17)

$$\|z\|_{\gamma,T} := \|\mathrm{e}^{\gamma t} z\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])},$$
$$|z|_{\gamma,T} := \|\mathrm{e}^{\gamma t} z_{|x_d=0}\|_{L^2(\mathbb{R}^{d-1}\times[0,T])},$$

we have by the Cauchy–Schwarz inequality,

$$\left|\int_{x_d>0} u^{\mathrm{T}} f' - \int_{x_d=0} u^{\mathrm{T}} {N_v}^{\mathrm{T}} g'\right|$$

$$\leq \|z\|_{\gamma,T} \|L_v u\|_{-\gamma,T} + \|M_v\|_{L^\infty} |z|_{\gamma,T} |B_v u|_{-\gamma,T} + \|z_{|t=0}\|_{L^2} \|u_{|t=0}\|_{L^2}.$$

On the other hand, the counterpart of step 1) for the backward problem in (9.2.55) implies that

$$\|z_{|t=0}\|_{L^2}^2 + \gamma \|z\|_{\gamma,T}^2 + |z|_{\gamma,T}^2 \lesssim \frac{1}{\gamma} \|f'\|_{\gamma,T}^2 + |g'|_{\gamma,T}^2,$$

hence for $\gamma \geq 1$,

$$\max(\|z_{|t=0}\|_{L^2}, \|z\|_{\gamma,T}, |z|_{\gamma,T}) \lesssim \frac{1}{\gamma} \|f'\|_{\gamma,T} + \frac{1}{\gamma^{1/2}} |g'|_{\gamma,T}.$$

Together with the previous inequality, this yields, for $\gamma \geq 1$,

$$\left|\int_{x_d>0} u^{\mathrm{T}} f' - \int_{x_d=0} (N_v u)^{\mathrm{T}} g'\right| \lesssim \left(\tfrac{1}{\gamma} \|f'\|_{\gamma,T} + \tfrac{1}{\gamma^{1/2}} |g'|_{\gamma,T}\right) \times \left(\|L_v u\|_{-\gamma,T} + |B_v u|_{-\gamma,T} + \|u_{|t=0}\|_{L^2} \right).$$

Since this holds true for any $f' \in \mathrm{e}^{-\gamma t} L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$ and $g' \in L^2(\mathbb{R}^{d-1} \times [0,T])$, this implies

$$\|u\|_{-\gamma,T} \lesssim \frac{1}{\gamma} \left(\|L_v u\|_{-\gamma,T} + |B_v u|_{-\gamma,T} + \|u_{|t=0}\|_{L^2} \right)$$

and

$$|N_v u|_{-\gamma,T} \lesssim \frac{1}{\gamma^{1/2}} \left(\|L_v u\|_{-\gamma,T} + |B_v u|_{-\gamma,T} + \|u_{|t=0}\|_{L^2} \right),$$

and *a fortiori*

$$\gamma\,\|u\|^2_{-\gamma,T} \lesssim \frac{1}{\gamma}\,\|L_v\,u\|^2_{-\gamma,T} \,+\, |B_v u|^2_{-\gamma,T} \,+\, \|u_{|t=0}\|^2_{L^2},$$

$$|N_v u|^2_{-\gamma,T} \lesssim \frac{1}{\gamma}\,\|L_v\,u\|^2_{-\gamma,T} \,+\, |B_v u|^2_{-\gamma,T} \,+\, \|u_{|t=0}\|^2_{L^2}$$

for $\gamma \geq 1$. Now, recall that by construction of N, at each point $w \in \mathbb{W}^0$, the square matrix whose rows are those of $N(w)$ and $B(w)$ is non-singular. Therefore, there exists a constant $\beta > 0$ (depending only on $\|v_{|x_d=0}\|_{L^\infty}$ for $v_{|x_d=0}$ with values in $\mathbb{W}^0$) such that

$$|u|^2_{-\gamma,T} \leq \beta\,(\,|N_v u|^2_{-\gamma,T} + |B_v u|^2_{-\gamma,T}\,) \lesssim \frac{1}{\gamma}\,\|L_v\,u\|^2_{-\gamma,T} + |B_v u|^2_{-\gamma,T} + \|u_{|t=0}\|^2_{L^2}$$

because of the bound for $|N_v u|^2_{-\gamma,T}$. Combining this with the bound for $\|u\|^2_{-\gamma,T}$ here above and going back to more explicit notations, we thus get

$$\gamma\,\|\mathrm{e}^{-\gamma t}\,u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} \,+\, \|\mathrm{e}^{-\gamma t}\,u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,T])}$$
$$\leq\, c\left(\tfrac{1}{\gamma}\,\|\mathrm{e}^{-\gamma t}\,L_v u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} \,+\, \|\mathrm{e}^{-\gamma t}\,B_v u\|^2_{L^2(\mathbb{R}^{d-1}\times[0,T])} \,+\, \|u_{|t=0}\|^2_{L^2}\right) \tag{9.2.56}$$

for all $\gamma \geq \gamma_0$, with γ_0 and c depending only on $\|v\|_{W^{1,\infty}}$. Finally, revisiting the computation of Step 1) using the Friedrichs symmetrizer we easily find that for u smooth enough and γ large enough,

$$\mathrm{e}^{-2\gamma t}\,\|u(t)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \,\lesssim\, \|u(0)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \,+\, \frac{1}{\gamma}\,\|L_v u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,t])}$$
$$+\,\gamma\,\|u\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,t])} \,+\, \|u_{|x_d=0}\|^2_{L^2(\mathbb{R}^{d-1}\times[0,t])}\,.$$

Here above the last two terms are controlled by the previous estimate (9.2.56). Therefore, the timewise estimate is a consequence of (9.2.56). Adding them together and taking γ proportional to $1/T$ we eventually obtain (9.2.53).

Step 3): For square-integrable initial data, the existence of a solution to the IBVP (9.2.52) readily follows from the density of H^1 in L^2. Indeed, take a sequence u_0^ε going to u_0 in $L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)$ and consider $u^\varepsilon \in \mathscr{C}([0,T];L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+))$, the solution given by step 2) of

$$L_v\,u^\varepsilon \,=\, f\,,\; Bu^\varepsilon_{|x_d=0} \,=\, g\,,\; u^\varepsilon_{|t=0} \,=\, u_0^\varepsilon\,.$$

The energy estimate (9.2.53) shows that

$$\|u^\varepsilon(t) - u^{\varepsilon'}(t)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \,\leq\, c\,\|u_0^\varepsilon - u_0^{\varepsilon'}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)}\,.$$

Hence (by the Cauchy criterion once more) u^ε is convergent in $\mathscr{C}([0,T];L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+))$, and the limit is a solution of (9.2.52), which satisfies the energy estimate (9.2.53) (by passing to the limit in (9.2.53) applied to u^ε). The uniqueness of this solution is a straightforward consequence of the uniqueness part in Theorem 9.18. □

9.2.5 *Coefficients of limited regularity*

This section aims at clarifying the intuitive idea that the more regular the coefficients and data, the more regular the solution. We begin with the Boundary Value Problem, by completing Theorem 9.17.

Theorem 9.20 *In the framework of Theorem 9.17, assume, moreover, that v belongs to $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ for some integer $m > (d+1)/2 + 1$, and that $v_{|x_d=0}$ belongs to $H^m(\mathbb{R}^{d-1} \times \mathbb{R})$. We also make the technical assumption that v can be approached in $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ by elements of $\mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ that take values in $\mathbb{W}$ and have traces at $x_d = 0$ taking values in $\mathbb{W}^0$.*

If the forcing term f belongs to to $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and the initial data g belongs to $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R})$ for all $\gamma \geq \gamma_0$, then the solution u of the BVP in (9.2.48) is also in $\mathscr{H}^m_\gamma$, as well as its trace on the boundary $\{x_d = 0\}$, and satisfies the energy estimate

$$\gamma \, \|u\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} \leq C_m \left(\frac{1}{\gamma} \, \|f\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|g\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} \right)$$

for all $\gamma \geq \gamma_m$, where $\gamma_m \geq \gamma_0$ and $C_m > 0$ depend continuously on the H^m norms of v and $v_{|x_d=0}$.

(We recall that the norm $\|u\|_{\mathscr{H}^m_\gamma}$ is equivalent, independently of γ, to $\|e^{-\gamma t} u\|_{H^m_\gamma}$; see Remark 9.9 for more details.)

Proof We begin with the special case when v is itself $\mathscr{C}^\infty$: the coefficients of L_v and B_v being infinitely smooth, the regularity of u and of its trace at $x_d = 0$ thus follows from Theorem 9.9, while the energy estimate follows from Theorem 9.7 and Remark 9.11. In other words, the present theorem with the reinforced assumption that v is $\mathscr{C}^\infty$ is an easy consequence of previous results.

For more general v, a natural idea is to regularize v and pass (once more) to the limit. Assume $v_\varepsilon \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ is going to v in $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ as ε goes to zero. Our technical assumption ensures that we can choose v_ε staying in the set of validity of the main assumptions (CH, NC, N, UKL). So we can apply the first step to the smooth v_ε for all $\varepsilon > 0$. Therefore, there exists a unique solution u_ε to the BVP

$$L_{v_\varepsilon} u_\varepsilon = f \text{ on } \{x_d > 0\}, \quad B_{v_\varepsilon} u_\varepsilon = g \text{ on } \{x_d = 0\}, \tag{9.2.57}$$

and u_ε belongs to $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ while $(u_\varepsilon)_{|x_d=0}$ belongs to $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R})$ for all γ greater than a number γ_m independent of ε, with the inequality

$$\gamma\,\|u_\varepsilon\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|(u_\varepsilon)_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}$$

$$\leq C_m\left(\frac{1}{\gamma}\,\|f\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|g\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}\right)$$

for C_m independent of ε. Consequently, there exist $\underline{u}\in\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ and $\underline{u}_0\in\mathscr{H}^m_\gamma(\mathbb{R}^d)$ such that u_ε converges weakly in $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ and strongly (up to extracting a subsequence) in $\mathrm{e}^{\gamma t}L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ to $\underline{u}$, while $(u_\varepsilon)_{|x_d=0}$ converges weakly in $\mathscr{H}^m_\gamma(\mathbb{R}^d)$ and strongly in $\mathrm{e}^{\gamma t}L^2(\mathbb{R}^d)$ to $\underline{u}_0$.

To conclude we now need to prove that $u=\underline{u}$ and $u_{|x_d=0}=\underline{u}_0$. Since $L_{v_\varepsilon}(u_\varepsilon-\underline{u})$ converges weakly to zero in $\mathscr{H}^{m-1}_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ and thus strongly (up to extracting a subsequence) in $\mathrm{e}^{\gamma t}L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$, Theorem 9.8 implies that $(u_\varepsilon)_{|x_d=0}$ converges strongly to $\underline{u}_{|x_d=0}$ in $\mathrm{e}^{\gamma t}L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$, hence $\underline{u}_0=\underline{u}_{|x_d=0}$. By passing to the limit in the approximate BVP (9.2.57) we thus see that $\underline{u}$ solves the same problem as u: by uniqueness this proves that $u=\underline{u}$ and $u_{|x_d=0}=\underline{u}_0$.

Finally, we obtain the $\mathscr{H}_\gamma$ estimate for u by taking the *lim inf* of the estimate we have for u_ε. □

We have a similar result for the Initial Boundary Value Problem with *zero initial data.*

Theorem 9.21 *In the framework of Theorem 9.18, we assume, moreover, that v belongs to $H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])$ for some integer $m>(d+1)/2+1$, and more precisely that*

$$v=\breve{v}_{|t\in[0,T]}\quad\textit{with}\ \breve{v}\in H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}),\ \breve{v}_{|t<\tau}\equiv 0$$

for some $\tau<T$, and additionally we suppose $\breve{v}$ is the limit in H^m of $\breve{v}_\varepsilon\in\mathscr{D}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ taking values in $\mathbb{W}$, such that $(\breve{v}_\varepsilon)_{|x_d=0}$ takes values in $\mathbb{W}^0$.

If $f\in H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])$ and $g\in H^m(\mathbb{R}^{d-1}\times[0,T])$ are such that $\partial_t^j f=0$, $\partial_t^j g=0$ at $t=0$ for all $j\in\{0,\dots,m-1\}$, then the solution u of the IBVP

$$Lu=f\ \textit{ for }\ x_d>0,\ t\in(0,T)\qquad\textit{and}\qquad Bu_{|x_d=0}=g\,,\ u_{|t=0}=0\,,$$

is also in H^m, as well as its trace on the boundary $\{x_d=0\}$, and satisfies $\partial_t^j u=0$ at $t=0$ for all $j\in\{0,\dots,m-1\}$, together with the energy estimate

$$\frac{1}{T}\,\|u\|^2_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|u_{|x_d=0}\|^2_{H^m(\mathbb{R}^{d-1}\times[0,T])}$$

$$\leq C_m\left(T\,\|f\|^2_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|g\|^2_{H^m(\mathbb{R}^{d-1}\times[0,T])}\right),$$

where $C_m>0$ depends only on the H^m norm of $\breve{v}$.

Proof The assumptions on f and g allow us to extend them, respectively, into $\breve{f} \in \mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and $\breve{g} \in \mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R})$, both vanishing for $t < 0$. By Theorem 9.20 (applied to the extended operator $L_{\breve{v}}$), the corresponding BVP admits a unique solution $\breve{u} \in \mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$, whose trace at $x_d = 0$ is in $\mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R})$. Furthermore, by Theorem 9.13 $\breve{u}$ vanishes for $t < 0$, (and $\breve{u}_{|t \leq T}$ does not depend on $\breve{f}_{|t>T}$ and $\breve{g}_{|t>T}$). Therefore, $u := \breve{u}_{|t\in[0,T]}$ is a solution of the homogeneous IVBP with forcing term f and boundary data g, and it belongs to $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$, while its trace at $x_d = 0$ belongs to $H^m(\mathbb{R}^{d-1} \times [0,T])$. The uniqueness of this solution follows from the uniqueness part in Theorem 9.18 (on the L^2 well-posedness). It remains to show the 'localized' H^m estimate. In fact, it will be a straightforward consequence of the estimate

$$\gamma \, \|\breve{u}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times(-\infty,T])} + \|\breve{u}_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times(-\infty,T])} \tag{9.2.58}$$

$$\lesssim \tfrac{1}{\gamma} \, \|L_{\breve{v}}\breve{u}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times(-\infty,T])} + \|B_{\breve{v}}\breve{u}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times(-\infty,T])} \, ,$$

and the fact that $\breve{u}$ vanishes for $t < 0$ and coincides with u on $[0,T]$. As regards the proof of the estimate (9.2.58), it can be deduced from the localized L^2 estimate (9.2.51) by the same method as in the proof of Theorem 9.7, with $(-\infty, T]$ as the time interval instead of $\mathbb{R}$. Indeed, the problem noted in Remark 9.10 about the bounded interval $[0,T]$ does not arise for the half-line $(-\infty, T]$. In other words, we do have the inequality analogous to (9.1.32), namely

$$\|\mathrm{e}^{-\gamma t} w\|_{L^2(\mathbb{R}^{d-1}\times(-\infty,T])} \leq \frac{1}{\gamma} \, \|\mathrm{e}^{-\gamma t}\partial_t w\|_{L^2(\mathbb{R}^{d-1}\times(-\infty,T])} \tag{9.2.59}$$

for all $w \in \mathscr{H}^1_\gamma(\mathbb{R}^{d-1} \times (-\infty, T])$. Indeed, the inequality (9.2.59) can be viewed as a L^1–L^2 convolution estimate, since we have

$$\mathrm{e}^{-\gamma t} w(t) \, H(T-t) = \int_{-\infty}^{+\infty} H(t-s) \, \mathrm{e}^{-\gamma(t-s)} \mathrm{e}^{-\gamma s} \partial_t w(s) \, H(T-s) \, \mathrm{d}s \, ,$$

where H denotes the Heaviside function, and the L^1 norm of $t \mapsto H(t) \, \mathrm{e}^{-\gamma t}$ is precisely $1/\gamma$.

Note: The multiplicative constant hidden in the sign $\lesssim$ in (9.2.58) depends a priori on $\|\breve{v}\|_{H^m}$. □

Finally, there is also a result for the general IBVP with coefficients of limited regularity, provided the operator is Friedrichs symmetrizable. The compatibility conditions needed are still $(\mathbf{CC}_p)$ (Section 9.2.3), with $A^j = A^j_v$ and $B = B_v$. We say initial data u_0, boundary data g and forcing term f are compatible up to order k if $(\mathbf{CC}_p)$ holds true for all $p \in \{0, \dots, k\}$.

Theorem 9.22 *In the framework of Theorem 9.19, assume, moreover, that v belongs to*

$$\mathscr{C}H^m_T := \{\, v \in \mathscr{D}'(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T]);\\ \partial_t^p v \in \mathscr{C}([0,T]; H^{m-p}(\mathbb{R}^{d-1} \times \mathbb{R}^+))p \leq m \,\}$$

for some integer $m > (d+1)/2 + 1$.

If $f \in H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])$, $g \in H^m(\mathbb{R}^{d-1} \times [0,T])$, and $u_0 \in H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+)$ are compatible up to order $m-1$, then the solution of the IBVP (9.2.52) *belongs to $\mathscr{C}H^m_T$.*

This result is implicitly contained in the lecture notes of Métivier [136]. The conclusion is as in Rauch and Massey's theorem, with slightly less regular data (H^m instead of $H^{m+1/2}$), and with much less regular coefficients. The drawback is the additional, Friedrichs-symmetrizability assumption (which is satisfied by 'most' physical systems anyway).

Proof The sketch of proof is roughly the same as for Theorem 9.19 (which gives solutions in $\mathscr{C}([0,T]; L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+))$ for L^2 data): 1) revisit the IBVP with zero initial data (Theorem 9.21) and show additional time-regularity of solutions; 2) consider initial data (smoother than in the statement) $u_0 \in H^{m+1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+)$, and reduce the problem to zero initial data by substracting an 'approximate solution' (whose counterpart in the proof of Theorem 9.19 is just the solution of a Cauchy problem in the whole space); 3) conclude by a density argument for initial data $u_0 \in H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+)$, thanks to appropriate (new) energy estimates.

Step 1) The key ingredient will be the refined estimate

$$\sum_{|\alpha| \leq m} \gamma^{2(m-|\alpha|)}\, \mathrm{e}^{-2\gamma\tau} \|\partial^\alpha w(\tau)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \tag{9.2.60}$$

$$+\gamma\, \|w\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times(-\infty,\tau])} + \|w_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times(-\infty,\tau])}$$

$$\lesssim \frac{1}{\gamma}\, \|L_v w\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times(-\infty,\tau])} + \|B_v w\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times(-\infty,\tau])}\,,$$

to be shown to be uniform for all γ large enough, for all $\tau > 0$ and all $w \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ vanishing for $t < 0$. In the inequality (9.2.60) here above, the simplifying notation $\lesssim$ means less than or equal to a constant depending only on the $W^{1,\infty}$ norm and the $\mathscr{H}^m_\gamma$ norm of $\breve{v}$, an extension of v as in Theorem 9.21. For simplicity we shall omit the $\breve{}$ sign. In addition, we introduce shortcuts for the sets of integration

$$D_\tau := \bar{\Omega} \times (-\infty,\tau] = \mathbb{R}^{d-1} \times \mathbb{R}^+ \times (-\infty,\tau]$$
$$\Gamma_\tau := \partial\Omega \times (-\infty,\tau] = \mathbb{R}^{d-1} \times (-\infty,\tau]\,.$$

The proof of (9.2.60) proceeds from the refined L^2 estimate in (9.2.54) applied to derivatives of w, as in the proof of (9.2.58) from the 'regular' L^2 estimate (9.2.51), which is itself similar to the proof of Theorem 9.7. There are two main steps: 1) estimate tangential derivatives and 2) estimate 'normal' derivatives. There is basically no novelty in the estimate of tangential derivatives: we apply (9.2.54) to $u_\alpha = \partial^\alpha u$, where ∂^α is a differential operator in the (y,t) directions only, and the d-uple is of length $|\alpha| \leq m$, and show that the right-hand side is bounded by

$$\widetilde{c}\,\Big(\tfrac{1}{\gamma}\,\|f\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} \;+\; \|g\|^2_{\mathscr{H}^m_\gamma(\Gamma_\tau)}$$

$$+\, \tfrac{1}{\gamma}\,\|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} \;+\; \tfrac{1}{\gamma^2}\,\|u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\Gamma_\tau)}\Big)\,,$$

for some positive $\widetilde{c}$ depending only on $\|\check{v}\|_{L^\infty}$ and $\|\check{v}\|_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))}$, hence

$$\sum_{|\alpha|\leq m} \mathrm{e}^{-2\gamma t}\,\gamma^{2(m-|\alpha|)}\,\|\partial^\alpha u(t)\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \;+\; \gamma\,\|u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))}\,,$$

$$+\,\|u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\Gamma_\tau)}$$

$$\leq\; 2\widetilde{c}\,\Big(\tfrac{1}{\gamma}\,\|L_v u\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} \;+\; \|B_v u_{|x_d=0}\|^2_{\mathscr{H}^m_\gamma(\Gamma_\tau)}\Big),$$

for γ large enough (by absorption of the error terms in the left-hand side). More care is needed in the estimate of $\|\partial^\beta u(t)\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)}$ when the operator ∂^β contains a derivative in the normal variable x_d (in this case β is a $(d+1)$-uple). Let us look at the case of a single derivative ∂_d. We take again a p-uple α, of length $|\alpha| \leq m-1$ this time. As in the proof of Theorem 9.7 we can write

$$\partial_d\partial^\alpha u \;=\; (A^d_v)^{-1}\,\Big(f_\alpha \;-\; \partial_t\partial^\alpha u \,+\, \sum_{j=1}^{d-1} A^j_v\,\partial_j\partial^\alpha u\Big)\,,$$

with $f_\alpha := L_v\,\partial^\alpha u$. The novelty here is that we need a *timewise* estimate of f_α. The trick is to use the L^2 estimates of both f_α and $\partial_t f_\alpha$. Indeed, by the same computation as in the proof of Theorem 9.7 we have

$$\gamma^{m-|\alpha|}\,\|\mathrm{e}^{-\gamma t}\,f_\alpha\|_{L^2(D_\tau)} \;\lesssim\; \|f\|_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} \;+\; \|u\|_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))}\,.$$

Furthermore,

$$\partial_t f_\alpha \;=\; f_\beta \,+\, \sum_{j=1}^{d} \partial_t(A^d_v)\,\partial_j u_\alpha\,,$$

with $\beta = (\alpha_0 + 1, \alpha')$ if $\alpha = (\alpha_0, \alpha')$, and f_β can be estimated as above, so we have

$$\gamma^{m-|\alpha|-1} \|e^{-\gamma t}\, \partial_t f_\alpha\|_{L^2(D_\tau)} \lesssim \|f\|_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} + \sum_{j=1}^{d} \|\partial_t(A_v^d)\|_{L^\infty} \|e^{-\gamma t}\, \partial_j u_\alpha\|\,,$$

where each term in the sum can be estimated thanks to part 1) on tangential derivatives. Therefore

$$\gamma^{m-|\alpha|-1} \|e^{-\gamma t}\, \partial_t f_\alpha\|_{L^2(D_\tau)} \lesssim \|f\|_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} + \|g\|_{\mathscr{H}^m_\gamma(\Gamma_\tau)}\,.$$

Consequently, the L^2 bounds for f_α and $\partial_t f_\alpha$ imply the timewise bound

$$\gamma^{m-|\alpha|-1/2}\, e^{-\gamma\tau}\, \|\, f_\alpha(\tau)\|_{L^2(\bar\Omega)} \lesssim \|f\|_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} + \|g\|_{\mathscr{H}^m_\gamma(\Gamma_\tau)}\,,$$

and finally, estimating the other terms in $\partial_d\partial^\alpha u$ thanks to part 1), we get

$$\gamma^{m-|\alpha|-1/2}\, e^{-\gamma\tau}\, \|\, \partial_d\partial^\alpha u(\tau)\|_{L^2(\bar\Omega)} \lesssim \|f\|_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} + \|g\|_{\mathscr{H}^m_\gamma(\Gamma_\tau)}\,,$$

hence

$$\gamma^{2(m-|\alpha|-1)}\, e^{-2\gamma\tau}\, \|\, \partial_d\partial^\alpha u(\tau)\|^2_{L^2(\bar\Omega)} \lesssim \frac{1}{\gamma} \|f\|^2_{L^2(\mathbb{R}^+;\mathscr{H}^m_\gamma(\Gamma_\tau))} + \|g\|^2_{\mathscr{H}^m_\gamma(\Gamma_\tau)}\,,$$

for γ large enough (≥ 1, which has been used for the g-term). The case of higher-order derivatives in x_d can be dealt with in a similar way, by induction (the details are left to the reader). This in turns provides estimates of $\|\partial^\beta u(t)\|_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+)}$ for all $(d+1)$-uple β of length $|\beta| \leq m$. Altogether these estimates prove (9.2.60).

Once we have (9.2.60), the usual *weak=strong* argument shows there is a sequence (u_ε) such that for all $p \in \{0,\dots,m\}$, $\partial_t^p u_\varepsilon$ goes to $\partial_t^p u$ in $\mathscr{C}([0,T]; H^{m-p}(\mathbb{R}^{d-1}\times\mathbb{R}^+))$, where u is the solution of IBVP with zero initial data given by Theorem 9.21.

Step 2) The crucial tool here will be the following lifting result.

Lemma 9.6 *Assume* $\mathbb{W}$ *and* $\mathbb{W}_0$ *are both convex and contain zero, and consider* $v \in \mathscr{D}'(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ *is such that* $v_{|t\in[0,T]} \in \mathscr{C}H^m_T$, $f \in H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])$, *and* $g \in H^m(\mathbb{R}^{d-1}\times[0,T])$ *with* $m > (d+1)/2+1$.

Then for any $u_0 \in H^{m+1/2}(\mathbb{R}^{d-1}\times\mathbb{R}^+)$ *taking values in* $\mathbb{W}$ *with a trace on* $x_d = 0$ *taking values in* $\mathbb{W}_0$*, compatible with* f *and* g *up to order* $m-1$*, there exists* $u_a \in H^{m+1}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$*, also taking values in* $\mathbb{W}$ *with a trace on* $x_d = 0$ *taking values in* $\mathbb{W}_0$*, vanishing for* $|t| \geq \varepsilon > 0$*, and such that*

$$(u_a)_{|t=0} = u_0\,, \quad \partial_t^p\big(f - L_v u_a\big)_{|t=0} \equiv 0 \quad , \quad \partial_t^p\Big(g - (B_v u_a)_{|x_d=0}\Big)_{|t=0} \equiv 0$$

for all $p \in \{0,\dots,m-1\}$*. Furthermore,* $f_a := f - L_v u_a$ *belongs to* $H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])$ *and* $g_a := g - (B_v u_a)_{|x_d=0}$ *belongs to* $H^m(\mathbb{R}^{d-1}\times[0,T])$.

Proof We first need to check that the induction formula in the compatibility conditions $(\mathbf{CC}_{m-1})$,

$$u_q(x) = \sum_{\ell=0}^{q-1} \binom{q-1}{\ell} P_\ell^0 \, u_{q-1-\ell}(x) + \partial_t^{q-1} f(x,0)$$

enables us to construct a sequence $(u_1, \dots, u_{m-1})$ such that for all $q \in \{0, \dots, m-1\}$, u_q belongs to $H^{m+1/2-q}(\mathbb{R}^{d-1} \times \mathbb{R}^+)$. Here the operators

$$P_\ell^0 := - \sum_{j=1}^{d} (\partial_t^\ell A^j \circ v)(x,0) \, \partial_j$$

have, by assumption on v, coefficients in $H^{m-\ell}(\mathbb{R}^{d-1} \times \mathbb{R}^+)$ for $\ell \geq 1$, and this is also true for P_0^0 up to a constant-coefficients operator. Therefore, thanks to Theorem C.10, the induction formula above does define u_q in $H^{m+1/2-q}(\mathbb{R}^{d-1} \times \mathbb{R}^+)$ if $u_0 \in H^{m+1/2-q}(\mathbb{R}^{d-1} \times \mathbb{R}^+)$.

Then by trace lifting (see, for instance, [1], pp. 216–217), we find $u_a \in H^{m+1}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ such that $\|u_a\|_{H^{m+1}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})} \lesssim \|u_0\|_{H^{m+1/2}(\mathbb{R}^{d-1} \times \mathbb{R})}$ and

$$\partial_t^q (u_a)_{|t=0} = u_q \quad \text{for all } \; q \in \{0, \dots, m-1\} .$$

Thanks to the convexity of $\mathbb{W}$ and W^0, up to multiplying u_a by a $\mathscr{C}^\infty$ cut-off function near $t=0$, we may assume, by continuity of u_a and $(u_a)_{|x_d=0}$ at $t=0$, that the range of u_a lies in $\mathbb{W}$ and the range of $(u_a)_{|x_d=0}$ lies in $\mathbb{W}^0$, as requested. The fact that $\partial_t^p (f_a)_{|t=0} \equiv 0$ for all $p \leq m-1$ directly follows from the construction of the u_q, and $\partial_t^p (g_a)_{|t=0} \equiv 0$ is a consequence of the equations

$$\partial_t^\ell g(x,0) = \sum_{q=0}^{\ell} \binom{\ell}{q} (\partial_t^{\ell-q} B_v)(x,0) \, u_q(x)$$

for $q \leq m-1$ contained in the compatibility conditions $(\mathbf{CC}_{m-1})$. Finally, the fact that f_a and g_a are H^m follow in a classical way from Proposition C.11 and Theorem C.12. □

Once we have the 'approximate solution' u_a, the 'approximate' forcing term f_a and the 'approximate' boundary date g_a, we can apply Step 1) to the IBVP with zero initial data

$$L_v u^h = f_a \,, \quad B_v(u^h)_{|x_d=0} = g_a \,, \quad (u^h)_{|t=0} = 0 \,.$$

By uniqueness of the solution u of the original IBVP, we have $u = u_a + u^h$, which therefore belongs to $\mathscr{C}H_T^m$ since both u^h and u_a do: for the latter, observe that $u_a \in H^{m+1}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ implies indeed

$$u_a \in H^{k+1}([0,T]; H^{m-k}(\mathbb{R}^{d-1} \times \mathbb{R}^+)) \hookrightarrow \mathscr{C}^k([0,T]; H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+))$$

for all $k \leq m$.

Step 3) It relies on two ingredients: the observation that u_0 can be approximated by smoother initial data u_0^k, still compatible with f and g up to order $m-1$; energy estimates of a new kind for the IBVP with zero forcing term f and zero boundary data g. We omit the details; see [136] for their proof in the more complicated framework of shock-waves stability. $\square$

PART III

NON-LINEAR PROBLEMS

This page intentionally left blank

10

THE CAUCHY PROBLEM FOR QUASILINEAR SYSTEMS

We turn to ‘realistic’ systems, which are non-linear and most often read

$$\frac{\partial f^0(u)}{\partial t} + \sum_{j=1}^{d} \frac{\partial f^j(u)}{\partial x_j} = c(u)\,, \tag{10.0.1}$$

where f^j and c are vector fields in $\mathbb{R}^n$. The time flux f^0 will be assumed throughout this chapter to be a diffeomorphism. Observing that each row in the left-hand side of (10.0.1) is the divergence with respect to (t, x) of a vector field $(f_i^0(u), f_i^1(u), \cdots, f_i^d(u))$ in $\mathbb{R}^{d+1}$, we say that (10.0.1) is in divergence form. In general, (10.0.1) is called a system of *balance* laws. When the source term c is not present (that is, is replaced by 0), (10.0.1) is called a system of *conservation* laws. As in the previous chapters, we shorten the notations and write (10.0.1) as

$$\partial_t f^0(u) + \sum_{j=1}^{d} \partial_j f^j(u) = c(u).$$

Among the well-known examples of systems of conservation laws, one inevitably encounters the Euler equations, governing the motion of a compressible inviscid and non-heat-conducting fluid (see [80, 195], and also Chapter 13). These read

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho\, \mathbf{u}) = 0\,, \\ \partial_t\, (\rho \mathbf{u}) + \nabla \cdot (\rho\, \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0\,, \\ \partial_t\, \left(\rho\, \left(\frac{1}{2}|\mathbf{u}|^2 + e\right)\right) + \nabla \cdot \left(\left(\rho\, \left(\frac{1}{2}|\mathbf{u}|^2 + e\right) + p\right) \mathbf{u}\right) = 0\,, \end{cases} \tag{10.0.2}$$

where $\rho > 0$ denotes the density of the fluid, $\mathbf{u}$ its velocity, $e > 0$ its internal energy per unit mass and $(\rho, e) \mapsto p(\rho, e) > 0$ is a given pressure law. The operator ∇ is acting in space only. The importance of Euler equations in both the physical applications and the development of the mathematical theory of conservation laws has prompted us to devote a whole part of the book (part IV) to these equations.

10.1 Smooth solutions

As a first approach to (10.0.1), we may look for smooth solutions. Smooth solutions of (10.0.1) are equivalently solutions of the *quasilinear* system

$$\partial_t u + \sum_{j=1}^{d} A^j(u)\, \partial_j u \;=\; b(u)\,, \tag{10.1.3}$$

where the $n \times n$ matrices A^j and the source term b smoothly depend on $u \in \mathbb{R}^n$. They are related to the fluxes f^j and the original source term c by

$$A^j(u) \;=\; \mathrm{d}f^0(u)^{-1}\,\mathrm{d}f^j(u)\,, \quad b(u) \;=\; \mathrm{d}f^0(u)^{-1}\,c(u).$$

Remark 10.1 It can be that a change of variables $u \mapsto \widetilde{u} = \varphi(u)$ leads to a more convenient quasilinear system than (10.1.3), that is, a system

$$\partial_t \widetilde{u} + \sum_{j=1}^{d} \widetilde{A}^j(\widetilde{u})\, \partial_j \widetilde{u} \;=\; \widetilde{b}(\widetilde{u}),$$

where the matrices $\widetilde{A}^j$ are sparser than the A^j. As an example, one may compare the quasilinear form of the Euler equations (10.0.2) in variables $(\rho, \mathbf{u}, e)$ to its counterpart in variables $(p, \mathbf{u}, s)$, where s is the physical entropy. Indeed, recalling that s is defined by the differential relation

$$T\,\mathrm{d}s \;=\; \mathrm{d}e \;-\; \frac{p}{\rho^2}\,\mathrm{d}\rho,$$

(where the integrating factor T is the temperature), we easily see that (10.0.2) is equivalent for smooth solutions to the rather simple-looking system

$$\begin{cases} \partial_t p \,+\, \rho c^2 \nabla\cdot\mathbf{u} \,+\, \mathbf{u}\cdot\nabla p \;=\; 0\,, \\ \partial_t\,\mathbf{u} \;+\; (\mathbf{u}\cdot\nabla)\mathbf{u} \,+\, \rho^{-1}\,\nabla p \;=\; 0\,, \\ \partial_t s \,+\, \mathbf{u}\cdot\nabla s \;=\; 0\,, \end{cases} \tag{10.1.4}$$

where $c^2 := \partial p/\partial\rho$ when p is regarded as a function of ρ and s. The function c is called the *sound speed.*

10.1.1 *Local well-posedness*

For general quasilinear systems of the form (10.1.3), the Cauchy problem can hardly be dealt with. However, there is a very important class of systems for which (local) H^s-well-posedness is known to hold true – for s large enough. This class is a natural generalization of Friedrichs-symmetrizable linear systems.

Definition 10.1 *Let $\mathscr{U}$ be an open subset of $\mathbb{R}^n$. The quasilinear system* (10.1.3) *is called* Friedrichs-symmetrizable *in $\mathscr{U}$ if there exists a $\mathscr{C}^\infty$ mapping*

$S : \mathscr{U} \to \mathbf{Sym}_n$ *such that* $S(u)$ *is positive-definite, and the matrices* $S(u)A^j(u)$ *are symmetric for all* $u \in \mathscr{U}$*.*

Example The system (10.1.4) admits the diagonal symmetrizer

$$\text{diag}\ (1/(\rho c^2), \rho, \cdots, \rho, 1)$$

in the domain $\{ \rho > 0\,,\, c^2 > 0\}$, that is, away from vacuum and where the sound speed is well-defined (which corresponds to hyperbolicity).

In more generality, systems admitting a strictly convex *entropy* are known to be Friedrichs-symmetrizable – see [46, 184] for an introduction to the notion of mathematical entropy and an extensive analysis of systems endowed with an entropy. For this reason, since mathematical entropies are often related to physical entropy (or energy), most 'physical' systems are Friedrichs-symmetrizable. In the case of Euler equations, various symmetrizations are discussed in Section 13.2.3. Also, see [71, 167], concerning the symmetrization of more complicated systems, or systems endowed with non-convex entropies [21, 46, 188].

The main result regarding the Cauchy problem for Friedrichs-symmetrizable systems is the following. It was proved by several authors independently [67, 94].

Theorem 10.1 *Let* $\mathscr{U}$ *be an open subset of* $\mathbb{R}^n$ *containing* 0*. We assume that* A^j *and* b *are* $\mathscr{C}^\infty$ *functions of* $u \in \mathscr{U}$*, all vanishing at* 0*, and that* (10.1.3) *is Friedrichs-symmetrizable in* $\mathscr{U}$*. We consider the Cauchy problem associated with* (10.1.3) *and initial data* $g \in H^s(\mathbb{R}^d)$ *with* $s > \frac{d}{2} + 1$ *taking values in* $\mathscr{U}$*. There exists* $T > 0$ *such that* (10.1.3) *has a unique classical solution* $u \in \mathscr{C}^1(\mathbb{R}^d \times [0, T])$ *achieving the initial data* $u(0) = g$*. Furthermore,* u *belongs to* $\mathscr{C}([0, T]; H^s) \cap \mathscr{C}^1([0, T]; H^{s-1})$*.*

Observe that this result does not apply straight to the Euler equations, which in general are not symmetrizable up to $\rho = 0$. But a slightly modified version of Theorem 10.1, allowing solutions that do not vanish at infinity, does apply: it suffices to replace 0 by some u_0, take $g \in u_0 + H^s$ and obtain $u(t) \in u_0 + H^s$. See Section 13.3 for more details on the Euler equations.

Proof The proof is based on the *iteration scheme*

$$S(u^k)\, \partial_t u^{k+1} + \sum_{j=1}^{d} S(u^k)\, A^j(u^k)\, \partial_j u^{k+1} \;=\; S(u^k)\, b(u^k). \tag{10.1.5}$$

Since we are looking for smooth solutions of (10.1.3), we are only concerned with smooth solutions of (10.1.5). By Theorem 2.12 in Chapter 2 we know that (10.1.5) admits a $\mathscr{C}^\infty$ solution u^{k+1} provided that u^k *and* the initial data are $\mathscr{C}^\infty$. This is why we are going to smooth back the initial data.

We introduce a mollifier ρ_k, being defined as usual by

$$\rho_k(x) = \varepsilon_k^{-d}\, \rho\left(\frac{x}{\varepsilon_k}\right),$$

with $\varepsilon_k > 0$ tending to 0 (in a way to be specified later) and $\rho \in \mathscr{D}(\mathbb{R}^d; \mathbb{R}^+)$ being supported in the unit ball, $\int \rho = 1$. We shall use the approximate initial data $g^k := g * \rho_k$, which is a $\mathscr{C}^\infty$ function that tends to g in H^s when ε_k goes to 0. Furthermore, we have a uniform estimate

$$\|g^k - g\|_{L^2} \leq C\,\varepsilon_k\,\|g\|_{H^1} \tag{10.1.6}$$

for ε_k small enough, say $\varepsilon_k \leq \bar{\varepsilon}$. From now on, this upper bound will be assumed to hold true.

We initialize the iteration scheme by taking u^0 independent of t as $u^0(t) := g^0$. Then we consider the induction process in which u^{k+1} is defined by (10.1.5) and $u^{k+1}(0) = g^{k+1}$. An additional difficulty is due to the restricted domain of definition of S, A^j and b. The process must be able to control the L^∞ norm of the iterates u^k so that they do not approach $\partial\mathscr{U}$ too closely. This will be done by using the following observation. Since g^0 tends to g in H^s when ε_0 goes to 0, by the classical Sobolev embedding

$$H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \tag{10.1.7}$$

for $s > d/2$, we can find ε_0, $\delta > 0$ and a relatively compact open subset $\mathscr{V}$ of $\mathscr{U}$ such that any smooth u satisfying the estimate

$$\|\, u - g^0 \,\|_{H^s} \leq \delta \tag{10.1.8}$$

only achieves values belonging to $\mathscr{V}$. Furthermore, another useful fact will be that for u satisfying (10.1.8), $\nabla_x u$ is bounded (this follows from the Sobolev embedding (10.1.7) applied to $s-1$, which is still greater than $d/2$ by assumption).

The relatively compact subset $\mathscr{V}$ will serve as a reference for energy estimates. We take $\beta > 0$ so that

$$\beta\, I_n \leq S(u) \leq \beta^{-1}\, I_n \tag{10.1.9}$$

for all $u \in \overline{\mathscr{V}}$.

Finally, other essential ingredients in the proof are the following Moser estimates [144].

Proposition 10.1

i) If u and v both belong to $L^\infty \cap H^s$ with $s > 0$ then their product also belongs to H^s and there exists $C > 0$ depending only on s such that

$$\|\, u\, v \,\|_{H^s} \leq C\, (\, \|u\|_{L^\infty}\, \|v\|_{H^s} + \|v\|_{L^\infty}\, \|u\|_{H^s}\,)\,. \tag{10.1.10}$$

ii) If u belongs to $H^s \cap L^\infty$ with $s > 0$ and b is a $\mathscr{C}^\infty$ function vanishing at 0 then the composed function $b(u)$ also belongs to H^s, and there exists a continuous function $C : [0, +\infty) \to [0, +\infty)$ (depending on s and b as parameters) such that

$$\|\, b(u) \,\|_{H^s} \leq C\,(\|u\|_{L^\infty})\; \|u\|_{H^s}\,. \tag{10.1.11}$$

iii) If $s > 1$ and α is a d-uple of length $|\alpha| \leq s$, there exists $C > 0$ such that for all u and a in H^s with ∇u and ∇a in L^∞

$$\| \, [\partial^\alpha \, , \, a \nabla] \, u \, \|_{L^2} \, \leq \, C \, (\, \|\nabla a\|_{L^\infty} \, \|u\|_{H^s} \, + \, \|\nabla u\|_{L^\infty} \, \|a\|_{H^s} \,) \, . \quad (10.1.12)$$

The estimates in (10.1.10) and (10.1.11) are classical (see [207], pp. 10–11, or [7], pp. 98–103). For completeness, their proof is given in Appendix C (see Proposition C.11 for *i)* and Theorem C.12 for *ii)*). The last one, (10.1.12), is actually a consequence of *i)* (see Proposition C.13 in Appendix C).

Note that the L^∞ assumption in *i)* (respectively, in *iii)*) automatically follows from the Sobolev embedding (10.1.7) if $s > d/2$ (respectively, if $s > d/2 + 1$).

We can now give a detailed proof of Theorem 10.1. There are two main steps, in terms of the so-called 'high norm' and 'low norm': 1) controlling the high norm $\| \, u^{k+1} \, - \, g^0 \, \|_{\mathscr{C}([0,T];H^s)}$ and, 2) showing a contraction property for the sequence $(u^k)_{k \in \mathbb{N}}$ in $\mathscr{C}([0,T];L^2)$. Both steps appear to work for T small enough. In the first one, we shall assume s is an integer in order to stay at a more elementary level. The result is true for any real number s greater than the critical regularity index $d/2 + 1$ though. (Also see Theorem 10.2 below.)

High-norm boundedness For simplicity, we assume that s is an integer. We proceed by induction. Initially, we have $u^0(t) = g^0$ and thus

$$\| \, u^0 \, - \, g^0 \, \|_{\mathscr{C}([0,T];H^s)} \, = \, 0 \, \leq \, \delta$$

for any T and δ. We assume that for all $\ell \leq k$, u^ℓ is defined inductively by (10.1.5) and $u^\ell(0) = g^\ell$ and satisfies the estimate

$$\| \, u^\ell \, - \, g^0 \, \|_{\mathscr{C}([0,T];H^s)} \, \leq \, \delta \, . \quad (10.1.13)$$

We are going to show the same estimate (10.1.13) for $\ell = k + 1$, provided that T is suitably chosen.

We introduce the notations $v^{k+1} := u^{k+1} - g^0$, $w_\alpha^{k+1} := \partial_x^\alpha v^{k+1}$ where α is any d-uple of length $|\alpha| \leq s$. By definition, v^{k+1} must solve the Cauchy problem

$$\begin{cases} \partial_t v^{k+1} \, + \, \displaystyle\sum_{j=1}^{d} A^j(u^k) \, \partial_j v^{k+1} \, = \, b(u^k) \, - \, \displaystyle\sum_{j=1}^{d} A^j(u^k) \, \partial_j \, g^0 \, , \\ v^{k+1}(0) \, = \, g^{k+1} \, - \, g^0 \, , \end{cases} \quad (10.1.14)$$

and, differentiating, w_α^{k+1} must solve

$$\begin{cases} \partial_t w_\alpha^{k+1} \, + \, \displaystyle\sum_{j=1}^{d} A^j(u^k) \, \partial_j w_\alpha^{k+1} \, = \, f_\alpha^{k+1} \, , \\ w_\alpha^{k+1}(0) \, = \, \partial_x^\alpha \, (\, g^{k+1} \, - \, g^0 \,) \, , \end{cases} \quad (10.1.15)$$

where the right-hand side f_α^{k+1} is defined by

$$f_\alpha^{k+1} := \partial_x^\alpha b(u^k) - \sum_{j=1}^d \partial_x^\alpha \left(A^j(u^k)\, \partial_j\, g^0 \right) - \sum_{j=1}^d [\, \partial_x^\alpha \, , \, A^j(u^k)\, \partial_j \,]\, v^{k+1}.$$

From (10.1.13) and the remark above, we know there is a constant C, depending only on $\overline{\varepsilon}$ and δ so that

$$\|u^\ell\|_{\mathscr{C}([0,T];H^s)} \leq C\,, \quad \|u^\ell\|_{L^\infty} \leq C\,, \quad \|\nabla_x u^\ell\|_{L^\infty} \leq C \tag{10.1.16}$$

for all $\ell \leq k$. By Proposition 10.1 (*i)* and *ii)*), this shows that the first terms in f_α^{k+1}, namely $\partial_x^\alpha b(u^k)$ and $\sum_{j=1}^d \partial_x^\alpha \left(A^j(u^k)\, \partial_j\, g^0 \right)$ are bounded in $\mathscr{C}([0,T];L^2)$ (by a constant depending only on $\overline{\varepsilon}$ and δ). Furthermore, (10.1.16), Proposition 10.1 (*iii)*) and the Sobolev inequality

$$\|\nabla_x v^{k+1}(t)\|_{L^\infty} \leq C'\, \|v^{k+1}(t)\|_{H^s}$$

show that

$$\left\| \sum_{j=1}^d [\, \partial_x^\alpha \, , \, A^j(u^k)\, \partial_j \,]\, v^{k+1} \right\|_{\mathscr{C}([0,T];L^2)} \leq C''\, \|v^{k+1}\|_{\mathscr{C}([0,T];H^s)},$$

where C'' is another constant depending only on $\overline{\varepsilon}$ and δ. For simplicity, we omit the primes. Thus we have an estimate

$$\|f_\alpha^{k+1}\|_{\mathscr{C}([0,T];L^2)} \leq C \left(1 + \|v^{k+1}\|_{\mathscr{C}([0,T];H^s)} \right). \tag{10.1.17}$$

A similar estimate also holds for $S(u^k)\, f_\alpha^{k+1}$ since u^k is bounded in L^∞. Now, using the estimates in (10.1.16) for $\ell = k$ and $\ell = k-1$ and recalling from (10.1.5) for $k-1$ that

$$\partial_t u^k = - \sum_{j=1}^d A^j(u^{k-1})\, \partial_j u^k + b(u^{k-1}),$$

we see that $\partial_t u^k$ is also bounded in L^∞. Together with the L^∞ bound for $\nabla_x u^k$, this implies

$$\| \, \partial_t S(u^k) + \sum_j \partial_j (\, S(u^k) A^j(u^k)\,) \,\|_{L^\infty} \leq C_0 \tag{10.1.18}$$

for some constant C_0 depending only on $\overline{\varepsilon}$ and δ. Therefore, we can apply the estimate (2.1.4) from Chapter 2 (Proposition 2.2) to the Cauchy problem (10.1.15) and λ such that $\beta(\lambda - 1) \geq C_0$. We find that

$$\beta^2\, \|w_\alpha^{k+1}(t)\|_{L^2}^2 \leq \mathrm{e}^{\lambda\, t}\, \|g^{k+1} - g^0\|_{H^s}^2 + \int_0^t \mathrm{e}^{\lambda\,(t-\tau)}\, \|f_\alpha^{k+1}(\tau)\|_{L^2}^2\, \mathrm{d}\tau.$$

Using (10.1.17) and summing on m all the inequalities for w_α^{k+1}, we get another constant C_s (depending only on s) so that

$$\beta^2 \sup_{t\in[0,T]} \|v^{k+1}(t)\|^2_{H^s} \leq C_s\, \mathrm{e}^{\lambda T} \left(\|g^{k+1} - g^0\|^2_{H^s} + 2T(1 + \sup_{t\in[0,T]} \|v^{k+1}(t)\|^2_{H^s})\right).$$

For $4C_s\,\mathrm{e}^{\lambda T}\, T \leq \beta^2$, we can absorb the H^s norm of v^{k+1} into the left-hand side. This yields the estimate

$$\beta^2 \sup_{t\in[0,T]} \|v^{k+1}(t)\|^2_{H^s} \;\leq\; 2\,C_s\, \mathrm{e}^{\lambda T}\, \left(\|g^{k+1} \,-\, g^0\|^2_{H^s} \,+\, 2\,T \right).$$

Since g^0 tends to g in H^s when ε_0 goes to 0, we can ensure that

$$\sup_{t\in[0,T]} \|v^{k+1}(t)\|_{H^s} \;\leq\; \delta$$

by choosing ε_0 small enough, up to diminishing T again.

Low-norm contraction Subtracting (10.1.5) for $k-1$ to the one for k, we see that $W^{k+1} := u^{k+1} - u^k$, solves the Cauchy problem

$$\begin{cases} S(u^k)\,\partial_t W^{k+1} \,+\, \displaystyle\sum_{j=1}^{d} S(u^k)\, A^j(u^k)\,\partial_j W^{k+1} \;=\; F^{k+1}\,, \\ \\ v^{k+1}(0) \;=\; g^{k+1} \,-\, g^k\,, \end{cases} \tag{10.1.19}$$

where

$$F^{k+1} = S(u^k)\, b(u^k) \,-\, S(u^{k-1})\, b(u^{k-1}) \,-\, (\, S(u^k) \,-\, S(u^{k-1})\,)\partial_t u^k$$
$$-\sum_{j=1}^{d} (\, S(u^k)\, A^j(u^k) \,-\, S(u^{k-1})\, A^j(u^{k-1})\,)\,\partial_j\, u^k\,.$$

In view of (10.1.16) for $\ell = k$ and $\ell = k-1$ and the fact that $\partial_t u^k$ is also bounded in L^∞, the mean-value theorem implies that

$$\|F^{k+1}\|_{\mathscr{C}([0,T];L^2)} \;\leq\; C\,\|W^k\|_{\mathscr{C}([0,T];L^2)}\,. \tag{10.1.20}$$

Applying the estimate (2.1.4) from Chapter 2 to the Cauchy problem (10.1.19), we have for $\beta(\lambda-1) \geq C_0$ (the bound C_0 being as in (10.1.18))

$$\beta^2 \sup_{t\in[0,T]} \|W^{k+1}(t)\|^2_{L^2} \;\leq\; \mathrm{e}^{\lambda T}\, \|g^{k+1} \,-\, g^k\|^2_{L^2} \,+\, C^2\, T\, \mathrm{e}^{\lambda T} \sup_{t\in[0,T]} \|W^k(t)\|^2_{L^2}.$$

Provided that T is small enough, more precisely if

$$2C^2 T \mathrm{e}^{\lambda T} \leq \beta^2,$$

we thus have the uniform estimate

$$\sup_{t\in[0,T]} \|u^{k+1}(t) - u^k(t)\|^2_{L^2} \leq \frac{1}{2} \sup_{t\in[0,T]} \|u^k(t) - u^{k-1}(t)\|^2_{L^2} + \frac{\mathrm{e}^{\lambda T}}{\beta^2} \|g^{k+1} - g^k\|^2_{L^2}.$$

Hence $(u^k)_{k\in\mathbb{N}}$ will be a Cauchy sequence in $\mathscr{C}([0,T];L^2)$ provided that the series

$$\sum_k \|g^{k+1} - g^k\|_{L^2}^2$$

is convergent. The estimate in (10.1.6) shows that this holds true for $\varepsilon^k = 2^{-k}\overline{\varepsilon}$, for instance. With this choice, the sequence $(u^k)_{k\in\mathbb{N}}$ has a limit u in $\mathscr{C}([0,T];L^2)$. It remains to prove additional regularity on u. (The method will generalize the proof of the continuity with values in H^1 of the solution in Theorem 2.9.)

Regularity and uniqueness Since $(u^k(t))_{k\in\mathbb{N}}$ is bounded in H^s (see (10.1.16)) and convergent in L^2 (for $t \leq T$), the limit $u(t)$ must be in H^s by weak compactness of bounded balls in H^s and uniqueness of the limit in the sense of distributions. Furthermore, by L^2–H^s interpolation, u^k is found to converge in $\mathscr{C}([0,T];H^{s'})$ for all $s' \in (0,s)$. In particular, we can choose s' greater than $1 + d/2$ (like s). Standard Sobolev embeddings then imply the convergence holds in $\mathscr{C}([0,T];\mathscr{C}_b^1(\mathbb{R}^d))$. Thus we can pass to the limit in the iteration scheme (10.1.5), which shows that $\partial_t u^k$ tends to $\partial_t u$ in $\mathscr{C}([0,T];\mathscr{C}_b(\mathbb{R}^d))$ and the limit u is a $\mathscr{C}^1$ solution of (10.1.3). The initial condition is trivially satisfied, by passing to the limit in the initial condition for u^k. It is not difficult to show that the $\mathscr{C}^1$ solution constructed by the iteration scheme (10.1.5) is the only one satisfying (10.1.8) in the time interval $[0,T]$. As a matter of fact, the L^2 norm of the difference between two solutions u and v can be estimated similarly as in the low-norm calculation on $u^{k+1} - u^k$. (We can even show the uniqueness of classical solutions in the wider class of entropy solutions, see [46] and Section 10.2 below.)

We already know that u belongs to $\mathscr{C}([0,T];H^{s'})$ for all $s' < s$. In fact, we can show that u belongs to $\mathscr{C}([0,T];H^s)$ (which automatically implies that u belongs to $\mathscr{C}^1([0,T];H^{s-1})$ by the equation in (10.1.3)). The proof is not obvious though.

As a first step, we can check that $u^k(t)$ converges uniformly on $[0,T]$ in H^s_w, the space H^s equipped with the weak topology. We just take any ϕ in H^{-s}, choose ψ in the dense subspace $H^{-s'}$ (for $s' < s$) close enough to ϕ, and make a standard splitting. Using the estimates in (10.1.16), we get

$$\sup_{t\in[0,T]} |\langle \phi, (u^k - u)(t)\rangle_{H^{-s},H^s}| \leq C\|\phi - \psi\|_{H^{-s}} + \sup_{t\in[0,T]} |\langle \psi, (u^k - u)(t)\rangle_{H^{-s'},H^{s'}}|.$$

The first term in the right-hand side can be made arbitrarily small, and the second term is known to tend to 0 since u^k converges to u in $\mathscr{C}([0,T];H^{s'})$. Hence the left-hand side also tends to 0.

Secondly, up to translating or/and reversing time, it is sufficient to prove the right continuity in H^s of the limit u at $t = 0$. Furthermore, by a straightforward $\varepsilon/3$ argument, we have that $u(t) - g$ tends to 0, as t tends to $0+$, in $H^{s'}$ for all $s' < s$, and so by the same splitting as before, $u(t)$ tends to g in H^s_w. In particular, we have

$$\liminf_{t\searrow 0} \|u(t)\|_{H^s} \geq \|g\|_{H^s}.$$

Hence, to prove the strong convergence, it just remains to show that

$$\|g\|_{H^s} \geq \limsup_{t \searrow 0} \|u(t)\|_{H^s}.$$

It appears that this inequality is easier to show using the equivalent norm on H^s, defined by

$$\|v\|_{s,g}^2 := \sum_{|\alpha|\leq s} \langle S(g)\, \partial_x^\alpha v \,,\, \partial_x^\alpha v \rangle.$$

More generally, because of (10.1.9), the norm $\|\cdot\|_{s,u}$ associated with any function u satisfying the estimate in (10.1.8) may serve as an equivalent norm. In particular, we see that $u(t)$ satisfies (10.1.8) by taking the *liminf* as ℓ tends to $+\infty$ in (10.1.13). And in fact, an additional useful observation is that

$$\limsup_{t \searrow 0} \|u(t)\|_{s,u(t)} = \limsup_{t \searrow 0} \|u(t)\|_{s,g}.$$

This follows from the pointwise continuity of u at $t = 0$ and its boundedness in H^s. Therefore, we are led to showing

$$\|g\|_{s,g} \geq \limsup_{t \searrow 0} \|u(t)\|_{s,u(t)}\,. \tag{10.1.21}$$

The third and last step is based on an energy estimate very similar to that previously computed in the high norm. For all d-uple m of length $|m| \leq s$, $u_\alpha^{k+1} := \partial_x^\alpha u^{k+1}$ solves the Cauchy problem

$$\begin{cases} \partial_t u_\alpha^{k+1} + \sum_{j=1}^{d} A^j(u^k)\, \partial_j u_\alpha^{k+1} = h_\alpha^{k+1}\,, \\ \\ u_\alpha^{k+1}(0) = \partial_x^\alpha\, g^{k+1}\,, \end{cases}$$

where the right-hand side h_α^{k+1} is defined by

$$h_\alpha^{k+1} := \partial_x^\alpha b(u^k) - \sum_{j=1}^{d} [\, \partial_x^\alpha \,,\, A^j(u^k)\, \partial_j\,]\, u^{k+1}.$$

Similarly as in (10.1.17), we have

$$\|h_\alpha^{k+1}\|_{\mathscr{C}([0,T];L^2)} \leq C\,(\,1 + \|u^{k+1}\|_{\mathscr{C}([0,T];H^s)}\,).$$

Now we compute that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle\, S(u^k)\, u_\alpha^{k+1} \,,\, u_\alpha^{k+1} \,\rangle &= \langle\, (\, \partial_t S(u^k) + \sum_j \partial_j (S(u^k) A^j(u^k))\,)\, u_\alpha^{k+1} \,,\, u_\alpha^{k+1} \,\rangle \\ &\quad + 2\, \mathrm{Re}\, \langle\, S(u^k)\, u_\alpha^{k+1} \,,\, h_\alpha^{k+1} \,\rangle\,. \end{aligned}$$

By (10.1.18) and the uniform boundedness of u^{k+1} in H^s, we find after integration that

$$\langle\, S(u^k(t))\, u_\alpha^{k+1}(t)\,,\, u_\alpha^{k+1}(t)\,\rangle \,\leq\, \langle\, S(g^k)\, g_\alpha^{k+1}\,,\, g_\alpha^{k+1}\,\rangle \,+\, C'\, t.$$

Summing on m we get

$$\|u^{k+1}(t)\|_{s,u^k(t)} \,\leq\, \|g^{k+1}\|_{s,g^k} \,+\, C'\, t,$$

where the right-hand side tends to $\|g\|_{s,g} \,+\, C'\, t$ as $k \to +\infty$ (because of the construction of g^k through mollification of g). As to the left-hand side, it is such that

$$\|u(t)\|_{s,u(t)} \,\leq\, \limsup_{k\to+\infty} \|u^{k+1}(t)\|_{s,u^k(t)}$$

because of the weak convergence of $u^{k+1}(t)$ in H^s and the uniform pointwise convergence of $u^k(t)$. Therefore, we have

$$\|u(t)\|_{s,u(t)} \,\leq\, \|g\|_{s,g} \,+\, C'\, t,$$

which yields (10.1.21). □

In fact, the local well-posedness result shown in Theorem 10.1 is also valid, under the same restriction of the regularity index, for systems admitting a *symbolic* symmetrizer in the following natural sense (see Definition 2.4 in Chapter 2).

Definition 10.2 *A* symbolic symmetrizer *associated with the quasilinear system (10.1.3) is a $\mathscr{C}^\infty$ mapping*

$$S : \mathbb{R}^n \times (\mathbb{R}^d\backslash\{0\}) \,\to \mathbf{M}_n(\mathbb{C}),$$

homogeneous degree 0 *in ξ such that*

$$S(u,\xi) \,=\, S(u,\xi)^* \,>\, 0 \;\; \textit{and} \;\; S(u,\xi)\, A(u,\xi) \,=\, A(u,\xi)^*\, S(u,\xi)$$

for all $(u,\xi) \in \mathbb{R}^n \times (\mathbb{R}^d\backslash\{0\})$, where $A(u,\xi) \,:=\, \sum_{j=1}^{d} \xi_j\, A^j(u).$

Theorem 10.2 *We assume that A^j and b are $\mathscr{C}^\infty$ functions of $u \in \mathbb{R}^n$, with $b(0) = 0$, and that* (10.1.3) *admits a symbolic symmetrizer. For all initial data $g \in H^s(\mathbb{R}^d)$ with $s > \frac{d}{2} + 1$, there exists $T > 0$ such that* (10.1.3) *has a unique solution $u \in \mathscr{C}([0,T]; H^s)$ such that $u(0) = g$.*

This theorem is shown for higher s by Taylor in [205]. Métivier [132] proves it under the optimal condition $s > \frac{d}{2} + 1$. In fact, he also allows the matrices A^j and the reaction term b to depend on (x,t) inside a compact set, and b to not necessarily vanish at 0. Moreover, the same precautions as in Theorem 10.1 would allow those matrices A^j and the reaction term b to be defined only on an open subset $\mathscr{U}$ of $\mathbb{R}^n$. We omit these refinements for simplicity.

Proof Unsurprisingly, the proof relies on the linear result in Theorem 2.7 and an iterative scheme. After initialization by $u^0 = g$, the scheme merely reads

$$L_{u^k} u^{k+1} = b(u^k)\,, \qquad u^{k+1}(0) = g,$$

where

$$L_u = \partial_t + \sum_j A^j(u(x,t))\,\partial_j.$$

This defines inductively $u^k \in \mathscr{C}([0,T];H^s) \cap \mathscr{C}^1([0,T];H^{s-1})$. Furthermore, at least for small T, we can control inductively the norms of u^k governing the constants K_k, C_k and γ_k in the energy estimate

$$\|u^{k+1}(t)\|_{H^s}^2 \le \mathrm{e}^{\gamma_k t} \left(K_k\, \|u(0)\|_{H^s}^2 + C_k\, t\, \|L_{u^k} u^{k+1}\|_{L^\infty(0,T;H^s)}^2 \right).$$

In other words, we can make K_k, C_k and γ_k independent of k, provided that $t \le T$ is small enough. Recall that K_k a priori depends on $\|u^k\|_{L^\infty([0,T];W^{1,\infty})}$ and C_k, γ_k a priori depend on

$$\sup\left(\|u^k\|_{L^\infty([0,T];H^s)}\,, \|\partial_t u^k\|_{L^\infty([0,T];H^{s-1})}\right).$$

We take $M_0 > \|g\|_{W^{1,\infty}}$, $M_1 > \|g\|_{H^s}$ and $M_2 > 0$ (to be enlarged later) and look for conditions under which

$$\|u^k\|_{L^\infty([0,T];W^{1,\infty})} \le M_0\,, \quad \|u^k\|_{L^\infty([0,T];H^s)} \le M_1 \quad \|\partial_t u^k\|_{L^\infty([0,T];H^{s-1})} \le M_2$$

independently of k.

Assume that this is the case for u^k. By Theorem C.12, we know there exists $c_1 = c_1(M_1)$ so that $\|b(u^k)\|_{H^s} \le c_1(M_1)$. Therefore, we have the estimate

$$\|u^{k+1}(t)\|_{H^s}^2 \le \mathrm{e}^{\gamma(\max(M_1,M_2))T} \left(K(M_0)\, \|g\|_{H^s}^2 + T\, C(\max(M_1,M_2))\, c_1(M_1)^2\right).$$

Hence, the first condition to keep $\|u^{k+1}(t)\|_{H^s}$ less than M_1 is

$$\mathrm{e}^{\gamma(\max(M_1,M_2))T} \left(K(M_0)\, \|g\|_{H^s}^2 + T\, C(\max(M_1,M_2))\, c_1(M_1)^2\right) \le M_1^2.$$

Assume that this is the case. Then, also by Theorem C.12 plus Proposition C.11 and the Sobolev embedding $H^{s-1} \hookrightarrow L^\infty$, there exists $c_2 = c_2(M_1)$ so that

$$\left\| \sum_j A^j(u^k)\, \partial_j u^{k+1} \right\|_{H^{s-1}} \le c_2(M_1).$$

Consequently, $\partial_t u^{k+1} = b(u^k) - \sum_j A^j(u^k)\, \partial_j u^{k+1}$ satisfies the estimate

$$\|\partial_t u^{k+1}\|_{L^\infty([0,T];H^{s-1})} \le c_1(M_1) + c_2(M_1).$$

Finally, we need an estimate of

$$\|u^{k+1}\|_{L^\infty([0,T];W^{1,\infty})} \le \|u^{k+1} - u^{k+1}(0)\|_{L^\infty([0,T];W^{1,\infty})} + \|g\|_{L^\infty([0,T];W^{1,\infty})}.$$

Since

$$\|u^{k+1} - u^{k+1}(0)\|_{L^\infty([0,T];W^{1,\infty})} \leq C_{s'} \|u^{k+1} - u^{k+1}(0)\|_{L^\infty([0,T];H^{s'})}$$

for $s' > d/2 + 1$, we can bound this term by L^2–H^s interpolation. For $d/2 + 1 < s' < s$, we get

$$\begin{aligned} &\|u^{k+1} - u^{k+1}(0)\|_{L^\infty([0,T];W^{1,\infty})} \\ &\leq C_{s',s}\, T^{1-s'/s}\, \|\partial_t u^{k+1}\|^{1-s'/s}_{L^\infty([0,T];L^2)} \|u^{k+1} - u^{k+1}(0)\|^{s'/s}_{L^\infty([0,T];H^s)} \\ &\leq 2^{s'/s}\, C_{s',s}\, T^{1-s'/s}\, M_2^{1-s'/s}\, M_1^{s'/s} \leq \widetilde{C}_{s,s'} \max(M_1, M_2)\, T^{1-s'/s}. \end{aligned}$$

In conclusion, the following successive choices appear to work. We take $M_0 > \|g\|_{W^{1,\infty}}$, $M_1 > \max(1, K(M_0)^{1/2})\, \|g\|_{H^s}$ and $M_2 \geq c_1(M_1) + c_2(M_1)$ and then choose T small enough so that

$$\|g\|_{W^{1,\infty}} + \widetilde{C}_{s,s'} \max(M_1, M_2)\, T^{1-s'/s} \leq M_0,$$

$$\mathrm{e}^{\gamma(\max(M_1,M_2))T} \left(K(M_0)\, \|g\|^2_{H^s} + T\, C(\max(M_1, M_2))\, c_1(M_1)^2 \right) \leq M_1^2.$$

Once we have these strong bounds on the sequence (u^k) we can show its convergence in $\mathscr{C}([0,T], H^{s-1})$. Indeed, we have by construction $(u^{k+1} - u^k)(0) = 0$ and

$$L_{u^k}(u^{k+1} - u^k) = b(u^k) - b(u^{k-1}) - (L_{u^k} - L_{u^{k-1}})\, u^k.$$

The right-hand side above can be estimated using Corollary C.3 (Appendix C). Hence, applying the a priori estimate of Theorem 2.4 with $m = s - 1$, we obtain

$$\|u^{k+1} - u^k\|^2_{\mathscr{C}([0,T];H^{s-1})} \leq C\, T\, \mathrm{e}^{\gamma T}\, \|u^k - u^{k-1}\|^2_{\mathscr{C}([0,T];H^{s-1})},$$

where C and γ are independent of k – they can be evaluated in terms of M_1, M_2. So the sequence (u^k) is convergent in $\mathscr{C}([0,T], H^{s-1})$ if T is so small that $C\, T\, \mathrm{e}^{\gamma T} < 1$.

The rest of the proof is the same as for Theorem 10.1. □

10.1.2 *Continuation of solutions*

Once we have local existence results like Theorems 10.1 and 10.2, we may wonder whether it is possible to continue, and for how long, local-in-time solutions. The continuation principle stated hereafter gives a simple answer to the first part of the question. Suppose T is the maximal time of existence of a solution $u \in \mathscr{C}([0,T); H^s(\mathbb{R}^d))$. There are roughly two alternatives. Either T is infinite or $\|u(t)\|_{W^{1,\infty}(\mathbb{R}^d)}$ is unbounded as t approaches T. In the latter case, a possibility is that $u(x,t)$ escapes any compact set of $\mathbb{R}^n$. In fact, if the matrices A^j and the reaction term b are well-defined only on an open subset $\mathscr{U}$ of $\mathbb{R}^n$, the correct statement is that, if T is finite, either $u(x,t)$ escapes any compact subset of $\mathscr{U}$ or $\|\nabla_x u(t)\|_{L^\infty(\mathbb{R}^d)}$ is unbounded. The first case is analogous to the finite time blow-up in ordinary differential equations. The second one corresponds to the formation of shocks, a phenomenon that is classically explained on the one-dimensional example of Burgers' equation, corresponding to $A^1(u) = u$. For more details on blow-up (in one space dimension), we recommend, for instance, the book by Whitham [219], p. 42–46. Also see the more recent book by Alinhac [6].

We give below a continuation result in the most general framework of systems admitting a symbolic symmetrizer. We shall use again para-differential calculus in the proof. Of course, a more elementary proof can be given for Friedrichs-symmetrizable systems (see, for instance, Majda [126], p. 46–48).

Theorem 10.3 *Let $\mathscr{U}$ be an open subset of $\mathbb{R}^n$ containing the closed ball $\overline{B(0;\omega)}$. We assume that A^j and b are $\mathscr{C}^\infty$ functions of $u \in \mathscr{U}$, with $b(0) = 0$, and that* (10.1.3) *has a symbolic symmetrizer in $\mathscr{U}$. If $u \in \mathscr{C}([0,T);H^s)$ with $0 < T < +\infty$ and $s > \frac{d}{2} + 1$ is a solution of* (10.1.3) *such that*

$$\|u\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))} \leq \omega$$

then u is continuable to a solution in $\mathscr{C}([0,T'];H^s)$ with $T' > T$.

Proof The main ingredient will be a uniform estimate of $\|u(t)\|_{H^s}$ for all $t < T$. In fact, this estimate is almost contained in the proof of Theorem 2.4 in Chapter 2 applied to $v = u$. For clarity, we give below the crucial points. We shall use the same notations as in the proof of that theorem. We first introduce the para-differential operator

$$P_u = \partial_t + \sum_j T_{A^j(u)}\,\partial_j .$$

For all $t < T$, $P_u u(t) - L_u u(t)$ belongs to H^s because of Proposition C.9 and Theorem C.12, and we have a uniform estimate

$$\|\,P_u u - L_u u\,\|_{H^s} \leq K\,\|u\|_{H^s},$$

where $K = K(\omega)$. This is the first point. Then we proceed as in the proof of Theorem 2.4. The new fact here is that $\|\partial_t u\|_{L^\infty(\mathbb{R}^d \times [0,T])}$ is bounded by a constant $C = C(\omega)$. This is simply due to the equation

$$\partial_t u = -\sum_j A^j(u)\partial_j u + b(u).$$

So, the estimate obtained at the end is

$$\|u(t)\|_{H^s}^2 \leq \frac{K_2(\omega)}{\beta(\omega)}\left(\mathrm{e}^{\gamma t}\,\|u(0)\|_{H^s}^2 + \int_0^t \mathrm{e}^{\gamma\,(t-\tau)}\,\|P_u u(\tau)\|_{H^s}^2\,\mathrm{d}\tau\right),$$

valid for all $\gamma \geq 1 + 2\,\beta(\omega)^{-1}\,(K_3(\omega) + C(\omega))$. Now, using that $P_u u = P_u u - L_u u + L_u u$, we get the estimate

$$\|u(t)\|_{H^s}^2 \leq \frac{K_2}{\beta}\left(\mathrm{e}^{\gamma\, t}\,\|u(0)\|_{H^s}^2 + 2\int_0^t \mathrm{e}^{\gamma\,(t-\tau)}\,(\,K + K_0\,)\,\|u(\tau)\|_{H^s}^2\,\mathrm{d}\tau\right),$$

where $K_0 = K_0(\omega)$ comes from Theorem C.12 applied to $F = b$. So in short, there is a constant C_0 depending only on ω and T such that

$$\|u(t)\|_{H^s}^2 \leq C_0\left(\|u(0)\|_{H^s}^2 + \int_0^t \|u(\tau)\|_{H^s}^2\,\mathrm{d}\tau\right).$$

By Gronwall's Lemma, this implies

$$\|u(t)\|_{H^s}^2 \leq C_0\,\mathrm{e}^{C_0\,T}\,\|u(0)\|_{H^s}^2.$$

Thus u belongs to $L^\infty([0,T];H^s)$. Furthermore, u belongs to $\mathscr{C}^1([0,T);H^{s-1})$ – because of the equation $\partial_t u = -\sum_j A^j(u)\,\partial_j u + b(u)$ and $u \in \mathscr{C}([0,T);H^s)$ – and thus to $\mathscr{C}^1([0,T);L^\infty)$ – by Sobolev embedding – and we have a uniform bound

$$\|\partial_t u\|_{L^\infty(\mathbb{R}^d\times[0,T])} \leq C.$$

Therefore, u is continuable to a $\mathscr{C}([0,T];L^\infty)$ function. Hence, by weak compactness of bounded sets in Hilbert spaces and uniqueness of the limit in the sense of distributions, we have

$$u(t) \rightharpoonup u(T) \text{ in } H_w^s \text{ when } t \nearrow T.$$

Consequently, we have $u \in L^\infty([0,T];H^s) \cap \mathscr{C}([0,T];H_w^s)$.

To show that u actually belongs to $\mathscr{C}([0,T];H^s)$, we can now make use of the uniqueness part of Theorem 2.7 in Chapter 2 – applied to $v = u$ and $f = b(u)$. Indeed, f belongs to $L^\infty([0,T];H^s) \cap \mathscr{C}([0,T];L^\infty)$ – like u – and thus to $L^\infty([0,T];H^s) \cap \mathscr{C}([0,T];H_w^s)$. So the theorem does apply, and the weak solution $u \in L^2([0,T];H^s)$ of $\partial_t u + \sum_j A^j(u)\,\partial_j u = f$ is a strong solution and belongs in fact to $\mathscr{C}([0,T];H^s)$.

Once we know this, we can apply the local existence result in Theorem 10.2 to $u(T)$ as initial data, and this completes the proof. □

10.2 Weak solutions

As suggested by our previous remarks and everyday experience of shocks in gas dynamics, for instance, blow-up in finite time does occur for generic initial data. This urges us to consider weak solutions, i.e. solutions in the sense of distributions that are not even continuous. For general quasilinear systems, the meaning of discontinuous solutions is unclear. However, for systems in divergence form (10.0.1), there is no ambiguity, and the associated Cauchy problems admit natural weak formulations (see, for instance, [46], p. 50–51 or [184], p. 86–87).

10.2.1 *Entropy solutions*

A well-known drawback of weak solutions is their non-uniqueness, and an appropriate notion that is likely to restore uniqueness is the one of *entropy solutions.* To define entropy solutions we need mathematical entropies, which are defined as follows.

Definition 10.3 *A function* $\eta \in \mathscr{C}^1(\mathscr{U};\mathbb{R})$, $\mathscr{U} \subset \mathbb{R}^n$, *is called a (mathematical)* entropy *of the system of balance laws* (10.0.1) *if and only if there exists* $q \in \mathscr{C}^1(\mathscr{U};\mathbb{R}^d)$ *such that*

$$\mathrm{d}q^j(u) \;=\; \mathrm{d}\eta(u)\cdot A^j(u)\,, \quad \forall\, u \in \mathscr{U}\,, \quad j = 1,\ldots,d,$$

where

$$A^j(u) \;=\; \mathrm{d}f^0(u)^{-1}\,\mathrm{d}f^j(u).$$

In this case, (η, q) *is called an entropy–entropy flux pair.*

Equivalently, (η, q) is an entropy–entropy flux pair if any smooth solution u of (10.0.1) satisfies the additional balance law

$$\partial_t\eta(u) \,+\, \sum_{j=1}^{d} \partial_j q^j(u) \;=\; \mathrm{d}\eta(u)\cdot b(u)\,, \quad b(u) \;=\; \mathrm{d}f^0(u)^{-1}\,c(u)\,.$$

Definition 10.4 *A weak solution* u *of* (10.0.1) *is called an* entropy solution *if it satisfies the inequality*

$$\partial_t\eta(u) \,+\, \sum_{j=1}^{d} \partial_j q^j(u) \;\leq\; \mathrm{d}\eta(u)\cdot b(u)$$

in the sense of distributions for any entropy–entropy flux pair (η, q) *with* η convex.

Of course, the convexity restriction plays an essential role here. If it were omitted, the requested inequality could only be an equality, which would be too restrictive – smooth solutions do satisfy the equality but discontinuous solutions in general do not.

Also note that Definition 10.4 might be helpless if no convex entropy exists, which is 'generically' the case for $n \geq 2$ and $nd \geq 3$ – there are too many equations for two unknowns. However, it appears that physics is *not* generic. For physical systems like the Euler equations considered in Chapter 13, there does exist at least one entropy, in connection with the thermodynamical entropy, and in standard cases it is convex[1]. But there is generally hardly more than one (nontrivial) convex entropy. So one may wonder whether a single one is sufficient to select a unique – supposedly good – solution. The answer is unfortunately no, in general. However, it appears that a single strictly convex entropy η is able to

[1]For examples of physical systems with only partially convex entropies, see [46], Section 5.3.

select the smooth solution if and as long as it exists. This important result is due to Dafermos. We reproduce it here for completeness. We shall use the shortcut *η-entropy* solution for a function u satisfying

$$\begin{cases} \partial_t f^0(u) + \sum_{j=1}^{d} \partial_j f^j(u) = c(u)\,, \\ \\ \partial_t \eta(u) + \sum_{j=1}^{d} \partial_j q^j(u) \le \mathrm{d}\eta(u) \cdot df^0(u)^{-1} \cdot c(u) \end{cases}$$

in the sense of distributions.

Theorem 10.4 (Dafermos) *Let us assume that* (10.0.1) *has an entropy–entropy flux pair* (η, q) *with* $\eta \in \mathscr{C}^2(\mathscr{U};\mathbb{R})$ *and* $\mathrm{d}^2\eta$ *positive-definite everywhere. Assume that* $\bar{u} \in \mathscr{C}^1(\mathbb{R}^d \times [0,T])$ *is a smooth solution of* (10.0.1) *and* $u \in L^\infty(\mathbb{R}^d \times [0,T])$ *is a η-entropy solution, both taking values in the same compact convex subset* $\mathscr{K}$ *of* $\mathscr{U}$*, such that* $u(0) - \bar{u}(0)$ *belongs to* $L^2(\mathbb{R}^d)$*. Then there exists* $C = C(T, \mathscr{K}, \|\nabla_x \bar{u}\|_{L^\infty})$ *such that*

$$\|u - \bar{u}\|_{L^\infty(0,T;L^2)} \le C\,\|u(0) - \bar{u}(0)\|_{L^2}.$$

Proof Interestingly, the proof of this 'energy estimate' involves new ingredients compared to the ones performed previously in this book, and in particular it incorporates, to some extent, Krushkov's idea of doubled unknowns. It also requires some information on the regularity in time of weak solutions, which we merely recall without proof from [46], p. 50.

Lemma 10.1 *For any weak solution* $u \in L^\infty(\mathbb{R}^d \times [0,T])$ *of* (10.0.1)*, for all* $t \in [0,T)$*, the mean value*

$$\frac{1}{\varepsilon}\int_t^{t+\varepsilon} u(\tau)\,\mathrm{d}\tau$$

has a limit in the weak-$*$ topology of $L^\infty(\mathbb{R}^d)$ *when* ε *goes to* 0*, and this limit is* $u(t)$ *for almost all* t *in* $[0,T]$.

Admitting this lemma, we can thus choose a representative of the weak solution u in the statement of the theorem such that $u(t)$ is the limit *everywhere.* This will be implicitly assumed in the computations below. In view of Lemma 10.1, we have in particular

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{P}} u(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau = \int_{\mathbb{P}} u(x,t)\,\mathrm{d}x \quad \forall t \in [0,T),$$

for any compact set $\mathbb{P} \subset \mathbb{R}^d$, and using that

$$\eta(u(x,\tau)) \ge \eta(u(x,t)) + \mathrm{d}\eta(u(x,t)) \cdot (u(x,\tau) - u(x,t))$$

by the convexity of η, we easily get

$$\liminf_{\varepsilon\searrow 0}\ \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\int_{\mathbb{P}_\varepsilon(\tau)} \eta(u(x,\tau))\,\mathrm{d}x\,\mathrm{d}\tau \ \geq \int_{\mathbb{P}_0(t)} \eta(u(x,t))\,\mathrm{d}x \quad \forall t\in[0,T),$$

if the set $\mathbb{P}_\varepsilon(\tau)$ depends continuously on (ε,τ).

Having these preliminary results at hand, we can now give a complete proof of Theorem 10.4, which is a slightly more detailed version of what can be found in [46], p. 66–68. Despite its technicality, it is interesting and easy to understand.

We assume for simplicity that $f^0(u) = u$ – in this case it is sometimes said that (10.0.1) is in *canonical form* – and define

$$\begin{aligned}
h(v,w) &= \eta(v) - \eta(w) - \mathrm{d}\eta(w)\cdot(v-w),\\
g^j(v,w) &= q^j(v) - q^j(w) - \mathrm{d}\eta(w)\cdot(f^j(v) - f^j(w)),\\
z^j(v,w) &= f^j(v) - f^j(w) - \mathrm{d}f^j(w)\cdot(v-w),\\
\zeta(v,w) &= \mathrm{d}\eta(v) - \mathrm{d}\eta(w) - \mathrm{d}^2\eta(w)\cdot(v-w)
\end{aligned}$$

for all $v,\ w\in\mathscr{K}$. By the strict convexity of η and Taylor expansions, we have the following uniform estimates on $\mathscr{K}$

$$\begin{aligned}
\beta\,\|v-w\|^2 \ \leq\ h(v,w) \ &\leq\ \beta^{-1}\,\|v-w\|^2,\\
\|g(v,w)\| := \textstyle\sum_j |g^j(v,w)| \ &\leq\ K\,\|v-w\|^2,\\
\|z(v,w)\| := \textstyle\sum_j \|z^j(v,w)\| \ &\leq\ K\,\|v-w\|^2,\\
\|\zeta(v,w)\| \ &\leq\ K\,\|v-w\|^2,
\end{aligned}$$

with $\beta > 0$ and $K > 0$.

Considering a Lipschitz continuous test function φ, we substract the equality

$$\int_{\mathbb{R}^d\times[0,T)}\Big(\eta(\bar u)\,\partial_t\varphi + \sum_j q^j(\bar u)\,\partial_j\varphi + \varphi\,d(\bar u)\Big) + \int_{\mathbb{R}^d}\eta(\bar u(0))\,\varphi(0) = 0,$$

to the inequality

$$\int_{\mathbb{R}^d\times[0,T)}\Big(\eta(u)\,\partial_t\varphi + \sum_j q^j(u)\,\partial_j\varphi + \varphi\,d(u)\Big) + \int_{\mathbb{R}^d}\eta(u(0))\,\varphi(0) \geq 0$$

where we have denoted $d(u) = \mathrm{d}\eta(u)\cdot b(u)$. Using the equalities

$$\int_{\mathbb{R}^d\times[0,T)}\Big(\bar u\,\partial_t\psi + \sum_j f^j(\bar u)\,\partial_j\psi_j + \psi_j\,b(\bar u)\Big) + \int_{\mathbb{R}^d}\bar u(0)\,\psi_j(0) = 0,$$

$$\int_{\mathbb{R}^d\times[0,T)}\Big(u\,\partial_t\psi_j + \sum_j f^j(u)\,\partial_j\psi_j + \psi_j\,b(u)\Big) + \int_{\mathbb{R}^d}u(0)\,\psi_j(0) = 0$$

for the family of test functions

$$\psi_j = \varphi \, \frac{\partial \eta}{\partial u_j}(\bar{u}) \,, \quad j = 1, \ldots, n,$$

we obtain the inequality

$$\int_{\mathbb{R}^d \times [0,T)} \Big(h(u,\bar{u}) \, \partial_t \varphi + \sum_j g^j(u,\bar{u}) \, \partial_j \varphi + \varphi \, (\, \mathrm{d}\eta(u) - \mathrm{d}\eta(\bar{u}) \,) \cdot b(u) \Big)$$
$$+ \int_{\mathbb{R}^d} h(u(0),\bar{u}(0)) \, \varphi(0)$$
$$\geq \int_{\mathbb{R}^d \times [0,T)} \varphi \, \Big(\partial_t(\mathrm{d}\eta(\bar{u})) \cdot (u - \bar{u}) + \sum_j \partial_j(\mathrm{d}\eta(\bar{u})) \cdot (f^j(u) - f^j(\bar{u})) \Big).$$

Using the fact that $\bar{u}$ is smooth and satisfies

$$\partial_t \bar{u} + \sum_{j=1}^{d} \mathrm{d}f^j(\bar{u}) \cdot \partial_j \bar{u} = b(\bar{u})$$

in the classical sense and that the Jacobian matrices $\mathrm{d}f^j(\bar{u})$ are $\mathrm{d}^2\eta(\bar{u})$-symmetric, we may rewrite the factor of φ in the integrand of the right-hand side above as

$$\mathrm{d}^2\eta(\bar{u}) \cdot (z^j(u,\bar{u}), \partial_j \bar{u}) + \mathrm{d}^2\eta(\bar{u}) \cdot (b(\bar{u}), u - \bar{u}).$$

Therefore, we are left with the inequality

$$\int_{\mathbb{R}^d \times [0,T)} \Big(h(u,\bar{u}) \, \partial_t \varphi + \sum_j g^j(u,\bar{u}) \, \partial_j \varphi \Big) + \int_{\mathbb{R}^d} h(u(0),\bar{u}(0)) \, \varphi(0)$$
$$\geq \int_{\mathbb{R}^d \times [0,T)} \varphi \, \Big(\, \mathrm{d}^2\eta(\bar{u}) \cdot (z^j(u,\bar{u}), \partial_j \bar{u}) - \zeta(u,\bar{u}) \cdot b(u)$$
$$- \mathrm{d}^2\eta(\bar{u}) \cdot (b(u) - b(\bar{u}), u - \bar{u}) \, \Big), \tag{10.2.22}$$

where the right-hand side is quadratic, i.e. bounded in absolute value by

$$C \, \|\varphi\|_{L^\infty} \int_{\mathrm{Supp} \ \varphi} h(u,\bar{u}),$$

where $C = C(K, \beta, \|u\|_{L^\infty}, \|\nabla \bar{u}\|_{L^\infty}, \|\mathrm{d}b\|_{L^\infty(\mathscr{K})})$.

