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Resolution of Singularities of Embedded Algebraic Surfaces

Second, Enlarged Edition

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Dedicated to
Professor Oscar Zariski

Without his blessings who can resolve the singularities?

Preface to the Second Edition

The common solutions of a finite number of polynomial equations in a finite number of variables constitute an algebraic variety. The degrees of freedom of a moving point on the variety is the dimension of the variety. A one-dimensional variety is a curve and a two-dimensional variety is a surface. A three-dimensional variety may be called a solid. Most points of a variety are simple points. Singularities are special points, or points of multiplicity greater than one. Points of multiplicity two are double points, points of multiplicity three are triple points, and so on. A nodal point of a curve is a double point where the curve crosses itself, such as the alpha curve. A cusp is a double point where the curve has a beak. The vertex of a cone provides an example of a surface singularity. A reversible change of variables gives a birational transformation of a variety. Singularities of a variety may be resolved by birational transformations.

In the last century, resolution of singularities of curves was achieved by Riemann, Noether and Dedekind by analytic, geometric and algebraic methods, respectively; for a historical overview of the resolution problem see my expository article [A8]; here items [A1] to [A17] refer to the Additional Bibliography and items [1] to [26] refer to the original Bibliography. Then, in case of characteristic zero, after several attempts by the Italian geometers such as Albanese [11] and Levi [16] at the turn of the century, surface desingularization was achieved by Zariski [A17], who soon followed it up by solid desingularization [25], which was brilliantly generalized by Hironaka [15] for higher-dimensional varieties. In my Ph.D. thesis [2], I gave a proof of surface desingularization in case of characteristic p, which later on I extended first to arithmetical surfaces [A2] and then to two-dimensional excellent schemes [A3]. Briefly, when the coefficients of the defining equations of an algebraic curve are integers, by reducing them modulo various prime numbers we get a family of algebraic curves over fields of different characteristics, and the resulting total object is called an arithmetical surface. A two-dimensional scheme is a further generalization of this concept.

The present book contains the geometric part of the proof of solid desingularization in characteristic $p \neq 2, 3, 5$ which I obtained in 1965; the algorithmic part is contained in my four previous articles [5] to [9]; the book does contain an alternative simple version of the algorithm for characteristic zero; half of the book can also be used as an introduction to birational algebraic geometry. I am thankful to Springer-Verlag for reprinting this book which was first published in 1966 by Academic Press. It may be hoped that this would stimulate other investigators to settle the general desingularization problem for higher-dimensional algebraic as well as arithmetical varieties. A discussion of this topic addressed to scientists and engineers may be found in my 1990 book [A7] which, according to the recent article [A11] by Hauser, "provides a description of the state of the art in resolution of singularities and related problems."

There is also the question of canonical processes of desingularization. Such a process for algebraic as well as arithmetical curves is described in my 1983 article [A4] and revisited in my 1997 article [A9]. Moreover, a discussion of such processes for higher-dimensional varieties in characteristic zero can be found in my monographs [A5] and [A6] of 1983 and 1988, respectively. These discussions together with various incarnations of the trick employed in item (10.24) of the present book have recently led me to discover a short proof of analytic desingularization in characteristic zero for any dimension on which I gave a lecture in various places in 1996-97. The text of that lecture is inserted as an Appendix to the present new edition of the book.

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16 July 1997

Preface to the First Edition

Some twenty years ago there appeared, in the *Annals of Mathematics*, the marvelous memoir of Zariski entitled: Reduction of singularities of algebraic three-dimensional varieties. Not only was a daring and ingenious solution of a difficult problem given in it, but so much of the technique invented for the solution has proved to be of such significance for algebraic geometry in general!

Hironaka's brilliantly energetic recent solution of the general resolution problem for zero characteristic constitutes, indeed, a high tribute to Zariski's memoir.

At present I am able to pay only a modest tribute to Zariski's memoir by giving a self-contained exposition of it. This then is the primary aim of the monograph.

A secondary aim is to partially extend some of the results to nonzero characteristic. The algorithm needed for such an extension has already been published in four papers, and it will not be repeated here. This monograph contains the geometric part of the argument. However, we do include an alternative simple version of the algorithm for zero characteristic thereby making the monograph self-contained for that case.

Finally, the matter is so arranged that about half of the monograph can be used as an introduction to certain foundational aspects of algebraic geometry.

My thanks are due to Annette Wortman for an excellent job of typing the manuscript. The work on this monograph was partially supported by the National Science Foundation under NSF-GP-4248-50-395 at Purdue University; I am grateful for this support.

September, 1966 S. S. A.
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Contents

§0. Introduction

Let k be a perfect ground field of characteristic p, and let X be a nonsingular irreducible three-dimensional projective algebraic variety over k. Then the principal results proved in this monograph are:

Global resolution. Given any algebraic surface Y over k embedded in X, there exists a sequence $X \to X_1 \to X_2 \to \dots \to X_m \to X'$ of monoidal transformations with nonsingular irreducible centers such that the total transform of Y in X' has only normal crossings and the proper transform of Y in X' is nonsingular.

Global principalization. Given any ideal I on X, there exists a sequence $X \to X_1 \to X_2 \to \dots \to X_m \to X'$ of monoidal transformations with nonsingular irreducible centers such that the inverse image of I on X' is locally principal.

Dominance. Given any irreducible projective algebraic variety X^* over k such that X^* is birationally equivalent to X, there exists a sequence $X \to X_1 \to X_2 \to \dots \to X_m \to X'$ of monoidal transformations with nonsingular irreducible centers such that X' dominates X^*.

Birational invariance. If k is algebraically closed and X^* is any nonsingular irreducible projective algebraic variety over k such that X^* is birationally equivalent to X, then $h^i(X) = h^i(X^*)$ for all $i \geqslant 0$, where $h^i(X)$ denotes the vector space dimension over k of the ith cohomology group of X with coefficients in the structure sheaf, and hence, in particular, the arithmetic genus of $X =$ the arithmetic genus of X^*.

Uniformization. Assume that either $p = 0$, or k is algebraically closed and $p \neq 2, 3, 5$. Let K be any three-dimensional algebraic function field over k and let W be any valuation ring of K containing k. Then there exists a projective model of K/k on which the center of W is at a simple point.

Birational resolution. Assume that either $p = 0$, or k is algebraically closed and $p \neq 2, 3, 5$. Let K be any three-dimensional

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algebraic function field over k. Then there exists a nonsingular projective model of K/k.

History. The following version of global resolution was proposed by Levi [16] and proved by Zariski [25]: if $p = 0$ and Y is any irreducible algebraic surface over k embedded in X, then there exists a sequence $X \to X_1 \to X_2 \to \ldots \to X_m \to X'$ of monoidal transformations with nonsingular irreducible centers such that the proper transform of Y in X' is nonsingular. For $p = 0$, Zariski [25] proved dominance. For $p = 0$, Zariski [23] proved uniformization for function fields of any dimension. For $p = 0$, Zariski [25] deduced birational resolution from uniformization and global resolution (in the form just mentioned). For $p = 0$, Hironaka [15] generalized all the above six results to varieties of any dimension. What we have called global principalization corresponds to what Hironaka [15] has called trivialization of a coherent sheaf of ideals.

We now describe the contents of the various chapters.

Chapter One. In this chapter we prove a certain local version of global resolution which may be called ***resolution***, and from it we deduce a certain local version of global principalization which may be called ***principalization***; it may be noted that for this deduction it is necessary to have resolution without assuming Y to be irreducible. In §1 and §2, we establish the terminology and make some general observations concerning the basic concepts. In §3 we prove a theorem (see (3.21)) which corresponds to what Zariski [25] has called the dominant character of a normal sequence, and which says that if the multiplicity of a given point of the embedded surface Y can be decreased by monoidal transformations of a certain type then it can also be decreased by monoidal transformation of a more restricted type; this has the effect of reducing the proof of resolution to an apparently weaker assertion. In §4 the proof of this weaker assertion is further reduced (see (4.22)) to a certain statement $(*)$ concerning monic polynomials in an indeterminate with coefficients in a two-dimensional regular local domain. The proof of resolution depends on the algorithm developed in the papers [5], [7], [8], and [9]; however, the matter is so arranged that this dependence is reduced to a single point; namely, a part of [9: Theorem 1.1] is restated as (5.1) which is nothing but the said statement ($*$). In §5, the main results of the chapter are deduced as direct consequences of (5.1), (4.22), (3.21), and the preliminary considerations made in §1 and §2. For the purpose of comparison, in §5 we give an alternative simple proof of (5.1) for $p = 0$ which does not in any way depend on the papers [5], [7], [8], and [9]; instead it uses the trick of killing the coefficient of Z^{e-1} in a polynomial of degree e in an indeterminate Z; this trick was effectively used by Hironaka in [15], and it was also used by Abhyankar and Zariski in [10]. As far as the case of $p = 0$ is concerned, the said alternative proof of (5.1) has the effect of making the entire monograph independent of the papers [5], [7], [8], and [9].

Chapter Two. §6 contains some generalities on the language of models. In §7 we show that resolution implies global resolution. In §8 we show that principalization implies global principalization, and that global principalization implies dominance. In deducing birational resolution from uniformization and global resolution, Zariski [25] made use of the theorem of Bertini on the variable singularities of a linear system; in doing so he had to apply global resolution to a generic member of a linear system and hence to a surface not defined over k but defined over a pure transcendental extension k^* of k. Now for $p \neq 0$ this approach causes two difficulties; namely, in the first place the said theorem of Bertini is then not valid and in the second place k^* will not be perfect. However, in §8 we show that retaining a part of Zariski's argument but replacing the use of Bertini's theorem by the use of global principalization (as suggested by Hironaka) and without extending the ground field k, it is possible to deduce birational resolution from uniformization for any p. We refer to Serre [22] for the definition of the cohomology groups and for the result that the h^i are finite and their alternating sum equals the arithmetic genus as classically defined in terms of the Hilbert polynomial; and we refer to Matsumura [17] for the result that: if k is algebraically closed and X and X^* are any irreducible nonsingular projective algebraic varieties over k such that X and X^* are birationally equivalent and X^* dominates X, then $h^i(X) \leqslant h^i(X^*)$ for all $i \geqslant 0$, and if moreover X^* is a monoidal transform of X with a nonsingular irreducible center then $h^i(X) = h^i(X^*)$ for all $i \geqslant 0$;

in view of these two references, birational invariance follows from dominance. In §9 we collect together the main results of this chapter, i.e., global resolution, global principalization, dominance, birational invariance, and the implication: uniformization $\Rightarrow$ birational resolution.

Chapter Three. As said above, for $p = 0$, Zariski [25] deduced birational resolution from uniformization and global resolution. In §10 we show that, for $p = 0$, uniformization can also be deduced from resolution; it may be noted that for this deduction it is necessary to have resolution for the total transform and without assuming Y to be irreducible. In §11 we state the resulting theorem: birational resolution for $p = 0$; thus, in view of the above-mentioned alternative proof of (5.1) for $p = 0$, we shall have completely reproved this theorem without using any results from the papers [5], [7], [8], and [9], and without appealing to Zariski's paper [23] on uniformization. Actually what we show in §10 is somewhat stronger; namely, assuming resolution, we prove uniformization under the hypothesis that: the residue field of the given valuation ring W is algebraic over k and there exists a projective model of K/k on which the center of W is at a point of multiplicity e such that $e! \not\equiv 0 \bmod p$. Consequently we would have birational resolution also for $p \neq 0$ if we could find a projective model of K/k such that every algebraic point of it has multiplicity $< p$. In §12 we show that it is possible to find such a model provided k is algebraically closed and $p \neq 2, 3, 5$. In §13 we state the resulting theorem: birational resolution when k is algebraically closed and $p \neq 2, 3, 5$. What we actually prove in §12 is this: assume that k is algebraically closed and let L be an algebraic function field over k of any dimension n; then there exists a projective model of L/k such that every rational point of it has multiplicity $\leqslant n!$. For $n = 2$ this theorem is due to Albanese [11] and Artin [12]. Our proof for any n is a straightforward generalization of Artin's proof; however, we do not fall back on any general intersection theory or exact sequences, but give a self-contained proof using only a few *leisurely* readable pages of Zariski-Samuel [28].

In (1.2) of §1 we collect together a few definitions and pertinent facts from Grothendieck's theory of excellent rings [14]. The use of these enables us to prove some of the results in a some what stronger form than described above; in particular also for algebroid varieties.

With an eye on arithmetical geometry, in most of this monograph, except in some crucial steps, the characteristic of a local domain is permitted to be different from the characteristic of its residue field.

We shall have occasion to use the following four known results: (0.1) and (0.2) are elementary facts about two-dimensional regular local domains and they are proved in [3: Lemma 12] and [6: Theorem 2] respectively; (0.3) is a generalization of Zariski's factorization theorem and is proved in [3: Theorem 3]; (0.4) is due to Hironaka and is proved in [15: Theorem 2 on page 220]; for terminology see §1.

(0.1). *Let $(R_i)_{0 \leqslant i < \infty}$ be an infinite sequence of two-dimensional regular local domains such that R_i is a quadratic transform of R_{i-1} for $0 < i < \infty$, and let $V = \bigcup_{i=0}^{\infty} R_i$. Then V is a valuation ring of the quotient field of R_0 and V dominates R_i and V is residually algebraic over R_i for $0 \leqslant i < \infty$. Moreover, if V' is any valuation ring of the quotient field of R_0 such that V' dominates R_i for $0 \leqslant i < \infty$ then $V' = V$.*

(0.2). *Let R_0 be a pseudogeometric two-dimensional regular local domain, let V be a valuation ring of the quotient field of R_0 such that V dominates R_0 and V is residually algebraic over R_0, let $(R_i)_{0<i<\infty}$ be the unique infinite sequence such that R_i is the quadratic transform of R_{i-1} along V for $0 < i < \infty$, and let $f_1, \ldots, f_q$ be any finite number of nonzero elements in V. Then there exists a nonnegative integer m and a basis (x, y) of the maximal ideal in R_m such that $f_j = g_j x^{a(j)} y^{b(j)}$ where g_j is a unit in R_m and $a(j)$ and $b(j)$ are nonnegative integers for $1 \leqslant j \leqslant q$.*

(0.3). *Let R and R^* be two-dimensional regular local domains such that R and R^* have the same quotient field and R^* dominates R. Then there exists a unique finite sequence $R_0, R_1, \ldots, R_m$, $(m \geqslant 0)$, of two-dimensional regular local domains such that $R_0 = R$, $R_m = R^*$, and R_i is a quadratic transform of R_{i-1} for $0 < i \leqslant m$.*

(0.4). *Let A be a noetherian domain such that A_P is regular for every prime ideal P in A. Let Q be a prime ideal in A such that $(A/Q)_{P'}$ is regular for every prime ideal P' in A/Q. Let J be an ideal in A. Then there exists an ideal H in A with $H \not\subset Q$ such that* $\mathrm{ord}_{A_P} J = \mathrm{ord}_{A_Q} J$ *for every prime ideal P in A for which $H \not\subset P$ and $Q \subset P$.*

We have tried to make the monograph fairly self-contained. Namely, with a few exceptions, we only use some well-known results from commutative algebra to be found in the books [4], [18], [27], and [28]; the only exceptions are: the above four results (0.1) to (0.4); the above-cited specific references to Grothendieck [14], Matsumura [17], and Serre [22]; and the said restatement (5.1) of [9: Theorem 1.1]. §1, §6, §10, and §12 could be used as a possible introduction to certain foundational aspects of algebraic geometry.

Most of the considerations of §1 may be used tacitly in the rest of the monograph. The logical interdependence of the remaining sections is thus:

$$\begin{tikzcd}
& & \S2 \arrow[d] \\
\S3 \arrow[r] \arrow[dr] & \S4 \arrow[r] & \S5 \arrow[dd] \\
\S6 \arrow[r] \arrow[dr] & \S7 \arrow[dr] & \\
& \S8 \arrow[r] & \S9 \arrow[d] \\
& \S10 \arrow[r] & \S11 \arrow[d] \\
& \S12 \arrow[r] & \S13
\end{tikzcd}$$

CHAPTER 1

Local Theory

§1. Terminology and preliminaries

(1.1). By a *ring* we mean a commutative ring with identity. A ring is said to be *normal* if it is integrally closed in its total quotient ring. By a *domain* we mean an integral domain. By a *prime ideal* (resp: a *maximal ideal*) in a ring A we mean an ideal P in A such that A/P is a domain (resp: a field); note that then $P \neq A$. For any ideal P in a ring A, by $\mathrm{rad}_A P$ or $\mathrm{rad}\, P$ we denote the radical of P in A. Let A be a ring and let P be an A-module; for any subset Q of P, by QA we denote the A-submodule of P generated by Q; for any elements $x_1, \ldots, x_n$ in P, by $(x_1, \ldots, x_n)A$ we denote the A-submodule of P generated by $x_1, \ldots, x_n$; elements $x_1, \ldots, x_n$ in P are said to form an *A-basis* (or simply, a *basis*) of P if $P = (x_1, \ldots, x_n)A$; P is said to be a *finite* A-module if P has a finite A-basis. For any subset P of a ring B and any element x in B, by xP or Px we denote the subset $\{xy: y \in P\}$ of B; note that if P is an A-submodule of B for a subring A of B then xP is an A-submodule of B, and if moreover $(x_1, \ldots, x_n)$ is an A-basis of P then $(xx_1, \ldots, xx_n)$ is an A-basis of xP. Given a ring A, let N be the set of all nonnegative integers n such that there exists a chain of distinct prime ideals $P_0 \subset P_1 \subset \cdots \subset P_n$ in A; we *define*: $\dim A = -\infty$ if $N = \varnothing$, $\dim A =$ the greatest integer in N if N is a nonempty finite set, and $\dim A = \infty$ if N is an infinite set.

By a *quasilocal ring* we mean a ring having exactly one maximal ideal. The maximal ideal in a quasilocal ring R is denoted by $M(R)$. A subset $\mathfrak{k}$ of a quasilocal ring R is said to be a *coefficient set* for R if $\mathfrak{k}$ contains 0 and 1 and for every $x \in R$ there exists a unique $x' \in \mathfrak{k}$ such that $x - x' \in M(R)$. Given quasilocal rings R and S, we say that *S dominates R* if R is a subring of S and $M(R) \subset M(S)$; note that then $M(R) = R \cap M(S)$. Given a quasilocal ring S and set E of quasilocal rings, we say that *S dominates E* if S dominates some element in E. Given sets E and E' of quasilocal rings, we say that E' *dominates* E if every element in E' dominates E, and we say that E' *properly dominates* E if E' dominates E and for every $R \in E$ there exists $S \in E'$ such that S dominates R. A set of quasilocal rings is said to be *normal* if every element in it is normal. The quotient ring of a ring A with respect to a prime ideal P in A is denoted by A_P. As a rule we consider only quotient rings of a domain A with respect to prime ideals in A, and we regard such quotient rings to be subrings of a fixed quotient field of A. The set of all quotient rings of a domain A with respect to the various prime ideals in A is denoted by $\mathfrak{V}(A)$; note that $R \to A \cap M(R)$ is a one-to-one inclusion-reversing map of $\mathfrak{V}(A)$ onto the set of all prime ideals in A and the inverse map is given by $P \to A_P$; a subset E of $\mathfrak{V}(A)$ is said to be *closed* in $\mathfrak{V}(A)$ if there exists an ideal Q in A such that $E = \{R \in \mathfrak{V}(A): QR \neq R\}$. Note that if A is a subring of a domain B and S is any element in $\mathfrak{V}(B)$ then S dominates exactly one element R in $\mathfrak{V}(A)$; namely, $R = A_{A \cap M(S)}$.

Given a domain B and a subring A of B, by $\operatorname{trdeg}_A B$ we denote the transcendence degree of the quotient field of B over the quotient field of A. Given a quasilocal ring R and a subring A of R, let h: $R \to R/M(R)$ be the canonical epimorphism and let k be the quotient field of $h(A)$ in $h(R)$; $\operatorname{trdeg}_k h(R)$ is called the *residual transcendence degree* of R over A and it is denoted by $\operatorname{restrdeg}_A R$; R is said to be *residually algebraic* (resp: *residually finite algebraic*, *residually separable algebraic*, *residually finite separable algebraic*, *residually purely inseparable*, *residually finite purely inseparable*) over A if $h(R)$ is an algebraic (resp: finite algebraic, separable algebraic, finite separable algebraic, purely inseparable, finite purely inseparable) extension of k (note that for a field K of characteristic zero, K is the only overfield of K which is regarded to be a purely inseparable extension of K); R is said to be *residually rational* over A if $h(R) = k$.

By a *local ring* we mean a noetherian quasilocal ring. Let R be a local ring; for any $x \in R$ we *define*: $\operatorname{ord}_R x = \max e$ such that $x \in M(R)^e$; note that then: $\operatorname{ord}_R x = \infty \Leftrightarrow x = 0$; for any $\varnothing \neq J \subset R$ we *define*: $\operatorname{ord}_R J = \max e$ such that $J \subset M(R)^e$; note that $\operatorname{ord}_R JR = \operatorname{ord}_R J = \min_{x \in J} \operatorname{ord}_R x$; also note that: $\operatorname{ord}_R J = \infty \Leftrightarrow J = \{0\}$; for any nonzero polynomial $f(Z) = \sum_i f_i Z^i$

in an indeterminate Z with coefficients f_i in R we *define*: $\operatorname{ord}_R f(Z) = \min(i + \operatorname{ord}_R f_i)$ where the minimum is taken over all i for which $f_i \neq 0$. The *embedding dimension* of a local ring R is denoted by emdim R, i.e., emdim R is the vector space dimension of $M(R)/M(R)^2$ as a vector space over $R/M(R)$; note that if H is any basis of $M(R)$ and emdim $R = n$ then n is the smallest nonnegative integer such that there exist n elements in H which form a basis of $M(R)$; note that by [28: Theorem 20 on page 288] we know that $\dim R =$ the smallest nonnegative integer d such that there exist d elements in R which generate an ideal which is primary for $M(R)$; hence in particular emdim $R \geqslant \dim R$. A local ring R is said to be *regular* if emdim $R = \dim R$. We may tacitly use the fact that every regular local ring is a unique factorization domain (see [28: Appendix 7]), and hence in particular it is normal. For any ideal J in a noetherian domain A, the set of all $R \in \mathfrak{B}(A)$ such that $JR \neq R$ and $R/(JR)$ is not regular is called the *singular locus* of (A, J) and is denoted by $\mathfrak{S}(A, J)$.

A local ring is said to be *analytically irreducible* if its completion is a domain. The completion R^* of a local ring R is regarded to be an overring of R; moreover, if R is analytically irreducible then the quotient field of R^* is regarded to be an overfield of the quotient field of R.

Note that given any local rings R and S and any homomorphism $f: R \to S$ such that $f(M(R)) \subset M(S)$, there exists a unique homomorphism $f^*: R^* \to S^*$, where R^* and S^* are the completions of R and S respectively, such that $f^*(M(R^*)) \subset M(S^*)$ and $f^*(x) = f(x)$ for all $x \in R$. The existence of f^* can be seen thus. Given $y \in R^*$ take a sequence (y_n) in R such that $y_n \to y$. Then (y_n) is a Cauchy sequence in R, and hence $(f(y_n))$ is a Cauchy sequence in S because $f(M(R)) \subset M(S)$. Therefore there exists $z \in S^*$ such that $f(y_n) \to z$. Clearly z depends only on y and not on the sequence (y_n). Define $f^*(y)$ to be z. It is easily checked that $f^*: R^* \to S^*$ is then a homomorphism such that $f^*(M(R^*)) \subset M(S^*)$ and $f^*(x) = f(x)$ for all $x \in R$. To prove the uniqueness let $g: R^* \to S^*$ be any homomorphism such that $g(M(R^*)) \subset M(S^*)$ and $g(x) = f(x)$ for all $x \in R$. Since $y - y_n \to 0$ and $g(M(R^*)) \subset M(S^*)$, we get that $g(y - y_n) \to 0$. Now $g(y - y_n) = g(y) - f(y_n)$ and hence $f(y_n) \to g(y)$. However, $f(y_n) \to f^*(y)$ and hence $g(y) = f^*(y)$.

Given local rings R and S such that R is a subring of S, we say that R is a *subspace* of S if R with its Krull topology is a subspace of S with its Krull topology; note that this is so if and only if S dominates R and there exists a sequence of nonnegative integers $a(n)$ such that $a(n)$ tends to infinity with n and $R \cap M(S)^n \subset M(R)^{a(n)}$ for all $n \geqslant 0$. By a theorem of Chevalley [28: Theorem 13 on page 270] it follows that if R is a complete local ring and S is a local ring dominating R then R is a subspace of S. Given local rings R and S such that S dominates R, let R^* and S^* be the completions of R and S respectively, and let $f^*\colon R^* \to S^*$ be the unique homomorphism such that $f^*(M(R^*)) \subset M(S^*)$ and $f^*(x) = x$ for all $x \in R$; note that then R is a subspace of S if and only if f^* is a monomorphism; namely, it is clear that if R is a subspace of S then f^* is a monomorphism; also R and S are always subspaces of R^* and S^* respectively, and hence the converse follows from the above-cited theorem of Chevalley.

Given a valuation v of a field K, by R_v we denote the *valuation ring* of v, i.e., $R_v = \{x \in K\colon v(x) \geqslant 0\}$. By a *valuation ring* of a field K we mean a subring V of K such that K is the quotient field of V and for every $0 \neq x \in K$ we have that either $x \in V$ or $1/x \in V$; note that a ring V is a valuation ring of a field K if and only if V is the valuation ring of some valuation of K. A valuation v of a field K is said to be *discrete* if the value group of v is an infinite cyclic group. Note that for any domain R we have that: R is the valuation ring of a discrete valuation of the quotient field of $R \Leftrightarrow R$ is a one-dimensional regular local domain $\Leftrightarrow R$ is a one-dimensional normal local domain (see [27: §6 and §7 of Chapter V]); also note that if R is any one-dimensional regular local domain with quotient field K and S is any subring of K contining R then either $S = K$ or $S = R$. Let R be a regular local domain with quotient field K; for any nonzero elements x and y in R we *define*: $\operatorname{ord}_R(x/y) = (\operatorname{ord}_R x) - (\operatorname{ord}_R y)$ (note that since R is regular, $\operatorname{ord}_R(x/y)$ is uniquely determined by R and x/y); note that if $\dim R \neq 0$ then ord_R is a discrete valuation of K and upon letting V be the valuation ring of ord_R we have that $M(R)^e V = M(V)^e$ and $M(R)^e = R \cap M(V)^e$ for every nonnegative integer e, and for any $x \in R$ we have that $xV = M(V)$ if and only if $\operatorname{ord}_R x = 1$, i.e., if and only if $x \in M(R)$ and $x \notin M(R)^2$ (see [28: Theorem 25 on page 301]).

By a *saturated chain* of prime ideals in a ring A we mean a chain of distinct prime ideals $P_0 \subset P_1 \subset \dots \subset P_n$ in A such that there does not exist any prime ideal P in A such that $P_i \subset P \subset P_{i+1}$ and $P_i \neq P \neq P_{i+1}$ for some i. A ring A is said to be *catenarian* if for every two prime ideals $Q \subset P$ in A the following condition is satisfied: let N be the set of all nonnegative integers n such that there exists a saturated chain of prime ideals $P_0 \subset P_1 \subset \dots \subset P_n$ in A with $P_0 = Q$ and $P_n = Q$; then N contains exactly one element.

By a *finitely generated ring extension* of a ring A we mean an overring B of A such that $B = A[x_1, \dots, x_n]$ for some finite number of elements $x_1, \dots, x_n$ in B. By an *affine ring* over a ring A we mean a domain which is a finitely generated ring extension of A. By a *spot* over a ring A we mean a quasilocal ring R such $R \in \mathfrak{V}(B)$ for some affine ring B over A; note that if R is a spot over a ring A and S is a spot over R then S is a spot over A. By a *function field* over a ring A we mean a field which is the quotient field of some affine ring over A, i.e., a field which is a spot over A.

A ring A is said to be *pseudogeometric* if for every prime ideal P in A and every finite algebraic extension K of the quotient field of A/P we have that the integral closure of A/P in K is a finite (A/P)-module. Note that every field is pseudogeometric, and every homomorphic image of a pseudogeometric ring is pseudogeometric. The following two results of Nagata [18: (17.9), (32.1), (36.5)] may be used tacitly.

(1.1.1). *Every complete local ring is pseudogeometric.*

(1.1.2). *For every pseudogeometric ring A we have that every finitely generated ring extension of A is pseudogeometric, and the quotient ring of A with respect to any multiplicative set in A is pseudogeometric; whence in particular, every spot over A is pseudogeometric.*

(1.2). We shall now recall some aspects of Grothendieck's theory of excellent rings [14: 5.6 and 7.8]. For the purpose of this monograph it will be enough to keep (1.2.6) in mind.

(1.2.1). For a noetherian ring A the following three conditions are equivalent.

(1) If $X_1, ..., X_n$ are any finite number of indeterminates then $A[X_1, ..., X_n]$ is catenarian (note that $A[X_1, ..., X_n] = A$ if $n = 0$).

(2) If Q is any ideal in A, B' is any finitely generated ring extension of A/Q, S is any multiplicative set in B', and B is the quotient ring of B' with respect to S, then B is catenarian.

(3) A is catenarian, and for every prime ideal P in A and every spot R over $B = A/P$ we have that $\dim B_{B \cap M(R)} + \operatorname{trdeg}_B R = \dim R + \operatorname{restrdeg}_B R$.

A ring A is said to be *universally catenarian* if A is noetherian and the above three conditions are satisfied.

(1.2.2). Given a local ring R, we say that the *formal fibers of R are geometrically regular* if for every prime ideal P in R and every finite algebraic extension K of the quotient field of $T = R/P$, upon letting T^* be the completion of T, we have that $(T^* \otimes_T K)_Q$ is a regular local ring for every prime ideal Q in $T^* \otimes_T K$.

(1.2.3). A ring A is said to be *excellent* if A is noetherian and the following three conditions are satisfied.

(1) A is universally catenarian.

(2) For every prime ideal P in A the formal fibers of A_P are geometrically regular.

(3) Given any prime ideal P in A and any finite purely inseparable extension K of the quotient field of A/P, there exists a subring B of K and a subset E of $\mathfrak{V}(B)$ such that $A/P \subset B$, B is a finite (A/P)-module, K is the quotient field of B, $\mathfrak{S}(B, \{0\}) \subset E$, E is closed in $\mathfrak{V}(B)$, and $E \neq \mathfrak{V}(B)$.

(1.2.4). *A local ring R is excellent if and only if R is universally catenarian and for every prime ideal P in R the formal fibers of R_P are geometrically regular.*

(1.2.5). *Let R be an excellent ring. Then there exists an ideal Q in R such that for any prime ideal P in R we have that R_P is regular if and only if $Q \not\subset P$. If R is local, then for any prime ideal P^* in* *the completion R^* of R we have that $R^*_{P^*}$ is regular if and only if $R_{R\cap P^*}$ is regular.*

(1.2.6).

(1) *Every complete local ring is excellent; whence in particular, every field is excellent. Every Dedekind domain of characteristic zero is excellent.*

(2) *For every excellent ring A we have that every homomorphic image of A is excellent, every finitely generated ring extension of A is excellent, and the quotient ring of A with respect to any multiplicative set in A is excellent; whence in particular, every spot over A is excellent.*

(3) *Every excellent ring is pseudogeometric.*

In view of [18: (17.9)], by (1.2.5) and (2) we get the following.

(4) *Let R be an excellent domain. If T is any affine ring over R and Q is any ideal in T then $\mathfrak{S}(T, Q)$ is closed in $\mathfrak{V}(T)$. If T is any regular spot over R and Q is any ideal in T then $\mathfrak{S}(T, Q)$ is closed in $\mathfrak{V}(T)$ and, upon letting T^* be the completion of T, we have that $\mathfrak{S}(T^*, QT^*) = \{S \in \mathfrak{V}(T^*): T_{T\cap M(S)} \in \mathfrak{S}(T, Q)\}$.*

(1.3). We may tacitly use the following result of Nagata and Zariski (see [18: (38.3)] or [15: Theorem 1 on page 218]).

(1.3.1). *For any nonempty subset J of a regular local domain R and any $S \in \mathfrak{V}(R)$ we have that $\text{ord}_R J \geqslant \text{ord}_S J$.*

We shall now give two proofs of the following elementary fact, one using (1.3.1) and the other without using (1.3.1).

(1.3.2). *Let $f(X_1, ..., X_n)$ be a nonzero polynomial of degree $\leqslant d$ in indeterminates $X_1, ..., X_n$ with coefficients in a field k, let Q be a prime ideal in $A = k[X_1, ..., X_n]$, let $R = A_Q$, and let $e = \text{ord}_R f(X_1, ..., X_n)$. Then $e \leqslant d$.*

PROOF. First suppose that Q is a maximal ideal in A. Then $M(R)^e \cap A = Q^e$ and hence $f(X_1, ..., X_n) \in Q^e$. Let k^* be an algebraic closure of k and let $A^* = k^*[X_1, ..., X_n]$. Then A^* is integral over A and hence there exists a maximal ideal Q^* in A^* such that $Q^* \cap A = Q$ (for instance see [4: Lemmas 1.19 and 1.20]). Now $Q^e \subset Q^{*e}$ and hence $f(X_1, ..., X_n) \in Q^{*e}$. By the Hilbert Nullstellensatz [28: Lemma on page 165],

$$Q^* = (X_1 - r_1, ..., X_n - r_n)A^*$$

with $r_1, ..., r_n$ in k^*. Therefore

$$f(X_1, ..., X_n) = \sum_{i_1+\dots+i_n=e} f_{i_1\dots i_n}(X_1, ..., X_n)(X_1 - r_1)^{i_1} \dots (X_n - r_n)^{i_n}$$

where $f_{i_1\dots i_n}(X_1, ..., X_n)$ are polynomials in $X_1, ..., X_n$ with coefficients in k^*. Let $g(X_1, ..., X_n)$ and $g_{i_1\dots i_n}(X_1, ..., X_n)$ be the polynomials in $X_1, ..., X_n$ with coefficients in k^* obtained by substituting $X_1 + r_1, ..., X_n + r_n$ for $X_1, ..., X_n$ in $f(X_1, ..., X_n)$ and $f_{i_1\dots i_n}(X_1, ..., X_n)$ respectively. Then $g(X_1, ..., X_n)$ is a nonzero polynomial of degree $\leqslant d$ in $X_1, ..., X_n$ with coefficients in k^*. Upon substituting $X_1 + r_1, ..., X_n + r_n$ for $X_1, ..., X_n$ in the above displayed formula we get that

$$g(X_1, ..., X_n) = \sum_{i_1+\dots+i_n=e} g_{i_1\dots i_n}(X_1, ..., X_n)X_1^{i_1} \dots X_n^{i_n}$$

and hence $g(X_1, ..., X_n)$ is either zero or is a polynomial of degree $\geqslant e$ in $X_1, ..., X_n$ with coefficients in k^*. Therefore $e \leqslant d$.

We shall deduce the general case from the special case proved above in two ways. Take a maximal ideal Q' in A containing Q and let $R' = A_{Q'}$. Then R' is regular (see [28: Remark on page 310]) and $R = R'_{QR'}$. Therefore $\operatorname{ord}_{R'} f(X_1, ..., X_n) \geqslant e$ by (1.3.1) and hence $e \leqslant d$ by the special case proved above. Alternatively, without using (1.3.1) we can argue thus. Let $h\colon A \to A/Q$ be the canonical epimorphism. Upon suitably relabeling $X_1, ..., X_n$ we may assume that $(h(X_{m+1}), ..., h(X_n))$ is a transcendence basis of $h(A)$ over $h(k)$. Let $A'' = k(X_{m+1}, ..., X_n)[X_1, ..., X_m]$ and $Q'' = QA''$. Then Q'' is a maximal ideal in A'', $R = A''_{Q''}$, and $f(X_1, ..., X_n)$ is a nonzero polynomial of degree $\leqslant d$ in $X_1, ..., X_m$ with coefficients in $k(X_{m+1}, ..., X_n)$. Therefore $e \leqslant d$ by the special case proved above.

The following generalization of [10: Lemma 3] is due to Sato [21: Lemma 1].

(1.3.3). *Let R be an n-dimensional local domain with $n > 1$. Let $x_1, ..., x_n$ be elements in R such that Q is primary for $M(R)$ where $Q = (x_1, ..., x_n)R$. Let $A = R[x_2/x_1, ..., x_n/x_1]$, and let h: $A \to A/(M(R)A)$ be the canonical epimorphism. Then $M(R)A$ is a prime ideal in A, $\dim A_{M(R)A} = 1$, $M(R)A = \mathrm{rad}(QA)$, $R \cap (M(R)A) = M(R)$, and the elements $h(x_2/x_1), ..., h(x_n/x_1)$ are algebraically independent over $h(R)$.*

PROOF. Clearly $QA = x_1A$. Since Q is primary for $M(R)$, there exists a positive integer e such that $M(R)^e \subset Q$, and then $(M(R)A)^e \subset x_1A$. Let $X_1, ..., X_n$ be indeterminates. Suppose if possible that $R \cap (M(R)A) \neq M(R)$; then we must have $M(R)A = A$ and hence $x_1A = A$; consequently $x_1y = 1$ for some $0 \neq y \in A$; since $0 \neq y \in A$, there exists a nonzero polynomial $f(X_2, ..., X_n)$ of some degree d in $X_2, ..., X_n$ with coefficients in R such that $y = f(x_2/x_1, ..., x_n/x_1)$; now $x_1^d = x_1^{d+1}y = x_1 f'(x_1, ..., x_n)$ where $f'(X_1, ..., X_n)$ is a nonzero homogeneous polynomial of degree d in $X_1, ..., X_n$ with coefficients in R; in particular $x_1^d \in M(R)Q^d$ which is a contradiction by [28: Theorem 21 on page 292]. Therefore $R \cap (M(R)A) = M(R)$, and hence $h(R)$ is isomorphic to the field $R/M(R)$. Suppose if possible that $h(x_2/x_1), ..., h(x_n/x_1)$ are algebraically dependent over $h(R)$; then there exists a nonzero polynomial $F(X_2, ..., X_n)$ of some degree u in $X_2, ..., X_n$ with coefficients in R at least one of which is not in $M(R)$ such that $F(x_2/x_1, ..., x_n/x_1) \in M(R)A$; since $F(x_2/x_1, ..., x_n/x_1) \in M(R)A$, there exists a polynomial $G(X_2, ..., X_n)$ in $X_2, ..., X_n$ with coefficients in $M(R)$ such that $F(x_2/x_1, ..., x_n/x_1) = G(x_2/x_1, ..., x_n/x_1)$; upon multiplying both sides of this equation by x_1^v for a suitable integer $v \geqslant u$ we get that $F'(x_1, ..., x_n) = G'(x_1, ..., x_n)$ where $F'(X_1, ..., X_n)$ is a nonzero homogeneous polynomial of degree v in $X_1, ..., X_n$ with coefficients in R at least one of which is not in $M(R)$, and $G'(X_1, ..., X_n)$ is either the zero polynomial or a nonzero homogeneous polynomial of degree v in $X_1, ..., X_n$ with coefficients in $M(R)$; in particular then $F'(x_1, ..., x_n) \in M(R)Q^v$ which is a contradiction by [28: Theorem 21 on page 292]. Therefore $h(x_2/x_1), ..., h(x_n/x_1)$ are algebraically independent over $h(R)$. Since $h(A) = h(R)[h(x_2/x_1), ..., h(x_n/x_1)]$, we get that $h(A)$ is a domain and hence $M(R)A$ is a prime ideal in A. Since $(M(R)A)^e \subset x_1A = QA$, by Krull's principal ideal theorem [27: Theorem 29 on page 238] we conclude that $M(R)A = \operatorname{rad}(QA)$ and $\dim A_{M(R)A} = 1$.

(1.4). Let R be an n-dimensional regular local domain. Recall that for any nonunit ideal P in R we have that R/P is regular if and only if there exists a basis $(x_1, ..., x_n)$ of $M(R)$ such that $(x_1, ..., x_m)R = P$ for some m (see [28: Theorem 26 on page 303]). Now let $(x_1, ..., x_n)$ be a basis of $M(R)$. Then

$$(0)R \subset (x_1)R \subset (x_1, x_2)R \subset ... \subset (x_1, ..., x_n)R$$

is a chain of distinct prime ideals in R. Therefore for any m with $0 < m \leqslant n$ upon letting $P = (x_1, ..., x_m)R$ and $S = R_P$ we get that $\dim R/P = n - m$, $\dim S = m$, and S is regular. Given any nonzero element w in R let d be the greatest integer such that $w \in P^d$; then $w = f(x_1, ..., x_m)$ where $f(X_1, ..., X_m)$ is a nonzero homogeneous polynomial of degree d in indeterminates $X_1, ..., X_m$ with coefficients in S at least one of which is not in $M(S)$; since $\dim S = m$ and $M(S) = (x_1, ..., x_m)S$, it follows that $w \in M(S)^d$ and $w \notin M(S)^{d+1}$ (see [28: Theorem 21 on page 292]), i.e., $\operatorname{ord}_S w = d$. Consequently, for every positive integer e we have that $M(S)^e \cap R = P^e$ and hence P^e is primary for P. Let S' be the valuation ring of ord_S. Then $M(S')^e = x_1^e S'$ and $M(S')^e \cap R = (M(S')^e \cap S) \cap R = M(S)^e \cap R = P^e$ for every nonnegative integer e. Let h: $R \to k$ and h': $R \to T$ be (ring) epimorphisms such that $\operatorname{Ker} h = M(R)$ and $\operatorname{Ker} h' = P$. Then clearly there exists a unique epimorphism h'': $T \to k$ such that $h(u) = h''(h'(u))$ for all $u \in R$. Now assume that $m > 1$. Let $A = R[x_2/x_1, ..., x_m/x_1]$. Let $B = T[X_2, ..., X_m]$ and $A^* = k[X_2, ..., X_m]$ where $X_2, ..., X_m$ are indeterminates. Then we have the following.

(1.4.1). *$P^eA = x_1^eA$, $P^eS' = x_1^eS' = M(S')^e$, $(P^eA) \cap R = P^e$, and $M(S')^e \cap A = P^eA$ for every nonnegative integer e. PA is a prime ideal in A and $S' = A_{PA}$. For any $0 \neq w \in A$ upon letting $d = \operatorname{ord}_S w$ we have that d is the greatest integer such that $w \in P^dA$, i.e., $w/x_1^d \in A$ and $w/x_1^{d+1} \notin A$. $M(R)A = (x_1, x_{m+1}, ..., x_n)A$, $M(R)A$ is a prime ideal in A, and $(M(R)A) \cap R = M(R)$. There exists a unique epimorphism H: $A \to A^*$ such that $H(x_i/x_1) = X_i$*

for $2 \leqslant i \leqslant m$ *and* $H(u) = h(u)$ *for all* $u \in R$; *there exists a unique epimorphism* $H'\colon A \to B$ *such that* $H'(x_i/x_1) = X_i$ *for* $2 \leqslant i \leqslant m$ *and* $H'(u) = h'(u)$ *for all* $u \in R$; *and there exists a unique epimorphism* $H''\colon B \to A^*$ *such that* $H''(X_i) = X_i$ *for* $2 \leqslant i \leqslant m$ *and* $H''(u) = h''(u)$ *for all* $u \in T$. *Moreover,* $\operatorname{Ker} H = M(R)A$, $\operatorname{Ker} H' = PA$, *and* $H(u) = H''(H'(u))$ *for all* $u \in A$.

(1.4.2). *Let* H *be as in* (1.4.1). *Let* $R' \in \mathfrak{V}(A)$ *such that* R' *dominates* R. *Let* $n' = \dim R'$, $t = \operatorname{restrdeg}_R R'$, $Q = A \cap M(R')$, $Q^* = H(Q)$, *and* $R^* = A^*_{Q^*}$. *Then we have the following.*

(1) $P^eR' = x_1^eR'$, $(P^eR') \cap A = P^eA$, *and* $M(S')^e \cap R' = P^eR'$ *for every nonnegative integer* e. PR' *is a prime ideal in* R' *and* $S' = R'_{PR'}$. *For any* $0 \neq w \in R'$ *upon letting* $d = \operatorname{ord}_S w$ *we have that* d *is the greatest integer such that* $w \in P^dR'$, i.e., $w/x_1^d \in R'$ *and* $w/x_1^{d+1} \notin R'$. $M(R)R' = (x_1, x_{m+1}, \ldots, x_n)R'$ *and* $M(R)R'$ *is a prime ideal in* R'.

(2) *For any* $0 \neq w \in R$ *such that* $\operatorname{ord}_S w = \operatorname{ord}_R w$, *upon letting* $d = \operatorname{ord}_R w$ *we have that* $\operatorname{ord}_{R'}(w/x_1^d) \leqslant d$.

(3) R' *and* R^* *are regular,* $\operatorname{restrdeg}_k R^* = t$, $\dim R^* = m - 1 - t$, *and* $n \geqslant n' = n - t \geqslant n - m + 1$. *If* D *is any subset of* A *such that* $H(D)R^* = M(R^*)$ *then* $DR' + (x_1, x_{m+1}, \ldots, x_n)R' = M(R')$. *If* m' *is an integer with* $1 \leqslant m' \leqslant m$ *such that* $x_i/x_1 \in M(R')$ *for* $2 \leqslant i \leqslant m'$ *then there exist elements* $y_1, \ldots, y_q$ *in* A, *where* $q = n' - n + m - m'$, *such that*

$$M(R') = (x_1, x_2/x_1, \ldots, x_{m'}/x_1, x_{m+1}, \ldots, x_n, y_1, \ldots, y_q)R'.$$

(4) *The following six conditions are equivalent:* (1′) R' *is residually algebraic over* R; (2′) $n' = n$; (3′) Q *is a maximal ideal in* A; (4′) Q^* *is a maximal ideal in* A^*; (5′) $\dim R^* = m - 1$; (6′) R^* *is residually algebraic over* k.

(5) R' *is residually separable algebraic over* R *if and only if* R^* *is residually separable algebraic over* k.

(6) R' *is residually rational over* R *if and only if* R^* *is residually rational over* k. *If there exists* $r_i \in R$ *such that* $(x_i/x_1) - r_i \in M(R')$ *for* $2 \leqslant i \leqslant m$ *then*

$$Q = (x_1, (x_2/x_1) - r_2, \ldots, (x_m/x_1) - r_m, x_{m+1}, \ldots, x_n)A,$$
$$M(R') = (x_1, (x_2/x_1) - r_2, \ldots, (x_m/x_1) - r_m, x_{m+1}, \ldots, x_n)R',$$

and R' is residually rational over R. If R' is residually rational over R then there exists $r_i \in R$ such that $(x_i/x_1) - r_i \in M(R')$ for $2 \leqslant i \leqslant m$. If R' is residually rational over R and $\mathfrak{k}$ is a coefficient set for R then there exists a unique $r_i \in \mathfrak{k}$ such that $(x_i/x_1) - r_i \in M(R')$ for $2 \leqslant i \leqslant m$.

(7) *There exists a unique epimorphism $H^*: R' \to R^*$ such that $H^*(u) = H(u)$ for all $u \in A$. Moreover,* $\operatorname{Ker} H^* = (\operatorname{Ker} H)R' = M(R)R' = (x_1, x_{m+1}, ..., x_n)R'$.

PROOF OF (1.4.1). Clearly $P^eA = x_1^eA$ and $P^eS' = x_1^eS' = M(S')^e$ for every nonnegative integer e. Now $\operatorname{ord}_S x_i = 1$ for $1 \leqslant i \leqslant m$ and hence $A \subset S'$. For any $0 \neq w \in A$ let $d = \operatorname{ord}_S w$, i.e., $d = \operatorname{ord}_{S'} w$; since $A \subset S'$ we must have $w/x_1^{d+1} \notin A$; since $w \in A$ there exists a nonnegative integer c such that $wx_1^c \in R$; then $\operatorname{ord}_S wx_1^c = d + c$ and hence $wx_1^c \in P^{d+c}$; consequently $wx_1^c \in x_1^{d+c}A$ and hence $w/x_1^d \in A$; thus d is the greatest integer such that $w \in P^dA$. For every nonnegative integer e we therefore get that $M(S')^e \cap A = P^eA$ and hence $(P^eA) \cap R = M(S')^e \cap R = P^e$. Since $PA = M(S') \cap A$ we get that PA is a prime ideal in A and hence A_{PA} is a one-dimensional regular local domain; clearly $A_{PA} \subset S'$ and A_{PA} and S' have the same quotient field; therefore $S' = A_{PA}$. Let $A_1 = S[x_2/x_1, ..., x_m/x_1]$. Then $A \subset A_1 \subset S'$ and upon replacing (R, P) by $(S, M(S))$ in the above argument we get that $M(S') \cap A_1 = M(S)A_1$ and hence $(M(S)A_1) \cap A = PA$. Let $h_1: A_1 \to A_1/M(S)A_1$ and $h_2: A \to A/PA$ be the canonical epimorphisms; then by (1.3.3) we know that $h_1(x_2/x_1), ..., h_1(x_m/x_1)$ are algebraically independent over $h_1(S)$, and hence *a fortiori* $h_2(x_2/x_1), ..., h_2(x_m/x_1)$ are algebraically independent over $h_2(R)$. Therefore there exists a unique epimorphism $H': A \to B$ such that $H'(x_i/x_1) = X_i$ for $2 \leqslant i \leqslant m$ and $H'(u) = h'(u)$ for all $u \in R$; moreover, $\operatorname{Ker} H' = \operatorname{Ker} h_2 = PA = x_1A$. Since $X_2, ..., X_m$ are indeterminates, there exists a unique epimorphism $H'': B \to A^*$ such that $H''(X_i) = X_i$ for $2 \leqslant i \leqslant m$ and $H''(u) = h''(u)$ for all $u \in T$; clearly

$$\begin{aligned}\operatorname{Ker} H'' = (\operatorname{Ker} h'')B &= (h'(x_1), ..., h'(x_n))B \\ &= (h'(x_1), h'(x_{m+1}), ..., h'(x_n))B\end{aligned}$$

and hence $H'^{-1}(\operatorname{Ker} H'') = (x_1, x_{m+1}, ..., x_n)A$. Let $H(u) = H''(H'(u))$ for all $u \in A$. Then $H: A \to A^*$ is an epimorphism such that $H(x_i/x_1) = X_i$ for $2 \leqslant i \leqslant m$ and $H(u) = h(u)$ for all $u \in R$; clearly H is the only such epimorphism and $\operatorname{Ker} H = H'^{-1}(\operatorname{Ker} H'') = (x_1, x_{m+1}, \ldots, x_n)A = M(R)A$, and hence in particular $M(R)A$ is a prime ideal in A. Since $\operatorname{Ker} H = M(R)A$, $\operatorname{Ker} h = M(R)$, and $H(u) = h(u)$ for all $u \in R$, it follows that $(M(R)A) \cap R = M(R)$.

PROOF OF (1.4.2). Now $R' = A_Q$ and $PA \subset M(R)A \subset Q$ and hence (1) follows from (1.4.1). By (1.4.1) we know that $\operatorname{Ker} H = M(R)A = (x_1, x_{m+1}, \ldots, x_n)A$; since also $M(R)A \subset Q$, we get (7). Let $h^*\colon R^* \to R^*/M(R^*)$ be the canonical epimorphism and let $h^{**}(u) = h^*(H^*(u))$ for all $u \in R'$. Then $h^{**}\colon R' \to R^*/M(R^*)$ is an epimorphism, $\operatorname{Ker} h^{**} = M(R')$, and $h^{**}(R) = h^*(k)$. Therefore it follows that $\operatorname{restrdeg}_k R^* = t$, and: R' is residually algebraic (resp: residually separable algebraic, residually rational) over R if and only if R^* is residually algebraic (resp: residually separable algebraic, residually rational) over k. Clearly Q is a maximal ideal in A if and only if Q^* is a maximal ideal in A^*; by the Hilbert Nullstellensatz [28: Lemma on page 165], we also get that Q^* is a maximal ideal in A^* if and only if R^* is residually algebraic over k. If there exists $r_i \in R$ such that $(x_i/x_1) - r_i \in M(R')$ for $2 \leqslant i \leqslant m$ then Q^* contains the maximal ideal $(X_2 - h(r_2), \ldots, X_m - h(r_m))A^*$ in A^* and hence $Q^* = (X_2 - h(r_2), \ldots, X_m - h(r_m))A^*$ and R^* is residually rational over k; since $Q = H^{-1}(Q^*)$ and $\operatorname{Ker} H = (x_1, x_{m+1}, \ldots, x_n)A$, we deduce that if there exists $r_i \in R$ such that $(x_i/x_1) - r_i \in M(R')$ for $2 \leqslant i \leqslant m$ then

$$Q = (x_1, (x_2/x_1) - r_2, \ldots, (x_m/x_1) - r_m, x_{m+1}, \ldots, x_n)A,$$
$$M(R') = (x_1, (x_2/x_1) - r_2, \ldots, (x_m/x_1) - r_m, x_{m+1}, \ldots, x_n)R',$$

and R' is residually rational over R. The last two statements in (6) are obvious. This completes the proof of (1), (5), (6), and (7); also in view of what we have shown so far, (4) would follow from (3). Given $0 \neq w \in R$ such that $\operatorname{ord}_S w = \operatorname{ord}_R w$ let $d = \operatorname{ord}_R w$; then $\operatorname{ord}_S w = d$ and hence $w = f(x_1, \ldots, x_m)$ where $f(X_1, \ldots, X_m)$ is a nonzero homogeneous polynomial of degree d in indeterminates $X_1, \ldots, X_m$ with coefficients in R at least one of which is not in P; since $\operatorname{ord}_R w = d$, we get that at least one of the coefficients of

$f(X_1, \ldots, X_m)$ is not in $M(R)$; let $F(X_2, \ldots, X_m)$ be the polynomial in $X_2, \ldots, X_m$ with coefficients in k obtained by applying h to the coefficients of $f(1, X_2, \ldots, X_m)$; then $w/x_1^d \in A$, $H(w/x_1^d) = F(X_2, \ldots, X_m)$, and $F(X_2, \ldots, X_m)$ is a nonzero polynomial of degree $\leqslant d$ in $X_2, \ldots, X_m$ with coefficients in k; therefore by (1.3.2) we get that $\mathrm{ord}_{R^*}H(w/x_1^d) \leqslant d$, i.e., $\mathrm{ord}_{R^*}H^*(w/x_1^d) \leqslant d$; clearly $\mathrm{ord}_{R'}(w/x_1^d) \leqslant \mathrm{ord}_{R^*}H^*(w/x_1^d)$ and hence $\mathrm{ord}_{R'}(w/x_1^d) \leqslant d$. This proves (2). It now only remains to prove (3). Now R^* is regular and $\dim R^* = m - 1 - \mathrm{restrdeg}_k R^*$ (see [28: Theorem 20 on page 193 and Remark on page 310]); since $\mathrm{restrdeg}_k R^* = t$, we get that $\dim R^* = m - 1 - t$. Let D be any subset of A such that $H(D)R^* = M(R^*)$; then $H(D)A^* = Q^* \cap N_1 \cap \ldots \cap N_s$ where N_j is a primary ideal in A^* with $N_j \not\subset Q^*$ for $1 \leqslant j \leqslant s$; since $\mathrm{Ker}\, H = (x_1, x_{m+1}, \ldots, x_n)A$, we get that

$$DA + (x_1, x_{m+1}, \ldots, x_n)A = Q \cap H^{-1}(N_1) \cap \ldots \cap H^{-1}(N_s);$$

now $H^{-1}(N_j)$ is a primary ideal in A and $H^{-1}(N_j) \not\subset Q$ for $1 \leqslant j \leqslant s$; therefore $DR' + (x_1, x_{m+1}, \ldots, x_n)R' = M(R')$. Let m' be any integer with $1 \leqslant m' \leqslant m$ such that $x_i/x_1 \in M(R')$ for $2 \leqslant i \leqslant m'$ (for instance $m' = 1$). Let $A' = k[X_{m'+1}, \ldots, X_m]$. Since $X_2, \ldots, X_m$ are indeterminates, there exists a unique epimorphism H_0: $A^* \to A'$ such that $H_0(X_i) = 0$ for $2 \leqslant i \leqslant m'$ and $H_0(u) = u$ for all $u \in A'$; note that then $\mathrm{Ker}\, H_0 = (X_2, \ldots, X_{m'})A^* \subset Q^*$. Let $Q' = H_0(Q^*)$. Then Q' is a prime ideal in A' and there exists a unique epimorphism H_1: $R^* \to A'_{Q'}$ such that $H_1(u) = H_0(u)$ for all $u \in A^*$; note that then $\mathrm{Ker}\, H_1 = (X_2, \ldots, X_{m'})R^*$. Let H_2: $A'_{Q'} \to A'_{Q'}/M(A'_{Q'})$ be the canonical epimorphism and let $H_3(u) = H_2(H_1(u))$ for all $u \in R^*$. Then H_3: $R^* \to A'_{Q'}/M(A'_{Q'})$ is an epimorphism, $\mathrm{Ker}\, H_3 = M(R^*)$, and $H_3(k) = H_2(k)$. Consequently $\mathrm{restrdeg}_k A'_{Q'} = \mathrm{restrdeg}_k R^*$ and hence $\mathrm{restrdeg}_k A'_{Q'} = t$. Therefore upon letting $q = m - m' - t$ we get that $A'_{Q'}$ is regular and $\dim A'_{Q'} = q$ (see [28: Theorem 20 on page 193 and Remark on page 310]). Consequently there exist elements $y_1, \ldots, y_q$ in A such that $(H_0(H(y_1)), \ldots, H_0(H(y_q)))A'_{Q'} = M(A'_{Q'})$. Upon taking $D = \{x_2/x_1, \ldots, x_{m'}/x_1, y_1, \ldots, y_q\}$ we get that $H(D)R^* = M(R^*)$ and hence $(x_1, x_2/x_1, \ldots, x_{m'}/x_1, x_{m+1}, \ldots, x_n, y_1, \ldots, y_q)R' = M(R')$. Therefore if we show that $\dim R' \geqslant n - t$ then it will follow that $\dim R' = n - t$, R' is regular, and

$q = n' - n + m - m'$, and this will complete the proof. Since $\dim R^* = m - t - 1$, there exist distinct prime ideals $Q_1 \subset Q_2 \subset \dots \subset Q_{m-t}$ in A^* such that $Q_1 = \{0\}$ and $Q_{m-t} = Q^*$. Now $H^{-1}(Q_1) \subset H^{-1}(Q_2) \subset \dots \subset H^{-1}(Q_{m-t})$ are distinct prime ideals in A, $H^{-1}(Q_1) = \operatorname{Ker} H$, and $H^{-1}(Q_{m-t}) = Q$. We shall find distinct nonzero prime ideals $P''_m \subset P''_{m+1} \subset \dots \subset P''_n$ in A with $P''_n = \operatorname{Ker} H$ and this will prove that $\dim R' \geqslant n - t$. Let H' and H'' be as in (1.4.1). By (1.4.1) we know that $H'^{-1}(\operatorname{Ker} H'') = \operatorname{Ker} H$ and $\operatorname{Ker} H' \neq \{0\}$. Consequently it suffices to find distinct prime ideals $P_m \subset P_{m+1} \subset \dots \subset P_n$ in B with $P_n = \operatorname{Ker} H''$ because then we can take $P''_j = H'^{-1}(P_j)$ for $m \leqslant j \leqslant n$. Let $P'_j = (h'(x_{m+1}), \dots, h'(x_j))T$ for $m < j \leqslant n$, and $P'_m = \{0\}$. Then $P'_m \subset P'_{m+1} \subset \dots \subset P'_n$ are distinct prime ideals in T and $P'_n = M(T)$. Let $h''_j \colon T \to T/P'_j$ be the canonical epimorphism and let $P_j = P'_j B$. Since $X_2, \dots, X_n$ are indeterminates, there exists a unique epimorphism $H''_j \colon B \to h''_j(T)[X_2, \dots, X_m]$ such that $H''_j(X_i) = X_i$ for $2 \leqslant i \leqslant m$ and $H''_j(u) = h''_j(u)$ for all $u \in T$; clearly $\operatorname{Ker} H''_j = P_j$ and hence P_j is a prime ideal in B and $P_j \cap T = P'_j$. Therefore $P_m \subset P_{m+1} \subset \dots \subset P_n$ are distinct prime ideals in B. Also $P_n = M(T)B = \operatorname{Ker} H''$.

(1.5). Given a local domain R and $S \in \mathfrak{B}(R)$, we say that S has a *simple point* at R if $R/(R \cap M(S))$ is regular.

Let R be an n-dimensional regular local domain. Given $E \subset \mathfrak{B}(R)$, we say that E has a ***normal crossing*** at R if there exists a basis $(x_1, \dots, x_n)$ of $M(R)$ such that for each $S \in E$ there exists a subset y_S of $\{x_1, \dots, x_n\}$ such that $y_S R = R \cap M(S)$. Given $E \subset \mathfrak{B}(R)$, we say that E has a ***strict normal crossing*** at R if E has a normal crossing at R and E contains at most two elements. Given a nonzero principal ideal I in R, we say that I has a ***normal crossing*** at R if $\{S' \in \mathfrak{B}(R) \colon \dim S' = 1 \text{ and } IS' \neq S'\}$ has a normal crossing at R; note that this is equivalent to saying that there exists a basis $(x_1, \dots, x_n)$ of $M(R)$ and nonnegative integers $a_1, \dots, a_n$ such that $I = x_1^{a_1} \dots x_n^{a_n} R$. Given $E \subset \mathfrak{B}(R)$ and a nonzero principal ideal I in R, we say that (E, I) has a ***normal crossing*** at R if $E \cup \{S' \in \mathfrak{B}(R) \colon \dim S' = 1 \text{ and } IS' \neq S'\}$ has a normal crossing at R; note that this is equivalent to saying that there exists a basis $(x_1, \dots, x_n)$ of $M(R)$, nonnegative integers $a_1, \dots, a_n$, and a subset y_S of $\{x_1, \dots, x_n\}$ for each $S \in E$, such that $I = x_1^{a_1} \dots x_n^{a_n} R$ and $R \cap M(S) = y_S R$ for each $S \in E$. Given $E \subset \mathfrak{V}(R)$ and a nonzero principal ideal I in R, we say that (E, I) has a *strict normal crossing* at R if (E, I) has a normal crossing at R and E contains at most two elements. Given $S \in \mathfrak{V}(R)$ and a nonzero principal ideal I in R, we say that (S, I) has a *normal crossing* at R if $(\{S\}, I)$ has a normal crossing at R. Given nonzero principal ideals J and I in R, we say that (J, I) has a *quasinormal crossing* at R if I has a normal crossing at R and for every nonzero principal prime ideal P in R with $J \subset P$ we have that PI has a normal crossing at R. Given a nonzero principal ideal I in R, we say that I has a *quasinormal crossing* at R if (I, R) has a quasinormal crossing at R. Note that for any nonzero principal ideal I in R the following four conditions are equivalent: (1) (I, I') has a quasinormal crossing at R for some nonzero principal ideal I' in R; (2) I has a quasinormal crossing at R; (3) for every nonzero principal prime ideal P in R with $I \subset P$ we have that R_P has a simple point at R; (4) $I = z_1 \ldots z_d R$ where $z_1, \ldots, z_d$ are elements in R with $\mathrm{ord}_R z_i = 1$ for $1 \leqslant i \leqslant d$ (we take $z_1 \ldots z_d R = R$ in case $d = 0$). Given $S \in \mathfrak{V}(R)$ and a nonzero principal ideal I in R, we say that (S, I) has a *pseudonormal crossing* at R if S has a simple point at R and for every nonzero principal prime ideal P in R with $I \subset P$ we have that $\{S, R_P\}$ has a normal crossing at R. Note that for any nonzero principal ideal I in R the following three conditions are equivalent: (1*) (S, I) has a pseudonormal crossing at R for some $S \in \mathfrak{V}(R)$; (2*) (R, I) has a pseudonormal crossing at R; (3*) I has a quasinormal crossing at R. Given $E \subset \mathfrak{V}(R)$ and a nonzero principal ideal I in R, we say that (E, I) has a *pseudonormal crossing* at R if I has a quasinormal crossing at R and for every $S \in E$ we have that (S, I) has a pseudonormal crossing at R.

For any ideal J in a regular local domain R, the set of all $S \in \mathfrak{V}(R)$ such that $\mathrm{ord}_S J = \mathrm{ord}_R J$ is called the *equimultiple locus* of (R, J) and is denoted by $\mathfrak{E}(R, J)$; for any nonnegative integer i, the set of all i-dimensional elements in $\mathfrak{E}(R, J)$ is denoted by $\mathfrak{E}^i(R, J)$.

Let J be a nonzero principal ideal in a regular local domain R. We say that (R, J) is *resolved* if there exists a nonnegative integer d and a nonzero principal ideal J' in R with $\mathrm{ord}_R J' \leqslant 1$ such that $J = J'^d$. We say that (R, J) is *unresolved* if (R, J) is not resolved. Note that if either $\dim R \leqslant 1$ or $\mathrm{ord}_R J \leqslant 1$ then (R, J) is resolved.

Also note that if $\mathrm{ord}_R J \neq 0$ (i.e., if $J \neq R$) then the following six conditions are equivalent: (1′) (R, J) is resolved; (2′) $\mathrm{ord}_R(\mathrm{rad}_R J) = 1$; (3′) $R/(\mathrm{rad}_R J)$ is regular; (4′) $J = (\mathrm{rad}_R J)^d$ where $d = \mathrm{ord}_R J$; (5′) $\mathfrak{E}^1(R, J) \neq \varnothing$; (6′) $\mathrm{rad}_R J$ is a prime ideal in R and upon letting S' be the quotient ring of R with respect to $\mathrm{rad}_R J$ we have that

$$\mathfrak{E}(R, J) = \{S \in \mathfrak{B}(R) \colon S \subset S'\} = \{S \in \mathfrak{B}(R) \colon JS \neq S\}.$$

Note that if (R, J) is resolved and I is a nonzero principal ideal in R such that I has a quasinormal crossing at R then JI has a quasinormal crossing at R. Finally note that if (R, J) is resolved and I is a nonzero principal ideal in R such that (J, I) has a quasinormal crossing at R then JI has a normal crossing at R.

We shall now prove some elementary results concerning the above concepts; these results will not be used tacitly.

(1.5.1). *Let R be an n-dimensional regular local domain with $n > 0$, let $(x_1, \ldots, x_n)$ be a basis of $M(R)$, let $I = x_1^{a_1} \ldots x_n^{a_n} R$ where $a_1, \ldots, a_n$ are nonnegative integers, let m be an integer with $1 \leqslant m \leqslant n$, and let $z \in (x_1, \ldots, x_m)R$ with $\mathrm{ord}_R z = 1$ such that (zR, I) has a quasinormal crossing at R. Then there exists an integer j with $1 \leqslant j \leqslant m$ such that upon letting $y_j = z$ and $y_i = x_i$ for all $i \neq j$ with $1 \leqslant i \leqslant n$ we have that $M(R) = (y_1, \ldots, y_n)R$, $(x_1, \ldots, x_m)R = (y_1, \ldots, y_m)R$, and $I = y_1^{a_1} \ldots y_n^{a_n} R$.*

PROOF. Let A be the set of all integers i with $1 \leqslant i \leqslant m$ such that $a_i \neq 0$, and let B be the set of all integers i with $m < i \leqslant n$ such that $a_i \neq 0$. Now $z \in (x_1, \ldots, x_m)R$ and clearly $x_i \notin (x_1, \ldots, x_m)R$ whenever $m < i \leqslant n$; consequently, if $zR = x_q R$ for some $q \in A \cup B$ then we must have $q \in A$ and hence it suffices to take $j = q$. So now assume that $zR \neq x_i R$ whenever $i \in A \cup B$. Since $\mathrm{ord}_R z = 1$ and (zR, I) has a quasinormal crossing at R, there exists a basis $(z_1, \ldots, z_n)$ of $M(R)$ such that $z_i R = x_i R$ whenever $i \in A \cup B$, and $z_e R = zR$ for some $e \notin A \cup B$ with $1 \leqslant e \leqslant n$. Let P be the ideal in R generated by the set of all x_i with $i \in A$, and let $Q = (z_1, \ldots, z_{e-1}, z_{e+1}, \ldots, z_n)R$. Now $Q \neq M(R)$ and hence $M(R) \not\subset Q + M(R)^2$; consequently $z \notin Q + M(R)^2$; since $P \subset Q$, we get that $z \notin P + M(R)^2$. Since $z \in (x_1, \ldots, x_m)R$, we get that $z = r_1x_1 + \cdots + r_mx_m$ with $r_1, ..., r_m$ in R; since $z \notin P + M(R)^2$, we get that $r_p \notin M(R)$ for some $p \notin A$ with $1 \leqslant p \leqslant m$. It suffices to take $j = p$.

(1.5.2). *Let R be a regular local domain. Let I be a nonzero principal ideal in R, and let $S \in \mathfrak{B}(R)$ such that (S, I) has a normal crossing at R. Let $z \in R \cap M(S)$ such that* $\mathrm{ord}_R z = 1$ *and (zR, I) has a quasinormal crossing at R. Then (S, zI) has a normal crossing at R.*

PROOF. Follows from (1.5.1).

(1.5.3). *Let J be a nonzero principal ideal in a regular local domain R. Assume that (S, JS) is resolved for some $S \in \mathfrak{E}(R, J)$* (note that by [18: (28.3)] we know that S is regular). *Then (R, J) is resolved.*

PROOF. If $J = R$ then we have nothing to show. So assume that $J \neq R$. Then $J = P_1^{u_1} \ldots P_n^{u_n}$ where $P_1, ..., P_n$ $(n > 0)$ are distinct nonzero principal prime ideals in R and $u_1, ..., u_n$ are positive integers. Now $JS = (P_1S)^{u_1} \ldots (P_nS)^{u_n}$. Since $S \in \mathfrak{E}(R, J)$, we get that

$$\sum_{i=1}^{n} u_i \,\mathrm{ord}_S P_i S = \mathrm{ord}_S JS = \mathrm{ord}_R J = \sum_{i=1}^{n} u_i \,\mathrm{ord}_R P_i,$$

and by (1.3.1) we know that $\mathrm{ord}_S P_i S \leqslant \mathrm{ord}_R P_i$ for $1 \leqslant i \leqslant n$. Therefore we must have $\mathrm{ord}_S P_i S = \mathrm{ord}_R P_i > 0$ for $1 \leqslant i \leqslant n$, and hence $P_1S, ..., P_nS$ are distinct nonzero principal prime ideals in S. Since (S, JS) is resolved, we conclude that $n = 1$ and $\mathrm{ord}_S P_1 S = 1$. Therefore $\mathrm{ord}_R P_1 = 1$, and hence (R, J) is resolved.

(1.5.4). *Let R be a pseudogeometric regular local domain such that* $\dim R \leqslant 3$, *and $\mathfrak{S}(R, P)$ is closed in $\mathfrak{B}(R)$ for every nonzero principal prime ideal P in R*; (see (1.2.6)). *Let J be a nonzero principal ideal in R such that (R, J) is unresolved. Then $\mathfrak{E}^2(R, J)$ is a finite set.*

PROOF. The assertion is obvious if $\dim R \neq 3$. So assume that $\dim R = 3$. Since (R, J) is unresolved, we have that $J = P_1^{u_1} \ldots P_n^{u_n}$

where n, $u_1, \ldots, u_n$ are positive integers, and $P_1, \ldots, P_n$ are distinct nonzero principal prime ideals in R. If $n > 1$ then $\{S \in \mathfrak{V}(R)$: $\dim S = 2$ and $P_1 + P_2 \subset R \cap M(S)\}$ is a finite set and it contains $\mathfrak{E}^2(R, J)$, and hence $\mathfrak{E}^2(R, J)$ is a finite set. So also assume that $n = 1$, and let $P = P_1$. Then $\mathfrak{E}^2(R, J) = \mathfrak{E}^2(R, P)$. Since (R, J) is unresolved, we must have $\operatorname{ord}_R P > 1$. By [18: (28.3)] we know that every element in $\mathfrak{V}(R)$ is regular; consequently $\mathfrak{E}^2(R, P) \subset \mathfrak{S}(R, P)$ and hence $\mathfrak{E}^2(R, J) \subset \mathfrak{S}(R, P)$. Clearly $\dim S' \geqslant 2$ for all $S' \in \mathfrak{S}(R, P)$, and by assumption $\mathfrak{S}(R, P)$ is closed in $\mathfrak{V}(R)$. Therefore $\{S' \in \mathfrak{S}(R, P)$: $\dim S' = 2\}$ is a finite set, and hence $\mathfrak{E}^2(R, J)$ is a finite set.

(1.6). Let A be a subring of a field K. By a *premodel* of K we mean a nonempty set of quasilocal domains with quotient field K. By an *irredundant premodel* of K we mean a premodel E of K such that no two distinct elements in E are dominated by the same valuation ring of K. Note that if R and R' are two elements in an irredundant premodel E of K and S is a quasilocal domain dominating R and R' then $R' = R$ (namely, upon identifying K with a subfield of the quotient field L of S and taking a valuation ring V of L dominating S we get that $V \cap K$ is a valuation ring of K dominating R and R'; hence $R = R'$); in this case we say that R is the *center* of S on E. By a *semimodel* (resp: *model*) of K/A (i.e., of K over A) we mean an irredundant premodel E of K such that there exists a family (resp: finite family) $(B_i)_{i\in I}$ of subrings B_i of K where each B_i is an overring of A (resp: affine ring over A) such that $E = \bigcup_{i\in I} \mathfrak{V}(B_i)$. Note that if B is any subring of K such that K is the quotient field of B and B is an overring of A (resp: affine ring over A) then $\mathfrak{V}(B)$ is a semimodel (resp: model) of K/A. Also note that for an irredundant premodel E of K we have that E is a semimodel of K/A if and only if for every $R \in E$ we have that $A \subset R$ and $\mathfrak{V}(R) = \{R' \in E\colon R \subset R'\}$. Also note that every model of K/A is a semimodel of K/A, and if A is noetherian then every element in E is a local ring. By a *complete semimodel* (resp: *complete model*) of K/A we mean a semimodel (resp: model) E of K/A such that every valuation ring of K containing A dominates E. Note that if K is the quotient field of A then $\mathfrak{V}(A)$ is a complete model of K/A. Also note that if E is a semimodel (resp: complete semimodel) of K/A then E dominates (resp: properly dominates) $\mathfrak{B}(A)$.

(1.7). Let A be a subring of a field K and let $(x_i)_{i\in I}$ be a family of elements in K such that $x_{i'} \neq 0$ for some $i' \in I$. We *define*

$$\mathfrak{W}(A; (x_i)_{i\in I}) = \bigcup_{j\in I, x_j \neq 0} \mathfrak{B}(A[(x_i/x_j)_{i\in I}])$$

where $A[(x_i/x_j)_{i\in I}]$ denotes the smallest subring of K which contains A and which contains x_i/x_j for all $i \in I$ (in case I is a finite set, say $I = \{1, 2, ..., n\}$, we may write $\mathfrak{W}(A; x_1, ..., x_n)$ instead of $\mathfrak{W}(A; (x_i)_{i\in I})$).

(1.7.1). *Note that for any $0 \neq x \in K$ we have that $\mathfrak{W}(A; (x_i)_{i\in I}) = \mathfrak{W}(A; x_i/x)_{i\in I})$. Taking any $i' \in I$ with $x_{i'} \neq 0$ and letting K' be the quotient field of $A[(x_i/x_{i'})_{i\in I}]$ we see that K' is the quotient field of $A[(x_i/x_j)_{i\in I}]$ for each $j \in I$ with $x_j \neq 0$; whence in particular, $\mathfrak{W}(A; (x_i)_{i\in I})$ is a premodel of K', $(x_i/x_{i'})_{i\in I}$ is a family of elements in K', and $\mathfrak{W}(A; (x_i)_{i\in I}) = \mathfrak{W}(A; (x_i/x_{i'})_{i\in I})$.*

We shall now prove the following.

(1.7.2). *Let $R \in \mathfrak{W}(A; (x_i)_{i\in I})$ and let S be a quasilocal ring such that S is a subring of K and S dominates R. Then there exists $j \in I$ such that $x_j \neq 0$ and $x_i/x_j \in S$ for all $i \in I$; moreover, for any such $j \in I$ we have that $R = B_Q$ where $B = A[(x_i/x_j)_{i\in I}]$ and $Q = B \cap M(S)$.*

PROOF. Since $R \in \mathfrak{W}(A; (x_i)_{i\in I})$, there exists $j' \in I$ with $x_{j'} \neq 0$ such that $R \in \mathfrak{B}(B')$ where $B' = A[(x_i/x_{j'})_{i\in I}]$, and then we have that $x_i/x_{j'} \in R$ for all $i \in I$ and $R = B'_{Q'}$ where $Q' = B' \cap M(R)$. Since S dominates R, we get that $x_i/x_{j'} \in S$ for all $i \in I$ and $Q' = B' \cap M(S)$. Therefore the first assertion follows by taking $j = j'$. To prove the second assertion, given any $j \in I$ such that $x_j \neq 0$ and $x_i/x_j \in S$ for all $i \in I$, let $B = A[(x_i/x_j)_{i\in I}]$ and $Q = B \cap M(S)$. Then $x_j/x_{j'}$ and $x_{j'}/x_j$ are both in S and hence they are units in S. Since $x_j/x_{j'} \in R$, S dominates R, and $x_j/x_{j'}$ is a unit in S, we get that $x_j/x_{j'}$ is a unit in R and hence $x_{j'}/x_j \in R$; consequently $x_i/x_j = (x_i/x_{j'})(x_{j'}/x_j) \in R$ for all $i \in I$ and hence

$B \subset R$; since S dominates R and $Q = B \cap M(S)$, we get that $Q = B \cap M(R)$ and hence $B_Q \subset R$. Again since $x_{j'}/x_j \in B_Q$, S dominates B_Q, and $x_{j'}/x_j$ is a unit in S, we get that $x_{j'}/x_j$ is a unit in B_Q and hence $x_j/x_{j'} \in B_Q$; consequently $x_i/x_{j'} = (x_i/x_j)(x_j/x_{j'}) \in B_Q$ for all $i \in I$ and hence $B' \subset B_Q$; since S dominates B_Q and $Q' = B' \cap M(S)$, we get that $Q' = B' \cap M(B_Q)$ and hence $B'_{Q'} \subset B_Q$. Since $R = B'_{Q'}$, we conclude that $R = B_Q$.

(1.7.3). *Let K' be as in* (1.7.1). *Then $\mathfrak{W}(A; (x_i)_{i\in I})$ is a semimodel of K'/A. If I is a finite set then $\mathfrak{W}(A; (x_i)_{i\in I})$ is a complete model of K'/A.*

PROOF. The first assertion follows from (1.7.1) and (1.7.2). To prove the second assertion assume that I is a finite set. In view of the first assertion it suffices to show that if V is any valuation ring of K' containing A then V dominates $\mathfrak{W}(A; (x_i)_{i\in I})$. Since I is a finite set, there exists $j \in I$ with $x_j \neq 0$ such that $x_i/x_j \in V$ for all $i \in I$. Let $B = A[(x_i/x_j)_{i\in I}]$ and $Q = B \cap M(V)$. Then V dominates B_Q and $B_Q \in \mathfrak{W}(A; (x_i)_{i\in I})$.

(1.8). Let A be a subring of a field K. By a *projective model* of K/A we mean a premodel E of K/A such that there exists a finite number of elements $x_1, \ldots, x_n$ in an overfield of K such that $x_i \neq 0$ for some i and $E = \mathfrak{W}(A; x_1, \ldots, x_n)$. By (1.7) it follows that if E is a projective model of K/A then E is a complete model of K/A and there exists a finite number of elements $x_1, \ldots, x_n$ in K such that $x_i \neq 0$ for some i and $E = \mathfrak{W}(A; x_1, \ldots, x_n)$.

(1.9). Let A be a subring of a field K and let P be a nonzero A-submodule of K. We *define*

$$\mathfrak{W}(A, P) = \bigcup_{0 \neq x \in P} \mathfrak{V}(A[Px^{-1}])$$

where $A[Px^{-1}]$ denotes the smallest subring of K which contains A and which contains y/x for all $y \in P$.

(1.9.1). *Note that for any $0 \neq x \in P$ we have that $P(A[Px^{-1}]) = x(A[Px^{-1}])$ and hence $PR = xR$ for all $R \in \mathfrak{V}(A[Px^{-1}])$; whence in* *particular, if P is an ideal in A then PR is a nonzero principal ideal in R for all $R \in \mathfrak{W}(A, P)$.*

(1.9.2). *Given any A-basis $(x_i)_{i \in I}$ of P we clearly have that $A[Px^{-1}] = A[(x_i/x)_{i \in I}]$ for all $0 \neq x \in K$; whence in particular, $\mathfrak{W}(A; (x_i)_{i \in I}) \subset \mathfrak{W}(A, P)$.*

We shall now prove the following.

(1.9.3). *For any A-basis $(x_i)_{i \in I}$ of P we have that $\mathfrak{W}(A; (x_i)_{i \in I}) = \mathfrak{W}(A, P)$.*

PROOF. In view of (1.9.2) it suffices to show that $\mathfrak{W}(A, P) \subset \mathfrak{W}(A; (x_i)_{i \in I})$. So let any $R \in \mathfrak{W}(A, P)$ be given. Then there exists $0 \neq x \in P$ such that $R = B_Q$ where $B = A[Px^{-1}] = A[(x_i/x)_{i \in I}] \subset R$ and $Q = B \cap M(R)$. Since $0 \neq x \in P$, there exists a nonempty finite subset I' of I such that $x = \sum_{i \in I'} r_i x_i$ with $r_i \in A$. Then $1 = \sum_{i \in I'} r_i(x_i/x)$ and $r_i \in R$ and $(x_i/x) \in R$ for all $i \in I'$, and hence there exists $j \in I'$ such that $x_j/x \notin M(R)$. Consequently $x_j \neq 0$, and x_j/x and x/x_j are units in R. In particular $x_i/x_j = (x_i/x)(x/x_j) \in R$ for all $i \in I$ and hence $B' \subset R$ where $B' = A[(x_i/x_j)_{i \in I}]$. Upon letting $Q' = B' \cap M(R)$ we get that $B'_{Q'} \in \mathfrak{W}(A; (x_i)_{i \in I})$ and R dominates $B'_{Q'}$. Since $x/x_j \in B'_{Q'}$, R dominates $B'_{Q'}$, and x/x_j is a unit in R, we get that x/x_j is a unit in $B'_{Q'}$ and hence $x_j/x \in B'_{Q'}$. Consequently $x_i/x = (x_i/x_j)(x_j/x) \in B'_{Q'}$ for all $i \in I$ and hence $B \subset B'_{Q'}$; since R dominates $B'_{Q'}$ and $Q = B \cap M(R)$, we get that $Q = B \cap M(B'_{Q'})$ and hence $B_Q \subset B'_{Q'}$, i.e., $R \subset B'_{Q'}$. Therefore $R = B'_{Q'}$, and hence $R \in \mathfrak{W}(A; (x_i)_{i \in I})$. Thus $\mathfrak{W}(A, P) \subset \mathfrak{W}(A; (x_i)_{i \in I})$.

(1.9.4). *$\mathfrak{W}(A, P)$ is a semimodel of a field over A, and if P is a finitely generated A-module then $\mathfrak{W}(A, P)$ is a projective model of a field over A; in particular, if P is a finitely generated ideal in A then $\mathfrak{W}(A, P)$ is a projective model of the quotient field of A over A. If $P = xA$ for some $0 \neq x \in K$ then $\mathfrak{W}(A, P) = \mathfrak{W}(A; x) = \mathfrak{V}(A)$.*

PROOF. Follows from (1.9.3) and (1.7).

(1.9.5). *If P is an ideal in A then $\{R \in \mathfrak{V}(A)\colon PR = R\} = \{R \in \mathfrak{W}(A, P)\colon PR = R\}$.*

PROOF. First let $R \in \mathfrak{B}(A)$ such that $PR = R$; then $P \not\subset M(R)$ and hence there exists $0 \neq x \in P$ such that $x \notin M(R)$; now $A[Px^{-1}] \subset R$ and hence, upon letting $Q = A[Px^{-1}] \cap M(R)$, we get that $R = (A[Px^{-1}])_Q \in \mathfrak{W}(A, P)$. Conversely let $R \in \mathfrak{W}(A, P)$ such that $PR = R$; now $\mathfrak{W}(A, P)$ dominates $\mathfrak{B}(A)$ and hence there exists $R' \in \mathfrak{B}(A)$ such that R dominates R'; since R dominates R' and $PR = R$, it follows that $PR' = R'$; therefore $R' \in \mathfrak{W}(A, P)$ by what we have already proved; since R dominates R as well as R', and, by (1.9.4), $\mathfrak{W}(A, P)$ is an irredundant premodel of the quotient field of A, we must have $R = R'$; consequently $R \in \mathfrak{B}(A)$.

(1.9.6). *If A is quasilocal and P is an ideal in A then the following three conditions are equivalent:* (1) *P is a principal ideal in A*; (2) $\mathfrak{W}(A, P) = \mathfrak{B}(A)$; (3) $A \in \mathfrak{W}(A, P)$.

PROOF. If P is a principal ideal in A then $P = xA$ for some $0 \neq x \in A$ and then $\mathfrak{W}(A, P) = \mathfrak{W}(A; x) = \mathfrak{B}(A)$; thus (1) implies (2). Clearly (2) implies (3). To show that (3) implies (1), assume that $A \in \mathfrak{W}(A, P)$; then there exists $0 \neq x \in P$ such that $A \in \mathfrak{B}(A[Px^{-1}])$; in particular $A[Px^{-1}] \subset A$ and hence $A[Px^{-1}] = A$; consequently $PA = P(A[Px^{-1}]) = x(A[Px^{-1}]) = xA$ and hence P is a principal ideal in A.

(1.9.7). *Assume that A is a regular local domain, and P is a prime ideal in A such that A/P is regular and* $\dim S > 1$ *where $S = A_P$. Let S' be the valuation ring of* ord_S, *and let $R' \in \mathfrak{W}(A, P)$ such that R' dominates A. Then* (1) *$S' \in \mathfrak{B}(R')$ and $S \notin \mathfrak{B}(R')$. Moreover,* (2) *if P_1 is a nonzero prime ideal in A such that A/P_1 is regular and $R' \in \mathfrak{W}(A, P_1)$ then $P_1 = P$.*

PROOF. By (1.4) we know that $S' \in \mathfrak{B}(R')$. Since $\dim S > 1 = \dim S'$, we get that $S \neq S'$. Since $S' \in \mathfrak{B}(R') \subset \mathfrak{W}(A, P)$, $\mathfrak{W}(A, P)$ is an irredundant premodel of the quotient field of A, and S' dominates S, we get that $S \notin \mathfrak{W}(A, P)$. This proves (1). To prove (2) let P_1 be a nonzero prime ideal in A such that A/P_1 is regular and $R' \in \mathfrak{W}(A, P_1)$. Let $S_1 = A_{P_1}$. Since $\dim S > 1$, we get that P is not a principal ideal in A and hence $R' \neq A$ by (1.9.6) because $R' \in \mathfrak{W}(A, P)$; since $R' \neq A$ and $R' \in \mathfrak{W}(A, P_1)$, again by (1.9.6) we get that P_1 is not a principal ideal in A and hence $\dim S_1 > 1$.

Now $S' \in \mathfrak{V}(R') \subset \mathfrak{W}(A, P_1)$, $\mathfrak{W}(A, P_1)$ is an irredundant premodel of the quotient field of A, $S' \neq S$, and S' dominates S; hence $S \notin \mathfrak{W}(A, P_1)$. Since $S \notin \mathfrak{W}(A, P_1)$ and $S \in \mathfrak{V}(A)$, by (1.9.5) we get that $P_1 S \neq S$ and hence $S \subset S_1$; by symmetry we get that $S_1 \subset S$. Therefore $S_1 = S$ and hence $P_1 = P$.

(1.10). Let R be a local domain, let $S \in \mathfrak{V}(R)$ with $\dim S > 0$, let J be an ideal in R, and let V be a valuation ring of the quotient field of R dominating R. By a *monoidal transform* of (R, S) we mean an element in $\mathfrak{W}(R, R \cap M(S))$ dominating R. Since $\mathfrak{W}(R, R \cap M(S))$ is a projective model of the quotient field of R over R, there exists a unique element R^* in $\mathfrak{W}(R, R \cap M(S))$ such that V dominates R^*; clearly R^* dominates R and hence R^* is a monoidal transform of (R, S); R^* is called the *monoidal transform* of (R, S) *along* V. Given a monoidal transform R' of (R, S), we define the (R, S, R')-*transform* of J to be the ideal in R' generated by the set of all elements r in R' such that $rx^d \in J$ for some nonnegative integer d and some element x in R' for which $xR' = (R \cap M(S))R'$. By a *monoidal transform* of (R, J, S) we mean a pair (R', J') where R' is a monoidal transform of (R, S) and J' is the (R, S, R')-transform of J. By the *monoidal transform* of (R, J, S) *along* V we mean the pair (R^*, J^*) where R^* is the monoidal transform of (R, S) along V and J^* is the (R, S, R^*)-transform of J. By a *quadratic transform* of R we mean a monoidal transform of (R, R). By the *quadratic transform* of R *along* V we mean the monoidal transform of (R, R) along V. In this chapter we shall use monoidal transforms (which are not quadratic transforms) only when R is regular, S has a simple point at R, and J is a nonzero principal ideal in R; note that in this case the considerations of (1.4) apply.

Given a regular local domain R, by an *iterated monoidal transform* of R we mean a local domain R^* such that there exist finite sequences $(R_i)_{0 \leqslant i \leqslant m}$ and $(S_i)_{0 \leqslant i < m}$ such that: m is a nonnegative integer; R_i is a local domain for $0 \leqslant i \leqslant m$; S_i is a positive-dimensional element in $\mathfrak{V}(R_i)$ having a simple point at R_i for $0 \leqslant i < m$; R_i is a monoidal transform of (R_{i-1}, S_{i-1}) for $0 < i \leqslant m$; $R_0 = R$; and $R_m = R^*$. Note that for any iterated monoidal transform R^* of a regular local domain R we have that: R^* is regular, R^* and R have the same quotient field, R^* is a spot over R, R^* dominates R, $\dim R^* + \operatorname{restrdeg}_R R^* = \dim R$ and $h(R^*)$ is a function field over $h(R)$ where $h\colon R^* \to R^*/M(R^*)$ is the canonical epimorphism; whence in particular the following three conditions are equivalent: (1) $\dim R^* = \dim R$; (2) R^* is residually algebraic over R; (3) R^* is residually finite algebraic over R. Also note that if R^* is an iterated monoidal transform of a regular local domain R such that $R^* \neq R$ then $0 < \dim R^* \leqslant \dim R$. Given a regular local domain R and a valuation ring V of the quotient field of R dominating R, by an *iterated monoidal transform of R along V* we mean an iterated monoidal transform R^* of R such that V dominates R^*.

(1.10.1). Let R be a regular local domain and let S be a positive-dimensional element in $\mathfrak{V}(R)$ having a simple point at R. Then $R \cap M(S)$ is a principal ideal in R if and only if $\dim S = 1$, and hence by (1.9.6) we get that the following three conditions are equivalent: (1) $\dim S = 1$; (2) R is a monoidal transform of (R, S); (3) R is the only monoidal transform of (R, S). Although we shall not make any use of it, we note the following consequence of (1.9.7): If R' is any monoidal transform of (R, S) such that $R' \neq R$ then S is uniquely determined by the pair (R, R'), i.e., if S_1 is any positive-dimensional element in $\mathfrak{V}(R)$ having a simple point at R such that R' is a monoidal transform of (R, S_1) then $S_1 = S$.

(1.10.2). Let R be a regular local domain, let S be a positive-dimensional element in $\mathfrak{V}(R)$ having a simple point at R, let J be a nonzero principal ideal in R, and let (R', J') be a monoidal transform of (R, J, S). We can then take $w \in R$ with $wR = J$ and $x \in R'$ with $xR' = (R \cap M(S))R'$, and then upon letting $d = \operatorname{ord}_S J$ we clearly have that $w/x^d \in R'$ and $(w/x^d)R' = J'$. Therefore by (1.4) we get that: *if $S \in \mathfrak{C}(R, J)$ and $\dim S > 1$ then $\operatorname{ord}_{R'} J' \leqslant \operatorname{ord}_R J$.* Also note that: *if $S \in \mathfrak{C}(R, J)$ and $\dim S = 1$ then $R' = R$ and hence $J' = R'$, i.e., $\operatorname{ord}_{R'} J' = 0$.*

(1.10.3). Let R be a regular local domain, let S be a positive-dimensional element in $\mathfrak{V}(R)$ having a simple point at R, and let J and I be nonzero principal ideals in R. Given a monoidal transform R' of (R, S), we define the *(R, S, R')-transform* of (J, I)

to be the pair (J', I') where J' is the (R, S, R')-transform of J and $I' = (IR')((R \cap M(S))R')^d$ where $d = \mathrm{ord}_S J$; note that then I' is the unique principal ideal in R' such that $J'I' = (JI)R'$. By a *monoidal transform* of (R, J, I, S) we mean a triple (R', J', I') where R' is a monoidal transform of (R, S) and (J', I') is the (R, S, R')-transform of (J, I). Given a valuation ring V of the quotient field of R dominating R, by the *monoidal transform* of (R, J, I, S) *along* V we mean the triple (R^*, J^*, I^*) where R^* is the monoidal transform of (R, S) along V and (J^*, I^*) is the (R, S, R^*)-transform of (J, I).

To avoid repetition we shall now prove some more results of an elementary nature concerning monoidal transforms; these results will not be used tacitly.

(1.10.4). *Let R be a regular local domain, let J be a nonzero principal ideal in R such that (R, J) is resolved, let S be a positive-dimensional element in $\mathfrak{C}(R, J)$ having a simple point at R, and let (R', J') be a monoidal transform of (R, J, S). Then (R', J') is resolved.*

PROOF. If $J = R$ then $J' = R'$ and we have nothing to show. So assume that $J \neq R$. Then $J = y^d R$ where $d = \mathrm{ord}_R J$ and $y \in R$ with $\mathrm{ord}_R y = 1$. Let $n = \dim R$ and $m = \dim S$. If $m = 1$ then $J' = R'$ by (1.10.2) and we have nothing to show. So also assume that $m > 1$. Since S has a simple point at R, there exists a basis $(y_1, ..., y_n)$ of $M(R)$ such that $R \cap M(S) = (y_1, ..., y_m)R$. Since $S \in \mathfrak{C}(R, J)$, we get that $y \in (y_1, ..., y_m)$ and hence there exists an integer j' with $1 \leqslant j' \leqslant m$ such that upon letting $(x_1, ..., x_n) = (y_1, ..., y_{j'-1}, y, y_{j'+1}, ..., y_n)$ we have that $M(R) = (x_1, ..., x_n)R$ and $R \cap M(S) = (x_1, ..., x_m)R$. Upon relabeling $(x_1, ..., x_m)$ we may assume that $x_i/x_1 \in R'$ for $2 \leqslant i \leqslant m$ and $J = x_j^d R$ for some j with $1 \leqslant j \leqslant m$. Now $J' = (x_j/x_1)^d R'$. If $x_j/x_i \notin M(R')$ then $J' = R'$ and we have nothing to show. So now assume that $x_j/x_1 \in M(R')$. Then we must have $2 \leqslant j \leqslant m$. Let $n' = \dim R'$. Then $n' \geqslant 2$ and there exists a basis $(z_1, ..., z_{n'})$ of $M(R')$ such that $z_1 = x_1$ and $z_2 = x_j/x_1$. In particular $\mathrm{ord}_{R'}(x_j/x_1) = 1$ and hence (R', J') is resolved.

(1.10.5). *Let R be a regular local domain, let J be a nonzero principal ideal in R such that (R, J) is unresolved, let S be a positive-*

dimensional element in $\mathfrak{E}(R, J)$ *having a simple point at* R, *and let* (R', J') *be a monoidal transform of* (R, J, S) *such that* $\mathrm{ord}_{R'}J' = \mathrm{ord}_R J$. *Then* $\dim S > 1$ *and* (R', J') *is unresolved.*

PROOF. Let $d = \mathrm{ord}_R J$. Then $d > 0$ and hence by (1.10.2) we get that $\dim S > 1$. We can take $w \in R$ and $x \in R'$ such that $wR = J$ and $xR' = (R \cap M(S))R'$. Then $w/x^d \in R'$, $(w/x^d)R' = J'$, $w/x^d \notin xR'$, and $\mathrm{ord}_{R'}x = 1$. Suppose if possible that (R', J') is resolved. Then $(w/x^d)R' = y^dR'$ with $y \in R'$ such that $\mathrm{ord}_{R'}y = 1$. Let $R^* = R'_{yR'}$. Then R^* is a one-dimensional regular local domain and $\mathrm{ord}_{R^*}(w/x^d) = d$. Also $x \notin yR'$ and hence $\mathrm{ord}_{R^*}w = d$ and $(R \cap M(S))R^* = R^*$. Now $R^* \in \mathfrak{V}(R') \subset \mathfrak{W}(R, R \cap M(S))$ and $(R \cap M(S))R^* = R^*$, and hence by (1.9.5) we get that $R^* \in \mathfrak{V}(R)$. Thus R^* is a one-dimensional regular local domain, $R^* \in \mathfrak{V}(R)$, and $\mathrm{ord}_{R^*}J = d = \mathrm{ord}_R J$; consequently $R \cap M(R^*)$ is a principal ideal in R with $\mathrm{ord}_R(R \cap M(R^*)) = 1$ and $J = (R \cap M(R^*))^d$. This contradicts the assumption that (R, J) is unresolved.

(1.10.6). *Let* R *be a regular local domain, let* J *and* I *be nonzero principal ideals in* R, *let* S *be a positive-dimensional element in* $\mathfrak{V}(R)$ *such that* (S, I) *has a normal crossing at* R, *and let* (R', J', I') *be a monoidal transform of* (R, J, I, S). *Then* I' *has a normal crossing at* R'.

PROOF. Let $d = \mathrm{ord}_S J$, $n = \dim R$, $m = \dim S$, and $n' = \dim R'$. Since (S, I) has a normal crossing at R, there exists a basis $(x_1, \ldots, x_n)$ of $M(R)$ and nonnegative integers $a(1), \ldots, a(n)$ such that $I = x_1^{a(1)} \ldots x_n^{a(n)}R$ and $R \cap M(S) = (x_1, \ldots, x_m)R$. Upon relabeling $x_1, \ldots, x_m$ we may assume that $x_i/x_1 \in M(R')$ for $2 \leqslant i \leqslant p$ and $x_i/x_1 \in R' - M(R')$ for $p < i \leqslant m$, where p is an integer with $1 \leqslant p \leqslant m$. Let $q = n' - n + m - p$. Then $q \geqslant 0$ and there exist elements $y_1, \ldots, y_q$ in R' such that $M(R') = (x_1, x_2/x_1, \ldots, x_p/x_1, x_{m+1}, \ldots, x_n, y_1, \ldots, y_q)R'$. Now

$$I' = x_1^{d+a(1)+\ldots+a(m)}(x_2/x_1)^{a(2)} \ldots (x_p/x_1)^{a(p)}x_{m+1}^{a(m+1)} \ldots x_n^{a(n)}R'$$

and hence I' has a normal crossing at R'.