The rest of the proof is based on a suitable choice of φ that will cancel out the left-hand side terms

$$\int_{\mathbb{R}^d \times [0,T)} g^j(u,\bar{u}) \, \partial_j \varphi \, .$$

In fact, we shall use a family of test functions φ_ε, whose definition depends on the rate $\lambda = 2K/\beta$ in the inequality

$$\|g(v,w)\| \,\leq\, \frac{\lambda}{2}\, h(v,w).$$

(Observe that λ should be homogeneous to a velocity.)

We fix $R > 0$, $t_0 \in (0,T)$, $0 < \varepsilon < T - t_0$, $0 \leq s \leq t_0$, and define $\varphi_{\varepsilon,s}$ by

$$\varphi_{\varepsilon,s}(x,t) \,=\, \begin{cases} 0 & \text{if } \ t \geq s+\varepsilon \\ & \text{or } \ \|x\| \,\geq\, R \,+\, \lambda(s+\varepsilon-t)\,, \\ & \\ 1 & \text{if } \ t \leq s \\ & \text{and } \ \|x\| \,\leq\, R \,+\, \lambda(s-t), \\ & \\ \theta_{\varepsilon,s}(t) & \text{if } \ s \leq t < s+\varepsilon \\ & \text{and } \ \|x\| \,\leq\, R \,+\, \lambda(s-t), \\ & \\ \theta_{\varepsilon,s}(t)\,\chi_{\varepsilon,s}(x,t) & \text{if } \ s \leq t < s+\varepsilon \\ & \text{and } \ R \,+\, \lambda(s-t) \,\leq\, \|x\| \,<\, R \,+\, \lambda(s+\varepsilon-t), \\ & \\ \chi_{\varepsilon,s}(x,t) & \text{if } \ t \leq s \\ & \text{and } \ R \,+\, \lambda(s-t) \,\leq\, \|x\| \,<\, R \,+\, \lambda(s+\varepsilon-t), \end{cases}$$

where

$$\theta_{\varepsilon,s}(t) \,=\, \frac{s+\varepsilon-t}{\varepsilon}\,, \quad \chi_{\varepsilon,s}(x,t) \,=\, \frac{R \,+\, \lambda(s+\varepsilon-t) - \|x\|}{\lambda\,\varepsilon}.$$

We first observe that

$$\int_{R\,+\,\lambda s \,\leq\, \|x\| \,<\, R\,+\,\lambda(s+\varepsilon)} h(u(0),\bar{u}(0))\,\chi_{\varepsilon,s}(x,0) \,=\, \mathcal{O}(\varepsilon)$$

(uniformly for $s \in [0,t_0]$) hence

$$\int_{\mathbb{R}^d} h(u(0),\bar{u}(0))\,\varphi_{\varepsilon,t}(0) \,=\, \int_{\|x\| \,\leq\, R\,+\,\lambda s} h(u(0),\bar{u}(0)) \,+\, \mathcal{O}(\varepsilon).$$

Next, we also have

$$\int_s^{s+\varepsilon} \int_{R\,+\,\lambda(s-t)\,\leq\,\|x\|\,<\,R\,+\,\lambda(s+\varepsilon-t)} h(u,\bar{u})\,\partial_t(\theta_{\varepsilon,s}\,\chi_{\varepsilon,s}) \,=\, \mathcal{O}(\varepsilon),$$

and therefore

$$\begin{aligned}\int_{\mathbb{R}^d\times[0,T]} h(u,\bar u)\,\partial_t\varphi_{\varepsilon,s} &= \mathcal{O}(\varepsilon) + \int_s^{s+\varepsilon}\int_{\|x\|\,\leq\, R+\lambda(s-t)} h(u,\bar u)\,\partial_t\theta_{\varepsilon,s}\\ &\quad + \int_0^s\int_{R+\lambda(s-t)\,\leq\,\|x\|\,<\,R+\lambda(s+\varepsilon-t)} h(u,\bar u)\,\partial_t\chi_{\varepsilon,s}\\ &= \mathcal{O}(\varepsilon) - \frac{1}{\varepsilon}\int_s^{s+\varepsilon}\int_{\|x\|\,\leq\, R+\lambda(s-t)} h(u,\bar u)\\ &\quad - \frac{1}{\varepsilon}\int_0^s\int_{R+\lambda(s-t)\,\leq\,\|x\|\,<\,R+\lambda(s+\varepsilon-t)} h(u,\bar u)\,.\end{aligned}$$

Similarly, we find that

$$\begin{aligned}&\int_{\mathbb{R}^d\times[0,T]} g^j(u,\bar u)\,\partial_j\varphi_{\varepsilon,s}\\ &\quad = \mathcal{O}(\varepsilon) - \frac{1}{\lambda\,\varepsilon}\int_0^s\int_{R+\lambda(s-t)\,\leq\,\|x\|\,<\,R+\lambda(s+\varepsilon-t)} g^j(u,\bar u)\,\frac{x_j}{\|x\|}.\end{aligned}$$

Now, using the fact that

$$\lambda\,h(u,\bar u) + \sum_j \frac{x_j}{\|x\|}\,g^j(u,\bar u) \,\geq\, \lambda\,h(u,\bar u) - \sum_j |g^j(u,\bar u)| \,\geq\, \frac{\lambda}{2}\,h(u,\bar u)$$

by definition of λ, and substituting the expansions obtained hereabove into the inequality (10.2.22) we arrive at

$$\begin{aligned}&\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\int_{\|x\|\,\leq\, R+\lambda(s-t)} h(u,\bar u) + \frac{1}{2\,\varepsilon}\int_0^s\int_{R+\lambda(s-t)\,\leq\,\|x\|\,<\,R+\lambda(s+\varepsilon-t)} h(u,\bar u)\\ &\quad\leq C_0\,\varepsilon + \int_{\|x\|\,\leq\, R+\lambda s} h(u(0),\bar u(0)) + C\int_0^{s+\varepsilon}\int_{\|x\|\leq R+\lambda(s+\varepsilon-t)} h(u,\bar u)\end{aligned} \tag{10.2.23}$$

for all $\varepsilon\in(0,T-t_0)$ and $s\in[0,t_0]$. We claim this implies an estimate of the form

$$\int_{\|x\|\,\leq\, R} h(u(t),\bar u(t)) \,\leq\, \widetilde{C}\int_{\|x\|\,\leq\, R+\lambda\,t_0} h(u(0),\bar u(0))$$

for all $t\in[0,t_0]$.

Indeed, setting

$$a(s) = \int_0^s\int_{\|x\|\leq R+\lambda(s-t)} h(u(x,t),\bar u(x,t))\,\mathrm{d}x\,\mathrm{d}t,$$

we have

$$a(s+\varepsilon)-a(s) = \int_s^{s+\varepsilon}\int_{\|x\|\,\leq\, R\,+\,\lambda(s+\varepsilon-t)} h(u,\bar{u})$$

$$+\int_0^s\int_{R\,+\,\lambda(s-t)\,\leq\,\|x\|\,<\,R\,+\,\lambda(s+\varepsilon-t)} h(u,\bar{u})$$

$$= \mathcal{O}(\varepsilon^2) + \int_s^{s+\varepsilon}\int_{\|x\|\,\leq\,R\,+\,\lambda(s-t)} h(u,\bar{u})$$

$$+\int_0^s\int_{R\,+\,\lambda(s-t)\,\leq\,\|x\|\,<\,R\,+\,\lambda(s+\varepsilon-t)} h(u,\bar{u}),$$

so that (10.2.23) implies

$$\frac{1}{2}\,\frac{a(s+\varepsilon)-a(s)}{\varepsilon} \leq \widetilde{C}_0\,\varepsilon + \int_{\|x\|\,\leq\,R\,+\,\lambda\,s} h(u(0),\bar{u}(0)) + C\,a(s+\varepsilon),$$

where $\widetilde{C}_0$ is the constant C_0 plus an upper bound of $\mathcal{O}(\varepsilon^2)/\varepsilon^2$. Applying our discrete Gronwall Lemma A.4, this yields

$$a(s) \leq \mathrm{e}^{2Cs}\int_{\|x\|\,\leq\,R\,+\,\lambda\,s} h(u(0),\bar{u}(0))$$

for all s. Returning to the original inequality (10.2.23), this shows in particular that

$$\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\int_{\|x\|\,\leq\,R\,+\,\lambda(s-t)} h(u,\bar{u}) \leq C_0\,\varepsilon + \int_{\|x\|\,\leq\,R\,+\,\lambda\,s} h(u(0),\bar{u}(0))$$

$$+\,C\,\mathrm{e}^{2C(s+\varepsilon)}\int_{\|x\|\,\leq\,R\,+\,\lambda\,(s+\varepsilon)} h(u(0),\bar{u}(0))\,.$$

Finally, using our preliminary remark to deal with the left-hand side when ε goes to 0 we obtain

$$\int_{\|x\|\,\leq\,R} h(u(s),\bar{u}(s)) \leq (1 + C\mathrm{e}^{2Cs})\int_{\|x\|\,\leq\,R\,+\,\lambda\,s} h(u(0),\bar{u}(0)).$$

Together with the known two-sided inequalities for h, this completes the proof. □

This is all we shall say about weak/entropy solutions in general, mainly because we do not know much more in several space dimensions! For recent, important results in one space dimension, see [24].

10.2.2 *Piecewise smooth solutions*

An amenable, and nonetheless interesting class of weak/entropy solutions is given by piecewise smooth solutions. They consist of smooth solutions separated by fronts of discontinuity, across which the so-called Rankine–Hugoniot condition is satisfied. We recall this condition below without justification, which can be found in many textbooks (see, for instance, [46], p. 10–11 or [184], p. 88–89). We also summarize hereafter the material needed for our subsequent analysis in Chapter 12.

Definition 10.5 *Suppose that Σ is a codimension one surface in $\mathbb{R}^d \times \mathbb{R}^+$ and that $(\mathbb{R}^d \times \mathbb{R}^+)\backslash\Sigma$ has two connected components Ω_- and Ω_+. Assume that $u_\pm \in \mathscr{C}^1(\Omega_\pm)$ solve the system of balance laws* (10.0.1) *on either side of Σ. Then the function*

$$\begin{aligned} u \,:\, (\mathbb{R}^d \times \mathbb{R}^+)\backslash\Sigma &\to \mathbb{R}^n \\ (x,t) \qquad &\mapsto u(x,t) \,=\, u_\pm(x,t) \ \text{ for } \ (x,t) \in \Omega_\pm \end{aligned}$$

is said to satisfy the Rankine–Hugoniot condition across Σ if and only if

$$N_0\,[f^0(u)] + \sum_{j=1}^{d} N_j\,[f^j(u)] \,=\, 0, \tag{10.2.24}$$

where N_0, N_1, ..., N_d denote the components in the directions t, x_1,..., x_d, respectively, of a vector N orthogonal to Σ. Here above, if N is pointing to Ω_+, *the brackets $[\cdot]$ stand for*

$$[f^j(u)](x,t) \,=\, \lim_{\varepsilon \searrow 0} \left(f^j(u_+((x,t)+\varepsilon N(x,t))) \,-\, f^j(u_-((x,t)-\varepsilon N(x,t))) \right)$$

at each point $(x,t) \in \Sigma$.

The Rankine–Hugoniot condition is known to be necessary and sufficient for u to be a *weak* solution of (10.0.1). Needless to say, the conservative, or divergence form of the right-hand side in (10.0.1) is crucial for this statement to make sense.

For completeness, let us recall the following basic characterization of piecewise smooth *entropy* solutions.

Proposition 10.2 *Suppose that Σ is a codimension one surface in $\mathbb{R}^d \times \mathbb{R}^+$ and that $(\mathbb{R}^d \times \mathbb{R}^+)\backslash\Sigma$ has two connected components Ω_- and Ω_+. A function*

$$\begin{aligned} u \,:\, (\mathbb{R}^d \times \mathbb{R}^+)\backslash\Sigma &\to \mathbb{R}^n \\ (x,t) \qquad &\mapsto u(x,t) \,=\, u_\pm(x,t) \ \text{ for } \ (x,t) \in \Omega_\pm\,, \end{aligned}$$

with $u_\pm \in \mathscr{C}^1(\Omega_\pm)$ is a η-entropy solution of the system of conservation laws

$$\frac{\partial f^0(u)}{\partial t} + \sum_{j=1}^{d} \frac{\partial f^j(u)}{\partial x_j} \,=\, 0 \tag{10.2.25}$$

if and only if $u_\pm$ *satisfy* (10.2.25) *in* $\Omega_\pm$*, the Rankine–Hugoniot* (10.2.24) *holds true across* Σ *and, moreover,*

$$N_0\,[\eta(u)] + \sum_{j=1}^{d} N_j\,[q^j(u)] \,\leq\, 0, \tag{10.2.26}$$

where N_0*,* N_1*, …,* N_d *denote the components in the directions* t*,* x_1*,…,* x_d*, respectively, of a vector* N *orthogonal to* Σ pointing to Ω_+*.*

Remark 10.2 This characterization of course does not depend on the choice of Ω_- and Ω_+: if these parts of the space $\mathbb{R}^d \times \mathbb{R}^+$ are exchanged, N is changed to $-N$ and the jumps are also changed into their opposite.

Many solutions of the type described in Proposition 10.2 have been observed experimentally (in gas dynamics, for instance) and numerically. But there is no global-in-time existence result for arbitrary jump front Σ.

The special case of a planar Σ reduces the problem to one space dimension. Solutions consisting of a planar front propagating with constant speed in constant, homogeneous states are easily found by examining the one-dimensional Rankine–Hugoniot condition. These solutions are called planar *shock* waves when (10.2.26) is a strict inequality. There are also planar contact discontinuities, for which (10.2.26) is in fact an equality. Planar fronts in gas dynamics are discussed in detail in Chapter 13.

Spherical shocks are already much more complicated, as they involve a non-constant speed and non-homogeneous states in general. As regards gas dynamics, for instance, spherical shocks have received attention for decades: this goes back to World War II and the atomic bomb research [73, 75, 106, 180, 204], in which basic solutions were obtained by means of similarity/dimensional analysis [181]; more recently the interest in the field has been renewed by (hopefully) more peaceful and nonetheless fascinating phenomena (e.g. sonoluminescence, cavitation) and various (potential) applications of shock focusing (extracorporeal therapies, nuclear fusion, etc.); current research concerns complex fluids (van der Waals or dusty gases, superfluids, etc.) and has incorporated group-analysis techniques. The interested reader may refer, for instance, to the collected papers in [198].

Regarding 'arbitrarily' curved fronts, as far as gas dynamics is concerned there is a wide literature on transonic (stationary) shocks (see, for instance, [30, 34, 220]) or other patterns, in particular those related to the reflexion of shocks (see, for instance, [182,225,226]). In more general, abstract settings, there are far fewer results on curved shocks. The main existence results are due to Majda [124–126], Métivier [133,136], Blokhin [17,18] and are based on a stability analysis of reference fronts (known to exist, such as planar ones, or assumed) and thus are local in time. The description of those results will be the main purpose of Chapter 12.

Even though they are beyond the scope this book, other interesting results on (abstract) multidimensional weak solutions are worth mentioning. In fact, they date back to the late 1980s! One of them is due to Métiver and deals with the interaction of two shocks [131]. Another study is due to Alinhac [3,4], and answers a question raised by Majda in [126] (p. 153–154) on the existence and structure of multidimensional rarefaction waves (initially discontinuous, but smoother for positive times). One of the main difficulties in constructing those waves is, 'the dominant signals in rarefaction fronts move at characteristic speeds', Majda said. Alinhac overcome it by using a Nash–Moser iterative scheme and *ad hoc* functional spaces (based on the Littlewood–Paley decomposition). It is notable that a similar difficulty arises for weak shocks. This case is paradoxical because, when the initial shock strength goes to zero we expect to recover a smooth solution, but if we apply Majda's result with brute force the existence time of the shock-front solution shrinks to 0, as the shock tends to be characteristic. It was resolved rather recently by Métivier and his coworker Francheteau (see [56, 133], and also the lecture notes [136]).

11

THE MIXED PROBLEM FOR QUASILINEAR SYSTEMS

Physical problems are not usually posed in the whole space: in fluid mechanics, for instance, a spatial domain typically has an entrance, an exit and walls; however, this kind of mixed-type and non-smooth boundary yields unsolved yet mathematical problems. The purpose of this chapter is to show how to deal with more regular non-linear initial boundary value problems (IBVP). The name IBVP refers explicitly to the initial data (at time zero) and the boundary data (on the boundary of the spatial domain). IBVPs are equivalently called *mixed problems* (regardless of the nature of boundary data) just because of the 'mixing' between initial data and boundary data: we will use either one of those names.

The general mixed problem for a system of balance laws reads

$$\begin{cases} \partial_t f^0(u) + \sum_{j=1}^{d} \partial_j f^j(u) \;=\; c(u)\,, & \Omega \times (0,T), \\ b(u) \;=\; \underline{b}\,, & \partial\Omega \times (0,T) \\ u_{|t=0} \;=\; u_0\,, & \Omega, \end{cases}$$

where Ω is a connected open subset of $\mathbb{R}^d$, the fluxes $f^j(u)$, the source term $c(u)$ and the boundary term $b(u)$ are expressed through supposedly smooth nonlinear vector-valued mappings f^j, c and b, respectively, and $\underline{b}$ and u_0 are smooth functions encoding, respectively, the boundary data and the initial data. We shall also assume that the level sets of the nonlinear mapping b are smooth submanifolds of the phase space $\mathbb{R}^n$.

We will consider only smooth solutions – recall that even the Cauchy problem is still most open for weak solutions in several space dimensions – so we may use the chain rule in the derivatives of fluxes and rewrite the PDEs in quasilinear form. More generally, we consider a quasilinear system of PDEs, not necessarily coming from equations in divergence form, and its associated initial boundary value problem

$$\begin{cases} \partial_t u + \sum_{j=1}^{d} A^j(u)\,\partial_j u \;=\; h(u)\,, & \Omega \times (0,T), \\ b(u) \;=\; \underline{b}\,, & \partial\Omega \times (0,T), \\ u_{|t=0} \;=\; u_0\,, & \Omega. \end{cases} \tag{11.0.1}$$

Here above, the A^j are $\mathscr{C}^\infty$ mappings on an open subset $\mathscr{U}$ of $\mathbb{R}^n$, with values in $\mathbf{M}_n(\mathbb{R})$, and the mappings $h : \mathscr{U} \to \mathbb{R}^n$ and $b : \mathscr{U} \to \mathbb{R}^p$ (with p a fixed integer), are also $\mathscr{C}^\infty$. In fact A^j, h and b may all depend on (x,t) as well but we avoid this refinement for the sake of simplicity. The boundary data $\underline{b} : \partial\Omega \times [0,T] \to \mathbb{R}^p$ and the initial data $u_0 : \Omega \to \mathbb{R}^n$ will be assumed to lie in Sobolev spaces: the main purpose of this chapter is indeed to solve the quasilinear IBVP (11.0.1) in Sobolev spaces of sufficiently high index (and for T small enough).

The resolution of (11.0.1) is a most open problem for general domains Ω, but under suitable assumptions on the geometry of Ω and on the boundary conditions, it is possible to prove well-posedness results: in the simpler one dimensional case – i.e. with Ω a segment of the real line – the (hardly obtainable) book by Li Ta Tsien and Yu Wen Ci [116] deals with IBVPs and other related problems; in higher dimensions our main references are the (unpublished) PhD thesis of Mokrane [140] and the lecture notes of Métivier [136] (see also the early work of Rauch and Massey [165], and more recent papers dealing with characteristic problems by Guès [76], the Japanese school [149–151, 192, 193] and Secchi and coworkers [43, 44, 176–179]).

11.1 Main results

11.1.1 *Structural and stability assumptions*

We enter now into detailed assumptions under which (11.0.1) is known to have a unique smooth solution (for T small enough, the non-linearities precluding global solutions in general).

A basic requirement is that the 'linearized' versions of the problem, including those with variable coefficients, fall into the framework of the results known for linear initial boundary value problems (LIBVP). Here come the restrictions on the domain Ω: Chapter 9 has shown we may hope to deal with either a half-space (up to a change of variables) or a smooth bounded Ω. We shall implicitly assume either one of these situations.

The LIBVP considered will be of the form

$$\begin{cases} \partial_t u + \displaystyle\sum_{j=1}^{d} A^j(v)\,\partial_j u \,=\, f\,, & \Omega \times (0,T), \\ \mathrm{d}b(v)\cdot u \,=\, g\,, & \partial\Omega \times (0,T) \\ u_{|t=0} \,=\, 0\,, & \Omega, \end{cases} \qquad (11.1.2)$$

with v a given function of possibly limited regularity, and f, g arbitrary data (in spaces to be specified afterwards).

Our starting point is the classical, even though not universal, constant hyperbolicity assumption:

(CH) the matrices

$$A(w,\xi) := \sum_{j=1}^{d} \xi_j \, A^j(w)$$

are diagonalizable with real eigenvalues of constant multiplicities on $\mathscr{U} \times \mathbb{S}^{d-1}$.

This condition is known to be violated, for instance, by the (challenging) system of ideal magnetohydrodynamics [17]. Nevertheless, **(CH)** is satisfied by many physically relevant systems, and in particular by the Euler equations of gas dynamics, the basic application considered in this book. For systems with variable multiplicities, we refer to the recent work by Métivier and Zumbrun [135], which goes far beyond the scope of this book.

We now import from Chapter 9 some assumptions on Ω and the matrix-valued function $B := \mathrm{d}b$.

(NC) for all $w \in \mathscr{U}$ and all normal vector ν to $\partial\Omega$, the matrix $A(w,\nu)$ is non-singular,

(N) the boundary matrix B is of constant, maximal rank and

$$\mathbb{R}^n = \ker B(w) \oplus E^s(A(w,\nu)) \quad \text{for all} \quad (w,\nu) \in \mathscr{U} \times \mathbb{S}^{d-1},$$

with ν an *outward* normal vector to $\partial\Omega$ and $E^s(A(w,\nu))$ the stable subspace of the (hyperbolic) matrix $A(w,\nu)$,

(T) the vector bundle $\ker B$ is trivializable, that is, $\ker B(w)$ admits a basis depending smoothly on $w \in \mathscr{U}$.

The latter is not very demanding: thanks to a classical differential topology result (see [85], p. 97), it is satisfied as soon as B is smooth (and of constant rank) and $\mathscr{U}$ is contractible (which is the case if it is a ball, for instance). On the other hand, assuming that the whole boundary is non-characteristic in such a strong sense as in **(NC)** is quite restrictive, especially when $\partial\Omega$ is a connected bounded manifold. Indeed, observe that for all $\xi \in \mathbb{S}^{d-1}$, $\det A(w,\xi) = (-1)^n \det A(w,-\xi)$ by homogeneity: assume then that $\partial\Omega$ is a smooth connected manifold and that there are two points x_1 and x_2 in $\partial\Omega$ where the normal vectors are opposite to each other; if, moreover, the dimension n of the phase space is odd, the mean-value theorem trivially implies that $\det A(w,.)$ vanishes on the connected set of normal vectors to $\partial\Omega$ along the path from x_1 to x_2. In some cases, the vanishing of $\det A(w,\nu)$ can even occur whatever the parity of n: in full gas dynamics, for instance, if $\partial\Omega$ is a sphere, the set of its normal vectors is $\mathbb{S}^{d-1}$, which intersects any hyperplane $\mathbf{u}^\perp$; this means the eigenvalue $\lambda_2 = \mathbf{u}\cdot\xi$ (with $\mathbf{u}$ the velocity of the fluid) does vanish at some points $\xi \in \mathbb{S}^{d-1}$, irrespective of the parity of d (or equivalently the parity of $n = d+2$); and even in isentropic gas dynamics, both eigenvalues $\mathbf{u}\cdot\xi \pm c\,|\xi|$ vanish at some $\pm\xi \in \mathbb{S}^{d-1}$ when the flow is *supersonic* ($|\mathbf{u}| \geq c$). These facts urge us to weaken **(NC)** by taking into account the boundary data. This is done in the following.

(NC$_b$) For all $(x,t) \in \partial\Omega \times [0,T]$, for all $w \in \mathscr{U}$ such that $b(w) = \underline{b}(x,t)$, the matrix $A(w,\nu(x))$ is non-singular (where $\nu(x)$ denotes the outward unit normal to $\partial\Omega$ at point x).

This is obviously more likely to be satisfied than **(NC)** (as we see in the example of full gas dynamics, for which **(NC$_b$)** is true provided that boundary data impose a non-singular and non-sonic velocity field normal to the boundary) and sufficient for our purpose. Even **(NC$_b$)** is not necessary though, but mixed problems with a (partly) characteristic boundary (for which **(NC$_b$)** is false) are much trickier; see [43,44,76,127,149–151,176–179,192,193]. The assumption **(N)** is to be weakened accordingly:

(N$_b$) the boundary matrix $B(w)$ is of constant, maximal rank for all $(x,t) \in \partial\Omega \times [0,T]$ and all $w \in \mathscr{U}$ such that $b(w) = \underline{b}(x,t)$, and

$$\mathbb{R}^n = \ker B(w) \oplus E^s(A(w,\nu(x))).$$

In geometrical terms, **(N$_b$)** means the level sets

$$\mathscr{M}_b(x,t) := \{ w \in \mathscr{U} \,;\, b(w) = \underline{b}(x,t) \}$$

are submanifolds of $\mathbb{R}^n$ of the same dimension for all $(x,t) \in \partial\Omega \times [0,T]$, and that for all $w \in \mathscr{M}_b(x,t)$ the tangent space $\mathrm{T}_w\mathscr{M}_b(x,t)$ is transverse to the stable subspace of the characteristic matrix $A(w,\nu(x))$.

Finally, we will of course need the uniform Kreiss–Lopatinskiĭ condition, a draft of which is the following.

(UKL) for all $(w,x,\xi,\tau) \in \mathscr{U} \times \mathrm{T}^*\partial\Omega \times \mathbb{C}$ with $\operatorname{Re}\tau > 0$, there exists $C > 0$ so that

$$\|V\| \leq C \,\|\, B(w)\, V\| \qquad \text{for all } V \in E_-(w,x,\xi,\tau),$$

where $E_-(w,x,\xi,\tau)$ is the stable subspace of

$$\mathcal{A}(w,x,\xi,\tau) := A(w,\nu(x))^{-1}\,(\,\tau\, I_n + i\, A(w,\xi)\,),$$

and $\nu(x)$ denotes the outward unit normal to $\partial\Omega$ at point x; *and* the same is true for $\operatorname{Re}\tau = 0$ once the subspace E_- has been extended by continuity.

Again, **(UKL)** is to be replaced by a weaker version **(UKL$_b$)**, obtained by asking the estimate only for those w that are in $\mathscr{M}_b(x,t)$ for some $t \in [0,T]$.

11.1.2 *Conditions on the data*

The resolution of (11.0.1) is possible in Sobolev spaces under two ‘technical’ conditions: 1) that 0 is a solution of the special IBVP with zero initial data and boundary data $\underline{b}(\cdot, t=0)$, which amounts to asking

$$h(0) = 0 \quad \text{and} \quad b(0) = \underline{b}(x,0) \quad \text{for all } x \in \partial\Omega$$

(recall that the vanishing of the source term at 0 was also asked for the Cauchy problem in Theorem 10.1); 2) that the boundary data and the initial data satisfy some compatibility conditions.

Of course, 1) assumes that zero belongs to the domain $\mathscr{U}$. It could be modified into

$$h(w) = 0 \quad \text{and} \quad b(w) = \underline{b}(x,0) \quad \text{for all } x \in \partial\Omega$$

for some fixed state $w \in \mathscr{U}$. This would yield *in fine* solutions in affine spaces $w + H^k$ instead of H^k. We set $w = 0$ just to simplify the presentation. The point 2) is undoubtedly crucial: we are bound to look for smooth solutions (because of non-linearities), and we know that the existence of smooth solutions even in the linear case does require compatibility conditions (see Section 9.2.3).

Definitely unpleasant to write down explicitly, compatibility conditions are nevertheless very natural. Indeed, assume that u is a smooth – in particular continuous up to the boundary – solution of (11.0.1), then necessarily

$$b(u_0(x)) = \underline{b}(x,0) \quad \text{for all } x \in \partial\Omega.$$

Now, if u is $\mathscr{C}^1$ up to the boundary,

$$\partial_t \underline{b}(x,0) = \mathrm{d}b(u_0(x)) \cdot \partial_t u(x,0) = \mathrm{d}b(u_0(x)) \cdot \left(h(u_0(x)) - \sum_{j=1}^{d} A^j(u_0(x))\partial_j u_0(x) \right)$$

for all $x \in \partial\Omega$. More generally, u being $\mathscr{C}^p$ up to the boundary implies

$$\partial_t^p \underline{b}(x,0) = C_p(u_0(x), \mathrm{D}u_0(x), ..., \mathrm{D}^p u_0(x)) \quad \text{for all } x \in \partial\Omega \tag{11.1.3}$$

for some complicated nonlinear function C_p, which can be computed by induction from $C_0(u) = b(u)$ through the formula

$$C_{p+1}(u, \mathrm{D}u, ..., \mathrm{D}^{p+1}u) = \sum_{k=0}^{p} \mathrm{d}_k C_p(u, \mathrm{D}u, ..., \mathrm{D}^p u) \cdot \mathrm{D}^k \left(h(u) - \sum_{j=1}^{d} A^j(u)\,\partial_j u \right),$$

where $\mathrm{d}_k C_p$ denotes the differential of C_p with respect to its $(k+1)$th argument (belonging to $\mathbb{R}^{n^k}$!), and D^k denotes the k-th order differentiation with respect to $x \in \mathbb{R}^d$.

11.1.3 *Local solutions of the mixed problem*

A rough statement of the main result in the theory of quasilinear mixed problems is,

> *existence and uniqueness of smooth solutions for smooth enough* and *compatible initial data and boundary data.*

In the accurate statement below, the regularity of the solution is a little weaker than the regularity of the data.

Theorem 11.1 *We assume* $\mathscr{U}$ *is convex,* $0 \in \mathscr{U}$, $h(0) = 0$, **(CH)**, **(T)**, *and we take* m *an integer greater than* $(d+1)/2 + 1$.

For all $\underline{b} \in H^{m+1/2}(\partial\Omega \times [0,T])$ *such that* $\underline{b}(\cdot, 0) \equiv b(0)$ *and all* $u_0 \in H^{m+1/2}(\bar{\Omega})$ *with values in* $\mathscr{U}$ *satisfying the compatibility conditions* (11.1.3) *for all* $p \in \{0, \dots, m-1\}$, *as well as* **(NC$_b$)**, **(N$_b$)**, *and* **(UKL$_b$)**, *there exists* $T > 0$ *so that the problem* (11.0.1) *admits a unique solution* $u \in H^m(\bar{\Omega} \times [0,T])$ *having a trace on* $\partial\Omega$ *that belongs to* $H^m(\partial\Omega \times [0,T])$.

This result was announced by Rauch and Massey [165], and actually proved by Mokrane [140]; see Section 11.2.2 for a rather detailed proof.

11.1.4 *Well-posedness of the mixed problem*

In the case of Friedrichs symmetrizability (and under other simplifying but less crucial assumptions) it is possible to improve Theorem 11.1 and obtain solutions with the same regularity as the data. This has been done by Métivier [136], in fact in the more complicated context of shock-waves stability, but we can state a simplified version for 'ordinary' mixed problems.

Theorem 11.2 *We assume* Ω *is the half-space* $\{x = (y, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^+\}$, *the domain* $\mathscr{U}$ *is convex and contains* 0, *we also assume* **(CH)** and *the existence of a Friedrichs symmetrizer* S_0 *on* $\mathscr{U}$ *(i.e.* $S_0 : \mathscr{U} \to \mathbf{SPD}_n$ *being* $\mathscr{C}^\infty$ *and such that* $S_0(w)A^j(w)$ *is symmetric for all* $j \in \{1, \dots, d\}$*).*

Additionally, we assume $h \equiv 0$ *and* $\underline{b} \equiv 0$.

If m *is an integer greater than* $d/2 + 1$, *for all* $u_0 : (y, z) \mapsto u_0(y, z)$ *in* $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+)$ *with values in* $\mathscr{U}$, *satisfying the compatibility conditions* (11.1.3) *for all* $p \in \{0, \dots, m-1\}$, *as well as* **(NC$_b$)**, **(N$_b$)**, *and* **(UKL$_b$)**, *then there exists* $T > 0$ *so that the problem* (11.0.1) *admits a unique solution* $u \in \mathscr{C}([0,T]; H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+))$ *such that* $\partial_t^p u \in \mathscr{C}([0,T]; H^{m-p}(\mathbb{R}^{d-1} \times \mathbb{R}^+))$ *for all* $p \in \{1, \dots, m\}$. *Furthermore, if the maximal time of existence of the solution* u *is a finite* T_* *then either* $u(x,t)$ *leaves every compact subset of* $\mathscr{U}$ *or*

$$\lim_{t \nearrow T_*} \|\nabla_x u(t)\|_{L^\infty(\mathbb{R}^{d-1} \times \mathbb{R}^+; \mathbb{R}^d)} = +\infty.$$

In particular, this theorem contains a blow-up criterion analogous to what is known for the Cauchy problem (see Theorem 10.3). Of course it does not tell us in advance if (and when) blow-up will take place. In one space dimension a few quantitative results are known, which provide either global solutions (for small enough data and boundary damping [115, 116]), or an estimate of the blow-up time (see, for instance, [14, 97]).

11.2 Proofs

For both Theorem 11.1 and Theorem 11.2 the sketch of the proof is most classical, and proceeds in two steps: 1) linearize and 2) use an iterative scheme. Details are more cumbersome. We will give only the proof of Theorem 11.1. Regarding Theorem 11.2, the reader is referred to Section 4 in Métivier's lectures notes [136].

11.2.1 *Technical material*

Topology

Let s be a real number greater than $d/2$ and consider a fixed $u \in H^s(\mathbb{R}^d)$ taking values in $\mathscr{U} \ni 0$. Then the closure of $u(\mathbb{R}^d)$ is a compact subset $\mathscr{K}$ of $\mathscr{U}$, and there exist $\rho > 0$ and $\mathscr{V} \subset\subset \mathscr{U}$ (which means $\overline{\mathscr{V}}$ is a compact subset of $\mathscr{U}$) such that $\mathscr{K} \subset \mathscr{V}$ and for all $v \in \mathscr{C}(\mathbb{R}^d)$,

$$\|v - u\|_{L^\infty(\mathbb{R}^d)} \leq \rho \quad \text{implies} \quad v(x) \in \mathscr{V} \text{ for all } x \in \mathbb{R}^d.$$

Hence, denoting by $\nu_{s,d}$ the norm of the Sobolev embedding

$$H^s(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d),$$

$$\|v - u\|_{H^s(\mathbb{R}^d)} \leq \rho/\nu_{s,d} \quad \text{implies} \quad v(x) \in \mathscr{V} \text{ for all } x \in \mathbb{R}^d.$$

The same property is true when $\mathbb{R}^d$ is replaced by $\bar{\Omega}$. (Recall that $H^s(\bar{\Omega})$ is made of functions having an extension in $H^s(\mathbb{R}^d)$.) From now on, we fix $\rho_0 > 0$ such that

$$\|v - u_0\|_{L^\infty(\Omega)} \leq \rho_0 \quad \text{implies} \quad v(x) \in \mathscr{V}_0 \subset\subset \mathscr{U} \text{ for all } x \in \Omega,$$

where u_0 is the initial data in the IBVP (11.0.1), supposed to belong at least to $\in H^s(\bar{\Omega})$ for $s > d/2$. (This assumption will be reinforced later.)

Calculus

The short way of writing compatibility conditions in (11.1.3) is convenient but it conceals some technical details needed for the proof of Theorem 11.1.

Here is a more explicit (though ugly) way, using repeatedly Faá di Bruno's formula for the nth derivative of a composition

$$(f \circ u)^{(n)} = \sum_{m=1}^{n} \sum_{i_1 + \cdots + i_m = n} c_{i_1,\ldots,i_m} \, (\mathrm{d}^m f \circ u) \cdot (u^{(i_1)}, \ldots, u^{(i_m)}).$$

(The actual value of coefficients $c_{i_1,\ldots,i_m}$ is all but important here.) The time derivatives of a $\mathscr{C}^p$ function u such that

$$\partial_t u = -\sum_j \widetilde{A}^j \, \partial_j u + \widetilde{h}$$

should satisfy the induction formula

$$\partial_t^i u \;=\; -\,\sum_{\ell=0}^{i-1} \binom{i-1}{\ell} \sum_{j=1}^{d} (\partial_t^\ell \widetilde{A}^j)\,\partial_j\,\partial_t^{i-1-\ell} u \;+\; \partial_t^{i-1}\widetilde{h}$$

for all $i \in \{1,\dots,p\}$. When $\widetilde{A}^j \;=\; A^j \circ u$ and $\widetilde{h} \;=\; h \circ u$, we may use Faá di Bruno's formula to expand the derivatives $\partial_t^\ell \widetilde{A}^j$ and $\partial_t^{i-1}\widetilde{h}$. Using also Faá di Bruno's formula to expand $\partial_t^p(b \circ u)$ for $p \geq 1$, we eventually find the non-linear mapping C_p introduced in Section 11.1.2 above alternatively reads

$$C_p(u, \mathrm{D}u, \dots, \mathrm{D}^p u) = \sum_{m=1}^{p} \sum_{i_1+\cdots+i_m=p} c_{i_1,\dots,i_m}\, \mathrm{d}^m b(u)\cdot(u_{i_1},\dots,u_{i_m}), \quad (11.2.4)$$

where the functions $u_i : x \mapsto u_i(x)$ are such that

$$\begin{cases} u_0 \;=\; u\,, \quad u_1 \;=\; h \circ u \;-\; \displaystyle\sum_{j=1}^{d} (A^j \circ u)\,\partial_j u\,, \\[2ex] u_i \;=\; \displaystyle\sum_{k=1}^{i-1} \sum_{\ell_1+\cdots+\ell_k=i-1} c_{\ell_1,\dots,\ell_k}\,(\mathrm{d}^k h \circ u)\cdot(u_{\ell_1},\dots,u_{\ell_k}) \\ \qquad -\displaystyle\sum_{j=1}^{d} (A^j \circ u)\,\partial_j u_{i-1} \\ -\displaystyle\sum_{\ell=1}^{i-1} \binom{i-1}{\ell} \sum_{j=1}^{d}\sum_{k=1}^{\ell} \sum_{\ell_1+\cdots+\ell_k=\ell} c_{\ell_1,\dots,\ell_k}\,(\mathrm{d}^k A^j \circ u)\cdot(u_{\ell_1},\dots,u_{\ell_k})\,\partial_j\, u_{i-1-\ell} \\[2ex] \text{for all } \; i \in \{2,\dots,p\}\,. \end{cases} \quad (11.2.5)$$

Lemma 11.1 *If $s > d/2$ and $u \in H^s(\bar{\Omega})$, then for p the largest integer less than s, there exists a finite sequence $(u_0, u_1, \dots, u_p)$ satisfying* (11.2.5) *and such that for all $i \in \{0,\dots,p\}$, $u_i \in H^{s-i}(\bar{\Omega})$. Moreover, using this sequence in* (11.2.4) *we get a well-defined function $x \mapsto C_p(u(x), \mathrm{D}u(x), \dots, \mathrm{D}^p u(x))$ that belongs to $H^{s-p}(\bar{\Omega})$.*

Proof By assumption, $u_0 = u$ belongs to $H^s(\bar{\Omega}) = H^{s-0}(\bar{\Omega})$. Let us see what happens with u_1. By assumption (our 'technical' condition 1)), h vanishes at 0, so that Theorem C.12 implies $h \circ u$ belongs to H^s. Furthermore, we may decompose the other terms into

$$(A^j \circ u)\,\partial_j u \;=\; (A^j \circ u \;-\; A^j(0))\,\partial_j u \;+\; A^j(0)\,\partial_j u,$$

where the latter obviously belongs to H^{s-1} and the former is a product of one term in H^s, thanks to Theorem C.12 again, and one term in H^{s-1}: since

$s + (s - 1) - (s - 1) > d/2$ by assumption, Theorem C.10 ensures the product is in H^{s-1}.

Let us now proceed by induction. We take $i \geq 2$ and assume that $u_k \in H^{s-k}(\bar{\Omega})$ for all $k \in \{0, \dots, i-1\}$. Then the first sum in u_i is found to be in H^{s-i} exactly as in the case $i = 1$. In the second term we find only products in $H^s \cdot H^{s-\ell_1} \cdots H^{s-\ell_k}$ (using the same trick as for $A^j(0)$ to cope with the non-zero $\mathrm{d}^k h(0)$), with

$$s + (s - \ell_1) + \cdots + (s - \ell_k) = s + ks - (i - 1) > s - (i - 1) + d/2$$

and $\min(s, s - \ell_1, \dots, s - \ell_k) = s - (i - 1)$, so that those products belong to $H^{s-(i-1)}$ by Theorem C.10. Finally, the quadruple sum in u_i involves products in $H^s \cdot H^{s-\ell_1} \cdots H^{s-\ell_k} \cdot H^{s-i+\ell}$, with

$$s + (s - \ell_1) + \cdots + (s - \ell_k) + (s - i + \ell) = s + ks - \ell + s - i + \ell > 2s - i + d/2$$

and $\min(s, s - \ell_1, \dots, s - \ell_k, s - i + \ell) = s - i + 1$, so that those products belong to H^{s-i} (since $s - i < \min(2s - i, s - i + 1)$).

To find that $C_p(u(x), \mathrm{D}u(x), \dots, \mathrm{D}^p u(x))$ belongs to H^{s-p} we use exactly the same argument as for the double sum in u_i. □

'Approximate solution'

The next technical step towards the proof of Theorem 11.1 consists in reducing the problem to an IBVP with zero initial data. This is done thanks to an appropriate lifting of the initial data u_0.

Lemma 11.2 *Under the assumptions of Theorem 11.1, there exists $T_0 > 0$ and $u_a \in H^{m+1}(\bar{\Omega} \times \mathbb{R})$ vanishing for $|t| \geq 2T_0$ so that*

$$(u_a)_{|t=0} = u_0, \quad \|u_a(x,t) - u_0(x)\| \leq \frac{\rho_0}{2} \quad \textit{for all } (x,t) \in \bar{\Omega} \times [-T_0, T_0],$$

and $f_0 := -L_{u_a} u_a + h \circ u_a$ and $g_0 := -(b \circ u_a)_{|\partial\Omega \times \mathbb{R}} + \underline{b}$ are such that

$$\partial_t^p f_0 \equiv 0, \quad \partial_t^p g_0 \equiv 0 \quad \textit{at } t = 0 \quad \textit{for all } p \in \{0, \dots, m-1\}.$$

Furthermore, f_0 belongs to $H^m(\bar{\Omega} \times \mathbb{R})$, g_0 belongs to $H^m(\partial\Omega \times \mathbb{R})$, and both vanish for $|t| \geq 2T_0$).

Proof By Lemma 11.1 we can construct $u_i \in H^{m+1/2-i}(\bar{\Omega})$ satisfying (11.2.5) for all $i \in \{1, \dots, m-1\}$ with $u = u_0$. Then, by trace lifting (see, for instance, [1], pp. 216–217), we find $u_a \in H^{m+1}(\bar{\Omega} \times \mathbb{R})$ such that $\|u_a\|_{H^{m+1}(\bar{\Omega} \times \mathbb{R})} \lesssim \|u_0\|_{H^{m+1/2}(\bar{\Omega})}$ and

$$\partial_t^i (u_a)_{|t=0} = u_i \quad \text{for all } i \in \{0, \dots, m-1\}.$$

By construction, u_a belongs to $\mathscr{C}(I; H^m(\bar{\Omega}))$ for all compact intervals I of $\mathbb{R}$. This implies, in particular, the existence of T_0 so that

$$\|u_a(t) - u_a(0)\|_{H^m(\bar{\Omega})} \leq \frac{\rho_0}{2\nu_{m,d}}$$

for all $t \in [-T_0, T_0]$, hence

$$\|u_a(t) - u_a(0)\|_{L^\infty(\Omega)} \leq \frac{\rho_0}{2},$$

which means

$$\|u_a(x,t) - u_0(x)\| \leq \frac{\rho_0}{2} \quad \text{for all } (x,t) \in \bar{\Omega} \times [-T_0, T_0].$$

This ensures, in particular, that $u_a(x,t)$ stays in $\mathscr{U}$ for all $(x,t) \in \bar{\Omega} \times [-T_0, T_0]$. Furthermore, multiplying u_a by a suitable $\mathscr{C}^\infty$ cut-off function in time, we may assume without loss of generality that u_a vanishes for $|t| \geq 2T_0$ (or any number greater than T_0) and, thanks to the convexity of $\mathscr{U}$ and the fact that 0 belongs to $\mathscr{U}$, that u_a stays in $\mathscr{U}$ for all $t \in \mathbb{R}$. This precaution allows us to speak about $A^j \circ u_a$, $h \circ u_a$ and $b \circ u_a$ and therefore to define $f_0 := -L_{u_a} u_a + h \circ u_a$ and $g_0 := -(b \circ u_a)_{|\partial\Omega \times \mathbb{R}} + \underline{b}$. That f_0 and g_0 are in H^m follows in a classical way from Proposition C.11 and Theorem C.12. The vanishing of $\partial_t^p f_0$ at $t = 0$ for $p \leq m-1$ follows from the construction – (11.2.5) – of the $u_i = \partial_t^i (u_a)_{|t=0}$. The vanishing of $\partial_t^p g_0$ at $t = 0$ for $p \leq m-1$ is a consequence of the compatibility conditions in (11.1.3) and the definition of the non-linear functions C_p in (11.2.4). Finally, that f_0 and g_0 vanish for $|t| \geq 2T_0$ follows from our 'technical' condition 1) and the fact that u_a does so. □

Once the 'approximate solution' u_a is available, the resolution of the IBVP (11.0.1) is equivalent to the resolution of the IBVP with zero initial data

$$\begin{cases} L_{u_a+v}(u_a+v) = h(u_a+v), & \Omega \times (0,T), \\ b(u_a+v) = \underline{b}, & \partial\Omega \times (0,T), \\ v_{|t=0} = 0, & \Omega. \end{cases} \tag{11.2.6}$$

Iterative scheme

The resolution of (11.2.6) will be done using the natural iterative scheme

$$\begin{cases} L_{u_a+v^k} v^{k+1} = -L_{u_a+v^k} u_a + h(u_a+v^k), & \Omega \times (-\infty, T], \\ B(u_a+v^k) v^{k+1} = B(u_a+v^k) v^k - b(u_a+v^k) + \underline{b}, & \partial\Omega \times (-\infty, T], \\ v^{k+1}_{|t<0} = 0, & \Omega. \end{cases} \tag{11.2.7}$$

In what follows we shall denote by I_T the time interval $(-\infty, T]$.

Let us introduce the following notations, extending those of Lemma 11.1 to a non-zero perturbation v of u_a:

$$f_v = -L_{u_a+v} u_a + h \circ (u_a+v), \quad g_v = B \circ (u_a+v) \cdot v - b \circ (u_a+v) + \underline{b}.$$

These are well-defined provided that $u_a + v$ only achieves values in $\mathscr{U}$: by the estimate of u_a in Lemma 11.1, this will be the case as long as $\|v\|_{L^\infty} \le \rho_0/2$. If v belongs to $H^m(\bar\Omega \times I_T)$ and its trace on $\partial\Omega$ belongs to $H^m(\partial\Omega \times I_T)$ then Proposition C.11 and Theorem C.12 imply that f_v also belongs to $H^m(\bar\Omega \times I_T)$ and g_v belongs to $H^m(\partial\Omega \times I_T)$. In fact, we have the following, more precise result, which will be useful to make the scheme in (11.2.7) work.

Lemma 11.3 *For all $T \in (0, T_0]$, for all $v \in H^m(\bar\Omega \times I_T)$, of norm less than $\rho_0/(2\nu_{m,d+1})$, having a trace in $H^m(\partial\Omega \times I_T)$ and such that $v_{|t<0} \equiv 0$, we have*

$$\partial_t^p (f_v)_{|t=0} = 0 \quad \textit{and} \quad \partial_t^p (g_v)_{|t=0} = 0 \quad \textit{for all} \;\; p \in \{0, ..., m-1\}.$$

Furthermore, for all $M \in (0, \rho_0/(2\nu_{m,d+1}))$ there exist $C_1 = C_1(M)$ and $C_2 = C_2(M)$ so that for all $T \in (0, T_0]$,

$$\|v\|_{H^m(\bar\Omega\times I_T)} \le M$$

implies

$$\|f_v\|_{H^m(\bar\Omega\times I_T)} \le C_1(M) \quad \textit{and} \quad \|g_v\|_{H^m(\partial\Omega\times I_T)} \le T\, C_2(M) + \varepsilon(T),$$

where $\varepsilon(T)$ is independent of M and goes to zero as T goes to zero.

Proof By definition of f_v, the derivative $\partial_t^p(f_v)$ at $t=0$ reduces to the sum of $\partial_t^p (f_0)_{|t=0}$, which is known to be zero by Lemma 11.2, and terms with derivatives of v up to order p in factor, which are zero at $t = 0$ by assumption. Therefore, $\partial_t^p (f_v)_{|t=0} = 0$. Similarly, $\partial_t^p (g_v)_{|t=0} = 0$ follows from the fact that $\partial_t^p (g_0)_{|t=0} = 0$.

Since $m > (d+1)/2$, the estimate of f_v in H^m is a straightforward consequence of Theorem C.12. The estimate of g_v is trickier. By a second-order Taylor expansion of b (recalling that $B = \mathrm{d}b$) we have

$$\begin{aligned} g_v(x,t) = &\int_0^1 (\theta - 1)\, \mathrm{d}^2 b(u_a(x,t) + \theta v(x,t)) \cdot (v(x,t), v(x,t))\, \mathrm{d}\theta \\ &+ \underline{b}(x,t) - b(u_a(x,t)) \end{aligned}$$

for all $(x,t) \in \partial\Omega \times [0, T_0]$. Let us define

$$\varepsilon(T) := \|\underline{b} - b \circ u_a\|_{H^m(\partial\Omega\times I_T)} \lesssim \|\underline{b} - b \circ u_a\|_{\mathscr{C}(I_T; H^{m-1}(\partial\Omega))}.$$

This does go to zero with T since (by construction of u_a) $\underline{b}(x,0) = b(u_a(x,0))$ for all $x \in \partial\Omega$. Regarding the other term in g_v we have, thanks to Proposition C.11 and Theorem C.12,

$$\left\| \int_0^1 (\theta - 1)\, \mathrm{d}^2 b(u_a + \theta v) \cdot (v, v)\, \mathrm{d}\theta \right\|_{H^m(\partial\Omega\times I_T)} \le C'\, \|v\|^2_{L^\infty(\partial\Omega\times I_T)}$$

for some new constant C' depending only on M (the H^m norm of v). Furthermore, since $m-1 > (d+1)/2$ by assumption, we have

$$\|v\|_{L^\infty(\partial\Omega\times I_T)} \leq C'\,\nu_{m-1,d}\,\|v\|_{H^{m-1}(\partial\Omega\times I_T)}.$$

Now, since all time derivatives of v up to order $m-1$ vanish at time $t=0$, we have by the Cauchy–Schwarz inequality

$$\|\partial_t^p\partial_x^\alpha v\|_{L^2(\partial\Omega\times I_T)} \leq T\,\|\partial_t^{p+1}\partial_x^\alpha v\|_{L^2(\partial\Omega\times I_T)}$$

for all $p\in\{0,\dots,m-1\}$ and all d-uple α of length less than or equal to $m-1-p$. This shows that

$$\|v\|_{H^{m-1}(\partial\Omega\times I_T)} \leq T\,\|v\|_{H^m(\partial\Omega\times I_T)},$$

hence the result aimed at, with $C_2 = C'\nu_{m-1,d}M$. □

11.2.2 *Proof of Theorem 11.1*

Construction of the sequence (v^k)

We fix $M\in(0,\rho_0/(2\nu_{m,d+1}))$. We set $v^0 = 0$, and construct v^k by induction. Assume that v^k has been constructed in such a way that for some $T\in(0,T_0]$,

$$\begin{cases} v^k\in H^m(\bar\Omega\times I_T)\,, \quad v^k_{|\partial\Omega}\in H^m(\partial\Omega\times I_T)\,, \\ \text{with } \|v^k\|_{H^m(\bar\Omega\times I_T)} \leq M\,, \\ \text{and } v^k_{|t<0} \equiv 0\,. \end{cases} \tag{11.2.8}$$

Then by Theorem 9.21, the IBVP in (11.2.7) admits a unique solution $v^{k+1}\in H^m(\bar\Omega\times I_T)$ having a trace in $H^m(\partial\Omega\times I_T)$ and satisfying the estimate

$$\frac{1}{T}\|v^{k+1}\|^2_{H^m(\bar\Omega\times I_T)} + \|v^{k+1}_{|x_d=0}\|^2_{H^m(\partial\Omega\times I_T)} \leq C\left(T\|f_{v^k}\|^2_{H^m(\bar\Omega\times I_T)} + \|g_{v^k}\|^2_{H^m(\partial\Omega)}\right),$$

where C depends only on the H^m norm of u_a+v, i.e. depends only on M $(+\|u_a\|_{H^m(\bar\Omega\times\mathbb{R})}$, which is fixed anyway). This implies in, particular,

$$\|v^{k+1}\|_{H^m(\bar\Omega\times I_T)} \leq \sqrt{C}\,\left(T\,\|f_{v^k}\|_{H^m(\bar\Omega\times I_T)} + \sqrt{T}\,\|g_{v^k}\|_{H^m(\partial\Omega)}\right),$$

or using Lemma 11.3,

$$\|v^{k+1}\|_{H^m(\bar\Omega\times I_T)} \leq \sqrt{C}\,\left(T\,C_1(M) + T^{3/2}\,C_2(M) + T^{1/2}\,\varepsilon(T)\right),$$

which we can assume to be less than or equal to M, up to diminishing T. This enables us to construct a whole sequence $(v^k)_{k\in\mathbb{N}}$ satisfying (11.2.8).

Convergence of the sequence (v^k)

From the uniform H^m bound in (11.2.8) we already know there is a subsequence of (v^k) converging weakly in H^m. The next point consists in proving the (whole) sequence converges in L^2. More precisely, we are going to prove that both (v^k) and the sequence of traces on $\partial\Omega$ converge in L^2.

In this respect, we look at the difference $w^k := v^{k+1} - v^k$, which satisfies

$$\begin{cases} L_{u^k}\, w^k \;=\; (L_{u^{k-1}} \;-\; L_{u^k})\, u^k \;+\; h(u^k) \;-\; h(u^{k-1})\,, & \Omega \times (0,T)\,, \\ B(u^k)\, w^k \;=\; B(u^{k-1})\, w^{k-1} \;+\; b(u^{k-1}) \;-\; b(u^k)\,, & \partial\Omega \times (0,T)\,, \\ w^{k+1}_{|t=0} \;=\; 0\,, & \Omega\,, \end{cases}$$

where we have denoted for simplicity $u^k := u_a + v^k$. Hence, by the L^2 estimate (9.2.51) in Theorem 9.18, there exists $c_k > 0$ depending only on $\|u^k\|_{W^{1,\infty}(\bar\Omega\times I_T)}$ so that

$$\begin{aligned} &\frac{1}{T}\,\|w^k\|^2_{L^2(\bar\Omega\times I_T)} \;+\; \|w^k_{|\partial\Omega}\|^2_{L^2(\partial\Omega\times I_T)} \\ &\quad\le\; c_k\, T\, \|(L_{u^{k-1}} \;-\; L_{u^k})\, u^k \;+\; h(u^k) \;-\; h(u^{k-1})\|^2_{L^2(\bar\Omega\times I_T)} \\ &\quad + c_k\, \|B(u^{k-1})\, w^{k-1} \;+\; b(u^{k-1}) \;-\; b(u^k)\|^2_{L^2(\partial\Omega\times I_T)}. \end{aligned}$$

Observing that

$$\|u^k\|_{W^{1,\infty}(\bar\Omega\times I_T)} \;\lesssim\; \|u^k\|_{H^m(\bar\Omega\times I_T)}$$

since $m > (d+1)/2 + 1$, we see the constant c_k depends in fact only on M. Merely by the mean value theorem we obtain $c' > 0$, depending only on the maximum of $\|u^k\|_{L^\infty(\bar\Omega\times I_T)}$ and $\|u^{k-1}\|_{L^\infty(\bar\Omega\times I_T)}$ such that

$$\begin{aligned} &\|(L_{u^{k-1}} \;-\; L_{u^k})\, u^k \;+\; h(u^k) \;-\; h(u^{k-1})\|_{L^2(\bar\Omega\times I_T)} \\ &\quad\le\; c'\,(1 \;+\; \|u^k\|_{W^{1,\infty}(\bar\Omega\times I_T)}\,)\; \|w^{k-1}\|_{L^2(\bar\Omega\times I_T)}. \end{aligned}$$

And by a second-order Taylor expansion of b we find that

$$\begin{aligned} &\|B(u^{k-1})\, w^{k-1} \;+\; b(u^{k-1}) \;-\; b(u^k)\|^2_{L^2(\partial\Omega\times I_T)} \\ &\quad\le\; c''\, \|w^{k-1}\|_{L^\infty(\partial\Omega\times I_T)}\, \|w^{k-1}\|_{L^2(\partial\Omega\times I_T)}, \end{aligned}$$

where c'' depends only on the maximum of $\|u^k\|_{L^\infty(\partial\Omega\times I_T)}$ and $\|u^{k-1}\|_{L^\infty(\partial\Omega\times I_T)}$. Now since $m > d/2$,

$$\|w^{k-1}\|_{L^\infty(\partial\Omega\times I_T)} \lesssim \|w^{k-1}\|_{H^m(\partial\Omega\times I_T)} \le 2\sqrt{C}(\sqrt{T}C_1(M) + TC_2(M) + \varepsilon(T))$$

thanks to Lemma 11.3 and the H^m estimate on $v^k_{|\partial\Omega}$ and $v^{k-1}_{|\partial\Omega}$. So, finally, we have

$$\begin{aligned} &\frac{1}{T}\,\|w^k\|^2_{L^2(\bar\Omega\times I_T)} \;+\; \|w^k_{|\partial\Omega}\|^2_{L^2(\partial\Omega\times I_T)} \\ &\quad\le \widetilde{c}_M\, T\, \|w^{k-1}\|^2_{L^2(\bar\Omega\times I_T)} \;+\; \widetilde{\varepsilon}_M(T)\, \|w^{k-1}_{|\partial\Omega}\|^2_{L^2(\partial\Omega\times I_T)} \end{aligned}$$

for $\widetilde{c}_M$ depending only on M and $\widetilde{\varepsilon}_M(T)$ going to zero as T goes to zero. Consequently, up to diminishing T so that all four numbers

$$\widetilde{\varepsilon}_M(T)\,,\ \widetilde{c}_M\,T\,,\ \widetilde{\varepsilon}_M(T)\,T\,,\ \widetilde{c}_M\,T^2$$

are less than or equal to 1/4, we have

$$\|w^k\|_{L^2(\bar{\Omega}\times I_T)} \leq 2^{-k}\,\|w^0\|_{L^2(\bar{\Omega}\times I_T)} \quad \text{and} \quad \|w^k_{|\partial\Omega}\|_{L^2(\partial\Omega\times I_T)} \leq 2^{-k}\,\|w^0\|_{L^2(\partial\Omega\times I_T)}.$$

This implies both (v^k) and $(v^k_{|\partial\Omega})$ are Cauchy sequences in L^2. Let us call v and $\underline{v}$ their respective limits.

Conclusion

As already said, the limit v of (v^k) is necessarily in $H^m(\bar{\Omega}\times I_T)$. Similarly, because of the uniform H^m bound $(\sqrt{C}\,(\,\sqrt{T_0}\,C_1(M) + T_0\,C_2(M) + \varepsilon(T_0)\,))$ for the traces, the limit $\underline{v}$ of $(v^k_{|\partial\Omega})$ is necessarily in $H^m(\partial\Omega\times I_T)$. Moreover, by L^2–H^m interpolation, (v^k) converges strongly to v in $H^s(\bar{\Omega}\times I_T)$ and $(v^k_{|\partial\Omega})$ converges strongly to $\underline{v}$ in $H^s(\partial\Omega\times I_T)$ for all $s\in[0,m)$. Therefore $\underline{v} = v_{|\partial\Omega}$ and v solves the IBVP (11.2.6), so that $u = u_a + v$ solves the original IBVP (11.0.1). □

12

PERSISTENCE OF MULTIDIMENSIONAL SHOCKS

Shock waves are of special importance in such diverse applications as aero- and gas dynamics, materials sciences, space sciences, geosciences, life sciences and medicine. Research in this field is a century old but still very active, as the existence of a research journal precisely entitled *Shock Waves* attests.

We know very well from everyday experience what shock wave means (especially in gas dynamics). However, it is a mathematical issue to prove the existence and/or the stability of (arbitrarily curved) shocks for general hyperbolic systems, and in particular for the Euler equations.

The formal, linearized stability of shock waves was addressed in the 1940s by several physicists and engineers. Then, the mathematical analysis of the fully non-linear problem waited for the independent works of Majda [124–126] and Blokhin [18] in the 1980s, and was more recently revisited by Métivier and co-workers [56, 133, 136, 140].

Following Freistühler [58, 59] in his work on non-classical shocks, we use here the term *persistence* (in particular in the title of this chapter) as a shortcut for existence-and-stability. Both notions are indeed closely related, and we can view the stability problem as a preliminary step to the existence problem: assume a special shock-wave solution is known (e.g. a planar shock propagating with constant speed, which is not difficult to find); one may ask whether a small initial pertubation (of the shock front and of the states on either side) will destroy its structure, or lead to a solution (local in time) still made of smooth regions separated by a (modified) shock front; when the reference shock falls into the latter case for a sufficiently large set of initial perturbations, it may be called 'structurally stable', and thus serve as a model to construct, in other words to show the *existence* of, a non-planar shock. Alternatively, one may put the problem slightly differently: consider a Cauchy problem where the initial data consist of two smooth regions separated by a given hypersurface; under what conditions (on the initial data) does this Cauchy problem admit a solution made of smooth regions separated by a (moving) shock front? The answer is twofold, as the initial data must satisfy compatibility *and* stability conditions. The necessity of compatibility conditions is easy to understand: even in one space dimension, two arbitrary, uniform states are not connected by a single shock wave in general. (The corresponding Cauchy problem is called a Riemann problem, and its solution involves, in general, as many waves as there are characteristic fields. See, for instance, [24, 46, 88, 184].) Those compatibility conditions come from the Rankine–Hugoniot jump conditions across fronts of discontinuities. Stability conditions are of a different nature: as shown by Majda, the stability of a planar shock wave (propagating with constant speed) is roughly speaking equivalent to the well-posedness of a (non-standard) constant-coefficients Initial Boundary Value Problem, which is itself encoded by the so-called (generalized) Lopatinskiĭ condition. It also turns out that a (arbitrarily curved) shock front is 'structurally stable' provided that all local jump discontinuities along the interface correspond to stable planar shock waves.

The purpose of this chapter is to explain how all this works. Before going into technical details, we can describe roughly the methodology. The general problem is a hyperbolic Cauchy problem with initial data discontinuous across a given hypersurface, and solutions are sought in a class of functions that are smooth on either side of a moving, unknown hypersurface. Thus various difficulties are involved:

- several space dimensions,
- non-linearity,
- free boundary.

The latter can be overcome in a standard way by fixing the free boundary through a change of variables (even though there is some arbitrariness in the choice of this change of variables). If, for instance, the unknown boundary stays close to a hyperplane, the free boundary problem (FBP) is easily changed into a mixed problem, or Initial Boundary Value Problem (IBVP) in a half-space. For clarity, we shall present most of the analysis in that framework, and come only at the end to shock fronts that are (smooth) compact manifolds (as in Majda's memoir [124]). The main novelty compared to Chapter 11 is that we have to deal with *non-standard* IBVP, in which the boundary conditions (coming from the Rankine–Hugoniot jump conditions across the front) are of differential type in the (unknown) front location. In fact, the front location can be eliminated and we get *pseudo-differential* boundary conditions in the main dependent variables, which can be treated almost as standard ones thanks to symbolic calculus. So the main point is the understanding of the elimination step.

The non-linearity of equations of course also plays an important role. In particular, it is in turn responsible for the smallness of the time of existence – as already seen for the Cauchy problem (Chapter 10) and for the regular IBVP (Chapter 11). However, the most important part of the job is in fact *linear*, the main problem being to deal with linear equations with coefficients of limited regularity: once the well-posedness of linearized problems about approximate solutions is proved, the solution of the non-linear problem is obtained (unsurprisingly) through a suitable iterative scheme – 'a straightforward adaptation of the standard proof of short-time existence for smooth solutions of the Cauchy problem', Majda stated [124].

As regards the several space dimensions, they can be tackled as in standard IBVP, by using a Fourier transform in the direction of the boundary, as far as the linearized, frozen coefficients problem is concerned: the resulting normal modes analysis is rather similar to what is done for standard IBVP; see Section 12.2 below.

12.1 From FBP to IBVP

12.1.1 *The non-linear problem*

Consider a system of conservation laws

$$\partial_t f^0(u) + \sum_{j=1}^{d} \partial_j f^j(u) \;=\; 0\,, \tag{12.1.1}$$

and the associated Rankine–Hugoniot condition

$$N_0\,[f^0(u)] \;+\; \sum_{j=1}^{d} N_j\,[f^j(u)] \;=\; 0\,. \tag{12.1.2}$$

A very general problem is the following.

(**FBP**) Find a codimension one surface Σ in $\mathbb{R}^d \times [0,T]$, splitting $(\mathbb{R}^d \times [0,T])\backslash\Sigma$ in two connected component Ω_- and Ω_+, and find u such that $u_{|\Omega_\pm} \in \mathscr{C}^1(\Omega_\pm)$ satisfy (12.1.1) in $\Omega_\pm$ and (12.1.2) – plus some admissibility condition, to be specified later – across Σ, with N_0, N_1, $\dots$, N_d being the components in the directions t, x_1,$\dots$, x_d of a vector N orthogonal to Σ, and the brackets $[\cdot]$ stand for jumps:

$$[f^j(u)](x,t) = \lim_{\varepsilon\searrow 0} \left(f^j(u((x,t)+\varepsilon N(x,t))) - f^j(u((x,t)-\varepsilon N(x,t)))\right),$$
$$(x,t) \in \Sigma\,.$$

We readily see that a necessary condition for having a solution to **(FBP)** is that the vector

$$\sum_{j=1}^{d} n_j\,[f^j(u_{|t=0})],$$

where $n_1, \dots, n_d$ are the components of a normal vector to the initial shock front $\Sigma_{|t=0}$, is parallel to $[f^0(u_{|t=0})]$. This is the first, natural compatibility condition.

There are many (almost) trivial solutions to **(FBP)**, which correspond to planar shock waves, with Σ a fixed hyperplane – corresponding to the propagation at constant speed of a hyperplane in the physical space – and $u_{|\Omega_\pm} \equiv u_\pm$ independent of t and x. The derivation of planar shock waves is mostly algebraic. Indeed, choose a direction of propagation, for instance x_d, and consider the so-called *Hugoniot set* passing through a reference state $w \in \mathscr{U}$ (the domain in $\mathbb{R}^n$

where the flux functions f^j are well-defined and smooth):

$$\mathscr{H}(w) := \{ u \in \mathscr{U} \,;\, f^d(u) - f^d(w) \parallel f^0(u) - f^0(w) \} .$$

(This set is studied in detail in Chapter 13 for the Euler equations of 'real' gas dynamics.) Assuming, for instance, the strict hyperbolicity of (12.1.1), we can easily show that $\mathscr{H}(w)$ is locally the union of n curves. Take u_+ on any of these curves and $u_- = w$. By definition of $\mathscr{H}(w)$ there exists $\sigma \in \mathbb{R}$ such that $f^d(u_+) - f^d(u_-) = \sigma (f^0(u_+) - f^0(u_-))$. Then take $\Sigma := \{ (x_1, \ldots, x_d, t) \,;\, x_d = \sigma t \}$ and you get a planar 'shock' wave propagating at speed σ in the direction x_d. At this stage, the 'shock' might be a contact discontinuity or any other kind of discontinuous wave. (We postpone, on purpose, the discussion of the admissibility of discontinuities.)

It is much more difficult in general to find solutions to **(FBP)** with non-planar Σ. The aim of this chapter is to show that such solutions do exist, for T small enough, provided we choose initial data satisfying: 1) *compatibility* conditions and 2) *stability* conditions. The compatibility conditions imply, in particular, that initially, at each point of the front, the states on either side of the front are connected by a planar shock wave in the direction normal to the front. The stability conditions require additionally that this shock be *uniformly stable* (in the sense of the uniform Kreiss–Lopatinskiĭ condition) with respect to multidimensional perturbations. We shall make this more precise below.

12.1.2 *Fixing the boundary*

For clarity, in what follows we seek solutions close to a *planar* reference shock. The case of arbitrary (compact) fronts of discontinuities (actually dealt with by Majda in [124]) is postponed to the end of this chapter.

Provided that the system (12.1.1) is invariant under rotation – again this is the case for the Euler equations of gas dynamics, for instance – we may choose co-ordinates, without loss of generality such that the direction of propagation of the reference shock is the last co-ordinate x_d. Then this reference shock can be represented by a mapping

$$\begin{array}{rl} \underline{u} \,:\, (\mathbb{R}^d \times \mathbb{R}^+) \backslash \underline{\Sigma} & \to \mathbb{R}^n \\ (x,t) & \mapsto \underline{u}_\pm \ \text{ if } \ x_d \gtrless \sigma t , \end{array}$$

with $\underline{\Sigma} := \{ (x_1, \ldots, x_d, t) \,;\, x_d = \sigma t \}$, and $f^d(\underline{u}_+) - f^d(\underline{u}_-) = \sigma (f^0(\underline{u}_+) - f^0(\underline{u}_-))$. We look for a perturbed shock

$$\begin{array}{rl} u \,:\, (\mathbb{R}^d \times \mathbb{R}^+) \backslash \Sigma & \to \mathbb{R}^n \\ (x,t) & \mapsto u(x,t) = u_\pm(x,t) \ \text{ if } \ x_d \gtrless \chi(x_1, \ldots, x_{d-1}, t) , \end{array}$$

with $\Sigma := \{ (x_1, \ldots, x_d, t) \,;\, x_d = \chi(x_1, \ldots, x_{d-1}, t) \}$ the perturbed shock front (supposedly close to $\underline{\Sigma}$), the unknowns $u_\pm$ and χ being such that both the interior equations (12.1.1) and the jump conditions (12.1.2) are satisfied.

It will turn out that u exists at least on a small finite time interval[1] $[0,T]$, if the reference shock is uniformly stable (in a sense to be specified later). For the proof, and even for the precise statement of this result, we need some more material.

The set of equations for $u_\pm$ and χ is

$$\begin{cases} \partial_t f^0(u_\pm) + \displaystyle\sum_{j=1}^{d} \partial_j f^j(u_\pm) \,=\, 0 & \text{for } x_d \gtrless \chi(x_1,\dots,x_{d-1},t)\,, \\ [f^0(u)]\,\partial_t\chi + \displaystyle\sum_{j=1}^{d-1} [f^j(u)]\,\partial_j\chi - [f^d(u)] \,=\, 0 & \text{at } x_d = \chi(x_1,\dots,x_{d-1},t)\,. \end{cases}$$

We introduce our usual shortcut $y := (x_1,\dots,x_{d-1})$ for the independent variables along the boundary. As long as the function χ stays close – in the class of $\mathscr{C}_b^1$ functions say – to the reference function $(y,t) \mapsto \sigma\, t$, both mappings

$$\Phi_\pm \,:\, (y,x_d,t) \mapsto (\,y\,,\,\pm(x_d - \chi(y,t))\,,\,t\,)$$

are diffeomorphisms from $\Omega_\pm = \{\,(y,x_d,t)\,;\, x_d \gtrless \chi(y,t)\}$ to the half-space

$$\Omega := \{\,(y,z,t)\,;\, z > 0\},$$

and both Φ_- and Φ_+ map Σ to the hyperplane $\{\,z = 0\,\}$. From now on, we replace both unknowns $u_\pm$ by $u_\pm \circ \Phi_\pm^{-1}$, which we still denote by $u_\pm$ for simplicity, and consider the IBVP on Ω obtained by this (unknown) change of variables for the *double-size* unknown $u = (u_-, u_+)$.

Notations Denoting (slightly differently from Chapter 10)

$$A^j(w) = \mathrm{d}f^j(w) \quad \text{for all } j \in \{0,\dots,d\} \quad \text{and } w \in \mathscr{U},$$

we introduce, for $\chi : (y,t) \mapsto \chi(y,t) \in \mathbb{R}$ at least differentiable once,

$$A_d(w,\mathrm{d}\chi) := A^d(w) - \sum_{j=1}^{d-1} (\partial_j\chi)\, A^j(w) - (\partial_t\chi)\, A^0(w).$$

This is only a shortcut for

$$A(w,-\partial_t\chi,-\partial_1\chi,\dots,-\partial_{d-1}\chi,1) \quad \text{where} \quad A(w,\xi) := \sum_{j=0}^{d} \xi_j\, A^j(w)$$

denotes the (generalized) characteristic matrix. For convenience, in what follows we denote indifferently ∂_0 or ∂_t the derivative with respect to t in the (y,z,t)-variables and for all $j \in \{1,\dots,d-1\}$, ∂_j now stands for the derivation with respect to y_j in the (y,z,t)-variables, as long as no confusion can occur.

[1]As said before, the short-time existence is due to the (high) non-linearity of the problem.

In these new variables, the interior equations read

$$\sum_{j=0}^{d-1} A^j(u_\pm)\,\partial_j u_\pm \,\pm\, A_d(u_\pm, \mathrm{d}\chi)\,\partial_z u_\pm \,=\, 0\,, \quad \text{for } z > 0\,, \tag{12.1.3}$$

and the boundary conditions are

$$\sum_{j=0}^{d-1} (\, f^j(u_+) \,-\, f^j(u_-)\,)\,\partial_j \chi \,=\, (\, f^d(u_+) \,-\, f^d(u_-)\,) \quad \text{at } z \,=\, 0\,. \tag{12.1.4}$$

Observe that the BVP in (12.1.3) and (12.1.4) for the unknown $(u_\pm, \chi)$ involves the unknown funtion $\mathrm{d}\chi$ both in the interior equations (12.1.3) and in the boundary conditions (12.1.4).

Remark 12.1 There has been some arbitrariness in the way we have changed the domains $\{\,(y, x_d, t)\,;\; x_d \gtrless \chi(y,t)\,\}$ into a half-space; for a discussion of alternatives, see [56]. One specific problem is that the resulting interior equations in (12.1.3) depend on (y,t) – through the derivatives of χ – for all values of x_d. In other words, the flattening of the boundary influences the far-field behaviour of equations. As pointed out by Métivier [136], an alternative consists in localizing the change of variables around the boundary. (A similar trick was used by Majda for compact boundaries, see Section 12.4.2.) More precisely, if $\varphi \in \mathscr{D}(\mathbb{R})$ is a positive cut-off function equal to 1 in say $[0,1]$, one may choose a positive κ, depending only on $\|\varphi'\|_{L^\infty}$ and $\|\chi\|_{L^\infty(\mathbb{R}^{d-1}\times[0,T])}$ so that

$$\Psi_\pm \,:\, (y,z,t) \,\mapsto\, (\,y\,,\, x_d \,=\, \pm\kappa\, z \,+\, \varphi(z)\,\chi(y,t)\,,\, t\,)$$

are diffeomorphisms from

$$\{\,(y,z,t)\,;\; z \,>\, 0\,,\; t \in (-T,T)\,\}$$

to

$$\{\,(y,x_d,t)\,;\; x_d \,\gtrless\, \chi(y,t)\,,\; t \in (-T,T)\,\},$$

and $\Psi_\pm$ both map the hyperplane $\{\,z \,=\, 0\,\}$ to the front $\{\,(y, x_d, t)\,;\; x_d \,=\, \chi(y,t)\,\}$. This choice does not alter the boundary conditions in (12.1.4), and the corresponding interior equations (which involve only the derivatives of $\Psi_\pm$) are (almost) the same (up to trivial rescaling and symmetry $z \mapsto x_d = \pm\kappa x$) as the original ones for z large enough. So this overcomes the problem of the far-field behaviour. The drawback is that interior equations look more complicated: for this reason we shall perform most of the analysis with the diffeomorphisms $\Phi_\pm^{-1}$ instead of $\Psi_\pm$.

12.1.3 *Linearized problems*

Once the non-linear problem is set on a fixed domain Ω, it can reasonably be linearized – the other way round would not have been possible. As we have in mind an iterative scheme, we are going to linearize about a special solution that is not necessarily the (planar) reference one. Substituting $u + \varepsilon\,\dot{u}$ for u and $\chi + \varepsilon\,\dot{\chi}$ for χ in (12.1.3) and (12.1.4), differentiating these equations with respect to ε and evaluating at $\varepsilon = 0$, we get the linearized problem

$$\begin{aligned}
&\sum_{j=0}^{d-1} A^j(u_\pm)\,\partial_j \dot{u}_\pm \,\pm\, A_d(u_\pm, \mathrm{d}\chi)\,\partial_z \dot{u}_\pm \\
&\quad + \sum_{j=0}^{d-1} (\mathrm{d}A^j(u_\pm)\cdot \dot{u}_\pm)\,\partial_j u_\pm \,\pm\, (\mathrm{d}_u A_d(u_\pm, \mathrm{d}\chi)\cdot \dot{u}_\pm)\,\partial_z u_\pm \\
&\quad \mp \sum_{j=0}^{d-1} (\partial_j \dot{\chi})\,A^j(u_\pm)\,\partial_z u_\pm \,=\, 0 \qquad \text{for } z > 0\,,
\end{aligned} \tag{12.1.5}$$

with the boundary conditions

$$\sum_{j=0}^{d-1} (\,f^j(u_+) \,-\, f^j(u_-)\,)\,\partial_j \dot{\chi} \,=\, A_d(u_+, \mathrm{d}\chi)\cdot \dot{u}_+ \,-\, A_d(u_-, \mathrm{d}\chi)\cdot \dot{u}_- \tag{12.1.6}$$

at $z = 0$.

Of course, *if* $u_\pm = \underline{u}_\pm$ are constant and $\chi = \underline{\chi}$, with $\underline{\chi}(y,t) := \sigma\,t$, (12.1.5) and (12.1.6) simplify and become

$$\sum_{j=0}^{d-1} A^j(\underline{u}_\pm)\,\partial_j \dot{u}_\pm \,\pm\, A_d(\underline{u}_\pm, \sigma, 0)\,\partial_z \dot{u}_\pm \,=\, 0\,, \tag{12.1.7}$$

$$\sum_{j=0}^{d-1} (\,f^j(\underline{u}_+) \,-\, f^j(\underline{u}_-)\,)\,\partial_j \dot{\chi} \,=\, A_d(\underline{u}_+, \sigma, 0)\cdot \dot{u}_+ \,-\, A_d(\underline{u}_-, \sigma, 0)\cdot \dot{u}_-\,. \tag{12.1.8}$$

Observe that the derivatives of $\dot{\chi}$ appear here only in the boundary conditions (12.1.8), whereas for the general problem (12.1.5) and (12.1.6) the derivatives of $\dot{\chi}$ do appear in the interior equations (unless we use Alinhac's trick, as described below).

The messy appearance of (12.1.5) and (12.1.6) can be lessened by using additional shorter notations. Let us first define the differential operators L_- and L_+ by

$$L_\pm(v, \mathrm{d}\chi) \,=\, \sum_{j=0}^{d-1} A^j(v)\,\partial_j \,\pm\, A_d(v, \mathrm{d}\chi)\,\partial_z.$$

Note: with these notations, the non-linear equations in (12.1.3) equivalently read

$$L_\pm(u_\pm, \mathrm{d}\chi)\,u_\pm \,=\, 0.$$

We also introduce a shortcut for the zeroth-order terms in (12.1.5):

$$D_\pm(v,\mathrm{d}\chi)\cdot\dot u \,=\, \sum_{j=0}^{d-1}(\mathrm{d}A^j(v)\cdot\dot u)\,\partial_j v \,\pm\, (\mathrm{d}_u A_d(v,\mathrm{d}\chi)\cdot\dot u)\,\partial_z v.$$

Then the interior equations in (12.1.5) read

$$L_\pm(u_\pm,\mathrm{d}\chi)\dot u_\pm \,+\, D_\pm(u_\pm,\mathrm{d}\chi)\cdot\dot u_\pm \,\mp\, A(u_\pm,\partial_t\dot\chi,\partial_1\dot\chi,\dots,\partial_{d-1}\dot\chi,0)\,\partial_z u_\pm \,=\, 0. \tag{12.1.9}$$

As pointed out by Alinhac [5], there is a way to replace the derivatives of $\dot\chi$ by zeroth-order contributions of $\dot\chi$ in these interior equations. This is done by changing the unknown $\dot u_\pm$ to the so-called *good unknown*

$$\dot v_\pm \,:=\, \dot u_\pm \,\mp\, \dot\chi\,\partial_z u_\pm.$$

Proposition 12.1 *With all the notations introduced above,*

$$\begin{aligned} &L_\pm(u_\pm,\mathrm{d}\chi)\dot u_\pm \,+\, D_\pm(u_\pm,\mathrm{d}\chi)\cdot\dot u_\pm \,\mp\, A(u_\pm,\partial_t\dot\chi,\partial_1\dot\chi,\dots,\partial_{d-1}\dot\chi,0)\,\partial_z u_\pm \\ &\quad = L_\pm(u_\pm,\mathrm{d}\chi)\dot v_\pm \,+\, D_\pm(u_\pm,\mathrm{d}\chi)\cdot\dot v_\pm \,+\, \dot\chi\,\partial_z\big(\,L_\pm(u_\pm,\mathrm{d}\chi)\,u_\pm\,\big)\end{aligned}$$

for all $u_\pm$, χ, $\dot u_\pm$, $\dot\chi$ for which both sides make sense.

Proof The computations for either one of the signs + or − are similar. We do it for the + sign, and to facilitate the reading we omit the subscript +. By definition of $\dot v \,=\, \dot u \,-\, \dot\chi\,\partial_z u$,

$$L(u,\mathrm{d}\chi)\dot v = L(u,\mathrm{d}\chi)\dot u - \dot\chi\,\partial_z\big(\,L(u,\mathrm{d}\chi)\,u,\big) + \dot\chi\,[\partial_z,\,L(u,\mathrm{d}\chi)]\,u - (L(u,\mathrm{d}\chi)\dot\chi)\partial_z u$$

and

$$D(u,\mathrm{d}\chi)\cdot\dot v \,=\, D(u,\mathrm{d}\chi)\cdot\dot u \,-\, \dot\chi\,D(u,\mathrm{d}\chi)\cdot(\partial_z u).$$

Summing these two equalities, and reordering terms we get

$$\begin{aligned} &L(u,\mathrm{d}\chi)\dot v \,+\, D(u,\mathrm{d}\chi)\cdot\dot v \,+\, \dot\chi\,\partial_z\big(\,L(u,\mathrm{d}\chi)\,u\,\big) \\ &\quad = L(u,\mathrm{d}\chi)\dot u \,+\, D(u,\mathrm{d}\chi)\cdot\dot u \,-\, (\,L(u,\mathrm{d}\chi)\,\dot\chi\,)\,\partial_z u \\ &\qquad + \dot\chi\,[\,\partial_z\,,\,L(u,\mathrm{d}\chi)\,]\,u \,-\, \dot\chi\,D(u,\mathrm{d}\chi)\cdot(\partial_z u).\end{aligned}$$

By definition of $L(u,\mathrm{d}\chi)$ and $D(u,\mathrm{d}\chi)$, the last two terms cancel out, and

$$L(u,\mathrm{d}\chi)\,\dot\chi \,=\, A(u,\partial_t\dot\chi,\partial_1\dot\chi,\dots,\partial_{d-1}\dot\chi,0),$$

hence the claimed equality. □

Therefore, (12.1.9) equivalently reads

$$L_\pm(u_\pm,\mathrm{d}\chi)\dot v_\pm \,+\, D_\pm(u_\pm,\mathrm{d}\chi)\cdot\dot v_\pm \,+\, \dot\chi\,\partial_z\big(\,L_\pm(u_\pm,\mathrm{d}\chi)u_\pm\,\big) \,=\, 0\,. \tag{12.1.10}$$

This is a first-order linear PDE in $(\dot v_\pm, \dot\chi)$, of principal part $L_\pm(u_\pm, \mathrm{d}\chi)\dot v_\pm$. The actual PDE on $\dot\chi$ comes from the boundary conditions in (12.1.6). We introduce the notation

$$B(u_-, u_+) \cdot (\xi_0, \ldots, \xi_d) \,=\, \sum_{j=0}^{d} \xi_j \,(\, f^j(u_+) \,-\, f^j(u_-)\,)\,,$$

in such a way that the non-linear boundary conditions (12.1.4) merely read

$$B(u_-, u_+) \cdot (\mathrm{d}\chi, -1) \,=\, 0,$$

where $\mathrm{d}\chi$ has been identified with the row vector $(\partial_t\chi, \partial_1\chi, \ldots, \partial_{d-1}\chi)$. The linearized boundary conditions (12.1.6) thus read

$$B(u_-, u_+) \cdot (\mathrm{d}\dot\chi, 0) \,=\, A_d(u_+, \mathrm{d}\chi) \cdot \dot u_+ \,-\, A_d(u_-, \mathrm{d}\chi) \cdot \dot u_-\,,$$

which we shorten even more into

$$b(u, \mathrm{d}\dot\chi) \,+\, M(u, \mathrm{d}\chi) \cdot (\dot u_-, \dot u_+) \,=\, 0\,, \tag{12.1.11}$$

with the obvious definitions

$$b(u, \mathrm{d}\dot\chi) \,:=\, B(u_-, u_+) \cdot (\mathrm{d}\dot\chi, 0),$$

$$M(u, \mathrm{d}\chi) \cdot (\dot u_-, \dot u_+) \,:=\, A_d(u_-, \mathrm{d}\chi) \cdot \dot u_- \,-\, A_d(u_+, \mathrm{d}\chi) \cdot \dot u_+.$$

12.2 Normal modes analysis

In the 1980s, Majda showed how to extend Kreiss' method to non-standard BVPs associated with the shock-persistence problem (here (12.1.3) and (12.1.4)), or more precisely their linearized versions (here (12.1.10) and (12.1.11)). As for standard BVP, a crucial, preliminary step is the so-called normal analysis of the constant coefficients problems (here (12.1.7) and (12.1.8)).

The main purpose of this whole section is to describe the normal modes analysis for (12.1.7) and (12.1.8). To simplify the writing, *in this section* we omit underlining the states of the reference planar shock: u_- and u_+ are to be understood as $\underline{u}_-$ and $\underline{u}_+$ in what follows.

12.2.1 *Comparison with standard IBVP*

In order to derive a generalized version of the Kreiss–Lopatinskiĭ condition we look for special solutions, or 'normal modes' of (12.1.7) and (12.1.8). This amounts to applying to (12.1.7) and (12.1.8) a Laplace transform in t and Fourier transform in y. The resulting equations are, if $\dot U_\pm$ and $\dot X$ denote the Fourier–Laplace transforms of $\dot u_\pm$ and $\dot\chi$, respectively,

$$A(u_\pm, \tau, i\eta, 0)\, \dot U_\pm \,\pm\, A_d(u_\pm, \sigma, 0)\, \partial_z \dot U_\pm \,=\, 0 \quad \text{ for } \; z \,>\, 0\,, \tag{12.2.12}$$

$$\dot X\, b(u, \tau, i\eta) \,-\, M(u, \sigma, 0) \cdot (\dot U_-, \dot U_+) \,=\, 0 \quad \text{ at } \; z \,=\, 0\,, \tag{12.2.13}$$

where we have used the shortcuts defined in the previous section, τ is the complex variable dual to t and $\eta \in \mathbb{R}^{d-1}$ is the wave vector associated to y. Furthermore, we have assumed that $\dot{u}_\pm$ and $\dot{\chi}$ equal zero at $t=0$: otherwise, there would have been a right-hand side in both (12.2.12) and (12.2.13); we shall introduce right-hand sides later.

We now start to make some assumptions. First, we suppose that the planar shock of speed σ between u_- and u_+ is not characteristic. This precisely means that both matrices

$$A_d(u_\pm, \sigma, 0) \;=\; A^d(u_\pm) \;-\; \sigma\, A^0(u_\pm)$$

are non-singular, so that the equations in (12.2.12) form a system of $2n$ independent differential equations, which can be rewritten as

$$\frac{\mathrm{d}\dot{U}}{\mathrm{d}z} \;=\; \mathcal{A}(u,\eta,\tau)\,\dot{U},$$

with

$$\dot{U} \;:=\; \begin{pmatrix} \dot{U}_- \\ \dot{U}_+ \end{pmatrix},$$

and

$$\mathcal{A}(u,\eta,\tau) := \begin{pmatrix} A_d(u_-,\sigma,0)^{-1}\,A(u_-,\tau,i\eta,0) & 0 \\ 0 & -\,A_d(u_+,\sigma,0)^{-1}\,A(u_+,\tau,i\eta,0) \end{pmatrix}.$$

The second assumption, which will turn out to be a consequence of a stronger one, is that $\dot{X}$ can be eliminated from the boundary condition (12.2.13). This amounts to requiring the *ellipticity* of the symbol $(\tau,\eta) \mapsto b(u,\tau,i\eta)$, that is,

$$b(u,\tau,i\eta) \;\neq\; 0 \quad \text{for all} \quad (\tau,\eta) \;\neq\; (0,0),$$

or equivalently that the jump vectors $[f^0(u)]$, $[f^1(u)]$, ..., $[f^{d-1}(u)]$ be independent in $\mathbb{R}^n$.

Remark 12.2 A byproduct of the ellipticity assumption on the symbol b is the necessary condition $d \leq n$ (i.e. the space dimension smaller than the size of the system). This precludes in particular *multidimensional scalar* conservation laws! Another observation is that ellipticity is obviously not uniform when the shock strength goes to zero: this problem was pointed out by Métivier [133], and overcome in [56, 133].

Introducing

$$\Pi(u,\tau,\eta) \;=\; I_n \;-\; \frac{b(u,\tau,i\eta)\,b(u,\tau,i\eta)^*}{\|b(u,\tau,i\eta)\|^2}$$

the orthogonal projection onto $b(u,\tau,i\eta)^\perp$, we see that the boundary condition in (12.2.13) is equivalent to

$$\Pi(u,\tau,\eta)\,M(u,\sigma,0)\,\dot U \;=\;0 \quad\text{and}\quad \dot X \;=\; \frac{b(u,\tau,i\eta)^*\,M(u,\sigma,0)\,\dot U}{|b(u,\tau,i\eta)|^2}$$

(where $M(u,\sigma,0)\,\dot U$ stands for $M(u,\sigma,0)\cdot(\dot U_-,\dot U_+)$, with a slight abuse of notation).

Therefore, solving the problem (12.2.12) and (12.2.13) is equivalent to solving the pure boundary value problem:

$$\begin{cases} \dfrac{\mathrm{d}\dot U}{\mathrm{d}z} \;=\; \mathcal{A}(u,\eta,\tau)\,\dot U & \text{for}\;\; z>0\,,\\[2ex] \Pi(u,\tau,\eta)\;M(u,\sigma,0)\,\dot U \;=\;0 & \text{at}\;\; z=0\,. \end{cases} \tag{12.2.14}$$

It is worth pausing for a while, and compare (12.2.14) to

$$\begin{cases} \dfrac{\mathrm{d}\dot V}{\mathrm{d}z} \;=\; -\,A^d(v)^{-1}\,A(v,\tau,i\eta,0)\dot V & \text{for}\;\; z>0\,,\\[2ex] C(v)\cdot\dot V\;=\;0 & \text{at}\;\; z=0\,, \end{cases}$$

derived by Fourier–Laplace transform from a standard IBVP

$$\begin{cases} \displaystyle\sum_{j=0}^{d-1} A^j(v)\,\partial_j\dot v \;+\; A^d(v)\,\partial_z\dot v \;=\;0 \quad \text{for}\;\; z>0\,,\\[2ex] C(v)\cdot\dot v\;=\;0 \qquad \text{at}\;\; z=0 \qquad\quad \text{and}\quad \dot v\;=\;0\quad\text{at}\;\; t=0\,. \end{cases}$$

Except for the size of the interior system (which is doubled in (12.2.14)), the main difference is that the boundary condition in (12.2.14) contains the 'frequencies' τ and η. This reflects the fact that the corresponding IBVP (obtained by an inverse Fourier–Laplace transform) has *pseudo-differential* boundary conditions. Despite this non-standard feature, one may derive a (generalized) Kreiss–Lopatinskiĭ condition by looking for solutions $\dot U\in L^2(\mathbb{R}^+)$ of (12.2.14). This is part of the normal modes analysis. In fact, the derivation of the *uniform* Kreiss–Lopatinskiĭ condition also requires the *neutral* modes analysis: neutral modes correspond to solutions of (12.2.14) for $\operatorname{Re}\tau\;=\;0$ that are not necessarily square-integrable and may oscillate in the z-direction, but not all of them are to be considered (this should be clear from Chapter 4, see also the discussion below).

As far as the normal modes analysis is concerned, we may avoid the cumbersome projection $\Pi(u,\tau,\eta)$ and work with the full system (12.2.12) and (12.2.13) on $(\dot U,\dot X)$.

A first question is, which values of (η,τ) ensure the hyperbolicity of the matrix $\mathcal{A}(u,\eta,\tau)$? A preliminary answer is, if the operators $L_\pm(u_\pm,\sigma,0)$ are hyperbolic in the t-direction (or equivalently, the operator $A^0(u)\,\partial_t\;+\;\sum_{j=1}^d A^j(u)\partial_j$ is hyperbolic in the t-direction, which also means the system of conservation laws (12.1.1) is hyperbolic) and if the $n \times n$ matrices $A_d(u_\pm, \sigma, 0)$ are non-singular (which we have already assumed), $\mathcal{A}(u, \eta, \tau)$ is hyperbolic at least for $\mathrm{Re}\ \tau > 0$. Indeed, the set of eigenvalues of the $2n \times 2n$ matrix $\mathcal{A}(u, \eta, \tau)$ is obviously the union of the eigenvalues of the $n \times n$ matrices

$$\mathcal{A}_\pm(u_\pm, \eta, \tau) \;:=\; \mp A_d(u_\pm, \sigma, 0)^{-1}\, A(u_\pm, \tau, i\eta)\,,$$

and the hyperbolicity of $L_\pm(u_\pm, \sigma, 0)$ prevents $\mathcal{A}_\pm(u_\pm, \eta, \tau)$ from having purely imaginary eigenvalues when τ is not purely imaginary itself; as already mentioned in Chapter 9, this observation dates back to Hersh [83].

From now on, we assume that the system of conservation laws (12.1.1) is hyperbolic and that the matrices $A_d(u_\pm, \sigma, 0)$ are non-singular.

The next question concerns the dimension of the stable subspace of $\mathcal{A}(u, \eta, \tau)$. The answer lies in the decomposition

$$\begin{aligned} E^s(\mathcal{A}(u, \eta, \tau)) &\simeq E^s(\mathcal{A}_-(u_-, \eta, \tau)) \times E^s(\mathcal{A}_+(u_-, \eta, \tau)) \\ &= E^s\big(A_d(u_-, \sigma, 0)^{-1} A(u_-, \tau, i\eta)\big) \times E^u\big(A_d(u_+, \sigma, 0)^{-1} A(u_+, \tau, i\eta)\big). \end{aligned}$$

The dimension of these spaces is constant over the connected set $\{\, (\tau, \eta)\,;\ \mathrm{Re}\ \tau > 0\,,\ \eta \in \mathbb{R}^{d-1}\,\}$. So it can be computed at $(\tau, 0)$. We easily see that a complex number ω is an eigenvalue of

$$A_d(v, \sigma, 0)^{-1}\, A(v, \tau, 0) \;=\; \tau\ (\, A^d(v) \;-\; \sigma\, A^0(v)\,)^{-1}\, A^0(v)$$

if and only if $\omega \;=\; \tau/(\lambda - \sigma)$, where λ is a root of

$$\det(\, A^d(v) \;-\; \lambda\, A^0(v)\,) \;=\; 0,$$

that is, λ is a *characteristic speed* of the operator

$$A^0(v)\, \partial_t \;+\; A^d(v) \partial_d.$$

More precisely, we have the following.

Proposition 12.2 *Assume that the operator*

$$L \;=\; A^0\, \partial_t \;+\; A^d \partial_d$$

is hyperbolic in the t-direction, and that both A^0 and $A^d \;-\; \sigma\, A^0$ are non-singular. Then the dimension of the stable subspace $E^s\big(\,(\, A^d \;-\; \sigma\, A^0\,)^{-1}\, A^0\,\big)$ is equal to the number, counted with multiplicity, of characteristic speeds of L less than σ.

Proof The hyperbolicity of L means that there exist a real diagonal matrix Λ and a real non-singular matrix P such that $A^d\, P \;=\; A^0\, P\, \Lambda$. Since $A^d \;-\; \sigma\, A^0$ is non-singular, we may assume without loss of generality that Λ splits into a first block, say of size k, with coefficients less than σ, and another one with coefficients greater than σ. By definition,

$$E^s\big(\,(\, A^d \;-\; \sigma\, A^0\,)^{-1}\, A^0\,\big) \;=\; \{\, h \;\in\; \mathbb{C}^n\,;\ \lim_{x \to +\infty}\ \mathrm{e}^{x(\, A^d \,-\, \sigma\, A^0\,)^{-1} A^0}\ h \;=\; 0\,\}.$$

Denoting by $\phi(x,h) := \mathrm{e}^{x(A^d - \sigma A^0)^{-1}A^0}\, h$ the flow of the differential equation

$$u' = (A^d - \sigma A^0)^{-1} A^0 u,$$

we see that $\varphi := P^{-1}\phi$ is the flow of

$$u' = (\Lambda - \sigma I_n)^{-1} u.$$

Indeed, since A^0 is non-singular,

$$(A^d - \sigma A^0)\,\partial_x\phi = A^0\phi \quad \Leftrightarrow \quad (\Lambda - \sigma I_n)\,\partial_x\varphi = \varphi.$$

Therefore, h belongs to $E^s\big((A^d - \sigma A^0)^{-1} A^0\big)$ if and only if $P^{-1}h$ belongs to $E^s(\Lambda - \sigma I_n)^{-1}$, which is clearly of dimension k. □

Corollary 12.1 *Assume that* (12.1.1) *is constantly hyperbolic. Then for all* $(\tau,\eta) \in \mathbb{C}\times\mathbb{R}^{d-1}$ *with* $\mathrm{Re}\,\tau > 0$*, the dimension of* $E^s(\mathcal{A}(u,\eta,\tau))$ *is equal to the number of characteristics* exiting *the shock, that is, the number of characteristic speeds of* $A^0(u_-)\,\partial_t + A^d(u_-)\partial_d$ *less than* σ *plus the number of characteristic speeds of* $A^0(u_+)\,\partial_t + A^d(u_+)\partial_d$ *greater than* σ*, all counted with multiplicity.*

12.2.2 *Nature of shocks*

The result of Corollary 12.1 is illustrated in Fig. 12.1 for a Lax shock, in which the dimension of $E^s(\mathcal{A}(u,\eta,\tau))$ is $(p-1)+n-p = n-1$. Indeed, we recall that Lax shocks [109] are defined as follows.

Definition 12.1 *Assume that* (12.1.1) *is constantly hyperbolic, and denote by*

$$\lambda_1(u,\nu) \le \cdots \le \lambda_n(u,\nu)$$

its characteristic speeds, that is, the roots (repeated according to their multiplicities) of $\det(A(u,\nu) - \lambda A^0(u))$*. A planar shock wave between* u_- *and* u_+*, propagating with speed* σ *in some direction* $\nu \in \mathbb{R}^d$*, is called a Lax shock if there is some integer* $p \in \{1,\dots,n\}$ *such that the* Lax shock inequalities

$$\lambda_p(u_+,\nu) < \sigma < \lambda_p(u_-,\nu) \quad \text{and} \quad \lambda_{p-1}(u_-,\nu) < \sigma < \lambda_{p+1}(u_+,\nu) \qquad (12.2.15)$$

are satisfied. (By convention, $\lambda_0 = -\infty$ *and* $\lambda_{n+1} = +\infty$ *when* $p=1$ *or* $p=n$*.)*

Remark 12.3 In space dimension $d=1$, a shock wave of the form

$$u(x,t) = u_\pm \quad \text{for} \quad x \gtrless \sigma t$$

satisfying (12.2.15) with $\nu = 1$ is called a p-shock. In space dimension $d \ge 2$, there is no 'natural' choice for left and right. Indeed, a shock between u_- and u_+ propagating with speed σ in the direction ν may be equivalently regarded as a shock between u_+ and u_- propagating with speed $-\sigma$ in the direction $-\nu$. In gas dynamics, for instance, in which the characteristic speeds are

$$\lambda_1(u,\nu) = \mathbf{u}\cdot\nu - c\,\|\nu\|\,, \quad \lambda_2(u,\nu) = \mathbf{u}\cdot\nu \quad \text{and} \quad \lambda_3(u,\nu) = \mathbf{u}\cdot\nu + c\,\|\nu\|$$

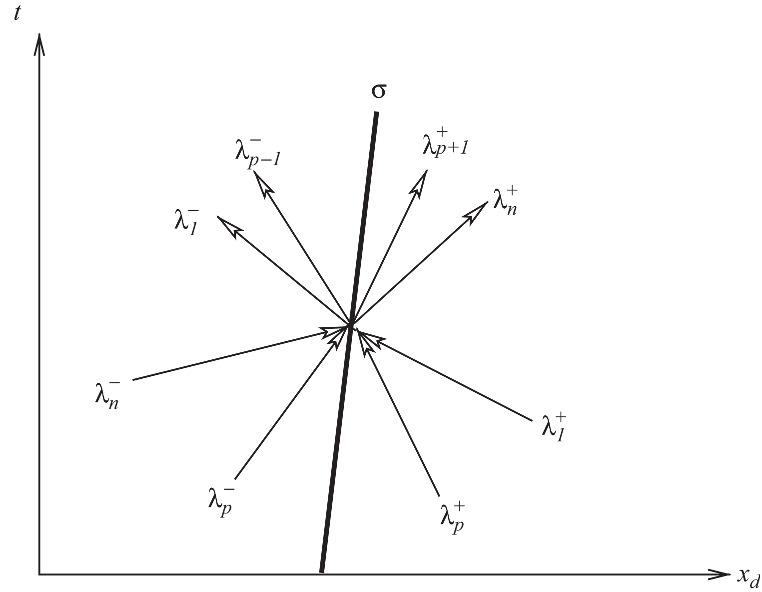

Figure 12.1: Characteristic speeds (λ_j) / shock speed (σ) for a p-Lax shock

(where $\mathbf{u}$ is the fluid velocity and c is the sound speed), the inequalities in (12.2.15) with $p = 1$ are equivalent, through the change of notations

$$(u_-, u_+, \sigma, \nu) \leftrightarrow (u_+, u_-, -\sigma, -\nu),$$

to the inequalities in (12.2.15) with $p = 3$. This shows that speaking of a p-shock in several space dimensions is meaningless. What is most important is to distinguish between the states on either side of the discontinuity by means of an intrinsic criterion (in gas dynamics, one may speak about the state *behind* the shock with respect to the flow of the gas, see Section 13.4).

Proposition 12.3 *Assume that* (12.1.1) *is constantly hyperbolic, and that the matrices* $A^0(u_\pm)$, $(\,A^d(u_\pm) - \sigma\, A^0(u_\pm)\,)$ *are non-singular. Then the dimension of* $E^s(\mathcal{A}(u,\eta,\tau))$ *for* $(\tau,\eta) \in \mathbb{C} \times \mathbb{R}^{d-1}$ *with* $\operatorname{Re}\tau > 0$ *is equal to* $n-1$ *if and only if* u *is a Lax shock.*

Proof The 'if' part has already been pointed out. The 'only if' part is also easy. If the dimension of $E^s(\mathcal{A}(u,\eta,\tau))$ is $n-1$, there must be an integer $p \in \{1,\ldots,d\}$ such that $\sigma < \lambda_p(u_-,\nu)$ (otherwise, there would be at least n characteristics exiting the shock, contradicting Corollary 12.1). Assume that p is the smallest one. Similarly, there must be an integer $q \in \{1,\ldots,d\}$ such that $\lambda_q(u_-,\nu) < \sigma$. Assume that q is the greatest one. Then we have

$$\lambda_q(u_+,\nu) < \sigma < \lambda_p(u_-,\nu) \quad \text{and} \quad \lambda_{p-1}(u_-,\nu) < \sigma < \lambda_{q+1}(u_+,\nu)$$

(with the convention $\lambda_0 = -\infty$ or $\lambda_{d+1} = +\infty$ if either $p = 1$ or $q = d$). These inequalities imply that the number of characteristics exiting the shock is $(p - 1) + n - q$, which is of course equal to $(n-1)$ only if $p = q$. In this case, the previous inequalities are precisely the Lax-shock inequalities (12.2.15). □

People aware of Majda's work and in particular of Proposition 12.3, used to consider non-Laxian shocks – now called non-classical shocks – as unstable and thus of no interest. At first glance, this is indeed a tempting conclusion. For, having $E^s(\mathcal{A}(u, \eta, \tau))$ of dimension $(n-1)$ is necessary for the *non-homogeonous* boundary value problem

$$\begin{cases} \dfrac{\mathrm{d}\dot{U}}{\mathrm{d}z} = \mathcal{A}(u, \eta, \tau)\, \dot{U} + F & \text{for } z > 0\,, \\ \dot{X}\, b(u, \tau, i\eta) - M(u, \sigma, 0)\, \dot{U} = G & \text{at } z = 0 \end{cases} \tag{12.2.16}$$

to be well-posed in $L^2(\mathbb{R}^+)$ – a condition needed for the well-posedness of the original, free boundary value problem: if $\dim E^s(\mathcal{A}(u, \eta, \tau)) \neq n-1$, either $E^s(\mathcal{A}(u, \eta, \tau))$ is too big, and the problem (12.2.16) suffers from non-uniqueness, or $E^s(\mathcal{A}(u, \eta, \tau))$ is too small and (12.2.16) has no solution in $L^2(\mathbb{R}^+)$. However, non-classical shocks are often physically relevant. So what is the trick?

When the dimension of $E^s(\mathcal{A}(u, \eta, \tau))$ is greater than $(n-1)$, the shock is called *undercompressive* – a term inspired from gas dynamics, meaning that there are more characteristics exiting the shock than for the usual, compressive shocks of Lax type. Then the homogeneous problem (12.2.16) with $F = 0$ and $G = 0$ admits non-trivial solutions $(\dot{U}, \dot{X}) \in L^2(\mathbb{R}^+) \times \mathbb{C}$. Those solutions are given by

$$\dot{U}(z) = \mathrm{e}^{z\, \mathcal{A}(u, \eta, \tau)}\, \dot{U}_0\,; \quad \dot{X}\, b(u, \tau, i\eta) = M(u, \sigma, 0)\, \dot{U}_0,$$

with $\dot{U}_0 \in E^s(\mathcal{A}(u, \eta, \tau))$. There are non-trivial ones just because the linear algebraic system of n equations

$$M(u, \sigma, 0)\, \dot{U} - \dot{X}\, b(u, \tau, i\eta) = 0 \tag{12.2.17}$$

is underdetermined in $E^s(\mathcal{A}(u, \eta, \tau)) \times \mathbb{C}$. In fact, this just means that the Rankine–Hugoniot conditions are not sufficient as jump conditions for under-compressive shocks: they should be supplemented with extra jump conditions (also called kinetic relations, see [113]), based on further modelling arguments; this is the role played by the so-called viscosity-capillarity criterion introduced in the 1980s by Slemrod [196], and independently by Truskinosky [214], for phase boundaries). It was pointed out by Freistühler [58–60] that well-chosen extra jump conditions could indeed restore stability of undercompressive shocks (for an application to subsonic liquid-vapour interfaces, see [9,10,12]). The detailed proof of the persistence of undercompressive shocks has been done by Coulombel [40], by adapting Métivier's method [136].

When $E^s(\mathcal{A}(u, \eta, \tau))$ has dimension less than $(n-1)$, the algebraic system (12.2.17) is overdetermined on $E^s(\mathcal{A}(u, \eta, \tau)) \times \mathbb{C}$ and the shock is termed *overcompressive*: it turns out – see [61, 120] – that the stability analysis of overcompressive shocks must include viscous effects, but this is far beyond the scope of this book.

In what follows, we concentrate on Lax shocks for simplicity. As regards undercompressive shocks, techniques are similar, but there are more jump conditions to deal with: the reader is referred in particular to [40, 58–60] for more details.

12.2.3 *The generalized Kreiss–Lopatinskiĭ condition*

A natural extension to (12.2.14) of the uniform Kreiss–Lopatinskiĭ condition is

(UKL$_0$) there exists $C > 0$ so that for all

$$(\eta,\tau) \in \mathbb{P}_1 := \{\, (\eta,\tau) \in \mathbb{R}^d \times \mathbb{C} \text{ with } \operatorname{Re}\tau \geq 0 \text{ and } \|\eta\|^2 + |\tau|^2 = 1 \,\},$$

$$\|\dot U\| \leq C \,\|\, \Pi(u,\tau,i\eta)\, M(u,\sigma,0)\, \dot U \,\| \qquad (12.2.18)$$

for all $\dot U \in E^s(\mathcal{A}(u,\eta,\tau))$, the stable subspace of $\mathcal{A}(u,\eta,\tau)$ if $\operatorname{Re}\tau > 0$, extended by continuity to imaginary values of τ.

As for standard IBVP, it is not easy to prove directly the estimate in (12.2.18). A preliminary step is of course to construct the stable subspace $E^s(\mathcal{A}(u,\eta,\tau))$, at first for $\operatorname{Re}\tau > 0$. A second step would be to formulate the existence of nontrivial solutions $\dot U \in E^s(\mathcal{A}(u,\eta,\tau))$ of the algebraic system

$$\Pi(u,\tau,i\eta)\, M(u,\sigma,0)\, \dot U = 0$$

as being equivalent to the vanishing of a (Lopatinskiĭ) determinant $\Delta^0(\tau,\eta)$, depending analytically on (τ,η) (and being $\mathscr{C}^\infty$ with respect to the parameters (u,σ)). Then, one would check (UKL$_0$) by showing that the function Δ^0, extended carefully by continuity up to the boundary of $\mathbb{P}_1$, does not have any zero in $\mathbb{P}_1$. Indeed, this would mean that for all $(\eta,\tau) \in \mathbb{P}_1$ the linear mapping $\Pi(u,\tau,i\eta)\, M(u,\sigma,0)$ is invertible when restricted to $E^s(\mathcal{A}(u,\eta,\tau))$, hence the inequality (12.2.18) with a uniform C on the compact set $\mathbb{P}_1$. (Observe this also implies (12.2.18) with the same constant C for all $(\tau,\eta) \neq (0,0)$ with $\operatorname{Re}\tau \geq 0$, since both $E^s(\mathcal{A}(u,\eta,\tau))$ and $\Pi(u,\tau,i\eta)$ are by definition homogeneous degree 0 in (τ,η).)

As we said before, handling the projection operator $\Pi(u,\tau,i\eta)$ is not very convenient. Furthermore, the condition (UKL$_0$) above is in fact not sufficient for the well-posedness of the complete problem (12.2.16). A more complete condition, which contains both (UKL$_0$) and the ellipticity of the symbol b is the following.

(UKL) there exists $C > 0$ so that for all $(\eta,\tau) \in \mathbb{P}_1$

$$\max(\, \|\dot U\|, |\dot X|\,) \leq C \,\|\, M(u,\sigma,0)\, \dot U - \dot X\, b(u,\tau,i\eta) \,\| \qquad (12.2.19)$$

for all $(\dot U, \dot X) \in E^s(\mathcal{A}(u,\eta,\tau)) \times \mathbb{C}$.

(Observe that unlike (12.2.18), the inequality (12.2.19) is not homogeneous, but this will be harmless.)

The condition (UKL) is what we call the uniform Kreiss–Lopatinskiĭ condition for the planar shock wave u of normal speed σ. As claimed above, (UKL) *implies* (by just taking $\dot{U} = 0$)

$$\| b(u, \tau, i\eta) \| \geq 1/C > 0,$$

hence also (12.2.18) by inserting

$$\dot{X} = -\frac{b(u, \tau, i\eta)^* M(u, \sigma, 0)\, \dot{U}}{\|b(u, \tau, i\eta)\|^2}$$

into (12.2.19).

In practice, the verification of (UKL) is far from being straightforward. Nevertheless, we claim it is mostly algebraic. The condition (UKL) is indeed equivalent to the non-existence, for all (η, τ) in the compact set $\mathbb{P}_1$, of non-trivial solutions in $E^s(\mathcal{A}(u, \eta, \tau)) \times \mathbb{C}$ to the algebraic system (12.2.17). This property can be formulated as the absence of zeroes in $\mathbb{P}_1$ of an analytic function Δ of (τ, η), depending smoothly on the shock wave u. And it can be shown that the zero set of Δ is contained in an algebraic manifold, say $\mathscr{M}$. (See Chapter 4 for theoretical explanations, and Chapter 15 for the example of gas dynamics.) The analytical parts in the verification of (UKL) are thus reduced to the determination of the continuous extension of Δ to purely imaginary values of τ, and the elimination of fake zeroes of Δ from $\mathbb{P}_1 \cap \mathscr{M}$. This is done in detail in Chapter 15 for the gas dynamics (for a more analytical approach on the same topic, see [92]). For more general systems, it is only known that *small* shocks are stable, as was pointed out by Métivier in [131] in the case $n = 2$ and proved in more generality in [133].

12.3 Well-posedness of linearized problems

12.3.1 *Energy estimates for the BVP*

In this section, we consider the linear BVP

$$\begin{cases} L_\pm(u_\pm, \mathrm{d}\chi)\, \dot{v}_\pm = f_\pm\,, & z > 0\,, \\ B(u) \cdot (\mathrm{d}\dot{\chi}, 0) + M(u, \mathrm{d}\chi) \cdot (\dot{v}_-, \dot{v}_+) = g\,, & z = 0\,. \end{cases}$$

It comes from (12.1.10) and (12.1.11) where we have sent the zeroth-order terms in $\dot{v}_\pm$ and $\dot{\chi}$ to the (arbitrary) right-hand sides $f_\pm$ and g. (Recall that the good unknowns $\dot{v}_\pm$ are merely related to $\dot{u}_\pm$ and $\dot{\chi}$ through the relation $\dot{v}_\pm = \dot{u}_\pm \mp \dot{\chi}\, \partial_z u_\pm$.) For simplicity, we just write this BVP as

$$\begin{cases} L(u, \mathrm{d}\chi)\, \dot{v} = f\,, & z > 0\,, \\ B(u) \cdot \mathrm{d}\dot{\chi} + M(u, \mathrm{d}\chi) \cdot \dot{v} = g\,, & z = 0\,, \end{cases} \tag{12.3.20}$$

where $\dot v = (\dot v_-, \dot v_+)$, the operator $L(u, \mathrm{d}\chi)$ is to be understood as

$$L(u, \mathrm{d}\chi) \;=\; \begin{pmatrix} L_-(u_-, \mathrm{d}\chi) & 0 \\ 0 & L_+(u_+, \mathrm{d}\chi) \end{pmatrix},$$

and $B(u) \cdot \mathrm{d}\dot\chi$ stands for $B(u) \cdot (\mathrm{d}\dot\chi, 0)$ by a slight abuse of notation.

Theorem 12.1 *We make the following main assumptions.*

(CH) *There exist open subsets of $\mathbb{R}^n$, say $\mathscr{U}_-$ and $\mathscr{U}_+$ containing, respectively $\underline{u}_-$ and $\underline{u}_+$, such that for all $w \in \mathscr{U}_\pm$ the matrix $A^0(w)$ is non-singular, and the operator $A^0(w)\,\partial_t + \sum_{j=1}^d A^j(w)\,\partial_j$ is* constantly hyperbolic *in the t-direction;*

(NC) *There exists $\sigma > 0$ so that*

$$f^d(\underline{u}_+) \,-\, f^d(\underline{u}_-) \;=\; \sigma\,(\,f^0(\underline{u}_+) \,-\, f^0(\underline{u}_-)\,)$$

and both matrices

$$A_d(\underline{u}_\pm, \sigma, 0) \;=\; A^d(\underline{u}_\pm) \;-\; \sigma\, A^0(\underline{u}_\pm)$$

are non-singular;

(N) *The associated discontinuous solution of* (12.1.1),

$$\underline{u} \;:\; (x,t) \;\mapsto\; \underline{u}_\pm \;\; \text{for} \;\; x_d \gtrless \sigma\, t\,,$$

is a Lax *shock (according to Definition 12.1);*

(UKL) *There exists $C > 0$ so that for all $(\eta, \tau) \in \mathbb{R}^{d-1} \times \mathbb{C}$ with* $\mathrm{Re}\,\tau \geq 0$ *and $|\tau|^2 + \|\eta\|^2 = 1$,*

$$\max(\,\|\dot U\|, |\dot X|\,) \;\leq\; C\,\|\, M(\underline{u}, \sigma, 0)\,\dot U \;-\; \dot X\, b(\underline{u}, \tau, i\eta)\,\| \qquad (12.3.21)$$

for all $(\dot U, \dot X) \in E^s(\mathcal{A}(\underline{u}, \eta, \tau)) \times \mathbb{C}$ (with M, b and $E^s(\mathcal{A})$ defined as in previous sections in terms of the fluxes f^j and their Jacobian matrices A^j).

The conclusion is that there exists $\rho > 0$ so that for all $\omega > 0$ there exist $C = C(\omega)$ and $\gamma_0 = \gamma_0(\omega)$ and for all compactly supported and Lipschitz-continuous $u_\pm : (y,z,t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R} \mapsto u_\pm(y,z,t) \in \mathbb{R}^n$ and $\mathrm{d}\chi : (y,t) \in \mathbb{R}^{d-1} \times \mathbb{R} \mapsto (\partial_t\chi(y,t), \nabla_y\chi(y,t)) \in \mathbb{R} \times \mathbb{R}^{d-1}$, with

$$\|u_\pm \,-\, \underline{u}_\pm\|_{L^\infty(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \;\leq\; \rho \quad \text{and} \quad \|\partial_t\chi \,-\, \sigma\|_{L^\infty(\mathbb{R}^d)} \;+\; \|\nabla_y\chi\|_{L^\infty(\mathbb{R}^d)} \leq \rho,$$

$$\|u_\pm\|_{W^{1,\infty}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \;\leq\; \omega \quad \text{and} \quad \|\mathrm{d}\chi\|_{W^{1,\infty}(\mathbb{R}^d)} \;\leq\; \omega,$$

for all $\gamma \geq \gamma_0$*, for all* $\dot v \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^{2n})$ *and all* $\dot\chi \in \mathscr{D}(\mathbb{R}^d; \mathbb{R})$*,*

$$\gamma \,\|\mathrm{e}^{-\gamma t}\, \dot v\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|\mathrm{e}^{-\gamma t}\, \dot v_{|z=0}\|^2_{L^2(\mathbb{R}^d)} \,+\, \|\mathrm{e}^{-\gamma t}\, \dot\chi\|^2_{H^1_\gamma(\mathbb{R}^d)} \tag{12.3.22}$$
$$\leq C \left(\frac{1}{\gamma}\, \|\mathrm{e}^{-\gamma t}\, L(u,\mathrm{d}\chi)\, \dot v\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \right.$$
$$\left. +\, \|\mathrm{e}^{-\gamma t}\, (\, B(u)\cdot \mathrm{d}\dot\chi \,+\, M(u,\mathrm{d}\chi)\cdot \dot v_{|z=0}\,)\|^2_{L^2(\mathbb{R}^d)} \right).$$

Proof The method of proof is very similar to what has been done for standard BVP (in Theorem 9.6). The main novelty is the additional unknown $\dot\chi$, the derivatives of which appear in the boundary terms: in the constant-coefficients case (i.e. actually for $u_\pm \equiv \underline{u}_\pm$ and $\chi(y,t) = \sigma t$), after elimination of the unknown $\dot\chi$, we are left with a BVP with *pseudo-differential* boundary conditions (see (12.2.14)); for more general $u_\pm$ and χ the idea is still to eliminate $\dot\chi$ and treat the reduced problem (almost) as a standard one, by means of a (generalized) Kreiss' symmetrizer of course, which will be possible after 'para-linearizing' the equations. Before going into detail, let us introduce additional (!) convenient notations.

We rewrite the differential operator $L(u,\mathrm{d}\chi)$ as

$$L(u,\mathrm{d}\chi) \,=\, \mathbb{A}^0(u)\,\partial_t \,+\, \sum_{j=1}^{d-1} \mathbb{A}^j(u)\,\partial_j \,+\, \mathbb{A}_d(u,\mathrm{d}\chi)\,\partial_z,$$

with

$$\mathbb{A}^j(u) \,=\, \begin{pmatrix} A^j(u_-) & 0 \\ 0 & A^j(u_+) \end{pmatrix}$$

for all $j \in \{0,\dots,d-1\}$ and

$$\mathbb{A}_d(u,\mathrm{d}\chi) \,=\, \begin{pmatrix} -\,A_d(u_-,\mathrm{d}\chi) & 0 \\ 0 & A_d(u_+,\mathrm{d}\chi) \end{pmatrix}$$

with (recall)

$$A_d(u,\mathrm{d}\chi) \,=\, A^d(u) \,-\, \sum_{j=1}^{d-1} (\partial_j\chi)\; A^j(u) \,-\, (\partial_t\chi)\; A^0(u).$$

To simplify the writing we shall use the shortcut $\mathbf{u}$ for $(u,\mathrm{d}\chi)$, or more precisely, $\mathbf{u}$ will stand for the Lipschitz continuous mapping

$$\begin{array}{rl} \mathbf{u}\,:\; \mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R} & \to \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^{d-1} \\ (y,z,t) & \mapsto (\,u_-(y,z,t), u_+(y,z,t), \partial_t\chi(y,t), \nabla_y\chi(y,t)\,)\,. \end{array}$$

We introduce the further notation

$$L^\gamma_{\mathbf{u}} \,:=\, L(\mathbf{u}(y,z,t)) \,+\, \gamma\,\mathbb{A}^0(u(y,z,t)),$$

and, recalling that the operators in the boundary conditions in (12.3.20) are defined by

$$B(u)\cdot \mathrm{d}\dot\chi \;=\; (\partial_t\dot\chi)\,(f^0(u_+)\;-\;f^0(u_-))\;+\;\sum_{j=1}^{d-1}(\partial_j\dot\chi)\,(f^j(u_+)\;-\;f^j(u_-)),$$

$$M(u,\mathrm{d}\chi)\cdot\dot v \;=\; A_d(u_-,\mathrm{d}\chi)\cdot\dot v_- \;-\; A_d(u_+,\mathrm{d}\chi)\cdot\dot v_+,$$

we introduce the boundary operator

$$B^\gamma_{\mathbf{u}}\,(\dot v,\dot\chi) = (\gamma\,\dot\chi+\partial_t\dot\chi)\,(f^0(u_+)-f^0(u_-))_{|z=0}+\sum_{j=1}^{d-1}(\partial_j\dot\chi)\,(f^j(u_+)-f^j(u_-))_{|z=0}$$

$$+\,(\,A_d(u_-,\mathrm{d}\chi)\cdot\dot v_- - A_d(u_+,\mathrm{d}\chi)\cdot\dot v_+\,)_{|z=0}\,.$$

With these notations, the energy estimate we want to prove equivalently reads

$$\gamma\,\|\widetilde v_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})}\;+\;\|(\widetilde v_\gamma)_{|z=0}\|^2_{L^2(\mathbb{R}^d)}\;+\;\|\widetilde\chi_\gamma\|^2_{H^1_\gamma(\mathbb{R}^d)}$$
$$\leq\; C\,\left(\frac{1}{\gamma}\,\|L^\gamma_{\mathbf{u}}\widetilde v_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})}\;+\;\|B^\gamma_{\mathbf{u}}\,(\widetilde v_\gamma,\widetilde\chi_\gamma)\|^2_{L^2(\mathbb{R}^d)}\right),$$

with

$$\widetilde v_\gamma \;:=\; \mathrm{e}^{-\gamma t}\,\dot v \quad\text{and}\quad \widetilde\chi_\gamma\;:=\;\mathrm{e}^{-\gamma t}\,\dot\chi.$$

Para-linearization of the equations We first observe that for u close enough to $\underline{u}$ and $\mathrm{d}\chi$ close enough to $(\sigma,0)$ in the L^∞ norm, $\mathbb{A}_d(\mathbf{u}(y,z,t))$ is non-singular for all (y,z,t). Hence the equality $\widetilde f_\gamma = L^\gamma_{\mathbf{u}}\widetilde v_\gamma$ equivalently reads

$$\partial_z\widetilde v_\gamma\;-\;P^\gamma_{\mathbf{u}}\,\widetilde v_\gamma\;=\;\mathbb{A}_d(\mathbf{u}(y,z,t))^{-1}\,\widetilde f_\gamma\,,$$

$$P^\gamma_{\mathbf{u}}\;:=\;-\,\mathbb{A}_d(\mathbf{u}(y,z,t))^{-1}\,(\,\mathbb{A}^0(u(y,z,t))\,(\,\gamma\;+\;\partial_t\,)\;+\;\sum_{j=1}^{d-1}\mathbb{A}^j(u(y,z,t))\partial_j\,).$$

This induces us to introduce

$$\mathcal{A}_{\mathbf{u}}(y,z,t,\eta,\tau)\;=\;-\,\mathbb{A}_d(\mathbf{u}(y,z,t))^{-1}\,\big(\,\tau\,\mathbb{A}^0(u(y,z,t))\;+\;i\,\sum_{j=1}^{d-1}\eta_j\,\mathbb{A}^j(u(y,z,t))\,\big).$$

One may observe, in particular, that for $\mathbf{u}\;=\;\underline{\mathbf{u}}\;:=\;(\underline{u},\sigma,0)$, $\mathcal{A}_{\mathbf{u}}$ is related to the symbol defined in Section 12.2 by

$$\mathcal{A}_{\underline{\mathbf{u}}}(y,z,t,\eta,\tau)\;=\;\mathcal{A}(\underline{u},\eta,\tau)\quad\text{for all}\;\;(y,z,t).$$

In more generality, for all fixed z, $\mathcal{A}_{\mathbf{u}}(y,z,t,\eta,\tau=\gamma+i\delta)$ can be regarded as a symbol with parameter γ in the variables (y,t), which can be associated with a para-differential operator with parameter $T^{\gamma}_{\mathcal{A}_{\mathbf{u}}}$. By Theorem C.20 we have the error estimate

$$\|P^{\gamma}_{\mathbf{u}}v \,-\, T^{\gamma}_{\mathcal{A}_{\mathbf{u}}}v\|_{L^2} \,\leq\, C(\omega)\,\|v\|_{L^2}.$$

On the other hand, the form of $B^{\gamma}_{\mathbf{u}}$ suggests the introduction of the symbol (belonging to Γ^1_1),

$$\begin{aligned} b_u(y,t,\eta,\tau) \,:=\, & \tau\,(f^0(u_+(y,0,t)) \,-\, f^0(u_-(y,0,t))) \\ & + i\sum_{j=1}^{d-1}\eta_j\,(f^j(u_+(y,0,t)) \,-\, f^j(u_-(y,0,t)))\,. \end{aligned}$$

Compared to the notation b introduced in Section 12.2 we have in the special case $u \,=\, \underline{u}$,

$$b_{\underline{u}}(y,t,\eta,\tau) \,=\, b(\underline{u},\tau,i\eta) \quad \text{for all} \;\; (y,t).$$

Finally, we shall use the notation

$$\mathcal{M}_{\mathbf{u}}(y,t) \,:=\, \begin{pmatrix} A_d(u_-(y,0,t),\mathrm{d}\chi) & 0 \\ 0 & -\,A_d(u_+(y,0,t),\mathrm{d}\chi) \end{pmatrix} \,(=\,-\,\mathbb{A}_d(\mathbf{u}(y,0,t))\,).$$

By Theorem C.20 we have the error estimate

$$\begin{aligned} & \|B^{\gamma}_{\mathbf{u}}(v,\psi) \,-\, T^{\gamma}_{\mathcal{M}_{\mathbf{u}}}v \,-\, T^{\gamma}_{b_u}\psi\|_{L^2} \\ & \leq\, C(\omega)\,(\,\|\psi\|_{L^2} \,+\, \|v\|_{H^{-1}_{\gamma}}\,) \,\leq\, \frac{C(\omega)}{\gamma}\,(\,\|\psi\|_{H^1_{\gamma}} \,+\, \|v\|_{L^2}\,)\,. \end{aligned}$$

Therefore, the searched energy estimate will be proved by absorption of the errors in the left-hand side if we show its para-linearized version

$$\begin{aligned} & \gamma\,\|v\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|v_{|z=0}\|^2_{L^2(\mathbb{R}^d)} \,+\, \|\psi\|^2_{H^1_{\gamma}(\mathbb{R}^d)} \\ & \leq\, C\,\left(\frac{1}{\gamma}\,\|\partial_z v \,-\, T^{\gamma}_{\mathcal{A}_{\mathbf{u}}}v\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|T^{\gamma}_{\mathcal{M}_{\mathbf{u}}}v_{|z=0} \,+\, T^{\gamma}_{b_u}\psi\|^2_{L^2(\mathbb{R}^d)}\right)\,. \end{aligned} \tag{12.3.23}$$

□

Proof of the para-linearized energy estimate: elimination of the front The assumption **(UKL)** implies, in particular, the existence of a constant $c>0$ such that (by homogeneity),

$$\|b(\underline{u},\tau,i\eta)\|^2 \,\geq\, c\,(\,|\tau|^2 \,+\, \|\eta\|^2) \quad \text{for all } (\eta,\tau)\in\mathbb{R}^{d-1}\times\mathbb{C}\backslash\{(0,0)\}$$
$$\text{with } \operatorname{Re}\tau\geq 0\,,$$

and for all (Lipschitz) continuous u such that $\|u - \underline{u}\|_{L^\infty} \leq \rho$ small enough,

$$\|b_u(y,t,\eta,\tau)\|^2 \geq \frac{c}{2}\,(\,|\tau|^2 + \|\eta\|^2) \quad \text{for all } \ (\eta,\tau) \in \mathbb{R}^{d-1} \times \mathbb{C}\backslash\{(0,0)\} \quad \text{with } \ \mathrm{Re}\,\tau \geq 0$$

and all $(y,t) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Therefore,

$$\Pi_u(y,t,\eta,\tau) := I_n - \frac{b_u(y,t,\eta,\tau)\,b_u(y,t,\eta,\tau)^*}{\|b_u(y,t,\eta,\tau)\|^2}$$

(consistently with the notation introduced in Section 12.2) is well-defined and homogeneous degree 0 in (η,τ) and thus belongs to Γ^0_1 if u is Lipschitz continuous with $\|u - \underline{u}\|_{L^\infty} \leq \rho$, and by Theorem C.22, $T^\gamma_{\Pi_u\,b_u} - T^\gamma_{\Pi_u}\,T^\gamma_{b_u}$ is of order $0+1-1=0$: this means there exists C depending only on $\omega = \|u-\underline{u}\|_{W^{1,\infty}}$ such that

$$\|T^\gamma_{\Pi_u\,b_u}\psi - T^\gamma_{\Pi_u}\,T^\gamma_{b_u}\psi\|_{L^2} \leq C\,\|\psi\|_{L^2}.$$

Since by definition $\Pi_u\,b_u$ is identically zero, this implies

$$\|\,T^\gamma_{\Pi_u}\,T^\gamma_{b_u}\psi\|_{L^2} \leq \frac{C}{\gamma}\,\|\psi\|_{H^1_\gamma}$$

for all smooth enough ψ. Also by Theorem C.22 we have another constant, still denoted by C, depending only on ω such that

$$\|T^\gamma_{\Pi_u\,\mathcal{M}_\mathbf{u}}v - T^\gamma_{\Pi_u}\,T^\gamma_{\mathcal{M}_\mathbf{u}}v\|_{L^2} \leq C\,\|v\|_{H^{-1}_\gamma} \leq \frac{C}{\gamma}\,\|v\|_{L^2}$$

for all smooth enough v.

Proof of the para-linearized energy estimate: estimate of the front The estimate $\|b_u(y,t,\eta,\gamma+i\delta)\| \geq \frac{c}{2}\,\lambda^{1,\gamma}(\delta,\eta)$ and Gårding's inequality (in Theorem C.23) show that

$$\mathrm{Re}\,\langle T^\gamma_{{b_u}^*b_u}\psi\,,\,\psi\rangle \geq \frac{c}{4}\,\|\psi\|^2_{H^1_\gamma}.$$

Theorems C.21 and C.22 show that $R^\gamma := T^\gamma_{{b_u}^*b_u} - (T^\gamma_{b_u})^*\,T^\gamma_{b_u}$ is an operator of order at most one; hence

$$\langle R^\gamma\psi\,,\,\psi\rangle \leq C\,\|\psi\|_{H^1_\gamma}\,\|\psi\|_{L^2} \leq \frac{C}{\gamma}\,\|\psi\|^2_{H^1_\gamma}.$$

Therefore, combining the two inequalities we get

$$\|\psi\|^2_{H^1_\gamma} \leq \frac{4}{c}\,\big(\,\|T^\gamma_{b_u}\psi\|^2_{L^2} + \frac{C}{\gamma}\,\|\psi\|^2_{H^1_\gamma}\,\big)\,;$$

hence for $\gamma \geq 8C/c$,

$$\|\psi\|^2_{H^1_\gamma} \leq \frac{4}{c}\,\|T^\gamma_{b_u}\psi\|^2_{L^2}$$

in which the right-hand side can be bounded in a trivial way:

$$\|T^\gamma_{b_u}\psi\|^2_{L^2} \leq 2\,\|T^\gamma_{b_u}\psi + T^\gamma_{\mathcal{M}_u} v_{|z=0}\|^2_{L^2} + 2\,\|T^\gamma_{\mathcal{M}_u} v_{|z=0}\|^2_{L^2}$$
$$\leq 2\,\|T^\gamma_{b_u}\psi + T^\gamma_{\mathcal{M}_u} v_{|z=0}\|^2_{L^2} + 2\,C\,\|v_{|z=0}\|^2_{L^2}$$

for a new C depending on ω.

Proof of the para-linearized energy estimate: the main estimate From the two steps above (elimination and estimate of the front), it should now be clear to the reader that (12.3.23) will follow (by absorption of the errors in the left-hand side) from the estimate on the reduced para-linearized problem

$$\gamma\,\|v\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|v_{|z=0}\|^2_{L^2(\mathbb{R}^d)} \tag{12.3.24}$$
$$\leq C\left(\frac{1}{\gamma}\,\|\partial_z v - T^\gamma_{\mathcal{A}_{\mathbf{u}}} v\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|T^\gamma_{\Pi_u\mathcal{M}_{\mathbf{u}}} v_{|z=0}\|^2_{L^2(\mathbb{R}^d)}\right).$$

Unsurprisingly, the proof of this estimate will be very similar to the proof of Theorem 9.6: it will rely on the construction of a generalized Kreiss symmetrizer for the reduced BVP

$$\begin{cases} \partial_z v - T^\gamma_{\mathcal{A}_{\mathbf{u}}} v = F\,, \\ T^\gamma_{\Pi_u\mathcal{M}_{\mathbf{u}}} v_{|z=0} = G\,. \end{cases}$$

Lemma 12.1 *Under the assumptions of Theorem 12.1, let* $\mathbf{u} = (u, \mathrm{d}\chi)$ *be Lipschitz continuous and close enough to* $\underline{\mathbf{u}} = (\underline{u}, \sigma, 0)$ *in the* L^∞ *norm (which means more precisely that the bound* ρ *in the statement of Theorem 12.1 is small enough). Then there exists a symbol*

$$\begin{aligned} \mathcal{R}_{\mathbf{u}}\,:\; \mathbb{X} := \{(y,z,t,\eta,\tau) \in \mathbb{R}^{2d+1}\times\mathbb{C}\,;\; z\geq 0\,,\, \mathrm{Re}\,\tau \geq 0\} &\to \mathbf{HPD}_{2n} \\ X = (y,z,t,\eta,\tau) &\mapsto \mathcal{R}_{\mathbf{u}}(X)\,, \end{aligned}$$

which belongs to Γ^0_1 *and is homogeneous degree* 0 *in* (η,τ)*, with an estimate*

$$\mathcal{R}_{\mathbf{u}}(y,0,t,\eta,\tau) \geq \alpha\,I_{2n} - \beta\,((\Pi_u\mathcal{M}_{\mathbf{u}})(y,t,\eta,\tau))^*\,(\Pi_u\mathcal{M}_{\mathbf{u}})(y,t,\eta,\tau) \tag{12.3.25}$$

for $\alpha > 0$ *and* $\beta > 0$ *depending only on the Lipschitz norm of* $\mathbf{u}$*, and additionally* $\mathcal{R}_{\mathbf{u}}(X)\,\mathcal{A}_{\mathbf{u}}(X)$ *decomposes for all* $X\in\mathbb{X}$ *into a* finite sum *of the form*

$$\mathcal{R}_{\mathbf{u}}(X)\,\mathcal{A}_{\mathbf{u}}(X) = \sum_j P_j(X)^*\left(\begin{array}{c|c} \gamma\,h_{0,j}(X) & 0 \\ \hline 0 & h_{1,j}(X)\end{array}\right) P_j(X),$$

- *with* $P_j \in \Gamma^0_1$*, homogeneous degree* 0 *in* (η,τ) *and* $\sum_j P_j(X)^*\,P_j(X) \geq C\,I_{2n}$,
- *and* $h_{0,j} \in \Gamma^0_1$*, homogeneous degree* 0 *in* (η,τ) *and* $\mathrm{Re}\,(h_{0,j}(X)) \geq C_j\,I_{k_j}$,
- *and* $h_{1,j} \in \Gamma^1_1$*, homogeneous degree* 1 *in* (η,τ) *and*

$$\mathrm{Re}\,(h_{1,j}(X)) \geq C_j\,\lambda^{\gamma,1}(\eta,\delta)\,I_{2n-k_j}.$$

We postpone the proof of this result and complete the proof of the estimate in (12.3.24). The outline is the same as in the proof of Theorem 9.6. We consider the family of operators

$$\mathrm{R}^\gamma_{\mathbf{u}} := \frac{1}{2}\,(\,T^\gamma_{\mathcal{R}_{\mathbf{u}}} + (T^\gamma_{\mathcal{R}_{\mathbf{u}}})^*).$$

By construction, $\mathrm{R}^\gamma_{\mathbf{u}}$ is a self-adjoint operator on L^2 and by Theorem C.21 and Remark C.2, there exists C depending only on (the Lipschitz norm of $\mathbf{u}$) ω so that

$$\|(\mathrm{R}^\gamma_{\mathbf{u}} v)_{|z=0} - (T^\gamma_{\mathcal{R}_{\mathbf{u}}} v)_{|z=0}\|_{L^2} \le \frac{C}{\gamma}\,\|v_{|z=0}\|_{L^2(\mathbb{R}^d,\mathrm{d}y\,\mathrm{d}t)}$$

for all smooth enough v. Hence, denoting by $\langle\cdot,\cdot\rangle$ the scalar product on $L^2(\mathbb{R}^d,\mathrm{d}y\,\mathrm{d}t)$, the inequality in (12.3.25) together with the error estimates in Theorems C.20 and C.22 and the Gårding inequality in Theorem C.23 imply

$$\langle\,\mathrm{R}^\gamma_{\mathbf{u}} v_{|z=0}\,,\,v_{|z=0}\,\rangle + \beta\,\mathrm{Re}\,\langle\,T^\gamma_{(\Pi_u\mathcal{M}_{\mathbf{u}})^*\Pi_u\mathcal{M}_{\mathbf{u}}}\,v_{|z=0}\,,\,v_{|z=0}\,\rangle \ge \frac{\alpha}{2}\,\|v_{|z=0}\|^2_{L^2(\mathbb{R}^d,\mathrm{d}y\,\mathrm{d}t)}$$

for γ large enough, hence by Theorems C.21 and C.22 again,

$$\langle\,\mathrm{R}^\gamma_{\mathbf{u}} v_{|z=0}\,,\,v_{|z=0}\,\rangle + \beta\,\|T^\gamma_{\Pi_u\mathcal{M}_{\mathbf{u}}}\,v_{|z=0}\|^2_{L^2(\mathbb{R}^d,\mathrm{d}y\,\mathrm{d}t)} \ge \frac{\alpha}{4}\,\|v_{|z=0}\|^2_{L^2(\mathbb{R}^d,\mathrm{d}y\,\mathrm{d}t)}$$

up to increasing γ.

On the other hand, by Theorems C.21 and C.22, there exists C' depending only on ω so that

$$\mathrm{Re}\,\langle\,\mathrm{R}^\gamma_{\mathbf{u}}\,T^\gamma_{\mathcal{A}_{\mathbf{u}}}\,v,\,v\,\rangle \ge \sum_j \mathrm{Re}\,\langle\gamma\,T^\gamma_{h_{0,j}}\,v_{0,j}\,,\,v_{0,j}\,\rangle + \mathrm{Re}\,\langle T^\gamma_{h_{1,j}}\,u_{1,j}\,,\,v_{1,j}\,\rangle - C'\|v\|^2_{L^2},$$

with $v_{0,j}$ and $v_{1,j}$ the two blocks (taking values in $\mathbb{C}^{k_j}$ and $\mathbb{C}^{2n-k_j}$, respectively) of $T^\gamma_{P_j}\,v$, and by Theorem C.23 we have the inequalities

$$\mathrm{Re}\,\langle\gamma\,T^\gamma_{h_{0,j}}\,v_{0,j}\,,\,v_{0,j}\,\rangle \ge \gamma\,\frac{C_j}{4}\,\|v_{0,j}\|^2_{L^2}\,,$$

$$\mathrm{Re}\,\langle T^\gamma_{h_{1,j}}\,v_{1,j}\,,\,v_{1,j}\,\rangle \ge \frac{C_j}{4}\,\|v_{1,j}\|^2_{H^{1/2}} \ge \gamma\,\frac{C_j}{4}\,\|v_{1,j}\|^2_{L^2}.$$

Therefore, using once more the Theorems C.21, C.22, and C.23 (the latter being applied to the degree 0 symbol $\sum_j P_j^* P_j$), we get new constants C_0 and C'' such that

$$\mathrm{Re}\,\langle\mathrm{R}^\gamma_{\mathbf{u}}\,T^\gamma_{\mathcal{A}_{\mathbf{u}}}\,v\,,\,v\rangle \ge (\,C_0\,\gamma - C''\,)\,\|v\|^2_{L^2},$$

hence for large enough $\gamma \ge 2C''/C_0$,

$$\mathrm{Re}\int\langle\,\mathrm{R}^\gamma_{\mathbf{u}}\,T^\gamma_{\mathcal{A}_{\mathbf{u}}}\,v\,,\,v\,\rangle\,\mathrm{d}z \ge \gamma\,\frac{C_0}{2}\,\|v\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}z)}\,. \qquad (12.3.26)$$

Now we prove the inequality in (12.3.24) by writing

$$\frac{\mathrm{d}}{\mathrm{d}z}\langle\, \mathrm{R}^\gamma_{\mathbf{u}}\, v\,,\, v\,\rangle = \langle\, \frac{\mathrm{dR}^\gamma_{\mathbf{u}}}{\mathrm{d}z}\, v\,,\, v\,\rangle + 2\,\mathrm{Re}\,\langle\, \mathrm{R}^\gamma_{\mathbf{u}}(\partial_z v - T^\gamma_{\mathcal{A}_{\mathbf{u}}}\, v)\,, v\,\rangle + 2\,\mathrm{Re}\,\langle\, \mathrm{R}^\gamma_{\mathbf{u}}\, T^\gamma_{\mathcal{A}_{\mathbf{u}}}\, v\,, v\,\rangle,$$

which implies after integration in z,

$$\frac{\alpha}{2}\,\|v_{|z=0}\|^2_{L^2(\mathbb{R}^d)} \;-\; \beta\,\mathrm{Re}\;\langle\, T^\gamma_{(\Pi_u\mathcal{M}_{\mathbf{u}})^*\Pi_u\mathcal{M}_{\mathbf{u}}}\, v_{|z=0}\,,\, v_{|z=0}\,\rangle$$
$$\leq\; (\, C_2 \,+\, \gamma\,(\theta\, C_1 \,-\, C_0)\,)\;\|v\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \;+\; \frac{C_1}{4\theta\gamma}\;\|\partial_z v \,-\, T^\gamma_{\mathcal{A}_{\mathbf{u}}}\, v\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}$$

for some new constants C_1 and C_2, and $\theta > 0$ arbitrary. Choosing $\theta = C_0/(2C_1)$, we get

$$\frac{\alpha}{2}\,\|v_{|z=0}\|^2_{L^2(\mathbb{R}^d)} \;+\; \gamma\,\frac{C_0}{4}\;\|v\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \;\leq\; \beta\,\mathrm{Re}\;\langle\, T^\gamma_{(\Pi_u\mathcal{M}_{\mathbf{u}})^*\Pi_u\mathcal{M}_{\mathbf{u}}}\, v_{|z=0}\,,\, v_{|z=0}\,\rangle$$
$$+\;\frac{C_1^2}{2C_0\gamma}\;\|\partial_z v \,-\, T^\gamma_{\mathcal{A}_{\mathbf{u}}}\, v\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)}$$

for all $\gamma \geq 4C_2/C_0$. Finally, using again Theorems C.21 and C.22 we obtain for γ large enough,

$$\frac{\alpha}{4}\,\|v_{|z=0}\|^2_{L^2(\mathbb{R}^d)} \;+\; \gamma\,\frac{C_0}{4}\;\|v\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} \;\leq\; \beta\;\|T^\gamma_{\Pi_u\mathcal{M}_{\mathbf{u}}}\, v_{|z=0}\|^2_{L^2(\mathbb{R}^d)}$$
$$+\;\frac{C_1^2}{2C_0\gamma}\;\|\partial_z v \,-\, T^\gamma_{\mathcal{A}_{\mathbf{u}}}\, v\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)},$$

which can be rewritten as (12.3.24) with $C = 4\max(\beta, C_1^2/(2C_0))/\min(\alpha, C_0)$. □

Sketch of proof of Lemma 12.1 To construct $\mathcal{R}_{\mathbf{u}}$ the idea is (as usual) to construct a local symmetrizer in the neighbourhood of each point $X = (y, z, t, \eta, \tau) \in \mathbb{X}$ with $|\tau|^2 + \|\eta\|^2 \;=\; 1$, to piece together these local symmetrizers by a partition of unity technique and then extend the resulting mapping to the whole set $\mathbb{X}$ by homogeneity in (η, τ).

By local symmetrizer at $\underline{X} \in \mathbb{X}_1$ we mean a matrix-valued function $r_{\mathbf{u}}$ defined on a neighbourhood $\mathscr{X} \subset \mathbb{X}_1$, associated with another matrix-valued function T, both being at least Lispchitz in (y, z, t) and $\mathscr{C}^\infty$ in (η, τ), such that

i) the matrix $r(X)$ is Hermitian and $T(X)$ is invertible for all $X \in \mathscr{X}$,

ii) the matrix $\mathrm{Re}\;(\, r(X)\, T(X)^{-1}\mathcal{A}_{\mathbf{u}}(X)T(X)\,)$ is block-diagonal, with blocks $h_0(X)$ and $h_1(X)$ such that $h_0(X)/\gamma$ is $\mathscr{C}^\infty$ and

$$\mathrm{Re}\;(\, h_0(X)\,) \geq C\,\gamma\, I_p\,, \quad \mathrm{Re}\;(\, h_1(X)\,) \geq C\;\; I_{n-p}\,, \tag{12.3.27}$$

for some $C > 0$ independent of $X \in \mathscr{X}$, *iii)* and additionally, *if* $\underline{X} \in \mathbb{X}^0$ (i.e. if $\underline{z} = 0$) there exist $\alpha > 0$ and $\beta > 0$ independent of $X \in \mathscr{X} \cap \mathbb{X}^0$ so that for all $V \in \mathbb{C}^{2n}$,

$$V^* \, r(y,0,t,\eta,\tau)\, V \;\geq\; \alpha\, \|V\|^2 \qquad (12.3.28)$$
$$- \beta\, \|\Pi_u(y,t,\eta,\tau)\mathcal{M}_{\mathbf{u}}(y,t)T(y,0,t,\eta,\tau)\, V\|^2.$$

The construction of a local symmetrizer at a point $\underline{X} \in \mathbb{X}_1$ such that $\mathbf{u}(\underline{y},\underline{z},\underline{t}) = \underline{\mathbf{u}}$ relies on the assumption (**UKL**) and more specifically on its consequence

($\mathbf{UKL}_0$) for all (τ,η) with $\mathrm{Re}\,\tau \geq 0$ and $|\tau|^2 + \|\eta\|^2 = 1$, for all $V \in E^s(\mathcal{A}(\underline{u},\eta,\tau))$,

$$\|V\| \;\leq\; C\, \|\Pi(\underline{u},\tau,i\eta)\, M(\underline{u},\sigma,0)V\|.$$

Details are basically the same as for the standard BVP. More precisely, we may consider the pair (η,τ) in the projection operator Π as parameters, and use the block-diagonal structure of $\mathcal{A}$ and ΠM together with the 'standard' construction of symmetrizers. This is made clear in the example of gas dynamics in Chapter 15.

By continuity, ($\mathbf{UKL}_0$) also implies that for $\mathbf{u}$ close enough to $\underline{\mathbf{u}}$ in the L^∞ norm,

$$\|V\| \;\leq\; \frac{C}{2}\, \|\Pi_{\mathbf{u}}(y,t,\eta,\tau)\, \mathcal{M}_{\mathbf{u}}(y,t,\eta,\tau)V\|$$

for all $(y,t) \in \mathbb{R}^d$, for all (τ,η) with $\mathrm{Re}\ \tau \geq 0$ and $|\tau|^2 + \|\eta\|^2 = 1$, and for all $V \in E^s(\mathcal{A}_{\mathbf{u}}(y,t,\eta,\tau))$. Consequently, the construction of local symmetrizers is in fact valid at all points $X \in \mathbb{X}_1$.

Then it suffices to piece together local symmetrizers as in the proof of Theorem 9.1: $\mathcal{R}_{\mathbf{u}}$ is defined on $\mathbb{X}_1$ as a finite sum of terms of the form $P(X)^* r(X) P(X)$ with $P(X) = \varphi(X)^{1/2}\, T(X)^{-1}$, φ coming from a partition of unity of $\mathbb{X}_1$ (of which the lack of compactness in the (y,z,t)-directions is why we require constant coefficients outside a compact set), and finally $\mathcal{R}_{\mathbf{u}}$ is extended by homogeneity of degree 0 in (η,τ). □

Remark 12.4 By density of $\mathscr{D}$ in H^1, the energy estimate (12.3.22) extends to all pairs $(\dot{v},\dot{\chi}) \in \mathrm{e}^{\gamma t} H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}) \times \mathrm{e}^{\gamma t} H^1(\mathbb{R}^d)$. (This is obviously enough to pass to the limit in the right-hand side of (12.3.22) applied to $\mathscr{D}$-approximations of $(\dot{v},\dot{\chi})$.)

The energy estimate (12.3.22) (in Theorem 12.1) can be used to derive higher-order energy estimates, which will be useful to eventually deal with the full non-linear problem. These higher-order estimates are as for standard BVP in terms of $\mathscr{H}^m_\gamma$ norms, defined by

$$\|w\|^2_{\mathscr{H}^m_\gamma} := \sum_{|\alpha| \leq m} \gamma^{2(m-|\alpha|)}\, \|\mathrm{e}^{-\gamma\, t}\, \partial^\alpha w\|^2_{L^2}.$$

Here above α stands for a d- or $(d+1)$-uple, (i.e. $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{d-1})$ or $(\alpha_0, \alpha_1, \dots, \alpha_d)$, with $\alpha_j \in \mathbb{N}$), and $|\alpha| = \sum_j \alpha_j$ (called the length of α).

Theorem 12.2 *Under the hypotheses of Theorem 12.1, assume, moreover, that $u - \underline{u}$ belongs to $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^{2n})$, $(u - \underline{u})_{|z=0}$ belongs to $H^m(\mathbb{R}^d; \mathbb{R}^{2n})$ and $(\partial_t \chi - \sigma, \nabla_y \chi)$ belongs to $H^m(\mathbb{R}^d; \mathbb{R}^d)$ for some integer $m > (d+1)/2 + 1$, with*

$$\|u - \underline{u}\|_{H^m} \le \mu\,,\ \|u_{|z=0} - \underline{u}_{|z=0}\|_{H^m} \le \mu \quad \textit{and} \quad \|(\partial_t \chi - \sigma, \nabla_y \chi)\|_{H^m} \le \mu.$$

Then there exist $\gamma_m = \gamma_m(\omega, \mu) \ge 1$ and $C_m = C_m(\omega, \mu) > 0$ such that, for all $\gamma \ge \gamma_m$, for all $\dot{v} \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^{2n})$ and all $\dot{\chi} \in \mathscr{D}(\mathbb{R}^d; \mathbb{R})$,

$$\begin{aligned} &\gamma\, \|\dot{v}\|^2_{L^2(\mathbb{R}^+; \mathscr{H}^m_\gamma(\mathbb{R}^d))} + \|\dot{v}_{|z=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} + \|\dot{\chi}\|^2_{\mathscr{H}^{m+1}_\gamma(\mathbb{R}^d)} \\ &\le C_m \left(\tfrac{1}{\gamma}\, \|L(u, \mathrm{d}\chi)\, \dot{v}\|^2_{L^2(\mathbb{R}^+; \mathscr{H}^m_\gamma(\mathbb{R}^d))} + \|B(u) \cdot \mathrm{d}\dot{\chi} + M(u, \mathrm{d}\chi) \cdot \dot{v}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} \right) \end{aligned} \tag{12.3.29}$$

$$\begin{aligned} &\gamma\, \|\dot{v}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})} + \|\dot{v}_{|z=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} + \|\dot{\chi}\|^2_{\mathscr{H}^{m+1}_\gamma(\mathbb{R}^d)} \\ &\le C_m \left(\tfrac{1}{\gamma}\, \|L(u, \mathrm{d}\chi)\, \dot{v}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})} + \|B(u) \cdot \mathrm{d}\dot{\chi} + M(u, \mathrm{d}\chi) \cdot \dot{v}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} \right). \end{aligned} \tag{12.3.30}$$

The proof is analogous to the proof of Theorem 9.7 for standard BVP. It starts with estimates of derivatives in the (y, t)-direction by using the L^2 estimate (12.3.22) (in Theorem 12.1) together with commutator estimates in $\mathscr{H}^s_\gamma$ norms: this yields (12.3.29). As a second step, estimates of derivatives in the z-direction are obtained by differentiating the equality

$$\partial_z \dot{v} = \mathbb{A}_d(u(y,z,t), \mathrm{d}\chi(y,t))^{-1} \left(L(u, \mathrm{d}\chi)\, \dot{v} - \sum_{j=1}^{d-1} \mathbb{A}^j(u(y,z,t))\, \partial_j\, \dot{v} \right),$$

and by repeated use of Lemma 9.3: this eventually leads to (12.3.30). We omit the (most technical) details and refer to [140], pp. 71–72, or [136], Section 4.6.

12.3.2 *Adjoint BVP*

In the previous section, we have performed a microlocal elimination of the unknown front to derive energy estimates. Here we are going to use a more algebraic approach, in order to define suitable adjoint problems.

Lemma 12.2 *If $b_1, \dots, b_d$ are $\mathscr{C}^\infty$ mappings $\mathscr{W} \to \mathbb{R}^n$, with $\mathscr{W}$ a contractible subset of $\mathbb{R}^N$, such that for all $u \in \mathscr{W}$ the family $(b_1(u), \dots, b_d(u))$ is independent,* *there exists* $Q \in \mathscr{C}^\infty(\mathscr{W}; \mathbf{GL}_n(\mathbb{R}))$ *so that for all* $u \in \mathscr{W}$ *and all* $\xi \in \mathbb{R}^d$,

$$Q(u) \sum_{j=1}^{d} \xi_j \, b_j(u) \;=\; (\xi_1, \,\cdots\, , \xi_d, \,, 0, \,\cdots\, , 0)^{\mathrm{T}}.$$

Proof If $B(u)$ denotes the matrix with columns $b_1(u), \ldots, b_d(u)$, the contractibility of $\mathscr{W}$ enables us to define $\mathscr{C}^\infty$ mappings $\ell_j : \mathscr{W} \to \mathbf{M}_{1\times n}(\mathbb{R})$ for $j \in \{d+1, \ldots, n\}$ such that for all $u \in \mathscr{W}$, $(\ell_{d+1}(u), \ldots, \ell_n(u))$ is a basis of $\ker B^{\mathrm{T}}$ (in other words, the vector bundle $\ker B^{\mathrm{T}}$ is trivializable, see [85] p. 97). Then for all $u \in \mathscr{W}$, the square matrix $\left(b_1(u), \ldots, b_d(u), \ell_{d+1}(u)^{\mathrm{T}}, \ldots, \ell_n(u)^{\mathrm{T}}\right)$ is invertible and its inverse $Q(u)$ answers the question by construction. □

Applying Lemma 12.2 to $b_j(u) = f^{j-1}(u_+) - f^{j-1}(u_-)$, $N = 2n$ and $\mathscr{W}$ a ball centred at $\underline{u}$ (of radius less than or equal to ρ say), we may rewrite the boundary conditions in (12.3.20) as

$$\begin{pmatrix} \nabla\dot{\chi} \\ 0 \end{pmatrix} \;+\; Q(u)\, M(\mathbf{u}) \cdot \dot{v} \;=\; Q(u)\, g.$$

Then, applying Lemma 9.4 in a ball $\mathscr{W}$ centred at $\underline{\mathbf{u}} = (\underline{u}, \sigma, 0, \ldots, 0)$ (in $\mathbb{R}^{2n+d}$) to $A = \mathbb{A}_d$ and $B \,:\, \mathbf{u} \mapsto Q(u)\, M(\mathbf{u})$ ($= -Q(u)\, \mathbb{A}_d(\mathbf{u})$, of rank n independently of $\mathbf{u}$), we find N, P and R in $\mathscr{C}^\infty(\mathscr{W}; \mathbf{M}_{n\times 2n}(\mathbb{R}))$ such that

$$\mathbb{R}^{2n} \;=\; \ker(Q(u)\, M(\mathbf{u})) \oplus \ker N(\mathbf{u})\,, \quad \mathbb{R}^{2n} \;=\; \ker P(\mathbf{u}) \,\oplus\, \ker R(\mathbf{u}),$$

$$\mathbb{A}_d(\mathbf{u}) = P(\mathbf{u})^{\mathrm{T}} Q(u) M(\mathbf{u}) + R(\mathbf{u})^{\mathrm{T}} N(\mathbf{u}), \quad \ker R(\mathbf{u}) = (\mathbb{A}_d(\mathbf{u}) \ker(Q(u) M(\mathbf{u})))^{\perp}$$

for all $\mathbf{u} \in \mathscr{W}$. This material will serve for the definition of an adjoint version of (12.3.20). Recalling that

$$L(\mathbf{u}(y,z,t)) \;=\; \mathbb{A}^0(u(y,z,t))\, \partial_t \;+\; \sum_{j=1}^{d-1} \mathbb{A}^j(u(y,z,t))\, \partial_j \;+\; \mathbb{A}_d(\mathbf{u}(y,z,t))\, \partial_z,$$

and denoting

$$\begin{aligned} &L(\mathbf{u}(y,z,t))^* \;= \\ &-(\mathbb{A}^0(u(y,z,t)))^{\mathrm{T}}\, \partial_t \;-\; \sum_{j=1}^{d-1} (\mathbb{A}^j(u(y,z,t)))^{\mathrm{T}}\, \partial_j \;-\; (\mathbb{A}_d(\mathbf{u}(y,z,t)))^{\mathrm{T}}\, \partial_z \\ &-\partial_t(\mathbb{A}^0(u(y,z,t)))^{\mathrm{T}} \;-\; \sum_{j=1}^{d-1} \partial_j(\mathbb{A}^j(u(y,z,t)))^{\mathrm{T}} \;-\; \partial_z(\mathbb{A}_d(\mathbf{u}(y,z,t)))^{\mathrm{T}}\;, \end{aligned}$$

we have for all smooth enough v and w,

$$\int_{z>0} \int_{\mathbb{R}^d} (\, w^{\mathrm{T}} L(\mathbf{u}) v - v^{\mathrm{T}} L(\mathbf{u})^* w) \;+\; \int_{\mathbb{R}^d} (w^{\mathrm{T}}\, \mathbb{A}_d(\mathbf{u})\, v)_{|z=0} \;=\; 0,$$

hence by definition of N, P and R,

$$\int_{z>0}\int_{\mathbb{R}^d} (\, w^{\mathrm{T}} L(\mathbf{u}) v - \, v^{\mathrm{T}} L(\mathbf{u})^* w) \; + \int_{\mathbb{R}^d} ((P(\mathbf{u})w)^{\mathrm{T}}\, Q(u)\, M(\mathbf{u})\, v)_{|z=0}$$

$$+ \int_{\mathbb{R}^d} ((R(\mathbf{u})w)^{\mathrm{T}}\, N(\mathbf{u})\, v)_{|z=0} \;=\; 0. \tag{12.3.31}$$

For $v = \dot{v}$ satisfying (12.3.20) together with some function $\dot{\chi}$, the latter equality equivalently reads

$$\int_{z>0}\int_{\mathbb{R}^d} (\, w^{\mathrm{T}} f - \, v^{\mathrm{T}} L(\mathbf{u})^* w) \; + \int_{\mathbb{R}^d} ((P(\mathbf{u})w)^{\mathrm{T}}\, Q(u)\, g\,)_{|z=0}$$

$$+ \int_{\mathbb{R}^d} (\, - (\nabla\dot{\chi})^{\mathrm{T}}\, P_1(\mathbf{u})w \, + \, (N(\mathbf{u})\, v)^{\mathrm{T}}\, R(\mathbf{u})w)_{|z=0} \;=\; 0,$$

where $P_1(\mathbf{u}) := \pi P(\mathbf{u})$, the first d rows among the n rows in $P(\mathbf{u})$. Finally, we may integrate by parts and rewrite

$$\int_{\mathbb{R}^d} (\, - (\nabla\dot{\chi})^{\mathrm{T}}\, P_1(\mathbf{u})w \;\;)_{|z=0} \;=\; \int_{\mathbb{R}^d} (\, \dot{\chi}\, \mathrm{div}_{(t,y)}(P_1(\mathbf{u})w) \;\;)_{|z=0}$$

provided that $\dot{\chi}$ is decaying sufficiently fast.

The computation here above urges us to consider the 'adjoint' problem

$$\begin{cases} L(\mathbf{u})^*\, w \;=\; 0\,, & z \;>\; 0\,, \\ R(\mathbf{u})w \;=\; 0\,,\;\; \mathrm{div}_{(t,y)}(P_1(\mathbf{u})w) \;=\; 0\,, & z \;=\; 0, \end{cases} \tag{12.3.32}$$

which is to some extent 'non-standard', though less than (12.3.20): the boundary operator in (12.3.32) is only (algebro)differential (instead of pseudo-differential in (12.3.20)). A crucial step towards the well-posedness of (12.3.20) is the proof of energy estimates for the adjoint BVP (12.3.32).

Theorem 12.3 *Under the assumptions of Theorem 12.1, the adjoint BVP in* (12.3.32) *meets the standard assumptions allowing* L^2 *energy estimates backward in time, namely for all* $\mathbf{u}$ *in a neighbourhood of* $\underline{\mathbf{u}} = (\underline{u}, \sigma, 0)$,

(CH_*) *The coefficient of* ∂_t *in* $L(\mathbf{u})^*$ *(i.e. the matrix* $\mathbb{A}^0(\mathbf{u})^{\mathrm{T}}$*) is non-singular and the two blocks in* $L(\mathbf{u})^*$ *are* constantly hyperbolic *in the* t*-direction;*

(NC_*) *The coefficient of* ∂_z *in* $L(\mathbf{u})^*$ *is non-singular (i.e. the matrix* $\mathbb{A}_d(\mathbf{u})^{\mathrm{T}}$ *is non-singular);*

(UKL_*) *Denoting*

$$\mathcal{A}_*(\mathbf{u}, \eta, \tau) \;=\; (\mathbb{A}_d(\mathbf{u})^{\mathrm{T}})^{-1}\, (\, \bar{\tau}\, \mathbb{A}^0(u)^{\mathrm{T}} \;-\; i \sum_{j=1}^{d-1} \eta_j\, \mathbb{A}^j(\mathbf{u})^{\mathrm{T}}\,),$$

there exists $C > 0$ *so that for all* $(\eta, \tau) \in \mathbb{R}^{d-1} \times \mathbb{C}$ *with* $\mathrm{Re}\,\tau \geq 0$ *and* $|\tau|^2 + \|\eta\|^2 = 1$,

$$\|W\| \leq C \,\|\, R(\mathbf{u}) W \| + \|\, (\bar{\tau}, -i\eta^{\mathrm{T}}) \cdot P_1(\mathbf{u})\, W \,\| \qquad (12.3.33)$$

for all $W \in E^s(\mathcal{A}_*(\mathbf{u}, \eta, \tau))$.

As a consequence, there exist $\rho > 0$, $C = C(\omega)$ *and* $\gamma_0 = \gamma_0(\omega)$ *so that for*

$$\|\mathbf{u} - \underline{\mathbf{u}}\|_{L^\infty} \leq \rho \quad \textit{and} \quad \|\mathbf{u} - \underline{\mathbf{u}}\|_{W^{1,\infty}} \leq \omega,$$

for all $\gamma \geq \gamma_0$ *and for all* $\dot{w} \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^{2n})$,

$$\gamma \,\|\mathrm{e}^{\gamma t}\, \dot{w}\|^2_{L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})} + \|\mathrm{e}^{\gamma t}\, \dot{w}_{|z=0}\|^2_{L^2(\mathbb{R}^d)}$$

$$\leq C \left(\tfrac{1}{\gamma} \,\|\mathrm{e}^{\gamma t}\, L(\mathbf{u})^*\, \dot{w}\|^2_{L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})} + \|\mathrm{e}^{\gamma t}\, R(\mathbf{u}) \dot{w}_{|z=0}\,\|^2_{L^2(\mathbb{R}^d)} \right.$$

$$\left. + \,\|\mathrm{e}^{\gamma t}\, \mathrm{div}_{(t,y)}(P_1(\mathbf{u}) \dot{w}_{|z=0}\,)\|^2_{H^{-1}_\gamma(\mathbb{R}^d)} \right).$$

Proof The first two properties, (CH$_*$) and (NC$_*$) are trivial consequences of **(CH)** and **(NC)**. For the proof of (UKL$_*$), we first need to check that

$$E^s(\mathcal{A}_*(\mathbf{u}, \eta, \tau)) = (\mathbb{A}_d(\mathbf{u})\, E^s(\mathcal{A}(\mathbf{u}, \eta, \tau)))^\perp, \qquad (12.3.34)$$

where

$$\mathcal{A}(\mathbf{u}, \eta, \tau) = -\,(\mathbb{A}_d(\mathbf{u}))^{-1}\, (\,\tau\, \mathbb{A}^0(u) + i \sum_{j=1}^{d-1} \eta_j\, \mathbb{A}^j(\mathbf{u})\,).$$

(As already done before, we make a 'subtle' distinction between the notations $\mathcal{A}$ and $\mathcal{A}_{\mathbf{u}}$: the mappings $\mathcal{A} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{d-1} \times \mathbb{C}^+ \to \mathbf{M}_{2n \times 2n}(\mathbb{C})$ and $\mathcal{A}_{\mathbf{u}} : \mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{C}^+ \to \mathbf{M}_{2n \times 2n}(\mathbb{C})$ are related by $\mathcal{A}(\mathbf{u}(y, z, t), \eta, \tau) = \mathcal{A}_{\mathbf{u}}(y, z, t, \eta, \tau)$ for all $(y, z, t, \eta, \tau) \in \mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{C}^+$.)

The proof of (12.3.34) is classical and was already done in Chapter 4 but we recall it for completeness. We first observe the matrices $\mathcal{A}(\mathbf{u}, \eta, \tau)$ and $\mathcal{A}_*(\mathbf{u}, \eta, \tau)$ are simultaneously hyperbolic, the latter $\mathcal{A}_*(\mathbf{u}, \eta, \tau)$ being conjugate to $-\mathcal{A}(\mathbf{u}, \eta, \tau)^*$: more precisely,

$$\mathcal{A}_*(\mathbf{u}, \eta, \tau) = -\,(\mathbb{A}_d(\mathbf{u})^{\mathrm{T}})^{-1}\, \mathcal{A}^*(\mathbf{u}, \eta, \tau)\, \mathbb{A}_d(\mathbf{u})^{\mathrm{T}}.$$

Hence the spaces $E^s(\mathcal{A}_*(\mathbf{u}, \eta, \tau))$ and $(\mathbb{A}_d(\mathbf{u})\, E^s(\mathcal{A}(\mathbf{u}, \eta, \tau)))^\perp$ are both of the same dimension, equal to $n+1$ since $E^s(\mathcal{A}(\mathbf{u}, \eta, \tau))$ is of dimension $n-1$ (by the assumption **(N)**). Furthermore, when $\mathcal{A}(\mathbf{u}, \eta, \tau)$ is hyperbolic, a standard ODE argument shows $E^s(\mathcal{A}_*(\mathbf{u}, \eta, \tau))$ is a subspace of $(\mathbb{A}_d(\mathbf{u})\, E^s(\mathcal{A}(\mathbf{u}, \eta, \tau)))^\perp$. Indeed, take $W = \psi(0)$ with ψ a solution of $\psi' = \mathcal{A}_* \psi$ tending to zero at $+\infty$.

For all $U = \phi(0)$ with ϕ a solution of $\phi' = \mathcal{A}\phi$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\psi(x)^* \mathbb{A}_d \, \phi(x) \,=\, (\mathcal{A}_* \psi(x))^* \mathbb{A}_d \phi(x) \,+\, \psi(x)^* \mathbb{A}_d \, \mathcal{A} \, \phi(x) \,=\, 0,$$

since $(\mathcal{A}_*)^* \, \mathbb{A}_d \,+\, \mathbb{A}_d \, \mathcal{A} \,=\, 0$, hence if, moreover, ϕ tends to zero at $+\infty$,

$$W^* \, \mathbb{A}_d \, U \,=\, \lim_{x \to +\infty} \, \psi(x)^* \mathbb{A}_d \, \phi(x) \,=\, 0.$$

This proves (12.3.34) when $\mathcal{A}(\mathbf{u}, \eta, \tau)$ is hyperbolic, in particular when $\mathrm{Re}\,\tau > 0$. By continuity, (12.3.34) is also true for $\mathrm{Re}\,\tau = 0$.

Now, suppose $W \in E^s(\mathcal{A}_*)$ is such that

$$R(\mathbf{u}) W \,=\, 0 \quad \text{and} \quad (\bar{\tau}, -i\eta^{\mathrm{T}}) \cdot P_1(\mathbf{u}) \, W \,=\, 0.$$

We are going to show that $P(\mathbf{u}) \, W \,=\, 0$, which will imply $W \,=\, 0$ (recall that by construction $\ker R(\mathbf{u})$ and $\ker P(\mathbf{u})$ are supplementary). To prove $P(\mathbf{u}) \, W \,=\, 0$, we compute $Y^* P(\mathbf{u}) \, W$ for an arbitrary $Y \in \mathbb{C}^{2n}$. A straightforward reformulation of the assumption **(UKL)** with our current notations shows there exists a unique pair $(\dot{X}, \dot{U}) \in \mathbb{C} \times E^s(\mathcal{A}(\mathbf{u}, \eta, \tau))$ such that

$$Y \,=\, \dot{X} \, (\tau, i\eta, 0, \ldots, 0)^{\mathrm{T}} \,+\, Q(u) \, M(\mathbf{u}) \, \dot{U}.$$

Therefore,

$$Y^* P(\mathbf{u}) \, W \,=\, \bar{\dot{X}} \, (\bar{\tau}, -i\eta^{\mathrm{T}}) \cdot P_1(\mathbf{u}) \, W \,+\, (Q(u) \, M(\mathbf{u}) \, \dot{U})^* \, P(\mathbf{u}) \, W,$$

where the first term is zero by assumption on W; hence

$$\overline{Y^* P(\mathbf{u}) \, W} \,=\, W^* P(\mathbf{u})^{\mathrm{T}} \, Q(u) \, M(\mathbf{u}) \, \dot{U} \,=\, W^* \mathbb{A}_d(\mathbf{u}) \, \dot{U} - W^* R(\mathbf{u})^{\mathrm{T}} \, N(\mathbf{u}) \dot{U} \,=\, 0,$$

since W belongs to $(\mathbb{A}_d E^s(\mathcal{A}))^{\perp}$ and $R(\mathbf{u}) W \,=\, 0$. Therefore, the mapping

$$W \in E^s(\mathcal{A}_*) \,\mapsto\, (R(\mathbf{u}) W, (\bar{\tau}, -i\eta^{\mathrm{T}}) \cdot P_1(\mathbf{u}) \, W) \in \mathbb{C}^n \times \mathbb{C}$$

is one-to-one, thus also onto since $\dim E^s(\mathcal{A}_*) \,=\, n+1$; the norm of the inverse mapping is uniformly bounded on the compact set $\{ \, (\eta, \tau) \in \mathbb{R}^{d-1} \times \mathbb{C} \, ; \; \mathrm{Re}\,\tau \geq 0 \, , \; |\tau|^2 + \|\eta\|^2 = 1 \, \}$.