(1.10.7). *Let* R *be a regular local domain, let* J *and* I *be nonzero principal ideals in* R *such that* (J, I) *has a quasinormal crossing* *at R, let S be a positive-dimensional element in $\mathfrak{C}(R, J)$ such that (S, I) has a normal crossing at R, and let (R', J', I') be a monoidal transform of (R, J, I, S). Then (J', I') has a quasinormal crossing at R'.*

PROOF. We can take $x \in R$ such that $\operatorname{ord}_R x = 1 = \operatorname{ord}_S x$ and $(R \cap M(S))R' = xR'$. Let $d = \operatorname{ord}_R J$. By (1.10.6) we know that I' has a normal crossing at R'. If $d = 0$ then $J' = R'$ and we have nothing more to show. So assume that $d \neq 0$. Now $J = z_1 \dots z_d R$ with $z_i \in R \cap M(S)$ such that $\operatorname{ord}_R z_i = 1 = \operatorname{ord}_S z_i$ and $z_i I$ has a normal crossing at R for $1 \leqslant i \leqslant d$. By (1.5.2) it follows that $(S, z_i I)$ has a normal crossing at R, and hence, upon letting (J_i', I_i') be the (R, S, R')-transform of $(x^{d-1}R, z_i I)$, by (1.10.6) we get that I_i' has a normal crossing at R' for $1 \leqslant i \leqslant d$. Now $J' = (z_1/x) \dots (z_d/x)R'$, and for $1 \leqslant i \leqslant d$ we have that $z_i/x \in R'$ and $(z_i/x)I' = I_i'$. Therefore (J', I') has a quasinormal crossing at R'.

(1.10.8). *Let R be a regular local domain, let J and I be nonzero principal ideals in R, let S be a positive-dimensional element in $\mathfrak{B}(R)$ such that (S, I) has a pseudonormal crossing at R, and let (R', J', I') be a monoidal transform of (R, J, I, S). Then I' has a quasinormal crossing at R'.*

PROOF. We can take $x \in R$ such that $\operatorname{ord}_R x = 1 = \operatorname{ord}_S x$ and $(R \cap M(S))R' = xR'$. Let $d = \operatorname{ord}_S J$ and $e = \operatorname{ord}_R I$. Then $I' = x^d(IR')$, and $I = z_1 \dots z_e R$ where $z_1, \dots, z_e$ are elements in R such that $\operatorname{ord}_R z_i = 1$ and $(S, z_i R)$ has a normal crossing at R for $1 \leqslant i \leqslant e$ (we take $z_1 \dots z_e R = R$ in case $e = 0$). Upon taking $(x^d R, R)$ for (J, I) in (1.10.6) we get that $x^d R'$ has a normal crossing at R'. For $1 \leqslant i \leqslant e$, upon taking $(R, z_i R)$ for (J, I) in (1.10.6) we get that $z_i R'$ has a normal crossing at R'. Since $I' = z_1 \dots z_e x^d R'$, it follows that I' has a quasinormal crossing at R'.

(1.10.9). *Let R be an n-dimensional regular local domain, let $(x_1, \dots, x_n)$ be a basis of $M(R)$, let S be the quotient ring of R with respect to $(x_1, \dots, x_m)R$ for some m with $1 \leqslant m \leqslant n$, let R' be a monoidal transform of (R, S) such that $x_i/x_1 \in R'$ for $1 \leqslant i \leqslant m$, let $S' \in \mathfrak{B}(R) \cap \mathfrak{B}(R')$ such that $S \subset S'$ and $S \neq S'$, and let $z \in R \cap M(S')$. Then $z/x_1 \in R' \cap M(S')$.*

PROOF. Now $z \in R \cap M(S') \subset R \cap M(S)$, $R \cap M(S) \not\subset M(S')$, $R' \subset S'$, and $(R \cap M(S))R' = x_1R'$. Therefore $z/x_1 \in R'$ and $x_1 \notin R' \cap M(S')$. Since $R' \cap M(S')$ is a prime ideal in R' and $z = (z/x_1)x_1 \in R' \cap M(S')$, we must have $z/x_1 \in R' \cap M(S')$.

(1.10.10). *Let R be an n-dimensional regular local domain, let $(x_1, \ldots, x_n)$ be a basis of $M(R)$, let S and S' be the quotient rings of R with respect to $(x_1, \ldots, x_m)R$ and $(x_2, \ldots, x_q)R$ respectively where $1 \leqslant m \leqslant q \leqslant n$, and let R' be a monoidal transform of (R, S). Then we have the following: (1) $S' \in \mathfrak{V}(R')$ if and only if $x_i/x_1 \in M(R')$ for $2 \leqslant i \leqslant m$. (2) If $S' \in \mathfrak{V}(R')$ then* $\dim R' = n$, *$M(R') = (x_1, x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_n)R'$, and $R' \cap M(S') = (x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_q)R'$.*

PROOF. Clearly $x_1/x_i \notin S'$ for $2 \leqslant i \leqslant m$; consequently, if $S' \in \mathfrak{V}(R')$ then $x_1/x_i \notin R'$ for $2 \leqslant i \leqslant m$; hence, if $S' \in \mathfrak{V}(R')$ then $x_i/x_1 \in M(R')$ for $2 \leqslant i \leqslant m$. Now assume that $x_i/x_1 \in M(R')$ for $2 \leqslant i \leqslant m$. Then $\dim R' = n$, $M(R') = (x_1, x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_n)R'$, and upon letting $A = R[x_2/x_1, \ldots, x_m/x_1]$ and $Q = A \cap M(R')$ we get that $A \subset R'$, Q is a prime ideal in A, $R' = A_Q$, and $Q = (x_1, x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_n)A$. Clearly $A \subset S'$ and hence upon letting $P = A \cap M(S')$ and $P' = (x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_q)A$ we get that $P' \subset P$, P is a prime ideal in A, and $S' = A_P$. Clearly $P' \subset Q$. Hence if we show that $P = P'$ then it will follow that $S' \in \mathfrak{V}(R')$ and $R' \cap M(S') = (x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_q)R'$. To show that $P = P'$ let any $0 \neq z \in P$ be given; since $0 \neq z \in A$, there exists a nonzero homogeneous polynomial $f(X_1, \ldots, X_m)$ of some degree e in indeterminates $X_1, \ldots, X_m$ with coefficients in R such that $zx_1^e = f(x_1, \ldots, x_m)$; now $zx_1^e \in P \cap R = M(S') \cap R = (x_2, \ldots, x_q)R$ and hence $f(x_1, \ldots, x_m) \in (x_2, \ldots, x_q)R$; clearly $f(x_1, \ldots, x_m) - x_1^e f(1, 0, \ldots, 0) \in (x_2, \ldots, x_m)R \subset (x_2, \ldots, x_q)R$ and hence $x_1^e f(1, 0, \ldots, 0) \in (x_2, \ldots, x_q)R$; also $x_1^e \notin (x_2, \ldots, x_q)R$ and hence $f(1, 0, \ldots, 0) \in (x_2, \ldots, x_q)R \subset P'$; clearly $z = f(1, x_2/x_1, \ldots, x_m/x_1)$ and $f(1, x_2/x_1, \ldots, x_m/x_1) - f(1, 0, \ldots, 0) \in (x_2/x_1, \ldots, x_m/x_1)A \subset P'$; therefore $z \in P'$. Thus $P \subset P'$ and hence $P = P'$.

(1.10.11). *Let R be an n-dimensional regular local domain, let R' be a quadratic transform of R, and let E be a set of $(n-1)$-*

dimensional elements in $\mathfrak{V}(R)$ *such that every subset of* E *having at most two elements has a normal crossing at* R. *Then* $E \cap \mathfrak{V}(R')$ *contains at most one element and* $E \cap \mathfrak{V}(R')$ *has a normal crossing at* R'.

PROOF. If $E \cap \mathfrak{V}(R') = \varnothing$ then we have nothing to show. So assume that $E \cap \mathfrak{V}(R') \neq \varnothing$ and take $S' \in E \cap \mathfrak{V}(R')$. By assumption S' has a simple point at R and hence there exists a basis $(x_1, ..., x_n)$ of $M(R)$ such that $R \cap M(S') = (x_2, ..., x_n)R$. Since $S' \in \mathfrak{V}(R')$, by (1.10.9) we get that $\dim R' = n$, $M(R') = (x_1, x_2/x_1, ..., x_n/x_1)R'$, and $R' \cap M(S') = (x_2/x_1, ..., x_n/x_1)R'$. Therefore S' has a simple point at R'. It now suffices to show that $E \cap \mathfrak{V}(R') = \{S'\}$. Suppose if possible that $E \cap \mathfrak{V}(R') \neq \{S'\}$ and take $S^* \in E \cap \mathfrak{V}(R')$ such that $S^* \neq S'$. By assumption $\{S', S^*\}$ has a normal crossing at R and hence there exists a basis $(y_1, ..., y_n)$ of $M(R)$ such that $R \cap M(S') = (y_2, ..., y_n)R$ and $R \cap M(S^*) = (y_1, ..., y_{n-1})R$. Since $S' \in \mathfrak{V}(R')$ and $S^* \in \mathfrak{V}(R')$, by (1.10.9) we get that $y_n/y_1 \in M(R')$ and $y_1/y_n \in M(R')$ which is a contradiction.

(1.10.12). *Let* R *be an* n*-dimensional regular local domain, let* J *and* I *be nonzero principal ideals in* R, *let* S *be a positive-dimensional element in* $\mathfrak{V}(R)$ *such that* (S, I) *has a pseudonormal crossing at* R, *let* (R', J', I') *be a monoidal transform of* (R, J, I, S), *and let* $S' \in \mathfrak{V}(R) \cap \mathfrak{V}(R')$ *such that* $\dim S' \geqslant n - 1$, $\{S, S'\}$ *has a normal crossing at* R, *and* (S', I) *has a pseudonormal crossing at* R. *Then* (S', I') *has a pseudonormal crossing at* R'.

PROOF. Let $d = \operatorname{ord}_S J$, $e = \operatorname{ord}_R I$, and $m = \dim S$. For a moment suppose that $m = 1$; then $R' = R$; we can take $x \in R$ such that $R \cap M(S) = xR$; then $\operatorname{ord}_R x = 1$ and $I' = x^d I$; since $\{S, S'\}$ has a normal crossing at R, we get that $(S', x^d R')$ has a normal crossing at R'; since $I' = x^d I$ and (S', I) has a pseudonormal crossing at R, we conclude that (S', I') has a pseudonormal crossing at R'. Henceforth assume that $m > 1$. Then by (1.9.7) we get that $S \notin \mathfrak{V}(R')$; since $S' \in \mathfrak{V}(R')$, we get that $S \notin \mathfrak{V}(S')$ and hence $R \cap M(S) \not\subset R \cap M(S')$. Therefore $\dim S' = n - 1$, and, since $\{S, S'\}$ has a normal crossing at R, there exists a basis $(x_1, ..., x_n)$ of $M(R)$ such that $R \cap M(S) = (x_1, ..., x_m)R$ and $R \cap M(S') = (x_2, ..., x_n)R$. Since $S' \in \mathfrak{V}(R')$, by (1.10.10) we get that $\dim R' = n$, $M(R') = (x_1, x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_n)R'$, and $R' \cap M(S') = (x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_n)R'$. Since (S, I) has a pseudonormal crossing at R and (S', I) has a pseudonormal crossing at R, we get that $I = z_1 \ldots z_e R$ where $z_1, \ldots, z_e$ are elements in R such that for $1 \leqslant i \leqslant e$ we have that $\operatorname{ord}_R z_i = 1$, $(S, z_i R)$ has a normal crossing at R, and $(S', z_i R)$ has a normal crossing at R (we take $z_1 \ldots z_e R = R$ in case $e = 0$). Now $I' = z_1 \ldots z_e x_1^d R'$ and clearly $(S', x_1^d R')$ has a normal crossing at R'. Therefore it suffices to show that $(S', z_i R')$ has a normal crossing at R' for $1 \leqslant i \leqslant e$. So let any i with $1 \leqslant i \leqslant e$ be given.

First suppose that $z_i \notin R \cap M(S')$. Since $z_i \in M(R)$, we can write $z_i = r_1 x_1 + \cdots + r_n x_n$ with $r_1, \ldots, r_n$ in R. Since $z_i \notin R \cap M(S')$, $\operatorname{ord}_R z_i = 1$, and $(S', z_i R)$ has a normal crossing at R, there exists a basis $(x'_1, \ldots, x'_n)$ of $M(R)$ such that $R \cap M(S') = (x'_2, \ldots, x'_n)R$ and $z_i R = x'_1 R$; it follows that $z_i \notin (R \cap M(S')) + M(R)^2$; since $R \cap M(S') = (x_2, \ldots, x_n)R$, we must have $r_1 \notin M(R)$ and hence $r_1 \notin M(R')$. Consequently $M(R') = (z_i, x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_n)R'$. Therefore $(S', z_i R')$ has a normal crossing at R'.

Next suppose that $z_i \in R \cap M(S')$ and $z_i \notin R \cap M(S)$. Since $z_i \in R \cap M(S')$, we can write $z_i = s_2 x_2 + \cdots + s_n x_n$ with $s_2, \ldots, s_n$ in R. Since $z_i \notin R \cap M(S)$, $\operatorname{ord}_R z_i = 1$, and $(S, z_i R)$ has a normal crossing at R, there exists a basis $(x_1^*, \ldots, x_n^*)$ of $M(R)$ such that $R \cap M(S) = (x_1^*, \ldots, x_m^*)R$ and $z_i R = x_q^* R$ for some q with $m < q \leqslant n$; it follows that $z_i \notin (R \cap M(S)) + M(R)^2$; since $R \cap M(S) = (x_1, \ldots, x_m)R$, we must have $s_p \notin M(R)$ for some p with $m < p \leqslant n$, and then $s_p \notin M(R')$. Consequently $M(R') = (x_1, x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_{p-1}, z_i, x_{p+1}, \ldots, x_n)R$ and $R' \cap M(S') = (x_2/x_1, \ldots, x_m/x_1, x_{m+1}, \ldots, x_{p-1}, z_i, x_{p+1}, \ldots, x_n)R'$. Therefore $(S', z_i R')$ has a normal crossing at R'.

Finally suppose that $z_i \in R \cap M(S')$ and $z_i \in R \cap M(S)$. Then we can write $z_i = t_1 x_1 + \cdots + t_m x_m$ and $z_i = t'_2 x_2 + \cdots + t'_n x_n$ with $t_1, \ldots, t_m, t'_2, \ldots, t'_n$ in R. From these two equations for z_i we get that $t_1 x_1 \in (x_2, \ldots, x_n)R$; now $x_1 \notin (x_2, \ldots, x_n)R$ and hence we must have $t_1 \in M(R)$; since $\operatorname{ord}_R z_i = 1$, from the first equation for z_i we now get that $t_a \notin M(R)$ for some a with $2 \leqslant a \leqslant m$. From the above two equations for z_i we get that $(t_a - t'_a)x_a \in (x_1, \ldots, x_{a-1}, x_{a+1}, \ldots, x_n)R$; now $x_a \notin (x_1, \ldots, x_{a-1}, x_{a+1}, \ldots, x_n)R$ and hence we must have $t_a - t'_a \in M(R)$; therefore $t'_a \notin M(R)$. Let

$y_a = z_i$, and let $y_j = x_j$ for all $j \neq a$ with $1 \leqslant j \leqslant n$. Then from the above two equations for z_i we deduce that $M(R) = (y_1, ..., y_n)R$, $R \cap M(S) = (y_1, ..., y_m)R$, and $R \cap M(S') = (y_2, ..., y_n)R$. Since $S' \in \mathfrak{B}(R')$, by (1.10.10) we get that $M(R') = (y_1, y_2/y_1, ..., y_m/y_1, y_m/y_1, y_{m+1}, ..., y_n)R'$ and $R' \cap M(S') = (y_2/y_1, ..., y_m/y_1, y_{m+1}, ..., y_n)R'$. Now $z_iR' = y_1(y_a/y_1)R'$ and hence (S', z_iR') has a normal crossing at R'.

(1.11). For any nonzero ideal I in a domain A, by I^{-1} we denote the set of all elements x in the quotient field K of A such that $xy \in A$ for all $y \in I$; note that then I^{-1} is an A-submodule of K; by II^{-1} we denote the set of all elements in K which can be expressed as a finite sum $x_1y_1 + \cdots + x_ny_n$ with $x_i \in I$ and $y_i \in I^{-1}$ for $1 \leqslant i \leqslant n$; note that then II^{-1} is an ideal in A and $I \subset II^{-1}$.

Let I be a nonzero ideal in a unique factorization domain A and let W be the set of all nonzero principal ideals in A containing I. Then W is a nonempty finite set and there exists a unique $P \in W$ such that $P \subset Q$ for all $Q \in W$; namely: take any $0 \neq f \in I$; then $fA = P_1^{a_1} ... P_n^{a_n}$ where $P_1, ..., P_n$ are distinct nonzero principal prime ideals in A and $a_1, ..., a_n$ are positive integers (as usual we take $P_1^{a_1} ... P_n^{a_n} = A$ in case $n = 0$); clearly $A \in W \subset \{P_1^{b_1} ... P_n^{b_n}: 0 \leqslant b_i \leqslant a_i \text{ for } 1 \leqslant i \leqslant n\}$ and hence W is a nonempty finite set; for $1 \leqslant i \leqslant n$ let c_i be the smallest nonnegative integer such that $I \subset P_i^{c_i}$, and let $P = P_1^{c_1} ... P_n^{c_n}$; then $P \in W$ and $P \subset Q$ for all $Q \in W$; also clearly P is the only such element in W. P is called the ***principal part*** of I in A and is denoted by $\text{prin}_A I$. Note that if A is noetherian then $\text{prin}_A I$ can also be defined thus: let $I = Q_1 \cap ... \cap Q_m$ be an irredundant primary decomposition of I in A where Q_i is primary for P_i; label $Q_1, ..., Q_m$ so that $\dim A_{P_i} = 1$ for $1 \leqslant i \leqslant m'$ and $\dim A_{P_i} \neq 1$ for $m' < i \leqslant m$; then $\text{prin}_A I = Q_1 \cap ... \cap Q_{m'}$ (we take $Q_1 \cap ... \cap Q_{m'} = A$ in case $m' = 0$).

We shall now prove some elementary results concerning the above two concepts; these results will not be used tacitly.

(1.11.1). *For any nonzero element x in a domain A we have that $(xA)^{-1} = x^{-1}A$ and $(xA)(xA)^{-1} = A$.*

PROOF. Obvious.

(1.11.2). *Let I be a nonzero ideal in a noetherian domain A and let B be the quotient ring of A with respect to a multiplicative set N in A $(0 \notin N)$. Then $I^{-1}B = (IB)^{-1}$ and $(II^{-1})B = (IB)(IB)^{-1}$.*

PROOF. For any $x \in I^{-1}$ we have that $Ix \subset A$ and hence $(IB)x \subset B$; consequently $I^{-1} \subset (IB)^{-1}$ and hence $I^{-1}B \subset (IB)^{-1}$. Conversely let $x \in (IB)^{-1}$; since A is noetherian, there exists a finite basis $(y_1, \ldots, y_n)$ of I; now $xy_i \in B$ for $1 \leqslant i \leqslant n$ and hence there exist elements $z, z_1, \ldots, z_n$ in A with $z \in N$ such that $xy_i = z_i/z$ for $1 \leqslant i \leqslant n$; then $(xz)y_i = z_i$ for $1 \leqslant i \leqslant n$ and hence $Ixz \subset A$; consequently $xz \in I^{-1}$ and hence $x \in I^{-1}B$. Thus $I^{-1}B = (IB)^{-1}$ and hence $(II^{-1})B = (IB)(IB)^{-1}$.

(1.11.3). *Let I be a nonzero ideal in a normal noetherian domain A. Then $(II^{-1})R = R$ for every one-dimensional element R in $\mathfrak{B}(A)$.*

PROOF. Now R is a principal ideal domain (see [27: §3, §6, and §7 of Chapter V]), and hence by (1.11.1) we get that $(IR)(IR)^{-1} = R$. By (1.11.2) we know that $(II^{-1})R = (IR)(IR)^{-1}$, and hence $(II^{-1})R = R$.

(1.11.4). *Let I be a nonzero ideal in a quasilocal domain R. Then: I is a principal ideal in $R \Leftrightarrow II^{-1} = R$.*

PROOF. By (1.11.1) we know that if I is a principal ideal in R then $II^{-1} = R$. Conversely suppose that $II^{-1} = R$. Then $1 = x_1y_1 + \cdots + x_ny_n$ with $x_i \in I$ and $y_i \in I^{-1}$ for $1 \leqslant i \leqslant n$. Now $x_iy_i \in R$ for all i, and hence x_jy_j is a unit in R for some j. In particular then $y_j \neq 0 \neq (x_jy_j)$ and $y_j^{-1} = x_j(x_jy_j)^{-1} \in I$. For every $z \in I$ we have that $zy_j \in R$ and $z = y_j^{-1}(zy_j)$. Therefore $I = y_j^{-1}R$.

(1.11.5). *Let A be any domain. Then for any ideal P in A we have that $P = \bigcap_{R \in \mathfrak{B}(A)} PR$. (Upon taking $P = A$ we get that $A = \bigcap_{R \in \mathfrak{B}(A)} R$).*

PROOF. Clearly $P \subset \bigcap_{R \in \mathfrak{B}(A)} PR$. Conversely, given any

$x \in \bigcap_{R \in \mathfrak{B}(A)} PR$, let $Q = \{y \in A: xy \in P\}$; then Q is an ideal in A and $Q \not\subset A \cap M(R)$ for all $R \in \mathfrak{B}(A)$; therefore $Q = A$ and hence $x \in P$.

(1.11.6). *Let I be a nonzero ideal in a noetherian domain A. Then: IR is a principal ideal in R for all $R \in \mathfrak{B}(A) \Leftrightarrow II^{-1} = A$.*

PROOF. By (1.11.2) we know that $(II^{-1})R = (IR)(IR)^{-1}$ for all $R \in \mathfrak{B}(A)$, and hence by (1.11.4) we get that: IR is a principal ideal in R for all $R \in \mathfrak{B}(A) \Leftrightarrow (II^{-1})R = R$ for all $R \in \mathfrak{B}(A)$. By (1.11.5) we get that: $(II^{-1})R = R$ for all $R \in \mathfrak{B}(A) \Leftrightarrow II^{-1} = A$.

(1.11.7). *Let I be a nonzero ideal in a unique factorization domain A, and let $x \in A$ such that $xA = \text{prin}_A I$. Then $I^{-1} = x^{-1}A$, $II^{-1} = Ix^{-1}$, and $(II^{-1})x = I$.*

PROOF. Now $Ix^{-1} \subset A$ and hence $x^{-1}A \subset I^{-1}$. Conversely, let y be any nonzero element in I^{-1}. We can write $x = rx_1^{a_1} \dots x_n^{a_n}$ and $y = sx_1^{b_1} \dots x_n^{b_n}$ where: r and s are units in A; $x_1, \dots, x_n$ are nonzero elements in R such that $x_1A, \dots, x_nA$ are distinct prime ideals in A; $a_1, \dots, a_n$ are nonnegative integers; and $b_1, \dots, b_n$ are integers. Since $xA = \text{prin}_A I$, there exists $z_i \in I$ such that $z_i/x_i^{a_i} \notin x_iA$; since $y \in I^{-1}$ and $z_i \in I$, we get that $yz_i \in A$ and hence $a_i + b_i \geqslant 0$. This being so for $1 \leqslant i \leqslant n$, we get that $y \in x^{-1}A$. Thus $I^{-1} = x^{-1}A$, and hence $II^{-1} = Ix^{-1}$ and $(II^{-1})x = I$.

(1.11.8). *Let I be a nonzero ideal in a unique factorization domain A. Then: I is a principal ideal in $A \Leftrightarrow II^{-1} = A$.*

PROOF. Follows from (1.11.1) and (1.11.7).

(1.11.9). *Let I be a nonzero ideal in a unique factorization domain A. Then $\text{prin}_A(II^{-1}) = A$.*

PROOF. We can take $x \in A$ such that $xA = \text{prin}_A I$. Then by (1.11.7) we have that $II^{-1} = Ix^{-1}$. Let z be any nonzero element in A such that $II^{-1} \subset zA$; then $Ix^{-1} \subset zA$ and hence $I \subset xzA$; consequently $xA = \text{prin}_A I \subset xzA$ and hence $zA = A$. Thus A is the only nonzero principal ideal in A containing II^{-1}, and hence $\text{prin}_A(II^{-1}) = A$.

(1.11.10). *Let I be a nonzero ideal in a unique factorization domain A, and let $0 \neq y \in A$. Then*: $Iy^{-1} \subset II^{-1} \Leftrightarrow I \subset yA \Leftrightarrow \text{prin}_A I \subset yA$. *Also*: $Iy^{-1} \subset II^{-1} \Rightarrow (Iy^{-1})(Iy^{-1})^{-1} = II^{-1}$.

PROOF. Now: $Iy^{-1} \subset II^{-1} \Rightarrow Iy^{-1} \subset A \Rightarrow I \subset yA \Rightarrow \text{prin}_A I \subset yA \Rightarrow I \subset yA \Rightarrow y^{-1} \in I^{-1} \Rightarrow Iy^{-1} \subset II^{-1}$. Assuming that $\text{prin}_A I \subset yA$, we shall show that $(Iy^{-1})(Iy^{-1})^{-1} = II^{-1}$ and this will complete the proof. We can take $x \in A$ such that $xA = \text{prin}_A I$. Then $xA \subset yA$ and hence $x/y \in A$; since $I \subset xA$, we get that $Iy^{-1} \subset (x/y)A$. For any nonzero element z in A we have that: $Iy^{-1} \subset zA \Rightarrow I \subset zyA \Rightarrow \text{prin}_A I \subset zyA \Rightarrow xA \subset zyA \Rightarrow (x/y)A \subset zA$. Therefore $\text{prin}_A(Iy^{-1}) = (x/y)A$, and hence by (1.11.7) we get that $(Iy^{-1})(Iy^{-1})^{-1} = (Iy^{-1})(x/y)^{-1} = Ix^{-1} = II^{-1}$.

(1.12). By a *semiresolver* we mean a sequence $(R_i, J_i, S_i)_{0 \leqslant i < m}$ where: either m is a positive integer or $m = \infty$; R_i is a regular local domain, J_i is a nonzero principal ideal in R_i, and S_i is a positive-dimensional element in $\mathfrak{C}(R_i, J_i)$ having a simple point at R_i for $0 \leqslant i < m$; (R_i, J_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, S_{i-1})$ for $0 < i < m$; and for $0 \leqslant i < m$ we have that: $\dim S_i = 2 \Leftrightarrow \mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i and $\mathfrak{C}^2(R_i, J_i) \neq \varnothing$.

By an *infinite semiresolver* we mean a semiresolver $(R_i, J_i, S_i)_{0 \leqslant i < m}$ where $m = \infty$ and (R_i, J_i) is unresolved for $0 \leqslant i < \infty$.

By an *finite semiresolver* we mean a system $[(R_i, J_i, S_i)_{0 \leqslant i < m}, (R_m, J_m)]$ where: m is a positive integer; $(R_i, J_i, S_i)_{0 \leqslant i < m}$ is a semiresolver such that (R_i, J_i) is unresolved for $0 \leqslant i < m$; R_m is a regular local domain and J_m is a nonzero principal ideal in R_m such that (R_m, J_m) is resolved; and (R_m, J_m) is a monoidal transform of $(R_{m-1}, J_{m-1}, S_{m-1})$.

By a *finite weak semiresolver* we mean a system $[(R_i, J_i, S_i)_{0 \leqslant i < m}, (R_m, J_m)]$ where: m is a positive integer; R_i is a regular local domain and J_i is a nonzero principal ideal in R_i for $0 \leqslant i \leqslant m$; S_i is a positive-dimensional element in $\mathfrak{C}(R_i, J_i)$ having a simple point at R_i for $0 \leqslant i < m$; (R_i, J_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, S_{i-1})$ for $0 < i \leqslant m$; and for $0 \leqslant i < m$ we have that: $\dim S_i = 2 \Rightarrow \mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i.

By a *resolver* we mean a sequence $(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$ where: either m is a positive integer or $m = \infty$; R_i is a regular local domain, J_i and I_i are nonzero principal ideals in R_i such that I_i has a quasinormal crossing at R_i, and S_i is a positive-dimensional element in $\mathfrak{C}(R_i, J_i)$ such that (S_i, I_i) has a pseudonormal crossing at R_i for $0 \leqslant i < m$; (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, S_{i-1})$ for $0 < i < m$; and for $0 \leqslant i < m$ we have that: $\dim S_i = 2 \Leftrightarrow \mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i and (S', I_i) has a pseudonormal crossing at R_i for some $S' \in \mathfrak{C}^2(R_i, J_i)$.

By an *infinite resolver* we mean a resolver $(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$ where $m = \infty$ and (R_i, J_i) is unresolved for $0 \leqslant i < \infty$.

By a *finite resolver* we mean a system $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ where: m is a positive integer; $(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$ is a resolver such that (R_i, J_i) is unresolved for $0 \leqslant i < m$; R_m is a regular local domain and J_m and I_m are nonzero principal ideals in R_m such that (R_m, J_m) is resolved and I_m has a quasinormal crossing at R_m; and (R_m, J_m, I_m) is a monoidal transform of $(R_{m-1}, J_{m-1}, I_{m-1}, S_{m-1})$ (note that then $(J_0 I_0)R_m = J_m I_m$ and hence $(J_0 I_0)R_m$ has a quasinormal crossing at R_m).

By a *finite weak resolver* we mean a system $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ where: m is a positive integer; R_i is a regular local domain and J_i and I_i are nonzero principal ideals in R_i such that I_i has a quasinormal crossing at R_i for $0 \leqslant i \leqslant m$; S_i is a positive-dimensional element in $\mathfrak{C}(R_i, J_i)$ such that (S_i, I_i) has a pseudonormal crossing at R_i for $0 \leqslant i < m$; (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, S_{i-1})$ for $0 < i \leqslant m$; and for $0 \leqslant i < m$ we have that: $\dim S_i = 2 \Rightarrow \mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i.

By an *infinite subresolver* we mean an infinite sequence $(R_i, J_i, I_i, L_i, S_i)_{0 \leqslant i < \infty}$ where: R_i is a regular local domain, J_i and I_i are nonzero principal ideals in R_i such that (R_i, J_i) is unresolved and $\mathfrak{C}^2(R_i, J_i)$ is a finite set, $\varnothing \neq L_i \subset \mathfrak{C}^2(R_i, J_i)$, and S_i is a positive-dimensional element in $\mathfrak{C}(R_i, J_i)$ such that (S_i, I_i) has a pseudonormal crossing at R_i for $0 \leqslant i < \infty$; (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, S_{i-1})$, $L_i = \{S \in \mathfrak{C}^2(R_i, J_i) \colon S \text{ dominates } L_{i-1}\}$, and $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_{i-1}} J_{i-1}$ for $0 < i < \infty$; and for $0 \leqslant i < \infty$ we have that: $\dim S_i = 2 \Leftrightarrow S_i \in L_i \Leftrightarrow \mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i and (L_i, I_i) has a pseudonormal crossing at R_i.

By a *detacher* we mean a sequence $(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$ where:

either m is a positive integer or $m = \infty$; R_i is a regular local domain, J_i and I_i are nonzero principal ideals in R_i such that (J_i, I_i) has a quasinormal crossing at R_i, and S_i is a positive-dimensional element in $\mathfrak{C}(R_i, J_i)$ such that (S_i, I_i) has a normal crossing at R_i for $0 \leqslant i < m$; (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, S_{i-1})$ for $0 < i < m$; and for $0 \leqslant i < m$ we have that: $\dim S_i = 2 \Leftrightarrow (\mathfrak{C}^2(R_i, J_i), I_i)$ has a strict normal crossing at R_i and $\mathfrak{C}^2(R_i, J_i) \neq \varnothing$.

By an *infinite detacher* we mean a detacher $(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$ where $m = \infty$ and (R_i, J_i) is unresolved for $0 \leqslant i < \infty$.

By a *finite detacher* we mean a system $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ where: m is a positive integer; $(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$ is a detacher such that (R_i, J_i) is unresolved for $0 \leqslant i < m$; R_m is a regular local domain and J_m and I_m are nonzero principal ideals in R_m such that (R_m, J_m) is resolved and (J_m, I_m) has a quasinormal crossing at R_m; and (R_m, J_m, I_m) is a monoidal transform of $(R_{m-1}, J_{m-1}, I_{m-1}, S_{m-1})$ (note that then $(J_0 I_0)R_m = J_m I_m$ and hence $(J_0 I_0)R_m$ has a normal crossing at R_m).

By a *principalizer* we mean a sequence $(R_i, I_i, S_i)_{0 \leqslant i < m}$ where: either m is a positive integer or $m = \infty$; R_i is a regular local domain, I_i is a nonzero ideal in R_i, and S_i is a positive-dimensional element in $\mathfrak{C}(R_i, I_i I_i^{-1})$ having a simple point at R_i for $0 \leqslant i < m$; R_i is a monoidal transform of (R_{i-1}, S_{i-1}) and $I_i = I_{i-1}R_i$ for $0 < i < m$; and for $0 \leqslant i < m$ we have that: $\dim S_i = 2 \Leftrightarrow \mathfrak{C}^2(R_i, I_i I_i^{-1})$ has a strict normal crossing at R_i and $\mathfrak{C}^2(R_i, I_i I_i^{-1}) \neq \varnothing$.

By an *infinite principalizer* we mean a principalizer $(R_i, I_i, S_i)_{0 \leqslant i < m}$ where $m = \infty$ and I_i is a nonprincipal ideal in R_i for $0 \leqslant i < \infty$.

By a *finite principalizer* we mean a system $[(R_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, I_m)]$ where: m is a positive integer; $(R_i, I_i, S_i)_{0 \leqslant i < m}$ is a principalizer such that I_i is a nonprincipal ideal in R_i for $0 \leqslant i < m$; R_m is a regular local domain and I_m is a nonzero principal ideal in R_m; and R_m is a monoidal transform of (R_{m-1}, S_{m-1}) and $I_m = I_{m-1}R_m$.

Let R be a regular local domain. We say that R is *strongly semiresolvable* if: there does not exist any infinite semiresolver $(R_i, J_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R. We say that R is *semiresolvable* if: given any iterated monoidal transform R' of R, any nonzero principal ideal J' in R' such that (R', J') is unresolved, and any valuation ring V of the quotient field of R such that V dominates R', there exists a finite semiresolver $[(R_i, J_i, S_i)_{0 \leqslant i < m}, (R_m, J_m)]$ such that $(R_0, J_0) = (R', J')$ and V dominates R_m. We say that R is *weakly semiresolvable* if: given any iterated monoidal R' of R, any nonzero principal ideal J' in R' such that (R', J') is unresolved, and any valuation ring V of the quotient field of R such that V dominates R', there exists a finite weak semiresolver $[(R_i, J_i, S_i)_{0 \leqslant i < m}, (R_m, J_m)]$ such that $(R_0, J_0) = (R', J')$, $\operatorname{ord}_{R'} J' = \operatorname{ord}_{R_i} J_i > \operatorname{ord}_{R_m} J_m$ for $0 \leqslant i < m$, and V dominates R_m. We say that R is *strongly resolvable* if: there does not exist any infinite resolver $(R_i, J_i, I_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R. We say that R is *resolvable* if: given any iterated monoidal transform R' of R, any nonzero principal ideals J' and I' in R' such that (R', J') is unresolved and I' has a quasinormal crossing at R', and any valuation ring V of the quotient field of R such that V dominates R', there exists a finite resolver $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ such that $(R_0, J_0, I_0) = (R', J', I')$ and V dominates R_m. We say that R is *weakly resolvable* if: given any iterated monoidal transform R' of R, any nonzero principal ideals J' and I' in R' such that (R', J') is unresolved and I' has a quasinormal crossing at R', and any valuation ring V of the quotient field of R such that V dominates R', there exists a finite weak resolver $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ such that $(R_0, J_0, I_0) = (R', J', I')$, $\operatorname{ord}_{R'} J' = \operatorname{ord}_{R_i} J_i > \operatorname{ord}_{R_m} J_m$ for $0 \leqslant i < m$, and V dominates R_m. We say that R is *strongly subresolvable* if: there does not exist any infinite subresolver $(R_i, J_i, I_i, L_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R_0. We say that R is *strongly detachable* if: there does not exist any infinite detacher $(R_i, J_i, I_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R. We say that R is *detachable* if: given any iterated monoidal transform R' of R, any nonzero principal ideals J' and I' in R' such that (R', J') is unresolved and (J', I') has a quasinormal crossing at R', and any valuation ring V of the quotient field of R such that V dominates R', there exists a finite detacher $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ such that $(R_0, J_0, I_0) = (R', J', I')$ and V dominates R_m. We say that R is *strongly principalizable* if: there does not exist any infinite principalizer $(R_i, I_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R. We say that R is *principalizable* if: given any iterated monoidal transform R' of R, any nonzero nonprincipal ideal I' in R', and any valuation ring V of the quotient field of R such that V dominates R', there exists a finite principalizer $[(R_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, I_m)]$ such that $(R_0, I_0) = (R', I')$ and V dominates R_m.

§2. Resolvers and principalizers

(2.1). *Let R be a regular local domain, let J be a nonzero principal ideal in R such that (R, J) is unresolved, and let V be a valuation ring of the quotient field of R such that V dominates R. Assume that there does not exist any infinite semiresolver $(R_i, J_i, S_i)_{0 \leqslant i < \infty}$ such that $(R_0, J_0) = (R, J)$ and V dominates R_i for $0 \leqslant i < \infty$. Then there exists a finite semiresolver $[(R_i, J_i, S_i)_{0 \leqslant i < m}, (R_m, J_m)]$ such that $(R_0, J_0) = (R, J)$ and V dominates R_m.*

PROOF. Let W be the set of all semiresolvers $(R_i, J_i, S_i)_{0 \leqslant i < m}$ such that $(R_0, J_0) = (R, J)$, and (R_i, J_i) is unresolved and V dominates R_i for $0 \leqslant i < m$. For each pair of elements $w = (R_i, J_i, S_i)_{0 \leqslant i < m}$ and $w' = (R_i', J_i', S_i')_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(R_i, J_i, S_i) = (R', J_i', S_i')$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property (i.e., given any nonempty ordered subset W' of W, there exists $w' \in W$ such that $w \leqslant w'$ foı all $w \in W'$). Also we get an element $(R_i, J_i, S_i)_{0 \leqslant i < 1}$ in W by taking $(R_0, J_0) = (R, J)$ and: $S_0 =$ some element in $\mathfrak{C}^2(R, J)$ if $\mathfrak{C}^2(R, J)$ has a strict normal crossing at R and $\mathfrak{C}^2(R, J) = \varnothing$; and $S_0 = R$ otherwise. Therefore $W \neq \varnothing$; and hence by Zorn's lemma W contains a maximal element $w = (R_i, J_i, S_i)_{0 \leqslant i < m}$. By assumption we must have $m \neq \infty$. Let (R_m, J_m) be the monoidal transform of $(R_{m-1}, S_{m-1}, J_{m-1})$ along V. Since w is a maximal element of W, we must have that (R_m, J_m) is resolved, because otherwise we would get an element $w' = (R_i, J_i, S_i)_{0 \leqslant i < m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking: $S_m =$ some element in $\mathfrak{C}^2(R_m, J_m)$ if $\mathfrak{C}^2(R_m, J_m)$ has a strict normal crossing at R_m and $\mathfrak{C}^2(R_m, J_m) \neq \varnothing$; and $S_m = R_m$

otherwise. Therefore $[(R_i, J_i, S_i)_{0 \leqslant i < m}, (R_m, J_m)]$ is a finite semiresolver having the required properties.

(2.2). *If R is a regular local domain such that R is strongly semiresolvable, then R is semiresolvable.*

PROOF. Follows from (2.1).

(2.3). *Let R be a regular local domain, let J and I be nonzero principal ideals in R such that (R, J) is unresolved and I has a quasinormal crossing at R, and let V be a valuation ring of the quotient field of R such that V dominates R. Assume that there does not exist any infinite resolver $(R_i, J_i, I_i, S_i)_{0 \leqslant i < \infty}$ such that $(R_0, J_0, I_0) = (R, J, I)$ and V dominates R_i for $0 \leqslant i < \infty$. Then there exists a finite resolver $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ such that $(R_0, J_0, I_0) = (R, J, I)$ and V dominates R_m.*

PROOF. Let W be the set of all resolvers $(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$ such that $(R_0, J_0, I_0) = (R, J, I)$, and (R_i, J_i) unresolved and V dominates R_i for $0 \leqslant i < m$. For each pair of elements $w = (R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$ and $w' = (R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(R_i, J_i, I_i, S_i) = (R'_i, J'_i, I'_i, S'_i)$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property. Also we get an element $(R_i, J_i, I_i, S_i)_{0 \leqslant i < 1}$ in W by taking $(R_0, J_0, I_0) = (R, J, I)$ and: S_0 = some element in $\mathfrak{C}^2(R, J)$ such that (S_0, I) has a pseudonormal crossing at R if $\mathfrak{C}^2(R, J)$ has a strict normal crossing at R and (S', I) has a pseudonormal crossing at R for some $S' \in \mathfrak{C}^2(R, J)$; and $S_0 = R$ otherwise. Therefore $W \neq \varnothing$ and hence by Zorn's lemma W contains a maximal element $w = (R_i, J_i, I_i, S_i)_{0 \leqslant i < m}$. By assumption we must have $m \neq \infty$. Let (R_m, J_m, I_m) be the monoidal transform of $(R_{m-1}, I_{m-1}, J_{m-1}, S_{m-1})$ along V. By (1.10.8) we know that I_m has a quasinormal crossing at R_m. Since w is a maximal element of W, we must have that (R_m, J_m) is unresolved, because otherwise we would get an element $w' = (R_i, J_i, I_i, S_i)_{0 \leqslant i < m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking: S_m = some element in $\mathfrak{C}^2(R_m, J_m)$ such that (S_m, I_m) has a pseudonormal crossing at R_m if $\mathfrak{C}^2(R_m, J_m)$ has a strict normal crossing at R_m and (S', I_m) has a pseudonormal crossing at R_m for some $S' \in \mathfrak{C}^2(R_m, J_m)$; and $S_m = R_m$ otherwise.

Therefore $[(R_i, J_i, I_i, S_i)_{0\leqslant i<m}, (R_m, J_m, I_m)]$ is a finite resolver having the required properties.

(2.4). *If R is a regular local domain such that R is strongly resolvable, then R is resolvable.*

PROOF. Follows from (2.3).

(2.5). *Let R be a regular local domain, let J and I be nonzero principal ideals in R such that (R, J) is unresolved and (J, I) has a quasinormal crossing at R, and let V be a valuation ring of the quotient field of R such that V dominates R. Assume that there does not exist any infinite detacher $(R_i, J_i, I_i, S_i)_{0\leqslant i<\infty}$ such that $(R_0, J_0, I_0) = (R, J, I)$ and V dominates R_i for $0 \leqslant i < \infty$. Then there exists a finite detacher $[(R_i, J_i, I_i, S_i)_{0\leqslant i<m}, (X_m, J_m, I_m)]$ such that $(R_0, J_0, I_0) = (R, J, I)$ and V dominates R_m.*

PROOF. Let W be the set of all detachers $(R_i, J_i, I_i, S_i)_{0\leqslant i<m}$ such that $(R_0, J_0, I_0) = (R, J, I)$, and (R_i, J_i) is unresolved and V dominates R_i for $0 \leqslant i < m$. For each pair of elements $w = (R_i, J_i, I_i, S_i)_{0\leqslant i<m}$ and $w' = (R'_i, J'_i, I'_i, S'_i)_{0\leqslant i<m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(R_i, J_i, I_i, S_i) = (R'_i, J'_i, I'_i, S'_i)$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property. Also we get an element $(R_i, J_i, I_i, S_i)_{0\leqslant i<1}$ in W by taking $(R_0, J_0, I_0) = (R, J, I)$ and: $S_0 =$ some element in $\mathfrak{E}^2(R, J)$ if $(\mathfrak{E}^2(R, J), I)$ has a strict normal crossing at R and $\mathfrak{E}^2(R, J) \neq \varnothing$; and $S_0 = R$ otherwise. Therefore $W \neq \varnothing$, and hence by Zorn's lemma W contains a maximal element $w = (R_i, J_i, I_i, S_i)_{0\leqslant i<m}$. By assumption we must have $m \neq \infty$. Let (R_m, J_m, I_m) be the monoidal transform of $(R_{m-1}, J_{m-1}, I_{m-1}, S_{m-1})$ along V. By (1.10.7) we know that (J_m, I_m) has a quasinormal crossing at R_m. Since w is a maximal element of W, we must have that (R_m, J_m) is resolved, because otherwise we would get an element $w' = (R_i, J_i, I_i, S_i)_{0\leqslant i<m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking: $S_m =$ some element in $\mathfrak{E}^2(R_m, J_m)$ if $(\mathfrak{E}^2(R_m, J_m), I_m)$ has a strict normal crossing at R_m and $\mathfrak{E}^2(R_m, J_m) \neq \varnothing$; and $S_m = R_m$ otherwise. Therefore $[(R_i, J_i, I_i, S_i)_{0\leqslant i<m}, (R_m, J_m, I_m)]$ is a finite detacher having the required properties.

(2.6). *If R is a regular local domain such that R is strongly detachable, then R is detachable.*

PROOF. Follows from (2.5).

(2.7). *Let R be a regular local domain, let I be a nonzero nonprincipal ideal in R, and let V be a valuation ring of the quotient field of R such that V dominates R. Assume that there does not exist any infinite principalizer $(R_i, I_i, S_i)_{0 \leqslant i < \infty}$ such that $(R_0, I_0) = (R, I)$ and V dominates R_i for $0 \leqslant i < \infty$. Then there exists a finite principalizer $[(R_i, I_i, S_i)_{0 \leqslant i < m}, (X_m, I_m)]$ such that $(R_0, I_0) = (R, I)$ and V dominates R_m.*

PROOF. Let W be the set of all principalizers $(R_i, I_i, S_i)_{0 \leqslant i < m}$ such that $(R_0, I_0) = (R, I)$, and I_i is a nonprincipal ideal in R_i and V dominates R_i for $0 \leqslant i < m$. For each pair of elements $w = (R_i, I_i, S_i)_{0 \leqslant i < m}$ and $w' = (R'_i, I'_i, S'_i)_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(R_i, I_i, S_i) = (R'_i, I'_i, S'_i)$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property. Also we get an element $(R_i, I_i, S_i)_{0 \leqslant i < 1}$ in W by taking $(R_0, I_0) = (R, I)$ and: $S_0 =$ some element in $\mathfrak{E}^2(R, II^{-1})$ if $\mathfrak{E}^2(R, II^{-1})$ has a strict normal crossing at R and $\mathfrak{E}^2(R, II^{-1}) \neq \varnothing$; and $S_0 = R$ otherwise. Therefore $W \neq \varnothing$, and hence by Zorn's lemma W contains a maximal element $w = (R_i, I_i, S_i)_{0 \leqslant i < m}$. By assumption we must have $m \neq \infty$. Let R_m be the monoidal transform of (R_{m-1}, S_{m-1}) along V and let $I_m = I_{m-1}R_m$. Since w is a maximal element of W, we must have that I_m is a nonprincipal ideal in R_m, because otherwise we would get an element $w' = (R_i, I_i, S_i)_{0 \leqslant i < m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking: $S_m =$ some element in $\mathfrak{E}^2(R_m, I_m I_m^{-1})$ if $\mathfrak{E}^2(R_m, I_m I_m^{-1})$ has a strict normal crossing at R_m and $\mathfrak{E}^2(R_m, I_m I_m^{-1}) \neq \varnothing$; and $S_m = R_m$ otherwise. Therefore $[(R_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, I_m)]$ is a finite principalizer having the required properties.

(2.8). *If R is a regular local domain such that R is strongly principalizable, then R is principalizable.*

PROOF. Follows from (2.7).

(2.9). *Let R be a regular local domain such that R is resolvable and detachable. Let R' be an iterated monoidal transform of R, let J' be a nonzero principal ideal in R', and let V be a valuation ring of the quotient field of R such that V dominates R'. Then there exists an iterated monoidal transform R^* of R' along V such that $J'R^*$ has a normal crossing at R^*.*

PROOF. If (R', J') is resolved then it suffices to take $R^* = R'$. So now assume that (R', J') is unresolved. Since R is resolvable, there exists a finite resolver $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ such that $(R_0, J_0, I_0) = (R', J', R')$ and V dominates R_m. Now $J'R_m = J_m I_m$ and hence $(J'R_m, R_m)$ has a quasinormal crossing at R_m. If $(R_m, J'R_m)$ is resolved then $J'R_m$ has a normal crossing at R_m and it suffices to take $R^* = R_m$. So now assume that $(R_m, J'R_m)$ is unresolved. Since R is detachable, there exists a finite detacher $[(R_i', J_i', I_i', S_i')_{0 \leqslant i < n}, (R^*, J^*, I^*)]$ such that $(R_0', J_0', I_0') = (R_m, J'R_m, R_m)$ and V dominates R^*. Now $J'R^* = J^*I^*$ and hence $J'R^*$ has a normal crossing at R^*.

(2.10). *Let R be a regular local domain such that R is principalizable. Let R' be an iterated monoidal transform of R, let V be a valuation ring of the quotient field of R such that V dominates R', and let $f_1, \ldots, f_q$ $(q > 0)$ be a finite number of nonzero elements in V. Then there exists an iterated monoidal transform R^* of R' along V such that $f_i \in R^*$ for $1 \leqslant i \leqslant q$.*

PROOF. We can take nonzero elements $F_0, F_1, \ldots, F_q$ in R' such that $f_i = F_i/F_0$ for $1 \leqslant i \leqslant q$. If $(F_0, \ldots, F_q)R'$ is a principal ideal in R' then take $R^* = R'$; if $(F_0, \ldots, F_q)R'$ is a nonprincipal ideal in R' then, since R' is principalizable, there exists a finite principalizer $[(R_i, I_i, S_i)_{0 \leqslant i < m}, (R^*, I^*)]$ such that $(R_0, I_0) = (R', (F_0, \ldots, F_q)R')$ and V dominates R^*. In both the cases R^* is an iterated monoidal transform of R' along V and $(F_0, \ldots, F_q)R^*$ is a nonzero principal ideal in R^*. Now $(F_0, \ldots, F_q)R^* = FR^*$ with $0 \neq F \in R^*$. In particular $F_i = r_i F$ for $0 \leqslant i \leqslant q$ and $F = s_0F_0 + \cdots + s_qF_q$ with $r_0, \ldots, r_q, s_0, \ldots, s_q$ in R^*. Now $1 = s_0r_0 + \cdots + r_qs_q$ and hence r_j is a unit in R^* for some j with $0 \leqslant j \leqslant q$. In particular then $(F_0, \ldots, F_q)R^* = F_jR^*$. Now $F_0/F_j \in R^* \subset V$, $F_j/F_0 \in V$, and V dominates R^*; consequently F_0/F_j

is a unit in R^* and hence $(F_0, ..., F_q)R^* = F_0R^*$. Therefore $f_i = F_i/F_0 \in R^*$ for $1 \leqslant i \leqslant q$.

(2.11). *Let R be a regular local domain such that R is resolvable, detachable, and principalizable. Let R' be an iterated monoidal transform of R and let V be a valuation ring of the quotient field of R such that V dominates R'. Then* (1) *given any nonzero ideal I' in R', there exists an iterated monoidal transform R^* of R' along V such that $I'R^*$ is a nonzero principal ideal in R^* having a normal crossing at R^*; and* (2) *given any finite number of nonzero elements $f_1, ..., f_q$ $(q > 0)$ in V there exists an iterated monoidal transform R^* of R' along V and a basis $(x_1, ..., x_n)$ of $M(R^*)$, where $n = \dim R^*$, such that $f_i = g_i x_1^{a(i,1)} \dots x_n^{a(i,n)}$ where g_i is a unit in R^* and $a(i,j)$ is a nonnegative integer for $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant n$.*

PROOF. To prove (1) let any nonzero ideal I' in R' be given. If I' is a principal ideal in R' then let $R'' = R'$; if I' is a nonprincipal ideal in R' then, since R is principalizable, there exists a finite principalizer $[(R_i, I_i, S_i)_{0 \leqslant i < m}, (R'', I'')]$ such that $(R_0, I_0) = (R', I')$ and V dominates R''. In both the cases R'' is an iterated monoidal transform of R', $I'R''$ is a nonzero principal ideal in R'', and V dominates R''. Since R is resolvable and detachable, by (2.9) there exists an iterated monoidal transform R^* of R'' along V such that $(I'R'')R^*$ has a normal crossing at R^*. It follows that R^* is an iterated monoidal transform of R' along V and $I'R^*$ is a nonzero principal ideal in R^* having a normal crossing at R^*.

To prove (2) let any finite number of nonzero elements $f_1, ..., f_q$ $(q > 0)$ in V be given. Since R is principalizable, by (2.10) there exists an iterated monoidal transform R^{**} of R' along V such that $f_i \in R^{**}$ for $1 \leqslant i \leqslant q$. Since R is resolvable and detachable, by (2.9) there exists an iterated monoidal transform R^* of R^{**} along V such that $((f_1 \dots f_q)R^{**})R^*$ has a normal crossing at R^*. It follows that R^* is an iterated monoidal transform of R' along V and there exists a basis $(x_1, ..., x_n)$ of $M(R^*)$, where $n = \dim R^*$, such that $f_i = g_i x_1^{a(i,1)} \dots x_n^{a(i,n)}$ where g_i is a unit in R^* and $a(i,j)$ is a nonnegative integer for $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant n$.

(2.12). *For any nonzero ideal I in any regular local domain R we have the following.*

(2.12.1). *There exists a finite number of nonzero principal ideals* $J_1, \ldots, J_n$ $(n > 0)$ *in* R *such that* $II^{-1} = J_1 + \cdots + J_n$ *and* $\operatorname{ord}_R J_q = \operatorname{ord}_R(II^{-1})$ *for* $1 \leqslant q \leqslant n$.

(2.12.2). *For any* $J_1, \ldots, J_n$ *as in* (2.12.1) *let* $J = J_1 \ldots J_n$. *Then: I is a principal ideal in* $R \Leftrightarrow (R, J)$ *is resolved.*

(2.12.3). *For any* $J_1, \ldots, J_n$ *as in* (2.12.1) *let* $J = J_1 \ldots J_n$. Then $\mathfrak{E}(R, II^{-1}) = \bigcap_{q=1}^{n} \mathfrak{E}(R, J_q) = \mathfrak{E}(R, J)$.

(2.12.4). *Let S be a positive-dimensional element in* $\mathfrak{E}(R, II^{-1})$ *having a simple point at R, let R′ be a monoidal transform of* (R, S) *such that* $\operatorname{ord}_{R'}(I'I'^{-1}) = \operatorname{ord}_R(II^{-1})$ *where* $I' = IR'$, *and for any* $J_1, \ldots, J_n$ *as in* (2.12.1) *let* J'_q *be the* (R, S, R')*-transform of* J_q *for* $1 \leqslant q \leqslant n$. *Then* $I'I'^{-1} = J'_1 + \cdots + J'_n$ *and* $\operatorname{ord}_{R'} J'_q = \operatorname{ord}_{R'}(I'I'^{-1})$ *for* $1 \leqslant q \leqslant n$.

(2.12.5). *Let S be a positive-dimensional element in* $\mathfrak{E}(R, II^{-1})$ *having a simple point at R, let R′ be a monoidal transform of* (R, S), *and let* $I' = IR'$. *Then* $\operatorname{ord}_{R'}(I'I'^{-1}) \leqslant \operatorname{ord}_R(II^{-1})$.

PROOF OF (2.12.1). Take any finite basis $(f_1, \ldots, f_n)$ $(n > 0)$ of II^{-1}. Upon relabeling $f_1, \ldots, f_n$ we may assume that $\operatorname{ord}_R f_1 = \operatorname{ord}_R(II^{-1})$. Let $g_1 = f_1$. For $2 \leqslant q \leqslant n$ let: $g_q = f_q$ if $\operatorname{ord}_R f_q = \operatorname{ord}_R(II^{-1})$, and $g_q = f_1 + f_q$ if $\operatorname{ord}_R f_q \neq \operatorname{ord}_R(II^{-1})$. Let $J_q = g_q R$ for $1 \leqslant q \leqslant n$. Then $J_1, \ldots, J_n$ have the required properties.

PROOF OF (2.12.2). First suppose that I is a principal ideal in R; then by (1.11.1) we have that $II^{-1} = R$ and hence $J_q = R$ for $1 \leqslant q \leqslant n$; consequently $J = R$ and hence (R, J) is resolved. Conversely suppose that (R, J) is resolved; let $d = \operatorname{ord}_R(II^{-1})$; then $\operatorname{ord}_R J = nd$ and hence $J = x^{nd}R$ where $x \in R$ with $\operatorname{ord}_R x = 1$; consequently $J_q = x^d R$ for $1 \leqslant q \leqslant n$, and hence $II^{-1} = x^d R$; therefore by (1.11.9) we get that $d = 0$, and hence $II^{-1} = R$; now by (1.11.8) we get that I is a principal ideal in R.

PROOF OF (2.12.3). This is obvious in view of (1.3.1). Concerning the second equality note that actually for any nonzero principal ideals $J_1', \ldots, J_m'$ in R we have that $\bigcap_{q=1}^{m} \mathfrak{E}(R, J_q') = \mathfrak{E}(R, J_1' \ldots J_m')$.

PROOF OF (2.12.4). Let $I^* = (II^{-1})R'$. Then $I^* = J_1R' + \cdots + J_nR'$. We can take $y \in R$ such that $yR = \text{prin}_R I$; then by (1.11.7) we have that $II^{-1} = Iy^{-1}$ and hence $I^* = I'y^{-1}$; now $I \subset yR$ and hence $I' \subset yR'$; consequently by (1.11.10) we get that $I'I'^{-1} = (I'y^{-1})(I'y^{-1})^{-1}$ and hence $I'I'^{-1} = I^*I^{*-1}$. Let $d = \text{ord}_R(II^{-1})$. Then $\text{ord}_{R'}(I^*I^{*-1}) = \text{ord}_{R'}(I'I'^{-1}) = d = \text{ord}_R J_q$ for $1 \leqslant q \leqslant n$. Now $(R \cap M(S))R' = xR'$ where $x \in R'$ with $\text{ord}_{R'} x = 1$. By (2.12.3) we know that $S \in \mathfrak{E}(R, J_q)$ and hence $J_qR' \subset x^dR'$, $J_q' = (J_qR')x^{-d}$, and $\text{ord}_{R'} J_q' \leqslant d$ for $1 \leqslant q \leqslant n$. Since $I^* = J_1R' + \cdots + J_nR'$ and $J_qR' \subset x^dR'$ for $1 \leqslant q \leqslant n$, we must have $I^* \subset x^dR'$ and hence $\text{prin}_{R'} I^* = zx^dR'$ with $0 \neq z \in R'$. By (1.11.7) we now get that $I^*I^{*-1} = I^*z^{-1}x^{-d}$; consequently $J_q'z^{-1} \subset R'$ for $1 \leqslant q \leqslant n$, and $I^*I^{*-1} = J_1'z^{-1} + \cdots + J_n'z^{-1}$. Since $\text{ord}_{R'}(I^*I^{*-1}) = d \geqslant \text{ord}_{R'} J_q'$ for $1 \leqslant q \leqslant n$, we conclude that z is a unit in R', $\text{ord}_{R'} J_q' = d = \text{ord}_{R'}(I'I'^{-1})$ for $1 \leqslant q \leqslant n$, and $J_1' + \cdots + J_n' = I^*I^{*-1} = I'I'^{-1}$.

PROOF OF (2.12.5). Take $J_1, \ldots, J_n$ as in (2.12.1), and let $I^* = (II^{-1})R'$. Then $I^* = J_1R' + \cdots + J_nR'$. We can take $y \in R$ such that $yR = \text{prin}_R I$; then by (1.11.7) we have that $II^{-1} = Iy^{-1}$, and hence $I^* = I'y^{-1}$; now $I \subset yR$ and hence $I' \subset yR'$; consequently by (1.11.10) we get that $I'I'^{-1} = (I'y^{-1})(I'y^{-1})^{-1}$, and hence $I'I'^{-1} = I^*I^{*-1}$. Let $d = \text{ord}_R(II^{-1})$. Then $\text{ord}_R J_q = d$ for $1 \leqslant q \leqslant n$. Now $(R \cap M(S))R' = xR'$ where $x \in R'$ with $\text{ord}_R x = 1$. By (2.12.3) we know that $S \in \mathfrak{E}(R, J_q)$, and hence upon letting J_q' be the (R, S, R')-transform of J_q we get that $J_qR' \subset x^dR'$, $J_q' = (J_qR')x^{-d}$, and $\text{ord}_{R'} J_q' \leqslant d$ for $1 \leqslant q \leqslant n$. Since $I^* = J_1R' + \cdots + J_nR'$ and $J_qR' \subset x^dR'$ for $1 \leqslant q \leqslant n$, we must have $I^* \subset x^dR'$ and hence by (1.11.10) we get that $I^*x^{-d} \subset I^*I^{*-1}$; now $J_1' = (J_1R')x^{-d} \subset I^*x^{-d} \subset I^*I^{*-1}$ and $\text{ord}_{R'} J_1' \leqslant d = \text{ord}_R(II^{-1})$; therefore $\text{ord}_{R'}(I'I'^{-1}) = \text{ord}_{R'}(I^*I^{*-1}) \leqslant \text{ord}_R(II^{-1})$.

(2.13). *Let R be a regular local domain such that R is strongly semiresolvable. Then R is strongly principalizable.*

PROOF. Given any principalizer $(R_i, I_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R, we want to show that I_j is a principal ideal in R_j for some nonnegative integer j. By (2.12.5) we know that $\text{ord}_{R_i}(I_i\bar{I}_i^{-1}) \leqslant \text{ord}_{R_a}(I_a\bar{I}_a^{-1})$ whenever $i \geqslant a$. Therefore there exist nonnegative integers b and d such that $\text{ord}_{R_i}(I_i\bar{I}_i^{-1}) = d$ whenever $i \geqslant b$. By (2.12.1) there exist nonzero principal ideals $J_{b,1}, ..., J_{b,n}$ $(n > 0)$ in R_b such that $I_b\bar{I}_b^{-1} = J_{b,1} + \cdots + J_{b,n}$ and $\text{ord}_{R_b}J_{b,q} = d$ for $1 \leqslant q \leqslant n$. For $1 \leqslant q \leqslant n$ let $(J_{i\,q})_{b<i<\infty}$ be the unique sequence such that $J_{i,q}$ is the (R_{i-1}, S_{i-1}, R_i)-transform of $J_{i-1,q}$ for $b < i < \infty$. In view of (2.12.4), by induction on i we see that $I_i\bar{I}_i^{-1} = J_{i,1} + \cdots + J_{i,n}$ and $\text{ord}_{R_i}J_{i,q} = d$ for $b \leqslant i < \infty$ and $1 \leqslant q \leqslant n$. Let $J_i = J_{i,1} \cdots J_{i,n}$ for $b \leqslant i < \infty$. Then J_i is a nonzero principal ideal in R_i for $b \leqslant i < \infty$, and J_i is the (R_{i-1}, S_{i-1}, R_i)-transform of J_{i-1} for $b < i < \infty$. By (2.12.3) we have that $\mathfrak{E}(R_i, J_i) = \mathfrak{E}(R_i, I_i\bar{I}_i^{-1})$ for $b \leqslant i < \infty$. It follows that $(R_{b+i}, J_{b+i}, S_{b+i})_{0 \leqslant i < \infty}$ is a semiresolver. Since R_b is an iterated monoidal transform of R and R is strongly semiresolvable, we conclude that (R_j, J_j) is resolved for some $j \geqslant b$. By (2.12.2) we now get that I_j is a principal ideal in R_j.

(2.14). *Let R be a three-dimensional regular local domain, let (x, y, z) be a basis of $M(R)$, let J be a nonzero principal ideal in R such that $J \subset zR$, let $I = x^a y^b z^c R$ where a, b, c are nonnegative integers, let (R', J', I') be a monoidal transform of (R, J, I, R) such that $\text{ord}_{R'}J' = \text{ord}_R J$, and let $E^* = \mathfrak{E}^2(R', J') - \mathfrak{E}^2(R, J)$. Then (E^*, I') has a normal crossing at R', E^* contains at most one element, and for every $S \in E^*$ we have that: either $y/x \in R'$ and $R' \cap M(S) = (x, z/x)R'$, or $x/y \in R'$ and $R' \cap M(S) = (y, z/y)R'$.*

PROOF. Now $J = zwR$ with $0 \neq w \in R$. Let $e = \text{ord}_R w$. Then $\text{ord}_R J = e + 1$, and hence by assumption $\text{ord}_{R'}J' = e + 1$. If $x/z \in R'$ and $y/z \in R'$ then we would get that $J' = (w/z^e)R'$ and $\text{ord}_{R'}(w/z^e) \leqslant e$ which would be a contradiction. Therefore we must have either $y/x \in R'$ and $z/x \in M(R')$, or $x/y \in R'$ and $z/y \in M(R')$. Consequently there exists a permutation (x', y') of (x, y) such that $y'/x' \in R'$ and $z/x' \in M(R')$. Let (a', b') be the corresponding permutation of (a, b). Then $I = x'^{a'} y'^{b'} z^c R$ and hence:

(1) $$I' = x'^{e+1+a'+b'+c}(y'/x')^{b'}(z/x')^{c'}R'.$$

Also (2): if $\dim R' \neq 3$ then $\dim R' = 2$, $y'/x' \notin M(R')$, and $M(R') = (x', z/x')R'$; and if $\dim R' = 3$ then $M(R') = (x', y^*, z/x')R'$ where $y^* = y'/x'$ in case $y'/x' \in M(R')$, and $y^* =$ some element in R' in case $y'/x' \notin M(R')$. By (1.9.5) we get that $M(R)R' \subset R' \cap M(S)$ for all $S \in \mathfrak{B}(R') - \mathfrak{B}(R)$; consequently $x' \in R' \cap M(S)$ for all $S \in \mathfrak{B}(R') - \mathfrak{B}(R)$; since $\mathrm{ord}_{R'} J' = \mathrm{ord}_R J$, we get that $\mathfrak{E}^2(R', J') \cap \mathfrak{B}(R) \subset \mathfrak{E}^2(R, J)$, and hence $E^* \subset \mathfrak{B}(R') - \mathfrak{B}(R)$; therefore:

$$(3) \qquad x' \in R' \cap M(S) \qquad \text{for all} \quad S \in E^*.$$

Now $J' = (z/x')(w/x'^e)R'$, $\mathrm{ord}_{R'} J' = e + 1$, $\mathrm{ord}_{R'}(z/x') = 1$, $w/x'^e \in R'$, and $\mathrm{ord}_{R'}(w/x'^e) \leqslant e$; consequently $\mathrm{ord}_{R'}(w/x'^e) = e$; it follows that $z/x' \in R' \cap M(S)$ for all $S \in \mathfrak{E}^2(R', J')$; therefore by (3) we get that $(x', z/x')R' \subset R' \cap M(S)$ for all $S \in E^*$, and hence by (2) we get that:

$$(4) \qquad (x', z/x')R' = R' \cap M(S) \qquad \text{for all} \quad S \in E^*.$$

By (1), (2), and (4) it follows that (E^*, I') has a normal crossing at R', and E^* contains at most the one element. Since (x', y') is a permutation of (x, y), by (4) we get that for every $S \in E^*$ we have that: either $y/x \in R'$ and $R' \cap M(S) = (x, z/x)R'$, or $x/y \in R'$ and $R' \cap M(S) = (y, z/y)R'$.

(2.15). *Let R be a regular local domain with* $\dim R \leqslant 3$. *Let J and I be nonzero principal ideals in R such that $J \neq R$, and (J, I) has a quasinormal crossing at R. Let (R', J', I') be a monoidal transform of (R, J, I, R) such that* $\mathrm{ord}_{R'} J' = \mathrm{ord}_R J$, *and let $E^* = \mathfrak{E}^2(R', J') - \mathfrak{E}^2(R, J)$. Then we have the following.*

(2.15.1). *(E^*, I') has a normal crossing at R', and E^* contains at most one element.*

(2.15.2). *Let $E \subset \mathfrak{E}^2(R, J)$ such that every subset of E containing at most two elements has a normal crossing at R, and (S, I) has a normal crossing at R for all $S \in E$. Let $E' = \mathfrak{E}^2(R', J') - (\mathfrak{E}^2(R, J) - E)$. Then (E', I') has a strist normal crossing at R'.*

(2.15.3). *If every subset of $\mathfrak{E}^2(R, J)$ containing at most two elements has a normal crossing at R', and (S, I) has a normal crossing* *at R for all $S \in \mathfrak{C}^2(R, J)$, then $(\mathfrak{C}^2(R', J'), I')$ has a strict normal crossing at R'.*

(2.15.4). *If $(\mathfrak{C}^2(R, J), I)$ has a normal crossing at R, then $(\mathfrak{C}^2(R', J'), I')$ has a strict normal crossing at R'.*

(2.15.5). *If $\mathfrak{C}^2(R, J)$ has a normal crossing at R, then $\mathfrak{C}^2(R', J')$ has a strict normal crossing at R'.*

PROOF OF (2.15.1). Since (J, I) has a quasinormal crossing at R and $J \neq R$, there exists a basis (x, y, z) of $M(R)$ such that $J \subset zR$ and $I = x^a y^b z^c R$ where a, b, c are nonnegative integers. Therefore our assertion follows from (2.14).

PROOF OF (2.15.2). By (1.10.6) we know that I' has a normal crossing at R'; also, if $\dim R' \neq 3$ then $\dim R' \leqslant 2$; therefore, if $\dim R' \neq 3$ then (E', I') has a strict normal crossing at R'. So assume that $\dim R' = 3$. Then $\dim R = 3$. Clearly $E^* \subset E'$. If $E^* = E$ then our assertion follows from (2.15.1). So also assume that $E^* \neq E'$ and take $S' \in E' - E^*$. Now $E' - E^* = E \cap \mathfrak{C}^2(R', J') \subset E \cap \mathfrak{B}(R')$, and hence by (1.10.11) we get that $E' - E^* = \{S'\}$, i.e., $E' = \{S'\} \cup E^*$. By assumption (S', I) has a normal crossing at R and hence there exists a basis (x, y', z') of $M(R)$ such that $R \cap M(S') = (y', z')R$ and $I = x^a y'^{b'} z'^{c'} R$ where a, b', c' are nonnegative integers. Now (J, I) has a quasinormal crossing at R, $J \neq R$, and $S' \in \mathfrak{C}^2(R, J)$; therefore there exists $z \in (y', z')R$ with $\mathrm{ord}_R z = 1$ such that $J \subset zR$ and (zR, I) has a quasinormal crossing at R; consequently by (1.5.1) there exists $y \in R$ such that $M(R) = (x, y, z)R$, $R \cap M(S') = (y, z)R$, and $I = x^a y^b z^c R$ where b and c are nonnegative integers. Since $S' \in \mathfrak{B}(R) \cap \mathfrak{B}(R')$, by (1.10.10) we get that $M(R') = (x, y/x, z/x)R'$ and $R' \cap M(S') = (y/x, z/x)R'$; clearly $x/y \notin R'$ and hence by (2.14) we get that E^* contains at most one element, and $R' \cap M(S) = (x, z/x)R'$ for every $S \in E^*$; now $I' = x^{d+a+b+c}(y/x)^b(z/x)^c R'$ where $d = \mathrm{ord}_R J$, and hence it follows that (E', I') has a strict normal crossing at R'.

PROOF OF (2.15.3). Take $E = \mathfrak{C}^2(R, J)$ in (2.15.2).

PROOF OF (2.15.4). Follows from (2.15.3).

PROOF OF (2.15.5). Now (J, R) has a quasinormal crossing at R, and $(\mathfrak{C}^2(R, J), R)$ has a normal crossing at R. Therefore our assertion follows by taking $I = R$ in (2.15.4).

(2.16). *Let R_0 be a regular local domain with* $\dim R_0 \leqslant 3$. *Let J_0 and I_0 be nonzero principal ideals in R_0 such that $J_0 \neq R_0$, $\mathfrak{C}^2(R_0, J_0)$ has a normal crossing at R_0, and (J_0, I_0) has a quasinormal crossing at R_0. Let $(R_i, J_i, I_i)_{0<i<\infty}$ be an infinite sequence such that for $0 < i < \infty$: R_i is a regular local domain, J_i and I_i are nonzero principal ideals in R_i, (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, R_{i-1})$, and* $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_0} J_0$. *Then there exists a nonnegative integer j such that $(\mathfrak{C}^2(R_i, J_i), I_i)$ has a strict normal crossing at R_i for all $i \geqslant j$.*

PROOF. By (1.10.7) we get that (J_i, I_i) has a quasinormal crossing at R_i for $0 \leqslant i < \infty$, and by (2.15.5) we get that $\mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i for $0 < i < \infty$. If $\dim R_j \neq 3$ for some nonnegative integer j, then for all $i \geqslant j$ we have that $\dim R_i \leqslant 2$ and hence $(\mathfrak{C}^2(R_i, J_i), I_i)$ has a strict normal crossing at R_i for all $i \geqslant j$. So now assume that $\dim R_i = 3$ for $0 \leqslant i < \infty$. Let $E = \bigcap_{p=0}^{\infty} \mathfrak{C}^2(R_p, J_p)$. Then there exists a positive integer q such that $E = \mathfrak{C}^2(R_0, J_0) \cap \mathfrak{C}^2(R_i, J_i)$ for all $i \geqslant q$. By induction on i we shall show that $((\mathfrak{C}^2(R_i, J_i) - \mathfrak{C}^2(R_0, J_0)), I_i)$ has a normal crossing at R_i for $0 \leqslant i < \infty$; this is obvious for $i = 0$; so now let $i > 0$ and assume that (E^*, I_{i-1}) has a normal crossing at R_{i-1} where $E^* = \mathfrak{C}^2(R_{i-1}, J_{i-1}) - \mathfrak{C}^2(R_0, J_0)$; upon taking $(R_{i-1}, J_{i-1}, I_{i-1}, E^*)$ for (R, J, I, E) in (2.15.2) we get that (E', I_i) has a normal crossing at R_i where $E' = \mathfrak{C}^2(R_i, J_i) - (\mathfrak{C}^2(R_{i-1}, J_{i-1}) - E^*)$; clearly $\mathfrak{C}^2(R_i, J_i) - \mathfrak{C}^2(R_0, J_0) \subset E'$ and hence $((\mathfrak{C}^2(R_i, J_i) - \mathfrak{C}^2(R_0, J_0)), I_i)$ has a normal crossing at R_i. This completes the induction on i. Hence in particular we get that $((\mathfrak{C}^2(R_i, J_i) - E), I_i)$ has a normal crossing at R_i for all $i \geqslant q$. Therefore if $E = \varnothing$ then it suffices to take $j = q$. So now assume that $E \neq \varnothing$ and take $S \in E$. Note that then $S \in \mathfrak{B}(R_i)$ for $0 \leqslant i < \infty$. Since $E \subset \mathfrak{B}(R_0) \cap \mathfrak{B}(R_1)$, by (1.10.11) we get that $E = \{S\}$. Since S has a simple point at R_0, there exists a basis (x, y', z') of $M(R_0)$ such that $R_0 \cap M(S) = (y', z')R_0$. Since I_0 has a normal crossing at R_0, there exists a basis (x_1, x_2, x_3) of

$M(R_0)$ such that $I_0 = x_1^{a_1}x_2^{a_2}x_3^{a_3}R_0$ where a_1, a_2, a_3 are nonnegative integers. Upon relabeling x_1, x_2, x_3 we may assume that there exists an integer u with $0 \leqslant u \leqslant 3$ such that $x_i^{a_i} \notin R_0 \cap M(S)$ for $0 \leqslant i \leqslant u$, and $x_i^{a_i} \in R_0 \cap M(S)$ for $u < i \leqslant 3$. Note that then $a_i > 0$ for $u < i \leqslant 3$, and hence $x_i \in R_0 \cap M(S)$ for $u < i \leqslant 3$. Since $\dim S = 2$, we get that $u \geqslant 1$, and if $u = 1$ then $R_0 \cap M(S) = (x_2, x_3)R_0$. Also, if $u = 2$ then $R_0 \cap M(S) = (y^*, x_3)R_0$ for some $y^* \in R_0$ (namely, since $\operatorname{ord}_{R_0} x_3 = 1$ and $x_3 \in R_0 \cap M(S) = (y', x')R_0$, we get that $x_3 = r_1 y' + r_2 z'$ where r_1 and r_2 are elements in R_0 at least one of which is not in $M(R_0)$; take $y^* = z'$ in case $r_1 \notin M(R_0)$, and $y^* = y'$ in case $r_1 \in M(R_0)$). Let $w = x_1^{a_1} \dots x_u^{a_u}$. Then $w \in R_0$ and $w \notin R_0 \cap M(S)$. Let

$$(y, z, b, c) = \begin{cases} (x_2, x_3, a_2, a_3) & \text{if } u = 1 \\ (y^*, x_3, 0, a_3) & \text{if } u = 2 \\ (y', z', 0, 0) & \text{if } u = 3 \end{cases}$$

Then $M(R_0) = (x, y, z)R_0$, $R_0 \cap M(S) = (y, z)R_0$, b and c are nonnegative integers, and $I_0 = wy^bz^cR_0$. Since $S \in \mathfrak{B}(R_i)$ for $0 \leqslant i < \infty$, by (1.10.10) we get that $M(R_i) = (x, y/x^i, z/x^i)R_i$ and $R_i \cap M(S) = (y/x^i, z/x^i)$ for $0 \leqslant i < \infty$. Let h: $R_0 \to R_0/(R_0 \cap M(S))$ be the canonical epimorphism, and let $v = \operatorname{ord}_{h(R_0)} h(w)$. Since $w \notin R_0 \cap M(S)$, we get that v is a nonnegative integer and $h(w) = h(r)h(x)^v$ where r is a unit in R_0. Consequently $w = rx^v + sy + tz$ with s and t in R_0. Let $r' = r + s(y/x^v) + t(z/x^v)$. Then $w = r'x^v$, and r' is a unit in R_i for all $i \geqslant v$. Let $d = \operatorname{ord}_{R_0} J_0$. Then $\operatorname{ord}_{R_i} J_i = d$ for $0 \leqslant i < \infty$, and hence

$$I_i = x^{v+i(d+b+c)}(y/x^i)^b(z/x^i)^cR_i \qquad \text{for all } \; i \geqslant v.$$

It follows that (E, I_i) has a normal crossing at R_i for all $i \geqslant v$. Let $j = 1 + \max(v, q)$. Then (E, I_{j-1}) has a normal crossing at R_{j-1}, and $((\mathfrak{C}^2(R_{j-1}, J_{j-1}) - E), I_{j-1})$ has a normal crossing at R_{j-1}; consequently (S', I_{j-1}) has a normal crossing at R_{j-1} for all $S' \in \mathfrak{C}^2(R_{j-1}, J_{j-1})$; also $\mathfrak{C}^2(R_{j-1}, J_{j-1})$ has a normal crossing at R_{j-1}; therefore by (2.15.3) we get that $(\mathfrak{C}^2(R_i, J_i), I_i)$ has a strict normal crossing at R_i for all $i \geqslant j$.

(2.17). *Let R be a regular local domain with* $\dim R \leqslant 3$. *Let J and I be nonzero principal ideals in R such that $J \neq R$,* $(\mathfrak{C}^2(R, J), I)$

has a strict normal crossing at R, and (J, I) has a quasinormal crossing at R. Let $S \in \mathfrak{C}^2(R, J)$ and let (R', J', I') be a monoidal transform of (R, J, I, S) such that $\operatorname{ord}_{R'} J' = \operatorname{ord}_R J$. Then $(\mathfrak{C}^2(R', J'), I')$ has a strict normal crossing at R'.

PROOF. By (1.10.6) we know that I' has a normal crossing at R'; hence if $\dim R' \neq 3$ then $(\mathfrak{C}^2(R', J'), I')$ has a strict normal crossing at R'. So assume that $\dim R' = 3$. Then $\dim R = 3$. Let

$$E^* = \{S^* \in \mathfrak{C}^2(R', J') : R \cap M(S) \subset R' \cap M(S^*)\}$$

and

$$E' = (\mathfrak{C}^2(R, J) - \{S\}) \cap \mathfrak{C}^2(R', J').$$

Since $\operatorname{ord}_{R'} J' = \operatorname{ord}_R J$, we get that $\mathfrak{B}(R) \cap \mathfrak{C}^2(R', J') \subset \mathfrak{C}^2(R, J)$; by (1.9.7) we know that $S \notin \mathfrak{B}(R')$; hence in view of (1.9.5) we get that $\mathfrak{C}^2(R', J') = E^* \cup E'$. Let $d = \operatorname{ord}_R J$. Then $\operatorname{ord}_{R'} J' = d$. We shall now divide the argument into two cases according as $E' = \varnothing$ or $E' \neq \varnothing$.

Case when $E' = \varnothing$. Now $\mathfrak{C}^2(R', J') = E^*$. Since (S, I) has a normal crossing at R, there exists a basis (x', y', z) of $M(R)$ such that $R \cap M(S) = (x', y')R$ and $I = x'^{a'} y'^{b'} z^c R$ where a', b', c are nonnegative integers. Now (J, I) has a quasinormal crossing at R, $J \neq R$, and $S \in \mathfrak{C}^2(R, J)$; consequently there exists $y \in (x', y')R$ with $\operatorname{ord}_R y = 1$ such that $J \subset yR$ and (yR, I) has a quasinormal crossing at R; hence by (1.5.1) there exists $x \in R$ such that $M(R) = (x, y, z)R$, $R \cap M(S) = (x, y)R$, and $I = x^a y^b z^c R$ where a and b are nonnegative integers. Now $J = yy^*R$ where $y^* \in R$ with $\operatorname{ord}_R y^* = d - 1 = \operatorname{ord}_S y^*$. If $x/y \in R'$ then we would get that $J' = (y^*/y^{d-1})R'$ and $\operatorname{ord}_{R'}(y^*/y^{d-1}) \leqslant d - 1$ which would be a contradiction. Consequently $x/y \notin R'$. Therefore $M(R') = (x, y/x, z)R'$, $I' = x^{d+a+b}(y/x)^b z^c R'$, and $x \in R' \cap M(S^*)$ for all $S^* \in \mathfrak{C}^2(R', J')$. Also $J' = (y/x)(y^*/x^{d-1})R'$, $\operatorname{ord}_{R'}(y/x) = 1$, $y^*/x^{d-1} \in R'$, $\operatorname{ord}_{R'}(y^*/x^{d-1}) \leqslant d - 1$, and $\operatorname{ord}_{R'} J' = d$; therefore $\operatorname{ord}_{R'}(y^*/x^{d-1}) = d - 1$; it follows that $y/x \in R' \cap M(S^*)$ for all $S^* \in \mathfrak{C}^2(R', J')$, and hence $(x, y/x)R' = R' \cap M(S^*)$ for all $S^* \in \mathfrak{C}^2(R', J')$. Therefore $(\mathfrak{C}^2(R', J'), I')$ has a strict normal crossing at R'.

Case when $E' \neq \varnothing$. Take $S' \in E'$. Then $\mathfrak{C}^2(R, J) = \{S, S'\}$, $S' \neq S$, and $\mathfrak{C}^2(R', J') = E^* \cup \{S'\}$. Since $(\mathfrak{C}^2(R, J), I)$ has a normal crossing at R, there exists a basis (x', y', z) of $M(R)$ such that $R \cap M(S) = (x', y')R$, $R \cap M(S') = (y', z)R$, and $I = x'^{a'}y'^{b'}z^cR$ where a', b', c are nonnegative integers. Now (J, I) has a quasinormal crossing at R, $J \neq R$, and $S \in \mathfrak{E}^2(R, J)$; consequently there exists $y \in (x', y')R$ with $\mathrm{ord}_R y = 1$ such that $J \subset yR$ and (yR, I) has a quasinormal crossing at R; hence by (1.5.1) there exists $x \in R$ such that $M(R) = (x, y, z)R$, $R \cap M(S) = (x, y)R$, and $I = x^a y^b z^c R$ where b and c are nonnegative integers. Since $S' \in \mathfrak{E}^2(R, J)$ and $J \subset yR$, we get that $y \in R \cap M(S')$. Thus $(y, z)R \subset R \cap M(S')$ and hence $R \cap M(S') = (y, z)R$. Since $S' \in \mathfrak{V}(R')$, by (1.10.10) we get that $M(R') = (x, y/x, z)R'$ and $R' \cap M(S') = (y/x, z)R'$. Now $I' = x^{d+a+b}(y/x)^b z^c R'$, and $x \in R' \cap M(S^*)$ for all $S^* \in E^*$. Also $J = yy^*R$ where $y^* \in R$ with $\mathrm{ord}_R y^* = d - 1 = \mathrm{ord}_S y^*$ and hence $J' = (y/x)(y^*/x^{d-1})R'$; now $\mathrm{ord}_{R'}(y/x) = 1$, $y^*/x^{d-1} \in R'$, $\mathrm{ord}_{R'}(y^*/x^{d-1}) \leqslant d - 1$, and $\mathrm{ord}_{R'} J' = d$; it follows that $y/x \in R' \cap M(S^*)$ for all $S^* \in \mathfrak{E}^2(R', J')$. Therefore $(x, y/x)R' = R' \cap M(S^*)$ for all $S^* \in E^*$. It follows that $(\mathfrak{E}^2(R', J'), I')$ has a strict normal crossing at R'.

(2.18). *Let R be a regular local domain with* $\dim R \leqslant 3$. *Let J and I be nonzero principal ideals in R such that (R, J) is unresolved and (J, I) has a quasinormal crossing at R. Let S be a positive-dimensional element in $\mathfrak{E}(R, J)$ such that (S, I) has a normal crossing at R, and let (R', J', I') be a monoidal transform of (R, J, I, S) such that* $\mathrm{ord}_{R'} J' = \mathrm{ord}_R J$. *Assume that if* $\dim S = 2$ *then* $(\mathfrak{E}^2(R, J), I)$ *has a strict normal crossing at R. Then we have the following.*

(2.18.1). *If $\mathfrak{E}^2(R, J)$ has a normal crossing at R, then $\mathfrak{E}^2(R', J')$ has a strict normal crossing at R'.*

(2.18.2). *If $(\mathfrak{E}^2(R, J), I)$ has a normal crossing at R, then $(\mathfrak{E}^2(R', J'), I')$ has a strict normal crossing at R'.*

PROOF. Since (R, J) is unresolved, we get that $2 \leqslant \dim S \leqslant 3$, and $J \neq R$. If $S = R$ then our assertions follow from (2.15.5) and (2.15.4) respectively. So now assume that $S \neq R$. Then we must have $S \in \mathfrak{E}^2(R, J)$ and $\dim R = 3$. Now by assumption $(\mathfrak{E}^2(R, J), I)$ has a strict normal crossing at R, and hence by (2.17) we get that $(\mathfrak{E}^2(R', J'), I')$ has a strict normal crossing at R'.

(2.19). *Let R be a regular local domain with* $\dim R \leqslant 3$ *such that R is strongly semiresolvable. Then R is strongly detachable.*

PROOF. Suppose if possible that R is not strongly detachable. Then there exists an infinite detacher $(R_i, J_i, I_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R. Note that now R_i is an iterated monoidal transform of R and (R_i, J_i) is unresolved for $0 \leqslant i < \infty$. Also $\operatorname{ord}_{R_i} J_i \leqslant \operatorname{ord}_{R_a} J_a$ whenever $i \geqslant a$, and hence there exist nonnegative integers b and d such that $\operatorname{ord}_{R_i} J_i = d$ whenever $i \geqslant b$. If $(\mathfrak{C}^2(R_q, J_q), I_q)$ has a normal crossing at R_q for some $q \geqslant b$ then by (2.18.2) we would get that $(\mathfrak{C}^2(R_i, J_i), I_i)$ has a normal crossing at R_i for all $i \geqslant q$ and from this it would follow that $(R_{q+i}, J_{q+i}, S_{q+i})_{0 \leqslant i < \infty}$ is an infinite semiresolver, which would contradict the assumption that R is strongly semiresolvable. Therefore for each $i \geqslant b$ we must have that $(\mathfrak{C}^2(R_i, J_i), I_i)$ does not have a normal crossing at R_i and hence $\dim S_i \neq 2$; consequently $S_i = R_i$ and $\dim R_i = 3$ for all $i \geqslant b$; hence in view of (2.16), for each $i \geqslant b$ we must have that $\mathfrak{C}^2(R_i, J_i)$ does not have a normal crossing at R_i. Consequently $(R_{b+i}, J_{b+i}, R_{b+i})_{0 \leqslant i < \infty}$ is an infinite semiresolver, which contradicts the assumption that R is strongly semiresolvable. Therefore R is strongly detachable.

(2.20). *Let R be a regular local domain, let J be a nonzero principal prime ideal in R, let S be a positive-dimensional element in $\mathfrak{B}(R)$ such that S has a simple point at R and $S \neq R_J$, and let (R', J') be a monoidal transform of (R, J, S). Then $J' \neq R'$ if and only if $R' \subset R_J$. Moreover, if $J' \neq R'$ then J' is a prime ideal in R' and $R'_{J'} = R_J$.*

PROOF. Let $d = \operatorname{ord}_S J$ and $Q = (R \cap M(S))R'$. Then Q is a nonzero principal prime ideal in R', $JR' = Q^d J'$, and $J' \not\subset Q$. Since $S \neq R_J$, we also have that $(R \cap M(S))R_J = R_J$.

First suppose that $R' \subset R_J$. Since $(R \cap M(S))R_J = R_J$, we get that $QR_J = R_J$. Since $JR' = Q^d J'$, we now get that $JR_J = J'R_J$. Clearly $JR_J \neq R_J$, and hence $J'R_J \neq R_J$. Therefore $J' \neq R'$.

Conversely suppose that $J' \neq R'$. Then $J' = Q_1 \dots Q_n$ where $Q_1, \dots, Q_n$ $(n > 0)$ are nonzero principal prime ideals in R'. Let $R_i = R'_{Q_i}$. Since $J' \not\subset Q$, we get that $Q \not\subset Q_i$ and hence $(R \cap M(S)) \not\subset Q_i$; consequently $(R \cap M(S))R_i = R_i$ and hence by

(1.9.5) we have that $R_i \in \mathfrak{B}(R)$; now $J \subset Q_i$, and hence $R_i \subset R_J$; since $\dim R_i = 1 = \dim R_J$, we must have $R_i = R_J$ and hence $Q_i R_J = M(R_J)$. This being so for $1 \leqslant i \leqslant n$, we get that $M(R_J) = JR_J = (Q^d Q_1 \dots Q_n)R_J \subset M(R_J)^n$ and hence $n = 1$.

(2.21). *Let R be a regular local domain such that R is semiresolvable. Let R' be an iterated monoidal transform of R, let J' be a nonzero principal prime ideal in R', and let V be a valuation ring of the quotient field of R'/J' such that V dominates R'/J'. Then there exists a regular spot R^* over R'/J' such that V dominates R^*.*

PROOF. Let $h: R' \to R'/J'$ be the canonical epimorphism. Now $R'_{J'}$ is a valuation ring of the quotient field L of R and there exists a unique epimorphism g of $R'_{J'}$ onto the quotient field K of R'/J' such that $g^{-1}(0) = M(R'_{J'})$ and $g(u) = h(u)$ for all $u \in R'$. Let $W = g^{-1}(V)$. Then W is a valuation ring of L, $g^{-1}(M(V)) = M(W)$, and W dominates R'. If (R', J') is resolved then R'/J' is regular and it suffices to take $R^* = R'/J'$. So assume that (R', J') is unresolved. Since R' is semiresolvable, there exists a finite semiresolver $[(R_i, J_i, S_i)_{0 \leqslant i < m}, (R_m, J_m)]$ such that $(R_0, J_0) = (R', J')$ and W dominates R_m. Now $R_m \subset W \subset R'_{J'}$, and hence by (2.20) we get that J_m is a prime ideal in R_m and $R_m \cap M(R'_{J'}) = J_m$. Let $R^* = g(R_m)$. Now (R_m, J_m) is resolved, and hence R^* is regular. Since R_m is a spot over R', we get that R^* is a spot over R'/J'. Since W dominates R_m, we also get that V dominates R^*.

§3. Dominant character of a normal sequence

(3.1). *Let R_0 be a pseudogeometric one-dimensional local domain and let $(R_i)_{0<i<\infty}$ be an infinite sequence of local domains such that R_i is a quadratic transform of R_{i-1} for $0 < i < \infty$. Then there exists a nonnegative integer j such that R_j is a one-dimensional regular local domain and $R_i = R_j$ for all $i \geqslant j$.*

PROOF. We can take a valuation ring V of the quotient field K of R_0 such that V dominates R_i for $0 \leqslant i < \infty$. Let T be the integral closure of R_0 in K. Then $T \subset V$. Let $P = T \cap M(V)$. Since R_0 is pseudogeometric, we have that T is a finite R_0-module and hence T is noetherian. Therefore by [27: §2, §3, §6, and §7 of Chapter V] we get that T_P is a one-dimensional regular local domain. Since V dominates T_P, we must have $V = T_P$. Given any $z \in V$, for each $i \geqslant 0$ let $u(i)$ be the smallest nonnegative integer such that $yz \in R_i$ for some $0 \neq y \in R_i$ with $\operatorname{ord}_V y = u(i)$; note that then $z \in R_i$ if and only if $u(i) = 0$; also note that $u(i) \geqslant u(i+1)$ for all $i \geqslant 0$. Given $i \geqslant 0$ take $0 \neq y \in R_i$ such that $yz \in R_i$ and $\operatorname{ord}_V y = u(i)$; we claim that $u(i+1) \leqslant \max(0, u(i) - 1)$; this is obvious when $u(i) = 0$ because then $u(i+1) = 0$; so assume that $u(i) \neq 0$; then $y \in M(R_i)$ and hence $yz \in M(R_i)$; now $M(R_i)R_{i+1} = xR_{i+1}$ for some $0 \neq x \in M(R_i)$; let $y' = y/x$; then $0 \neq y' \in R_{i+1}$, $y'z = (yz)/x \in R_{i+1}$, and $\operatorname{ord}_V y' \leqslant u(i) - 1$; hence $u(i+1) \leqslant u(i) - 1$. Thus $u(i+1) \leqslant \max(0, u(i) - 1)$ for all $i \geqslant 0$. Therefore $u(u(0)) = 0$ and hence $z \in R_{u(u(0))}$. Thus $\bigcup_{i=0}^{\infty} R_i = V$. In particular $T \subset \bigcup_{i=0}^{\infty} R_i$. Since T is a finite R_0-module, there exists a nonnegative integer j such that $T \subset R_i$ for all $i \geqslant j$. For each $i \geqslant j$ we then have that $T \subset R_i \subset V$ and $T \cap M(R_i) = T \cap M(V) = P$; consequently $T_P \subset R_i \subset V = T_P$ and hence $R_i = T_P$.

(3.2). *Let R_0 be a pseudogeometric local domain, let $(R_i)_{0<i<\infty}$ be an infinite sequence of local domains such that R_i is a quadratic transform of R_{i-1} for $0 < i < \infty$, and let $S \in \bigcap_{i=0}^{\infty} \mathfrak{V}(R_i)$ such that* $\dim S = (\dim R_0) - 1$. *Then there exists a nonnegative integer j such that S has a simple point at R_i for all $i \geqslant j$.*

PROOF. If $R_j = R_{j+1}$ for some $j \geqslant 0$ then by (1.9.6) we get that $M(R_j)$ is a principal ideal in R_j and hence for each $i \geqslant j$ we have that R_i is a one-dimensional regular local domain and hence S has a simple point at R_i. So now assume that $R_i \neq R_{i+1}$ for all $i \geqslant 0$. Let $R = \bigcup_{i=0}^{\infty} R_i$. Then R is a quasilocal domain, $M(R) = \bigcup_{i=0}^{\infty} M(R_i)$, $R \subset S$, and $R \cap M(S)$ is a prime ideal in R. Let $h\colon R \to R/(R \cap M(S))$ be the canonical epimorphism. Then $h(R)$ is a domain and for each $i \geqslant 0$ we have that $h(R_i)$ is a subring of $h(R)$ and $h(R_i)$ is isomorphic to $R_i/(R_i \cap M(S))$; also

$\dim h(R_0) = 1$ and $h(R_0)$ is pseudogeometric. Let any $i \geqslant 0$ be given; since R_{i+1} is a quadratic transform of R_i, there exists $0 \neq x \in M(R_i)$ such that $R_i[M(R_i)x^{-1}] \subset R_{i+1}$ and $R_{i+1} = A_Q$ where $A = R_i[M(R_i)x^{-1}]$ and $Q = A \cap M(R_{i+1})$; since $R_i \neq R_{i+1}$ and $S \in \mathfrak{V}(R_i) \cap \mathfrak{V}(R_{i+1})$, we get that $M(R_i) \not\subset M(S)$; since $M(R_i)R_{i+1} = xR_{i+1}$ and $R_{i+1} \subset S$, we must have $x \notin M(S)$; therefore $0 \neq h(x) \in M(h(R_i))$, $h(A) = h(R_i)[M(h(R_i))h(x)^{-1}]$, $h(Q)$ is a prime ideal in $h(A)$, $M(h(R_i)) \subset h(Q)$, and $h(R_{i+1}) = h(A)_{h(Q)}$; consequently $h(R_{i+1})$ is a quadratic transform of $h(R_i)$. Thus $h(R_{i+1})$ is a quadratic transform of $h(R_i)$ for all $i \geqslant 0$ and hence by (3.1) there exists a nonnegative integer j such that for each $i \geqslant j$ we have that $h(R_i)$ is regular and hence S has a simple point at R_i.