The attentive reader will have noticed that (UKL_*) is simply the uniform Kreiss–Lopatinskiĭ condition *backward* in time for the BVP (12.3.32). Indeed, freezing the coefficients in the principal part of $L(\mathbf{u})^*$ and performing a Fourier–Laplace transform in the direction $(y, -t)$ the BVP (12.3.32) we get the ODE problem

$$\begin{cases} \dfrac{\mathrm{d}\dot{W}}{\mathrm{d}z} \,=\, \mathcal{A}_*(\mathbf{u}, \eta, \tau) \, \dot{W} & \text{for} \;\; z > 0 \, , \\[2ex] R(\mathbf{u}) \dot{W} \,=\, 0 \quad \text{and} \quad (\bar{\tau}, -i\eta^{\mathrm{T}}) \cdot P_1(\mathbf{u}) \, \dot{W} \,=\, 0 & \text{at} \;\; z = 0 \, . \end{cases}$$

Thanks to the properties (CH_*), (NC_*) and (UKL_*), the same method of proof as for the original BVP (12.3.20) (in Theorem 12.1), replacing there the interior symbol $\mathcal{A}_{\mathbf{u}}$ by $\mathcal{A}_*(\mathbf{u}(y,z,t),\eta,\tau)$, and the boundary symbol $\Pi_{\mathbf{u}}\,\mathcal{M}_{\mathbf{u}}$ by

$$\begin{pmatrix} R(\mathbf{u}(y,z,t)) \\ (\bar{\tau},-i\eta^{\mathrm{T}})\cdot P_1(\mathbf{u}(y,z,t)) \end{pmatrix},$$

shows the announced backward weighted estimate for the adjoint BVP (12.3.32): the H^{-1}_{γ} norm in the right-hand side shows up because when the inequality (12.3.33) in (UKL$_*$) is extended to all pairs (τ,η), it yields, for homogeneity reasons,

$$\|W\| \,\leq\, C\,\|\,R(\mathbf{u})W\| \,+\, \lambda^{-1,\gamma}(\delta,\eta)\,\|\,(\bar{\tau},-i\eta^{\mathrm{T}})\cdot P_1(\mathbf{u})\,W\,\|.$$

□

12.3.3 *Well-posedness of the BVP*

Theorem 12.4 *Under the assumptions of Theorem 12.1, there exist* $\rho>0$ *and* $\gamma_0=\gamma_0(\omega)$ *so that for*

$$\|\mathbf{u}-\underline{\mathbf{u}}\|_{L^\infty}\leq\rho \quad \textit{and} \quad \|\mathbf{u}\|_{W^{1,\infty}}\leq\omega,$$

for all $\gamma\geq\gamma_0$*, for all* $f\in \mathrm{e}^{\gamma t}\,L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ *and all* $g\in \mathrm{e}^{\gamma t}\,L^2(\mathbb{R}^{d-1}\times\mathbb{R})$*, there is one and only one solution* $(\dot{v},\dot{\chi})\in \mathrm{e}^{\gamma t}\,L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})\times \mathrm{e}^{\gamma t}\,H^{1/2}(\mathbb{R}^{d-1}\times\mathbb{R})$ *of the BVP in* (12.3.20)*. Furthermore,* $\dot{\chi}$ *belongs in fact to* $\mathrm{e}^{\gamma t}\,H^1(\mathbb{R}^{d-1}\times\mathbb{R})$*, the trace of* $\dot{v}$ *at* $z=0$ *belongs to* $\mathrm{e}^{\gamma t}\,L^2(\mathbb{R}^{d-1}\times\mathbb{R})$*, and* $(\widetilde{v}_\gamma,\widetilde{\chi}_\gamma)\,:=\,\mathrm{e}^{-\gamma t}(\dot{v},\dot{\chi})$ *enjoys the estimate*

$$\begin{aligned}\gamma\,\|\widetilde{v}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|(\widetilde{v}_\gamma)_{|z=0}\|^2_{L^2(\mathbb{R}^d)} \,+\, \|\widetilde{\chi}_\gamma\|^2_{H^1_\gamma(\mathbb{R}^d)} \\ \leq\, C\,\left(\frac{1}{\gamma}\,\|\widetilde{f}_\gamma\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \,+\, \|\widetilde{g}_\gamma\|^2_{L^2(\mathbb{R}^d)}\right)\end{aligned} \tag{12.3.35}$$

for some constant $C=C(\omega)$*.*

Note: the estimate (12.3.35) here above is simply another way of writing (12.3.22) (in Theorem 12.1).

Proof Noting that (12.3.20) is equivalent to $L^\gamma_{\mathbf{u}}\,\widetilde{v}_\gamma \,=\, \widetilde{f}_\gamma$ in the interior and

$$\begin{pmatrix} (\gamma\,+\,\partial_t)\widetilde{\chi}_\gamma \\ \nabla_y\widetilde{\chi}_\gamma \\ 0 \end{pmatrix} \,+\, Q(u)\,M(\mathbf{u})\cdot\widetilde{v}_\gamma \,=\, Q(u)\,\widetilde{g}_\gamma$$

at the boundary $z\,=\,0$, we may reformulate the conclusion of Theorem 12.4 as follows: for γ large enough, for all $f\in L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ and all $g\in L^2(\mathbb{R}^{d-1}\times\mathbb{R})$, there is one and only one $(v,\psi)\in L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})\times H^{1/2}(\mathbb{R}^{d-1}\times\mathbb{R})$

such that

$$L^\gamma_{\mathbf{u}} v = f \quad \text{and} \quad \begin{pmatrix} (\gamma + \partial_t)\psi \\ \nabla_y \psi \\ 0 \end{pmatrix} + Q(u)\, M(\mathbf{u}) \cdot v_{|z=0} = Q(u)\, g\,, \qquad (12.3.36)$$

and this solution satisfies an estimate of the form

$$\gamma\, \|v\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|v_{|z=0}\|^2_{L^2(\mathbb{R}^d)} + \|\psi\|^2_{H^1_\gamma(\mathbb{R}^d)} \lesssim \frac{1}{\gamma}\, \|f\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R}^d)}\,.$$

Existence of a weak solution Unsurprisingly, we are going to use the adjoint BVP (12.3.32) introduced in the previous section, or more precisely the equivalent problem

$$\begin{cases} (L^\gamma_{\mathbf{u}})^*\, \mathrm{e}^{\gamma t}\, w = 0\,, & z > 0\,, \\ R(\mathbf{u})\mathrm{e}^{\gamma t}\, w = 0\,,\ \mathrm{div}_{-\gamma}(P_1(\mathbf{u})\mathrm{e}^{\gamma t}\, w) = 0\,, & z = 0\,, \end{cases}$$

where the differential operator $\mathrm{div}_{-\gamma}$ is defined by

$$\mathrm{div}_{-\gamma}(p_0\,, p_1, \dots, p_{d-1}) = (-\gamma + \partial_t)\, p_0 + \sum_{j=1}^{d-1} \partial_j p_j.$$

From this point of view the 'dual' energy estimate of Theorem 12.3 equivalently reads

$$\gamma\, \|w\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|w_{|z=0}\|^2_{L^2(\mathbb{R}^d)} \le C\, \Big(\frac{1}{\gamma}\, \|(L^\gamma_{\mathbf{u}})^*\, w\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|R(\mathbf{u}) w_{|z=0}\,\|^2_{L^2(\mathbb{R}^d)} + \|\mathrm{div}_{-\gamma}(P_1(\mathbf{u}) w_{|z=0}\,)\|^2_{H^{-1}_\gamma(\mathbb{R}^d)} \Big)\,.$$

The resolution of the BVP in (12.3.36) follows the same lines as the proof of Theorem 9.17. We consider the set

$$\mathscr{E} := \{\, w \in \mathscr{D}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})\,;\ R(\mathbf{u}) w_{|z=0} = 0\,,\ \mathrm{div}_{-\gamma}(P_1(\mathbf{u}) w_{|z=0}) = 0 \}.$$

Theorem 12.3 shows that for all $w \in \mathscr{E}$,

$$\gamma\, \|w\|^2_{L^2} + \|w_{|z=0}\|^2_{L^2} \le \frac{C}{\gamma}\, \|(L^\gamma_{\mathbf{u}})^*\, w\|^2_{L^2}.$$

This allows the definition of a bounded linear form ℓ on $(L^\gamma_{\mathbf{u}})^* \mathscr{E}$ by

$$\ell((L^\gamma_{\mathbf{u}})^* w) = \int_{z>0} \int_{\mathbb{R}^d} w^{\mathrm{T}} f + \int_{\mathbb{R}^d} ((P(\mathbf{u}) w)^{\mathrm{T}}\, Q(u)\, g\,)_{|z=0}.$$

Indeed, we have

$$\begin{aligned} &\left| \int_{z>0} \int_{\mathbb{R}^d} w^{\mathrm{T}} f \,+\, \int_{\mathbb{R}^d} ((P(\mathbf{u})w)^{\mathrm{T}}\, Q(u)\, g\,)_{|z=0} \right| \\ &\leq \, \|f\|_{L^2}\, \|w\|_{L^2} \,+\, \|P\circ \mathbf{u}\|_{L^\infty}\, \|Q\circ u\|_{L^\infty}\, \|g\|_{L^2}\, \|w_{|z=0}\|_{L^2} \\ &\lesssim \, \Big(\frac{1}{\gamma}\, \|f\|_{L^2} \,+\, \frac{1}{\gamma^{1/2}}\, \|g\|_{L^2} \Big)\, \|L^*\, v\|_{L^2}. \end{aligned}$$

Therefore, by the Hahn–Banach theorem, ℓ extends to a continuous form on L^2, and by the Riesz theorem, there exists $v \in L^2$ such that

$$\ell((L^\gamma_{\mathbf{u}})^* w) \,=\, \int_{z>0} \int_{\mathbb{R}^d} v^{\mathrm{T}} (L^\gamma_{\mathbf{u}})^*\, w.$$

We thus get in particular, by definition of ℓ,

$$\int_{z>0} \int_{\mathbb{R}^d} (\, v^{\mathrm{T}} (L^\gamma_{\mathbf{u}})^*\, w \,-\, f^{\mathrm{T}}\, w\,) \,=\, 0$$

for all $w \in \mathscr{D}(\mathbb{R}^{d-1} \times (0,+\infty) \times \mathbb{R})$, hence $L^\gamma_{\mathbf{u}} v \,=\, f$ (in the sense of distributions). Consequently, using the identity (12.3.31), or more precisely its modified version obtained with $L^\gamma_{\mathbf{u}}$ instead of $L(\mathbf{u})$, we find that

$$\int_{\mathbb{R}^d} ((P(\mathbf{u})w)^{\mathrm{T}}\, Q(u)\, g\,)_{|z=0} \,=\, \int_{\mathbb{R}^d} ((P(\mathbf{u})w)^{\mathrm{T}}\, Q(u)\, M(\mathbf{u})\, v\,)_{|z=0} \qquad (12.3.37)$$

for all $w \in \mathscr{E}$, and by extension this is true for all $w \in H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ such that $R(\mathbf{u})w \,=\, 0$ and $\mathrm{div}_{-\gamma}(P_1(\mathbf{u})w) \,=\, 0$.

In order to recover the boundary condition in (12.3.36), we first use a (standard) trace-lifting argument. For all $\varphi \in H^{1/2}(\mathbb{R}^{d-1} \times \mathbb{R}; \mathbb{R}^{2n})$ there exists $\Phi \in H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^{2n})$ so that $\Phi_{|z=0} \,=\, \varphi$. Therefore, for all $\theta \in H^{1/2}(\mathbb{R}^d; \mathbb{R}^n)$ there exists $w \in H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^{2n})$ so that $R(\mathbf{u})\, w_{|z=0} \,=\, 0$ and $P(\mathbf{u})\, w_{z=0} \,=\, \theta$: indeed, by construction

$$q \,:\, (y,z,t) \,\mapsto\, \begin{pmatrix} P(\mathbf{u}(y,z,t)) \\ R(\mathbf{u}(y,z,t)) \end{pmatrix}^{-1}$$

belongs to $W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}; \mathbf{GL}_{2n}(\mathbb{R}))$, so it suffices to define $w \,=\, q\,\Phi$, where Φ is obtained by trace lifting from $\varphi \,:\, (y,t) \,\mapsto\, (\theta(y,t), 0, \ldots, 0)$. In particular, for all $\theta_2 \in H^{1/2}(\mathbb{R}^d; \mathbb{R}^{n-d})$ there exists $w_2 \in H^1(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R}; \mathbb{R}^{2n})$ so that $R(\mathbf{u})\,(w_2)_{|z=0} \,=\, 0$, $P_1(\mathbf{u})\,(w_2)_{|z=0} \,=\, 0$, and $P_2(\mathbf{u})\,(w_2)_{|z=0} \,=\, \theta_2$, where $P_2(\mathbf{u}) \,=\, (I_n \,-\, \pi)\, P(\mathbf{u})$ denotes the last $(n-d)$ rows in $P(\mathbf{u})$. Applying (12.3.37) to $w = w_2$ we get

$$\int_{\mathbb{R}^d} {\theta_2}^{\mathrm{T}}\, Q_2(u)\, (\, g \,-\, M(\mathbf{u})\, v\,)_{|z=0} \,=\, 0,$$

where naturally $Q_2(u) = (I_n - \pi)\, Q(u)$. This implies

$$Q_2(u)\,(\,g - M(\mathbf{u})\,v\,)_{|z=0} = 0.$$

Similarly, for all $\theta_1 \in H^{1/2}(\mathbb{R}^d;\mathbb{R}^d)$ there exists $w_1 \in H^1(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R};\mathbb{R}^{2n})$ so that $R(\mathbf{u})\,(w_1)_{|z=0} = 0$, $P_1(\mathbf{u})\,(w_1)_{|z=0} = \theta_1$, and $P_2(\mathbf{u})\,(w_1)_{|z=0} = 0$, and if additionally $\mathrm{div}_{-\gamma}\theta_1 = 0$ we can apply (12.3.37) to $w = w_1$, which gives

$$\int_{\mathbb{R}^d} {\theta_1}^{\mathrm{T}}\, Q_1(u)\,(\,g - M(\mathbf{u})\,v\,)_{|z=0} = 0.$$

This implies the existence of $\psi \in H^{1/2}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$ such that

$$Q_1(u)\,(\,g - M(\mathbf{u})\,v\,)_{|z=0} = \begin{pmatrix} (\gamma + \partial_t)\psi \\ \nabla_y\psi \end{pmatrix}$$

thanks to the following simple result.

Lemma 12.3 *For all $\gamma \geq 1$, the mapping*

$$\begin{array}{lll} \nabla^\gamma : & H^{1/2}(\mathbb{R}^d;\mathbb{R}) \to H^{-1/2}(\mathbb{R}^d;\mathbb{R}^d) \\ & \psi \qquad\qquad\quad \mapsto \nabla^\gamma\psi := \begin{pmatrix} (\gamma + \partial_t)\psi \\ \nabla_y\psi \end{pmatrix} \end{array}$$

has range

$$(\ker \mathrm{div}_{-\gamma})^\perp = \{\, v \in H^{1/2}(\mathbb{R}^d;\mathbb{R}^d)\ ;\ \langle v\,,\,\theta\,\rangle_{(H^{-1/2},H^{1/2})} = 0$$
$$\text{for all }\ \theta \in H^{1/2}(\mathbb{R}^d;\mathbb{R}^d)\,;\ \mathrm{div}_{-\gamma}\theta = 0\,\}\,.$$

Proof This is a Fourier-transform exercise. Indeed, the range of ∇^γ is obviously a subset of $(\mathrm{kerdiv}_{-\gamma})^\perp$, which can also be written as

$$\left\{ v \in H^{1/2}(\mathbb{R}^d;\mathbb{R}^d)\,;\ \widehat{v}(\delta,\eta) \parallel \begin{pmatrix} \gamma + i\delta \\ i\eta \end{pmatrix} \right\},$$

and for all v in this set we have

$$\widehat{v}(\delta,\eta) = \frac{(\gamma - i\delta, -i\eta^{\mathrm{T}})\cdot\widehat{v}(\delta,\eta)}{\gamma^2 + \delta^2 + \|\eta\|^2}\begin{pmatrix} \gamma + i\delta \\ i\eta \end{pmatrix},$$

which equivalently means that

$$v = \nabla^\gamma\left((-\gamma^2 + \Delta)^{-1}\,\mathrm{div}_{-\gamma}v\right),$$

where $(-\gamma^2 + \Delta)^{-1}\,\mathrm{div}_{-\gamma}v$ does belong to $H^{1/2}(\mathbb{R}^d;\mathbb{R})$. □

Weak=strong argument As ‘usual’, we consider a mollifying operator R_ε in the (y,t)-edirections and consider $v_\varepsilon := R_\varepsilon v$ and $\psi_\varepsilon := R_\varepsilon\psi$ (where (v,ψ) is the weak solution found here above), $f_\varepsilon := R_\varepsilon\mathbb{A}_d(\mathbf{u})^{-1} f$ (where f is a given source term in L^2) and $g_\varepsilon := R_\varepsilon g$ (where g is a L^2 data on the boundary). These regularized functions satisfy the following properties

$$\begin{array}{lll} f_\varepsilon \in L^2(\mathbb{R}^+; H^{+\infty}(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}^{2n})) & \text{and} & f_\varepsilon \xrightarrow{L^2(\mathbb{R}^+; L^2(\mathbb{R}^d\times\mathbb{R};\mathbb{R}^{2n}))} \mathbb{A}_d(\mathbf{u})^{-1} f\,, \\ v_\varepsilon \in L^2(\mathbb{R}^+; H^{+\infty}(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}^{2n})) & \text{and} & v_\varepsilon \xrightarrow{L^2(\mathbb{R}^+; L^2(\mathbb{R}^d\times\mathbb{R};\mathbb{R}^{2n}))} v\,, \\ (v_\varepsilon)_{|z=0} \in H^{+\infty}(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}^{2n}) & \text{and} & (v_\varepsilon)_{|z=0} \xrightarrow{H^{-1/2}(\mathbb{R}^d\times\mathbb{R};\mathbb{R}^{2n})} v_{|z=0}\,, \\ \psi_\varepsilon \in H^{+\infty}(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}) & \text{and} & \psi_\varepsilon \xrightarrow{H^{1/2}_\gamma(\mathbb{R}^d\times\mathbb{R};\mathbb{R})} \psi\,, \\ g_\varepsilon \in H^{+\infty}(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}^n) & \text{and} & g_\varepsilon \xrightarrow{L^2(\mathbb{R}^d\times\mathbb{R};\mathbb{R}^n)} g\,. \end{array}$$

Furthermore, we have (see Lemma 9.5)

$$[P^\gamma_{\mathbf{u}}, R_\varepsilon](v) \xrightarrow{L^2(\mathbb{R}^+; L^2(\mathbb{R}^d\times\mathbb{R};\mathbb{R}^{2n}))} 0 \quad \text{and} \quad [B^\gamma_{\mathbf{u}}, R_\varepsilon](v_{|z=0}, \psi) \xrightarrow{L^2(\mathbb{R}^d\times\mathbb{R};\mathbb{R}^n)} 0$$

(where the operator $P^\gamma_{\mathbf{u}}$ is such that $f = L^\gamma_{\mathbf{u}} v$ equivalently reads $\partial_z v - P^\gamma_{\mathbf{u}}\, v = \mathbb{A}_d(\mathbf{u})^{-1} f$). Now we easily see that $\partial_z v_\varepsilon = f_\varepsilon + P^\gamma_{\mathbf{u}}\, v_\varepsilon - [P^\gamma_{\mathbf{u}}, R_\varepsilon](v)$ belongs to $L^2(\mathbb{R}^+; L^2(\mathbb{R}^d \times \mathbb{R}; \mathbb{R}^{2n}))$, hence v_ε is in $H^1(\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R})$. By Remark 12.4, the energy estimate (12.3.22) (in Theorem 12.1) thus applies to the pair $(v_\varepsilon, \psi_\varepsilon)$: more precisely we have

$$\gamma\, \|v_\varepsilon\|^2_{L^2} + \|(v_\varepsilon)_{|z=0}\|^2_{L^2} + \|\psi_\varepsilon\|^2_{H^1_\gamma} \lesssim \frac{1}{\gamma}\, \|\partial_z v_\varepsilon - P^\gamma_{\mathbf{u}}\, v_\varepsilon\|^2_{L^2} + \|B^\gamma_{\mathbf{u}}\,(v_\varepsilon, \psi_\varepsilon)\|^2_{L^2}.$$

By linearity of the operators $P^\gamma_{\mathbf{u}}$ and $B^\gamma_{\mathbf{u}}$, this inequality also applies to pairs $(v_\varepsilon - v_{\varepsilon'}, \psi_\varepsilon - \psi_{\varepsilon'})$, and together with the above limiting properties this shows that $(v_\varepsilon)_{|z=0}$ is a Cauchy sequence in L^2 and ψ_ε is a Cauchy sequence in H^1_γ. By uniqueness of limits in the sense of distributions, this implies that $v_{|z=0}$ is the limit of $(v_\varepsilon)_{|z=0}$ in L^2, and ψ is the limit of ψ_ε in H^1_γ. Then, by passing to the limit in the estimate for $(v_\varepsilon, \psi_\varepsilon)$, we get

$$\gamma\, \|v\|^2_{L^2} + \|v_{|z=0}\|^2_{L^2} + \|\psi\|^2_{H^1_\gamma} \lesssim \frac{1}{\gamma}\, \|\partial_z v - P^\gamma_{\mathbf{u}}\, v\|^2_{L^2} + \|B^\gamma_{\mathbf{u}}\,(v, \psi)\|^2_{L^2},$$

or equivalently,

$$\gamma\, \|v\|^2_{L^2} + \|v_{|z=0}\|^2_{L^2} + \|\psi\|^2_{H^1_\gamma} \lesssim \frac{1}{\gamma}\, \|L^\gamma_{\mathbf{u}} v\|^2_{L^2} + \|B^\gamma_{\mathbf{u}}\,(v, \psi)\|^2_{L^2}\,, \qquad (12.3.38)$$

which is another way of writing (12.3.35). This completes the proof of Theorem 12.4. □

Now, when the coefficients $(\mathbf{u})$ and the data f, g enjoy more regularity, we can prove more regularity on the solution. This is the purpose of the following result (analogous to Theorem 9.20 for standard BVP).

Theorem 12.5 *We still make the assumptions of Theorem 12.1, and assume, moreover, (as in Theorem 12.2) that* $u - \underline{u}$, $(u - \underline{u})_{|z=0}$ *and* $(\partial_t \chi - \sigma, \nabla_y \chi)$ *are in* H^m *for some integer* $m > (d+1)/2 + 1$. *Then the solution* $(\dot{v}, \dot{\chi})$ *of the BVP* (12.3.20) *given by Theorem 12.4 is such that* $\dot{v}$ *belongs to* $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times$

$\mathbb{R};\mathbb{R}^{2n})$, $\dot v_{|z=0}$ belongs to $H^m(\mathbb{R}^d;\mathbb{R}^{2n})$, and $(\partial_t\dot\chi,\nabla_y\dot\chi)$ belongs to $H^m(\mathbb{R}^d;\mathbb{R}^d)$, and altogether they satisfy the estimate

$$\gamma\,\|\dot v\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|\dot v_{|z=0}\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)} + \|\dot\chi\|^2_{\mathscr{H}^{m+1}_\gamma(\mathbb{R}^d)}$$

$$\leq C_m(\mu)\left(\frac{1}{\gamma}\,\|f\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} + \|g\|^2_{\mathscr{H}^m_\gamma(\mathbb{R}^d)}\right)$$

for $\gamma\geq\gamma_m(\mu)\geq 1$, where μ is an upper bound for the H^m norm of the coefficients; more precisely,

$$\|u-\underline{u}\|_{H^m}\leq\mu\,,\ \|u_{|z=0}-\underline{u}_{|z=0}\|_{H^m}\leq\mu\quad\textit{and}\quad\|(\partial_t\chi-\sigma,\nabla_y\chi)\|_{H^m}\leq\mu\,.$$

Proof As for Theorem 9.20 on standard BVP, it is in two steps: 1) the proof for infinitely smooth coefficients and 2) the extension to H^m coefficients by passing to the limit in regularized problems (to which the first step applies).

Step 1) Assume here that u belongs to $\mathscr{D}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R};\mathbb{R}^{2n})$ and $(\partial_t\chi-\sigma,\nabla_y\chi)$ belongs to $\mathscr{D}(\mathbb{R}^d;\mathbb{R}^d)$.

We already know from Theorem 12.4 that $v:=\mathrm{e}^{-\gamma t}\dot v$ belongs to $L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$, $v_{|z=0}$ belongs to $L^2(\mathbb{R}^d)$, and $\psi:=\mathrm{e}^{-\gamma t}\dot\chi$ belongs to $H^1(\mathbb{R}^d)$. Our aim is to prove enough regularity on (v,ψ) to be allowed to use Theorem 12.2 and thus obtain $\mathcal{H}^m_\gamma$ estimates on $(\dot v,\dot\chi)$ in terms of μ. A first idea is to proceed by induction and use the same method as in the proof of Theorem 9.10. So, assume that v belongs to $H^k(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$, $v_{|z=0}$ belongs to $H^k(\mathbb{R}^d)$, and ψ belongs to $H^{k+1}(\mathbb{R}^d)$ for $0\leq k\leq m-1$. We want to show that v belongs to $H^{k+1}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$, $v_{|z=0}$ belongs to $H^{k+1}(\mathbb{R}^d)$, and ψ belongs to $H^{k+2}(\mathbb{R}^d)$. In fact, thanks to Proposition 2.3 applied with z instead of t, it is sufficient to show that v belongs to $L^2(\mathbb{R}^+;H^{k+1}(\mathbb{R}^d))$. We are going to use several ingredients: the characterization of H^{k+1} functions given by Proposition 9.2; the two-sided inequality provided by Proposition 9.3; the commutators estimates given by Theorem 9.11; the energy estimate (12.3.38). We introduce a smoothing operator R_ε in the (y,t) directions satisfying the requirements of Proposition 9.3 and define $\varphi_\varepsilon:=R_\varepsilon(v)\in L^2(\mathbb{R}^+;H^{+\infty}(\mathbb{R}^d))$, $\psi_\varepsilon:=R_\varepsilon(\psi)\in H^{+\infty}(\mathbb{R}^d)$. By Proposition 2.3 again, φ_ε belongs at least to $H^1(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})$, and therefore (by Remark 12.4) we can apply the energy estimate (12.3.38) to the pair $(\varphi_\varepsilon,\psi_\varepsilon)$. This gives for γ large enough (and in particular $\gamma\geq 1$),

$$\gamma\,\|\varphi_\varepsilon\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} + \|(\varphi_\varepsilon)_{|x_d=0}\|^2_{L^2(\mathbb{R}^d)} + \|\psi_\varepsilon\|^2_{H^1(\mathbb{R}^d)}$$

$$\lesssim\frac{1}{\gamma}\,\|L^\gamma_{\mathbf{u}}\varphi_\varepsilon\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}^+)} + \|B^\gamma_{\mathbf{u}}((\varphi_\varepsilon)_{|x_d=0},\psi_\varepsilon)\|^2_{L^2(\mathbb{R}^d)}\,.$$

Now, Theorem 9.11 provides some bounds for the commutators $[B^\gamma_{\mathbf{u}},R_\varepsilon]$ and $[P^\gamma_{\mathbf{u}},R_\varepsilon]$, with (as usual) $P^\gamma_{\mathbf{u}}=\partial_z-(\mathbb{A}^d)^{-1}L^\gamma_{\mathbf{u}}$. Since $(\mathbb{A}^d)^{-1}$ is uniformly bounded, this eventually gives

$$\gamma \int_0^{+\infty} \|v\|_{k,\theta}^2 \, \mathrm{d}x_d \, + \, \|v_{|x_d=0}\|_{k,\theta}^2 \, + \, \|\psi\|_{k+1,\theta}^2 \lesssim \tfrac{1}{\gamma} \int_0^{+\infty} \|v\|_{k,\theta}^2 \, \mathrm{d}x_d$$
$$+ \gamma \int_0^{+\infty} \|v\|_{m-1,\theta}^2 \, \mathrm{d}x_d \, + \, \tfrac{1}{\gamma} \int_0^{+\infty} n_{k,\theta}(L_{\mathbf{u}}^\gamma v(x_d)) \, \mathrm{d}x_d$$
$$+ n_{k,\theta}(B_{\mathbf{u}}^\gamma(v_{|x_d=0}, \psi)) \, + \, \|v_{|x_d=0}\|_{k-1,\theta}^2 \, + \, \|\psi\|_{k,\theta}^2 \, .$$

For γ large enough the first term in the right-hand side can be absorbed in the left-hand side. Therefore, applying Propostions 9.2 and 9.3, we get

$$\gamma \int_0^{+\infty} \|v\|_{k,\theta}^2 \, \mathrm{d}x_d \, + \, \|v_{|x_d=0}\|_{k,\theta}^2 \, + \, \|\psi\|_{k+1,\theta}^2 \lesssim \frac{1}{\gamma} \, \|L_{\mathbf{u}}^\gamma v\|_{L^2(\mathbb{R}^+;H^{k+1}(\mathbb{R}^d))}^2$$

$$+ \, \|B_{\mathbf{u}}^\gamma(v_{|x_d=0}, \psi)\|_{H^{k+1}(\mathbb{R}^d)}^2 + \gamma \, \|v\|_{L^2(\mathbb{R}^+;H^{k+1}(\mathbb{R}^d))}^2 + \|v_{|x_d=0}\|_{H^k(\mathbb{R}^d)}^2 + \|\psi\|_{k+1}^2.$$

By assumption (since $k \leq m-1$), the right-hand side is finite, and of course is independent of θ, Proposition 9.2 thus shows that v belongs to $L^2(\mathbb{R}^+; H^{k+1}(\mathbb{R}^d))$ (hence v belongs to $H^{k+1}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ by Proposition 2.3), $v_{|x_d=0}$ belongs to $H^{k+1}(\mathbb{R}^d)$, and ψ belongs to $H^{k+2}(\mathbb{R}^d)$. Therefore, the induction process shows that v belongs to $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$, $v_{|x_d=0}$ belongs to $H^m(\mathbb{R}^d)$, and ψ belongs to $H^{m+1}(\mathbb{R}^d)$. The regularity obtained in this way for (v, ψ) is not sufficient yet to apply Theorem 12.2 and get $\mathscr{H}_\gamma^m$ estimates. However, we can construct $f_\gamma^\varepsilon \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and $g_\gamma^\varepsilon \in \mathscr{D}(\mathbb{R}^{d-1} \times \mathbb{R})$ such that f_γ^ε goes to f in $\mathscr{H}_\gamma^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and g_γ^ε goes to g in $\mathscr{H}_\gamma^m(\mathbb{R}^{d-1} \times \mathbb{R})$, and by the first step the solution $(v_\gamma^\varepsilon, \chi_\gamma^\varepsilon)$ of the BVP

$$L_{\mathbf{u}} \, v_\gamma^\varepsilon \, = \, f_\gamma^\varepsilon \ \text{ for } \ z > 0 \, , \quad B_{\mathbf{u}} \, (v_\gamma^\varepsilon, \chi_\gamma^\varepsilon) \, = \, g_\gamma^\varepsilon \ \text{ at } \ z = 0$$

is smooth enough to satisfy the $\mathscr{H}_\gamma^m$ estimate (12.3.30): by linearity, this is also true for the differences $(v_\gamma^\varepsilon - v_\gamma^{\varepsilon'}, \chi_\gamma^\varepsilon - \chi_\gamma^{\varepsilon'})$; hence the convergence of $(v_\gamma^\varepsilon)_{\varepsilon>0}$ in $\mathscr{H}_\gamma^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$, and of $((v_\gamma^\varepsilon)_{|x_d})_{\varepsilon>0}$ and $(\chi_\gamma^\varepsilon)_{\varepsilon>0}$ in $\mathscr{H}_\gamma^m(\mathbb{R}^{d-1} \times \mathbb{R})$; the limit must be $(\dot{v}, \dot{\chi})$ since it is a solution of the same BVP, and by passing to the limit in the estimate (12.3.30) applied to $(v_\gamma^\varepsilon, \chi_\gamma^\varepsilon)$ we get the estimate for $(\dot{v}, \dot{\chi})$.

Step 2): It is a matter of smoothing coefficients and passing once more to the limit. We omit the details, which are identical to those in the proof of Theorem 9.20 for the standard BVP. □

12.3.4 *The IBVP with zero initial data*

In this section we fix $T \in \mathbb{R}$ and we denote by I_T the half-line $(-\infty, T]$.

Theorem 12.6 *Under the hypotheses* **(CH)***,* **(NC)***,* **(N)** *and* **(UKL)** *of Theorem 12.1, for all Lipschitz-continuous* $u_\pm : (y, z, t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+ \times I_T \mapsto u_\pm(y, z, t) \in \mathbb{R}^n$ *and* $\mathrm{d}\chi : (y, t) \in \mathbb{R}^{d-1} \times I_T \mapsto (\partial_t \chi(y, t), \nabla_y \chi(y, t)) \in$

$\mathbb{R} \times \mathbb{R}^{d-1}$, *with*

$$\|u_\pm - \underline{u}_\pm\|_{L^\infty(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)} \leq \rho \quad \textit{and} \quad \|\mathrm{d}\chi - (\sigma, 0)\|_{L^\infty(\mathbb{R}^{d-1}\times I_T)} \leq \rho,$$

$$\|u_\pm\|_{W^{1,\infty}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \leq \omega \quad \textit{and} \quad \|\mathrm{d}\chi\|_{W^{1,\infty}(\mathbb{R}^{d-1}\times I_T)} \leq \omega,$$

for all $f \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times I_T; \mathbb{R}^{2n})$ *and all* $g \in L^2(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^n)$ *such that*

$$f_{|t<0} = 0\,,\; g_{|t<0} = 0,$$

there exists a unique pair $(\dot{v}, \dot{\chi}) \in L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times I_T; \mathbb{R}^{2n}) \times H^1(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^n)$ *such that*

$$L(u, \mathrm{d}\chi)\dot{v} = f\,,\; B(u)\cdot \mathrm{d}\dot{\chi} + M(u, \mathrm{d}\chi)\cdot \dot{v}_{|z=0} = g\,,\text{ and } \dot{v}_{|t<0} = 0\,, \quad \dot{\chi}_{|t<0} = 0.$$

Furthermore, $\dot{v}_{|z=0} = 0$ *belongs to* $L^2(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^{2n})$*, and there exist* $\gamma_0 \geq 1$ *and* $c > 0$ *depending continuously on* ω *such that for all* $\gamma \geq \gamma_0$,

$$\begin{aligned} &\gamma\,\|\mathrm{e}^{-\gamma t}\,\dot{v}\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)} + \|\mathrm{e}^{-\gamma t}\,\dot{v}_{|z=0}\|^2_{L^2(\mathbb{R}^d)} + \|\mathrm{e}^{-\gamma t}\,\dot{\chi}\|^2_{H^1_\gamma(\mathbb{R}^{d-1}\times I_T)} \\ &\leq c\left(\tfrac{1}{\gamma}\,\|\mathrm{e}^{-\gamma t}\,f\|^2_{L^2(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)} + \|\mathrm{e}^{-\gamma t}\,g\|^2_{L^2(\mathbb{R}^{d-1}\times I_T)}\right). \end{aligned} \tag{12.3.39}$$

The proof is analogous to the proof of Theorem 9.18 for the 'standard' IBVP with zero initial data: it relies on Theorem 12.4 and on the following support theorem.

Theorem 12.7 *In the framework of Theorem 12.4, if both* f *and* g *vanish for* $t < t_0$ *then so do* $\dot{v}$ *and* $\dot{\chi}$.

(To prove this result, proceed as for Theorem 9.13, using the energy estimate (12.3.35) shown by Theorem 12.4; also see [140], p. 63–64.)

A refined version of Theorem 12.6, with non-zero initial data but for Friedrichs-symmetrizable systems, is proved by Métivier in [136], Section 3.

For smoother coefficients, we have the following.

Theorem 12.8 *In the framework of Theorem 12.6, assume, moreover, that* $u - \underline{u}$ *belongs to* $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times I_T; \mathbb{R}^{2n})$*,* $(u - \underline{u})_{|z=0}$ *belongs to* $H^m(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^{2n})$ *and* $(\partial_t\chi - \sigma, \nabla_y\chi)$ *belongs to* $H^m(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^d)$ *for some integer* $m > (d+1)/2 + 1$*, with* $(u - \underline{u})_{|t<\tau} \equiv 0$*,* $(\partial_t\chi - \sigma, \nabla_y\chi)_{|t<\tau} \equiv 0$ *for some* $\tau < T$*, and*

$$\|u - \underline{u}\|_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)} \leq \mu\,,\; \|u_{|z=0} - \underline{u}\|_{H^m(\mathbb{R}^{d-1}\times I_T)} \leq \mu$$

$$\|(\partial_t\chi - \sigma, \nabla_y\chi)\|_{H^m(\mathbb{R}^{d-1}\times I_T)} \leq \mu.$$

Also assume that f belongs to $H^m(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^{2n})$ and g belongs $H^m(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^d)$, with still

$$f_{|t<0} = 0\,,\; g_{|t<0} = 0.$$

Then $\dot{v}$ belongs to $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times I_T; \mathbb{R}^{2n})$, $\dot{v}_{|z=0}$ belongs to $H^m(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^{2n})$, $\dot{\chi}$ belongs $H^{m+1}(\mathbb{R}^{d-1} \times I_T; \mathbb{R}^d)$, and they satisfy an estimate

$$\frac{1}{T}\,\|\dot{v}\|^2_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|\dot{v}_{|z=0}\|^2_{H^m(\mathbb{R}^{d-1}\times[0,T])} + \|\dot{\chi}_{|z=0}\|^2_{H^{m+1}(\mathbb{R}^{d-1}\times[0,T])}$$

$$\leq T\,\|f\|^2_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times[0,T])} + \|g\|^2_{H^m(\mathbb{R}^{d-1}\times[0,T])}$$

for $C > 0$ depending only on and continously on μ.

The proof relies on Theorem 12.5 for the BVP: it is completely analogous to the proof of Theorem 9.21 (in which the counterpart of Theorem 12.5 is Theorem 9.20) and is therefore omitted.

12.4 Resolution of non-linear IBVP

12.4.1 *Planar reference shocks*

The local-in-time existence (and uniqueness) of perturbed shocks front solutions near uniformly stable planar Lax shocks was first shown by Majda [124]. We will give below a revisited, improved version of Majda's theorem, which is due to Métivier and coworkers, as can be found (partially) in [140] (Section 4) and in Métivier's lecture notes [136].

Before giving a precise statement, we simplify a little the problem by assuming the normal speed of the reference, planar shock is $\sigma = 0$ (this just amounts to making the change of frame $x_d \mapsto x_d - \sigma t$); in other words, we assume the reference shock location is given by $x_d = \underline{\chi}(y,t) = 0$. Then we use Remark 12.1 on the way of fixing the unknown boundary (which is supposedly close to the hyperplane $\{x_d = 0\}$). Hence, instead of (12.1.3) and (12.1.4), we are led to consider the (slightly more complicated) BVP

$$\sum_{j=0}^{d-1} A^j(u_\pm)\,\partial_j u_\pm + \mathbf{A}_d(u_\pm, \mathrm{d}\Psi_\pm)\,\partial_z u_\pm = 0\,, \quad \text{for } z > 0, \tag{12.4.40}$$

$$\sum_{j=0}^{d-1} (\,f^j(u_+) - f^j(u_-)\,)\,\partial_j\chi = (\,f^d(u_+) - f^d(u_-)\,) \quad \text{at } z = 0\,, \tag{12.4.41}$$

$$\Psi_\pm(y,z,t) = \pm\kappa\, z + \varphi(z)\,\chi(y,t)\,, \quad \text{for } z \geq 0\,, \tag{12.4.42}$$

where $\varphi \in \mathscr{D}(\mathbb{R})$ is a cut-off function (equal to one on some interval $[0, z_0]$), and

$$\mathbf{A}_d(v, \xi_0, \xi_1, \ldots, \xi_{d-1}, \xi_d) := \frac{1}{\xi_d}\,A_d(v, \xi_0, \xi_1, \ldots, \xi_{d-1}).$$

Clearly, a solution of (12.4.40)–(12.4.42) with

$$\|\chi\|_{L^\infty} \le \eta := \frac{\kappa}{2\|\varphi'\|_{L^\infty}}$$

gives a solution of the original FBP, as formulated in Section 12.1.

To actually solve (12.4.40)–(12.4.42) with given initial data $u_0 = (u_-^0, u_+^0)$ and χ_0, we shall (unsurprisingly) need compatibility conditions, which (unsurprisingly) are uglier than for standard IBVP. The principle of their derivation is, nevertheless, very simple, as we now explain.

On the one hand, thanks to Lemma 12.2, the jump conditions in (12.4.40) can be rewritten equivalently as

$$\begin{cases} \partial_t \chi &= q(u) \\ \nabla_y \chi &= r(u) \\ 0 &= s(u) \end{cases} \quad \text{with} \quad \begin{pmatrix} q(u) \\ r(u) \\ s(u) \end{pmatrix} = F(u) := Q(u_-, u_+)\,(\,f^d(u_+) - f^d(u_-)\,),$$

where $Q \in \mathscr{C}^\infty(\mathscr{W}; \mathbf{GL}_n(\mathbb{R}))$, $\mathscr{W}$ being a neighbourhood of $(\underline{u}_-, \underline{u}_+)$. If both χ and u are smooth enough, the first equality ($\partial_t \chi = q(u_{|z=0})$) implies by Faá di Bruno's formula,

$$\partial_t^{p+1} \chi = \sum_{m=1}^{p} \sum_{i_1+\cdots+i_m=p} c_{i_1,\ldots,i_m}\, (\mathrm{d}^m q \circ u_{|z=0}) \cdot (\partial_t^{i_1} u, \ldots, \partial_t^{i_m} u)_{|z=0}.$$

On the other hand, the interior equations in (12.4.40) can be rewritten in a more compact way

$$\mathbb{A}^0(u)\,\partial_t u + \sum_{j=0}^{d-1} \mathbb{A}^j(u)\,\partial_j u + \mathbb{A}_d(u, \mathrm{d}\Psi)\,\partial_z u = 0,$$

with our usual blockwise definitions of $\mathbb{A}^j$ for $j \le d-1$, and a *revisited* definition for $\mathbb{A}_d$ (which would coincide with the former definition in the special case $\kappa = 1$ and $\varphi \equiv 1$), namely

$$\mathbb{A}_d(u, \mathrm{d}\Psi) = \begin{pmatrix} \mathbf{A}_d(u_-, \mathrm{d}\Psi_-) & 0 \\ 0 & \mathbf{A}_d(u_+, \mathrm{d}\Psi_+) \end{pmatrix}.$$

Since $\mathbb{A}^0(u)$ is invertible, the interior equations are thus equivalent to

$$\partial_t u = -\sum_{j=0}^{d-1} (\mathbb{A}^0(u))^{-1}\,\mathbb{A}^j(u)\,\partial_j u - (\mathbb{A}^0(u))^{-1}\,\mathbb{A}_d(u, \mathrm{d}\Psi)\,\partial_z u,$$

which we may differentiate p times if u and Ψ (or equivalently χ) are smooth enough. This yields

$$\partial_t^{p+1} u = -\sum_{\ell=0}^{p} \binom{p}{\ell} \sum_{j=0}^{d} \partial_t^\ell(\mathbb{B}_j(u, \mathrm{d}\Psi))\,\partial_j\,\partial_t^{p-\ell} u,$$

where (to save room and hopefully the reader's nerves) we have used unified notations, namely $\partial_0 = \partial_t$, $\partial_d = \partial_z$, $\mathbb{B}_j(u, \mathrm{d}\Psi) := (\mathbb{A}^0(u))^{-1}\, \mathbb{A}^j(u)$ for $j \leq d-1$ and $\mathbb{B}_d(u, \mathrm{d}\Psi) := (\mathbb{A}^0(u))^{-1}\, \mathbb{A}_d(u, \mathrm{d}\Psi)$.

Applying again Faá di Bruno's formula to the derivatives $\partial_t^\ell(\mathbb{B}_j(u, \mathrm{d}\Psi))$, we are led to the following formulation of compatibility conditions up to order s.

($\mathbf{CC}_s$) A pair of initial data (u_0, χ_0) is said to be compatible up to order s if the functions (u_p, χ_p) defined inductively by

$$\begin{cases} \chi_1 \;=\; q \circ (u_0)_{|z=0}\,, \quad u_1 \;=\; -\displaystyle\sum_{j=0}^{d} \mathbb{B}_j(u_0, \mathrm{d}\Psi_0)\, \partial_j\, u_0\,, \\[2ex] \chi_{p+1} \;=\; \displaystyle\sum_{m=1}^{p} \sum_{\ell_1+\cdots+\ell_m=p} c_{\ell_1,\ldots,\ell_m}\, (\mathrm{d}^m q \circ (u_0)_{|z=0}) \cdot (u_{\ell_1}, \ldots, u_{\ell_m})_{|z=0}\,, \\[2ex] u_{p+1} \;=\; -\displaystyle\sum_{j=1}^{d} \mathbb{B}^j(u_0, \mathrm{d}\Psi_0)\; \partial_j u_p \\ -\displaystyle\sum_{\ell=1}^{p} \binom{p}{\ell} \sum_{j=1}^{d} \sum_{k=1}^{\ell} \sum_{\ell_1+\cdots+\ell_k=\ell} c_{\ell_1,\ldots,\ell_k}\; \mathrm{d}^k\mathbb{B}^j(\mathbf{u}_0) \cdot (\mathbf{u}_{\ell_1}, \ldots, \mathbf{u}_{\ell_k})\; \partial_j\, u_{p-\ell}, \end{cases} \tag{12.4.43}$$

with the (obvious) notations

$$\mathbf{u}_\ell \;=\; (u_\ell, \mathrm{d}\Psi_\ell)\,,\;\; \Psi_\ell \;=\; (\Psi_\ell^-, \Psi_\ell^+)\,,\;\; \Psi_\ell^\pm(y,z,t) \;=\; \pm\kappa\, z \;+\; \varphi(z)\, \chi_\ell(y,t),$$

are such that, for all $p \in \{0, \ldots, s\}$,

$$\begin{cases} \nabla_y \chi_p \;=\; \displaystyle\sum_{m=1}^{p} \sum_{\ell_1+\cdots+\ell_m=p} c_{\ell_1,\ldots,\ell_m}\, (\mathrm{d}^m r \circ (u_0)_{|z=0}) \cdot (u_{\ell_1}, \ldots, u_{\ell_m})_{|z=0}\,, \\[2ex] 0 \;=\; \displaystyle\sum_{m=1}^{p} \sum_{\ell_1+\cdots+\ell_m=p} c_{\ell_1,\ldots,\ell_m}\, (\mathrm{d}^m s \circ (u_0)_{|z=0}) \cdot (u_{\ell_1}, \ldots, u_{\ell_m})_{|z=0}\,. \end{cases} \tag{12.4.44}$$

Theorem 12.9 *Under the assumptions of Theorem 12.1 with $\sigma = 0$, there exists $\rho > 0$ so that for all $u_0 \;=\; (u_-^0, u_+^0) \in \underline{u} + H^{m+1/2}(\mathbb{R}^{d-1} \times \mathbb{R}^+; \mathbb{R}^{2n})$ and all $\chi_0 \in H^{m+1/2}(\mathbb{R}^{d-1})$ with m an integer greater than $(d+1)/2 + 1$, compatible up to order $m-1$, and such that*

$$\|u_\pm^0 \;-\; \underline{u}_\pm\|_{L^\infty(\mathbb{R}^{d-1}\times\mathbb{R}^+)} \;\leq\; \rho \quad \textit{and} \quad \|\nabla_y \chi_0\|_{L^\infty(\mathbb{R}^{d-1})} \;\leq\; \rho,$$

there exists $T > 0$ *and a solution* $(u, \chi) = (u_-, u_+, \chi)$ *of* (12.4.40)–(12.4.42) *such that*

$$u_\pm = u^0_\pm \quad \text{and} \quad \chi = \chi_0 \quad \text{at } t = 0,$$

with $u \in \underline{u} + H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0, T]; \mathbb{R}^{2n})$ *taking values in* $\mathscr{U} := \mathscr{U}_- \times \mathscr{U}_+$*, with* $\kappa = 4\|\varphi'\|_{L^\infty} \|\chi_0\|_{L^\infty}$*, and* $\chi \in H^m(\mathbb{R}^{d-1} \times [0, T])$ *taking values in the ball of radius* $\eta = 2\|\chi_0\|_{L^\infty}$*.*

Proof It is very similar to the proof of Theorem 11.1 on the 'standard' IBVP. It starts with the construction of an 'approximate solution', and uses an iterative scheme to solve the IBVP with zero initial data (obtained by subtracting the approximate solution): thanks to our knowledge of linear IBVP with zero initial data (here Theorem 12.8), the iterative scheme can be shown to converge on small enough time intervals.

On the one hand, the interior equations in (12.4.40) merely read

$$\mathbf{L}(u, \mathrm{d}\Psi)u = 0, \text{ with } \mathbf{L}(u, \mathrm{d}\Psi) := \mathbb{A}^0(u)\,\partial_t + \sum_{j=1}^{d-1} \mathbb{A}^j(u)\,\partial_j + \mathbb{A}_d(u, \mathrm{d}\Psi)\,\partial_z,$$

the matrices $\mathbb{A}^j$ and $\mathbb{A}_d$ being defined as above. This induces us to consider the iterative scheme

$$\mathbf{L}(u^a + v^k, \mathrm{d}\Psi^a + \mathrm{d}\Phi^k)v^{k+1} = -\mathbf{L}(u^a + v^k, \mathrm{d}\Psi^a + \mathrm{d}\Phi^k)u^a\,, \tag{12.4.45}$$

where (u^a, Ψ^a) is an 'approximate solution' (to be specified in Lemma 12.4 below) of the IBVP associated with (12.4.40)–(12.4.42) and the initial data (u_0, χ_0),

On the other hand, the boundary conditions (12.4.40) can be written as

$$J\nabla\chi + F(u_{|z=0}) = 0\,, \tag{12.4.46}$$

with J the constant $n \times d$ matrix consisting of the identity $I_d \in \mathbf{M}_{n\times n}(\mathbb{R})$ and a zero block underneath, and F defined as above (thanks to Lemma 12.2) by $F(u) = Q(u_-, u_+)\,(\,f^d(u_+) - f^d(u_-)\,)$ for $u \in \mathscr{W}$, a contractible open subset of $\mathscr{U}$. (In the following we shall assume, up to reducing it, that $\mathscr{W}$ is convex, for example, a ball centered at $\underline{u}$.) By linearity of (12.4.46) in χ, we are led to consider the induction formula

$$\begin{aligned} &J\nabla\chi^{k+1} + \mathrm{d}F((u^a + v^k)_{|z=0})v^{k+1}_{|z=0} \qquad (12.4.47)\\ &\quad = -\,J\,\nabla\chi^a + \mathrm{d}F((u^a + v^k)_{|z=0})v^k_{|z=0} - F((u^a + v^k)_{|z=0}), \end{aligned}$$

with $\chi^k = \Phi^k_{|z=0}$, $\chi^a = \Phi^a_{|z=0}$.

Remark 12.5 As pointed out by Métivier [136], it is possible to make a change of unknowns $u \mapsto \widetilde{u}$ so that the boundary conditions (12.4.46) become linear also in $\widetilde{u}$. Indeed, on the manifold

$$\mathscr{M} := \{(u_-, u_+, \xi) \in \mathscr{W} \times \mathbb{R}^d\,;\ \|\xi\| \le \rho \text{ and } J\xi + F(u_-, u_+) = 0\,\}$$

the mapping $(u_-, u_+, \xi) \mapsto (\widetilde{u}_- := F(u_-, u_+), \widetilde{u}_+ := u_+, \widetilde{\xi} := \xi)$ is (for ρ small enough) a local diffeomorphism (whose differential at (u_-, u_+, ξ) is the mapping $(\dot{u}_-, \dot{u}_+, \dot{\xi}) \mapsto (-Q(u_-, u_+)\, A_d(u_-, \xi) \cdot \dot{u}_-, \dot{u}_+, \dot{\xi}))$. So, up to reducing $\mathscr{W}$ and ρ, we may assume it is a diffeomorphism on $\mathscr{M}$. Then (12.4.46) is equivalent for $(u_-, u_+) \in \mathscr{W}$ and $\|\nabla\chi\| \leq \rho$ to

$$J\nabla\chi + \Pi\, \widetilde{u} = 0,$$

with $\widetilde{u} := (\widetilde{u}_-, \widetilde{u}_+)$ and Π merely the projection operator $\Pi : (\widetilde{u}_-, \widetilde{u}_+) \in \mathbb{R}^{2n} \mapsto \widetilde{u}_- \in \mathbb{R}^n$. This way of rewriting the boundary conditions (12.4.46) would simplify the iterative scheme (12.4.47) into

$$J\nabla\chi^{k+1} + \Pi\widetilde{v}^{k+1}_{|z=0} = -J\,\nabla\chi^a - \Pi\widetilde{u}^a_{|z=0}.$$

However, we refrain from using this simplification because it is too specific to Lax shocks.

Before studying the iterative scheme (12.4.45) and (12.4.47) we must specify the approximate solution (u^a, Ψ^a).

Lemma 12.4 *Under the assumptions of Theorem 12.9, choose $\rho_0 > 0$ so that*

$$\|w - \underline{u}\|_{\mathbb{R}^{2n}} \leq \rho_0 \quad \text{implies} \quad w \in \mathscr{W} \subset \mathscr{U}.$$

There exists $T_0 > 0$ and $u^a \in \underline{u} + H^{m+1}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$, taking values in $\mathscr{U}$, $\chi^a \in H^{m+1}(\mathbb{R}^d)$ with both $u^a - \underline{u}$ and χ^a vanishing for $|t| \geq 2T_0$, such that

$$(u^a)_{|t=0} = u_0\,, \quad (\chi^a)_{|t=0} = \chi_0,$$

$$\|u^a(y,z,t) - u_0(y,z)\| \leq \frac{\rho_0}{2}\,, \quad \|\chi^a(y,t) - \chi_0(y)\| \leq \frac{\eta}{2}$$

for all $(y,z,t) \in \mathbb{R}^{d-1} \times \mathbb{R}^+ \times [-T_0, T_0]$ and additionally, $f_0 := -\mathbf{L}(u^a, \mathrm{d}\Psi^a)u^a$, with

$$\Psi^a = (\Psi^a_-, \Psi^a_+)\,,\ \Psi^a_\pm(y,z,t) = \pm\kappa\, z + \varphi(z)\,\chi^a(y,t)\,, \tag{12.4.48}$$

and $g_0 := -J\,\nabla\chi^a - F(u^a_{|z=0})$ are such that

$$\partial_t^p f_0 \equiv 0, \quad \partial_t^p g_0 \equiv 0 \quad \text{at } t = 0 \quad \text{for all } p \in \{0, \dots, m-1\}.$$

Furthermore, f_0 belongs to $H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$, g_0 belongs to $H^m(\mathbb{R}^d)$, and both vanish for $|t| \geq 2T_0$).

Proof Similarly as in Lemma 11.1 we can construct $u_i \in H^{m+1/2-i}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and $\chi_i \in H^{m+1/2-i}(\mathbb{R}^d)$ satisfying (12.4.43) for all $i \in \{1, \dots, m-1\}$. Then, by trace lifting (see, for instance, [1], pp. 216–217), we find $u^a \in \underline{u} + H^{m+1}(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times \mathbb{R})$ and $\chi^a \in H^{m+1}(\mathbb{R}^d)$ such that

$$\|u^a - \underline{u}\|_{H^{m+1}(\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R})} \lesssim \|u_0 - \underline{u}\|_{H^{m+1/2}(\mathbb{R}^{d-1}\times\mathbb{R}^+)}\,,$$

$$\|\chi_a\|_{H^{m+1}(\mathbb{R}^d)} \lesssim \|\chi_0\|_{H^{m+1/2}(\mathbb{R}^{d-1})},$$

and

$$\partial_t^i(u^a)_{|t=0} \;=\; u_i\,, \quad \partial_t^i(\chi^a)_{|t=0} \;=\; \chi_i \quad \text{for all } \; i\in\{0,\dots,m-1\}.$$

By Sobolev embeddings we find $T_0 > 0$ so that

$$\|u^a(x,t) \;-\; u_0(x)\| \;\leq\; \frac{\rho_0}{2} \quad \text{and} \quad \|\chi^a(y,t) \;-\; \chi_0(y)\| \;\leq\; \frac{\eta}{2}$$

for $|t| \leq T_0$. Furthermore, if $\varphi_0 \in \mathscr{D}(\mathbb{R})$ is a cut-off function such that $\varphi_0(t) \;=\; 1$ for $|t| \leq T_0$ and $\varphi_0(t) \;=\; 0$ for $|t| \leq 2T_0$, we may replace u^a by $(1-\varphi_0)\,\underline{u} \;+\; \varphi_0 u^a$, which takes values in the convex set $\mathscr{W}$. Similarly, we may replace χ^a by $\varphi_0\chi^a$, and define Ψ^a by (12.4.48). That $f_0 \;:=\; -\,\mathbf{L}(u^a, \mathrm{d}\Psi^a)u^a$ belongs to $H^m(\mathbb{R}^{d-1}\times \mathbb{R}^+\times\mathbb{R})$ follows from Proposition C.11 and Theorem C.12: observe that $f_0 \;=\; -\,\mathbf{L}(u^a, \mathrm{d}\Psi^a)(u^a-\underline{u})$ and use that $u^a-\underline{u}$ and $\mathrm{d}\Psi^a$ both belong to $H^m(\mathbb{R}^{d-1}\times \mathbb{R}^+\times\mathbb{R})$. Similarly, we find that $g_0 \;:=\; -\,J\,\nabla\chi^a \;-\; F(u^a_{|z=0})$ belongs to $H^m(\mathbb{R}^d)$ because $\nabla\chi^a$ and $u^a_{|z=0} \;-\; \underline{u}$ do so and $F(\underline{u}) \;=\; 0$. Finally, the time derivatives of f^a and g^a vanish at $t=0$ up to order $m-1$ thanks to the compatibility conditions in (12.4.43) and (12.4.44) with $s\leq m-1$. □

We can now analyse the iterative scheme (12.4.45) and (12.4.47) with the pair (u^a, Ψ^a) given by Lemma 12.4. We introduce the following shorcuts:

$$\mathbf{v} := (v,\Phi)\,, \quad f_{\mathbf{v}} \;:=\; -\,\big(\mathbf{1}_{t>0}\big)\,\mathbf{L}(u^a+v, \mathrm{d}\Psi^a+\mathrm{d}\Phi)u^a,$$

and $g_v \;:=\; (\mathbf{1}_{t>0})\,(\,-\,J\,\nabla\chi^a \;+\; \mathrm{d}F((u^a+v)_{|z=0})\cdot v_{|z=0} \;-\; F((u^a+v)_{|z=0})\,)$.

Lemma 12.5 *Under the assumptions of Lemma 12.4, there exists $M>0$ (depending on ρ_0 and η) so that for all $T\in(0,T_0]$, denoting by I_T the half-line $(-\infty,T]$, for all $v\in H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)$ having a trace $v_{|z=0}\in H^m(\mathbb{R}^{d-1}\times I_T)$ and for all $\Phi\in H^{m+1}(\mathbb{R}^{d-1}\times I_T)$ with*

$$v_{|t<0} \;\equiv\; 0\,, \quad \Phi_{|t<0} \;\equiv\; 0,$$

and $\|v\|_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)} \leq M\,, \quad \|v_{|z=0}\|_{H^m(\mathbb{R}^{d-1}\times I_T)} \leq M\,, \quad \|\Phi\|_{H^{m+1}(\mathbb{R}^{d-1}\times I_T)} \leq M$, *we have $f_{\mathbf{v}}\in H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)$ and $g_v\in H^m(\mathbb{R}^{d-1}\times I_T)$ and there exist continuous functions $M\mapsto C_1(M)$, $M\mapsto C_2(M)$ and $T\mapsto\varepsilon(T)$, the latter going to zero as T goes to zero, so that*

$$\|f_{\mathbf{v}}\|_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)} \;\leq\; C_1(M) \quad \textit{and} \quad \|g_v\|_{H^m(\mathbb{R}^{d-1}\times I_T)} \;\leq\; T\,C_2(M) \;+\; \varepsilon(T).$$

We omit the proof, which is mostly similar to that of Lemma 11.3 (also, see [140] pp. 94–96). □

From now on, we fix M as in Lemma 12.5 and we show how to complete the proof of Theorem 12.9. Setting $v^0 \;\equiv\; 0$ and $\Phi^0 \;\equiv\; 0$, thanks to (12.4.45) and (12.4.47), Lemma 12.5 and Theorem 12.8, we can construct by induction a sequence (v^k, Φ^k) such that, up to diminishing T,

$$\begin{cases} v^k \in H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times I_T), \quad v^k_{|z=0} \in H^m(\mathbb{R}^{d-1} \times I_T), \quad \Phi^k \in H^{m+1}(\mathbb{R}^{d-1} \times I_T) \\ \text{with } \ \|v^k\|_{H^m(\mathbb{R}^{d-1}\times\mathbb{R}^+\times I_T)} \leq M\,, \quad \|v^k_{|z=0}\|_{H^m(\mathbb{R}^{d-1}\times I_T)} \leq M\,, \\ \|\Phi^k\|_{H^{m+1}(\mathbb{R}^{d-1}\times I_T)} \leq M\,, \text{ and } \ v^k_{|t<0} \equiv 0\,, \quad \Phi^k_{|t<0} \equiv 0\,. \end{cases} \tag{12.4.49}$$

Then, thanks to Theorem 12.6, we can show (v^k), $(v^k_{|z=0})$ and (Φ^k) are, for T small enough, Cauchy sequences in $L^2(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times I_T)$, $L^2(\mathbb{R}^{d-1} \times I_T)$ and $H^1(\mathbb{R}^{d-1} \times I_T)$, respectively. By standard weak compactness and interpolation arguments, we conclude that their limits are in the appropriate Sobolev spaces and solve the original problem. For more details, the reader may refer to Section 11.2.2, which is mostly similar. □

For Friedrichs-symmetrizable systems, Métivier has improved Theorem 12.9 as follows.

Theorem 12.10 *In the framework of Theorem 12.9, assume, moreover, that the operator $A^0(w)\,\partial_t + \sum_{j=1}^d A^j(w)\,\partial_j$ is Friedrichs symmetrizable for w in neighbourhoods of $\underline{u}_\pm$. Then there is a unique solution in $\mathscr{C}H^m_T \times H^{m+1}(\mathbb{R}^d \times [0,T])$, with*

$$\mathscr{C}H^m_T := \{\, v \in \mathscr{D}'(\mathbb{R}^{d-1} \times \mathbb{R}^+ \times [0,T])\,;\ \partial_t^p v \in \mathscr{C}([0,T]; H^{m-p}(\mathbb{R}^{d-1} \times \mathbb{R}^+)) \\ p \leq m\,\},$$

even for data $u_0 \in H^m(\mathbb{R}^{d-1} \times \mathbb{R}^+; \mathbb{R}^{2n})$, $\chi_0 \in H^m(\mathbb{R}^{d-1})$.

The proof is based on refined energy estimates, see [136] (Section 4).

12.4.2 *Compact shock fronts*

To stay as close as possible to earlier notations, we consider now a compact codimension one surface Σ_0 in $\mathbb{R}^d$, and denote by n the outward unit normal vector to Σ_0. We also consider initial data u_0 being smooth (in a sense to be specified) outside Σ_0 and experiencing a jump discontinuity across Σ_0 compatible (at least at zeroth order) with the Rankine-Hugoniot condition. This means that for all $y \in \Sigma_0$ there exists $\sigma(y) \in \mathbb{R}$ such that

$$\sum_{j=1}^d n_j(y)\,[f^j(u_0)](y) \,=\, \sigma(y)\,[f^0(u_0)](y).$$

Here above,

$$[f^j(u)](y) \,=\, \lim_{\varepsilon \searrow 0} \big(\, f^j(u_0(y+\varepsilon n(y))) \,-\, f^j(u(y-\varepsilon n(y)))\,\big).$$

Our main purpose is to study the persistence of the front of discontinuity under (short) time evolution, as in Majda's second memoir [124]. In other words, we want to find a solution to (**FBP**) with $u_{|t=0} = u_0$ and $\Sigma_{|t=0} = \Sigma_0$. This will indeed be possible under some further (higher-order) compatibility conditions and the uniform stability of initial discontinuities. The meaning of the latter should be clear to the reader from the previous sections. However, the statement of higher-order compatibility conditions requires some more work, and is (as usual) quite technical.

Fixing the boundary

A preliminary step is (as in Section 12.1.2) to make a change of variables to fix the unknown front. More precisely, as in Remark 12.1, we can make (locally in time) a change of variables that does not influence the far-field equations and sends all front locations $\Sigma(t)$ to the fixed (initial) surface Σ_0. This can be done explicitly by considering tubular neighbourhoods of Σ_0, $\mathscr{X}_0 \subset \mathscr{X}_1 \subset \mathscr{X}_2$ say, such that for all $x \in \mathscr{X}_2$ there exists a unique $(y, \theta) \in \Sigma_0 \times \mathbb{R}$ such that $x = y + \theta\, n(y)$, and a cut-off function φ that is equal to one on $\mathscr{X}_0$ and zero outside $\mathscr{X}_1$. Since $\Sigma(t)$ is expected to remain in $\mathscr{X}_2$ for t small enough, it may be described as

$$\Sigma(t) \,=\, \{\, x \,=\, y \,+\, \chi(y,t)\, n(y)\,;\; y \in \Sigma_0 \,\},$$

for some unknown function ('the front location') χ. Now for T small enough, we can define the mapping

$$\begin{aligned} \Phi \;:\; \mathbb{R}^d \times (-T,T) &\to \mathbb{R}^d \times (-T,T) \\ (x,t) \qquad &\mapsto (\,\widetilde{x} \,:=\, x \,-\, \rho(x)\, \chi(Y(x),t)\, n(Y(x))\,,\, \widetilde{t} \,:=\, t)\,, \end{aligned}$$

where for $x \in \mathscr{X}_2$, $Y(x) \in \Sigma_0$ is uniquely determined by the requirement that $x \,-\, Y(x)$ be parallel to $n(Y(x))$, and $Y(x)$ is extended arbitrarily for x outside $\mathscr{X}_2$: since ρ is identically zero near the boundary of $\mathscr{X}_2$, Φ inherits in the whole space the regularity of Y in $\mathscr{X}_2$. By construction Φ maps every subset of the form $\Sigma(t) \times \{t\}$ onto $\Sigma_0 \times \{t\}$, up to diminishing T so that $\Sigma(t)$ lies in fact in $\mathscr{X}_0$ for all $t \in (-T,T)$. Furthermore, Φ is a global diffeomorphism provided that

$$\sup\{\, |\chi(y,t)| \,+\, \|\mathrm{d}_y \chi(y,t)\|\,;\, y \in \Sigma_0\,,\, t \in (-T,T)\}$$

is small enough, which is indeed what we expect for T small enough.

Modified bulk equations

The inverse mapping Φ^{-1} is of the form $(\widetilde{x}, \widetilde{t}\,) \,\mapsto\, (x \,=\, \psi(\widetilde{x}, \widetilde{t}\,), t \,=\, \widetilde{t}\,)$. (We shall give more details on ψ later.)

Therefore, it is a calculus exercise to check that the quasilinear system

$$A^0(u)\, \partial_t u \,+\, \sum_{j=1}^{d} A^j(u)\, \partial_{x_j} u \,=\, 0$$

is satisfied by $u(x,t) = \widetilde{u}(\widetilde{x},\widetilde{t})$ outside $\Sigma(t)$ if and only if, outside Σ_0,

$$A^0(\widetilde{u})\,\partial_{\widetilde{t}}\widetilde{u} + \sum_{j=1}^{d} \widetilde{A}^j(\widetilde{x},\widetilde{t},\widetilde{u})\,\partial_{\widetilde{x}_j}\widetilde{u} = 0,$$

with

$$\widetilde{A}^j(\widetilde{x},\widetilde{t},\widetilde{u}) = A^j(\widetilde{u}) - \sum_{k=1}^{d} c_{j,k}(\widetilde{x},\widetilde{t})\;A^k(\widetilde{u}) - c_{j,0}(\widetilde{x},\widetilde{t})\;A^0(\widetilde{u}),$$

where, for $j,k \in \{1,\dots,d\}$,

$$c_{j,k}(\widetilde{x},\widetilde{t}) = \partial_{x_k}(\rho(x)\chi(Y(x),\widetilde{t})n_j(Y(x)))_{|x=\psi(\widetilde{x},\widetilde{t})},$$

$$c_{j,0}(\widetilde{x},\widetilde{t}) = (\rho(x)\,\partial_t\chi(Y(x),\widetilde{t})n_j(Y(x)))_{|x=\psi(\widetilde{x},\widetilde{t})}.$$

What is hidden in these formulae is that ψ, as Φ and thus also Φ^{-1} depends on the unknown front χ. In fact, we can make this dependence more explicit.

Assume indeed that the tubular neighbourhoods $\mathscr{X}_i$, $i = 0,1,2$ are given by

$$\mathscr{X}_i = \{\, x = y + \theta\, n(y)\,;\; y \in \Sigma_0\,,\, \theta \in (-\varepsilon_i, \varepsilon_i)\},$$

for $0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 = \varepsilon_0 + \varepsilon_1$, and that

$$\sup\{\,|\chi(y,t)|\,;\, y \in \Sigma_0\,,\, t \in (-T,T)\} \le \varepsilon_0.$$

With the mapping Y introduced above, and Θ defined on $\mathscr{X}_2$ by

$$\Theta(x) = (x - Y(x))\cdot n(Y(x)),$$

(in such a way that $x = Y(x) + \Theta(x)\,n(Y(x))$), we may define the cut-off ρ by $\rho(x) = \varphi(\Theta(x))$, where φ is a $\mathscr{C}^\infty$ cut-off function $\mathbb{R} \to [0,1]$ being equal to one on $(-\varepsilon_0,\varepsilon_0)$ and equal to zero outside $(-\varepsilon_1,\varepsilon_1)$. For x outside $\mathscr{X}_1$, $\widetilde{x} = x$, and for $x \in \mathscr{X}_1$, $\widetilde{x}$ belongs to $\mathscr{X}_2$ and by construction $Y(x) = Y(\widetilde{x})$. Therefore, to find a representation for the mapping ψ, it just remains to look at $\Theta(x)$ in terms of $\widetilde{x}$. By definition, for $x \in \mathscr{X}_1$,

$$\widetilde{x} = Y(x) + (\Theta(x) - \varphi(\Theta(x))\,\chi(Y(x),t)\,)\,n(Y(x)),$$

hence (using the fact that $t = \widetilde{t}$ and $Y(x) = Y(\widetilde{x})$)

$$\Theta(\widetilde{x}) = \Theta(x) - \varphi(\Theta(x))\,\chi(Y(\widetilde{x}),\widetilde{t}\,)\,).$$

Now, if $\varepsilon_0 < 1$ and $|z| < \varepsilon_0$, the function $\theta \mapsto \theta - \varphi(\theta)\,z$ is an increasing diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$. Therefore, by the implicit function theorem and a connectedness argument, there exists a $\mathscr{C}^\infty$ mapping $g : \mathbb{R}\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}$ such that, for all $(\widetilde{\theta},\theta,z) \in \mathbb{R}\times\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)$,

$$\theta - \varphi(\theta)\,z = \widetilde{\theta} \quad \Leftrightarrow \quad \theta = g(\widetilde{\theta},z).$$

Consequently, for $x \in \mathscr{X}_1$,

$$\Theta(x) \;=\; g(\Theta(\widetilde{x}), \chi(Y(\widetilde{x}), \widetilde{t}\,))$$

and therefore

$$\psi(\widetilde{x}, \widetilde{t}\,) \;=\; Y(\widetilde{x}) \;+\; g(\Theta(\widetilde{x}), \chi(Y(\widetilde{x}), \widetilde{t}\,))\, n(Y(\widetilde{x})).$$

Denoting

$$\begin{array}{rl} \widetilde{\chi} \,:\, & \mathbb{R}^d \times (-T, T) \to \mathbb{R} \\ & (\widetilde{x}, \widetilde{t}\,) \qquad \mapsto \widetilde{\chi}(\widetilde{x}, \widetilde{t}\,) \;:=\; \chi(Y(\widetilde{x}), \widetilde{t}\,)\,, \end{array}$$

the discussion above shows that the coefficients $c_{j,k}$ are in fact of the form

$$c_{j,k} \;=\; \widetilde{c}_{j,k}(\widetilde{x}, \widetilde{t}, \widetilde{\chi}(\widetilde{x}, \widetilde{t}\,), \mathrm{d}\widetilde{\chi}(\widetilde{x}, \widetilde{t}\,))$$

for some (complicated) non-linear mappings

$$\widetilde{c}_{j,k} \;:\; \mathbb{R}^d \times (-T, T) \times \mathbb{R} \times (\mathbb{R}^{d+1})' \to \mathbb{R}$$

independent of the unknown front χ. So finally, if we assume additionally (as usual) that $A^0(\widetilde{u})$ is everywhere non-singular, the bulk equations in the new variables read,

$$\partial_{\widetilde{t}}\widetilde{u} \;+\; \sum_{j=1}^{d} A_j(\widetilde{x}, \widetilde{t}, \widetilde{u}, \widetilde{\chi}, \mathrm{d}\widetilde{\chi})\, \partial_{\widetilde{x}_j}\widetilde{u} \;=\; 0\,, \qquad (\widetilde{x}, \widetilde{t}\,) \in (\mathbb{R}^d \backslash \Sigma_0) \times (-T, T)\,, \tag{12.4.50}$$

with

$$A_j(\widetilde{x}, \widetilde{t}, \widetilde{u}, \widetilde{\chi}, \mathrm{d}\widetilde{\chi}) := A^0(\widetilde{u})^{-1} \left(A^j(\widetilde{u}) - \sum_{k=0}^{d} \widetilde{c}_{j,k}(\widetilde{x}, \widetilde{t}, \widetilde{\chi}(\widetilde{x}, \widetilde{t}), \mathrm{d}\widetilde{\chi}(\widetilde{x}, \widetilde{t}))\, A^k(\widetilde{u}) \right).$$

Before dropping the tildas definitively, we have to write also the jump conditions in the new variables.

Jump equations

For simplicity, let us rewrite the Rankine–Hugoniot condition (12.1.2) as

$$N_0\, [f^0(u)] \;+\; N^{\mathrm{T}}[f(u)] \;=\; 0,$$

where N^{T} stands for the row vector of components $(N_1, \ldots, N_d)$ and $f(u)$ stands for the $d \times n$ matrix whose column vectors are $f^1(u), \ldots, f^d(u)$. By assumption, this condition is satisfied by $u = u_0$ at any point $y \in \Sigma_0$, with $N_0 \;=\; -\sigma(y)$ and $N \;=\; n(y)$. The N_j are of course to be modified when the discontinuity surface moves along

$$\Sigma(t) \;=\; \{\, x \;=\; y \;+\; \chi(y, t)\, n(y)\,;\; y \in \Sigma_0 \,\}.$$

To be more precise, let us denote by W the Weingarten map on Σ_0, characterized by

$$W(y_0)\cdot h \;=\; \frac{\mathrm{d}}{\mathrm{d}s}(n(y(s)))_{|s=0}$$

for $s\mapsto y(s)$ any curve on Σ_0 such that $y(0) = y_0$ and $y'(0) = h$, and by $\nabla_y\chi(y,t)$ the tangential gradient (along Σ_0) of the mapping $y\mapsto\chi(y,t)$: by definition, for all $y\in\Sigma_0$, $\nabla_y\chi(y,t)$ is in the tangent space $T_y\chi_0$, viewed as a subspace of $\mathbb{R}^d$, such that the linear form $\mathrm{d}_y\chi(y,t)$ on $T_y\chi_0$ is given by

$$\mathrm{d}_y\chi(y,t)\cdot h \;=\; \langle\nabla_y\chi(y,t), h\rangle$$

for all $h\in T_y\chi_0$, where $\langle\,,\,\rangle$ stands for the standard inner product in $\mathbb{R}^d$ (which gives the first fundamental form on $T_y\chi_0$). Then it is a standard differential geometry exercise to show (using the symmetry of the Weingarten map with respect to the first fundamental form) that the normal vector to $\Sigma = \cup_t\Sigma(t)\times t$ at point $(y + \chi(y,t)\,n(y),t)$ is parallel to

$$\begin{pmatrix} n(y) \;-\; (1 \,+\, \chi(y,t)\,W(y))^{-1}\,\nabla_y\chi(y,t) \\ -\partial_t\chi(y,t)\end{pmatrix}.$$

Therefore, the Rankine-Hugoniot condition (12.1.2) equivalently reads

$$-\partial_t\chi(y,t)\,[f^0(u)](y,t) + \big(n(y) - (1+\chi(y,t)\,W(y))^{-1}\,\nabla_y\chi(y,t)\big)^{\mathrm{T}}[f(u)](y,t) = 0$$

for all $(y,t)\in\Sigma_0\times(-T,T)$. Noting that by construction, $\widetilde{\chi}(\widetilde{x},\widetilde{t}\,) = \chi(\widetilde{x},\widetilde{t}\,)$ for $\widetilde{x}\in\Sigma_0$, we can rewrite this jump condition in the new co-ordinates as

$$\begin{aligned}&\big(n(\widetilde{x}) \;-\; (1 \,+\, \widetilde{\chi}(\widetilde{x},\widetilde{t}\,)\,W(\widetilde{x}))^{-1}\,\nabla_y\widetilde{\chi}(\widetilde{x},\widetilde{t}\,)\big)^{\mathrm{T}}[f(\widetilde{u})](\widetilde{x},\widetilde{t}\,)\\ &\quad= \partial_{\widetilde{t}}\widetilde{\chi}(\widetilde{x},\widetilde{t}\,)\,[f^0(\widetilde{u})](\widetilde{x},\widetilde{t}\,)\,, \qquad (\widetilde{x},\widetilde{t}\,)\in\Sigma_0\times(-T,T)\,.\end{aligned} \tag{12.4.51}$$

Our original problem is clearly equivalent to finding $(\widetilde{u},\widetilde{\chi})$ solving (12.4.50)–(12.4.51) with the initial data $\widetilde{u}(\widetilde{x},0) = u_0(\widetilde{x})$, $\widetilde{\chi}(\widetilde{x},0) = 0$.

From now on, we drop all tildas.

Higher-order compatibility conditions

Our next task is to find higher-order compatibility conditions between the initial maps u_0 and σ. We recall that 'zeroth-order' compatibility conditions are

$$-\,\sigma\,(\,f^0(u_0^+) \,-\, f^0(u_0^-)\,) \,+\, n(x)^{\mathrm{T}}\,(\,f(u_0^+) \,-\, f^0(u_0^-)\,) \,=\, 0 \qquad \text{on}\;\;\Sigma_0\,. \tag{12.4.52}$$

As usual, higher-order compatibility conditions will be obtained by differentiating successively the bulk equations (12.4.50) and the jump conditions (12.4.51), and by equating the results on $\Sigma_0\times\{0\}$. For convenience we rewrite all these equations without the tilda, omitting a number of obvious independent variables but introducing the distinction between u^- the restriction of u to the inside D^- of Σ_0 and u^+ the restriction of u to the outside D^+ of Σ_0:

$$\begin{cases} \partial_t u^\pm + \displaystyle\sum_{j=1}^d A_j(x,t,u^\pm,\chi,\mathrm{d}\chi)\,\partial_{x_j} u^\pm = 0\,, \quad x \in D^\pm\,, \\ \big(n - (1+\chi W)^{-1}\nabla_y\chi\big)^{\mathrm{T}}[f(u)] = (\partial_t\chi)\,[f^0(u)] \quad \text{on } \Sigma_0\,. \end{cases} \tag{12.4.53}$$

We shall not write down explicitly all compatibility conditions, which are quite terrible (see [124], pp. 24–25 for some details). We concentrate on the first-order ones. Assume that $(u^\pm,\chi)$ is a solution of (12.4.53) with $u^\pm$ of class $\mathscr{C}^1$ on $\overline{D^\pm}\times[0,T)$ and χ of class $\mathcal{C}^2$ on $\mathbb{R}^d\times[0,T)$, with $u^\pm_{|t=0} = u_0^\pm$ and $\partial_t\chi_{|\Sigma_0\times\{0\}} = \sigma$ satisfying the zeroth-order compatibility condition in (12.4.52). In addition, we assume that for all $y\in\Sigma_0$, the discontinuity between $u_0^-(y)$ and $u_0^+(y)$ is a 1D-stable Lax shock (of speed $\sigma(y)$) in the direction $n(y)$. We are going to exhibit $(n-1)$ independent equations on $u_1^\pm := \partial_t u^\pm_{|t=0}$ and $\chi_1 := \partial_t\chi_{|t=0}$ valid on $\Sigma_0\times\{0\}$.

On the one hand, in view of the definition of A_j, the first equation in (12.4.53) imposes that

$$u_1^\pm(x) = -\sum_{j=1}^d \Big(A^0(u_0^\pm(x))^{-1}A^j(u_0^\pm(x)) - \sigma(x)\,n_j(x)\Big)\,\partial_{x_j}u_0^\pm(x) \tag{12.4.54}$$

for $x\in\Sigma_0$. On the other hand, differentiating once with respect to t the second equation in (12.4.53) and evaluating at $t=0$, we get

$$\begin{aligned} &-(\partial^2_{tt}\chi(x,0))\,\big(f^0(u_0^+(x)) - f^0(u_0^+(x))\big) \\ &\big(A(u_0^+(x),n(x)) - \sigma(x)A^0(u_0^+(x))\big)\,\partial_t u^+(x,0) \\ &-\big(A(u_0^-(x),n(x)) - \sigma(x)A^0(u_0^-(x))\big)\,\partial_t u^-(x,0) \\ &= (\nabla_y\chi_1(x,0))^{\mathrm{T}}\big(f(u_0^+(x)) - f(u_0^+(x))\big) \end{aligned} \tag{12.4.55}$$

for $x\in\Sigma_0$, where we have used our usual notation $A(u,n) = \sum_{j=1}^d n_j\,A^j(u)$. Now, the 1D-stability of the Lax shock $(u_0^-(x),u_0^+(x),\sigma(x))$ can be used in the following way: introducing $P^+(x)$ the eigenprojector onto the unstable subspace $E^+(x)$ of $A(u_0^+(x),n(x)) - \sigma(x)A^0(u_0^+(x))$ and $P^-(x)$ the eigenprojector onto the stable subspace $E^-(x)$ of $A(u_0^-(x), n(x)) - \sigma(x)A^0(u_0^-(x))$, the mapping

$$\begin{aligned} L(x) \,:\, \mathbb{R} \times E^+(x) \times E^-(x) &\to \mathbb{R}^n \\ (\beta, v^+, v^-) &\mapsto -\beta\left(f^0(u_0^+(x)) - f^0(u_0^+(x))\right) \\ &\quad + \left(A(u_0^+(x), n(x)) - \sigma(x)A^0(u_0^+(x))\right) v^+ \\ &\quad - \left(A(u_0^-(x), n(x)) - \sigma(x)A^0(u_0^-(x))\right) v^- \end{aligned}$$

is an isomorphism. Therefore, if we denote by $B(x)$, $V^+(x)$, $V^-(x)$ the components of the inverse mapping $L(x)^{-1}$, (12.4.55) equivalently reads

$$\begin{cases} \partial^2_{tt}\chi(x,0) \,=\, B(x)(q(x))\,, \\ P^+(x)(u_1^+(x)) \,=\, V^+(x)(q(x))\,, \\ P^-(x)(u_1^-(x)) \,=\, V^-(x)(q(x))\,, \end{cases} \tag{12.4.56}$$

where

$$\begin{aligned} q(x) \,:=\, &\left(\nabla_y \chi_1(x,0)\right)^{\mathrm{T}}\left(f(u_0^+(x)) - f(u_0^+(x))\right) \\ &- \left(A(u_0^+(x), n(x)) - \sigma(x)A^0(u_0^+(x))\right)\,(I - P^+(x))(u_1^+(x)) \\ &+ \left(A(u_0^-(x), n(x)) - \sigma(x)A^0(u_0^-(x))\right)\,(I - P^-(x))(u_1^-(x))\,. \end{aligned}$$

The first equation in (12.4.56) is a ‘bonus’ here (it would serve to define χ_2 if we were to derive second-order compatibility conditions). The others make the announced $n - p + p - 1 = n - 1$ first-order compatibility conditions.

More generally, the compatibility conditions to order s (as in $(\mathbf{CC}_s)$) consist of

- the definition (by induction) of a sequence $(u_p^\pm, \chi_p)$, $p \in \{1, \cdots, s\}$, depending only on $u_0^\pm$, σ and their derivatives,
- for each $p \in \{1, \cdots, s\}$, $(n-1)$ equations on $u_p^\pm$ and $\nabla_y \chi_p$ in terms of $u_\ell^\pm$ and χ_ℓ, $\ell \in \{1, \cdots, p\}$, on Σ_0.

Main persistence result

The following is a slight adaptation of Majda’s theorem ([124], p. 8), where the ‘block-structure condition’ has been replaced by the more explicit constant hyperbolicity assumption: indeed, as shown by Métivier [134], constant hyperbolicity and Friedrichs symmetrizability together imply the block structure condition of Majda. The Sobolev index m is supposed to be a large enough integer: Majda assumed it to be at least equal to $2[d/2] + 7$; the work of Métivier suggests that $m > (d+1)/2 + 1$ is sufficient. Finally, the initial data is assumed for simplicity to vanish at infinity, but any data asymptotically constant at infinity would also work (as stated by Majda in [124]).

Theorem 12.11 (Majda) *Structural assumptions are the following.*

(CH) *There exist open subsets of* $\mathbb{R}^n$, *say* $\mathscr{U}_-$ *and* $\mathscr{U}_+ \ni 0$ *such that for all* $w \in \mathscr{U}_\pm$ *the matrix* $A^0(w)$ *is non-singular, and the operator* $A^0(w)\,\partial_t + \sum_{j=1}^d A^j(w)\,\partial_j$ *is* constantly hyperbolic *in the* t*-direction;*

(S) *The operator* $A^0(w)\,\partial_t + \sum_{j=1}^d A^j(w)\,\partial_j$ *is Friedrichs symmetrizable in* $\mathscr{U}_\pm$.

We assume that Σ_0 *is* $\mathscr{C}^\infty$ *compact manifold in* $\mathbb{R}^n$ *and denote by* D^- *its inside and* D^+ *its outside. We consider* $u_0^\pm \in H^{m+1}(D^\pm)$, *taking values in* $\mathscr{U}_\pm$, *and* $\sigma \in H^{m+1}(\Sigma_0)$ *satisfying*

- *all compatibility conditions up to order* $m-1$, *including* (12.4.52),
- *for all* $x \in \Sigma_0$, *the planar discontinuity between* $u_0^-(x)$ *and* $u_+^0(x)$, *of speed* $\sigma(x)$ *in the direction* $n(x)$, *is a uniformly stable Lax shock in the sense of the Kreiss–Lopatinskiĭ condition.*

Then there exists $T > 0$, $C > 0$, *and a solution* (u^-, u^+, χ) *of* (12.4.53) *such that*

$$u_\pm = u_\pm^0 \quad \textit{and} \quad \chi = 0, \quad \partial_t \chi_{|\Sigma_0} = \sigma \quad \textit{at } t = 0,$$

and $u^\pm \in H^m(D^\pm \times [0,T])$, $\chi \in H^{m+1}(\mathbb{R}^d \times [0,T])$.

The proof is most technical, but the ideas are basically the same as for Theorem 12.9:

- derivation of an 'approximate' solution;
- convergence of an iterative scheme, thanks to the analysis of linear IBVP with zero initial data.