(3.3). *Let R_0 be an n-dimensional regular local domain and let $(R_i)_{0<i<\infty}$ be an infinite sequence of regular local domains such that R_i is a quadratic transform of R_{i-1} for $0 < i < \infty$. Let E be a finite set of $(n-1)$-dimensional elements in $\mathfrak{V}(R_0)$ and let $E' = E \cap (\bigcap_{i=0}^{\infty} \mathfrak{V}(R_i))$. Then we have the following.*

(3.3.1). *Assume that every element in E has a simple point at R_0. Then E' contains at most one element and there exists a nonnegative integer j such that for all $i \geqslant j$ we have that $E' = E \cap \mathfrak{V}(R_i)$ and E' has a normal crossing at R_i.*

(3.3.2). *Assume that R_0 is pseudogeometric. Then E' contains at most one element and there exists a nonnegative integer j such that for all $i \geqslant j$ we have that $E' = E \cap \mathfrak{V}(R_i)$ and E' has a normal crossing at R_i.*

PROOF OF (3.3.1). Clearly $E \cap \mathfrak{V}(R_b) \subset E \cap \mathfrak{V}(R_a)$ whenever $b \geqslant a$, and hence there exists a nonnegative integer c such that $E' = E \cap \mathfrak{V}(R_i)$ for all $i \geqslant c$. Consequently if $E' = \varnothing$ then we have nothing to show. So now assume that $E' \neq \varnothing$ and take $S \in E'$. Then there exists a basis $(x_1, ..., x_n)$ of $M(R_0)$ such that $R_0 \cap M(S) = (x_2, ..., x_n)R_0$. Repeatedly applying (1.10.10) we get that for all $i \geqslant 0$ we have that $\dim R_i = n$, $M(R_i) = (x_1, x_2/x_1^i, ..., x_n/x_1^i)R_i$, and $R_i \cap M(S) = (x_2/x_1^i, ..., x_n/x_1^i)R_i$. In particular S has a simple point at R_i for all $i \geqslant 0$. It now suffices to show that $E' = \{S\}$. Suppose if possible that $E' \neq \{S\}$ and take $S' \in E'$ with $S' \neq S$. For any $i > 0$ and any $z \in R_{i-1} \cap M(S')$, by (1.10.9) we have that $z/x_1 \in R_i \cap M(S')$. Therefore for any $z \in R_0 \cap M(S')$ we get that $z/x_1^i \in R_i \cap M(S')$ for all $i \geqslant 0$ and hence $z/x_1^i \in M(R_i)$ for all $i \geqslant 0$. Since $S' \neq S$, there exists $z \in R_0 \cap M(S')$ such that $z \notin (x_2, ..., x_n)R_0$. Let h: $R_0 \to R_0/(x_2, ..., x_n)R_0$ be the canonical epimorphism and let $i = \mathrm{ord}_{h(R_0)}h(z)$. Then i is a nonnegative integer and $z = r_1x_1^i + r_2x_2 + \cdots + r_nx_n$ where $r_1, ..., r_n$ are elements in R_0 such that $r_1 \notin M(R_0)$. Now $z/x_1^i = r_1 + r_2(x_2/x_1^i) + \cdots + r_n(x_n/x_1^i)$, $r_q \in R_i$ for $1 \leqslant q \leqslant n$, $x_q/x_1 \in M(R_i)$ for $2 \leqslant q \leqslant n$, and $r_1 \notin M(R_i)$. Therefore $z/x_1^i \notin M(R_i)$ which is a contradiction.

PROOF OF (3.3.2). Clearly $E \cap \mathfrak{V}(R_b) \subset E \cap \mathfrak{V}(R_a)$ whenever $b \geqslant a$, and hence there exists a nonnegative integer c such that $E' = E \cap \mathfrak{V}(R_i)$ for all $i \geqslant c$. Consequently if $E' = \varnothing$ then we have nothing to show. So now assume that $E' \neq \varnothing$ and take $S \in E'$. If $\dim R_d \neq n$ for some d then R_i is an $(n-1)$-dimensional regular local domain for all $i \geqslant d$, and hence $E' = \{S\}$ and E' has a normal crossing at R_i for all $i \geqslant d$. So also assume that $\dim R_i = n$ for all $i \geqslant 0$. By (3.2) there exists $e \geqslant c$ such that every element in E' has a simple point at R_e. Now by (3.3.1) we get that E' contains at most one element and there exists $j \geqslant e$ such that E' has a normal crossing at R_i for all $i \geqslant j$.

(3.4). *Let R_0 be an n-dimensional regular local domain and let J_0 and I_0 be nonzero principal ideals in R_0 such that I_0 has a quasinormal crossing at R_0. Let $(R_i, J_i, I_i)_{0<i<\infty}$ be an infinite sequence such that for $0 < i < \infty$: R_i is a regular local domain, J_i and I_i are nonzero principal ideals in R_i, and (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, R_{i-1})$. Let E be a finite set of $(n-1)$-dimensional elements in $\mathfrak{V}(R_0)$, and let $E' = E \cap (\bigcap_{i=0}^{\infty} \mathfrak{V}(R_i))$. Then we have the following.*

(3.4.1). *Assume that every element in E has a simple point at R_0. Then E' contains at most one element and there exists a nonnegative integer j such that for all $i \geqslant j$ we have that $E' = E \cap \mathfrak{V}(R_i)$ and (E', I_i) has a pseudonormal crossing at R_i.*

(3.4.2). *Assume that R_0 is pseudogeometric. Then E' contains at most one element and there exists a nonnegative integer j such that for all $i \geqslant j$ we have that $E' = E \cap \mathfrak{B}(R_i)$ and (E', I_i) has a pseudonormal crossing at R_i.*

PROOF OF (3.4.1). By (1.10.8) we know that I_i has a quasinormal crossing at R_i for all $i \geqslant 0$. By (3.3.1) we know that E' contains at most one element and there exists a nonnegative integer j' such that $E' = E \cap \mathfrak{B}(R_i)$ for all $i \geqslant j'$. If $E' = \varnothing$ then it suffices to take $j = j'$. So now assume that $E' \neq \varnothing$ and take $S \in E'$. Then $E' = \{S\}$ and there exists a basis $(x_1, ..., x_n)$ of $M(R_0)$ such that $R_0 \cap M(S) = (x_2, ..., x_n)R_0$. Repeatedly applying (1.10.10) we get that for all $i \geqslant 0$ we have that $\dim R_i = n$, $M(R_i) = (x_1, x_2/x_1^i, ..., x_n/x_1^i)R_i$, and $R_i \cap M(S) = (x_2/x_1^i, ..., x_n/x^i)R_i$. Now $I_0 = z_1 ... z_a R_0$ where $z_1, ..., z_a$ are elements in R_0 with $\operatorname{ord}_{R_0} z_q = 1$ for $1 \leqslant q \leqslant a$ (we take $z_1 ... z_a R_0 = R_0$ in case $a = 0$). Upon relabeling $z_1, ..., z_a$ we may assume that $z_q \in (x_2, ..., x_n)R_0$ for $1 \leqslant q \leqslant b$, and $z_q \notin (x_2, ..., x_n)R_0$ for $b < q \leqslant a$ where b is an integer with $0 \leqslant b \leqslant a$. Let $I_0' = z_1 ... z_b R_0$ and $z = z_{b+1} \cdots z_a$ (we take $z_1 ... z_b R_0 = R_0$ in case $b = 0$, and $z_{b+1} ... z_a = 1$ in case $b = a$). Then $z \notin (x_2, ..., x_n)R_0$. Let $h\colon R_0 \to R_0/(x_2, ..., x_n)R_0$ be the canonical epimorphism and let $e = \operatorname{ord}_{h(R_0)} h(z)$. Then e is a nonnegative integer and $z = r_1 x_1^e + r_2 x_2 + \cdots + r_n x_n$ where $r_1, ..., r_n$ are elements in R_0 with $r_1 \notin M(R_0)$. It follows that z/x_1^e is a unit in R_i for all $i \geqslant e$. Let $(I_i')_{0<i<\infty}$ be the unique infinite sequence such that (R_i, J_i, I_i') is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}', R_{i-1})$ for $0 < i < \infty$. Let q be any integer with $1 \leqslant q \leqslant b$; since $z_q \in (x_2, ..., x_n)R_0$ and $\operatorname{ord}_{R_0} z_q = 1$, we get that $z_q = s_2 x_2 + \cdots + s_n x_n$ where $s_2, ..., s_n$ are elements in R_0 such that $s_p \notin M(R_0)$ for some p; now $M(R_0) = (x_1, x_2, ..., x_{p-1}, z_q, x_{p+1}, ..., x_n)R_0$ and $R_0 \cap M(S) = (x_2, ..., x_{p-1}, z_q, x_{p+1}, ..., x_n)R_0$; consequently $(S, z_q R_0)$ has a pseudonormal crossing at R_0. This being so for $1 \leqslant q \leqslant b$, it follows that (S, I_0') has a pseudonormal crossing at R_0. Therefore, upon applying (1.10.12) repeatedly, we get that (S, I_i') has a pseudonormal crossing at R_i for all $i \geqslant 0$. Clearly $I_i = zI_i'$ for all $i \geqslant 0$, and hence (S, I_i) has a pseudonormal crossing at R_i for all $i \geqslant e$. It now suffices to take $j = \max(j', e)$.

PROOF OF (3.4.2). By (1.10.8) we know that I_i has a quasinormal crossing at R_i for all $i \geqslant 0$. Clearly $E \cap \mathfrak{V}(R_b) \subset E \cap \mathfrak{V}(R_a)$ whenever $b \geqslant a$, and hence there exists a nonnegative integer c such that $E' = E \cap \mathfrak{V}(R_i)$ for all $i \geqslant c$. Consequently if $E' = \varnothing$ then we have nothing to show. So now assume that $E' \neq \varnothing$ and take $S \in E'$. If $\dim R_d \neq n$ for some d then R_i is an $(n-1)$-dimensional regular local domain for all $i \geqslant d$, and hence $E' = \{S\}$ and (E', I_i) has a pseudonormal crossing at R_i for all $i \geqslant d$. So also assume that $\dim R_i = n$ for all $i \geqslant 0$. By (3.2) there exists $e \geqslant c$ such that every element in E' has a simple point at R_e. Now by (3.4.1) we get that E' contains at most one element and there exists $j \geqslant e$ such that (E', I_i) has a pseudonormal crossing at R_i for all $i \geqslant j$.

(3.5). *Let R_0 be a pseudogeometric two-dimensional regular local domain and let J_0 be a nonzero principal ideal in R_0. Let $(R_i, J_i)_{0<i<\infty}$ be an infinite sequence such that for $0 < i < \infty$: R_i is a regular local domain, J_i is a nonzero principal ideal in R_i, and (R_i, J_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, R_{i-1})$. Then there exists a nonnegative integer j such that (R_i, J_i) is resolved for all $i \geqslant j$.*

PROOF. Take $w \in R_0$ such that $J_0 = wR_0$. Then for each $i \geqslant 0$ there exist elements w_i and z_i in R_i such that $w = w_i z_i$ and $J_i = w_i R_i$. If (R_j, J_j) is resolved for some j then by (1.10.4) we have that (R_i, J_i) is resolved for all $i \geqslant j$. Hence it suffices to show that (R_i, J_i) is resolved for some $i \geqslant 0$. Suppose if possible that (R_i, J_i) is unresolved for all $i \geqslant 0$. Then $\dim R_i = 2$ and $\operatorname{ord}_{R_i} J_i \geqslant 2$ for all $i \geqslant 0$. Now $\operatorname{ord}_{R_{i+1}} J_{i+1} \leqslant \operatorname{ord}_{R_i} J_i$ for all $i \geqslant 0$, and hence there exists a nonnegative integer j such that $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_j} J_j$ for all $i \geqslant j$. By (0.1) and (0.2) (alternatively see [8: Lemmas 3.7 and 3.14(4)]) there exists $i \geqslant j$ and a basis (x, y) of $M(R_i)$ such that $w = rx^{u'}y^{v'}$ where r is a unit in R_i and u' and v' are nonnegative integers. Since $w = w_i z_i$ and $J_i = w_i R_i$, it follows that $J_i = x^u y^v R_i$ where u and v are nonnegative integers. Upon relabeling x and y we may assume that $y/x \in R_{i+1}$. Then $J_{i+1} = (y/x)^v R_{i+1}$. Since (R_{i+1}, J_{i+1}) is unresolved, we must have $y/x \in M(R_{i+1})$ and hence $M(R_{i+1}) = (x, y/x)R_{i+1}$. Since (R_i, J_i) is unresolved, we must have $u > 0$ and $v > 0$. Now $\operatorname{ord}_{R_{i+1}} J_{i+1} = v < u + v = \operatorname{ord}_{R_i} J_i$ which is a contradiction.

(3.6). *Let R_0 be a pseudogeometric regular local domain, let J_0 be a nonzero principal ideal in R_0, let T_0 be an element in $\mathfrak{E}^2(R_0, J_0)$ having a simple point at R_0, and let S_0 be a positive-dimensional element in $\mathfrak{E}(R_0, J_0)$ having a simple point at R_0 such that $S_0 \subset T_0$. Let $(R_i, J_i, T_i, S_i)_{0<i<\infty}$ be an infinite sequence such that for $0 < i < \infty$: R_i is a regular local domain, J_i is a nonzero principal ideal in R_i, T_i is an element in $\mathfrak{E}(R_i, J_i)$ having a simple point at R_i, S_i is a positive-dimensional element in $\mathfrak{E}(R_i, J_i)$ having a simple point at R_i, $S_i \subset T_i$, (R_i, J_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, S_{i-1})$, and T_i dominates T_{i-1}. Assume that $S_i = T_i$ for infinitely many distinct values of i. Then there exists a nonnegative integer j such that (R_i, J_i) is resolved for all $i \geqslant j$.*

PROOF. Let N be the set of all nonnegative integers i such that $S_i = T_i$. By assumption N is an infinite set. We can arrange all the integers in N in the form of an infinite sequence $a(0) < a(1) < a(2) < \dots$.

Let i be any given nonnegative integer. Let $d = \operatorname{ord}_{R_i} J_i$, $P = R_i \cap M(S_i)$, and $Q = PT_i$. Take $w \in R_i$ such that $J_i = wR_i$. Then there exists $0 \neq x \in P$ such that $R_{i+1} \in \mathfrak{V}(R_i[Px^{-1}])$, $PR_{i+1} = xR_{i+1}$, and $J_{i+1} = (w/x^d)R_{i+1}$. Now clearly $J_i T_i = wT_i$, $0 \neq x \in Q$, $T_{i+1} \in \mathfrak{V}(T_i[Qx^{-1}])$, $QT_{i+1} = xT_{i+1}$, and $J_{i+1}T_{i+1} = (w/x^d)T_{i+1}$. First suppose that $i \notin N$; then $Q = T_i$ and hence $T_{i+1} = xT_{i+1}$; since T_{i+1} dominates T_i, we get that x is a unit in T_i and hence $T_i[Qx^{-1}] = T_i$; since $T_{i+1} \in \mathfrak{V}(T_i[Qx^{-1}])$ and $J_{i+1}T_{i+1} = (w/x^d)T_{i+1}$, we conclude that $(T_{i+1}, J_{i+1}T_{i+1}) = (T_i, J_iT_i)$. Next suppose that $i \in N$; then $Q = M(T_i)$ and hence $(T_{i+1}, J_{i+1}T_{i+1})$ is a monoidal transform of (T_i, J_iT_i, T_i).

It follows that $(T_{a(i)}, J_{a(i)}T_{a(i)})_{0 \leqslant i < \infty}$ is an infinite sequence such that for $0 \leqslant i < \infty$: $T_{a(i)}$ is a regular local domain, $J_{a(i)}T_{a(i)}$ is a nonzero principal ideal in $T_{a(i)}$, and $(T_{a(i+1)}, J_{a(i+1)}T_{a(i+1)})$ is a monoidal transform of $(T_{a(i)}, J_{a(i)}T_{a(i)}, T_{a(i)})$. Also $T_{a(0)} = T_0$, and hence $T_{a(0)}$ is pseudogeometric and $\dim T_{a(0)} = 2$. Therefore by (3.5) there exists a nonnegative integer j' such that $(T_{a(i)}, J_{a(i)}T_{a(i)})$ is resolved for all $i \geqslant j'$. Let $j = a(j')$. Then by (1.5.3) we get that (R_i, J_i) is resolved for all $i \geqslant j$.

(3.7). *Let R be a regular domain with* $\dim R \leqslant 3$, *let J be a nonzero nonunit principal ideal in R, let $d = \operatorname{ord}_R J$, let (R', J')*

be a monoidal transform of (R, J, R) *such that* $\operatorname{ord}_{R'} J' = d$, *and let* $E' = \{S' \in \mathfrak{C}^2(R', J') \colon M(R) \subset M(S')\}$. *Then we have the following.*

(3.7.1). *Assume that* $\dim R = \dim R' = 3$ *and* $E' \neq \varnothing$. *Let* $S' \in E'$, *let* $\mathfrak{k}$ *be a coefficient set for* R, *and let* (x_1, x_2, x_3) *be a basis of* $M(R)$ *such that* $x_2/x_1 \in R'$ *and* $x_3/x_1 \in R'$. *Then there exist elements* r_1, r_2, r_3 *in* $\mathfrak{k}$ *such that* $R' \cap M(S') = (x_1, r_1 + r_2(x_2/x_1) + r_3(x_3/x_1))R'$ *and* $M(R') = (x_1, r_1 + r_2(x_2/x_1) + r_3(x_3/x_1), t)R'$ *for some* $t \in R'$ *(whence in particular* S' *has a simple point at* R'*). Moreover, if* S'' *is any two-dimensional element in* $\mathfrak{V}(R')$ *such that* $M(R) \subset M(S'')$ *and* $J' \subset M(S'')$ *then* $S'' = S'$ *(whence in particular* $E' = \{S'\}$*).*

(3.7.2). *Let* E *be a set of two-dimensional elements in* $\mathfrak{V}(R)$ *such that every subset of* E *containing at most two elements has a normal crossing at* R. *Then* $(E \cap \mathfrak{V}(R')) \cup E'$ *has a strict normal crossing at* R', *and* $E \cap \mathfrak{V}(R')$ *contains at most one element.*

(3.7.3). *If* $\mathfrak{C}^2(R, J)$ *has a strict normal crossing at* R *then* $\mathfrak{C}^2(R', J')$ *has a strict normal crossing at* R', *and at most one element in* $\mathfrak{C}^2(R', J')$ *dominates* $\mathfrak{C}^2(R, J)$.

(3.7.4). *Let* I *be a nonzero principal ideal in* R *such that* I *has a quasinormal crossing at* R, *and let* I' *be the unique nonzero principal ideal in* R' *such that* (R', J', I') *is a monoidal transform of* (R, J, I, R). *Then* (E', I') *has a pseudonormal crossing at* R'. *Moreover, if* $\mathfrak{C}^2(R, J)$ *has a strict normal crossing at* R *and* $(\mathfrak{C}^2(R, J), I)$ *has a pseudonormal crossing at* R, *then* $\mathfrak{C}^2(R', J')$ *has a strict normal crossing at* R' *and* $(\mathfrak{C}^2(R', J'), I')$ *has a pseudonormal crossing at* R'.

PROOF OF (3.7.1). Let $h \colon R \to R/M(R)$ be the canonical epimorphism, let $A = R[x_2/x_1, x_3/x_1]$, let X_2, X_3 be indeterminates, let $A^* = h(R)[X_2, X_3]$, and let $H \colon A \to A^*$ be the unique epimorphism such that $H(x_2/x_1) = X_2$, $H(x_3/x_1) = X_3$, and $H(u) = h(u)$ for all $u \in R$. Now $\operatorname{Ker} H = x_1 A$, $x_1 A \subset (A \cap M(S')) \subset (A \cap M(R'))$ are distinct prime ideals in A, and S' is the quotient ring of A with respect to $(A \cap M(S'))$. Therefore $H(A \cap M(S'))$ is a nonzero principal prime ideal in A^* and, upon letting S^* be the quotient ring of A^* with respect to $H(A \cap M(S'))$, there exists a unique epimorphism $H^*: S' \to S^*$ such that $H^*(u) = H(u)$ for all $u \in A$. Take $w \in R$ such that $wR = J$. Then

$$w = \sum_{a+b+c=d} r_{abc} x_1^a x_2^b x_3^c$$

where r_{abc} are elements in R at least one of which is not in $M(R)$. Let $w' = w/x_1^d$. Then $w' \in A$ and

$$H(w') = \sum_{a+b+c=d} h(r_{abc}) X_2^b X_3^c \in A^*.$$

Therefore $H(w')$ is a nonzero polynomial of degree $\leqslant d$ in X_2, X_3 with coefficients in $h(R)$. Let $d^* = \operatorname{ord}_{S^*} H(w')$. Since $H(A \cap M(S'))$ is a nonzero principal prime ideal in A^*, we get that $H(w') \in H(A \cap M(S'))^{d^*}$. Now $J' = w'R'$ and hence $\operatorname{ord}_{S'} w' = d$; also $\operatorname{ord}_{S^*} H(w') = \operatorname{ord}_{S^*} H^*(w') \geqslant \operatorname{ord}_{S'} w'$ and hence $d^* \geqslant d$. Thus $H(A \cap M(S'))$ is a nonzero principal prime ideal in A^*, $H(w')$ is a nonzero polynomial of degree $\leqslant d$ in X_2, X_3 with coefficients in $h(R)$, $H(w') \in H(A \cap M(S'))^{d^*}$, and $d^* \geqslant d$; consequently we must have: $d^* = d$, $H(w')A^* = H(A \cap M(S'))^d$, $H(A \cap M(S'))$ is the only principal prime ideal in A^* which contains $H(w')$, and there exist elements r_1, r_2, r_3 in $\mathfrak{k}$ such that $H(A \cap M(S')) = (h(r_1) + h(r_2)X_2 + h(r_3)X_3)A^*$, and hence $R' \cap M(S') = (x_1, r_1 + r_2(x_2/x_1) + r_3(x_3/x_1))R'$. Now $H(A \cap M(R'))$ is a maximal ideal in A^*, $h(r_1) + h(r_2)X_2 + h(r_3)X_3 \in H(A \cap M(R'))$, and at least one of the two elements $h(r_2)$ and $h(r_3)$ is not zero; consequently $H(A \cap M(R')) = (h(r_1) + h(r_2)X_2 + h(r_3)X_3, H(t))A^*$ for some $t \in A$; now $M(R') = (x_1, r_1 + r_2(x_2/x_1) + r_3(x_3/x_1), t)R'$ and hence S' has a simple point at R'. Given any two-dimensional element S'' in $\mathfrak{V}(R')$ such that $M(R) \subset M(S'')$ and $J' \subset M(S'')$ we have that $w' \in (A \cap M(S''))$ and hence $H(w') \in H(A \cap M(S''))$; also $x_1 A \subset (A \cap M(S'')) \subset (A \cap M(R'))$ are distinct prime ideals in A and hence $H(A \cap M(S''))$ is a nonzero principal ideal in A^*; consequently $H(A \cap M(S'')) = H(A \cap M(S'))$ and hence $(A \cap M(S'')) = (A \cap M(S'))$; therefore $S'' = S'$.

PROOF OF (3.7.2). If $\dim R' \neq 3$ then our assertion is trivial. So now assume that $\dim R' = 3$. Then $\dim R = 3$. If $E' = \varnothing$ then our assertion follows from (1.10.11); and if $E \cap \mathfrak{V}(R') = \varnothing$ then our assertion follows from (3.7.1). So now assume that

$E' \neq \varnothing \neq E \cap \mathfrak{B}(R')$ and take $S' \in E'$ and $S \in E \cap \mathfrak{B}(R')$. Then $E' = \{S'\}$ by (3.7.1) and $E \cap \mathfrak{B}(R') = \{S\}$ by (1.10.11), and hence it suffices to show that $\{S, S'\}$ has a normal crossing at R'. Now S has a simple point at R and hence there exists a basis (x_1, x_2, x_3) of $M(R)$ such that $R \cap M(S) = (x_2, x_3)R$. Since $S \in \mathfrak{B}(R')$, by (1.10.10) we get that $M(R') = (x_1, x_2/x_1, x_3/x_1)R'$ and $R' \cap M(S) = (x_2/x_1, x_3/x_1)R'$. By (3.7.1) there exist elements r and s in R at least one of which is not in $M(R)$ such that $R' \cap M(S') = (x_1, r(x_2/x_1) + s(x_3/x_1))R'$. Let $y_1 = x_1$ and $y_2 = r(x_2/x_1) + s(x_3/x_1)$. Let $y_3 = x_3/x_1$ if $r \notin M(R)$, and $y_3 = x_2/x_1$ if $r \in M(R)$. Then $M(R') = (y_1, y_2, y_3)R'$, $R' \cap M(S) = (y_2, y_3)$, and $R' \cap M(S') = (y_1, y_2)R'$. Therefore $\{S, S'\}$ has a normal crossing at R'.

PROOF OF (3.7.3). By (1.9.5) we know that if S' is any element in $\mathfrak{B}(R')$ such that $S' \notin \mathfrak{B}(R)$ then $M(R) \subset M(S')$; since $\operatorname{ord}_{R'} J' = \operatorname{ord}_R J$, we also get that $\mathfrak{E}^2(R', J') \cap \mathfrak{B}(R) \subset \mathfrak{E}^2(R, J)$; consequently $\mathfrak{E}^2(R', J') \subset (\mathfrak{E}^2(R, J) \cap \mathfrak{B}(R')) \cup E'$. Therefore upon taking $E = \mathfrak{E}^2(R, J)$ our assertion follows from (3.7.2).

PROOF OF (3.7.4). By (1.9.5) we know that if S' is any element in $\mathfrak{B}(R')$ such that $S' \notin \mathfrak{B}(R)$ then $M(R) \subset M(S')$; since $\operatorname{ord}_{R'} J' = \operatorname{ord}_R J$, we also get that $\mathfrak{E}^2(R', J') \cap \mathfrak{B}(R) \subset \mathfrak{E}^2(R, J)$; consequently $\mathfrak{E}^2(R', J') \subset (\mathfrak{E}^2(R, J) \cap \mathfrak{B}(R')) \cup E'$. By (3.7.3) we know that if $\mathfrak{E}^2(R, J)$ has a strict normal crossing at R then $\mathfrak{E}^2(R', J')$ has a strict normal crossing at R'. By (1.10.8) we know that I' has a quasinormal crossing at R'. Finally by (1.10.12) we get that if S' is any element in $\mathfrak{E}^2(R, J) \cap \mathfrak{B}(R')$ such that (S', I) has a pseudonormal crossing at R then (S', I') has a pseudonormal crossing at R'. Therefore our assertions would follow by showing that if $\dim R' = 3$ and $E' \neq \varnothing$ then (E', I') has a pseudonormal crossing at R'. So assume that $\dim R' = 3$ and $E' \neq \varnothing$. Then $\dim R = 3$. Take $S \in E'$. Then by (3.7.1) we get that $E' = \{S\}$ and S has a simple point at R'. Now $I = z_1 \dots z_e R$ where $z_1, \dots, z_e$ are elements in R with $\operatorname{ord}_R z_i = 1$ for $1 \leqslant i \leqslant e$ (we take $z_1 \dots z_e R = R$ in case $e = 1$). Let $P = M(R)R'$. Then P is a nonzero principal ideal in R' and $I' = (z_1 R') \dots (z_e R')P^d$. We can take a basis (x_1, x_2, x_3) of $M(R)$ such that $x_2/x_1 \in R'$ and $x_3/x_1 \in R'$. Then $P = x_1 R'$, and by (3.7.1) there exist elements s' and t' in R' such that $M(R') = (x_1, s', t')R'$ and $R' \cap M(S) = (x_1, s')R'$.

Therefore (S, P) has a pseudonormal crossing at R'. Hence it suffices to show that (S, z_iR') has a pseudonormal crossing at R' for $1 \leqslant i \leqslant e$. So let any i with $1 \leqslant i \leqslant e$ be given. Since $\mathrm{ord}_R z_i = 1$, there exist elements w_1 and w_2 in R such that $M(R) = (z_i, w_1, w_2)R$. If $w_1/z_i \in R'$ and $w_2/z_i \in R'$ then $z_iR' = P$ and we have nothing more to show. So assume that either $w_1/z_i \notin R'$ or $w_2/z_i \notin R'$. Let $(y_1, y_3) = (w_1, w_2)$ in case $w_2/w_1 \in R'$, and $(y_1, y_3) = (w_2, w_1)$ in case $w_2/w_1 \notin R'$. Then $M(R) = (y_1, z_i, y_3)R$, $z_i/y_1 \in M(R')$, and $y_3/y_1 \in R'$. Let $\mathfrak{k}$ be a coefficient set for R. Then by (3.7.1) there exist elements r_1, r_2, r_3 in $\mathfrak{k}$ such that $R' \cap M(S) = (y_1, r_1 + r_2(z_i/y_1) + r_3(y_3/y_1))R'$ and $M(R') = (y_1, r_1 + r_2(z_i/y_1) + r_3(y_3/y_1), t)R'$ for some $t \in R'$. First suppose that $r_3 = 0$; then $r_1 = 0 \neq r_2$, and hence $R' \cap M(S) = (y_1, z_i/y_1)R'$ and $M(R') = (y_1, z_i/y_1, t)R'$; since $z_iR' = (y_1R')((z_i/y_1)R')$, we see that (S, z_iR') has a pseudonormal crossing at R'. Next suppose that $r_3 \neq 0$; then we have that $M(R') = (y_1, r_1 + r_2(z_i/y_1) + r_3(y_3/y_1), z_i/y_1)R'$; since $z_iR' = (y_1R')((z_i/y_1)R')$, we see that (S, z_iR') has a pseudonormal crossing at R'.

(3.8). *Let R_0 be a regular local domain with* $\dim R_0 \leqslant 3$, *and let J_0 be a nonzero nonunit principal ideal in R_0. Let $(R_i, J_i)_{0<i<\infty}$ be an infinite sequence such that for $0 < i < \infty$: R_i is a regular local domain, J_i is a nonzero principal ideal in R_i, (R_i, J_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, R_{i-1})$, and* $\mathrm{ord}_{R_i} J_i = \mathrm{ord}_{R_0} J_0$. *Then we have the following.*

(3.8.1). *Let E be any set of two-dimensional elements in $\mathfrak{V}(R_0)$ such that E has a strict normal crossing at R_0. Then $\mathfrak{C}^2(R_i, J_i) - (\mathfrak{V}(R_0) - E)$ has a strict normal crossing at R_i for all $i \geqslant 0$.*

(3.8.2). *Assume that $\mathfrak{C}^2(R_0, J_0)$ is a finite set and every element in $\mathfrak{C}^2(R_0, J_0)$ has a simple point at R_0. Then there exists a nonnegative integer j such that $\mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i for all $i \geqslant j$.*

(3.8.3). *Assume that R_0 is pseudogeometric and $\mathfrak{C}^2(R_0, J_0)$ is* *a finite set. Then there exists a nonnegative integer j such that $\mathfrak{E}^2(R_i, J_i)$ has a strict normal crossing at R_i for all $i \geqslant 0$.*

(3.8.4). *Assume that (R_0, J_0) is unresolved, R_0 is pseudogeometric, and $\mathfrak{S}(R_0, P)$ is closed in $\mathfrak{V}(R_0)$ for every nonzero principal prime ideal P in R_0* (see (1.2.6)). *Then there exists a nonnegative integer j such that $\mathfrak{E}^2(R_i, J_i)$ has a strict normal crossing at R_i for all $i \geqslant j$.*

PROOF OF (3.8.1). We shall make induction on i. The assertion is trivial for $i = 0$. So let $i > 0$ and assume that E^* has a strict normal crossing at R_{i-1} where $E^* = \mathfrak{E}^2(R_{i-1}, J_{i-1}) - (\mathfrak{V}(R_0) - E)$. Let $E' = \{S' \in \mathfrak{E}^2(R_i, J_i): M(R_{i-1}) \subset M(S')\}$. Upon taking (R_{i-1}, J_{i-1}, E^*) for (R, J, E) in (3.7.2) we get that $(E^* \cap \mathfrak{V}(R_i)) \cup E'$ has a strict normal crossing at R_i. By (1.9.5) we know that if S' is any element in $\mathfrak{V}(R_i)$ such that $S' \notin \mathfrak{V}(R_{i-1})$ then $M(R_{i-1}) \subset M(S')$; since $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_{i-1}} J_{i-1}$, we also get that $\mathfrak{E}^2(R_i, J_i) \cap \mathfrak{V}(R_{i-1}) \subset \mathfrak{E}^2(R_{i-1}, J_{i-1})$; consequently $\mathfrak{E}^2(R_i, J_i) \subset \mathfrak{E}^2(R_{i-1}, J_{i-1}) \cup E'$. Therefore $\mathfrak{E}^2(R_i, J_i) - (\mathfrak{V}(R_0) - E) \subset (E^* \cap \mathfrak{V}(R_i)) \cup E'$, and hence $\mathfrak{E}^2(R_i, J_i) - (\mathfrak{V}(R_0) - E)$ has a strict normal crossing at R_i.

PROOF OF (3.8.2). Since $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_0} J_0$, we get that $\mathfrak{E}^2(R_i, J_i) \cap \mathfrak{V}(R_0) \subset \mathfrak{E}^2(R_0, J_0) \cap \mathfrak{V}(R_i)$ for all $i \geqslant 0$. Let $E = \mathfrak{E}^2(R_0, J_0) \cap (\bigcap_{i=0}^{\infty} \mathfrak{V}(R_i))$. If $\dim R_0 = 3$ then by (3.3.1) and if $\dim R_0 \neq 3$ then obviously E contains at most one element and there exists a nonnegative integer j such that for each $i \geqslant j$ we have that $E = \mathfrak{E}^2(R_0, J_0) \cap \mathfrak{V}(R_i)$. Since $\mathfrak{E}^2(R_i, J_i) \cap \mathfrak{V}(R_0) \subset \mathfrak{E}^2(R_0, J_0) \cap \mathfrak{V}(R_i)$ for all $i \geqslant 0$, and $E = \mathfrak{E}^2(R_0, J_0) \cap \mathfrak{V}(R_i)$ for all $i \geqslant j$, we get that $\mathfrak{E}^2(R_i, J_i) \subset \mathfrak{E}^2(R_i, J_i) - (\mathfrak{V}(R_0) - E)$ for all $i \geqslant j$. Now E has a strict normal crossing at R_0, and hence by (3.8.1) we get that $\mathfrak{E}^2(R_i, J_i) - (\mathfrak{V}(R_0) - E)$ has a strict normal crossing at R_i for all $i \geqslant 0$. Therefore $\mathfrak{E}^2(R_i, J_i)$ has a strict normal crossing at R_i for all $i \geqslant j$.

PROOF OF (3.8.3). Since $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_0} J_0$, we get that $\mathfrak{E}^2(R_i, J_i) \cap \mathfrak{V}(R_0) \subset \mathfrak{E}^2(R_0, J_0) \cap \mathfrak{V}(R_i)$ for all $i \geqslant 0$. Upon taking $E = \varnothing$, by (3.8.1) we get that $\mathfrak{E}^2(R_i, J_i) - \mathfrak{V}(R_0)$ has a strict normal crossing at R_i for all $i \geqslant 0$. If $\dim R_0 = 3$ then by (3.3.2) and if $\dim R_0 \neq 3$ then obviously there exists a nonnegative integer b such that every element in $\mathfrak{C}^2(R_0, J_0) \cap \mathfrak{B}(R_b)$ has a simple point at R_b. It follows that $\mathfrak{C}^2(R_b, J_b)$ is a finite set and every element in $\mathfrak{C}^2(R_b, J_b)$ has a simple point at R_b. Therefore by (3.8.2) there exists an integer $j \geqslant b$ such that $\mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i for all $i \geqslant j$.

PROOF OF (3.8.4). Follows from (1.5.4) and (3.8.3).

(3.9). *Let R_0 be a regular local domain with* $\dim R_0 \leqslant 3$, *let J_0 be a nonzero nonunit principal ideal in R_0, and let I_0 be a nonzero principal ideal in R_0 such that I_0 has a quasinormal crossing at R_0. Let $(R_i, J_i, I_i)_{0<i<\infty}$ be an infinite sequence such that for $0 < i < \infty$: R_i is a regular local domain, J_i and I_i are nonzero principal ideals in R_i, (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, R_{i-1})$, and* $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_0} J_0$. *Then we have the following.*

(3.9.1). *Let E be any set of two-dimensional elements in $\mathfrak{B}(R_0)$ such that (E, I_0) has a pseudonormal crossing at R_0. Then $(\mathfrak{C}^2(R_i, J_i) - (\mathfrak{B}(R_0) - E), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant 0$.*

(3.9.2). *Assume that $\mathfrak{C}^2(R_0, J_0)$ is a finite set and every element in $\mathfrak{C}^2(R_0, J_0)$ has a simple point at R_0. Then there exists a nonnegative integer j such that $\mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i and $(\mathfrak{C}^2(R_i, J_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$.*

(3.9.3). *Assume that R_0 is pseudogeometric and $\mathfrak{C}^2(R_0, J_0)$ is a finite set. Then there exists a nonnegative integer j such that $\mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i and $(\mathfrak{C}^2(R_i, J_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$.*

(3.9.4). *Assume that (R_0, J_0) is unresolved, R_0 is pseudogeometric, and $\mathfrak{S}(R_0, P)$ is closed in $\mathfrak{B}(R_0)$ for every nonzero principal prime ideal P in R_0* (see (1.2.6)). *Then there exists a nonnegative integer j such that $\mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i and $(\mathfrak{C}^2(R_i, J_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$.*

PROOF OF (3.9.1). We shall make induction on i. The assertion is trivial for $i = 0$. So let $i > 0$ and assume that (E^*, I_{i-1})

has a pseudonormal crossing at R_{i-1} where $E^* = \mathfrak{E}^2(R_{i-1}, J_{i-1}) - (\mathfrak{B}(R_0) - E)$. By (1.10.8) and (1.10.12) we get that $(E^* \cap \mathfrak{B}(R_i), I_i)$ has a pseudonormal crossing at R_i. Let $E' = \{S' \in \mathfrak{E}^2(R_i, J_i): M(R_{i-1}) \subset M(S')\}$. Then by (3.7.4) we get that (E', I_i) has a pseudonormal crossing at R_i. By (1.9.5) we know that if S' is any element in $\mathfrak{B}(R_i)$ such that $S' \notin \mathfrak{B}(R_{i-1})$ then $M(R_{i-1}) \subset M(S')$; since $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_{i-1}} J_{i-1}$, we also get that $\mathfrak{E}^2(R_i, J_i) \cap \mathfrak{B}(R_{i-1}) \subset \mathfrak{E}^2(R_{i-1}, J_{i-1})$; consequently $\mathfrak{E}^2(R_i, J_i) \subset \mathfrak{E}^2(R_{i-1}, J_{i-1}) \cup E'$. Therefore $\mathfrak{E}^2(R_i, J_i) - (\mathfrak{B}(R_0) - E) \subset (E^* \cap \mathfrak{B}(R_i)) \cup E'$, and hence $(\mathfrak{E}^2(R_i, J_i) - (\mathfrak{B}(R_0) - E), I_i)$ has a pseudonormal crossing at R_i.

PROOF OF (3.9.2). In view of (3.8.2) it suffices to show that there exists a nonnegative integer j such that $(\mathfrak{E}^2(R_i, J_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$. Since $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_0} J_0$, we get that $\mathfrak{E}^2(R_i, J_i) \cap \mathfrak{B}(R_0) \subset \mathfrak{E}^2(R_0, J_0) \cap \mathfrak{B}(R_i)$ for all $i \geqslant 0$. Upon taking $E = \varnothing$, by (3.9.1) we get that $(\mathfrak{E}^2(R_i, J_i) - \mathfrak{B}(R_0), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant 0$. If $\dim R_0 = 3$ then by (3.4.1) and if $\dim R \neq 3$ then obviously there exists a nonnegative integer j such that $(\mathfrak{E}^2(R_0, J_0) \cap \mathfrak{B}(R_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$. It follows that $(\mathfrak{E}^2(R_i, J_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$.

PROOF OF (3.9.3). In view of (3.8.3) it suffices to show that there exists a nonnegative integer j such that $(\mathfrak{E}^2(R_i, J_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$. Since $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_0} J_0$, we get that $\mathfrak{E}^2(R_i, J_i) \cap \mathfrak{B}(R_0) \subset \mathfrak{E}^2(R_0, J_0) \cap \mathfrak{B}(R_i)$ for all $i \geqslant 0$. Upon taking $E = \varnothing$, by (3.9.1) we get that $(\mathfrak{E}^2(R_i, J_i) - \mathfrak{B}(R_0), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant 0$. If $\dim R_0 = 3$ then by (3.4.2) and if $\dim R_0 \neq 3$ then obviously there exists a nonnegative integer j such that $(\mathfrak{E}^2(R_0, J_0) \cap \mathfrak{B}(R_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$. It follows that $(\mathfrak{E}^2(R_i, J_i), I_i)$ has a pseudonormal crossing at R_i for all $i \geqslant j$.

PROOF OF (3.9.4). Follows from (1.5.4) and (3.9.3).

(3.10). *Let R be a three-dimensional regular local domain, let J be a nonzero nonunit principal ideal in R, let $d = \operatorname{ord}_R J$, let S be an element in $\mathfrak{E}^2(R, J)$ having a simple point at R, let*

(R', J') be a monoidal transform of (R, J, S), and let $E' = \{S' \in \mathfrak{C}^2(R', J'): R \cap M(S) \subset M(S')\}$. Then we have the following.

(3.10.1). *Let (x_1, x_2, x_3) be a basis of $M(R)$ such that $R \cap M(S) = (x_1, x_2)R$ and $x_2/x_1 \in R'$, let $h: R \to R/M(R)$ be the canonical epimorphism, let $\mathfrak{k}$ be a coefficient set for R, let X_1, X_2, X_3 be indeterminates, let $w \in R$ such that $wR = J$, and let r_{abc} be the unique elements in $\mathfrak{k}$ such that*

$$w - \sum_{a+b+c=d} r_{abc} x_1^a x_2^b x_3^c \in M(R)^{d+1}.$$

Assume that $\operatorname{ord}_{R'} J' = d$. Then there exist unique elements r and s in $\mathfrak{k}$ with $s \neq 0$ such that

$$\sum_{a+b+c=d} h(r_{abc}) X_1^a X_2^b X_3^c = h(s)(X_2 + h(r)X_1)^d.$$

Moreover, r is the unique element in $\mathfrak{k}$ such that $(x_2/x_1) + r \in M(R')$. Also $M(R') = (x_1, (x_2/x_1) + r, x_3)R'$, $\dim R' = 3$, and R' is residually rational over R.

(3.10.2). *If $J' \subset M(R')$ then $\dim R' = 3$. If $\operatorname{ord}_{R'} J' = d$ then R' is residually rational over R. If $\operatorname{ord}_{R'} J' = d$ and (R^*, J^*) is any monoidal transform of (R, J, S) such that $J^* \subset M(R^*)$ then $R^* = R'$.*

(3.10.3). *Assume that $\operatorname{ord}_{R'} J' = d$ and $E' \neq \varnothing$, and take $S' \in E'$. Then S' dominates S, and S' has a simple point at R'. Moreover, if (x_1, x_2, x_3) is any basis of $M(R)$ such that $R \cap M(S) = (x_1, x_2)R$ and $x_2/x_1 \in M(R')$ then $R' \cap M(S') = (x_1, (x_2/x_1) + r)R'$ for some $r \in x_3 R'$. Finally, if S'' is any two-dimensional element in $\mathfrak{V}(R')$ such that $R \cap M(S) \subset M(S'')$ and $J' \subset M(S'')$ then $S'' = S'$ (whence in particular $E' = \{S'\}$).*

(3.10.4). *Assume that $J' \subset M(R')$ and there exists $S_1 \in \mathfrak{C}^2(R, J)$ with $S_1 \neq S$ such that $\{S, S_1\}$ has a normal crossing at R. Then $\dim R' = 3$, $\operatorname{ord}_{R'} J' = d$, $S_1 \in \mathfrak{C}(R', J')$, and $E' \cup \{S_1\}$ has a strict normal crossing at R'. Let (R'', J'') be a monoidal transform of (R', J', S_1) such that $J'' \subset M(R'')$, let V be a valuation ring of the quotient field of R such that V dominates R'', and let (R^*, J^*) be the monoidal transform of (R, J, S_1) along V. Then $\dim R'' = 3 = \dim R^*$,*

$\operatorname{ord}_{R^*} J^* = d$, $S \in \mathfrak{E}(R^*, J^*)$, *S has a simple point at* R^*, *and* (R'', J'') *is the monoidal transform of* (R^*, J^*, S) *along* V.

(3.10.5). *Assume that* $\mathfrak{E}^2(R, J)$ *has a strict normal crossing at* R *and* $\operatorname{ord}_{R'} J' = d$. *Then* $\mathfrak{E}^2(R', J')$ *has a strict normal crossing at* R', $\mathfrak{E}^2(R', J') = (\mathfrak{E}^2(R, J) \cup E') - \{S\}$, $S \notin \mathfrak{B}(R')$, E' *contains at most one element, every element in* E' *dominates* S, *and every element in* $\mathfrak{E}^2(R', J')$ *dominates exactly one element in* $\mathfrak{E}^2(R, J)$.

(3.10.6). *Assume that* $\operatorname{ord}_{R'} J' = d$, *let* (R'', J'') *be a monoidal transform of* (R', J', R') *such that* $J'' \subset M(R'')$, *let* V *be a valuation ring of the quotient field of* R *such that* V *dominates* R'', *and let* (R^*, J^*) *be the monoidal transform of* (R, J, R) *along* V. *Then* $\operatorname{ord}_{R^*} J^* = d$, *and* $\mathfrak{B}(R^*)$ *contains exactly one two-dimensional element* S^* *such that* $M(R) \subset M(S^*)$ *and* $J^* \subset M(S^*)$. *Moreover,* S^* *has a simple point at* R^*, $S^* \in \mathfrak{E}(R^*, J^*)$, *and upon letting* (R^{**}, J^{**}) *be the monoidal transform of* (R^*, J^*, S^*) *along* V *we have that:* (1) *if* $S \notin \mathfrak{B}(R^*)$ *then* $2 \leqslant \dim R^* = \dim R^{**} \leqslant 3$ *and* $(R'', J'') = (R^{**}, J^{**})$; *and* (2) *if* $S \in \mathfrak{B}(R^*)$ *then* $\dim R^* = \dim R^{**} = \dim R'' = 3$, $S \in \mathfrak{E}(R^*, J^*)$, *S has a simple point at* R^*, $\operatorname{ord}_{R^{**}} J^{**} = d$, $S \in \mathfrak{E}(R^{**}, J^{**})$, *S has a simple point at* R^{**}, *and* (R'', J'') *is the monoidal transform of* (R^{**}, J^{**}, S) *along* V.

PROOF OF (3.10.1) AND (3.10.2). Since S has a simple point at R, there exists a basis (x_1, x_2, x_3) of $M(R)$ such that $R \cap M(S) = (x_1, x_2)R$ and $x_2/x_1 \in R'$. Now let (x_1, x_2, x_3) be any such basis of $M(R)$. Let h, $\mathfrak{k}$, X_1, X_2, X_3, w, and r_{abc} be as in the statement of (3.10.1).

Since $S \in \mathfrak{E}(R, J)$, there exist elements r'_{ab} in R such that

$$w = \sum_{a+b=d} r'_{ab} x_1^a x_2^b . \tag{1}$$

Let r''_{ab} be the unique element in $\mathfrak{k}$ such that

$$r'_{ab} - r''_{ab} \in M(R). \tag{2}$$

Then

$$w - \sum_{a+b=d} r''_{ab} x_1^a x_2^b \in M(R)^{d+1}$$

and hence for all nonnegative integer a, b, c with $a + b + c = d$ we have that

(3) $r_{abc} = r''_{ab}$ if $c = 0$, and $r_{abc} = 0$ if $c \neq 0$.

Since $\operatorname{ord}_R w = d$, we also get that

(4) $h(r''_{ab}) \neq 0$ for some nonnegative integers a, b with $a + b = d$.

Let $A = R[x_2/x_1]$, let $H\colon A \to h(R)[X_2]$ be the unique epimorphism such that $H(x_2/x_1) = X_2$ and $H(u) = h(u)$ for all $u \in R$. Let $w' = w/x_1^d$. Then $w' \in A$ and $w'R' = J'$. By (1) and (2) we get that

$$H(w') = \sum_{a+b=d} h(r''_{ab})X_2^b. \tag{5}$$

By (4) and (5) we get that $H(w') \neq 0$; now $H(A \cap M(R'))$ is a prime ideal in $h(R)[X_2]$; if $J' \subset M(R')$ then $w' \in A \cap M(R')$ and hence $H(w') \in H(A \cap M(R'))$. Therefore if $J' \subset M(R')$ then $H(A \cap M(R')$ is a maximal ideal in $h(R)[X_2]$ and hence $\dim R' = 3$ and $A \cap M(R')$ is a maximal ideal in A.

Henceforth assume that $\operatorname{ord}_{R'} J' = d$. Then in particular $J' \subset M(R')$ and hence $\dim R' = 3$, $H(A \cap M(R'))$ is a maximal ideal in $h(R)[X_2]$, and $A \cap M(R')$ is a maximal ideal in A. Since $\operatorname{ord}_{R'} w' = \operatorname{ord}_{R'} J' = d$ and $A \cap M(R')$ is a maximal ideal in A, we get that $w' \in (A \cap M(R'))^d$ and hence $H(w') \in (H(A \cap M(R')))^d$. By (4) and (5) we know that $H(w')$ is a nonzero polynomial of degree $\leqslant d$ in X_2 with coefficients in $h(R)$; since $H(w') \in (H(A \cap M(R')))^d$ and $H(A \cap M(R'))$ is a maximal ideal in $h(R)[X_2]$, we deduce that there exist elements r and s in $\mathfrak{k}$ with $s \neq 0$ such that

$$H(A \cap M(R')) = (X_2 + h(r))h(R)[X_2] \tag{6}$$

and

$$H(w') = h(s)(X_2 + h(r))^d. \tag{7}$$

By (3), (5), and (7) we get that r and s are the unique elements in $\mathfrak{k}$ with $s \neq 0$ such that

$$\sum_{a+b+c=d} h(r_{abc})X_1^a X_2^b X_3^c = h(s)(X_2 + h(r)X_1)^d.$$

Now $\operatorname{Ker} H = (x_1, x_3)A$ and hence by (6) we get that

$$A \cap M(R') = (x_1, (x_2/x_1) + r, x_3)A \tag{8}$$

and hence $M(R') = (x_1, (x_2/x_1) + r, x_3)R'$. In particular $(x_2/x_1) + r \in M(R')$ and hence r is the only such element in $\mathfrak{k}$. By (6) we also get that the quotient ring of $h(R)[X_2]$ with respect to $H(A \cap M(R'))$ is residually rational over $h(R)$, and hence R' is residually rational over R.

By (6) and (7) we get that $H(A \cap M(R'))$ is the only prime ideal in $h(R)[X_2]$ which contains $H(w')$. Now $\operatorname{Ker} H = M(R)A$, and hence we get that $A \cap M(R')$ is the only prime ideal in A which contains $M(R)$ and w'. It follows that: if (R^*, J^*) is any monoidal transform of (R, J, S) such that $J^* \subset M(R^*)$ and $x_2/x_1 \in R^*$ then $R^* = R'$ (note that the only assumptions used in proving this are that (R', J') is a monoidal transform of (R, J, S) such that $\operatorname{ord}_{R'} J' = d$, and (x_1, x_2, x_3) is a basis of $M(R)$ such that $R \cap M(S) = (x_1, x_2)R$ and $x_2/x_1 \in R'$).

Finally let (R^*, J^*) be any monoidal transform of (R, J, S) such that $J^* \subset M(R^*)$. We shall show that then $R^* = R'$ and this will complete the proof. In view of what we have said in the above paragraph, it suffices to show that $x_2/x_1 \in R^*$. Suppose if possible that $x_2/x_1 \notin R^*$. Then $x_1/x_2 \in M(R^*)$. Let $y_1 = x_2 + rx_1 + x_1$ and $y_2 = x_2 + rx_1$. Then $M(R) = (y_1, y_2, x_3)R$, $R \cap M(S) = (y_1, y_2)R$, $y_2/y_1 \in R'$, and $y_2/y_1 \in R^*$. Therefore by what we have said in the above paragraph we get that $R^* = R'$, and hence $x_2/x_1 \notin R'$ which is a contradiction.

Proof of (3.10.3). Since S has a simple point at R, there exists a basis (x_1, x_2', x_3) of $M(R)$ such that $R \cap M(S) = (x_1, x_2')R$ and $x_2'/x_1 \in R'$; by (3.10.1) there exists $r' \in R$ such that $(x_2'/x_1) + r' \in M(R')$ and then upon letting $x_2 = x_2' + r'x_1$ we have that $M(R) = (x_1, x_2, x_3)R$, $R \cap M(S) = (x_1, x_2)R$, and $x_2/x_1 \in M(R')$. Now let (x_1, x_2, x_3) be any basis of $M(R)$ such that $R \cap M(S) = (x_1, x_2)R$ and $x_2/x_1 \in M(R')$. Then by (3.10.1) we know that $\dim R' = 3$ and $M(R') = (x_1, x_2/x_1, x_3)R'$. Let $\mathfrak{k}$ be a coefficient set for R and let $h\colon R \to R/M(R)$ be the canonical epimorphism. Take $w^* \in R$ such that $w^*R = J$. Since $S \in \mathfrak{E}(R, J)$, there exist elements $r_0^*, ..., r_d^*$ in R such that

$$w^* = \sum_{a=0}^{d} r_a^* x_1^a x_2^{d-a} .$$

Let r_a^{**} be the unique element in $\mathfrak{k}$ such that $r_a^* - r_a^{**} \in M(R)$. Then

$$w^* - \sum_{a=0}^{d} r_a^{**} x_1^a x_2^{d-a} \in M(R)^{d+1}.$$

Since $x_2/x_1 \in M(R')$, by (3.10.1) we therefore get that

$$\sum_{a=0}^{d} h(r_a^{**}) X_1^a X_2^{d-a} = h(s) X_2^d \qquad \text{with} \quad 0 \neq s \in \mathfrak{k}$$

where X_1, X_2 are indeterminates. Upon letting $r_a = r_a^*/s$ and $w = w^*/s$ we then get that $w \in R$, $wR = J$, $r_a \in M(R)$ for $1 \leqslant a \leqslant d$, and

$$w = x_2^d + \sum_{a=1}^{d} r_a x_1^a x_2^{d-a}.$$

Since $r_a \in M(R)$, we get that

$$w = x_2^d + \sum_{a=1}^{d} (s_a x_1 + s_a' x_2 + s_a'' x_3) x_1^a x_2^{d-a}$$

with s_a, s_a', s_a'' in R. Hence upon letting $w' = w/x_1^d$ we get that $w' \in R'$, $w'R' = J'$, and $w' - (x_2/x_1)^d \in (x_1, x_3)R'$. Since $S' \in \mathfrak{E}(R', J')$, we get that $\text{ord}_{S'} w' = d$. Let h': $R' \to R'/x_1R'$ be the canonical epimorphism. Then $h'(R')$ is a two-dimensional regular local domain and $M(h'(R')) = (h'(x_2/x_1), h'(x_3))h'(R')$. Now $R \cap M(S) \subset M(S')$ and $(R \cap M(S))R' = x_1R'$, and hence $x_1R' \subset R' \cap M(S')$; since dim $S' = 2$, we get that $h'(R' \cap M(S'))$ is a nonzero nonmaximal ideal in $h'(R')$; therefore $h'(R' \cap M(S'))$ is a nonzero principal prime ideal in $h'(R')$ and hence there exists $t \in R' \cap M(S')$ such that $h'(R' \cap M(S')) = h'(t)h'(R')$; it follows that $R' \cap M(S') = (x_1, t)R'$. Let S^* be the quotient ring of $h'(R')$ with respect $h'(R' \cap M(S'))$. Then there exists a unique epimorphism $h^*\colon S' \to S^*$ such that $h^*(u) = h'(u)$ for all $u \in R'$. Now $\operatorname{ord}_{S^*}h'(w') = \operatorname{ord}_{S^*}h^*(w') \geqslant \operatorname{ord}_{S'}w' = d$; since $h'(R' \cap M(S'))$ is the nonzero principal prime ideal $h'(t)h'(R')$ in $h'(R')$, we get that $h'(w') \in (h'(t)h'(R'))^d$; consequently there exists $t^* \in R'$ such that $h'(w') = h'(t^*)h'(t)^d$, i.e., $w' - t^*t^d \in x_1R'$. Let $h''\colon R' \to R'/(x_1, x_3)R'$ be the canonical epimorphism. Then $h''(R')$ is a one-dimensional regular local domain and $M(h''(R')) = h''(x_2/x_1)h''(R')$; since $w' - t^*t^d \in x_1R'$ and $w' - (x_2/x_1)^d \in (x_1, x_3)R'$, we get that $h''(t^*)h''(t)^d = h''(x_2/x_1)^d$; since $t \in R' \cap M(S')$, we get that $h''(t) \in M(h''(R'))$; therefore $h''(t^*) \notin M(h''(R'))$ and there exists $t' \in R'$ such that $h''(t') \notin M(h''(R'))$ and $h''(t)/h''(t') = h''(x_2/x_1)$; it follows that $t' \notin M(R')$ and upon letting $s' = t/t'$ we get that $s' \in R'$, $R' \cap M(S') = (x_1, s')R'$, and $s' - (x_2/x_1) \in (x_1, x_3)R'$; consequently there exists $r \in x_3R'$ such that upon letting $s = (x_2/x_1) + r$ we have that $R' \cap M(S') = (x_1, s)R'$ and $s - s' \in x_1R'$; finally, upon letting $s^* = t^*t'^d$ we get that $s^* \in R'$, $s^* \notin M(R')$, and $w' - s^*s^d \in x_1R'$. Thus we have found $s^* \in R'$ with $s^* \notin M(R')$ and $r \in x_3R'$ such that upon letting $s = (x_2/x_1) + r$ we have that $R' \cap M(S') = (x_1, s)R'$ and $w' - s^*s^d \in x_1R'$; in particular, $h'(R' \cap M(S'))$ is a nonzero principal prime ideal in $h'(R')$ and $h'(w')h'(R') = (h'(R' \cap M(S')))^d$. Now $(x_1, s, x_3)R' = M(R')$ and hence S' has a simple point at R'; since $S' \neq R'$, we also get that $x_3 \notin M(S')$ and hence $M(R) \not\subset R \cap M(S')$; since $R \cap M(S) \subset R \cap M(S')$, $\dim S = 2$, and $R \cap M(S')$ is a prime ideal in R, we conclude that $R \cap M(S) = R \cap M(S')$ and hence S' dominates S. Finally, let S'' be any two-dimensional element in $\mathfrak{V}(R')$ such that $R \cap M(S) \subset M(S'')$ and $J' \subset M(S'')$; since $J' \subset M(S')$, we get that $w' \in R' \cap M(S'')$ and hence $h'(w') \in h'(R' \cap M(S''))$; since $\dim S'' = 2$, $R \cap M(S) \subset M(S'')$, and $(R \cap M(S))R' = x_1R'$, we get that $h'(R' \cap M(S''))$ is a nonzero principal prime ideal in $h'(R')$; since $h'(w')h'(R') = (h'(R' \cap M(S')))^d$, $h'(w') \in h'(R' \cap M(S''))$, and $h'(R' \cap M(S'))$ and $h'(R' \cap M(S''))$ are nonzero principal prime ideals in $h'(R')$, we conclude that $h'(R' \cap M(S'')) = h'(R' \cap M(S'))$; therefore $R' \cap M(S'') = R' \cap M(S')$ and hence $S'' = S'$.

Proof of (3.10.4). Since $\{S, S_1\}$ has a normal crossing at R, there exists a basis (x_1, x_2, x_3) of $M(R)$ such that $R \cap M(S) = (x_1, x_2)R$ and $R \cap M(S_1) = (x_2, x_3)R$. Take $w \in R$ such that

$wR = J$. By induction on q ($0 \leqslant q \leqslant d$) we shall show that there exist elements $r(q, a)$ in R for $0 \leqslant a < q$ such that $w - w_q \in x_2^q R$ where

$$w_q = \sum_{a=0}^{q-1} r(q, a) x_2^a (x_1 x_3)^{d-a}.$$

For $q = 0$ the sum is considered to be equal to zero and hence our assertion is trivial for $q = 0$. Now let $q > 0$ and suppose we have found $r(q-1, a)$ for $0 \leqslant a < q-1$. Let $h^*\colon R \to R/x_2R$ be the canonical epimorphism. Then $h^*(R)$ is a two-dimensional regular local domain and $M(h^*(R)) = (h^*(x_1), h^*(x_3))h^*(R)$. Now $w - w_{q-1} \in x_2^{q-1}R$ and $\operatorname{ord}_S w = d = \operatorname{ord}_{S_1} w$; also clearly $\operatorname{ord}_S w_{q-1} \geqslant d \leqslant \operatorname{ord}_{S_1} w_{q-1}$. Therefore

$$\operatorname{ord}_S((w - w_{q-1})/x_2^{q-1}) \geqslant d - q + 1 \leqslant \operatorname{ord}_{S_1}((w - w_{q-1})/x_2^{q-1})$$

and hence

$$(w - w_{q-1})/x_2^{q-1} \in ((x_1, x_2)R)^{d-q+1} \cap ((x_2, x_3)R)^{d-q+1}.$$

Consequently

$$h^*((x - w_{\ -1})/x_2^{q-1}) \in h^*(x_1)^{d-q+1}h^*(R) \cap h^*(x_3)^{d-q+1}h^*(R).$$

Now

$$h^*(x_1)^{d-q+1}h^*(R) \cap h^*(x_3)^{d-q+1}h^*(R) = h^*(x_1x_3)^{d-q+1}h^*(R)$$

and hence there exists $r(q, q-1) \in R$ such that

$$h^*((w - w_{q-1})/x_2^{q-1}) = h^*(r(q, q-1)(x_1x_3)^{d-q+1}),$$

i.e.,

$$((w - w_{q-1})/x_2^{q-1}) - (r(q, q-1)(x_1x_3)^{d-q+1}) \in x_2R.$$

Upon letting $r(q, a) = r(q-1, a)$ for $0 \leqslant a < q-1$, we get that $w - w_q \in x_2^q R$. This completes the induction on q. Let $r_a = r(d, a)$

for $0 \leqslant a < d$ and let $r_d = (w - w_d)/x_2^d$. Then $r_a \in R$ for $0 \leqslant a \leqslant d$, and

$$(1) \qquad w = \sum_{a=0}^{d} r_a x_2^a (x_1 x_3)^{d-a}.$$

Since $\operatorname{ord}_R w = d$, we get that

$$(2) \qquad r_d \notin M(R).$$

Now $x_3 \in M(R')$. If $x_1/x_2 \in R'$ then $(w/x_2^d)R' = J'$ and by (1) and (2) we would get that $w/x_2^d \notin M(R')$ which would contradict the assumption that $J' \subset M(R')$. Therefore $x_2/x_1 \in M(R')$ and hence by (1.10.10) we get that $\dim R' = 3$, $M(R') = (x_1, x_2/x_1, x_3)R'$, $S_1 \in \mathfrak{B}(R')$, and $R' \cap M(S_1) = (x_2/x_1, x_3)R'$; in particular S_1 has a simple point at R'. Also $w/x_1^d \in R'$ and $(w/x_1^d)R' = J'$. By (1) we get that

$$(3) \qquad w/x_1^d = \sum_{a=0}^{d} r_a (x_2/x_1)^a x_3^{d-a} \in ((x_2/x_1, x_3)R')^d$$

and hence $\operatorname{ord}_{R'} J' = d$ and $S_1 \in \mathfrak{C}(R', J')$. In virtue of (3.10.3) we also get that $E' \cup \{S_1\}$ has a strict normal crossing at R'. If $x_3/(x_2/x_1) \in M(R'')$ then $((w/x_1^d)/(x_2/x_1)^d)R'' = J''$ and by (2) and (3) we would get that $(w/x_1^d)/(x_2/x_1)^d \notin M(R'')$ which would contradict the assumption that $J'' \subset M(R'')$. Therefore $(x_2/x_1)/x_3 \in R''$ and hence $R'' = B'_{Q'}$ where $B' = R'[(x_2/x_1)/x_3]$ and $Q' = B' \cap M(V)$; also $(w/x_1^d)/x_3^d \in R''$ and $((w/x_1^d)/x_3^d)R'' = J''$, i.e., $(w/(x_1x_3)^d)R'' = J''$. Since $J'' \subset M(R'')$, by (3.10.2) we get that $\dim R'' = 3$. Now $R' = B_Q$ where $B = R[x_2/x_1]$ and $Q = B \cap M(V)$; since $R'' = B'_{Q'}$, we get that $R'' = B''_{Q''}$ where $B'' = R[x_2/(x_1x_3)]$ and $Q'' = B'' \cap M(V)$. Since $(x_2/x_1)/x_3 \in R'' \subset V$ we get that $x_2/x_3 \in M(V)$ and $(x_2/x_3)/x_1 \in V$. Since $x_2/x_3 \in M(V)$, we get that $x_2/x_3 \in M(R^*)$ and $R^* = A_P$ where $A = R[x_2/x_3]$ and $P = A \cap M(V)$. Since $x_2/x_3 \in M(R^*)$, by (1.10.10) we get that $\dim R^* = 3$, $M(R^*) = (x_1, x_2/x_3, x_3)R^*$, $S \in \mathfrak{B}(R^*)$, and $R^* \cap M(S) = (x_1, x_2/x_3)R^*$; in particular S has a simple point at R^*. Also $w/x_3^d \in R^*$ and $(w/x_3^d)R^* = J^*$. By (1) we get that

$$w/x_3^d = \sum_{a=0}^{d} r_a (x_2/x_3)^a x_1^{d-a} \in ((x_1, x_2/x_3)R^*)^d$$

and hence $\operatorname{ord}_{R^*} J^* = d$ and $S \in \mathfrak{C}(R^*, J^*)$. Let (R^{**}, J^{**}) be the monoidal transform of (R^*, J^*, S) along V. Since $(x_2/x_3)/x_1 \in V$, we get that $w/(x_1x_3)^d = (w/x_3^d)/x_1^d \in R^{**}$, $J^{**} = ((w/x_3^d)/x_1^d)R^{**} = (w/(x_1x_3)^d)R^{**}$, and $R^{**} = A^*_{P^*}$ where $A^* = R^*[(x_2/x_3)/x_1]$ and $P^* = A^* \cap M(V)$. Since $R^* = A_P$ and $R^{**} = A^*_{P^*}$, it follows that $R^{**} = B''_{Q''}$. Therefore $(R^{**}, J^{**}) = (R'', J'')$.

PROOF OF (3.10.5). By (1.9.5) we know that if S' is any element in $\mathfrak{B}(R')$ such that $S' \notin \mathfrak{B}(R)$ then $R \cap M(S) \subset M(S')$; since $\operatorname{ord}_{R'} J' = \operatorname{ord}_R J$, we also get that $\mathfrak{C}^2(R', J') \cap \mathfrak{B}(R) \subset \mathfrak{C}^2(R, J)$; consequently $\mathfrak{C}^2(R', J') \subset \mathfrak{C}^2(R, J) \cup E'$. By (1.9.7) we know that $S \notin \mathfrak{B}(R')$. Therefore our assertion follows from (3.10.3) and (3.10.4).

PROOF OF (3.10.6). Since S has a simple point at R, there exists a basis (x_1, x_2', x_3) of $M(R)$ such that $R \cap M(S) = (x_1, x_2')R$ and $x_2'/x_1 \in R'$. By (3.10.1) there exists $r \in R$ such that $(x_2'/x_1) + r \in M(R')$. Let $x_2 = x_2' + rx_1$. Then (x_1, x_2, x_3) is a basis of $M(R)$, $R \cap M(S) = (x_1, x_2)R$, and $x_2/x_1 \in M(R')$. Again by (3.10.1) we get that $\dim R' = 3$ and $M(R') = (x_1, x_2/x_1, x_3)R'$. Take $w \in R$ such that $wR = J$. Let $w' = w/x_1^d$. Then $w' \in R'$ and $w'R' = J'$, and hence $w' \in M(R')^d$. Let $\mathfrak{k}$ be a coefficient set for R, let $h\colon R \to R/M(R)$ be the canonical epimorphism, and let X_1, X_2, X_3 be indeterminates. Since $S \in \mathfrak{C}(R, J)$, there exist elements r_{ab} in R such that

$$z = \sum_{a+b=d} r_{ab} x_1^a x_2^b. \tag{3}$$

Let r'_{ab} be the unique element in $\mathfrak{k}$ such that $r_{ab} - r'_{ab} \in M(R)$. Then

$$w - \sum_{a+b=d} r'_{ab} x_1^a x_2^b \in M(R)^{d+1}$$

and hence by (3.10.1) we get that

$$\sum_{a+b=d} h(r'_{ab}) X_1^a X_2^b = h(s) X_2^d \qquad \text{with} \quad 0 \neq s \in \mathfrak{k}.$$

Therefore $r_{0d} \notin M(R)$ and $r_{ab} \in M(R)$ whenever $a \neq 0$. Consequently

$$w = r_{0d} x_2^d + \sum_{a=1}^{d} (r_a x_1 + r'_a x_2 + r''_a x_3) x_1^a x_2^{d-a}$$

with r_a, r'_a, r''_a in R, and hence

$$(4) \qquad w' - r_{0d}(x_2/x_1)^d \in (x_1, x_3)R'.$$

Since $w' \in M(R')^d$, there exist elements r_{abc} in R' such that

$$(5) \qquad w' = \sum_{a+b+c=d} r_{abc} x_1^a (x_2/x_1)^b x_3^c.$$

By (4) and (5) we get that $(r_{0d} - r_{0d0})(x_2/x_1)^d \in (x_1, x_3)R'$; now $r_{0d} \notin M(R')$, $\dim R' = 3$, and $M(R') = (x_1, x_2/x_1, x_3)R'$, and hence we must have

$$(6) \qquad r_{0d0} \notin M(R').$$

Suppose if possible that $x_1/(x_2/x_1) \in M(V)$ and $x_3/(x_2/x_1) \in M(V)$; then $x_1/(x_2/x_1) \in M(R'')$ and $x_3/(x_2/x_1) \in M(R'')$; consequently $w'/(x_2/x_1)^d \in R''$ and $(w'/(x_2/x_1)^d)R'' = J''$, and hence $w'/(x_2/x_1)^d \in M(R'')$; however, in view of (5) and (6) we get that $w'/(x_2/x_1)^d \notin M(R'')$ which is a contradiction. Therefore

$$(7) \qquad \text{either} \quad (x_2/x_1)/x_1 \in V \quad \text{or} \quad (x_2/x_1)/x_3 \in V.$$

We shall now divide the argument into two cases according as $x_3/x_1 \in V$ or $x_3/x_1 \notin V$.

Case when $x_3/x_1 \in V$. By (7) we get that $(x_2/x_1)/x_1 \in V$. Consequently $R'' = B'_{Q'}$ where $B' = R'[(x_2/x_1)/x_1, x_3/x_1]$ and $Q' = B' \cap M(V)$; also $w'/x_1^d \in R''$ and $(w'/x_1^d)R'' = J''$. Now $R' = B_Q$ where $B = R[x_2/x_1]$ and $Q = B \cap M(V)$. Therefore $R'' = B''_{Q''}$ where $B'' = R[x_2/x_1^2, x_3/x_1]$ and $Q'' = B'' \cap M(V)$. Now $x_2/x_1 \in M(V)$ and $x_3/x_1 \in V$; consequently $R^* = A_P$ where $A = R[x_2/x_1, x_3/x_1]$ and $P = A \cap M(V) = A \cap M(R^*)$; also $w'R^* = J^*$. Since $x_3/x_1 \in R^*$, it follows that $S \notin \mathfrak{V}(R^*)$. Let H: $A \to h(R)[X_2, X_3]$ be the unique epimorphism such that $H(x_2/x_1) = X_2$, $H(x_3/x_1) = X_3$, and $H(u) = h(u)$ for all $u \in R$. Then $X_2 \in H(A \cap M(R^*))$ and hence it follows that: $2 \leqslant \dim R^* \leqslant 3$; if $\dim R^* = 2$ then $M(R^*) = (x_1, x_2/x_1)R^*$; and if $\dim R^* = 3$ then there exists a unique monic polynomial $f(X_3)$ of positive degree in X_3 with coefficients in $\mathfrak{k}$ such that $M(R^*) = (x_1, x_2/x_1, f(x_3/x_1))R^*$. Now $(x_1, x_2/x_1)R^*$ is a prime ideal in R^* and upon letting S^* be the quotient ring of R^* with respect to $(x_1, x_2/x_1)R^*$ we have that S^* is a two-dimensional element in $\mathfrak{B}(R^*)$, S^* has a simple point at R^*, and $M(R) \subset M(S^*)$. Now $R' \subset R^*$, $w' \in M(R')^d$, and $M(R')R^* = (x_1, x_2/x_1, x_3)R^* = (x_1, x_2/x_1)R^*$; since $w'R^* = J^*$, we conclude that $\mathrm{ord}_{S^*}J^* \geqslant d$; therefore $\mathrm{ord}_{S^*}J^* = d = \mathrm{ord}_{R^*}J^*$ and $S^* \in \mathfrak{C}(R^*, J^*)$. By (3.7.1) it follows that S^* is the only two-dimensional element in $\mathfrak{B}(R^*)$ whose maximal ideal contains $M(R)$ and J^*. Since $(x_2/x_1)/x_1 \in V$, we get that $R^{**} = A^*_{P^*}$ where $A^* = R^*[(x_2/x_1)/x_1]$ and $P^* = A^* \cap M(V)$; also $w'/x_1^d \in R^{**}$ and $(w'/x_1^d)R^{**} = J^{**}$. Since $R^* = A_P$ and $R^{**} = A^*_{P^*}$, it follows that $R^{**} = B''_{Q''}$. Therefore $(R^{**}, J^{**}) = (R'', J'')$. Finally by (3.10.2) we get that $\dim R^{**} = \dim R^*$.

Case when $x_3/x_1 \notin V$. Now $x_1/x_3 \in M(V)$ and hence $x_2/x_3 \in M(V)$. Therefore $\dim R^* = 3$, $M(R^*) = (x_1/x_3, x_2/x_3, x_3)R^*$, and $R^* = A_P$ where $A = R[x_1/x_3, x_2/x_3]$ and $P = A \cap M(V)$; also $w/x_3^d \in R^*$ and $(w/x_3^d)R^* = J^*$. By (1.10.10) we get that $S \in \mathfrak{B}(R^*)$ and $R \cap M(S) = (x_1/x_3, x_2/x_3)R^*$, and hence S has a simple point at R^*; also $\mathrm{ord}_S J^* = \mathrm{ord}_S(z/x_3^d) = \mathrm{ord}_S z = d$, and hence $\mathrm{ord}_{R^*}J^* = d$ and $S \in \mathfrak{C}(R^*, J^*)$. In a moment we shall show that

$$w/x_3^d \in ((x_2/x_3, x_3)R)^d. \tag{8}$$

First, assuming (8) we shall complete the proof. Let S^* be the quotient ring of R^* with respect to $(x_2/x_3, x_3)R^*$. Then S^* is a two-dimensional element in $\mathfrak{B}(R^*)$, S^* has a simple point at R^*, $M(R) \subset M(S^*)$, and by (8) we get that $S^* \in \mathfrak{C}(R^*, J^*)$. By (3.7.1) it follows that S^* is the only two-dimensional element in $\mathfrak{B}(R^*)$ whose maximal ideal contains $M(R)$ and J^*. By (7) we get $(x_2/x_3)/x_3 \in M(V)$; therefore $\dim R^{**} = 3$, $M(R^{**}) = (x_1/x_3, x_2/x_3^2, x_3)R^{**}$, and $R^{**} = A^*_{P^*}$ where $A^* = R^*[x_2/x_3^2]$ and $P^* = A^* \cap M(V)$; also $w/x_3^{2d} = (w/x_3^d)/x_3^d \in R^*$ and $(w/x_3^{2d})R^{**} = J^{**}$. By (1.10.10) we get that $S \in \mathfrak{B}(R^{**})$ and $R^{**} \cap M(S) = (x_1/x_3, x_2/x_3^2)R^{**}$, and hence S has a simple point at R^{**}; also $\mathrm{ord}_S J^{**} = \mathrm{ord}_S(w/x_3^{2d}) = \mathrm{ord}_S w = d$ and hence $\mathrm{ord}_{R^{**}}J^{**} = d$ and $S \in \mathfrak{C}(R^{**}, J^{**})$. Let (R'^*, J'^*) be the monoidal transform of (R^{**}, J^{**}, S) along V. Now $(x_2/x_3^2)/(x_1/x_3) = x_2/(x_1x_3)$ and hence by (7) we get that $(x_2/x_3^2)/(x_1/x_3) \in V$. Therefore $R'^* = A^{**}_{P^{**}}$ where $A^{**} = R^{**}[x_2/(x_1x_3)]$ and $P^{**} = A^{**} \cap M(V)$; also $w/(x_1x_3)^d = (w/(x_3^{2d}))/(x_1/x_3)^d \in R'^*$ and $(w/(x_1x_3)^d)R'^* = J'^*$. Since

$R^* = A_P$, $R^{**} = A^*_{P^*}$, and $R'^* = A^{**}_{P^{**}}$, we get that $R'^* = A''_{P''}$ where $A'' = R[x_1/x_3, x_2/(x_1x_3)]$ and $P'' = A'' \cap M(V)$. Now $M(R') = (x_1, x_2/x_1, x_3)R'$ and $x_1/x_3 \in V$, and hence by (7) we get that $(x_2/x_1)/x_3 \in V$; therefore $R'' = B'_{Q'}$ where $B' = R'[(x_1/x_3, x_2/(x_1x_3)]$ and $Q' = B' \cap M(V)$; also $w/(x_1x_3)^d = (w/x_1^d)/x_3^d = w'/x_3^d \in R''$ and $(w/(x_1x_3)^d)R'' = J''$. Also $R' = B_Q$ where $B = R[x_2/x_1]$ and $Q = B \cap M(V)$. Since $R' = B_Q$ and $R'' = B'_{Q'}$, it follows that $R'' = A''_{P''}$. Therefore $(R'^*, J'^*) = (R'', J'')$, i.e., (R'', J'') is the monoidal transform of (R^{**}, J^{**}, S) along V; since $J'' \subset M(R'')$, by (3.10.2) we get that $\dim R'' = 3$.

We shall now prove (8). There exist unique elements s_{abc} in $\mathfrak{k}$ such that in the completion R_0 of R we have

$$w = \sum s_{abc} x_1^a x_2^b x_3^c$$

where the sum is over all nonnegative integers a, b, c. We shall show that

$$(9) \qquad s_{abc} = 0 \quad \text{whenever} \quad a + b < d.$$

Suppose if possible that (9) is not true and let e be the smallest integer such that $s_{abc} \neq 0$ for some (a, b, c) with $a + b = e$. Take (a', b') such that $a' + b' = e$ and $s_{a'b'c} \neq 0$ for some c. Let c' be the smallest integer such that $s_{a'b'c'} \neq 0$. For all nonnegative integers a and b let

$$s_{ab} = \sum_{c=0}^{\infty} s_{abc} x_3^c \in R_0 .$$

Then $s_{a'b'}/x_3^{c'}$ is a unit in R_0, and

$$w = \sum s_{ab} x_1^a x_2^b$$

where the sum is over all nonnegative integers a, b with $a + b \geqslant e$. Let

$$y = \sum_{a+b=e} s_{ab} x_1^a x_2^b .$$

Then $w - y \in ((x_1, x_2)R_0)^{e+1}$. Let S_0 be the quotient ring of R_0 with respect to $(x_1, x_2)R_0$. Then S_0 is a two-dimensional regular local domain and $M(S_0) = (x_1, x_2)S_0$. Since $s_{a'b'}/x_3^{c'}$ is a unit in R_0, we get that $s_{a'b'}$ is a unit in S_0 and hence $y \notin M(S_0)^{e+1}$. Therefore $y \notin ((x_1, x_2)R_0)^{e+1}$ and hence $w \notin ((x_1, x_2)R_0)^{e+1}$. Since $e < d$, we get that $w \notin ((x_1, x_2)R)^d$. This is a contradiction because $S \in \mathfrak{E}(R, J)$.

Thus (9) is proved. Let

$$(10) \qquad w_1 = \sum_{a+b+c<2d} s_{abc}x_1^a x_2^b x_3^c \qquad \text{and} \qquad w_2 = w - w_1 .$$

Then $w_2 \in R \cap M(R_0)^{2d} = M(R)^{2d}$. Since $S \in \mathfrak{E}(R, J)$, we have that $w \in ((x_1, x_2)R)^d$; by (9) we also have that $w_1 \in ((x_1, x_2)R)^d$; therefore $w_2 \in ((x_1, x_2)R)^d$.

For any integers $m \geqslant n \geqslant 0$ we claim that $D'_{mn} = D_{mn}$ where $D'_{mn} = M(R)^m \cap ((x_1, x_2)R)^n$ and D_{mn} is the ideal in R generated by all monomials $x_1^a x_2^b x_3^c$ for which $a + b \geqslant n$ and $a + b + c = m$. Clearly $D_{mn} \subset D'_{mn}$. So let $t \in D'_{mn}$ be given. We want to show that $t \in D_{mn}$. By induction on q we shall show that for any nonnegative integer q there exists $t'_q \in D_{mn}$ and $t_q \in R$ such that $t = t'_q + t_q x_3^q$. For $q = 0$ it suffices to take $t'_q = 0$ and $t_q = t$. Now let $q > 0$ and suppose we have found t'_{q-1} and t_{q-1}. Then

$$t_{q-1}x_3^{q-1} \in D'_{mn} = M(R)^m \cap ((x_1, x_2)R)^n$$

and hence

$$t_{q-1} \in M(R)^{q'} \qquad \text{where} \quad q' = \max(n, m - q + 1).$$

Let $h'\colon R \to R/x_3R$ be the canonical epimorphism. Then $h'(R)$ is a two-dimensional regular local domain, $h'(M(R)) = (h'(x_1), h'(x_2))h'(R') = M(h'(R'))$, and $h'(t_{q-1}) \in (h'(M(R)))^{q'}$. Therefore $t_{q-1} = t''_q + t_q x_3$ where $t''_q \in ((x_1, x_2)R)^{q'}$ and $t_q \in R$. It follows that $t''_q x_3^{q-1} \in D_{mn}$ and hence upon letting $t'_q = t'_{q-1} + t''_q x_3^{q-1}$ we get that $t = t'_q + t_q x_3^q$ with $t'_q \in D_{mn}$ and $t_q \in R$. This completes the induction on q. Therefore

$$t \in \bigcap_{q=0}^{\infty} (D_{mn} + x_3^q R) \subset \bigcap_{q=0}^{\infty} (D_{mn} + M(R)^q) = D_{mn} .$$

Thus the claim is proved. Therefore upon taking $m = 2d$ and $n = d$ we can find elements t_{abc} in R such that

$$(11) \qquad w_2 = \sum_{a+b+c=2d,\, a+b \geqslant d} t_{abc}x_1^a x_2^b x_3^c .$$

Let W be the set of all triples (a, b, c) of nonnegative integers a, b, c such that $a + b + c < 2d$, $a + b \geqslant d$, $a + 2b + c < 2d$, and $s_{abc} \neq 0$. Let W' be the set of all triples (a, b, c) of nonnegative integers a, b, c such that $a + b + c \leqslant 2d$, $a + b \geqslant d$, and $a + 2b + c \geqslant 2d$; for any (a, b, c) in W' let $s'_{abc} = s_{abc}$ if $a + b + c < 2d$ and $s'_{abc} = t_{abc}$ if $a + b + c = 2d$. Then W and W' are finite sets and by (9), (10), and (11) we get that

$$w = \sum_{(a,b,c)\in W} s_{abc}x_1^a x_2^b x_3^c + \sum_{(a,b,c)\in W'} s'_{abc}x_1^a x_2^b x_3^c .$$

Now $w/x_1^d = w' \in M(R')^d$ and $M(R') = (x_1, x_2/x_1, x_3)R'$. Clearly

$$x_1^{-d} \sum_{(a,b,c)\in W'} s'_{abc}x_1^a x_2^b x_3^c = \sum_{(a,b,c)\in W'} x_1^{a+b-d}(x_2/x_1)^b x_3^c \in M(R')^d$$

because $(a + b - d) + b + c \geqslant d$ for all $(a, b, c) \in W'$. Therefore

$$x_1^{-d} \sum_{(a,b,c)\in W} s_{abc}x_1^a x_2^b x_3^c \in M(R')^d ,$$

i.e.,

$$\sum_{(a,b,c)\in W} s_{abc}x_1^{a+b-d}(x_2/x_1)^b x_3^c \in M(R')^d. \tag{12}$$

For all $(a, b, c) \in W$ we have that $(a + b - d) + b + c < d$ and $0 \neq s_{abc} \in \mathfrak{k}$, and for all $(a, b, c) \neq (a', b', c')$ in W we have that $(a + b - d, b, c) \neq (a' + b' - d, b', c')$; also, by (3.10.2), $\mathfrak{k}$ is a coefficient set for R'; consequently by (12) we get $W = \varnothing$. Therefore

$$w = \sum_{(a,b,c)\in W'} s'_{abc}x_1^a x_2^b x_3^c$$

and hence

$$w/x_3^d = \sum_{(a,b,c)\in W'} s'_{abc}(x_1/x_3)^a(x_2/x_3)^b x_3^{a+b+c-d}. \tag{13}$$

Now $b + (a + b + c - d) \geqslant d$ for all $(a, b, c) \in W'$, and hence (8) follows from (13).

(3.11). *Let R be a regular local domain with* $\dim R \leqslant 3$, *let J be a nonzero principal ideal in R such that $\mathfrak{E}^2(R, J)$ has a strict* *normal crossing at R, let S be a positive-dimensional element in $\mathfrak{E}(R, J)$ having a simple point at R, and let (R', J') be a monoidal transform of (R, J, S) such that* $\operatorname{ord}_{R'} J' = \operatorname{ord}_R J$. *Then $\mathfrak{E}^2(R', J')$ has a strict normal crossing at R'.*

PROOF. If either $\dim R \neq 3$, or $\dim S = 1$, or $J = R$, then the assertion is trivial. If $\dim R = 3$, $\dim S > 1$, and $J \neq R$, then the assertion follows from (3.7.3) and (3.10.5).

(3.12). *Let $(R_i, J_i, I_i, L_i, S_i)_{0 \leqslant i < \infty}$ be an infinite subresolver such that* $\dim R_0 \leqslant 3$ *and R_0 is pseudogeometric. Then* $\dim R_i = 3$ *for $0 \leqslant i < \infty$, and there exists a nonnegative integer j such that for all $i \geqslant j$ we have that $\mathfrak{E}^2(R_i, J_i)$ has a strict normal crossing at R_i and $S_i \in \mathfrak{E}^2(R_i, J_i)$.*

PROOF. Since $\dim R_0 \leqslant 3$ and (R_i, J_i) is unresolved for $0 \leqslant i < \infty$, we get that $2 \leqslant \dim S_i \leqslant \dim R_i \leqslant 3$ for $0 \leqslant i < \infty$. Since R_0 is pseudogeometric and (R_i, J_i) is unresolved for $0 \leqslant i < \infty$, by (3.5) we get that $\dim R_i = 3$ for $0 \leqslant i < \infty$. Since $L_i \neq \varnothing$, we can fix $T_i \in L_i$ for $0 \leqslant i < \infty$. In view of (3.9.3) we get that $S_i \in L_i$ for infinitely many distinct values of i. In particular, there exists a nonnegative integer j such that $S_j \in L_j$. Now $\mathfrak{E}^2(R_j, I_j)$ has a strict normal crossing at R_j, and hence by (3.11) we get that $\mathfrak{E}^2(R_i, J_i)$ has a strict normal crossing at R_i for all $i \geqslant j$. Suppose if possible that $S_{q-1} \notin \mathfrak{E}^2(R_{q-1}, J_{q-1})$ for some $q > j$. Then $S_{q-1} = R_{q-1}$ and hence by (3.7.3) we get that $L_q = \{T_q\}$. In view of (3.7.3) and (3.10.5), by induction on i we now see that $L_{q+i} = \{T_{q+i}\}$ for $0 \leqslant i < \infty$. Thus R_q is a pseudogeometric regular local domain, J_q is a nonzero principal ideal in R_q, T_q is an element in $\mathfrak{E}^2(R_q, J_q)$ having a simple point at R_q, S_q is a positive-dimensional element in $\mathfrak{E}(R_q, J_q)$ having a simple point at R_q, $S_q \subset T_q$, and $(R_{q+i}, J_{q+i}, T_{q+i}, S_{q+i})_{0<i<\infty}$ is an infinite sequence such that for $0 < i < \infty$: R_{q+i} is a regular local domain, J_{q+i} is a nonzero principal ideal in R_{q+i}, T_{q+i} is an element in $\mathfrak{E}(R_{q+i}, J_{q+i})$ having a simple point at R_{q+i}, S_{q+i} is a positive-dimensional element in $\mathfrak{E}(R_{q+i}, J_{q+i})$ having a simple point at R_{q+i}, $S_{q+i} \subset T_{q+i}$, (R_{q+i}, J_{q+i}) is a monoidal transform of $(R_{q+i-1}, J_{q+i-1}, S_{q+i-1})$, and T_{q+i} dominates T_{q+i-1}. Also $S_{q+i} = T_{q+i}$ for infinitively many distinct values of i, and hence by (3.6) we get that (R_p, J_p) is resolved for some $p \geqslant q$. This is a contradiction. Therefore $S_i \in \mathfrak{C}^2(R_i, J_i)$ for all $i \geqslant j$.

(3.13). *Let d and n be positive integers, and let $[(R_i, J_i, S_i)_{0 \leqslant i < n}, (R_n, J_n)]$ be a system such that: R_i is a regular local domain and J_i is a nonzero principal ideal in R_i for $0 \leqslant i \leqslant n$;* $\operatorname{ord}_{R_i} J_i = d$ *for $0 \leqslant i < n$; $S_i \in \mathfrak{C}^2(R_i, J_i)$ and S_i has a simple point at R_i for $0 \leqslant i < n$; (R_i, J_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, S_{i-1})$ for $0 < i \leqslant n$; and* $\dim R_0 = 3$. *Then we have the following.*

(3.13.1). $\dim R_i = 3$ *for $0 < i < n$. If $J_n \subset M(R_n)$ then* $\dim R_n = 3$.

(3.13.2). *If $\mathfrak{C}^2(R_0, J_0) = \{S_0\}$ then $\mathfrak{C}^2(R_i, J_i) = \{S_i\}$ and S_i dominates S_{i-1} for $0 < i < n$.*

PROOF. By induction on i, the assertions follow from (3.10.2) and (3.10.5).

(3.14). *Let d and n be positive integers, and let $[(R_i, J_i, S_i)_{0 \leqslant i < n}, (R_n, J_n)]$ be a system such that: R_i is a regular local domain and J_i is a nonzero principal ideal in R_i for $0 \leqslant i \leqslant n$;* $\operatorname{ord}_{R_i} J_i = d$ *for $0 \leqslant i < n$; $J_n \subset M(R_n)$; $S_i \in \mathfrak{C}^2(R_i, J_i)$ and S_i has a simple point at R_i for $0 \leqslant i < n$; (R_i, J_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, S_{i-1})$ for $0 < i \leqslant n$;* $\dim R_0 = 3$; *$\mathfrak{C}^2(R_0, J_0)$ has a normal crossing at R_0; and $\mathfrak{C}^2(R_0, J_0)$ contains exactly two distinct elements S and S^*. Let W be the set of all integers i with $0 \leqslant i < n$ such that S_i dominates S, and let m be the number of elements in W. Let W^* be the set of all integers i with $0 \leqslant i < n$ such that S_i dominates S^*, and let m^* be the number of elements in W^*. Then we have the following.*

(1) *For $0 < i < n$ we have that* $\dim R_i = 3$, *$\mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i, and each element in $\mathfrak{C}^2(R_i, J_i)$ dominates exactly one element in $\mathfrak{C}^2(R_{i-1}, J_{i-1})$.* (2) *For $0 \leqslant j \leqslant i < n$ we have that each element in $\mathfrak{C}^2(R_i, J_i)$ dominates exactly one element in $\mathfrak{C}^2(R_j, J_j)$, and S_i dominates exactly one element in $\mathfrak{C}^2(R_j, J_j)$ (whence in particular $m + m^* = n$).* (3) *If b is any integer with $0 \leqslant b \leqslant n$ such that $j \notin W$ whenever $0 \leqslant j < b$, then $S \in \mathfrak{C}^2(R_i, J_i)$*

whenever $0 \leqslant i \leqslant b$. (3*) *If* b *is any integer with* $0 \leqslant b \leqslant n$ *such that* $j \notin W^*$ *whenever* $0 \leqslant j < b$, *then* $S^* \in \mathfrak{C}^2(R_i, J_i)$ *whenever* $0 \leqslant i \leqslant b$. (4) *If* $m = 0$ *then* $\dim R_n = 3$, $\operatorname{ord}_{R_n} J_n = d$, $S \in \mathfrak{C}^2(R_n, J_n)$, *and* $\mathfrak{C}^2(R_n, J_n)$ *has a strict normal crossing at* R_n. (4*) *If* $m^* = 0$ *then* $\dim R_n = 3$, $\operatorname{ord}_{R_n} J_n = d$, $S^* \in \mathfrak{C}^2(R_n, J_n)$, *and* $\mathfrak{C}^2(R_n, J_n)$ *has a strict normal crossing at* R_n. (5) *If* $\mathfrak{C}^2(R_n, J_n) = \varnothing$ *then* $m \neq 0 \neq m^*$.

Now assume that $m \neq 0$ *and let* $a(0), a(1), \ldots, a(n-1)$ *be the unique permutation of* $0, 1, \ldots, n-1$ *such that:* $a(i) \in W$ *if and only if* $0 \leqslant i < m$; $a(j) < a(i)$ *whenever* $0 \leqslant j < i < m$; $a(i) \in W^*$ *if and only if* $m \leqslant i < n$; *and* $a(j) < a(i)$ *whenever* $m \leqslant j < i < n$. *Then we have the following.*

(6) $S_{a(i)}$ *dominates* $S_{a(i-1)}$ *whenever either* $0 < i < m$ *or* $m < i < n$. (7) $S_{a(0)} = S$. (8) *If* $m^* \neq 0$ *then* $S_{a(m)} = S^*$. (9) *There exists a sequence* $(R_i', J_i')_{0 \leqslant i < n}$ *such that:* R_i' *is a three-dimensional regular local domain and* J_i' *is a nonzero principal ideal in* R_i' *with* $\operatorname{ord}_{R_i'} J_i' = d$ *for* $0 \leqslant i < n$; $S_{a(i)} \in \mathfrak{C}^2(R_i', J_i')$ *and* $\mathfrak{C}^2(R_i', J_i')$ *has a strict normal crossing at* R_i' *for* $0 \leqslant i < n$; $(R_0', J_0') = (R_0, J_0)$; (R_i', J_i') *is a monoidal transform of* $(R_{i-1}', J_{i-1}', S_{a(i-1)})$ *for* $0 < i < n$; *and* (R_n, J_n) *is a monoidal transform of* $(R_{n-1}', J_{n-1}', S_{a(n-1)})$.

PROOF. In view of (3.10.2) and (3.10.5), (1) follows by induction on i. (2) follows from (1). In view of (1) and (3.10.4), (3) and (3*) follow by induction on i. (4) and (4*) follow from (1), (3), (3*), (3.10.4), and (3.10.5). (5) follows from (4) and (4*). Now assume that $m \neq 0$ and let $a(0), a(1), \ldots, a(n-1)$ be the said permutation of $0, 1, \ldots, n-1$. (6), (7), and (8) follow from (1), (2), (3), and (3*). In proving (9) we shall tacitly use (1) and (2). We shall now prove (9) by induction on n. In case of $n = 1$ we must have $S_{a(0)} = S_0$ and it suffices to take $(R_0', J_0') = (R_0, J_0)$. Now let $n > 1$ and assume that the assertion is true for all values of n smaller than the given one. If $m^* = 0$ then $a(i) = i$ for $0 \leqslant i < n$, and it suffices to take $(R_i', J_i') = (R_i, J_i)$ for $0 \leqslant i < n$. So now assume that $m^* \neq 0$. Let $c(0), c(1), \ldots, c(n-2)$ be the permutation of $0, 1, \ldots, n-2$ defined thus: if $n-1 \notin W$ then $c(i) = a(i)$ for $0 \leqslant i \leqslant n-2$; and if $n-1 \in W$ then $c(i) = a(i)$ for $0 \leqslant i \leqslant m-2$, and $c(i) = a(i+1)$ for $m-1 \leqslant i \leqslant n-2$.

Upon applying the induction hypothesis to the system $[(R_i, J_i, S_i)_{0 \leqslant i < n-1}, (R_{n-1}, J_{n-1})]$ we find a sequence $(R_i^*, J_i^*)_{0 \leqslant i < n-1}$ such that: R_i^* is a three-dimensional regular local domain and J_i^* is a nonzero principal ideal in R_i^* with $\operatorname{ord}_{R_i^*} J^* = d$ for $0 \leqslant i < n-1$; $S_{c(i)} \in \mathfrak{C}^2(R_i^*, J_i^*)$ and $\mathfrak{C}^2(R_i^*, J_i^*)$ has a strict normal crossing at R_i^* for $0 \leqslant i < n-1$; $(R_0^*, J_0^*) = (R_0, J_0)$; (R_i^*, J_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, S_{c(i-1)})$ for $0 < i < n-1$; and (R_{n-1}, J_{n-1}) is a monoidal transform of $(R_{n-2}^*, J_{n-2}^*, S_{c(n-2)})$ (note that if $i \in W^*$ for $0 \leqslant i < n-1$ then, without using the induction hypothesis, we are simply taking $(R_i^*, J_i^*) = (R_i, J_i)$ for $0 \leqslant i < n-1$). If $n-1 \notin W$ then it suffices to take $(R_i', J_i') = (R_i^*, J_i^*)$ for $0 \leqslant i < n-1$, and $(R_{n-1}', J_{n-1}') = (R_{n-1}, J_{n-1})$. So now assume that $n-1 \in W$. Let $q(i) = c(i)$ for $0 \leqslant i < n-2$, and $q(n-2) = n-1$. Now (R_{n-1}, J_{n-1}) is a monoidal transform of $(R_{n-2}^*, J_{n-2}^*, S_{a(n-1)})$, $S_{q(n-2)} \in \mathfrak{C}^2(R_{n-1}, J_{n-1})$, and $S_{q(n-2)}$ does not dominate $S_{a(n-1)}$. Consequently $S_{a(n-1)} \neq S_{q(n-2)}$, and by (3.10.5) we deduce that $\mathfrak{C}^2(R_{n-2}^*, J_{n-2}^*) = \{S_{a(n-1)}, S_{q(n-2)}\}$. Let V be a valuation ring of the quotient field of R_n such that V dominates R_n, and let (R_{n-1}', J_{n-1}') be the monoidal transform of $(R_{n-2}^*, J_{n-2}^*, S_{q(n-2)})$ along V. Then by (3.10.4) and (3.10.5) we get that R_{n-1}' is a three-dimensional regular local domain, J_{n-1}' is a nonzero principal ideal in R_{n-1}' with $\operatorname{ord}_{R_{n-1}'} J_{n-1}' = d$, $S_{a(n-1)} \in \mathfrak{C}^2(R_{n-1}', J_{n-1}')$, $\mathfrak{C}^2(R_{n-1}', J_{n-1}')$ has a strict normal crossing at R_{n-1}', and (R_n, J_n) is a monoidal transform of $(R_{n-1}', J_{n-1}', S_{a(n-1)})$. Upon applying the induction hypothesis to the system $[(R_i^*, J_i^*, S_{q(i)})_{0 \leqslant i < n-1}, (R_{n-1}', J_{n-1}')]$ we find a sequence $(R_i', J_i')_{0 \leqslant i < n-1}$ such that: R_i' is a three-dimensional regular local domain and J_i' is a nonzero principal ideal in R_i' with $\operatorname{ord}_{R_i'} J_i' = d$ for $0 \leqslant i < n-1$; $S_{a(i)} \in \mathfrak{C}^2(R_i', J_i')$ and $\mathfrak{C}^2(R_i', J_i')$ has a strict normal crossing at R_i' for $0 \leqslant i < n-1$; $(R_0', J_0') = (R_0^*, J_0^*)$; and (R_i', J_i') is a monoidal transform of $(R_{i-1}', J_{i-1}', S_{a(i-1)})$ for $0 < i \leqslant n-1$. It follows that the sequence $(R_i', J_i')_{0 \leqslant i < n}$ has the required properties. This completes the induction.