Here, the linear IBVP (as we can guess from the form of (12.4.50)) involve coefficients depending on (x,t) in a $\mathscr{C}^\infty$ manner as well as on $(u^k(x,t), \chi^k(x,t))$, (an element of the iterative scheme) of limited regularity. For simplicity in this book, we have not dealt with IBVP (nor even Cauchy problems) with coefficients of the form $A(x,t,v(x,t))$ say, but only either of the form $A(x,t)$ or of the form $A(v(x,t))$. However, coefficients of the form $A(x,t,v(x,t))$ do not conceal any additional, fundamental difficulty. Another difference between Theorem 12.9 and Theorem 12.11 is that in the latter we have to deal with IBVP in a (fixed) compact domain instead of a half-space: as seen in Chapter 9 for standard IBVP, this is mainly a matter of co-ordinate charts.

Anyway, we shall not produce the complete proof of Theorem 12.11 here. The reader may refer to [124] (which relies on [125] for the analysis of linearized problems). In [124] (p. 26–28), Majda also shows, interestingly, how to construct a large class of compatible initial data. This relies on an important observation on the structure of compatibility conditions: as we can see in (12.4.54),

$$(u_1^{\pm})_{|\Sigma_0} \;=\; -\sum_{j=1}^{d}\Big(A^0(u_0^{\pm})^{-1}\,A^j(u_0^{\pm},n)\;-\;\sigma\Big)\,\frac{\partial u_0^{\pm}}{\partial n}\;+\;\text{tangential derivatives},$$

and a similar (though more intricate) fact is true for all the $u_p^{\pm}$, $p \geq 2$ (which we did not define here, see [124], p. 24).

Remark 12.6 A similar observation on the structure of $(\mathbf{CC}_s)$ is used by Métivier in [136] (Proposition 4.2.4) to prove that the existence result in Theorem 12.9 is valid with slightly less regular initial data, namely with $u_0 - \underline{u} \in H^m$ instead of $H^{m+1/2}$.

PART IV

APPLICATIONS TO GAS DYNAMICS

This page intentionally left blank

13

THE EULER EQUATIONS FOR REAL FLUIDS

The mathematical theory of gas dynamics is often 'limited' to polytropic ideal gases (see, for instance, [50,55,117], etc). Here, we are interested in more general compressible fluids, which can be gases (e.g. non-polytropic ideal gases) but also liquids or even liquid–vapour mixtures (e.g. van der Waals fluids). We call them *real* because of their possible complex thermodynamical behaviour. (This was initially motivated by the PhD thesis of Jaouen [90].) Still, we are aware that they are not that real, as long as dissipation due to viscosity and/or heat conduction is neglected: except for the discussion on the admissibility of shock waves (in Section 13.4), our subsequent analysis does not take into account dissipation phenomena, to stay within the theory of hyperbolic PDEs. Our aim is to investigate, for inviscid and non-heat-conducting fluids, the Initial Boundary Value Problem and the stability of shock waves, by means of the methods described in the previous parts of the book.

The present chapter is devoted to generalities on the thermodynamics and the equations of motion for real compressible fluids (in the zero viscosity/heat conduction limit), and to basic results regarding smooth solutions and (planar) shock waves. Some material is most classical and some is inspired from an important but not so well-known paper by Menikoff and Plohr [130]. Boundary conditions and stability of shocks will be addressed in separate chapters (Chapter 14 and Chapter 15, respectively).

13.1 Thermodynamics

We consider a fluid whose specific internal energy e is everywhere uniquely (and smoothly) determined by its specific volume v and its specific entropy s. This amounts to assuming the fluid is endowed with what we call (after Menikoff and Plohr [130]) a *complete equation of state* $e = e(v, s)$.

The fundamental thermodynamics relation is

$$\mathrm{d}e = -p\,\mathrm{d}v + T\,\mathrm{d}s\,, \tag{13.1.1}$$

where p is the pressure and T the temperature. To avoid confusion when performing changes of thermodynamic variables we adopt a physicists' convention: throughout the chapter, we shall specify after a vertical bar the variable maintained constant in partial derivatives with respect to thermodynamic variables.

Using this convention, we may rewrite (13.1.1) as

$$p = -\left.\frac{\partial e}{\partial v}\right|_s, \quad T = \left.\frac{\partial e}{\partial s}\right|_v. \tag{13.1.2}$$

We shall use four thermodynamic *non-dimensional* quantities.

- The first one is the *adiabatic exponent*, defined by

$$\gamma := -\frac{v}{p}\left.\frac{\partial p}{\partial v}\right|_s. \tag{13.1.3}$$

We have adopted here the same convention as in [130]. Except in the case of polytropic gases, this coefficient γ differs from the widely used ratio of heat capacities.

- Another important one is called the Grüneisen coefficient, defined by

$$\Gamma := -\frac{v}{T}\left.\frac{\partial T}{\partial v}\right|_s. \tag{13.1.4}$$

- By (13.1.2), γ and Γ are related to the Hessian of e through the equalities

$$\gamma\,\frac{p}{v} = \frac{\partial^2 e}{\partial v^2}, \quad \Gamma\,\frac{T}{v} = -\frac{\partial^2 e}{\partial s \partial v}. \tag{13.1.5}$$

The last relevant quantity regarding $D^2 e$ is

$$\delta := \frac{p\,v}{T^2}\left.\frac{\partial T}{\partial s}\right|_v = \frac{p\,v}{T^2}\frac{\partial^2 e}{\partial s^2}. \tag{13.1.6}$$

It is related to the heat capacity at constant volume

$$c_v := \frac{T}{\left.\frac{\partial^2 e}{\partial s^2}\right|_v} = \left.\frac{\partial e}{\partial T}\right|_v \tag{13.1.7}$$

through the simple formula

$$c_v\,\delta = \frac{p\,v}{T}. \tag{13.1.8}$$

- Finally, a 'higher-order' coefficient will be of interest. Following [130], we denote

$$\mathcal{G} := -\frac{1}{2}\,v\,\frac{\left.\frac{\partial^3 e}{\partial v^3}\right|_s}{\left.\frac{\partial^2 e}{\partial v^2}\right|_s}. \tag{13.1.9}$$

Standard thermodynamics requires that γ be positive and e be a convex function of (v, s), which amounts to asking $\gamma\,\delta - \Gamma^2 \geq 0$. The positivity of γ

means that p is increasing with the density $\rho := 1/v$, thus allowing the definition of *sound speed* c by

$$c := \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_s} = \sqrt{\gamma \frac{p}{\rho}}\,. \tag{13.1.10}$$

Using c and ρ, we may rewrite $\mathcal{G}$ as

$$\mathcal{G} = \frac{1}{c} \left.\frac{\partial(\rho\, c)}{\partial \rho}\right|_s\,. \tag{13.1.11}$$

We shall see that the sign of $\mathcal{G}$ plays a crucial role in both the non-linearity of the acoustic fields and in the admissibility of shock waves.

The following definition refers to independent works of Bethe [16] and Weyl [218] in the 1940s (also see [69]).

Definition 13.1 *We call a Bethe–Weyl fluid any fluid endowed with a complete equation of state, with e bounded by below, such that the pressure and temperature defined in* (13.1.2) *are positive, and the associated coefficients γ, Γ, δ and $\mathcal{G}$ (defined in* (13.1.3), (13.1.4), (13.1.6) *and* (13.1.9), *respectively) satisfy*

$$\gamma > 0\,, \quad \gamma\,\delta \geq \Gamma^2\,, \quad \Gamma > 0\,, \quad \mathcal{G} > 0 \tag{13.1.12}$$

and

$$\lim_{\rho \to \rho_{max}} p(\rho, s) = \infty\,. \tag{13.1.13}$$

As mentioned above, the first two requirements in (13.1.12) come from standard thermodynamics. The condition $\Gamma > 0$ is not imposed by thermodynamics, but it often holds true (even though there are simple counterexamples, like water near 0°C, as pointed out by Bethe). It ensures in particular that the isentropes do not cross each other in the (v, p)-plane. And the condition $\mathcal{G} > 0$ means that these isentropes are strictly convex. If this is not the case, the fluid may exhibit weird features, as was pointed out by Thompson and Lambrakis [209] (we warn the reader that $\mathcal{G}$ is denoted Γ in that paper). Note that when $\Gamma > 0$, i.e. when p is increasing with entropy at constant volume, the fact that δ is positive is equivalent to

$$\left.\frac{\partial T}{\partial p}\right|_v > 0\,. \tag{13.1.14}$$

This condition is at least consistent with everyday experience (with air, water, etc.).

Refined conditions were later introduced by Smith [197], in connection with the resolution of the Riemann problem[1]. We retain here the weakest of Smith' conditions.

Definition 13.2 *We call a Smith fluid a Bethe–Weyl fluid satisfying the additional condition*

$$\Gamma < 2\gamma. \tag{13.1.15}$$

Example

Ideal gas The ideal gas law is known to be

$$p\,v = R\,T, \tag{13.1.16}$$

where R is a constant. (If v stands for the molar volume instead of the specific volume, R is universal and equals approximately $8.3144\,\mathrm{J}\cdot\mathrm{K}^{-1}\cdot\mathrm{mol}^{-1}$.) We readily see that (13.1.14) is satisfied for $v > 0$. Then it is easy to check that (13.1.16) is compatible with the fundamental relation (13.1.1), or equivalently (13.1.2), provided that

$$e = \varepsilon\left(v^{-R}\exp(s)\right),$$

where ε is *any* (smooth) function that is bounded by below on $(0, +\infty)$. With a complete equation of state of this form, an easy calculation shows that

$$\gamma = \Gamma + 1 \quad \text{and} \quad \delta = \Gamma.$$

Therefore, the first three inequalities in (13.1.12) are altogether equivalent to $\gamma > 1$. Note that in view of (13.1.8) we have

$$\Gamma = \delta = \frac{R}{c_v}.$$

Thus (13.1.15) is trivially satisfied. Regarding $\mathcal{G}$, there is no simple expression for general functions ε. The next example concerns power functions ε, for which quite a nice expression of $\mathcal{G}$ is available.

Polytropic gas Polytropic gases are merely ideal gases for which c_v is constant and ε is the power function

$$\varepsilon(u) = u^{1/c_v}.$$

In this case, Γ, γ and δ are all constant and we have

$$e = c_v\,T \quad \text{and} \quad p = (\gamma - 1)\,\rho\,e.$$

The latter equality is the most famous example of an *incomplete equation of state*, giving p in terms of ρ and e. Incomplete equations of state, or simply pressure

[1]The Riemann problem for a one-dimensional hyperbolic system of conservation laws is a special Cauchy problem, in which the initial data are step functions. Its resolution is crucial both for theoretical and numerical purposes.

laws, are sufficient for the closure of the Euler equations (which by definition encode the conservation of mass, momentum and total energy, see Section 13.2 below). The present pressure law, widely used in the mathematical theory of gas dynamics, is often termed the γ-law for an obvious reason. Note that it is only for polytropic gases that our γ coincides with the ratio of heat capacities (see [130], p. 79b for further discussion on this topic). In addition, we find that for polytropic gases

$$\mathcal{G} \,=\, \frac{\gamma+1}{2},$$

which is obviously positive: isentropes are indeed convex for polytropic gases. As to the asymptotic condition (13.1.13), it holds true with $\rho_{\max} \,=\, +\infty$, thanks to the positivity of R/c_v.

Van der Waals fluid The van der Waals law is a modification of (13.1.16) that takes into account the finite size of molecules by imposing a positive minimum value – the so-called covolume $b > 0$ – for the specific (or the molar) volume, and also some intermolecular forces through an additional term depending on a parameter $a \geq 0$. (See, for instance, [168] for more details.) The van der Waals law reads

$$p \,=\, \frac{RT}{v-b} \,-\, \frac{a}{v^2}\,, \tag{13.1.17}$$

which obviously satisfies (13.1.14) for $v > b$. Furthermore, (13.1.17) is compatible with (13.1.1) and (13.1.7) with

$$e \,=\, (\,(v-b)^{-R}\,\exp(s)\,)^{1/c_v} \,-\, \frac{a}{v},$$

provided that c_v is constant. Even though the constancy of the heat capacity c_v is notoriously false (see [168], p. 263) near critical temperature (below which liquid and vapour phases can coexist, and whose theoretical value is $T_c \,=\, 8a/(27bR)$, c_v may reasonably be considered as constant far away from the critical point. In this case, we easily see that (13.1.13) holds true with $\rho_{\max} \,=\, 1/b$, and an equivalent way of writing e is,

$$e \,=\, c_v\,T \,-\, \frac{a}{v}.$$

Some algebra then shows that

$$\Gamma \,=\, \frac{R}{c_v}\,\frac{v}{v-b}\,, \quad \delta \,=\, \Gamma \,-\, \frac{a/v}{e\,+\,a/v},$$

$$\gamma \,=\, (\,\frac{R}{c_v} \,+\, 1\,)\,\frac{v}{v-b} \,+\, \Big(\,(\,\frac{R}{c_v} \,+\, 1\,)\,\frac{v}{v-b} \,-\, 2\Big)\,\frac{a/v^2}{p}$$

and

$$\mathcal{G} = \frac{1}{2\gamma}\left\{ (\frac{R}{c_v} + 1)(\frac{R}{c_v} + 2)\frac{v^2}{(v-b)^2} \right.$$
$$\left. + \left((\frac{R}{c_v} + 1)(\frac{R}{c_v} + 2)\frac{v^2}{(v-b)^2} - 6 \right) \frac{a/v^2}{p} \right\}.$$

The sets $\{\gamma = 0\}$ (critical points of the isentropes) and $\{\mathcal{G} = 0\}$ (inflexion points of the isentropes) in the (v, p)-plane are depicted in Fig. 13.1, together with three kinds of isentropes (a convex one, a non-convex monotone one and a non-monotone one), for parameters a, b and c_v corresponding (roughly) to water (for which the van der Waals law is widely used, in particular in nuclear engineering, even though it is known to be a poor approximation of reality: the adequation coefficient, defined as the ratio of the theoretical compression factor $p_c v_c/(RT_c) = 3/8$ and the real one, is only about 60%!).

Note The van der Waals law (13.1.17) in the special case $a = 0$ does not support phase boundaries. For this reason, fluids endowed with such a law are usually referred to as van der Waals *gases*. We point out that *dusty gases* (as

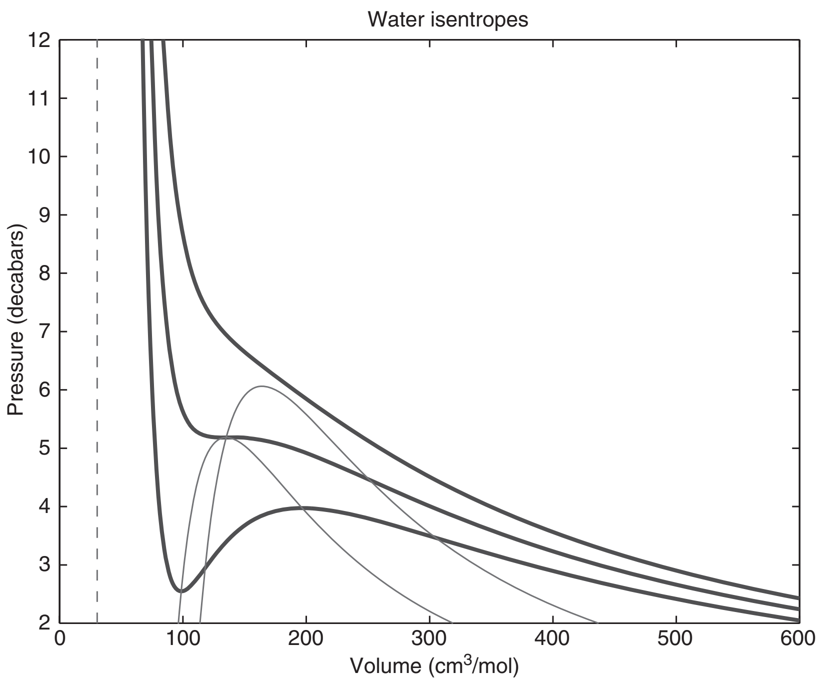

Figure 13.1: Water isentropes (thick solid lines) with the van der Waals law (a, b taken from [216], and $c_v/R = 3.1$).

in [91], for instance) follow a similar law: the Mie–Grüneisen-type pressure law used in [91] is indeed of the form (13.1.17) with $a = 0$.

13.2 The Euler equations

13.2.1 *Derivation and comments*

The motion of a compressible, inviscid and non-heat-conducting, fluid is governed by the Euler equations, consisting of the mass, momentum and energy conservation laws (see, for instance, [47]),

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0\,, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0\,, \\ \partial_t \left(\rho \left(\tfrac{1}{2}\|\mathbf{u}\|^2 + e\right)\right) + \nabla \cdot \left(\left(\rho \left(\tfrac{1}{2}\|\mathbf{u}\|^2 + e\right) + p\right) \mathbf{u}\right) = 0\,. \end{cases} \tag{13.2.18}$$

This system of $(d+2)$ equations contains $(d+3)$ unknowns, the density $\rho \in \mathbb{R}^+$, the velocity $\mathbf{u} \in \mathbb{R}^d$, the internal energy $e \in \mathbb{R}$ and the pressure $p \in \mathbb{R}$. This obviously makes too few equations for too many unknowns. The system (13.2.18) has to be closed by adding a suitable equation of state, for example an incomplete equation of state, or pressure law, $(\rho, e) \mapsto p(\rho, e)$.

Simplified models are obtained by retaining only the mass and momentum conservation laws, assuming that the motion is either isentropic or isothermal. In this case, the pressure law reduces to $\rho \mapsto p(\rho)$. If a complete equation of state is available, the isentropic pressure law is given by

$$p = - \left.\frac{\partial e}{\partial v}\right|_s ,$$

and the isothermal pressure law is given by

$$p = - \left.\frac{\partial f}{\partial v}\right|_T ,$$

where $f := = e - T s$ is the specific free energy.

We observe that an incomplete equation of state is sufficient to give sense to the quantities γ, Γ and $\mathcal{G}$ originally defined in (13.1.3), (13.1.4) and (13.1.9). As a matter of fact, if (13.1.1) holds then $(v, s) \mapsto (v, e)$ is a local diffeomorphism provided that $T > 0$, and the partial derivatives in the old variables are given in terms of the new variables by

$$\left.\frac{\partial}{\partial v}\right|_s = \left.\frac{\partial}{\partial v}\right|_e - p \left.\frac{\partial}{\partial e}\right|_v , \quad \left.\frac{\partial}{\partial s}\right|_v = T \left.\frac{\partial}{\partial e}\right|_v . \tag{13.2.19}$$

Therefore, alternative formulae are

$$\gamma = -\frac{v}{p}\left(\left.\frac{\partial p}{\partial v}\right|_e - p\left.\frac{\partial p}{\partial e}\right|_v\right), \quad \Gamma = v\left.\frac{\partial p}{\partial e}\right|_v, \tag{13.2.20}$$

$$\mathcal{G} = \frac{1}{2}\left\{\gamma + 1 - \frac{v}{\gamma}\left(\left.\frac{\partial \gamma}{\partial v}\right|_e - p\left.\frac{\partial \gamma}{\partial e}\right|_v\right)\right\}, \tag{13.2.21}$$

which also make sense if it is just an incomplete equation of state that is given. Equation (13.2.21) may seem less obvious to the reader than (13.2.20). It is merely obtained by rewriting

$$\mathcal{G} = -\frac{v^2}{2\gamma p}\left.\frac{\partial}{\partial v}\left(\frac{\gamma p}{v}\right)\right|_s.$$

13.2.2 *Hyperbolicity*

A classical and elementary manipulation shows that, for smooth solutions, (13.2.18) is equivalent to

$$\begin{cases} \partial_t \rho + \mathbf{u}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{u} = 0, \\ \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \rho^{-1}\nabla p = 0, \\ \partial_t e + \mathbf{u}\cdot\nabla e + \rho^{-1} p\,\nabla\cdot\mathbf{u} = 0. \end{cases} \tag{13.2.22}$$

Therefore, the hyperbolicity of (13.2.18) is equivalent to the uniform real diagonalizability of the matrix

$$A(\mathbf{U};\mathbf{n}) := \begin{pmatrix} \mathbf{u}\cdot\mathbf{n} & \rho\,\mathbf{n}^{\mathrm{T}} & 0 \\ \rho^{-1} p'_\rho\,\mathbf{n} & (\mathbf{u}\cdot\mathbf{n})\,I_d & \rho^{-1} p'_e\,\mathbf{n} \\ 0 & \rho^{-1} p\,\mathbf{n}^{\mathrm{T}} & \mathbf{u}\cdot\mathbf{n} \end{pmatrix}$$

for all $\mathbf{U} = (\rho, \mathbf{u}, e)$ and $\mathbf{n} \in \mathbb{R}^d\backslash\{0\}$. We have denoted for simplicity

$$p'_\rho = \left.\frac{\partial p}{\partial \rho}\right|_e, \quad p'_e = \left.\frac{\partial p}{\partial e}\right|_\rho.$$

If $(\dot\rho, \dot{\mathbf{u}}, \dot e)^{\mathrm{T}}$ is an eigenvector of $A(\mathbf{U};\mathbf{n})$ associated with an eigenvalue $\lambda(\mathbf{U};\mathbf{n})$, then

$$(\mathbf{u}\cdot\mathbf{n} - \lambda(\mathbf{U};\mathbf{n}))\,\dot\rho + \rho\,\mathbf{n}\cdot\dot{\mathbf{u}} = 0,$$
$$\rho^{-1} p'_\rho\,\dot\rho\,\mathbf{n} + (\mathbf{u}\cdot\mathbf{n} - \lambda(\mathbf{U};\mathbf{n}))\,\dot{\mathbf{u}} + \rho^{-1} p'_e\,\dot e\,\mathbf{n} = 0,$$
$$\rho^{-1} p\,\mathbf{n}\cdot\dot{\mathbf{u}} + (\mathbf{u}\cdot\mathbf{n} - \lambda(\mathbf{U};\mathbf{n}))\,\dot e = 0.$$

We thus see that $\lambda(\mathbf{U};\mathbf{n}) = \mathbf{u}\cdot\mathbf{n}$ is an eigenvalue of $A(\mathbf{U};\mathbf{n})$, with a d-dimensional eigenspace given by

$$\{\,(\dot\rho,\dot{\mathbf{u}},\dot e)^{\mathrm{T}}\,;\; \mathbf{n}\cdot\dot{\mathbf{u}} \,=\, 0\,,\, p'_\rho\,\dot\rho \,+\, p'_e\,\dot e \,=\, 0\,\}.$$

For eigenvectors associated with other eigenvalues, we must have

$$p\,\dot\rho \,=\, \rho^2\,\dot e,$$

as we see from the first and last equations above. Taking the inner product by $\mathbf{n}$ of the intermediate equation, and eliminating $\mathbf{n}\cdot\dot{\mathbf{u}}$ by means of the first equation, we get

$$(\,p'_\rho\,|\mathbf{n}|^2 \,-\, (\mathbf{u}\cdot\mathbf{n} \,-\, \lambda)^2\,)\,\dot\rho \,+\, p'_e\,|\mathbf{n}|^2\,\dot e \,=\, 0.$$

This yields the dispersion relation

$$(\mathbf{u}\cdot\mathbf{n} \,-\, \lambda(\mathbf{U};\mathbf{n}))^2 \,=\, |\mathbf{n}|^2\,(\,p'_\rho \,+\, p\,p'_e/\rho^2\,)\,, \tag{13.2.23}$$

of which the solutions $\lambda(\mathbf{U};\mathbf{n})$ are real if and only if

$$p'_\rho \,+\, p\,p'_e/\rho^2 \,\geq\, 0.$$

Recalling the definition of γ in (13.2.20), this amounts to requiring $\gamma \geq 0$. If $\gamma > 0$ then the solutions of (13.2.23) are distinct,

$$\lambda(\mathbf{U};\mathbf{n}) \,=\, \mathbf{u}\cdot\mathbf{n} \,\pm\, c\,|\mathbf{n}|,$$

where c is the sound speed defined as in (13.1.10) by

$$c \,=\, \sqrt{p'_\rho \,+\, p\,p'_e/\rho^2} \,=\, \sqrt{\gamma\,\frac{p}{\rho}}.$$

Each of the eigenvalues $\mathbf{u}\cdot\mathbf{n} \,\pm\, c\,|\mathbf{n}|$ of $A(\mathbf{U},\mathbf{n})$ has a one-dimensional eigenspace, which is spanned by

$$(\,\rho\,,\,\pm\,c\,\mathbf{n}/|\mathbf{n}|\,,\,p/\rho\,)^{\mathrm{T}}.$$

These very standard results are stated for later use in the following.

Proposition 13.1 *The system* (13.2.18), *endowed with a positive pressure law such that*

$$\gamma \,:=\, \frac{\rho}{p}\,\left(\,p'_\rho \,+\, p\,p'_e/\rho^2\,\right) \,>\, 0,$$

is constantly hyperbolic, and strictly hyperbolic in dimension $d = 1$. Its eigenvalues in the direction $\mathbf{n}$ *are*

$$\lambda_1(\mathbf{U};\mathbf{n}) \,:=\, \mathbf{u}\cdot\mathbf{n} \,-\, c\,|\mathbf{n}|\,,\quad \lambda_2(\mathbf{U};\mathbf{n}) \,:=\, \mathbf{u}\cdot\mathbf{n}\,,\quad \lambda_3(\mathbf{U};\mathbf{n}) \,:=\, \mathbf{u}\cdot\mathbf{n} \,+\, c\,|\mathbf{n}|\,, \tag{13.2.24}$$

where $c = \sqrt{\gamma p/\rho}$*, with associated eigenvectors*

$$r_1(\mathbf{U};\mathbf{n}) := \begin{pmatrix} \rho \\ \\ -c\,\dfrac{\mathbf{n}}{|\mathbf{n}|} \\ \\ p/\rho \end{pmatrix}, \quad r_2(\mathbf{U};\mathbf{n}) := \begin{pmatrix} -\,\dot{\alpha}\,p'_e \\ \\ \dot{\mathbf{u}} \\ \\ \dot{\alpha}\,p'_\rho \end{pmatrix} \text{ with } \dot{\mathbf{u}}\cdot\mathbf{n} = 0\,,$$

$$r_3(\mathbf{U};\mathbf{n}) := \begin{pmatrix} \rho \\ \\ c\,\dfrac{\mathbf{n}}{|\mathbf{n}|} \\ \\ p/\rho \end{pmatrix}. \tag{13.2.25}$$

In addition, the characteristic field (λ_2, r_2) *is* linearly degenerate*, that is,* $\mathrm{d}\lambda_2 \cdot r_2 \equiv 0$ *(where* d *stands for differentiation with respect to* $\mathbf{U}$*), and the fields* (λ_1, r_1) *and* (λ_3, r_3) *(also called acoustic fields) are* genuinely non-linear*, that is,* $\mathrm{d}\lambda_{1,3} \cdot r_{1,3} \neq 0$*, if and only if*

$$\mathcal{G} := \frac{1}{c}\left(\frac{\partial}{\partial\rho} + \frac{p}{\rho^2}\frac{\partial}{\partial e}\right)(\rho\,c) \neq 0.$$

Proof It just remains to check the nature of characteristic fields. On the one hand, it is clear that $\mathrm{d}\lambda_2 \cdot r_2 \equiv 0$. On the other hand, we easily compute that

$$\mathrm{d}\lambda_{1,3} \cdot r_{1,3} = \rho\,c'_\rho + c + \frac{p}{\rho}\,c'_e = c\,\mathcal{G}.$$

(We have set here $|\mathbf{n}| = 1$ to simplify the writing.) Of course the present definition of $\mathcal{G}$ is consistent with (13.1.11) and (13.2.19). □

13.2.3 *Symmetrizability*

There are several ways to symmetrize the Euler equations (13.2.18).

Handmade symmetrization in non-conservative variables The most elementary way is the following, which makes sense if a complete equation of state is given that satisfies (13.1.1), and if

$$\left.\frac{\partial p}{\partial\rho}\right|_s > 0.$$

In this case, $(p, \mathbf{u}, s)$ are independent variables and we may rewrite the quasilinear system (13.2.22) in those new variables. We find

$$\begin{cases} \partial_t p + \mathbf{u} \cdot \nabla p + \rho\, c^2 \nabla \cdot \mathbf{u} = 0\,, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \rho^{-1} \nabla p = 0\,, \\ \partial_t s + \mathbf{u} \cdot \nabla s = 0\,, \end{cases} \tag{13.2.26}$$

where c denotes as usual the sound speed (see (13.1.10)). The characteristic matrix of this system reads

$$\begin{pmatrix} \mathbf{u} \cdot \mathbf{n} & \rho\, c^2\, \mathbf{n}^{\mathrm{T}} & 0 \\ \rho^{-1}\, \mathbf{n} & (\mathbf{u} \cdot \mathbf{n})\, I_d & 0 \\ 0 & 0 & \mathbf{u} \cdot \mathbf{n} \end{pmatrix},$$

which is 'almost' symmetric, up to the *diagonal symmetrizer* $\mathrm{diag}((\rho c^2)^{-1}, \rho, \cdots, \rho, 1)$.

Special symmetrization for polytropic gases A special symmetrization procedure was found by Makino *et al.* [128] for polytropic gases. Their motivation was to deal with compactly supported solutions, and their symmetrizer supported vacuum regions (unlike the one hereabove). To explain this in more detail, we consider again the Euler equations in the variables $(p, \mathbf{u}, s)$, as in (13.2.26). For polytropic gases, we may rewrite the pressure law as

$$p = (\gamma - 1)\, \rho^\gamma \exp(s/c_v),$$

which implies $p = 0$ for $\rho = 0$ as soon as $\gamma > 0$. For $\gamma > 1$, the sound speed $c = \sqrt{\gamma p/\rho}$ is also well-defined up to $\rho = 0$. Furthermore, the apparently singular term in the velocity equation can be rewritten as

$$\rho^{-1} \nabla p = \gamma \exp(s/(\gamma\, c_v))\, \nabla \left(\left(\frac{p}{\gamma - 1} \right)^{(\gamma-1)/\gamma} \right),$$

and the quantity

$$\pi := \left(\frac{p}{\gamma - 1} \right)^{(\gamma-1)/\gamma}$$

is well-defined up to $p = 0$. In view of the equation on p in (13.2.26), we have

$$\partial_t \pi + \mathbf{u} \cdot \nabla \pi + (\gamma - 1)\, \pi\, \nabla \cdot \mathbf{u} = 0.$$

The characteristic matrix of the system made of the equations on π, $\mathbf{u}$ and s is

$$\begin{pmatrix} \mathbf{u}\cdot\mathbf{n} & (\gamma-1)\,\pi\,\mathbf{n}^{\mathrm{T}} & 0 \\ \gamma\,\mathrm{e}^{s/(\gamma\,c_v)}\,\mathbf{n} & (\mathbf{u}\cdot\mathbf{n})\,I_d & 0 \\ 0 & 0 & \mathbf{u}\cdot\mathbf{n} \end{pmatrix},$$

which is again 'almost' symmetric. But as in the previous general situation, the obvious diagonal symmetrizer blows up at vacuum, since it involves $1/\pi$. The trick of Makino *et al.* consists in replacing π by a suitable power of π. For any $\alpha \in (0,1)$, $\widetilde{\pi} := \pi^\alpha$ is well-defined up to 0, and

$$\nabla\pi \;=\; \frac{1}{\alpha}\,\pi^{1-\alpha}\,\nabla(\pi^\alpha) \;=\; \frac{1}{\alpha}\,\widetilde{\pi}^{(1-\alpha)/\alpha}\,\nabla\widetilde{\pi}$$

also makes sense. Furthermore, $\widetilde{\pi}$ satisfies the equation

$$\partial_t\widetilde{\pi} \,+\, \mathbf{u}\cdot\nabla\widetilde{\pi} \,+\, \alpha\,(\gamma-1)\,\widetilde{\pi}\,\nabla\cdot\mathbf{u} \;=\; 0.$$

We see that the power of $\widetilde{\pi}$ in front of $\nabla\cdot\mathbf{u}$ is 1, which is the same as in front of $\nabla\widetilde{\pi}$ in the velocity equation only if $\alpha = 1/2$. With this choice, the characteristic matrix for the equations on $\widetilde{\pi}$, $\mathbf{u}$ and s is

$$\begin{pmatrix} \mathbf{u}\cdot\mathbf{n} & \frac{1}{2}\,(\gamma-1)\,\widetilde{\pi}\,\mathbf{n}^{\mathrm{T}} & 0 \\ 2\,\gamma\,\mathrm{e}^{s/(\gamma\,c_v)}\,\widetilde{\pi}\,\mathbf{n} & (\mathbf{u}\cdot\mathbf{n})\,I_d & 0 \\ 0 & 0 & \mathbf{u}\cdot\mathbf{n} \end{pmatrix},$$

which admits the diagonal symmetrizer

$$\mathrm{diag}\,\left(\frac{2}{\gamma-1}\,,\,\frac{1}{2\gamma}\,\mathrm{e}^{-s/(\gamma\,c_v)}\,,\,\frac{1}{2\gamma}\,\mathrm{e}^{-s/(\gamma\,c_v)}\,,\,1\right),$$

independently of $\widetilde{\pi}$.

Entropy symmetrization in conservative variables A more sophisticated way makes use of a mathematical entropy, as suggested by the general theory of hyperbolic systems of conservation laws (see [184], p. 83–84). The mathematical entropy to be used is $S := \rho\,s$, the physical entropy per unit volume. As regards its convexity properties, we have the following characterization.

Proposition 13.2 *If e is a convex function of* (v,s) *satisfying* (13.1.1) *with p and T positive, then $S := \rho\,s$ is a concave mathematical entropy of* (13.2.18).

Proof The fact that S is a mathematical entropy follows from the last equation in (13.2.26) and the mass continuity equation: indeed, S is easily found to be a conserved quantity for smooth solutions of (13.2.18) (endowed with a complete equation of state), that is, (13.1.1) implies for smooth solutions of (13.2.18),

$$\partial_t S \,+\, \nabla \cdot (\, S \, \mathbf{u} \,) \;=\; 0\,. \tag{13.2.27}$$

The concavity of S is more delicate to obtain, since it has to hold true in the (complicated) conservative variables

$$\mathbf{W} \;=\; (\,\rho\,,\, \rho\,\mathbf{u}\,,\, \frac{1}{2}\,\rho\,\|\mathbf{u}\|^2 \,+\, \rho\, e\,).$$

A very simplifying remark is based on the following fact. Any concave function can be viewed as the infimum of a family of affine functions. This shows that S is a concave function of $\mathbf{W}$ for $\rho \in (0, +\infty)$ if and only if $s \,=\, v\, S$ is a concave function of

$$(\, v\,,\, \mathbf{u}\,,\, \frac{1}{2}\,\|\mathbf{u}\|^2 \,+\, e\,).$$

The last component of these variables,

$$\varepsilon \;=\; \frac{1}{2}\,\|\mathbf{u}\|^2 \,+\, e,$$

is obviously a convex function of $(v, \mathbf{u}, s)$. We claim that, together with the monotonicity of e with respect to s (since $T > 0$), this implies that s is a concave function of $(v, \mathbf{u}, \varepsilon)$. (This generalizes the result that the reciprocal of an increasing convex function is concave.) As a matter of fact, $(v, \mathbf{u}, s) \;\mapsto (\, v\,,\, \mathbf{u}\,,\, \varepsilon(v, \mathbf{u}, s)\,)$ is a local diffeomorphism and we have

$$\mathrm{d}s \;=\; T^{-1}\,(\,p\,\mathrm{d}v \,-\, \mathbf{u}\,\mathrm{d}\mathbf{u} \,+\, \mathrm{d}\varepsilon\,).$$

(In this equality and similar ones below, $\mathbf{u}$ should be viewed as a row vector.) A short computation then shows that

$$\mathrm{D}^2 s(v, \mathbf{u}, \varepsilon) \cdot (\dot v, \dot{\mathbf{u}}, \dot\varepsilon)^{[2]} \;=\; -\,\frac{1}{T}\,\left(\,|\dot{\mathbf{u}}|^2 \,+\, \mathrm{D}^2 e(v, s) \cdot (\dot v, \dot s)^{[2]}\,\right)$$

with

$$\dot s \;=\; T^{-1}\,(\,p\,\dot v \,-\, \mathbf{u} \cdot \dot{\mathbf{u}} \,+\, \dot\varepsilon\,).$$

Hence the Hessian of s as a function of $(v, \mathbf{u}, \varepsilon)$ is clearly negative. □

Once we know that S is a concave function of $(\rho\,,\, \mathbf{m} \,:=\, \rho\mathbf{u}\,,\, \mathcal{E} \,:=\, \rho\varepsilon\,)$, we may consider its Legendre transform $S^\star$. By (13.1.1), we have

$$T\,\mathrm{d}S \;=\; \mathrm{d}E \,-\, g\,\mathrm{d}\rho,$$

where $E = \rho\, e$ denotes the internal energy per unit volume and $g \,:=\, e - sT + pv$ denotes the chemical potential. Rewriting

$$E = \mathcal{E} - \frac{1}{2\,\rho}\,\|\mathbf{m}\|^2,$$

we arrive at

$$T\,\mathrm{d}S \;=\; (\,\frac{1}{2}\,\|\mathbf{u}\|^2\;-\;g\,)\,\mathrm{d}\rho\;-\;\mathbf{u}\,\mathrm{d}\mathbf{m}\;+\;\mathrm{d}\mathcal{E}.$$

Therefore, by definition of the Legendre transform,

$$S^\star \;=\; \frac{1}{T}\,\left\{\,(\,\frac{1}{2}\,\|\mathbf{u}\|^2\;-\;g\,)\,\rho\;-\;\mathbf{u}\cdot\mathbf{m}\;+\;\mathcal{E}\,\right\}\;-\;S.$$

After some simplifications, it appears that

$$S^\star \;=\; -\,\frac{p}{T}.$$

In addition, we have

$$\mathrm{d}S^\star \;=\; \rho\,\mathrm{d}q\;+\;\mathbf{m}\,\mathrm{d}\mathbf{n}\;+\;r\,\mathrm{d}\mathcal{E},$$

where

$$(\,q\,,\,\mathbf{n}\,,\,r\,)\;:=\;\frac{1}{T}\,\left(\frac{1}{2}\,\|\mathbf{u}\|^2\;-\;g\,,\,-\,\mathbf{u}\,,\,1\right)$$

denote the dual variables of $(\rho\,,\,\mathbf{m}\,,\,\mathcal{E}\,)$. Then it is easy to check that (13.2.18) also reads

$$\begin{aligned}
\partial_t\left(\frac{\partial S^\star}{\partial q}\right)\;+\;\nabla\cdot\left(\frac{\partial(S^\star\mathbf{u})}{\partial q}\right)\;&=\;0\,,\\
\partial_t\left(\frac{\partial S^\star}{\partial \mathbf{n}}\right)\;+\;\nabla\cdot\left(\frac{\partial(S^\star\mathbf{u})}{\partial \mathbf{n}}\right)\;&=\;0\,,\\
\partial_t\left(\frac{\partial S^\star}{\partial r}\right)\;+\;\nabla\cdot\left(\frac{\partial(S^\star\mathbf{u})}{\partial r}\right)\;&=\;0\,.
\end{aligned}\tag{13.2.28}$$

Viewed in the variables $(\,q\,,\,\mathbf{n}\,,\,r\,)$, the equations in (13.2.28) form a typical Friedrichs-symmetric system: it admits a straightforward energy estimate and it fits with Definition 2.1 when rewritten, in conservative variables, as

$$\partial_t\begin{pmatrix}\rho\\ \mathbf{m}\\ \mathcal{E}\end{pmatrix}\;+\;\sum_{\alpha=1}^{d}\mathrm{D}^2(S^\star\mathbf{u})\;\mathrm{D}^2S\;\partial_\alpha\begin{pmatrix}\rho\\ \mathbf{m}\\ \mathcal{E}\end{pmatrix}\;=\;0\,,\tag{13.2.29}$$

where $\mathrm{D}^2(S^\star\mathbf{u})$ denotes the Hessian of $(S^\star\mathbf{u})$ as a function of $(\,q\,,\,\mathbf{n}\,,\,r\,)$, and D^2S the Hessian of S as a function of $(\rho\,,\,\mathbf{m}\,,\,\mathcal{E}\,)$. These matrices are symmetric by the Schwarz Lemma. Hence $-\,\mathrm{D}^2S$ is a symmetrizer of (13.2.29) in the sense of Definition 2.1. And the system (13.2.29) is simply the quasilinear form of the Euler equations (13.2.18) in conservative variables.

13.3 The Cauchy problem

The Friedrichs symmetrizers described in Section 13.2 allow us, in principle, to apply the general theory of Chapter 10 for the local existence of H^k solutions, $k > d/2 + 1$. One has to be careful with vacuum though, because it is not allowed in the general symmetrization procedure, and supposedly H^k solutions should of course vanish at infinity.

For polytropic gases, Chemin proved the existence of smooth solutions involving vacuum outside a compact set by using the procedure of Makino *et al.* [128]; see [32] for a detailed analysis.

For more general materials, it is possible to prove the existence of smooth solutions *away from vacuum* by modifying slightly Theorem 10.1. We may impose non-zero conditions at infinity, as already mentioned in Chapter 10, and look for solutions in the affine space $\mathbf{W}_0 + H^k$. (By a change of frame, the velocity at infinity, $\mathbf{u}_0$, may be taken equal to 0.)

Theorem 13.1 *Considering the Euler equations* (13.2.18) *endowed with a complete equation of state $e = e(v, s)$ such that, in some open domain*

$$\mathscr{U} \subset \{ (v, \mathbf{u}, s) \in (0, +\infty) \times \mathbb{R}^d \times \mathbb{R} \},$$

$$\frac{\partial^2 e}{\partial v^2} = - \left.\frac{\partial p}{\partial v}\right|_s > 0,$$

we assume that

$$g \in \mathbf{V}_0 + H^k(\mathbb{R}^d), \quad \mathbf{V}_0 = (v_0, \mathbf{u}_0, s_0) \in \mathscr{U},$$

with $k > 1 + d/2$ and g is valued in $\mathscr{K} \subset\subset \mathscr{U}$. Then there exists $T > 0$ and a unique classical solution $\mathbf{V} \in \mathscr{C}^1(\mathbb{R}^d \times [0, T])$ of the Cauchy problem associated with (13.2.18) *and the initial data $u(0) = g$. Furthermore, $\mathbf{V} - \mathbf{V}_0$ belongs to $\mathscr{C}([0, T]; H^k) \cap \mathscr{C}^1([0, T]; H^{k-1})$.*

There is a wide literature on the continuation of smooth solutions, mostly for polytropic gases though. In some cases, global smooth solutions arise [74, 183]. On the other hand, there are numerous blow-up results [5, 32, 194].

Still, the understanding of the Cauchy problem for (multidimensional) Euler equations (with general pressure laws) is a wide open question. This makes the study of special, piecewise smooth, solutions, like curved shocks as in Chapter 15 interesting. In the next section, we review some basic facts on (planar) shock waves in gas dynamics.

13.4 Shock waves

13.4.1 *The Rankine–Hugoniot condition*

A function $\mathbf{U} = (\rho, \mathbf{u}, e)$ of class $\mathscr{C}^1$ outside a moving interface Σ is a weak solution of (13.2.18) if and only if it satisfies (13.2.18) outside Σ and if the Rankine–Hugoniot jump conditions hold across Σ. If $\mathbf{n} \in \mathbb{R}^d$ denotes a (unit) normal vector to Σ and σ the normal speed of propagation of Σ, the Rankine–Hugoniot condition associated with (13.2.18) reads

$$\begin{cases} [\rho(\mathbf{u}\cdot\mathbf{n} - \sigma)] = 0, \\ [\rho(\mathbf{u}\cdot\mathbf{n} - \sigma)\mathbf{u} + p\,\mathbf{n}] = 0, \\ \left[\rho(\mathbf{u}\cdot\mathbf{n} - \sigma)(\frac{1}{2}\|\mathbf{u}\|^2 + e) + p\,\mathbf{u}\cdot\mathbf{n}\right] = 0, \end{cases} \tag{13.4.30}$$

where the brackets $[\cdot]$ stand as usual for jumps across Σ. The first equation in (13.4.30) implies

$$j := \rho_l(\mathbf{u}_l\cdot\mathbf{n} - \sigma) = \rho_r(\mathbf{u}_r\cdot\mathbf{n} - \sigma), \tag{13.4.31}$$

where by convention, the subscripts l and r are such that the vector $\mathbf{n}$ points to the state indexed by r, and j thus corresponds to the *mass-transfer flux* from the state indexed by l to the state indexed by r.

We may distinguish between two kinds of discontinuities, depending on the value of j. The first kind corresponds to discontinuities moving accordingly with the fluid, also called *contact discontinuities*, for which the mass-transfer flux j is in fact equal to zero. Their normal speed of propagation is $\sigma = \mathbf{u}_{r,l}\cdot\mathbf{n}$. When their tangential velocity does experience a jump across Σ, which causes a singularity of the vorticity $\nabla\times\mathbf{u}$, they are called *vortex sheets*.

The second kind of discontinuity corresponds to $j \neq 0$. We call them *dynamical* discontinuities. The fact that j is non-zero implies the tangential velocity must be continuous and only the normal velocity $\mathbf{u}\cdot\mathbf{n} =: u$ experiences a jump across Σ. Furthermore, some manipulations of the Rankine–Hugoniot equations (13.4.30) yield the relation

$$[e] + \langle p\rangle\,[v] = 0, \tag{13.4.32}$$

where $\langle p\rangle$ denotes the arithmetic mean value of pressures on both sides. We recall the derivation of (13.4.32) for completeness. First, we observe that (13.4.31) implies

$$[u] = j\,[v]. \tag{13.4.33}$$

Then, substituting in the momentum jump condition $j\,[u] + [p] = 0$, we get

$$[p] = -j^2\,[v]. \tag{13.4.34}$$

So that a necessary condition for such a discontinuity to exist is $[p]\,[v] < 0$. Additionally, the energy jump condition can be rewritten as

$$j\,\langle u\rangle\,[u] + j\,[e] + [p\,u] = 0.$$

Using (13.4.33) and (13.4.34), we see that

$$j\,\langle u\rangle\,[u] + [p\,u] = j\,\langle p\rangle\,[v].$$

Therefore, we have

$$j\,(\,[e]\,+\,\langle p\rangle\,[v]\,)\;=\;0,$$

which proves (13.4.32) since $j \neq 0$.

Two different states are thus connected by a dynamical discontinuity if and only if there exists $j \in \mathbb{R}\backslash\{0\}$ so that (13.4.32), (13.4.33) and (13.4.34) hold. Observe that (13.4.32) and (13.4.34) are purely thermodynamic. We call the set of all possible states connected to a reference state by a dynamical discontinuity a *Hugoniot locus.*

13.4.2 *The Hugoniot adiabats*

The structure of Hugoniot loci is of great importance in the theory of shock waves, and relies heavily on the so-called Hugoniot adiabats, characterized by (13.4.32). Given a reference specific volume v_0, the corresponding density $\rho_0 = 1/v_0$, and a reference energy e_0, we define $p_0 := p(\rho_0, e_0)$ and consider the set of states $(v = 1/\rho, e)$ such that

$$e\,-\,e_0\,+\,\frac{p(\rho,e)+p_0}{2}\left(\frac{1}{\rho}\,-\,\frac{1}{\rho_0}\right)\;=\;0\,. \tag{13.4.35}$$

This set is called a *Hugoniot adiabat.* By the implicit function theorem, it is *locally* a smooth curve, parametrized by ρ, in the (ρ, e)-plane. Its *global* behaviour is crucial in the resolution of Riemann problems.

Examples

- For a Bethe–Weyl fluid, using the fact that $v \mapsto p(v, s)$ is a global diffeomorphism, Hugoniot adiabats are equivalently defined in the (p, s)-plane by the zero set of the function

  $$h_0(p,s)\;:=\;e(v(p,s),s)\,-\,e_0\,+\,\frac{p+p_0}{2}\,(v(p,s)\,-\,v_0)\,. \tag{13.4.36}$$

  This will be used below in the discussion of admissibility criteria.
- In the case of polytropic gases with $\gamma \geq 1$, the Hugoniot adiabats are merely hyperboles in the (ρ, p)-plane, given by

  $$\rho\;=\;\rho_0\,\frac{(\gamma+1)\,p_0\,+\,(\gamma-1)\,p}{(\gamma+1)\,p\,+\,(\gamma-1)\,p_0}.$$

13.4.3 *Admissibility criteria*

For real compressible fluids, the admissibility of shocks is still a controversial topic. The purpose of this section is to compare several admissibility criteria under various assumptions on the equation of state.

We start with a most elementary observation on the variations of thermodynamic quantities across dynamical discontinuities: because of (13.4.32) and

(13.4.34) we have indeed

$$\text{sign}\,[\rho] \;=\; \text{sign}\,[p] \;=\; \text{sign}\,[e].$$

This allows us to speak about *compressive* discontinuities without ambiguity, in that both the density and the pressure should increase along particle paths. On the contrary, density and pressure decrease along particle paths for *expansive* (or rarefaction) discontinuities.

Entropy criterion

A first admissibility criterion comes from the second principle of thermodynamics, or equivalently from the Lax entropy criterion associated with the mathematical entropy $S = \rho\, s$, which is concave (see Proposition 13.2) if e is a convex function of (v, s), a 'minimal' thermodynamical stability assumption that we make from now on. The entropy criterion requires that

$$j\,[s] \;\geq\; 0\,. \tag{13.4.37}$$

(Observe that the inequality (13.4.37) is trivially satisfied (and is an equality) for static discontinuites.) It is shown below that for dynamical discontinuities in Smith fluids, the entropy criterion (13.4.37) is equivalent to compressivity.

Proposition 13.3 (Henderson–Menikoff) *For a Smith fluid, we have*

$$\textit{sign}\,[p] \;=\; \textit{sign}\,[s]$$

on Hugoniot loci, which means that the function h_0 *defined in* (13.4.36) *does not vanish for* (p, s) *such that* $(p - p_0)\,(s \;-\; s_0) \;<\; 0.$

Proof By (13.1.2) and (13.1.5), we have

$$v\,\mathrm{d}p \;=\; -\,\gamma\,p\,\mathrm{d}v \;+\; \Gamma\,T\,\mathrm{d}s\,. \tag{13.4.38}$$

So, substituting $\mathrm{d}v$ into

$$\mathrm{d}h_0 \;=\; \mathrm{d}e \;+\; \frac{v - v_0}{2}\,\mathrm{d}p \;+\; \frac{p + p_0}{2}\,\mathrm{d}v \;=\; T\,\mathrm{d}s \;+\; \frac{v - v_0}{2}\,\mathrm{d}p \;+\; \frac{p_0 - p}{2}\,\mathrm{d}v \tag{13.4.39}$$

we obtain

$$\mathrm{d}h_0 \;=\; \left(1 \;+\; \Gamma\,\frac{p_0 - p}{2\gamma p}\right) T\,\mathrm{d}s \;+\; \frac{1}{2}\left(v - v_0 \;+\; \frac{v}{\gamma\,p}\,(p - p_0)\right)\mathrm{d}p\,. \tag{13.4.40}$$

Therefore, because of Smith's condition in (13.1.15), h_0 is increasing with s at constant p. We thus have

$$(\,h_0(p, s) \;-\; h_0(p, s_0)\,)\,(s \;-\; s_0) \;>\; 0.$$

Now, if we look at $h_{00} := h_0(\cdot, s_0)$, preferably in the v variable, we see that it has an inflexion point at $v = v_0$, and its second derivative reduces to

$$\partial^2_{vv} h_{00} \,=\, \frac{1}{2}\,(v - v_0)\,\frac{\partial^2 p}{\partial v^2}.$$

By the convexity of p, this shows that

$$(\,h_{00}(v) \,-\, h_{00}(v_0)\,)\,(v \,-\, v_0) \,>\, 0,$$

or equivalently

$$(\,h_0(p, s_0) \,-\, h_0(p_0, s_0)\,)\,(p \,-\, p_0) \,<\, 0.$$

To conclude, assume that $(p - p_0)\,(s \,-\, s_0) \,<\, 0$. Then

$$h_0(p, s) \,=\, (\,h_0(p, s) \,-\, h_0(p, s_0)\,) \,+\, (\,h_0(p, s_0) \,-\, h_0(p_0, s_0)\,)$$

is the sum of two terms of the same sign as $(s \,-\, s_0)$, which cannot vanish unless $s \,=\, s_0$. □

Internal structure of shocks

Another, physically relevant, admissibility criterion is based on dissipative effects due to viscosity and/or heat conductivity. Indeed, dynamical 'discontinuities' in fluids are not exactly sharp: because of dissipative phenomena they have an internal structure, also called a *shock layer*. Mathematically, a shock layer is a smooth solution of the full equations of motion – the (compressible) Navier–Stokes–Fourier equations, consisting of the Euler equations supplemented with viscous terms in the stress tensor and with a heat-flux term in the energy equation – which is 'almost' a solution of the Euler equations outside a thin region of high gradients. A more precise definition is delicate to formulate, and it is a tough task to actually show the existence of arbitrarily curved shock layers (see the recent series of papers by Guès *et al.* [77–79]). But for *planar* discontinuities, the search for a layer is a much simpler, ODE problem, which was addressed a long time ago by Gilbarg [69]. Planar shock layers are indeed to be sought as heteroclinic travelling wave solutions of the Navier–Stokes–Fourier equations

$$\begin{cases} \partial_t \rho \,+\, \nabla\cdot(\rho\,\mathbf{u}) \,=\, 0\,, \\[2ex] \partial_t\,(\rho\mathbf{u}) \,+\, \nabla\cdot(\rho\,\mathbf{u}\otimes\mathbf{u}) \,+\, \nabla p \,=\, \nabla\,(\lambda\,\nabla\cdot\mathbf{u}) \,+\, \nabla\cdot(\mu\,(\nabla\mathbf{u} \,+\, {}^t(\nabla\mathbf{u})))\,, \\[2ex] \partial_t\,\left(\rho\,\left(\frac{1}{2}\|\mathbf{u}\|^2 \,+\, e\right)\right) \,+\, \nabla\cdot\left(\left(\rho\,\left(\frac{1}{2}\|\mathbf{u}\|^2 \,+\, e\right) \,+\, p\,\right)\mathbf{u}\right) \,=\, \nabla\cdot(\,\kappa\,\nabla T\,) \\[2ex] \;+\,\nabla\cdot(\,(\lambda\,\nabla\cdot\mathbf{u} \,+\, \mu\,(\nabla\mathbf{u} \,+\, (\nabla\mathbf{u})^{\mathrm{T}})\,)\,\mathbf{u}\,)\,. \end{cases} \tag{13.4.41}$$

Here, λ and μ are viscosity coefficients and κ is heat conductivity, and these three coefficients may depend (in a smooth way) on ρ.

Theorem 13.2 (Gilbarg) *For Bethe–Weyl fluids, all planar compressive discontinuities admit shock layers for any λ, μ, κ such that $\lambda + 2\mu > 0$ and $\kappa > 0$.*

We do not reproduce the proof here, which is rather long but very clear in [69]. The interested reader will easily check that our definition of Bethe–Weyl fluids fits with the assumptions I,Ia., II, III, IV ([69] p. 271) of Gilbarg. To recast our problem in Gilbarg's framework, which is purely one-dimensional, we first write the ODE system governing travelling waves of speed σ in a given direction $\mathbf{n}$. Decomposing the velocity field $\mathbf{u}$ into its normal component $u := \mathbf{u} \cdot \mathbf{n}$ and its tangential component $\check{\mathbf{u}}$, we get the equations

$$\begin{cases} (\rho\,(u-\sigma))' = 0\,, \\ (\rho\,(u-\sigma)\,\check{\mathbf{u}})' = (\mu\,\check{\mathbf{u}}')'\,, \\ (\rho\,(u-\sigma)\,u)' + p' = ((\lambda+2\mu)\,u')'\,, \\ \big(\rho\,(u-\sigma)\,(e+\tfrac{1}{2}\,\|\mathbf{u}\|^2)\big)' + (p\,u)' = (\kappa\,T' + (\lambda+2\mu)\,u\,u')'\,, \end{cases} \tag{13.4.42}$$

where $'$ denotes the derivative with respect to $\xi := \mathbf{x}\cdot\mathbf{n} - \sigma t$. For solutions of (13.4.42), we have

$$\rho\,(u-\sigma) \equiv \textit{const.} =: j\,, \tag{13.4.43}$$

consistently with the notation previously introduced (see (13.4.31)). Regarding the tangential velocity, we have

$$\mu\,(\check{\mathbf{u}})' - j\,\check{\mathbf{u}} \equiv \textit{const.}$$

Thus for non-zero mass-tranfer flux j, we must have

$$\check{\mathbf{u}}(-\infty) = \check{\mathbf{u}}(+\infty) =: \check{\mathbf{u}}^\infty,$$

which is again consistent with the Rankine–Hugoniot condition (13.4.30). Furthermore, the only homoclinic orbit of the ODE

$$\mu\,(\check{\mathbf{u}})' = j\,(\check{\mathbf{u}} - \check{\mathbf{u}}^\infty)$$

is its fixed point istself, that is, we must have $\check{\mathbf{u}} \equiv \check{\mathbf{u}}^\infty$. Taking this into account, as well as (13.4.43), the last equation in (13.4.42) equivalently reads

$$(\rho\,u - \sigma)\,(e + \tfrac{1}{2}\,u^2) + (p\,u) - \kappa\,T' - (\lambda+2\mu)\,u\,u' \equiv \textit{const.}$$

Additionally, by change of frame, we can assume without loss of generality that $\sigma = 0$. In this way, we have exactly recovered the equations solved by Gilbarg (namely, (1), (2) and (3) p. 257 in [69]).

In view of Proposition 13.3 and Theorem 13.2, for Smith fluids the entropy criterion (13.4.37) is equivalent to compressivity and ensures the existence of a planar shock layer.

More generally, what can we say for Bethe–Weyl fluids (of which Smith fluids are special cases) about the relationship between compressivity, the entropy criterion (13.4.37) and the existence of a planar shock layer? An easy result is the following.

Proposition 13.4 *For Bethe–Weyl fluids, dissipative shock layers converging fast enough to their endstates, that is, heteroclinic orbits between hyperbolic fixed points of* (13.4.42) *with* $\kappa \geq 0$ *and* $\lambda + 2\mu \geq 0$*, satisfy the entropy criterion in* (13.4.37)*.*

Proof For simplicity we denote by $\mathbf{D}$ the diffusion tensor

$$\mathbf{D} := (\lambda \nabla \cdot \mathbf{u}) I_d + \mu (\nabla \mathbf{u} + {}^t(\nabla \mathbf{u})).$$

Along smooth solutions of (13.4.41), because of (13.1.1) the entropy per unit volume S satisfies

$$\partial_t S + \nabla \cdot (S \mathbf{u}) = \frac{1}{T} (\nabla \cdot (\kappa \nabla T) + \mathbf{D} : \nabla \mathbf{u}), \tag{13.4.44}$$

which is reminiscent from (13.2.27), with additional terms due to dissipation. Looking at travelling-wave solutions of (13.4.44) for which $\rho(u - \sigma) \equiv j$ and $\check{\mathbf{u}} \equiv \check{\mathbf{u}}^\infty$, we find that

$$j T s' = (\kappa T')' + (\lambda + 2\mu)(u')^2 \tag{13.4.45}$$

along heteroclinic orbits of (13.4.42). (This equation can of course be obtained directly from (13.4.42), using (13.1.1) in the form $T s' = e' + p v'$.) Integrating (13.4.45) then yields, if the orbit converges sufficiently fast towards its endstates,

$$j [s] = \int_{-\infty}^{+\infty} \left(\kappa \left(\frac{T'}{T} \right)^2 + \frac{1}{T} (\lambda + 2\mu)(u')^2 \right) \mathrm{d}\xi, \tag{13.4.46}$$

which is obviously non-negative. □

So, for Bethe–Weyl fluids a reasonable admissibility criterion is the existence of a shock layer, and under the additional condition (13.1.15), it is equivalent to compressivity.

For other kinds of fluids, like van der Waals fluids below a critical temperature, for instance, alternate admissibility criteria are needed. As a matter of fact, on the one hand, it is well-known that dynamical phase transitions can be compressive (condensation) but also expansive (vapourization). On the other hand, layers for dynamical phase transitions, also called diffuse interfaces, cannot be obtained from the purely dissipative system (13.4.41). Some dispersion has to be added in connection with capillarity effects. This approach was proposed independently by Slemrod [196] and Truskinovsky [214] in the 1980s and led to the so-called viscosity–capillarity criterion. We shall not enter into more detail here (see [10]).

The Liu criterion

Even with monotone isentropes, the admissibility of shocks is a tough problem in lack of convexity (that is, without the assumption $\mathcal{G} > 0$). It was investigated in the 1970s by Tai Ping Liu [119] and independently by Ling Hsiao and coworkers [224]. This led them to introduce what is now called the Liu criterion and is a generalization of the Oleĭnik criterion – originally introduced for scalar conservation laws – formulated in terms of shock speeds along Hugoniot loci.

Definition 13.3 (Liu) *A discontinuity between $\mathbf{W}_l$ and $\mathbf{W}_r$, of speed σ, satisfies the Liu criterion if*

- *for all $\mathbf{W} \in \mathscr{H}(\mathbf{W}_l; \mathbf{W}_r)$, the part of the Hugoniot locus issued from $\mathbf{W}_l$ and arriving at $\mathbf{W}_r$, the corresponding speed $\sigma(\mathbf{W}_l; \mathbf{W})$ satisfies*

$$\sigma = \sigma(\mathbf{W}_l; \mathbf{W}_r) \leq \sigma(\mathbf{W}_l; \mathbf{W}),$$

- *for all $\mathbf{W} \in \mathscr{H}(\mathbf{W}_r; \mathbf{W}_l)$, the part of the Hugoniot locus issued from $\mathbf{W}_r$ and arriving at $\mathbf{W}_l$, the corresponding speed $\sigma(\mathbf{W}_r; \mathbf{W})$ satisfies*

$$\sigma(\mathbf{W}_r; \mathbf{W}) \leq \sigma(\mathbf{W}_r; \mathbf{W}_l) = \sigma.$$

Using this criterion, we have the following result, analogous to Theorem 13.2 without the restriction $\mathcal{G} > 0$.

Theorem 13.3 (Liu–Pego) *For fluids with positive pressure and temperature satisfying*

$$\gamma > 0\,, \quad \delta > 0\,, \quad \Gamma > 0,$$

all discontinuities satisfying Liu's criterion admit purely viscous shock layers, that is, with identically zero heat conductivity κ. If, additionally,

$$\Gamma \geq \delta,$$

this is true for arbitrary heat conductivity: discontinuities satisfying Liu's criterion admit general shock layers.

We refer to the original papers [119, 157] for the proof. The first part was proved by Liu [119], and the second one by Pego [157]. To help the reader with the way those authors stated their assumptions, we just point out the following equivalences, where the right-hand inequalities are precisely the ones required by

Liu and Pego. For positive pressure and temperature

$$\gamma > 0 \iff \left.\frac{\partial p}{\partial v}\right|_s < 0\,,$$

$$\delta > 0 \iff \left.\frac{\partial e}{\partial T}\right|_v > 0\,,$$

$$\delta\,\Gamma > 0 \iff \left.\frac{\partial p}{\partial T}\right|_v > 0\,,$$

$$\Gamma \geq \delta \iff \left.\frac{\partial e}{\partial v}\right|_T \geq 0\,.$$

The latter inequality is in fact an equality for ideal gases, and a strict inequality for van der Waals fluids.

Interestingly, the Liu criterion allows *expansive* discontinuities. Taking $v_l < v_r$ on the same adiabat and such that the graph of p is above its chord on the interval $[v_l\,,\,v_r]$, and u_l, u_r compatible with (13.4.33) and $j := \sqrt{-(p_r - p_l)/(v_r - v_l)} > 0$, we obtain an expansive discontinuity of speed

$$\sigma = u_{l,r} - v_{l,r}\sqrt{-\frac{p_r - p_l}{v_r - v_l}}.$$

And so for intermediate states $\mathbf{W}$,

$$\sigma(\mathbf{W}_l;\mathbf{W}) = u_l - v_l\sqrt{-\frac{p - p_l}{v - v_l}} \geq \sigma,$$

while

$$\sigma(\mathbf{W}_r;\mathbf{W}) == u_r - v_r\sqrt{-\frac{p - p_r}{v - v_r}} \leq \sigma.$$

This means that the discontinuity meets the Liu criterion, though being expansive.

However, expansive discontinuities may lack internal structure: for instance, as was pointed out by Pego [157], when heat conductivity dominates viscosity and the condition $\Gamma \geq \delta$ fails, there are expansive discontinuities having *no* layer, though meeting Liu's criterion.

The Lax shock criterion

As a byproduct of the Liu criterion, we obtain the celebrated (weak) Lax shock inequalities

$$\lambda_i(\mathbf{W}_r;\mathbf{n}) \leq \sigma(\mathbf{W}_l;\mathbf{W}_r) \leq \lambda_i(\mathbf{W}_l;\mathbf{n}),$$

either for $i = 1$ or for $i = 3$. This is a standard result in the theory of hyperbolic conservation laws, relying on the fact that shock speeds tend to characteristic speeds as the amplitudes of shocks go to 0. It can be checked in a most direct way for fluids. As a matter of fact, two states $\mathbf{W}_0$ and $\mathbf{W}_1$ connected by a dynamical discontinuity of speed σ are such that

$$\sigma = u_0 \pm v_0 \sqrt{-\frac{p_1 - p_0}{v_1 - v_0}} = u_1 \pm v_1 \sqrt{-\frac{p_1 - p_0}{v_1 - v_0}}.$$

Assume for instance that $\pm$ here above is a $+$. Then by a connectedness argument we have

$$\sigma(\mathbf{W}_0; \mathbf{W}) = u_0 + v_0 \sqrt{-\frac{p - p_0}{v - v_0}}$$

along $\mathscr{H}(\mathbf{W}_0; \mathbf{W}_1)$, and

$$\sigma(\mathbf{W}_1; \mathbf{W}) = u_1 + v_1 \sqrt{-\frac{p - p_1}{v - v_1}}$$

along $\mathscr{H}(\mathbf{W}_1; \mathbf{W}_0)$. And from (13.4.38) and (13.4.39) we have

$$\left(v + \frac{1}{2}\Gamma\,(v - v_0) \right) \mathrm{d}p = \left(\frac{1}{2}\Gamma\,(p - p_0) - \gamma\, p \right) \mathrm{d}v \tag{13.4.47}$$

along $\{h_0 = 0\}$, which contains $\mathscr{H}(\mathbf{W}_0; \mathbf{W}_1)$. Therefore,

$$-\frac{p - p_0}{v - v_0} \longrightarrow \gamma_0 \frac{p_0}{v_0} \quad \text{when} \quad \mathbf{W} \xrightarrow{\mathscr{H}(\mathbf{W}_0;\mathbf{W}_1)} \mathbf{W}_0,$$

and so

$$\sigma(\mathbf{W}_0; \mathbf{W}) \longrightarrow u_0 + \sqrt{\gamma_0 p_0 v_0} = u_0 + c_0 = \lambda_3(\mathbf{W}_0; \mathbf{n}).$$

For the same reason,

$$\sigma(\mathbf{W}_1; \mathbf{W}) \longrightarrow u_1 + \sqrt{\gamma_1 p_1 v_1} = u_1 + c_1 = \lambda_3(\mathbf{W}_1; \mathbf{n})$$

when

$$\mathbf{W} \xrightarrow{\mathscr{H}(\mathbf{W}_1;\mathbf{W}_0)} \mathbf{W}_1.$$

More precisely, i-Lax shock inequalities read

$$\lambda_i(\mathbf{W}_r; \mathbf{n}) < \sigma < \lambda_i(\mathbf{W}_l; \mathbf{n}), \quad \lambda_{i-1}(\mathbf{W}_l; \mathbf{n}) < \sigma < \lambda_{i+1}(\mathbf{W}_r; \mathbf{n}).$$

or equivalently, for $i = 1$,

$$u_r - c_r < \sigma < u_l - c_l\,, \quad \sigma < u_r\,, \tag{13.4.48}$$

and for $i = 3$,

$$u_r + c_r < \sigma < u_l + c_l\,, \quad \sigma > u_l\,. \tag{13.4.49}$$

A discontinuity satisfying either one of these sets of inequalities is called a Lax shock. More precisely, in one space dimension, a discontinuity satisfying the i-Lax shock inequalities is usually called an i-shock. However, in several space dimensions this distinction is irrelevant, because the choice of left and right is arbitrary, and exchanging $\mathbf{W}_l$ and $\mathbf{W}_r$ amounts to changing $\mathbf{n}$ into $-\mathbf{n}$, hence the 1-Lax shock inequalities into the 3-ones. (Recall that the notation u stands for $\mathbf{u} \cdot \mathbf{n}$, and that σ is a normal speed and therefore depends also on the direction of $\mathbf{n}$.) Still, it is possible to distinguish between the two states of a Lax shock from an intrinsic point of view, and more precisely, we may speak of the state *behind* the shock and the state ahead of the shock. Indeed, the motion of the shock with respect of the fluid flow on either side of the shock goes from the state indexed by r to the state indexed by l if we have (13.4.48) whereas it goes from l to r if we have (13.4.49). In other words, the state behind the shock is indexed by r in the first case and by l in the second one. In both cases, the Lax shock inequalities imply that the state behind the shock is *subsonic*, and the other one, *supersonic*, according to whether the Mach number

$$M_{l,r} := \frac{|u_{l,r} - \sigma|}{c_{l,r}} \tag{13.4.50}$$

is less than or greater than one. Indeed, (13.4.48) implies $M_r < 1 < M_l$ and (13.4.49) implies $M_l < 1 < M_r$. To summarize, Lax shocks are characterized by

- a non-zero mass-transfer flux across the discontinuity,
- a subsonic state behind the discontinuity, and a supersonic state ahead of the discontinuity.

Now, what is the relationship between the Lax shock criterion and the other criteria? For concave S, a standard Taylor expansion shows that *weak* shocks satisfying the entropy criterion (13.4.37) are necessarily Lax shocks. (See [109].) What can we say for shocks of arbitrary strength? The answer is not easy. One difficulty is that Mach numbers involve slopes of isentropes, since by (13.4.31) and the definition of sound speed,

$$M^2 = \frac{j^2}{- \left.\dfrac{\partial p}{\partial v}\right|_s}, \tag{13.4.51}$$

and isentropes are different from shock curves. However, it is possible to reformulate the Lax shock criterion in terms of the slopes of Hugoniot adiabats, thanks to the following result, which is in fact the continuation of Proposition 13.3.

Proposition 13.5 *For a Smith fluid, the Hugoniot adiabat,* $\{h_0(p,s) = 0\}$, *issued from* (p_0, s_0) *(where* h_0 *is defined as in* (13.4.36)*), is a curve parametrized* *by p in the (v,p)-plane. The non-dimensional quantity*

$$R := \frac{p-p_0}{v-v_0}\,\frac{\partial v}{\partial p}\bigg|_{h_0}. \tag{13.4.52}$$

achieves the value 1 *on that curve if and only if*

$$M^2 = \frac{(p-p_0)/(v-v_0)}{\dfrac{\partial p}{\partial v}\bigg|_s} \tag{13.4.53}$$

does so at the same point.

(Observe that, thanks to (13.4.34), the equality in (13.4.53) is equivalent to (13.4.51).)

Proof It relies on (13.4.47), valid along $\{h_0 = 0\}$. Under the assumption (13.1.15), the coefficient of dv in (13.4.47) is always non-zero, which proves the first claim and yields the formula

$$R = \frac{p-p_0}{v-v_0}\,\frac{v + \frac{1}{2}\Gamma\,(v-v_0)}{\frac{1}{2}\Gamma\,(p-p_0) - \gamma\,p}.$$

Hence, $R = 1$ if and only if

$$\frac{p-p_0}{v-v_0} = -\frac{\gamma\,p}{v}.$$

Since

$$-\frac{\gamma\,p}{v} = \frac{\partial p}{\partial v}\bigg|_s$$

by (13.1.10), we get the conclusion. □

Of course the equalities $R = 1$ and $M^2 = 1$ occur simultaneously at p_0. But we also know (by Taylor expansion) that M^2 is less than 1 on the compression branch $\{\,p > p_0\,\}$ of the Hugoniot locus close to the reference state. So by Proposition 13.5 this property persists as long as R does not achieve 1 along the corresponding branch of the Hugoniot adiabat. Similarly, M^2 is greater than 1 on the expansion branch. More precisely, using the fact that $(p - p_0)\,(v - v_0) < 0$ along the Hugoniot locus, we can see that $(M^2-1)\,(R-1) \geq 0$, with equality only at (p_0, v_0).

Corollary 13.1 *For a Smith fluid, Lax shocks are characterized by $R < 1$ behind the discontinuity and $R > 1$ ahead.*

14

BOUNDARY CONDITIONS FOR EULER EQUATIONS

We now turn to the Initial Boundary Value Problem (IBVP) for Euler equations. We provide below a classification of IBVPs according to various physical situations, and discuss possible boundary conditions ensuring well-posedness. (For similar discussions in the case of *viscous* compressible fluids, the reader may refer in particular to [152, 159, 201]; also see the review paper by Higdon [84].) We shall also give an explicit and elementary construction of Kreiss symmetrizers for uniformly stable Boundary Value Problems.

14.1 Classification of fluids IBVPs

As far as smooth domains are concerned, a crucial issue is the well-posedness of IBVPs in half-spaces (obtained using co-ordinate charts). To fix ideas, we consider IBVPs in the half-space $\{ x_d \geq 0 \}$ (without loss of generality, the Euler equations being invariant by rotations).

We recall from Section 13.2 (Proposition 13.1) that the characteristic speeds of the Euler equations (13.2.18) in the direction $\mathbf{n} = (0, \ldots, 0, 1)^{\mathrm{T}}$ are

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c,$$

where $u := \mathbf{u} \cdot \mathbf{n}$ is the last component of $\mathbf{u}$ and c is the sound speed. When the boundary is a *wall*, u is clearly zero. Otherwise, if $u \neq 0$, we can distinguish between *incoming* flows, for which $u > 0$, and *outgoing* flows, for which $u < 0$. Another distinction to be made concerns the Mach number

$$M := \frac{|u|}{c}.$$

The flow is said to be *subsonic* if $M < 1$ and *supersonic* if $M > 1$.

This yields the following classification, when c is non-zero.

Non-characteristic problems

Out-Supersonic ($u < 0$ and $M > 1$, hence $\lambda_1, \lambda_2, \lambda_3 < 0$). There is *no* incoming characteristic. No boundary condition should be prescribed.

Out-Subsonic ($u < 0$ and $M < 1$, hence $\lambda_1, \lambda_2 < 0, \lambda_3 > 0$). There is *one* incoming characteristic. One and only one boundary condition should be prescribed.

In-Subsonic ($u > 0$ and $M < 1$, hence $\lambda_1 < 0$, λ_2, $\lambda_3 > 0$). There are $(d+1)$ incoming characteristics, counting with multiplicity. This means that $(d+1)$ independent boundary conditions are needed.

In-Supersonic ($u > 0$ and $M > 1$, hence λ_1, λ_2, $\lambda_3 > 0$). *All* characteristics are incoming. This means that all components of the unknown $\mathbf{W}$ should be prescribed on the boundary.

Characteristic problems

Slip walls ($u = 0$, hence $\lambda_1 < 0$, $\lambda_2 = 0$, $\lambda_3 > 0$). One and only one boundary condition $b(\mathbf{W})$ should be prescribed. For the IBVP to be *normal*, the 2-eigenfield should be tangent to the level set of b.

Out-Sonic ($u = -c$, hence λ_1, $\lambda_2 < 0$, $\lambda_3 = 0$). No boundary condition should be prescribed.

In-Sonic ($u = c$, hence $\lambda_1 = 0$, λ_2, $\lambda_3 > 0$). A set of $(d+1)$ boundary conditions $b_1(\mathbf{W}), \dots b_{d+1}(\mathbf{W})$ should be prescribed, and the 1-eigenfield should be tangent to the level sets of $b_1, \dots b_{d+1}$.

14.2 Dissipative initial boundary value problems

In this section, we look for dissipative boundary conditions. This notion depends on the symmetrization used. For concreteness, we use the simplest symmetrization, in $(p, \mathbf{u}, s)$ variables, given in Section 13.2.3. We recall indeed that, away from vacuum, the Euler equations can be rewritten as

$$S(p,\mathbf{u},s)\,(\,\partial_t + A(p,\mathbf{u},s;\nabla)\,)\begin{pmatrix} p \\ \mathbf{u} \\ s \end{pmatrix} = 0,$$

where $S(p, \mathbf{u}, s)$ is symmetric positive-definite and

$$S(p,\mathbf{u},s)\,A(p,\mathbf{u},s;\mathbf{n}) = \begin{pmatrix} \dfrac{\mathbf{u}\cdot\mathbf{n}}{\rho c^2} & \mathbf{n}^{\mathrm{T}} & 0 \\ \mathbf{n} & \rho\,(\mathbf{u}\cdot\mathbf{n})\,I_d & 0 \\ & & \\ 0 & 0 & \mathbf{u}\cdot\mathbf{n} \end{pmatrix}.$$

In what follows we take $\mathbf{n} = (0, \dots, 0, 1)^{\mathrm{T}}$. According to Definition 9.2, dissipativeness of a set of boundary conditions encoded by a non-linear mapping $b : (p, \mathbf{u}, s) \mapsto b(p, \mathbf{u}, s)$ requires that $-SA^d$ be non-negative on the tangent bundle of the manifold $\mathscr{B} = \{\, b(p, \mathbf{u}, s) = \underline{b}\,\}$, while strict dissipativeness requires that $-SA^d$ be coercive on the same bundle.

A straightforward computation shows that for $\dot{U} = (\dot{p}, \dot{\check{\mathbf{u}}}, \dot{u}, \dot{s})^{\mathrm{T}}$,

$$(SA^d\dot{U},\dot{U}) = \rho\,u\left(\frac{1}{u^2}\,(\,u\,\dot{u} + v\,\dot{p}\,)^2 - \frac{v^2}{u^2}\,(1 - M^2)\,\dot{p}^2 + \|\dot{\check{\mathbf{u}}}\|^2 + v\,\dot{s}^2\right) \tag{14.2.1}$$

if $u \neq 0$.

We can now review the different cases.

Supersonic outflow ($u < 0$ and $M > 1$). We see that $-SA^d$ is coercive on the whole space. So this case is harmless.

Subsonic outflow ($u < 0$ and $M < 1$). The restriction of $-SA^d$ to the hyperplane $\{ \dot{p} = 0\}$ is obviously coercive. Thus a strictly dissipative condition is obtained by prescribing the pressure p at the boundary. Another possible, simple, choice is to prescribe the normal velocity u, since $-SA^d$ is also coercive when restricted to the hyperplane $\{ \dot{u} = 0\}$.

Subsonic inflow ($u > 0$ and $M < 1$). This is the most complicated case. Prescribing the pressure among the boundary conditions would obviously be a bad idea, for the same reason as it is a good one for subsonic outflows. On the other hand, the easiest way to cancel some bad terms is to prescribe the tangential velocity $\check{\mathbf{u}}$ and the entropy s, which leaves only one boundary condition to be determined in such a way that $u\,\mathrm{d}u + v\,\mathrm{d}p = 0$ on the tangent bundle of $\mathscr{B}$. Recalling that, by (13.1.1), the specific enthalpy $h = e + p\,v$ is such that

$$\mathrm{d}h = T\,\mathrm{d}s + v\,\mathrm{d}p\,,$$

we see that the above requirement is achieved by $\frac{1}{2}\,u^2 + h$ along isentropes. Hence a strictly dissipative set of boundary conditions is

$$(\,\frac{1}{2}\,u^2 + h\,,\,\check{\mathbf{u}}\,,\,s\,)\,.$$

Other boundary conditions may be exhibited that are more relevant from a physical point of view – for instance using concepts of *total pressure* and *total temperature*, see [11].

Supersonic inflow ($u > 0$ and $M > 1$). We see that SA^d (instead of $-SA^d$) is coercive. But the tangent spaces are reduced to $\{0\}$. So this case is also harmless.

Slip walls ($u = 0$). The kernel of A^d is the d-dimensional subspace $\{\,(0, \check{\mathbf{u}}, 0, \dot{s})\}$, which is part of the tangent subspace $\{\,\dot{u} = 0\}$ associated with the natural boundary condition on u – as required by the normality criterion. The matrix SA^d is null on $\{\,\dot{u} = 0\}$, which means that the boundary condition is dissipative but of course not strictly dissipative.

Out-Sonic ($u = -c$). The matrix $-SA^d$ is non-negative but has isotropic vectors (defined by $u\,\dot{u} + v\,\dot{p} = 0$, $\dot{\check{\mathbf{u}}} = 0$, $\dot{s} = 0$).

In-Sonic ($u = c$). The only possible choice of dissipative boundary conditions is the one described for subsonic inflows, which cancels all remaining terms in $(SA^d\dot{U}, \dot{U})$ (since $-(1 - M^2)\,\dot{p}^2$ is zero). The normality criterion is met by those boundary conditions because

$$\mathrm{Ker}\ SA^d = \{(\dot{p}, \dot{u}, \dot{s})\,;\ u\,\dot{u} + v\,\dot{p} = 0\,,\ \dot{\check{\mathbf{u}}} = 0\,,\ \dot{s} = 0\,\}\,.$$

We now turn to a more systematic testing of boundary conditions, which is known (and will be shown) to be less restrictive.

14.3 Normal modes analysis

Our purpose is to discuss boundary conditions from the Kreiss–Lopatinskiĭ point of view, for general fluids equipped with a complete equation of state. To get simpler computations, we choose the specific volume v and the specific entropy s as thermodynamic variables, and rewrite the Euler equations as

$$\begin{cases} \partial_t v + \mathbf{u}\cdot\nabla v - v\,\nabla\cdot\mathbf{u} = 0\,, \\ \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + v\,p'_v\,\nabla v + v\,p'_s\,\nabla s = 0\,, \\ \partial_t s + \mathbf{u}\cdot\nabla s = 0\,, \end{cases} \tag{14.3.2}$$

with the short notations

$$p'_v = \left.\frac{\partial p}{\partial v}\right|_s, \quad p'_s = \left.\frac{\partial p}{\partial s}\right|_v .$$

Later, we shall make the connection with the non-dimensional coefficients γ and Γ, in that

$$p'_v = -\gamma\,\frac{p}{v}, \quad p'_s = \Gamma\,\frac{T}{v}.$$

Alternatively, we recall that

$$p'_v = -\frac{c^2}{v^2},$$

where c is the sound speed defined in (13.1.10). Our minimal assumption is that c is real (positive). Furthermore, we have seen in Section 14.1 that boundary conditions for supersonic flows are either trivial or absent. A normal modes analysis is irrelevant in those cases. And sonic IBVP are so degenerate that a normal modes analysis is also useless. So from now on we concentrate on the subsonic case, assuming that

$$0 < |u| < c. \tag{14.3.3}$$

14.3.1 *The stable subspace of interior equations*

Linearizing (14.3.2) about a reference state $(v, \mathbf{u} = (\check{\mathbf{u}}, u), s)$, we get

$$\begin{cases} (\,\partial_t + \check{\mathbf{u}}\cdot\check{\nabla} + u\,\partial_z\,)\,\dot v - v\,\nabla\cdot\dot{\mathbf{u}} = 0\,, \\ (\,\partial_t + \check{\mathbf{u}}\cdot\check{\nabla} + u\,\partial_z\,)\,\dot{\mathbf{u}} + v\,p'_v\,\nabla\dot v + v\,p'_s\,\nabla\dot s = 0\,, \\ (\,\partial_t + \check{\mathbf{u}}\cdot\check{\nabla} + u\,\partial_z\,)\,\dot s = 0\,, \end{cases} \tag{14.3.4}$$

where z stands for the co-ordinate x_d, normal to the boundary, and $\check{\nabla}$ is the gradient operator along the boundary. The tangential co-ordinates will be denoted by $\mathbf{y} \in \mathbb{R}^{d-1}$. By definition, for $\mathrm{Re}\,\tau > 0$ and $\boldsymbol{\eta} \in \mathbb{R}^{d-1}$, the sought stable subspace, $E_-(\tau, \boldsymbol{\eta})$ is the space spanned by vectors $(\dot{v}, \dot{\mathbf{u}}, s)$ such that there exists a mode ω of positive real part for which

$$\exp(\,\tau\, t\,)\ \exp(\,i\,\boldsymbol{\eta}\cdot\mathbf{y})\ \exp(\,-\,\omega\, z\,)\,(\dot{v}, \dot{\mathbf{u}}, s)$$

solves (14.3.4). We are thus led to the system

$$\begin{cases} (\,\tau\,+\,i\,\boldsymbol{\eta}\cdot\check{\mathbf{u}}\,-\,u\,\omega\,)\,\dot{v}\,-\,v\,(i\,\boldsymbol{\eta}\cdot\dot{\check{\mathbf{u}}})\,+\,v\,\omega\,\dot{u}\,=\,0\,, \\ (\,\tau\,+\,i\,\boldsymbol{\eta}\cdot\check{\mathbf{u}}\,-\,u\,\omega\,)\,\dot{\check{\mathbf{u}}}\,+\,v\,p'_v\,i\,\dot{v}\,\boldsymbol{\eta}\,+\,v\,p'_s\,i\,\dot{s}\,\boldsymbol{\eta}\,=\,0\,, \\ (\,\tau\,+\,i\,\boldsymbol{\eta}\cdot\check{\mathbf{u}}\,-\,u\,\omega\,)\,\dot{u}\,-\,v\,p'_v\,\omega\,\dot{v}\,-\,v\,p'_s\,\omega\,\dot{s}\,=\,0\,, \\ (\,\tau\,+\,i\,\boldsymbol{\eta}\cdot\check{\mathbf{u}}\,-\,u\,\omega\,)\,\dot{s}\,=\,0\,, \end{cases} \tag{14.3.5}$$

where we have used the obvious (although ugly) notation $\dot{\mathbf{u}} = (\dot{\check{\mathbf{u}}}, \dot{u})$. To simplify the writing we introduce

$$\widetilde{\tau}\,:=\,\tau\,+\,i\,\boldsymbol{\eta}\cdot\check{\mathbf{u}}\,.$$

(Observe that $\mathrm{Re}\,\widetilde{\tau}\,=\,\mathrm{Re}\,\tau$.) The only non-trivial modes are thus obtained

- for $\omega\,=\,\widetilde{\tau}/u$ and

$$u\,(i\,\boldsymbol{\eta}\cdot\dot{\check{\mathbf{u}}})\,-\,\widetilde{\tau}\,\dot{u}\,=\,0 \quad \text{and} \quad p'_v\,\dot{v}\,+\,p'_s\,\dot{s}\,=\,0\,, \tag{14.3.6}$$

- for ω solution of the dispersion relation

$$(\,\widetilde{\tau}\,-\,u\,\omega\,)^2\,+\,v^2\,p'_v\,(\,\omega^2\,-\,\|\boldsymbol{\eta}\|^2\,)\,=\,0\,,$$

  and

$$\begin{cases} (\,\widetilde{\tau}\,-\,u\,\omega\,)\,\dot{\check{\mathbf{u}}}\,+\,v\,p'_v\,i\,\dot{v}\,\boldsymbol{\eta}\,=\,0\,, \\ (\,\widetilde{\tau}\,-\,u\,\omega\,)\,\dot{u}\,-\,v\,p'_v\,\omega\,\dot{v}\,=\,0 \\ \dot{s}\,=\,0\,. \end{cases} \tag{14.3.7}$$

We see that the dispersion equation, which also reads

$$(\,\widetilde{\tau}\,-\,u\,\omega\,)^2\,=\,c^2\,(\,\omega^2\,-\,\|\boldsymbol{\eta}\|^2\,) \tag{14.3.8}$$

has no purely imaginary root when $\mathrm{Re}\,\tau > 0$. Looking at the easier case $\boldsymbol{\eta} = 0$ and using our usual continuity argument, we find that because of the subsonicity condition (14.3.3), (14.3.8) has exactly one root of positive real part, which we denote by ω_+, and one root of negative real part, ω_-. By definition, the stable subspace $E_-(\tau, \boldsymbol{\eta})$ involved in the Kreiss–Lopatinskiĭ condition is made of normal modes with $\mathrm{Re}\,\omega > 0$ for $\mathrm{Re}\,\tau > 0$. (Recall that – with our notation – decaying modes at $z = +\infty$ are obtained for $\operatorname{Re}\omega > 0$.) So the root ω_- does not contribute to $E_-(\tau, \boldsymbol{\eta})$, and we only need to consider ω_+ and, *if* $u > 0$, $\omega_0 := \widetilde{\tau}/u$. To simplify again the writing, we simply denote ω_+ by ω when no confusion is possible. (We shall come back to the notations $\omega_\pm$ in Section 14.4.)