(3.15). *Let $(R_i, J_i, S_i)_{0 \leqslant i < \infty}$ be an infinite sequence such that: R_i is a regular local domain and J_i is a nonzero principal ideal in R_i for $0 \leqslant i < \infty$; $S_i \in \mathfrak{C}^2(R_i, J_i)$ and S_i has a simple point at R_i for $0 \leqslant i < \infty$; and (R_i, J_i) is a monoidal transform of*

$(R_{i-1}, J_{i-1}, S_{i-1})$ *for* $0 < i < \infty$. *Assume that* R_0 *is pseudo-geometric,* dim $R_0 = 3$, (R_0, J_0) *is unresolved, and* $\mathfrak{E}^2(R_0, J_0)$ *has a strict normal crossing at* R_0. *Let* $d = \operatorname{ord}_{R_0} J_0$. *Then* $\operatorname{ord}_{R_i} J_i \neq d$ *for some nonnegative integer* i.

PROOF. Since (R_0, J_0) is unresolved, we have that $d > 1$. Suppose if possible that $\operatorname{ord}_{R_i} J_i = d$ for $0 \leqslant i < \infty$. Then by (1.10.5) we have that (R_i, J_i) is unresolved for $0 \leqslant i < \infty$. If $\mathfrak{E}^2(R_0, J_0) = \{S_0\}$ then by (3.13.2) we would get that S_i dominates S_{i-1} for $0 < i < \infty$, and this would contradict (3.6). Therefore $\mathfrak{E}^2(R_0, J_0) \neq \{S_0\}$ and hence $\mathfrak{E}^2(R_0, J_0)$ contains exactly two distinct elements. By (3.14) there exists $S \in \mathfrak{E}^2(R_0, J_0)$ such that the set W of all nonnegative integers i for which S_i dominates S is an infinite set. Let a be the unique order-preserving one-to-one map of the set of all nonnegative integers onto W. Then by (3.14) we have that $S_{a(i)}$ dominates $S_{a(j)}$ whenever $0 \leqslant j \leqslant i$. Given any nonnegative integer q, upon taking $n = a(q) + 1$, by (3.14) we find a sequence $(R_{q,i}, J_{q,i})_{0 \leqslant i \leqslant q}$ such that: $R_{q,i}$ is a regular local domain and $J_{q,i}$ is a nonzero principal ideal in $R_{q,i}$ with $\operatorname{ord}_{R_{q,i}} J_{q,i} = d$ for $0 \leqslant i \leqslant q$; $S_{a(i)} \in \mathfrak{E}^2(R_{q,i}, J_{q,i})$ and $S_{a(i)}$ has a simple point at $R_{q,i}$ for $0 \leqslant i \leqslant q$; $(R_{q,0}, J_{q,0}) = (R_0, J_0)$; $(R_{q,i}, J_{q,i})$ is a monoidal transform of $(R_{q,i-1}, J_{q,i-1}, S_{a(i-1)})$ for $0 < i \leqslant q$; and $R_{a(q)+1}$ dominates $R_{q,q}$. We can take a valuation ring V of the quotient field of R_0 such that V dominates R_i for $0 \leqslant i < \infty$. Then for each $q \geqslant 0$ we have that V dominates $R_{q,q}$ and hence $(R_{q,i}, J_{q,i})$ is the monoidal transform of $(R_{q,i-1}, J_{q,i-1}, S_{a(i-1)})$ along V for $0 < i \leqslant q$. Therefore we must have $(R_{q,i}, J_{q,i}) = (R_{p,i}, J_{p,i})$ whenever $0 \leqslant i \leqslant q \leqslant p$. Consequently, upon letting $(R'_i, J'_i) = (R_{i,i}, J_{i,i})$ for $0 \leqslant i < \infty$, we get an infinite sequence $(R'_i, J'_i)_{0 \leqslant i < \infty}$ such that: R'_i is a regular local domain and J'_i is a nonzero principal ideal in R'_i with $\operatorname{ord}_{R'_i} J'_i = d$ for $0 \leqslant i < \infty$; $S_{a(i)} \in \mathfrak{E}^2(R'_i, J'_i)$ and $S_{a(i)}$ has a simple point at R'_i for $0 \leqslant i < \infty$; $(R'_0, J'_0) = (R_0, J_0)$; and (R'_i, J'_i) is a monoidal transform of $(R'_{i-1}, J'_{i-1}, S_{a(i-1)})$ for $0 < i < \infty$. By (1.10.5) we get that (R'_i, J'_i) is unresolved for $0 \leqslant i < \infty$. Now $S_{a(i)}$ dominates $S_{a(i-1)}$ for $0 < i < \infty$, and hence, in view of (3.6), we are led to a contradiction.

(3.16). *Let* d *and* n *be positive integers, and let* $[(R_i, J_i, S_i)_{0 \leqslant i < n}, (R_n, J_n)]$ *be a system such that:* R_i *is a regular local domain and* J_i

is a nonzero principal ideal in R_i *for* $0 \leqslant i \leqslant n$; $\operatorname{ord}_{R_i} J_i = d$ *for* $0 \leqslant i < n$; $S_i \in \mathfrak{C}^2(R_i, J_i)$ *and* S_i *has a simple point at* R_i *for* $0 \leqslant i < n$; (R_i, J_i) *is a monoidal transform of* $(R_{i-1}, J_{i-1}, S_{i-1})$ *for* $0 < i \leqslant n$; *and* $\dim R_0 = 3$. *Then we have the following.*

(3.16.1). *Assume that* $\mathfrak{C}^2(R_0, J_0)$ *has a strict normal crossing at* R_0, $J_n \subset M(R_n)$, *and* $\mathfrak{C}^2(R_n, J_n) = \varnothing$. *Let* S *be any element in* $\mathfrak{C}^2(R_0, J_0)$. *Then there exists a sequence* $(R_i', J_i', S_i')_{0 \leqslant i < n}$ *such that:* R_i' *is a three-dimensional regular local domain and* J_i' *is a nonzero principal ideal in* R_i' *with* $\operatorname{ord}_{R_i'} J_i' = d$ *for* $0 \leqslant i < n$; $\mathfrak{C}^2(R_i', J_i')$ *has a strict normal crossing at* R_i' *for* $0 \leqslant i < n$; $S_i' \in \mathfrak{C}^2(R_i', J_i')$ *for* $0 \leqslant i < n$; $(R_0', J_0', S_0') = (R_0, J_0, S)$; (R_i', J_i') *is a monoidal transform of* $(R_{i-1}', J_{i-1}', S_{i-1}')$ *for* $0 < i < n$; *and* (R_n, J_n) *is a monoidal transform of* $(R_{n-1}', J_{n-1}', S_{n-1}')$.

(3.16.2). *Assume that* $\operatorname{ord}_{R_n} J_n = d$. *Let* (R_{n+1}, J_{n+1}) *be a monoidal transform of* (R_n, J_n, R_n) *such that* $J_{n+1} \subset M(R_{n+1})$, *let* V *be a valuation ring of the quotient field of* R_0 *such that* V *dominates* R_{n+1}, *and let* (R_0', J_0') *be the monoidal transform of* (R_0, J_0, R_0) *along* V. *Then* $2 \leqslant \dim R_0' = \dim R_{n+1} \leqslant 3$, $\operatorname{ord}_{R_0'} J_0' = d$, *and there exists exactly one two-dimensional element* S_0' *in* $\mathfrak{B}(R_0')$ *such that* $M(R_0) \subset M(S_0')$ *and* $J_0' \subset M(S_0')$. *Moreover,* $S_0' \in \mathfrak{C}^2(R_0', J_0')$, S_0' *has a simple point at* R_0', *and there exists a positive integer* m *and a sequence* $(R_i', J_i', S_i')_{0 < i < m}$ *such that:* R_i' *is a regular local domain with* $\dim R_i' = \dim R_0'$ *and* J_i' *is a nonzero principal ideal in* R_i' *with* $\operatorname{ord}_{R_i'} J_i' = d$ *for* $0 < i < m$; $S_i' \in \mathfrak{C}^2(R_i', J_i')$ *and* S_i' *has a simple point at* R_i' *for* $0 < i < m$; (R_i', J_i') *is a monoidal transform of* $(R_{i-1}', J_{i-1}', S_{i-1}')$ *for* $0 < i < m$; *and* (R_{n+1}, J_{n+1}) *is a monoidal transform of* $(R_{m-1}', J_{m-1}', S_{m-1}')$.

(3.16.3). *Assume that* $\mathfrak{C}^2(R_0, J_0)$ *has a strict normal crossing at* R_0, $\operatorname{ord}_{R_n} J_n = d$, *and* $\mathfrak{C}^2(R_n, J_n) = \varnothing$. *Let* (R_{n+1}, J_{n+1}) *be a monoidal transform of* (R_n, J_n, R_n) *such that* $J_{n+1} \subset M(R_{n+1})$. *Let* S *be an element in* $\mathfrak{C}(R_0, J_0)$ *with* $\dim S \geqslant 2$. *Let* (R', J') *be a monoidal transform of* (R_0, J_0, S). *Assume that there exists a valuation ring* V *of the quotient field of* R_0 *such that* V *dominates* R' *and* V *dominates* R_{n+1}. *Then* $\operatorname{ord}_{R'} J' = d$, *and there exists a positive integer* e *and a semiresolver* $(R_i', J_i', S_i')_{0 \leqslant i < e}$ *such that* $\operatorname{ord}_{R_i'} J_i' = d$

for $0 \leqslant i < e$, $(R'_0, J'_0) = (R', J')$, *and* (R_{n+1}, J_{n+1}) *is a monoidal transform of* $(R'_{e-1}, J'_{e-1}, S'_{e-1})$.

PROOF OF (3.16.1). We shall make induction on n. First consider the case of $n = 1$; if $S \neq S_0$ then by (3.10.4) we would get that $S \in \mathfrak{E}^2(R_n, J_n)$ which would be a contradiction; therefore $S = S_0$ and hence it suffices to take $(R'_0, J'_0, S'_0) = (R_0, J_0, S_0)$. Now let $n > 1$ and assume that the assertion is true for all values of n smaller than the given one. By (3.10.2) and (3.10.5) we get that $\dim R_i = 3$ and $\mathfrak{E}^2(R_i, J_i)$ has a strict normal crossing at R_i for $0 \leqslant i < n$. Hence if $S = S_0$ then it suffices to take $(R'_i, J'_i, S'_i) = (R_i, J_i, S_i)$ for $0 \leqslant i < n$. So now assume that $S \neq S_0$. Then by (3.10.4) we get that $S \in \mathfrak{E}^2(R_1, J_1)$. Therefore upon applying the induction hypothesis to the system $[(R_i, J_i, S_i)_{1 \leqslant i < n}, (R_n, J_n)]$ we can find a sequence $(R_i^*, J_i^*, S_i^*)_{1 \leqslant i < n}$ such that: R_i^* is a three-dimensional regular local domain and J_i^* is a nonzero principal ideal in R_i^* with $\operatorname{ord}_{R_i^*} J_i^* = d$ for $1 \leqslant i < n$; $\mathfrak{E}^2(R_i^*, J_i^*)$ has a strict normal crossing at R_i^* for $1 \leqslant i < n$; $S_i^* \in \mathfrak{E}^2(R_i^*, J_i^*)$ for $1 \leqslant i < n$; $(R_i^*, J_i^*, S_i^*) = (R_1, J_1, S)$; (R_i^*, J_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, S_{i-1}^*)$ for $1 < i < n$; and (R_n, J_n) is a monoidal transform of $(R_{n-1}^*, J_{n-1}^*, S_{n-1}^*)$. Let $(R_n^*, J_n^*) = (R_n, J_n)$. Let $(R'_0, J'_0, S'_0) = (R_0, J_0, S)$. Take a valuation ring V of the quotient field of R_0 such that V dominates R_2^*. Let (R'_1, J'_1) be the monoidal transform of (R'_0, J'_0, S'_0) along V, and let $S'_1 = S_0$. Then by (3.10.4) and (3.10.5) we get that $\dim R'_1 = 3$, $\operatorname{ord}_{R'_1} J'_1 = d$, $\mathfrak{E}^2(R'_1, J'_1)$ has a strict normal crossing at R'_1, $S'_1 \in \mathfrak{E}^2(R'_1, J'_1)$, and (R_2^*, J_2^*) is a monoidal transform of (R'_1, J'_1, S'_1). It suffices to take $(R'_i, J'_i, S'_i) = (R_i^*, J_i^*, S_i^*)$ for $2 \leqslant i < n$.

PROOF OF (3.16.2). We shall make induction on n.

First consider the case of $n = 1$. By (3.10.6) we get that $2 \leqslant \dim R'_0 = \dim R_2 \leqslant 3$, $\operatorname{ord}_{R'_0} J'_0 = d$, and there exists exactly one two-dimensional element S'_0 in $\mathfrak{V}(R'_0)$ such that $M(R_0) \subset M(S'_0)$ and $J'_0 \subset M(S'_0)$. Moreover, by (3.10.6) we get that $S'_0 \in \mathfrak{E}^2(R'_0, J'_0)$, S'_0 has a simple point at R'_0, and upon letting (R', J') be the monoidal transform of (R'_0, J'_0, S'_0) along V we have that: (1) if $S_0 \notin \mathfrak{V}(R'_0)$ then $(R', J') = (R_2, J_2)$; and (2) if $S_0 \in \mathfrak{V}(R'_0)$ then $\dim R' = \dim R'_0$, $\operatorname{ord}_{R'} J' = d$, $S_0 \in \mathfrak{E}^2(R', J')$, S_0 has a simple point at R', and (R_2, J_2) is the monoidal transform of (R', J', S_0) along V. In case (1) it suffices to take $m = 1$. In case (2) it suffices to take $m = 2$ and $(R_1', J_1', S_1') = (R', J', S_0)$.

Now let $n > 1$ and assume that the assertion is true for all values of n smaller than the given one. By (3.10.2) we have that $\dim R_1 = 3$, and hence upon applying the induction hypothesis to the system $[(R_i, J_i, S_i)_{1 \leqslant i < n}, (R_n, J_n)]$ we can find a positive integer q and a sequence $(R_i^*, J_i^*, S_i^*)_{0 \leqslant i < q}$ such that: R_i^* is a regular local domain with $\dim R_i^* = \dim R_{n+1}$ and J^* is a nonzero principal ideal in R_i^* with $\operatorname{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i < q$; $S_i^* \in \mathfrak{C}^2(R_i^*, J_i^*)$ and S_i^* has a simple point at R_i^* for $0 \leqslant i < q$; (R_0^*, J_0^*) is a monoidal transform of (R_1, J_1, R_1); (R_i^*, J_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, S_{i-1}^*)$ for $0 < i < q$; and (R_{n+1}, J_{n+1}) is a monoidal transform of $(R_{q-1}^*, J_{q-1}^*, S_{q-1}^*)$. Now (R_0^*, J_0^*) is a monoidal transform of (R_1, J_1, R_1), and $\operatorname{ord}_{R_0^*} J_0^* = d$; hence in particular $J_0^* \subset M(R_0^*)$; therefore by (3.10.6) we get that $2 \leqslant \dim R_0' = \dim R_0^* \leqslant 3$, $\operatorname{ord}_{R_0'} J_0' = d$, and there exists exactly one element S_0' in $\mathfrak{B}(R_0')$ such that $M(R_0) \subset M(S_0')$ and $J_0' \subset M(S_0')$. Moreover, by (3.10.6) we get that $S_0' \in \mathfrak{C}^2(R_0', J_0')$, S_0' has a simple point at R_0', and upon letting (R', J') be the monoidal transform of (R_0', J_0', S_0') along V we have that: (1*) if $S_0 \notin \mathfrak{B}(R_0')$ then $(R', J') = (R_0^*, J_0^*)$; and (2*) if $S_0 \in \mathfrak{B}(R_0')$ then $\dim R' = \dim R_0'$, $\operatorname{ord}_{R'} J' = d$, $S_0 \in \mathfrak{C}^2(R', J')$, S_0 has a simple point at R', and (R_0^*, J_0^*) is the monoidal transform of (R', J', S_0) along V. In case (1*) it suffices to take $m = q + 1$ and $(R_i', J_i', S_i') = (R_{i-1}^*, J_{i-1}^*, S_{i-1}^*)$ for $1 \leqslant i < m$. In case (2*) it suffices to take $m = q + 2$, $(R_1', J_1', S_1') = (R', J', S_0)$, and $(R_i', J_i', S_i') = (R_{i-2}^*, J_{i-2}^*, S_{i-2}^*)$ for $2 \leqslant i < m$.

PROOF OF (3.16.3). First suppose that $S \neq R_0$. Then $S \in \mathfrak{C}^2(R_0, J_0)$ and hence by (3.16.1) there exists a sequence $(R_i^*, J_i^*, S_i^*)_{0 \leqslant i \leqslant n}$ such that: R_i^* is a regular local domain and J_i^* is a nonzero principal ideal in R^* with $\operatorname{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i \leqslant n$; $\mathfrak{C}^2(R_i^*, J_i^*)$ has a strict normal crossing at R_i^* for $0 \leqslant i < n$; $S_i^* \in \mathfrak{C}^2(R_i^*, J_i^*)$ for $0 \leqslant i < n$; $(R_0^*, J_0^*, I_0^*) = (R_0, J_0, S)$; (R_i^*, J_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, S_{i-1}^*)$ for $0 < i \leqslant n$; and $(R_n^*, J_n^*, S_n^*) = (R_n, J_n, R_n)$. Clearly $(R', J') = (R_1^*, J_1^*)$, and hence $\operatorname{ord}_{R'} J' = d$. It suffices to take $e = n$ and $(R_i', J_i', S_i') = (R_{i+1}^*, J_{i+1}^*, S_{i+1}^*)$ for $0 \leqslant i < e$.

Next suppose that $S = R_0$. Then by (3.16.2) we get that $\mathrm{ord}_{R'}J' = d$ and there exists a positive integer m and a sequence $(R_i^*, J_i^*, S_i^*)_{0 \leqslant i < m}$ such that: R_i^* is a regular local domain and J_i^* is a nonzero principal ideal in R_i^* with $\mathrm{ord}_{R_i^*}J_i^* = d$ for $0 \leqslant i < m$; $S_i^* \in \mathfrak{C}^2(R_i^*, J_i^*)$ and S_i^* has a simple point at R_i^* for $0 \leqslant i < m$; $(R_0^*, J_0^*) = (R', J')$; (R_i^*, J_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, S_{i-1}^*)$ for $0 < i < m$; and (R_{n+1}, J_{n+1}) is a monoidal transform of $(R_{m-1}^*, J_{m-1}^*, S_{m-1}^*)$. By (3.11) we get that $\mathfrak{C}^2(R_i^*, J_i^*)$ has a strict normal crossing at R_i^* for $0 \leqslant i < m$. It suffices to take $e = m$ and $(R_i', J_i', S_i') = (R_i^*, J_i^*, S_i^*)$ for $0 \leqslant i < e$.

(3.17). *Let $(R_i, J_i, S_i)_{0 \leqslant i < \infty}$ be an infinite semiresolver such that $\mathrm{ord}_{R_i}J_i = \mathrm{ord}_{R_0}J_0$ for $0 \leqslant i < \infty$, R_0 is pseudogeometric, and $\dim R_0 \leqslant 3$. Let $d = \mathrm{ord}_{R_0}J_0$. Then we have the following.*

(3.17.1). *There exists a unique nonnegative integer n such that $S_i \neq R_i$ for $0 \leqslant i < n$ and $S_n = R_n$.*

(3.17.2). *Let n be as in (3.17.1). Let S be a positive-dimensional element in $\mathfrak{C}(R_0, J_0)$ such that S has a simple point at R_0, and if $\dim S = 2$ then $\mathfrak{C}^2(R_0, J_0)$ has a strict normal crossing at R_0. Let (R', J') be a monoidal transform of (R_0, J_0, S). Assume that there exists a valuation ring V of the quotient field of R_0 such that V dominates R' and V dominates R_{n+1}. Then $\mathrm{ord}_{R'}J' = d$, and there exists a positive integer q and a semiresolver $(R_i', J_i', S_i')_{0 \leqslant i < q}$ such that $\mathrm{ord}_{R_i'}J_i' = d$ for $0 \leqslant i < q$, $(R_0', J_0') = (R', J')$, and $(R_{q-1}', J_{q-1}', S_{q-1}') = (R_{n+1}, J_{n+1}, S_{n+1})$.*

(3.17.3). *Let $[(R_i^*, J_i^*, S_i^*)_{0 \leqslant i < m}, (R_m^*, J_m^*)]$ be a finite weak semiresolver such that $(R_0^*, J_0^*) = (R_0, J_0)$. Assume that there exists a valuation ring V of the quotient field of R_0 such that V dominates R_m^* and V dominates R_i for $0 \leqslant i < \infty$. Then $\mathrm{ord}_{R_i^*}J_i^* = d$ for $0 \leqslant i \leqslant m$, and there exist positive integers q and e and a semiresolver $(R_i', J_i', S_i')_{0 \leqslant i < q}$ such that $\mathrm{ord}_{R_i'}J_i' = d$ for $0 \leqslant i < q$, $(R_0', J_0') = (R_m^*, J_m^*)$, and $(R_{q-1}', J_{q-1}', S_{q-1}') = (R_e, J_e, S_e)$.*

PROOF OF (3.17.1). The uniqueness is obvious. Since (R_i, J_i) is unresolved for $0 \leqslant i < \infty$, we get that $2 \leqslant \dim S_i \leqslant 3$ for

$0 \leqslant i < \infty$; in view of (3.5) we also get that $\dim R_i = 3$ for $0 \leqslant i < \infty$. Therefore the existence follows from (3.15).

PROOF OF (3.17.2). Since (R_i, J_i) is unresolved for $0 \leqslant i < \infty$, we get that $d > 1$, $2 \leqslant \dim S \leqslant 3$, and $2 \leqslant \dim S_i \leqslant 3$ for $0 \leqslant i < \infty$; in view of (3.5) we also get that $\dim R_i = 3$ for $0 \leqslant i < \infty$. Therefore $S_i \in \mathfrak{E}^2(R_i, J_i)$ for $0 \leqslant i < n$. If $S = S_0$ then $(R', J') = (R_1, J_1)$, and hence $\operatorname{ord}_{R'} J' = d$ and it suffices to take $q = n + 1$ and $(R'_i, J'_i, S'_i) = (R_{i+1}, J_{i+1}, S_{i+1})$ for $0 \leqslant i < q$. So now assume that $S \neq S_0$. Then we must have $n > 0$, $S_0 \in \mathfrak{E}^2(R_0, J_0)$, and $\mathfrak{E}^2(R_0, J_0)$ has a strict normal crossing at R_0. By (3.11) we now get that $\mathfrak{E}^2(R_n, J_n)$ has a strict normal crossing at R_n; since $\dim S_n \neq 2$, we conclude that $\mathfrak{E}^2(R_n, J_n) = \varnothing$. Therefore by (3.16.3) we get that $\operatorname{ord}_{R'} J' = d$, and there exists a positive integer e and a semiresolver $(R'_i, J'_i, S'_i)_{0 \leqslant i < e}$ such that $\operatorname{ord}_{R'_i} J'_i = d$ for $0 \leqslant i < e$, $(R'_0, J'_0) = (R', J')$, and (R_{n+1}, J_{n+1}) is a monoidal transform of $(R'_{e-1}, J'_{e-1}, S'_{e-1})$. It now suffices to take $q = e + 1$ and $(R'_{q-1}, J'_{q-1}, S'_{q-1}) = (R_{n+1}, J_{n+1}, S_{n+1})$.

PROOF OF (3.17.3). We shall make induction on m.

First consider the case of $m = 1$. By (3.17.1) there exists a unique nonnegative integer n such that $S_i \neq R_i$ for $0 \leqslant i < n$ and $S_n = R_n$. Let $S = S_0^*$. Then S is a positive-dimensional element in $\mathfrak{E}(R_0, J_0)$ such that S has a simple point at R_0, and if $\dim S = 2$ then $\mathfrak{E}^2(R_0, J_0)$ has a strict normal crossing at R_0. Also (R_m^*, J_m^*) is a monoidal transform of (R_0, J_0, S), V dominates R_m^*, and V dominates R_{n+1}. Therefore by (3.17.2) we get that $\operatorname{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i \leqslant m$, and there exists a positive integer q and a semiresolver $(R'_i, J'_i, S'_i)_{0 \leqslant i < q}$ such that $\operatorname{ord}_{R'_i} J'_i = d$ for $0 \leqslant i < q$, $(R'_0, J'_0) = (R_m^*, J_m^*)$, and $(R'_{q-1}, J'_{q-1}, S'_{q-1}) = (R_{n+1}, J_{n+1}, S_{n+1})$. It suffices to take $e = n + 1$.

Now let $m > 1$ and assume that the assertion is true for all values of m smaller than the given one. Upon applying the induction hypothesis to the finite weak semiresolver $[(R_i^*, J_i^*, S_i^*)_{0 \leqslant i < m-1}, (R_{m-1}^*, J_{m-1}^*)]$ we get that $\operatorname{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i \leqslant m - 1$, and there exist positive integers a and b and a semiresolver $(R''_i, J''_i, S''_i)_{0 \leqslant i < a}$ such that $\operatorname{ord}_{R''_i} J'' = d$ for $0 \leqslant i < a$, $(R''_0, J''_0) = (R_{m-1}^*, J_{m-1}^*)$, and $(R''_{a-1}, J''_{a-1}, S''_{a-1}) = (R_b, J_b, S_b)$.

Now R_0'' is pseudogeometric and $\dim R_0'' \leqslant 3$. By (1.10.5) we also get that (R_i'', J_i'') is unresolved for $0 \leqslant i < a$. Let $(R_i'', J_i'', S_i'') = (R_{b-a+1+i}, J_{b-a+1+i}, S_{b-a+1+i})$ for $a \leqslant i < \infty$. Then $(R_i'', J_i'', S_i'')_{0 \leqslant i < \infty}$ is an infinite semiresolver. Therefore by (3.17.1) there exists a unique nonnegative integer n such that $S_i'' \neq R_i''$ for $0 \leqslant i < n$ and $S_n'' = R_n''$. Let $S = S_{m-1}^*$. Then S is a positive-dimensional element in $\mathfrak{E}(R_0'', J_0'')$ such that S has a simple point at R_0'', and if $\dim S = 2$ then $\mathfrak{E}^2(R_0'', J_0'')$ has a strict normal crossing at R_0''. Also (R_m^*, J_m^*) is a monoidal transform of (R_0'', J_0'', S), V dominates R_m^*, and V dominates R_{n+1}''. Therefore by (3.17.2) we get that $\operatorname{ord}_{R_m^*} J_m^* = d$, and there exists a positive integer c and a semiresolver $(R_i', J_i', S_i')_{0 \leqslant i < c}$ such that $\operatorname{ord}_{R_i'} J_i' = d$ for $0 \leqslant i < c$, $(R_0', J_0') = (R_m^*, J_m^*)$, and $(R_{c-1}', J_{c-1}', S_{c-1}') = (R_{n+1}'', J_{n+1}'', S_{n+1}'')$. If $n + 1 \geqslant a - 1$ then it suffices to take $q = c$ and $e = b - a + n + 2$. If $n + 1 < a - 1$ then it suffices to take $e = b$, $q = c + a - n - 2$, and $(R_i', J_i', S_i') = (R_{n+2-c+i}'', J_{n+2-c+i}'', S_{n+2-c+i}'')$ for $c \leqslant i < q$.

(3.18). *Let d and n be positive integers and let $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < n}, (R_n, J_n, I_n)]$ be a system such that: R_i is a regular local domain and J_i and I_i are nonzero principal ideals in R_i for $0 \leqslant i \leqslant n$; $\operatorname{ord}_{R_i} J_i = d$ for $0 \leqslant i < n$; $S_i \in \mathfrak{E}^2(R_i, J_i)$ and (S_i, I_i) has a pseudonormal crossing at R_i for $0 \leqslant i < n$; (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, S_{i-1})$ for $0 < i \leqslant n$; and* $\dim R_0 = 3$. *Then we have the following.*

(3.18.1). *Assume that $\mathfrak{E}^2(R_0, J_0)$ has a strict normal crossing at R_0, $J_n \subset M(R_n)$, and there does not exist any element S' in $\mathfrak{E}^2(R_n, J_n)$ such that (S', I_n) has a pseudonormal crossing at R_n. Let S be any element in $\mathfrak{E}^2(R_0, J_0)$ such that (S, I_0) has a pseudonormal crossing at R_0. Then there exists a sequence $(R_i', J_i', I_i', S_i')_{0 \leqslant i < n}$ such that: R_i' is a three-dimensional regular local domain and J_i' and I_i' are nonzero principal ideals in R_i' with $\operatorname{ord}_{R_i'} J_i' = d$ for $0 \leqslant i < n$; $\mathfrak{E}^2(R_i', J_i')$ has a strict normal crossing at R_i' for $0 \leqslant i < n$; $S_i' \in \mathfrak{E}^2(R_i', J_i')$ and (S_i', I_i') has a pseudonormal crossing at R_i' for $0 \leqslant i < n$; $(R_0', J_0', I_0', S_0') = (R_0, J_0, I_0, S)$; (R_i', J_i', I_i') is a monoidal transform of $(R_{i-1}', J_{i-1}', I_{i-1}', S_{i-1}')$ for $0 < i < n$; and (R_n, J_n, I_n) is a monoidal transform of $(R_{n-1}', J_{n-1}', I_{n-1}', S_{n-1}')$.*

(3.18.2). *Assume that* $\operatorname{ord}_{R_n} J_n = d$. *Let* $(R_{n+1}, J_{n+1}, I_{n+1})$ *be a monoidal transform of* (R_n, J_n, I_n, R_n) *such that* $J_{n+1} \subset M(R_{n+1})$, *let* V *be a valuation ring of the quotient field of* R_0 *such that* V *dominates* R_{n+1}, *and let* (R'_0, J'_0, I'_0) *be the monoidal transform of* (R_0, J_0, I_0, R_0) *along* V. *Then* $2 \leqslant \dim R'_0 = \dim R_{n+1} \leqslant 3$, $\operatorname{ord}_{R'_0} J'_0 = d$, *and there exists exactly one two-dimensional element* S'_0 *in* $\mathfrak{B}(R'_0)$ *such that* $M(R_0) \subset M(S'_0)$ *and* $J'_0 \subset M(S'_0)$. *Moreover,* $S'_0 \in \mathfrak{C}^2(R'_0, J'_0)$, (S'_0, I'_0) *has a pseudonormal crossing at* R'_0, *and there exists a positive integer* m *and a sequence* $(R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < m}$ *such that:* R'_i *is a regular local domain with* $\dim R'_i = \dim R'_0$ *and* J'_i *and* I'_i *are nonzero principal ideals in* R'_i *with* $\operatorname{ord}_{R'_i} J'_i = d$ *for* $0 < i < m$; $S'_i \in \mathfrak{C}^2(R'_i, J'_i)$ *and* (S'_i, I'_i) *has a pseudonormal crossing at* R'_i *for* $0 < i < m$; (R'_i, J'_i, I'_i) *is a monoidal transform of* $(R'_{i-1}, J'_{i-1}, I'_{i-1}, S'_{i-1})$ *for* $0 < i < m$; *and* $(R_{n+1}, J_{n+1}, I_{n+1})$ *is a monoidal transform of* $(R'_{m-1}, J'_{m-1}, I'_{m-1}, S'_{m-1})$.

(3.18.3). *Assume that* $\mathfrak{C}^2(R_0, J_0)$ *has a strict normal crossing at* R_0, $\operatorname{ord}_{R_n} J_n = d$, *and there does not exist any element* S' *in* $\mathfrak{C}^2(R_n, J_n)$ *such that* (S', I_n) *has a pseudonormal crossing at* R_n. *Let* $(R_{n+1}, J_{n+1}, I_{n+1})$ *be a monoidal transform of* (R_n, J_n, I_n, R_n) *such that* $J_{n+1} \subset M(R_{n+1})$. *Let* S *be an element in* $\mathfrak{C}(R_0, J_0)$ *with* $\dim S \geqslant 2$ *such that* (S, I_0) *has a pseudonormal crossing at* R_0. *Let* (R', J', I') *be a monoidal transform of* (R_0, J_0, I_0, S). *Assume that there exists a valuation ring* V *of the quotient field of* R_0 *such that* V *dominates* R' *and* V *dominates* R_{n+1}. *Then* $\operatorname{ord}_{R'} J' = d$, *and there exists a positive integer* e *and a resolver* $(R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < e}$ *such that* $\operatorname{ord}_{R'_i} J'_i = d$ *for* $0 \leqslant i < e$, $(R'_0, J'_0, I'_0) = (R', J', I')$, *and* $(R_{n+1}, J_{n+1}, I_{n+1})$ *is a monoidal transform of* $(R'_{e-1}, J'_{e-1}, I'_{e-1}, S'_{e-1})$.

PROOF OF (3.18.1). We shall make induction on n. First consider the case of $n = 1$; if $S \neq S_0$ then by (3.10.4) and (1.10.12) we would get that $S \in \mathfrak{C}^2(R_n, J_n)$ and (S, I_n) has a pseudonormal crossing at R_n, which would contradict our assumption; therefore $S = S_0$ and hence it suffices to take $(R'_0, J'_0, I'_0, S'_0) = (R_0, J_0, I_0, S_0)$. Now let $n > 1$ and assume that the assertion is true for all values of n smaller than the given one. By (3.10.2) and (3.10.5) we get that $\dim R_i = 3$ and $\mathfrak{C}^2(R_i, J_i)$ has a strict normal crossing at R_i for $0 \leqslant i < n$. Hence if $S = S_0$ then it suffices to take $(R'_i, J'_i, I'_i, S'_i) = (R_i, J_i, I_i, S_i)$ for $0 \leqslant i < n$. So now assume that $S \neq S_0$. Then by (3.10.4) and (1.10.12) we get that $S \in \mathfrak{E}^2(R_1, J_1)$ and (S, I_1) has a pseudonormal crossing at R_1. Therefore upon applying the induction hypothesis to the system $[(R_i, J_i, I_i, S_i)_{1 \leqslant i < n}, (R_n, J_n, I_n)]$ we can find a sequence $(R_i^*, J_i^*, I_i^*, S_i^*)_{1 \leqslant i < n}$ such that: R_i^* is a three-dimensional regular local domain and J_i^* and I_i^* are nonzero principal ideals in R_i^* with $\operatorname{ord}_{R_i^*} J_i^* = d$ for $1 \leqslant i < n$; $\mathfrak{E}^2(R_i^*, J_i^*)$ has a strict normal crossing at R_i^* for $1 \leqslant i < n$; $S_i^* \in \mathfrak{E}^2(R_i^*, J_i^*)$ and (S_i^*, I_i^*) has a pseudonormal crossing at R_i^* for $1 \leqslant i < n$; $(R_1^*, J_1^*, I_1^*, S_1^*) = (R_1, J_1, I_1, S)$; (R_i^*, J_i^*, I_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, I_{i-1}^*, S_{i-1}^*)$ for $1 < i < n$; and (R_n, J_n, I_n) is a monoidal transform of $(R_{n-1}^*, J_{n-1}^*, I_{n-1}^*, S_{n-1}^*)$. Let $(R_n^*, J_n^*, I_n^*) = (R_n, J_n, I_n)$. Let $(R_0', J_0', I_0', S_0') = (R_0, J_0, I_0, S)$. Take a valuation ring V of the quotient field of R_0 such that V dominates R_2^*. Let (R_1', J_1', I_1') be the monoidal transform of (R_0', J_0', I_0', S_0') along V, and let $S_1' = S_0$. Then by (3.10.4), (3.10.5), and (1.10.12) we get that $\dim R_1' = 3$, $\operatorname{ord}_{R_1'} J_1' = d$, $\mathfrak{E}^2(R_1', J_1')$ has a strict normal crossing at R_1', $S_1' \in \mathfrak{E}^2(R_1', J_1')$, (S_1', I_1') has a pseudonormal crossing at R_1', and (R_2^*, J_2^*, I_2^*) is a monoidal transform of (R_1', J_1', I_1', S_1'). It now suffices to take $(R_i', J_i', I_i', S_i') = (R_i^*, J_i^*, I_i^*, S_i^*)$ for $2 \leqslant i < n$.

PROOF OF (3.18.2). We shall make induction on n.

First consider the case of $n = 1$. By (3.10.6) we get that $2 \leqslant \dim R_0' = \dim R_2 \leqslant 3$, $\operatorname{ord}_{R_0'} J_0' = d$, and there exists exactly one two-dimensional element S_0' in $\mathfrak{B}(R_0')$ such that $M(R_0) \subset M(S_0')$ and $J_0' \subset M(S_0')$. Moreover, by (1.10.12), (3.7.2), (3.7.4), and (3.10.6) we get that $S_0' \in \mathfrak{E}^2(R_0', J_0')$, (S_0', I_0') has a pseudonormal crossing at R_0', and upon letting (R', J', I') be the monoidal transform of (R_0', J_0', I_0', S_0') along V we have that: (1) if $S_0 \notin \mathfrak{B}(R_0')$ then $(R', J', I') = (R_2, J_2, I_2)$; and (2) if $S_0 \in \mathfrak{B}(R_0)$ then $\dim R' = \dim R_0'$, $\operatorname{ord}_{R'} J' = d$, $\{S_0, S_0'\}$ has a normal crossing at R_0', (S_0, I_0') has a pseudonormal crossing at R_0', $S_0 \in \mathfrak{E}^2(R', J')$, (S_0, I') has a pseudonormal crossing at R', and (R_2, J_2, I_2) is the monoidal transform of (R', J', I', S_0) along V. In case (1) it suffices to take $m = 1$. In case (2) it suffices to take $m = 2$ and $(R_1', J_1', I_1', S_1') = (R', J', I', S_0)$.

Now let $n > 1$ and assume that the assertion is true for all values of n smaller than the given one. By (3.10.2) we have that

$\dim R_1 = 3$, and hence upon applying the induction hypothesis to the system $[(R_i, J_i, I_i, S_i)_{1 \leqslant i < n}, (R_n, J_n, I_n)]$ we can find a positive integer q and a sequence $(R_i^*, J_i^*, I_i^*, S_i^*)_{0 \leqslant i < q}$ such that: R_i^* is a regular local domain with $\dim R_i^* = \dim R_{n+1}$ and J_i^* and I_i^* are nonzero principal ideals in R_i^* with $\mathrm{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i < q$; $S_i^* \in \mathfrak{C}^2(R_i^*, J_i^*)$ and (S_i^*, I_i^*) has a pseudonormal crossing at R_i^* for $0 \leqslant i < q$; (R_0^*, J_0^*, I_0^*) is a monoidal transform of (R_1, J_1, I_1, R_1); (R_i^*, J_i^*, I_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, I_{i-1}^*, S_{i-1}^*)$ for $0 < i < q$; and $(R_{n+1}, J_{n+1}, I_{n+1})$ is a monoidal transform of $(R_{q-1}^*, J_{q-1}^*, I_{q-1}^*, S_{q-1}^*)$. Now (R_0^*, J_0^*, I_0^*) is a monoidal transform of (R_1, J_1, I_1, R_1) and $\mathrm{ord}_{R_0^*} J_0^* = d$; hence in particular $J_0^* \subset M(R_0^*)$; therefore by (3.10.6) we get that $2 \leqslant \dim R_0' = \dim R_0^* \leqslant 3$, $\mathrm{ord}_{R_0'} J_0' = d$, and there exists exactly one two-dimensional element S_0' in $\mathfrak{V}(R_0')$ such that $M(R_0) \subset M(S_0')$ and $J_0' \subset M(S_0')$. Moreover, by (1.10.12), (3.7.2), (3.7.4), and (3.10.6) we get that $S_0' \in \mathfrak{C}^2(R_0', J_0')$, (S_0', I_0') has a pseudonormal crossing at R_0', and upon letting (R', J', I') be the monoidal transform of (R_0', J_0', I_0', S_0') along V we have that: (1*) if $S_0 \in \mathfrak{V}(R_0')$ then $(R', J', I') = (R_0^*, J_0^*, I_0^*)$; and (2*) if $S_0 \notin \mathfrak{V}(R_0')$ then $\dim R' = \dim R_0'$, $\mathrm{ord}_{R'} J' = d$, $\{S_0, S_0'\}$ has a normal crossing at R_0', (S_0, I_0') has a pseudonormal crossing at R_0', $S_0 \in \mathfrak{C}^2(R', J')$, (S_0, I') has a pseudonormal crossing at R', and (R_0^*, J_0^*, I_0^*) is the monoidal transform of (R', J', I', S_0) along V. In case (1*) it suffices to take $m = q + 1$ and $(R_i', J_i', I_i', S_i') = (R_{i-1}^*, J_{i-1}^*, I_{i-1}^*, S_{i-1}^*)$ for $1 \leqslant i < m$. In case (2*) it suffices to take $m = q + 2$, $(R_1', J_1', I_1', S_1') = (R', J', I', S_0)$, and $(R_i', J_i', I_i', S_i') = (R_{i-2}^*, J_{i-2}^*, I_{i-2}^*, S_{i-2}^*)$ for $2 \leqslant i < m$.

PROOF OF (3.18.3). First suppose that $S \neq R_0$. Then $S \in \mathfrak{C}^2(R_0, J_0)$ and hence by (3.18.1) there exists a sequence $(R_i^*, J_i^*, I_i^*, S_i^*)_{0 \leqslant i \leqslant n}$ such that: R_i^* is a regular local domain and J_i^* and I_i^* are nonzero principal ideals in R_i^* with $\mathrm{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i \leqslant n$; $\mathfrak{C}^2(R_i^*, J_i^*)$ has a strict normal crossing at R_i^* for $0 \leqslant i < n$; $S_i^* \in \mathfrak{C}^2(R_i^*, J_i^*)$ and (S_i^*, I_i^*) has a pseudonormal crossing at R_i^* for $0 \leqslant i < n$; $(R_0^*, J_0^*, I_0^*, S_0^*) = (R_0, J_0, I_0, S)$; (R_i^*, J_i^*, I_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, I_{i-1}^*, S_{i-1}^*)$ for $0 < i \leqslant n$; and $(R_n^*, J_n^*, I_n^*, S_n^*) = (R_n, J_n, I_n, R_n)$. Clearly $(R', J', I') = (R_1^*, J_1^*, I_1^*)$, and hence $\mathrm{ord}_{R'} J' = d$. By (1.10.8) we get that I_n has a quasinormal crossing at R_n, and hence

(R_n, I_n) has a pseudonormal crossing at R_n. It suffices to take $e = n$ and $(R'_i, J'_i, I'_i, S'_i) = (R^*_{i+1}, J^*_{i+1}, I^*_{i+1}, S^*_{i+1})$ for $0 \leqslant i < e$.

Next suppose that $S = R_0$. Then by (3.18.2) we get that $\operatorname{ord}_{R'} J' = d$ and there exists a positive integer m and a sequence $(R^*_i, J^*_i, I^*_i, S^*_i)_{0 \leqslant i < m}$ such that: R^*_i is a regular local domain and J^*_i and I^*_i are nonzero principal ideals in R^*_i with $\operatorname{ord}_{R^*_i} J^*_i = d$ for $0 \leqslant i < m$; $S^*_i \in \mathfrak{C}^2(R^*_i, J^*_i)$ and (S^*_i, I^*_i) has a pseudonormal crossing at R^*_i for $0 \leqslant i < m$; $(R^*_0, J^*_0, I^*_0) = (R', J', I')$; (R^*_i, J^*_i, I^*_i) is a monoidal transform of $(R^*_{i-1}, J^*_{i-1}, I^*_{i-1}, S^*_{i-1})$ for $0 < i < m$; and $(R_{n+1}, J_{n+1}, I_{n+1})$ is a monoidal transform of $(R^*_{m-1}, J^*_{m-1}, I^*_{m-1}, S^*_{m-1})$. By (3.11) we get that $\mathfrak{C}^2(R^*_i, J^*_i)$ has a strict normal crossing at R^*_i for $0 \leqslant i < m$. It suffices to take $e = m$ and $(R'_i, J'_i, I'_i, S'_i) = (R^*_i, J^*_i, I^*_i, S^*_i)$ for $0 \leqslant i < e$.

(3.19). *Let $(R_i, J_i, I_i, S_i)_{0 \leqslant i < \infty}$ be an infinite resolver such that* $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_0} J_0$ *for* $0 \leqslant i < \infty$, *R_0 is pseudogeometric, and* $\dim R_0 \leqslant 3$. *Let* $d = \operatorname{ord}_{R_0} J_0$. *Then we have the following.*

(3.19.1). *There exists a unique nonnegative integer n such that $S_i \neq R_i$ for $0 \leqslant i < n$ and $S_n = R_n$.*

(3.19.2). *Let n be as in* (3.19.1). *Let S be a positive-dimensional element in $\mathfrak{C}(R_0, J_0)$ such that (S, I_0) has a pseudonormal crossing at R_0, and if* $\dim S = 2$ *then $\mathfrak{C}^2(R_0, J_0)$ has a strict normal crossing at R_0. Let (R', J', I') be a monoidal transform of (R_0, J_0, I_0, S). Assume that there exists a valuation ring V of the quotient field of R_0 such that V dominates R' and V dominates R_{n+1}. Then* $\operatorname{ord}_{R'} J' = d$, *and there exists a positive integer q and a resolver $(R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < q}$ such that* $\operatorname{ord}_{R'_i} J'_i = d$ *for $0 \leqslant i < q$, $(R'_0, J'_0, I'_0) = (R', J', I')$, and $(R'_{q-1}, J'_{q-1}, I'_{q-1}, S'_{q-1}) = (R_{n+1}, J_{n+1}, I_{n+1}, S_{n+1})$.*

(3.19.3). *Let $[(R^*_i, J^*_i, I^*_i, S^*_i)_{0 \leqslant i < m}, (R^*_m, J^*_m, I^*_m)]$ be a finite weak resolver such that $(R^*_0, J^*_0, I^*_0) = (R_0, J_0, I_0)$. Assume that there exists a valuation ring V of the quotient field of R_0 such that V dominates R^*_m and V dominates R_i for $0 \leqslant i < \infty$. Then* $\operatorname{ord}_{R^*_i} J^*_i = d$ *for $0 \leqslant i \leqslant m$, and there exist positive integers q and e and a resolver $(R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < q}$ such that* $\operatorname{ord}_{R'_i} J'_i = d$ *for $0 \leqslant i < q$, $(R'_0, J'_0, I'_0) = (R^*_m, J^*_m, I^*_m)$, and $(R'_{q-1}, J'_{q-1}, I'_{q-1}, S'_{q-1}) = (R_e, J_e, I_e, S_e)$.*

PROOF OF (3.19.1). The uniqueness is obvious. Since (R_i, J_i) is unresolved for $0 \leqslant i < \infty$, we get that $2 \leqslant \dim S_i \leqslant 3$ for $0 \leqslant i < \infty$; in view of (3.5) we also get that $\dim R_i = 3$ for $0 \leqslant i < \infty$. Therefore the existence follows from (3.15).

PROOF OF (3.19.2). Since (R_i, J_i) is unresolved for $0 \leqslant i < \infty$, we get that $d > 1$, $2 \leqslant \dim S \leqslant 3$, and $2 \leqslant \dim S_i \leqslant 3$ for $0 \leqslant i < \infty$; in view of (3.5) we also get that $\dim R_i = 3$ for $0 \leqslant i < \infty$. Therefore $S_i \in \mathfrak{C}^2(R_i, J_i)$ for $0 \leqslant i < n$. If $S = S_0$ then $(R', J', I') = (R_1, J_1, I_1)$, and hence $\operatorname{ord}_{R'} J' = d$ and it suffices to take $q = n + 1$ and $(R'_i, J'_i, I'_i, S'_i) = (R_{i+1}, J_{i+1}, I_{i+1}, S_{i+1})$ for $0 \leqslant i < q$. So now assume that $S \neq S_0$. Then we must have $n > 0$, $S_0 \in \mathfrak{C}^2(R_0, J_0)$, and $\mathfrak{C}^2(R_0, J_0)$ has a strict normal crossing at R_0. By (3.11) we now get that $\mathfrak{C}^2(R_n, J_n)$ has a strict normal crossing at R_n; since $\dim S_n \neq 2$, we conclude that there does not exist any element S' in $\mathfrak{C}^2(R_n, J_n)$ such that (S', I_n) has a pseudonormal crossing at R_n. Therefore by (3.18.3) we get that $\operatorname{ord}_{R'} J' = d$, and there exists a positive integer e and a resolver $(R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < e}$ such that $\operatorname{ord}_{R'_i} J'_i = d$ for $0 \leqslant i < e$, $(R'_0, J'_0, I'_0) = (R', J', I')$, and $(R_{n+1}, J_{n+1}, I_{n+1})$ is a monoidal transform of $(R'_{e-1}, J'_{e-1}, I'_{e-1}, S'_{e-1})$. It now suffices to take $q = e + 1$ and $(R'_{q-1}, J'_{q-1}, I'_{q-1}, S'_{q-1}) = (R_{n+1}, J_{n+1}, I_{n+1}, S_{n+1})$.

PROOF OF (3.19.3). We shall make induction on m.

First consider the case of $m = 1$. By (3.19.1) there exists a unique nonnegative n such that $S_i \neq R_i$ for $0 \leqslant i < n$ and $S_n = R_n$. Let $S = S_0^*$. Then S is a positive-dimensional element in $\mathfrak{C}(R_0, J_0)$ such that (S, I_0) has a pseudonormal crossing at R_0, and if $\dim S = 2$ then $\mathfrak{C}^2(R_0, J_0)$ has a strict normal crossing at R_0. Also (R_m^*, J_m^*, I_m^*) is a monoidal transform of (R_0, J_0, I_0, S), V dominates R_m^*, and V dominates R_{n+1}. Therefore by (3.19.2) we get that $\operatorname{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i \leqslant m$, and there exists a positive integer q and a resolver $(R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < q}$ such that $\operatorname{ord}_{R'_i} J'_i = d$ for $0 \leqslant i < q$, $(R'_0, J'_0, I'_0) = (R_m^*, J_m^*, I_m^*)$, and $(R'_{q-1}, J'_{q-1}, I'_{q-1}, S'_{q-1}) = (R_{n+1}, J_{n+1}, I_{n+1}, S_{n+1})$. It suffices to take $e = n + 1$.

Now let $m > 1$ and assume that the assertion is true for all values of m smaller than the given one. Upon applying the induction hypothesis to the finite weak resolver $[(R_i^*, J_i^*, I_i^*, S_i^*)_{0 \leqslant i < m-1}, (R_{m-1}^*, J_{m-1}^*, I_{m-1}^*)]$ we get that $\operatorname{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i \leqslant m - 1$, and there exist positive integers a and b and a resolver $(R_i'', J_i'', I_i'', S_i'')_{0\leqslant i<a}$ such that $\mathrm{ord}_{R_i''}J_i'' = d$ for $0 \leqslant i < a$, $(R_0'', J_0'', I_0'') = (R_{m-1}^*, J_{m-1}^*, I_{m-1}^*)$, and $(R_{a-1}'', J_{a-1}'', I_{a-1}'', S_{a-1}'') = (R_b, J_b, I_b, S_b)$. Now R_0'' is pseudogeometric and $\dim R_0'' \leqslant 3$. By (1.10.5) we also get that (R_i'', J_i'') is unresolved for $0 \leqslant i < a$. Let $(R_i'', J_i'', I_i'', S_i'') = (R_{b-a+1+i}, J_{b-a+1+i}, I_{b-a+1+i}, S_{b-a+1+i})$ for $a \leqslant i < \infty$. Then $(R_i'', J_i'', I_i'', S_i'')_{0\leqslant i<\infty}$ is an infinite resolver. Therefore by (3.19.1) there exists a unique nonnegative integer n such that $S_i'' \neq R_i''$ for $0 \leqslant i < n$ and $S_n'' = R_n''$. Let $S = S_{m-1}^*$. Then S is a positive-dimensional element in $\mathfrak{C}(R_0'', J_0'')$ such that (S, I_0'') has a pseudonormal crossing at R_0'', and if $\dim S = 2$ then $\mathfrak{C}^2(R_0'', J_0'')$ has a strict normal crossing at R_0''. Also (R_m^*, J_m^*, I_m^*) is a monoidal transform of (R_0'', J_0'', I_0'', S), V dominates R_m^*, and V dominates R_{n+1}''. Therefore by (3.19.2) we get that $\mathrm{ord}_{R_m^*}J_m^* = d$, and there exists a positive integer c and a resolver $(R_i', J_i', I_i', S_i')_{0\leqslant i<c}$ such that $\mathrm{ord}_{R_i'}J_i' = d$ for $0 \leqslant i < c$, $(R_0', J_0', I_0') = (R_m^*, J_m^*, I_m^*)$, and $(R_{c-1}', J_{c-1}', I_{c-1}', S_{c-1}') = (R_{n+1}'', J_{n+1}'', I_{n+1}'', S_{n+1}'')$. If $n + 1 \geqslant a - 1$ then it suffices to take $q = c$ and $e = b - a + n + 2$. If $n + 1 < a - 1$ then it suffices to take $e = b$, $q = c + a - n - 2$, and $(R_i', J_i', I_i', S_i') = (R_{n+2-c+i}'', J_{n+2-c+i}'', I_{n+2-c+i}'', S_{n+2-c+i}'')$ for $c \leqslant i < q$.

(3.20). *Let R be a regular local domain such that R is weakly resolvable. Then R is weakly semiresolvable.*

PROOF. Let R' be any iterated monoidal transform of R, let J' be any nonzero principal ideal in R' such that (R', J') is unresolved, and let V be any valuation ring of the quotient field of R such that V dominates R'. Now R' is a nonzero principal ideal in R', and R' has a quasinormal crossing at R'. Since R is weakly resolvable, there exists a finite weak resolver $[(R_i, J_i, I_i, S_i)_{0\leqslant i<m}, (R_m, J_m, I_m)]$ such that $(R_0, J_0, I_0) = (R', J', R')$, $\mathrm{ord}_{R'}J' = \mathrm{ord}_{R_i}J_i > \mathrm{ord}_{R_m}J_m$ for $0 \leqslant i < m$, and V dominates R_m. Clearly $[(R_i, J_i, S_i)_{0\leqslant i<m}, (R_m, J_m)]$ is a finite weak semiresolver. It follows that R is weakly semiresolvable.

(3.21). *Let R be a pseudogeometric regular local domain. Then we have the following.*

(3.21.1). *Assume that* $\dim R \leqslant 2$. *Then R is strongly semiresolvable, semiresolvable, weakly semiresolvable, strongly resolvable, resolvable, and weakly resolvable.*

(3.21.2). *Assume that* $\dim R \leqslant 3$. *Then R is strongly subresolvable.*

(3.21.3). *Assume that* $\dim R \leqslant 3$ *and R is weakly semiresolvable. Then R is strongly semiresolvable.*

(3.21.4). *Assume that* $\dim R \leqslant 3$ *and R is weakly resolvable. Then R is strongly resolvable, weakly semiresolvable, and strongly semiresolvable.*

PROOF OF (3.21.1). By (3.5) it follows that R is strongly semiresolvable and strongly resolvable.

To prove that R is semiresolvable and weakly semiresolvable, let R' be any iterated monoidal transform of R, let J' be any nonzero principal ideal in R' such that (R', J') is unresolved, and let V be any valuation ring of the quotient field of R such that V dominates R'. Then R' is pseudogeometric and $\dim R' = 2$. Let $(R_i, J_i)_{0 \leqslant i < \infty}$ be the unique infinite sequence such that $(R_0, J_0) = (R', J')$, and (R_i, J_i) is the monoidal transform of $(R_{i-1}, J_{i-1}, R_{i-1})$ along V for $0 < i < \infty$. By (3.5) there exists a positive integer m such that (R_i, J_i) is unresolved for $0 \leqslant i < m$, and (R_m, J_m) is resolved. Clearly $[(R_i, J_i, R_i)_{0 \leqslant i < m}, (R_m, J_m)]$ is a finite semiresolver. Now $\operatorname{ord}_{R_i} J_i \geqslant \operatorname{ord}_{R_j} J_j$ whenever $0 \leqslant j \leqslant i < \infty$. Therefore in view of (1.10.5) there exists an integer n with $0 < n \leqslant m$ such that $\operatorname{ord}_{R'} J' = \operatorname{ord}_{R_i} J_i > \operatorname{ord}_{R_n} J_n$ for $0 \leqslant i < n$. Clearly $[(R_i, J_i, R_i)_{0 \leqslant i < n}, (R_n, J_n)]$ is a finite weak semiresolver. It follows that R is a semiresolvable and weakly semiresolvable.

To prove that R is resolvable and weakly resolvable, let R' be any iterated monoidal transform of R, let J' and I' be any nonzero principal ideals in R' such that (R', J') is unresolved and I' has a quasinormal crossing at R', and let V be any valuation ring of the quotient field of R such that V dominates R'. Then R' is pseudogeometric and $\dim R' = 2$. Let $(R_i, J_i, I_i)_{0 \leqslant i < \infty}$ be the unique infinite sequence such that $(R_0, J_0, I_0) = (R', J', I')$, and

(R_i, J_i, I_i) is the monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, R_{i-1})$ along V for $0 < i < \infty$. By (1.10.8) we get that I_i has a quasinormal crossing at R_i for $0 \leqslant i < \infty$. By (3.5) there exists a positive integer m such that (R_i, J_i) is unresolved for $0 \leqslant i < m$, and (R_m, J_m) is resolved. Clearly $[(R_i, J_i, I_i, R_i)_{0 \leqslant i < m}, (R_m, J_m, I_m)]$ is a finite resolver. Now $\operatorname{ord}_{R_i} J_i \geqslant \operatorname{ord}_{R_j} J_j$ whenever $0 \leqslant j \leqslant i < \infty$. Therefore in view of (1.10.5) there exists an integer n with $0 < n \leqslant m$ such that $\operatorname{ord}_{R'} J' = \operatorname{ord}_{R_i} J_i > \operatorname{ord}_{R_n} J_n$ for $0 \leqslant i < n$. Clearly $[(R_i, J_i, I_i, R_i)_{0 \leqslant i < n}, (R_n, J_n, I_n)]$ is a finite weak resolver. It follows that R is resolvable and weakly resolvable.

PROOF OF (3.21.2). Follows from (3.12) and (3.15).

PROOF OF (3.21.3). Suppose if possible that R is not strongly semiresolvable. Then there exists an infinite semiresolver $(R_i, J_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R. Now $\operatorname{ord}_{R_i} J_i \geqslant \operatorname{ord}_{R_j} J_j$ whenever $0 \leqslant j \leqslant i < \infty$, and hence there exists a nonnegative integer n such that $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_n} J_n$ whenever $n \leqslant i < \infty$. We can take a valuation ring V of the quotient field of R_0 such that V dominates R_i for $0 \leqslant i < \infty$. Now R_n is an iterated monoidal transform of R, R_n is pseudogeometric, $\dim R_n \leqslant 3$, and $(R_{n+i}, J_{n+i}, S_{n+i})_{0 \leqslant i < \infty}$ is an infinite semiresolver. Since R is weakly semiresolvable, there exists a finite weak semiresolver $[(R_i^*, J_i^*, S_i^*)_{0 \leqslant i < m}, (R_m^*, J_m^*)]$ such that $(R_0^*, J_0^*) = (R_n, J_n)$, $\operatorname{ord}_{R_n} J_n > \operatorname{ord}_{R_m^*} J_m^*$, and V dominates R_m^*. By (3.17.3) we get that $\operatorname{ord}_{R_m^*} J_m^* = \operatorname{ord}_{R_n} J_n$. This is a contradiction.

PROOF OF (3.21.4). By (3.20) and (3.21.3) it follows that R is weakly semiresolvable and strongly semiresolvable.

Suppose if possible that R is not strongly resolvable. Then there exists an infinite resolver $(R_i, J_i, I_i, S_i)_{0 \leqslant i < \infty}$ such that R_0 is an iterated monoidal transform of R. Now $\operatorname{ord}_{R_i} J_i \geqslant \operatorname{ord}_{R_j} J_j$ whenever $0 \leqslant j \leqslant i < \infty$, and hence there exists a nonnegative integer n such that $\operatorname{ord}_{R_i} J_i = \operatorname{ord}_{R_n} J_n$ whenever $n \leqslant i < \infty$. We can take a valuation ring V of the quotient field of R_0 such that V dominates R_i for $0 \leqslant i < \infty$. Now R_n is an iterated monoidal transform of R, R_n is a pseudogeometric, $\dim R_n \leqslant 3$, and $(R_{n+i}, J_{n+i}, I_{n+i}, S_{n+i})_{0 \leqslant i < \infty}$ is an infinite resolver. Since R is weakly resolvable, there exists a finite weak resolver $[(R_i^*, J_i^*, I_i^*, S_i^*)_{0\leqslant i<m}, (R_m^*, J_m^*, I_m^*)]$ such that $(R_0^*, J_0^*, I_0^*) = (R_n, J_n, I_n)$, $\operatorname{ord}_{R_n} J_n > \operatorname{ord}_{R_m^*} J_m^*$, and V dominates R_m^*. By (3.19.3) we get that $\operatorname{ord}_{R_m^*} J_m^* = \operatorname{ord}_{R_n} J_n$. This is a contradiction.

§4. Unramified local extensions

(4.1). *Let R and R' be local rings such that R' dominates R, and $M(R)R'$ is primary for $M(R')$. Then for any nonunit ideal Q' in R' we have that* $\dim(R'/Q') \leqslant \dim(R/(R \cap Q'))$.

PROOF. Let h: $R' \to R'/Q'$ be the canonical epimorphism and let $n = \dim(R/(R \cap Q'))$. Now $h(R)$ is isomorphic to $R/(R \cap Q')$, and hence there exist elements $x_1, \ldots, x_n$ in $M(h(R))$ such that $(x_1, \ldots, x_n)h(R)$ is primary for $M(h(R))$. Since $M(R)R'$ is primary for $M(R')$, we get that $M(h(R))(R'/Q')$ is primary for $M(R'/Q')$ and hence $(x_1, \ldots, x_n)(R'/Q')$ is primary for $M(R'/Q')$. Therefore $\dim(R'/Q') \leqslant n$.

(4.2). *Let R and R' be regular local domains such that* $\dim R' = \dim R$, *R' dominates R, and $M(R)R'$ is primary for $M(R')$. Let $S' \in \mathfrak{V}(R')$. Then we have the following.*

(4.2.1). $\dim S' \geqslant \dim R_{R \cap M(S')}$.

(4.2.2). *Assume that* $\dim S' = \dim R_{R \cap M(S')}$. *Then $(R \cap M(S'))S'$ is primary for $M(S')$.*

(4.2.3). *Assume that* $\dim S' = \dim R_{R \cap M(S')}$. *Then:* $(R \cap M(S'))S' = M(S') \Leftrightarrow S' \notin \mathfrak{S}(R', (R \cap M(S'))R')$.

PROOF. (4.2.1) follows from (4.1) in view of the fact that for any regular local domain R and any $S \in \mathfrak{V}(R)$ we have that $\dim S + \dim(R/(R \cap M(S))) = \dim R$ (see [18: (34.5)]). Now assume that $\dim S' = \dim R_{R \cap M(S')}$. Suppose if possible that there exists a prime ideal Q' in R' such that $(R \cap M(S'))R' \subset Q' \subset R' \cap M(S')$ and $Q' \neq R' \cap M(S')$; now $R \cap Q' = R \cap M(S')$ and hence by (4.2.1) we get that $\dim R'_{Q'} \geqslant \dim R_{R \cap M(S')}$; consequently

$\dim R'_{Q'} \geqslant \dim S'$; since $Q' \subset R' \cap M(S')$, we must have $Q' = R' \cap M(S')$; this is contradiction. Therefore $R' \cap M(S')$ is a minimal prime ideal of $(R \cap M(S')R'$ in R', and hence $(R \cap M(S'))S'$ is primary for $M(S')$. This proves (4.2.2). (4.2.3) follows from (4.2.2).

(4.3). REMARK. In (4.3) we shall make some observations which will be used tacitly in the rest of §4. Let R and R' be regular local domains such that $\dim R' = \dim R$, R' dominates R, and $M(R)R' = M(R')$. Then for any nonempty subset Q of R we clearly have that $\operatorname{ord}_{R'}Q = \operatorname{ord}_R Q$. Let S be any element in $\mathfrak{V}(R)$ having a simple point at R. Then clearly $(R \cap M(S))R'$ is a prime ideal in R' and upon letting S' be the quotient ring of R' with respect to $(R \cap M(S))R'$ we get that S' has a simple point at R' and $\dim S' = \dim S$. By (4.2.1) it follows that S' dominates S, and S' is the only element in $\mathfrak{V}(R')$ such that: $\dim S' = \dim S$ and S' dominates S. Since $M(S)S' = M(S')$, for any nonzero principal ideal J in R we have that: $S \in \mathfrak{C}(R, J) \Leftrightarrow S' \in \mathfrak{C}(R', JR')$. Finally note that if $S_1, \ldots, S_n$ are any distinct elements in $\mathfrak{V}(R)$ such that $\{S_1, \ldots, S_n\}$ has a normal crossing at R then, upon letting S'_i be the unique element in $\mathfrak{V}(R')$ such that $\dim S'_i = \dim S_i$ and S'_i dominates S_i, we get that $S'_1, \ldots, S'_n$ are distinct elements in $\mathfrak{V}(R')$ and $\{S'_1, \ldots, S'_n\}$ has a normal crossing at R'.

(4.4). *Let R and R' be regular local domains such that* $\dim R' = \dim R$, *R' dominates R, and $M(R)R' = M(R')$. Assume that* (1) *for every $S' \in \mathfrak{V}(R')$ we have that $S' \notin \mathfrak{S}(R', (R \cap M(S'))R')$. Then for every nonzero principal ideal J in R we have the following.*

(4.4.1). *Assume that (R', JR') is resolved. Then (R, J) is resolved.*

(4.4.2). *Assume that (R, J) is unresolved and let S' be any element in $\mathfrak{C}^2(R', JR')$. Then $R_{R \cap M(S')} \in \mathfrak{C}^2(R, J)$.*

(4.4.3). *Assume that $\mathfrak{C}^2(R, J)$ has a strict normal crossing at R, let $S_1, \ldots, S_n$ $(0 \leqslant n \leqslant 2)$ be the distinct elements in $\mathfrak{C}^2(R, J)$, and let S'_i be the unique element in $\mathfrak{V}(R')$ such that* $\dim S'_i = 2$ *and S'_i dominates S_i. Then $S'_1, \ldots, S'_n$ are distinct, $\mathfrak{C}^2(R', JR') = \{S'_1, \ldots, S'_n\}$, and $\mathfrak{C}^2(R', JR')$ has a strict normal crossing at R'.*

PROOF OF (4.4.1). We have nothing to show if $J = R$. So assume that $J \neq R$. Let $d = \mathrm{ord}_R J$. Then $JR' = x^d R'$ where $0 \neq x \in R'$ such that $\mathrm{ord}_{R'} x = 1$. Now $J = P_1^{u_1} \dots P_n^{u_n}$ where $n, u_1, \dots, u_n$ are positive integers and $P_1, \dots, P_n$ are distinct nonzero principal prime ideals in R. Let $e_i = \mathrm{ord}_R P_i$. Then $P_i R' = x^{e_i} R'$ and in particular $P_i \subset R \cap xR'$ for $1 \leqslant i \leqslant n$. Now $R \cap xR'$ is a prime ideal in R and by (4.2.1) we know that $\dim R_{R \cap xR'} \leqslant 1$; since $P_i \subset R \cap xR'$ for $1 \leqslant i \leqslant n$, we get that $n = 1$ and $P_1 = R \cap xR'$. By (1) we get that $R'_{xR'} \notin \mathfrak{S}(R', P_1 R')$ and hence $e_1 = 1$. Therefore (R, J) is resolved.

PROOF OF (4.4.2). Note that S' and $R_{R \cap M(S')}$ are regular by [18: (28.3)], and clearly S' dominates $R_{R \cap M(S')}$. Since (R, J) is unresolved, we have that $J \neq R$ and hence $J = P_1^{u_1} \dots P_n^{u_n}$ where $n, u_1, \dots, u_n$ are positive integers and $P_1, \dots, P_n$ are distinct nonzero principal prime ideals in R. Clearly $S' \in \mathfrak{C}(R', P_i R')$ for $1 \leqslant i \leqslant n$; consequently $P_i R' \subset M(S')$ and hence $P_i \subset R \cap M(S')$ for $1 \leqslant i \leqslant n$. Suppose if possible that $P_1 = R \cap M(S')$; then we must have $n = 1$; since (R, J) is unresolved, we get that $\mathrm{ord}_R P_1 > 1$ and hence $\mathrm{ord}_{R'} P_1 R' > 1$; since $S' \in \mathfrak{C}(R', P_1 R')$, we deduce that $S' \in \mathfrak{S}(R', P_1 R')$; this contradicts (1). Therefore $P_1 \neq R \cap M(S')$; since $P_1 \subset R \cap M(S')$, we get that $\dim R_{R \cap M(S')} \geqslant 2$ and hence by (4.2.1) we get that $\dim R_{R \cap M(S')} = 2$. Therefore in view of (1), by (4.2.3) we get that $(R \cap M(S'))S' = M(S')$ and hence $M(R_{R \cap M(S')})S' = M(S')$. Consequently $R_{R \cap M(S')} \in \mathfrak{C}^2(R, J)$.

PROOF OF (4.4.3). If $\dim R < 3$ then our assertion is trivial. If $\dim R \geqslant 3$ then (R, J) is unresolved and hence our assertion follows from (4.4.2).

(4.5). Let A be the formal power series ring $k[[X_1, \dots, X_n]]$ in indeterminates $X_1, \dots, X_n$ over a field k. Upon letting

$$D_j^* f = \sum i_j f_{i_1 \dots i_n} X_1^{i_1} \cdots X_{j-1}^{i_{j-1}} X_j^{i_j - 1} X_{j+1}^{i_{j+1}} \cdots X_n^{i_n}$$

for every

$$f = \sum f_{i_1 \dots i_n} X_1^{i_1} \cdots X_n^{i_n} \in A \qquad (f_{i_1 \dots i_n} \in k),$$

we get a derivation D_j^* of A; let $\mathfrak{D}^*$ denote the set of these n derivations $D_1^*, ..., D_n^*$ of A; for any derivation D of k we get a derivation D^{**} of A by taking

$$D^{**}f = \sum (Df_{i_1 \cdots i_n}) X_1^{i_1} \cdots X_n^{i_n} ;$$

let $\mathfrak{D}^{**}$ denote the set of all these derivations D^{**} of A as D varies over the set of all derivations of k; let $\mathfrak{D} = \mathfrak{D}^*$ if k is of zero characteristic, and $\mathfrak{D} = \mathfrak{D}^* \cup \mathfrak{D}^{**}$ if k is of nonzero characteristic. Let k' be a separable algebraic extension of k, and let $A' = k'[[X_1, ..., X_n]]$ which is regarded as an overring of A. Let $\mathfrak{D}'$ be the set of derivations of A' obtained by replacing k by k' in the above definition of $\mathfrak{D}$. Then given any $D \in \mathfrak{D}$ there exists a unique $D' \in \mathfrak{D}'$ such that $D'f = Df$ for all $f \in A$; let H: $\mathfrak{D} \to \mathfrak{D}'$ be the map defined by taking $H(D) = D'$ for all $D \in \mathfrak{D}$. The following is contained in Nagata's Jacobian Criterion [18: (46.3)].

(4.5.1). *Let Q be an ideal in A, let $P^* \subset P$ be prime ideals in A such that $A_P \notin \mathfrak{S}(A, Q)$ and P^* is a minimal prime ideal of Q in A, and let $e =$ dim R_{P^*}. Then there exist elements $w_1, ..., w_e$ in Q and elements $D_1, ..., D_e$ in $\mathfrak{D}$ such that* $\det(D_i w_j)_{i,j=1,...,e} \notin P$ (for $e = 0$ we take the value of the determinant to be one).

The following is also implicitly contained in the proof of [18: (46.3)]; for the sake of completeness we shall prove it here.

(4.5.2). *Let P be a prime ideal in A, let $w_1, ..., w_e$ be elements in P, and let $D_1, ..., D_e$ be elements in $\mathfrak{D}$ such that* $\det(D_i w_j)_{i,j=1,...,e} \notin P$. *Then $A_P/(w_1, ..., w_e)A_P$ is regular. Moreover, if P^* is a prime ideal in A such that $(w_1, ..., w_e)A \subset P^* \subset P$ and* dim $A_{P^*} = e$ *then $(w_1, ..., w_e)A_P = P^* A_P$.*

PROOF. Now A_P is regular by [18: (28.3)] and hence it suffices to show that if $r_1, ..., r_e$ are any elements in A_P such that $r_1 w_1 + \cdots + r_e w_e \in M(A_P)^2$ then $r_j \in M(A_P)$ for all j. So let $r_1, ..., r_e$ be any elements in A_P such that $r_1 w_1 + \cdots + r_e w_e \in M(A_P)^2$. Then there exist elements $v_1, ..., v_b$ in P, elements u_{pq} in A, and an element t in A with $t \notin P$, such that upon letting $s_j = r_j t$ we have that $s_j \in A$ for all j and $s_1 w_1 + \cdots + s_e w_e = v$ where

$$v = \sum_{p,q=1,...,b} u_{pq} v_p v_q .$$

Now $D_i(s_1w_1 + \cdots + s_ew_e) = s_1(D_iw_1) + \cdots + s_e(D_iw_e) + (D_is_1)w_1 + \cdots + (D_is_e)w_e$, $(D_is_1)w_1 + \cdots + (D_is_e)w_e \in P$, and $D_iv \in P$. Therefore $s_1(D_iw_1) + \cdots + s_e(D_iw_e) \in P$ for $1 \leqslant i \leqslant e$; since $\det(D_iw_j)_{i,j=1,\dots,e} \notin P$, it follows that $s_j \in P$ for all j, and hence $r_j \in M(A_P)$ for all j.

From (4.5.1) and (4.5.2) we shall now deduce (4.5.3).

(4.5.3). *Let Q be any ideal in A. Then* $\mathfrak{S}(A', QA') \subset \{S' \in \mathfrak{V}(A')$: $A_{A\cap M(S')} \in \mathfrak{S}(A, Q)\}$.

PROOF. Let S' be any element in $\mathfrak{V}(A')$ such that upon letting $P = A \cap M(S')$ we have that $A_P \notin \mathfrak{S}(A, Q)$. We want to show that then $S' \notin \mathfrak{S}(A', QA')$. This is obvious if $Q \not\subset P$. So assume that $Q \subset P$. Since $A_P \notin \mathfrak{S}(A, Q)$, we get that QA_P is a prime ideal in A_P and hence $Q = P^* \cap Q_1 \cap \dots \cap Q_m$ where P^* is a minimal prime ideal of Q in A, $P^* \subset P$, and $Q_1, \dots, Q_m$ are primary ideals in A such that $Q_b \not\subset P$ for $1 \leqslant b \leqslant m$. Let $e = \dim A_{P^*}$. Since $A_P \notin \mathfrak{S}(A, Q)$, by (4.5.1) there exist elements $w_1, \dots, w_e$ in Q and elements $D_1, \dots, D_e$ in $\mathfrak{D}$ such that $\det(D_iw_j)_{i,j=1,\dots,e} \notin P$. By (4.5.2) we now get that $(w_1, \dots, w_e)A_P = P^*A_P$; consequently $(w_1, \dots, w_e)A_P = QA_P$ and hence $(w_1, \dots, w_e)S' = QS'$. Let $D_i' = H(D_i)$. Then $D_1', \dots, D_e'$ are elements in $\mathfrak{D}'$ and $\det(D_i'w_j)_{i,j=1,\dots,e} \notin A' \cap M(S')$; consequently by (4.5.2) we get that $S'/(w_1, \dots, w_e)S'$ is regular, and hence $S' \notin \mathfrak{S}(A', QA')$.

(4.6). *Let R and R' be regular local domains such that* $\dim R' = \dim R$, *R' dominates R, R' is residually separable algebraic over R, $M(R)R' = M(R')$, and the characteristic of $R/M(R)$ is the same as the characteristic of R. Let R^* and R'^* be the completions of R and R' respectively. Assume that:* (1) *for every ideal Q in R we have that* $\mathfrak{S}(R^*, QR^*) = \{S \in \mathfrak{V}(R^*)$: $R_{R\cap M(S)} \in \mathfrak{S}(R, Q)\}$; *and* (1′) *for every ideal Q' in R' we have that* $\mathfrak{S}(R'^*, Q'R'^*) = \{S' \in \mathfrak{V}(R'^*)$: $R'_{R'\cap M(S')} \in \mathfrak{S}(R', Q')\}$ (see (1.2.6)). *Then we have the following.*

(4.6.1). *If Q is any ideal in R then* $\mathfrak{S}(R', QR') \subset \{S' \in \mathfrak{V}(R')$: $R_{R\cap M(S')} \in \mathfrak{S}(R, Q)\}$.

(4.6.2). *If J is any nonzero principal ideal in R such that (R, J) is unresolved then (R', JR') is unresolved.*

(4.6.3). *If J is any nonzero principal ideal in R such that (R, J) is unresolved and S' is any element in $\mathfrak{E}^2(R', JR')$ then $R_{R \cap M(S')} \in \mathfrak{E}^2(R, J)$.*

(4.6.4). *If J is any nonzero principal ideal in R such that $\mathfrak{E}^2(R, J)$ has a strict normal crossing at R then $\mathfrak{E}^2(R', JR')$ has a strict normal crossing at R' and, upon letting $S_1, \ldots, S_n$ $(0 \leqslant n \leqslant 2)$ be the distinct elements in $\mathfrak{E}^2(R, J)$ and S_i' the unique element in $\mathfrak{V}(R')$ such that* $\dim S_i' = 2$ *and S_i' dominates S_i, we have that $S_1', \ldots, S_n'$ are distinct and $\mathfrak{E}^2(R', JR') = \{S_1', \ldots, S_n'\}$.*

PROOF. For any $S' \in \mathfrak{V}(R')$ we clearly have that $R_{R \cap M(S')} \notin \mathfrak{S}(R, R \cap M(S'))$, and hence (4.6.2), (4.6.3), and (4.6.4) would follow from (4.4) and (4.6.1). Therefore it suffices to prove (4.6.1). Let h: $R^* \to R'^*$ be the unique homomorphism such that $h(M(R^*)) \subset M(R'^*)$ and $h(u) = u$ for all $u \in R$. Now $\dim h(R^*) \leqslant \dim R^* = \dim R'^*$ and $M(h(R^*))R'^* = M(R'^*)$. Therefore we must have $\dim h(R^*) = \dim R^*$ and hence h is a monomorphism. Consequently we may identify R^* with a subring of R'^*. Let $n = \dim R$ and let $(x_1, \ldots, x_n)$ be a basis of $M(R)$. By Cohen's structure theorem [28: Theorem 27 on page 304] there exists a subfield k of R^* such that k is a coefficient set for R^*. By Zorn's lemma there exists a subfield k' of R'^* such that $k \subset k'$ and k' is not contained in any subfield of R'^* other than k'. Now R'^* is residually separable algebraic over R^* and hence k' is separable algebraic over k and by Hensel's lemma [28: Theorem 17 on page 279] it follows that k' is a coefficient set for R'^*. Therefore there exists a unique k'-isomorphism h': $R'^* \to k'[[X_1, \ldots, X_n]]$ such that $h'(x_i) = X_i$ for $1 \leqslant i \leqslant n$, where $k'[[X_1, \ldots, X_n]]$ is the formal power series ring in indeterminates $X_1, \ldots, X_n$ with coefficients in k'. Via h' let us identify R'^* with $k'[[X_1, \ldots, X_n]]$. Note that then R^* gets identified with $k[[X_1, \ldots, X_n]]$. To prove (4.6.1) let Q be any ideal in R and let S' be any element in $\mathfrak{S}(R', QR')$; since R'^* is the completion of R', we can find $S'^* \in \mathfrak{V}(R'^*)$ such that $R' \cap M(S'^*) = R' \cap M(S')$ (see [28: Corollary 1 on page 269]); then by (1′) we get that $S'^* \in \mathfrak{S}(R'^*, QR'^*)$; let S^* and S be the quotient rings of R^* and R with respect to $R^* \cap M(S'^*)$ and $R \cap M(S'^*)$ respectively; then by (4.5.3) we get that $S^* \in \mathfrak{S}(R^*, QR^*)$, and hence by (1) we get that $S \in \mathfrak{S}(R, Q)$, i.e., $R_{R \cap M(S')} \in \mathfrak{S}(R, Q)$.

(4.7). *Let k be a field, let k' be a separable algebraic extension of k, let $B = k[X_1, ..., X_n]$ and $B' = k'[X_1, ..., X_n]$ where $X_1, ..., X_n$ are indeterminates, and let Q be a prime ideal in B. Then there exists a prime ideal Q' in B' such that $Q' \cap B = Q$, and for any such Q' we have that $B'_{Q'}$ dominates B_Q, $B'_{Q'}$ is residually separable algebraic over B_Q, and $M(B_Q)B'_{Q'} = M(B'_{Q'})$.*

PROOF. Now B' is integral over B and hence there exists a prime ideal Q' in B' such that $Q' \cap B = Q$ (for instance see [4: Lemma 1.20]). Now let Q' be any such. Then $B'_{Q'}$ clearly dominates B_Q. Let $h: B'_{Q'} \rightarrow B'_{Q'}/M(B'_{Q'})$ be the canonical epimorphism. To prove that $B'_{Q'}$ is residually separable algebraic over B_Q and $M(B_Q)B'_{Q'} = M(B'_{Q'})$ it suffices to show that for any $x \in B'_{Q'}$ and $y \in M(B'_{Q'})$ we have that $h(x)$ is separable algebraic over $h(B_Q)$ and $y \in M(B_Q)B'_{Q'}$. So let any $x \in B'_{Q'}$ and $y \in M(B'_{Q'})$ be given. Then there exists a finite separable algebraic extension k^* of k contained in k' such that $x \in B^*_{Q^*}$ and $y \in M(B^*_{Q^*})$ where $B^* = k^*[X_1, ..., X_n]$ and $Q^* = Q' \cap B^*$. We can take a primitive element z of k^* over k and then upon letting $f(Z)$ be the minimal monic polynomial of z over k, where Z is an indeterminate, and upon letting d be the Z-discriminant of $f(Z)$ we have that $0 \neq d \in k$ and hence $d \notin Q$. Now B is integrally closed in its quotient field $k(X_1, ..., X_n)$, B^* is the integral closure of B in $k^*(X_1, ..., X_n)$, z is a primitive element of $k^*(X_1, ..., X_n)$ over $k(X_1, ..., X_n)$, and $f(Z)$ is the minimal monic polynomial of z over $k(X_1, ..., X_n)$. Therefore by Krull's Diskriminantsatz (see [4: Lemma 1.17, Lemma 1.28, and Theorem 1.44]) we get that $B^*_{Q^*}$ is residually separable algebraic over B_Q and $M(B_Q)B^*_{Q^*} = M(B^*_{Q^*})$. It follows that $h(x)$ is separable algebraic over $h(B_Q)$ and $y \in M(B_Q)B'_{Q'}$.

(4.8). *Let R and R' be regular local domains such that* $\dim R' = \dim R$, *R' dominates R, R' is residually separable algebraic over R, and $M(R)R' = M(R')$. Let S be a positive-dimensional element in $\mathfrak{V}(R)$ having a simple point at R, let R_1 be a monoidal transform of (R, S), and let S' be the unique element in $\mathfrak{V}(R')$ such that* $\dim S' = \dim S$ *and S' dominates S. Then there exists a monoidal transform R'_1 of (R', S') such that R'_1 dominates R_1. Moreover, for any such R'_1 we have the following:* (1) $\dim R'_1 = R_1$, *R'_1 is residually separable algebraic over R_1,* $M(R_1)R'_1 = M(R'_1)$; (2) *if J is any nonzero* *principal ideal in* R *and* J_1 *is the* (R, S, R_1)*-transform of* J, *then* J_1R_1' *is the* (R', S', R_1')*-transform of* JR'; *and* (3) *if* J *and* I *are any nonzero principal ideals in* R *and* (J_1, I_1) *is the* (R, S, R_1)*-transform of* (J, I), *then* (J_1R_1', I_1R_1') *is the* (R', S', R_1')*-transform of* (JR', IR').

PROOF. Let $n = \dim R$ and $m = \dim S$. Then there exists a basis $(x_1, \ldots, x_n)$ of $M(R)$ such that $R \cap M(S) = (x_1, \ldots, x_m)R$ and $x_i/x_1 \in R_1$ for $2 \leqslant i \leqslant m$. Now $(x_1, \ldots, x_n)$ is a basis of $M(R')$ and $R' \cap M(S') = (x_1, \ldots, x_m)R'$. If $m = 1$ then we have nothing to show. So henceforth assume that $m > 1$. Let $A = R[x_2/x_1, \ldots, x_m/x_1]$ and $A' = R'[x_2/x_1, \ldots, x_m/x_1]$. Let $h'\colon R' \to k'$ be an epimorphism such that $\operatorname{Ker} h' = M(R')$, and let $k = h'(R)$. Then k is a subfield of k', k' is separable algebraic over k, and upon letting $h(u) = h'(u)$ for all $u \in R$ we get an epimorphism $h\colon R \to k$ such that $\operatorname{Ker} h = M(R)$. Let $B = k[X_2, \ldots, X_m]$ and $B' = k'[X_2, \ldots, X_m]$ where $X_2, \ldots, X_m$ are indeterminates. Let $H'\colon A' \to B'$ be the unique epimorphism such that $H'(x_i/x_1) = X_i$ for $2 \leqslant i \leqslant m$ and $H'(u) = h'(u)$ for all $u \in R'$. Upon letting $H(u) = H'(u)$ for all $u \in A$ we get an epimorphism $H\colon A \to B$ such that $H(x_2/x_i) = X_i$ for $2 \leqslant i \leqslant m$ and $H(u) = h(u)$ for all $u \in R$. Now $\operatorname{Ker} H = M(R)A$ and $\operatorname{Ker} H' = M(R')A'$. Let $P = A \cap M(R_1)$. Then P is a prime ideal in A, $\operatorname{Ker} H \subset P$, and $R_1 = A_P$. Let $Q = H(P)$. Then Q is a prime ideal in B.

By (4.7) there exists a prime ideal Q' in B' such that $B \cap Q' = Q$ and then upon letting $P' = H'^{-1}(Q')$ and $R_1' = A'_{Q'}$ we get that R_1' is a monoidal transform of (R', S') and R_1' dominates R_1.

Conversely let R_1' be any monoidal transform of (R', S') such that R_1' dominates R_1. Then $x_i/x_1 \in R_1'$ for $2 \leqslant i \leqslant m$ and hence upon letting $P' = A' \cap M(R_1')$ we get that P' is a prime ideal in A', $\operatorname{Ker} H' \subset P'$, $R_1' = A'_{P'}$, and $A \cap P' = P$. Let $Q' = H'(P')$. Then Q' is a prime ideal in B' such that $B \cap Q' = Q$. Let $H'^*\colon R_1' \to B'_{Q'}$ be the unique epimorphism such that $H'^*(u) = H'(u)$ for all $u \in A'$. Then upon letting $H^*(u) = H'^*(u)$ for all $u \in R_1$ we get that $H^*\colon R_1 \to B_Q$ is an epimorphism and $H^*(u) = H(u)$ for all $u \in A$. Let $h^*\colon B_Q \to B_Q/M(B_Q)$ and $h'^*\colon B'_{Q'} \to B'_{Q'}/M(B'_{Q'})$ be the canonical epimorphisms, and let t be the transcendence degree of $h^*(B_Q)$ over $h^*(k)$. Then $\dim R_1 = n - t$. By (4.7), $B'_{Q'}$ is residually algebraic over B_Q and hence t is the transcendence degree of

$h'^*(B'_{Q'})$ over $h'^*(k')$; consequently $\dim R'_1 = n - t$ and hence $\dim R'_1 = \dim R_1$. Let $h'_1(u) = h'^*(H'^*(u))$ for all $u \in R'_1$. Then $h'_1\colon R'_1 \to B'_{Q'}/M(B'_{Q'})$ is an epimorphism, $\operatorname{Ker} h'_1 = M(R'_1)$, and $h'_1(R_1) = h'^*(B_Q)$; since, by (4.7), $B'_{Q'}/M(B'_{Q'})$ is a separable algebraic extension of $h'^*(B_Q)$, we get that R'_1 is residually separable algebraic over R_1. Now $H'(P)B'_{Q'} = QB'_{Q'} = M(B_Q)B'_{Q'}$ and by (4.7) we get that $M(B_Q)B'_{Q'} = M(B'_{Q'})$; consequently $H'(P)B'_{Q'} = M(B'_{Q'})$ and hence $PR'_1 + (x_1, x_{m+1}, \ldots, x_n)R'_1 = M(R'_1)$; also $H(P)B_Q = M(B_Q)$ and hence $PR_1 + (x_1, x_{m+1}, \ldots, x_n)R_1 = M(R_1)$; therefore $M(R_1)R'_1 = M(R'_1)$. Now let J be any given nonzero principal ideal in R, let J_1 be the (R, S, R_1)-transform of J and let J'_1 be the (R', S', R'_1)-transform of JR'; let $d = \operatorname{ord}_S J$; then $d = \operatorname{ord}_{S'} JR'$ and hence upon taking $w \in R$ with $wR = J$ we get that $J_1 = (w/x_1^d)R_1$ and $J'_1 = (w/x_1^d)R'_1$; therefore $J'_1 = J_1R'_1$. Finally, let I be any given nonzero principal ideal in R, let I_1 be the nonzero principal ideal in R' such that (J_1, I_1) is the (R, S, R_1)-transform of (J, I), and let I'_1 be the nonzero principal ideal in R'_1 such that (J'_1, I'_1) is the (R', S', R'_1)-transform of (JR', IR'); then $I_1 = x_1^d(IR_1)$ and $I'_1 = x_1^d((IR')R'_1)$, and hence $I'_1 = I_1R'_1$.

(4.9). *Let R and R' be regular local domains such that* $\dim R' = \dim R$, *R' dominates R, R' is residually separable algebraic over R, and $M(R)R' = M(R')$. Let S be a positive-dimensional element in $\mathfrak{V}(R)$ having a simple point at R. Let S' be the unique element in $\mathfrak{V}(R')$ such that* $\dim S' = \dim S$ *and S' dominates S* (note that then S' has a simple point at R'). *Let $(R_i, S_i)_{0 \leqslant i < n}$ be an infinite sequence such that either n is a positive integer or $n = \infty$, $(R_0, S_0) = (R, S)$, and for $0 < i < n$: R_i is a regular local domain; S_i is a positive-dimensional element in $\mathfrak{V}(R_i)$ having a simple point at R_i; and R_i is a monoidal transform of (R_{i-1}, S_{i-1}). Then we have the following.*

(4.9.1). *There exists a sequence $(R'_i, S'_i)_{0 \leqslant i < n}$ such that $(R'_0, S'_0) = (R', S')$ and for $0 < i < n$: R'_i is a regular local domain;* $\dim R'_i = \dim R_i$; *R'_i dominates R_i; R'_i is residually separable algebraic over R_i; $M(R_i)R'_i = M(R'_i)$; S'_i is the unique element in $\mathfrak{V}(R'_i)$ such that* $\dim S'_i = \dim S_i$ *and S'_i dominates S_i* (note that then S'_i has a simple point at R'_i); *and R'_i is a monoidal transform of (R'_{i-1}, S'_{i-1}).*

(4.9.2). *If $(R'_i, S'_i)_{0 \leqslant i < n}$ is any sequence as in* (4.9.1), *and J_i is any nonzero principal ideal in R_i for $0 \leqslant i < n$ such that (R_i, J_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, S_{i-1})$ for $0 < i < n$, then $(R'_i, J_iR'_i)$ is a monoidal transform of $(R'_{i-1}, J_{i-1}R'_{i-1}, S'_{i-1})$ for $0 < i < n$.*

(4.9.3). *If $(R'_i, S'_i)_{0 \leqslant i < n}$ is any sequence as in* (4.9.1), *and J_i and I_i are any nonzero principal ideals in R_i for $0 \leqslant i < n$ such that (R_i, J_i, I_i) is a monoidal transform of $(R_{i-1}, J_{i-1}, I_{i-1}, S_{i-1})$ for $0 < i < n$, then $(R'_i, J_iR'_i, I_iR'_i)$ is a monoidal transform of $(R'_{i-1}, J_{i-1}R'_{i-1}, I_{i-1}R'_{i-1}, S'_{i-1})$ for $0 < i < n$.*

PROOF. (4.9.2) and (4.9.3) follow from (4.8). To prove (4.9.1) let W be the set of all sequences $(R'_i, S'_i)_{0 \leqslant i < m}$ such that either m is a positive integer or $m = \infty$, $m \leqslant n$, $(R'_0, S'_0) = (R', S')$, and for $0 < i < m$: R'_i is a regular local domain; $\dim R'_i = \dim R_i$; R'_i dominates R_i; R'_i is residually separable algebraic over R_i; $M(R_i)R'_i = M(R'_i)$; S'_i is the unique element in $\mathfrak{B}(R'_i)$ such that $\dim S'_i = S_i$ and S'_i dominates S_i; and R'_i is a monoidal transform of (R'_{i-1}, S'_{i-1}). For each pair of elements $w = (R'_i, S'_i)_{0 \leqslant i < m}$ and $w^* = (R^*_i, S^*_i)_{0 \leqslant i < m^*}$ in W define: $w \leqslant w^* \Leftrightarrow m \leqslant m^*$ and $(R'_i, S'_i) = (R^*_i, S^*_i)$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property. Also we get an element $(R'_i, S'_i)_{0 \leqslant i < 1}$ in W by taking $(R'_0, S'_0) = (R', S')$. Therefore $W \neq \varnothing$, and hence by Zorn's lemma W contains a maximal element $(R'_i, S'_i)_{0 \leqslant i < m}$; in view of (4.8) we must have $m = n$.

(4.10). *Let S be an n-dimensional regular local domain, let $(x_1, \ldots, x_n)$ be a basis of $M(S)$, and let $f(Z)$ be a monic polynomial of degree $e > 1$ in an indeterminate Z with coefficients in S. Let $r \in S$ and let $s = tx_1^{a_1} \ldots x_n^{a_n}$ where t is a unit in S and $a_1, \ldots, a_n$ are nonnegative integers. Let $g(Z) = s^{-e}f(sZ + r)$. Assume that $g(Z) \in S[Z]$ and $0 < \operatorname{ord}_S g(Z) < e$. Then we have the following.*

(4.10.1). *Let $r' \in S$ and let $s' = t'x_1^{b}$ where $0 \neq t' \in S$ and b is a nonnegative integer. Assume that $s'^{-e}f(s'Z + r') \in S[Z]$. Then $b \leqslant a_1$ and $(r' - r)/x_1^b \in S$.*

(4.10.2). *Let $r' \in S$ and let $s' = t'x_1^{b_1} \ldots x_n^{b_n}$ where t' is a unit in S and $b_1, \ldots, b_n$ are nonnegative integers. Let $g'(Z) = s'^{-e}f(s'Z + r')$,*

$r^* = (r' - r)/s'$, $c_i = a_i - b_i$, $t^* = t/t'$, and $s^* = t^* x_1^{c_1} \dots x_n^{c_n}$. Note that then $c_1, \dots, c_n$ are integers, t^* is a unit in S, and $g(Z) = s^{*-e} g'(s^* Z - r^*)$. Assume that $g'(Z) \in S[Z]$. Then $r^* \in S$ and $c_i \geqslant 0$ for $1 \leqslant i \leqslant n$.