If $u < 0$ (outflow case), $E_-(\tau, \boldsymbol{\eta})$ is a line, spanned by the solution $\mathbf{e}(\tau, \boldsymbol{\eta}) = (\dot{v}, \dot{\check{\mathbf{u}}}, \dot{u}, \dot{s})^{\mathrm{T}}$ of (14.3.7) defined by

$$\mathbf{e}(\tau, \boldsymbol{\eta}) = \begin{pmatrix} v\,(\widetilde{\tau} - u\,\omega) \\ i\,c^2\,\boldsymbol{\eta} \\ -\,c^2\,\omega \\ 0 \end{pmatrix}. \tag{14.3.9}$$

If $u > 0$ (inflow case), $E_-(\tau, \boldsymbol{\eta})$ is a hyperplane. For convenience, we introduce the additional notation

$$a := u\,\widetilde{\tau} + \omega\,(c^2 - u^2). \tag{14.3.10}$$

An elementary manipulation of (14.3.8) then shows that

$$a\,(\widetilde{\tau} - u\,\omega) = c^2\,(\widetilde{\tau}\,\omega - u\,\|\boldsymbol{\eta}\|^2).$$

Using this relation and combining (14.3.6) and (14.3.7) together, we get the very simple description

$$E_-(\tau, \boldsymbol{\eta}) = \ell(\tau, \boldsymbol{\eta})^{\perp},$$
$$\ell(\tau, \boldsymbol{\eta}) := \left(a\,,\,-i\,v\,u\,\boldsymbol{\eta}^{\mathrm{T}}\,,\,v\,\widetilde{\tau}\,,\,a\,p'_s/p'_v\right).$$

(Observe that ℓ is homogeneous degree 1 in $(\tau, \boldsymbol{\eta})$, like a.) This description has the advantage of unifying the treatment of regular points and Jordan points $\widetilde{\tau} = u\,\|\boldsymbol{\eta}\|$ – where ω coincides with ω_0. In the particular 'one-dimensional' case, i.e. with $\boldsymbol{\eta} = 0$, one easily checks that

$$\omega = \frac{\tau}{u + c}, \quad a = \tau\,c \quad \text{and} \quad \ell(\tau, 0) := \tau\,\left(c\,,\,0\,,\,v\,,\,c p'_s/p'_v\right).$$

Omitting the null coefficient, we recover in the latter a left eigenvector associated with the incoming eigenvalue $\lambda_1 = u - c$ of the one-dimensional Euler equations

$$\begin{cases} \partial_t v + u\,\partial_x v - v\,\partial_x u = 0\,, \\[2ex] \partial_t\, u + u\,\partial_x u + v\,p'_v\,\partial_x v + v\,p'_s\,\partial_x s = 0\,, \\[2ex] \partial_t s + u\,\partial_x s = 0\,. \end{cases}$$

14.3.2 *Derivation of the Lopatinskiĭ determinant*

Once we have the description of $E_-(\tau, \boldsymbol{\eta})$ we easily arrive at the Lopatinskiĭ condition. We consider the two cases separately.

Outflow case If $u < 0$, one boundary condition $b(v, \mathbf{u}, s)$ is required. The existence of non-trivial normal modes in the line $E_-(\tau, \boldsymbol{\eta})$ is thus equivalent to

$$\Delta(\tau, \boldsymbol{\eta}) := \mathrm{d}b \cdot \mathbf{e}(\tau, \boldsymbol{\eta}) \neq 0 .$$

We see in particular that this condition does not depend on $\partial b/\partial s$. By definition of $\mathbf{e}$,

$$\Delta(\tau, \boldsymbol{\eta}) = v\,(\widetilde{\tau} - u\,\omega)\,\frac{\partial b}{\partial v} + i\,c^2\,\mathrm{d}_{\check{\mathbf{u}}} b \cdot \boldsymbol{\eta} - c^2\,\omega\,\frac{\partial b}{\partial u} .$$

We thus recover (as pointed out in Section 14.2) that prescribing the pressure ensures the uniform Lopatinskiĭ condition, since for $b(v, \mathbf{u}, s) = p(v, s)$ we have

$$\Delta(\tau, \boldsymbol{\eta}) = v\,(\widetilde{\tau} - u\,\omega)\,p'_v \neq 0 \quad \text{for} \quad \mathrm{Re}\,\tau \geq 0\,,\ (\tau, \boldsymbol{\eta}) \neq (0,0) .$$

This is less obvious with the alternative boundary condition $b(v, \mathbf{u}, s) = u$, because in this case

$$\Delta(\tau, \boldsymbol{\eta}) = c^2\,\omega\,,$$

and it demands a little effort to check that ω does not vanish. For clarity, we state this point in the following.

Proposition 14.1 *For $0 > u > -c$, the root ω_+ of* (14.3.8) *that is of positive real part for* $\mathrm{Re}\,\widetilde{\tau} > 0$ *has a continuous extension to* $\mathrm{Re}\,\widetilde{\tau} = 0$ *that does not vanish for* $(\tau, \boldsymbol{\eta}) \neq (0,0)$.

Proof The only points where it could happen that ω_+ vanishes are such that $\widetilde{\tau}^2 = -c^2\,\|\boldsymbol{\eta}\|^2$. In particular, $\omega_+ = 0$ implies $\widetilde{\tau} \in i\,\mathbb{R}$, and also $-\widetilde{\tau}^2 \geq (c^2 - u^2)\,\|\boldsymbol{\eta}\|^2$ – otherwise, ω_+ is of positive real part – in which case both roots of (14.3.8) are purely imaginary. To determine which one is ω_+, we use the Cauchy–Riemann equations, which imply that

$$\frac{\partial(\mathrm{Im}\ \omega_+)}{\partial(\mathrm{Im}\ \widetilde{\tau})} > 0 .$$

Using this selection criterion, it is not difficult to see that

$$\omega_\pm = \frac{-\,u\,\widetilde{\tau} \pm i\,c\,\mathrm{sign}(\mathrm{Im}\ \widetilde{\tau})\,\sqrt{-\,\widetilde{\tau}^2 - (c^2 - u^2)\,\|\boldsymbol{\eta}\|^2}}{c^2 - u^2} . \tag{14.3.11}$$

(The same formulae hold for $u > 0$.) For $\widetilde{\tau}^2 = -\,c^2\,\|\boldsymbol{\eta}\|^2$, this gives (using the fact that u is negative)

$$\omega_- = 0 \quad \text{and} \quad \omega_+ = \frac{-\,2\,u\,\widetilde{\tau}}{c^2 - u^2} ,$$

the latter being non-zero unless $(\tau, \boldsymbol{\eta}) = (0,0)$. □

More generally, we can find alternative boundary conditions that satisfy the uniform Lopatinskiĭ condition without being dissipative. For instance, take $\alpha \in (0,1)$ and

$$b(v, \mathbf{u}, s) = \frac{\alpha}{2} u^2 + h(p(v,s), s),$$

where $h = e + p\,v$ is the specific enthalpy. We find that

$$\Delta(\tau, \boldsymbol{\eta}) = -c^2 (\widetilde{\tau} - (1 - \alpha) u \omega) \neq 0 \quad \text{for} \quad \operatorname{Re} \tau \geq 0, \; (\tau, \boldsymbol{\eta}) \neq (0,0).$$

This means that the uniform Lopatinskiĭ condition is satisfied. Nevertheless, the quadratic form defined in (14.2.1) may be non-definite on the tangent hyperplane $\{\alpha\, u\, \dot{u} + v\, \dot{p} + T\, \dot{s} = 0\}$. More specifically, this happens for

$$\alpha \in \left(\frac{1}{1 + \sqrt{1 - M^2}}, 1 \right).$$

Inflow case If $u < 0$, $(d+1)$ boundary conditions $b_1(v, \mathbf{u}, s), \ldots, b_{d+1}(v, \mathbf{u}, s)$ are needed. The existence of non-trivial normal modes in the hyperplane $E_-(\tau, \boldsymbol{\eta})$ is thus equivalent to

$$\Delta(\tau, \boldsymbol{\eta}) := \det \begin{pmatrix} \mathrm{d}b_1 \\ \vdots \\ \mathrm{d}b_{d+1} \\ \ell(\tau, \boldsymbol{\eta}) \end{pmatrix} \neq 0.$$

We may consider, for example, as the first d conditions

$$b_1 = \check{\mathbf{u}}_1, \ldots, b_{d-1} = \check{\mathbf{u}}_{d-1}, \quad b_d = s.$$

Then, up to a minus sign,

$$\Delta(\tau, \boldsymbol{\eta}) = -v\, \widetilde{\tau}\, \frac{\partial b_{d+1}}{\partial v} + a\, \frac{\partial b_{d+1}}{\partial u}.$$

In particular, for

$$b_{d+1}(v, u, s) = \frac{\alpha}{2} u^2 + h(p(v,s), s),$$

we have

$$\Delta(\tau, \boldsymbol{\eta}) = (c^2 + \alpha\, u^2)\, \widetilde{\tau} + \alpha\, u \omega (c^2 - u^2) \neq 0 \quad \text{for} \quad \operatorname{Re} \tau \geq 0, (\tau, \boldsymbol{\eta}) \neq (0,0),$$

provided that $\alpha > 0$. For α large enough, this gives again an example of boundary conditions satisfying the uniform Lopatinskiĭ condition without being dissipative.

14.4 Construction of a Kreiss symmetrizer

The system (14.3.5) can be put into the abstract form

$$(\mathcal{A}(\tau,\boldsymbol{\eta}) + \omega\, I_{d+2})\,\dot{U} = 0\,, \quad \dot{U} = \begin{pmatrix} \dot{v} \\ \dot{\mathbf{u}} \\ \dot{u} \\ \dot{s} \end{pmatrix}.$$

We do not really need the explicit form of the $(d+2)\times(d+2)$ matrix $\mathcal{A}(\tau,\boldsymbol{\eta})$. We already know the eigenmodes of $\mathcal{A}(\tau,\boldsymbol{\eta})$ from the calculation in Section 14.3.1. Its eigenvalues are $-\omega_0$, of geometric multiplicity d, $-\omega_-$ and $-\omega_+$, with associated eigenvectors defined by (14.3.9), i.e.

$$\mathbf{e}_\pm(\tau,\boldsymbol{\eta}) = \begin{pmatrix} v\,(\widetilde{\tau} - u\,\omega_\pm) \\ i\,c^2\,\boldsymbol{\eta} \\ -\,c^2\,\omega_\pm \\ 0 \end{pmatrix}.$$

The construction of a Kreiss' symmetrizer in the neighbourhood of points $(\tau,\boldsymbol{\eta})$ such that $\mathrm{Re}\,\tau > 0$ follows the general line described in Chapter 5, basically using the Lyapunov matrix theorem. It is more interesting to look at the construction about points with $\mathrm{Re}\,\tau = 0$, or equivalently $\mathrm{Re}\,\widetilde{\tau} = 0$. It will appear to be much more elementary than for general abstract systems.

We recall that a local construction amounts to finding local coordinates, i.e. locally invertible matrices $T(\tau,\boldsymbol{\eta})$, and Hermitian matrices $\widetilde{\mathcal{R}}(\tau,\boldsymbol{\eta})$, such that in those co-ordinates the new matrix $\widetilde{\mathcal{A}} = T^{-1}\,\mathcal{A}\,T$ enjoys a local estimate

$$\mathrm{Re}\,(\widetilde{\mathcal{R}}\,\widetilde{\mathcal{A}}) \gtrsim \gamma\,I\,, \quad \gamma = \mathrm{Re}\,\tau\,, \tag{14.4.12}$$

and the boundary matrix $\widetilde{B} = B\,T$ satisfies

$$\widetilde{\mathcal{R}} + C\,\widetilde{B}^*\,\widetilde{B} \geq \beta\,I \tag{14.4.13}$$

for β and $C > 0$. For this construction, we of course assume that the Lopatinskiĭ condition holds at the point $(\tau_0,\boldsymbol{\eta}_0)$ considered, which means that we have an algebraic estimate

$$\|P_0\,U\|^2 \lesssim \|(I - P_0)\,U\|^2 + \|\,\widetilde{B}\,U\,\|^2\,, \quad \forall U \in \mathbb{C}^{d+2}\,,$$

where P_0 denotes a projector onto $E_-(\tau_0,\boldsymbol{\eta}_0)$. If P is a smoothly defined projector in the neighbourhood of $(\tau_0,\boldsymbol{\eta}_0)$ such that $P_0 = P(\tau_0,\boldsymbol{\eta}_0)$, this also implies the locally uniform estimate

$$\|P\,U\|^2 \lesssim \|(I - P)\,U\|^2 + \|\,\widetilde{B}\,U\,\|^2\,, \quad \forall U \in \mathbb{C}^{d+2}\,. \tag{14.4.14}$$

This will be our working assumption, with a projector P to be specified.

We also assume (14.3.3). For $\mathrm{Re}\,\tau = 0$, we always have $\mathrm{Re}\,\omega_0 = 0$, and we have locally uniform bounds on $\mathrm{Re}\,\omega_\pm$ depending on the location of $(\tau,\boldsymbol{\eta})$ with respect to *glancing points*, where

$$-\widetilde{\tau}^2 = (c^2 - u^2)\,\|\boldsymbol{\eta}\|^2\,, \quad \omega_- = \omega_+\,.$$

- If $-\widetilde{\tau}^2 < (c^2 - u^2)\,\|\boldsymbol{\eta}\|^2$,

$$\mathrm{Re}\,\omega_- \lesssim -1 \quad \text{and} \quad \mathrm{Re}\,\omega_+ \gtrsim 1\,,$$

- whereas, if $-\widetilde{\tau}^2 > (c^2 - u^2)\,\|\boldsymbol{\eta}\|^2$, we only have

$$\mathrm{Re}\,\omega_- \lesssim -\gamma \quad \text{and} \quad \mathrm{Re}\,\omega_+ \gtrsim \gamma\,. \tag{14.4.15}$$

  Both ω_- and ω_+ remain bounded away from ω_0 though.

Of course the weaker estimates (14.4.15) are satisfied in the first case. So we can address the two open cases simultaneously. The glancing points will be dealt with separately. Away from glancing points, ω_0, ω_+ and ω_- are separated, so we can choose a smooth basis of and consider the diagonal matrix similar to $\mathcal{A}$,

$$\widetilde{\mathcal{A}} = \begin{pmatrix} -\,\omega_+ & & \\ & -\,\omega_0\,I_d & \\ & & -\,\omega_- \end{pmatrix}.$$

The construction of a matrix $\widetilde{\mathcal{R}}$ satisfying (14.4.12) and (14.4.13) thus depends on the sign of $\mathrm{Re}\,\omega_0$. We shall use the spectral projectors $\Pi_\pm$ according to $\omega_\pm$.

Outflow $(u < 0$ hence $\mathrm{Re}\,\omega_0 \lesssim -\gamma)$ E_- is one-dimensional and spanned by the first vector in the new basis, and the basic estimate deriving from the Lopatinskiĭ condition is (14.4.14) with $P = \Pi_+$, i.e.

$$\|U_1\|^2 \lesssim \sum_{j=2}^{d+2} \|U_j\|^2 + \|\,\widetilde{B}\,U\,\|^2.$$

Then, taking

$$\widetilde{\mathcal{R}} = \begin{pmatrix} -\,1 & & \\ & \mu\,I_d & \\ & & \mu \end{pmatrix}$$

with μ large enough ensures (14.4.12) and (14.4.13).

Inflow $(u > 0$ hence $\mathrm{Re}\,\omega_0 \gtrsim \gamma)$ E_- is the hyperplane spanned by the first $(d+1)$ vectors of the new basis, and the estimate deriving from the Lopatinskiĭ condition is (14.4.14) with $P = I - \Pi_-$, i.e.

$$\sum_{j=1}^{d+1} \|U_j\|^2 \lesssim \|U_{d+2}\|^2 + \|\,\widetilde{B}\,U\,\|^2\,.$$

Then, taking

$$\widetilde{\mathcal{R}} = \begin{pmatrix} -1 & & \\ & -I_d & \\ & & \mu \end{pmatrix}$$

with μ large enough ensures (14.4.12) and (14.4.13).

The construction of a symmetrizer about glancing points is, in general, a hard piece of algebra/algebraic geometry. Here, it will be greatly simplified by the special structure of our system, and more specifically the nice form of the eigenvectors $\mathbf{e}_\pm$, which depend linearly on $\omega_\pm$. There is a very simple way to get a Jordan basis at glancing points and extend it smoothly in their neighbourhoods. It merely consists in considering

$$\mathbf{e}_1 := i\,\frac{\mathbf{e}_+ + \mathbf{e}_-}{2} \quad \text{and} \quad \mathbf{e}_2 := \frac{\mathbf{e}_+ - \mathbf{e}_-}{\omega_+ - \omega_-}$$

instead of $\mathbf{e}_+$ and $\mathbf{e}_-$. Both $\mathbf{e}_1$ and $\mathbf{e}_2$ are well-defined and independent of each other in the neighbourhood of glancing points, $\mathbf{e}_2$ is even constant

$$\mathbf{e}_2 = \begin{pmatrix} -v\,u \\ 0 \\ -c^2 \\ 0 \end{pmatrix},$$

and $\mathbf{e}_1$ does belong to E_- *at* glancing points. (The interest of the factor i in $\mathbf{e}_1$ will appear afterwards.) It is not difficult to find the reduced 2×2 matrix $\mathbf{a}$ of $\mathcal{A}$ on the invariant plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. To do the calculation even more easily, we may use the standard notations

$$\langle\omega\rangle = \frac{\omega_+ + \omega_-}{2} \quad , \quad \langle\mathbf{e}\rangle = \frac{\mathbf{e}_+ + \mathbf{e}_-}{2}\,, \text{ etc.},$$

$$[\omega] = \omega_+ - \omega_- \quad , \quad [\mathbf{e}] = \mathbf{e}_+ - \mathbf{e}_-\,, \text{ etc.}$$

On the one hand, we have

$$\mathcal{A}\,[\mathbf{e}] = -\,[\omega\,\mathbf{e}]$$

and thus, by standard manipulation,

$$\mathcal{A}\,[\mathbf{e}] = -\,[\omega]\,\langle\mathbf{e}\rangle - [\omega]\,\langle\omega\rangle\,\frac{[\mathbf{e}]}{[\omega]}\,.$$

On the other hand, we have

$$\mathcal{A}\,\langle\mathbf{e}\rangle = -\,\langle\omega\,\mathbf{e}\rangle = -\,\langle\omega\,\rangle\,\langle\mathbf{e}\rangle + (\langle\omega\,\rangle^2 - \langle\omega^2\,\rangle)\,\frac{[\mathbf{e}]}{[\omega]}\,.$$

Therefore,

$$\mathbf{a} = \begin{pmatrix} -\langle \omega \rangle & i \\ i\,(\langle \omega \rangle^2 - \langle \omega^2 \rangle) & -\langle \omega \rangle \end{pmatrix}.$$

Recalling from (13.2.23) that

$$\omega_+ + \omega_- = -\frac{2\,u\,\widetilde{\tau}}{c^2 - u^2} \quad \text{and} \quad \omega_+\,\omega_- = -\frac{\widetilde{\tau}^2 + c^2\,\|\boldsymbol{\eta}\|^2}{c^2 - u^2}, \tag{14.4.16}$$

the matrix $\mathbf{a}$ is explicitly given by

$$\mathbf{a} = \begin{pmatrix} \dfrac{u\,\widetilde{\tau}}{c^2 - u^2} & i \\[2ex] -\,i\,c^2\,\dfrac{\widetilde{\tau}^2 + (c^2 - u^2)\,\|\boldsymbol{\eta}\|^2}{(c^2 - u^2)^2} & \dfrac{u\,\widetilde{\tau}}{c^2 - u^2} \end{pmatrix}.$$

(Note that, at glancing points, $\mathbf{a}/i$ is exactly a 2×2 Jordan block: we have performed here an explicit calculation of the reduction pointed out by Ralston [160] in an abstract framework.) Completing $\{\mathbf{e}_1, \mathbf{e}_2\}$ into a whole basis of $\mathbb{C}^{d+2}$ by means of independent eigenvectors of $\mathcal{A}$ associated with $-\omega_0$, we get the reduced – block-diagonal – matrix in that basis

$$\widetilde{\mathcal{A}} = \begin{pmatrix} -\,\omega_0\,I_d & \\ & \mathbf{a} \end{pmatrix}.$$

Then we look for a local symmetrizer $\widetilde{\mathcal{R}}$ that has the same structure as $\widetilde{\mathcal{A}}$. We shall use here the projectors $\Pi_{1,2}$ onto the co-ordinate axes spanned by $\mathbf{e}_{1,2}$.

Outflow ($u < 0$ hence $\mathrm{Re}\,\omega_0 \lesssim -\gamma$) From the Lopatinskiĭ condition we have (14.4.14) with $P = \Pi_1$, i.e.

$$\|U_{d+1}\|^2 \lesssim \sum_{j=1}^{d} \|U_j\|^2 + \|U_{d+2}\|^2 + \|\,\widetilde{B}\,U\,\|^2.$$

So we look for

$$\widetilde{\mathcal{R}} = \begin{pmatrix} \mu\,I_d & \\ & \mathbf{r} \end{pmatrix},$$

with

$$\mathrm{Re}\,(\mathbf{r}\,\mathbf{a}) \gtrsim \gamma\,I_2 \quad \text{and} \quad \mathbf{r} \gtrsim \begin{pmatrix} -1 & 0 \\ 0 & \mu \end{pmatrix}. \tag{14.4.17}$$

Inflow ($u > 0$ hence $\mathrm{Re}\,\omega_0 \gtrsim \gamma$.) From the Lopatinskiĭ condition we have (14.4.14) with $P = I - \Pi_2$, i.e.

$$\sum_{j=1}^{d+1} \|U_j\|^2 \lesssim \|U_{d+2}\|^2 + \|\,\widetilde{B}\,U\,\|^2,$$

and we look for

$$\widetilde{\mathcal{R}} = \begin{pmatrix} -I_d & \\ & \mathbf{r} \end{pmatrix},$$

with again (14.4.17).

We now perform the construction of $\mathbf{r}$ in the two cases simultaneously. We first expand $\mathbf{a}$, writing as before $\tau = \gamma + i\,\delta$. Denoting for simplicity

$$a := \frac{u}{c^2 - u^2} \neq 0, \quad b := \frac{2\,c^2}{(c^2 - u^2)^2} > 0, \quad \text{and} \quad \varepsilon := c^2\,\frac{\delta^2 - (c^2 - u^2)\|\boldsymbol{\eta}\|^2}{(c^2 - u^2)^2},$$

we have

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2\,,$$

$$\mathbf{a}_0 = i\begin{pmatrix} a\,\delta & 1 \\ \varepsilon & a\,\delta \end{pmatrix}, \; \mathbf{a}_1 = \gamma\begin{pmatrix} a & 0 \\ b\,\delta & a \end{pmatrix}, \; \mathbf{a}_2 = -\frac{i}{2}\gamma^2\begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

The matrix $\mathbf{a}_2$ only contributes to higher-order terms in γ. Noting that the lower-left corner of $\mathbf{a}_1/\gamma$,

$$b\,\delta \sim b\sqrt{c^2 - u^2}\,\|\boldsymbol{\eta}\|$$

is non-zero, we consider a matrix $\mathbf{r}$ of the form

$$\mathbf{r} = \begin{pmatrix} \varepsilon\,g & \dfrac{1}{b\delta} + i\,\gamma\,a\,g \\ \dfrac{1}{b\delta} - i\,\gamma\,a\,g & h \end{pmatrix},$$

with g, h some real numbers to be specified. Then

$$\mathrm{Re}\,(\mathbf{r}\,\mathbf{a}) = \gamma\begin{pmatrix} 1 & b\,\delta\,h + \dfrac{a}{b\delta} \\ b\,\delta\,h + \dfrac{a}{b\delta} & a\,(g + h) \end{pmatrix} + \mathcal{O}(\gamma^2)\,.$$

So by the Cauchy–Schwarz inequality, the first requirement in (14.4.17) will hold for small enough γ, provided that g is chosen of the same sign as a, with $|g| \gg h^2 > h$. Observing that, *at* the glancing point,

$$\mathbf{r} = \begin{pmatrix} 0 & 1/(b\delta) \\ 1/(b\delta) & h \end{pmatrix},$$

the second requirement in (14.4.17) will hold with μ of the same order as h (large), in a small enough neighbourhood of the glancing point.

15

SHOCK STABILITY IN GAS DYNAMICS

The general theory (Chapter 12) has shown that the stability of shocks requires that the number of outgoing characteristics (counting with multiplicity) be equal to the number of jump conditions minus one: recall indeed that there is one degree of freedom for the unknown front. With Rankine–Hugoniot jump conditions, a necessary condition to have this equality is the Lax shock criterion.

In this chapter, we deal with the stability of Lax shocks for the full Euler equations, as was done by Majda [126] (and independently by Blokhin [18]). For the stability analysis of undercompressive shocks (submitted to generalized jump conditions), and in particular of subsonic liquid–vapour phase boundaries, the reader may refer to the series of papers [10, 12, 39–41] (also see [215]). Otherwise, we recommend the very nice review paper by Barmin and Egorushkin [8] (in which the reader will find, if not afraid of the cyrillic alphabet, numerous interesting references to the Soviet literature), addressing also the stability of viscous (Lax) shocks.

15.1 Normal modes analysis

Regarding the stability of shocks in gas dynamics, normal modes analysis dates back to the 1940s (and the atomic bomb research): stability conditions were derived in particular by Bethe [16], Erpenbeck [53], and independently by D′yakov [52], Iordanskiĭ [89], Kontorovich [101], etc. It was completed by Majda and coworkers, who paid attention to *neutral* modes. However, they did not publish the complete analysis for the full Euler equations (Majda referring in [126] to unpublished computations by Oliger and Sundström), but only for the isentropic Euler equations. We provide below this analysis, from a mostly algebraic point of view, in which the isentropic case shows up as a special, easier case – corresponding to a section of the algebraic manifold considered. (For a more analytical point of view, see [92].)

With the full Euler equations in space dimension d, the number of outgoing characteristics for Lax shocks is $d+1$. Furthermore, it is easy to check that outgoing characteristics correspond to the state *behind* (according to the terminology introduced in Chapter 13) the shock only: depending on the choice of the normal vector, pointing to the region where states are indexed by r, outgoing characteristics would be denoted λ_2^r and λ_3^r, or λ_1^l and λ_2^l. Or equivalently, the stable subspace to be considered in the normal modes analysis only involves modes associated with the state *behind* the shock. We shall (arbitrarily) use the subscript l for the state past the shock, and the subscript r for the state behind, the latter being most often omitted for simplicity.

From now on, we consider a reference *subsonic* state $\mathbf{W} = \mathbf{W}_r$, and a *supersonic* state $\mathbf{W}_l$ on the corresponding Hugoniot locus such that the mass-transfer flux j is positive (consistently with the Lax shock inequalities in (13.4.48)). By a change of frame, we can assume without loss of generality that the normal propagation speed of the shock σ is zero, and that the tangential velocity of the fluid $\check{\mathbf{u}}$ is zero (recall that the tangential velocity is continuous across the shock front.) In this way, we merely have

$$j = \rho u = \rho_l u_l > 0 \quad \text{and} \quad 0 < M = \frac{u}{c} < 1 . \qquad (15.1.1)$$

15.1.1 *The stable subspace for interior equations*

The story is the same as for Initial Boundary Value Problems, since we only need to consider the equations *behind* the shock. This simplification always occurs for *extreme shocks*. (It does not occur for the phase transitions studied in [9, 10, 12] for instance, but the method is analogous.)

We briefly recall the derivation of the stable subspace associated with interior equations, for readers who would have skipped Section 14.3. There is one additional simplification here, due to the null tangential velocity $\check{\mathbf{u}}$, which amounts to replacing $\widetilde{\tau}$ by τ. Linearizing the Euler equations written in the variables $(v, \mathbf{u}, s)$, (14.3.2), about our constant reference state $(v, \mathbf{u} = (0, u), s)$, we get

$$\begin{cases} (\partial_t + u\,\partial_z)\,\dot{v} - v\,\nabla\cdot\dot{\mathbf{u}} = 0 , \\ (\partial_t + u\,\partial_z)\,\dot{\mathbf{u}} + v\,p'_v\,\nabla\dot{v} + v\,p'_s\,\nabla\dot{s} = 0 , \\ (\partial_t + u\,\partial_z)\,\dot{s} = 0 , \end{cases} \qquad (15.1.2)$$

where z denotes the co-ordinate normal to the shock front. By definition, for Re $\tau > 0$ and $\boldsymbol{\eta} \in \mathbb{R}^{d-1}$, the sought stable subspace, $E_-(\tau, \boldsymbol{\eta})$ is the space spanned by vectors $(\dot{v}, \dot{\mathbf{u}}, s)$ such that there exists a mode ω of positive real part for which

$$\exp(\tau\,t)\;\exp(i\,\boldsymbol{\eta}\cdot\mathbf{y})\;\exp(-\omega\,z)\,(\dot{v}, \dot{\mathbf{u}}, s)$$

solves (15.1.2). Here, we have denoted $\mathbf{y} := (x_1, \ldots, x_{d-1})$. We are thus led to the system

$$\begin{cases} (\tau - u\,\omega)\,\dot{v} - v\,(i\,\boldsymbol{\eta}\cdot\dot{\check{\mathbf{u}}}) + v\,\omega\,\dot{u} = 0 , \\ (\tau - u\,\omega)\,\dot{\check{\mathbf{u}}} + v\,p'_v\,i\,\dot{v}\,\boldsymbol{\eta} + v\,p'_s\,i\,\dot{s}\,\boldsymbol{\eta} = 0 , \\ (\tau - u\,\omega)\,\dot{u} - v\,p'_v\,\omega\,\dot{v} - v\,p'_s\,\omega\,\dot{s} = 0 , \\ (\tau - u\,\omega)\,\dot{s} = 0 , \end{cases}$$

where $\dot{\mathbf{u}} = (\dot{\check{\mathbf{u}}}, \dot{u})$. Because of (15.1.1), the only non-trivial solutions with Re $\omega > 0$ correspond to either $\omega = \tau/u$ or the only root of positive real part of the dispersion equation

$$(\tau - u\omega)^2 = c^2(\omega^2 - \|\boldsymbol{\eta}\|^2). \tag{15.1.3}$$

And the corresponding invariant subspace admits the simple characterization

$$\begin{aligned} E_-(\tau,\boldsymbol{\eta}) &= \ell(\tau,\boldsymbol{\eta})^\perp, \\ \ell(\tau,\boldsymbol{\eta}) &:= \left(a, -i\,v\,u\,\boldsymbol{\eta}^{\mathrm{T}}, v\,\tau, a\,p'_s/p'_v\right), \end{aligned} \tag{15.1.4}$$

where

$$a := u\tau + \omega(c^2 - u^2). \tag{15.1.5}$$

15.1.2 *The linearized jump conditions*

With

$$\sigma = \frac{\partial_t X}{\sqrt{1 + \|\check{\nabla} X\|^2}} \quad \text{and} \quad \mathbf{n} = \frac{1}{\sqrt{1+\|\check{\nabla} X\|^2}} \begin{pmatrix} -\check{\nabla} X \\ 1 \end{pmatrix}$$

the first $(d+1)$ equations in the Rankine–Hugoniot conditions (13.4.30) may be rewritten as

$$\left\{ \begin{aligned} &\left[\frac{u - \check{\mathbf{u}}\cdot\check{\nabla} X}{v}\right] = \left[\frac{1}{v}\right]\partial_t X, \\ &\left[\frac{u - \check{\mathbf{u}}\cdot\check{\nabla} X}{v}\check{\mathbf{u}}\right] - [p]\,\check{\nabla} X = \partial_t X\left[\frac{\check{\mathbf{u}}}{v}\right], \\ &\left[\frac{u - \check{\mathbf{u}}\cdot\check{\nabla} X}{v}u\right] + [p] = \partial_t X\left[\frac{u}{v}\right]. \end{aligned} \right.$$

As to the last equation in (13.4.30), it may equivalently be replaced by the purely thermodynamical equation (13.4.32).

Again, we emphasize that we need not perturb the state past the shock, which simplifies the linearized jump conditions below. This is because we ultimately look for solutions of the linearized jump conditions that belong to the stable subspace $E_-(\tau,\boldsymbol{\eta})$, which does not depend on the state past the shock, $(v_l, \mathbf{u}_l, s_l)$. This simplification is specific to extreme Lax shocks. However, the method is similar for other kinds of shocks.

Linearizing all jump conditions about $v_r = v$, $\check{\mathbf{u}}_r = 0$, $u_r = u$, $s_r = s$ (with v_l, $\check{\mathbf{u}}_l = 0$, u_l, s_l kept fixed) and $X = 0$, and looking for front modes of the form

$\exp(\tau\,t\,)\;\exp(\,i\,\boldsymbol{\eta}\cdot\mathbf{y})$, we obtain the equations

$$\begin{cases} -\,\dfrac{u}{v^2}\,\dot{v}\,+\,\dfrac{1}{v}\,\dot{u}\,=\,[\rho]\,\tau\,\dot{X}\,, \\[2ex] \dfrac{u}{v}\,\dot{\check{\mathbf{u}}}\,=\,i\,[p]\,\dot{X}\,\boldsymbol{\eta}\,, \\[2ex] (p'_v\,-\,u^2/v^2)\,\dot{v}\,+\,2\,\dfrac{u}{v}\,\dot{u}\,+\,p'_s\,\dot{s}\,=\,0\,, \\[2ex] \frac{1}{2}\,(\,p'_v\,[v]\,-\,[p]\,)\,\dot{v}\,+\,(\,T\,+\,\frac{1}{2}\,p'_s\,[v]\,)\,\dot{s}\,=\,0\,, \end{cases} \tag{15.1.6}$$

where the jumps now stand for jumps between the reference states. The last equation has been obtained by linearizing (13.4.32), using the fundamental relation (13.1.1). Note that the Rankine–Hugoniot condition applied to the reference states implies

$$[p]\,=\,-\,\frac{u^2}{v^2}\,[v]. \tag{15.1.7}$$

(This is (13.4.34) with $j\,=\,u/v$.) So we shall substitute $-\,\dfrac{u^2}{v^2}\,[v]$ for $[p]$ in the system (15.1.6).

15.1.3 *The Lopatinskiĭ determinant*

The existence of non-trivial vectors belonging to the space $E_-(\tau,\boldsymbol{\eta})$ defined in (15.1.4) and satisfying the system (15.1.6) is clearly equivalent to the existence of non-trivial solutions of a linear, purely algebraic, system. To simplify the computation, we can first eliminate $\dot{\check{\mathbf{u}}}$ by means of the second equation in (15.1.6), which allows us to substitute $u^2\,[v]\,\|\boldsymbol{\eta}\|^2\,\dot{X}$ for $i\,u\,v\,\boldsymbol{\eta}\cdot\dot{\check{\mathbf{u}}}$ in the equation $\ell(\tau,\boldsymbol{\eta})\,\cdot\,(\,\dot{v},\dot{\check{\mathbf{u}}},\dot{u},\dot{s}\,)\,=\,0$ (defining $E_-(\tau,\boldsymbol{\eta})$). Using again the relation $c^2\,=\,-\,v^2\,p'_v$, we readily obtain the linear system

$$\begin{pmatrix} u & \dfrac{1}{v} & 0 & -\,[\rho]\,\tau \\[2ex] (c^2\,+\,u^2) & 2\,\dfrac{u}{v} & p'_s & 0 \\[2ex] \frac{1}{2}\,(\,c^2\,-\,u^2\,)\,[v] & 0 & T\,+\,\frac{1}{2}\,p'_s\,[v] & 0 \\[2ex] -\,a\,v^2 & v\,\tau & -\,a\,p'_s\,\dfrac{v^2}{c^2} & -\,u^2\,[v]\,\|\boldsymbol{\eta}\|^2 \end{pmatrix} \begin{pmatrix} -\dot{v}/v^2 \\[2ex] \dot{u} \\[2ex] \dot{s} \\[2ex] \dot{X} \end{pmatrix} = \begin{pmatrix} 0 \\[2ex] 0 \\[2ex] 0 \\[2ex] 0 \end{pmatrix}. \tag{15.1.8}$$

The existence of non-trivial solutions of (15.1.8) is then equivalent to the vanishing of its determinant, which we denote by $\Delta(\tau, \boldsymbol{\eta})$.

'One-dimensional' case The expansion of $\Delta(\tau, 0)$ is straightforward. Using the fact that $a = \tau\, c$ for $\boldsymbol{\eta} = 0$, we have

$$\Delta(\tau, 0) = \tfrac{1}{2}\, v\, [v]\, [\rho]\, \tau^2 \begin{vmatrix} c^2 + u^2 & 2\,\dfrac{u}{v} & p_s' \\ c^2 - u^2 & 0 & p_s' + \frac{2\,T}{[v]} \\ -\,c\,v & 1 & -\,p_s'\, v/c \end{vmatrix}$$

$$= \tau^2\, T\, \tfrac{[v]}{v_l}\, c\,(c + u)\ \left(1 + M + M^2\, \tfrac{p_s'}{T}\, [v]\,\right).$$

(Recall that $M = u/c$ denotes the Mach number.) Rewriting $p_s'/T = \Gamma/v$, the one-dimensional stability condition thus reads

$$1 + M + \Gamma\, M^2\, \frac{[v]}{v} \neq 0\,. \tag{15.1.9}$$

This condition holds true at least for weak shocks, as expected from the general theory, or can be checked directly. For, the Mach number M admits the finite limit 1 and thus

$$1 + M + \Gamma\, M^2\, \frac{[v]}{v} > 0 \tag{15.1.10}$$

for $|[v]|$ small enough. The condition (15.1.9) might break down for large shocks though. It depends of course on the equation of state.

An easy consequence of the previous argument is the following.

Proposition 15.1 *Lax shocks of arbitrary strength are stable in one space dimension, provided that $\gamma \geq \Gamma \geq 0$ (at least at the endstates).*

Proof Surprisingly enough, the inequality (15.1.10) trivially holds true for an expansive shock, that is, if $[v] > 0$. But more 'standard' shocks are compressive. And if $[v] < 0$, we need an upper bound for $\Gamma\, M^2/v$ to prove that (15.1.10) holds true. This bound can be found by noting that

$$c^2 = \gamma\, p\, v \geq \Gamma\, p\, v$$

by assumption. Therefore,

$$\Gamma\, M^2\, \frac{[v]}{v} = \frac{\Gamma\, v}{c^2}\, \frac{u^2}{v^2}\, [v] \geq \frac{1}{p}\, j^2\, [v] = -\, \frac{[p]}{p}$$

because of the Rankine–Hugoniot condition in (13.4.34). And consequently,

$$1 + M + \Gamma\, M^2\, \frac{[v]}{v} \geq M + \frac{p_l}{p} > 0\,.$$

□

Observe that ideal gases trivially satisfy $\gamma \geq \Gamma > 0$ (provided that $c_v > 0$). For van der Waals gases, a short computation shows that $\gamma \geq \Gamma$ if and only if

$$\frac{R}{c_v} \geq 2\,\frac{v-b}{b} - \frac{v^2}{a(v-b)}\,RT\,.$$

This is obviously true for large enough temperatures, a rough bound being the critical temperature $T_c := \frac{8a}{27bR}$, but false for 'small' temperatures.

In view of (13.2.20), an equivalent statement of the strict inequalities $\gamma > \Gamma > 0$ is

$$\left.\frac{\partial p}{\partial v}\right|_e < 0 \quad \text{and} \quad \left.\frac{\partial p}{\partial e}\right|_v > 0\,.$$

It is notable that this is precisely the assumptions that enabled Liu [118] to show the unique solvability of the Riemann problem (using his admissibility criterion for shocks).

Multidimensional case Assuming that $\boldsymbol{\eta} \neq 0$, we may introduce

$$V := \frac{\tau}{i\,\|\boldsymbol{\eta}\|} \quad \text{and} \quad A := \frac{a}{i\,\|\boldsymbol{\eta}\|}$$

to simplify the writing. In fact, it will be clearer to handle only quantities homogeneous to velocities. This is why we set

$$A = c\,W\,.$$

Recalling the definition of a, (14.3.10), where ω is the root of positive real part of (13.2.23), we find that W has the simple representation

$$W^2 = V^2 - (c^2 - u^2)\,, \quad \operatorname{Im} W < 0 \text{ when } \operatorname{Im} V < 0\,, \tag{15.1.11}$$

which extends analytically to $\{V \in \mathbb{R}\,;\ V^2 < c^2 - u^2\}$. By a standard argument, using Cauchy–Riemann equations, we also find that W has a continuous extension to $\{V \in \mathbb{R}\,;\ V^2 \geq c^2 - u^2\}$, given by

$$W = \operatorname{sign}(V)\,\sqrt{V^2 - (c^2 - u^2)}\,.$$

Using these notations, the genuinely multidimensional Lopatinskiĭ determinant reads

$$\Delta(\tau,\boldsymbol{\eta}) := -\,\|\boldsymbol{\eta}\|^2 \begin{vmatrix} u & \frac{1}{v} & 0 & -[\rho]\,V \\ (c^2+u^2) & 2\,\frac{u}{v} & p'_s & 0 \\ \frac{1}{2}\,(c^2-u^2)\,[v] & 0 & T + \frac{1}{2}\,p'_s\,[v] & 0 \\ -\,c\,W\,v^2 & v\,V & -\,W\,p'_s\,\frac{v^2}{c} & u^2\,[v] \end{vmatrix}\,,$$

that is,

$$\Delta(\tau,\boldsymbol{\eta}) \;:=\; -\,\|\boldsymbol{\eta}\|^2\,\frac{[v]^2}{2}\,\begin{vmatrix} u & \dfrac{1}{v} & 0 & \dfrac{V}{v_l} \\[2ex] c^2\,+\,u^2 & 2\,\dfrac{u}{v} & p_s' & 0 \\[2ex] c^2\,-\,u^2 & 0 & p_s'\,+\,\frac{2T}{[v]} & 0 \\[2ex] -\,c\,v\,W & V & -\,W\,\dfrac{v}{c}\,p_s' & u^2 \end{vmatrix}.$$

The expansion of $\Delta(\tau,\boldsymbol{\eta})$ results in

$$\Delta(\tau,\boldsymbol{\eta}) \;=\; -\,\|\boldsymbol{\eta}\|^2\,T\,\frac{[v]}{v_l}\,c^2\,u^2\,\Big\{(\,2\,+\,\Gamma\,M^2\,\tfrac{[v]}{v}\,)\,(\,V/u\,+\,W/c\,)\,V/u \qquad (15.1.12)$$
$$-\,(\,1\,-\,M^2\,)\,(\,(V/u)^2\,+\,v_l/v\,)\,\Big\},$$

where we have substituted $T\,\Gamma/v$ for p_s', as in the 1D-case.

From the expression of the Lopatinskiĭ determinant in (15.1.12), we see that multidimensional stability conditions must involve the two non-dimensional quantities

$$r \;:=\; \frac{v_l}{v} \quad \text{and} \quad k \;:=\; 2\,+\,\Gamma\,M^2\,\frac{[v]}{v}\,.$$

It is to be noted that k is related in a simple way to the ratio R defined in Proposition 13.5,

$$R \;=\; \frac{[p]}{[v]}\;\frac{v\,+\,\frac{1}{2}\,\Gamma\,[v]}{\frac{1}{2}\,\Gamma\,[p]\,-\,\gamma\,p}\,.$$

Indeed, using (15.1.7), we find that

$$R \;=\; 1\,-\,\frac{2}{k}\,(1\,-\,M^2)\,.$$

15.2 Stability conditions

15.2.1 *General result*

Thanks to the preliminary work of the previous section, we can derive a complete hierarchy of stability conditions. As expected from the theory in [186], the one-dimensional stability threshold in (15.1.9) turns out to be a transition point from strong multidimensional instability to weak stability of $\mathcal{WR}$ type. Attentive readers will also note that the transition from neutral to uniform stability occurs at a *glancing* point – in that the neutral mode found for $k = 1 + M^2(r-1)$ corresponds to $\omega = \tau/u$ (see Chapter 13 for a more detailed discussion of glancing points). (This kind of transition was pointed out in an abstract setting in [13].) To summarize, we have the following.

Theorem 15.1 (Majda) *The stability of a Lax shock depends on the Mach number* behind *the shock*

$$M = \frac{|\mathbf{u}\cdot\mathbf{n} - \sigma|}{c} \in (0,1)$$

and on the volumes behind *and* past *the shock, respectively denoted by* v *and* v_0*, through the quantities*

$$r := \frac{v_0}{v} \quad \textit{and} \quad k := 2 + \Gamma M^2 \frac{v - v_0}{v}. \tag{15.2.13}$$

- *One-dimensional stability is equivalent to*

$$k \neq 1 - M. \tag{15.2.14}$$

- *Weak (non-uniform) multidimensional stability is equivalent to*

$$1 - M < k \leq 1 + M^2 (r - 1). \tag{15.2.15}$$

- *Uniform multidimensional stability is equivalent to*

$$1 + M^2 (r - 1) < k. \tag{15.2.16}$$

Remark 15.1 If the lower inequality in (15.2.15) is violated, and more precisely if

$$k < 1 - M,$$

the shock is *violently unstable* with respect to multidimensional perturbations, even though it is stable to one-dimensional perturbations. As mentioned by Majda in [125], this striking result was known since the work of Erpenbeck [53] (also see the earlier work of D′yakov [52]).

Proof The one-dimensional stability condition has already been obtained above, in the equivalent form (15.1.9).

It will turn out that multidimensional stability conditions can be infered from the properties of a second-order algebraic curve. We strongly advocate this mostly algebraic, and elementary point of view, which is a convenient alternative to the analytical approach of Jenssen and Lyng [92], for instance. Of course, all methods require some care and are a little lengthy.

The expression in (15.1.12) of the Lopatinskiĭ determinant in terms of W, defined in (15.1.11), shows that multidimensional stability is encoded by the zeroes of the function

$$f(z) := k\,(z + g(z))\,z - (1 - M^2)\,(z^2 + r), \tag{15.2.17}$$

where

$$g(z)^2 = M^2 z^2 - (1 - M^2), \quad \operatorname{Im} g(z) < 0 \tag{15.2.18}$$

on the lower half-plane $\{\mathrm{Im}\, z < 0\}$. On the real axis, g (and thus also f) is extended by continuity, which implies that

$$z\, g(z) > 0 \quad \text{for all} \quad z \in \mathbb{R}\,; \quad z^2 > (1 - M^2)/M^2\,. \tag{15.2.19}$$

Of course, g is analytic up to $z \in \mathbb{R}$ if $z^2 < (1 - M^2)/M^2$. And on that part of the real axis, g is purely imaginary. So it is clear that f cannot vanish for $z \in \mathbb{R}$ and $z^2 < (1 - M^2)/M^2$. Besides (15.2.19), it will also be useful to have in mind that

$$z\, g(z) < 0 \quad \text{for all} \quad z \in i\,\mathbb{R}^-\,. \tag{15.2.20}$$

We shall obtain stability conditions mainly by algebraic arguments, using the following observation. If z is a zero of f, then $x = z^2$ must be a zero of the polynomial $p(\cdot, k)$ defined by

$$p(x,k) = \Big((\,k - (\,1 - M^2\,)\,)\,x - (\,1 - M^2\,)\,r \Big)^2 - k^2\, x\,(\,M^2\, x - (1 - M^2)\,)\,. \tag{15.2.21}$$

For simplicity, we have stressed here only the dependence of p upon the parameter k, even though it is also obviously a polynomial in r and M^2. This simplifies the writing and is confusionless since r and M^2 are kept fixed in the discussion below. Readers gifted in algebra will have checked that p is the *resultant* with respect to y of two polynomials $F(x,y)$ and $G(x,y)$,

$$F(x,y) = k\,(\,x + y\,) - (\,1 - M^2\,)\,(\,x + r\,)\,, \tag{15.2.22}$$

which is obtained by replacing z^2 by x and $z\,g(z)$ by y in $f(z)$, and

$$G(x,y) = y^2 - x\,(M^2\, x - (1 - M^2))\,, \tag{15.2.23}$$

which vanishes simultaneously with $y^2 - z^2\, g(z)^2$.

Conversely, if x is a zero of $p(\cdot,k)$ and $z^2 = x$ then, either

$$(\,k - (\,1 - M^2\,)\,)\,z^2 - (\,1 - M^2\,)\,r = -\,k\,z\,g(z)\,,$$

which means that $f(z) = 0$, or

$$(\,k - (\,1 - M^2\,)\,)\,z^2 - (\,1 - M^2\,)\,r = +\,k\,z\,g(z)$$

and z is *not* a zero of f (unless $z\,g(z) = 0$). If x is *real*, the inequalities (15.2.19) and (15.2.20) will enable us to decide which is the case. This discussion can be summarized as follows, where for convenience we denote

$$x_0 := \frac{(\,1 - M^2\,)\,r}{k - (\,1 - M^2\,)}\,.$$

- If x is a negative real root of $p(\cdot,k)$ and $z^2 = x$, $z \in i\,\mathbb{R}^-$ is a zero of f if and only if

$$(\,k - (\,1 - M^2\,)\,)\,(\,x - x_0\,) > 0\,.$$

- If x is a positive real root of $p(\cdot,k)$ and $z^2 = x > x_* := (1-M^2)/M^2$, $z \in \mathbb{R}$ is a zero of f if and only if

$$(\,k\,-\,(\,1\,-\,M^2\,)\,)\,(\,x\,-\,x_0\,)\;<\;0\,.$$

Therefore, the conclusion will depend on

▷ the sign of $k\,-\,(\,1\,-\,M^2\,)$

and on the position of $x_0(k)$ with respect to the real roots of $p(\cdot,k)$. From the definition (15.2.21) of $p(\cdot,k)$, we see that

$$\begin{aligned} p(x_0(k),k) \;&=\; -\,k^2\,x_0(k)\,(\,M^2\,x_0(k)\,-\,(1\,-\,M^2)\,) \\ &=\; -\,k^2\,\frac{(\,1\,-\,M^2\,)^2\,r}{(\,k\,-\,(\,1\,-\,M^2\,)\,)^2}\,(\,M^2\,r\,-\,k\,+\,1\,-\,M^2\,), \end{aligned}$$

and we easily compute that the dominant coefficient of $p(\cdot,k)$ is

$$a_2 \;=\; (1\,-\,M^2)\,(\,k\,-\,1\,-\,M\,)\,(\,k\,-\,1\,+\,M\,)\,.$$

Consequently, the position of $x_0(k)$ with respect to the real roots of $p(\cdot,k)$ is determined by

▷ the sign of $(\,k\,-\,1\,-\,M\,)\,(\,k\,-\,1\,+\,M\,)\,(\,M^2\,r\,-\,k\,+\,1\,-\,M^2\,)$.

Since $M \in (0,1)$ we obviously have

$$1\,-\,M\;<\;1\,-\,M^2\;<\;1\,+\,M\,.$$

So there will be a priori only two cases to consider, depending on the position of $1\,+\,M$ with respect to $M^2\,r\,+\,1\,-\,M^2$.

Case 1 $\;r\,M\;<\;1\,+\,M$, hence

$$1\,-\,M\;<\;1\,-\,M^2\;<\;M^2\,r\,+\,1\,-\,M^2\;<\;1\,+\,M\,,$$

Case 2 $\;r\,M\;>\;1\,+\,M$, hence

$$1\,-\,M\;<\;1\,-\,M^2\;<\;1\,+\,M\;<\;M^2\,r\,+\,1\,-\,M^2\,.$$

Regarding the other coefficients of $p(\cdot,k)$, at the zeroth order we have

$$a_0\;=\;(1\,-\,M^2)^2\,r^2\;>\;0\,,$$

and

$$a_1(k)\;=\;(1\,-\,M^2)\,(\,k^2\,-\,2\,r\,(\,k\,-\,(\,1\,-\,M^2\,)\,)\,)\,.$$

Thus it is only

$$\text{sign}\;a_2(k)\;=\;\text{sign}\;\{\,(\,k\,-\,1\,-\,M\,)\,(\,k\,-\,1\,+\,M\,)\,\}$$

that determines the sign of the product of the roots of $p(\cdot,k)$. The sign of a_1 is important at singular points of a_2, in order to localize the infinite branches of the algebraic curve $\mathscr{P} := \{ (x,k) \, ; \, p(x,k) = 0 \}$. In fact, we have

$$a_1(1 - M) = (1 - M^2)(1 - M)(1 - M + 2rM) > 0,$$

but the sign of

$$a_1(1 + M) = (1 - M^2)(1 + M)(1 + M - 2rM)$$

depends on r. This leads us to split Case 1 into

Case 1a $rM < (1 + M)/2$, hence $a_1(1 + M) > 0$,

Case 1b $(1 + M)/2 < rM < 1 + M$, hence $a_1(1 + M) < 0$.

We now have all the ingredients to infer the needed qualitative features of $\mathscr{P}$. There is always a parabolic branch with vertical asymptotes $x = -1$ and $x = 0$. Another remark is that $x_0(k)$ is a root of $p(\cdot, k)$ only at $k = 0$, where $x_0 = -r$, and at $k = M^2 r + 1 - M^2$, where $x_0 = x_* = (1 - M^2)/M^2$. (The latter will appear to be a transition point from neutral stability to uniform stability.)

We shall consider successively Cases 1a, 1b, and 2, and arrive at the conclusion that

- a necessary and sufficient condition for f to have only real roots is (15.2.15);
- a necessary and sufficient condition for f to have no root at all is (15.2.16).

The reader may refer to the corresponding figures (Figs. 15.1–15.3) in order to visualize the tedious but elementary arguments involved in the discussion below.

Case 1a

- for $k < 1 - M$, we have $a_2(k) > 0$ and $a_2(k)\, p(x_0(k), k) < 0$. Therefore, $p(\cdot, k)$ has two real roots of the same sign and $x_0(k) < 0$ is in between. Consequently, the smallest root, $\underline{x}$ of p satisfies

  $$(k - (1 - M^2))(\underline{x} - x_0) > 0$$

  and thus yields a root $z \in i\,\mathbb{R}^-$ of f. This root corresponds to a *strongly unstable* mode. When k goes to $1 - M$, $\underline{x}$ goes to $-\text{sign}\,(a_1 a_2)\,\infty = -\infty$.
- for $1 - M < k < 1 - M^2$, we have $a_2(k) < 0$ and $a_2(k)\, p(x_0(k), k) > 0$. Therefore, $p(\cdot, k)$ has two real roots of opposite signs (one going to $+\infty$ when k goes to $1 - M$) and $x_0(k)$ is smaller than both of them. The negative root of p, still denoted $\underline{x}$ for simplicity, now satisfies

  $$(k - (1 - M^2))(\underline{x} - x_0) < 0,$$

  which shows it does not give rise to a root of f. The positive one satisfies the same inequality, and thus its square root (and the opposite) is a real zero of f. This zero corresponds to a *neutral* mode.
- for $1 - M^2 < k < M^2 r + 1 - M^2$, we still have $a_2(k) < 0$ and $a_2(k)\, p(x_0(k), k) > 0$ and thus two real roots of opposite signs for $p(\cdot, k)$.

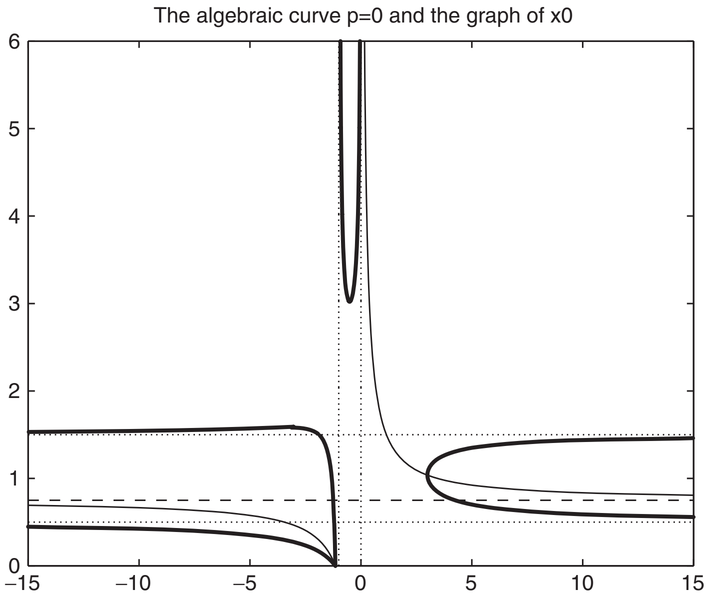

Figure 15.1: The curve $\mathscr{P}$ (thick solid line) in an example of Case 1a ($M = 0.5$, $r = 1.15$).

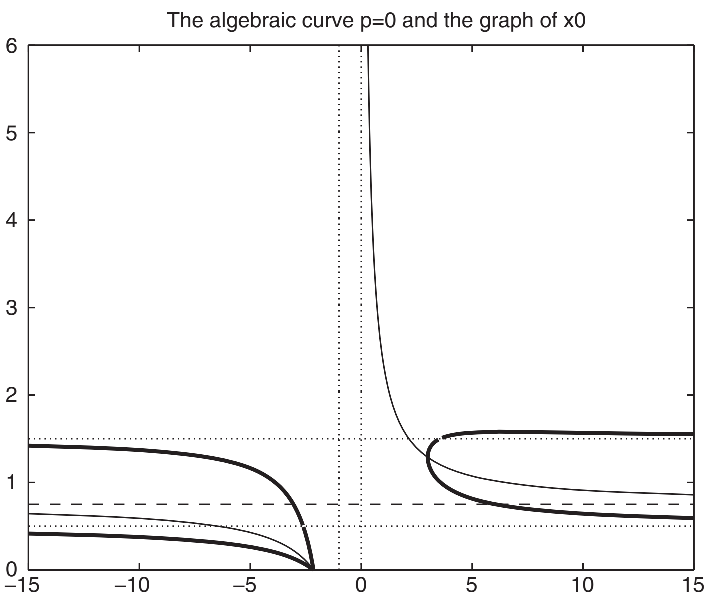

Figure 15.2: The curve $\mathscr{P}$ (thick solid line) in an example of Case 1b ($M = 0.5$, $r = 4$); the upper branch is out of range.

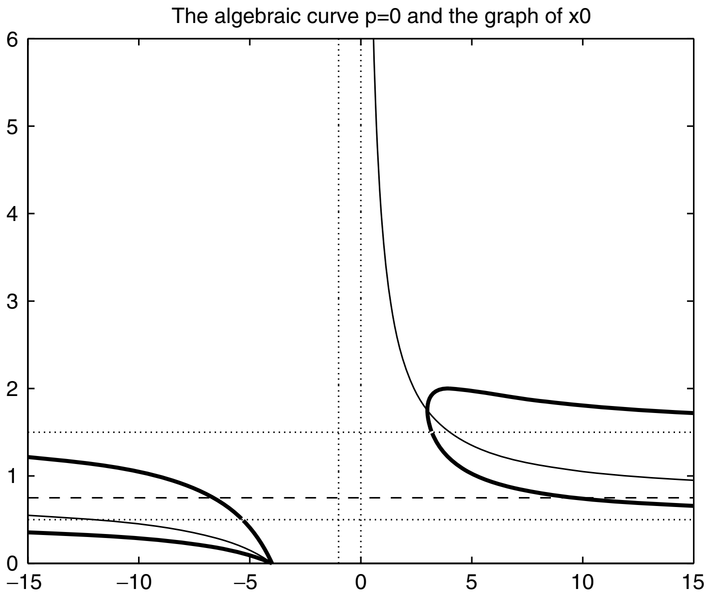

Figure 15.3: The curve $\mathscr{P}$ (thick solid line) in an example of Case 2 ($M = 0.5$, $r = 2.15$); the upper branch is out of range.

But $x_0(k)$ is now greater than both of them. Therefore, we are in the same situation as before. The positive root of p yields a real root of f.

- for $M^2r+1-M^2<k<1+M$, we have $a_2(k) < 0$ and $a_2(k)\,p(x_0(k),k) < 0$. Therefore, $p(\cdot,k)$ has two real roots of opposite signs and $x_0(k)$ is in between. The negative one is such that

$$(k - (1 - M^2))(\underline{x} - x_0) < 0,$$

and it is the contrary for the positive one. Thus, none of them yields a root of f. The positive root goes to $-\text{sign}\,(a_1a_2)\,\infty = +\infty$ when k goes to $1 + M$.

- for $1 + M < k$, we have $a_2(k) > 0$ and $a_2(k)\,p(x_0(k),k) > 0$. Therefore, $p(\cdot,k)$ has either two reals roots of the same sign or two conjugated complex roots. And $p(\cdot,k)$ has one real root going to $-\text{sign}\,(a_1a_2)\,\infty = -\infty$ when k goes to $1 + M$. Therefore, there exists $\varepsilon > 0$ such that for $1 + M < k < 1 + M + \varepsilon$, $p(\cdot,k)$ has two negative real roots, and $x_0(k)$ is greater than both of them. (This is also the case for large enough k due to the parabolic branch mentioned above.) As before, none of them yields a root of f. In general, p has conjugated complex roots on some interval $[K,L]$ with $K \geq 1 + M + \varepsilon$. But these cannot give rise to any complex zero of f, by Rouché's theorem and a connectedness argument.

Case 1b

- for $k < M^2 r + 1 - M^2$, the situation is exactly as in Case 1a.
- for $M^2 r + 1 - M^2 < k < 1 + M$, as in Case 1a, $p(\cdot, k)$ has two real roots of opposite signs, $x_0(k)$ is in between, and none of them yields a root of f. The only difference is that it is the negative root that goes to infinity when k goes to $1 + M$.
- for $1 + M < k$, as in Case 1a, $p(\cdot, k)$ has either two reals roots of the same sign or two conjugated complex roots. This time $p(\cdot, k)$ has one real root going to $-\text{sign}\,(a_1 a_2)\,\infty = +\infty$ when k goes to $1 + M$. Therefore, there exists $\varepsilon > 0$ such that for $1 + M < k < 1 + M + \varepsilon$, $p(\cdot, k)$ has two positive real roots, and $x_0(k)$ is less than both of them. As before, none of them yields a root of f. For larger k, the conclusion that f keeps having no roots follows in the same way as in Case 1a.

Case 2

- for $k<1+M$, the situation is similar as in Case 1 when $k<M^2r+1-M^2$.
- for $1 + M < k < M^2 r + 1 - M^2$, $a_2(k) > 0$ and $a_2(k)\,p(x_0(k), k) < 0$. Therefore, $p(\cdot, k)$ has positive real roots and $x_0(k)$ is in between. Only the smallest one yields a (real) root of f, because

$$(k - (1 - M^2))(\underline{x} - x_0) < 0.$$

  The greatest one goes to $-\text{sign}\,(a_1 a_2)\,\infty = +\infty$ when k goes to $1 + M$.
- for $M^2 r + 1 - M^2 < k$, as in Case 1b, $p(\cdot, k)$ has two positive real roots, and $x_0(k)$ is less than both of them, as long as k is not too large. None of them yields a root of f. The conclusion is the same as in other cases.

□

15.2.2 *Notable cases*

The uniform stability condition (15.2.16) may equivalently be rewritten as

$$(\Gamma + 1)\, M^2 \,\frac{v_0 - v}{v} < 1 .$$

It is obviously satisfied if $v_0 < v$, which characterizes *expansive shocks*, as soon as $\Gamma \geq -1$. Using that $M^2 = v^2 j^2/c^2$ and the jump relation (13.4.34), another statement of the uniform stability condition is

$$(\Gamma + 1)\, v\, \frac{p - p_0}{c^2} < 1 .$$

Ideal gases Since for ideal gases,

$$\Gamma + 1 = \gamma = \frac{v\, c^2}{p} ,$$

the uniform stability condition here above is obviously always satisfied. Therefore, we have the following.

Theorem 15.2 *All Lax shocks are uniformly stable in ideal gases.*

This result has been well-known since Majda's work [125]. It was, nevertheless, questioned, wrongly, very recently (see [42] for more details).

Weak shocks As noted by Métivier in [133], the uniform stability condition (15.2.16) is satisfied for shocks of small enough amplitude. As a matter of fact, k tends to 2 and r tends to 1 when the amplitude of the shock goes to 0.

Isentropic case We claim that stability conditions for isentropic gas dynamics are a byproduct of stability conditions for complete gas dynamics, because isentropic stability conditions are merely obtained by suppressing the penultimate row and column of Δ. This just amounts to setting $\Gamma = 0$, or equivalently $k = 2$, in the complete stability conditions. Therefore, we see from (15.2.14) that the one-dimensional condition reduces to $1 + M \neq 0$, which is always satisfied. Accordingly, the first inequality in the weak multidimensional stability condition (15.2.15) is trivially satisfied. And the uniform multidimensional stability condition reduces to

$$M^2\,(\,r\,-\,1\,)\,<\,1\,, \tag{15.2.24}$$

or equivalently

$$\frac{p\,-\,p_0}{-\,v\,p'_v}\,<\,1\,.$$

This condition holds true in particular for the γ-law $p\,=\,\mathrm{cst}\,v^{-\gamma}$ with $\gamma\,\geq\,1$.

15.2.3 *Kreiss symmetrizers*

The explicit construction of Kreiss' symmetrizers given in Chapter 14 for regular Initial Boundary Value problems is easily generalized to the shock stability problem.

To stay close to the notations of Chapter 12, we denote here $U\,=\,(v,\mathbf{u},s)$ the set of thermodynamic and kinematic dependent variables, and

$$\begin{aligned}\mathbf{U}\,:\,\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R} &\to \mathbb{R}^{d+2}\times\mathbb{R}^{d+2}\times\mathbb{R}\times\mathbb{R}^{d-1}\\ (y,z,t) &\mapsto (\,U_-(y,z,t),U_+(y,z,t),\partial_t\chi(y,t),\nabla_y\chi(y,t)\,)\end{aligned}$$

the sought shock solution, where χ is the unknown front and $U_\pm\,=\,(v_\pm,\mathbf{u}_\pm,s_\pm)$ correspond to the unknown states on either side (both defined on $\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}$ after a suitable change of variables, see Chapter 12). For $X\,=\,(\mathbf{y},z,t,\boldsymbol{\eta},\tau)\in\mathbb{R}^{d-1}\times\mathbb{R}^+\times\mathbb{R}^{d-1}\times\mathbb{C}$, we denote by $\mathcal{A}_{\mathbf{U}}(X)$ the $2(d+2)\times 2(d+2)$ matrix obtained, as in the abstract framework of Chapter 12, through the following successive transformations of the Euler equations on both sides of the unknown front:

i) fixing of the free boundary,
ii) linearization about $\mathbf{U}$,
iii) freezing of coefficients at point $(\mathbf{y}, z, t)$,
iv) Fourier–Laplace transform, which amounts to replacing ∂_t by its symbol τ and ∂_{y_j} by its symbol $i\eta_j$ for $j \in \{1, \dots, d-1\}$.

There is no need to write the matrix $\mathcal{A}_{\mathbf{U}}(X)$ explicitly, for we know it is block-diagonal,

$$\mathcal{A}_{\mathbf{U}}(X) \,=\, \begin{pmatrix} \mathcal{A}_l(X) & \\ & \mathcal{A}_r(X) \end{pmatrix},$$

and we know from Section 14.4 reduced forms of the blocks $\mathcal{A}_{l,r}(X)$. Indeed, if we denote by $U_l = U_-(\mathbf{y}, z, t)$ the supersonic state and by $U_r = U_+(\mathbf{y}, z, t)$ the subsonic state of the fluid with respect to the shock front, for $\mathrm{Re}\,\tau > 0$ the dispersion equation

$$(\tau \,-\, u_l\,\omega)^2 \,=\, c_l^2\,(\omega \,-\, \|\boldsymbol{\eta}\|^2)$$

has two roots of positive real parts, which we denote by ω_1^l and ω_2^l, while the analogous dispersion equation for the subsonic state

$$(\tau \,-\, u_r\,\omega)^2 \,=\, c_r^2\,(\omega \,-\, \|\boldsymbol{\eta}\|^2)$$

has one root of positive real part, which we denote by ω_+^r, and one of negative real part, which we denote by ω_-^r. Denoting also $\omega_0^{l,r} := \tau/u_{l,r}$, we have that $\mathcal{A}_l(X)$ is similar to

$$\widetilde{\mathcal{A}}_l(X) \,=\, \begin{pmatrix} \omega_0^l\,I_d & & \\ & \omega_1^l & \\ & & \omega_2^l \end{pmatrix} \,\geq\, 0\,,$$

while $\mathcal{A}_r(X)$ is similar to

$$\widetilde{\mathcal{A}}_r(X) \,=\, \begin{pmatrix} -\omega_0^r\,I_d & \\ & \mathbf{a} \end{pmatrix},$$

where the 2×2 block $\mathbf{a}$ is either $\mathrm{diag}(-\omega_+^r, -\omega_-^l)$ if ω_+^r and ω_-^r are distinct (i.e. if $\tau^2 \,+\, (c_r^2 \,-\, u_r^2)\,\|\boldsymbol{\eta}\|^2) \neq 0$) or a Jordan block. Therefore, we can find a Kreiss symmetrizer of the form

$$\mathcal{R}_{\mathbf{U}}(X) \,=\, \begin{pmatrix} \mu\,I_{d+2} & \\ & \mathcal{R}_r(X) \end{pmatrix},$$

where the block $\mathcal{R}_r(X)$ is of the same form as the Kreiss symmetrizer constructed in Section 14.4 for the standard, subsonic inflow IBVP. More precisely, $\mathcal{R}_r(X)$ is locally similar (in the same basis as $\mathcal{A}_r(X)$) to

$$\widetilde{\mathcal{R}}_r(X) \,=\, \begin{pmatrix} -I_d & \\ & \mathbf{r} \end{pmatrix},$$

where $\mathbf{r}$ is merely $\operatorname{diag}(-1, \mu)$ if ω_+^r and ω_-^r are distinct. Here above, of course, the parameter μ has to be chosen large enough to absorb 'bad' terms.

This construction completes the proof of Lemma 12.1 in the case of gas dynamics.

15.2.4 *Weak stability*

In the case of weak multidimensional stability (i.e. with (15.2.15)), it is still possible to derive energy estimates, but of course with a loss of derivatives. This question was investigated in detail by Coulombel [38, 39]. Here we just want to address the order of vanishing of the Lopatinskiĭ determinant, which is crucially related to those weak energy estimates.

The order of vanishing of the Lopatinskiĭ determinant is indeed tightly related to the order of the corresponding root x of the polynomial p defined in (15.2.21).

Proposition 15.2 *If z is a double root of the function f defined by* (15.2.17) *and* (15.2.18)*, with $r \neq 0$ and $M^2 < 1$, then z^2 is a double root of the polynomial $p(\cdot, k)$ defined in* (15.2.21)*.*

Proof We first note that $z = 0$ is not possible since $r \neq 0$. A little algebra then shows that $f(z) = f'(z) = 0$ implies

$$a_2(k)\, z^2 + \frac{a_1(k)}{2} = 0\,,$$

where $a_m(k)$ denotes as before the coefficient of order m of $p(\cdot, k)$. Hence z^2 can only be a double root of the second-order polynomial $p(\cdot, k)$. Beyond this elementary proof, a cleverer one makes use of the fact that p is a resultant. We recall indeed that p is the resultant with respect to the y variable of the polynomials F and G defined in (15.2.22) and (15.2.23). A basic property of resultants says there exist polynomials $R(x, y)$ and $Q(x, y)$ (in fact, $Q(x)$ since F is of degree 1 in y) such that

$$p(x) = F(x, y)\, R(x, y) + G(x, y)\, Q(x)\,.$$

In particular, this implies the identity

$$p(z^2) = f(z)\; R(\, z^2\,,\, z\, g(z)\,)\,.$$

Differentiating once, we obtain that

$$2\, z\, p'(z^2) = f'(z)\; R(\, z^2\,,\, z\, g(z)\,)$$

if $f(z) = 0$, and thus $p'(z^2) = 0$ if $f'(z)$ is also zero (recall that $z \neq 0$ whenever $f(z) = 0$). □

PART V

APPENDIX

This page intentionally left blank

A

BASIC CALCULUS RESULTS

The celebrated Gronwall Lemma is used repeatedly in this book. We state our most useful versions of it for convenience.

Lemma A.1 (Basic Gronwall Lemma) *If u and f are smooth functions of $t \in [0,T]$ such that*

$$u(t) \leq C_0 + C_1 \int_0^t (\, u(\tau) + f(\tau)\,) \,\mathrm{d}\tau \qquad \forall t \in [0,T]\,,$$

with $C_0 \in \mathbb{R}$ and $C_1 > 0$ then

$$u(t) \leq \mathrm{e}^{C_1 t} \left(C_0 + C_1 \int_0^t f(\tau) \,\mathrm{d}\tau \right) \qquad \forall t \in [0,T]\,.$$

Proof The only trick in the proof is to show the final estimate for the right-hand side

$$v(t) := C_0 + C_1 \int_0^t (\, u(\tau) + f(\tau)\,) \,\mathrm{d}\tau$$

of the original one. Since

$$v'(t) = C_1 (\, u(t) + f(t)\,) \leq C_1 (\, v(t) + f(t)\,)$$

we easily get the inequality

$$\mathrm{e}^{-C_1 t}\, v(t) \leq v(0) + C_1 \int_0^t \mathrm{e}^{-C_1 \tau} f(\tau) \,\mathrm{d}\tau\,,$$

of which the claimed estimate is only a rougher version. □

A slightly more elaborate version that we often use is the following.

Lemma A.2 (Gronwall Lemma) *If u and f are smooth functions of $t \in [0,T]$ such that*

$$u(t) \leq C_0 \,\mathrm{e}^{\gamma t} + C_1 \int_0^t \mathrm{e}^{\gamma (t-\tau)} (\, u(\tau) + f(\tau)\,) \,\mathrm{d}\tau \qquad \forall t \in [0,T]$$

with $C_0 \in \mathbb{R}$ and $C_1 > 0$ then

$$u(t) \leq C_0 \,\mathrm{e}^{(C_1+\gamma)\, t} + C_1 \int_0^t \mathrm{e}^{(C_1+\gamma)\,(t-\tau)} f(\tau) \,\mathrm{d}\tau \qquad \forall t \in [0,T]\,.$$

Lemma A.3 ('Multidimensional' Gronwall Lemma) *Assume $\mathcal{L} \subset \mathbb{R}^{d+1}$ is a lens foliated by hypersurfaces $\mathcal{H}_\theta$ and denote*

$$\mathcal{L}_\theta = \bigcup_{\varepsilon \in [0,\theta]} \mathcal{H}_\varepsilon \subset \mathcal{L}$$

for $\theta \in [0,1]$. If u is a smooth function in the neighbourhood of $\mathcal{L}$ such that

$$\int_{\mathcal{H}_\theta} |u| \leq C \left(\int_{\mathcal{H}_0} |u| + \int_{\mathcal{L}_\theta} |u| \right) \qquad \forall \theta \in [0,1]$$

then there exists C' depending only on C and $\mathcal{L}$ such that

$$\int_{\mathcal{H}_1} |u| \leq C' \int_{\mathcal{H}_0} |u| \, .$$

Proof The proof relies on the same trick as before but requires a little multi-dimensional calculus. Introducing parametric equations $x = X(y,\theta)$, $t = T(y,\theta)$ ($y \in \Omega \subset \mathbb{R}^d$) for $\mathcal{H}_\theta$ we have

$$\int_{\mathcal{H}_\theta} |u| = \int_\Omega |u(X(y,\theta),T(y,\theta))| \sqrt{|\mathrm{d}_y X|^2 + |\mathrm{d}_y T|^2} \, \mathrm{d}y \, ,$$

$$\int_{\mathcal{L}_\theta} u = \int_0^\theta \int_\Omega |u(X(y,\varepsilon),T(y,\varepsilon))| \, |J(y,\varepsilon)| \, \mathrm{d}y \, \mathrm{d}\varepsilon \, , \qquad J = \begin{vmatrix} \mathrm{d}_y X & \partial_\theta X \\ \mathrm{d}_y T & \partial_\theta T \end{vmatrix} .$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\mathcal{L}_\theta} |u| = \int_\Omega |u(X(y,\theta),T(y,\varepsilon))| \, |J(y,\theta)| \, \mathrm{d}y$$

$$\leq \max_{\bar{\Omega} \times [0,1]} \frac{|J|}{\sqrt{|\mathrm{d}_y X|^2 + |\mathrm{d}_y T|^2}} \int_{\mathcal{H}_\theta} |u| \, .$$

Then we easily get the wanted estimate with

$$C' = \max_{\bar{\Omega} \times [0,1]} \exp \left(C \, \frac{|J|}{\sqrt{|\mathrm{d}_y X|^2 + |\mathrm{d}_y T|^2}} \right) .$$

□

Lemma A.4 (Discrete Gronwall Lemma) *If a is a non-negative continuous function of $s \in [0,t]$ and b is a non-decreasing function of $s \in [0,t]$ such that*

$$\frac{a(s+\varepsilon) - a(s)}{\varepsilon} \leq C \left(\varepsilon + b(s) + a(s+\varepsilon) \right)$$

for all (ε, s) with $0 < \varepsilon \leq \varepsilon_0 \in (0,t)$, $s \in [0, t-\varepsilon]$, then

$$a(t) \leq \mathrm{e}^{C\,t} \left(a(0) + b(t) \right) .$$

Proof The proof is fully elementary. Take $D > C$ and consider $n \in \mathbb{N}$ such that $\varepsilon_n := \frac{t}{n+1}$ satisfies

$$\mathrm{e}^{-D\,\varepsilon_n} \le 1 - C\,\varepsilon_n\,.$$

For all $s \in [0, t - \varepsilon_n]$, we have

$$a(s+\varepsilon_n) \le \frac{1}{1 - C\varepsilon_n}\,a(s) + \frac{C\varepsilon_n}{1 - C\varepsilon_n}\,(\varepsilon_n + b(s))\,.$$

Therefore,

$$\begin{aligned}
\mathrm{e}^{-D\,t}\,a(t) - a(0) &= \sum_{k=0}^{n} \mathrm{e}^{-D\,(k+1)\,\varepsilon_n}\,a((k+1)\,\varepsilon_n) - \mathrm{e}^{-D\,k\,\varepsilon_n}\,a(k\,\varepsilon_n)\\
&\le \sum_{k=0}^{n} \mathrm{e}^{-D\,k\,\varepsilon_n}\left(\frac{\mathrm{e}^{-D\,\varepsilon_n}}{1 - C\varepsilon_n} - 1\right) a(k\,\varepsilon_n)\\
&\quad + \sum_{k=0}^{n} \mathrm{e}^{-D\,(k+1)\,\varepsilon_n}\,\frac{C\varepsilon_n}{1 - C\varepsilon_n}\,(\varepsilon_n + b(k\,\varepsilon_n))\\
&\le \frac{C}{1 - C\varepsilon_n}(\varepsilon_n + b(t)) \int_0^x \mathrm{e}^{-D\,s}\mathrm{d}s \le \frac{C}{D(1 - C\varepsilon_n)}(\varepsilon_n + b(t)).
\end{aligned}$$

We get the final estimate by letting n go to ∞ and then D go to C. □

B

FOURIER AND LAPLACE ANALYSIS

B.1 Fourier transform

There are several possible conventions for the definition of a Fourier transform, depending on where the number 2π shows up.

We adopt the following one. For all $v \in L^1(\mathbb{R}^d)$ its Fourier transform $\mathscr{F} v = \widehat{v}$ is the continuous and bounded function defined by

$$\mathscr{F} v(\eta) = \widehat{v}(\eta) := \int_{\mathbb{R}^d} \mathrm{e}^{-i\eta\cdot x} v(x)\,\mathrm{d}x$$

for all $\eta \in \mathbb{R}^d$.

For any d-uple $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, we adopt the shortcut

$$\partial^\alpha \;=\; \partial_1{}^{\alpha_1} \cdots \partial_d{}^{\alpha_d}$$

for differential operators of order $|\alpha| := \sum_{k=1}^d \alpha_k$, either in the original variables (x) or in the frequency variables (η).

For convenience, we state the following standard results.

Theorem B.1

- *The Fourier transform $\mathscr{F}$ restricted to the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ is an automorphism.*
- *By duality, it can be also defined on $\mathscr{S}'(\mathbb{R}^d)$, where it is still an automorphism.*
- *For all $v \in \mathscr{S}(\mathbb{R}^d)$ and all d-uple α, we have*

  $$\mathscr{F}[\partial^\alpha v](\eta) \;=\; (i\,\eta)^\alpha\; \widehat{v}(\eta)$$

  for all $\eta \in \mathbb{R}^d$. And this formula extends to $\mathscr{S}'(\mathbb{R}^d)$ in the sense that for $v \in \mathscr{S}'(\mathbb{R}^d)$, the distribution $\mathscr{F}[\partial^\alpha v]$ is $\widehat{v}$ times the polynomial function $\eta \mapsto i\,\eta^\alpha$.
- **(Plancherel's identity)** *For all $v \in \mathscr{S}(\mathbb{R}^d)$,*

  $$\|\widehat{v}\|_{L^2(\mathbb{R}^d)} \;=\; (2\pi)^{d/2} \|v\|_{L^2(\mathbb{R}^d)}\,.$$

- *By density, $\mathscr{F}$ extends to an automorphism of $L^2(\mathbb{R}^d)$, which still satisfies Plancherel's identity.*

- **(Inversion formula)** *If both* v *and* $\widehat{v}$ *are integrable, one recovers* v *from* $\widehat{v}$ *through the inversion formula*

$$v(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\eta\cdot x}\widehat{v}(\xi)d\xi\,.$$

Another useful result is the following, of which the easy part is the direct one.

Theorem B.2 (Paley–Wiener) *If* $v \in \mathscr{D}(\mathbb{R}^d)$ *then* $\widehat{v}$ *extends to an analytic function* V *on* $\mathbb{C}^d$*. Furthermore, if* $K = \mathrm{Supp}\,v$*, for all* $p \in \mathbb{N}$*, there exists* $C_p > 0$ *so that*

$$|V(\zeta)| \leq \frac{C_p}{(1 + |\zeta|)^p} \exp\Big(\max_{x\in K}(x \cdot \mathrm{Im}\ \zeta)\Big)$$

for all $\zeta \in \mathbb{C}^d$*. Conversely, if* V *is an analytic function on* $\mathbb{C}^d$ *satisfying the above estimate for some convex compact set* K*, there exists* $v \in \mathscr{D}(\mathbb{R}^d)$ *with support included in* K *such that* $\widehat{v} = V|_{\mathbb{R}^d}$*.*

The 'dual' result also holds true.

Theorem B.3 (Paley–Wiener–Schwartz) *If* v *is a compactly supported distribution then* $\widehat{v}$ *extends to an analytic function* V *on* $\mathbb{C}^d$ *through the formula*

$$V(\zeta) = \langle\, v\,,\, e^{-i\,\zeta\cdot}\,\rangle\,.$$

Furthermore, if $K = \mathrm{Supp}\,v$*, there exist* $p \in \mathbb{N}$ *and* $C_p > 0$ *so that*

$$|V(\zeta)| \leq C_p\,(1 + |\zeta|)^p \exp\Big(\max_{x\in K}(x \cdot \mathrm{Im}\ \zeta)\Big)$$

for all $\zeta \in \mathbb{C}^d$*. Conversely, if* V *is an analytic function on* $\mathbb{C}^d$ *satisfying the above estimate for some convex compact set* K*, there exists* $v \in \mathcal{E}'(\mathbb{R}^d)$ *with support included in* K *such that* $\widehat{v} = V|_{\mathbb{R}^d}$*.*

B.2 Laplace transform

The Laplace transform applies to functions of one variable $t \in (0, +\infty)$.

Definition B.1 *If* f *is a measurable function of* $t \in (0, +\infty)$ *and if there exists* $a \in \mathbb{R}$ *so that* $t \mapsto e^{-\,a\,t}\,f(t)$ *is square-integrable, the Laplace transform of* f *is the function* $\mathcal{L}[f]$*, holomorphic in the half-plane* $\{\,\tau\,;\ \mathrm{Re}\ \tau > a\}$*, defined by*

$$\mathcal{L}[f](\tau) = \int_0^{+\infty} f(t)\,e^{-\,\tau\,t}\,dt\,.$$

The Laplace transform may be interpreted in terms of a Fourier transform. Indeed, for all $\gamma > a$ and $\delta \in \mathbb{R}$,

$$\mathcal{L}[f](\gamma + i\,\delta) = \widehat{(g\,f)}\,(\delta)\,,$$

where $g(t) = \mathbf{1}_{\mathbb{R}^+}(t)\, \mathrm{e}^{-\gamma t}$. Therefore, the theorem of Paley–Wiener has a counterpart in Laplace transforms theory. This is the main result we need regarding Laplace transforms in this book.

Theorem B.4 (Paley–Wiener) *If F is an holomorphic function in the right half-plane $\{\tau\,;\ \mathrm{Re}\,\tau > a\}$, which is square-integrable on each vertical line $\{\tau\,;\ \mathrm{Re}\,\tau = \gamma\}$, $\gamma > a$, and if there exists $C > 0$ so that*

$$\sup_{\gamma > a} \int_{-\infty}^{+\infty} |\,F(\gamma + i\,\delta)\,|^2\, \mathrm{d}\delta \leq C\,,$$

then there exists $f \in L^2(\mathbb{R}^+\{\mathrm{e}^{-a\,t}\,\mathrm{d}t\})$ (meaning that $t \mapsto \mathrm{e}^{-a\,t} f(t)$ is square-integrable) such that $F = \mathcal{L}[f]$. If, additionally, F is integrable on the vertical line $\{\tau\,;\ \mathrm{Re}\,\tau = \gamma\}$, $\gamma > a$, we recover f through the inversion formula

$$f(t) = \frac{1}{2\,i\,\pi} \int_{\mathrm{Re}\ \tau = \gamma} F(\tau)\, \mathrm{e}^{\tau\, t}\, \mathrm{d}s\,.$$

B.3 Fourier–Laplace transform

Fourier and Laplace transformations can be extended to vector-valued functions – in fact functions with values in Banach spaces. From a practical point of view, this amounts to saying that for functions of several variables one may perform Fourier or Laplace transformations in some variables only and keep the same regularity/decay properties in the other variables.

More precisely, let u be a function of $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$ that belongs to

$$L^2(\mathbb{R}^d \times \mathbb{R}^+\{\mathrm{e}^{-a\,t}\,\mathrm{d}t\,\mathrm{d}x\})\,.$$

For almost all $t \in \mathbb{R}^+$, the function $u(t) : x \in \mathbb{R}^d \mapsto u(x,t)$ is square-integrable and thus admits a Fourier transform $\widehat{u(t)} \in L^2(\mathbb{R}^d)$ such that the function $(\xi, t) \mapsto \widehat{u(t)}(\xi)$ belongs to

$$L^2(\mathbb{R}^d \times \mathbb{R}^+\{\mathrm{e}^{-a\,t}\,\mathrm{d}t\,\mathrm{d}\xi\})\,.$$

Therefore, for almost all $\xi \in \mathbb{R}^d$, the function $t \in \mathbb{R}^+ \mapsto \mathrm{e}^{-a\,t}\,\widehat{u(t)}(\xi)$ is square-integrable and thus the Laplace transform

$$\mathcal{L}[t \mapsto \widehat{u(t)}(\xi)]$$

is well-defined and holomorphic on the half-plane $\{\tau\,;\ \mathrm{Re}\,\tau > a\}$. The function

$$(\xi, \tau) \mapsto \mathcal{L}[t \mapsto \widehat{u(t)}(\xi)](\tau)$$

is what we call the Fourier–Laplace transform of u.

C

PSEUDO-/PARA-DIFFERENTIAL CALCULUS

The aim of this appendix is to facilitate the reading of the book for those who are not familiar with para-differential or even pseudo-differential calculus. Just a basic background on distributions theory and Fourier analysis is assumed. As many textbooks deal with pseudo-differential calculus (see for instance [7, 31, 87, 205]), we recall here only basic definitions and useful results for our concern, mostly without proof. Para-differential calculus is much less widespread. Originally developed by Bony [20] and Meyer [138], it has been used since then in various contexts, in particular by Gérard and Rauch [68] for non-linear hyperbolic equations and more recently by Métivier and coworkers for hyperbolic initial boundary value problems – see in particular the lectures notes [136]. Other helpful references on para-differential calculus are [33, 88, 206]. We detail in this appendix the most accessible part of the theory of para-differential operators, among which we find para-products. Some useful results are gathered together with their complete (and most often elementary) proof, using the Littlewood–Paley decomposition. The rest of the theory is presented heuristically, together with a collection of results used elsewhere in the book. Additionally, we borrow from [31] and [136] versions of pseudo-differential and para-differential calculus with a parameter, which are needed for initial boundary value problems.

We use standard notations from differential calculus. To any d-uple $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{N}^d$, we associate the differential operator of order $|\alpha| := \sum_{k=1}^{d} \alpha_k$

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}\,.$$

When no confusion can occur this operator is simply denoted by ∂^α. This notation should not be mixed up with the one used throughout the book

$$\partial_\alpha = \frac{\partial}{\partial x_\alpha}$$

for $\alpha \in \{1, \cdots, d\}$. To avoid confusion, the indices lying in $\{1, \cdots, d\}$ are here preferably denoted by roman letters (typically j).

C.1 Pseudo-differential calculus

C.1.1 *Symbols and approximate symbols*

Pseudo-differential operators are defined through their *symbol*, which is a function depending on $x \in \mathbb{R}^d$ and on the 'dual' variable, or frequency, $\xi \in \mathbb{R}^d$. This function is not polynomial in ξ, except for (standard) differential operators – for instance the symbol of ∂^α is

$$(i)^{|\alpha|}\,\xi^\alpha \;=\; (i\xi_1)^{\alpha_1}\cdots(i\xi_d)^{\alpha_d}\,.$$

In the classical theory, symbols are scalar-valued (in $\mathbb{C}$). As far as we are concerned, matrix-valued symbols are also of interest. This extension costs nothing, but some little care in Lemma C.1 below and in the handling of commutators.

From now on, we fix some integers d and N that will be omitted in the notations if no confusion can occur.

Definition C.1 *For any real number m, we define the set $\mathbf{S}^m$ of functions $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{N\times N})$ such that for all d-uples α and β there exists $C_{\alpha,\beta} > 0$ so that*

$$\|\,\partial_x^\alpha\,\partial_\xi^\beta\,a(x,\xi)\,\| \;\leq\; C_{\alpha,\beta}\,(\,1\,+\,\|\xi\|\,)^{m-|\beta|}\,. \tag{C.1.1}$$

Symbols belonging to $\mathbf{S}^m$ are said to be of *order* m. The set of symbols of all orders is

$$\mathbf{S}^{-\infty} \;:=\; \bigcap_m \mathbf{S}^m\,.$$

Basic examples

Differential symbols. Functions of the form

$$a(x,\xi) \;=\; \sum_{|\alpha|\,\leq\,m} a_\alpha(x)\,(i\xi)^\alpha\,,$$

where all the coefficients a_α are $\mathscr{C}^\infty$ and bounded, as well as all their derivatives, belong to $\mathbf{S}^m$.

'Homogeneous' functions. A function $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d\backslash\{0\})$ that is bounded as well as all its derivatives in x and homogeneous degree m in ξ is 'almost' a symbol of order m. This means that it becomes a symbol provided that we remove the singularity at $\xi = 0$. As a matter of fact, considering a $\mathscr{C}^\infty$ function χ vanishing in a neighbourhood of 0 and such that $\chi(\xi) = 1$ for $\|\xi\| \geq 1$, we have the result that

$$\widetilde{a}(x,\xi) \;=\; \chi(\xi)\,a(x,\xi)$$

belongs to $\mathbf{S}^m$. Any other symbol constructed in this way differs from $\widetilde{a}$ by a symbol in $\mathbf{S}^{-\infty}$. For convenience we shall denote $\dot{\mathbf{S}}^m$ the set of such functions a.

Sobolev symbols Some special symbols are extensively used in the theory, which we refer to as Sobolev symbols since they are naturally involved in Sobolev norms. Denoting

$$\lambda^s(\xi) \,:=\, (\,1\,+\,\|\xi\|^2\,)^{s/2}$$

it is easily seen that λ^s is a symbol of order s. The important point is that the Sobolev space H^s can be equipped with the norm

$$\|\,u\,\|_{H^s} \,=\, \|\,\lambda^s\,\widehat{u}\,\|_{L^2}\,.$$

Additionally, the following result will be of interest in the proof of Theorem C.4.

Lemma C.1 *For all $a \in \mathbf{S}^0$, respectively $a \in \dot{\mathbf{S}}^0$, such that $a(x,\xi)$ is Hermitian and uniformly positive-definite for $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, respectively, for $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d\backslash\{0\}$, there exists $b \in \mathbf{S}^0$, respectively $b \in \dot{\mathbf{S}}^0$, such that $b(x,\xi)^*\, b(x,\xi) = a(x,\xi)$.*

Proof The proof proceeds in the same way in both cases ($a \in \mathbf{S}^0$ or $a \in \dot{\mathbf{S}}^0$). By assumption, $a(x,\xi)$ lies in a bounded subset of the cone of Hermitian positive-definite matrices and thus the set of eigenvalues of $a(x,\xi)$ is included in some real interval $[\alpha,\beta] \subset (0,+\infty)$. In particular, there exists a positively oriented contour Γ lying in $\mathbb{C}\backslash(-\infty,0]$ that is symmetric with respect to the real axis and contains $[\alpha,\beta]$ in its interior. Therefore, considering the holomorphic complex square root $\sqrt{\ }$ in $\mathbb{C}\backslash(-\infty,0]$, the Dunford–Taylor integral

$$b(x,\xi) \,:=\, \frac{1}{2i\pi}\,\int_\Gamma \sqrt{z}\,(\,z\,-\,a(x,\xi)\,)^{-1}\,\mathrm{d}z$$

answers the question. As a matter of fact, by the symmetry of Γ, $b(x,\xi)$ is obviously Hermitian, and

$$b^*\,b \,=\, \frac{-1}{4\pi^2}\,\int_\Gamma\int_{\Gamma'} \sqrt{z}\sqrt{z'}\,(\,z\,-\,a\,)^{-1}\,(\,z'\,-\,a\,)^{-1}\,\mathrm{d}z\,\mathrm{d}z'$$

for another contour Γ' enjoying the same properties as Γ and containing it in its interior. By the well-known resolvent equation, we thus have

$$b^*\,b \,=\, \frac{1}{(2i\pi)^2}\,\int_\Gamma\int_{\Gamma'} \sqrt{z}\sqrt{z'}\,\frac{(\,z\,-\,a\,)^{-1}\,-\,(\,z'\,-\,a\,)^{-1}}{z'-z}\,\mathrm{d}z\,\mathrm{d}z'\,.$$

On the one hand, for $z' \in \Gamma'$, the function $z \mapsto \sqrt{z}/(z'-z)$ is holomorphic in the interior of Γ and thus

$$\int_\Gamma \frac{\sqrt{z}}{z'\,-\,z}\,\mathrm{d}z \,=\, 0\,.$$

On the other hand, by Cauchy's formula we have

$$\sqrt{z} \;=\; \frac{1}{2i\pi}\int_{\Gamma'}\frac{\sqrt{z'}}{z' - z}\,\mathrm{d}z'$$

for $z \in \Gamma$. This eventually proves that

$$b^*\,b \;=\; \frac{1}{2i\pi}\int_{\Gamma} z\,(\,z \,-\, a\,)^{-1}\,\mathrm{d}z \;=\; a\,.$$

In view of the smoothness of the mapping $(z, a) \mapsto (\,z \,-\, a\,)^{-1}$, it is clear by the chain rule and Lebesgue's theorem that b is as smooth as a. It is also easily shown by induction that b satisfies the estimate (C.1.1) with $m = 0$, if a belongs to $\mathbf{S}^0$. If a belongs to $\dot{\mathbf{S}}^0$ instead, it is obvious that b is also homogeneous degree 0 in ξ. □

C.1.2 *Definition of pseudo-differential operators*

The introduction of pseudo-differential operators is based on the following observation. If $a \in \mathbf{S}^m$ is polynomial in ξ, like in the first basic example given above, it is naturally associated with the differential operator

$$\mathrm{Op}(a) \;=\; \sum_{|\alpha|\,\leq\, m} a_\alpha(x)\,\partial^\alpha$$

in the sense that

$$(\mathrm{Op}(a)u)(x) \;=\; \mathscr{F}^{-1}(a(x,\cdot)\,\widehat{u})$$

for all $u \in \mathscr{S}$ and $x \in \mathbb{R}^d$. But this formula can be used to define operators associated with more general symbols. This is the purpose of the following.

Proposition C.1 *Let a be a symbol of order m. Then there exists a continuous linear operator on $\mathscr{S}$, denoted by* $\mathrm{Op}(a)$*, such that*

$$(\mathrm{Op}(a)u)(x) \;=\; \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi}\,a(x,\xi)\,\widehat{u}(\xi)\,\mathrm{d}\xi \qquad \text{(C.1.2)}$$

for all $u \in \mathscr{S}$. Furthermore, the mapping $a \mapsto$ $\mathrm{Op}(a)$ *is one-to-one.*

Observe that for 'constant-coefficient symbols', that is, symbols independent of x, (C.1.2) reduces to the same formula as for differential operators

$$\mathrm{Op}(a)u \;=\; \mathscr{F}^{-1}(a\,\widehat{u})\,.$$

In short, for symbols a depending only on ξ, we have by definition

$$\mathrm{Op}(a) \;=\; \mathscr{F}^{-1}a\,\mathscr{F}\,.$$

Definition C.2 *The set of pseudo-differential operators of order m is*

$$\mathbf{OPS}^m \;:=\; \{\,\mathrm{Op}(a)\,;\; a \in \mathbf{S}^m\,\} \subset \mathscr{B}(\mathscr{S})\,.$$

For $a \in \mathbf{S}^m$, the operator $\mathrm{Op}(a)$ *is called a pseudo-differential operator of order m and symbol a.*

It is more subtle to show that pseudo-differential operators extend to operators on $\mathscr{S}'$. By a standard duality argument, this amounts to showing the following.

Theorem C.1 *The adjoint of a pseudo-differential operator of order m is a pseudo-differential operator of order m. Furthermore, the symbol of the adjoint operator* $\mathrm{Op}(a)^*$ *differs from a^* (where $a^*(x,\xi) := a(x,\xi)^*$ merely in the sense of matrices) by a symbol of order $m-1$, which means that*

$$(\mathrm{Op}(a))^* - \mathrm{Op}(a^*) \in \mathbf{OPS}^{m-1} \tag{C.1.3}$$

for all $a \in \mathbf{S}^m$.

The proof of this theorem is a very fine piece of analysis. The interested reader may refer to [7, 87, 205].

C.1.3 *Basic properties of pseudo-differential operators*

The first important property of pseudo-differential operators is the following.

Theorem C.2 *Let P be a pseudo-differential operator of order m, extended to $\mathscr{S}'$ by the formula*

$$\langle Pu, \phi \rangle = \langle u, P^*\phi \rangle$$

for all $u \in \mathscr{S}'$ and $\phi \in \mathscr{S}$. Then, for all $s \in \mathbb{R}$, P belongs to $\mathscr{B}(H^s; H^{s-m})$.

A straightforward example. For all $s \in \mathbb{R}$, let Λ^s denote the pseudo-differential operator of symbol λ^s as defined in Section C.1.1. Then for all real numbers s we have

$$\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}.$$

Therefore, we have for all $m \in \mathbb{R}$

$$\|\Lambda^m u\|_{H^{s-m}} = \|\Lambda^{s-m}\Lambda^m u\|_{L^2} = \|\Lambda^s u\|_{L^2} = \|u\|_{H^s}.$$

In fact, using the operators Λ^s and Λ^{-m}, all cases of Theorem C.2 can be deduced from the case $m = s = 0$ and the other following basic result.

Theorem C.3 *If P and Q are pseudo-differential operators of order m and n, respectively, then*

i) the composed operator PQ is a pseudo-differential operator of order $m+n$, and its symbol differs from the product of symbols by a lower-order term, which means that

$$\mathrm{Op}(a)\,\mathrm{Op}(b) - \mathrm{Op}(ab) \in \mathbf{OPS}^{m+n-1} \tag{C.1.4}$$

for all $a \in \mathbf{S}^m$ and $b \in \mathbf{S}^n$.

ii) if one of the operators is scalar-valued, the commutator

$$[P, Q] := P\,Q - Q\,P$$

is of order $m + n - 1$ *(and its symbol differs from the Poisson bracket of symbols*

$$\{a\,,\,b\} := \sum_j \frac{\partial a}{\partial \xi_j}\,\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\,\frac{\partial b}{\partial \xi_j}$$

by a lower-order term).

The proofs of Theorems C.2 and C.3, as well as more complete results, can be found in the textbooks quoted above [7,31,87,205]. Note that apart from the bracketed statement, *ii)* is a trivial consequence *i)*.

Finally, other important results are the Gårding inequality, which relates the positivity of an operator (up to a lower-order error) to the positivity of its symbol, and the sharp form of Gårding's inequality, which applies to non-negative symbols. We begin with the standard form of Gårding's inequality (for matrix-valued symbols) and its elementary proof.

Theorem C.4 (Gårding inequality) *If* A *is a pseudo-differential operator of symbol* $a \in \mathbf{S}^m$*, or* A *is associated with* $a \in \dot{\mathbf{S}}^m$ *by a low-frequency cut-off, such that for some positive* α

$$a(x, \xi) + a(x, \xi)^* \;\geq\; \alpha\,\lambda^m(\xi)\,I_N$$

(in the sense of Hermitian matrices) for all $x \in \mathbb{R}^d$ *and* $\|\xi\|$ *large, then there exists* C *so that*

$$\mathrm{Re}\,\langle\,Au\,,\,u\,\rangle \;\geq\; \frac{\alpha}{4}\,\|u\|^2_{H^{m/2}} - C\,\|u\|^2_{H^{m/2-1}} \tag{C.1.5}$$

for all $u \in H^{m/2}$.

Proof Replacing A by $\Lambda^{-m/2}\,A\,\Lambda^{-m/2}$, we can suppose without loss of generality that $m = 0$.

• We have $\mathrm{Re}\,\langle\,Au\,,\,u\,\rangle = \mathrm{Re}\,\langle\,\frac{1}{2}\,(A + A^*)u\,,\,u\,\rangle$, and by (C.1.3) we know that

$$(A + A^*) - \mathrm{Op}(a + a^*) \in \mathbf{OPS}^{-1}\,.$$

Therefore, there exists $c > 0$ so that

$$\begin{aligned}\langle\,(A + A^*)u\,,\,u\,\rangle &\geq \langle\,\mathrm{Op}(a + a^*)u\,,\,u\,\rangle - c\,\|u\|_{H^{-1}}\,\|u\|_{L^2}\\ &\geq \langle\,\mathrm{Op}(a + a^*)u\,,\,u\,\rangle - \frac{\alpha}{16}\,\|u\|^2_{L^2} - \frac{4c^2}{\alpha}\,\|u\|^2_{H^{-1}}\,.\end{aligned}$$

Thus the result will be proved if we show that

$$\langle \operatorname{Op}(a + a^*)u \,,\, u \rangle \;\geq\; \frac{9\alpha}{16}\, \|u\|_{L^2}^2 \;-\; C\, \|u\|_{H^{-1}}^2 \,.$$

In some sense this reduces the problem to Hermitian symbols.

• By assumption, the Hermitian symbol $\widetilde{a} := a + a^* \,-\, \alpha' I_N$, with $\alpha' = 3\alpha/4$, is positive-definite. By Lemma C.1, there exists $b \in \mathbf{S}^0$ such that $b^* b \,=\, \widetilde{a}$. Denoting $B = \operatorname{Op}(b)$ and $\widetilde{A} = \operatorname{Op}(a + a^* \,-\, \alpha' I_N)$, we know from Theorems C.1 and C.3 *i)* that

$$B^* B \,-\, \widetilde{A} \in \mathbf{OPS}^{-1} \,.$$

Consequently, there exists $\widetilde{c} > 0$ so that

$$\langle \widetilde{A} u,\, u\rangle \geq \langle B^* B u,\, u\rangle - \widetilde{c} \|u\|_{H^{-1}} \|u\|_{L^2} \geq \|Bu\|_{L^2}^2 - \frac{\alpha'}{4} \|u\|_{L^2}^2 - \frac{\widetilde{c}^2}{\alpha'} \|u\|_{H^{-1}}^2,$$

which implies that

$$\langle \operatorname{Op}(a + a^*)u \,,\, u \rangle \;\geq\; \frac{3\alpha'}{4}\, \|u\|_{L^2}^2 \;-\; \frac{\widetilde{c}^2}{\alpha'}\, \|u\|_{H^{-1}}^2 \,.$$

• Finally, we have the inequality in (C.1.5) with $C = (4c^2 + 3\widetilde{c}^2)/(2\alpha)$. □

We complete this section by stating without proof the sharp Gårding inequality, which amounts to allowing $\alpha = 0$ in the standard one. In other words, it shows that non-negative symbols imply a gain of derivatives: an operator of order s with non-negative symbol satisfies a lower bound as though it were of order $s - 1$. The sharp Gårding inequality was originally proved by Hörmander [86] for scalar operators and by Lax and Nirenberg [112] for matrix-valued symbols. The proof was later simplified by several authors; it can be found in [88, 205, 210].

Theorem C.5 (Sharp Gårding inequality) *If A is a pseudo-differential operator of symbol $a \in \mathbf{S}^m$, or A is associated with $a \in \dot{\mathbf{S}}^m$ by a low frequency cut-off, such that for some positive α*

$$a(x,\xi) + a(x,\xi)^* \;\geq\; 0$$

(in the sense of Hermitian matrices) for all $x \in \mathbb{R}^d$ and $\|\xi\|$ large, then there exists C so that

$$\operatorname{Re}\, \langle\, Au \,,\, u \,\rangle \;\geq\; -\,C\, \|u\|_{H^{(m-1)/2}}^2 \qquad \text{(C.1.6)}$$

for all $u \in H^{m/2}$.

C.2 Pseudo-differential calculus with a parameter

The introduction of a parameter γ is intended to deal with weighted-in-time estimates, typically in $L^2(\mathbb{R}, \mathrm{e}^{-\gamma t}\, \mathrm{d}t)$.

We shall consider symbols that depend uniformly on a parameter $\gamma \in [1, +\infty)$. To avoid overcomplicated notations, we shall use, as far as possible, the same notations as in standard pseudo-differential calculus. The main difference is that λ now stands for the rescaled weight

$$\lambda^s(\xi,\gamma) := (\gamma^2 + \|\xi\|^2)^{s/2}.$$

Alternatively, we shall sometimes denote $\lambda^{s,\gamma} = \lambda^s(\cdot,\gamma)$. Associated with $\lambda^{s,\gamma}$ is an equivalent norm on H^s, namely $\|\lambda^{s,\gamma}\,\widehat{u}\,\|_{L^2}$. To avoid confusion with the standard norm, we shall denote

$$\|u\|_{H^s_\gamma} = \|\lambda^{s,\gamma}\,\widehat{u}\,\|_{L^2}.$$

One may also observe that

$$\|u\|_{H^s} \leq \|u\|_{H^s_\gamma} \leq \gamma^s\,\|u\|_{H^s}$$

for $s > 0$ (and the converse for $s < 0$). Another useful remark about these weighted norms is that

$$\|u\|_{H^s_\gamma} \leq \gamma^{s-m}\,\|u\|_{H^m_\gamma}$$

for $s \leq m$.

Remark C.1 For $m \in \mathbb{N}$, there exist $c_m > 0$ and $C_m > 0$ so that

$$c_m \sum_{|\alpha|\leq m} \gamma^{2(m-|\alpha|)}\,\|\partial^\alpha u\|^2_{L^2} \leq \|u\|^2_{H^m_\gamma} \leq C_m \sum_{|\alpha|\leq m} \gamma^{2(m-|\alpha|)}\,\|\partial^\alpha u\|^2_{L^2}$$

for all $\gamma \geq 1$ and $u \in H^m$.

Definition C.3 *For any real number m, we define the set $\mathbf{S}^m$ of functions $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d \times [1,+\infty[;\mathbb{C}^{N\times N})$ such that for all d-uples α and β there exists $C_{\alpha,\beta} > 0$ so that*

$$\|\partial_x^\alpha\,\partial_\xi^\beta\,a(x,\xi,\gamma)\| \leq C_{\alpha,\beta}\,\lambda^{m-|\beta|}(\xi,\gamma). \tag{C.2.7}$$

(One may relax the smoothness with respect to γ assumption, which is of no use actually.)

Basic examples. Of course the first example is given by functions λ^s themselves. Clearly, λ^m belongs to $\mathbf{S}^m$. More generally, if $(\xi,\gamma) \mapsto a(\xi,\gamma)$ is $\mathscr{C}^\infty$ on $(\mathbb{R}^d \times \mathbb{R}^+)\backslash\{(0,0)\}$ and homogeneous degree m, then its restriction to $\mathbb{R}^d \times [1,+\infty)$ belongs to $\mathbf{S}^m$. (Compared to the usual symbols, the singularity at the origin is eliminated by taking $\gamma \geq 1$.)

Needless to say, Lemma C.1 extends to this setting in a straightforward way. Also, Proposition C.1 and Theorem C.1 enable us to define $\mathrm{Op}^\gamma(a)$ in such a way that

$$(\mathrm{Op}^\gamma(a)u)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{ix\cdot\xi}\,a(x,\xi,\gamma)\,\widehat{u}(\xi)\,\mathrm{d}\xi \tag{C.2.8}$$

for all $u \in \mathscr{S}$. And Theorem C.2 applies to the operator Op^γ for all $\gamma \geq 1$. However, one may wish to keep track of the dependence on γ in the estimates. This leads to the following.

Definition C.4 *A family of pseudo-differential operators $\{P^\gamma\}_{\gamma \geq 1}$ is said to be of order m if P^γ belongs to* $\mathbf{OPS}^m$ *for all $\gamma \geq 1$ and for all s there exists a constant C,* independent of *γ, so that*

$$\|P^\gamma u\|_{H_\gamma^{s-m}} \leq C\, \|u\|_{H_\gamma^s} .$$

The basic example of a family of order m is precisely

$$\{\Lambda^{m,\gamma}\}_{\gamma \geq 1} ,$$

where $\Lambda^{m,\gamma}$ is by definition the pseudo-differential operator of symbol $\lambda^{m,\gamma}$. As a matter of fact, the little calculation made in Section C.1.3 just becomes

$$\|\Lambda^{m,\gamma}\, u\|_{H_\gamma^{s-m}} \;=\; \|\Lambda^{s-m,\gamma}\Lambda^{m,\gamma}\, u\|_{L^2} \;=\; \|\Lambda^{s,\gamma}\, u\|_{L^2} \;=\; \|u\|_{H_\gamma^s} .$$

More generally, it can be shown that $\{\mathrm{Op}^\gamma(a)\}_{\gamma \geq 1}$ is a family of order m for any $a \in \mathbf{S}^m$. We actually have a collection of results of this kind – extending Theorems C.1 and C.3 – that we summarize in the following.

Theorem C.6 *If a and b belong to $\mathbf{S}^m$ and $\mathbf{S}^n$, respectively, then*

i) $\{\mathrm{Op}^\gamma(a)\}_{\gamma \geq 1}$ *is a family of order m,*
ii) $\{\,(\mathrm{Op}^\gamma(a))^* \,-\, (\mathrm{Op}^\gamma(a^*))\,\}_{\gamma \geq 1}$ *is a family of order $m-1$,*
iii) $\{\mathrm{Op}^\gamma(a) \circ \mathrm{Op}^\gamma(b) \,-\, \mathrm{Op}^\gamma(ab)\}_{\gamma \geq 1}$ *is a family of order $m+n-1$,*
iv) $\{\,[\,\mathrm{Op}^\gamma(a)\,,\, \mathrm{Op}^\gamma(b)\,] \,-\, \mathrm{Op}^\gamma([a,b])\,\}_{\gamma \geq 1}$ *is a family of order $m+n-1$.*

The proof of course relies on the fact that the estimates in (C.2.7) are independent of γ. Let us sketch the proof of *i)*. We need a bound for $\mathrm{Op}^\gamma(a)$ in $\mathscr{B}(H_\gamma^s; H_\gamma^{s-m})$ that is uniform in γ. So we go back to the usual estimation of $\mathrm{Op}^\gamma(a)$, and pay attention to the dependence on γ. The case $m = s = 0$ is rather easy, because the norm of $\mathrm{Op}^\gamma(a)$ as an operator on L^2 only depends on bounds on derivatives of a (even though this fact is not so easy to prove, it is well-known), which are independent of γ by assumption. For arbitrary m and s, the proof amounts to playing with commutators involving $\Lambda^{s,\gamma}$ and $\Lambda^{-m,\gamma}$, thus reducing the problem to the case $m = s = 0$. So it is closely related to the proof of *iii)*. The details are left to the reader.

Remark C.2 Since

$$\|\, u \,\|_{H_\gamma^s} \;\leq\; \gamma^{s-p}\, \|\, u \,\|_{H_\gamma^p}$$

for $s \leq p$, a family $\{P^\gamma\}_{\gamma \geq 1}$ of order m satisfies

$$\|P^\gamma u\|_{H_\gamma^{s-m}} \;\leq\; C\, \gamma^{s-p}\, \|u\|_{H_\gamma^p}$$

for $s \leq p$. In particular, for a family of negative order, we can take $p = s - m$ and obtain

$$\|P^\gamma u\|_{H^p_\gamma} \leq C\,\gamma^m\,\|u\|_{H^p_\gamma}\,.$$

We complete this section by parameter versions of Gårding's inequality.

Theorem C.7 (Gårding inequality with parameter) *If $a \in \mathbf{S}^m$ is such that for some positive α,*

$$a(x,\xi,\gamma) + a(x,\xi,\gamma)^* \geq \alpha\,\lambda^m(\xi,\gamma)\,I_N$$

(in the sense of Hermitian matrices) for all $(x,\xi,\gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times [1,+\infty)$, then there exists $\gamma_0 \geq 1$ so that for all $\gamma \geq \gamma_0$ and $u \in H^{m/2}$,

$$\mathrm{Re}\,\langle\,\mathrm{Op}^\gamma(a)u\,,\,u\,\rangle \geq \frac{\alpha}{4}\,\|u\|^2_{H^{m/2}_\gamma}\,. \tag{C.2.9}$$

Compated to the standard Gårding's inequality in (C.1.5), the inequality (C.2.9) does not contain any remainder term. This is made possible by absorbing the errors in the main term for γ large enough.

Proof The proof parallels that of Theorem C.4.

• By Theorem C.6 *ii)* and the Cauchy–Schwarz inequality, there exists $c > 0$ so that

$$\langle\,(\mathrm{Op}^\gamma(a) + \mathrm{Op}^\gamma(a)^*)u\,,\,u\,\rangle \geq \langle\,\mathrm{Op}^\gamma(a+a^*)u\,,\,u\,\rangle - c\,\|u\|_{H^{2m-1-m}_\gamma}\,\|u\|_{H^m_\gamma}\,.$$

• The symbol $\widetilde{a} := a + a^* - \alpha'\,\lambda^{2m}\,I_N$, with $\alpha' < \alpha$, is positive-definite. By Lemma C.1, there exists $b \in \mathbf{S}^m$ such that $b^*b = \widetilde{a}$. Then, denoting $\widetilde{A}^\gamma = \mathrm{Op}^\gamma(a + a^* - \alpha'\,\lambda^{2m}\,I_N)$, by Theorem C.6 *iii) ii)* and the Cauchy–Schwarz inequality there exists $\widetilde{c} > 0$ so that

$$\langle\,\widetilde{A}^\gamma u\,,\,u\,\rangle \geq \langle\,\mathrm{Op}^\gamma(b)^*\,\mathrm{Op}^\gamma(b)u\,,\,u\,\rangle - \widetilde{c}\,\|u\|_{H^{m-1}_\gamma}\,\|u\|_{H^m_\gamma}\,,$$

which implies that

$$\langle\,\mathrm{Op}^\gamma(a+a^*)u\,,\,u\,\rangle \geq \alpha'\,\|u\|^2_{H^m_\gamma} - \widetilde{c}\,\|u\|_{H^{m-1}_\gamma}\,\|u\|_{H^m_\gamma}\,.$$

• Therefore, by Young's inequality,

$$\langle\,(\mathrm{Op}^\gamma(a) + \mathrm{Op}^\gamma(a)^*)u\,,\,u\,\rangle \geq \frac{\alpha'}{2}\,\|u\|^2_{H^m_\gamma} - \frac{2}{\alpha'}\,(c+\widetilde{c})^2\,\|u\|^2_{H^{m-1}_\gamma}\,.$$

The conclusion just follows from the fact that

$$\frac{\alpha'}{2}\,\lambda^{2m,\gamma} - \frac{2}{\alpha'}\,(c+\widetilde{c})^2\,\lambda^{2(m-1),\gamma} \geq \frac{\alpha'}{3}\,\lambda^{2m,\gamma}$$

for γ large enough. □

A sharpened version of Theorem C.7 is the following.

Theorem C.8 (Sharp Gårding inequality with parameter) *If $a \in \mathbf{S}^m$ satisfies*

$$a(x,\xi,\gamma) + a(x,\xi,\gamma)^* \ \geq\ 0$$

(in the sense of Hermitian matrices) for all $(x,\xi,\gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times [1,+\infty)$, then there exist $\gamma_0 \geq 1$ and $C > 0$ so that for all $\gamma \geq \gamma_0$ and all $u \in H^{m/2}$,

$$\mathrm{Re}\, \langle\, \mathrm{Op}^\gamma(a)u\,,\, u\,\rangle \ \geq\ -\,C\,\|u\|^2_{H_\gamma^{(m-1)/2}}\,. \tag{C.2.10}$$

The proof is sketched in [125] (pp. 82–84, in the case $m = 1$), by adapting the approach of Nagase [146].

C.3 Littlewood–Paley decomposition

C.3.1 *Introduction*

Littlewood–Paley decomposition is a well-known tool in modern analysis, of which various versions are available [57,211,212]. Here we adopt the presentation of Meyer [137], and also Gérard and Rauch [68]. For a recent, PDE oriented presentation, see also [33].

We consider a reference cut-off function $\psi \in \mathscr{D}(\mathbb{R}^d)$, monotonically decaying along rays and so that

$$\begin{array}{lll} \psi(\xi) \,=\, 1 & \text{if} & \|\xi\| \,\leq\, 1/2\,, \\ 0 \,\leq\, \psi(\xi) \,\leq\, 1 & \text{if} & 1/2 \,\leq\, \|\xi\| \,\leq\, 1\,, \\ \psi(\xi) \,=\, 0 & \text{if} & \|\xi\| \,\geq\, 1\,. \end{array} \tag{C.3.11}$$

This function is associated with the other cut-off function $\phi \in \mathscr{D}(\mathbb{R}^d)$ defined by

$$\phi(\xi) \,:=\, \psi(\xi/2) \,-\, \psi(\xi)\,.$$

The monotonicity property of ψ implies that $\phi(\xi)$ is everywhere non-negative. The other main feature of ϕ is that it is supported by the set $\{\,1/2 \,\leq\, \|\xi\| \,\leq\, 2\,\}$.

Now we consider the functions ϕ_q defined by

$$\phi_q(\xi) \,:=\, \phi(\,2^{-q}\,\xi\,)$$

for all $q \in \mathbb{N}$. By construction we have

$$\mathrm{Supp}\,\phi_q \,\subset\, \{\,2^{q-1} \,\leq\, \|\xi\| \,\leq\, 2^{q+1}\,\}\,.$$

Then it is elementary to show the following.

Proposition C.2 *For the functions ψ and ϕ_q defined as above, we have*

$$\phi_p\,\phi_q \,\equiv\, 0 \quad \text{if} \quad |p-q| \,\geq\, 2\,, \tag{C.3.12}$$

$$\psi \,+\, \sum_{q\geq 0} \phi_q \,\equiv\, 1\,, \tag{C.3.13}$$

$$\frac{1}{2} \leq \psi^2 + \sum_{q \geq 0} \phi_q^2 \leq 1 . \tag{C.3.14}$$

Observe that because of (C.3.12) the sums in (C.3.13) and (C.3.14) are locally finite. Indeed for all $\xi \in \mathbb{R}^d$ there are at most two indices q such that $\phi_q(\xi) \neq 0$.

For convenience we also denote $\phi_{-1} := \psi$.

All functions ϕ_q for $q \geq -1$ can be viewed as constant-coefficient symbols in $\mathbf{S}^{-\infty}$ and thus associated with pseudo-differential operators, denoted by Δ_q. Equivalently, Δ_q is defined on $\mathscr{S}'$ by

$$\Delta_q := \mathscr{F}^{-1} \phi_q \mathscr{F} , \tag{C.3.15}$$

where $\mathscr{F}$ denotes the Fourier transform on $\mathscr{S}'$.

The interest of these operators is that, for all $u \in \mathscr{S}'$, we have because of (C.3.13)

$$u = \sum_{q \geq -1} \Delta_q u ,$$

where the convergence of the series holds true in $\mathscr{S}'$, and the terms $\Delta_q u$ are $\mathscr{C}^\infty$ functions (since the ϕ_q are compactly supported).

We introduce the notation for partial sums

$$S_q := \sum_{p=-1}^{q-1} \Delta_p .$$

By convention, we put $\Delta_p = 0$ for $p \leq -2$ and $S_q = 0$ for $q \leq -1$.

By definition we have for all $u \in \mathscr{S}'$

$$\mathscr{F}(\Delta_q u) = \phi_q \, \widehat{u} \quad \text{and} \quad \mathscr{F}(S_q u) = \psi_q \, \widehat{u} \tag{C.3.16}$$

with the rescaled functions ψ_q being defined similarly as the ϕ_q by

$$\psi_q(\xi) := \psi(2^{-q} \xi)$$

for $q \geq 0$.

A first interesting property of the operators Δ_q is that the L^∞ norms of $\Delta_q u$, $S_q u$ and their derivatives are all controlled by the L^∞ norm of u. The cost of one derivative is found to be 2^q.

Proposition C.3 (Bernstein) *For all $m \in \mathbb{N}$, there exists $C_m > 0$ so that for all $u \in L^\infty$, for all d-uple α, $|\alpha| \leq m$, for all $q \geq -1$,*

$$\| \partial^\alpha (\Delta_q u) \|_{L^\infty} \leq C_m \, 2^{q|\alpha|} \, \| u \|_{L^\infty} \quad \textit{and} \quad \| \partial^\alpha (S_q u) \|_{L^\infty} \leq C_m \, 2^{q|\alpha|} \, \| u \|_{L^\infty} . \tag{C.3.17}$$

Proof By (C.3.16) we have

$$\Delta_q u = \mathscr{F}^{-1}\phi_q * u \quad \text{and} \quad S_q u = \mathscr{F}^{-1}\psi_q * u .$$

All functions $\mathscr{F}^{-1}\phi_q$ are integrable because of the regularity of ϕ_q, with the additional invariance property

$$\| \mathscr{F}^{-1}\phi_q \|_{L^1} = \| \mathscr{F}^{-1}\phi \|_{L^1}$$

for all $q \geq 0$. We also easily compute that

$$\| \partial^\alpha (\mathscr{F}^{-1}\phi_q) \|_{L^1} = 2^{q|\alpha|} \| \partial^\alpha (\mathscr{F}^{-1}\phi) \|_{L^1} .$$

The same is of course true for ψ_q. Then a basic convolution inequality yields the conclusion with

$$C = \max_{|\alpha| \leq m} (\| \partial^\alpha (\mathscr{F}^{-1}\psi) \|_{L^1}, \| \partial^\alpha (\mathscr{F}^{-1}\phi) \|_{L^1}) .$$

□

C.3.2 *Basic estimates concerning Sobolev spaces*

All results displayed in this section but the very last are concerned with the most classical Sobolev spaces H^s on the whole space $\mathbb{R}^d$.

First, we note that if u belongs to H^s the equality $u = \sum \Delta_q u$ holds true not only in $\mathscr{S}'$ but also in H^s. As a matter of fact, we have

$$\mathscr{F}(S_q u - u) \xrightarrow{q \to \infty} 0 \quad \text{and} \quad |\lambda^s(\xi) \mathscr{F}(S_q u - u)(\xi)|^2 \leq (1 + \|\psi\|^2_{L^\infty}) \, |\lambda^s(\xi) \widehat{u}(\xi)|^2.$$

Thus by Lebesgue's theorem we have

$$\lim_{q \to \infty} \|S_q u - u\|_{H^s} = 0 .$$

Furthermore, the operators Δ_q appear to give rise to equivalent norms on the Sobolev spaces.

Proposition C.4 *For all $s \in \mathbb{R}$, there exist $C_s > 1$ such that for all $u \in H^s$*

$$\frac{1}{C_s} \sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|^2_{L^2} \leq \|u\|^2_{H^s} \leq C_s \sum_{q \geq -1} 2^{2qs} \|\Delta_q u\|^2_{L^2} . \qquad \text{(C.3.18)}$$

Proof We begin with the case $s = 0$. We claim that the estimate in (C.3.18) works with $C_0 = 2$. One may remark that the equality, that is (C.3.18) with $C_0 = 1$, could be true if the Δ_q were pairwise orthogonal. But we only have, in view of (C.3.12),

$$\langle \Delta_p u \, , \, \Delta_q u \rangle = 0 \quad \text{provided that } |p - q| \geq 2 . \qquad \text{(C.3.19)}$$

The inequalities in (C.3.18) can be viewed as measuring the default of orthogonality. Their proof is almost straightforward. As a matter of fact, the inequalities in (C.3.14) imply that

$$\sum_{q \geq -1} | \phi_q(\xi) \, \widehat{u}(\xi) |^2 \leq | \widehat{u}(\xi) |^2 \leq 2 \sum_{q \geq -1} | \phi_q(\xi) \, \widehat{u}(\xi) |^2$$

for all $u \in L^2$ and almost all $\xi \in \mathbb{R}^d$. Integrating in ξ we get, in view of the definition (C.3.15) of Δ_q,

$$\sum_{q\geq -1} \| \widehat{\Delta_q u} \|_{L^2}^2 \leq \| \widehat{u} \|_{L^2}^2 \leq 2 \sum_{q\geq -1} \| \widehat{\Delta_q u} \|_{L^2}^2$$

and we just conclude by Plancherel's theorem.

The general case is not much more difficult. The inequalities in (C.3.14) imply that

$$\sum_{q\geq -1} | \lambda^s(\xi)\, \phi_q(\xi)\, \widehat{u}(\xi) |^2 \leq | \lambda^s(\xi)\, \widehat{u}(\xi) |^2 \leq 2 \sum_{q\geq -1} | \lambda^s(\xi)\, \phi_q(\xi)\, \widehat{u}(\xi) |^2 .$$

Assume, for instance, that s is positive. Then for $q \geq 0$ and for

$$\xi \in \operatorname{Supp} \phi_q \subset \{ 2^{q-1} \leq \|\xi\| \leq 2^{q+1} \}$$

we have

$$2^{-2s}\, 2^{2qs} \leq \lambda^{2s}(\xi) = (1 + \|\xi\|^2)^s \leq 2^{3s}\, 2^{2qs} ,$$

while for

$$\xi \in \operatorname{Supp} \phi_{-1} \subset \{ \|\xi\| \leq 1 \}$$

we have

$$2^{2s}\, 2^{-2s} = 1 \leq \lambda^{2s}(\xi) \leq 2^s = 2^{3s}\, 2^{-2s} .$$

Therefore, the inequalities in (C.3.14) holds true with $C_s = 2^{3s+1}$. When s is negative the estimates on λ^{2s} are reversed and thus (C.3.14) holds true with $C_s = 2^{-3s+1}$.

□

In particular, this proposition shows that for all $u \in H^s$ and all $q \geq -1$

$$\| \Delta_q u \|_{L^2} \leq \sqrt{C_s}\, 2^{-qs}\, \| u \|_{H^s} . \tag{C.3.20}$$

Of course the constant $\sqrt{C_s}$ becomes 1 if we replace the usual H^s norm by the equivalent norm

$$\|u\|_{H^s} = \left(\sum_{q\geq -1} 2^{2qs}\, \|\Delta_q u\|_{L^2}^2 \right)^{1/2} . \tag{C.3.21}$$

As regards the operation of Δ_q on L^2, we actually have much more information than that obtained by setting $s = 0$ in the inequality (C.3.20). We have similar estimates as in Proposition C.3.

Proposition C.5 *For all $m \in \mathbb{N}$, there exists $C_m > 0$ so that for all $u \in L^2$, for all d-uple α, $|\alpha| \le m$, for all $q \ge -1$,*

$$\| \partial^\alpha (\Delta_q u) \|_{L^2} \le C_m 2^{q|\alpha|} \| u \|_{L^2} \quad \textit{and} \quad \| \partial^\alpha (S_q u) \|_{L^2} \le C_m 2^{q|\alpha|} \| u \|_{L^2} . \tag{C.3.22}$$

In other words, we have a kind of 'symmetric' counterpart of (C.3.20). For all positive integers s, there exists $C > 0$ so that for all $q \ge -1$ and $u \in L^2$

$$\|\Delta_q u\|_{H^s} \le C\, 2^{qs} \|u\|_{L^2} \quad \text{and} \quad \|S_q u\|_{H^s} \le C\, 2^{qs} \|u\|_{L^2} . \tag{C.3.23}$$

(Note that the constant C here depends on s. It is strictly increasing with s.)

The proof of Proposition C.5 is exactly the same as that of Proposition C.3, replacing the $L^1 - L^\infty$ convolution estimates by $L^1 - L^2$ convolution estimates.

Another noteworthy remark is that the L^∞ norms of $\Delta_q u$ and $S_q u$ can be controlled even for unbounded u (to which Proposition C.3 does not apply), provided that u belongs to some H^s (which is *not* embedded in L^∞ for $s \le d/2$!), as shown in the following.

Proposition C.6 *For all $s \in \mathbb{R}$, there exists $C > 0$ so that for all $u \in H^s(\mathbb{R}^d)$ and all $q \ge -1$,*

$$\| \Delta_q u \|_{L^\infty} \le C\, 2^{-q(s-d/2)} \| u \|_{H^s} \quad \textit{and} \quad \|S_q u\|_{H^s} \le C\, 2^{-q(s-d/2)} \| u \|_{H^s} . \tag{C.3.24}$$

Proof The proof is somewhat analogous to that of Proposition C.3. Since both $\widehat{\Delta_q u}$ and $\widehat{S_q u}$ are supported by the ball $\{ \|\xi\| \le 2^{q+1} \}$ we have, for instance,

$$\widehat{\Delta_q u} = \psi_{q+2}\, \widehat{\Delta_q u} ,$$

and similarly for $\widehat{S_q u}$. Therefore,

$$\Delta_q u = \mathscr{F}^{-1} \psi_{q+2} * \Delta_q u .$$

Now, to get the correct estimate we just have to pay attention to the fact that the L^2 norm is not invariant by the rescaling. We have indeed

$$\|\psi_q\|_{L^2} = 2^{qd/2} \|\psi\|_{L^2}$$

and thus Plancherel's theorem and a basic convolution inequality yield

$$\| \Delta_q u \|_{L^\infty} \le 2^{(q+2)d/2} \|\psi\|_{L^2} \| \Delta_q u \|_{L^2} .$$

Together with (C.3.20) this gives (C.3.24) for $\Delta_q u$ with $C = 2^d \|\psi\|_{L^2} \sqrt{C_s}$. The same computation shows the inequality for $S_q u$. □

A straightforward consequence of this proposition is, of course, the well-known Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty$ for $s > d/2$. For, the inequality in (C.3.24) shows the series $\sum \Delta_q u$ is normally convergent in L^∞ if u belongs to $H^s(\mathbb{R}^d)$ and $s > d/2$, and its sum must be u (by uniqueness of limits in the space of distributions).

Remark C.3 By a similar calculation as in the proof of Proposition C.6, we have L^2 estimates of $\Delta_q u$ for $u \in L^1$. Namely, there exists $C > 0$ so that

$$\| \Delta_q u \|_{L^2} \leq C\, 2^{q\, d/2} \| u \|_{L^1}$$

for all $u \in L^1(\mathbb{R}^d)$ and $q \geq -1$. Indeed, by definition of $\Delta_q u$, Plancherel's theorem shows that

$$\| \Delta_q u \|_{L^2} = \| \phi_q \, \widehat{u} \|_{L^2} \leq \| \phi_q \|_{L^2} \| \widehat{u} \|_{L^\infty} ,$$

for $q \geq 0$ (for $q = -1$ just replace ϕ_q by ψ) and

$$\|\phi_q\|_{L^2} = 2^{qd/2} \|\phi\|_{L^2} ,$$

while, of course, $\| \widehat{u} \|_{L^\infty} \leq \| u \|_{L^1}$. As a consequence of these estimates and Proposition C.4 we find the embedding $L^1(\mathbb{R}^d) \hookrightarrow H^{-s}(\mathbb{R}^d)$ for $s > \frac{d}{2}$.

To complete this section, we prove an additional result in the same spirit as Proposition C.6, which gives an estimate of $\| \Delta_q u \|_{L^\infty}$ in terms of $\| \Delta_q u \|_{W^{m,\infty}}$ (instead of $\| \Delta_q u \|_{L^2}$ in the proof of Proposition C.6).

Proposition C.7 *For all $m \in \mathbb{N}$, there exists $C_m > 0$ so that for all $u \in L^\infty$, and all $q \geq 0$,*

$$\| \Delta_q u \|_{L^\infty} \leq C_m \, 2^{-qm} \sum_{|\alpha|=m} \| \partial^\alpha (\Delta_q u) \|_{L^\infty}. \tag{C.3.25}$$

Proof There is nothing to prove for $m = 0$. Let us assume $m \geq 1$. We consider some function $\chi \in \mathscr{D}(\mathbb{R}^d)$ vanishing near 0 and being equal to 1 on the support of ϕ (for instance take $\chi(\xi) = \psi(\xi/4) - \psi(2\xi)$), so that $\phi = \chi\, \phi$. With obvious notations we also have

$$\phi_q = \chi_q \, \phi_q$$

for all $q \geq 0$. Since χ vanishes near 0 we can define for all d-uples α of length m a function $\chi^\alpha \in \mathscr{D}(\mathbb{R}^d)$ by

$$\chi^\alpha(\xi) = \frac{(i\xi)^\alpha}{\displaystyle\sum_{|\alpha|=m} (i\xi^\alpha)^2} \, \chi(\xi) .$$

By construction we have

$$\chi(\xi) = \sum_{|\alpha|=m} (i\xi)^\alpha \, \chi^\alpha(\xi) ,$$

and

$$\chi_q(\xi) = 2^{-qm} \sum_{|\alpha|=m} (i\xi)^\alpha \, \chi_q^\alpha(\xi) ,$$

with still the obvious notation $\chi_q^\alpha(\xi) = \chi^\alpha(2^{-q}\xi)$. This easily implies that

$$\Delta_q u \,=\, 2^{-qm} \sum_{|\alpha|=m} \mathscr{F}^{-1}\chi_q^\alpha * \partial^\alpha(\Delta_q u)\,.$$

The result follows again from a convolution inequality and the identities

$$\|\, \mathscr{F}^{-1}\chi_q^\alpha \,\|_{L^1} \,=\, \|\, \mathscr{F}^{-1}\chi^\alpha \,\|_{L^1}\,.$$

We find that

$$\|\, \Delta_q u \,\|_{L^\infty} \,\leq\, 2^{-qm} \max_{|\alpha|=m} \|\, \mathscr{F}^{-1}\chi^\alpha \,\|_{L^1} \sum_{|\alpha|=m} \|\, \partial^\alpha(\Delta_q u) \,\|_{L^\infty}\,.$$

□

The proof here above would obviously fail for $q = -1$, because $\Delta_{-1}u$ does involve small frequencies. However, by Proposition C.3,

$$\|\Delta_{-1}u\|_{L^\infty} \,\leq\, C_0\,\|u\|_{L^\infty} \,\leq\, C_0\,2^k\|u\|_{L^\infty}$$

for all $k \in \mathbb{N}$. Therefore, using the commutation property $\partial^\alpha\,\Delta_q \,=\, \Delta_q\,\partial^\alpha$, a consequence of Proposition C.7 is the following.

Corollary C.1 *For all $k \in \mathbb{N}$, there exists $C_k > 0$ so that for all $u \in W^{k,\infty}$,*

$$W^{k,\infty} \,=\, \{\, u\,;\ \|\, \partial_x^\alpha\, u \,\|_{L^\infty} \,<\, \infty \ \ \forall \alpha \in \mathbb{N}^d\,;\ |\alpha| \,\leq\, k \,\}\,,$$

$$\forall q \geq -1\,, \quad \|\, \Delta_q u \,\|_{L^\infty} \,\leq\, C_k\,2^{-qk}\,\|\, u \,\|_{W^{k,\infty}}. \tag{C.3.26}$$

C.3.3 *Para-products*

The operators Δ_q are also convenient tools to define *para-products*. The para-product by (a not necessarily smooth) function u is intended to operate on Sobolev spaces (and on Hölder spaces) when the standard product does not. Para-products were originally introduced by Bony [20]. He actually defined two kinds of para-products, one based on the dyadic Littlewood–Paley decomposition, as presented below, and one based on a continuous spectral decomposition, and showed that they essentially enjoy the same properties.

To motivate the definition, let us consider two tempered distributions u and v, and *formally* write

$$\begin{aligned} u\,v &= \sum_{p,q\geq -1} \Delta_p u\,\Delta_q v \\ &= \sum_{p\geq 2} \Delta_p u \sum_{q=-1}^{p-3} \Delta_q v \,+\, \sum_{q\geq 2}\sum_{p=-1}^{q-3} \Delta_p u\,\Delta_q v \,+\, \sum_{|p-q|\leq 2} \Delta_p u\,\Delta_q v. \end{aligned}$$

There is some arbitrariness in this decomposition. The idea is to separate terms involving frequencies of the same order (last sum) from terms where the frequencies of u dominate those of v (first sum) or vice versa (middle sum).

Denoting as before by S_q the truncated sums

$$S_q \,=\, \sum_{p=-1}^{q-1} \Delta_p$$

for $q \geq 0$ and setting by convention $S_q \,=\, 0$ for $q \leq -1$, we introduce the para-product of v by u as

$$T_u v \,:=\, \sum_{q \,\geq\, -1} S_{q-2} u \, \Delta_q v \,=\, \sum_{q \,\geq\, 2} S_{q-2} u \, \Delta_q v \, . \tag{C.3.27}$$

This definition is still formal for arbitrary distributions. However, observing that

$$\mathrm{Supp}\, \mathscr{F}(S_{q-2} u \, \Delta_q v) \subset \mathrm{Supp} \sum_{p=-1}^{q-3} \phi_p * \phi_q \subset \{\|\xi\| \leq 2^{q-2}\} + \{2^{q-1} \leq \|\xi\| \leq 2^{q+1}\}$$

and hence

$$\mathrm{Supp}\, \mathscr{F}(\, S_{q-2} u \, \Delta_q v \,) \,\subset\, \left\{ \frac{1}{4}\, 2^q \,\leq\, \|\xi\| \,\leq\, \frac{9}{4}\, 2^q \right\}, \tag{C.3.28}$$

it will be easy to give sense to (C.3.27) for a wide range of u and v.

Remark C.4 In the special, apparently trivial case when u is constant, the para-product $T_u v$ is not exactly the usual product $u\,v$, but differs from it by a $\mathscr{C}^\infty$ function. Indeed, $\widehat{u} \,=\, u\,\delta$ and thus $\Delta_{-1} u \,=\, u$ while $\Delta_q u \,=\, 0$ for $q \geq 0$. Therefore, $T_u v \,=\, u \sum_{q\,\geq\,2} \Delta_q v$ and

$$u\,v \,-\, T_u v \,=\, u \sum_{|q|\,\leq\,1} \Delta_q v \, .$$

This means that the operator $(u \,-\, T_u)$ is infinitely smoothing when u is constant. We shall see in Theorem C.13 that for any Lipschitz function u the operator $(u \,-\, T_u)$ is still smoothing, to a limited extent though.

In general, we formally have the symmetric decomposition

$$u\,v \,=\, T_v u \,+\, T_u v \,+\, R(u,v), \tag{C.3.29}$$

where the remainder term is

$$R(u,v) \,:=\, \sum_{|p-q|\,\leq\,2} \Delta_p u \, \Delta_q v \, , \tag{C.3.30}$$

and will appear to be the smoothest term when it is well-defined.

As regards the para-product T_u, it does operate on H^s for all s provided that u belongs to L^∞, as shown in Proposition C.8 below. The para-product by u is a typical example of a *para-differential operator* (of order 0). See [20] for more details, or Section C.4 for a sketch.

Proposition C.8 *For all s there exists $C > 0$ so that for all $u \in L^\infty$ and all $v \in H^s$*

$$\|T_u v\|_{H^s} \leq C \|u\|_{L^\infty} \|v\|_{H^s} . \tag{C.3.31}$$

Proof We begin with a remark on the meaning of the definition in (C.3.27) for $u \in L^\infty$ and $v \in H^s$. By (C.3.17) and (C.3.20) we have

$$\|S_{q-2} u \, \Delta_q v\|_{L^2} \leq C \|u\|_{L^\infty} 2^{-qs} \|v\|_{H^s} .$$

Thus if $s > 0$ the series in (C.3.27) is normally convergent in L^2. However, the following computations justify a posteriori the definition of $T_u v$ for all s, since they show that the series in (C.3.27) is convergent (though not normally, in general) in H^s.

Using the equivalent norm (C.3.21) from Proposition C.4, the estimate (C.3.31) equivalently reads

$$\sum_{p \geq -1} 2^{2ps} \|\Delta_p T_u v\|_{L^2}^2 \leq C^2 \|u\|_{L^\infty}^2 \sum_{p \geq -1} 2^{2ps} \|\Delta_p v\|_{L^2}^2 .$$

To prove this, it is to be noted that

$$\Delta_p (S_{q-2} u \, \Delta_q v) \equiv 0 \quad \text{for } |p-q| \geq 4 . \tag{C.3.32}$$

Equation (C.3.32) is easy to check, since by (C.3.28) the support of $\mathscr{F}(S_{q-2} u \, \Delta_q v)$ is clearly disjoint from $\{ 2^{p-1} \leq \|\xi\| \leq 2^{p+1} \}$ for $|p-q| \geq 4$.

Therefore, and this is the crucial point in the proof, only a finite number of terms from the sum in (C.3.27) are to persist under the operation of Δ_p. We have

$$\Delta_p T_u v = \sum_{q=p-3}^{p+3} \Delta_p (S_{q-2} u \, \Delta_q v) ,$$

which implies by the Cauchy–Schwarz inequality that

$$2^{2ps} \|\Delta_p T_u v\|_{L^2}^2 \leq 7 \times 2^{6|s|} \sum_{q=p-3}^{p+3} 2^{2qs} \|\Delta_p (S_{q-2} u \, \Delta_q v)\|_{L^2}^2 .$$

Now all terms in this sum are easily estimated. By (C.3.22) we have

$$\|\Delta_p (S_{q-2} u \, \Delta_q v)\|_{L^2} \leq C \|S_{q-2} u \, \Delta_q v\|_{L^2}$$

and by (C.3.17)

$$\|S_{q-2} u \, \Delta_q v\|_{L^2} \leq C \|u\|_{L^\infty} \|\Delta_q v\|_{L^2} .$$

Collecting and summing these successive inequalities, we arrive at the aimed result

$$\sum_{p\geq -1} 2^{2ps}\,\|\Delta_p T_u v\|_{L^2}^2 \;\leq\; 7^2 \times 2^{6|s|}\, C^4\, \|u\|_{L^\infty}^2 \sum_{q\geq -1} 2^{2qs}\,\|\Delta_q v\|_{L^2}^2 \,.$$

□

Actually, this proof can be refined and extended to a more general framework. Let r be a rational integer. If $r \geq 2$ then we can find $k \in \mathbb{N}$ so that we have similarly as in (C.3.32)

$$\Delta_p\,(\,S_{q-r}u\;\Delta_q v\,) \;\equiv\; 0 \quad \text{for } |p-q| \;\geq\; k+1\,.$$

This is elementary by looking at

$$\operatorname{Supp}\mathscr{F}(\,S_{q-r}u\;\Delta_q v\,) \;\subset\; \{\,\|\xi\| \;\leq\; 2^{q-r}\,\} \,+\, \{\,2^{q-1} \;\leq\; \|\xi\| \;\leq\; 2^{q+1}\,\}\,.$$

But we point out that for $r \leq 1$, the support of $\mathscr{F}(\,S_{q-r}u\;\Delta_q v\,)$ is not bounded away from 0 and thus we only have $\Delta_p\,(\,S_{q-r}u\;\Delta_q v\,) \equiv 0$ for large p. More precisely, there exists $k \in \mathbb{N}$ so that

$$\Delta_p\,(\,S_{q-r}u\;\Delta_q v\,) \;\equiv\; 0 \quad \text{for } p-q \;\geq\; k+1\,. \tag{C.3.33}$$

However, these properties are sufficient to prove the analogous Proposition C.8, under some restriction on s though.

Proposition C.9 *For all $r \in \mathbb{Z}$ and $s > 0$ there exists $C > 0$ so that for $u \in L^\infty$ and $v \in H^s$*

$$\left\|\sum_{q\,\geq\,-1} S_{q-r}u\;\Delta_q v\right\|_{H^s} \;\leq\; C\,\|u\|_{L^\infty}\,\|v\|_{H^s}\,. \tag{C.3.34}$$

Proof The proof is very similar to that of Proposition C.8, except that we must make use of a finer Cauchy–Schwarz inequality (in ℓ^2 instead of $\mathbb{R}^7$!) and thus require $s > 0$. To facilitate the reading, we denote

$$S(u,v) \;=\; \sum_{q\,\geq\,-1} S_{q-r}u\;\Delta_q v\,.$$

Because of (C.3.33) we have

$$\Delta_p\,S(u,v) \;=\; \sum_{q\,=\,p-k}^{+\infty} \Delta_p\,(\,S_{q-r}u\;\Delta_q v\,)\,,$$

which implies by the Cauchy–Schwarz inequality that

$$\|\Delta_p\,S(u,v)\|_{L^2}^2 \;\leq\; \left(\sum_{q\,=\,p-k}^{+\infty} 2^{-qs}\right)\left(\sum_{q\,=\,p-k}^{+\infty} 2^{qs}\,\|\Delta_p\,(\,S_{q-r}u\;\Delta_q v\,)\|_{L^2}^2\right)\,.$$

The first factor is clearly bounded by $c_{s,k}\, 2^{-ps}$ with $c_{s,k} = \sum_{l\geq -k} 2^{-ls}$ for positive s. And we know from (C.3.22) and (C.3.17) that

$$\|\Delta_p\,(\,S_{q-r}u\;\Delta_q v\,)\|_{L^2} \;\leq\; C^2\,\|u\|_{L^\infty}\,\|\Delta_q v\|_{L^2}\,.$$

Therefore, we find that

$$\sum_{p\geq -1} 2^{2ps}\,\|\Delta_p\,S(u,v)\|_{L^2}^2 \;\leq\; c_{s,k}\,C^4\,\|u\|_{L^\infty}^2 \sum_{p\geq -1}\;\sum_{q\geq p-k} 2^{(p+q)s}\,\|\Delta_q v\|_{L^2}^2\,.$$

The conclusion then follows from the observation that

$$\sum_{p\geq -1}\;\sum_{q\geq p-k} 2^{(p+q)s}\,\|\Delta_q v\|_{L^2}^2 \;=\; \left(\sum_{l\geq -k} 2^{-ls}\right)\sum_{q\geq -1} 2^{2qs}\,\|\Delta_q v\|_{L^2}^2\,.$$

(The constant C in (C.3.34) is thus $c_{s,k}\,C^2$ with our present notations.) □

As a consequence of this proposition, we get in particular an error estimate for $T_v u$, provided that both u and v belong to $L^\infty \cap H^s$, $s>0$.

Proposition C.10 *For all $s>0$, there exists $C>0$ such that for all u and v in $L^\infty\cap H^s$, we have*

$$\|\,u\,v\;-\;T_v u\,\|_{H^s} \;\leq\; C\,\|u\|_{L^\infty}\,\|v\|_{H^s}\,.$$

Proof The assumption $s>0$ ensures that for $u,v\in H^s$ the series $\sum \Delta_p u$ and $\sum \Delta_q v$ are normally convergent in L^2 (because of (C.3.20)). This justifies the formula

$$u\,v \;=\; \sum_{p,q\geq -1} \Delta_p u\;\Delta_q v\,,$$

and thus by definition of $T_v u$:

$$u\,v\;-\;T_v u \;=\; \sum_{q\geq -1} S_{q+3}u\;\Delta_q v\,.$$

Consequently, Proposition C.9 applied to $r=-3$ yields the result. □

This in turn leads to a very simple proof of the following.

Proposition C.11 *For all $s>0$ there exists $C>0$ such that for all u and v in $L^\infty\cap H^s$, the product uv also belongs to H^s and*

$$\|\,u\,v\,\|_{H^s} \;\leq\; C\;(\,\|u\|_{L^\infty}\,\|v\|_{H^s} \;+\; \|v\|_{L^\infty}\,\|u\|_{H^s}\,)\;.$$

Proof Just sum the estimate of $T_v u$ obtained in Proposition C.8 with the error estimate in Proposition C.10. □

An alternative form of Proposition C.11 is the following.

Proposition C.12 *For each* integer $s > 0$ *there exists* $C > 0$ *such that for all* u *and* v *in* $L^\infty \cap H^s$ *and all d-uples* α, β *with* $|\alpha| + |\beta| = s$ *we have*

$$\| (\partial^\alpha u)\ (\partial^\beta v) \|_{L^2} \leq C\ (\|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{L^\infty} \|u\|_{H^s})\ . \tag{C.3.35}$$

This proposition can actually be proved in a more classical way, by using the Hölder inequality together with the *Gagliardo–Nirenberg inequality* [64,148]. The latter holds indeed for each positive integer s, and gives a constant $C > 0$ so that for $u \in L^\infty \cap H^s$ and $|\alpha| \leq s$

$$\| \partial^\alpha u \|_{L^{2s/|\alpha|}} \leq C \|u\|_{L^\infty}^{1-|\alpha|/s} \|u\|_{H^s}^{|\alpha|/s}\ . \tag{C.3.36}$$

An easy consequence of Proposition C.12 is the following commutator estimate.

Proposition C.13 *If* $s > 1$ *and* α *is a d-uple of length* $|\alpha| \leq s$*, there exists* $C > 0$ *such that for all* u *and* a *in* H^s *with* ∇u *and* ∇a *in* L^∞

$$\| [\, \partial^\alpha\, ,\, a\, \nabla]\, u \|_{L^2} \leq C\ (\|\nabla a\|_{L^\infty} \|u\|_{H^s} + \|\nabla u\|_{L^\infty} \|a\|_{H^s})\ .$$

Proof We first note that the assumptions on a and u imply by Proposition C.11 that $a\nabla u$ belongs to H^{s-1} and so

$$[\, \partial^\alpha\, ,\, a\, \nabla]\, u = \partial^\alpha(a\nabla u) - a\, \nabla(\partial^\alpha u) \in H^{-1}$$

for $|\alpha| \leq s$. Our purpose is to show that in fact $[\, \partial^\alpha\, ,\, a\, \nabla]\, u$ belongs to L^2 (which is in general the best one can expect since for $a \in \mathscr{C}^\infty$ the commutator $[\, \partial^\alpha\, ,\, a\, \nabla]$ is a differential operator of order $|\alpha|$).

We consider an index $j \in \{1, \cdots, d\}$ and want to estimate the L^2 norm of $[\, \partial^\alpha\, ,\, a\, \partial_j]\, u$. By the Leibniz rule there exist coefficients c_α^β with $c_\alpha^0 = 1$ so that

$$\partial^\alpha\, (\, a\, \partial_j\, u) = \sum_{|\beta| \leq |\alpha|} c_\alpha^\beta\ (\partial^\beta a)\ (\partial^{\alpha-\beta} \partial_j\, u)\, .$$

Consequently, we have

$$[\, \partial^\alpha\, ,\, a\, \partial_j]\, u = \sum_{1 \leq |\beta| \leq |\alpha|} c_\alpha^\beta\, (\partial^\beta a)\ (\partial^{\alpha-\beta} \partial_j\, u)\, .$$

In the latter sum, all terms are of the form

$$(\partial^{\beta_k} \partial_k a)\ (\partial^{\alpha-\beta} \partial_j\, u)$$

for some index $k \in \{1, \cdots, d\}$ and d-uple β_k of length $|\beta_k| = |\beta| - 1$. Both $\partial_k a$ and $\partial_j u$ belong to $L^\infty \cap H^{|\alpha|-1}$ and thus meet the assumptions of Proposition C.12 with $s = |\alpha| - 1$. This yields the estimate

$$\|(\partial^{\beta_k} \partial_k a)\ (\partial^{\alpha-\beta} \partial_j u)\|_{L^2} \leq C\, (\|\partial_k a\|_{L^\infty} \|\partial_j u\|_{H^{|\alpha|-1}} + \|\partial_j u\|_{L^\infty} \|\partial_k a\|_{H^{|\alpha|-1}})\, ,$$

where the right-hand side is clearly bounded by $\|\nabla a\|_{L^\infty} \|u\|_{H^s} + \|\nabla u\|_{L^\infty} \|a\|_{H^s}$ for $|\alpha| \leq s$. Hence, by summing on β and k we get the desired estimate. □

To illustrate the power of para-products, let us just show the following result on the remainder R, where we see that the regularity of $R(u,v)$ is 'almost' the one of u plus the one of v.

Theorem C.9 *For all s and t with $s+t>0$, there exists $C>0$ so that for all $u\in H^s$ and all $v\in H^t$, $R(u,v)$ is well-defined by* (C.3.30) *and meets the estimate*

$$\|R(u,v)\|_{H^{s+t-d/2}} \leq C\,\|u\|_{H^s}\,\|v\|_{H^t}\,. \tag{C.3.37}$$

Proof • At first, we check that the assumption $s+t>0$ ensures that R is well-defined and

$$R(u,v) = \sum_{q\geq -1} R_q(u,v) \quad \text{with} \quad R_q(u,v) := \sum_{r=q-2}^{q+2} \Delta_r u\,\Delta_q v\,.$$

As a matter of fact, we have

$$\|R_q(u,v)\|_{L^1} \leq \sum_{r=q-2}^{q+2} \|\Delta_r u\|_{L^2}\,\|\Delta_q v\|_{L^2} \leq C \sum_{r=q-2}^{q+2} 2^{-rs}\,\|u\|_{H^s}\,2^{-qt}\,\|v\|_{H^t}$$

by (C.3.20), hence

$$\|R_q(u,v)\|_{L^1} \leq 5\times 2^{2|s|}\,C\,\|u\|_{H^s}\,\|v\|_{H^t}\,2^{-q(s+t)}\,,$$

which shows that the series $\sum R_q(u,v)$ is normally convergent in L^1.

• To prove the estimate in (C.3.37) we must evaluate the L^2 norm of $\Delta_p\,R(u,v)$. Similarly as in the proof of Proposition C.9, we note that $\Delta_p\,R(u,v)$ only involves some terms $\Delta_p\,R_q(u,v)$. This is due to the fact (already used in (C.3.33)) that there is an integer k such that

$$\Delta_p\,(\,\Delta_r u\,\Delta_q v\,) \equiv 0 \quad \text{for } p-q \geq k+1 \quad \text{and } |r-q|\leq 2\,. \tag{C.3.38}$$

Therefore, we have

$$\Delta_p\,R(u,v) = \sum_{q\geq p-k} \Delta_p\,R_q(u,v)\,.$$

For all $p\geq -1$ we have

$$\Delta_p\,R_q(u,v) = \sum_{r=q-2}^{q+2} \mathscr{F}^{-1}\phi_p * (\,\Delta_r u\,\Delta_q v\,)$$

and thus a standard convolution inequality and Plancherel's theorem show that

$$\|\,\Delta_p\,R_q(u,v)\,\|_{L^2} \leq \|\,\phi_p\|_{L^2} \sum_{r=q-2}^{q+2} \|\,\Delta_r u\,\Delta_q v\,\|_{L^1}\,.$$

Since for $p\geq 0$ we have

$$\|\phi_p\|_{L^2} = 2^{pd/2}\,\|\phi\|_{L^2}$$

the latter inequality implies that for all $p \geq -1$

$$\| \Delta_p R_q(u,v) \|_{L^2} \leq c\, 2^{pd/2} \sum_{r=q-2}^{q+2} \| \Delta_r u \|_{L^2} \, \|\Delta_q v\|_{L^2} ,$$

with $c := \max(\|\phi\|_{L^2}, 2^{d/2}\|\psi\|_{L^2})$.

Like in Proposition C.9 we can apply the Cauchy–Schwarz inequality to obtain

$$\|\Delta_p R(u,v)\|_{L^2}^2 \leq c_{t+s,k}\, 2^{-p(t+s)} \sum_{q \geq p-k} 2^{q(t+s)} \|\Delta_p R_q(u,v)\|_{L^2}^2 ,$$

with $c_{t+s,k} := \sum_{l=-k}^{+\infty} 2^{-l(t+s)}$. Consequently, we have

$$\|\Delta_p R(u,v)\|_{L^2}^2 \leq C'\, 2^{-p(t+s-d)} \sum_{q \geq p-k} 2^{q(t+s)} \|\Delta_q v\|_{L^2}^2 \sum_{r=q-2}^{q+2} \| \Delta_r u\|_{L^2}^2 ,$$

with $C' := 5 \times c^2\, c_{t+s,k}$ and thus

$$2^{2p(s+t-d/2)} \|\Delta_p R(u,v)\|_{L^2}^2 \leq 5 \times 2^{2|s|}\, C' \sum_{q \geq p-k} 2^{(p-q)(t+s)}\, 2^{2qt} \|\Delta_q v\|_{L^2}^2 \times \sum_{r=q-2}^{q+2} 2^{2rs} \| \Delta_r u\|_{L^2}^2 .$$

So we have by (C.3.18)

$$\sum_{p \geq -1} 2^{2p(s+t-d/2)} \|\Delta_p R(u,v)\|_{L^2}^2 \leq C'' \|v\|_{H^t}^2 \|u\|_{H^s}^2 ,$$

with $C'' := 5 \times 2^{2|s|}\, C'\, c_{t+s,k}\, C_t\, C_s$. This finally proves the estimate in (C.3.37) with $C = \sqrt{C'' C_{t+s-d/2}}$. □

Remark C.5 This result on the remainder $R(u,v)$ gives a slightly bigger index than in the classical result recalled below for the full product uv.

Theorem C.10 *For all s and t with $s+t>0$, if $u \in H^s$ and $v \in H^t$ then the product belongs to H^r for all $r \leq \min(s,t)$ such that $r < s+t-d/2$. Furthermore, there exists C (depending only on r, s, t and d) such that*

$$\|u\, v\|_{H^r} \leq C\, \|u\|_{H^s} \|v\|_{H^t} .$$

In the case $r=s=t$, this result may be viewed as a consequence of Proposition C.11; an alternative, elementary proof, which also gives the result in Sobolev spaces on any smooth domain Ω, may be found in [1], p. 115–117. For more general values of r, s and t (but $\Omega = \mathbb{R}^d$), Theorem C.10 can be deduced from Theorem C.9 and the following additional result on para-products.

Proposition C.14 *For all s and t, if $u \in H^s$ and $v \in H^t$ then the para-product $T_u v$ is well-defined and belongs to H^r for all $r < s+t-d/2$. Furthermore, there exists $C > 0$ independent of u and v so that*

$$\|T_u v\|_{H^r} \leq C \|u\|_{H^s} \|v\|_{H^t} . \tag{C.3.39}$$

Proof It is very similar to the proof of Proposition C.8, replacing the use of the estimate of $\|S_q u\|_{L^\infty}$ in (C.3.17) by the estimate in (C.3.24): the condition $r < s+t-d/2$ is here to ensure the convergence of the series $\sum_p 2^{-2p(s+t-d/2-r)}$, hence of $\sum_p 2^{2ps} \|\Delta_p T_u v\|^2_{L^2}$. Details are left to the reader. □

An easy consequence of Theorem C.10 is the following commutator estimate.

Corollary C.2 *If m is an integer greater than $d/2+1$ and α is a d-uple of length $|\alpha| \in [1, m]$, there exists $C > 0$ such that for all a in H^m and all $u \in H^{|\alpha|-1}$,*

$$\| [\partial^\alpha , a] u \|_{L^2} \leq C \|a\|_{H^m} \|u\|_{H^{|\alpha|-1}} .$$

C.3.4 *Para-linearization*

Proposition C.8 and Theorem C.9 show in particular that for all $s > 0$, if $u \in H^s \cap L^\infty$ then

$$u^2 = 2 T_u u + R(u,u) = T_{2u} u + R(u,u),$$

with the uniform estimates

$$\|T_{2u} u\|_{H^s} \leq C \|u\|_{L^\infty} \|u\|_{H^s} , \quad \|R(u,u)\|_{H^{2s-d/2}} \leq C \|u\|^2_{H^s} .$$

A very strong result from para-differential calculus says that this decomposition of $F(u) = u^2$ can be generalized to *any* $\mathscr{C}^\infty$ function F vanishing at 0, under the only constraint that $s > d/2$.

Theorem C.11 (Bony–Meyer) *If $F \in \mathscr{C}^\infty(\mathbb{R})$, $F(0) = 0$, if $s > d/2$ then for all $u \in H^s(\mathbb{R}^d)$ we have*

$$F(u) = T_{F'(u)} u + R(u), \tag{C.3.40}$$

with $R(u) \in H^{2s-d/2}$.

(Note that the assumption $s > d/2$ automatically implies $u \in L^\infty$ if $u \in H^s(\mathbb{R}^d)$.)

Equation (C.3.40) is often referred to as the para-linearization formula of Bony. Historically, Bony proved that the remainder term $R(u)$ belongs to $H^{2s-d/2-\varepsilon}$ for $\varepsilon > 0$ [20], and Meyer proved the actual result with $\varepsilon = 0$ [138].

In particular, (C.3.40) shows that $F(u)$ belongs to H^s. We do not intend to give the extensive proof of Theorem C.11. We 'directly' show that $F(u)$ enjoys the same estimate as its para-linearized counterpart $T_{F'(u)} u$.

Theorem C.12 *If $F \in \mathscr{C}^\infty(\mathbb{R})$, $F(0) = 0$, if $s > d/2$ then there exists a continuous function $C : [0, +\infty) \to [0, +\infty)$ such that for all $u \in H^s(\mathbb{R}^d)$*

$$\| F(u) \|_{H^s} \le C(\|u\|_{L^\infty}) \|u\|_{H^s}.$$

Proof The assumption $s > d/2$ obviously implies that each $u \in H^s(\mathbb{R}^d)$ necessarily belongs to $L^\infty \cap L^2$.

• We begin by showing the estimate for $F(S_0 u)$ instead of $F(u)$. To do so, it is sufficient to bound $\|\partial^\alpha F(S_0 u)\|_{L^2}$ for all d-uples α of length $|\alpha| \le m$ with $m - 1 \le s < m$. For $|\alpha| = 0$ this is almost trivial. By Propositions C.3 and C.5 we have

$$\|S_0 u\|_{L^\infty} \le C \|u\|_{L^\infty} \quad \text{and} \quad \|S_0 u\|_{L^2} \le C \|u\|_{L^2}$$

and thus the mean value theorem applied to F in the ball of radius $R = \|u\|_{L^\infty}$ implies that

$$\|F(S_0 u)\|_{L^2} \le C_R \|u\|_{L^2}$$

for some $C_R > 0$ depending continuously on R. For $|\alpha| \ge 1$, the chain rule shows there exist coefficients c_α^b, with $b = \{\beta^1, \cdots, \beta^n\}$ being a family of d-uples of positive length and of sum $\beta^1 + \cdots + \beta^n = \alpha$, so that

$$\partial^\alpha (F(S_0 u)) = \sum_{\substack{1 \le n \le |\alpha| \\ \beta^1 + \cdots + \beta^n = \alpha \\ |\beta^i| \ge 1}} c_\alpha^b \, F^{(n)}(S_0 u) \, \partial^{\beta^1}(S_0 u) \cdots \partial^{\beta^n}(S_0 u).$$

By Propositions C.3 and C.5 we have

$$\|\partial^{\beta^i}(S_0 u)\|_{L^\infty} \le C \|u\|_{L^\infty} \quad \text{and} \quad \|\partial^{\beta^i}(S_0 u)\|_{L^2} \le C \|u\|_{L^2}.$$

Therefore, using L^∞ bounds for the successive derivatives $F^{(n)}$, $n \le m$, on the ball of radius R we obtain a uniform estimate

$$\|\partial^\alpha (F(S_0 u)) \|_{L^2} \le C_R^m \|u\|_{L^2}$$

for all α with $|\alpha| \le m$. In particular, up to modifying C_R^m, we have

$$\| F(S_0 u) \|_{H^s} \le C_R^m \|u\|_{L^2}$$

for all $s \le m$.

• The other main part of the proof consists in bounding the 'error' $F(u) - F(S_0 u)$. Since $S_p u$ is known to tend to u in H^s, we *formally* have

$$F(u) - F(S_0 u) = \sum_{p=0}^{\infty} (F(S_{p+1} u) - F(S_p u)).$$

To justify this decomposition we must show that the series involved is convergent in H^s. At first, we note that

$$F(S_{p+1}u) \;-\; F(S_pu) \;=\; G(S_pu, \Delta_pu)\,\Delta_pu,$$

where

$$G(v,w) \;=\; \int_0^1 F'(v+tw)\,\mathrm{d}t$$

is a $\mathscr{C}^\infty$ function of both its arguments. By Proposition C.3 and a piece of calculus we can bound $G(S_pu, \Delta_pu)$ in L^∞ in exactly the same way we bounded $F(S_0u)$ in L^2. Thus we find another constant depending continuously on R, still denoted by C_R^m, so that

$$\|\partial^\alpha\,(\,G(S_pu,\Delta_pu)\,)\,\|_{L^\infty} \;\leq\; C_R^m\,2^{p|\alpha|}$$

for all α with $|\alpha| \leq m$. Then a fine result, postponed to Lemma C.2 below, enables us to conclude. As a matter of fact, Lemma C.2 applies to $M_p = G(S_pu, \Delta_pu)$ and shows that

$$\left\|\sum_{p=0}^{\infty}(\,F(S_{p+1}u) \;-\; F(S_pu)\,)\right\|_{H^s} \;=\; \left\|\sum_{p\geq 0} G(S_pu,\Delta_pu)\,\Delta_pu\right\|_{H^s} \;\leq\; c\,C_R^m\,\|u\|_{H^s}\,.$$

- We have

$$F(u) \;=\; F(S_0u) \;+\; \sum_{p=0}^{\infty}(\,F(S_{p+1}u) \;-\; F(S_pu)\,)\,.$$

Collecting and summing the estimates of $F(S_0u)$ and the series $\sum(\,F(S_{p+1}u) - F(S_pu)\,)$ we find that

$$\|\,F(u)\,\|_{H^s} \;\leq\; C\,(\|u\|_{L^\infty})\;\;\|u\|_{H^s}\,,$$

with $C\,(\|u\|_{L^\infty}) \;=\; (1+c)\,C_R^m$. (We recall that $R \geq \|u\|_{L^\infty}$.)

□

Lemma C.2 (Meyer) *Let $\{M_p\}_{p\geq 0}$ be a sequence of $\mathscr{C}^\infty$ functions enjoying the uniform estimates*

$$\|\partial^\alpha\,M_p\,\|_{L^\infty} \;\leq\; c_m\,2^{p|\alpha|} \tag{C.3.41}$$

for all α with $|\alpha| \leq m$. Then for all $0 < s < m$, there exists c so that for all $u \in H^s$ the series $\sum M_p\,\Delta_pu$ is convergent in H^s and

$$\left\|\sum_{p\geq 0} M_p\;\Delta_pu\right\|_{H^s} \;\leq\; c\,c_m\,\|u\|_{H^s}\,. \tag{C.3.42}$$

Proof The proof resembles the one of Proposition C.9, in that the sequence $\{M_q\}$ satisfies by assumption the same estimates as S_qu (derived in Proposition

C.3). However, there is an additional difficulty due to the fact that, unlike $S_q u$, $\widehat{M_p}$ is *not* supposed to be compactly supported. This is why we first perform a frequency decomposition of M_p. For this we use the dilated functions $\phi_q(2^{-p-3}\cdot)$ and define

$$M_{p,q} \,:=\, \mathscr{F}^{-1}(\phi_q(2^{-p-3}\cdot)\,\widehat{M_p})$$

for all $q \geq -1$. Observe that, for $q \geq 0$, we merely have

$$M_{p,q} \,:=\, \Delta_{q+p+3}\, M_p\,,$$

of which the spectrum is included in

$$\{\,\xi\,;\; 2^{p+q+2} \,\leq\, \|\xi\| \,\leq\, 2^{p+q+4}\}\,,$$

and that the first term of the expansion,

$$M_{p,-1} \,=\, \mathscr{F}^{-1}(\psi(2^{-p-3}\cdot)\,\widehat{M_p})\,,$$

has a spectrum included in

$$\{\,\xi\,;\; \|\xi\| \,\leq\, 2^{p+3}\}\,.$$

Because of (C.3.13) (evaluated at $2^{-p-3}\xi$), we have

$$M_p \,=\, \sum_{q\geq -1} M_{p,q}$$

in the sense of $\mathscr{S}'$. In fact, this series is normally convergent in L^∞, since by Proposition C.7,

$$\|\, M_{p,q}\,\|_{L^\infty} \,\leq\, C_m \sum_{|\alpha|=m} \|\partial^\alpha\, M_{p,q}\,\|_{L^\infty}\, 2^{-(p+q+3)m}$$

for $q \geq 0$, and by Proposition C.3 applied to $\partial^\alpha M_p$

$$\|\partial^\alpha\, M_{p,q}\,\|_{L^\infty} \,\leq\, C_m\, \|\partial^\alpha\, M_p\,\|_{L^\infty}\,,$$

(we have used here the fact that $[\partial^\alpha, \Delta_r] = 0$ for all r) and so the assumption (C.3.41) implies that

$$\|\, M_{p,q}\,\|_{L^\infty} \,\leq\, \widetilde{C}_m\, 2^{-qm}\,.$$

• Let us now look at the two parameters family $\{M_{p,q}\Delta_p u\}_{p\geq 0, q\geq -1}$. By (C.3.20) and the previous inequality we have

$$\sum_{p\geq 0}\sum_{q\geq -1} \|M_{p,q}\,\Delta_p u\|_{L^2} \,\leq\, \widetilde{C}_m \sum_{p\geq 0}\sum_{q\geq -1} 2^{-qm}2^{-ps}\, \|u\|_{H^s} \,<\, \infty\,,$$

which justifies the interchanging formula

$$\sum_{p\geq 0}\sum_{q\geq -1} M_{p,q}\,\Delta_p u \,=\, \sum_{q\geq -1}\sum_{p\geq 0} M_{p,q}\,\Delta_p u\,.$$

This equivalently reads

$$\sum_{p\geq 0} M_p\,\Delta_p u \;=\; \sum_{q\geq -1} \Sigma_q\,,$$

with

$$\Sigma_q \;:=\; \sum_{p\geq 0} M_{p,q}\,\Delta_p u\,.$$

- We can now estimate Σ_q in H^s.

We begin with the special case $q=-1$. We have

$$\operatorname{Supp}\mathscr{F}(\,M_{p,-1}\;\Delta_p u\,)\;\subset\;\{\,\|\xi\|\;\leq\;2^{p+4}\,\}$$

and thus

$$\Delta_r\,(\,M_{p,-1}\;\Delta_p u\,)\;\equiv\;0\quad\text{for}\quad r\geq p+5\,.$$

Therefore,

$$\Delta_r\Sigma_{-1}\;=\;\sum_{p=r-4}^{+\infty}\Delta_r\,(\,M_{p,-1}\;\Delta_p u\,)$$

for all $r\geq -1$. By exactly the same procedure as in the proof of Proposition C.9 and the uniform estimate

$$\|\,M_{p,-1}\,\|_{L^\infty}\;\leq\;C\,\|\,M_p\,\|_{L^\infty}\;\leq\;C\,C_0,$$

we show that

$$\sum_{r\geq -1} 2^{2rs}\,\|\Delta_r\Sigma_{-1}\|_{L^2}^2\;\leq\;c_{s,4}^2\,C^2\,C_0^2\sum_{p\geq -1}2^{2ps}\,\|\Delta_p u\|_{L^2}^2\,,$$

with $c_{s,4}=\sum_{l\geq -4}2^{-ls}$. Using the equivalent norm in (C.3.21) this precisely means that

$$\|\Sigma_{-1}\|_{H^s}\;\leq\;c_{s,4}\,C\;C_0\,\|u\|_{H^s}\,.$$

The general case $q\geq 0$ is no more difficult. We have

$$\operatorname{Supp}\mathscr{F}(\,M_{p,q}\;\Delta_p u\,)\;\subset\;\{\,2^{p+q+1}\;\leq\;\|\xi\|\;\leq\;2^{p+q+5}\,\}$$

and thus

$$\Delta_r\,(\,M_{p,q}\;\Delta_p u\,)\;\equiv\;0\quad\text{for}\quad r\geq p+q+6\quad\text{or}\quad r\leq p+q-1\,.$$

Consequently,

$$\Delta_r\Sigma_q\;=\;\sum_{p=r-q-5}^{r-q}\Delta_r\,(\,M_{p,q}\;\Delta_p u\,)$$

and

$$\| \Delta_r (M_{p,q} \Delta_p u) \|_{L^2} \leq \widetilde{C}_m 2^{-qm} \| \Delta_p u \|_{L^2} .$$

By the Cauchy–Schwarz inequality, we obtain

$$\sum_{r \geq -1} 2^{2rs} \| \Delta_r (M_{p,q} \Delta_p u) \|_{L^2}^2 \leq 6^2 \times 2^{8s} \widetilde{C}_m^2 2^{-2q(m-s)} \sum_{p \geq -1} 2^{2ps} \| \Delta_p u \|_{L^2}^2 ,$$

which means that

$$\|\Sigma_q\|_{H^s} \leq 6 \times 2^{4s} \widetilde{C}_m \|u\|_{H^s} 2^{-2q(m-s)} .$$

• The conclusion then follows from the summation on q of the estimates for $\|\Sigma_q\|_{H^s}$. □

Let us mention an easy consequence of Theorem C.12 and Proposition C.11.