PROOF. (4.10.2) follows from (4.10.1). We shall now prove (4.10.1). Let $G(Z) = t^e g(t^{-1}Z)$. Then

(1) $$G(Z) = Z^e + G_1 Z^{e-1} + \cdots + G_e \quad \text{with} \quad G_i \in S.$$

Also $\mathrm{ord}_S G(Z) = \mathrm{ord}_S g(Z)$ and hence

(2) $$G_e \in M(S)$$

and there exists an integer d with $1 \leqslant d \leqslant e$ such that

(3) $$\mathrm{ord}_S G_d < d \quad \text{and} \quad \mathrm{ord}_S G_i \geqslant i \quad \text{whenever} \quad 1 \leqslant i < d.$$

Let $a = a_1$. Let $u = x_2^{a_2} \dots x_n^{a_n}$ if $n > 1$, and $u = 1$ if $n = 1$. Let $f'(Z) = s'^{-e} f(s'Z + r')$ and $f^*(Z) = t'^e f'(t'^{-1}Z)$. Then $f^*(Z) \in S[Z]$ and

(4) $$f^*(Z) = (u x_1^{a-b})^e G(u^{-1} x_1^{b-a} Z + (r' - r) u^{-1} x_1^{-a}).$$

Let $R = S_{x_1 S}$. By (1) and (4) we get that $f^*(0) x_1^{eb} = G^*$ where

$$G^* = (r' - r)^e + \sum_{i=1}^{e} G_i u^i x_1^{ia} (r' - r)^{e-i};$$

since $f^*(Z) \in S[Z]$, we get that $\mathrm{ord}_R f^*(0) x_1^{eb} \geqslant eb$; also, if $\mathrm{ord}_R(r' - r) < a$ then clearly $\mathrm{ord}_R(r' - r)^e = \mathrm{ord}_R G^*$; therefore we get that

(5) $$\text{if} \quad \mathrm{ord}_R(r' - r) < a \quad \text{then} \quad \mathrm{ord}_R(r' - r) \geqslant b.$$

By (5) we get that if $b \leqslant a$ then $\mathrm{ord}_R(r' - r) \geqslant b$, i.e., $(r' - r)/x_1^b \in S$. Hence it suffices to show that $b \leqslant a$. Suppose if possible that $b > a$. Then by (5) we get that $\mathrm{ord}_R(r' - r) \geqslant a$, i.e., $(r' - r)/x_1^a \in S$. Let $h\colon S \to S/x_1 S$ be the canonical epimorphism. Then there exist elements r^* and r'' in S such that $(r' - r)/x_1^a = r^* + r'' x_1$ and $\mathrm{ord}_S r^* = \mathrm{ord}_{h(S)} h((r' - r)/x_1^a)$; note that then $\mathrm{ord}_S r^* = \mathrm{ord}_{h(S)} h(r^*)$. Let

$$F'(Z) = x_1^{(b-a-1)e} f^*(x_1^{1+a-b} Z) \quad \text{and} \quad F(Z) = F'(Z - r'').$$

Since $f^*(Z) \in S[Z]$ and $b > a$, we get that $F'(Z) \in S[Z]$ and hence $F(Z) \in S[Z]$. Therefore

$$(6) \qquad F(Z) = Z^e + F_1 Z^{e-1} + \cdots + F_e \quad \text{with} \quad F_i \in S.$$

By (4) and the definition of $F'(Z)$ and $F(Z)$ we get that

$$(7) \qquad F(Z) = u^e x_1^{-e} G(u^{-1} x_1 Z + r^* u^{-1}).$$

By (1), (6), and (7) we get that

$$G_e u^e = G(0) u^e = x_1^e F(-r^* x_1^{-1}) = (-1)^e r^{*e} + \sum_{i=1}^{e} (-1)^{e-i} F_i x_1^i r^{*e-i}$$

and hence $\operatorname{ord}_{h(S)} h(r^{*e}) = \operatorname{ord}_{h(S)} h(G_e u^e)$; consequently in view of (2) we get that $\operatorname{ord}_{h(S)} h(r^*) > \operatorname{ord}_{h(S)} h(u)$; now $\operatorname{ord}_{h(S)} h(r^*) = \operatorname{ord}_S r^*$ and $\operatorname{ord}_{h(S)} h(u) \geqslant \operatorname{ord}_S u$; therefore

$$(8) \qquad \operatorname{ord}_S r^* > \operatorname{ord}_S u.$$

For $0 \leqslant i < j \leqslant e$ let W_{ij} be the elements in S such that

$$(Z + 1)^{e-i} = Z^{e-i} + \sum_{j=i+1}^{e} W_{ij} Z^{e-j}.$$

Then by (1), (6), and (7) we get that

$$(9) \qquad F_d x_1^d = G_d u^d + W_{0d} r^{*d} + \sum_{i=1}^{d-1} W_{id} G_i u^i r^{*d-i}.$$

Let $p = \operatorname{ord}_S F_d$ and $q = \operatorname{ord}_S G_d$. Then by (3), (8), and (9) we get (10) and (11):

$$(10) \qquad q < d \quad \text{and} \quad p + d = \operatorname{ord}_S F_d x_1^d = \operatorname{ord}_S G_d u^d;$$

$$(11) \qquad F_d x_1^d - G_d u^d \in M(S)^{p+d+1}.$$

By (10) it follows that $n > 1$; now $\operatorname{ord}_S u = a_2 + \cdots + a_n$ and hence by (10) we get that $p + d = q + d(a_2 + \cdots + a_n)$. Since $\operatorname{ord}_S F_d = p$, there exists a nonzero homogeneous polynomial $P(X_1, \ldots, X_n)$ of degree p in indeterminates $X_1, \ldots, X_n$ with coefficients in S at least one of which is not in $M(S)$ such that

$F_d = P(x_1, ..., x_n)$. Since $\text{ord}_S G_d = q$, there exists a nonzero homogeneous polynomial $Q(X_1, ..., X_n)$ of degree q in $X_1, ..., X_n$ with coefficients in S at least one of which is not in $M(S)$ such that $G_d = Q(x_1, ..., x_n)$. Let

$$A(X_1, ..., X_n) = X_1^d P(X_1, ..., X_n)$$

and

$$B(X_1, ..., X_n) = X_2^{da_2} \cdots X_n^{da_n} Q(X_1, ..., X_n).$$

Then $A(X_1, ..., X_n)$ and $B(X_1, ..., X_n)$ are nonzero homogeneous polynomials of degree $p + d$ in $X_1, ..., X_n$ with coefficients in S, and $F_d x_1^d = A(x_1, ..., x_n)$ and $G_d u^d = B(x_1, ..., x_n)$. Therefore by (11) we get that

$$A(x_1, ..., x_n) - B(x_1, ..., x_n) \in M(S)^{p+d+1}. \tag{12}$$

Let $h'\colon S \to S/M(S)$ be the canonical epimorphism. Let $P'(X_1, ..., X_n)$, $Q'(X_1, ..., X_n)$, $A'(X_1, ..., X_n)$, $B'(X_1, ..., X_n)$ be the homogeneous polynomials in $X_1, ..., X_n$ with coefficients in $h'(S)$ obtained by applying h' to the coefficients of $P(X_1, ..., X_n)$, $Q(X_1, ..., X_n)$, $A(X_1, ..., X_n)$, $B(X_1, ..., X_n)$ respectively. Then by (12) we get that

$$A'(X_1, ..., X_n) = B'(X_1, ..., X_n)$$

and hence

$$X_1^d P'(X_1, ..., X_n) = X_2^{da_2} \cdots X_n^{da_n} Q'(X_1, ..., X_n).$$

This is a contradiction because $q < d$ and $Q'(X_1, ..., X_n)$ is a nonzero homogeneous polynomial of degree q in $X_1, ..., X_n$ with coefficients in $h'(S)$.

(4.11). *Let S be an n-dimensional regular local domain and let R be an $(n + 1)$-dimensional regular local domain such that R dominates S, R is residually rational over S, and $M(R) = zR + M(S)R$ with $z \in R$. Then we have the following.*

(4.11.1). *$(zR + M(R)^b) \cap S = M(S)^b$ for every nonnegative integer b.*

(4.11.2). *If s is any element in S then* $\operatorname{ord}_S s = \operatorname{ord}_R s = \operatorname{ord}_{h(R)} h(s)$ *where h: $R \to R/zR$ is the canonical epimorphism.*

(4.11.3). *If $f(Z)$ is any polynomial in an indeterminate Z with coefficients in the quotient field of S such that $f(z) \in R$, then $f(Z) \in S[Z]$.*

PROOF. Let R^* and S^* be the completion of R and S respectively. Then there exists a unique homomorphism h': $S^* \to R^*$ such that $h'(M(S^*)) \subset M(R^*)$ and $h'(u) = u$ for all $u \in S$. Now $\dim h'(S^*) \leqslant n$, $\dim R^* = n + 1$, and $M(R^*) = zR^* + M(h'(S^*))R^*$. Therefore $\dim h'(S^*) = n$ and hence h' is a monomorphism. Consequently we may identify S^* with a subring of R^*. Note that then S^* is a subspace of R^*. We shall first show that: (1) *given any $r \in R^*$ there exists a unique sequence of elements $r_0, r_1, r_2, \ldots$ in S^* such that $r = r_0 + r_1 z + r_2 z^2 + \cdots$.* Let $\mathfrak{k}$ be a coefficient set for S^* and let $(x_1, \ldots, x_n)$ be a basis of $M(S^*)$. Then $\mathfrak{k}$ is a coefficient set for R^* and $(z, x_1, \ldots, x_n)$ is a basis of $M(R^*)$. Consequently, given any $r \in R^*$ there exist elements in $r_{ji_1 \ldots i_n}$ in $\mathfrak{k}$ such that

$$r = \sum r_{ji_1 \ldots i_n} z^j x_1^{i_1} \cdots x_n^{i_n}$$

where the sum is over all nonnegative integers $j, i_1, \ldots, i_n$. Upon letting

$$r_j = \sum r_{ji_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

where the sum is over all nonnegative integers $i_1, \ldots, i_n$ we get that $r_j \in S^*$ for $j = 0, 1, 2, \ldots$, and $r = r_0 + r_1 z + r_2 z^2 + \cdots$. To prove the uniqueness let $r_0', r_1', r_2', \ldots$ be any elements in S^* such that $r = r_0' + r_1' z + r_2' z^2 + \cdots$. Since $r_j' \in S^*$, there exist elements $r'_{ji_1 \ldots i_n}$ in $\mathfrak{k}$ such that

$$r_j' = \sum r'_{ji_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

where the sum is over all nonnegative integers $i_1, \ldots, i_n$. Then

$$\sum r_{ji_1 \ldots i_n} z^j x_1^{i_1} \cdots x_n^{i_n} = r = \sum r'_{ji_1 \ldots i_n} z^j x_1^{i_1} \cdots x_n^{i_n}$$

where the sums are over all nonnegative integers $j, i_1, \ldots, i_n$; since all the elements $r_{ji_1 \ldots i_n}$ and $r'_{ji_1 \ldots i_n}$ are in $\mathfrak{k}$ we get that $r_{ji_1 \ldots i_n} =$

$r'_{ji_1\ldots i_n}$ for all $j, i_1, \ldots, i_n$, and hence $r_j = r'_j$ for all j. This completes to proof of (1).

To prove (4.11.1) let b be any nonnegative integer; clearly $M(S)^b \subset (zR + M(R)^b) \cap S$ and hence it suffices to show that $(zR + M(R)^b) \cap S \subset M(S)^b$; so let any $s \in (zR + M(R)^b) \cap S$ be given; now $zR + M(R)^b \subset zR^* + ((x_1, \ldots, x_n)R^*)^b$ and hence

$$s = s'z + \sum_{i_1+\cdots+i_n=b} s'_{i_1\ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

where s' and $s'_{i_1\ldots i_n}$ are elements in R^*; by (1)

$$s' = \sum_{j=1}^{\infty} s'_j z^{j-1} \qquad \text{and} \qquad s'_{i_1\ldots i_n} = \sum_{j=0}^{\infty} s'_{ji_1\ldots i_n} z^j$$

with $s'_j \in S^*$ and $s'_{ji_1\ldots i_n} \in S^*$; upon letting

$$s_0 = \sum_{i_1+\cdots+i_n=b} s'_{0i_1\ldots i_n} x_1^{i_1} \cdots x_n^{i_n}$$

and

$$s_j = s'_j + \sum_{i_1+\cdots+i_n=b} s'_{ji_1\ldots i_n} x_1^{i_1} \cdots x_n^{i_n} \qquad \text{for } j > 0$$

we get that $s_j \in S^*$ for all $j \geqslant 0$ and $s = s_0 + s_1 z + s_2 z^2 + \cdots$; since $s \in S \subset S^*$, by the uniqueness part of (1) we get that $s = s_0$; clearly $s_0 \in M(S^*)^b$ and hence $s \in S \cap M(S^*)^b = M(S)^b$; thus $(zR + M(R)^b) \cap S \subset M(S)^b$. This completes the proof of (4.11.1). (4.11.2) follows from (4.11.1).

To prove (4.11.3) let $f(Z)$ be any polynomial in an indeterminate Z with coefficients in the quotient field of S such that $f(z) \in R$; then $f(Z) = (f_0/t) + (f_1/t)Z + \cdots + (f_e/t)Z^e$ where e is a nonnegative integer, t is a nonzero element in S, and $f_0, \ldots, f_e$ are elements in S; since $f(z) \in R$, by (1) there exist elements $g_0, g_1, g_2, \ldots$ in S^* such that $f(z) = g_0 + g_1 z + g_2 z^2 + \cdots$; now $f_0 + f_1 z + \cdots + f_e z^e = tf(z) = (tg_0) + (tg_1)z + (tg_2)z^2 + \cdots$ and hence by the uniqueness part of (1) we get that $f_j = tg_j$ for $0 \leqslant j \leqslant e$; thus $f_j \in (tS^*) \cap S = tS$ for $0 \leqslant j \leqslant e$ and hence $f_j/t \in S$ for $0 \leqslant j \leqslant e$, i.e., $f(Z) \in S[Z]$.

(4.12). *Let n be an integer with $n \geqslant 2$. Let R and R' be n-dimensional regular local domains such that R' dominates R and*

$M(R)R' = M(R')$. Let S be an $(n-1)$-dimensional element in $\mathfrak{B}(R)$ having a simple point at R. Let S' be the unique $(n-1)$-dimensional element in $\mathfrak{B}(R')$ such that S' dominates S. Let I be a nonzero principal ideal in R such that I has a quasinormal crossing at R. Let $I' = IR'$. Then we have the following.

(4.12.1). *Assume that (S', I') has a pseudonormal crossing at R'. Then (S, I) has a pseudonormal crossing at R.*

(4.12.2). *Assume that there exists a basis $(x_1, ..., x_n)$ of $M(R')$, a nonzero principal ideal L' in R', and nonnegative integers $q_1, ..., q_n$ such that: $R' \cap M(S') = (x_1, ..., x_{n-1})R'$, $I' = x_1^{q_1} ... x_n^{q_n} L'$, and* $\mathrm{ord}_{S'}L' = \mathrm{ord}_{R'}L'$. *Then (S, I) has a pseudonormal crossing at R.*

PROOF OF (4.12.1). Now there exists a basis $(y_1, ..., y_n)$ of $M(R)$ such that $R \cap M(S) = (y_1, ..., y_{n-1})R$. Note that then $(y_1, ..., y_n)$ is a basis of $M(R')$ and $R' \cap M(S') = (y_1, ..., y_{n-1})R'$. Since I has a quasinormal crossing at R, it suffices to show that if w is any element in R such that $\mathrm{ord}_R w = 1$ and $I \subset wR$ then (S, wR) has a pseudonormal crossing at R. First suppose that $w \in R \cap M(S)$; then $w = r_1 y_1 + \cdots + r_{n-1} y_{n-1}$ where $r_1, ..., r_{n-1}$ are elements in R such that $y_j \notin M(R)$ for some j; now $M(R) = (y_1, ..., y_{j-1}, w, y_{j+1}, ..., y_n)R$ and $R \cap M(S) = (y_1, ..., y_{j-1}, w, y_{j+1}, ..., y_{n-1})R$, and hence (S, wR) has a pseudonormal crossing at R. Next suppose that $w \notin R \cap M(S)$; now $\mathrm{ord}_{R'} w = 1$ and $I' \subset wR'$; since (S', I') has a pseudonormal crossing at R', there exists a basis of $(z_1, ..., z_n)$ of $M(R')$ such that $R' \cap M(S') = (z_1, ..., z_{n-1})R'$ and $wR' = z_i R'$ for some i with $1 \leqslant i \leqslant n$; since S' dominates S and $w \notin R \cap M(S)$, we must have $i = n$; therefore $w \notin (z_1, ..., z_{n-1})R' + M(R')^2$, and hence $w \notin (R \cap M(S)) + M(R)^2$; since $w \in M(R)$, we have that $w = s_1 y_1 + \cdots + s_n y_n$ with $s_1, ..., s_n$ in R; since $w \notin (R \cap M(S)) + M(R)^2$, we must have $s_n \notin M(R)$; consequently $M(R) = (y_1, ..., y_{n-1}, w)R$ and hence (S, wR) has a pseudonormal crossing at R.

PROOF OF (4.12.2). Since I has a quasinormal crossing at R, we have that I' has a quasinormal crossing at R'. In view of (4.12.1) it suffices to show that (S', I') has a pseudonormal crossing at R'. Clearly $(S', x_i R')$ has a pseudonormal crossing at R' for $1 \leqslant i \leqslant n$, and hence it suffices to show that if w is any element in R' such that $\operatorname{ord}_{R'}w = 1$ and $L' \subset wR'$ then (S', wR') has a pseudonormal crossing at R'. Since $\operatorname{ord}_{R'}L' = \operatorname{ord}_{S'}L'$, we must have $\operatorname{ord}_{S'}w = \operatorname{ord}_{R'}w = 1$. Therefore $w = r_1x_1 + \cdots + r_{n-1}x_{n-1}$ where $r_1, \ldots, r_{n-1}$ are elements in R' such that $r_j \notin M(R')$ for some j. Now

$$M(R') = (x_1, \ldots, x_{j-1}, w, x_{j+1}, \ldots, x_n)R'$$

and

$$R' \cap M(S') = (x_1, \ldots, x_{j-1}, w, x_{j+1}, \ldots, x_{n-1})R',$$

and hence (S', wR') has a pseudonormal crossing at R'.

(4.13). *Let R be an n-dimensional regular local domain with $n > 0$ such that $R/M(R)$ is infinite, let P be a nonzero principal ideal in R, and let $e = \operatorname{ord}_R P$. Then there exists a basis $(x_1, \ldots, x_n)$ of $M(R)$ such that $P \not\subset (x_1, \ldots, x_{n-1}, x_n^{e+1})R$.*

PROOF. Let $(y_1, \ldots, y_{n-1}, x_n)$ be any basis of $M(R)$, let $Z_1, \ldots, Z_n$ be indeterminates, and let $h\colon R \to R/M(R)$ be the canonical epimorphism. Then there exist nonzero homogeneous polynomials $f(Z_1, \ldots, Z_n)$ and $f'(Z_1, \ldots, Z_n)$ of degree e in $Z_1, \ldots, Z_n$ with coefficients in R and $R/M(R)$ respectively such that $P = f(y_1, \ldots, y_{n-1}, x_n)R$ and upon applying h to the coefficients of $f(Z_1, \ldots, Z_n)$ we get $f'(Z_1, \ldots, Z_n)$. Now $f'(Z_1, \ldots, Z_{n-1}, 1)$ is a nonzero polynomial in $Z_1, \ldots, Z_{n-1}$ with coefficients in the infinite field $R/M(R)$, and hence there exist elements $r_1, \ldots, r_{n-1}$ in R such that $f'(h(r_1), \ldots, h(r_{n-1}), 1) \neq 0$. Let $x_i = y_i - r_ix_n$ for $1 \leqslant i < n$. Then $y_i = x_i + r_ix_n$ for $1 \leqslant i < n$, and hence $(x_1, \ldots, x_n)$ is a basis of $M(R)$. Let

$$g(Z_1, \ldots, Z_n) = f(Z_1 + r_1Z_n, \ldots, Z_{n-1} + r_{n-1}Z_n, Z_n),$$

and

$$g'(Z_1, \ldots, Z_n) = f'(Z_1 + h(r_1)Z_n, \ldots, Z_{n-1} + h(r_{n-1})Z_n, Z_n).$$

Then $g(Z_1, \ldots, Z_n)$ and $g'(Z_1, \ldots, Z_n)$ are nonzero homogeneous polynomials of degree e in $Z_1, \ldots, Z_n$ with coefficients in R and $R/M(R)$ respectively, and upon applying h to the coefficients of $g(Z_1, \ldots, Z_n)$ we get $g'(Z_1, \ldots, Z_n)$. Now $g(x_1, \ldots, x_n) = f(y_1, \ldots,$

$y_{n-1}, x_n)$ and hence $P = g(x_1, ..., x_n)R$. Also $g'(0, ..., 0, 1) = f'(h(r_1), ..., h(r_{n-1}), 1)$ and hence $g'(0, ..., 0, 1) \neq 0$. Let h': $R \to R/(x_1, ..., x_{n-1})R$ be the canonical epimorphism. Then $h'(R)$ is a one-dimensional regular local domain, $M(h'(R)) = h'(x_n R)$, and $h'(x_n^e R) \not\subset h'(x_n^{e+1}R) = h'((x_1, ..., x_{n-1}, x_n^{e+1})R)$. Now $h'(P) = h'(g(x_1, ..., x_n)R) = h'(x_n^e R)$, and hence $P \not\subset (x_1, ..., x_{n-1}, x_n^{e+1})R$.

(4.14). *Let R be a three-dimensional regular local domain such that $R/M(R)$ is algebraically closed, let $\mathfrak{k}$ be a coefficient set for R, let P be a nonzero nonunit principal ideal in R, let $e = \operatorname{ord}_R P$, let (x, y, z) be a basis of $M(R)$ such that $P \not\subset (x, y, z^{e+1})R$, and let (R', P') be a monoidal transform of (R, P, R) such that $\dim R' = 3$ and $\operatorname{ord}_{R'} P' = e$. Then there exists a unique basis (x', y', z') of $M(R')$ such that: if $y/x \in R'$ then $x = x'$, $(y/x') - y' \in \mathfrak{k}$, and $(z/x') - z' \in \mathfrak{k}$; and if $y/x \notin R'$ then $x = y'x'$, $y = y'$, and $(z/y') - z' \in \mathfrak{k}$. Moreover, for any such basis (x', y', z') of $M(R')$ we have that $P' \not\subset (x', y', z'^{e+1})R'$.*

PROOF. Take $w \in R$ such that $wR = P$. Since $\operatorname{ord}_R P = e$, there exist elements t_{abc} in R such that

$$w = \sum_{a+b+c=e} t_{abc} x^a y^b z^c.$$

Let h: $R \to R/(x, y)R$ be the canonical epimorphism. Now $wR = P \not\subset (x, y, z^{e+1})R = h^{-1}(M(h(R))^{e+1})$ and hence $h(w) \notin M(h(R))^{e+1}$; if $t_{00e} \in M(R)$ then we would get that $h(w) \in M(h(R))^{e+1}$; therefore $t_{00e} \notin M(R)$ and hence $t_{00e} \notin M(R')$. Suppose if possible that $x/z \in M(R')$ and $y/z \in M(R')$; then $M(R') = (x/z, y/z, z)R'$ and $P' = (w/z^e)R'$; since $\operatorname{ord}_{R'} P' = e > 0$, we get that $w/z^e \in M(R')$; however,

$$w/z^e = \sum_{a+b+c=e} t_{abc}(x/z)^a (y/z)^b$$

and hence $w/z^e \notin M(R')$ because $t_{00e} \notin M(R')$; this is a contradiction. Therefore either $x/z \notin M(R')$ or $y/z \notin M(R')$. It follows that: if $y/x \in R'$ then $z/x \in R'$ and there exists a unique basis (x', y', z') of $M(R')$ such that $x = x'$, $(y/x') - y' \in \mathfrak{k}$, and $(z/x') - z' \in \mathfrak{k}$; and if $y/x \notin R'$ then $x/y \in M(R')$ and $z/y \in R'$ and there exists a unique basis (x', y', z') of $M(R')$ such that $x = y'x'$, $y = y'$, and $(z/y') - z' \in \mathfrak{k}$.

If $y/x \in R'$ then let $(x^*, y^*, z^*) = (x, xy', xz')$, and if $y/x \notin R'$ then let $(x^*, y^*, z^*) = (y, x, yz')$. Then $(x^*, y^*, z^*)R = M(R)$, $(x^*, y^*)R = (x, y)R$, $(x^*, y^*/x^*, z^*/x^*)R' = M(R')$, $z^*/x^* = z'$, $(x^*, y^*/x^*)R' = (x', y')R'$, and $M(R)R' = x^*R'$; also $(w/x^{*e})R' = P'$ and hence $w/x^{*e} \in M(R')^e$. Since $w \in M(R)^e$, there exist elements t^*_{abc} in R such that

$$w = \sum_{a+b+c=e} t^*_{abc} x^{*a} y^{*b} z^{*c} .$$

Now $wR = P \not\subset (x, y, z^{e+1})R = h^{-1}(M(h(R))^{e+1})$ and hence $h(w) \notin M(h(R))^{e+1}$; since $(x^*, y^*)R = (x, y)R$, we get that if $t^*_{00e} \in M(R)$ then $h(w) \in M(h(R))^{e+1}$; therefore $t^*_{00e} \notin M(R)$ and hence $t^*_{00e} \notin M(R')$. Now

$$w/x^{*e} = \sum_{a+b+c=e} t^*_{abc} (y^*/x^*)^b (z^*/x^*)^c.$$

Since $w/x^{*e} \in M(R')^e$, we must have $t^*_{abc} \in M(R')$ whenever $a \neq 0$, and hence $t^*_{abc} \in M(R)$ whenever $a \neq 0$; also $M(R)R' = x^*R'$ and hence $t^*_{abc} \in x^*R'$ whenever $a \neq 0$. Let h': $R' \to R'/(x^*, y^*/x^*)R'$ be the canonical epimorphism. Now $h'(R')$ is a one-dimensional regular local domain, $M(h'(R')) = h'((z^*/x^*)R')$, and $h'((z^*/x^*)^e R') \not\subset h'((z^*/x^*)^{e+1} R') = h'((x^*, y^*/x^*, (z^*/x^*)^{e+1})R')$; also $h'(t^*_{00e})$ is a unit in $h'(R')$ and hence $h'((w/x^{*e})R') = h'((z^*/x^*)^e R')$; since $P' = (w/x^{*e})R'$, we conclude that $P' \not\subset (x^*, y^*/x^*, (z^*/x^*)^{e+1})R'$; since $(x^*, y^*/x^*)R' = (x', y')R'$ and $z^*/x^* = z'$, we thus get that $P' \not\subset (x', y', z'^{e+1})R'$.

(4.15). *Let R_0 be a three-dimensional regular local domain such that $R_0/M(R_0)$ is algebraically closed, let $\mathfrak{k}$ be a coefficient set for R_0, let P_0 be a nonzero nonunit principal ideal in R_0, let $e = \operatorname{ord}_{R_0} P_0$, and let (x, y, z) be a basis of $M(R_0)$ such that $P_0 \not\subset (x, y, z^{e+1})R_0$. Let $(R_i, P_i)_{0<i<n}$ be a sequence such that: either n is a positive integer or $n = \infty$; R_i is a three-dimensional regular local domain and P_i is a nonzero principal ideal in R_i with $\operatorname{ord}_{R_i} P_i = e$ for $0 < i < n$; and (R_i, P_i) is a monoidal transform of $(R_{i-1}, P_{i-1}, R_{i-1})$ for $0 < i < n$. Then there exists a unique sequence $(x_i, y_i, z_i)_{0 \leqslant i<n}$ such that $(x_0, y_0, z_0) = (x, y, z)$ and for $0 < i < n$: (x_i, y_i, z_i) is a basis of $M(R_i)$; $P_i \not\subset (x_i, y_i, z_i^{e+1})R_i$; if $y_{i-1}/x_{i-1} \in R_i$ then*

$x_{i-1} = x_i$, $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$, *and* $(z_{i-1}/x_i) - z_i \in \mathfrak{k}$; *and if* $y_{i-1}/x_{i-1} \notin R_i$ *then* $x_{i-1} = y_i x_i$, $y_{i-1} = y_i$, *and* $(z_{i-1}/y_i) - z_i \in \mathfrak{k}$.

PROOF. Note that $R_i/M(R_i)$ is algebraically closed and $\mathfrak{k}$ is a coefficient set for R_i for $0 \leqslant i < n$. In view of (4.14), the uniqueness follows by an obvious induction. To prove the existence let W be the set of all sequences $(x_i, y_i, z_i)_{0 \leqslant i < m}$ such that either m is a positive integer or $m = \infty$, $m \leqslant n$, $(x_0, y_0, z_0) = (x, y, z)$, and for $0 < i < m$: (x_i, y_i, z_i) is a basis of $M(R_i)$; $P_i \not\subset (x_i, y_i, z_i^{e+1})R_i$; if $y_{i-1}/x_{i-1} \in R_i$ then $x_{i-1} = x_i$, $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$, and $(z_{i-1}/x_i) - z_i \in \mathfrak{k}$; and if $y_{i-1}/x_{i-1} \notin R_i$ then $x_{i-1} = y_i x_i$, $y_{i-1} = y_i$, and $(z_{i-1}/y_i) - z_i \in \mathfrak{k}$. For each pair of elements $w = (x_i, y_i, z_i)_{0 \leqslant i < m}$ and $w' = (x'_i, y'_i, z'_i)_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(x_i, y_i, z_i) = (x'_i, y'_i, z'_i)$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property. Also we get an element $(x_i, y_i, z_i)_{0 \leqslant i < 1}$ in W by taking $(x_0, y_0, z_0) = (x, y, z)$. Therefore $W \neq \varnothing$, and hence by Zorn's lemma W contains a maximal element $(x_i, y_i, z_i)_{0 \leqslant i < m}$; in view of (4.14) we must have $m = n$.

(4.16). *Let* $(R, J, I, L, P, x, y, z, p, q, c, d, e)$ *be a system such that: R is a three-dimensional regular local domain; $R/M(R)$ is algebraically closed; J, I, L, and P are nonzero principal ideals in R; (x, y, z) is a basis of $M(R)$; p, q, and c are nonnegative integers; d is a positive integer;* $e = c + d$; $\operatorname{ord}_R J = d$; $\operatorname{ord}_R L = c$; $I = x^p y^q L$; $P = JL$; *and* $P \not\subset (x, y, z^{e+1})R$. *Let m be a nonnegative integer. Let* $(R_i^*, J_i^*, I_i^*, P_i^*)_{m \leqslant i < \infty}$ *be an infinite sequence such that* $(R_m^*, J_m^*, I_m^*, P_m^*) = (R, J, I, P)$ *and for* $m < i < \infty$: R_i^* *is a three-dimensional regular local domain;* J_i^*, I_i^*, *and* P_i^* *are nonzero principal ideals in* R_i^*; $\operatorname{ord}_{R_i^*} J_i^* = d$; (R_i^*, J_i^*, I_i^*) *is a monoidal transform of* $(R_{i-1}^*, J_{i-1}^*, I_{i-1}^*, R_{i-1}^*)$; *and* (R_i^*, P_i^*) *is a monoidal transform of* $(R_{i-1}^*, P_{i-1}^*, R_{i-1}^*)$. *Then we have the following.*

(4.16.1). *Assume that* $\operatorname{ord}_{R_j^*} P_j^* \neq e$ *for some j with* $m < j < \infty$. *Also assume that I has a quasinormal crossing at R. Then there exists an integer $n > m$, a nonzero principal ideal L_n^* in R_n^*, a basis (x^*, y^*, z^*) of $M(R_n^*)$, and nonnegative integers p^*, q^*, and c^* such that:* $I_n^* = x^{*p^*} y^{*q^*} L_n^*$, $\operatorname{ord}_{R_n^*} L_n^* = c^* < c$, *and* $J_n^* L_n^* \not\subset (x^*, y^*, z^{*d+c^*+1})R_n^*$.

(4.16.2). *Assume that* $\operatorname{ord}_{R_i^*}P_i^* = e$ *for* $m < i < \infty$, R *is complete, and* $R/M(R)$ *has the same characteristic as* R. *Then there exists an integer* $m' \geqslant m$, *a field* $\mathfrak{k}$, *and an infinite sequence* $(R_i, J_i, I_i, L_i, S_i, f_i(Z), x_i, y_i, z_i, r_i, s_i, t_i, p_i, q_i, a_i, b_i)_{m' \leqslant i < \infty}$ *having the following description. For* $m' \leqslant i < \infty$: R_i *is a three-dimensional regular local domain;* $R_i/M(R_i)$ *is algebraically closed;* R_i *dominates* R_i^*; R_i *is residually rational over* R_i^*; $M(R_i) = M(R_i^*)R_i$; $J_i = J_i^*R_i$; $I_i = I_i^*R_i$; L_i *is a nonzero principal ideal in* R_i; $\operatorname{ord}_{R_i}L_i = c$; (x_i, y_i, z_i) *is a basis of* $M(R_i)$; p_i *and* q_i *are nonnegative integers; and* $I_i = x_i^{p_i}y_i^{q_i}L_i$. *For* $m' \leqslant i < \infty$: S_i *is a two-dimensional regular local domain;* $S_i/M(S_i)$ *is algebraically closed;* $\mathfrak{k}$ *is a subfield of* S_i; $\mathfrak{k}$ *is a coefficient set for* S_i; R_i *dominates* S_i; R_i *is residually rational over* S_i; (x_i, y_i) *is a basis of* $M(S_i)$; $r_i \in S_i$; t_i *is a unit in* S_i; a_i *and* b_i *are nonnegative integers;* $s_i = t_ix_i^{a_i}y_i^{b_i}$; $f_i(Z)$ *is a monic polynomial of degree* e *in an indeterminate* Z *with coefficients in* S_i; $J_iL_i = f_i(z_i)R_i$; $z_{m'} = s_iz_i + r_i$; *and* $f_i(Z) = s_i^{-e}f_{m'}(s_iZ + r_i)$. $S_{m'}$ *is isomorphic to the ring of formal power series in two indeterminates with coefficients in* $R/M(R)$. *For* $m' < i < \infty$: S_i *is a quadratic transform of* S_{i-1}; *if* $y_{i-1}/x_{i-1} \in S_i$ *then* $x_{i-1} = x_i$ *and* $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$; *and if* $y_{i-1}/x_{i-1} \notin S_i$ *then* $x_{i-1} = y_ix_i$ *and* $y_{i-1} = y_i$. *Finally,* $x_{i+1} \neq x_i$ *for infinitely many distinct values of* i *with* $m' \leqslant i < \infty$.

PROOF OF (4.16.1). Note that $R_i^*/M(R_i^*)$ is algebraically closed for $m \leqslant i < \infty$. Take any coefficient set $\mathfrak{k}$ for R. Then $\mathfrak{k}$ is a coefficient set for R_i^* for $m \leqslant i < \infty$. Now $\operatorname{ord}_{R_i^*}P_i^* \leqslant \operatorname{ord}_{R_a^*}P_a^*$ whenever $m \leqslant a \leqslant i < \infty$. Therefore there exists an integer $n > m$ such that $\operatorname{ord}_{R_i^*}P_i^* = e > \operatorname{ord}_{R_n^*}P_n^*$ for $m \leqslant i < n$. By (4.15) there exists a sequence $(x_i, y_i, z_i)_{m \leqslant i < n}$ such that $(x_0, y_0, z_0) = (x, y, z)$ and for $m < i < n$: (x_i, y_i, z_i) is a basis of $M(R_i^*)$; $P_i^* \not\subset (x_i, y_i, z_i^{e+1})R_i^*$; if $y_{i-1}/x_{i-1} \in R_i^*$ then $x_{i-1} = x_i$, $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$, and $(z_{i-1}/x_i) - z_i \in \mathfrak{k}$; and if $y_{i-1}/x_{i-1} \notin R_i^*$ then $x_{i-1} = y_ix_i$, $y_{i-1} = y_i$, and $(z_{i-1}/y_i) - z_i \in \mathfrak{k}$. Let $(L_i^*)_{m \leqslant i < \infty}$ be the unique sequence such that: L_i^* is a nonzero principal ideal in R_i^* for $m \leqslant i < \infty$; $L_m^* = L$; and (R_i^*, L_i^*) is a monoidal transform of $(R_{i-1}^*, L_{i-1}^*, R_{i-1}^*)$ for $m < i < \infty$. By induction on i we see that $P_i^* = J_i^*L_i^*$ for $m \leqslant i < \infty$, and hence upon letting $c^* = \operatorname{ord}_{R_n^*}L_n^*$ we get that $\operatorname{ord}_{R_i^*}L_i^* = c > c^*$ for $m \leqslant i < n$. Let $D_m = 1$, and for $m < i < n$ let: $D_i = x_i$ if $y_{i-1}/x_{i-1} \in R_i^*$, and

$D_i = y_i$ if $y_{i-1}/x_{i-1} \notin R_i^*$. Then $M(R_{i-1}^*)R_i^* = D_iR_i^*$ for $m < i < n$, and hence by induction on i we get that $I_i^* = x^py^qD_m^eD_{m+1}^e \dots D_i^eL_i^*$ for $m \leqslant i < n$. By induction on i we also get that $x^py^qD_m^eD_{m+1}^e \dots D_i^eR_i^* = x_i^{u_i}y_i^{v_i}R_i^*$ for $m \leqslant i < n$ where u_i and v_i are nonnegative integers. In particular, upon letting $u = u_{n-1}$, $v = v_{n-1}$, and $R^* = R_{n-1}^*$, we get that $I_{n-1}^* = x_{n-1}^uy_{n-1}^vL_{n-1}^*R^*$. It follows that $I_n^* = x_{n-1}^uy_{n-1}^v(M(R^*)R_n^*)^eL_n^*$. By (1.10.8) we get that I_i^* has a quasinormal crossing at R_i^* for $m \leqslant i < \infty$, and hence L_i^* has a quasinormal crossing at R_i^* for $m \leqslant i < \infty$. In particular $L_{n-1}^* = Q_1 \dots Q_c$ where $Q_1, \dots, Q_c$ are nonzero principal ideals in R^* with $\operatorname{ord}_{R^*}Q_b = 1$ for $1 \leqslant b \leqslant c$. Let $Q_1^*, \dots, Q_c^*$ be the nonzero principal ideals in R_n^* such that (R_n^*, Q_b^*) is a monoidal transform of (R^*, Q_b, R^*) for $1 \leqslant b \leqslant c$. Then $\operatorname{ord}_{R_n^*}Q_b^* \leqslant \operatorname{ord}_{R^*}Q_b$ for $1 \leqslant b \leqslant c$. Now $L_n^* = Q_1^* \dots Q_c^*$, and hence upon relabeling $Q_1, \dots, Q_c$ we may assume that $\operatorname{ord}_{R_n^*}Q_b^* = 1$ for $1 \leqslant b \leqslant c^*$, and $Q_b^* = R_n^*$ for $c^* < b \leqslant c$. Let $A = Q_{c^*+1} \dots Q_c$ and $B = Q_1 \dots Q_{c^*}J_{n-1}^*$. Then A and B are nonzero principal ideals in R^*, $\operatorname{ord}_{R^*}A = c - c^*$, $\operatorname{ord}_{R^*}B = d + c^*$, and $AB = P_{n-1}^*$; let h: $R^* \to R^*/(x_{n-1}, y_{n-1})R^*$ be the canonical epimorphism; then $h(R^*)$ is a one-dimensional regular local domain and $h((x_{n-1}, y_{n-1}, z_{n-1}^a)R^*) = M(h(R^*))^a$ for every nonnegative integer a; since $P_{n-1}^* \not\subset (x_{n-1}, y_{n-1}, z_{n-1}^{e+1})R^*$, we conclude that $A \not\subset (x_{n-1}, y_{n-1}, z_{n-1}^{c-c^*+1})R^*$ and $B \not\subset (x_{n-1}, y_{n-1}, z_{n-1}^{d+c^*+1})R^*$. Now $\operatorname{ord}_{R^*}B = d + c^* = \operatorname{ord}_{R_n^*}J_n^*L_n^*$ and $(R_n^*, J_n^*L_n^*)$ is a monoidal transform of (R^*, B, R^*); since $B \not\subset (x_{n-1}, y_{n-1}, z_{n-1}^{d+c^*+1})R^*$, by (4.14) there exists a basis (x^*, y^*, z^*) of $M(R_n^*)$ such that: $J_n^*L_n^* \not\subset (x^*, y^*, z^{*d+c^*+1})R_n^*$; if $y_{n-1}/x_{n-1} \in R_n^*$ then $x_{n-1} = x^*$, $(y_{n-1}/x^*) - y^* \in \mathfrak{k}$, and $(z_{n-1}/x^*) - z^* \in \mathfrak{k}$; and if $y_{n-1}/x_{n-1} \notin R_n^*$ then $x_{n-1} = y^*x^*$, $y_{n-1} = y^*$, and $(z_{n-1}/y^*) - z^* \in \mathfrak{k}$. Now $M(R^*)R_n^* = x^*R_n^*$ if $y_{n-1}/x_{n-1} \in R_n^*$, and $M(R^*)R_n^* = y^*R_n^*$ if $y_{n-1}/x_{n-1} \notin R_n^*$. It follows that $x_{n-1}^uy_{n-1}^v(M(R^*)R_n^*)^e = x^{*p^*}y^{*q^*}R_n^*$ where p^* and q^* are nonnegative integers. Since $I_n^* = x_{n-1}^uy_{n-1}^v(M(R^*)R_n^*)^eL_n^*$, we conclude that $I_n^* = x^{*p^*}y^{*q^*}L_n^*$.

PROOF OF (4.16.2). By Cohen's structure theorem [28: Theorem 27 on page 304] there exists a subfield $\mathfrak{k}$ of R such that $\mathfrak{k}$ is a coefficient set for R. Note that then $\mathfrak{k}$ is isomorphic to $R/M(R)$, and for $m < i < \infty$ we have that $R_i^*/M(R_i^*)$ is algebraically closed and $\mathfrak{k}$ is a coefficient set for R_i^*. By (4.15) there exists a sequence

$(x_i', y_i', z_i)_{m \leqslant i < \infty}$ such that $(x_0', y_0', z_0) = (x, y, z)$ and for $m < i < \infty$: (x_i', y_i', z_i) is a basis of $M(R_i^*)$; $P_i^* \not\subset (x_i', y_i', z_i^{e+1})R_i^*$; if $y_{i-1}'/x_{i-1}' \in R_i^*$ then $x_{i-1}' = x_i'$, $(y_{i-1}'/x_i') - y_i' \in \mathfrak{k}$, and $(z_{i-1}/x_i') - z_i \in \mathfrak{k}$; and if $y_{i-1}'/x_{i-1}' \notin R_i^*$ then $x_{i-1}' = y_i'x_i'$, $y_{i-1}' = y_i'$, and $(z_{i-1}/y_i') - z_i \in \mathfrak{k}$. Let $(L_i^*)_{m \leqslant i < \infty}$ be the unique sequence such that: L_i^* is a nonzero principal ideal in R_i^* for $m \leqslant i < \infty$; $L_m^* = L$; and (R_i^*, L_i^*) is a monoidal transform of $(R_{i-1}^*, L_{i-1}^*, R_{i-1}^*)$ for $m < i < \infty$. By induction on i we see that $P_i^* = J_i^* L_i^*$ for $m \leqslant i < \infty$, and hence $\operatorname{ord}_{R_i^*} L_i^* = c$ for $m \leqslant i < \infty$. Let $D_m = 1$, and for $m < i < \infty$ let: $D_i = x_i'$ if $y_{i-1}'/x_{i-1}' \in R_i^*$, and $D_i = y_i'$ if $y_{i-1}'/x_{i-1}' \notin R_i^*$. Then $M(R_{i-1}^*)R_i^* = D_i R_i^*$ for $m < i < \infty$, and hence by induction on i we get that $I_i^* = x^p y^q D_m^e D_{m+1}^e \dots D_i^e L_i^*$ for $m \leqslant i < \infty$. By induction on i we also get that $x^p y^q D_m^e D_{m+1}^e \dots D_i^e R_i^* = x_i'^{u_i} y_i'^{v_i} R_i^*$ for $m \leqslant i < \infty$ where u_i and v_i are nonnegative integers. We shall now prove the following:

(1) There exists an integer $m' \geqslant m$ and a sequence $(R_i, J_i, I_i, L_i, P_i, x_i, y_i, p_i, q_i)_{m' \leqslant i < \infty}$ having the following description. For $m' \leqslant i < \infty$: R_i is a three-dimensional regular local domain; $R_i/M(R_i)$ is algebraically closed; R_i dominates R_i^*; R_i is residually rational over R_i^*; $M(R_i) = M(R_i^*)R_i$; $J_i = J_i^* R_i^*$; $I_i = I_i^* R_i$; $L_i = L_i^* R_i$; $P_i = P_i^* R_i$; L_i is a nonzero principal ideal in R_i; $\operatorname{ord}_{R_i} L_i = c$; (x_i, y_i, z_i) is a basis of $M(R_i)$; $(x_i, y_i)R_i = (x_i', y_i')R_i$; p_i and q_i are nonnegative integers; and $I_i = x_i^{p_i} y_i^{q_i} L_i$. $R_{m'}$ is the completion of $R_{m'}^*$. For $m' < i < \infty$: (R_i, P_i) is a monoidal transform of $(R_{i-1}, P_{i-1}, R_{i-1})$; if $y_{i-1}/x_{i-1} \in R_i$ then $x_{i-1} = x_i$, $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$, and $(z_{i-1}/x_i) - z_i \in \mathfrak{k}$; and if $y_{i-1}/x_{i-1} \notin R_i$ then $x_{i-1} = y_i x_i$, $y_{i-1} = y_i$, and $(z_{i-1}/y_i) - z_i \in \mathfrak{k}$. Finally, $x_{i+1} \neq x_i$ for infinitely many distinct values of i with $m' \leqslant i < \infty$.

First suppose that $x_{i+1}' \neq x_i'$ for infinitely many distinct values of i with $m \leqslant i < \infty$. Let $m' = m$, let $R_{m'}$ be the completion of $R_{m'}^*$, and let $P_{m'} = P_{m'}^* R_{m'}$. By (4.9) there exists a sequence $(R_i, P_i)_{m' < i < \infty}$ such that for $m' < i < \infty$: R_i is a three-dimensional regular local domain; R_i dominates R_i^*; R_i is residually rational over R_i^*; $M(R_i) = M(R_i^*)R_i$; $P_i = P_i^* R_i$; and (R_i, P_i) is a monoidal transform of $(R_{i-1}, P_{i-1}, R_{i-1})$. It suffices to take $(J_i, I_i, L_i, x_i, y_i, p_i, q_i) = (J_i^* R_i, I_i^* R_i, L_i^* R_i, x_i', y_i', u_i, v_i)$ for $m' \leqslant i < \infty$.

Next suppose that there exists an integer $n \geqslant m$ such that

$y'_{i-1} = x'_{i-1}y'_i$ whenever $n < i < \infty$. Let $m' = n$, let $R_{m'}$ be the completion of $R^*_{m'}$, and let $P_{m'} = P^*_{m'}R_{m'}$. By (4.9) there exists a sequence $(R_i, P_i)_{m'<i<\infty}$ such that for $m' < i < \infty$: R_i is a three-dimensional regular local domain; R_i dominates R^*_i; R_i is residually rational over R^*_i; $M(R_i) = M(R^*_i)R_i$; $P_i = P^*_iR_i$; and (R_i, P_i) is a monoidal transform of $(R_{i-1}, P_{i-1}, R_{i-1})$. It suffices to take $(J_i, I_i, L_i, x_i, y_i, p_i, q_i) = (J^*_iR_i, I^*_iR_i, L^*_iR_i, y'_i, x'_i, v_i, u_i)$ for $m' \leqslant i < \infty$. Note that now actually $x_{i+1} \neq x_i$ whenever $m' \leqslant i < \infty$.

Finally suppose that $x'_{i+1} \neq x'_i$ for only finitely many distinct values of i with $m \leqslant i < \infty$, and there does not exist any integer $n \geqslant m$ such that $y'_{i-1} = x'_{i-1}y'_i$ whenever $n < i < \infty$. Then there exists an integer $m' > m$ such that $y'_{m'-1}R^*_{m'} = x'_{m'-1}R^*_{m'}$, and $y'_{i-1}/x'_{i-1} \in R^*_i$ whenever $m' \leqslant i < \infty$. It follows that $v_i = 0$ for $m' \leqslant i < \infty$. Let $R_{m'}$ be the completion of $R^*_{m'}$, and let $P_{m'} = P^*_{m'}R_{m'}$. By (4.9) there exists a sequence $(R_i, P_i)_{m'<i<\infty}$ such that for $m' < i < \infty$: R_i is a three-dimensional regular local domain; R_i dominates R^*_i; R_i is residually rational over R^*_i; $M(R_i) = M(R^*_i)R_i$; $P_i = P^*_iR_i$; and (R_i, P_i) is a monoidal transform of $(R_{i-1}, P_{i-1}, R_{i-1})$. Let $x = x'_{m'}$. Then $x'_i = x$ for $m' \leqslant i < \infty$, and upon letting $r^*_i = y'_ix^{-1} - y'_{i+1}$ we get that $r^*_i \in \mathfrak{k}$ for $m' \leqslant i < \infty$. Now $R_{m'}$ being complete, for $m' \leqslant i < \infty$ we get an element r'_i in $R_{m'}$ by setting: $r'_i = r^*_i + r^*_{i+1}x + r^*_{i+2}x^2 + \cdots$. Let $y = y'_{m'} - xr'_{m'}$. By induction on i we see that $y'_i - xr'_i = yx^{m'-i}$ for $m' \leqslant i < \infty$. Let $x_i = yx^{m'-i}$ and $y_i = x$ for $m' \leqslant i < \infty$. Then $(x_i, y_i)R_i = (x'_i, y'_i)R_i$ for $m' \leqslant i < \infty$, and hence (x_i, y_i, z_i) is a basis of $M(R_i)$ for $m' \leqslant i < \infty$. Also clearly $y_{i-1}/x_{i-1} \notin R_i$, $x_{i-1} = y_ix_i$, and $(z_{i-1}/y_i) - z_i \in \mathfrak{k}$ for $m' < i < \infty$. In particular, $x_{i+1} \neq x_i$ whenever $m' \leqslant i < \infty$. It suffices to take $(J_i, I_i, L_i, p_i, q_i) = (J^*_iR_i, I^*_iR_i, L^*_iR_i, 0, u_i)$ for $m' \leqslant i < \infty$.

This completes the proof of (1). Note that $P_i = J_iL_i$ and $\operatorname{ord}_{R_i}P_i = e$ for $m' \leqslant i < \infty$. Let T be the ring of formal power series in indeterminates X, Y, Z with coefficients in $\mathfrak{k}$, and let S be the ring of formal power series in X, Y with coefficients in $\mathfrak{k}$ where we regard S to be a subring of $\mathfrak{k}$. Then there exists a unique $\mathfrak{k}$-isomorphism h of T onto $R_{m'}$ such that $h(X) = x_{m'}$, $h(Y) = y_{m'}$, and $h(Z) = z_{m'}$. Let $S_{m'} = h(S)$. Then $S_{m'}$ is a two-dimensional regular local domain, $S_{m'}/M(S_{m'})$ is algebraically closed, $\mathfrak{k}$ is a subfield of $S_{m'}$, $\mathfrak{k}$ is a coefficient set for $S_{m'}$, $R_{m'}$ dominates $S_{m'}$, $R_{m'}$ is residually rational over $S_{m'}$, $(x_{m'}, y_{m'})$ is a basis of $M(S_{m'})$, and $S_{m'}$ is isomorphic to the ring of formal power series in two indeterminates with coefficients in $R/M(R)$. For $m' < i < \infty$ let S_i be the quotient ring of $S_{m'}[x_i, y_i]$ with respect to $(S_{m'}[x_i, y_i]) \cap M(R_i)$. By induction on i we see that for $m' < i < \infty$: S_i is a two-dimensional regular local domain; $S_i/M(S_i)$ is algebraically closed; $\mathfrak{k}$ is a subfield of S_i; $\mathfrak{k}$ is a coefficient set for S_i; R_i dominates S_i; R_i is residually rational over S_i; (x_i, y_i) is a basis of $M(S_i)$; S_i is a quadratic transform of S_{i-1}; if $y_{i-1}/x_{i-1} \in S_i$ then $x_{i-1} = x_i$ and $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$; and if $y_{i-1}/x_{i-1} \notin S_i$ then $x_{i-1} = y_i x_i$ and $y_{i-1} = y_i$. Since $R_{m'}$ is the completion of $R^*_{m'}$, $(x_{m'}, y_{m'})R_{m'} = (x'_{m'}, y'_{m'})R_{m'}$, and $P_{m'} = P^*_{m'}R_{m'}$, we get that $((x_{m'}, y_{m'}, z_{m'}^{e+1})R_{m'}) \cap R^*_{m'} = (x'_{m'}, y'_{m'}, z_{m'}^{e+1})R^*_{m'}$ and $P_{m'} \cap R^*_{m'} = P^*_{m'}$; since $P^*_{m'} \not\subset (x'_{m'}, y'_{m'}, z_{m'}^{e+1})R^*_{m'}$, we conclude that $P_{m'} \not\subset (x_{m'}, y_{m'}, z_{m'}^{e+1})R_{m'}$. Therefore upon letting $P' = h^{-1}(P_{m'})$ we get that P' is a nonzero principal ideal in T with $\operatorname{ord}_T P' = e$ and $P' \not\subset (X, Y, Z^{e+1})T$; consequently by the Weierstrass Preparation Theorem [28: Corollary 1 on page 145] there exists a monic polynomial $f(Z)$ of degree e in Z with coefficients in S such that $P' = f(Z)T$. Let $f_{m'}(Z)$ be the monic polynomial of degree e in Z with coefficients in $S_{m'}$ obtained by applying h to the coefficients of $f(Z)$. Then $P_{m'} = f_{m'}(z_{m'})R_{m'}$ and hence $J_{m'}L_{m'} = f_{m'}(z_{m'})R_{m'}$. Let $r_{m'} = 0 = a_{m'} = b_{m'}$ and $s_{m'} = 1 = t_{m'}$; then $z_{m'} = s_{m'}z_{m'} + r_{m'}$ and $f_{m'}(Z) = s_{m'}^{-e}f_{m'}(s_{m'}Z + r_{m'})$. For $m' < i < \infty$ we have that: if $y_{i-1}/x_{i-1} \in R_i$ then $M(R_{i-1})R_i = x_iR_i$, and if $y_{i-1}/x_{i-1} \notin R_i$ then $M(R_{i-1})R_i = y_iR_i$. Therefore by induction on i we get that for $m' < i < \infty$: there exists an element r_i in S_i, a unit t_i in S_i, and nonnegative integers a_i and b_i, such that upon letting $s_i = t_i x_i^{a_i} y_i^{b_i}$ we have that $z_{m'} = s_i z_i + r_i$ and $P_i = s_i^{-e} f_{m'}(z_{m'})R_i$. Let $f_i(Z) = s_i^{-e} f_{m'}(s_iZ + r_i)$ for $m' < i < \infty$. Then for $m' < i < \infty$ we have that: $f_i(Z)$ is a monic polynomial of degree e in Z with coefficients in the quotient field of S_i, and $P_i = f_i(z_i)R_i$; in particular $f_i(z_i) \in R_i$ and hence by (4.11.3) we get that $f_i(Z) \in S_i[Z]$; since $J_iL_i = P_i$, we also get that $J_iL_i = f_i(z_i)R_i$.

(4.17). *Let R be a three-dimensional regular local domain, let (x, y, z) be a basis of $M(R)$, let R' be a monoidal transform of $(R, R_{(x,z)R})$ such that $z/x \in M(R')$, let h: $R \to R/zR$ and h': $R' \to R'/(z/x)R'$ be the canonical epimorphisms, and let r be an* *element in R such that* $\operatorname{ord}_R r = \operatorname{ord}_{h(R)} h(r)$. *Then* $\operatorname{ord}_R r = \operatorname{ord}_{R'} r = \operatorname{ord}_{h'(R')} h'(r)$.

PROOF. Now R' is a three-dimensional regular local domain dominating R, $(x, y, z/x)R' = M(R')$, $h(R)$ and $h'(R')$ are two-dimensional regular local domains, and $(h(x), h(y))h(R) = M(h(R))$ and $(h'(x), h'(y))h'(R') = M(h'(R'))$. If $r = 0$ then we have nothing to show. So assume that $r \neq 0$. Let $e = \operatorname{ord}_R r$. Since R' dominates R, we get that $\operatorname{ord}_{R'} r \geqslant e$, and clearly $\operatorname{ord}_{h'(R')} h'(r) \geqslant \operatorname{ord}_{R'} r$. By assumption $\operatorname{ord}_{h(R)} h(r) = \operatorname{ord}_R r$, and hence

$$h(r) = \sum_{i+j=e} h(r_{ij}) h(x)^i h(y)^j$$

where r_{ij} are elements in R at least one of which is not in $M(R)$. Now

$$r = sz + \sum_{i+j=e} r_{ij} x^i y^j \qquad \text{with} \quad s \in R,$$

and hence

$$h'(r) = \sum_{i+j=e} h'(r_{ij}) h'(x)^i h'(y)^j .$$

Also $h'(r_{ij}) \in h'(R')$ for all (i, j), and $h'(r_{ij}) \notin M(h'(R'))$ for some (i, j). Therefore $\operatorname{ord}_{h'(R')} h'(r) = e$. It follows that $\operatorname{ord}_R r = \operatorname{ord}_{R'} r = \operatorname{ord}_{h'(R')} h'(r)$.

(4.18). *Let R be a three-dimensional regular local domain, let (x, y, z) be a basis of $M(R)$, let $h\colon R \to R/zR$ be the canonical epimorphism, and let $w = g_0 z^e + g_1 z^{e-1} + \cdots + g_e$ where e is a positive integer and $g_0, \ldots, g_e$ are elements in R such that* $\operatorname{ord}_R g_j = \operatorname{ord}_{h(R)} h(g_j)$ *for $0 \leqslant j \leqslant e$, and* $\operatorname{ord}_R g_{j'} < j'$ *for some j' with $0 \leqslant j' \leqslant e$. Then* $\operatorname{ord}_R w < e$.

PROOF. Let d be the greatest integer with $0 \leqslant d \leqslant e$ such that $\operatorname{ord}_R g_d < d$. Let $w' = g_0 z^d + g_1 z^{d-1} + \cdots + g_d$. Then $\operatorname{ord}_R(w - w' z^{e-d}) \geqslant e$. Also $\operatorname{ord}_R w' \leqslant \operatorname{ord}_{h(R)} h(w') = \operatorname{ord}_{h(R)} h(g_d) < d$, and hence $\operatorname{ord}_R w' z^{e-d} < e$. Therefore $\operatorname{ord}_R w < e$.

(4.19). *Let R be a three-dimensional regular local domain, let J be a nonzero nonunit principal ideal in R, let $d = \operatorname{ord}_R J$, let c be a* *nonnegative integer, let* $e = d + c$, *and let* I *be a nonzero principal ideal in* R *such that* I *has a quasinormal crossing at* R. *Assume that there exists a basis* (x, y, z) *of* $M(R)$, *a nonzero principal ideal* L *in* R, *elements* $w, g_1, \ldots, g_e$ *in* R, *and nonnegative integers* p *and* q *such that:* $\operatorname{ord}_R L = c$; $I = x^p y^q L$; $JL = wR$; $\operatorname{ord}_R g_j = \operatorname{ord}_{h(R)} h(g_j)$ *for* $1 \leqslant j \leqslant e$ *where* $h\colon R \to R/M(R)$ *is the canonical epimorphism;* $g_e \in M(R)$; $\operatorname{ord}_R g_{j'} < j'$ *for some* j' *with* $1 \leqslant j' \leqslant e$; *and*

$$w = z^e + \sum_{j=1}^{e} g_j x^j z^{e-j}.$$

Let $S = R_{(x,z)R}$, *let* (R', J', I') *be a monoidal transform of* (R, J, I, S) *such that* $\operatorname{ord}_{R'} J' = d$, *let* L' *be the* (R, S, R')*-transform of* L, *and let* $c^* = \operatorname{ord}_{R'} L'$. *Then* $\dim R' = 3$, R' *is residually rational over* R, $I' = x^{p+e} y^q L'$, $c^* < c$, *and there exists* $z^* \in R'$ *such that* $M(R') = (x, y, z^*)R'$ *and* $J'L' \not\subset (x, y, z^{*d+c^*+1})R'$.

PROOF. Now $\operatorname{ord}_R J = d$, $\operatorname{ord}_R L = c$, $e = d + c$, and clearly $\operatorname{ord}_S JL = \operatorname{ord}_S w \geqslant e$; therefore $\operatorname{ord}_S J = d$, $\operatorname{ord}_S L = c$, and $\operatorname{ord}_S JL = e$. Also clearly $J'L'$ is the (R, S, R')-transform of JL. Suppose if possible that $z/x \notin R'$; then $x/z \in M(R')$, $J'L' = (w/z^e)R'$, and $w/z^e = 1 + g_1(x/z) + \cdots + g_e(x/z)$; consequently $J'L' = R'$ and hence $J' = R'$; this is a contradiction because $\operatorname{ord}_{R'} J' = d > 0$. Therefore $z/x \in R'$. Consequently $J'L' = (w/x^e)R'$ and $I' = x^{p+e} y^q L'$. Let $\mathfrak{k}$ be a coefficient set for R. Suppose if possible that $c^* = c$; then $\operatorname{ord}_{R'} J'L' = e = \operatorname{ord}_R JL$; for $1 \leqslant j \leqslant e$ let r_j be the unique element in $\mathfrak{k}$ such that $g_j - r_j \in M(R)$; since $g_e \in M(R)$, we get that $r_e = 0$ and hence

$$w - \left(z^e + \sum_{j=1}^{e-1} r_j x^j z^{e-j}\right) \in M(R)^{e+1};$$

consequently by (3.10.1) we get that $\dim R' = 3$ and $M(R') = (x, y, z/x)R'$; let $h'\colon R' \to R'/(z/x)R'$ be the canonical epimorphism; then by (4.17) we get that $\operatorname{ord}_R g_j = \operatorname{ord}_{R'} g_j = \operatorname{ord}_{h'(R')} h'(g_j)$ for $1 \leqslant j \leqslant e$; it follows that $g_e \in M(R')$ and $\operatorname{ord}_{R'} g_{j'} < j'$; now

$$w/x^e = (z/x)^e + \sum_{j=1}^{e} g_j (z/x)^{e-j}$$

and hence by (4.18) we get that $\operatorname{ord}_{R'}(w/x^e) < e$; consequently $\operatorname{ord}_{R'} J'L' < e$ which is a contradiction. Therefore $c^* < c$. Since $\operatorname{ord}_{R'} J' = \operatorname{ord}_R J > 0$, by (3.10.2) we get that $\dim R' = 3$ and R' is residually rational over R; since $z/x \in R'$, there exists $t \in R$ such that $(z/x) - t \in M(R')$. Let $z' = z - tx$ and $z^* = z'/x$. Then $M(R) = (x, y, z')R$, $R \cap M(S) = (x, z')R$, and $M(R') = (x, y, z^*)R'$. Since I has a quasinormal crossing at R, we have that $L = L_1 \dots L_c$ where $L_1, \dots, L_c$ are nonzero principal ideals in R with $\operatorname{ord}_R L_j = 1$ for $1 \leqslant j \leqslant c$. Let L_j' be the (R, S, R')-transform of L_j for $1 \leqslant j \leqslant c$. Since $\operatorname{ord}_R L = \operatorname{ord}_S L$, we get that $\operatorname{ord}_S L_j = \operatorname{ord}_R L_j$ for $1 \leqslant j \leqslant c$. Therefore $\operatorname{ord}_{R'} L_j' \leqslant \operatorname{ord}_R L_j$ for $1 \leqslant j \leqslant c$. Now $L' = L_1' \dots L_c'$ and $\operatorname{ord}_{R'} L' = c^*$. Therefore upon relabeling $L_1, \dots, L_c$ we may assume that $\operatorname{ord}_{R'} L_j' = 1$ for $1 \leqslant j \leqslant c^*$, and $L_j' = R'$ for $c^* < j \leqslant c$. Let $Q = L_1 \dots L_{c^*} J$, and take $w' \in R$ such that $w'R = Q$. Then $\operatorname{ord}_R Q = d + c^* = \operatorname{ord}_S Q = \operatorname{ord}_{R'} J'L'$, $J'L'$ is the (R, S, R')-transform of Q, and $J'L' = (w'/x^{d+c^*})R'$. Since $\operatorname{ord}_S Q = d + c^*$, we have that

$$w' = \sum_{j=0}^{d+c^*} g_j' x^j z'^{d+c^*-j} \qquad \text{with} \quad g_j' \in R.$$

For $0 \leqslant j \leqslant d + c^*$ let r_j' be the unique element in $\mathfrak{k}$ such that $g_j' - r_j' \in M(R)$. Then

$$w' - \left(\sum_{j=0}^{d+c^*} r_j' x^j z'^{d+c^*-j} \right) \in M(R)^{d+c^*+1}.$$

Since $z'/x \in M(R')$, by (3.10.1) we now deduce that $r_0' \neq 0$, and $r_j' = 0$ for $1 \leqslant j \leqslant d + c^*$. Therefore g_0' is a unit in R', and for $1 \leqslant j \leqslant d + c^*$ we have that $g_j' = s_j x + s_j^* y + s_j' z'$ with s_j, s_j^*, s_j' in R. Therefore

$$w'/x^{d+c^*} = g_0' z^{*d+c^*} + w_1 x + w_2 y \qquad \text{with} \quad w_1 \in R' \quad \text{and} \quad w_2 \in R'.$$

Let $h^*\colon R' \to R'/(x, y)R'$ be the canonical epimorphism. Then $h^*(R')$ is a one-dimensional regular local domain, $M(h^*(R')) = h^*(z^* R')$, $h^*(z^{*d+c^*} R') \not\subset h^*(z^{*d+c^*+1} R') = h^*((x, y, z^{*d+c^*+1})R')$, $h^*(g_0') \notin M(h^*(R'))$, and $h^*(w'/x^{d+c^*}) = h^*(g_0') h^*(z^{*d+c^*})$; since $J'L' = (w'/x^{d+c^*})R'$, we conclude that $J'L' \not\subset (x, y, z^{*d+c^*+1})R'$.

(4.20). *Let R^{**} and R be three-dimensional regular local domains such that: R dominates R^{**}; R is residually separable algebraic over R^{**}; $M(R^{**})R = M(R)$; $R^{**}/M(R^{**})$ has the same characteristic as R^{**}; for every iterated monoidal transform T of R^{**} and every ideal Q in T we have that $\mathfrak{S}(T^*, QT^*) = \{S \in \mathfrak{V}(T^*)\colon T_{T \cap M(S)} \in \mathfrak{S}(T, Q)\}$ where T^* is the completion of T; and for every iterated monoidal transform T of R and every ideal Q in T we have that $\mathfrak{S}(T^*, QT^*) = \{S \in \mathfrak{V}(T^*)\colon T_{T \cap M(S)} \in \mathfrak{S}(T, Q)\}$ where T^* is the completion of T* (note that by (1.2.6) we know that if R is complete then the last condition is automatically satisfied). *Let V be a valuation ring of the quotient field of R^{**} such that V dominates R^{**}. Let J^{**} be a nonzero nonunit principal ideal in R^{**} such that $\mathfrak{C}^2(R^{**}, J^{**})$ has a strict normal crossing at R^{**}. Let I^{**} be a nonzero principal in R^{**} such that I^{**} has a quasinormal crossing at R^{**}. Let $d = \operatorname{ord}_{R^{**}} J^{**}$, let c be a nonnegative integer, and let $e = d + c$. Let $J = J^{**}R$ and $I = I^{**}R$. Assume that there exists a basis (x, y, z) of $M(R)$, a nonzero principal ideal L in R, elements $w, g_1, \ldots, g_e$ in R, and nonnegative integers p, q, a, b such that: $\operatorname{ord}_R L = c$; $I = x^p y^q L$; $JL = wR$; $\operatorname{ord}_R g_j = \operatorname{ord}_{h(R)} h(g_j)$ for $1 \leqslant j \leqslant e$ where $h\colon R \to R/zR$ is the canonical epimorphism; $g_e \in M(R)$; $\operatorname{ord}_R g_{j'} < j'$ for some j' with $1 \leqslant j' \leqslant e$; and*

$$w = z^e + \sum_{j=1}^{e} g_j x^{aj} y^{bj} z^{e-j}.$$

Then one of the following two conditions is satisfied.

(1) *There exists a finite weak resolver $[(R_i^*, J_i^*, I_i^*, S_i^*)_{0 \leqslant i < u}, (R_u^*, J_u^*, I_u^*)]$ such that: $(R_0^*, J_0^*, I_0^*) = (R^{**}, J^{**}, I^{**})$; $\dim S_i^* = 2$ for $0 \leqslant i < u$; $\dim R_i^* = 3$ and $\operatorname{ord}_{R_i^*} J_i^* = d > \operatorname{ord}_{R_u^*} J_u^*$ for $0 \leqslant i < u$; and V dominates R_u^*.*

(2) *There exists a finite weak resolver $[(R_i^*, J_i^*, I_i^*, S_i^*)_{0 \leqslant i < u}, (R_u^*, J_u^*, I_u^*)]$ and a system $(R^*, J^*, I^*, L^*, x^*, y^*, z^*, p^*, q^*, c^*)$ such that: $(R_0^*, J_0^*, I_0^*) = (R^{**}, J^{**}, I^{**})$; $\dim S_i^* = 2$ for $0 \leqslant i < u$; $\dim R_i^* = 3$ and $\operatorname{ord}_{R_i^*} J_i^* = d$ for $0 \leqslant i \leqslant u$; $\mathfrak{C}^2(R_u^*, J_u^*)$ has a strict normal crossing at R_u^*; V dominates R_u^*; R^* is a three-dimensional complete regular local domain; $R^*/M(R^*)$ is isomorphic to $R/M(R)$; R^* dominates R_u^*; R^* is residually separable algebraic over R_u^*; $M(R^*) = M(R_u^*)R^*$; $J^* = J_u^* R^*$; $I^* = I_u^* R^*$; L^* is a*

nonzero principal ideal in R^*; $\operatorname{ord}_{R^*}L^* = c^* < c$; (x^*, y^*, z^*) *is a basis of* $M(R^*)$; p^* *and* q^* *are nonnegative integers;* $I^* = x^{*p^*}y^{*q^*}L^*$; *and* $J^*L^* \not\subset (x^*, y^*, z^{*d+c^*+1})R^*$.

PROOF. In view of (4.18) we have that $a + b \geqslant 1$. We shall make induction on $a + b$.

First consider the case when $a + b = 1$. Upon relabeling x and y we may assume that $a = 1$ and $b = 0$. Let $(R_0^*, J_0^*, I_0^*) = (R^{**}, J^{**}, I^{**})$. Let $S = R_{(x,z)R}$. Then $\operatorname{ord}_S w \geqslant e = \operatorname{ord}_R JL = \operatorname{ord}_R w$, and hence $\operatorname{ord}_S w = e$ and $S \in \mathfrak{C}^2(R, JL)$; consequently $S \in \mathfrak{C}^2(R, L)$ and $S \in \mathfrak{C}^2(R, J)$. Since $S \in \mathfrak{C}^2(R, J)$, by (4.6.4) there exists $S_0^* \in \mathfrak{C}^2(R_0^*, J_0^*)$ such that S is the unique two-dimensional element in $\mathfrak{V}(R)$ dominating S_0^*. Let (R_1^*, J_1^*, I_1^*) be the monoidal transform of $(R_0^*, J_0^*, I_0^*, S_0^*)$ along V. By (4.12.2) we get that (S_0^*, I_0^*) has a pseudonormal crossing at R_0^*, and hence by (1.10.8) we get that I_1^* has a quasinormal crossing at R_1^*. Now $\operatorname{ord}_{R_1^*} J_1^* \leqslant d$. Hence if $\operatorname{ord}_{R_1^*} J_1^* \neq d$ then upon taking $u = 1$ we have that condition (1) is satisfied. So now assume that $\operatorname{ord}_{R_1^*} J_1^* = d$. Then by (3.11) we get that $\mathfrak{C}^2(R_1^*, J_1^*)$ has a strict normal crossing at R_1^*. By (4.8) there exists a regular local domain R' and nonzero principal ideals J' and I' in R' such that: $\dim R' = \dim R_1^*$; R' dominates R_1^*; R' is residually separable algebraic over R_1^*; $M(R') = M(R_1^*)R'$; $J' = J_1^*R'$; $I' = I_1^*R'$; and (R', J', I') is a monoidal transform of (R, J, I, S). Now $\operatorname{ord}_R J = \operatorname{ord}_{R_0^*} J_0^* = d = \operatorname{ord}_{R_1^*} J_1^* = \operatorname{ord}_{R'} J'$. Since I_0^* has a quasinormal crossing at R_0^*, we also have that I has a quasinormal crossing at R. Therefore by (4.19) there exists a basis (x^*, y^*, z^*) of $M(R')$, a nonzero principal ideal L' in R', and nonnegative integers p^*, q^*, c^* such that: $I' = x^{*p^*}y^{*q^*}L'$; $\operatorname{ord}_{R'} L' = c^* < c$; and $J'L' \not\subset (x^*, y^*, z^{*d+c^*+1})R'$. By (4.19) we also know that $\dim R' = 3$ and R' is residually rational over R. Let R^* be the completion of R' .Let $J^* = J'R^*$, $I^* = I'R^*$, and $L^* = L'R^*$. Then $I^* = x^{*p^*}y^{*q^*}L^*$, $\operatorname{ord}_{R^*} L^* = c^*$, and $J^*L^* \not\subset (x^*, y^*, z^{*d+c^*+1})R^*$. Upon taking $u = 1$ we thus have that condition (2) is satisfied.

Now let $a + b > 1$ and assume that the assertion is true for all values of $a + b$ smaller than the given one. Upon relabeling x and y we may assume that $a > 0$. Let $(R_0^*, J_0^*, I_0^*) = (R^{**}, J^{**}, I^{**})$. Let $S = R_{(x,z)R}$. Then $\operatorname{ord}_R w \geqslant e = \operatorname{ord}_R JL = \operatorname{ord}_S w$, and hence $\operatorname{ord}_S w = e$ and $S \in \mathfrak{C}^2(R, JL)$; consequently $S \in \mathfrak{C}^2(R, L)$ and

$S \in \mathfrak{E}^2(R, J)$. Since $S \in \mathfrak{E}^2(R, J)$, by (4.6.4) there exists $S_0^* \in \mathfrak{E}^2(R_0^*, J_0^*)$ such that S is the unique two-dimensional element in $\mathfrak{B}(R)$ dominating S_0^*. Let (R_1^*, J_1^*, I_1^*) be the monoidal transform of $(R_0^*, J_0^*, I_0^*, S_0^*)$ along V. By (4.12.2) we get that (S_0^*, I_0^*) has a pseudonormal crossing at R_0^*, and hence by (1.10.8) we get that I_0^* has a quasinormal crossing at R_1^*. Now $\operatorname{ord}_{R_1^*} J_1^* \leqslant d$. Hence if $\operatorname{ord}_{R_1^*} J_1^* \neq d$ then upon letting $u = 1$ we have that condition (1) is satisfied. So now assume that $\operatorname{ord}_{R_1^*} J_1^* = d$. Then by (3.11) we get that $\mathfrak{E}^2(R_1^*, J_1^*)$ has a strict normal crossing at R_1^*. By (4.8) there exists a regular local domain R' and nonzero principal ideals J' and I' in R' such that: $\dim R' = \dim R_1^*$; R' dominates R_1^*; R' is residually separable algebraic over R_1^*; $M(R') = M(R_1^*)R'$; $J' = J_1^* R'$; $I' = I_1^* R'$; and (R', J', I') is a monoidal transform of (R, J, I, S). Now $\operatorname{ord}_{R'} J' = \operatorname{ord}_{R_1^*} J_1^* = d > 0$ and hence in particular $J' \neq R'$. Let L' be the (R, S, R')-transform of L. Then $J'L'$ is the (R, S, R')-transform of JL. Suppose if possible that $x/z \in R'$; then $J'L' = (w/z^e)R'$; now

$$w/z^e = 1 + \sum_{j=1}^{e} g_j x^{aj-j} y^{bj} (x/z)^j$$

and hence $w/z^e \notin M(R')$ because $a + b > 1$; consequently $J'L' = R'$ and hence $J' = R'$; this is a contradiction. Therefore $z/x \in M(R')$. Consequently $\dim R' = 3$, R' is residually rational over R, $M(R') = (x, y, z/x)R'$, $J'L' = (w/x^e)R'$, and $I' = x^{p+e} y^q L'$. Now

$$w/x^e = (z/x)^e + \sum_{j=1}^{e} g_j x^{(a-1)j} y^{bj} (z/x)^{e-j}.$$

Let $h'\colon R' \to R'/(z/x)R'$. Then by (4.17) we get that $\operatorname{ord}_{R'} g_j = \operatorname{ord}_{h'(R')} h'(g_j)$ for $1 \leqslant j \leqslant e$, and $\operatorname{ord}_{R'} g_{j'} = \operatorname{ord}_R g_{j'} < j'$. Also $g_e \in M(R')$. Since $(a-1) + b < a + b$, by the induction hypothesis we conclude that one of the following two conditions is satisfied.

(1′) There exists a finite weak resolver $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < v}, (R_v, J_v, I_v)]$ such that: $(R_0, J_0, I_0) = (R_1^*, J_1^*, I_1^*)$; $\dim S_i = 2$ for $0 \leqslant i < v$; $\dim R_i = 3$ and $\operatorname{ord}_{R_i} J_i = d > \operatorname{ord}_{R_v} J_v$ for $0 \leqslant i < v$; and V dominates R_v.

(2′) There exists a finite weak resolver $[(R_i, J_i, I_i, S_i)_{0 \leqslant i < v}, (R_v, J_v, I_v)]$ and a system $(R^*, J^*, I^*, L^*, x^*, y^*, z^*, p^*, q^*, c^*)$

such that: $(R_0, J_0, I_0) = (R_1^*, J_1^*, I_1^*)$; $\dim S_i = 2$ for $0 \leqslant i < v$; $\dim R_i$ and $\operatorname{ord}_{R_i} J_i = d$ for $0 \leqslant i \leqslant v$; $\mathfrak{C}^2(R_v, J_v)$ has a strict normal crossing at R_v; V dominates R_v; R^* is a three-dimensional complete regular local domain; $R^*/M(R^*)$ is isomorphic to $R'/M(R')$; R^* dominates R_v; R^* is residually separable algebraic over R_v; $M(R^*) = M(R_v)R^*$; $J^* = J_vR^*$; $I^* = I_vR^*$; L^* is a nonzero principal ideal in R^*; $\operatorname{ord}_{R^*} L^* = c^* < c$; (x^*, y^*, z^*) is a basis of $M(R^*)$; p^* and q^* are nonnegative integers; $I^* = x^{*p^*}y^{*q^*}L^*$; and $J^*L^* \not\subset (x^*, y^*, z^{*d+c^*+1})R^*$.

If condition (1′) is satisfied then upon taking $u = v + 1$, $S_i^* = S_{i-1}$ for $1 \leqslant i < u$, and $(R_i^*, J_i^*, I_i^*) = (R_{i-1}, J_{i-1}, I_{i-1})$ for $2 \leqslant i \leqslant u$, we get that condition (1) is satisfied. If condition (2′) is satisfied then upon taking $u = v + 1$, $S_i^* = S_{i-1}$ for $1 \leqslant i < u$, and $(R_i^*, J_i^*, I_i^*) = (R_{i-1}, J_{i-1}, I_{i-1})$ for $2 \leqslant i \leqslant u$, we get that condition (2) is satisfied.

(4.21). *Let R'' be a three-dimensional regular local domain such that $R''/M(R'')$ is a perfect field having the same characteristic as R''. Let R' be an iterated monoidal transform of R'', let J' be a nonzero principal ideal in R' such that (R', J') is unresolved, let $d = \operatorname{ord}_{R'} J'$, let I' be a nonzero principal ideal in R' such that I' has a quasinormal crossing at R', and let V be a valuation ring of the quotient field of R'' such that V dominates R'. Consider the following four conditions where in the second condition c is an integer, and in the third and the fourth conditions c is a nonnegative integer.*

(1) *There exists a finite weak resolver $[(R_i', J_i', I_i', S_i')_{0 \leqslant i < m}, (R_m', J_m', I_m')]$ such that: $(R_0', J_0', I_0') = (R', J', I')$; $\operatorname{ord}_{R_i'} J_i' = d > \operatorname{ord}_{R_m'} J_m'$ for $0 \leqslant i < m$; and V dominates R_m'.*

(2_c) *There exists a finite weak resolver $[(R_i', J_i', I_i', S_i')_{0 \leqslant i < m}, (R_m', J_m', I_m')]$ and a system $(R, J, I, L, x, y, z, p, q, c')$ such that: $(R_0', J_0', I_0') = (R', J', I')$; $\dim R_i' = 3$ and $\operatorname{ord}_{R_i'} J_i' = d$ for $0 \leqslant i \leqslant m$; $\mathfrak{C}^2(R_m', J_m')$ has a strict normal crossing at R_m'; R is a three-dimensional complete regular local domain; $R/M(R)$ is algebraically closed; R dominates R_m'; R is residually separable algebraic over R_m'; $M(R) = M(R_m')R$; $J = J_m'R$; $I = I_m'R$; L is a nonzero principal ideal in R; $\operatorname{ord}_R L = c' \leqslant c$; (x, y, z) is a basis of $M(R)$; p and q are nonnegative integers; $I = x^py^qL$; and $JL \not\subset (x, y, z^{d+c'+1})R$.*

(3$_c$) *There exists a finite weak resolver* $[(R'_i, J'_i, I'_i, S'_i)_{0\leqslant i<m}, (R'_m, J'_m, I'_m)]$, *an integer* $m' \geqslant m$, *a field* $\mathfrak{k}$, *and infinite sequences* $(R'_i, J'_i, I'_i)_{m\leqslant i<\infty}$ *and* $(R_i, J_i, I_i, L_i, S_i, f_i(Z), x_i, y_i, z_i, r_i, s_i, t_i, p_i, q_i, a_i, b_i)_{m'\leqslant i<\infty}$ *having the following description.* $(R'_0, J'_0, I'_0) = (R', J', I')$; $\dim R'_i = 3$ *and* $\operatorname{ord}_{R'_i} J'_i = d$ *for* $0 \leqslant i \leqslant m$; R'_i *is a three-dimensional regular local domain and* J'_i *and* I'_i *are nonzero principal ideals in* R'_i *for* $m < i < \infty$; $\operatorname{ord}_{R'_i} J'_i = d$ *and* I'_i *has a quasinormal crossing at* R'_i *for* $m < i < \infty$; (R'_i, J'_i, I'_i) *is a monoidal transform of* $(R'_{i-1}, J'_{i-1}, I'_{i-1}, R'_{i-1})$ *for* $m < i < \infty$; $\mathfrak{C}^2(R'_i, J'_i)$ *has a strict normal crossing at* R'_i *for* $m \leqslant i < \infty$; *and* V *dominates* R'_i *for* $0 \leqslant i < \infty$. *For* $m' \leqslant i < \infty$: R_i *is a three-dimensional regular local domain;* $R_i/M(R_i)$ *is algebraically closed;* R_i *dominates* R'_i; R_i *is residually separable algebraic over* R'_i; $M(R_i) = M(R'_i)R_i$; $J_i = J'_i R_i$; $I_i = I'_i R_i$; L_i *is a nonzero principal ideal in* R_i; $\operatorname{ord}_{R_i} L_i = c$; (x_i, y_i, z_i) *is a basis of* $M(R_i)$; p_i *and* q_i *are nonnegative integers; and* $I_i = x_i^{p_i} y_i^{q_i} L_i$. *For* $m' \leqslant i < \infty$: S_i *is a two-dimensional regular local domain;* $S_i/M(S_i)$ *is algebraically closed;* $\mathfrak{k}$ *is a subfield of* S_i; $\mathfrak{k}$ *is a coefficient set for* S_i; R_i *dominates* S_i; R_i *is residually rational over* S_i; (x_i, y_i) *is a basis of* $M(S_i)$; $r_i \in S_i$; t_i *is a unit in* S_i; a_i *and* b_i *are nonnegative integers;* $s_i = t_i x_i^{a_i} y_i^{b_i}$; $f_i(Z)$ *is a monic polynomial of degree* e *in an indeterminate* Z *with coefficients in* S_i *where* $e = d + c$; $J_i L_i = f_i(z_i) R_i$; $z_{m'} = s_i z_i + r_i$; *and* $f_i(Z) = s_i^{-e} f_{m'}(s_i Z + r_i)$. $S_{m'}$ *is isomorphic to the ring of formal power series in two indeterminates with coefficients in an algebraic closure of* $R''/M(R'')$. *For* $m' < i < \infty$: S_i *is a quadratic transform of* S_{i-1}; *if* $y_{i-1}/x_{i-1} \in S_i$ *then* $x_{i-1} = x_i$ *and* $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$; *and if* $y_{i-1}/x_{i-1} \notin S_i$ *then* $x_{i-1} = y_i x_i$ *and* $y_{i-1} = y_i$. *Finally,* $x_{i+1} \neq x_i$ *for infinitely many distinct values of* i *with* $m' \leqslant i < \infty$.

(4$_c$) *There exists a finite weak resolver* $[(R'_i, J'_i, I'_i, S'_i)_{0\leqslant i<m}, (R'_m, J'_m, I'_m)]$, *an integer* $n \geqslant m$, *a sequence* $(R'_i, J'_i, I'_i)_{m<i\leqslant n}$, *and a system* $(R, J, I, L, x, y, z, w, p, q, a, b, e, g_1, \ldots, g_e)$ *such that:* $(R'_0, J'_0, I'_0) = (R', J', I')$; $\dim R'_i = 3$ *and* $\operatorname{ord}_{R'_i} J'_i = d$ *for* $0 \leqslant i \leqslant m$; R'_i *is a three-dimensional regular local domain and* J'_i *and* I'_i *are nonzero principal ideals in* R'_i *for* $m < i \leqslant n$; $\operatorname{ord}_{R'_i} J'_i = d$ *and* I'_i *has a quasinormal crossing at* R'_i *for* $m < i \leqslant n$; (R'_i, J'_i, I'_i) *is a monoidal transform of* $(R'_{i-1}, J'_{i-1}, I'_{i-1}, R'_{i-1})$ *for* $m < i \leqslant n$; $\mathfrak{C}^2(R'_n, J'_n)$ *has a strict normal crossing at* R'_n; V *dominates* R'_n; R *is a three-dimensional complete regular local domain;* $R/M(R)$ *is* *algebraically closed;* R *dominates* R'_n; R *is residually separable algebraic over* R'_n; $M(R) = M(R'_n)R$; $J = J'_nR$; $I = I'_nR$; L *is a nonzero principal ideal in* R; $\operatorname{ord}_R L = c$; $e = d + c$; (x, y, z) *is a basis of* $M(R)$; $w, g_1, \ldots, g_e$ *are elements in* R; p, q, a, b *are nonnegative integers;* $I = x^p y^q L$; $JL = wR$; $\operatorname{ord}_R g_j = \operatorname{ord}_{h(R)} h(g_j)$ *for* $1 \leqslant j \leqslant e$ *where* h: $R \to R/zR$ *is the canonical epimorphism;* $g_e \in M(R)$; $\operatorname{ord}_R g_{j'} < j'$ *for some* j' *with* $1 \leqslant j' \leqslant e$; *and*

$$w = z^e + \sum_{j=1}^{e} g_j x^{aj} y^{bj} z^{e-j} .$$

Also consider the following condition concerning $R''/M(R'')$.

$(*)$ *Let* S_0 *be the ring of formal power series in two indeterminates with coefficients in an algebraic closure of* $R''/M(R'')$. *Let* (x_0, y_0) *be any basis of* $M(S_0)$, *let* $\mathfrak{k}$ *be any coefficient set for* S_0, *let* e *be any positive integer, and let* $f(Z)$ *be any monic polynomial of degree* e *in an indeterminate* Z *with coefficients in* S_0. *Let* $(S_i, x_i, y_i)_{0<i<\infty}$ *be any infinite sequence such that for* $0 < i < \infty$: S_i *is a two-dimensional regular local domain;* S_i *is a quadratic transform of* S_{i-1}; (x_i, y_i) *is a basis of* $M(S_i)$; *if* $y_{i-1}/x_{i-1} \in S_i$ *then* $x_{i-1} = x_i$ *and* $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$; *and if* $y_{i-1}/x_{i-1} \notin S_i$ *then* $x_{i-1} = y_i x_i$ *and* $y_{i-1} = y_i$. *Assume that* $x_{i+1} \neq x_i$ *for infinitely many distinct values of* i. *Then there exists a nonnegative integer* n *and an element* r *in the completion* S^* *of* S_n *such that either:* $f(Z + r) = Z^e$, *or: there exist nonnegative integers* u *and* v *such that upon letting* $g(Z) = (x_n^u y_n^v)^{-e} f(x_n^u y_n^v Z + r)$ *we have that* $g(Z) \in S^*[Z]$ *and* $0 < \operatorname{ord}_{S^*} g(Z) < e$.

Then we have the following.

(4.21.1). *Assume that* R'' *is pseudogeometric, and for every iterated monoidal transform* T *of* R'' *and every nonzero principal prime ideal* P *in* T *we have that* $\mathfrak{S}(T, P)$ *is closed in* $\mathfrak{V}(T)$ (see (1.2.6)). *Then either condition* (1) *is satisfied, or condition* (2_c) *is satisfied for some nonnegative integer* c.

(4.21.2). *Assume that* R'' *is pseudogeometric, and let* c *be a nonnegative integer such that condition* (2_c) *is satisfied. Then either condition* (1) *is satisfied, or condition* (2_{c-1}) *is satisfied, or condition* (3_c) *is satisfied.*

(4.21.3). *Assume that condition* $(*)$ *is satisfied, and for every iterated monoidal transform T of R'' and every ideal Q in T we have that* $\mathfrak{S}(T^*, QT^*) = \{S \in \mathfrak{V}(T^*)\colon T_{T \cap M(S)} \in \mathfrak{S}(T, Q)\}$ *where T^* is the completion of T* (see (1.2.6)). *Let c be a nonnegative integer such that condition* (3_c) *is satisfied. Then condition* (4_c) *is satisfied.*

(4.21.4). *Assume that for every iterated monoidal transform T of R'' and every ideal Q in T we have that* $\mathfrak{S}(T^*, QT^*) = \{S \in \mathfrak{V}(T^*)\colon T_{T \cap M(S)} \in \mathfrak{S}(T, Q)\}$ *where T^* is the completion of T* (see (1.2.6)). *Let c be a nonnegative integer such that condition* (4_c) *is satisfied. Then either condition* (1) *is satisfied, or condition* (2_{c-1}) *is satisfied.*

(4.21.5). *Assume that: R'' is pseudogeometric; for every iterated monoidal transform T of R'' and every nonzero principal prime ideal P in T we have that $\mathfrak{S}(T, P)$ is closed in $\mathfrak{V}(T)$; and for every iterated monoidal transform T of R'' and every ideal Q in T we have that* $\mathfrak{S}(T^*, QT^*) = \{S \in \mathfrak{V}(T^*)\colon T_{T \cap M(S)} \in \mathfrak{S}(T, Q)\}$ *where T^* is the completion of T* (see (1.2.6)). *Also assume that condition* $(*)$ *is satisfied. Then condition* (1) *is satisfied.*

PROOF OF (4.21.1). Let $(R'_i, J'_i, I'_i)_{0 \leqslant i < \infty}$ be the infinite sequence such that $(R'_0, J'_0, I'_0) = (R', J', I')$, and (R'_i, J'_i, I'_i) is the monoidal transform of $(R'_{i-1}, J'_{i-1}, I'_{i-1}, R'_{i-1})$ along V for $0 < i < \infty$. Note that then $\operatorname{ord}_{R'_i} J'_i \leqslant \operatorname{ord}_{R'_j} J'_j$ whenever $0 \leqslant j \leqslant i < \infty$, and by (1.10.8) we also have that I'_i has a quasinormal crossing at R_i for $0 \leqslant i < \infty$.

First suppose that $\operatorname{ord}_{R'_j} J'_j \neq d$ for some j with $0 \leqslant j < \infty$. Then there exists a unique positive integer m such that $\operatorname{ord}_{R'_i} J'_i = d > \operatorname{ord}_{R'_m} J'_m$ for $0 \leqslant i < m$. Upon taking $S'_i = R'_i$ for $0 \leqslant i < m$ we get that condition (1) is satisfied.

So now assume that $\operatorname{ord}_{R'_i} J'_i = d$ for $0 \leqslant i < \infty$. Then by (1.10.5) we know that (R'_i, J'_i) is unresolved for $0 \leqslant i < \infty$, and hence by (3.21.1) we get that $\dim R'_i = 3$ for $0 \leqslant i < \infty$. Now R'_i is residually algebraic over R'_{i-1} for $0 < i < \infty$, and hence $R'_i/M(R'_i)$ is perfect for $0 \leqslant i < \infty$. By (3.8.4) there exists a positive integer m such that $\mathfrak{C}^2(R'_m, J'_m)$ has a strict normal crossing at R'_m. Let $S'_i = R'_i$ for $0 \leqslant i < m$. Then $[(R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < m},$

$(R'_m, J'_m, I'_m)]$ is a finite weak resolver. Let T be the completion of R'_m. In view of Cohen's structure theorem [28: Theorem 27 on page 304] we may identify T with the ring of formal power series in indeterminates X, Y, Z with coefficients in a field k which is isomorphic to $R'_m/M(R'_m)$. Let R be the ring of formal power series in X, Y, Z with coefficients in an algebraic closure of k, where we regard R to be an overring of T. Let $J = J'_m R$, $I = I'_m R$, $L = I$, $c' = c = \operatorname{ord}_R I$, and $e = d + c$. Now $\operatorname{ord}_R J = \operatorname{ord}_{R'_m} J'_m$ and hence $\operatorname{ord}_R JL = e$. By (4.13) there exists a basis (x, y, z) of $M(R)$ such that $JL \not\subset (x, y, z^{d+c'+1})R$. Upon taking $p = 0 = q$ we also have that $I = x^p y^q L$. Therefore condition (2_c) is satisfied.

PROOF OF (4.21.2). If $c' < c$ then condition (2_{c-1}) is satisfied and we have nothing more to show. So assume that $c' = c$. Let $(R'_i, J'_i, I'_i)_{m<i<\infty}$ be the infinite sequence such that (R'_i, J'_i, I'_i) is the monoidal transform of $(R'_{i-1}, J'_{i-1}, I'_{i-1}, R'_{i-1})$ along V for $m < i < \infty$. Note that then $\operatorname{ord}_{R'_i} J'_i \leqslant \operatorname{ord}_{R'_j} J'_j$ whenever $m \leqslant j \leqslant i < \infty$, and by (1.10.8) we also have that I'_i has a quasinormal crossing at R'_i for $m \leqslant i < \infty$. For a moment suppose that $\operatorname{ord}_{R'_j} J'_j \neq d$ for some j with $m < j < \infty$; then there exists a unique integer $n > m$ such that $\operatorname{ord}_{R'_i} J'_i = d > \operatorname{ord}_{R'_n} J'_n$ for $0 \leqslant i < n$; upon letting $S'_i = R'_i$ for $m \leqslant i < n$ we now get that $[(R'_i, J'_i, I'_i, S'_i)_{0 \leqslant i < n}, (R'_n, J'_n, I'_n)]$ is a finite weak resolver, and hence condition (1) is satisfied. So henceforth assume that $\operatorname{ord}_{R'_i} J'_i = d$ for $m < i < \infty$. Then by (1.10.5) we know that (R'_i, J'_i) is unresolved for $m \leqslant i < \infty$, and hence by (3.21.1) we get $\dim R'_i = 3$ for $m \leqslant i < \infty$. By (3.11) we also get that $\mathfrak{C}^2(R'_i, J'_i)$ has a strict normal crossing at R'_i for $m \leqslant i < \infty$. Since I'_m has a quasinormal crossing at R'_m, we get that I has a quasinormal crossing at R. Let $P = JL$ and $e = d + c$. Note that then $\operatorname{ord}_R J = d$, $\operatorname{ord}_R P = e$, and $P \not\subset (x, y, z^{e+1})R$. By (4.9) there exists an infinite sequence $(R_i^*, J_i^*, I_i^*)_{m \leqslant i < \infty}$ such that $(R_m^*, J_m^*, I_m^*) = (R, J, I)$ and for $m < i < \infty$: R_i^* is a three-dimensional regular local domain; R_i^* dominates R'_i; R_i^* is residually separable algebraic over R'_i; $M(R_i^*) = M(R'_i)R_i^*$; $J_i^* = J'_i R_i^*$; $I_i^* = I'_i R_i^*$; and (R_i^*, J_i^*, I_i^*) is a monoidal transform of $(R_{i-1}^*, J_{i-1}^*, I_{i-1}^*, R_{i-1}^*)$. Note that now $\operatorname{ord}_{R_i^*} J_i^* = d$ and $R_i^*/M(R_i^*)$ is algebraically closed for $m \leqslant i < \infty$. Let $(P_i^*)_{m \leqslant i < \infty}$ be the unique infinite sequence such that: P_i^* is a nonzero principal ideal in R_i^* for $m \leqslant i < \infty$; $P_m^* = P$; and (R_i^*, P_i^*) is a monoidal transform of $(R_{i-1}^*, P_{i-1}^*, R_{i-1}^*)$ for $m < i < \infty$.

First suppose that $\operatorname{ord}_{R_j^*} P_j^* \neq e$ for some j with $m < j < \infty$. Then by (4.16.1) there exists an integer $n > m$, a nonzero principal ideal L_n^* in R_n^*, a basis (x^*, y^*, z^*) of $M(R_n^*)$, and nonnegative integers p^*, q^*, and c^* such that: $I_n^* = x^{*p^*}y^{*q^*}L_n^*$, $\operatorname{ord}_{R_n^*} L_n^* = c^* < c$, and $J_n^* L_n^* \not\subset (x^*, y^*, z^{*d+c^*+1})R_n^*$. Let R^* be the completion of R_n^*. Then R^* is a three-dimensional complete regular local domain, $R^*/M(R^*)$ is algebraically closed, (x^*, y^*, z^*) is a basis of $M(R^*)$, R^* dominates R_n', R^* is residually separable algebraic over R_n', and $M(R^*) = M(R_n')R^*$. Let $J^* = J_n'R^*$, $I^* = J_n'R^*$, and $L^* = L_n^*R^*$. Then L^* is a nonzero principal ideal in R^*, $\operatorname{ord}_{R^*} L^* = c^*$, $I^* = x^{*p^*}y^{*q^*}L^*$, and $J^*L^* \not\subset (x^*, y^*, z^{*d+c^*+1})R^*$. Let $S_i' = R_i'$ for $m \leqslant i < n$. Then $[(R_i', J_i', I_i', S_i')_{0 \leqslant i < n}, (R_n', J_n', I_n')]$ is a finite weak resolver. Since $c^* < c$, we conclude that condition (2_{c-1}) is satisfied.

Next suppose that $\operatorname{ord}_{R_i^*} P_i^* = e$ for $m < i < \infty$. Then by (4.16.2) there exists an integer $m' \geqslant m$, a field $\mathfrak{k}$, and an infinite sequence $(R_i, J_i, I_i, L_i, S_i, f_i(Z), x_i, y_i, z_i, r_i, s_i, t_i, p_i, q_i, a_i, b_i)_{m' \leqslant i < \infty}$ having the description given in (4.16.2). It follows that condition (3_c) is satisfied.

PROOF OF (4.21.3). By $(*)$ there exists an integer $n \geqslant m'$ and an element r in the completion S^* of S_n such that either: $f_{m'}(Z + r) = Z^e$, or: there exist nonnegative integers u and v such that upon letting $g(Z) = (x_n^u y_n^v)^{-e} f_{m'}(x_n^u y_n^v Z + r)$ we have that $g(Z) \in S^*[Z]$ and $0 < \operatorname{ord}_{S^*} g(Z) < e$. Let R^* be the completion of R_n. Then there exists a unique homomorphism h': $S^* \to R^*$ such that $h'(M(S^*)) \subset M(R^*)$ and $h'(s) = s$ for all $s \in S_n$; now $\dim h'(S^*) \leqslant 2$, $\dim R^* = 3$, and $M(R^*) = M(h'(S^*))R^* + z_n R^*$; consequently $\dim h'(S^*) = 2$ and hence h' is a monomorphism; therefore we may identify S^* with a subring of R^*. Let $J^* = J_n R^*$, $I^* = I_n R^*$, $L^* = L_n R^*$, $(x^*, y^*, z') = (x_n, y_n, z_n)$, $(p^*, q^*) = (p_n, q_n)$, $f(Z) = f_{m'}(Z)$, $g'(Z) = f_n(Z)$, and $(r', s', t', a', b') = (r_n, s_n, t_n, a_n, b_n)$.

Note that then: R^* is a three-dimensional complete regular local domain; $R^*/M(R^*)$ is algebraically closed; (x^*, y^*, z') is a basis of $M(R^*)$; J^*, I^*, and L^* are nonzero principal ideals in R^*;

$\mathrm{ord}_{R^*}J^* = d$; $\mathrm{ord}_{R^*}L^* = c$; $e = d + c$; p^*, q^*, a', and b' are nonnegative integers; $I^* = x^{*p^*}y^{*q^*}L^*$; R^* dominates R_n'; R^* is residually separable algebraic over R_n'; $M(R^*) = M(R_n')R^*$; $J^* = J_n'R^*$; $I^* = I_n'R^*$; S^* is a two-dimensional complete regular local domain; R^* dominates S^*; R^* is residually rational over S^*; (x^*, y^*) is a basis of $M(S^*)$; $f(Z)$ and $g'(Z)$ are monic polynomials of degree e in an indeterminate Z with coefficients in S^*; $r' \in S^*$; t' is a unit in S^*; $s' = t'x^{*a'}y^{*b'}$; $g'(Z) = s'^{-e}f(s'Z + r')$; $J^*L^* = g'(z')R^*$; and $r \in S^*$. Also, either: (′) $f(Z + r) = Z^e$, or: (″) there exist nonnegative integers u and v such that upon letting $g(Z) = (x^{*u}y^{*v})^{-e}f(x^{*u}y^{*v}Z + r)$ we have that $g(Z) \in S^*[Z]$ and $0 < \mathrm{ord}_{S^*}g(Z) < e$. Let $r^* = (r' - r)/s'$. Then r^* is in the quotient field of S^*.

By (1.10.5) we have that (R_n', J_n') is unresolved; consequently by (4.6.2) we get that (R^*, J^*) is unresolved, and from this it follows that (R^*, J^*L^*) is unresolved. Suppose if possible that we have (′); then $(z' + r^*)^e = g'(z') \in R^*$ and hence by (4.11.3) we get that $r^{*e} \in S^*$; since S^* is normal, we deduce that $r^* \in S^*$; since z' and $g'(z')$ are in $M(R^*)$, we must have $r^* \in M(R^*)$ and hence $r^* \in M(S^*)$; consequently $M(R^*) = (x^*, y^*, z' + r^*)R^*$; since $J^*L^* = g'(z')R^*$, we get that (R^*, J^*L^*) is resolved; this is a contradiction. Therefore we must have (″).

Let $a = u - a'$, $b = v - b'$, and $s^* = t'^{-1}x^{*a}y^{*b}$. Then by (4.10.2) we get that $r^* \in S^*$, a and b are nonnegative integers, and $g(Z) = s^{*-e}g'(s^*Z - r^*)$. Since $\mathrm{ord}_{R^*}J^*L^* = e$ and $J^*L^* = g'(z')R^*$, we get that $g'(z') \in M(R^*)^e$; consequently we must have $g'(Z) - Z^e \in M(S^*)[Z]$, and hence $g'(s^*z' - r^*) - (s^*z' - r^*)^e \in M(R^*)$; also $g'(s^*z' - r^*) = s^{*e}g(z') \in M(R^*)$, and hence we get that $r^* \in M(R^*)$. Therefore $r^* \in M(S^*)$, and hence upon letting $z^* = z' + r^*$ we get that $M(R^*) = (x^*, y^*, z^*)R^*$. Let $g^*(Z) = t'^{-e}g(t'Z)$. Then $g^*(Z) \in S^*[Z]$ and $\mathrm{ord}_{S^*}g^*(Z) = \mathrm{ord}_{S^*}g(Z)$. Consequently

$$g^*(Z) = Z^e + g_1Z^{e-1} + \cdots + g_e$$

where $g_1, \ldots, g_e$ are elements in S^* such that $g_e \in M(S^*)$, and $g_{j'} \notin M(S^*)^{j'}$ for some j' with $1 \leqslant j' \leqslant e$. In particular then $g_1, \ldots, g_e$ are elements in R^* with $g_e \in M(R^*)$, and in view of (4.11.2) we get that $\mathrm{ord}_{R^*}g_j = \mathrm{ord}_{h(R^*)}g_j$ for $1 \leqslant j \leqslant e$ where h: $R^* \to R^*/z^*R^*$ is the canonical epimorphism, and $\mathrm{ord}_{R^*}g_{j'} < j'$.

Let $w = g'(z')$. Then $J^*L^* = wR^*$. Also $g'(z') = x^{*ae}y^{*be}g^*(x^{*-a}y^{*-b}z^*)$, and hence

$$w = z^{*e} + \sum_{j=1}^{e} g_j x^{*aj}y^{*bj}z^{*e-j}.$$

It follows that condition (4_c) is satisfied.

PROOF OF (4.21.4). By (4.20) we get that one of the following two conditions is satisfied.

(1′) There exists a finite weak resolver $[(R_i^*, J_i^*, I_i^*, S_i^*)_{0\leqslant i<u}, (R_u^*, J_u^*, I_u^*)]$ such that: $(R_0^*, J_0^*, I_0^*) = (R_n', J_n', I_n')$; $\operatorname{ord}_{R_i^*}J_i^* = d > \operatorname{ord}_{R_u^*}J_u^*$ for $0 \leqslant i < u$; and V dominates R_u^*.

(2′) There exists a finite weak resolver $[(R_i^*, J_i^*, I_i^*, S_i^*)_{0\leqslant i<u}, (R_u^*, J_u^*, I_u^*)]$ and a system $(R^*, J^*, I^*, L^*, x^*, y^*, z^*, p^*, q^*, c^*)$ such that: $(R_0^*, J_0^*, I_0^*) = (R_n', J_n', I_n')$; $\dim R_i^* = 3$ and $\operatorname{ord}_{R_i^*}J_i^* = d$ for $0 \leqslant i \leqslant u$; $\mathfrak{C}^2(R_u^*, J_u^*)$ has a strict normal crossing at R_u^*; V dominates R_u^*; R^* is a three-dimensional complete regular local domain; $R^*/M(R^*)$ is isomorphic to $R/M(R)$; R^* dominates R_u^*; R^* is residually separable algebraic over R_u^*; $M(R^*) = M(R_u^*)R^*$; $J^* = J_u^*R^*$; $I^* = I_u^*R^*$; L^* is a nonzero principal ideal in R^*; $\operatorname{ord}_{R^*}L^* = c^* < c$; (x^*, y^*, z^*) is a basis of $M(R^*)$; p^* and q^* are nonnegative integers; $I^* = x^{*p^*}y^{*q^*}L^*$; and $J^*L^* \not\subset (x^*, y^*, z^{*d+c^*+1})R^*$.

First suppose that condition (1′) is satisfied. Let $v = n + u$, $S_i' = R_i'$ for $m \leqslant i < n$, $S_i' = S_{i-n}^*$ for $n \leqslant i < v$, and $(R_i', J_i', I_i') = (R_{i-n}^*, J_{i-n}^*, I_{i-n}^*)$ for $n < i \leqslant v$. Then $[(R_i', J_i', I_i', S_i')_{0\leqslant i<v}, (R_v', J_v', I_v')]$ is a finite weak resolver. It follows that condition (1) is satisfied.

Next suppose that condition (2′) is satisfied. Let $v = n + u$, $S_i' = R_i'$ for $m \leqslant i < n$, $S_i' = S_{i-n}^*$ for $n \leqslant i < v$, and $(R_i', J_i', I_i') = (R_{i-n}^*, J_{i-n}^*, I_{i-n}^*)$ for $n < i \leqslant v$. Then $[(R_i', J_i', I_i', S_i')_{0\leqslant i<v}, (R_v', J_v', I_v')]$ is a finite weak resolver. Since $c^* < c$, it follows that condition (2_{c-1}) is satisfied.

PROOF OF (4.21.5). By (4.21.1) there exists a nonnegative integer c^* such that either condition (1) is satisfied or condition (2_{c^*}) is satisfied. By (4.21.2), (4.21.3), and (4.21.4) we get that if c is a nonnegative integer such that condition (2_c) is satisfied then either condition (1) is satisfied or condition (2_{c-1}) is satisfied. Therefore by induction on j we get that for $0 \leqslant j \leqslant c^* + 1$ we have that either condition (1) is satisfied or condition (2_{c^*-j}) is satisfied. In particular, upon taking $j = c^* + 1$, we get that either condition (1) is satisfied or condition (2_{-1}) is satisfied. However, clearly condition (2_{-1}) can never be satisfied. Therefore condition (1) is satisfied.

(4.22). *Let R be a three-dimensional pseudogeometric regular local domain such that: $R/M(R)$ is a perfect field having the same characteristic as R; for every iterated monoidal transform T of R and every nonzero principal prime ideal P in T we have that $\mathfrak{S}(T, P)$ is closed in $\mathfrak{B}(T)$; and for every iterated monoidal transform T of R and every ideal Q in T we have that $\mathfrak{S}(T^*, QR^*) = \{S \in \mathfrak{B}(T^*): T_{T\cap M(S)} \in \mathfrak{S}(T, Q)\}$ where T^* is the completion of T* (see (1.2.6)). *Assume that the following condition is satisfied.*

$(*)$ *Let S_0 be the ring of formal power series in two indeterminates with coefficients in an algebraic closure of $R/M(R)$. Let (x_0, y_0) be any basis of $M(S_0)$, let $\mathfrak{k}$ be any coefficient set for S_0, let e be any positive integer, and let $f(Z)$ be any monic polynomial of degree e in an indeterminate Z with coefficients in S_0. Let $(S_i, x_i, y_i)_{0<i<\infty}$ be any infinite sequence such that for $0 < i < \infty$: S_i is a two-dimensional regular local domain; S_i is a quadratic transform of S_{i-1}; (x_i, y_i) is a basis of $M(S_i)$; if $y_{i-1}/x_{i-1} \in S_i$ then $x_{i-1} = x_i$ and $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$; and if $y_{i-1}/x_{i-1} \notin S_i$ then $x_{i-1} = y_i x_i$ and $y_{i-1} = y_i$. Assume that $x_{i+1} \neq x_i$ for infinitely many distinct values of i. Then there exists a nonnegative integer n and an element r in the completion S^* of S_n such that either: $f(Z + r) = Z^e$, or: there exist nonnegative integers u and v such that upon letting $g(Z) = (x_n^u y_n^v)^{-e} f(x_n^u y_n^v Z + r)$ we have that $g(Z) \in S^*[Z]$ and $0 < \operatorname{ord}_{S^*} g(Z) < e$.*

Then R is weakly resolvable.

PROOF. Let R' be any iterated monoidal transform of R, let J' and I' be any nonzero principal ideals in R' such that (R', J') is unresolved and I' has a quasinormal crossing at R', and let V be any valuation ring of the quotient field of R such that V dominates R'. We want to show that then there exists a finite weak resolver $[(R_i, J_i, I_i, S_i)_{0\leqslant i<m}, (R_m, J_m, I_m)]$ such that $(R_0, J_0, I_0) =$

(R', J', I'), $\mathrm{ord}_{R_i} J_i = \mathrm{ord}_{R'} J' > \mathrm{ord}_{R_m} J_m$ for $0 \leqslant i < m$, and V dominates R_m. This however follows from (4.21.5).

(4.23). REMARK. If in (4.22) we only wanted to prove the weaker assertion that R is weakly semiresolvable then, upon disregarding several considerations of this section and simplifying some of the remaining considerations, we can make a simpler proof. The reader may find it instructive to extract such a simpler proof of the said weaker assertion.

§5. Main results

In [9: Theorem 1.1] we proved the following.

(5.1). *Let S_0 be a two-dimensional regular local domain such that $S_0/M(S_0)$ is an algebraically closed field having the same characteristic as S_0. Let (x_0, y_0) be a basis of $M(S_0)$, let $\mathfrak{k}$ be a coefficient set for S_0, and let $f(Z)$ be a monic polynomial of degree $e > 0$ in an indeterminate Z with coefficients in S_0. Let $(S_i, x_i, y_i)_{0<i<\infty}$ be an infinite sequence such that for $0 < i < \infty$: S_i is a two-dimensional regular local domain; S_i is a quadratic transform of S_{i-1}; (x_i, y_i) is a basis of $M(S_i)$; if $y_{i-1}/x_{i-1} \in S_i$ then $x_{i-1} = x_i$ and $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$; and if $y_{i-1}/x_{i-1} \notin S_i$ then $x_{i-1} = y_i x_i$ and $y_{i-1} = y_i$. Assume that $x_{i+1} \neq x_i$ for infinitely many distinct values of i. Then there exists a nonnegative integer n and an element r in the completion S^* of S_n such that either: $f(Z + r) \doteq Z^e$, or: there exist nonnegative integers u and v such that upon letting $g(Z) = (x_n^u y_n^v)^{-e} f(x_n^u y_n^v Z + r)$ we have that $g(Z) \in S^*[Z]$ and $0 < \mathrm{ord}_{S^*} g(Z) < e$.*

In (5.2), (5.3), and (5.4) we shall state and prove the main results of this chapter.

(5.2). *Let R be a pseudogeometric regular local domain with* $\dim R \leqslant 3$. *Assume that if* $\dim R = 3$ *then the following three conditions are satisfied:* (1) *$R/M(R)$ is a perfect field having the same characteristic as R*; (2) *for every iterated monoidal transform T of R and every nonzero principal prime ideal P in T we have that $\mathfrak{S}(T, P)$ is closed in $\mathfrak{V}(T)$*; and (3) *for every iterated monoidal* *transform T of R and every ideal Q in T we have that* $\mathfrak{S}(T^*, QT^*) = \{S \in \mathfrak{V}(T^*)\colon T_{T \cap M(S)} \in \mathfrak{S}(T, Q)\}$ *where T^* is the completion of T* (see (1.2.6)).

Then we have the following.

(5.2.1). *R is weakly semiresolvable, semiresolvable, strongly semiresolvable, weakly resolvable, resolvable, strongly resolvable, detachable, strongly detachable, principalizable, and strongly principalizable.*

(5.2.2). *Let R' be any iterated monoidal transform of R, let I' be any nonzero ideal in R', and let V be any valuation ring of the quotient field of R such that V dominates R'. Then there exists an iterated monoidal transform R^* of R along V such that $I'R^*$ is a nonzero principal ideal in R^* having a normal crossing at R^*.*

(5.2.3). *Let R' be any iterated monoidal transform of R, let V be any valuation ring of the quotient field of R such that V dominates R', and let $f_1, \ldots, f_q$ $(q > 0)$ be any finite number of nonzero elements in V. Then there exists an iterated monoidal transform R^* of R' along V and a basis $(x_1, \ldots, x_n)$ of $M(R^*)$, where $n = \dim R^*$, such that $f_i = g_i x_1^{a(i,1)} \ldots x_n^{a(i,n)}$ where g_i is a unit in R^* and $a(i, j)$ is a nonnegative integer for $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant n$.*

(5.2.4). *Let R' be any iterated monoidal transform of R, let J' be any nonzero principal prime ideal in R', and let V be any valuation ring of the quotient field of R'/J' such that V dominates R'/J'. Then there exists a regular spot R^* over R'/J' such that V dominates R^*.*

PROOF. In view of (5.1) this follows from (2.2), (2.4), (2.6), (2.8), (2.11), (2.13), (2.19), (2.21), (3.21), and (4.22).

(5.3). *Let R be a pseudogeometric regular local domain such that: $R/M(R)$ is a perfect field having the same characteristic as R; for every regular spot over T over R with* $\dim T \leqslant 3$ *and every nonzero principal prime ideal P in T we have that $\mathfrak{S}(T, P)$ is closed in $\mathfrak{V}(T)$; and for every regular spot T over R with* $\dim T \leqslant 3$ *and every ideal Q*

in T we have that $\mathfrak{S}(T^*, QT^*) = \{S \in \mathfrak{V}(T^*)\colon T_{T \cap M(S)} \in \mathfrak{S}(T, Q)\}$ where T^* is the completion of T (see (1.2.6)). *Let K be a function field over R such that* $\dim R + \operatorname{trdeg}_R K \leqslant 2$. *Let V be a valuation ring of K such that V dominates R. Then there exists a regular spot R^* over R with quotient field K such that V dominates R^*.*

PROOF. There exists a finite number of elements $x_1, ..., x_n$ in K such that K is the quotient field of $R[x_1, ..., x_n]$. We shall prove our assertion by induction on n. If $n = 0$ then it suffices to take $R^* = R$. Now let $n > 0$ and assume that the assertion is true for all values of n smaller than the given one. Let L be the quotient field of $R[x_1, ..., x_{n-1}]$. Then $\dim R + \operatorname{trdeg}_R L \leqslant 2$, and hence by the induction hypothesis there exists a regular spot A over R with quotient field L such that V dominates R. For a moment suppose that x_n is transcendental over L; let $x = x_n$ if $x_n \in V$, and $x = 1/x_n$ if $x_n \notin V$; let R^* be the quotient ring of $A[x]$ with respect to $M(V) \cap A[x]$; then R^* is a spot over R with quotient field K and V dominates R^*; since x is transcendental over L, by [18: (14.8) and (28.3)] we have that R^* is regular. So now assume that x_n is algebraic over L. Then there exists $0 \neq r \in A$ such that upon letting $y = rx_n$ we have that y is integral over A. Note that now $K = L(y)$ and $A[y] \subset V$. Clearly A dominates R, and hence by [28: Proposition 2 on page 326] we get that $\dim A \leqslant 2$, and either $\dim A \leqslant 1$ or A is residually algebraic over R. Let Z be an indeterminate and let $h\colon A[Z] \to A[y]$ be the unique epimorphism such that $h(Z) = y$ and $h(u) = u$ for all $u \in A$. Let $f(Z)$ be the minimal monic polynomial of y over L. Since A is normal, we get that $f(Z) \in A[Z]$ and $h^{-1}(0) = f(Z)A[Z]$. Let B be the quotient ring of $A[y]$ with respect to $M(V) \cap A[y]$, let R' be the quotient ring of $A[Z]$ with respect to $h^{-1}(M(V) \cap A[y])$, and let $J' = f(Z)R'$. Then B is a spot over A with quotient field K, V dominates B, R' is a spot over A, R' dominates A, J' is a nonzero principal prime ideal in R', and R'/J' is isomorphic to B. By [28: Proposition 2 on page 326] we get that $\dim R' \leqslant 3$, and either $\dim R' \leqslant 2$ or R' is residually algebraic over R. It follows that if $\dim R' = 3$ then $R'/M(R')$ is a perfect field having the same characteristic as R'. By [18: (14.8) and (28.3)] we have that R' is regular. Therefore by (5.2.4) there exists a regular spot R^* over B with quotient field K such that V dominates R^*. Clearly R^* is a spot over R.

(5.4). *Let R be a complete local domain such that $R/M(R)$ is a perfect field having the same characteristic as R, let K be a function field over R such that* $\dim R + \operatorname{trdeg}_R K \leqslant 2$, *and let V be a valuation ring of K such that V dominates R. Then there exists a regular spot R^* over R with quotient field K such that V dominates R^*.*

PROOF. By Cohen's structure theorem [18: (31.6)] there exists a complete regular local domain R' such that $\dim R' = \dim R$, R dominates R', R is residually rational over R', and R is a finite R'-module. In view of (1.2.6), by (5.3) there exists a regular spot R^* over R' with quotient field K such that V dominates R'. Since R^* is normal, we get that R^* is a spot over R.

We shall now give an *alternative simple proof of* (5.1) *for the case when S_0 is of zero characteristic.* In this proof we shall only use (0.1) and the trick of killing the coefficient of Z^{e-1} in $f(Z)$; in particular, we shall not use any results from the papers [5], [7], [8], and [9]. Thus for the case when R is of zero characteristic we shall have given a simpler proof of (5.2), (5.3), and (5.4) which is independent of the papers [5], [7], [8], and [9]. What we shall prove is actually slightly stronger than the case of (5.1) when S_0 is of zero characteristic and is as follows:

(5.5). *Let S_0 be a two-dimensional regular local domain such that $S_0/M(S_0)$ is algebraically closed. Let (x_0, y_0) be a basis of $M(R_0)$, let $\mathfrak{k}$ be a coefficient set for S_0, let e be a positive integer which is not divisible by the characteristic of $S_0/M(S_0)$, and let $f(Z)$ be a monic polynomial of degree e in an indeterminate Z with coefficients in S_0. Let $(S_i, x_i, y_i)_{0<i<\infty}$ be an infinite sequence such that for $0 < i < \infty$: S_i is a two-dimensional regular local domain; S_i is a quadratic transform of S_{i-1}; (x_i, y_i) is a basis of $M(S_i)$; if $y_{i-1}/x_{i-1} \in S_i$ then $x_{i-1} = x_i$ and $(y_{i-1}/x_i) - y_i \in \mathfrak{k}$; and if $y_{i-1}/x_{i-1} \notin S_i$ then $x_{i-1} = y_i x_i$ and $y_{i-1} = y_i$. Assume that $x_{i+1} \neq x_i$ for infinitely many distinct values of i. Then there exists a positive integer n and an element r in S_n such that either: $f(Z + r) = Z^e$, or: there exist nonnegative integers u and v such that upon letting $g(Z) = (x_n^u y_n^v)^{-e} f(x_n^u y_n^v Z + r)$ we have that $g(Z) \in S_n[Z]$ and $0 < \operatorname{ord}_{S_n} g(Z) < e$.*

First we shall prove the following.

(5.6). *Let (S, S', x, y, y', G, a) be a system such that: S and S' are two-dimensional regular local domains; S' is a quadratic transform of S; $M(S) = (x, y)S$; $M(S') = (x, y')S'$; $(y/x) - y' \in S$; $0 \neq G \in S$; and $a = \operatorname{ord}_S G$. Then $G = x^a(tx + t'y'^b)$ where $t \in S'$, t' is a unit in S', and b is an integer with $0 \leqslant b \leqslant a$.*

PROOF. Clearly $M(S) = (x, xy')S$ and hence $G = r_0x^a + r_1x^{a-1}(xy') + \cdots + r_a(xy')^a$ where $r_0, r_1, \ldots, r_a$ are elements in S at least one of which is not in $M(S)$. Let b be the smallest integer with $0 \leqslant b \leqslant a$ such that $r_b \notin M(S)$. Now $M(S)S' = xS'$ and hence upon letting $t = (r_0/x) + (r_1/x)y' + \cdots + (r_{b-1}/x)y'^{b-1}$ we get that $t \in S'$ (we take $t = 0$ in case $b = 0$). Let $t' = r_b + r_{b+1}y' + \cdots + r_a y'^{a-b}$. Then t' is a unit in S', and $G = x^a(tx + t'y'^b)$.

Although the following assertion follows from [8: Lemmas 3.7 and 3.14.(3)] and [9: Lemma 1.2], we shall deduce it directly from (5.6).

(5.7). *Let $(S_i, x_i, y_i)_{0 \leqslant i < \infty}$ and $\mathfrak{k}$ be as in (5.5), let m be a nonnegative integer, and let $0 \neq F \in S_m$. Then there exists an integer $c \geqslant m$ such that $F = sx_c^p y_c^q$ where s is a unit in S_c and p and q are nonnegative integers.*

PROOF. Let W be the set of all integers $i \geqslant m$ such that $y_i/x_i \in S_{i+1}$ and $y_{i+1}/x_{i+1} \notin S_{i+2}$.

First suppose that W is a finite set. Since by assumption $x_{i+1} \neq x_i$ for infinitely many distinct values of i, it follows that then there exists an integer $j \geqslant m$ such that $y_i = y_j$ and $x_i = x_j y_j^{i-i}$ for $j \leqslant i < \infty$. Let p be the greatest nonnegative integer such that $F \in x_j^p S_j$, and let $d = \operatorname{ord}_{h(S)} h(Fx_j^{-p})$ where $h\colon S_j \to S_j/x_jS_j$ is the canonical epimorphism. Then d is a nonnegative integer and $F = x_j^p(s^*x_j + s'y_j^d)$ where $s^* \in S_j$, and s' is a unit in S_j. Let $c = j + d$, $q = pd + d$, and $s = s^*x_jy_j^{-d} + s'$. Then s is a unit in S_c and $F = sx_c^py_c^q$.

Next suppose that W is an infinite set. Let $F_m = F$ and define $F_{i+1} \in S_{i+1}$ for $m \leqslant i < \infty$ by the following recurrence relation: $F_i = z^dF_{i+1}$ where $d = \operatorname{ord}_{S_i} F_i$, and $z = x_{i+1}$ in case $y_i/x_i \in S_{i+1}$, and $z = y_{i+1}$ in case $y_i/x_i \notin S_{i+1}$. Then $F = s_ix_i^{p_i}y_i^{q_i}F_i$ for $m \leqslant i < \infty$ where s_i is a unit in S_i and p_i and q_i are nonnegative integers. Therefore it suffices to show that $\operatorname{ord}_{S_c} F_c = 0$ for some integer $c \geqslant m$. Let $a_i = \operatorname{ord}_{S_i} F_i$ for $m \leqslant i < \infty$. Then by (5.6) (or alternatively by (1.10.2)) we get that $a_{i+1} \leqslant a_i$ for $m \leqslant i < \infty$. We shall show that if i is any integer in W such that $a_i \neq 0$ then $a_{i+2} < a_i$; since W is an infinite set, this will imply that $a_c = 0$ for some integer $c \geqslant m$. So let any integer i in W be given such that $a_i \neq 0$. Then by (5.6) we get that $F_{i+1} = tx_{i+1} + t'y_{i+1}^b$ where $t \in S_{i+1}$, t' is a unit in S_{i+1}, and b is an integer with $0 \leqslant b \leqslant a_i$. If $a_{i+1} \neq a_i$ then $a_{i+2} \leqslant a_{i+1} < a_i$. So now assume that $a_{i+1} = a_i$. Then we must have $b = a_i$ and $t \in M(S_{i+1})^{b-1}$. Now $F_{i+2} = y_{i+2}^{-b} F_{i+1} = ty_{i+2}^{1-b} x_{i+2} + t'$, $ty_{i+2}^{1-b} \in S_{i+2}$, $x_{i+2} \in M(S_{i+2})$, and t' is a unit in S_{i+2}. Therefore $a_{i+2} = 0$ and hence $a_{i+2} < a_i$.

From (5.7) we shall now deduce the following.

(5.8). *Let $(S_i, x_i, y_i)_{0 \leqslant i < \infty}$ and $\mathfrak{k}$ be as in* (5.5), *let m be a nonnegative integer, let $0 \neq F \in S_m$, and let d be a positive integer. Then there exists an integer $n \geqslant m$ and nonnegative integers u and v such that $(x_n^u y_n^v)^{-d} F \in S_n$ and* $\operatorname{ord}_{S_n}((x_n^u y_n^v)^{-d} F) < d$.

PROOF. By (5.7) there exists an integer $c \geqslant m$ such that $F = s x_c^p y_c^q$ where s is a unit in S_c and p and q are nonnegative integers. If $y_{n-1}/x_{n-1} \in S_n$ and $y_{n-1}/x_{n-1} \neq y_n$ for some integer $n > c$ then clearly $F = s' x_n^w$ where s' is a unit in S_n and w is a nonnegative integer, and hence it suffices to take $v = 0$ and $u =$ the greatest nonnegative integer such that $ud \leqslant w$. So now assume that for $c < i < \infty$ we have that: if $y_{i-1}/x_{i-1} \in S_i$ then $y_{i-1}/x_{i-1} = y_i$. Let $(p_c, q_c) = (p, q)$ and define a pair of nonnegative integers (p_i, q_i) for $c < i < \infty$ by the following recurrence relation: $(p_i, q_i) = (p_{i-1} + q_{i-1}, q_{i-1})$ if $y_{i-1}/x_{i-1} \in S_i$ and $(p_i, q_i) = (p_{i-1}, p_{i-1} + q_{i-1})$. Then $F = s x_i^{p_i} y_i^{q_i}$ for $c \leqslant i < \infty$. For $c \leqslant i < \infty$ let u_i, v_i, a_i, b_i be the unique nonnegative integers such that $p_i = du_i + a_i$, $a_i < d$, $q_i = dv_i + b_i$, and $b_i < d$. Then it is clear that if i is any integer with $i > c$ such that $a_{i-1} + b_{i-1} \geqslant d$ then $a_i + b_i < a_{i-1} + b_{i-1}$. From this it follows that there exists an integer n with $c \leqslant n < c + d$ such that $a_n + b_n < d$. It now suffices to take $u = u_n$ and $v = v_n$.

Finally, from (0.1) and (5.8) we shall now deduce (5.5).

PROOF OF (5.5). Let t be the coefficient of Z^{e-1} in $f(Z)$. *Since e is not divisible by the characteristic* of $S_0/M(S_0)$, we get that

$t/e \in S_0$ and $f(Z - (t/e)) = Z^e + F_2 Z^{e-2} + F_3 Z^{e-3} + \cdots + F_e$ with $F_2, F_3, \ldots, F_e$ in S_0. If $F_j = 0$ for $2 \leqslant j \leqslant e$ then it suffices to take $n = 0$ and $r = -(t/e)$. So now assume that $F_j \neq 0$ for some j with $2 \leqslant j \leqslant e$. Then $e \geqslant 2$. Let $V = \bigcup_{i=0}^{\infty} S_i$. By (0.1) we know that V is a valuation ring of the quotient field of S_0, and V dominates S_i for $0 \leqslant i < \infty$. Let $a = e!$. Since V is a valuation ring of the quotient field of S_0, there exists an integer d with $2 \leqslant d \leqslant e$ such that upon letting $F = F_d$ we have that $F \neq 0$ and $F_j^{a/j}/F^{a/d} \in V$ for $2 \leqslant j \leqslant e$. Since $V = \bigcup_{i=0}^{\infty} S_i$, there exists a nonnegative integer m such that $F_j^{a/j}/F^{a/d} \in S_m$ for $2 \leqslant j \leqslant e$. By (5.8) there exists an integer $n \geqslant m$ and nonnegative integers u and v such that $(x_n^u y_n^v)^{-d} F \in S_n$ and $\operatorname{ord}_{S_n}((x_n^u y_n^v)^{-d} F) < d$. Let $g'(Z) = (x_n^u y_n^v)^{-e} f(x_n^v y_n^v Z - (t/e))$. Since S_n is normal, we get that $g'(Z) = Z^e + G_2' Z^{e-2} + G_3' Z^{e-3} + \cdots + G_e'$ with $G_2', G_3', \ldots, G_e'$ in S_n, and $0 \leqslant \operatorname{ord}_{S_n} g'(Z) < e$. If $0 < \operatorname{ord}_{S_n} g'(Z)$ then it suffices to take $r = -(t/e)$. So now assume that $0 = \operatorname{ord}_{S_n} g'(Z)$. Then $G_e' \notin M(S_n)$. Now $S_n/M(S_n)$ is algebraically closed and hence there exists a unit s in S_n such that $g'(s) \in M(S_n)$. Let $r = s x_n^u y_n^v - (t/e)$. Then $r \in S_n$. Let $g(Z) = (x_n^u y_n^v)^{-e} f(x_n^u y_n^v Z + r)$. Then $g(Z) = g'(Z + s)$ and hence $g(Z) = Z^e + G_1 Z^{e-1} + G_2 Z^{e-2} + \cdots + G_e$ with $G_1, G_2, \ldots, G_e$ in S_n. Since $g(Z) = g'(Z + s)$, we also get that $G_e = g'(s) \in M(S_n)$, and $G_1 = es$. Again *since e is not divisible by the characteristic of* $S_0/M(S_0)$, we conclude that $es \notin M(S_n)$. Therefore $0 < \operatorname{ord}_{S_n} g(Z) < e$.

CHAPTER 2

Global Theory

In this chapter k will be a noetherian domain and K will be a function field over k. We *define:* $\dim_k K = \dim k + \operatorname{trdeg}_k K$ (if $\dim k = \infty$ then we take $\dim_k K = \infty$). Most of the considerations of §6 may be used tacitly in the rest of this chapter.

§6. Terminology and preliminaries

(6.1). Let X be a topological space and let $Y \subset X$.

X is said to be *irreducible* if $X \neq \varnothing$ and X cannot be expressed as the union of two closed subsets of X different from X. Y is said to be *irreducible* if Y is irreducible in the induced topology. By an *irreducible component* of Y we mean an irreducible subset Z of Y such that Z is not contained in any irreducible subset of Y other than Z. It is easily seen that if Z is any irreducible subset of Y then the closure of Z in Y is irreducible; consequently every irreducible component of Y is closed in Y. By Zorn's lemma it follows that if Z is any irreducible subset of Y then Z is contained in some irreducible component of Y; since $\{y\}$ is irreducible for all $y \in Y$, it follows that Y is the union of its irreducible components. The following is easily proved.

(6.1.1). *Assume that X is the union of a finite family $(X_i)_{i \in I}$ of closed subsets X_i of X such that each X_i has only finitely many irreducible components. Then we have that: if X' is any irreducible component of X then X' is an irreducible component of X_i for some $i \in I$. In particular X has only finitely many irreducible components.*

X is said to be *quasicompact* if every open covering of X contains a finite subcovering. Again, Y is said to be *quasicompact* if Y is quasicompact in the induced topology.

S. S. Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces*

The following four conditions are easily seen to be equivalent: (1) every nonempty family of open subsets of X contains a maximal element; (2) there does not exist any infinite sequence $X_1 \subset X_2 \subset \ldots$ of distinct open subsets of X; (3) every nonempty family of closed subsets of X contains a minimal element; (4) there does not exist any infinite sequence $X_1 \supset X_2 \supset \ldots$ of distinct closed subsets of X. X is said to be ***noetherian*** if these conditions are satisfied. Again, Y is said to be ***noetherian*** if Y is noetherian in the induced topology. We note the following.

(6.1.2). *If X is noetherian then every subset of X is noetherian. If X is the union of a finite family of noetherian subsets then X is noetherian. X is noetherian if and only if every open subset of X is quasicompact. If X is noetherian then X has only finitely many irreducible components.*

Everything, except possibly the last statement, is obvious. Suppose if possible that X is noetherian and has infinitely many irreducible components. Let F be the set of all closed subsets of X having infinitely many components. Then $F \neq \varnothing$ and hence F contains a minimal element Z. Now $Z \neq \varnothing$ and Z is not irreducible, and hence $Z = Z_1 \cup Z_2$ where Z_1 and Z_2 are closed subsets of X different from Z. Since Z is a minimal element of F we get that $Z_i \notin F$ and hence Z_i has only finitely many irreducible components for $i = 1, 2$. Consequently by (6.1.1) we get that Z has only finitely many irreducible components which is a contradiction.

(6.2). In (6.2) (and only in (6.2)) we relax the assumptions on K/k; namely, we only assume that K is a field and k is a subring of K.

By $\mathfrak{R}(K/k)$ we denote the *Riemann-Zariski space* of K/k, i.e., $\mathfrak{R}(K/k)$ is the set of all valuation rings of K which contain k. By $\mathfrak{R}'(K/k)$ we denote the set of all quasilocal rings with quotient field K which contain k. We topologize $\mathfrak{R}'(K/k)$ by designating that a subset Y of $\mathfrak{R}'(K/k)$ is open if and only if there exists a family $(B_i)_{i\in I}$ of finite subsets B_i of K such that $Y = \{R \in \mathfrak{R}'(K/k)$: $B_i \subset R$ for some $i \in I\}$. Every subset of $\mathfrak{R}'(K/k)$ is to be regarded as a topological space with the topology induced by this topology of $\mathfrak{R}'(K/k)$; in particular this is so for $\mathfrak{R}(K/k)$ and for every model of K/k. For geometric visualization, elements in any subset X of $\mathfrak{R}'(K/k)$ may be called ***points*** of X; thus: by a ***closed point*** of X we mean an element R in X such that $\{R\}$ is a closed subset of X; by a ***normal point*** of X we mean an element R in X such that R is a normal domain; and by a ***regular point*** of X we mean an element R in X such that R is a regular local domain. Also, given any $X \subset \mathfrak{R}'(K/k)$ and $R \in \mathfrak{R}'(K/k)$ we may say that X ***passes through*** R to mean that $R \in X$. The following two results are proved in [28: Lines 1 and 2 on page 116] and [28: Theorem 40 on page 113] respectively.

(6.2.1). *For any $R \in X \subset \mathfrak{R}'(K/k)$ we have that $\{R' \in X: R' \subset R\} =$ closure of R in X.*

(6.2.2). *$\mathfrak{R}(K/k)$ is quasicompact.*

For any $X \subset \mathfrak{R}'(K/k)$, by (6.2.1) we get that there exists at most one point R of X such that X is the closure of $\{R\}$ in X; when R exists it is called the ***generic point*** of X; note that if R exists and Y is any nonempty open subset of X then R is the generic point of Y and Y is irreducible. In view of (6.2.1) we also get the following.

(6.2.3). *For any semimodel X of K/k we have the following: (1) X is irreducible and K is the generic point of X. (2) For any $V \in \mathfrak{R}(K/k)$ we have that: V dominates $X \Leftrightarrow R \subset V$ for some $R \in X$. (3) For any $R \in X$ we have that: $\{R' \in X: R \in \mathfrak{V}(R')\} = \{R' \in X: R' \subset R\} =$ closure of $\{R\}$ in X; and $\mathfrak{V}(R) =$ intersection of all open subsets of X containing R.*

For any $X \subset \mathfrak{R}'(K/k)$ we *define:* $\mathfrak{R}(X) = \{V \in \mathfrak{R}(K/k): V$ dominates $X\}$. Note that if X is a semimodel of K/k then, as noted above, $\mathfrak{R}(X) = \{V \in \mathfrak{R}(K/k): R \subset V$ for some $R \in X\}$; moreover, X is complete $\Leftrightarrow \mathfrak{R}(X) = \mathfrak{R}(K/k)$. If A is any affine ring over k with quotient field K then clearly $\mathfrak{R}(K/A) = \mathfrak{R}(\mathfrak{V}(A))$ and the topology of $\mathfrak{R}(K/A)$ induced by the topology of $\mathfrak{R}(K/k)$ coincides with the topology of $\mathfrak{R}(K/A)$ as the Riemann-Zariski space of K/A. Therefore by (6.2.2) we get the following.

(6.2.4). *If X is any model of K/k then $\mathfrak{R}(X)$ is quasicompact.*

For any two subsets X and X' of $\mathfrak{R}'(K/k)$ such that X is an irredundant premodel of K and X' dominates X, and any $R' \in X'$ we *define:* $[X', X](R')$ = center of R' on X; the resulting map $[X', X]: X' \to X$ is called the ***domination map*** of X' into X. Note that if X is any model of K/k and X' is any complete model of K/k dominating X then $[X', X](X') = X$. For any two subsets X and X' of $\mathfrak{R}'(K/k)$ we define: $\mathfrak{F}(X', X)$ = (*fundamental locus on X for the pair* (X', X)) = $\{R \in X: R$ does not dominate $X'\}$; note that if X' is a model of K/k then $\mathfrak{F}(X', X) = \{R \in X: R' \not\subset R$ for all $R' \in X'\}$; also note that if X and X' are models of K/k and X' dominates X then $\mathfrak{F}(X', X) = X - X'$. The following three results are proved in Lemmas 3, 7, and 6 of [28: §17 of Chapter VI] respectively.

(6.2.5). *If X is any model of K/k and X' is any subset of $\mathfrak{R}'(K/k)$ dominating X then $[X', X]$ is a continuous map of X' into X.*

(6.2.6). *If X is any complete model of K/k then there exists a projective model X' of K/k such that X' dominates X.*

(6.2.7). *Let X and X' be any two models of K/k. Then there exists a unique model X^* of K/k such that: X^* dominates X and X', and if Y is any subset of $\mathfrak{R}'(K/k)$ dominating X and X' then Y dominates X^*.* (X^* is called the *join* of X and X' and is denoted by $X + X'$.) *If $X = \bigcup_{i\in I} \mathfrak{V}(A_i)$ and $X' = \bigcup_{j\in J} \mathfrak{V}(B_j)$ where $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ are finite families of affine rings over k then $X^* = \bigcup_{i\in I, j\in J} \mathfrak{V}(A_{ij})$ where A_{ij} is the smallest subring of K containing A_i and B_j. If X and X' are complete (resp: projective) models of K/k then X^* is a complete (resp: projective) model of K/k.*

From the first characterization of the join we get the following.

(6.2.8). *Let X and X' be any two models of K/k. Then $\mathfrak{F}(X', X) = \mathfrak{F}(X + X', X)$. If $R \in X$, $R' \in X'$, $R^* \in X + X'$, and $V \in \mathfrak{R}(K/k)$ such that V dominates R, R', and R^* then the following four conditions are equivalent:* (1) $R^* = R$; (2) $R \notin \mathfrak{F}(X', X)$; (3) *$R$ dominates R'*; (4) $R' \subset R$.

The following two results are proved in Lemmas 1 and 2 of [28: §17 of Chapter VI] respectively.

(6.2.9). *Let A be any subring of K with quotient field K such that $k \subset A$, and let $Y \subset \mathfrak{V}(A)$. Then: Y is closed in $\mathfrak{V}(A) \Leftrightarrow$ there exists an ideal H in A such that $Y = \{R \in \mathfrak{V}(A)\colon HR \neq R\}$.* (Thus we see that the definition of a closed subset of $\mathfrak{V}(A)$ given here agrees with the definition given in (1.1).)

(6.2.10). *Let X be any model of K/k. If A is any affine ring over k such that $\mathfrak{V}(A) \subset X$ then $\mathfrak{V}(A) = \{R \in X\colon A \subset R\}$ and $\mathfrak{V}(A)$ is open in X. If $Y \subset X = \bigcup_{i \in I} \mathfrak{V}(A_i)$ where $(A_i)_{i \in I}$ is a finite family of affine rings over k then: Y is open (resp: closed) in $X \Leftrightarrow Y \cap \mathfrak{V}(A_i)$ is open (resp: closed) in $\mathfrak{V}(A_i)$ for all $i \in I$.*

For any element x in any domain A we clearly have that if $x \neq 0$ then $\{R \in \mathfrak{V}(A)\colon xR = R\} = \mathfrak{V}(A[x^{-1}])$, and if $x = 0$ then $\{R \in \mathfrak{V}(A)\colon xR = R\} = \varnothing$; also for any basis H' of any ideal H in A we have that $\mathfrak{V}(A) - \{R \in \mathfrak{V}(A)\colon HR \neq R\} = \bigcup_{x \in H'} \{R \in \mathfrak{V}(R)\colon xR = R\}$. Therefore by (6.2.9) we get the following.

(6.2.11). *Let A be any subring of K with quotient field K such that $k \subset A$, and let Y be any subset of $\mathfrak{V}(A)$. Then: Y is open in $\mathfrak{V}(A) \Leftrightarrow$ there exists a family $(x_i)_{i \in I}$ of nonzero elements in A such that $Y = \bigcup_{i \in I} \mathfrak{V}(A[x_i^{-1}])$. If A is noetherian then: Y is open in $\mathfrak{V}(A) \Leftrightarrow$ there exists a finite family $(x_i)_{i \in I}$ of nonzero elements in A such that $Y = \bigcup_{i \in I} \mathfrak{V}(A[x_i^{-1}])$.*

By (6.2.10) and (6.2.11) we get the following.

(6.2.12). *Let Y be any subset of any model X of K/k. Then: Y is open in $X \Leftrightarrow$ there exists a family $(A_i)_{i \in I}$ of affine rings A_i over k such that $Y = \bigcup_{i \in I} \mathfrak{V}(A_i)$. If k is noetherian and $Y \neq \varnothing$ then: Y is open in $X \Leftrightarrow Y$ is a model of K/k.*

By (6.2.4), (6.2.5), and (6.2.12) we get the following.

(6.2.13). *Let $Y \subset \mathfrak{R}'(K/k)$. If Y is a closed subset of some model of K/k then $\mathfrak{R}(Y)$ is quasicompact. If k is noetherian and Y is a closed subset of some open subset of some model of K/k then $\mathfrak{R}(Y)$ is quasicompact.*

Let A be any subring of K with quotient field K such that $k \subset A$. Given any closed subset Y of $\mathfrak{V}(A)$ let $P = \{x \in A: xR \neq R$ for all $R \in Y\}$; then clearly P is an ideal A and $Y \subset \{R \in \mathfrak{V}(A): PR \neq R\}$; by (6.2.9) there exists an ideal H in A such that $Y = \{R \in \mathfrak{V}(A): HR \neq R\}$; clearly $H \subset P$ (actually it can be shown that $P = \operatorname{rad} H$; however we shall not use this fact) and hence $\{R \in \mathfrak{V}(A): PR \neq R\} \subset \{R \in \mathfrak{V}(A) : HR \neq R\}$; consequently: (1) $Y = \{R \in \mathfrak{V}(A): PR \neq R\}$. Clearly if $Y \neq \varnothing$ then $P \neq A$. If $P \neq A$ and P is not a prime ideal in A then there exist elements x_1 and x_2 in A such that $x_1 \notin P$, $x_2 \notin P$, $x_1x_2 \in P$, and then upon letting $Y_i = \{R \in \mathfrak{V}(A): (P + x_iA)R \neq R\}$ by (1) and (6.2.9) we get that Y_1 and Y_2 are closed subsets of $\mathfrak{V}(A)$ such that $Y = Y_1 \cup Y_2$ and clearly $Y_1 \neq Y \neq Y_2$. Thus: (2) if Y is irreducible then P is a prime ideal in A. By (6.2.1) we get that: (3) if Q is any prime ideal in A then $\{R \in \mathfrak{V}(A): QR \neq R\}$ is an irreducible closed subset of $\mathfrak{V}(A)$ and A_Q is its generic point. By (1), (2), and (3) we get that: (4) for any $Z \subset \mathfrak{V}(A)$ we have that Z is an irreducible closed subset of $\mathfrak{V}(A) \Leftrightarrow Z$ is the closure of $\{R\}$ in $\mathfrak{V}(A)$ for some $R \in \mathfrak{V}(A)$ (and then R is the generic point of Z). By (1) we also get that: (5) if A is noetherian then $\mathfrak{V}(A)$ is noetherian. In view of (4) and (5), by (6.1.1), (6.1.2), (6.2.1), and (6.2.10) we get (6.2.14) and (6.2.15).

(6.2.14). *Let $Z \subset \mathfrak{N}'(K/k)$. Assume that Z is a closed subset of some model of K/k (note that if k is noetherian then, in view of* (6.2.12), *this is equivalent to assuming that Z is a closed subset of some open subset of some model of K/k). Then: $R \to$ closure of $\{R\}$ in Z is a one-to-one map of Z onto the set of all irreducible closed subsets of Z; every irreducible closed subset of Z, and hence in particular every irreducible component of Z, has a generic point; and $\{R \in Z$: R is the generic point of some irreducible component of $Z\} = \{R \in Z$: $R \not\subset R'$ for every $R' \in Z$ with $R' \neq R\}$.*

(6.2.15). *If k is noetherian and Z is any subset of $\mathfrak{N}'(K/k)$ such that Z is contained in some model of K/k then Z is noetherian and quasicompact and has only finitely many irreducible components.*

In view of (6.2.1), (6.2.14), and (6.2.15) we get the following.

(6.2.16). *Assume that k is noetherian. Let Z be any subset of $\mathfrak{R}'(K/k)$ such that Z is a closed subset of some open subset of some model of K/k. Let Z' be any open subset of Z, and let $Z_1, ..., Z_n$ be the irreducible components of Z labeled so that $Z' \cap Z_i \neq \varnothing$ for $1 \leqslant i \leqslant m$ and $Z' \cap Z_i = \varnothing$ for $m < i \leqslant n$. Then $Z' \cap Z_1, ..., Z' \cap Z_m$ are exactly all the distinct irreducible components of Z', and the generic point of Z_i is the generic point of $Z' \cap Z_i$ for $1 \leqslant i \leqslant m$.*

For any models X and X' of K/k such that X' dominates X and for any closed subset Z of X we define: $[X', X]$-*transform of* $Z =$ closure of $Z \cap X'$ in X'; note that $Z \cap X' = Z - \mathfrak{F}(X', X)$. In view of (6.2.1), (6.2.14), and (6.2.15) we get the following.

(6.2.17). *Assume that k is noetherian. Let X and X' be any models of K/k such that X' dominates X and let Z be any closed subset of X such that upon letting $(S_i)_{i \in I}$ be the generic points of the irreducible components of Z we have that $S_i \in X'$ (i.e., $S_i \notin \mathfrak{F}(X', X)$) for all $i \in I$. Then $(S_i)_{i \in I}$ are the generic points of the irreducible components of the $[X', X]$-transform Z' of Z (i.e., $Z' = \bigcup_{i \in I}$ closure of $\{S_i\}$ in X'), and $[X', X](Z') \subset Z$. (If X and X' are complete models of K/k then $[X', X]$ is a closed map by* [28: Lemma 5 of §17 of Chapter VI] *and hence $[X', X](Z') = Z$;* we shall not use this remark in this monograph.)

By (6.2.17) we get the following.

(6.2.18). *Assume that k is noetherian. Let $(X_i, Z_i)_{0 \leqslant i \leqslant m}$ be a sequence where m is a nonnegative integer, X_i is a model of K/k, and Z_i is a closed subset of X_i for $0 \leqslant i \leqslant m$, and for $0 < i \leqslant m$ we have that X_i dominates X_{i-1}, $\mathfrak{F}(X_i, X_{i-1})$ does not pass through the generic point of any irreducible component of Z_{i-1}, and Z_i is the $[X_i, X_{i-1}]$-transform of Z_{i-1}. Then $\mathfrak{F}(X_m, X_0)$ does not pass through the generic point of any irreducible component of Z_0, and Z_m is the $[X_m, X_0]$-transform of Z_0.*

The following observation will not be used in this monograph.

(6.2.19). *Let Z be an irreducible closed subset of a model X of K/k, let S be the generic point of Z, let h be the canonical epimorphism* *of S onto $S/M(S)$, let $Z' = \{h(R): R \in Z\}$, and let f be the map of Z onto Z' given by taking $f(R) = h(R)$ for all $R \in Z$. Then Z' is a model of $h(S)/h(k)$ and f is a homeomorphism of Z onto Z'. Moreover, if X is a complete (resp: projective) model of K/k then Z' is a complete (resp: projective) model of $h(S)/h(k)$.*

PROOF. Clearly $Z' \subset \mathfrak{R}'(h(S)/h(k))$. Fix any valuation ring V of K dominating S. Then there exists an epimorphism g of V onto an overfield L of $h(S)$ such that $g(x) = h(x)$ for all $x \in S$ and $g^{-1}(0) = M(V)$.

Let any $W \in \mathfrak{R}(h(S)/h(k))$ be given. Then there exists a valuation ring W' of L such that W' dominates W. Now $g^{-1}(W') \in \mathfrak{R}(K/k)$ and $M(g^{-1}(W')) = g^{-1}(M(W'))$. If R is any point of X such that $g^{-1}(W')$ dominates R then clearly $R \in Z$ and W dominates $h(R)$. Also, if R is any point of Z such that W dominates $h(R)$ then clearly $g^{-1}(W')$ dominates R. It follows that f is a one-to-one map of Z onto Z', Z' is an irredundant premodel of $h(S)$, and if X is a complete model of K/k then $\mathfrak{R}(h(S)/h(k))$ dominates Z'.

Now $X = \mathfrak{V}(A_1) \cup \cdots \cup \mathfrak{V}(A_n)$ where $A_1, ..., A_n$ are affine rings over k. Upon relabeling $A_1, ..., A_n$ we may assume that $\mathfrak{V}(A_i) \cap Z \neq \varnothing$ for $1 \leqslant i \leqslant m$ and $\mathfrak{V}(A_i) \cap Z = \varnothing$ for $m < i \leqslant n$. Then $h(A_1), ..., h(A_m)$ are affine rings over $h(k)$ and $Z' = \mathfrak{V}(h(A_1)) \cup \cdots \cup \mathfrak{V}(h(A_m))$. Therefore Z' is a model of $h(S)/h(k)$, and if X is a complete model of K/k then Z' is a complete model of $h(S)/h(k)$. Let $f_i(R) = f(R)$ for all $R \in \mathfrak{V}(A_i) \cap Z$. Then by (6.2.9) we get that f_i is a homeomorphism of $\mathfrak{V}(A_i) \cap Z$ onto $\mathfrak{V}(h(A_i))$ for $1 \leqslant i \leqslant m$. Therefore by (6.2.10) we get that f is a homeomorphism of Z onto Z'.

Finally assume that X is a projective model of K/k. Then there exist nonzero elements $x_1, ..., x_r$ in K such that $X = \mathfrak{V}(B_1) \cup \cdots \cup \mathfrak{V}(B_r)$ where $B_i = k[x_1/x_i, ..., x_r/x_i]$. Upon relabeling $x_1, ..., x_r$ we may assume that x_i/x_1 is a unit in V for $1 \leqslant i \leqslant s$ and $x_i/x_1 \in M(V)$ for $s < i \leqslant r$. Let $y_i = x_i/x_1$. Then y_i is a unit in V for $1 \leqslant i \leqslant s$, $y_i \in M(V)$ for $s < i \leqslant r$, and $B_i = k[y_1/y_i, ..., y_r/y_i]$ for $1 \leqslant i \leqslant r$. Now $S \in \mathfrak{V}(B_i)$ for $1 \leqslant i \leqslant s$ and $\mathfrak{V}(B_i) \cap Z = \varnothing$ for $s < i \leqslant r$, and hence $y_i \in S$ for $1 \leqslant i \leqslant r$, $y_i \notin M(S)$ for $1 \leqslant i \leqslant s$, $y_i \in M(S)$ for $s < i \leqslant r$, and $Z' = \mathfrak{V}(h(B_1)) \cup \cdots \cup \mathfrak{V}(h(B_s))$. Upon letting $z_i = h(y_i)$ we get that $z_1, ..., z_r$ elements in $h(S)$, $z_i \neq 0$ for $1 \leqslant i \leqslant s$, $z_i = 0$ for $s < i \leqslant r$, and $h(B_i) =$

$h(k)[z_1/z_i, \dots, z_r/z_i]$ for $1 \leqslant i \leqslant s$. Therefore Z' is a projective model of $h(S)/h(k)$.

(6.3). Recall that henceforth k is a noetherian domain and K is a function field over k.

Let $Z \subset \mathfrak{R}'(K/k)$. Assume that Z is a closed subset of some open subset of some model of K/k (note that by (6.2.12) this is equivalent to assuming that Z is a closed subset of some model of K/k). For any $R \in Z$ we define:

$$\mathfrak{L}(R, Z) = (\textit{local ring of } R \textit{ on } Z) = R \Big/ \Big(\bigcap_{S \in \mathfrak{V}(R) \cap Z} (R \cap M(S)) \Big) ;$$

note that upon letting $S_1, \dots, S_m$ be the generic points of the irreducible components of Z passing through R we have that

$$(1') \qquad \bigcap_{S \in \mathfrak{V}(R) \cap Z} (R \cap M(S)) = \bigcap_{i=1}^{m} (R \cap M(S_i))$$

and hence

$$(2') \qquad \dim \mathfrak{L}(R, Z) = \max_{1 \leqslant i \leqslant m} \dim(R/(R \cap M(S_i))) ;$$

consequently: (3′) $\mathfrak{L}(R, Z)$ is a field $\Leftrightarrow \dim \mathfrak{L}(R, Z) = 0 \Leftrightarrow R$ is the generic point of some irreducible component of Z. We define: $\mathfrak{S}(Z) =$ (*singular locus of* Z) $= \{R \in Z\colon \mathfrak{L}(R, Z)$ is not regular$\}$; note that then:

$$Z - \mathfrak{S}(Z) = \{R \in Z\colon \text{only one irreducible component } Z' \text{ of } Z \text{ passes through } R \text{ and } R/(R \cap M(S')) \text{ is regular where } S' \text{ is the generic point of } Z'\};$$

in particular $\mathfrak{S}(Z)$ does not pass through the generic point of any irreducible component of Z, and hence if $Z \neq \varnothing$ then $\mathfrak{S}(Z) \neq Z$. Z is said to be *nonsingular* if $\mathfrak{S}(Z) = \varnothing$. Given any regular point R of Z, we say that Z has a *normal crossing* at R if, upon letting $S_1, \dots, S_m$ be the generic points of the irreducible components of Z passing through R, we have that $\{R \cap M(S_1), \dots, R \cap M(S_m)\}$ has a normal crossing at R. Note that for any regular point R of Z we have that: $R \notin \mathfrak{S}(Z) \Leftrightarrow Z$ has a normal crossing at R and only one irreducible component of Z passes through R. Given any regular point R of Z, we say that Z has a *strict normal crossing* at R if Z has a normal crossing at R and at most two irreducible components of Z pass through R. In case every point of Z is regular, we say that Z has *only normal crossings* (resp: *only strict normal crossings*) if Z has a normal crossing (resp: strict normal crossing) at each of its points. In case every point of Z is regular, we say that Z is *unlooped* if Z has only strict normal crossings and there does not exist any infinite sequence $(Z_i, R_i)_{0 \leqslant i < \infty}$ such that for $0 \leqslant i < \infty$: Z_i is an irreducible component of Z, $Z_i \neq Z_{i+1}$, $R_i \in Z_i \cap Z_{i+1}$, and $R_i \neq R_{i+1}$. We *define:*

$$\operatorname{codim} Z = \begin{cases} \min_{R \in Z} \dim R & \text{if } Z \neq \varnothing \\ \infty & \text{if } Z = \varnothing \end{cases}$$

and

$$\dim Z = \begin{cases} \max_{R \in Z} \dim \mathfrak{L}(R, Z) & \text{if } Z \neq \varnothing \\ -\infty & \text{if } Z = \varnothing . \end{cases}$$

Given a nonnegative integer d we say that Z is pure *d-dimensional* (resp: *pure d-codimensional*) if $Z \neq \varnothing$ and $\dim Z' = d$ (resp: $\operatorname{codim} Z' = d$) for every irreducible component Z' of Z. By a *surface* (resp: *curve*) in Z we mean a pure 2-dimensional (resp: pure 1-dimensional) closed subset of Z.

(6.3.1). *Let $Z \subset \mathfrak{R}'(K/k)$. Assume that Z is a closed subset of some open subset of some model of K/k* (note that by (6.2.12) this is equivalent to assuming that Z is a closed subset of some model of K/k). *Then we have the following:* (1) *If Z is irreducible and S is the generic point of Z then* $\operatorname{codim} Z = \dim S < \dim R$ *for every $R \in Z$ with $R \neq S$.* (2) *If $Z \neq \varnothing$ and $Z_1, \ldots, Z_n$ are the irreducible components of Z then* $\operatorname{codim} Z = \min_{1 \leqslant i \leqslant n} \operatorname{codim} Z_i$. (3) *If Z is the union of a nonempty family $(Z_i)_{i \in I}$ of closed subsets Z_i of Z then* $\operatorname{codim} Z = \min_{I \in i} \operatorname{codim} Z_i$. (4) *If $Z \neq \varnothing$ and Z' is a closed subset of Z such that Z' does not contain any irreducible component of Z then* $\operatorname{codim} Z' > \operatorname{codim} Z$. (5) *If Z is irreducible and S is the generic point of Z then* $\dim Z = \max_{R \in Z} \dim(R/(R \cap M(S)))$. (6) *If $Z \neq \varnothing$ and $Z_1, \ldots, Z_n$ are the irreducible components of Z then*

$\dim Z = \max_{1 \leqslant i \leqslant n} \dim Z_i$. (7) *If Z is the union of a nonempty finite family $(Z_i)_{i \in I}$ of closed subsets Z_i of Z then* $\dim Z = \max_{i \in I} \dim Z_i$. (8) *If $Z \neq \varnothing$,* $\dim Z < \infty$, *and Z' is a closed subset of Z such that Z' does not contain any irreducible component of Z then* $\dim Z' < \dim Z$. (9) *If Z is irreducible then:* $\dim Z = 0 \Leftrightarrow Z$ *contains only one point $\Leftrightarrow$ every point of Z is a closed point of Z.* (10) $\dim Z \leqslant 0 \Leftrightarrow$ *every irreducible component of Z contains only one point $\Leftrightarrow$ every point of Z is a closed point of Z.* (11) *If* $\dim Z \leqslant 0$ *then Z is a finite set and every point of Z is a closed point of Z.*

PROOF. (1), (2), (3), and (4) are obvious. (5) and (6) follow from (2′). (7) follows from (6) and (6.1.1). For the special case when Z is irreducible, (8) follows from (5) and (6); in the general case, upon letting $Z_1, \ldots, Z_n$ be the irreducible components of Z, by (7) and the special case we get that $\dim Z = \max_{i \leqslant 1 \leqslant n} \dim Z_i$, $\dim Z' = \max_{i \leqslant 1 \leqslant n} \dim(Z_i \cap Z')$, and $\dim(Z_i \cap Z') < \dim Z_i$ for for $1 \leqslant i \leqslant n$. (9) follows from (5). (10) follows from (9) by noting that if $Z \neq \varnothing$ and $Z_1, \ldots, Z_n$ are the irreducible components of Z then by (7) we get that: $\dim Z = 0 \Leftrightarrow \dim Z_i = 0$ for $1 \leqslant i \leqslant n$. (11) follows from (10).

(6.3.2). *For any spot R over any local domain S such that R dominates S we have that* $\dim R + \operatorname{restrdeg}_S R \leqslant \dim S + \operatorname{trdeg}_S R$.

This is proved in [28: Proposition 2 of Appendix 1].

(6.3.3). *For any model X of K/k we have the following:* (1) *If Z is any closed subset of X then:* $\operatorname{codim} Z = 0 \Leftrightarrow Z = X$. (2) $\dim X = \max_{R \in X} \dim R \leqslant \dim_k K$. (3) $\mathfrak{S}(X) = \{R \in X$: *$R$ is not regular*$\}$. (4) *If* $\dim X < \infty$ *and Z is any nonempty closed subset of X then* $\dim Z + \operatorname{codim} Z \leqslant \dim X$.

PROOF. (1) and (3) are obvious. (2) follows from (6.3.2). To prove (4) we can take $R \in Z$ such that $\dim \mathfrak{L}(R, Z) = \dim Z$ and then by (2′) we get that $\dim \mathfrak{L}(R, Z) = \dim R/(R \cap M(S))$ for the generic point S of some irreducible component of Z passing through R; clearly $\operatorname{codim} Z \leqslant \dim S$ and $\dim S + \dim R/(R \cap M(S)) \leqslant \dim R \leqslant \dim X$.

(6.3.4). *Let L be any field and let L' be any pure transcendental extension of L (not necessarily of finite transcendence degree). Then there exists a valuation ring V of L' such that $L \subset V$ and V is residually rational over L.*

PROOF. We can take an ordered set I and a one-to-one map x of I into L' such that $(x(i))_{i \in I}$ is a transcendence basis of L' over L, and $L' = L((x(i))_{i \in I})$. Let J be the set of all maps n of I into the set of all nonnegative integers such that $n(i) \neq 0$ for only finitely many values of i. Given any two distinct elements n and n' in J, there exists a unique element q in I such that $n(q) \neq n'(q)$, and $n(i) = n'(i)$ for all $i \in I$ with $i > q$; we define: $n > n'$ or $n' > n$ according as $n(q) > n'(q)$ or $n'(q) > n(q)$. This makes J into an ordered set. Let R be the set of elements in L' which can be expressed in the form

$$\Big(\sum_{n \in J'} a'(n) \prod x(i)^{n(i)}\Big) \Big/ \Big(\sum_{n \in J^*} a^*(n) \prod x(i)^{n(i)}\Big)$$

where J' and J^* are nonempty finite subsets of J, $a'(n)$ and $a^*(n)$ are nonzero elements in L, and, upon letting n' and n^* be the smallest elements in J' and J^* respectively, we have that $n' \geqslant n^*$ (note that the product $\prod x(i)^{n(i)}$ is taken to be 1 in case $n(i) = 0$ for all $i \in I$, and otherwise it is taken over all $i \in I$ with $n(i) \neq 0$). Let $V = R \cup \{0\}$. It can easily be seen that then V is a valuation ring of L', $L \subset V$, and V is residually rational over L.

(6.3.5). *Let L be any field and let L' be any overfield of L. Then there exists a valuation ring V of L' such that $L \subset V$ and V is residually algebraic over L.*

PROOF. Follows from (6.3.4).

(6.3.6). *Let A be any domain, let L be any overfield of A, and let W be any valuation ring of L with $A \subset W$. Then given any prime ideal P in A with $A \cap M(W) \subset P$* (note that we can always find a maximal ideal P in A with $A \cap M(W) \subset P$), *there exists a valuation ring V of L such that $A \subset V \subset W$, $A \cap M(V) = P$, and V is residually algebraic over A.*

PROOF. Let h: $W \to W/M(W)$ be the canonical epimorphism. Then $h(A)$ is a subring of the field $h(W)$, and $h(P)$ is a prime ideal in $h(A)$. Consequently there exists a valuation ring V' of $h(W)$ such that $h(A) \subset V'$ and $h(A) \cap M(V') = h(P)$. Let h': $V' \to V'/M(V')$ be the canonical epimorphism, and let L^* be the quotient field of $h'(h(A))$ in $h'(V')$. Then by (6.3.5) there exists a valuation ring V^* of $h(V')$ such that $L^* \subset V^*$ and V^* is residually algebraic over L^*. It suffices to take $V = h^{-1}(h'^{-1}(V^*))$.

The following observation will not be used in this monograph.

(6.3.7). *If k is universally catenarian and* $\dim k < \infty$ *then for every complete model X of K/k we have that* $\dim X = \dim_k K$.

PROOF. Take $S \in \mathfrak{B}(k)$ such that $\dim S = \dim k$. By (6.3.6) there exists a valuation ring V of K such that V dominates S and V is residually algebraic over S. Let R be the center of V on X. Then R dominates S and R is residually algebraic over S. Since k is universally catenarian we get that $\dim R = \dim S + \operatorname{trdeg}_S R$. Therefore by part (2) of (6.3.3) we get that $\dim X = \dim_k K$.

(6.4). Let $X \subset \mathfrak{R}'(K/k)$. By a *preideal* on X we mean a function I which associates to each $R \in X$ an ideal in R which we denote by IR. For any preideal I on X we *define* the preideal $\operatorname{rad} I$ on X by the formula: $(\operatorname{rad} I)R = \operatorname{rad}(IR)$ for all $R \in X$. For any preideals I and I' on X we *define:* $I \subset I' \Leftrightarrow IR \subset I'R$ for all $R \in X$. For any preideals $I_1, \ldots, I_n$ on X we *define* the preideals $I_1 \cdots I_n$, $I_1 + \cdots + I_n$, and $I_1 \cap \cdots \cap I_n$ on X by the formulas: $(I_1 \cdots I_n)R = (I_1R) \cdots (I_nR)$, $(I_1 + \cdots + I_n)R = (I_1R) + \cdots + (I_nR)$, and $(I_1 \cap \cdots \cap I_n)R = (I_1R) \cap \cdots \cap (I_nR)$ for all $R \in X$. A preideal I on X is said to be *principal* if IR is a principal ideal in R for all $R \in X$. By 1_X we denote the preideal on X given by the formula: $1_XR = R$ for all $R \in X$. By 0_X we denote the preideal on X given by the formula: $0_XR = \{0\}$ for all $R \in X$. A preideal I on X is said to be *zero* if $I = 0_X$. A preideal I on X is said to be *nonzero* if $I \neq 0_X$. For any preideal I on X such that $IR \neq \{0\}$ for all $R \in X$ we *define* the preideal II^{-1} on X by the formula: $(II^{-1})R = (IR)(IR)^{-1}$ for all $R \in X$; note that then $I \subset II^{-1}$, and $(II^{-1})R \neq \{0\}$ for all

$R \in X$. For any $Z \subset X$ we *define* the preideal $\mathfrak{I}(Z, X)$ on X by the formula:

$$\mathfrak{I}(Z, X)R = \bigcap_{S \in \mathfrak{V}(R) \cap Z} (R \cap M(S)) \qquad \text{for all} \quad R \in X;$$

note that by convention the intersection of the empty family of ideals in any ring is the unit ideal and hence: $\mathfrak{I}(Z, X)R = R \Leftrightarrow \mathfrak{V}(R) \cap Z = \varnothing$. For any preideal I on X we define: $\mathfrak{Z}(I) =$ (*zero-set of* I) $= \{R \in X\colon IR \neq R\}$.

(6.4.1). *For any* $X \subset \mathfrak{R}'(K/k)$ *we have the following:* (1) *If* Z *is any subset of* X *then* $\operatorname{rad} \mathfrak{I}(Z, X) = \mathfrak{I}(Z, X)$. (2) *If* I *is any preideal on* X *then* $\mathfrak{Z}(I) = \mathfrak{Z}(\operatorname{rad} I)$. (3) *If* Z *and* Z' *are any subsets of* X *with* $Z \subset Z'$ *then* $\mathfrak{I}(Z', X) \subset \mathfrak{I}(Z, X)$. (4) *If* I *and* I' *are any preideals on* X *with* $I \subset I'$ *then* $\mathfrak{Z}(I') \subset \mathfrak{Z}(I)$. (5) *If* $Z_1, \ldots, Z_n$ *are any subsets of* X *then* $\mathfrak{I}(Z_1 \cup \cdots \cup Z_n, X) = \mathfrak{I}(Z_1, X) \cap \cdots \cap I(Z_n, X)$. (6) *If* $I_1, \ldots, I_n$ *are any preideals on* X *then* $\mathfrak{Z}(I_1 + \cdots + I_n) = \mathfrak{Z}(I_1) \cap \cdots \cap \mathfrak{Z}(I_n)$ *and* $\mathfrak{Z}(I_1 \cdots I_n) = \mathfrak{Z}(I_1 \cap \cdots \cap I_n) = \mathfrak{Z}(I_1) \cup \cdots \cup \mathfrak{Z}(I_n)$.

The proof is obvious.

For any subsets X and X' of $\mathfrak{R}'(K/k)$ such that X is an irredundant premodel of K and X' dominates X, and any preideal I on X we *define* the preideal IX' on X' by the formula: $(IX')R' = (IR)R'$ for all $R' \in X'$ where R is center of R' on X.

(6.4.2). *Let* X *and* X' *be any subsets of* $\mathfrak{R}'(K/k)$ *such that* X *is an irredundant premodel of* K *and* X' *dominates* X, *let* I *be any preideal on* X, *and let* $I' = IX'$. *Then* $\mathfrak{Z}(I') = [X', X]^{-1}(\mathfrak{Z}(I))$. *If moreover* $IR \neq \{0\}$ *for all* $R \in X$, *then* $I'R' \neq \{0\}$ *for all* $R' \in X'$, $(II^{-1})X' \subset I'I'^{-1}$, *and* $\mathfrak{Z}(I'I'^{-1}) \subset \mathfrak{Z}((II^{-1})X') = [X', X]^{-1}(\mathfrak{Z}(II^{-1}))$.

The proof is again obvious.

Given any $X \subset \mathfrak{R}'(K/k)$, any subring A of K with $\mathfrak{V}(A) \subset X$, and any preideal I on X we *define*: $A \cap I = \bigcap_{R \in \mathfrak{V}(A)} IR$; by (1.11.5) we get that: $A \cap I$ is an ideal in A; if P is any ideal in A such that $PR = IR$ for all $R \in \mathfrak{V}(A)$, then $P = A \cap I$; in particular there exists at most one ideal P in A such that $PR = IR$ for all $R \in \mathfrak{V}(A)$.

(6.4.3). *For any preideal I on any model X of K/k the following two conditions are equivalent:* (1) *If A is any affine ring over k such that $\mathfrak{B}(A) \subset X$ then $(A \cap I)R = IR$ for all $R \in \mathfrak{B}(A)$.* (2) *For each $R' \in X$ there exists an affine ring $B(R')$ over k such that $\mathfrak{B}(B(R')) \subset X$ and $(B(R') \cap I)R = IR$ for all $R \in \mathfrak{B}(B(R'))$.*

PROOF. Obviously (1) implies (2). Now assume (2) and let A be any affine ring over k such that $\mathfrak{B}(A) \subset X$. By (6.2.10), (6.2.11), and (6.2.15) we can find a finite number of nonzero elements $x_1, ..., x_n$ in A and elements $R_1, ..., R_n$ in $\mathfrak{B}(A)$ such that $\mathfrak{B}(A) = \bigcup_{i=1}^{n} \mathfrak{B}(A[x_i^{-1}])$ and $\mathfrak{B}(A[x_i^{-1}]) \subset \mathfrak{B}(B(R_i))$ for $1 \leqslant i \leqslant n$. Since $\mathfrak{B}(A[x_i^{-1}]) \subset \mathfrak{B}(B(R_i))$, by (1.11.5) we get that $B(R_i) \subset A[x_i^{-1}]$; since $(B(R_i) \cap I)R = IR$ for all $R \in \mathfrak{B}(B(R_i))$, we get that $((B(R_i) \cap I)A[x_i^{-1}])S = IS$ for all $S \in \mathfrak{B}(A[x_i^{-1}])$; consequently by (1.11.5) we get that $(A[x_i^{-1}] \cap I)S = IS$ for all $S \in \mathfrak{B}(A[x_i^{-1}])$. Clearly $(A \cap I)R \subset IR$ for all $R \in \mathfrak{B}(A)$. Now let any $R \in \mathfrak{B}(A)$ and any $z \in IR$ be given. We shall show that then $z \in (A \cap I)R$, and this will complete the proof. Upon relabeling $x_1, ..., x_n$ we may assume that $R \in A[x_1^{-1}]$. Now $(A[x_1^{-1}] \cap I)R = IR$ and hence there exists $y \in A[x_1^{-1}]$ with $y \notin M(R)$ such that $zy \in A[x_1^{-1}] \cap I$. Let $A_i = A[x_i^{-1}, x_1^{-1}]$. Then $\mathfrak{B}(A_i) = \mathfrak{B}(A[x_i^{-1}]) \cap \mathfrak{B}(A[x_1^{-1}])$. Since $\mathfrak{B}(A_i) \subset \mathfrak{B}(A[x_1^{-1}])$, we get that $zy \in A_i \cap I$. Since $\mathfrak{B}(A_i) \subset \mathfrak{B}(A[x_i^{-1}])$ and $(A[x_i^{-1}] \cap I)S = IS$ for all $S \in \mathfrak{B}(A[x_i^{-1}])$, we get that $(A[x_i^{-1}] \cap I)A_i$ is an ideal in A_i and $((A[x_i^{-1}] \cap I)A_i)S = IS$ for all $S \in \mathfrak{B}(A_i)$. Therefore by (1.11.5) we get $(A[x_i^{-1}] \cap I)A_i = A_i \cap I$ and hence $zy \in (A[x_i^{-1}] \cap I)A_i$; since $A_i = (A[x_i^{-1}])[x_1^{-1}]$, there exists a positive integer m_i such that $zyx_1^{m_i} \in A[x_i^{-1}] \cap I$. Let $m = \max(m_1, ..., m_n)$. Then $zyx_1^{m} \in A[x_i^{-1}] \cap I$ for $1 \leqslant i \leqslant n$. Since $\mathfrak{B}(A) = \bigcup_{i=1}^{n} \mathfrak{B}(A[x_i^{-1}])$, we get that $zyx_1^{m} \in A \cap I$. Now $y \in R$, $y \notin M(R)$, $x_1 \in R$, and $x_1 \notin M(R)$. Therefore $z \in (A \cap I)R$.

By an *ideal* on a model X of K/k we mean a preideal I on X satisfying the conditions of (6.4.3); note that clearly 1_X and 0_X are ideals on X.

(6.4.4). *Let $I, I_1, ..., I_n$ be any ideals on any model X of K/k and let A be any affine ring over k such that $\mathfrak{B}(A) \subset X$. Then* rad I,

$I_1 \cdots I_n$, $I_1 + \cdots + I_n$, and $I_1 \cap \cdots \cap I_n$ are ideals on X, and $\mathrm{rad}(A \cap I) = A \cap (\mathrm{rad}\, I)$, $(A \cap I_1) \cdots (A \cap I_n) = A \cap (I_1 \cdots I_n)$, $(A \cap I_1) + \cdots + (A \cap I_n) = A \cap (I_1 + \cdots + I_n)$, *and* $(A \cap I_1) \cap \cdots \cap (A \cap I_n) = A \cap (I_1 \cap \cdots \cap I_n)$.

In view of (1.11.5), this follows from the fact that if P, P_1, ..., P_n are any ideals in any domain A and R is any element in $\mathfrak{V}(A)$ then $(\mathrm{rad}\, P)R = \mathrm{rad}(PR)$, $(P_1 \cdots P_n)R = (P_1R) \cdots (P_nR)$, $(P_1 + \cdots + P_n)R = (P_1R) + \cdots + (P_nR)$, and $(P_1 \cap \cdots \cap P_n)R = (P_1R) \cap \cdots \cap (P_nR)$.

(6.4.5). *Let I be any ideal on any model X of K/k. Then for all $R \in X$ and $S \in \mathfrak{V}(R)$ we have that $IS = (IR)S$.*

The proof is obvious.

(6.4.6). *For any model X of K/k we have the following: (1) If Z is any closed subset of X then $\mathfrak{I}(Z, X)$ is an ideal on X and $\mathfrak{Z}(\mathfrak{I}(Z, X)) = Z$. (2) If I is any ideal on X then $\mathfrak{Z}(I)$ is a closed subset of X and $\mathfrak{I}(\mathfrak{Z}(I), X) = \mathrm{rad}\, I$. (3) $Z \to \mathfrak{I}(Z, X)$ is a one-to-one inclusion-reversing map of the set of all closed subsets Z of X onto the set of all ideals I on X such that $\mathrm{rad}\, I = I$, and the inverse map is given by $I \to \mathfrak{Z}(I)$.*

PROOF OF (1). It suffices to show that if Z is any closed subset of X and A is any affine ring over k such that $\mathfrak{V}(A) \subset X$ then $(A \cap \mathfrak{I}(Z, X))R = \mathfrak{I}(Z, X)R$ for all $R \in \mathfrak{V}(A)$ and $\mathfrak{V}(A) \cap \mathfrak{Z}(\mathfrak{I}(Z, X)) = \mathfrak{V}(A) \cap Z$. Now $\mathfrak{V}(A) \cap Z$ is closed in $\mathfrak{V}(A)$ and hence by (6.2.9) there exists an ideal H in A such that $\mathfrak{V}(A) \cap Z = \{R \in \mathfrak{V}(A): HR \neq R\}$. Let $P = \mathrm{rad}\, H$. Then $\mathfrak{V}(A) \cap Z = \{R \in \mathfrak{V}(A): PR \neq R\}$. First suppose that $P = A$; then $\mathfrak{V}(A) \cap Z = \varnothing$ and hence $\mathfrak{I}(Z, X)R = R = AR$ for all $R \in \mathfrak{V}(A)$; therefore by (1.11.5) we get that $A = A \cap \mathfrak{I}(Z, X)$, and clearly $\mathfrak{V}(A) \cap \mathfrak{Z}(\mathfrak{I}(Z, X)) = \varnothing$. Next suppose that $P \neq A$; then $P = P_1 \cap \cdots \cap P_n$ where P_1, ..., P_n are prime ideals in A, and for all $R \in \mathfrak{V}(A)$ we clearly have that $\mathfrak{I}(Z, X)R = (P_1R) \cap \cdots \cap (P_nR) = PR$; therefore by (1.11.5) we get that $P = A \cap \mathfrak{I}(Z, X)$, and $(A \cap \mathfrak{I}(Z, X))R = \mathfrak{I}(Z, X)R$ for all $R \in \mathfrak{V}(A)$; now $\mathfrak{V}(A) \cap Z = \{R \in \mathfrak{V}(A): PR \neq R\} = \{R \in \mathfrak{V}(A): \mathfrak{I}(Z, X)R \neq R\} = \mathfrak{V}(A) \cap \mathfrak{Z}(\mathfrak{I}(Z, X))$.

PROOF OF (2). In view of (6.2.10), it suffices to show that if I is any ideal on X and A is any affine ring over k such that $\mathfrak{V}(A) \subset X$ then $\mathfrak{V}(A) \cap \mathfrak{Z}(I)$ is closed in $\mathfrak{V}(A)$ and $\mathfrak{I}(\mathfrak{Z}(I), X)R = \operatorname{rad}(IR)$ for all $R \in \mathfrak{V}(A)$. Now $A \cap I$ is an ideal in A and $(A \cap I)R = IR$ for all $R \in \mathfrak{V}(A)$. Therefore $\mathfrak{V}(A) \cap \mathfrak{Z}(I) = \{R \in \mathfrak{V}(A): (A \cap I)R \neq R\}$ and hence by (6.2.9) we get that $\mathfrak{V}(A) \cap \mathfrak{Z}(I)$ is closed in $\mathfrak{V}(A)$. Let $P = \operatorname{rad}(A \cap I)$. Then $\mathfrak{V}(A) \cap \mathfrak{Z}(I) = \{R \in \mathfrak{V}(A): PR \neq R\}$. If $P = A$ then for all $R \in \mathfrak{V}(A)$ we clearly have that $\mathfrak{I}(\mathfrak{Z}(I), X)R = R = \operatorname{rad}(IR)$. If $P \neq A$ then $P = P_1 \cap \cdots \cap P_n$ where $P_1, \ldots, P_n$ are prime ideals in A, and for all $R \in \mathfrak{V}(A)$ we clearly have that $\mathfrak{I}(\mathfrak{Z}(I), X))R = (P_1R) \cap \cdots \cap (P_nR) = PR = \operatorname{rad}(IR)$.

PROOF OF (3). Follows from (1), (2), and (6.4.1).

(6.4.7). *Let I be any ideal on any model X of K/k. Then the following five conditions are equivalent:* (1) $I \neq 0_X$; (2) $IR \neq \{0\}$ *for all* $R \in X$; (3) $K \notin \mathfrak{Z}(I)$; (4) $\mathfrak{Z}(I) \neq X$; (5) $\operatorname{codim} \mathfrak{Z}(I) > 0$ (note that by (6.4.6) we know that $\mathfrak{Z}(I)$ is a closed subset of X). *Moreover, if $I \neq 0_X$ then II^{-1} is a nonzero ideal on X and for every affine ring A over k with $\mathfrak{V}(A) \subset X$ we have that* $(A \cap I)(A \cap I)^{-1} = A \cap (II^{-1})$.

PROOF. The first assertion follows by noting that given any $R \in X$ there exists an affine ring A over k with $R \in \mathfrak{V}(A) \subset X$ and then $(A \cap I)R = IR$, $K \in \mathfrak{V}(A)$, and $(A \cap I)K = IK$, and hence: $IR = \{0\} \Leftrightarrow A \cap I = \{0\} \Leftrightarrow IK = \{0\}$. The second assertion follows from (1.11.5) and (1.11.2).

(6.4.8). *Let I be any preideal on any subset X of $\mathfrak{R}'(K/k)$ such that $IR \neq \{0\}$ for all $R \in X$. Then $\mathfrak{Z}(II^{-1}) = \{R \in X: IR$ is not principal$\}$. In particular, $\mathfrak{Z}(II^{-1}) = \varnothing \Leftrightarrow I$ is principal.*

This follows from (1.11.4).

(6.4.9). *Let I be any nonzero ideal on any model X of K/k. Then* $\dim R \geqslant 2$ *for every normal point R of $\mathfrak{Z}(II^{-1})$* (note that every point of $\mathfrak{Z}(II^{-1}) - \mathfrak{S}(X)$ is normal). *In particular, if X is normal then* $\operatorname{codim} \mathfrak{Z}(II^{-1}) \geqslant 2$ (note that by (6.4.6) and (6.4.7) we know that $\mathfrak{Z}(II^{-1})$ is a closed subset of X; also note that if X is nonsingular then X is normal).

This follows from (6.4.8) by noting that every one-dimensional normal local domain is a principal ideal domain (see [27: §6 and §7 of Chapter V]).

(6.4.10). *Let I be any nonzero principal ideal on any model X of K/k. Then $\mathfrak{Z}(I)$ is either empty or pure* 1*-codimensional* (note that by (6.4.6) we know that $\mathfrak{Z}(I)$ is a closed subset of X).

This follows from Krull's principal ideal theorem [27: Theorem 29 on page 238].

(6.4.11). *Let X and X' be any models of K/k such that X' dominates X, and let I be any ideal on X. Then IX' is an ideal on X'. Moreover, $I = 0_X \Leftrightarrow IX' = 0_{X'}$.*

PROOF. Given any $R' \in X'$ let $R = [X', X](R')$. We can take an affine ring A over k such that $R \in \mathfrak{V}(A) \subset X$. By (6.2.5) and (6.2.12) there exists an affine ring A' over k such that $R' \in \mathfrak{V}(A') \subset [X', X]^{-1}(\mathfrak{V}(A))$. Now $A \subset [X', X](S) \subset S$ for all $S \in [X', X]^{-1}(\mathfrak{V}(A))$, and hence in particular $A \subset S$ for all $S \in \mathfrak{V}(A')$. Therefore by (1.11.5) we get $A \subset A'$. Consequently $(A \cap I)A'$ is an ideal in A', and clearly $((A \cap I)A')S = I'S$ for all $S \in \mathfrak{V}(A')$. Therefore by (1.11.5) we get that $(A \cap I)A' = A' \cap I'$, and $(A' \cap I')S = I'S$ for all $S \in \mathfrak{V}(A')$. This shows that IX' is an ideal on X'. By (6.4.7) we get that $I = 0_X \Leftrightarrow IX' = 0_{X'}$.

(6.4.12). *Let X and X' be any models of K/k such that X' dominates X, and let I be any nonzero ideal on X such that IX' is principal. Then $[X', X]^{-1}(\mathfrak{Z}(I))$ is a closed subset of X', and $[X', X]^{-1}(\mathfrak{Z}(I))$ is either empty or pure* 1*-codimensional.*

This follows from (6.4.2), (6.4.10), and (6.4.11).

(6.4.13). *Let X and X' be any models of K/k such that X' dominates X, and let Z be any closed subset of X. Then* $\mathfrak{I}([X', X]^{-1}(Z), X') = \operatorname{rad}(\mathfrak{I}(Z, X)X')$.

PROOF. By (6.4.6) we get that $\mathfrak{I}(Z, X)$ is an ideal on X, and $\mathfrak{Z}(\mathfrak{I}(Z, X)) = Z$. Hence by (6.4.2) we get that $[X', X]^{-1}(Z) = \mathfrak{Z}(\mathfrak{I}(Z, X)X')$, and by (6.4.11) we get that $\mathfrak{I}(Z, X)X'$ is an ideal on X'. Therefore by (6.4.6) we get that $\mathfrak{I}([X', X]^{-1}(Z), X') =$ rad$(\mathfrak{I}(Z, X)X')$.

(6.4.14). *Let X and X' be any models of K/k such that X' dominates X, let Y be any subset of X, let Z be any closed subset of X, let I be any nonzero ideal on X, let $Y' = [X', X]^{-1}(Y)$, let $Z' = [X', X]^{-1}(Z)$, and let $I' = IX'$. Assume that every irreducible component of $\mathfrak{Z}(II^{-1})$ having a nonempty intersection with $Z \cap Y$ is contained in Z. Then every irreducible component of $\mathfrak{Z}(I'I'^{-1})$ having a nonempty intersection with $Z' \cap Y'$ is contained in Z'.* (Note that by (6.4.6), (6.4.7), and (6.4.11) we know that $\mathfrak{Z}(II^{-1})$ is a closed subset of X, and $\mathfrak{Z}(I'I'^{-1})$ is a closed subset of X'.)

PROOF. Let Z^* be any irreducible component of $\mathfrak{Z}(I'I'^{-1})$ having a nonempty intersection with $Z' \cap Y'$. Take $R' \in Z^* \cap Z' \cap Y'$ and let $R = [X', X](R')$. Since $R' \in \mathfrak{Z}(I'I\mathrm{l}^{-1})$, by (6.4.2) we get that $R \in \mathfrak{Z}(II^{-1})$. Since $R' \in Z' \cap Y'$, we also get that $R \in Z \cap Y$. Let S^* be the generic point of Z^*. Now $S^* \in \mathfrak{V}(R')$, and R' dominates R; therefore upon letting $S = R_{R \cap M(S^*)}$ we get that $S \in \mathfrak{V}(R) \subset X$, and S^* dominates S; consequently $[X', X](S^*) = S$; since $S^* \in \mathfrak{Z}(I'I'^{-1})$, by (6.4.2) we get that $S \in \mathfrak{Z}(II^{-1})$. Let Z_1 be an irreducible component of $\mathfrak{Z}(II^{-1})$ passing through S. Now Z_1 is a closed subset of X, and hence $R \in Z_1$; therefore $Z_1 \cap Z \cap Y = \varnothing$, and hence by assumption we get that $Z_1 \subset Z$; consequently $S \in Z$, and hence $S^* \in Z'$. By (6.2.5) we know that Z' is a closed subset of X', and hence $Z^* \subset Z'$.

(6.5). For any ideal J on any model X of K/k we define: $\mathfrak{S}(J) =$ (*singular locus of* J) $= \{R \in \mathfrak{Z}(J)$: $R/(JR)$ is not regular$\}$. In view of (6.4.4) and (6.4.6) we get the following.

(6.5.1). *For any model X of K/k we have the following:* (1) *If Z is any closed subset of X then $\mathfrak{S}(Z) = \mathfrak{S}(\mathfrak{I}(Z, X))$.* (2) *If J is any ideal on X then $\mathfrak{S}(J) = \mathfrak{S}(\mathrm{rad}\, J) \cup \{R \in X$: $JR \neq \mathrm{rad}(JR)\}$.* (3) *If J is any ideal on X and A is any affine ring over k with $\mathfrak{V}(A) \subset X$ then $\mathfrak{S}(J) \cap \mathfrak{V}(A) = \mathfrak{S}(A, A \cap J)$.*

Let J and I be any nonzero principal ideals on any nonsingular model X of K/k. Given any $R \in X$ and $E \subset \mathfrak{V}(R)$, we say that (E, I)

has a *normal crossing* at R if (E, IR) has a normal crossing at R. Given any $R \in X$ and $E \subset \mathfrak{V}(R)$, we say that (E, I) has a *strict normal crossing* at R if (E, IR) has a strict normal crossing at R. We say that I has *only normal crossings* if for each $R \in X$ we have that IR has a normal crossing at R. We say that (J, I) has *only quasinormal crossings* if for each $R \in X$ we have that (JR, IR) has a quasinormal crossing at R. We say that I has *only quasinormal crossings* if for each $R \in X$ we have that IR has a quasinormal crossing at R; note that this is equivalent to saying that $(I, 1_X)$ has only quasinormal crossings. Given $R \in X$ and $S \in \mathfrak{V}(R)$, we say that (S, I) has a *pseudonormal crossing* at R if (S, IR) has a pseudonormal crossing at R. Given $R \in X$ and $E \subset \mathfrak{V}(R)$, we say that (E, I) has a *pseudonormal crossing* at R if (E, IR) has a pseudonormal crossing at R. We *define:* $\mathfrak{S}^*(J) = \{R \in X\colon (R, JR)$ is unresolved$\}$. J is said to be *resolved* if $\mathfrak{S}^*(J) = \varnothing$. In view of (6.4.4) and (6.4.6) we get the following.

(6.5.2). *For any nonzero principal ideals J and I on any nonsingular model X of K/k we have the following:* (1) $\mathfrak{S}^*(J) = \mathfrak{S}^*(\operatorname{rad} J) = \mathfrak{S}(\operatorname{rad} J) = \mathfrak{S}(\mathfrak{Z}(J))$. (2) *$I$ has only normal crossings* $\Leftrightarrow$ rad *I has only normal crossings* $\Leftrightarrow$ *$\mathfrak{Z}(I)$ has only normal crossings.* (3) *I has only quasinormal crossings* $\Leftrightarrow$ rad *I has only quasinormal crossings* $\Leftrightarrow$ *each irreducible component of $\mathfrak{Z}(I)$ is nonsingular.* (4) *If I has only normal crossings then I has only quasinormal crossings.* (5) *If (J, I) has only quasinormal crossings and $\mathfrak{S}^*(J) = \varnothing$ then JI has only normal crossings.* (6) *If I has only quasinormal crossings and $\mathfrak{S}^*(J) = \varnothing$ then JI has only quasinormal crossings.*

In view of (6.2.10), (6.4.6), (6.5.1), and part (1) of (6.5.2) we get the following.

(6.5.3). *Assume that for every affine ring A over k with quotient field K and every ideal Q in A we have that $\mathfrak{S}(A, Q)$ is closed in $\mathfrak{V}(A)$* (see (1.2.6)). *Let X be any model of K/k. Then we have the following:* (1) *If Z is any closed subset X then $\mathfrak{S}(Z)$ is closed in X.* (2) *If J is any ideal on X then $\mathfrak{S}(J)$ is closed in X.* (3) *If X is nonsingular and J is any nonzero principal ideal on X then $\mathfrak{S}^*(J)$ is closed in X and* codim $\mathfrak{S}^*(J) \geqslant 2$.

For any ideal J on any model X of K/k and any $Z \subset X$ we *define:*

$$\operatorname{ord}_Z J = \begin{cases} \max_{R \in Z} \operatorname{ord}_R JR & \text{if } Z \neq \varnothing \\ -\infty & \text{if } Z = \varnothing \end{cases}$$

and: $\mathfrak{E}^*(Z, J) = \{R \in Z\colon \operatorname{ord}_R JR = \operatorname{ord}_Z J\}$; note that: (1) if $Z \neq \varnothing$ then $\operatorname{ord}_Z J$ is either a nonnegative integer or ∞; (2) if $Z \neq \varnothing$ and $J = 0_X$ then $\operatorname{ord}_Z J = \infty$ and $\mathfrak{E}^*(Z, J) = Z$; (3) if $Z \neq \varnothing$ and $J \neq 0_X$ then: $\mathfrak{E}^*(Z, J) \neq \varnothing \Leftrightarrow \operatorname{ord}_Z J$ is a nonnegative integer. For any regular point R of any model X of K/k and any ideal J on X we *define*: $\mathfrak{E}(R, J) = \mathfrak{E}(R, JR)$ and $\mathfrak{E}^i(R, J) = \mathfrak{E}^i(R, JR)$ for every nonnegative integer i; note that by (1.3.1) we then have that $\mathfrak{E}(R, J) = \mathfrak{E}^*(\mathfrak{V}(R), J)$. From (0.4) we now deduce the following.

(6.5.4). *Assume that for every affine ring A over k with quotient field K and every ideal Q in A we have that $\mathfrak{S}(A, Q)$ is closed in $\mathfrak{V}(A)$* (see (1.2.6)). *Let J be any ideal on any model X of K/k, let Y be any open subset of X with $\mathfrak{S}(X) \cap Y = \varnothing$, and let Z be any nonempty closed subset of Y. Then $\mathfrak{E}^*(Z, J)$ is a nonempty closed subset of Z.*

PROOF. Let W be the set of all nonempty closed subsets Z^* of Z such that $\mathfrak{E}^*(Z^*, J) = \mathfrak{E}^*(Z, J)$. Then $W \neq \varnothing$ and hence by (6.2.15), W contains a minimal element Z'. Let $Z_1, ..., Z_n$ be the irreducible components of Z', and let S_i be the generic point of Z_i. In view of (6.2.12) and (6.5.3) we can find an affine ring A_i over k such that $S_i \in \mathfrak{V}(A_i) \subset Y$ and $\mathfrak{V}(A_i) \cap \mathfrak{S}(Z_i) = \varnothing$. Upon taking $(A_i, A_i \cap J, A_i \cap M(S_i))$ for (A, J, Q) in (0.4), we can find an ideal H_i in A_i with $H_i S_i = S_i$ such that $\operatorname{ord}_R JR = \operatorname{ord}_{S_i} JS_i$ for all $R \in \mathfrak{V}(A_i)$ for which $R \subset S_i$ and $H_i R = R$; upon letting $Z_i' = \{R \in \mathfrak{V}(A_i)\colon R \subset S_i \text{ and } H_i R \neq R\} \cup (Z_i - \mathfrak{V}(A_i))$, in view of (6.2.9) and (6.2.12) we get that Z_i' is a closed subset of Z_i with $S_i \notin Z_i'$, and clearly $\operatorname{ord}_R JR = \operatorname{ord}_{S_i} JS_i$ for all $R \in Z_i - Z_i'$. Upon relabeling $Z_1, ..., Z_n$ we may assume that $\operatorname{ord}_{Z_1} J = \cdots = \operatorname{ord}_{Z_m} J > \operatorname{ord}_{Z_i} J$ for $m < i \leqslant n$; clearly $Z_1 \cup \cdots \cup Z_m$ is a nonempty closed subset of Z' and $\mathfrak{E}^*(Z_1 \cup \cdots \cup Z_m, J) = \mathfrak{E}^*(Z', J)$; since Z' is a minimal element of W, we must have $Z_1 \cup \cdots \cup Z_m = Z'$ and hence $m = n$. Thus $\operatorname{ord}_{Z_i} J = \operatorname{ord}_{Z'} J$ for

$1 \leqslant i \leqslant n$. If $\mathrm{ord}_{S_i} JS_i < \mathrm{ord}_{Z'} J$ for some i then, upon letting $Z'' = Z_1 \cup \cdots \cup Z_{i-1} \cup Z'_i \cup Z_{i+1} \cup \cdots \cup Z_n$, we would get that Z'' is a nonempty closed subset of Z', $Z'' \neq Z'$, and $\mathfrak{E}^*(Z'', J) = \mathfrak{E}^*(Z', J)$, and this would contradict the assumption that Z' is a minimal element of W. Therefore $\mathrm{ord}_{S_i} JS_i = \mathrm{ord}_{Z'} J$ for $1 \leqslant i \leqslant n$. Consequently by (1.3.1) we get that $\mathrm{ord}_R JR = \mathrm{ord}_{Z'} J$ for all $R \in Z'$, and hence $Z' = \mathfrak{E}^*(Z', J) = \mathfrak{E}^*(Z, J)$.