Corollary C.3 *If $F \in \mathscr{C}^\infty(\mathbb{R})$ and $s > d/2$, then there exists a continuous function $C : (0, +\infty) \to (0, +\infty)$ such that for all u and v in H^s,*

$$\|F(u) - F(v)\|_{H^s} \leq C(\max(\|u\|_{H^s}, \|v\|_{H^s})) \|u - v\|_{H^s} .$$

Proof Without loss of generality, we may assume $F'(0)$ equals 0. By Taylor's formula and Proposition C.11, we have

$$\begin{aligned}\|F(u) - F(v)\|_{H^s} &\leq \int_0^1 \|F'(v + \theta(u - v)) (u - v)\|_{H^s} \mathrm{d}\theta \\ &\leq C_1 \Bigg(\max_{|w| \leq \max(\|u\|_{L^\infty}, \|v\|_{L^\infty})} |F'(w)| \|u - v\|_{H^s} \\ &\qquad + \max_{\theta \in [0,1]} \|F'(v + \theta(u - v))\|_{H^s} \|u - v\|_{L^\infty} \Bigg) .\end{aligned}$$

The first term inside parentheses is already of the wanted form, by the Sobolev embedding $H^s \hookrightarrow L^\infty$. And in the second one we have

$$\|F'(v + \theta(u - v))\|_{H^s} \|u - v\|_{L^\infty} \leq C_0(\|v + \theta(u - v)\|_{L^\infty}) \|v + \theta(u - v)\|_{H^s} \|u - v\|_{L^\infty}$$

by Theorem C.12, which yields the wanted inequality using again the Sobolev embedding $H^s \hookrightarrow L^\infty$.

□

C.3.5 *Further estimates*

A useful result that was not pointed out yet is the smoothing effect of the operator $(a - T_a)$ when a is at least Lipschitz.

Theorem C.13 *For all $k \in \mathbb{N}$, there exists $C > 0$ such that for all $a \in W^{k,\infty}$ and all u in L^2,*

$$\| a u - T_a u \|_{H^k} \leq C \|a\|_{W^{k,\infty}} \|u\|_{L^2} .$$

The case $k = 0$ (with no smoothing effect!) is a trivial consequence of Proposition C.8 and the triangular inequality. The difficult case is of course for $k \geq 1$. A detailed proof can be found in [136] or [38] – theses references deal in fact with the operator with parameter T_a^γ but the method works for T_a.

A straightforward consequence of Theorem C.13 is the following.

Corollary C.4 *There exists $C > 0$ such that for all $a \in W^{1,\infty}$ and u in L^2, for $j \in \{1, \dots, d\}$,*

$$\| a\, \partial_j u \,-\, T_a \partial_j u \, \|_{L^2} \,\leq\, C\, \|a\|_{W^{1,\infty}} \, \|u\|_{L^2}\, .$$

Proof Observe that

$$a\, \partial_j u \,-\, T_a \partial_j u \,=\, \partial_j(\, a\, u \,-\, T_a u\,) \,-\, (\, (\partial_j a)\, u \,-\, T_{\partial_j a} \partial_j u\,)\, .$$

The second term is obviously bounded by $\|\partial_j a\|_{L^\infty} \, \|u\|_{L^2}$ (according to Proposition C.8 and the triangular inequality, or Theorem C.13 with $k = 0$!) And the first one is bounded by $\|a\|_{W^{k,\infty}} \, \|u\|_{L^2}$ according to Theorem C.13 with $k = 1$. □

A further useful result in this direction is the following.

Theorem C.14 *If R_ε is a smoothing operator defined as the convolution operator by a kernel ρ_ε satisfying the standard properties of mollifiers, namely $\rho_\varepsilon(x) = \varepsilon^{-d}\, \rho(x/\varepsilon)$ with $\rho \in \mathscr{D}(\mathbb{R}^d; \mathbb{R}^+)$ and $\int_{\mathbb{R}^d} \rho = 1$, then for all $a \in W^{1,\infty}$ and u in L^2, for $j \in \{1, \dots, d\}$,*

$$\| \, R_\varepsilon\, (a\, \partial_j u) \,-\, a\, \partial_j (R_\varepsilon u) \, \|_{L^2} \,\leq\, C\, \|a\|_{W^{1,\infty}} \, \|u\|_{L^2} \tag{C.3.43}$$

and

$$\lim_{\varepsilon \to 0} \| \, R_\varepsilon\, (a\, \partial_j u) \,-\, a\, \partial_j (R_\varepsilon u) \, \|_{L^2} \,=\, 0\, . \tag{C.3.44}$$

Proof We can prove separately the results for the commutators $[\, R_\varepsilon\, , \, T_a \partial_j\,]$ and $[\, R_\varepsilon\, , \, (a - T_a) \partial_j\,]$. The estimate for the latter comes directly from Corollary C.4 and the boundedness of R_ε on L^2. The estimate of $[\, R_\varepsilon\, , \, T_a \partial_j\,]$ is, in fact, a consequence of para-differential calculus (see Section C.4 below, Theorem C.17), once we observe that $(R_\varepsilon)_{\varepsilon \in (0,1)}$ is a family of pseudo-differential operators of order 0. Indeed, each operator R_ε is infinitely smoothing and R_ε goes to identity when $\varepsilon \to 0$. To be more precise, the symbol of R_ε, $\widehat{\rho_\varepsilon} = \widehat{\rho}(\varepsilon \cdot)$, belongs to $\mathscr{S}$ and thus to $\mathbf{S}^{-\infty}$. However, in the estimate

$$\| \, \partial_\xi^\beta \, \widehat{\rho_\varepsilon}(\xi) \, \| \,\leq\, C_{m,\beta}(\varepsilon)\, (\, 1 \,+\, \|\xi\| \,)^{m - |\beta|} \quad \forall \xi \in \mathbb{R}^d\, ,$$

the constant $C_{m,\beta}(\varepsilon)$ is uniformly bounded for $\varepsilon \in (0,1)$ only if $m \geq 0$. So Theorem C.17 shows that $[\, R_\varepsilon\, , \, T_a \partial_j\,]$ is a family of para-differential operators of order $0 + 1 - 1 = 0$. Regarded as operators on L^2, their norms are uniformy controlled by $\|a\|_{W^{1,\infty}}$. This shows the estimate for $[\, R_\varepsilon\, , \, T_a \partial_j\,]$. Summing with the estimate for $[\, R_\varepsilon\, , \, (a - T_a) \partial_j\,]$ we get (C.3.43).

The proof of the limit in (C.3.44) then proceeds in a classical way, approximating any function u of L^2 by a sequence of smoother functions u_n (H^1 is sufficient) for which we know the limit holds true and applying (C.3.43) to $u - u_n$. □

Remark C.6 A slightly more general statement of (C.3.44) is that for all $a \in W^{1,\infty}$ and all $u \in H^{-1}$,

$$\lim_{\varepsilon \to 0} \| [\, a \,,\, R_\varepsilon \,] \, u \, \|_{L^2} \;=\; 0 \,.$$

Another smoothing result, which admits a proof much more elementary than Theorems C.13 and C.14, holds true in Sobolev spaces of large negative index.

Proposition C.15 *If $s > \frac{d}{2} + 1$, there exists $C > 0$ so that for all $a \in H^s(\mathbb{R}^d)$ and all $\varphi \in \mathscr{D}(\mathbb{R}^d)$,*

$$\| \, a \, \varphi \;-\; T_a \varphi \, \|_{H^{-s+1}} \;\leq\; C \, \|a\|_{H^s} \, \|\varphi\|_{H^{-s}} \,.$$

Proof In fact, we shall prove that $a \, \varphi \;-\; T_a \varphi$ belongs to L^1 – and use Remark C.3 to conclude. The assumptions show that both series $\sum_q \Delta_q a$ and $\sum_p \Delta_p \varphi$ are normally convergent in L^2, which allows us to write

$$a \, \varphi \;=\; \sum_{p,q \geq -1} \Delta_p \varphi \; \Delta_q a \,,$$

and thus by definition of $T_a \varphi$:

$$a \, u \;-\; T_a \varphi \;=\; \sum_{q \geq -1} \, \sum_{p=-1}^{q+2} \Delta_p \varphi \; \Delta_q a \,.$$

By Proposition C.4, $\| \, \Delta_q a \, \|_{L^2} \;=\; 2^{-\,q\,s} \, \alpha_q$ and $\| \, \Delta_p \varphi \, \|_{L^2} \;=\; 2^{p\,s} \, \beta_p$ with $\sum_p \beta_p^2 \;\leq\; C_s \, \|\varphi\|_{H^{-s}}^2$. Therefore, by the Cauchy–Schwarz inequality in L^2,

$$\begin{aligned}
\sum_{q \geq -1} \, \sum_{p=-1}^{q+2} \|\Delta_p \varphi \; \Delta_q a\|_{L^1} \;&\leq\; \sum_{q \geq -1} \, \sum_{p=-1}^{q+2} \|\Delta_p \varphi\|_{L^2} \; \|\Delta_q a\|_{L^2} \\
&=\; \sum_{q \geq -1} \, \sum_{p=-1}^{q+2} 2^{(p-q)\,s} \; \beta_p \; \alpha_q \\
&=\; \sum_{k \geq -2} 2^{-\,k\,s} \, \sum_{p \geq -1} \beta_p \; \alpha_{p+k} \\
&\leq\; \sum_{k \geq -2} 2^{-\,k\,s} \, \|\beta\|_{\ell^2} \; \|\alpha\|_{\ell^2} \;\leq\; \widetilde{C}_s \, \|a\|_{H^s} \; \|\varphi\|_{H^{-s}} \,,
\end{aligned}$$

since $s > 0$. This proves that $a \, u \;-\; T_a \varphi$ belongs to L^1 and

$$\| \, a \, \varphi \;-\; T_a \varphi \, \|_{L^1} \;\leq\; \widetilde{C}_s \, \|a\|_{H^s} \, \|\varphi\|_{H^{-s}} \,.$$

Finally, since $s > \frac{d}{2} + 1$ we have the embedding $L^1(\mathbb{R}^d) \hookrightarrow H^{-s+1}(\mathbb{R}^d)$ (see Remark C.3). This completes the proof. □

C.4 Para-differential calculus

The tools introduced in the previous section provide a basis for what is called para-differential calculus, involving operators whose 'symbol' has a limited regularity in x. In particular, the operators T_a encountered in para-products are special cases of para-differential operators.

The purpose of this section is not to develop the whole theory but only some major aspects that are used elsewhere in the book. We shall use again the notation

$$\lambda^s(\xi) := (1 + \|\xi\|^2)^{s/2}$$

for all $s \in \mathbb{R}$.

C.4.1 *Construction of para-differential operators*

Definition C.5 *For any real number m and any natural integer k, we define the set Γ_k^m of functions, also called symbols, $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}^{N\times N}$ such that*

- *for almost all $x \in \mathbb{R}^d$, the mapping $\xi \in \mathbb{R}^d \mapsto a(x,\xi)$ is $\mathscr{C}^\infty$,*
- *for all d-uple β and all $\xi \in \mathbb{R}^d$, the mapping $x \in \mathbb{R}^d \mapsto \partial_\xi^\beta a(x,\xi)$ belongs to $W^{k,\infty}$ and there exists $C_\beta > 0$ so that for all $\xi \in \mathbb{R}^d$,*

$$\| \partial_\xi^\beta a(\cdot,\xi) \|_{W^{k,\infty}} \leq C_\beta \lambda^{m-|\beta|}(\xi) . \tag{C.4.45}$$

Of course, by Definition C.1 we have $\mathbf{S}^m \subset \Gamma_k^m$ for all k. The novelty is that functions with rather poor regularity in x are allowed. In particular, $W^{k,\infty}$ functions of x only may be viewed as symbols in Γ_k^0.

Symbols belonging to Γ_k^m are said to be of *order* m and *regularity* k. Unlike infinitely smooth symbols in $\mathbf{S}^m$, functions in Γ_k^m are not naturally associated with bounded operators $H^s \to H^{s-m}$. But this will be the case for the subclass Σ_k^m of symbols in Γ_k^m satisfying the additional, spectral property:

$$\mathrm{Supp}\,(\mathscr{F}(a(\cdot,\xi))) \subset B(0;\varepsilon\lambda^1(\xi)) \tag{C.4.46}$$

for some $\varepsilon \in (0,1)$ *independent* of ξ, see Theorem C.15 below. One may argue that since their Fourier transform is compactly supported such symbols are necessarily $\mathscr{C}^\infty$ in x. And we want to handle non-smooth symbols. So where is the trick? In fact, it relies on a special smoothing procedure, associating any symbol $a \in \Gamma_k^m$ with a symbol $\sigma \in \Sigma_k^m$. We shall give more details below. Let us start with the study of operators associated with symbols in Σ_k^m, $k \geq 0$.

Theorem C.15 *For all $a \in \Gamma_0^m$ satisfying* (C.4.46), *consider*

$$\begin{aligned} \mathrm{Op}(a) :\ & \mathscr{F}^{-1}(\mathscr{E}') \longrightarrow \mathscr{C}_b^\infty \\ & u \mapsto \mathrm{Op}(a)\,u\,;\ (\mathrm{Op}(a)\,u)(x) = \tfrac{1}{(2\pi)^d} \langle \mathrm{e}^{i\,x\,\cdot}\, a(x,\cdot)\,,\, \widehat{u} \rangle_{(\mathscr{C}^\infty,\mathscr{E}')}\,, \end{aligned}$$

where $\mathscr{E}'$ *denotes the space of temperate distributions having compact support*[1]*, and the unusual ordering* $(\mathscr{C}^\infty, \mathscr{E}')$ *is just meant to account for matrix-valued* a*. This definition of* $\mathrm{Op}(a)$ *coincides with* (C.1.2) *if* a *belongs to* $\mathbf{S}^m$*. Furthermore, for all* $s \in \mathbb{R}$*,* $\mathrm{Op}(a)$ *extends in a unique way into a bounded operator from* H^s *to* H^{s-m}*.*

This is a fundamental result, which we admit here. Its proof shows in particular that $\mathscr{F}(\mathrm{Op}(a)\,u)$ is compactly supported.

Remark C.7 The set Σ_0^m is strictly bigger than $\mathbf{S}^m$, as those functions $a \in \Sigma_0^m$ only satisfy a weakened version of (C.1.1), namely

$$|\,\partial_x^\alpha\,\partial_\xi^\beta\,a(x,\xi)\,| \;\leq\; C_{\alpha,\beta}\,\lambda^{m-|\beta|+|\alpha|}(\xi)\,. \tag{C.4.47}$$

This estimate is a consequence of what is known as Berstein's Lemma, in fact the special, easy case stated below.

Lemma C.3 *If* $u \in L^\infty$ *and* $\mathrm{Supp}\,\widehat{u} \subset B(0;\varepsilon\lambda)$ *with* $\varepsilon > 0$ *and* $\lambda > 0$*, then for all d-uple* α*, there exists* $C_{\varepsilon,\alpha} > 0$ *(independent of* λ*) so that*

$$\|\partial^\alpha u\|_{L^\infty} \;\leq\; C_{\varepsilon,\alpha}\,\lambda^{|\alpha|}\,\|u\|_{L^\infty}\,.$$

Proof Just write

$$\widehat{u}(\xi) \;=\; \varphi_\varepsilon(\xi/\lambda)\,\widehat{u}(\xi),$$

with φ_ε a smooth compactly supported function such that $\varphi_\varepsilon \equiv 1$ on $B(0;\varepsilon)$, and proceed with a basic convolution estimate like in the proof of Proposition C.3. □

Let us now describe the smoothing procedure for symbols in Γ_k^m, which amounts to a frequency cut-off depending on the ξ-variable.

Definition C.6 *A* $\mathscr{C}^\infty$ *function* $\chi : (\eta,\xi) \mapsto \chi(\eta,\xi) \in \mathbb{R}^+$ *is called an* admissible frequency cut-off *if there exist* $\varepsilon_{1,2}$ *with* $0 < \varepsilon_1 < \varepsilon_2 < 1$ *so that*

$$\begin{cases} \chi(\eta,\xi) \;=\; 1\,, \text{ if } \;\|\eta\| \;\leq\; \varepsilon_1\,\|\xi\| \quad \text{and } \;\|\xi\| \;\geq\; 1\,,\\ \chi(\eta,\xi) \;=\; 0\,, \text{ if } \;\|\eta\| \;\geq\; \varepsilon_2\,\lambda^1(\xi) \text{ or } \;\|\xi\| \;\leq\; \varepsilon_2\,,\end{cases}$$

and if for all d-uples α *and* β *there exists* $C_{\alpha,\beta} > 0$ *so that*

$$|\,\partial_\eta^\alpha\,\partial_\xi^\beta\,\chi(\eta,\xi)\,| \;\leq\; C_{\alpha,\beta}\,\lambda^{-|\alpha|-|\beta|}(\xi)\,. \tag{C.4.48}$$

Example. If ψ and ϕ are as in the Littlewood–Paley decomposition of Section C.3, the function χ defined by

$$\chi(\eta,\xi) \;=\; \sum_{p\,\geq\,0}\,\psi(2^{2-p}\,\eta)\,\phi(2^{-p}\,\xi) \;=\; \sum_{p\,\geq\,0}\,\psi_{p-2}(\eta)\,\phi_p(\xi)$$

[1] The dual space of $\mathscr{C}^\infty$.

is an admissible frequency cut-off. Indeed, recall first that there are at most two indices p for which $\phi_p(\xi)$ is non-zero. So the sum is locally finite. Furthermore, recalling that

$$\operatorname{Supp}\psi_{p-2} \subset \{\,\eta\,;\ \|\eta\| \leq 2^{p-2}\,\} \quad \text{and} \quad \operatorname{Supp}\phi_p \subset \{\,\xi\,;\ 2^{p-1} \leq \|\xi\| \leq 2^{p+1}\,\},$$

it is easy to check that χ vanishes as requested with $\varepsilon_2 = 1/2$. Indeed, for $\|\xi\| \leq 1/2$, $\phi_p(\xi) = 0$ for all $p \geq 0$, which implies $\chi(\eta,\xi) = 0$ wherever η is. Additionally, for all (η,ξ) we have

$$\chi(\eta,\xi) = \sum_{p\,;\,\|\eta\|\leq 2^{p-2}} \psi_{p-2}(\eta)\,\phi_p(\xi)\,,$$

and therefore $\chi(\eta,\xi) = 0$ as soon as $\|\xi\| \leq 2\,\|\eta\|$. *A fortiori*, this means that $\chi(\eta,\xi) = 0$ for $\|\eta\| \geq \frac{1}{2}\,\lambda^1(\xi)$. On the 'contrary', since $\psi \equiv 1$ on the sphere of radius 1/2, if $\|\xi\| \geq 16\,\|\eta\|$ then $\psi_{p-2}(\eta) = 1$ for all $p \geq 0$ such that $\phi_p(\xi) \neq 0$. And if $\|\xi\| \geq 1$, $\psi(\xi) = 0$ and thus $\sum_{p\geq 0} \phi_p(\xi) = 1$. This shows that $\chi(\eta,\xi) = 1$ if $\|\xi\| \geq 16\,\|\eta\|$ and $\|\xi\| \geq 1$. So the first requirement on χ holds true with $\varepsilon_1 = 1/16$. The inequalities in (C.4.48)[2] are trivially satisfied for $\|\xi\| < 1/2$. Otherwise, for $\|\xi\| \geq 1/2$, let us rewrite

$$\chi(\eta,\xi) = \sum_{p\,;\,\|\xi\|\leq 2^{p+2}} \psi_{p-2}(\eta)\,\phi_p(\xi)\,,$$

hence

$$\partial_\eta^\alpha\,\partial_\xi^\beta\,\chi(\eta,\xi) = \sum_{p\,;\|\xi\|\leq 2^{p+2}} 2^{(2-p)|\alpha|-p|\beta|}\,\partial_\eta^\alpha\psi(2^{2-p}\,\eta)\,\partial_\xi^\beta\phi(2^{-p}\xi)\,.$$

For $1 \leq 2\|\xi\| \leq 2^{p+2}$ we have

$$2^{-p} \leq \frac{2}{\|\xi\|} \leq \frac{8}{\lambda^1(\xi)},$$

so, recalling that the sum is locally finite we find that

$$|\,\partial_\eta^\alpha\,\partial_\xi^\beta\,\chi(\eta,\xi)\,| \leq C\,2^{2|\alpha|}\,\Big(\frac{8}{\lambda^1(\xi)}\Big)^{-|\alpha|-|\beta|}\,\|\partial_\eta^\alpha\psi\,\partial_\xi^\beta\phi\|_{L^\infty}\,.$$

Proposition C.16 *Let χ be an admissible frequency cut-off according to Definition C.6 and consider the operator*

$$R^\chi\,:\,a \in \Gamma_k^m \mapsto \sigma \in \mathscr{C}^\infty\,;\ \sigma(\cdot,\xi) = K^\chi(\cdot,\xi) *_x a(\cdot,\xi)\,,$$

where the kernel K^χ is defined by

$$K^\chi(\cdot,\xi) = \mathscr{F}^{-1}(\chi(\cdot,\xi))\,.$$

[2]Observe that they mean χ belongs to $\mathbf{S}^0$ as a function of $2d$ variables.

Then R^χ maps into

$$\Sigma_k^m \,=\, \{\, a \in \Gamma_k^m\,;\, \mathrm{Supp}\,(\, \mathscr{F}(a(\cdot,\xi))\,)\,\subset\, B(0;\varepsilon_2\lambda^1(\xi))\,\}\,.$$

Furthermore, if $k \geq 1$, for all $a \in \Gamma_k^m$, $a \,-\, R^\chi(a)$ belongs to Γ_{k-1}^{m-1}.

In other words, the symbol $\sigma = R^\chi(a)$ is related to a in Fourier space by

$$\mathscr{F}(\sigma(\cdot,\xi)) \,=\, \chi(\cdot,\xi)\, \mathscr{F}(a(\cdot,\xi))$$

for all $\xi \in \mathbb{R}^d$. In particular, if a is independent of x, $\mathscr{F}(a(\cdot,\xi)) \,=\, a(\xi)\,\delta$ hence $\mathscr{F}(\sigma(\cdot,\xi)) \,=\, \chi(0,\xi)\, a(\xi)\,\delta$. So we see that if $\chi(0,\xi)$ were equal to 1 for all ξ we would have $\sigma \,=\, a$. This is not exactly the case[3], but σ and a differ by a compactly supported function of ξ. In terms of operators, this means that $\mathrm{Op}(\sigma)$ differs from the Fourier multiplier associated with a by an infinitely smoothing operator, which is harmless in terms of para-differential calculus.

Proof Take $a \in \Gamma_k^m$ and consider $\sigma = R^\chi(a)$. Since $\mathrm{Supp}\,\chi(\cdot,\xi) \subset B(0;\varepsilon_2\lambda^1(\xi))$, by construction

$$\mathrm{Supp}\,(\, \mathscr{F}(\sigma(\cdot,\xi))\,)\,\subset\, B(0;\varepsilon_2\lambda^1(\xi))\,.$$

The fact that σ belongs to Γ_k^m requires an L^1 estimate of the kernel K^χ, namely

$$\|\partial_\xi^\beta K^\chi(\cdot,\xi)\,\|_{L^1(\mathbb{R}^d)} \,\leq\, C_\beta\,\lambda^{-|\beta|}(\xi)\,,$$

which is left to the reader. Once we know this, we immediately obtain

$$\|\sigma(\cdot,\xi)\|_{L^\infty} \,\leq\, C_0\,\|a(\cdot,\xi)\|_{L^\infty} \,\leq\, \widetilde{C}_0\,\lambda^m(\xi),$$

since a belongs to Γ_k^m. The estimates of partial derivatives $\partial_x^\alpha\,\sigma$ then follow from the observation that

$$\partial_x^\alpha\, R^\chi(a) \,=\, R^\chi(\partial_x^\alpha a)\,.$$

Finally, the estimates of crossed partial derivatives $\partial_x^\alpha\,\partial_\xi^\beta\,\sigma$ use the bilinearity of $*$ and the Leibniz formula.

One may observe that for $\sigma \in \Sigma_k^m$ with the number ε in (C.4.46) less than $\varepsilon_1/2$, $R^\chi(\sigma)$ is 'almost' equal to σ. Indeed, since $\chi(\eta,\xi) \,=\, 1$ for $\|\eta\| \,\leq\, \varepsilon_1\,\|\xi\|$ and $\|\xi\| \,\geq 1$, (C.4.46) with $\varepsilon \,\leq \varepsilon_1/2$ implies

$$\mathscr{F}(R^\chi(\sigma)(\cdot,\xi)) \,=\, \mathscr{F}(\sigma(\cdot,\xi))$$

for $\|\xi\| \,\geq 1$. So in terms of operators it means that $\mathrm{Op}(R^\chi(\sigma) \,-\, \sigma)$ is infinitely smoothing.

For $a \in \Gamma_1^m$, let us show now that $a \,-\, R^\chi(a)$ belongs to Γ_0^{m-1}. By triangular inequality, we already know that

$$\|\partial_\xi^\beta\,(a \,-\, R^\chi(a))\|_{W^{1,\infty}} \,\leq\, C_\beta\,\lambda^{m-|\beta|}(\xi)$$

[3]This problem is, in fact, overcome when using para-differential calculus with a parameter, see Section C.5.

and we want to show that

$$\|\partial_\xi^\beta \, (a \,-\, R^\chi(a))\|_{L^\infty} \,\leq\, \widetilde{C}_\beta \, \lambda^{m-1-|\beta|}(\xi)\,.$$

For convenience, we shall denote $b \,=\, \partial_\xi^\beta \,(a \,-\, R^\chi(a))$. There is nothing to do for $\|\xi\| \,\leq\, 1$ since λ^1 is bounded on the unit ball! It is more delicate to obtain a bound of $\|b(\cdot,\xi)\|_{L^\infty}$ for $\|\xi\| \,>\, 1$. Littewood–Paley decomposition will be of help again. Indeed, by definition of R^χ we have

$$\mathscr{F}(b(\cdot,\xi)) \,=\, \partial_\xi^\beta \Big((\,1 \,-\, \chi(\cdot,\xi)\,)\,\mathscr{F}(a(\cdot,\xi)) \Big)\,,$$

which vanishes identically on $B(0;\varepsilon_1\,\|\xi\|)$ for $\|\xi\| \geq 1$. Therefore, recalling that $\Delta_q \,=\, \mathscr{F}^{-1}\,\phi_q\,\mathscr{F}$ with $\operatorname{Supp}\phi_q \,\subset\, B(0;2^{q+1})$, $\Delta_q(b(\cdot,\xi)) \,\equiv\, 0$ for q and ξ such that

$$\varepsilon_1\,\|\xi\| \,>\, 2^{q+1} \quad \text{and} \quad \|\xi\| \,\geq\, 1\,.$$

Consequently, when $\|\xi\| \,\geq\, 1$ the Littlewood–Paley decomposition of $b(\cdot,\xi)$ reads

$$b(\cdot,\xi) \,=\, \sum_{q\,;\,\varepsilon_1\,\|\xi\|\,\leq\,2^{q+1}} \Delta_q(b(\cdot,\xi))$$

and this sum is locally finite. Furthermore, by Corollary C.1,

$$\|\Delta_q(b(\cdot,\xi))\|_{L^\infty} \,\leq\, 2^{-q}\,\|\,b(\cdot,\xi)\,\|_{W^{1,\infty}}\,,$$

and for $1 \leq \|\xi\| \leq 2^{q+1}$ we have

$$2^{-q} \,\leq\, \frac{2}{\|\xi\|} \,\leq\, \frac{4}{\lambda^1(\xi)}\,.$$

This implies

$$\|\,b(\cdot,\xi)\,\|_{L^\infty} \,\leq\, C\,\frac{4}{\lambda^1(\xi)}\,\|\,b(\cdot,\xi)\,\|_{W^{1,\infty}} \,\leq\, 4\,C\,C_\beta\,\lambda^{m-|\beta|-1}(\xi)\,.$$

□

A straightforward consequence of Proposition C.16 is the following.

Corollary C.5 *If χ_1 and χ_2 are two admissible cut-off functions, for all $a \in \Gamma_1^m$, $R^{\chi_1}(a) \,-\, R^{\chi_2}(a)$ belongs to Γ_0^{m-1}.*

Definition C.7 *Let χ be an admissible frequency cut-off according to Definition C.6. To any symbol $a \in \Gamma_k^m$ we associate the so-called* para-differential operator*, said to be of order m,*

$$T_a^\chi \,:=\, \operatorname{Op}(R^\chi(a))\,.$$

In particular, Corollary C.5 shows that for $a \in \Gamma_1^m$, T_a^χ are unique *modulo* operators of order $m-1$.

Another interesting point is the following.

Remark C.8 If χ is constructed through Littlewood–Paley decomposition as explained above, and $k \geq 1$, for any function of x only, $a \in W^{k,\infty}$ viewed as a symbol in Γ^0_k, the operators T^χ_a coincide with the para-product operator T_a *up to an infinitely smoothing operator*. Indeed, if

$$\chi(\eta, \xi) \,=\, \sum_{p \geq 0} \psi(2^{2-p}\,\eta)\,\phi(2^{-p}\,\xi) \,=\, \sum_{p \geq 0} \psi_{p-2}(\eta)\,\phi_p(\xi)\,,$$

then

$$\mathscr{F}(R^\chi(a)(\cdot,\xi)) \,=\, (\sum_{|p|\leq 1} \psi_{p-2}(\cdot)\,\phi_p(\xi))\,\mathscr{F}(a) \,+\, \sum_{p\geq 2} \mathscr{F}(S_{p-2}(a))\,\phi_p(\xi)\,,$$

or equivalently

$$R^\chi(a)(x,\xi) \,=\, \sum_{|p|\leq 1} \mathscr{F}^{-1}(\,\psi_{p-2}\,\mathscr{F}(a)\,)(x)\,\phi_p(\xi) \,+\, \sum_{p\geq 2} S_{p-2}(a)(x)\,\phi_p(\xi)\,.$$

In terms of operators, this means that for all $u \in \mathscr{F}^{-1}(\mathcal{E}')$,

$$T^\chi_a u \,=\, \mathrm{Op}(b)\,u \,+\, T_a\,u,$$

where the last term is the usual para-product, while

$$b(x,\xi) \,=\, \sum_{|p|\leq 1} \mathscr{F}^{-1}(\,\psi_{p-2}\,\mathscr{F}(a)\,)(x)\,\phi_p(\xi)$$

satisfies (C.4.46) with $\varepsilon = 1/2$ and is compactly supported in ξ.

C.4.2 *Basic results*

We omit below the superscript χ, all results being valid for any admissible frequency cut-off χ.

Theorem C.16 *For all $a \in \Gamma^m_1$, the adjoint operator $(T_a)^*$ is of order m and $(T_a)^* \,-\, T_{a^*}$ is of order $m-1$.*

Theorem C.17 *For all $a \in \Gamma^m_1$ and $b \in \Gamma^n_1$, the product ab belongs to Γ^{m+n}_1 and $T_a \circ T_b \,-\, T_{a\,b}$ is a para-differential operator of order $m+n-1$, associated with a symbol in Γ^{m+n-1}_0. In particular, if the symbols a and b commute – for example, if at least one of the operators is scalar-valued – the commutator $[\,T_a\,,\,T_b\,]$ is of order $m+n-1$.*

Proposition C.17 *If $a \in \Gamma^{2m}_1$, there exists $C > 0$ such that for all $u \in H^m$,*

$$|\mathrm{Re}\,\langle T_a\,u\,,\,u\rangle| \,\leq\, C\,\|u\|^2_{H^m}\,.$$

Proof We have

$$|\mathrm{Re}\,\langle T_a\,u\,,\,u\rangle| \,=\, |\mathrm{Re}\,\langle \lambda^{-m}\,\widehat{T_a\,u}\,,\,\lambda^m\widehat{u}\rangle| \,\leq\, \|\,(\Lambda^{-m}\circ T_a)(u)\,\|_{L^2}\,\|\,\Lambda^m u\,\|_{L^2}\,.$$

Since $\| \Lambda^m u \|_{L^2} = \|u\|_{H^m}$, the final estimate follows from the fact that $\Lambda^{-m} \circ T_a$ is an operator of order $-m + 2m = m$, which is a consequence of Theorem C.17 and the fact that $T_{\lambda^m} - \Lambda^m$ is infinitely smoothing. □

Theorem C.18 (Gårding inequality) *If $a \in \Gamma_1^{2m}$ is such that for some positive α,*

$$a(x,\xi) + a(x,\xi)^* \geq \alpha\, \lambda^{2m}(\xi)\, I_N$$

(in the sense of Hermitian matrices) for all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, then there exists $C > 0$ so that for all $u \in H^m$,

$$\mathrm{Re}\, \langle\, T_a u\,,\, u\,\rangle \;\geq\; \frac{\alpha}{4}\, \|u\|_{H^m}^2 \;-\; C\, \|u\|_{H^{m-1/2}}^2\,. \tag{C.4.49}$$

One may also state a sharpened version of Gårding's inequality in this context, but for smoother symbols (at least $\mathscr{C}^2$ in x).

C.5 Para-differential calculus with a parameter

The final refinement in this overview of modern analysis tools concerns families of para-differential operators depending on one parameter, as extensions of pseudo-differential operators with parameter.

As in Section C.2, we denote

$$\lambda^{s,\gamma}(\xi) \;=\; (\gamma^2 \,+\, \|\xi\|^2)^{s/2}\,,$$

and define parameter-dependent symbols of limited regularity as follows.

Definition C.8 *For any real number m and any natural integer k, the set Γ_k^m consists of functions, $a : \mathbb{R}^d \times \mathbb{R}^d \times [1,+\infty) \to \mathbb{C}^{N\times N}$ that are $\mathscr{C}^\infty$ in ξ and such that for all d-uple β, there exists $C_\beta > 0$ so that for all $(\xi,\gamma) \in \mathbb{R}^d \times [1,+\infty)$,*

$$\|\, \partial_\xi^\beta\, a(\cdot,\xi,\gamma)\, \|_{W^{k,\infty}} \;\leq\; C_\beta\; \lambda^{m-|\beta|,\gamma}(\xi)\,. \tag{C.5.50}$$

The subset Σ_k^m is made of symbols $a \in \Gamma_k^m$ satisfying the spectral requirement

$$\mathrm{Supp}\,(\, \mathscr{F}(a(\cdot,\xi,\gamma))\,)\; \subset\; B(0;\varepsilon\lambda^{1,\gamma}(\xi)) \tag{C.5.51}$$

for some $\varepsilon \in (0,1)$ independent *of (ξ,γ).*

The analogue of Theorem C.15 is the following fundamental result.

Theorem C.19 *Any symbol $a \in \Sigma_0^m$ can be associated with a family of operators denoted by $\{\mathrm{Op}^\gamma(a)\}_{\gamma\geq 1}$, defined on temperate distributions with a compact spectrum by*

$$\begin{aligned} \mathrm{Op}^\gamma(a) \;:\; \mathscr{F}^{-1}(\mathcal{E}') &\longrightarrow \mathscr{C}_b^\infty \\ u \quad &\mapsto\; \mathrm{Op}^\gamma(a)\, u; (\mathrm{Op}^\gamma(a)\, u)(x) = \tfrac{1}{(2\pi)^d} \langle\, \mathrm{e}^{ix\cdot} a(x,\cdot,\gamma), \widehat{u} \rangle_{(\mathscr{C}^\infty,\mathcal{E}')}. \end{aligned}$$

This definition of $\mathrm{Op}^\gamma(a)$ *coincides with* (C.2.8) *if a belongs to* $\mathbf{S}^m$*. Furthermore, for all $s \in \mathbb{R}$ and $\gamma \geq 1$,* $\mathrm{Op}^\gamma(a)$ *extends in a unique way into a bounded operator* *from H^s to H^{s-m}, and there exists $C_s > 0$* independent *of γ and u so that*

$$\| \operatorname{Op}^\gamma(a)\, u \,\|_{H^{s-m}_\gamma} \;\le\; C_s \,\| \, u \,\|_{H^s_\gamma} \,.$$

The proof, which we omit here, makes use of a parameter version of Littlewood–Paley decomposition, based on cut-off functions in the (ξ, γ)-space. Namely, taking $\psi \in \mathscr{D}(\mathbb{R}^d \times \mathbb{R})$ with $\psi(\xi, \gamma) \;=\; \Psi((\gamma^2 \,+\, \|\xi\|^2)^{1/2})$ and Ψ monotically decaying such that

$$\Psi(r) \;=\; 1 \text{ if } \quad r \,\le\, 1/2\,, \qquad \Psi(r) \;=\; 0 \text{ if } \quad r \,\ge\, 1\,,$$

and denoting

$$\begin{aligned} &\psi^\gamma_q(\xi) = \psi(2^{-q}\,\xi, 2^{-q}\,\gamma)\,, \quad \phi(\xi,\gamma) \;:=\; \psi(\xi/2,\gamma/2) \,-\, \psi(\xi,\gamma), \\ &\phi^\gamma_q(\xi) = \phi(2^{-q}\,\xi, 2^{-q}\,\gamma)\,, \end{aligned}$$

we may define operators S^γ_q and Δ^γ_q of symbols, respectively, ψ^γ_q and ϕ^γ_q. Observing that $\Delta^\gamma_q = 0$ for $\gamma \ge 2^{q+1}$, and in particular $\Delta^\gamma_{-1} = 0$ for $\gamma \ge 1$, we easily check that

$$\sum_{p\ge 0} \Delta^\gamma_q \;=\; \mathrm{id}$$

in $\mathscr{S}'$. Furthermore, the analogue of Proposition C.4 for the standard H^s norm is the following for the H^s_γ norm.

Proposition C.18 *For all $s \in \mathbb{R}$, $u \in H^s(\mathbb{R})$ if and only if*

$$\sum_{p\ge 0} 2^{2ps} \|\Delta^\gamma_p u\|^2_{L^2} \;<\; \infty$$

for all $\gamma \ge 1$. In addition, there exists $C_s > 1$ so that

$$\frac{1}{C_s} \sum_{p\ge 0} \, 2^{2ps} \|\Delta^\gamma_p u\|^2_{L^2} \;\le\; \|u\|_{H^s_\gamma} \;\le\; C_s \sum_{p\ge 0} \, 2^{2ps} \|\Delta^\gamma_p u\|^2_{L^2}$$

for all $\gamma \ge 1$.

Knowing Theorem C.19, it is then possible to define a family of operators associated with all symbols a in Γ^m_k. The procedure is the same as in standard (that is, without parameter) para-differential calculus. The basic tool is a so-called admissible cut-off function.

Definition C.9 *A $\mathscr{C}^\infty$ function $\chi : (\eta, \xi, \gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times [1, +\infty) \mapsto \chi(\eta, \xi, \gamma) \in \mathbb{R}^+$ is termed* an admissible frequency cut-off *if there exist $\varepsilon_{1,2}$ with $0 \,<\, \varepsilon_1 \,<\, \varepsilon_2 \,<\, 1$ so that*

$$\begin{cases} \chi(\eta,\xi,\gamma) \;=\; 1\,, \text{ if } \; \|\eta\| \,\le\, \varepsilon_1\, \lambda^1(\xi,\gamma)\,, \\ \chi(\eta,\xi,\gamma) \;=\; 0\,, \text{ if } \; \|\eta\| \,\ge\, \varepsilon_2\, \lambda^1(\xi,\gamma)\,, \end{cases}$$

and if for all d-uples α and β there exists $C_{\alpha,\beta} > 0$ so that

$$|\,\partial_\eta^\alpha\,\partial_\xi^\beta\,\chi(\eta,\xi,\gamma)\,| \;\leq\; C_{\alpha,\beta}\,\lambda^{-|\alpha|-|\beta|}(\xi,\gamma)\,. \tag{C.5.52}$$

Example. If ψ and ϕ are as in the Littlewood–Paley decomposition with parameter described above, the function χ defined by

$$\chi(\eta,\xi,\gamma) \;=\; \sum_{p\,\geq\,0}\,\psi(2^{2-p}\,\eta,0)\;\phi(2^{-p}\,\xi,2^{-p}\,\gamma)$$

is an admissible frequency cut-off with $\varepsilon_1 = 1/16$ and $\varepsilon_2 = 1/2$. The verification is left to the reader.

We now continue the machinery of Section C.4.

Proposition C.19 *Let χ be an admissible frequency cut-off according to Definition C.9 and consider the operator*

$$R^\chi\;:\;a\;\in\;\Gamma_k^m\;\mapsto\;\sigma\;\in\;\mathscr{C}^\infty\;;\;\sigma(\cdot,\xi,\gamma)\;=\;K^\chi(\cdot,\xi,\gamma)\;*_x\;a(\cdot,\xi,\gamma)\,,$$

where the kernel K^χ is defined by

$$K^\chi(\cdot,\xi,\gamma)\;=\;\mathscr{F}^{-1}(\chi(\cdot,\xi,\gamma))\,.$$

Then R^χ maps into

$$\Sigma_k^m\;=\;\{\,a\in\Gamma_k^m\,;\;\mathrm{Supp}\,(\,\mathscr{F}(a(\cdot,\xi,\gamma))\,)\;\subset\;B(0;\varepsilon_2\lambda^{1,\gamma}(\xi))\,\}\,.$$

Furthermore, if $k\geq 1$, for all $a\in\Gamma_k^m$, $a\;-\;R^\chi(a)$ belongs to Γ_{k-1}^{m-1}.

Note: Since $\chi(0,\xi,\gamma)\;\equiv\;1$, $R^\chi(a)\;=\;a$ for all symbols a depending only on ξ.

Definition C.10 *If χ is an admissible frequency cut-off, to any symbol $a\in\Gamma_k^m$ we associate the family of para-differential operators $\{T_a^{\chi,\gamma}\}_{\gamma\geq 1}$ defined by*

$$T_a^{\chi,\gamma}\;:=\;\mathrm{Op}^\gamma(R^\chi(a))\,.$$

Remark C.9 If the symbol a is a function of x only, $a\;\in\;W^{k,\infty}$, it can be viewed as a symbol in Γ_k^0 and $T_a^{\chi,\gamma}u$ is a parameter version of the para-product of a and u. More precisely, if the cut-off function χ is based on the Littlewood–Paley decomposition with parameter in the way explained above, we have

$$T_a^{\chi,\gamma}u\;=\;\sum_{p\geq 0}\;S_{p-2}^0 a\;\Delta_p^\gamma u\,,$$

where $S_p^0\;:=\;\mathscr{F}^{-1}\,\psi_p^0\,\mathscr{F}$ with $\psi_p^0(\xi)\;:=\;\psi(2^{-p}\,\xi,0)\;=\;\Psi(2^{-p}\,\|\xi\|)$ (as in the standard Littlewood–Paley decomposition [4]).

For simplicity, we shall now omit the dependence on χ and just denote T_a^γ.

Remark C.10 For a symbol of the form

$$a(x,\xi)\;=\;p(\xi)\,b(x,\gamma)\,,$$

[4]Except for the definitions of S_{-2}, S_{-1}, which were taken to be zero in Section C.3.3

the regularized symbol is given by $R^\chi(a)(x,\xi,\gamma) = p(\xi)\, R^\chi(b)(x,\xi,\gamma)$. Applying this in particular to polynomials p, we see that for any d-uple α,

$$T_b^\gamma\, \partial^\alpha u = T^\gamma_{(i^{|\alpha|}\xi^\alpha\, b)}\, u\,.$$

It is important for the applications to be able to estimate the error done when replacing products by para-products. Such estimates are given in Theorem C.13 (and its corollary) for standard para-products. We have similar results for T_a^γ, which show that $a - T_a^\gamma$ is of order -1 as soon as a is Lipschitz.

Theorem C.20 *There exists $C > 0$ so that, for all $a \in W^{1,\infty}$ and $u \in L^2(\mathbb{R}^d)$, for all $\gamma \geq 1$,*

$$\gamma\, \|\, a\, u - T_a^\gamma\, u\,\|_{L^2} \leq C\, \|a\|_{W^{1,\infty}}\, \|u\|_{L^2}\,,$$

$$\|\, a\, \partial_j\, u - T_a^\gamma\, \partial_j\, u\,\|_{L^2} = \|\, a\, \partial_j\, u - T^\gamma_{i\,\xi_j\, a}\, u\,\|_{L^2} \leq C\, \|a\|_{W^{1,\infty}}\, \|u\|_{L^2}\,,$$

$$\|\, a\, u - T_a^\gamma\, u\,\|_{H^1_\gamma} \leq C\, \|a\|_{W^{1,\infty}}\, \|u\|_{L^2}\,.$$

Proof The first inequality is easy to show. The factor γ comes from the fact that $\Delta_q^\gamma u = 0$ for $\gamma \geq 2^{q+1}$. Indeed, the fact that a is Lipschitz implies, by Corollary C.1 for the standard Littlewood–Paley decomposition, that

$$\|\Delta_q^0 a\|_{L^\infty} \lesssim 2^{-q}\, \|a\|_{W^{1,\infty}}\,,$$

and therefore that the series $\sum \Delta_q a$ is normally convergent in L^∞. Take $u \in \mathscr{S}$. Then the series $\sum \Delta_p^\gamma u$ is normally convergent in L^2 and $u = \sum \Delta_p^\gamma u$. Therefore,

$$a\, u - T_a^\gamma\, u = \sum_{q\geq -1} \sum_{p\geq 0} \Delta_q^0 a\; \Delta_p^\gamma u - \sum_{p\geq 0} S^0_{p-2} a\; \Delta_p^\gamma u = \sum_{2^{q+3}\geq\gamma} \Delta_q^0 a\, S^\gamma_{q+3} u\,.$$

(Recall that $S_q^\gamma = 0$ for $\gamma \geq 2^q$.) Hence

$$\|a\, u - T_a^\gamma\, u\|_{L^2} \lesssim \sum_{2^{q+3}\geq\gamma} 2^{-q}\, \|a\|_{W^{1,\infty}}\, \|u\|_{L^2} \lesssim \frac{1}{\gamma}\, \|a\|_{W^{1,\infty}}\, \|u\|_{L^2}\,.$$

Here, we have used the fact that

$$\|\, S_q^\gamma u\,\| \lesssim \|u\|_{L^2}\,,$$

which comes from the definition of S_q^γ, a $L^1 - L^2$ convolution estimate and a uniform bound for $\|\mathscr{F}^{-1}(\psi_q^\gamma)\|_{L^1}$. The derivation of the latter bound comes from the observation that

$$\sup_{1\leq\,\gamma\leq 2^q} \|\mathscr{F}_\xi^{-1}(\psi_q^\gamma)\|_{L^1(\mathbb{R}^d)} \leq \|\mathscr{F}^{-1}_{(\gamma,\xi)}(\psi_q)\|_{L^1(\mathbb{R}\times\mathbb{R}^d)} = \|\mathscr{F}^{-1}_{(\gamma,\xi)}(\psi)\|_{L^1(\mathbb{R}\times\mathbb{R}^d)}\,.$$

We omit the proof of the second inequality, which is much trickier (see [38, 136]).

The third inequality is an easy consequence of the first two. Indeed, by definition,

$$\|f\|^2_{H^1_\gamma} \leq \gamma^2 \|f\|^2_{L^2} + \|\nabla f\|^2_{L^2},$$

for all $f \in H^1_\gamma$, and

$$\begin{aligned}\|\partial_j(a\,u - T^\gamma_a u)\|^2_{L^2} &\leq 3\,\|a\,\partial_j u - T^\gamma_a \partial_j u\|^2_{L^2} + 3\,\|(\partial_j a)\,u\|^2_{L^2} + 3\,\|T^\gamma_{\partial_j a} u\|^2_{L^2} \\ &\leq 3\,C^2\,\|a\|^2_{W^{1,\infty}}\,\|u\|^2_{L^2} + 3(1+C_0^2)\,\|\partial_j a\|^2_{L^\infty}\,\|u\|^2_{L^2},\end{aligned}$$

where C_0 comes from the basic estimate

$$\|T^\gamma_b u\|_{L^2} \leq C_0\,\|b\|_{L^\infty}\,\|u\|_{L^2}.$$

Therefore,

$$\begin{aligned}\|\,a\,u - T^\gamma_a u\,\|^2_{H^1_\gamma} &\leq \gamma^2\,\|\,a\,u - T^\gamma_a u\,\|^2_{L^2} + \sum_j \|\partial_j(a\,u - T^\gamma_a u)\|^2_{L^2} \\ &\leq (\,(1+3d)\,C^2 + 3d(1+C_0^2))\,\|a\|^2_{W^{1,\infty}}\,\|u\|^2_{L^2}.\end{aligned}$$

□

Other basic results, similar to those in pseudo-differential calculus with parameter, are the following.

Theorem C.21 *For all $a \in \Gamma^m_1$, the family of adjoint operators $\{(T^\gamma_a)^*\}_{\gamma\geq 1}$ is of order m and the family $\{(T_a)^* - T_{a^*}\}_{\gamma\geq 1}$ is of order (less than or equal to) $m-1$.*

Theorem C.22 *For all $a \in \Gamma^m_1$ and $b \in \Gamma^n_1$, the product $a\,b$ belongs to Γ^{m+n}_1 and the family $\{T^\gamma_a \circ T^\gamma_b - T^\gamma_{a\,b}\}_{\gamma\geq 1}$ is of order (less than or equal to) $m+n-1$.*

Theorem C.23 (Gårding inequality) *If $a \in \Gamma^{2m}_1$ is such that for some positive α,*

$$a(x,\xi,\gamma) + a(x,\xi,\gamma)^* \geq \alpha\,\lambda^{2m,\gamma}(\xi)\,I_N$$

(in the sense of Hermitian matrices) for all $(x,\xi,\gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times [1,+\infty)$, then there exists $\gamma_0 \geq 1$ so that for all $\gamma \geq \gamma_0$ and all $u \in H^m$,

$$\mathrm{Re}\,\langle\, T^\gamma_a u\,,\,u\,\rangle \geq \frac{\alpha}{4}\,\|u\|^2_{H^m_\gamma}. \tag{C.5.53}$$

Again, we omit the sharp form of Gårding's inequality, which is valid only for smoother symbols.

REFERENCES

[1] R. A. Adams. *Sobolev spaces*, volume 65 of *Pure and applied mathematics*. Academic Press, New York–London, 1975.

[2] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Comm. Pure Appl. Math.*, 17:35–92, 1964.

[3] S. Alinhac. Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Comm. Partial Differential Equations*, 14(2):173–230, 1989.

[4] S. Alinhac. Unicité d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Indiana Univ. Math. J.*, 38(2):345–363, 1989.

[5] S. Alinhac. Temps de vie des solutions régulières des équations d'Euler compressibles axisymétriques en dimension deux. *Invent. Math.*, 111(3):627–670, 1993.

[6] S. Alinhac. *Blowup for nonlinear hyperbolic equations*. Progress in Nonlinear Differential Equations and their Applications, 17. Birkhäuser Boston Inc., Boston, MA, 1995.

[7] S. Alinhac and P. Gérard. *Opérateurs pseudo-différentiels et théorème de Nash–Moser*. InterEditions, Paris, 1991.

[8] A. A. Barmin and S. A. Egorushkin. Stability of shock waves. *Adv. Mech.*, 15(1–2):3–37, 1992.

[9] S. Benzoni-Gavage. Stability of multi-dimensional phase transitions in a van der Waals fluid. *Nonlinear Anal.*, 31(1–2):243–263, 1998.

[10] S. Benzoni-Gavage. Stability of subsonic planar phase boundaries in a van der Waals fluid. *Arch. Ration. Mech. Anal.*, 150(1):23–55, 1999.

[11] S. Benzoni-Gavage, J.-F. Coulombel, and S. Aubert. Boundary conditions for Euler equations. *AIAA Journal*, 41(1):56–63, 2003.

[12] S. Benzoni-Gavage and H. Freistühler. Effects of surface tension on the stability of dynamical liquid-vapor interfaces. *Arch. Ration. Mech. Anal.*, 174(1):111–150, 2004.

[13] S. Benzoni-Gavage, F. Rousset, D. Serre, and K. Zumbrun. Generic types and transitions in hyperbolic initial-boundary value problems. *Proc. Royal Soc. Edinburgh, A*, 132:1073–1104, 2002.

[14] P. Bergeret. Classification of smooth solutions to 2×2 hyperbolic systems with boundary damping. *Math. Methods Appl. Sci.*, 20(18):1563–1598, 1997.

[15] J. Bergh and J. Löfström. *Interpolation spaces. An introduction*. Grundlehren der mathematischen Wissenschaften, Vol. 223. Springer-Verlag, Berlin, 1976.

[16] H. A. Bethe. On the theory of shock waves for an arbitrary equation of state [Rep. No. 545, Serial No. NDRC-B-237, Office Sci. Res. Develop., U. S. Army Ballistic Research Laboratory, Aberdeen Proving Ground, MD, 1942]. In *Classic papers in shock compression science*, pages 421–492. Springer, New York, 1998.

[17] A. Blokhin and Y. Trakhinin. Stability of strong discontinuities in fluids and MHD. In S. Friedlander and D. Serre, editors, *Handbook of mathematical fluid dynamics, Vol. I*, pages 545–652. North-Holland, Amsterdam, 2002.
[18] A. M. Blokhin. Estimation of the energy integral of a mixed problem for gas dynamics equations with boundary conditions on the shock wave. *Siberian Math. J.*, 22(4):501–523, 1982.
[19] G. Boillat and T. Ruggeri. Hyperbolic principal subsystems: Entropy convexity and sub-characteristic conditions. *Arch. Rational Mech. Anal.*, 137:305–320, 1997.
[20] J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. Ec. Norm. Super., IV. Ser.*, 14:209–246, 1981.
[21] Y. Brenier. Hydrodynamic structure of the augmented Born–Infeld equations. *Arch. Rational Mech. Anal.*, 172:65–91, 2004.
[22] P. Brenner. The Cauchy problem for symmetric hyperbolic systems in L_p. *Math. Scand.*, 19:27–37, 1966.
[23] P. Brenner. The Cauchy problem for systems in L_p and $L_{p,\alpha}$. *Ark. Mat.*, 11:75–101, 1973.
[24] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.
[25] A. Bressan and P. Goatin. Oleĭnik type estimates and uniqueness for $n \times n$ conservation laws. *J. Diff. Equations*, 156:26–49, 1999.
[26] A. Bressan and M. Lewicka. A uniqueness condition for hyperbolic systems of conservation laws. *Discrete Cont. Dyn. Syst.*, 6:673–682, 2000.
[27] A. Bressan, Tai-Ping Liu, and Tong Yang. L^1-stability estimates for $n \times n$ conservation laws. *Arch. Rat. Mech. Anal.*, 149:1–22, 1999.
[28] H. Brezis. *Analyse fonctionnelle. Théorie et applications*. Masson, Paris, 1983.
[29] L. Caffarelli, L Nirenberg, and J. Spruck. The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian. *Acta Math.*, 155:261–271, 1985.
[30] S. Čanić, B. L. Keyfitz, and Eun Heui Kim. Free boundary problems for the unsteady transonic small disturbance equation: transonic regular reflection. *Methods Appl. Anal.*, 7(2):313–335, 2000.
[31] J. Chazarain and A. Piriou. *Introduction to the Theory of Linear Partial Differential Equations*. North-Holland Publishing Co., Amsterdam, 1982. Translated from the French.
[32] J.-Y. Chemin. Dynamique des gaz à masse totale finie. *Asymptotic Anal.*, 3(3):215–220, 1990.
[33] J.-Y. Chemin. *Perfect incompressible fluids*. The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.
[34] Gui-Qiang Chen and M. Feldman. Free boundary problems and transonic shocks for the Euler equations in unbounded domains. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 3(4):827–869, 2004.
[35] Shui-Nee Chow and J. K. Hale. *Methods of bifurcation theory*. Grundlehren der mathematischen Wissenschaften, Vol. 251. Springer-Verlag, Heidelberg, 1982.

[36] P. G. Ciarlet. *Mathematical elasticity. Vol. I. Three-dimensional elasticity.* North-Holland Publishing Co., Amsterdam, 1988.
[37] B. D. Coleman and E. H. Dill. Thermodynamic restrictions on the constitutive equations of electromagnetic theory. *Z. Angew. Math. Phys.*, 22:691–702, 1971.
[38] J.-F. Coulombel. *Stabilité multidimensionnelle d'interfaces dynamiques; application aux transitions de phase liquide-vapeur.* PhD thesis, École normale supérieure de Lyon, 2002.
[39] J.-F. Coulombel. Weak stability of non uniformly stable multidimensional shocks. *SIAM J. Math. Analysis*, 34(1):142–172, 2002.
[40] J.-F. Coulombel. Stability of multidimensional undercompressive shock waves. *Interfaces Free Bound.*, 5(4):360–390, 2003.
[41] J.-F. Coulombel. Weakly stable multidimensional shocks. *Ann. Inst. H. Poincaré (Analyse non linéaire)*, 21:401–443, 2004.
[42] J.-F. Coulombel, S. Benzoni-Gavage, and D. Serre. Note on a paper 'Shock wave instability and the carbuncle phenomenon: same intrinsic origin?' [J. Fluid Mech. **417** (2000), 237–263], by J.-Ch. Robinet, J. Gressier, G. Casalis and J.-M. Moschetta. *J. Fluid Mech.*, 469:401–405, 2002.
[43] J.-F. Coulombel and P. Secchi. The stability of compressible vortex sheets in two space dimensions. *Indiana Univ. Math. J.*, 53:941–1012, 2004.
[44] J.-F. Coulombel and P. Secchi. Nonlinear compressible vortex sheets in two space dimensions. *Preprint*, 2005.
[45] C. M. Dafermos. Quasilinear hyperbolic systems with involutions. *Arch. Rational Mech. Anal.*, 94(4):373–389, 1986.
[46] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics.* Springer-Verlag, Berlin, 2000.
[47] S. R. de Groot and P. Mazur. *Nonequilibrium thermodynamics.* Dover Publications Inc., New York, 1984. Reprint of the 1962 original.
[48] S. Demoulini, D. Stuart, and A. Tzavaras. A variational approximation scheme for three-dimensional elastodynamics with polyconvex energy. *Arch. Rat. Mech. Anal.*, 157:325–344, 2001.
[49] R. J. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rat. Mech. Anal.*, 82:27–70, 1983.
[50] R. J. DiPerna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, 91(1):1–30, 1983.
[51] N. Dunford and J. T. Schwartz. *Linear operators. Part I. General theory.* John Wiley & Sons Inc., New York, 1988. With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original.
[52] S. P. D′yakov. On the stability of shock waves. *Ž. Eksper. Teoret. Fiz.*, 27:288–295, 1954.
[53] J. J. Erpenbeck. Stability of step shocks. *Phys. Fluids*, 5:1181–1187, 1962.
[54] Haitao Fan and M. Slemrod. Dynamic flows with liquid/vapor phase transitions. In S. Friedlander and D. Serre, editors, *Handbook of mathematical fluid dynamics, Vol. I*, pages 373–420. North-Holland, Amsterdam, 2002.
[55] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.*, 3(4):358–392, 2001.

[56] J. Francheteau and G. Métivier. Existence de chocs faibles pour des systèmes quasi-linéaires hyperboliques multidimensionnels. *Astérisque*, 268:viii+198, 2000.

[57] M. Frazier, B. Jawerth, and G. Weiss. *Littlewood–Paley theory and the study of function spaces*, volume 79 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1991.

[58] H. Freistühler. The persistence of ideal shock waves. *Appl. Math. Lett.*, 7(6):7–11, 1994.

[59] H. Freistühler. A short note on the persistence of ideal shock waves. *Arch. Math. (Basel)*, 64(4):344–352, 1995.

[60] H. Freistühler. Some results on the stability of non-classical shock waves. *J. Partial Differential Equations*, 11(1):25–38, 1998.

[61] H. Freistühler and Tai-Ping Liu. Nonlinear stability of overcompressive shock waves in a rotationally invariant system of viscous conservation laws. *Comm. Math. Phys.*, 153(1):147–158, 1993.

[62] S. Friedland, J. W. Robbin, and J. H. Sylvester. On the crossing rule. *Comm. Pure Appl. Math.*, 37:19–37, 1984.

[63] K. O. Friedrichs. Symmetric hyperbolic linear differential equations. *Comm. Pure Appl. Math.*, 7:345–392, 1954.

[64] E. Gagliardo. Ulteriori proprietà di alcune classi di funzioni in più variabili. *Ricerche Mat.*, 8:24–51, 1959.

[65] L. Gårding. Linear hyperbolic partial differential equations with constant coefficients. *Acta Math.*, 85:1–62, 1951.

[66] L. Gårding. An inequality for hyperbolic polynomials. *J. Math. Mech.*, 8:957–965, 1959.

[67] L. Gårding. Problème de Cauchy pour les systèmes quasi-linéaires d'ordre un strictement hyperboliques. In *Les Équations aux Dérivées Partielles (Paris, 1962)*, pages 33–40. Éditions du Centre National de la Recherche Scientifique, Paris, 1963.

[68] P. Gérard and J. Rauch. Propagation de la régularité locale de solutions d'équations hyperboliques non linéaires. *Ann. Inst. Fourier (Grenoble)*, 37(3):65–84, 1987.

[69] D. Gilbarg. The existence and limit behavior of the one-dimensional shock layer. *Amer. J. Math.*, 73:256–274, 1951.

[70] J. Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.*, 18:697–715, 1965.

[71] S. K. Godunov. Lois de conservation et intégrales d'énergie des équations hyperboliques. In *Nonlinear hyperbolic problems (St. Etienne, 1986)*, volume 1270 of *Lecture Notes in Math.*, pages 135–149. Springer, Berlin, 1987.

[72] S. K. Godunov. *Ordinary differential equations with constant coefficients*, volume 169 of *Translations of Math. Monographs*. Amer. Math. Soc., Providence, 1997. Translated from the 1994 Russian original.

[73] H. H. Goldstine and J. von Neumann. Blast wave calculation. *Comm. Pure Appl. Math.*, 8:327–353, 1955.

[74] M. Grassin. Global smooth solutions to Euler equations for a perfect gas. *Indiana Univ. Math. J.*, 47(4):1397–1432, 1998.

[75] G. Guderley. Starke kugelige und zylindrische Verdichtungsstösse in der Nähe des Kugelmittelpunktes bzw. der Zylinderachse. *Luftfahrtforschung*, 19:302–311, 1942.
[76] O. Guès. Problème mixte hyperbolique quasi-linéaire caractéristique. *Comm. Partial Differential Equations*, 15(5):595–645, 1990.
[77] O. Guès, G. Métivier, M. Williams, and K. Zumbrun. Boundary layer and long time stability for multidimensional viscous shocks. *Discrete Contin. Dyn. Syst.*, 11(1):131–160, 2004.
[78] O. Guès, G. Métivier, M. Williams, and K. Zumbrun. Existence and stability of multidimensional shock fronts in the vanishing viscosity limit. *Arch. Ration. Mech. Anal.*, 175(2):151–244, 2005.
[79] O. Guès, G. Métivier, M. Williams, and K. Zumbrun. Multidimensional viscous shocks. I. Degenerate symmetrizers and long time stability. *J. Amer. Math. Soc.*, 18(1):61–120, 2005.
[80] M. E. Gurtin. *An introduction to continuum mechanics*. Academic Press Inc., New York, 1981.
[81] J. Harris. *Algebraic geometry. A first course*. Springer-Verlag, New York, 1992.
[82] J. W. Helton and V. Vinnikov. Linear matrix inequality representation of sets. Tech. report, Math. Depart., UCSD, 2002.
[83] R. Hersh. Mixed problems in several variables. *J. Math. Mech.*, 12:317–334, 1963.
[84] R. L. Higdon. Initial-boundary value problems for linear hyperbolic systems. *SIAM Rev.*, 28(2):177–217, 1986.
[85] M. W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.
[86] L. Hörmander. Pseudo-differential operators and non-elliptic boundary problems. *Ann. of Math. (2)*, 83:129–209, 1966.
[87] L. Hörmander. *The analysis of linear partial differential operators. III. Pseudo-differential operators*. Springer-Verlag, Berlin, 1994. Corrected reprint of the 1985 original.
[88] L. Hörmander. *Lectures on nonlinear hyperbolic differential equations*. Springer-Verlag, Berlin, 1997.
[89] S. V. Iordanskiĭ. On the stability of a plane stationary shock wave. *Prikl. Mat. Meh.*, 21:465–472, 1957.
[90] S. Jaouen. *Étude mathématique et numérique de stabilité pour des modèles hydrodynamiques avec transition de phase*. PhD thesis, Université Pierre et Marie Curie (Paris VI), 2001.
[91] J. Jena and V. D. Sharma. Self-similar shocks in a dusty gas. *Internat. J. Non-Linear Mech.*, 34(2):313–327, 1999.
[92] K. Jenssen and G. Lyng. The Lopatinski determinant for multi-dimensional Euler equations. In S. Friedlander and D. Serre, editors, *Handbook of mathematical fluid dynamics. Vol. III*, Amsterdam, 2004. North-Holland.
[93] K. Kasahara and M. Yamaguti. Strongly hyperbolic systems of linear partial differential equations with constant coefficients. *Mem. Coll. Sci. Univ. Kyoto Ser. A. Math.*, 33:1–23, 1960/1961.
[94] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, 58(3):181–205, 1975.
[95] T. Kato. *Perturbation theory for linear operators*. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[96] S. Klainerman. A commuting vectorfields approach to Strichartz-type inequalities and applications to quasi-linear wave equations. *Internat. Res. Math. Notices*, 5:221–274, 2001.

[97] S. Klainerman and A. Majda. Formation of singularities for wave equations including the nonlinear vibrating string. *Comm. Pure Appl. Math.*, 33(3):241–263, 1980.

[98] A. Klyachko. Stable bundles, representation theory and Hermitian operators. *Selecta Math. (N. S.)*, 4(3):419–445, 1998.

[99] A. Knutson and T. Tao. The honeycomb model of $GL_n(\mathbb{C})$ tensor products. I. Proof of the saturation conjecture. *J. Amer. Math. Soc.*, 12(4):1055–1090, 1999.

[100] K. Kojima and M. Taniguchi. Mixed problem for hyperbolic equations in a domain with a corner. *Funkcial. Ekvac.*, 23(2):171–195, 1980.

[101] V. M. Kontorovič. Stability of shock waves in relativistic hydrodynamics. *Soviet Phys. JETP*, 34 (7):127–132, 1958.

[102] H.-O. Kreiss. Über Matrizen die beschränkte Halbgruppen erzeugen. *Math. Scand.*, 7:71–80, 1959.

[103] H.-O. Kreiss. Initial boundary value problems for hyperbolic systems. *Comm. Pure Appl. Math.*, 23:277–298, 1970.

[104] H.-O. Kreiss and J. Lorenz. *Initial-boundary value problems and the Navier–Stokes equations.* Academic Press Inc., Boston, MA, 1989.

[105] S. Kružkov. Generalized solutions of the Cauchy problem in the large for nonlinear equations of first order. *Dokl. Akad. Nauk SSSR*, 187:29–32, 1969.

[106] L. D. Landau and E. M. Lifshitz. *Fluid mechanics.* Translated from the Russian by J. B. Sykes and W. H. Reid. Course of Theoretical Physics, Vol. 6. Pergamon Press, London, 1959.

[107] D. Lannes. Dispersive effects for nonlinear geometrical optics with rectification. *Asymptotic Analysis*, 18:111–146, 1998.

[108] J. Laurens and H. Le Ferrand. Systèmes hyperboliques d'équations aux dérivées partielles linéaires: régularité $L^2_{\rm loc}$ et matrices diagonalisables. *C. R. Acad. Sci. Paris Sér. I*, 332:311–314, 2001.

[109] P. D. Lax. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[110] P. D. Lax. Differential equations, difference equations and matrix theory. *Comm. Pure Appl. Math.*, 11:175–194, 1958.

[111] P. D. Lax. The multiplicity of eigenvalues. *Bull. Amer. Math. Soc.*, 6:213–214, 1982.

[112] P. D. Lax and L. Nirenberg. On stability for difference schemes: A sharp form of Gårding's inequality. *Comm. Pure Appl. Math.*, 19:473–492, 1966.

[113] P. G. LeFloch. *Hyperbolic systems of conservation laws.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2002. The theory of classical and nonclassical shock waves.

[114] A. S. Lewis, P. A. Parrilo, and M. V. Ramana. The Lax conjecture is true. *Proceedings of the Amer. Math. Soc.*, 133(9):2495–2499, 2005.

[115] Ta Tsien Li. *Global classical solutions for quasilinear hyperbolic systems*, volume 32 of *RAM: Research in Applied Mathematics.* Masson, Paris, 1994.

[116] Ta Tsien Li and Wen Ci Yu. *Boundary value problems for quasilinear hyperbolic systems.* Duke University Mathematics Department, Durham, N.C., 1985.

[117] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 2. Compressible models.* The Clarendon Press Oxford University Press, New York, 1998. Oxford Science Publications.

[118] Tai Ping Liu. The Riemann problem for general systems of conservation laws. *J. Differential Equations*, 18:218–234, 1975.

[119] Tai Ping Liu. The entropy condition and the admissibility of shocks. *J. Math. Anal. Appl.*, 53(1):78–88, 1976.

[120] Tai Ping Liu. Nonlinear stability and instability of overcompressive shock waves. In J. E. Dunn, R. Fosdick, and M. Slemrod, editors, *Shock induced transitions and phase structures in general media*, volume 52 of *IMA Vol. Math. Appl.*, pages 159–167. Springer, New York, 1993.

[121] Ya. B. Lopatinskiĭ. On a method of reducing boundary problems for a system of differential equations of elliptic type to regular integral equations (Russian). *Ukrain. Mat. Ž.*, 5:123–151, 1953.

[122] Ya. B. Lopatinskiĭ. The mixed Cauchy–Dirichlet type problem for equations of hyperbolic type. *Dopovīdī Akad. Nauk Ukraïn. RSR Ser. A*, 1970:592–594, 668, 1970.

[123] Ya. B. Lopatinskiĭ. *Teoriya obshchikh granichnykh zadach.* ‘Naukova Dumka’, Kiev, 1984.

[124] A. Majda. The existence of multidimensional shock fronts. *Mem. Amer. Math. Soc.*, 43(281):v+93, 1983.

[125] A. Majda. The stability of multidimensional shock fronts. *Mem. Amer. Math. Soc.*, 41(275):iv+95, 1983.

[126] A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables.* Springer-Verlag, New York, 1984.

[127] A. Majda and S. Osher. Initial-boundary value problems for hyperbolic equations with uniformly characteristic boundary. *Comm. Pure Appl. Math.*, 28(5):607–675, 1975.

[128] T. Makino, S. Ukai, and S. Kawashima. Sur la solution à support compact de l’équation d’Euler compressible. *Japan J. Appl. Math.*, 3(2):249–257, 1986.

[129] L. Mencherini and S. Spagnolo. Uniformly symmetrizable 3×3 matrices. *Linear Algebra and its Applications*, 382:25–38, 2004.

[130] R. Menikoff and B. J. Plohr. The Riemann problem for fluid flow of real materials. *Rev. Mod. Phys.*, 61(1):75–130, 1989.

[131] G. Métivier. Interaction de deux chocs pour un système de deux lois de conservation, en dimension deux d’espace. *Trans. Amer. Math. Soc.*, 296(2):431–479, 1986.

[132] G. Métivier. Problèmes de Cauchy et ondes non linéaires. In *Journées “Équations aux dérivées partielles” (Saint Jean de Monts, 1986)*, volume No. I, 29. École Polytech., Palaiseau, 1986.

[133] G. Métivier. Stability of multi-dimensional weak shocks. *Comm. Partial Differential Equations*, 15(7):983–1028, 1990.

[134] G. Métivier. The block structure condition for symmetric hyperbolic systems. *Bull. London Math. Soc.*, 32(6):689–702, 2000.

[135] G. Métivier and K. Zumbrun. Hyperbolic boundary value problems for symmetric systems with variable multiplicities. *J. Differential Equations*, 211(1):61–134, 2005.

[136] Guy Métivier. Stability of multidimensional shocks. In *Advances in the theory of shock waves*, volume 47 of *Progr. Nonlinear Differential Equations Appl.*, pages 25–103. Birkhäuser Boston, Boston, MA, 2001.

[137] Y. Meyer. Régularité des solutions des équations aux dérivées partielles non linéaires (d'après J.-M. Bony). In *Bourbaki Seminar, Vol. 1979/80*, pages 293–302. Springer, Berlin, 1981.

[138] Y. Meyer. Remarques sur un théorème de J.-M. Bony. In *Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980)*, volume Suppl. 1, pages 1–20, 1981.

[139] S. Miyatake. Mixed problems for hyperbolic equations of second order with first order complex boundary operators. *Japan. J. Math. (N.S.)*, 1(1):109–158, 1975/76.

[140] A. Mokrane. *Problèmes mixtes hyperboliques non linéaires.* PhD thesis, Université de Rennes I, 1987.

[141] A. Morando and D. Serre. On the L^2 well-posedness of an initial boundary value problem for the 3-D linear elasticity. *Commun. Math. Sci.*, 3:575–586, 2005.

[142] A. Morando and D. Serre. A result of L^2 well-posedness concerning the system of elasticity in 2-D. *Commun. Math. Sci.*, 3:317–334, 2005.

[143] C. B. Morrey, Jr. *Multiple integrals in the calculus of variations.* Springer-Verlag New York, Inc., New York, 1966.

[144] J. Moser. A rapidly convergent iteration method and non-linear partial differential equations. I. *Ann. Scuola Norm. Sup. Pisa (3)*, 20:265–315, 1966.

[145] F. Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5(3):489–507, 1978.

[146] M. Nagase. A new proof of sharp Gårding inequality. *Funkcial. Ekvac.*, 20(3):259–271, 1977.

[147] R. Natalini. Recent results on hyperbolic relaxation problems. In *Analysis of systems of conservation laws (Aachen, 1997)*, pages 128–198. Chapman & Hall/CRC, Boca Raton, FL, 1999.

[148] L. Nirenberg. On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa (3)*, 13:115–162, 1959.

[149] T. Nishitani and M. Takayama. Characteristic initial-boundary value problems for symmetric hyperbolic systems. *Osaka J. Math.*, 35(3):629–657, 1998.

[150] T. Ohkubo. Regularity of solutions to hyperbolic mixed problems with uniformly characteristic boundary. *Hokkaido Math. J.*, 10:93–123, 1981.

[151] T. Ohkubo. Well-posedness for quasi-linear hyperbolic mixed problems with characteristic boundary. *Hokkaido Math. J.*, 18:79–123, 1989.

[152] J. Oliger and A. Sundström. Theoretical and practical aspects of some initial boundary value problems in fluid dynamics. *SIAM J. Appl. Math.*, 25 (3):419–446, 1978.

[153] S. Osher. Initial-boundary value problems for hyperbolic systems in regions with corners. I. *Trans. Amer. Math. Soc.*, 176:141–164, 1973.

[154] S. Osher. Initial-boundary value problems for hyperbolic systems in regions with corners. II. *Trans. Amer. Math. Soc.*, 198:155–175, 1974.

[155] Y. Oshime. Canonical forms of 3×3 strongly hyperbolic systems with real constant coefficients. *J. Math. Kyoto Univ.*, 31(4):937–982, 1991.

[156] A. Pazy. *Semigroups of linear operators and applications to partial differential equations.* Springer-Verlag, New York, 1983.

[157] R. L. Pego. Nonexistence of a shock layer in gas dynamics with a nonconvex equation of state. *Arch. Rational Mech. Anal.*, 94(2):165–178, 1986.

[158] I. Petrowsky. Über das Cauchysche Problem für Systeme von partiellen Differentialgleichungen. *Rec. Math. Moscou, n. Ser.*, 2:815–868, 1937.

[159] T. J. Poinsot and S. K. Lele. Boundary conditions for direct simulations of compressible viscous flows. *J. Comput. Phys.*, 101(1):104–129, 1992.

[160] J. V. Ralston. Note on a paper of Kreiss. *Comm. Pure Appl. Math.*, 24(6):759–762, 1971.

[161] J. Rauch. *Energy inequalities for hyperbolic initial boundary value problems.* PhD thesis, New York University, 1971.

[162] J. Rauch. L^2 is a continuable initial condition for Kreiss' mixed problems. *Comm. Pure Appl. Math.*, 25:265–285, 1972.

[163] J. Rauch. Symmetric positive systems with boundary characteristic of constant multiplicity. *Trans. Amer. Math. Soc.*, 291(1):167–187, 1985.

[164] J. Rauch. Boundary value problems with nonuniformly characteristic boundary. *J. Math. Pures Appl. (9)*, 73(4):347–353, 1994.

[165] J. B. Rauch and F. Massey. Differentiability of solutions to hyperbolic initial-boundary value problems. *Trans. Amer. Math. Soc.*, 189:303–318, 1974.

[166] J.-Ch. Robinet, J. Gressier, G. Casalis, and J.-M. Moschetta. Shock wave instability and the carbuncle phenomenon: same intrinsic origin? *J. Fluid Mech.*, 417:237–263, 2000.

[167] E. I. Romensky. Hyperbolic systems of thermodynamically compatible conservation laws in continuum mechanics. *Math. Comput. Modelling*, 28(10):115–130, 1998.

[168] J. S. Rowlinson and B. Widom. *Molecular theory of capillarity.* Oxford University Press, 1982.

[169] W. Rudin. *Real and complex analysis.* McGraw–Hill, New York, 1966.

[170] M. Sablé-Tougeron. Existence pour un problème de l'élastodynamique Neumann non linéaire en dimension 2. *Arch. Rational Mech. Anal.*, 101(3):261–292, 1988.

[171] R. Sakamoto. Mixed problems for hyperbolic equations. I. Energy inequalities. *J. Math. Kyoto Univ.*, 10:349–373, 1970.

[172] R. Sakamoto. Mixed problems for hyperbolic equations. II. Existence theorems with zero initial datas and energy inequalities with initial datas. *J. Math. Kyoto Univ.*, 10:403–417, 1970.

[173] R. Sakamoto. L^2-well-posedness for hyperbolic mixed problems. *Publ. Res. Inst. Math. Sci.*, 8:265–293, 1972.

[174] R. Sakamoto. *Hyperbolic boundary value problems.* Cambridge University Press, Cambridge, 1982. Translated from the Japanese by Katsumi Miyahara.

[175] L. Sarason. Hyperbolic and other symmetrizable systems in regions with corners and edges. *Indiana Univ. Math. J.*, 26(1):1–39, 1977.

[176] P. Secchi. The initial-boundary value problem for linear symmetric hyperbolic systems with characteristic boundary of constant multiplicity. *Differential Integral Equations*, 9(4):671–700, 1996.

[177] P. Secchi. Well-posedness of characteristic symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, 134(2):155–197, 1996.

[178] P. Secchi. A symmetric positive system with nonuniformly characteristic boundary. *Differential Integral Equations*, 11(4):605–621, 1998.

[179] P. Secchi. Full regularity of solutions to a nonuniformly characteristic boundary value problem for symmetric positive systems. *Adv. Math. Sci. Appl.*, 10(1):39–55, 2000.

[180] L. I. Sedov. *Metody podobiya i razmernosti v mehanike*. Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, Leningrad, 1951.

[181] L. I. Sedov. *Similarity and dimensional methods in mechanics*. Mir, Moscow, 1982. Translated from the Russian by V. I. Kisin.

[182] D. Serre. Écoulements de fluides parfaits en deux variables indépendantes de type espace. Réflexion d'un choc plan par un dièdre compressif. *Arch. Rational Mech. Anal.*, 132(1):15–36, 1995.

[183] D. Serre. Solutions classiques globales des équations d'Euler pour un fluide parfait compressible. *Ann. Inst. Fourier (Grenoble)*, 47(1):139–153, 1997.

[184] D. Serre. *Systems of conservation laws. Vol. I. Hyperbolicity, entropies, shock waves*. Cambridge University Press, Cambridge, 1999. Translated from the 1996 French original by I. N. Sneddon.

[185] D. Serre. *Systems of conservation laws. Vol. II. Geometric structures, oscillations, and initial-boundary value problems*. Cambridge University Press, Cambridge, 2000. Translated from the 1996 French original by I. N. Sneddon.

[186] D. Serre. La transition vers l'instabilité pour les ondes de choc multi-dimensionnelles. *Transactions of the A. M. S.*, 353:5071–5093, 2001.

[187] D. Serre. *Matrices: Theory and Applications*. Graduate Text in Mathematics 216. Springer-Verlag, New York, 2002.

[188] D. Serre. Hyperbolicity of the non-linear models of Maxwell's equations. *Arch. Rational Mech. Anal.*, 172:309–331, 2004.

[189] D. Serre. Second-order initial-boundary value problems of variational type. *J. Funct. Anal.*, 236:409–446, 2006.

[190] D. Serre. Second-order initial-boundary value problems of variational type: The incompressible case. *Rend. Circ. Mat. Palermo. Serie II*, Suppl. 78:285–312, 2006.

[191] D. Serre. Solvability of BVPs for the wave operator through filtering. *Methods and Appl. of Anal.*, 12:253–266, 2005.

[192] Y. Shizuta. On the final form of the regularity theorem for solutions to the characteristic initial boundary value problem for symmetric hyperbolic systems. *Proc. Japan Acad. Ser. A Math. Sci.*, 76(4):47–50, 2000.

[193] Y. Shizuta and Y. Tanaka. On a refinement of the regularity theorem for solutions to the characteristic initial boundary value problem for linear symmetric hyperbolic systems. *Proc. Japan Acad. Ser. A Math. Sci.*, 76(3):31–34, 2000.

[194] T. Sideris. Formation of singularities in three-dimensional compressible fluids. *Comm. Math. Phys.*, 101(4):475–485, 1985.

[195] M. Šilhavý. *The mechanics and thermodynamics of continuous media*. Springer-Verlag, Berlin, 1997.

[196] M. Slemrod. Admissibility criteria for propagating phase boundaries in a van der Waals fluid. *Arch. Rational Mech. Anal.*, 81:301–315, 1983.

[197] R. G. Smith. The Riemann problem in gas dynamics. *Trans. Amer. Math. Soc.*, 249(1):1–50, 1979.

[198] R. C. Srivastava, D. Leutloff, K. Takayama, and H. Grönig, editors. *Shock focusing effect in medical science and sonoluminescence*, Physics and Astronomy, Berlin Heidelberg, 2003. Springer-Verlag.

[199] G. Strang. On strong hyperbolicity. *J. Math. Kyoto Univ.*, 6(3):397–417, 1967.
[200] R. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.
[201] J. C. Strikwerda. Initial boundary value problems for incompletely parabolic systems. *Comm. Pure Appl. Math.*, 30(6):797–822, 1977.
[202] L. Tartar. Compensated compactness and applications to partial differential equations. In *Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV*, pages 136–212. Pitman, Boston, Mass., 1979.
[203] D. Tataru. Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation. *Trans. Amer. Math. Soc.*, 353(2):795–807, 2001.
[204] G. I. Taylor. The formation of a blast wave by a very intense explosion. *Proc. Roy. Soc. London. Ser. A.*, 201:159–174, 175–186, 1950.
[205] M. E. Taylor. *Pseudo differential operators*. Lecture Notes in Mathematics, Vol. 416. Springer-Verlag, Berlin, 1974.
[206] M. E. Taylor. *Pseudodifferential operators and nonlinear PDE*, volume 100 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1991.
[207] M. E. Taylor. *Partial differential equations. III*, volume 117 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1997. Nonlinear equations, Corrected reprint of the 1996 original.
[208] R. Temam. *Navier–Stokes equations. Theory and numerical analysis*. North-Holland Publishing Co., Amsterdam, revised edition, 1979. With an appendix by F. Thomasset.
[209] P. A. Thompson and K. C. Lambrakis. Negative shock waves. *J. Fluid Mech.*, 60:187–208, 1973.
[210] F. Trèves. *Introduction to pseudodifferential and Fourier integral operators. Vol. 1.* Plenum Press, New York, 1980. Pseudodifferential operators, The University Series in Mathematics.
[211] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.
[212] H. Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.
[213] L. Truskinovsky. The structure of an isothermic phase jump. *Dokl. Akad. Nauk SSSR*, 285:309–315, 1985.
[214] L. Truskinovsky. Kinks versus shocks. In J. E. Dunn, R. Fosdick, and M. Slemrod, editors, *Shock induced transitions and phase structures in general media*, volume 52 of *IMA Vol. Math. Appl.*, pages 185–227. Springer, New York, 1993.
[215] Ya-Guang Wang and Zhouping Xin. Stability and existence of multidimensional subsonic phase transitions. *Acta Math. Appl. Sin. Engl. Ser.*, 19(4):529–558, 2003.
[216] R. C. Weast and M. J. Astle, editors. *CRC Handbook of Chemistry and Physics*. CRC Press Inc., Boca Raton, Florida 33431, 1980.
[217] H. F. Weinberger. Remark on the preceeding paper of Lax. *Comm. Pure Appl. Math.*, 11:195–196, 1958.
[218] H. Weyl. Shock waves in arbitrary fluids [Commun. Pure Appl. Math. **ii** (1949), no. 2-3, 103–122]. In *Classic papers in shock compression science*, pages 497–519. Springer, New York, 1998.
[219] G. B. Whitham. *Linear and nonlinear waves*. John Wiley & Sons Inc., New York, 1999. Reprint of the 1974 original, A Wiley-Interscience Publication.

[220] Zhouping Xin and Huicheng Yin. Transonic shock in a nozzle. I. Two-dimensional case. *Comm. Pure Appl. Math.*, 58(8):999–1050, 2005.

[221] M. Yamaguti and K. Kasahara. Sur le système hyperbolique à coefficients constants. *Proc. Japan Acad.*, 35:547–550, 1959.

[222] Wen-An Yong. Basic aspects of hyperbolic relaxation systems. In *Advances in the theory of shock waves*, pages 259–305. Birkhäuser Boston, Boston, MA, 2001.

[223] Wen-An Yong. Basic structures of hyperbolic relaxation systems. *Proc. Roy. Soc. Edinburgh Sect. A*, 132:1259–1274, 2002.

[224] Tong Zhang and Ling Xiao. Riemann problem and discontinuous initial value problem for a typical nonconvex quasilinear hyperbolic system. *Acta Math. Sinica*, 20(3):229–231, 1977.

[225] Yuxi Zheng. *Systems of conservation laws.* Progress in Nonlinear Differential Equations and their Applications, 38. Birkhäuser Boston Inc., Boston, MA, 2001. Two-dimensional Riemann problems.

[226] Yuxi Zheng. A global solution to a two-dimensional Riemann problem involving shocks as free boundaries. *Acta Math. Appl. Sin. Engl. Ser.*, 19(4):559–572, 2003.

This page intentionally left blank

INDEX