(6.5.5). *Assume that for every affine ring A over k with quotient field K and every ideal Q in A we have that $\mathfrak{S}(A, Q)$ is closed in* $\mathfrak{V}(A)$ (see (1.2.6)). *Let X be any model of K/k, let Y be any open subset of X such that $\mathfrak{S}(X) \cap Y = \varnothing$, let Z be any closed subset of Y, let J be any ideal on X, and let $Z^* = \mathfrak{S}(Z) \cup \{R \in Z - \mathfrak{S}(Z)$: $S \notin \mathfrak{E}(R, J)$ where S is the generic point of the irreducible component of Z passing through $R\}$. Then Z^* is a closed subset of Z. Moreover, if* $\dim_k K \leqslant 3$ *and* codim $Z \geqslant 2$ *then Z^* consists of a finite number of closed points of X.*

PROOF. Let $Z_1, \ldots, Z_n$ be the irreducible components of Z, let S_i be the generic point of Z_i, and let $Z_i^* = \mathfrak{S}(Z_i) \cup \{R \in Z_i - \mathfrak{S}(Z_i)$: $S_i \notin \mathfrak{E}(R, J)\}$. Clearly $Z_i \not\subset Z^*$ for $1 \leqslant i \leqslant n$, and hence, in view of (6.3.1) and (6.3.3), the second assertion follows from the first assertion. Also $Z^* = Z^* \cup \cdots \cup Z_n^*$, and hence in proving the first assertion, without loss of generality, we may assume that Z is irreducible. Let S be the generic point of Z, and let $Z' = \{R \in Z$: $\mathrm{ord}_R JR > \mathrm{ord}_S JS\}$. Then by (1.3.1) we get that $Z^* = \mathfrak{S}(Z) \cup Z'$, and by (6.5.3) we know that $\mathfrak{S}(Z)$ is a closed subset of Z. Therefore it suffices to show that Z' is a closed subset of Z. If J is zero then $Z' = \varnothing$. So now also assume that J is nonzero. Let $d = \mathrm{ord}_Z J$. Then by (6.5.4) we get that d is a nonnegative integer. Let $Z'_i = \{R \in Z$: $\mathrm{ord}_R JR > d - i\}$. By induction we shall show that Z'_i is a closed subset of Z for $0 \leqslant i \leqslant d$. Clearly $Z'_0 = \varnothing$. So let $0 < i \leqslant d$ and assume that Z'_{i-1} is a closed subset of Z. If $Z'_i = Z'_{i-1}$ then we have nothing to show. So now suppose that $Z'_i \neq Z'_{i-1}$. Then $Y - Z'_{i-1}$ is an open subset of X with $(Y - Z'_{i-1}) \cap \mathfrak{S}(X) = \varnothing$, $Z - Z'_{i-1}$ is a nonempty closed subset of $Y - Z'_{i-1}$, and $\mathfrak{E}^*(Z - Z'_{i-1}, J) = Z'_i - Z'_{i-1}$. Therefore by (6.5.4) we get that $Z'_i - Z'_{i-1}$ is a closed subset of $Z - Z'_{i-1}$. It follows that Z'_i is a closed subset of Z. This completes the induction.

Now $Z' = Z'_e$ where $e = d - \mathrm{ord}_S JS$, and hence Z' is a closed subset of Z.

(6.5.6). *Let J be any nonzero principal ideal on any nonsingular model X of K/k. Then for any $R \in X$ we have the following:* (1) *If $R \in \mathfrak{S}^*(J)$ then $\mathfrak{C}(R, J) \subset \mathfrak{S}^*(J)$.* (2) *If $R \in \mathfrak{C}^*(\mathfrak{S}^*(J), J)$ then $\mathfrak{C}(R, J) = \mathfrak{V}(R) \cap \mathfrak{C}^*(\mathfrak{S}^*(J), J)$.*

PROOF. (1) follows from (1.5.3). (2) follows from (1).

(6.5.7). *Assume that* $\dim_k K \leqslant 3$. *Let X be any nonsingular model of K/k, let Z be any closed subset of X with* $\operatorname{codim} Z \geqslant 2$, *let I be any nonzero principal ideal on X such that I has only quasinormal crossings, and let $Z' = \{R \in Z - \mathfrak{S}(Z)$: (S, I) does not have a pseudonormal crossing at R where S is the generic point of the irreducible component of Z passing through $R\}$. Then Z' consists of a finite number of closed points of X.*

PROOF. Let $Z_1, \ldots, Z_n$ be the irreducible components of Z, let S_i be the generic point of Z_i, and let $Z'_i = \{R \in Z_i - \mathfrak{S}(Z_i)$: (S_i, I) does not have a pseudonormal crossing at $R\}$. Then clearly $Z' \subset Z'_1 \cup \cdots \cup Z'_n$ and $\operatorname{codim} Z_i \geqslant 2$ for $1 \leqslant i \leqslant n$, and hence, without loss of generality, we may assume that Z is irreducible. Let S be the generic point of Z. By (6.4.6) we know that $\mathfrak{Z}(I)$ is a closed subset of X. Let $Y_1, \ldots, Y_t$ be the irreducible components of $\mathfrak{Z}(I)$ passing through S, and let Y be the union of the remaining irreducible components of $\mathfrak{Z}(I)$. In view of (6.3.1) and (6.3.3) we get that $Z \cap Y$ consists of a finite number of closed points of X. Therefore it suffices to show that $Z' \subset Z \cap Y$, i.e., (S, I) has a pseudonormal crossing at R for all $R \in Z - \mathfrak{S}(Z) - Y$. So let any $R \in Z - \mathfrak{S}(Z) - Y$ be given. Clearly (S, I) has a pseudonormal crossing at S. So now assume that $R \neq S$. Then in view of (6.3.3) we get that $\dim S = 2$ and $\dim R = 3$. Since IR has a quasinormal crossing at R, we get that $IR = x_1^{a_1} \cdots x_t^{a_t} R$ where $a_1, \ldots, a_t$ are positive integers and $x_1, \ldots, x_t$ are elements in R such that $\mathrm{ord}_R x_i = 1$ and $x_i R = \mathfrak{I}(Y_i, X)R$ for $1 \leqslant i \leqslant t$ (we take $x_1^{a_1} \cdots x_t^{a_t} R = R$ in case $t = 0$). Also there exists a basis (x, y, z) of $M(R)$ such that $R \cap M(S) = (x, y)R$. Since $S \in Y_i$, we get that $x_i \in (x, y)R$, i.e., $x_i = u_i x + v_i y$ with $u_i \in R$ and $v_i \in R$. Since $\mathrm{ord}_R x_i = 1$, we must have either $u_i \notin M(R)$ or $v_i \notin M(R)$. Let $y_i = y$ in case $u_i \notin M(R)$, and $y_i = x$ in case $u_i \in M(R)$. Then (x_i, y_i, z) is a basis of $M(R)$ and $R \cap M(S) = (x_i, y_i)R$. This being so for $1 \leqslant i \leqslant t$, we conclude that (S, I) has a pseudonormal crossing at R.

(6.5.8). *Assume that* $\dim_k K \leqslant 3$. *Let X be any nonsingular model of K/k, let Z be any closed subset of X with* $\operatorname{codim} Z \geqslant 2$, *let I be any nonzero principal ideal on X such that I has only normal crossings, and let $Z' = \{R \in Z - \mathfrak{S}(Z)\colon (S, I)$ does not have a normal crossing at R where S is the generic point of the irreducible component of Z passing through $R\}$. Then Z' consists of a finite number of closed points of X.*

PROOF. Let $Z_1, \ldots, Z_n$ be the irreducible components of Z, let S_i be the generic point of Z_i, and let $Z_i' = \{R \in Z_i - \mathfrak{S}(Z_i)\colon (S_i, I)$ does not have a normal crossing at $R\}$. Then clearly $Z' \subset Z_1' \cup \cdots \cup Z_n'$ and $\operatorname{codim} Z_i \geqslant 2$ for $1 \leqslant i \leqslant n$, and hence, without loss of generality, we may assume that Z is irreducible. Let S be the generic point of Z. By (6.4.6) we know that $\mathfrak{Z}(I)$ is a closed subset of X. Let $Y_1, \ldots, Y_t$ be the irreducible components of $\mathfrak{Z}(I)$ passing through S, and let Y be the union of the remaining irreducible components of $\mathfrak{Z}(I)$. In view of (6.3.1) and (6.3.3) we get that $Z \cap Y$ consists of a finite number of closed points of X. Therefore it suffices to show that $Z' \subset Z \cap Y$, i.e., (S, I) has a normal crossing at R for all $R \in Z - \mathfrak{S}(Z) - Y$. So let any $R \in Z - \mathfrak{S}(Z) - Y$ be given. Clearly (S, I) has a normal crossing at S. So now assume that $R \neq S$. Then in view of (6.3.3) we get that $\dim S = 2$ and $\dim R = 3$. First suppose that $t = 0$; then $IR = R$ and hence (S, I) has a normal crossing at R. Next suppose that $t = 1$; then $IR = x^a R$ where a is positive integer and x is an element in R such that $\operatorname{ord}_R x = 1$ and $xR = \mathfrak{I}(Y_1, X)R$; also there exists a basis (x', y', z') of $M(R)$ such that $R \cap M(S) = (x', y')R$; since $S \in Y_1$ we get that $x \in (x', y')R$, i.e., $x = ux' + vy'$ with $u \in R$ and $v \in R$; since $\operatorname{ord}_R x = 1$ we must have either $u \notin M(R)$ or $v \notin M(R)$; upon relabeling x' and y' we may assume that $u \notin M(R)$; then (x, y', z') is a basis of $M(R)$ and $R \cap M(S) = (x, y')R$; therefore (S, I) has a normal crossing at R. Finally suppose that $t \geqslant 2$; then there exists a basis (x, y, z) of $M(R)$ such that

$xR = \mathfrak{I}(Y_1, X)R, yR = \mathfrak{I}(Y_2, X)R,$ and $IR = x^a y^b z^c R$ where a and b are positive integers and c is a nonnegative integer; since $S \in Y_1 \cap Y_2$, we get that $(x, y)R \subset R \cap M(S)$; since $\dim S = 2$, we must have $R \cap M(S) = (x, y)R$; therefore (S, I) has a normal crossing at R.

(6.6). Let X be any model of K/k and let I be any nonzero ideal on X.

We *define:*

$$\mathfrak{W}(X, I) = \bigcup_{R \in X} \{R' \in \mathfrak{W}(R, IR) \colon R' \text{ dominates } R\}.$$

(6.6.1). *$\mathfrak{W}(X, I)$ is an irredundant premodel of K, $\mathfrak{W}(X, I)$ properly dominates X, and for all $R \in X$ we have $[\mathfrak{W}(X, I), X]^{-1}(R) = \{R' \in \mathfrak{W}(R, IR) \colon R'$ dominates $R\}$. If $\mathfrak{R}(K/k)$ dominates X then $\mathfrak{R}(K/k)$ dominates $\mathfrak{W}(X, I)$.*

This follows by noting that if P is any nonzero ideal in any noetherian domain A with quotient field L then $\mathfrak{W}(A, P)$ is a projective model of L/A and hence $\mathfrak{W}(A, P)$ is a complete model of L/A.

(6.6.2). *Let A be any noetherian subring of K such that $\mathfrak{V}(A) \subset X$ and let P be any nonzero ideal in A such that $PR = IR$ for all $R \in \mathfrak{V}(A)$. Then $[\mathfrak{W}(X, I), X]^{-1}(\mathfrak{V}(A)) = \mathfrak{W}(A, P) = \bigcup_{R \in \mathfrak{V}(A)} \mathfrak{W}(R, IR)$.*

PROOF. We can take nonzero elements $x_1, \ldots, x_n$ in A such that $(x_1, \ldots, x_n)A = P$. Then

$$\mathfrak{W}(A, P) = \bigcup_{i=1}^{n} \mathfrak{V}(A[x_1/x_i, \ldots, x_n/x_i])$$

and

$$\mathfrak{W}(R, IR) = \bigcup_{i=1}^{n} \mathfrak{V}(R[x_1/x_i, \ldots, x_n/x_i]) \qquad \text{for all} \quad R \in \mathfrak{V}(A).$$

Hence our assertion follows by noting that for $1 \leqslant i \leqslant n$ we clearly have that

$$\bigcup_{R \in \mathfrak{V}(A)} \{R' \in \mathfrak{V}(R[x_1/x_i, ..., x_n/x_i]) \colon R' \text{ dominates } R\}$$
$$= \mathfrak{V}(A[x_1/x_i, ..., x_n/x_i])$$
$$= \bigcup_{R \in \mathfrak{V}(A)} \mathfrak{V}(R[x_1/x_i, ..., x_n/x_i]).$$

By (6.6.2) we get (6.6.3) and (6.6.4).

(6.6.3). $[\mathfrak{W}(X, I), X]^{-1}(\mathfrak{V}(R)) = \mathfrak{W}(R, IR)$ *for all* $R \in X$, *and hence* $\mathfrak{W}(X, I) = \bigcup_{R \in X} \mathfrak{W}(R, IR)$.

(6.6.4). *If A is any affine ring over k such that $\mathfrak{V}(A) \subset X$ then* $[\mathfrak{W}(X, I), X]^{-1}(\mathfrak{V}(A)) = \mathfrak{W}(A, A \cap I) = \bigcup_{R \in \mathfrak{V}(A)} \mathfrak{W}(R, IR)$.

By (6.6.1) and (6.6.4) we get the following.

(6.6.5). *$\mathfrak{W}(X, I)$ is a model of K/k. If $X = \bigcup_{i=1}^{n} \mathfrak{V}(A_i)$ where $A_1, ..., A_n$ are affine rings over k then $\mathfrak{W}(X, I) = \bigcup_{i=1}^{n} \mathfrak{W}(A_i, A_i \cap I)$. If X is a complete model of K/k then $\mathfrak{W}(X, I)$ is a complete model of K/k.*

Next we prove the following.

(6.6.6). *Let J be any nonzero ideal on X. Then* $\mathfrak{W}(X, I) + \mathfrak{W}(X, J) = \mathfrak{W}(X, IJ) = \mathfrak{W}(\mathfrak{W}(X, I), J\mathfrak{W}(X, I))$ (note that by (6.4.4), (6.4.11), and (6.6.5) we know that: IJ is a nonzero ideal on X; $\mathfrak{W}(X, I)$, $\mathfrak{W}(X, J)$, and $\mathfrak{W}(X, IJ)$ are models of K/k; and $J\mathfrak{W}(X, I)$ is a nonzero ideal on $\mathfrak{W}(X, I)$).

PROOF. Now $X = \bigcup_{d=1}^{e} \mathfrak{V}(A_d)$ where $A_1, ..., A_e$ are affine rings over k. We can take nonzero elements $(x_{di})_{1 \leqslant i \leqslant m(d)}$ in A_d which form a basis of $A_d \cap I$, and nonzero elements $(y_{dj})_{1 \leqslant j \leqslant n(d)}$ in A_d which form a basis of $A_d \cap J$. Then $(x_{di}y_{dj})_{1 \leqslant i \leqslant m(d), 1 \leqslant j \leqslant n(d)}$ are nonzero elements in A_d, and by (6.4.4) we know that they form a basis of $A_d \cap (IJ)$. Let

$$\begin{aligned} E_{di} &= A_d[(x_{da}/x_{di})_{1 \leqslant a \leqslant m(d)}], \\ F_{dj} &= A_d[(y_{db}/y_{dj})_{1 \leqslant b \leqslant n(d)}], \\ G_{dij} &= A_d[((x_{da}y_{db})/(x_{di}y_{dj}))_{1 \leqslant a \leqslant m(d), 1 \leqslant b \leqslant n(d)}]. \end{aligned}$$

Then by (6.6.5) we get that:

$$\mathfrak{W}(X, I) = \bigcup_{d=1}^{e} \mathfrak{W}(A_d, A_d \cap I), \qquad \mathfrak{W}(A_d, A_d \cap I) = \bigcup_{i=1}^{m(d)} \mathfrak{V}(E_{di});$$

$$\mathfrak{W}(X, J) = \bigcup_{d=1}^{e} \mathfrak{W}(A_d, A_d \cap J), \qquad \mathfrak{W}(A_d, A_d \cap J) = \bigcup_{j=1}^{n(d)} \mathfrak{V}(F_{dj});$$

$$\mathfrak{W}(X, IJ) = \bigcup_{d=1}^{e} \bigcup_{i=1}^{m(d)} \bigcup_{j=1}^{n(d)} \mathfrak{V}(G_{dij}).$$

Clearly G_{dij} is the smallest subring of K which contains E_{di} and F_{dj}. Therefore by (6.2.7) and (6.2.12) we get that $\mathfrak{W}(A_d, A_d \cap I)$ and $\mathfrak{W}(A_d, A_d \cap J)$ are models of K/k for $1 \leqslant d \leqslant e$, and

$$\begin{aligned} \mathfrak{W}(X, IJ) &= \bigcup_{d=1}^{e} (\mathfrak{W}(A_d, A_d \cap I) + \mathfrak{W}(A_d, A_d \cap J)) \\ &\subset \mathfrak{W}(X, I) + \mathfrak{W}(X, J). \end{aligned}$$

In particular, $\mathfrak{W}(X, IJ)$ dominates $\mathfrak{W}(X, I)$ and $\mathfrak{W}(X, J)$. Let any $R \in \mathfrak{R}'(K/k)$ be given such that R dominates $\mathfrak{W}(X, I)$ and $\mathfrak{W}(X, J)$; then R dominates X; let R^* be the center of R on X; then $R^* \in \mathfrak{V}(A_d)$ for some d; by (6.6.2) it follows that R dominates $\mathfrak{W}(A_d, A_d \cap I)$ and $\mathfrak{W}(A_d, A_d \cap J)$; consequently R dominates $\mathfrak{W}(A_d, A_d \cap I) + \mathfrak{W}(A_d, A_d \cap J)$, and hence R dominates $\mathfrak{W}(X, IJ)$. Therefore $\mathfrak{W}(X, IJ) = \mathfrak{W}(X, I) + \mathfrak{W}(X, J)$. Now $(A_d \cap J)E_{di}$ is an ideal in E_{di}, and $((A_d \cap J)E_{di})S = (J\mathfrak{W}(X, I))S$ for all $S \in \mathfrak{V}(E_{di})$; consequently by (1.11.5) we get that $E_{di} \cap (J\mathfrak{W}(X, I)) = (A_d \cap J)E_{di}$, and hence the elements $(y_{dj})_{1 \leqslant j \leqslant n(d)}$ form a basis of $E_{di} \cap (J\mathfrak{W}(X, I))$; clearly

$$G_{dij} = E_{di}[(y_{db}/y_{dj})_{1 \leqslant b \leqslant n(d)}],$$

and hence by (6.6.5) we get that

$$\mathfrak{W}(\mathfrak{W}(X, I), J\mathfrak{W}(X, I)) = \bigcup_{d=1}^{e} \bigcup_{i=1}^{m(d)} \bigcup_{j=1}^{n(d)} \mathfrak{V}(G_{dij}).$$

Therefore $\mathfrak{W}(X, IJ) = \mathfrak{W}(\mathfrak{W}(X, I), J\mathfrak{W}(X, I))$.

(6.6.7). *Let X' be any model of K/k dominating X. Then* $\mathfrak{W}(X', IX') = \mathfrak{W}(X, I) + X'$ (note that by (6.4.11) and (6.6.5) we know that IX' is a nonzero ideal on X', and $\mathfrak{W}(X, I)$ and $\mathfrak{W}(X', IX')$ are models of K/k).

PROOF. Now $X = \bigcup_{d=1}^{e} \mathfrak{V}(A_d)$ where $A_1, \ldots, A_e$ are affine rings over k. Note that K is a point of every model of K/k, and K is not dominated by any point of $\mathfrak{R}'(K/k)$ other than K. Therefore, upon letting $X'_d = [X', X]^{-1}(\mathfrak{V}(A_d))$, by (6.2.5) and (6.2.12) we get that X'_d is a model of K/k and $X'_d = \bigcup_{p=1}^{q(d)} \mathfrak{V}(B_{dp})$ where B_{dp} is an affine ring over k for $1 \leqslant p \leqslant q(d)$. By (1.11.5) it follows that $A_d \subset B_{dp}$, and hence $(A_d \cap I)B_{dp} = B_{dp} \cap (IX')$. We can take nonzero elements $(x_{di})_{1 \leqslant i \leqslant m(d)}$ in A_d which form a basis of $A_d \cap I$. Let

$$A_{di} = A_d[(x_{da}/x_{di})_{1 \leqslant a \leqslant m(d)}],$$
$$B_{dpi} = B_{dp}[(x_{da}/x_{di})_{1 \leqslant a \leqslant m(d)}].$$

Then by (6.6.5) we get that:

$$\mathfrak{W}(X, I) = \bigcup_{d=1}^{e} \mathfrak{W}(A_d, A_d \cap I), \qquad \mathfrak{W}(A_d, A_d \cap I) = \bigcup_{i=1}^{m(d)} \mathfrak{V}(A_{di});$$

$$\mathfrak{W}(X', IX') = \bigcup_{d=1}^{e} \bigcup_{p=1}^{q(d)} \bigcup_{i=1}^{m(d)} \mathfrak{V}(B_{dpi}).$$

Clearly B_{dpi} is the smallest subring of K which contains A_{di} and B_{dp}. Therefore by (6.2.7) and (6.2.12) we get that $\mathfrak{W}(A_d, A_d \cap I)$ is a model of K/k for $1 \leqslant d \leqslant e$, and

$$\mathfrak{W}(X', IX') = \bigcup_{d=1}^{e} (\mathfrak{W}(A_d, A_d \cap I) + X'_d) \subset \mathfrak{W}(X, I) + X'.$$

In particular, $\mathfrak{W}(X', IX')$ dominates $\mathfrak{W}(X, I)$ and X'. Let any $R \in \mathfrak{R}'(K/k)$ be given such that R dominates $\mathfrak{W}(X, I)$ and X'; then R dominates X; let R^* be the center of R on X; then $R^* \in \mathfrak{V}(A_d)$ for some d; by (6.6.2) it follows that R dominates $\mathfrak{W}(A_d, A_d \cap I)$, and clearly R dominates X'_d; consequently R dominates $\mathfrak{W}(A_d, A_d \cap I) + X'_d$, and hence R dominates $\mathfrak{W}(X', IX')$. Therefore $\mathfrak{W}(X', IX') = \mathfrak{W}(X, I) + X'$.

(6.6.8). *For any subset X' of $\mathfrak{R}'(K/k)$ dominating X, upon letting $I' = IX'$, we have that $\mathfrak{F}(\mathfrak{W}(X, I), X') = \mathfrak{Z}(I'I'^{-1}) = \{R' \in X' : I'R'$ is not principal$\}$, and: X' dominates $\mathfrak{W}(X, I) \Leftrightarrow I'$ is principal.* (Note that then in particular $I\mathfrak{W}(X, I)$ is principal, and $\mathfrak{W}(X, I)$ can be characterized as the unique model X^* of K/k having the following two properties: (1) IX^* is principal; and (2) if X' is any model of K/k such that X' dominates X and IX' is principal, then X' dominates X^*; this characterization of $\mathfrak{W}(X, I)$ is due to Hironaka.)

PROOF. Clearly, X' dominates $\mathfrak{W}(X, I) \Leftrightarrow \mathfrak{F}(\mathfrak{W}(X, I), X') = \varnothing$. Therefore, in view of (6.4.8), it suffices to show that for any $R' \in X'$, upon letting $R = [X', X](R')$, we have that: R' dominates $\mathfrak{W}(X, I) \Leftrightarrow (IR)R'$ is principal. Clearly $I\mathfrak{W}(X, I)$ is principal, and hence we get that if R' dominates $\mathfrak{W}(X, I)$ then $(IR)R'$ is principal. Conversely, suppose that $(IR)R'$ is principal. Let $(x_1, ..., x_n)$ be any basis of IR. Then $(x_1, ..., x_n)R'$ is a nonzero principal ideal in R', and hence there exists i such that $x_i \neq 0$ and $x_j/x_i \in R'$ for $1 \leqslant j \leqslant n$. Let $A = R[x_1/x_i, ..., x_n/x_i]$. Then $A \subset R'$, R' dominates $A_{A \cap M(R')}$, and $A_{A \cap M(R')} \in \mathfrak{W}(R, IR)$. By (6.6.2) we know that $\mathfrak{W}(R, IR) \subset \mathfrak{W}(X, I)$, and hence R' dominates $\mathfrak{W}(X, I)$.

(6.6.9). *Let $X^* = \mathfrak{W}(X, I)$. Then we have the following:* (1) *$[X^*, X]^{-1}(\mathfrak{Z}(I))$ is a closed subset of X^*, and $[X^*, X]^{-1}(\mathfrak{Z}(I))$ is either empty or pure* 1*-codimensional* (note that by (6.6.1) and (6.6.5) we know that X^* is a model of K/k dominating X). (2) *$\mathfrak{F}(X^*, X) = \mathfrak{Z}(II^{-1}) = \{R \in X : IR$ is not principal$\}$.* (3) *$\mathfrak{F}(X^*, X)$ is a closed subset of X.* (4) *$[X^*, X]^{-1}(\mathfrak{F}(X^*, X))$ is a closed subset of X^*.* (5) *If X is normal then* codim $\mathfrak{F}(X^*, X) > 1$ (note that if X is nonsingular then X is normal). (6) *If every point of X is* *a unique factorization domain* X *then* $* = \mathfrak{W}(X, II^{-1})$, *and* $[X^*, X^{-1}]$ $(\mathfrak{F}(X^*, X))$ *is either empty or pure* 1-*codimensional* (note that by (6.4.7) we know that II^{-1} is a nonzero ideal on X; also note that if X is nonsingular then every point of X is a unique factorization domain).

PROOF. (1) follows from (6.4.12) and (6.6.8). (2) follows from (6.6.8), or alternatively, also from (1.9.6) and (6.4.8). (3) follows from (2), (6.4.6), and (6.4.7). (4) follows from (3) and (6.2.5). (5) follows from (2) and (6.4.9). In view of (1) and (2), to prove (6) it suffices to show that if every point of X is a unique factorization domain then $X^* = \mathfrak{W}(X, II^{-1})$. In turn, to show this it is enough to prove that if R is any point of X such that R is a unique factorization domain, then $\mathfrak{W}(R, P) = \mathfrak{W}(R, PP^{-1})$ where $P = IR$. Since R is a unique factorization domain, there exists $0 \neq x \in R$ such that $xR = \mathrm{prin}_R P$. By (1.11.7) we get that $(PP^{-1})x = P$. Let $(y_1, ..., y_n)$ be any basis of PP^{-1}. Then $(y_1 x, ..., y_n x)$ is a basis of P. Now $\mathfrak{W}(R, P) = \mathfrak{W}(R; y_1 x, ..., y_n x)$ and $\mathfrak{W}(R, PP^{-1}) = \mathfrak{W}(R; y_1, ..., y_n)$. Clearly $\mathfrak{W}(R; y_1 x, ..., y_n x) = \mathfrak{W}(R; y_1, ..., y_n)$, and hence $\mathfrak{W}(R, P) = \mathfrak{W}(R, PP^{-1})$.

(6.7). We now study the operation $\mathfrak{W}$ for projective models.

(6.7.1). *Let* X *be any projective model of* K/k, *and let* I *be any nonzero ideal on* X. *Then* $\mathfrak{W}(X, I)$ *is a projective model of* K/k.

PROOF. Now there exist nonzero elements $x_1, ..., x_n$ in K such that $X = \mathfrak{V}(A_1) \cup \cdots \cup \mathfrak{V}(A_n)$ where $A_i = k[x_1/x_i, ..., x_n/x_i]$. Let H_i be the set of all nonzero homogeneous polynomials $f(W_1, ..., W_n)$ in indeterminates $W_1, ..., W_n$ with coefficients in k such that $f(x_1/x_i, ..., x_n/x_i) \in A_i \cap I$. Let $H = H_1 \cap \cdots \cap H_n$.

Let i be any integer with $1 \leqslant i \leqslant n$ and let $f^*(W_1, ..., W_n)$ be any element in H_i. We claim that then $W_i^e f^*(W_1, ..., W_n) \in H$ for some nonnegative integer e. It suffices to show that given any integer j with $1 \leqslant j \leqslant n$ there exists a nonnegative integer $e(j)$ such that $W_i^{e(j)} f^*(W_1, ..., W_n) \in H_j$, because then it would be enough to take $e = \max(e(1), ..., e(n))$. Since $0 \neq x_j/x_i \in A_i$, we get that $\mathfrak{V}(B) \subset \mathfrak{V}(A_i)$ where $B = A_i[(x_j/x_i)^{-1}]$, and hence $f^*(x_1/x_i, ..., x_n/x_i)R \in IR$ for all $R \in \mathfrak{V}(B)$. Also $0 \neq x_i/x_j \in A_j$

and clearly $B = A_j[(x_i/x_j)^{-1}]$; therefore $\mathfrak{V}(B) \subset \mathfrak{V}(A_j)$ and $(A_j \cap I)B$ is an ideal in B such that $((A_j \cap I)B)R = IR$ for all $R \in \mathfrak{V}(B)$, and hence by (1.11.5) we get that $f^*(x_1/x_i, ..., x_n/x_i) \in (A_j \cap I)B$; since $B = A_j[(x_i/x_j)^{-1}]$, there exists a nonnegative integer $e(j)$ such that $(x_i/x_j)^{e(j)}f^*(x_1/x_i, ..., x_n/x_i) \in A_j \cap I$. Let e' be the degree of $f^*(W_1, ..., W_n)$ in $W_1, ..., W_n$, and let $f(W_1, ..., W_n) = W_i^{e(j)}f^*(W_1, ..., W_n)$. Then $f(x_1/x_j, ..., x_n/x_j) = (x_i/x_j)^{e'+e(j)}f^*(x_1/x_i, ..., x_n/x_i) \in A_j \cap I$ and hence $f(W_1, ..., W_n) \in H_j$.

Now $A_1 \cap I \neq \{0\}$ and hence by what we have proved in the above paragraph there exists $h(W_1, ..., W_n) \in H$ such that $h(x_1, ..., x_n) \neq 0$. Since $k[W_1, ..., W_n]$ is noetherian, there exists a finite number of elements $f_1(W_1, ..., W_n), ..., f_t(W_1, ..., W_n)$ $(t > 0)$ in H which form a basis of the ideal in $k[W_1, ..., W_n]$ generated by H. Upon relabeling $f_1, ..., f_t$ we may assume that $f_q(x_1, ..., x_n) \neq 0$ for $1 \leqslant q \leqslant s$ and $f_q(x_1, ..., x_n) = 0$ for $s < q \leqslant t$ where s is an integer with $1 \leqslant s \leqslant t$. Let $d(q)$ be the degree of $f_q(W_1, ..., W_n)$ in $W_1, ..., W_n$. Take a nonnegative integer d such that $d \geqslant d(q)$ for $1 \leqslant q \leqslant s$. Let $m = ns$. Let $g_{p+nq-n}(W_1, ..., W_n) = W_p^{d-d(q)}f_q(W_1, ..., W_n)$ for $1 \leqslant p \leqslant n$ and $1 \leqslant q \leqslant s$. Then for $1 \leqslant j \leqslant m$ we have that $g_j(W_1, ..., W_n)$ is a nonzero homogeneous polynomial of degree d in $W_1, ..., W_n$ with coefficients in k, $g_j(W_1, ..., W_n) \in H$, and $g_j(x_1, ..., x_n) \neq 0$. Let $y_j = g_j(x_1, ..., x_n)$ and $y_{ij} = g_j(x_1/x_i, ..., x_n/x_i)$. Then $0 \neq y_j \in K$, $0 \neq y_j/x_i^d = y_{ij} \in A_i$, and $y_{ib}/y_{ij} = (x_i^d y_{ib})/(x_i^d y_{ij}) = y_b/y_j$ for $1 \leqslant i \leqslant n$, $1 \leqslant b \leqslant m$, and $1 \leqslant j \leqslant m$. Let i be any integer with $1 \leqslant i \leqslant n$. We claim that then $(y_{i1}, ..., y_{im})A_i = A_i \cap I$. Since $g_j(W_1, ..., W_n) \in H$ and $y_{ij} = g_j(x_1/x_i, ..., x_n/x_i)$, we get that $y_{ij} \in A_i \cap I$ for $1 \leqslant j \leqslant m$. Conversely, let y be any nonzero element in $A_i \cap I$. Since $0 \neq y \in A_i$, there exists a nonzero homogeneous polynomial $f^*(W_1, ..., W_n)$ of some degree e' in $W_1, ..., W_n$ with coefficients in k such that $f^*(x_1/x_i, ..., x_n/x_i) = y$. Now $f^*(W_1, ..., W_n) \in H_i$ and hence by what we have proved in the above paragraph there exists a nonnegative integer e such that upon letting $f(W_1, ..., W_n) = W_i^e f^*(W_1, ..., W_n)$ we have that $f(W_1, ..., W_n) \in H$. Since $f(W_1, ..., W_n) \in H$, there exist elements $F_q(W_1, ..., W_n)$ in $k[W_1, ..., W_n]$ such that

$$f(W_1, ..., W_n) = \sum_{q=1}^{t} F_q(W_1, .., W_n)f_q(W_1, ..., W_n).$$

Upon multiplying both sides by W_i^d we get that

$$g(W_1, \ldots, W_n) = \sum_{j=1}^{m} G_j(W_1, \ldots, W_n) g_j(W_1, \ldots, W_n)$$

$$+ \sum_{q=s+1}^{t} F'_q(W_1, \ldots, W_n) f_q(W_1, \ldots, W_n)$$

where

$$g(W_1, \ldots, W_n) = W_i^d f(W_1, \ldots, W_n),$$

$$F'_q(W_1, \ldots, W_n) = W_i^d F_q(W_1, \ldots, W_n) \qquad \text{for} \quad s < q \leqslant t,$$

$$G_{i+nq-n}(W_1, \ldots, W_n) = W_i^{d(q)} F_q(W_1, \ldots, W_n) \qquad \text{for} \quad 1 \leqslant q \leqslant s,$$

and

$$G_{p+nq-n}(W_1, \ldots, W_n) = 0 \qquad \text{for} \quad 1 \leqslant q \leqslant s \quad \text{and} \quad 1 \leqslant p \leqslant n \quad \text{with} \quad p \neq i.$$

Now

$$f_q(x_1/x_i, \ldots, x_n/x_i) = x_i^{-d(q)} f_q(x_1, \ldots, x_n) = 0 \qquad \text{for} \quad s < q \leqslant t,$$

$$g_j(x_1/x_i, \ldots, x_n/x_i) = y_{ij} \qquad \text{for} \quad 1 \leqslant j \leqslant m,$$

and

$$g(x_1/x_i, \ldots, x_n/x_i) = y.$$

Therefore upon substituting $(x_1/x_i, \ldots, x_n/x_i)$ for $(W_1, \ldots, W_n)$ we get that

$$y = \sum_{j=1}^{m} G_j(x_1/x_i, \ldots, x_n/x_i) y_{ij}$$

and hence $y \in (y_{i1}, \ldots, y_{im}) A_i$.

Thus we have found nonzero elements y_j and y_{ij} in K for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$ such that $(y_{i1}, \ldots, y_{im})A_i = A_i \cap I$ and $y_{ib}/y_{ij} = y_b/y_j$ for $1 \leqslant i \leqslant n$, $1 \leqslant b \leqslant m$, and $1 \leqslant j \leqslant m$. By (6.6.5) we know that $\mathfrak{W}(X, I) = \mathfrak{W}(A_1, A_1 \cap I) \cup \cdots \cup \mathfrak{W}(A_n, A_n \cap I)$ and clearly $\mathfrak{W}(A_i, A_i \cap I) = \mathfrak{V}(A_{i1}) \cup \cdots \cup$

$\mathfrak{V}(A_{im})$ where $A_{ij} = A_i[y_{i1}/y_{ij}, \ldots, y_{im}/y_{ij}]$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Therefore

$$\mathfrak{W}(X, I) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \mathfrak{V}(A_{ij}).$$

Since $y_{ib}/y_{ij} = y_b/y_j$ for $1 \leqslant i \leqslant n$, $1 \leqslant b \leqslant m$, and $1 \leqslant j \leqslant m$, we get that

$$A_{ij} = k[((x_a y_b)/(x_i y_j))_{a=1,\cdots,n;b=1,\cdots,m}]$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Therefore

$$\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \mathfrak{V}(A_{ij}) = \mathfrak{W}(k; (x_a y_b)_{a=1,\cdots,n;b=1,\cdots,n})$$

and hence $\mathfrak{W}(X, I)$ is a projective model of K/k.

We now prove the following converse of (6.7.1).

(6.7.2). *Let X and X^* be any projective models of K/k. Then there exists a nonzero ideal I on X such that $\mathfrak{W}(X, I) = X + X^*$* (note that $X + X^* = X^* \Leftrightarrow X^*$ dominates X). *Moreover, for any such I and any model X' of K/k dominating X, upon letting $I' = IX'$, we have the following*: (1) *I' is a nonzero ideal on X', and $\mathfrak{W}(X', I') = X' + X^*$*; (2) $\mathfrak{F}(X^*, X') = \mathfrak{Z}(I'I'^{-1}) = \{R' \in X'$: *$I'R'$ is not principal*$\}$; (3) *X' dominates $X^* \Leftrightarrow I'$ is principal.*

PROOF. Now there exist nonzero elements $x_1, \ldots, x_n, z_1, \ldots, z_m$ in K such that $X = \mathfrak{V}(A_1) \cup \cdots \cup \mathfrak{V}(A_n)$ and $X^* = \mathfrak{V}(B_1) \cup \cdots \cup \mathfrak{V}(B_m)$ where $A_i = k[x_1/x_i, \ldots, x_n/x_i]$ and $B_j = k[z_1/z_j, \ldots, z_m/z_j]$. Since K is the quotient field of A_1, there exists a nonnegative integer d and nonzero homogeneous polynomials $f_1(W_1, \ldots, W_n)$, $\ldots, f_m(W_1, \ldots, W_n)$ of degree d in indeterminates $W_1, \ldots, W_n$ with coefficients in k such that $z_j = f_j(x_1, \ldots, x_n)/x_1^d$ for $1 \leqslant j \leqslant m$. Let $y_j = f_j(x_1, \ldots, x_n)$ for $1 \leqslant j \leqslant m$. Then $y_1, \ldots, y_m$ are nonzero elements in K and $B_j = k[y_1/y_j, \ldots, y_m/y_j]$ for $1 \leqslant j \leqslant m$. Now $0 \neq y_j/x_i^d = f_j(x_1/x_i, \ldots, x_n/x_i) \in A_i$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Let $P_i = (y_1/x_i^d, \ldots, y_m/x_i^d)A_i$, and let A_{ij} be the smallest subring of K containing A_i and B_j. Then $A_{ij} = A_i[(y_1/x_i^d)/(y_j/x_i^d), \ldots,$

$(y_m/x_i^d)/(y_j/x_i^d)]$ for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Therefore by (6.2.7) we get that

$$X + X^* = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \mathfrak{V}(A_{ij}) = \bigcup_{i=1}^{n} \mathfrak{W}(A_i, P_i).$$

If u and v are any integers with $1 \leqslant u \leqslant n$ and $1 \leqslant v \leqslant n$ and R is any point in $\mathfrak{V}(A_u) \cap \mathfrak{V}(A_v)$ then x_u/x_v is a unit in R and hence $P_uR = P_vR$. Therefore we get a nonzero preideal I on X by taking $IR = P_iR$ for $1 \leqslant i \leqslant n$ and all $R \in \mathfrak{V}(A_i)$. By (1.11.5) we get that I is an ideal on X and $A_i \cap I = P_i$ for $1 \leqslant i \leqslant n$. Therefore by (6.6.5) we get that $\mathfrak{W}(X, I) = \mathfrak{W}(A_1, P_1) \cup \cdots \cup \mathfrak{W}(A_n, P_n)$, and hence $\mathfrak{W}(X, I) = X + X^*$. The rest now follows from (6.2.8), (6.4.11), (6.6.7), and (6.6.8).

In view of (6.2.8) and (6.6.9), by (6.7.2) we get the following result due to Zariski [24].

(6.7.3). *For any projective models X and X^* of K/k we have the following*: (1) $\mathfrak{F}(X^*, X)$ *is a closed subset of* X. (2) *If X is normal then* codim $\mathfrak{F}(X^*, X) > 1$ (note that if X is nonsingular then X is normal). (3) *If X^* dominates X then* $[X^*, X]^{-1}(\mathfrak{F}(X^*, X))$ *is a closed subset of X^*.* (4) *If X^* dominates X and every point of X is a unique factorization domain then* $[X^*, X]^{-1}(\mathfrak{F}(X^*, X))$ *is either empty or pure* 1*-codimensional* (note that if X is nonsingular then every point of X is a unique factorization domain).

(6.8). For any closed subset T of any model X of K/k with $T \neq X$, by *the monoidal transform of X with center T* we mean $\mathfrak{W}(X, \mathfrak{I}(T, X))$; note that then upon letting $X' = \mathfrak{W}(X, \mathfrak{I}(T, X))$, in view of (1.4), (6.2.5), (6.4.6), (6.6.1), (6.6.5), (6.6.8), and Krull's principal ideal theorem [27: Theorem 29, page 238], we get: (1) X' is a model of K/k; (2) X' properly dominates X; (3) $[X', X]$ is a continuous map of X' onto X; (4) $[X', X]^{-1}(X - \mathfrak{S}(X) - \mathfrak{S}(T)) \subset X' - \mathfrak{S}(X')$; (5) $\mathfrak{F}(X', X) \subset T$; and (6) if Z is any closed subset of any open subset Y of X such that $T \cap Y \subset Z$ and codim $Z \geqslant 2$ then $\mathfrak{F}(X', X) \cap Y = T \cap Y$.

Given any nonsingular model X of K/k and any model X^* of K/k, we say that X^* is an *iterated monoidal transform of X with nonsingular irreducible centers* if exists a nonnegative integer m, a model X_i of K/k for $0 \leqslant i \leqslant m$, and a nonsingular irreducible closed subset T_i of X_i with $T_i \neq X_i$ for $0 \leqslant i < m$, such that: $X_0 = X$, $X_m = X^*$, and X_{i+1} is the monoidal transform of X_i with center T_i for $0 \leqslant i < m$; note that then X^* is nonsingular and X^* properly dominates X.

(6.9). Let X be any nonsingular model of K/k, let T be any nonsingular closed subset of X with $T \neq X$, let X' be the monoidal transform of X with center T, and let J and I be any nonzero principal ideals on X (note that by (6.8) we know that then X' is a nonsingular model of K/k, and X' properly dominates X). Let J' be the principal preideal on X' defined thus: if $R' \in X' - [X', X]^{-1}(T)$ then let $J'R' = JR$ where $R = [X', X](R')$; and if $R' \in [X', X]^{-1}(T)$ then let $J'R' = (R, S, R')$-transform of JR where $R = [X', X](R')$ and S is the generic point of the irreducible component of T passing through R. We say that (X', J') is *the monoidal transform of* (X, J) *with center* T. Let I' be the unique principal preideal on X' such that $(JI)X' = J'I'$. We say that (X', J', I') is *the monoidal transform of* (X, J, I) *with center* T. Note that for any $R' \in X'$ upon letting $R = [X', X](R)$ we have that: if $R \notin T$ then $R' = R$ and $(J'R', I'R') = (JR, IR)$; and if $R \in T$ then $(J'R', I'R')$ is the (R, S, R')-transform of (J, I) where S is the generic point of the irreducible component of T passing through R. Let J^* be the unique principal preideal on X' such that $JX' = J'J^*$, i.e., such that (X', J', J^*) is the monoidal transform of $(X, J, 1_X)$ with center T; note that then $I' = (IX')J^*$.

(6.9.1). *Let* $T_1, \ldots, T_n$ *be the irreducible components of* T *labeled so that* codim $T_i > 1$ *for* $1 \leqslant i \leqslant m$ *and* codim $T_i = 1$ *for* $m < i \leqslant n$. *Let* S_i *be the generic point of* T_i, *let* $d_i = \operatorname{ord}_{S_i} JS_i$, *let* S_i' *be the valuation ring of* ord_{S_i}, *let* $T_i' = [X', X]^{-1}(T_i)$, *and let* $T' = [X', X]^{-1}(T)$. *Then we have the following:* (1) T' *is a nonsingular closed subset of* X'; $T_1', \ldots, T_n'$ *are the irreducible components of* T'; S_i' *is the generic point of* T_i' *for* $1 \leqslant i \leqslant n$; *and* $\mathfrak{F}(X', X) = T_1 \cup \ldots \cup T_m$. (2) $J^* = \prod_{i=1}^{n} \mathfrak{Z}(T_i', X')^{d_i}$ *where we take* $\mathfrak{Z}(T_i', X')^{d_i} = 1_{X'}$ *in case* $d_i = 0$, *and* $\prod_{i=1}^{n} \mathfrak{Z}(T_i', X')^{d_i} = 1_{X'}$ *in case* $n = 0$. (3) J^*, J', *and* I' *are nonzero principal ideals on* X'. (4) $\mathfrak{Z}(J') - T' = \mathfrak{Z}(J) - T$, *and* $\mathfrak{Z}(J') =$ *closure of* $\mathfrak{Z}(J') - T'$ *in*

X'. (5) *If* codim $T > 1$ *then* $\mathfrak{F}(X', X) = T$, $\mathfrak{F}(X', X)$ *does not pass through the generic point of any irreducible component of* $\mathfrak{Z}(J)$, *and* $\mathfrak{Z}(J')$ *is the* $[X', X]$*-transform of* $\mathfrak{Z}(J)$ (note that by (6.4.6) we know that $\mathfrak{Z}(J)$ is a closed subset of X).

PROOF. By (6.2.5) we get that $T', T_1, \ldots, T_n$ are closed subsets of X'. Let any i with $1 \leqslant i \leqslant n$ be given; now $\mathfrak{I}(T, X)R = R \cap M(S_i)$ for all $R \in T_i$, and hence $T_i' = \bigcup_{R \in T_i} Y(R)$ where $Y(R) = \{R' \in \mathfrak{W}(R, R \cap M(S_i))\colon R' \text{ dominates } R\}$; for every $R \in T_i$, by (1.4) we get that $S_i' \in Y(R)$, and $S_i' \in \mathfrak{V}(R')$ and $R'/(R' \cap M(S_i'))$ is regular for all $R' \in Y(R)$; it follows that T_i' is the closure of S_i' in X', T_i' is irreducible, S_i' is the generic point of T_i', T_i' is nonsingular, and $\mathfrak{I}(T_i', X')R' = R' \cap M(S_i')$ and $J^*R' = (\mathfrak{I}(T_i', X')^{d_i})R'$ for all $R' \in T_i'$. Now $X' \neq T' = T_1' \cup \cdots \cup T_n'$, $\dim S_i' = 1$ for all i, and $T_i' \cap T_j' = \varnothing$ whenever $i \neq j$. It follows that $T_1', \ldots, T_n'$ are the irreducible components of T', T' is nonsingular, and $J^* = \prod_{i=1}^{n} \mathfrak{I}(T_i', X')^{d_i}$. Clearly $T_1 \cup \cdots \cup T_m = \{R \in X\colon \mathfrak{I}(T, X)R \text{ is not principal}\}$, and hence by (6.6.9) we also get that $\mathfrak{F}(X', X) = T_1 \cup \cdots \cup T_m$. This completes the proof of (1) and (2). In view of (1) and (2), by (6.4.4) and (6.4.6) we get that J^* is a nonzero principal ideal on X'; since $I' = (IX')J^*$, in view of (6.4.4) and (6.4.11) we also get that I' is a nonzero principal ideal on X'. Let A be any affine ring over k with $\mathfrak{V}(A) \subset X'$; by (6.4.11) we know that JX' is an ideal on X' and hence, upon letting $P = A \cap (JX')$, we get that P is an ideal in A and $PR' = (JX')R'$ for all $R' \in \mathfrak{V}(A)$; since J^* is an ideal on X', upon letting $P^* = A \cap J^*$ we get that P^* is an ideal in A and $P^*R' = J^*R'$ for all $R' \in \mathfrak{V}(A)$; let $P' = (P : P^*)$; then P' is an ideal in A; since the operation (:) commutes with the operation of forming a quotient ring, we get that $(P : P^*)R' = ((PR') : (P^*R'))$ for all $R' \in \mathfrak{V}(A)$; clearly, $((JX')R' : (J^*R')) = J'R'$ for all $R' \in X'$, and hence $P'R' = J'R'$ for all $R' \in \mathfrak{V}(A)$. Thus J' is a nonzero principal ideal on X'. This completes the proof of (3). Clearly $\mathfrak{Z}(J) - T = \mathfrak{Z}(J') - T'$. By (3) and (6.4.6) we know that $\mathfrak{Z}(J')$ is closed in X', and hence $\mathfrak{Z}(J')$ contains the closure of $\mathfrak{Z}(J) - T$ in X'. Conversely, given any $R' \in \mathfrak{Z}(J') \cap T'$, let $R = [X', X](R')$; then $R \in T$, and hence $R \in T_i$ for a unique value of i; now $(JR)R' = (J^*R')(J'R')$,

$J^*R' = (R' \cap M(S_i'))^{d_i}$, $(R \cap M(S_i))R' = R' \cap M(S_i')$, $R' \cap M(S_i')$ is a nonzero principal prime ideal in R', $J'R' \not\subset R' \cap M(S_i')$, and $J'R'$ is a nonzero nonunit principal ideal in R'; consequently there exists $S' \in \mathfrak{V}(R')$ such that $(J'R')S' \neq S'$ and $R' \cap M(S_i') \not\subset R' \cap M(S')$; since $(J'R')S' = J'S'$, we get that $S' \in \mathfrak{Z}(J')$; since S_i' is the generic point of T_i', and T_i' is the only irreducible component of T' passing through R', we get that $S' \notin T'$; thus $S' \in \mathfrak{V}(R')$ and $S' \in \mathfrak{Z}(J') - T'$, and hence $R' \in$ closure of $\mathfrak{Z}(J') - T'$ in X'. This proves (4). By (6.4.10) we know that $\mathfrak{Z}(J)$ is either empty or pure 1-codimensional, and hence (5) follows from (1) and (4).

(6.9.2). *Assume that (J, I) has only quasinormal crossings and for every $R \in T$, upon letting S be the generic point of the irreducible component of T passing through R, we have that $S \in \mathfrak{C}(R, J)$ and (S, I) has a normal crossing at R. Then (J', I') has only quasinormal crossings.*

PROOF. Follows from (1.10.7).

(6.9.3). *Assume that I has only quasinormal crossings and for every $R \in T$, upon letting S be the generic point of the irreducible component of T passing through R, we have that (S, I) has a pseudonormal crossing at R. Then I' has only quasinormal crossings.*

PROOF. Follows from (1.10.8).

(6.9.4). *If $T \subset \mathfrak{S}^*(J)$ then $\mathfrak{S}^*(J') \subset [X', X]^{-1}(\mathfrak{S}^*(J))$. If $T \subset \mathfrak{C}^*(\mathfrak{S}^*(J), J)$ then $\mathrm{ord}_{\mathfrak{S}^*(J')} J' \leqslant \mathrm{ord}_{\mathfrak{S}^*(J)} J$. If $T \subset \mathfrak{C}^*(\mathfrak{S}^*(J), J)$ and $\mathrm{ord}_{\mathfrak{S}^*(J')} J' = \mathrm{ord}_{\mathfrak{S}^*(J)} J$ then*

$$\mathfrak{C}^*(\mathfrak{S}^*(J'), J') \subset [X', X]^{-1}(\mathfrak{C}^*(\mathfrak{S}^*(J), J)).$$

PROOF. The first assertion is obvious. The second and the third assertions follow from (1.10.2).

(6.10). K/k is said to be *uniformizable* if: given any $V \in \mathfrak{R}(K/k)$ such that V is residually algebraic over k and $k \cap M(V)$ is a maximal ideal in k, there exists a regular spot R over k with quotient field K such that V dominates R.

(6.10.1). *K/k is uniformizable if and only if: given any $V \in \mathfrak{R}(K/k)$ there exists a regular spot R over k with quotient field K such that V dominates R.*

By [18: (28.3)] we know that if R is any regular local domain then every element in $\mathfrak{V}(R)$ is regular. Therefore our assertion follows from (6.3.6).

(6.10.2). *If there exists a finite number of models $X_1, \ldots, X_n$ of K/k such that $\mathfrak{R}(K/k) = \bigcup_{i=1}^{n} [\mathfrak{R}(K/k), X_i]^{-1}(X_i - \mathfrak{S}(X_i))$, then K/k is uniformizable. Conversely, if K/k is uniformizable and for every affine ring A over k with quotient field K and every ideal Q in A we have that $\mathfrak{S}(A, Q)$ is closed in $\mathfrak{V}(A)$* (see 1.2.6)), *then there exists a finite number of projective models $X_1, \ldots, X_n$ of K/k such that $\mathfrak{R}(K/k) = \bigcup_{i=1}^{n} [\mathfrak{R}(K/k), X_i]^{-1}(X_i - \mathfrak{S}(X_i))$.*

This follows from (6.2.2), (6.2.5), (6.5.3), and (6.10.1).

§7. Global resolvers

Assume that $\dim_k K \leqslant 3$; note that then for any model X of K/k, by (6.3.3) we have that $\dim X = \max_{R \in X} \dim R \leqslant 3$. Also assume that for every affine ring A over k with quotient field K and every ideal Q in A we have that $\mathfrak{S}(A, Q)$ is closed in $\mathfrak{V}(A)$ (see (1.2.6)).

(7.1). DEFINITION. By a *global semiresolver* of K/k we mean a sequence $(X_i, J_i, T_i)_{0 \leqslant i < m}$ where: (1) either m is a positive integer or $m = \infty$; (2) for $0 \leqslant i < m$: X_i is a nonsingular model of K/k, J_i is a nonzero principal ideal on X_i, and T_i is a nonsingular closed subset of X_i with $T_i \subset \mathfrak{S}^*(J_i)$ such that for every $R \in T_i$, upon letting S be the generic point of the irreducible component of T_i passing through R, we have that $S \in \mathfrak{E}(R, J_i)$ and: $\dim S = 2 \Leftrightarrow \mathfrak{E}^2(R, J_i)$ has a strict normal crossing at R and $\mathfrak{E}^2(R, J_i) \neq \varnothing$; and (3) for $0 < i < m$: (X_i, J_i) is the monoidal transform of (X_{i-1}, J_{i-1}) with center T_i.

By an *infinite global semiresolver* of K/k we mean a global semiresolver $(X_i, J_i, T_i)_{0 \leqslant i < m}$ of K/k where $m = \infty$ and $T_i \neq \varnothing$ for infinitely many distinct values of i.

By a *finite global semiresolver* of K/k we mean a system $[(X_i, J_i, T_i)_{0 \leqslant i \leqslant m}, (X', J')]$ where: m is a positive integer; $(X_i, J_i, T_i)_{0 \leqslant i < m}$ is a global semiresolver of K/k such that for $0 \leqslant i < m$ we have that $T_i \subset \mathfrak{E}^*(\mathfrak{S}^*(J_i), J_i)$ and either $T_i = \varnothing$ or T_i is irreducible; X' is a nonsingular model of K/k and J' is a nonzero principal ideal on X' such that $\mathfrak{S}^*(J') = \varnothing$; and (X', J') is the monoidal transform of (X_{m-1}, J_{m-1}) with center T_{m-1}.

K/k is said to be *globally semiresolvable* if: given any nonsingular model X of K/k and any nonzero principal ideal J on X, there exists a finite global resolver $[(X_i, J_i, T_i)_{0 \leqslant i < m}, (X', J')]$ of K/k such that $(X_0, J_0) = (X, J)$. K/k is said to be *globally strongly semiresolvable* if there does not exist any infinite global semiresolver of K/k. K/k is said to be *locally strongly semiresolvable* if every regular spot over k with quotient field K is strongly semiresolvable.

(7.2). *Let X be any nonsingular model of K/k and let J be any nonzero principal ideal on X. Then we have the following.*

(7.2.1). *Assume that $\mathfrak{S}^*(J) \neq \varnothing$. Then there exists a nonsingular irreducible closed subset T of X such that $T \subset \mathfrak{S}^*(J)$, $T \subset \mathfrak{E}^*(\mathfrak{S}^*(J), J)$, and, upon letting S be the generic point of T, for every $R \in T$ we have that $S \in \mathfrak{E}(R, J)$ and:* $\dim S = 2 \Leftrightarrow \mathfrak{E}^2(R, J)$ *has a strict normal crossing at R and $\mathfrak{E}^2(R, J) \neq \varnothing$.*

(7.2.2). *Assume that there does not exist any infinite global semiresolver $(X_i, J_i, T_i)_{0 \leqslant i < \infty}$ of K/k such that $(X_0, J_0) = (X, J)$. Then there exists a finite global semiresolver $[(X_i, J_i, T_i)_{0 < i < m}, (X', J')]$ of K/k such that $(X_0, J_0) = (X, J)$.*

PROOF OF (7.2.1). By (6.5.3) we know that $\mathfrak{S}^*(J)$ is a closed subset of X and $\operatorname{codim} \mathfrak{S}^*(J) \geqslant 2$. Therefore by (6.5.4) we get that $\mathfrak{E}^*(\mathfrak{S}^*(J), J)$ is a nonempty closed subset of $\mathfrak{S}^*(J)$ and $\operatorname{codim} \mathfrak{E}^*(\mathfrak{S}^*(J), J) \geqslant 2$. Let $S_1, \ldots, S_n$ $(n > 0)$ be the generic points of the irreducible components of $\mathfrak{E}^*(\mathfrak{S}^*(J), J)$. Then $2 \leqslant \dim S_i \leqslant 3$ for $1 \leqslant i \leqslant n$. If $\dim S_i = 3$ for some i, then

$\{S_i\}$ is a nonsingular irreducible closed subset of X and by (6.5.6) we get that $\mathfrak{C}^2(S_i, J) = \emptyset$, and hence it suffices to take $T = \{S_i\}$. Now assume that $\dim S_i = 2$ for $1 \leqslant i \leqslant n$. Then by (6.5.6) we get that $\mathfrak{C}^2(R, J) = \mathfrak{V}(R) \cap \{S_1, ..., S_n\}$ for all $R \in \mathfrak{C}^*(S^*(J), J)$. If there exists $R \in \mathfrak{C}^*(S^*(J), J)$ such that $\mathfrak{V}(R) \cap \{S_1, ..., S_n\}$ does not have a strict normal crossing at R then $\dim R = 3$ and $\{R\}$ is a nonsingular irreducible closed subset of X, and hence it suffices to take $T = \{R\}$. If $\mathfrak{V}(R) \cap \{S_1, ..., S_n\}$ has a strict normal crossing at R for all $R \in \mathfrak{C}^*(\mathfrak{S}^*(J), J)$, then it suffices to take $T =$ closure of $\{S_i\}$ in X for some i.

PROOF OF (7.2.2). Let W be the set of all global semiresolvers $(X_i, J_i, T_i)_{0 \leqslant i < m}$ of K/k such that $(X_0, J_0) = (X, J)$ and $T_i \subset \mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i)$ and T_i is irreducible for $0 \leqslant i < m$. If $\mathfrak{S}^*(J) = \emptyset$ then we get a finite global semiresolver $[(X_i, J_i, T_i)_{0 \leqslant i < 1}, (X', J')]$ of K/k of the required type by taking $(X_0, J_0) = (X, J) = (X', J')$ and $T_0 = \emptyset$. So now assume that $\mathfrak{S}^*(J) \neq \emptyset$. Then there exists T as in (7.2.1) and we get an element $(X_i, J_i, T_i)_{0 \leqslant i < 1}$ in W by taking $(X_0, J_0, T_0) = (X, J, T)$. Therefore W is nonempty. For each pair of elements $w = (X_i, J_i, T_i)_{0 \leqslant i < m}$ and $w' = (X'_i, J'_i, T'_i)_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(X_i, J_i, T_i) = (X'_i, J'_i, T'_i)$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property and hence by Zorn's lemma W contains a maximal element $w = (X_i, J_i, T_i)_{0 \leqslant i < m}$. By assumption we must have $m \neq \infty$. Let (X', J') be the monoidal transform of (X_{m-1}, J_{m-1}) with center T_{m-1}. Then by (6.8) and (6.9.1) we have that X' is a nonsingular model of K/k and J' is a nonzero principal ideal on X'. Suppose if possible that $\mathfrak{S}^*(J') \neq \emptyset$; then by (7.2.1) there exists a nonsingular irreducible closed subset T' of X' such that $T' \subset \mathfrak{S}^*(J')$, $T' \subset \mathfrak{C}^*(\mathfrak{S}^*(J'), J')$, and, upon letting S be the generic point of T', for every $R \in T'$ we have that $S \in \mathfrak{C}(R, J')$ and: $\dim S = 2 \Leftrightarrow \mathfrak{C}^2(R, J')$ has a strict normal crossing at R and $\mathfrak{C}^2(R, J') \neq \emptyset$; we now get an element $w' = (X'_i, J'_i, T'_i)_{0 \leqslant i < m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking $(X'_i, J'_i, T'_i) = (X_i, J_i, T_i)$ for $0 \leqslant i < m$ and $(X'_m, J'_m, T'_m) = (X', J', T')$; this is a contradiction because w is a maximal element of W. Therefore $\mathfrak{S}^*(J') = \emptyset$ and hence $[(X_i, J_i, T_i)_{0 \leqslant i < m}, (X', J')]$ is a finite global semiresolver of K/k with $(X_0, J_0) = (X, J)$.

(7.3). *If K/k is globally strongly semiresolvable then K/k is globally semiresolvable.*

PROOF. Follows from (7.2.2).

(7.4). *For any global semiresolver $(X_i, J_i, T_i)_{0 \leqslant i < \infty}$ of K/k we have the following.*

(7.4.1). *Given any nonnegative integer n and any $R \in \bigcap_{i=n}^{\infty} X_i$, there exists an open subset D of X_n with $R \in D$ such that $D \subset X_i$ and $D \cap T_i = \varnothing$ for all $i \geqslant n$.*

(7.4.2). *Assume that there does not exist any infinite semiresolver $(R_j, P_j, S_j)_{0 \leqslant j < \infty}$ with $R_0 \mathfrak{S} \mathfrak{S}^*(J_0)$. Then there exists a nonnegative integer m such that $T_i = \varnothing$ for all $i \geqslant m$.*

PROOF OF (7.4.1). In view of (6.5.3) and (6.8) we have that $\mathfrak{S}^*(J_i)$ is a closed subset of X_i with codim $\mathfrak{S}^*(J_i) \geqslant 2$ and $\mathfrak{F}(X_{i+1}, X_i) = T_i \subset \mathfrak{S}^*(J_i)$ for $0 \leqslant i < \infty$. Since $R \in \bigcap_{i=n}^{\infty} X_i$, we get that $R \notin T_i$ for all $i \geqslant n$. For each $i \geqslant n$ let G_i be the union of the irreducible components of $\mathfrak{S}^*(J_i)$ passing through R, let H_i be the union of the remaining irreducible components of $\mathfrak{S}^*(J_i)$, let

$$G_i^* = \mathfrak{S}(G_i) \cup \{R' \in G_i - \mathfrak{S}(G_i) \colon S \notin \mathfrak{E}(R', J_i) \text{ where } S \text{ is the generic point of the irreducible component of } G_i \text{ passing through } R'\},$$

and let $D_i = X_i - ((G_i^* - \{R\}) \cup H_i)$. Then $R \in D_i$, and in view of (6.5.5) we get that D_i is an open subset of X_i. For any open subset E of X_i with $R \in E$, in view of (6.2.16) we get that:

$$\begin{aligned}&(\mathfrak{S}^*(J_i) \cap E \cap D_i) - \{R\}\\ &= \{R' \in (\mathfrak{S}^*(J_i) \cap E) - (\mathfrak{S}(\mathfrak{S}^*(J_i) \cap E) \cup \{R\}) \colon \text{upon letting } F\\ &\quad \text{be the irreducible component of } \mathfrak{S}^*(J_i) \cap E \text{ passing through } R',\\ &\quad \text{and } S \text{ be the generic point of } F, \text{ we have that } R \in F, \text{ and}\\ &\quad S \in \mathfrak{E}(R', J_i)\};\end{aligned}$$

let us refer to this observation as $[i, E]$. Since $R \notin T_i$, by (6.5.6) and $[i, X_i]$ we see that $D_i \cap T_i = \varnothing$; consequently $D_i \subset X_{i+1}$ and:

$$(1_i) \qquad J_{i+1}D_i = J_iD_i \quad \text{and} \quad \mathfrak{S}^*(J_{i+1}) \cap D_i = \mathfrak{S}^*(J_i) \cap D_i .$$

In view of (6.2.5) we have that D_i is an open subset of X_{i+1}, and hence by (1_i), $[i, D_i]$, and $[i+1, D_i]$ we get that:

$$(2_i) \qquad (\mathfrak{S}^*(J_i) \cap D_i) - \{R\} = (\mathfrak{S}^*(J_{i+1}) \cap D_i \cap D_{i+1}) - \{R\}.$$

Since $T_i \subset \mathfrak{S}^*(J_i)$, we get that $X_i - \mathfrak{S}^*(J_i) \subset X_{i+1} - \mathfrak{S}^*(J_{i+1})$; also $X_{i+1} - \mathfrak{S}^*(J_{i+1}) \subset D_{i+1}$, and hence $X_i - \mathfrak{S}^*(J_i) \subset D_{i+1}$; therefore by (2_i) we get that $D_i \subset D_{i+1}$. Thus D_i is an open subset of X_i with $R \in D_i$ and $D_i \cap T_i = \varnothing$ for all $i \geqslant n$, and also $D_i \subset D_{i+1}$ for all $i \geqslant n$. It suffices to take $D = D_n$.

PROOF OF (7.4.2). In view of (6.5.3) and (6.8) we have that $\mathfrak{F}(X_{i+1}, X_i) = T_i \subset \mathfrak{S}^*(J_i)$ for $0 \leqslant i < \infty$, and hence $\mathfrak{F}(X_i, X_0) \subset \mathfrak{S}^*(J_0)$ for $0 \leqslant i < \infty$. Given any $V \in \mathfrak{R}(X_0)$, let R_i' be the center of V on X_i for $0 \leqslant i < \infty$, and let $(a(j))_{0 \leqslant j \leqslant n}$ be the unique sequence such that: either n is a nonnegative integer or $n = \infty$; $a(j)$ is a nonnegative integer for $0 \leqslant j < n$; $a(0) = 0$; $a(j-1) < a(j)$ and $R'_{a(j-1)} = R_i' \neq R'_{a(j)}$ whenever $0 < j < n$ and $a(j-1) \leqslant i < a(j)$; if $n \neq \infty$ then $a(n)$ is a nonnegative integer and $R'_{a(n)} = R_i'$ whenever $a(n) \leqslant i < \infty$; and if $n = \infty$ then $a(n) = \infty$. For $0 \leqslant j < n$, upon letting $R_j = R'_{a(j)}$, we get that $R_j \in T_{a(j+1)-1}$; let S_j be the generic point of the irreducible component of $T_{a(j+1)-1}$ passing through R_j, and let $P_j = J_{a(j+1)-1}R_j$. Suppose if possible that $n = \infty$; then $(R_j, P_j, S_j)_{0 \leqslant j < \infty}$ is an infinite semiresolver; since $\mathfrak{F}(X_i, X_0) \subset \mathfrak{S}^*(J_0)$ for $0 \leqslant i < \infty$, we get that $R_0 \in \mathfrak{S}^*(J_0)$; this contradicts our assumption. Therefore $n \neq \infty$. Let $n(V) = n$. Thus for each $V \in \mathfrak{R}(X_0)$ we have found a nonnegative integer $n(V)$ such that upon letting $R(V)$ be the center of V on $X_{n(V)}$, we have that $R(V) \in \bigcap_{i=n(V)}^{\infty} X_i$. By (7.4.1) there exists an open subset $D(V)$ of $X_{n(V)}$ with $R(V) \in D(V)$ such that $D(V) \subset X_i$ and $D(V) \cap T_i = \varnothing$ for all $i \geqslant n(V)$. For each $V \in \mathfrak{R}(X_0)$ we clearly have that $V \in [\mathfrak{R}(X_0), X_{n(V)}]^{-1}(D(V))$, and by (6.2.5) we get that $[\mathfrak{R}(X_0), X_{n(V)}]^{-1}(D(V))$ is an open subset of $\mathfrak{R}(X_0)$; now $\mathfrak{R}(X_0)$ is quasicompact by (6.2.13), and hence there exists a finite number of elements $V_1, \ldots, V_q$ in $\mathfrak{R}(X_0)$ such that

$$\mathfrak{R}(X_0) = \bigcup_{j=1}^{q} [\mathfrak{R}(X_0), X_{n(V_j)}]^{-1}(D(V_j)).$$

Let m be any nonnegative integer such that $m \geqslant n(V_j)$ for $1 \leqslant j \leqslant q$. Then clearly $T_i = \varnothing$ for all $i \geqslant m$.

(7.5). *If K/k is locally strongly semiresolvable then K/k is globally semiresolvable and globally strongly semiresolvable.*

PROOF. Follows from (7.3) and (7.4.2).

(7.6). *Let $[(X_i, J_i, T_i)_{0 \leqslant i < m}, (X', J')]$ be any finite global semiresolver of K/k. Then $\mathfrak{F}(X', X_0)$ and $\mathfrak{Z}(J_0)$ are closed subsets of X_0, $\mathfrak{F}(X', X_0) = \mathfrak{S}^*(J_0) = \mathfrak{S}(\mathfrak{Z}(J_0)) \subset \mathfrak{Z}(J_0)$, $\mathfrak{F}(X', X_0)$ does not pass through the generic point of any irreducible component of $\mathfrak{Z}(J_0)$, $\mathfrak{Z}(J')$ is the $[X', X_0]$-transform of $\mathfrak{Z}(J_0)$, and $\mathfrak{Z}(J')$ is nonsingular.*

PROOF. Follows from (6.2.18), (6.4.6), (6.5.2), (6.5.3), (6.9.1), and (6.9.4).

(7.7). *Assume that K/k is globally semiresolvable. Let X be any nonsingular model of K/k and let Z be any closed subset of X such that either $Z = X$ or Z is pure* 1*-codimensional* (note that the assumptions about Z are satisfied if Z is a surface in X, and they are also satisfied if $\dim_k K \leqslant 2$ and Z is a curve in X). *Then there exists a nonsingular model X' of K/k such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{S}(Z)$, and the $[X', X]$-transform of Z is nonsingular* (note that by (6.5.3) we know that $\mathfrak{S}(Z)$ is a closed subset of Z, and clearly $\mathfrak{S}(Z)$ does not pass through the generic point of any irreducible component of Z).

PROOF. If $Z = X$ then it suffices to take $X' = X$. So now assume that $Z \neq X$ and let $J = \mathfrak{J}(Z, X)$. By (6.4.6) we have that J is a nonzero ideal on X and $Z = \mathfrak{Z}(J)$. Clearly J is principal. Since K/k is globally semiresolvable, there exists a finite global semiresolver $[(X_i, J_i, T_i)_{0 \leqslant i < m}, (X', J')]$ of K/k with $(X_0, J_0) = (X, J)$. Now X' is an iterated monoidal transform of X with nonsingular irreducible centers, and by (7.6) we get that $\mathfrak{F}(X', X) = \mathfrak{S}(Z)$, and the $[X', X]$-transform of Z is nonsingular.

(7.8). DEFINITION. By a *global detacher* of K/k we mean a sequence $(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}$ where: (1) either m is a positive integer or $m = \infty$; (2) for $0 \leqslant i < m$: X_i is a nonsingular model of K/k, J_i and I_i are nonzero principal ideals on X_i such that (J_i, I_i) has only quasinormal crossings, and T_i is a nonsingular closed subset of X_i with $T_i \subset \mathfrak{S}^*(J_i)$ such that for every $R \in T_i$, upon letting S be the generic point of the irreducible component of T_i passing through R, we have that $S \in \mathfrak{E}(R, J_i)$, (S, I_i) has a normal crossing at R, and: $\dim S = 2 \Leftrightarrow (\mathfrak{E}^2(R, J_i), I_i)$ has a strict normal crossing at R and $\mathfrak{E}^2(R, J_i) \neq \varnothing$; and (3) for $0 < i < m$: (X_i, J_i, I_i) is the monoidal transform of $(X_{i-1}, J_{i-1}, I_{i-1})$ with center T_i.

By an *infinite global detacher* of K/k we mean a global detacher $(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}$ of K/k where $m = \infty$ and $T_i \neq \varnothing$ for infinitely many distinct values of i.

By a *finite global detacher* of K/k we mean a system $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}, (X', J', I')]$ where : m is a positive integer; $(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}$ is a global detacher of K/k such that for $0 \leqslant i < m$ we have that $T_i \subset \mathfrak{E}^*(\mathfrak{S}^*(J_i), J_i)$ and either $T_i = \varnothing$ or T_i is irreducible; X' is a nonsingular model of K/k and J' and I' are nonzero principal ideals on X' such that $\mathfrak{S}^*(J') = \varnothing$ and (J', I') has only quasinormal crossings; and (X', J', I') is the monoidal transform of $(X_{m-1}, J_{m-1}, I_{m-1})$ with center T_{m-1} (note that by (6.4.4) we have that $J'I'$ is a nonzero principal ideal on X', and clearly $J'I'$ has only normal crossings; also note that $(J_0 I_0)X' = J'I'$).

K/k is said to be *globally detachable* if: given any nonsingular model X of K/k and any nonzero principal ideals J and I on X such that (J, I) has only quasinormal crossings, there exists a finite global detacher $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}, (X', J', I')]$ of K/k such that $(X_0, J_0, I_0) = (X, J, I)$. K/k is said to be *globally strongly detachable* if there does not exist any infinite global detacher of K/k. K/k is said to be *locally strongly detachable* if every regular spot over k with quotient field K is strongly detachable.

(7.9). *Let X be any nonsingular model of K/k and let J and I be any nonzero principal ideals on X such that (J, I) has only quasinormal crossings. Then we have the following.*

(7.9.1). *Assume that $\mathfrak{S}^*(J) \neq \varnothing$. Then there exists a nonsingular irreducible closed subset T of X such that $T \subset \mathfrak{S}^*(J)$, $T \subset \mathfrak{C}^*(\mathfrak{S}^*(J), J)$, and, upon letting S be the generic point of T, for every $R \in T$ we have that $S \in \mathfrak{C}(R, J)$, (S, I) has a normal crossing at R, and:* $\dim S = 2 \Leftrightarrow (\mathfrak{C}^2(R, J), I)$ *has a strict normal crossing at R and $\mathfrak{C}^2(R, J) \neq \varnothing$.*

(7.9.2). *Assume that there does not exist any infinite global detacher $(X_i, J_i, I_i, T_i)_{0 \leqslant i < \infty}$ of K/k such that $(X_0, J_0, I_0) = (X, J, I)$. Then there exists a finite global detacher $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}, (X', J', I')]$ of K/k such that $(X_0, J_0, I_0) = (X, J, I)$.*

PROOF OF (7.9.1). By (6.5.3) we know that $\mathfrak{S}^*(J)$ is a closed subset of X and $\operatorname{codim} \mathfrak{S}^*(J) \geqslant 2$. Therefore by (6.5.4) we get that $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ is a nonempty closed subset of $\mathfrak{S}^*(J)$ and $\operatorname{codim} \mathfrak{C}^*(\mathfrak{S}^*(J), J) \geqslant 2$. Let $S_1, \ldots, S_n$ $(n > 0)$ be the generic points of the irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$. Then $2 \leqslant \dim S_i \leqslant 3$ for $1 \leqslant i \leqslant n$. Since (J, I) has only quasinormal crossings, we get that (R, I) has a normal crossing at R for all $R \in X$. If $\dim S_i = 3$ for some i, then $\{S_i\}$ is a nonsingular irreducible closed subset of X and by (6.5.6) we get that $\mathfrak{C}^2(S_i, J) = \varnothing$, and hence it suffices to $T = \{S_i\}$. So now assume that $\dim S_i = 2$ for $1 \leqslant i \leqslant n$. Then by (6.5.6) we get that $\mathfrak{C}^2(R, J) = \mathfrak{V}(R) \cap \{S_1, \ldots, S_n\}$ for all $R \in \mathfrak{C}^*(\mathfrak{S}^*(J), J)$. If there exists $R \in \mathfrak{C}^*(\mathfrak{S}^*(J), J)$ such that $(\mathfrak{V}(R) \cap \{S_1, \ldots, S_n\}, I)$ does not have a strict normal crossing at R then $\dim R = 3$ and $\{R\}$ is a nonsingular irreducible closed subset of X, and hence it suffices to take $T = \{R\}$. If $(\mathfrak{V}(R) \cap \{S_1, \ldots, S_n\}, I)$ has a strict normal crossing at R for all $R \in \mathfrak{C}^*(\mathfrak{S}^*(J), J)$, then it suffices to take $T =$ closure of $\{S_i\}$ in X for some i.

PROOF OF (7.9.2). Let W be the set of all global detachers $(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}$ of K/k such that $(X_0, J_0, I_0) = (X, J, I)$ and $T_i \subset \mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i)$ and T_i is irreducible for $0 \leqslant i < m$. If $\mathfrak{S}^*(J) = \varnothing$ then we get a finite global detacher $[(X_i, J_i, I_i,$

$T_i)_{0\leqslant i<1}$, $(X', J', I')]$ of K/k of the required type by taking $(X_0, J_0, I_0) = (X, J, I) = (X', J', I')$ and $T_0 = \varnothing$. So now assume that $\mathfrak{S}^*(J) \neq \varnothing$. Then there exists T as in (7.9.1) and we get an element $(X_i, J_i, I_i, T_i)_{0\leqslant i<1}$ in W by taking $(X_0, J_0, I_0, T_0) = (X, J, I, T)$. Therefore W is nonempty. For each pair of elements $w = (X_i, J_i, I_i, T_i)$ and $w' = (X_i', J_i', I_i', T_i')_{0\leqslant i<m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(X_i, J_i, I_i, T_i) = (X_i', J_i', I_i', T_i')$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property and hence by Zorn's lemma W contains a maximal element $w = (X_i, J_i, I_i, T_i)_{0\leqslant i<m}$. By assumption we must have $m \neq \infty$. Let (X', J', I') be the monoidal transform of $(X_{m-1}, J_{m-1}, I_{m-1})$ with center T_{m-1}. Then by (6.8), (6.9.1), and (6.9.2), we have that X' is a nonsingular model of K/k and J' and I' are nonzero principal ideals on X' such that (J', I') has only quasinormal crossings. Suppose if possible that $\mathfrak{S}^*(J') \neq \varnothing$; then by (7.9.1) there exists a nonsingular irreducible closed subset T' of X' such that $T' \subset \mathfrak{S}^*(J')$, $T' \subset \mathfrak{C}^*(\mathfrak{S}^*(J'), J')$, and, upon letting S be the generic point of T', for every $R \in T'$ we have that $S \in \mathfrak{C}(R, J')$, (S, I') has a normal crossing at R, and: $\dim S = 2 \Leftrightarrow (\mathfrak{C}^2(R, J'), I')$ has a strict normal crossing at R and $\mathfrak{C}^2(R, J') \neq \varnothing$; we now get an element $w' = (X_i', J_i', I_i', T_i')_{0\leqslant i<m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking $(X_i', J_i', I_i', T_i') = (X_i, J_i, I_i, T_i)$ for $0 \leqslant i < m$ and $(X_m', J_m', I_m', T_m') = (X', J', I', T')$. This is a contradiction because w is a maximal element of W. Therefore $\mathfrak{S}^*(J') = \varnothing$ and hence $[(X_i, J_i, I_i, T_i)_{0\leqslant i<m}, (X', J', I')]$ is a finite global detacher of K/k with $(X_0, J_0, I_0) = (X, J, I)$.

(7.10). *If K/k is globally strongly detachable then K/k is globally detachable.*

PROOF. Follows from (7.9.2).

(7.11). *For any global detacher $(X_i, J_i, I_i, T_i)_{0\leqslant i<\infty}$ of K/k we have the following.*

(7.11.1). *Given any nonnegative integer n and any $R \in \bigcap_{i=n}^{\infty} X_i$, there exists an open subset D of X_n with $R \in D$ such that $D \subset X_i$ and $D \cap T_i = \varnothing$ for all $i \geqslant n$.*

(7.11.2). *Assume that there does not exist any infinite detacher $(R_j, P_j, Q_j, S_j)_{0 \leqslant j < \infty}$ with $R_0 \in \mathfrak{S}^*(J_0)$. Then there exists a non-negative integer m such that $T_i = \varnothing$ for all $i \geqslant m$.*

PROOF OF (7.11.1). In view of (6.5.3) and (6.8) we have that $\mathfrak{S}^*(J_i)$ is a closed subset of X_i with $\operatorname{codim} \mathfrak{S}^*(J_i) \geqslant 2$ and $\mathfrak{F}(X_{i+1}, X_i) = T_i \subset \mathfrak{S}^*(J_i)$ for $0 \leqslant i < \infty$. Since $R \in \bigcap_{i=n}^{\infty} X_i$, we get that $R \notin T_i$ for all $i \geqslant n$. For each $i \geqslant n$ let G_i be the union of the irreducible components of $\mathfrak{S}^*(J_i)$ passing through R, let H_i be the union of the remaining irreducible components of $\mathfrak{S}^*(J_i)$, let

$$G_i^* = \mathfrak{S}(G_i) \cup \{R' \in G_i - \mathfrak{S}(G_i): S \in \mathfrak{C}(R', J_i) \text{ where } S \text{ is the generic point of the irreducible component of } G_i \text{ passing through } R'\},$$

let

$$G_i' = \{R' \in G_i - \mathfrak{S}(G_i): (S, I_i) \text{ does not have a normal crossing at } R' \text{ where } S \text{ is the generic point of the irreducible component of } G_i \text{ passing through } R'\},$$

and let $D_i = X_i - (((G_i^* \cup G_i') - \{R\}) \cup H_i)$. Then $R \in D_i$, and in view of (6.5.5) and (6.5.8) we get that D_i is an open subset of X_i. For any open subset E of X_i with $R \in E$, in view of (6.2.16) we get that:

$$\begin{aligned}(\mathfrak{S}^*(J_i) \cap E \cap D_i) - \{R\} \\ = \{R' \in (\mathfrak{S}^*(J_i) \cap E) - (\mathfrak{S}(\mathfrak{S}^*(J_i) \cap E) \cup \{R\})&: \text{ upon letting } F \text{ be the irreducible component of } \mathfrak{S}^*(J_i) \cap E \text{ passing through } R', \\ &\text{and } S \text{ be the generic point of } F, \text{ we have that } R \in F, \\ &S \in \mathfrak{C}(R', J_i), \text{ and } (S, I_i) \text{ has a normal crossing at } R'\};\end{aligned}$$

let us refer to this observation as $[i, E]$. Since $R \notin T_i$, by (6.5.6) and $[i, X_i]$ we see that $D_i \cap T_i = \varnothing$; consequently $D_i \subset X_{i+1}$ and:

$$(1_i) \qquad \begin{aligned} &I_{i+1}D_i = I_iD_i, \; J_{i+1}D_i = J_iD_i, \quad \text{and} \\ &\mathfrak{S}^*(J_{i+1}) \cap D_i = \mathfrak{S}^*(J_i) \cap D_i. \end{aligned}$$

In view of (6.2.5) we have that D_i is an open subset of X_{i+1}, and hence by (1_i), $[i, D_i]$, and $[i+1, D_i]$ we get that:

$$(2_i) \qquad (\mathfrak{S}^*(J_i) \cap D_i) - \{R\} = (\mathfrak{S}^*(J_{i+1}) \cap D_i \cap D_{i+1}) - \{R\}.$$

Since $T_i \subset \mathfrak{S}^*(J_i)$, we get that $X_i - \mathfrak{S}^*(J_i) \subset X_{i+1} - \mathfrak{S}^*(J_{i+1})$; also $X_{i+1} - \mathfrak{S}^*(J_{i+1}) \subset D_{i+1}$, and hence $X_i - \mathfrak{S}^*(J_i) \subset D_{i+1}$; therefore by (2_i) we get that $D_i \subset D_{i+1}$. Thus D_i is an open subset of X_i with $R \in D_i$ and $D_i \cap T_i = \varnothing$ for all $i \geqslant n$, and also $D_i \subset D_{i+1}$ for all $i \geqslant n$. It suffices to take $D = D_n$.

PROOF OF (7.11.2). In view of (6.5.3) and (6.8) we have that $\mathfrak{F}(X_{i+1}, X_i) = T_i \subset \mathfrak{S}^*(J_i)$ for $0 \leqslant i < \infty$, and hence $\mathfrak{F}(X_i, X_0) \subset \mathfrak{S}^*(J_0)$ for $0 \leqslant i < \infty$. Given any $V \in \mathfrak{R}(X_0)$, let R_i' be the center of V on X_i for $0 \leqslant i < \infty$, and let $(a(j))_{0 \leqslant j \leqslant n}$ be the unique sequence such that: either n is a nonnegative integer or $n = \infty$; $a(j)$ is a nonnegative integer for $0 \leqslant j < n$; $a(0) = 0$; $a(j-1) < a(j)$ and $R'_{a(j-1)} = R_i' \neq R'_{a(j)}$ whenever $0 < j < n$ and $a(j-1) \leqslant i < a(j)$; if $n \neq \infty$ then $a(n)$ is a nonnegative integer and $R'_{a(n)} = R_i'$ whenever $a(n) \leqslant i < \infty$; and if $n = \infty$ then $a(n) = \infty$. For $0 \leqslant j < n$, upon letting $R_j = R'_{a(j)}$, we get that $R_j \in T_{a(j+1)-1}$; let S_j be the generic point of the irreducible component of $T_{a(j+1)-1}$ passing through R_j, let $P_j = J_{a(j+1)-1}R_j$, and let $Q_j = I_{a(j+1)-1}R_j$. Suppose if possible that $n = \infty$; then $(R_j, P_j, Q_j, S_j)_{0 \leqslant j < \infty}$ is an infinite detacher; since $\mathfrak{F}(X_i, X_0) \subset \mathfrak{S}^*(J_0)$ for $0 \leqslant i < \infty$, we get that $R_0 \in \mathfrak{S}^*(J_0)$; this contradicts our assumption. Therefore $n \neq \infty$. Let $n(V) = n$. Thus for each $V \in \mathfrak{R}(X_0)$ we have found a nonnegative integer $n(V)$ such that, upon letting $R(V)$ be the center of V on $X_{n(V)}$, we have that $R(V) \in \bigcap_{i=n(V)}^{\infty} X_i$. By (7.11.1) there exists an open subset $D(V)$ of $X_{n(V)}$ with $R(V) \in D(V)$ such that $D(V) \subset X_i$ and $D(V) \cap T_i = \varnothing$ for all $i \geqslant n(V)$. For each $V \in \mathfrak{R}(X_0)$ we clearly have that $V \in [\mathfrak{R}(X_0), X_{n(V)}]^{-1}(D(V))$, and by (6.2.5) we get that $[\mathfrak{R}(X_0), X_{n(V)}]^{-1}(D(V))$ is an open subset of $\mathfrak{R}(X_0)$; now $\mathfrak{R}(X_0)$ is quasicompact by (6.2.13), and hence there exists a finite number of elements $V_1, \ldots, V_q$ in $\mathfrak{R}(X_0)$ such that

$$\mathfrak{R}(X_0) = \bigcup_{j=1}^{q} [\mathfrak{R}(X_0), X_{n(V_j)}]^{-1}(D(V_j)).$$

Let m be any nonnegative integer such that $m \geqslant n(V_j)$ for $1 \leqslant j \leqslant q$. Then clearly $T_i = \varnothing$ for all $i \geqslant m$.

(7.12). *If K/k is locally strongly detachable then K/k is globally detachable and globally strongly detachable.*

PROOF. Follows from (7.10) and (7.11.2).

(7.13). DEFINITION. By a *global subresolver* of K/k we mean a sequence $(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}$ where: (1) either m is a positive integer or $m = \infty$; (2) for $0 \leqslant i < m$: X_i is a nonsingular model of K/k, J_i and I_i are nonzero principal ideals on X_i such that I_i has only quasinormal crossings, Z_i is a pure 2-codimensional closed subset of X_i with $Z_i \subset \mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i)$, and T_i is a nonsingular closed subset of Z_i such that for every $R \in T_i$, upon letting S be the generic point of the irreducible component of T_i passing through R, we have that (S, I_i) has a pseudonormal crossing at R and: $\dim S = 2 \Leftrightarrow \mathfrak{C}^2(R, J_i)$ has a strict normal crossing at R and $(Z_i \cap \mathfrak{C}^2(R, J_i), I_i)$ has a pseudonormal crossing at R; and (3) for $0 < i < m$: (X_i, J_i, I_i) is the monoidal transform of $(X_{i-1}, J_{i-1}, I_{i-1})$ with center T_i, $\operatorname{ord}_{\mathfrak{S}^*(J_i)} J_i = \operatorname{ord}_{\mathfrak{S}^*(J_{i-1})} J_{i-1}$, and Z_i is the closure in X_i of $\{S \in \mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i): S$ dominates a two-dimensional point of $Z_{i-1}\}$.

By an *infinite global subresolver of K/k* we mean a global subresolver $(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}$ of K/k where $m = \infty$ and $T_i \neq \varnothing$ for infinitely many distinct values of i.

By a *finite global subresolver* of K/k we mean a system $[(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}, (X', J', I')]$ where: m is a positive integer; $(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}$ is a global subresolver of K/k such that T_i is irreducible for $0 \leqslant i < m$; X' is a nonsingular model of K/k and J' and I' are nonzero principal ideals on X' such that I' has only quasinormal crossings; (X', J', I') is the monoidal transform of $(X_{m-1}, J_{m-1}, I_{m-1})$ with center T_{m-1}; and for every $S \in \mathfrak{S}^*(J')$ such that S dominates a two-dimensional point of Z_{m-1} we have that $\operatorname{ord}_S J'S < \operatorname{ord}_{\mathfrak{S}^*(J_{m-1})} J_{m-1}$.

K/k is said to be *globally subresolvable* if: given any nonsingular model X of K/k, any nonzero principal ideals J and I on X such that I has only quasinormal crossings, and any pure 2-codimensional closed subset Z of X with $Z \subset \mathfrak{C}^*(\mathfrak{S}^*(J), J)$, there exists a finite global subresolver $[(X_i, J_i, I_i, Z_i, T_i)_{0\leqslant i<m}, (X', J', I')]$ of K/k such that $(X_0, J_0, I_0, Z_0) = (X, I, J, Z)$. K/k is said to be *globally strongly subresolvable* if there does not exist any infinite global subresolver of K/k. K/k is said to be *locally strongly subresolvable* if every regular spot over k with quotient field K is strongly subresolvable.

(7.14). *Let X be any nonsingular model of K/k, let J and I be any nonzero principal ideals on X such that I has only quasinormal crossings, and let Z be any pure 2-codimensional closed subset of X with $Z \subset \mathfrak{C}^*(\mathfrak{S}^*(J), J)$. Then we have the following.*

(7.14.1). *There exists a nonsingular irreducible closed subset T of Z such that, upon letting S be the generic point of T, for every $R \in T$ we have that (S, I) has a pseudonormal crossing at R and:* $\dim S = 2 \Leftrightarrow \mathfrak{C}^2(R, J)$ *has a strict normal crossing at R and $(Z \cap \mathfrak{C}^2(R, J), I)$ has a pseudonormal crossing at R.*

(7.14.2). *Assume that there does not exist any infinite global subresolver $(X_i, J_i, I_i, Z_i, T_i)_{0\leqslant i<\infty}$ of K/k such that $(X_0, J_0, I_0, Z_0) = (X, I, J, Z)$. Then there exists a finite global subresolver $[(X_i, J_i, I_i, Z_i, T_i)_{0\leqslant i<m}, (X', J', I')]$* of *$K/k$ such that $(X_0, J_0, I_0, Z_0) = (X, J, I, Z)$.*

PROOF OF (7.14.1). If there exists $R \in Z$ such that either $\mathfrak{C}^2(R, J)$ does not have a strict normal crossing at R or $(Z \cap \mathfrak{C}^2(R, J), I)$ does not have a pseudonormal crossing at R then $\dim R = 3$, $\{R\}$ is a nonsingular irreducible closed subset of Z, and (R, I) has a pseudonormal crossing at R, and hence it suffices to take $T = \{R\}$. If for every $R \in Z$ we have that $\mathfrak{C}^2(R, J)$ has a strict normal crossing at R and $(Z \cap \mathfrak{C}^2(R, J), I)$ has a pseudonormal crossing at R, then it suffices to take $T =$ any irreducible component of Z.

PROOF OF (7.14.2). Let W be the set of all global subresolvers $(X_i, J_i, I_i, Z_i, T_i)_{0\leqslant i<m}$ of K/k such that $(X_0, J_0, I_0, Z_0) = (X, J, I, Z)$ and T_i is irreducible for $0 \leqslant i < m$. We get an element $(X_i, J_i, I_i, Z_i, T_i)_{0\leqslant i<1}$ in W by taking $(X_0, J_0, I_0, Z_0, T_0) = (X, J, I. Z, T)$ where T is as in (7.14.1). Thus W is nonempty. For each pair of elements $w = (X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}$ and $w' = (X'_i, J'_i, I'_i, Z'_i, T'_i)_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(X_i, J_i, I_i, Z_i, T_i) = (X'_i, J'_i, I'_i, Z'_i, T'_i)$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property and hence by Zorn's lemma W contains a maximal element $w = (X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}$. By assumption we must have $m \neq \infty$. Let (X', J', I') be the monoidal transform of $(X_{m-1}, J_{m-1}, I_{m-1})$ with center T_{m-1}. Then by (6.8), (6.9.1), (6.9.3), and (6.9.4), we have that X' is a nonsingular model of K/k and J' and I' are nonzero principal ideals on X' such that I' has only quasinormal crossings and, upon letting $d = \operatorname{ord}_{\mathfrak{S}^*(J_{m-1})} J_{m-1}$ and $e = \operatorname{ord}_{\mathfrak{S}^*(J')} J'$, we have that $e \leqslant d$. If $e < d$ then $[(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}, (X', J', I')]$ is a finite global subresolver of K/k with $(X_0, J_0, I_0, Z_0) = (X, J, I, Z)$. So now assume that $e = d$. Let Z' be the closure in X' of $\{S \in \mathfrak{C}^*(\mathfrak{S}^*(J'), J')$: such that S dominates a two-dimensional point of $Z_{m-1}\}$. By (6.5.3) and (6.5.4) we get that $\mathfrak{C}^*(\mathfrak{S}^*(J'), J')$ is a closed subset of X' with codim $\mathfrak{C}^*(\mathfrak{S}^*(J'), J') \geqslant 2$. Therefore $Z' \subset \mathfrak{C}^*(\mathfrak{S}^*(J'), J')$, and either $Z' = \varnothing$ or Z' is pure 2-codimensional. Suppose if possible that $Z' \neq \varnothing$; then by (7.14.1) there exists a nonsingular irreducible closed subset T' of Z' such that, upon letting S' be the generic point of T', for every $R' \in T'$ we have that (S', I') has a pseudonormal crossing at R' and: $\dim S' = 2 \Leftrightarrow \mathfrak{C}^2(R', J')$ has a strict normal crossing at R' and $(Z' \cap \mathfrak{C}^2(R', J'), I')$ has a pseudonormal crossing at R'; we now get an element $w' = (X'_i, J'_i, I'_i, Z'_i, T'_i)_{0 \leqslant i < m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking $(X'_i, J'_i, I'_i, Z'_i, T'_i) = (X_i, J_i, I_i, Z_i, T_i)$ for $0 \leqslant i < m$ and $(X'_m, J'_m, I'_m, Z'_m, T'_m) = (X', J', I', Z', T')$. This is a contradiction because w is a maximal element of W'. Therefore $Z' = \varnothing$ and hence $[(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}, (X', J', I')]$ is a finite global subresolver of K/k with $(X_0, J_0, I_0, Z_0) = (X, J, I, Z)$.

(7.15). *If K/k is globally strongly subresolvable then K/k is globally subresolvable.*

PROOF. Follows from (7.14.2).

(7.16). *For any global subresolver $(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < \infty}$ of K/k we have the following.*

(7.16.1). *Given any nonnegative integer n and any $R \in \bigcap_{i=n}^{\infty} X_i$, there exists an open subset D of X_n with $R \in D$ such that $D \subset X_i$ and $D \cap T_i = \varnothing$ for all $i \geqslant n$.*

(7.16.2). *Assume that there does not exist any infinite subresolver $(R_j, P_j, Q_j, L_j, S_j)_{0 \leqslant j < \infty}$ with $R_0 \in Z_0$. Then there exists a nonnegative integer m such that $T_i = \varnothing$ for all $i \geqslant m$.*

PROOF OF (7.18.1). In view of (6.5.3) and (6.8) we have that $\mathfrak{S}^*(J_i)$ is a closed subset of X_i with $\operatorname{codim} \mathfrak{S}^*(J_i) \geqslant 2$ and $\mathfrak{F}(X_{i+1}, X_i) = T_i \subset \mathfrak{S}^*(J_i)$ for $0 \leqslant i < \infty$. Since $R \in \bigcap_{i=n}^{\infty} X_i$, we get that $R \notin T_i$ for all $i \geqslant n$. For each $i \geqslant n$ let G_i be the union of the irreducible components of $\mathfrak{S}^*(J_i)$ passing through R, let H_i be the union of the remaining irreducible components of $\mathfrak{S}^*(J_i)$, let

$$G_i^* = \mathfrak{S}(G_i) \cup \{R' \in G_i - \mathfrak{S}(G_i) \colon S \notin \mathfrak{C}(R', J_i) \text{ where } S \text{ is the generic point of the irreducible component of } G_i \text{ passing through } R'\},$$

let

$$G_i' = \{R' \in G_i - \mathfrak{S}(G_i) \colon (S, I_i) \text{ does not have a pseudonormal crossing at } R' \text{ where } S \text{ is the generic point of the irreducible component of } G_i \text{ passing through } R'\},$$

and let $D_i = X_i - (((G_i^* \cup G_i') - \{R\}) \cup H_i)$. Then $R \in D_i$, and in view of (6.5.5) and (6.5.7) we get that D_i is an open subset of X_i. For any open subset E of X_i with $R \in E$, in view of (6.2.16) we get that:

$$\begin{aligned}
&(\mathfrak{S}^*(J_i) \cap E \cap D_i) - \{R\} \\
&= \{R' \in (\mathfrak{S}^*(J_i) \cap E) - (\mathfrak{S}(\mathfrak{S}^*(J_i) \cap E) \cup \{R\}) \colon \text{upon letting } F \\
&\quad \text{be the irreducible component of } \mathfrak{S}^*(J_i) \cap E \text{ passing through} \\
&\quad R', \text{ and } S \text{ be the generic point of } F, \text{ we have that } R \in F,\ S \in \\
&\quad \mathfrak{C}(R', J_i), \text{ and } (S, I_i) \text{ has a pseudonormal crossing at } R'\};
\end{aligned}$$

let us refer to this observation as $[i, E]$. Since $R \notin T_i$, by (6.5.6) and $[i, X_i]$ we see that $D_i \cap T_i = \varnothing$; consequently $D_i \subset X_{i+1}$ and:

$$(1_i) \qquad \begin{array}{l} I_{i+1}D_i = I_iD_i\,, \qquad J_{i+1}D_i = J_iD_i\,, \qquad \text{and} \\ \mathfrak{S}^*(J_{i+1}) \cap D_i = \mathfrak{S}^*(J_i) \cap D_i\,. \end{array}$$

In view of (6.2.5) we have that D_i is an open subset of X_{i+1}, and hence by (1_i), $[i, D_i]$, and $[i+1, D_i]$ we get that:

$$(2_i) \qquad (\mathfrak{S}^*(J_i) \cap D_i) - \{R\} = (\mathfrak{S}^*(J_{i+1}) \cap D_i \cap D_{i+1}) - \{R\}.$$

Since $T_i \subset \mathfrak{S}^*(J_i)$, we get that $X_i - \mathfrak{S}^*(J_i) \subset X_{i+1} - \mathfrak{S}^*(J_{i+1})$; also $X_{i+1} - \mathfrak{S}^*(J_{i+1}) \subset D_{i+1}$, and hence $X_i - \mathfrak{S}^*(J_i) \subset D_{i+1}$; therefore by (2_i) we get that $D_i \subset D_{i+1}$. Thus D_i is an open subset of X_i with $R \in D_i$ and $D_i \cap T_i = \varnothing$ for all $i \geqslant n$, and also $D_i \subset D_{i+1}$ for all $i \geqslant n$. It suffices to take $D = D_n$.

PROOF OF (7.16.2). In view of (6.8) we have that $\mathfrak{F}(X_{i+1}, X_i) = T_i$ and $Z_{i+1} \subset [X_{i+1}, X_i]^{-1}(Z_i)$ for $0 \leqslant i < \infty$, and hence $\mathfrak{F}(X_i, X_0) \subset Z_0$ for $0 \leqslant i < \infty$. Given any $V \in \mathfrak{R}(X_0)$, let R'_i be the center of V on X_i for $0 \leqslant i < \infty$, and let $(a(j))_{0 \leqslant j < n}$ be the unique sequence such that: either n is a nonnegative integer or $n = \infty$; $a(j)$ is a nonnegative integer for $0 \leqslant j < n$; $a(0) = 0$; $a(j-1) < a(j)$ and $R'_{a(j-1)} = R'_i \neq R'_{a(j)}$ whenever $0 < j < n$ and $a(j-1) \leqslant i < a(j)$; if $n \neq \infty$ then $a(n)$ is a nonnegative integer and $R'_{a(n)} = R'_i$ whenever $a(n) \leqslant i < \infty$; and if $n = \infty$ then $a(n) = \infty$. For $0 \leqslant j < n$, upon letting $R_j = R'_{a(j)}$, we get that $R_j \in T_{a(j+1)-1}$; let S_j be the generic point of the irreducible component of $T_{a(j+1)-1}$ passing through R_j, let $P_j = J_{a(j+1)-1}R_j$, let $Q_j = I_{a(j+1)-1}R_j$, and let L_j be the set of all two-dimensional points of $\mathfrak{V}(R_j) \cap Z_{a(j+1)-1}$. Suppose if possible that $n = \infty$; then in view of (6.5.3) and (6.5.6) we see that $(R_j, P_j, Q_j, L_j, S_j)_{0 \leqslant j < \infty}$ is an infinite subresolver; since $\mathfrak{F}(X_i, X_0) \subset Z_0$ for $0 \leqslant i < \infty$, we get that $R_0 \in Z_0$; this contradicts our assumption. Therefore $n \neq \infty$. Let $n(V) = n$. Thus for each $V \in \mathfrak{R}(X_0)$ we have found a nonnegative integer $n(V)$ such that, upon letting $R(V)$ be the center of V on $X_{n(V)}$, we have that $R(V) \in \bigcap_{i=n}^{\infty} X_i$. By (7.16.1) there exists an open subset $D(V)$ of $X_{n(V)}$ with $R(V) \in D(V)$ such that

$D(V) \subset X_i$ and $D(V) \cap T_i = \varnothing$ for all $i \geqslant n(V)$. For each $V \in \mathfrak{R}(X_0)$ we clearly have that $V \in [\mathfrak{R}(X_0), X_{n(V)}]^{-1}(D(V))$, and by (6.2.5) we get that $[\mathfrak{R}(X_0), X_{n(V)}]^{-1}(D(V))$ is an open subset of $\mathfrak{R}(X_0)$; now $\mathfrak{R}(X_0)$ is quasicompact by (6.2.13), and hence there exists a finite number of elements $V_1, \ldots, V_q$ in $\mathfrak{R}(X_0)$ such that

$$\mathfrak{R}(X_0) = \bigcup_{j=1}^{q} [\mathfrak{R}(X_0), X_{n(V_j)}]^{-1}(D(V_j)).$$

Let m be any nonnegative integer such that $m \geqslant n(V_j)$ for $1 \leqslant j \leqslant q$. Then clearly $T_i = \varnothing$ for all $i \geqslant m$.

(7.17). *If K/k is locally strongly subresolvable then K/k is globally subresolvable and globally strongly subresolvable.*

PROOF. Follows from (7.15) and (7.16.2).

(7.18). *If k is pseudogeometric then K/k is locally strongly subresolvable, globally subresolvable, and globally strongly subresolvable.*

PROOF. Follows from (3.21.2) and (7.17).

(7.19). *Let J be a nonzero principal ideal on a nonsingular model X of K/k, and let J' be a nonzero principal ideal on a nonsingular model X' of K/k* (note that by (6.5.3) and (6.5.4) we know that then $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ is a closed subset of X with codim $\mathfrak{C}^*(\mathfrak{S}^*(J), J) \geqslant 2$, and $\mathfrak{C}^*(\mathfrak{S}^*(J'), J')$ is a closed subset of X' with codim $\mathfrak{C}^*(\mathfrak{S}^*(J'), J') \geqslant 2$). *Let Z be the union of a certain number of 2-codimensional irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$, and let Y be the union of the remaining irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$. Let Z' be the union of a certain number of 2-codimensional irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J'), J')$, and let Y' be the union of the remaining irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J'), J')$. Let T be a nonsingular closed subset of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ such that for every 2-codimensional irreducible component T^* of T and every $R \in T^*$ we have that $\mathfrak{C}^2(R, J)$ has a strict normal crossing at R. Assume that (X', J') is the monoidal transform of (X, J) with center T,* $\mathrm{ord}_{\mathfrak{S}^*(J')} J' = \mathrm{ord}_{\mathfrak{S}^*(J)} J$, *and Z' is the closure in X' of $\{S' \in \mathfrak{C}^*(\mathfrak{S}^*(J'), J')$: S' dominates a two-dimensional point of $Z\}$. Then we have the following.*

(7.19.1). *For any $R' \in X'$ and $S' \in \mathfrak{V}(R')$, upon letting $R = [X', X](R')$ and $S = [X', X](S')$, we have that $S \in \mathfrak{V}(R)$.*

(7.19.2). *Assume that Y has only strict normal crossings, and let R' be any point of Y'. Then Y' has a strict normal crossing at R'. Moreover, if S_1' and S_2' are any points of $Y' \cap \mathfrak{V}(R')$ such that $S_1' \neq S_2'$ and $S_1' \neq R' \neq S_2'$, then $[X', X](S_1') \neq [X', X](S_2')$ and either* $\dim S_1' = 2$ *or* $\dim S_2' = 2$.

(7.19.3). *If Y has only strict normal crossings then Y' has only strict normal crossings.*

(7.19.4). *Let R_1' and R_2' be any points of $\mathfrak{C}^*(\mathfrak{S}^*(J'), J')$ such that* $\dim R_1' = 3 = \dim R_2'$, $R_1' \neq R_2'$, *and* $[X', X](R_1') = [X', X](R_2')$. *Then upon letting $R = [X', X](R_1')$ we have that* $\dim R = 3$, *and $\{R\}$ is an irreducible component of T.*

(7.19.5). *Assume that Y has only strict normal crossings, and let R_1', R_2', S' be any points of Y' such that $R_1' \neq R_2'$, $S' \in \mathfrak{V}(R_1') \cap \mathfrak{V}(R_2')$, and $[X', X](R_1') = [X', X](R_2')$. Then $[X', X](S') = [X', X](R_1')$.*

(7.19.6). *Assume that Y' has only strict normal crossings, and let R_1', R_2', S_1', S_2', S_3' be any points of Y' such that $R_1' \neq R_2'$, $S_1' \neq S_2' \neq S_3'$,* $\dim S_1' = \dim S_2' = \dim S_3' = 2$, $S_1' \in \mathfrak{V}(R_1')$, $S_2' \in \mathfrak{V}(R_1') \cap \mathfrak{V}(R_2')$, $S_3' \in \mathfrak{V}(R_2')$, *and* $[X', X](R_1') = [X', X](R_2')$. *Then* $[X', X](S_1') \neq [X', X](S_3')$.

(7.19.7). *If Y is unlooped then Y' is unlooped.*

PROOF OF (7.19.1). Now $R_{R \cap M(S')} \in \mathfrak{V}(R) \subset X$, and S' dominates $R_{R \cap M(S')}$. Therefore $S = R_{R \cap M(S')}$ and hence $S \in \mathfrak{V}(R)$.

PROOF OF (7.19.2). Let $S_1', ..., S_n'$ $(n > 0)$ be the generic points of the irreducible components of Y' passing through R'. If $R' = S_1'$ then $Y' \cap \mathfrak{V}(R') = \{R'\}$ and our assertion is obvious. So now assume that $R' \neq S_1'$. Then $\dim R' = 3$, $\dim S_i' = 2$ for $1 \leqslant i \leqslant n$, and $Y' \cap \mathfrak{V}(R') = \{R', S_1', ..., S_n'\}$. Let $R = [X', X](R')$, and $S_i = [X', X](S_i')$ for $1 \leqslant i \leqslant n$. Then $\dim R = 3$, and in view of (6.9.4) and (7.19.1) we get that $R \in \mathfrak{E}^*(\mathfrak{S}^*(J), J)$, and $S_i \in \mathfrak{V}(R) \cap \mathfrak{E}^*(\mathfrak{S}^*(J), J)$ and $2 \leqslant \dim S_i \leqslant 3$ for $1 \leqslant i \leqslant n$. Upon relabeling $S'_1, \ldots, S'_n$ we may assume that $\dim S_i = 2$ (i.e., $S_i \neq R$) for $1 \leqslant i \leqslant m$, and $\dim S_i = 3$ (i.e., $S_i = R$) for $m < i \leqslant n$, where m is an integer with $0 \leqslant m \leqslant n$. Then for $1 \leqslant i \leqslant m$ we have that S_i is the generic point of an irreducible component of Y passing through R, and hence $\{S_1, \ldots, S_m\}$ has a strict normal crossing at R. We shall show that $S_1, \ldots, S_m$ are all distinct, $n - m \leqslant 1$, and $\{S'_1, \ldots, S'_n\}$ has a strict normal crossing at R'; this will complete the proof. This is obvious in case $R \notin T$ because then $R' = R$ and $S'_i = S_i$ for $1 \leqslant i \leqslant n$. So now assume that $R \in T$, and let S be the generic point of the irreducible component of T passing through R. Note that then: $2 \leqslant \dim S \leqslant 3$; S has a simple point at R; JR is a nonzero nonunit principal ideal in R; (R', JR') is a monoidal transform of (R, JR, S); $\operatorname{ord}_{R'} JR' = \operatorname{ord}_R JR$; $\{S_1, \ldots, S_m\} \subset \mathfrak{E}^2(R, JR)$; $\{S_1, \ldots, S_n\} \subset \mathfrak{E}^2(R', JR')$; for $1 \leqslant i \leqslant m$ we have that S'_i dominates S_i; and for $m < i \leqslant n$ we have that S'_i dominates R.

First suppose that $\dim S \neq 2$. Then $S = R$; consequently for $1 \leqslant i \leqslant m$ we have that $S_i \notin T$ and hence $S'_i = S_i$; in particular, $S_1, \ldots, S_m$ are all distinct and $\{S_1, \ldots, S_m\} = \{S'_1, \ldots, S'_m\} \subset \mathfrak{V}(R) \cap \mathfrak{V}(R')$. Therefore by (3.7.1) and (3.7.2) we get that $n - m \leqslant 1$ and $\{S'_1, \ldots, S'_n\}$ has a strict normal crossing at R'.

Next suppose that $\dim S = 2$. For $m < i \leqslant n$ we have that $(S'_i, J'S'_i)$ is a monoidal transform of (R, JR, S) and $J'S'_i \neq S'_i$, and hence by (3.10.2) we get that $\dim S'_i = 3$. Consequently we must have $m = n$. By assumption we know that $\mathfrak{E}^2(R, JR)$ has a strict normal crossing at R; therefore by (3.10.5) we conclude that $S_1, \ldots, S_m$ are all distinct and $\{S'_1, \ldots, S'_n\}$ has a strict normal crossing at R'.

PROOF OF (7.19.3). This follows from (7.19.2).

PROOF OF (7.19.4). Since $[X', X](R'_1) = R = [X', X](R'_2)$ and $R'_1 \neq R'_2$, we must have $R \in T$. Also clearly $\dim R = 3$. Let S be the generic point of the irreducible component of T passing through R. Then $2 \leqslant \dim S \leqslant 3$. Suppose if possible that $S \neq R$. Then $\dim S = 2$. Now S has a simple point at R, JR is a nonzero nonunit principal ideal in R, and for $i = 1, 2$ we have that $(R'_i,$

$J'R_i'$) is a monoidal transform of (R, JR, S) such that $\operatorname{ord}_{R_i'}J_i'R_i' = \operatorname{ord}_R JR$. Since $R_1' \neq R_2'$, this contradicts (3.10.2). Therefore $S = R$ and hence $\{R\}$ is an irreducible component of T.

PROOF OF (7.19.5). Let $R = [X', X](R_1')$ and $S = [X', X)(S')$. Suppose if possible that $S \neq R$. Then $R_1' \neq S' \neq R_2'$ and hence $\dim R_1' = 3 = \dim R_2'$ and $\dim S' = 2$. By (7.19.4) we now get that $\dim R = 3$ and $\{R\}$ is an irreducible component of T. Therefore R_1' and R_2' are quadratic transforms of R. By (6.9.4) and (7.19.1) we have that $S \in \mathfrak{V}(R) \cap \mathfrak{E}^*(\mathfrak{S}^*(J), J)$. Since $S \neq R$, we get that $\dim S = 2$. Since $S' \in Y'$ and $\dim S' = 2$, we must have $S' \notin Z'$. Therefore $S \notin Z$ and hence $S \in Y$. Consequently S must be the generic point of an irreducible component of Y passing through R, and hence by assumption S has a simple point at R. Therefore there exists a basis (x, y, z) of $M(R)$ such that $R \cap M(S) = (y, z)R$. Since $\{R\}$ is an irreducible component of T, we also get that $S \notin T$ and hence $S' = S$. Let $A = R[y/x, z/x]$ and let $P = (x, y/x, z/x)A$. Then P is a maximal ideal in A (for instance see (1.3.3)). Now for $i = 1, 2$, we have that $S \in \mathfrak{V}(R_i')$ and hence by (1.10.10) we get that $M(R_i') = (x, y/x, z/x)R_i'$; therefore in view of (1.7.2) we get that $R_i' \in \mathfrak{V}(A)$; clearly $P \subset A \cap M(R_i')$; since P is a maximal ideal in A, we must have $R_i' = A_P$. Thus $R_1' = R_2'$ which is a contradiction.

PROOF OF (7.19.6). Let $R = [X', X](R_1')$, and $S_i = [X', X](S_i')$ for $i = 1, 2, 3$. Clearly $\dim R_1' = 3 = \dim R_2'$, and hence by (7.19.4) we get that $\dim R = 3$ and $\{R\}$ is an irreducible component of T. By (7.19.2) we get that $S_1 \neq S_2$, and by (7.19.5) we get that $S_2 = R$. Therefore $S_1 \neq R$. By (7.19.1) we have that $S_1 \in \mathfrak{V}(R)$; since $\{R\}$ is an irreducible component of T, we conclude that $S_1 \notin T$, and hence $S_1' = S_1$. Suppose if possible that $S_1 = S_3$. Since $S_1' = S_1$, we must then have $S_3' = S_1'$. In particular then $S_1' \in \mathfrak{V}(R_1') \cap \mathfrak{V}(R_2')$, and hence by (7.19.5) we get that $[X', X](S_1') = [X', X](R_1')$, i.e., $S_1 = R$. This is a contradiction.

PROOF OF (7.19.7). Assume that Y has only strict normal crossings and there exists an infinite sequence $(Y_i', R_i')_{0 \leqslant i < \infty}$ such that for $0 \leqslant i < \infty$: Y_i' is an irreducible component of Y', $Y_i' \neq Y_{i+1}'$, $R_i' \in Y_i' \cap Y_{i+1}'$, and $R_i' \neq R_{i+1}'$. We shall show that then there exists an infinite sequence $(Y_j^*, R_j^*)_{0\leqslant j<\infty}$ such that for $0 \leqslant j < \infty$: Y_j^* is an irreducible component of Y, $Y_j^* \neq Y_{j+1}^*$, $R^* \in Y_j^* \cap Y_{j+1}^*$, and $R_j^* \neq R_{j+1}^*$. In view of (7.19.3) this will complete the proof. For $1 \leqslant i < \infty$ let S_i' be the generic point of Y_i', let $S_i = [X', X](S_i')$, let Y_i be the closure of $\{S_i\}$ in X, and let $R_i = [X', X](R_i')$. For $1 \leqslant i < \infty$ we clearly have that $\dim R_i' = 3$ and $\dim S_i' = 2$, and hence $\dim R_i = 3$ and $2 \leqslant \dim S_i \leqslant 3$. In view of (6.9.4) we get that $Y_i \subset \mathfrak{C}^*(\mathfrak{S}^*(J), J)$ for $1 \leqslant i < \infty$, and in view of (7.19.1) we get that $S_i \in \mathfrak{V}(R_i)$ for $1 \leqslant i < \infty$, i.e., $R_i \in Y_i$ for $1 \leqslant i < \infty$. Let N be the set of all integers i with $1 \leqslant i < \infty$ such that $\dim S_i = 2$. By (7.19.2) we get that N is an infinite set. Arrange all the integers in N in the form of a sequence $a(0) < a(1) < a(2) < \ldots$. For each $i \in N$ we have that Y_i is an irreducible component of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$; since $\dim S_i' = 2$ and $S_i' \in Y'$, we also get that $S_i' \notin Z'$ and hence $S_i \notin Z$; consequently Y_i must be an irreducible component of Y. It now suffices to show that $S_{a(j+1)} \in \mathfrak{V}(R_{a(j)})$, $S_{a(j)} \neq S_{a(j+1)}$, and $R_{a(j)} \neq R_{a(j+1)}$ for $0 \leqslant j < \infty$, because then we would get an infinite sequence $(Y_j^*, R_j^*)_{0\leqslant i<\infty}$ having the desired properties by taking $(Y_j^*, R_j^*) = (Y_{a(j)}, R_{a(j)})$ for $0 \leqslant j < \infty$. So let any integer j with $0 \leqslant j < \infty$ be given.

First suppose that $a(j+1) = a(j) + 1$. Then by (7.19.1) we get that $S_{a(j+1)} \in \mathfrak{V}(R_{a(j)})$, and by (7.19.2) we get that $S_{a(j)} \neq S_{a(j+1)}$. Since $\dim S_{a(j+1)} = 2$ and $\dim R_{a(j)} = 3$, we get that $S_{a(j+1)} \neq R_{a(j)}$; hence by (7.19.5) we must have $R_{a(j)} \neq R_{a(j+1)}$.

Next suppose that $a(j+1) \neq a(j) + 1$. Then $\dim S_{a(j)+1} = 3$ and hence by (7.19.2) we get that $a(j+1) = a(j) + 2$. By (7.19.1) we have that $S_{a(j)+1} \in \mathfrak{V}(R_{a(j)}) \cap \mathfrak{V}(R_{a(j)+1})$; since $\dim S_{a(j)+1} = 3$, we deduce that $R_{a(j)} = R_{a(j)+1}$. Again by (7.19.1) we have that $S_{a(j+1)} \in \mathfrak{V}(R_{a(j)+1})$; since $R_{a(j)} = R_{a(j)+1}$, we conclude that $S_{a(j+1)} \in \mathfrak{V}(R_{a(j)})$. By (7.19.6) we get that $S_{a(j)} \neq S_{a(j+1)}$. Since $\dim S_{a(j+1)} = 2$ and $\dim R_{a(j)+1} = 3$, we get that $S_{a(j+1)} \neq R_{a(j)+1}$; hence by (7.19.5) we must have $R_{a(j)+1} \neq R_{a(j+1)}$; since $R_{a(j)} = R_{a(j)+1}$, we conclude that $R_{a(j)} \neq R_{a(j+1)}$.

(7.20). *Assume that K/k is globally subresolvable. Let X be any nonsingular model of K/k and let J and I be any nonzero principal ideals on X such that $\mathfrak{S}^*(J) \neq \varnothing$ and I has only quasinormal crossings. Then there exists a nonsingular model X' of K/k and* *nonzero principal ideals J' and I' on X' such that: I' has only quasi-normal crossings, X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) \subset \mathfrak{C}^*(\mathfrak{S}^*(J), J)$, $(JI)X' = J'I'$, $J(X \cap X') = J'(X \cap X')$, $\mathfrak{Z}(J')$ is the $[X', X]$-transform of $\mathfrak{Z}(J)$, $\mathrm{ord}_{\mathfrak{S}^*(J')} J' \leqslant \mathrm{ord}_{\mathfrak{S}^*(J)} J$, and either $\mathrm{ord}_{\mathfrak{S}^*(J')} J' < \mathrm{ord}_{\mathfrak{S}^*(J)} J$ or $\mathfrak{C}^*(\mathfrak{S}^*(J'), J')$ is a nonempty unlooped closed subset of X'* (note that by (6.4.6) we know that $\mathfrak{Z}(J)$ is a closed subset of X).

PROOF. By (6.5.3) and (6.5.4) we know that $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ is a nonempty closed subset of X with $\operatorname{codim} \mathfrak{C}^*(J), J) \geqslant 2$. Let Z be the union of all the 2-codimensional irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$, and let Y be the union of the remaining irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$. Then clearly Y is unlooped. Therefore if $Z = \varnothing$ then it suffices to take $(X', J', I') = (X, J, I)$. So now assume that $Z \neq \varnothing$. Since K/k is globally subresolvable, there exists a finite global subresolver $[(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}, (X', J', I')]$ of K/k with $(X_0, J_0, I_0, Z_0) = (X, J, I, Z)$. In view of (6.2.18), (6.9.1), and (6.9.4) we get that $\mathfrak{F}(X', X) \subset \mathfrak{C}^*(\mathfrak{S}^*(J), J)$, $J(X \cap X') = J'(X \cap X')$, and $\mathfrak{Z}(J')$ is the $[X', X]$-transform of $\mathfrak{Z}(J)$. By (6.9.4) we have that $\mathrm{ord}_{\mathfrak{S}^*(J')} J' \leqslant \mathrm{ord}_{\mathfrak{S}^*(J)} J$. If $\mathrm{ord}_{\mathfrak{S}^*(J')} J' < \mathrm{ord}_{\mathfrak{S}^*(J)} J$ then we have nothing more to show. So assume that $\mathrm{ord}_{\mathfrak{S}^*(J')} J' = \mathrm{ord}_{\mathfrak{S}^*(J)} J$. Let $X_m = X'$ and $J_m = J'$. In view of (6.5.3) and (6.5.4) we then have that $\mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i)$ is a nonempty closed subset of X_i with $\operatorname{codim} \mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i) \geqslant 2$ for $0 \leqslant i \leqslant m$. Let Z_m be the closure in X_m of $\{S' \in \mathfrak{C}^*(\mathfrak{S}^*(J_m), J_m)$: S' dominates a two-dimensional point of $Z_{m-1}\}$. Since $[(X_i, J_i, I_i, Z_i, T_i)_{0 \leqslant i < m}, (X', J', I')]$ is a finite global subresolver of K/k, we must then have $Z_m = \varnothing$. For $0 \leqslant i \leqslant m$ we clearly have that Z_i is the union of a certain number of 2-codimensional irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i)$; let Y_i be the union of the remaining irreducible components of $\mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i)$. Then $Y_m = \mathfrak{C}^*(\mathfrak{S}^*(J_m), I_m)$ and $Y_0 = Y$. In particular Y_0 is unlooped and hence, in view of (7.19.7), by induction on i we see that Y_i is unlooped for $0 \leqslant i \leqslant m$. Upon taking $i = m$ we conclude that $\mathfrak{C}^*(\mathfrak{S}^*(J'), J')$ is unlooped.

(7.21). DEFINITION. By a *global resolver* of K/k we mean a sequence $(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}$ where: (1) either m is a positive integer or $m = \infty$; (2) for $0 \leqslant i < m$: X_i is a nonsingular model of K/k, J_i and I_i are nonzero principal ideals on X_i such that I_i has only quasinormal crossings, and T_i is a nonsingular closed subset of X_i with $T_i \subset \mathfrak{S}^*(J_i)$ such that for every $R \in T_i$, upon letting S be the generic point of the irreducible component of T_i passing through R, we have that $S \in \mathfrak{C}(R, J_i)$, (S, I_i) has a pseudonormal crossing at R, and: $\dim S = 2 \Leftrightarrow \mathfrak{C}^2(R, J_i)$ has a strict normal crossing at R and (S', I_i) has a pseudonormal crossing at R for some $S' \in \mathfrak{C}^2(R, J_i)$; and (3) for $0 < i < m$: (X_i, J_i, I_i) is the monoidal transform of $(X_{i-1}, J_{i-1}, I_{i-1})$ with center T_{i-1}.

By an *infinite global resolver* of K/k we mean a global resolver $(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}$ of K/k where $m = \infty$ and $T_i \neq \varnothing$ for infinitely many distinct values of i.

By a *finite global resolver* of K/k we mean a system $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}, (X_m, J_m, I_m)]$ where: (1′) m is a positive integer; (2′) for $0 \leqslant i \leqslant m$: X_i is a nonsingular model of K/k, and J_i and I_i are nonzero principal ideals on X_i such that I_i has only quasinormal crossings; (3′) for $0 \leqslant i < m$: $\mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i)$ is a nonempty unlooped closed subset of X_i, and T_i is a nonsingular irreducible closed subset of $\mathfrak{C}^*(\mathfrak{S}^*(J_i), J_i)$ such that, upon letting S be the generic point of T_i, for every $R \in T_i$ we have that (S, I_i) has a pseudonormal crossing at R and: $\dim S = 2 \Leftrightarrow (S', I_i)$ has a pseudonormal crossing at R for some $S' \in \mathfrak{C}^2(R, J_i)$; (4′) for $0 < i \leqslant m$: (X_i, J_i, I_i) is the monoidal transform of $(X_{i-1}, J_{i-1}, I_{i-1})$ with center T_{i-1}; and (5′) for $0 \leqslant i < m$: $\operatorname{ord}_{\mathfrak{S}^*(J_0)} J_0 = \operatorname{ord}_{\mathfrak{S}^*(J_i)} J_i > \operatorname{ord}_{\mathfrak{S}^*(J_m)} J_m$.

K/k is said to be *globally resolvable* if: given any nonsingular model X and K/k and any nonzero principal ideals J and I on X such that $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ is a nonempty unlooped closed subset of X and I has only quasinormal crossings, there exists a finite global resolver $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}, (X_m, J_m, I_m)]$ of K/k such that $(X_0, J_0, I_0) = (X, J, I)$. K/k is said to be *globally strongly resolvable* if there does not exist any infinite global resolver of K/k. K/k is said to be *locally strongly resolvable* if every regular spot over k with quotient field K is strongly resolvable.

(7.22). *Let X be any nonsingular model of K/k and let J and I be any nonzero principal ideals on X such that $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ is a nonempty unlooped closed subset of X and I has only quasinormal crossings. Then we have the following.*

(7.22.1). *There exists a nonsingular irreducible closed subset T of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ such that, upon letting S be the generic point of T, for every $R \in T$ we have that (S, I) has a pseudonormal crossing at R and:* $\dim S = 2 \Leftrightarrow (S', I)$ *has a pseudonormal crossing at R for some $S' \in \mathfrak{C}^2(R, J)$.*

(7.22.2). *Assume that there does not exist any infinite global resolver $(X_i, J_i, I_i, T_i)_{0 \leqslant i < \infty}$ of K/k such that $(X_0, J_0, I_0) = (X, J, I)$. Then there exists a finite global resolver $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}, (X_m, J_m, I_m)]$ of K/k such that $(X_0, J_0, I_0) = (X, J, I)$.*

PROOF OF (7.22.1). By (6.5.3) we have that $\operatorname{codim} \mathfrak{C}^*(\mathfrak{S}^*(J), J) \geqslant 2$. For each irreducible component Z of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ let $H(Z)$ be the set of all points R of Z such that (S, I) has a pseudonormal crossing at R where S is the generic point of Z. For a moment suppose that there exists an irreducible component Z of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ such that $H(Z) = Z$; since $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ is unlooped, we have that Z is nonsingular; let S be the generic point of Z; if $\dim S \neq 3$ then $\dim S = 2$ and $S \in \mathfrak{C}^2(R, J)$ for all $R \in Z$, and hence it suffices to take $T = Z$; if $\dim S = 3$ then $Z = \{S\}$ and by (6.5.6) we have that $\mathfrak{C}^2(S, J) = \varnothing$, and hence again it suffices to take $T = Z$. So henceforth assume that for every irreducible component Z of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ we have that $H(Z) \neq Z$. Note that then $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ is pure 2-codimensional. Let W be the set of all sequences $(Z_i, R_i)_{0 \leqslant i < m}$ such that: either m is a positive integer or $m = \infty$; Z_i is an irreducible component of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ and $R_i \in Z_i - H(Z_i)$ for $0 \leqslant i < m$; and $R_{i-1} \in H(Z_i)$ for $0 < i < m$ (note that then $Z_{i-1} \neq Z_i$ and $R_{i-1} \neq R_i$ for $0 < i < m$). For each pair of elements $w = (Z_i, R_i)_{0 \leqslant i < m}$ and $w' = (Z'_i, R'_i)_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(Z_i, R_i) = (Z'_i, R'_i)$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property. Upon taking Z_0 to be any irreducible component of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ and R_0 to be any point of $Z_0 - H(Z_0)$ we get an element $(Z_i, R_i)_{0 \leqslant i < 1}$ in W. Therefore $W \neq \varnothing$, and hence by Zorn's lemma W contains a maximal element $w = (Z_i, R_i)_{0 \leqslant i < m}$. Since $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$ is unlooped, we must have $m \neq \infty$. Now $\dim R_{m-1} = 3$, $\{R_{m-1}\}$ is a nonsingular irreducible closed subset of $\mathfrak{C}^*(\mathfrak{S}^*(J), J)$, and (R_{m-1}, I) has a pseudonormal crossing at R_{m-1}. Suppose if possible that there exists $S' \in \mathfrak{C}^2(R_{m-1}, J)$

such that (S', I) has a pseudonormal crossing at R_{m-1}; let Z' be the closure of S' in X; then by (6.5.6) we get that Z' is an irreducible component of $\mathfrak{E}^*(\mathfrak{S}^*(J), J)$; in particular $H(Z') \neq Z'$ and hence we can take $R' \in Z' - H(Z')$; now we get an element $w' = (Z_i', R_i')_{0 \leqslant i < m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking $(Z_i', R_i') = (Z_i, R_i)$ for $0 \leqslant i < m$ and $(Z_m', R_m') = (Z', R')$; this is a contradiction because w is a maximal element of W. Therefore it suffices to take $T = \{R_{m-1}\}$.

PROOF OF (7.22.2). Let W be the set of all sequences $(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}$ where: (1) either m is a positive integer or $m = \infty$; (2) for $0 \leqslant i < m$: X_i is a nonsingular model of K/k, J_i and I_i are nonzero principal ideals on X_i such that $\mathfrak{E}^*(\mathfrak{S}^*(J_i), J_i)$ is a nonempty unlooped closed subset of X_i and I_i has only quasinormal crossings, $\operatorname{ord}_{\mathfrak{S}^*(J_i)} J_i = \operatorname{ord}_{S^*(J)} J$, and T_i is a nonsingular irreducible closed subset of $\mathfrak{E}^*(\mathfrak{S}^*(J_i), J_i)$ such that, upon letting S be the generic point of T_i, for every $R \in T_i$ we have that (S, I_i) has a pseudonormal crossing at R and: $\dim S = 2 \Leftrightarrow (S', I_i)$ has a pseudonormal crossing at R for some $S' \in \mathfrak{E}^2(R, J_i)$; (3) for $0 < i < m$: (X_i, J_i, I_i) is a monoidal transform of $(X_{i-1}, J_{i-1}, I_{i-1})$ with center T_{i-1}; and (4) $(X_0, J_0, I_0) = (X, J, I)$. Upon taking $(X_0, J_0, I_0, T_0) = (X, J, I, T)$ where T is as in (7.22.1) we get an element $(X_i, J_i, I_i, T_i)_{0 \leqslant i < 1}$ in W, and hence $W \neq \varnothing$. For each pair of elements $w = (X_i, J_i, I_i, T_i)$ and $w' = (X_i', J_i', I_i', T_i')_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(X_i, J_i, I_i, T_i) = (X_i', J_i', I_i', T_i')$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property and hence by Zorn's lemma W contains a maximal element $w = (X_i, J_i, I_i, T_i)_{0 \leqslant i < m}$. In view of (6.5.6) we see that every element in W is a global resolver of K/k. Therefore by assumption we must have $m \neq \infty$. Let (X_m, J_m, I_m) be the monoidal transform of $(X_{m-1}, J_{m-1}, I_{m-1})$ with center T_{m-1}. Then by (6.8), (6.9.1), (6.9.3), and (6.9.4) we have that X_m is a nonsingular model of K/k, J_m and I_m are nonzero principal ideals on X_m, I_m has only quasinormal crossings, and $\operatorname{ord}_{\mathfrak{S}^*(J_m)} \leqslant \operatorname{ord}_{\mathfrak{S}^*(J)} J$. Suppose if possible that $\operatorname{ord}_{\mathfrak{S}^*(J_m)} = \operatorname{ord}_{\mathfrak{S}^*(J)} J$; then by (6.5.3), (6.5.4), (6.5.6), and (7.19.7) we get that $\mathfrak{E}^*(\mathfrak{S}^*(J_m), J_m)$ is a nonempty unlooped closed subset of X_m; hence by (7.22.1) there exists a nonsingular irreducible closed subset T_m of $\mathfrak{E}^*(\mathfrak{S}^*(J_m), J_m)$ such that, upon letting S be the generic point

point of T_m, for every $R \in T_m$ we have that (S, I_m) has a pseudonormal crossing at R and: $\dim S = 2 \Leftrightarrow (S', I_m)$ has a pseudonormal crossing at R for some $S' \in \mathfrak{C}^2(R, J_m)$; we now get an element $w' = (X_i, J_i, I_i, T_i)_{0 \leqslant i < m+1}$ in W with $w \leqslant w'$ and $w \neq w'$; this is a contradiction because w is a maximal element of W. Therefore $\operatorname{ord}_{\mathfrak{S}^*(J_m)} J_m < \operatorname{ord}_{\mathfrak{S}^*(J)} J$, and hence $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}, (X_m, J_m, I_m)]$ is a finite global resolver of K/k with $(X_0, J_0, I_0) = (X, J, I)$.

(7.23). *If K/k is globally strongly resolvable then K/k is globally resolvable.*

PROOF. Follows from (7.22.2).

(7.24). *For any global resolver $(X_i, J_i, I_i, T_i)_{0 \leqslant i < \infty}$ of K/k we have the following.*

(7.24.1). *Given any nonnegative integer n and any $R \in \bigcap_{i=n}^{\infty} X_i$, there exists an open subset D of X_n with $R \in D$ such that $D \subset X_i$ and $D \cap T_i = \varnothing$ for all $i \geqslant n$.*

(7.24.2). *Assume that there does not exist any infinite resolver $(R_j, P_j, Q_j, S_j)_{0 \leqslant j < \infty}$ with $R_0 \in \mathfrak{S}^*(J_0)$. Then there exists a nonnegative integer m such that $T_i = \varnothing$ for all $i \geqslant m$.*

PROOF OF (7.24.1). In view of (6.5.3) and (6.8) we have that $\mathfrak{S}^*(J_i)$ is a closed subset of X_i with $\operatorname{codim} \mathfrak{S}^*(J_i) \geqslant 2$ and $\mathfrak{F}(X_{i+1}, X_i) = T_i \subset \mathfrak{S}^*(J_i)$ for $0 \leqslant i < \infty$. Since $R \in \bigcap_{i=n}^{\infty} X_i$, we get that $R \in T_i$ for all $i \geqslant n$. For each $i \geqslant n$ let G_i be the union of the irreducible components of $\mathfrak{S}^*(J_i)$ passing through R, let H_i be the union of the remaining irreducible components of $\mathfrak{S}^*(J_i)$, let

$$G_i^* = \mathfrak{S}(G_i) \cup \{R' \in G_i - \mathfrak{S}(G_i) \colon S \notin \mathfrak{C}(R', J_i) \text{ where } S \text{ is the generic point of the irreducible component of } G_i \text{ passing through } R'\},$$

let

$G_i' = \{R' \in G_i - \mathfrak{S}(G_i)$: (S, I_i) does not have a pseudonormal crossing at R' where S is the generic point of the irreducible component of G_i passing through $R'\}$,

and let $D_i = X_i - (((G_i^* \cup G_i') - \{R\}) \cup H_i)$. Then $R \in D_i$, and in view of (6.5.5) and (6.5.7) we get that D_i is an open subset of X_i. For any open subset E of X_i with $R \in E$, in view of (6.2.16) we get that:

$(\mathfrak{S}^*(J_i) \cap E \cap D_i) - \{R\}$

$= \{R' \in (\mathfrak{S}^*(J_i) \cap E) - (\mathfrak{S}(\mathfrak{S}^*(J_i) \cap E) \cup \{R\})$: upon letting F be the irreducible component of $\mathfrak{S}^*(J_i) \cap E$ passing through R', and S be the generic point of F, we have that $R \in F$, $S \in \mathfrak{E}(R', J_i)$, and (S, I_i) has a pseudonormal crossing at $R'\}$;

let us refer to this observation as $[i, E]$. Since $R \notin T_i$, by (6.5.6) and $[i, X_i]$ we see that $D_i \cap T_i = \varnothing$; consequently $D_i \subset X_{i+1}$ and:

$$(1_i) \qquad \begin{aligned} &I_{i+1}D_i = I_iD_i\,,\ J_{i+1}D_i = J_iD_i\,, \quad \text{and} \\ &\mathfrak{S}^*(J_{i+1}) \cap D_i = \mathfrak{S}^*(J_i) \cap D_i\,. \end{aligned}$$

In view of (6.2.5) we have that D_i is an open subset of X_{i+1}, and hence by (1_i), $[i, D_i]$, and $[i+1, D_i]$ we get that:

$$(2_i) \qquad (\mathfrak{S}^*(J_i) \cap D_i) - \{R\} = (\mathfrak{S}^*(J_{i+1}) \cap D_i \cap D_{i+1}) - \{R\}.$$

Since $T_i \subset \mathfrak{S}^*(J_i)$, we get that $X_i - \mathfrak{S}^*(J_i) \subset X_{i+1} - \mathfrak{S}^*(J_{i+1})$; also $X_{i+1} - \mathfrak{S}^*(J_{i+1}) \subset D_{i+1}$, and hence $X_i - \mathfrak{S}^*(J_i) \subset D_{i+1}$; therefore by (2_i) we get that $D_i \subset D_{i+1}$. Thus D_i is an open subset of X_i with $R \in D_i$ and $D_i \cap T_i = \varnothing$ for all $i \geqslant n$, and also $D_i \subset D_{i+1}$ for all $i \geqslant n$. It suffices to take $D = D_n$.

PROOF OF (7.24.2). In view of (6.5.3) and (6.8) we have that $\mathfrak{F}(X_{i+1}, X_i) = T_i \subset \mathfrak{S}^*(J_i)$ for $0 \leqslant i < \infty$, and hence $\mathfrak{F}(X_i, X_0) \subset \mathfrak{S}^*(J_0)$ for $0 \leqslant i < \infty$. Given any $V \in \mathfrak{N}(X_0)$, let R_i' be the center of V on X_i for $0 \leqslant i < \infty$, and let $(a(j))_{0 \leqslant j < n}$ be the unique sequence such that: either n is a nonnegative integer or $n = \infty$;

$a(j)$ is a nonnegative integer for $0 \leqslant j < n$; $a(0) = 0$; $a(j-1) < a(j)$ and $R'_{a(j-1)} = R'_i \neq R'_{a(j)}$ whenever $0 < j < n$ and $a(j-1) \leqslant i < a(j)$; if $n \neq \infty$ then $a(n)$ is a nonnegative integer and $R'_{a(n)} = R'_i$ whenever $a(n) \leqslant i < \infty$; and if $n = \infty$ then $a(n) = \infty$. For $0 \leqslant j < n$, upon letting $R_j = R'_{a(j)}$, we get that $R_j \in T_{a(j+1)-1}$; let S_j be the generic point of the irreducible component of $T_{a(j+1)-1}$ passing through R_j, let $P_j = J_{a(j+1)-1}R_j$, and let $Q_j = I_{a(j+1)-1}R_j$. Suppose if possible that $n = \infty$; then $(R_j, P_j, Q_j, S_j)_{0 \leqslant j < \infty}$ is an infinite resolver; since $\mathfrak{F}(X_i, X_0) \subset \mathfrak{S}^*(J_0)$ for $0 \leqslant i < \infty$, we get that $R_0 \in \mathfrak{S}^*(J_0)$; this contradicts our assumption. Therefore $n \neq \infty$. Let $n(V) = n$. Thus for each $V \in \mathfrak{R}(X_0)$ we have found a nonnegative integer $n(V)$ such that, upon letting $R(V)$ be the center of V on $X_{n(V)}$, we have that $R(V) \in \bigcap_{i=n(V)}^{\infty} X_i$. By (7.24.1) there exists an open subset $D(V)$ of $X_{n(V)}$ with $R(V) \in D(V)$ such that $D(V) \subset X_i$ and $D(V) \cap T_i = \varnothing$ for all $i \geqslant n(V)$. For each $V \in \mathfrak{R}(X_0)$ we clearly have that $V \in [\mathfrak{R}(X_0), X_{n(V)}]^{-1}(D(V))$, and by (6.2.5) we get that $[\mathfrak{R}(X_0), X_{n(V)}]^{-1}(D(V))$ is an open subset of $\mathfrak{R}(X_0)$; now $\mathfrak{R}(X_0)$ is quasicompact by (6.2.13), and hence there exists a finite number of elements $V_1, \ldots, V_q$ in $\mathfrak{R}(X_0)$ such that

$$\mathfrak{R}(X_0) = \bigcup_{j=1}^{q} [\mathfrak{R}(X_0), X_{n(V_j)}]^{-1}(D(V_j)).$$

Let m be any nonnegative integer such that $m \geqslant n(V_j)$ for $1 \leqslant j \leqslant q$. Then clearly $T_i = \varnothing$ for all $i \geqslant m$.

(7.25). *If K/k is locally strongly resolvable then K/k is globally resolvable and globally strongly resolvable.*

PROOF. Follows from (7.23) and (7.24.2).

(7.26). *Assume that K/k is globally subresolvable and globally resolvable. Let X be any nonsingular model of K/k and let J and I be any nonzero principal ideals on X such that I has only quasinormal crossings. Then there exists a nonsingular model X^* of K/k and nonzero principal ideals J^* and I^* on X^* such that: $\mathfrak{S}^*(J^*) = \varnothing$, I^* has only quasinormal crossings, X^* is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X^*, X) = \mathfrak{S}^*(J)$,*

$(JI)X^* = J^*I^*$, $J(X \cap X^*) = J^*(X \cap X^*)$, *and* $\mathfrak{Z}(J^*)$ *is the* $[X^*, X]$-*transform of* $\mathfrak{Z}(J)$ (note that by (6.4.6) we know that $\mathfrak{Z}(J)$ is a closed subset of X).

PROOF. If $\mathfrak{S}^*(J) = \varnothing$ then it suffices to take $(X^*, J^*, I^*) = (X, J, I)$. So assume that $\mathfrak{S}^*(J^*) \neq \varnothing$, and let $d = \mathrm{ord}_{\mathfrak{S}^*(J)} J$. Then by (6.5.3) and (6.5.4) we get that d is a positive integer. By induction on j we shall first show that if j is any integer with $0 \leqslant j \leqslant d$ then there exists a nonsingular model X^* of K/k and nonzero principal ideals J^* and I^* on X^* such that: $\mathrm{ord}_{\mathfrak{S}^*(J^*)} \leqslant d - j$, I^* has only quasinormal crossings, X^* is iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X^*, X) \subset \mathfrak{S}^*(J)$, $(JI)X^* = J^*I^*$, $J(X \cap X^*) = J^*(X \cap X^*)$, and $\mathfrak{Z}(J^*)$ is the $[X^*, X]$-transform of $\mathfrak{Z}(J)$.

For $j = 0$ it suffices to take $(X^*, J^*, I^*) = (X, J, I)$. Now let $0 < j \leqslant d$ and assume that the above assertion is true for all values of j smaller than the given one. Then by the induction hypothesis there exists a nonsingular model X'' of K/k and nonzero principal ideals J'' and I'' on X'' such that: $\mathrm{ord}_{\mathfrak{S}^*(J'')} \leqslant d - j + 1$, I'' has only quasinormal crossings, X'' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X'', X) \subset \mathfrak{S}^*(J)$, $(JI)X'' = J''I''$, $J(X \cap X'') = J''(X \cap X'')$, and $\mathfrak{Z}(J'')$ is the $[X'', X]$-transform of $\mathfrak{Z}(J)$. If $\mathfrak{S}^*(J'') = \varnothing$ then $\mathrm{ord}_{\mathfrak{S}^*(J'')} J'' = -\infty$ and hence it suffices to take $(X^*, J^*, I^*) = (X'', J'', I'')$. So now assume that $\mathfrak{S}^*(J'') \neq \varnothing$. Since K/k is globally subresolvable, by (7.20) there exists a nonsingular model X' of K/k and nonzero principal ideals J' and I' on X' such that: I' has only quasinormal crossings, X' is an iterated monoidal transform of X'' with nonsingular irreducible centers, $\mathfrak{F}(X', X'') \subset \mathfrak{S}^*(J'')$, $(J''I'')X' = J'I'$, $J''(X'' \cap X') = J'(X'' \cap X')$, $\mathfrak{Z}(J')$ is the $[X', X'']$-transform of $\mathfrak{Z}(J'')$, $\mathrm{ord}_{\mathfrak{S}^*(J')} \leqslant d - j + 1$, and either $\mathrm{ord}_{\mathfrak{S}^*(J')} J' \leqslant d - j$ or $\mathfrak{E}^*(\mathfrak{S}^*(J'), J')$ is a nonempty unlooped closed subset of X'. Now X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) \subset \mathfrak{S}^*(J)$, and $J(X \cap X') = J'(X \cap X')$. By (6.5.2) we know that $\mathfrak{S}^*(J) = \mathfrak{S}(\mathfrak{Z}(J))$ and $\mathfrak{S}^*(J'') = \mathfrak{S}^*(\mathfrak{Z}(J''))$; consequently $\mathfrak{F}(X'', X)$ does not pass through the generic point of any irreducible component of $\mathfrak{Z}(J)$, and $\mathfrak{F}(X', X'')$ does not pass through the generic point of any irreducible component $\mathfrak{Z}(J'')$; therefore by (6.2.18) we get that $\mathfrak{Z}(J')$ is the

$[X', X]$-transform of $\mathfrak{Z}(J)$. Hence if $\operatorname{ord}_{\mathfrak{S}^*(J')} J' \leqslant d - j$ then it suffices to take $(X^*, J^*, I^*) = (X', J', I')$. So now assume that $\operatorname{ord}_{\mathfrak{S}^*(J')} J' = d - j + 1$. Then $\mathfrak{E}^*(\mathfrak{S}^*(J'), J')$ is a nonempty unlooped closed subset of X'. Since K/k is globally resolvable, there exists a finite global resolver $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m},$ $(X^*, J^*, I^*)]$ of K/k with $(X_0, J_0, I_0) = (X', J', I')$. Now $\operatorname{ord}_{\mathfrak{S}^*(J^*)} J^* \leqslant d - j$, X^* is an iterated monoidal transform of X with nonsingular irreducible centers, and $(JI)X^* = J^*I^*$. In view of (6.2.18), (6.9.1), and (6.9.4) we also get that $\mathfrak{F}(X^*, X) \subset \mathfrak{S}^*(J)$, $J(X \cap X^*) = J^*(X \cap X^*)$, and $\mathfrak{Z}(J^*)$ is the $[X^*, X]$-transform of $\mathfrak{Z}(J)$.

This completes the induction on j. Upon taking $j = d$ we find a nonsingular model X^* of K/k and nonzero principal ideals J^* and I^* on X^* such that: $\operatorname{ord}_{\mathfrak{S}^*(J^*)} \leqslant 0$, I^* has only quasinormal crossings, X^* is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X^*, X) \subset \mathfrak{S}^*(J)$, $(JI)X^* = J^*I^*$, $J(X \cap X^*) = J^*(X \cap X^*)$, and $\mathfrak{Z}(J^*)$ is the $[X^*, X]$-transform of $\mathfrak{Z}(J)$. Since $\operatorname{ord}_{\mathfrak{S}^*(J^*)} \leqslant 0$, we must have $\mathfrak{S}^*(J^*) = \varnothing$ and hence $\mathfrak{F}(X^*, X) = \mathfrak{S}^*(J)$.

(7.27). *Assume that K/k is globally subresolvable, globally resolvable, and globally detachable. Let X be any nonsingular model of K/k and let J be any nonzero principal ideal on X. Then there exists a nonsingular model X' on K/k and nonzero principal ideals J' and I' on X' such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{S}^*(J)$, $JX' = J'I'$, $J(X \cap X') = J'(X \cap X')$, JX' is a nonzero principal ideal on X', JX' has only normal crossings, $\mathfrak{Z}(JX') = [X', X]^{-1}(\mathfrak{Z}(J))$, $\mathfrak{Z}(JX')$ is a closed subset of X', $\mathfrak{Z}(JX')$ has only normal crossings, $\mathfrak{S}^*(J') = \varnothing$, $\mathfrak{Z}(J')$ is the $[X', X]$-transform of $\mathfrak{Z}(J)$, and $\mathfrak{Z}(J')$ is nonsingular* (note that by (6.4.6), (6.5.2), and (6.5.3) we know that $\mathfrak{Z}(J)$ is a closed subset of X, $\mathfrak{S}^*(J) = \mathfrak{S}(\mathfrak{Z}(J))$, $\mathfrak{S}^*(J)$ is a closed subset of $\mathfrak{Z}(J)$, and $\mathfrak{S}^*(J)$ does not pass through the generic point of any irreducible component of $\mathfrak{Z}(J)$).

PROOF. Upon taking $I = 1_X$ in (7.26) we find a nonsingular model X^* of K/k and nonzero principal ideals J^* and I^* on X^* such that: $\mathfrak{S}^*(J^*) = \varnothing$, I^* has only quasinormal crossings, X^* is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X^*, X) = \mathfrak{S}^*(J)$, $JX^* = J^*I^*$, $J(X \cap X^*) = J^*(X \cap X^*)$, and $\mathfrak{Z}(J^*)$ is the $[X^*, X]$-transform of $\mathfrak{Z}(J)$. By (6.4.4) we get that J^*I^* is a nonzero principal ideal on X^*, and clearly $(J^*I^*, 1_{X^*})$ has only quasinormal crossings. Since K/k is globally detachable, there exists a finite global detacher $[(X_i, J_i, I_i, T_i)_{0 \leqslant i < m}, (X', J'', I'')]$ of K/k with $(X_0, J_0, I_0) = (X^*, J^*I^*, 1_{X^*})$. Now X^* is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{S}^*(J'') = \varnothing$, and $JX' = J''I''$. In view of (6.2.5), (6.4.2), (6.4.4), and (6.5.2) we now get that JX' is a nonzero principal ideal on X', JX' has only normal crossings, $\mathfrak{Z}(JX') = [X', X]^{-1}(\mathfrak{Z}(J))$, $\mathfrak{Z}(JX')$ is a closed subset of X', and $\mathfrak{Z}(JX')$ has only normal crossings. Clearly $\mathfrak{S}^*(J^*I^*) \subset [X^*, X]^{-1} (\mathfrak{S}^*(J))$; so in view of (6.8) and (6.9.4) we get $\mathfrak{F}(X', X) = \mathfrak{S}^*(J)$. Let $[(J_i', L_i')_{0 \leqslant i < m}, (J', L')]$ be the unique system such that: J_i' and L_i' are nonzero principal ideals on X_i for $0 \leqslant i < m$, and J' and L' are nonzero principal ideals on X'; $J_0' = J^*$; (X_i, J_i') is the monoidal transform of (X_{i-1}, J_{i-1}') with center T_{i-1} for $0 < i < m$; (X', J') is the monoidal transform of (X_{m-1}, J_{m-1}') with center T_{m-1}; $L_0' = I^*$; (X_i, L_i') is the monoidal transform of (X_{i-1}, L_{i-1}') with center T_{i-1} for $0 < i < m$; and (X', L') is the monoidal transform of (X_{m-1}, L_{m-1}') with center T_{m-1}. Then clearly $J_i = J_i'L_i'$ for $0 \leqslant i < m$, and $J'' = J'L'$. Since $\mathfrak{S}^*(J^*I^*) \subset [X^*, X]^{-1}(\mathfrak{S}^*(J))$, in view of (6.2.18), (6.9.1), and (6.9.4) we get that $J(X \cap X') = J'(X \cap X')$, and $\mathfrak{Z}(J')$ is the $[X', X]$-transform of $\mathfrak{Z}(J)$. Since $J'' = J'L'$ and $\mathfrak{S}^*(J'') = \varnothing$, we get that $\mathfrak{S}^*(J') = \varnothing$ and hence by (6.5.2) we get that $\mathfrak{Z}(J')$ is nonsingular. Let $I' = L'I''$. Then by (6.4.4) we have that I' is a nonzero principal ideal on X', and clearly $JX' = J'I'$.

(7.28). *Assume that K/k is globally subresolvable, globally resolvable, and globally detachable. Let X be any nonsingular model of K/k and let Z be any closed subset of X such that either $Z = X$ or Z is pure 1-codimensional* (note that the assumptions about Z are satisfied if Z is a surface in X, and they are also satisfied if $\dim_k K \leqslant 2$ and Z is a curve in X). *Then there exists a nonsingular model X' of K/k such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{S}(Z)$, $[X', X]^{-1}(Z)$ is a closed subset of X', $[X', X]^{-1}(Z)$ has only normal crossings, and the $[X', X]$-transform of $\mathfrak{Z}(J)$ is nonsingular* (note that by (6.5.3) we know that $\mathfrak{S}(Z)$ is a closed subset of Z, and clearly $\mathfrak{S}(Z)$ does not pass through the generic point of any irreducible component of Z).

PROOF. If $Z = X$ then it suffices to take $X' = X$. So now assume that $Z \neq X$ and let $J = \mathfrak{I}(Z, X)$. By (6.4.6) we have that J is a nonzero ideal on X and $Z = \mathfrak{Z}(J)$. Clearly J is principal, and hence our assertion follows from (7.27).

§8. Global principalizers

Assume that $\dim_k K \leqslant 3$; note that then for any model X of K/k, by (6.3.3) we have that $\dim X = \max_{R \in X} \dim R \leqslant 3$. Also assume that for every affine ring A over k with quotient field K and every ideal Q in A we have that $\mathfrak{S}(A, Q)$ is closed in $\mathfrak{V}(A)$ (see (1.2.6)). Note that for any nonzero ideal I on any model X of K/k, by (6.4.6) and (6.4.7) we have that II^{-1} is a nonzero ideal on X and $\mathfrak{Z}(II^{-1})$ is a closed subset of X.

(8.1). DEFINITION. By a *global principalizer* of K/k we mean a sequence $(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}$ where: (1) either m is a positive integer or $m = \infty$; (2) for $0 \leqslant i < m$: X_i is a model of K/k, I_i is a nonzero ideal on X_i, Y_i is an open subset of X_i, and Z_i and T_i are closed subsets of X_i such that $\mathfrak{S}(X_i) \cap Y_i = \varnothing = \mathfrak{S}(T_i) \cap Y_i$, $T_i \subset \mathfrak{Z}(I_i I_i^{-1}) \cap Z_i$, every irreducible component of $\mathfrak{Z}(I_i I_i^{-1})$ having a nonempty intersection with $Z_i \cap Y_i$ is contained in Z_i, and for every $R \in T_i \cap Y_i$, upon letting S be the generic point of the irreducible component of T_i passing through R, we have that $S \in \mathfrak{C}(R, I_i I_i^{-1})$ and: $\dim S = 2 \Leftrightarrow \mathfrak{C}^2(R, I_i I_i^{-1})$ has a strict normal crossing at R and $\mathfrak{C}^2(R, I_i I_i^{-1}) \neq \varnothing$; and (3) for $0 < i < m$: X_i is the monoidal transform of X_{i-1} with center T_{i-1}, $I_i = I_{i-1} X_i$, $Y_i = [X_i, X_{i-1}]^{-1}(Y_{i-1})$, and $Z_i = [X_i, X_{i-1}]^{-1}(Z_{i-1})$.

By an *infinite global principalizer* of K/k we mean a global principalizer $(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}$ of K/k where $m = \infty$ and $T_i \cap Y_i \neq \varnothing$ for infinitely many distinct values of i.

By a *finite global principalizer* of K/k we mean a system $[(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}, (X', I', Y', Z')]$ where: m is a positive integer;

$(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}$ is a global principalizer of K/k such that for $0 \leqslant i < m$ we have that $T_i \cap Y_i \subset \mathfrak{E}^*(Z_i \cap Y_i, I_i I_i^{-1})$ and either $T_i = \varnothing$ or T_i is irreducible and $T_i \cap Y_i \neq \varnothing$; X' is a model of K/k, I' is a nonzero ideal on X', Y' is an open subset of X', and Z' is a closed subset of X' such that $\mathfrak{S}(X') \cap Y' = \varnothing = \mathfrak{Z}(I'I'^{-1}) \cap Z' \cap Y'$; and X' is the monoidal transform of X_{m-1} with center T_{m-1}, $I' = I_{m-1}X'$, $Y' = [X', X_{m-1}]^{-1}(Y_{m-1})$, and $Z' = [X', X_{m-1}]^{-1}(Z_{m-1})$.

K/k is said to be *globally principalizable* if: given any model X of K/k, any nonzero ideal I on X, any open subset Y of X with $\mathfrak{S}(X) \cap Y = \varnothing$, and any closed subset Z of X such that every irreducible component of $\mathfrak{Z}(II^{-1})$ having a nonempty intersection with $Z \cap Y$ is contained in Z, there exists a finite global principalizer $[(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}, (X', I', Y', Z')]$ of K/k such that $(X_0, I_0, Y_0, Z_0) = (X, I, Y, Z)$. K/k is said to be *globally strongly principalizable* if there does not exist any infinite global principalizer of K/k. K/k is said to be *locally strongly principalizable* if every regular spot over k with quotient field K is strongly principalizable.

(8.2). *Let X be a model of K/k, I a nonzero ideal on X, Y an open subset of X with $\mathfrak{S}(X) \cap Y = \varnothing$, and Z a closed subset of X such that every irreducible component of $\mathfrak{Z}(II^{-1})$ having a nonempty intersection with $Z \cap Y$ is contained in Z. Then we have the following.*

(8.2.1). *Assume that $\mathfrak{Z}(II^{-1}) \cap Z \cap Y \neq \varnothing$. Then there exists an irreducible closed subset T of X such that $\mathfrak{S}(T) \cap Y = \varnothing$, $T \subset \mathfrak{Z}(II^{-1}) \cap Z$, $\varnothing \neq T \cap Y \subset \mathfrak{E}^*(Z \cap Y, II^{-1})$, and, upon letting S be the generic point of T, for every $R \in T \cap Y$ we have that $S \in \mathfrak{E}(R, II^{-1})$ and:* $\dim S = 2 \Leftrightarrow \mathfrak{E}^2(R, II^{-1})$ *has a strict normal crossing at R and $\mathfrak{E}^2(R, II^{-1}) \neq \varnothing$.*

(8.2.2). *Assume that there does not exist any infinite global principalizer $(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < \infty}$ of K/k such that $(X_0, I_0, Y_0, Z_0) = (X, I, Y, Z)$. Then there exists a finite global principalizer $[(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}, (X', I', Y', Z')]$ of K/k such that $(X_0, I_0, Y_0, Z_0) = (X, I, Y, Z)$.*

PROOF OF (8.2.1). Now $Z \cap Y$ is a nonempty closed subset of Y and hence by (6.5.4) we get that $\mathfrak{E}^*(Z \cap Y, II^{-1})$ is a nonempty closed subset of $Z \cap Y$; since $\mathfrak{Z}(II^{-1}) \cap Z \cap Y \neq \varnothing$, upon letting $d = \operatorname{ord}_{Z \cap Y} II^{-1}$ we get that $0 < d < \infty$ and $\mathfrak{C}^*(Z \cap Y, II^{-1}) \subset \mathfrak{Z}(II^{-1})$, and hence by (6.4.9) we get that $\dim R \geqslant 2$ for all $R \in \mathfrak{C}^*(Z \cap Y, II^{-1})$. For any $R \in \mathfrak{C}^*(Z \cap Y, II^{-1})$ and $S \in \mathfrak{C}(R, II^{-1})$ we have that $S \in \mathfrak{Z}(II^{-1}) \cap Y$; consequently there exists an irreducible component Z' of $\mathfrak{Z}(II^{-1})$ with $S \in Z'$; now $R \in Z' \cap Z \cap Y$ and hence by assumption $Z' \subset Z$; therefore $S \in Z \cap Y$ and hence $S \in \mathfrak{C}^*(Z \cap Y, II^{-1})$. Thus for every $R \in \mathfrak{C}^*(Z \cap Y, II^{-1})$ we have $\mathfrak{C}(R, II^{-1}) = \mathfrak{V}(R) \cap \mathfrak{C}^*(Z \cap Y, II^{-1})$. Let $S_1, \ldots, S_n$ $(n > 0)$ be the generic points of the irreducible components of $\mathfrak{C}^*(Z \cap Y, II^{-1})$. If $\dim S_i \neq 2$ for some i then $\dim S_i = 3$, $\{S_i\}$ is a nonsingular irreducible closed subset of X, and $\mathfrak{C}(S_i, II^{-1}) = \{S_i\}$, and hence it suffices to take $T = \{S_i\}$. So now assume that $\dim S_i = 2$ for $1 \leqslant i \leqslant n$. Then for each $R \in \mathfrak{C}^*(Z \cap Y, II^{-1})$ we have that $\mathfrak{C}^2(R, II^{-1}) = \mathfrak{V}(R) \cap \{S_1, \ldots, S_n\}$. If there exists $R \in \mathfrak{C}^*(Z \cap Y, II^{-1})$ such that $\mathfrak{V}(R) \cap \{S_1, \ldots, S_n\}$ does not have a strict normal crossing at R then $\dim R = 3$ and $\{R\}$ is a nonsingular irreducible closed subset of X, and hence it suffices to take $T = \{R\}$. If $\mathfrak{V}(R) \cap \{S_1, \ldots, S_n\}$ has a strict normal crossing at R for all $R \in \mathfrak{C}^*(Z \cap Y, II^{-1})$, then it suffices to take T = closure of $\{S_i\}$ in X for some i.

PROOF OF (8.2.2). Let W be the set of all global principalizers $(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}$ of K/k such that $(X_0, I_0, Y_0, Z_0) = (X, I, Y, Z)$ and $\varnothing \neq T_i \cap Y_i \subset \mathfrak{C}^*(Z_i \cap Y_i, I_i I_i^{-1})$ and T_i is irreducible for $0 \leqslant i < m$. If $\mathfrak{Z}(II^{-1}) \cap Z \cap Y = \varnothing$ then we get a finite global principalizer $[(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < 1}, (X', I', Y', Z')]$ *of* K/k of the required type by taking $(X_0, I_0, Y_0, Z_0) = (X, I, Y, Z) = (X', I', Y', Z')$ and $T_0 = \varnothing$. So now assume that $\mathfrak{Z}(II^{-1}) \cap Z \cap Y \neq \varnothing$. Then there exists T as in (8.2.1) and we get an element $(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < 1}$ in W by taking $(X_0, I_0, Y_0, Z_0, T_0) = (X, I, Y, Z, T)$. Therefore W is nonempty. For each pair of elements $w = (X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}$ and $w' = (X_i', I_i', Y_i', Z_i', T_i')_{0 \leqslant i < m'}$ in W define: $w \leqslant w' \Leftrightarrow m \leqslant m'$ and $(X_i, I_i, Y_i, Z_i, T_i) = (X_i', I_i', Y_i', Z_i', T_i')$ for $0 \leqslant i < m$. Then W becomes a partially ordered set having the Zorn property and hence by Zorn's lemma W contains a maximal element $w = (X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}$. By assumption we must have $m \neq \infty$. Let X' be the monoidal transform of X with center T_{m-1}, let $I' = I_{m-1}X'$, let $Y' = [X', X_{m-1}]^{-1}(Y_{m-1})$, and let $Z' = [X', X_{m-1}]^{-1}(Z_{m-1})$. Then by (6.4.11), (6.4.14), and (6.8), we have that X' is a model of K/k, I' is a nonzero ideal on X', Y' is an open subset of X' with $\mathfrak{S}(X') \cap Y' = \varnothing$, and Z' is a closed subset of X' such that every irreducible component of $\mathfrak{Z}(I'I'^{-1})$ having a nonempty intersection with $Z' \cap Y'$ is contained in Z'. Suppose if possible that $\mathfrak{Z}(I'I'^{-1}) \cap Z' \cap Y' \neq \varnothing$; then by (8.2.1) there exists an irreducible closed subset T' of X' such that $\mathfrak{S}(T') \cap Y' = \varnothing$, $T' \subset \mathfrak{Z}(I'I'^{-1}) \cap Z'$, $\varnothing \neq T' \cap Y' \subset \mathfrak{C}^*(Z' \cap Y', I'I'^{-1})$, and, upon letting S be the generic point of T', for every $R \in T' \cap Y'$ we have that $S \in \mathfrak{C}(R, I'I'^{-1})$ and: $\dim S = 2 \Leftrightarrow \mathfrak{C}^2(R, I'I'^{-1})$ has a strict normal crossing at R and $\mathfrak{C}^2(R, I'I'^{-1}) \neq \varnothing$; we now get an element $w' = (X'_i, I'_i, Y'_i, Z'_i, T'_i)_{0 \leqslant i < m+1}$ in W with $w \leqslant w'$ and $w \neq w'$ by taking $(X'_i, I'_i, Y'_i, Z'_i, T'_i) = (X_i, I_i, Y_i, Z_i, T_i)$ for $0 \leqslant i < m$ and $(X'_m, I'_m, Y'_m, Z'_m, T'_m) = (X', I', Y', Z', T')$; this is a contradiction because w is a maximal element of W. Therefore $\mathfrak{Z}(I'I'^{-1}) \cap Z' \cap Y' = \varnothing$ and hence $[(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}, (X', I', Y', Z')]$ is a finite global principalizer of K/k with $(X_0, I_0, Y_0, Z_0) = (X, I, Y, Z)$.

(8.3). *If K/k is globally strongly principalizable then K/k is globally principalizable.*

PROOF. Follows from (8.2.2).

(8.4). *For any global principalizer $(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < \infty}$ of K/k we have the following.*

(8.4.1). *Given any nonnegative integer n and any $R \in \bigcap_{i=n}^{\infty} Y_i$, there exists an open subset D of Y_n with $R \in D$ such that $D \subset Y_i$ and $D \cap T_i = \varnothing$ for all $i \geqslant n$.*

(8.4.2). *Assume that there does not exist any infinite principalizer $(R_j, Q_j, S_j)_{0 \leqslant j < \infty}$ with $R_0 \in \mathfrak{Z}(I_0 I_0^{-1}) \cap Z_0 \cap Y_0$. Then there exists a nonnegative integer m such that $T_i \cap Y_i = \varnothing$ for all $i \geqslant m$.*

PROOF OF (8.4.1). In view of (6.4.9) and (6.8) we have that $\mathfrak{Z}(I_i I_i^{-1}) \cap Z_i \cap Y_i$ is a closed subset of Y_i with codim $\mathfrak{Z}(I_i I_i^{-1}) \cap Z_i \cap Y_i \geqslant 2$ and $\mathfrak{F}(X_{i+1}, X_i) \cap Y_i = T_i \cap Y_i \subset$

$\mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap Y_i$ for $0 \leqslant i < \infty$. Since $R \in \bigcap_{i=n}^{\infty} Y_i$, we get that $R \notin T_i$ for all $i \geqslant n$. For each $i \geqslant n$ let G_i be the union of the irreducible components of $\mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap Y_i$ passing through R, let H_i be the union of the remaining irreducible components of $\mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap Y_i$, let

$$G_i^* = \mathfrak{S}(G_i) \cup \{R' \in G_i - \mathfrak{S}(G_i) \colon S \notin \mathfrak{C}(R', I_iI_i^{-1}) \text{ where } S \text{ is the generic point of the irreducible component of } \mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap Y_i \text{ passing through } R'\},$$

and let $D_i = Y_i - ((G_i^* - \{R\}) \cup H_i)$. Then $R \in D_i$, and in view of (6.5.5) we get that D_i is an open subset of Y_i. For any open subset E of Y_i with $R \in E$, in view of (6.2.16) we get that:

$$\begin{aligned}(\mathfrak{Z}(I_iI_i^{-1}) &\cap Z_i \cap E \cap D_i) - \{R\} \\ = \{R' &\in (\mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap E) - (\mathfrak{S}(\mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap E) \cup \{R\}) \colon \text{upon letting } F \text{ be the irreducible component of } \mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap E \\ &\text{passing through } R', \text{ and } S \text{ be the generic point of } F, \text{ we have that } R \in F, \text{ and } S \in \mathfrak{C}(R', I_iI_i^{-1})\};\end{aligned}$$

let us refer to this observation as $[i, E]$. Since $R \notin T_i$, by $[i, Y_i]$ we see that $D_i \cap T_i = \varnothing$, and hence $\mathfrak{F}(X_{i+1}, X_i) \cap D_i = \varnothing$. Consequently $D_i \subset Y_{i+1}$, $I_{i+1}D_i = I_iD_i$, and $Z_{i+1} \cap D_i = Z_i \cap D_i$. Therefore:

$$(1_i) \qquad \begin{aligned}&(I_{i+1}I_{i+1}^{-1})D_i = (I_iI_i^{-1})\, D_i \quad \text{and} \\ &\mathfrak{Z}(I_{i+1}I_{i+1}^{-1}) \cap Z_{i+1} \cap D_i = \mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap D_i .\end{aligned}$$

In view of (6.2.5) we have that D_i is an open subset of Y_{i+1}, and hence by (1_i), $[i, D_i]$, and $[i+1, D_i]$ we get that:

$$(2_i) \qquad \begin{aligned}&(\mathfrak{Z}(I_iI_i^{-1}) \cap Z_i \cap D_i) - \{R\} \\ &= (\mathfrak{Z}(I_{i+1}I_{i+1}^{-1}) \cap Z_{i+1} \cap D_i \cap D_{i+1}) - \{R\}.\end{aligned}$$

Since $\mathfrak{F}(X_{i+1}, X_i) \cap Y_i \subset \mathfrak{Z}(I_iI^{-1}) \cap Z_i$, we get that

$$Y_i - (\mathfrak{Z}(I_iI_i^{-1}) \cap Z_i) \subset Y_{i+1} - (\mathfrak{Z}(I_{i+1}I_{i+1}^{-1}) \cap Z_{i+1});$$

also

$$D_i \subset Y_i \quad \text{and} \quad Y_{i+1} - (\mathfrak{Z}(I_{i+1}I_{i+1}^{-1}) \cap Z_{i+1}) \subset D_{i+1}\,;$$

therefore

$$(3_i) \qquad D_i - (\mathfrak{Z}(I_i I_i^{-1}) \cap Z_i) \subset D_{i+1}\,.$$

By (2_i) and (3_i) we get that $D_i \subset D_{i+1}$. Thus D_i is an open subset of Y_i with $R \in D_i$ and $D_i \cap T_i = \varnothing$ for all $i \geqslant n$, and also $D_i \subset D_{i+1}$ for all $i \geqslant n$. It suffices to take $D = D_n$.

PROOF OF (8.4.2). In view of (6.4.2), (6.4.9), and (6.8), we have that $\mathfrak{F}(X_{i+1}, X_i) \cap Y_i = T_i \cap Y_i$ and $\mathfrak{F}(X_i, X_0) \subset \mathfrak{Z}(I_0 I_0^{-1}) \cap Z_0 \cap Y_0$ for $0 \leqslant i < \infty$. Given any $V \in \mathfrak{R}(Y_0)$, let R_i' be the center of V on Y_i for $0 \leqslant i < \infty$, and let $(a(j))_{0 \leqslant j \leqslant n}$ be the unique sequence such that: either n is a nonnegative integer or $n = \infty$; $a(j)$ is a nonnegative integer for $0 \leqslant j < n$; $a(0) = 0$; $a(j-1) < a(j)$ and $R'_{a(j-1)} = R_i' \neq R'_{a(j)}$ whenever $0 < j < n$ and $a(j-1) \leqslant i < a(j)$; if $n \neq \infty$ then $a(n)$ is a nonnegative integer and $R'_{a(n)} = R_i'$ whenever $a(n) \leqslant i < \infty$; and if $n = \infty$ then $a(n) = \infty$. For $0 \leqslant j < n$, upon letting $R_j = R'_{a(j)}$, we get that $R_j \in T_{a(j+1)-1}$; let S_j be the generic point of the irreducible component of $T_{a(j+1)-1}$ passing through R_j, and let $Q_j = I_{a(j+1)-1}R_j$. Suppose if possible that $n = \infty$; then $(R_j, Q_j, S_j)_{0 \leqslant j < \infty}$ is an infinite principalizer; since $\mathfrak{F}(X_i, X_0) \subset \mathfrak{Z}(I_0 I_0^{-1}) \cap Z_0 \cap Y_0$ for $0 \leqslant i < \infty$, we get that $R_0 \in \mathfrak{Z}(I_0 I_0^{-1}) \cap Z_0 \cap Y_0$; this contradicts our assumption. Therefore $n \neq \infty$. Let $n(V) = n$. Thus for each $V \in \mathfrak{R}(Y_0)$ we have found a nonnegative integer $n(V)$ such that, upon letting $R(V)$ be the center of V on $Y_{n(V)}$, we have that $R(V) \in \bigcap_{i=n(V)}^{\infty} Y_i$. By (8.4.1) there exists an open subset $D(V)$ of $Y_{n(V)}$ with $R(V) \in D(V)$ such that $D(V) \subset Y_i$ and $D(V) \cap T_i = \varnothing$ for all $i \geqslant n(V)$. For each $V \in \mathfrak{R}(Y_0)$ we clearly have that $V \in [\mathfrak{R}(Y_0), Y_{n(V)}]^{-1}(D(V))$, and by (6.2.5) we get that $[\mathfrak{R}(Y_0), Y_{n(V)}]^{-1}(D(V))$ is an open subset of $\mathfrak{R}(Y_0)$; now $\mathfrak{R}(Y_0)$ is quasicompact by (6.2.13), and hence there exists a finite number of elements $V_1, \ldots, V_q$ in $\mathfrak{R}(Y_0)$ such that

$$\mathfrak{R}(Y_0) = \bigcup_{j=1}^{q} [\mathfrak{R}(Y_0), Y_{n(V_j)}]^{-1}(D(V_j)).$$

Let m be any nonnegative integer such that $m \geqslant n(V_j)$ for $1 \leqslant j \leqslant q$. Then clearly $T_i \cap Y_i = \varnothing$ for all $i \geqslant m$.

(8.5). *If K/k is locally strongly principalizable then K/k is globally principalizable and globally strongly principalizable.*

PROOF. Follows from (8.3) and (8.4.2).

(8.6). *Assume that K/k is globally principalizable. Let X be any nonsingular model of K/k and let I be any nonzero ideal on X. Then there exists a nonsingular model X' of K/k such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{Z}(II^{-1})$, and IX' is a nonzero principal ideal on X.*

PROOF. Since K/k is globally principalizable, there exists a finite global principalizer $[(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}, (X', I', Y', Z')]$ of K/k with $(X_0, I_0, Y_0, Z_0) = (X, I, X, X)$. Now X' is a nonsingular model of K/k, X' is an iterated monoidal transform of X with nonsingular irreducible centers, $IX = I'$, I' is a nonzero ideal on X', and $\mathfrak{Z}(I'I'^{-1}) = \varnothing$; therefore in view of (6.4.8) we get that IX' is a nonzero principal ideal on X'. By (6.4.2) we also get that $\mathfrak{F}(X', X) = \mathfrak{Z}(II^{-1})$.

(8.7). *Assume that K/k is globally principalizable. Let X be any nonsingular projective model of K/k and let X^* be any projective model of K/k. Then there exists a nonsingular projective model X' of K/k such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{F}(X^*, X)$, and X' dominates X^*.*

PROOF. By (6.7.2) there exists a nonzero ideal I on X such that $\mathfrak{W}(X, I) = X + X^*$. By (8.6) there exists a nonsingular model X' of K/k such that X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{Z}(II^{-1})$, and IX' is a nonzero principal ideal on X'. By (6.7.1) and (6.7.2) it follows that X' is a projective model of K/k, $\mathfrak{F}(X', X) = \mathfrak{F}(X^*, X)$, and X' dominates X^*.

(8.8). *Assume that K/k is globally principalizable. Let X be any nonsingular projective model of K/k, and let $X_1, \ldots, X_n$ be any* *finite number of complete models of K/k. Then there exists a nonsingular projective model X' of K/k such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, and X' dominates X_i for $1 \leqslant i \leqslant n$.*

PROOF. By (6.2.6) and (6.2.7) there exists a projective model X^* of K/k such that X^* dominates X_i for $1 \leqslant i \leqslant n$. The assertion now follows from (8.7).

(8.9). *Assume that K/k is globally principalizable, and let X and X^* be any projective models of K/k such that X^* dominates X. Then there exists a projective model X' of K/k such that X' dominates X^*, $[X', X]^{-1}(X - \mathfrak{S}(X)) \subset X' - \mathfrak{S}(X')$, and upon letting*

$$Z^* = \mathfrak{F}(X', X^*) \cap ([X^*, X]^{-1}(\mathfrak{S}(X) \cap \mathfrak{F}(X^*, X)))$$

we have that Z^* is a closed subset of X^* and Z^* contains every irreducible component of $\mathfrak{F}(X', X^*)$ having a nonempty intersection with $Z^* - \mathfrak{S}(X^*)$ (note that by (6.7.3) we know that $\mathfrak{F}(X', X^*)$ is a closed subset of X^*).

PROOF. By (6.7.2) there exists a nonzero ideal I on X such that $\mathfrak{W}(X, I) = X^*$. Let $Y = X - \mathfrak{S}(X)$ and $Z = X$. By (6.5.3) we know that Y is an open subset of X. Since K/k is globally principalizable, there exists a finite global principalizer $[(X_i, I_i, Y_i, Z_i, T_i)_{0 \leqslant i < m}, (X_m, I_m, Y_m, Z_m)]$ of K/k with $(X_0, I_0, Y_0, Z_0) = (X, I, Y, Z)$. Let $X' = X_m + X^*$. Then by (6.2.7) and (6.7.1) we know that X_m and X' are projective models of K/k. By (6.8) we also have that $[X_m, X]^{-1}(X - \mathfrak{S}(X)) \subset X_m - \mathfrak{S}(X_m)$. Now $Z_m = X_m$, $Y_m = [X_m, X]^{-1}(X - \mathfrak{S}(X))$, and $\mathfrak{Z}(I_m I_m^{-1}) \cap Z_m \cap Y_m = \varnothing$; therefore by (6.6.8) we get that $([X_m, X]^{-1}(X - \mathfrak{S}(X))) \cap \mathfrak{F}(X^*, X_m) = \varnothing$; consequently $[X_m, X]^{-1}(X - \mathfrak{S}(X)) \subset X'$, and hence $[X_m, X]^{-1}(X - \mathfrak{S}(X)) \subset X' - \mathfrak{S}(X')$. By (6.5.3) and (6.7.3) we know that $\mathfrak{S}(X)$ and $\mathfrak{F}(X^*, X)$ are closed subsets of X, and hence by (6.2.5) we get that Z^* is a closed subset of X^*. Let G be any irreducible component of $\mathfrak{F}(X', X^*)$ having a nonempty intersection with $Z^* - \mathfrak{S}(X^*)$, take $R^* \in G \cap (Z^* - \mathfrak{S}(X^*))$, and let S^* be the generic point of G. We shall show that $S^* \in Z^*$ and this will complete the proof. By (6.5.3) we know that $X^* -$

$\mathfrak{S}(X^*)$ is an open subset of X^*, and therefore $S^* \in X^* - \mathfrak{S}(X^*)$; since $S^* \in \mathfrak{F}(X', X^*)$, we get that $\dim S^* \geqslant 2$. If $S^* = R^*$ then we have nothing to show. So assume that $S^* \neq R^*$. Then we must have $\dim S^* = 2$ and $\dim R^* = 3$. Let $R = [X^*, X](R^*)$ and $S = R_{R \cap M(S^*)}$. Then $S \in \mathfrak{B}(R) \subset X$ and S^* dominates S. Therefore $S = [X^*, X](S^*)$. Since $S^* \in X^* - \mathfrak{S}(X^*)$, we get that if $S \in \mathfrak{S}(X)$ then $S \in \mathfrak{F}(X^*, X)$ and hence $S^* \in Z^*$. So also assume that $S \in X - \mathfrak{S}(X)$; we shall show that this leads to a contradiction and that will complete the proof. Since $R^* \in Z^*$, we get that $R \in \mathfrak{S}(X)$; consequently $S \neq R$, and hence $\dim S \leqslant 2$. Take any $V \in [\mathfrak{R}(K/k), X^*]^{-1}(S^*)$, and let S_i be the center of V on X_i for $0 \leqslant i \leqslant m$. Then $S_0 = S \in Y$, and hence for $0 < i \leqslant m$ we have that S_i is a regular local domain, $S_i \in Y_i$, S_i dominates S_{i-1}, and $\dim S_i \leqslant \dim S_{i-1}$. Since $\dim S \leqslant 2$, we get that $\dim S_i \leqslant 2$ for $0 \leqslant i \leqslant m$. Now $S^* \in \mathfrak{F}(X', X^*)$, and hence by (6.2.8) we get that $S^* \in \mathfrak{F}(X_m, X^*)$; consequently S^* does not dominates S_m. Since S^* dominates S_0, there exists an integer n with $0 \leqslant n < m$ such that S^* dominates S_n, and S^* does not dominate S_{n+1}. Then in particular $S_{n+1} \neq S_n$ and hence $S_n \in \mathfrak{F}(X_{n+1}, X_n)$. In view of (6.4.9), (6.6.8), and (6.8), we have that $\mathfrak{Z}(I_n I_n^{-1}) \cap Y_n$ is a closed subset of Y_n with $\operatorname{codim} \mathfrak{Z}(I_n I_n^{-1}) \cap Y_n \geqslant 2$, $\mathfrak{F}(X_{n+1}, X_n) \cap Y_n = T_n \cap Y_n$, and $\mathfrak{F}(X^*, X_n) = \mathfrak{Z}(I_n I_n^{-1})$; consequently we get that $S_n \in T_n \cap Y_n$, $\operatorname{codim} T_n \cap Y_n \geqslant 2$, $S_n \in \mathfrak{F}(X^*, X_n)$, and $S^* \neq S_n$. Now S_n is a regular local domain, $\dim S_n \leqslant 2$, S^* dominates S_n, S^* and S_n have the same quotient field, and $\dim S^* = 2$; therefore we must have $\dim S_n = 2$. Consequently S_n is the generic point of the irreducible component of T_n passing through S_n, and hence S_{n+1} is the quadratic transform of S_n along V. Therefore by (0.3) we get that S^* dominates S_{n+1}. This is a contradiction.

(8.10). *Assume that K/k is globally principalizable, and let $X_1, \ldots, X_n$ be any finite number of projective models of K/k. Then there exists a projective model X' of K/k such that for $1 \leqslant i \leqslant n$ we have that X' dominates X_i and $[X', X_i]^{-1}(X_i - \mathfrak{S}(X_i)) \subset X' - \mathfrak{S}(X')$.*

PROOF. The general case follows from the case of $n = 2$ by a straightforward induction. So assume that $n = 2$. Let $X = X_1 +$

X_2. By (6.2.7) we know that X is a projective model of K/k. Since X dominates X_1, by (8.9) there exists a projective model X^* of K/k such that X^* dominates X, and

$$(1)\qquad [X^*, X_1]^{-1}(X_1 - \mathfrak{S}(X_1)) \subset X^* - \mathfrak{S}(X^*).$$

Now X^* dominates X_2, and hence again by (8.9) there exists a projective model X'' of K/k such that X'' dominates X^*,

$$(2)\qquad [X'', X_2]^{-1}(X_2 - \mathfrak{S}(X_2)) \subset X'' - \mathfrak{S}(X''),$$

and upon letting

$$(3)\qquad Z^* = \mathfrak{F}(X'', X^*) \cap ([X^*, X_2]^{-1}(\mathfrak{S}(X_2) \cap \mathfrak{F}(X^*, X_2)))$$

we have that Z^* is a closed subset of X^*, and Z^* contains every irreducible component of $\mathfrak{F}(X'', X^*)$ having a nonempty intersection with $Z^* - \mathfrak{S}(X^*)$. Let $Y^* = X^* - \mathfrak{S}(X^*)$. By (6.5.3) we know that Y^* is an open subset of X^*. By (6.7.2) there exists a nonzero ideal I^* on X^* such that $\mathfrak{W}(X^*, I^*) = X''$, and then by (6.6.8) we have that $\mathfrak{F}(X'', X^*) = \mathfrak{Z}(I^*I^{*-1})$. Since K/k is globally principalizable, there exists a finite global principalizer $[(X_i^*, I_i^*, Y_i^*, Z_i^*, T_i^*)_{0 \leq i < m}, (X^{*\prime}, I^{*\prime}, Y^{*\prime}, Z^{*\prime})]$ of K/k with $(X_0^*, I_0^*, Y_0^*, Z_0^*) = (X^*, I^*, Y^*, Z^*)$. Clearly

$$(4)\qquad \mathfrak{F}(X^{*\prime}, X^*) \subset Z^*,$$

and by (6.8) we have that

$$(5)\qquad [X^{*\prime}, X^*]^{-1}(X^* - \mathfrak{S}(X^*)) \subset X^{*\prime} - \mathfrak{S}(X^{*\prime}).$$

Now $\mathfrak{Z}(I^{*\prime}I^{*\prime -1}) \cap Z^{*\prime} \cap Y^{*\prime} = \varnothing$, $Z^{*\prime} = [X^{*\prime}, X^*]^{-1}(Z^*)$, $Y^{*\prime} = [X^{*\prime}, X^*]^{-1}(X^* - \mathfrak{S}(X^*))$, and by (6.6.8) we have that $\mathfrak{F}(X'', X^{*\prime}) = \mathfrak{Z}(I^{*\prime}I^{*\prime -1})$; consequently

$$(6)\qquad \mathfrak{F}(X'', X^{*\prime}) \cap ([X^{*\prime}, X^*]^{-1}(Z^* - \mathfrak{S}(X^*))) = \varnothing.$$

Let $X' = X^{*\prime} + X''$. Then by (6.2.7) and (6.7.1) we get that X' is a projective model of K/k. Given any $R' \in X'$ let R_1, R_2, R^*, R'', and $R^{*\prime}$ be the centers of R' on X_1, X_2, X^*, X'', and $X^{*\prime}$ respectively. We shall show that if either $R_1 \notin \mathfrak{S}(X_1)$ or $R_2 \notin \mathfrak{S}(X_2)$ then $R' \notin \mathfrak{S}(X')$, and this will complete the proof. First suppose that $R_2 \notin \mathfrak{S}(X_2)$; then $R^* \notin Z^*$ by (3), and hence $R^{*\prime} = R^*$ by (4); also $R'' \notin \mathfrak{S}(X'')$ by (2); since X'' dominates X^*, we get that R'' dominates R^*, and hence R'' dominates $R^{*\prime}$; consequently $R' = R''$ and $R' \notin \mathfrak{S}(X')$. Next suppose that $R_1 \notin \mathfrak{S}(X_1)$ and $R^* \notin \mathfrak{F}(X'', X^*)$; then $R^* \notin \mathfrak{S}(X^*)$ by (1); since X'' dominates X^*, we get that $R'' = R^*$; also $R^* \notin Z^*$ by (3), and hence $R^{*\prime} = R^*$ by (4); thus $R'' = R^{*\prime} = R^* \notin \mathfrak{S}(X^*)$, and hence $R' = R^*$ and $R' \notin \mathfrak{S}(X')$. Finally suppose that $R_1 \notin \mathfrak{S}(X_1)$, $R_2 \in \mathfrak{S}(X_2)$, and $R^* \in \mathfrak{F}(X'', X^*)$; then $R^* \notin \mathfrak{S}(X^*)$ by (1), and hence in particular $R^* \neq R_2$; since X^* dominates X_2 , we must have $R_2 \in \mathfrak{F}(X^*, X_2)$, and hence $R^* \in Z^*$ by (3); consequently $R^{*\prime} \notin \mathfrak{S}(X^{*\prime})$ by (5), and $R^{*\prime}$ dominates R'' by (6); therefore $R' = R^{*\prime}$ and $R' \notin \mathfrak{S}(X')$.

(8.11). *If K/k is globally principalizable and uniformizable then there exists a nonsingular projective model of K/k.*

PROOF. Follows from (6.10.2) and (8.10).

§9. Main results

In view of (5.2.1), by (6.3.2), (7.5), (7.12), (7.18), (7.25), (7.27), (7.28), (8.5), (8.6), (8.7), (8.8), and (8.11) we get the following.

(9.1). *Assume that k is pseudogeometric,* $\dim_k K \leqslant 3$, *and for every affine ring A over k with quotient field K and every ideal Q in A we have that $\mathfrak{S}(A, Q)$ is closed in $\mathfrak{V}(A)$* (see (1.2.6)). *Also assume that if* $\dim_k K = 3$ *then the following three conditions are satisfied*: (1) *for every maximal ideal N in k we have that k/N is a perfect field having the same characteristic as k*; (2) *for every regular spot R over k with quotient field K and every nonzero principal prime ideal P in R we have that $\mathfrak{S}(R, P)$ is closed in $\mathfrak{V}(R)$*; *and* (3) *for every regular spot R over k with quotient field K and every ideal Q in R we have that* $\mathfrak{S}(R^*, QR^*) = \{S \in \mathfrak{V}(R^*): R_{R \cap M(S)} \in \mathfrak{S}(R, Q)\}$ *where R^* is the completion of R* (see (1.2.6)).

Then we have the following.

(9.1.1). *K/k is locally strongly semiresolvable, globally semiresolvable, globally strongly semiresolvable, locally strongly detachable,* *globally detachable, globally strongly detachable, locally strongly resolvable, globally resolvable, globally strongly resolvable, locally strongly principalizable, globally principalizable, and globally strongly principalizable.*

(9.1.2). *Let X be any nonsingular model of K/k and let J be any nonzero principal ideal on X. Then there exists a nonsingular model X' of K/k and nonzero principal ideals J' and I' on X' such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{S}^*(J)$, $JX' = J'I'$, $J(X \cap X') = J'(X \cap X')$, JX' is a nonzero principal ideal on X', JX' has only normal crossings, $\mathfrak{Z}(JX') = [X', X]^{-1}(\mathfrak{Z}(J))$, $\mathfrak{Z}(JX')$ is a closed subset of X', $\mathfrak{Z}(JX')$ has only normal crossings, $\mathfrak{S}^*(J') = \varnothing$, $\mathfrak{Z}(J')$ is the $[X', X]$-transform of $\mathfrak{Z}(J)$, and $\mathfrak{Z}(J')$ is nonsingular* (note that $\mathfrak{Z}(J)$ is a closed subset of X, $\mathfrak{S}^*(J) = \mathfrak{S}(\mathfrak{Z}(J))$, $\mathfrak{S}^*(J)$ is a closed subset of $\mathfrak{Z}(J)$, and $\mathfrak{S}^*(J)$ does not pass through the generic point of any irreducible component of $\mathfrak{Z}(J)$).

(9.1.3). *Let X be any nonsingular model of K/k and let Z be any closed subset of X such that either $Z = X$ or Z is pure 1-codimensional* (note that the assumptions about Z are satisfied if Z is a surface in X, and they are also satisfied if $\dim_k K \leqslant 2$ and Z is a curve in X). *Then there exists a nonsingular model X' of K/k such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{S}(Z)$, $[X', X]^{-1}(Z)$ is a closed subset of X', $[X', X]^{-1}(Z)$ has only normal crossings, and the $[X', X]$-transform of Z is nonsingular* (note that $\mathfrak{S}(Z)$ is a closed subset of Z, and $\mathfrak{S}(Z)$ does not pass through the generic point of any irreducible component of Z).

(9.1.4). *If X is any nonsingular model of K/k and I is any nonzero ideal on X then there exists a nonsingular model X' of K/k such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{Z}(II^{-1})$, and IX' is a nonzero principal ideal on X'.*

(9.1.5). *If X is any nonsingular projective model of K/k and X^* is any projective model of K/k then there exists a nonsingular projective model X' of K/k such that: X' is an iterated monoidal transform of*

X with nonsingular irreducible centers, $\mathfrak{F}(X', X) = \mathfrak{F}(X^*, X)$, and X' dominates X^*.

(9.1.6). *If X is any nonsingular projective model of K/k and $X_1, \ldots, X_n$ are any finite number of complete models of K/k then there exists a nonsingular projective model X' of K/k such that: X' is an iterated monoidal transform of X with nonsingular irreducible centers, and X' dominates X_i for $1 \leqslant i \leqslant n$.*

(9.1.7). *If K/k is uniformizable then there exists a nonsingular projective model of K/k.*

In view of (5.3), by (9.1) we get the following.

(9.2). *Assume that: k is pseudogeometric;* $\dim_k K \leqslant 2$; *for every maximal ideal N in k we have that k_N is regular and k/N is a perfect field having the same characteristic as k; for every affine ring A over k with quotient field K and every ideal Q in A we have that $\mathfrak{S}(A, Q)$ is closed in $\mathfrak{V}(A)$; for every regular spot R over k with* $\dim R \leqslant 3$ *and every nonzero principal prime ideal P in R we have that $\mathfrak{S}(R, P)$ is closed in $\mathfrak{V}(R)$; and for every regular spot R over k with* $\dim R \leqslant 3$ *and every ideal Q in R we have that $\mathfrak{S}(R^*, QR^*) = \{S \in \mathfrak{V}(R^*)\colon R_{R \cap M(S)} \in \mathfrak{S}(R, Q)\}$ where R^* is the completion of R* (see (1.2.6)). *Then there exists a nonsingular projective model of K/k. Moreover,* (9.1.1) *to* (9.1.6) *hold for K/k.*

In view of (1.2.6) and (5.4), by (9.1) we get the following.

(9.3). *Assume that k is a complete local domain,* $\dim_k K \leqslant 2$, *and $k/M(k)$ is a perfect field having the same characteristic as k. Then there exists a nonsingular projective model of K/k. Moreover,* (9.1.1) *to* (9.1.6) *hold for K/k.*

Note that, in view of the alternative proof of (5.1) for the case when S_0 is of zero characteristic given in §5, the proof of the above three results ((9.1), (9.2), and (9.3)) for the case when k is of zero characteristic has been made independent of the papers [5], [7], [8], and [9].

For the sake of completeness, in connection with (9.2) and (9.3) we note the following classical result.

(9.4). *Assume that k is pseudogeometric and* $\dim_k K \leqslant 1$. *Then there exists a nonsingular projective model of K/k.*

PROOF. We can take a finite number of nonzero elements $x_1, \ldots, x_n$ in K with $x_1 = 1$ such that K is the quotient field of $k[x_1, \ldots, x_n]$. Let $A_i = k[x_1/x_i, \ldots, x_n/x_i]$ and let B_i be the integral closure of A_i in K. Then B_i is a finite A_i-module for $1 \leqslant i \leqslant n$, and hence upon letting $X^* = \mathfrak{V}(B_1) \cup \cdots \cup \mathfrak{V}(B_n)$ we get that X^* is a normal complete model of K/k (for instance see [4: Lemmas 1.17 and 1.28]). By (6.3.2) we know that $\dim X^* \leqslant 1$, and hence it follows that X^* is nonsingular and $X^* = \mathfrak{R}(K/k)$ (for instance see [27: §6 and §7 of Chapter V]). By (6.2.6) there exists a projective model X of K/k such that X dominates X^*; since $X^* = \mathfrak{R}(K/k)$, we must have $X = X^*$ (alternatively, we can use (9.1.7)).

(9.5). *Arithmetic genus.* Assume that k is an algebraically closed field, and let $n = \dim_k K$. Given any projective model X of K/k, let $h^i(X)$ denote the vector space dimension over k of the ith cohomology group of X with coefficients in the structure sheaf. These groups have been defined by Serre in [22] where it is also shown that $h^i(X)$ is finite for all i. We define:

$$\begin{aligned} p_a(X) &= \textit{the arithmetic genus } \text{of } X \\ &= h^n(X) - h^{n-1}(X) + \cdots + (-1)^{n-1}h^1(X). \end{aligned}$$

Since X is a projective model of K/k, there exists a finite number of elements $x_1, \ldots, x_m$ in K with $x_1 \neq 0$ such that $X = \mathfrak{W}(k; x_1, \ldots, x_m)$. The classical definition of the arithmetic genus is relative to such a representation of X and is thus. Let $Y, Y_1, \ldots, Y_m$ be indeterminates, and let $y_j = (x_j/x_1)Y$ for $1 \leqslant j \leqslant m$; for each nonnegative integer q let $A_q = \{0\} \cup \{u \in k[y_1, \ldots, y_m]\colon u = f(y_1, \ldots, y_m)$ for some nonzero homogeneous polynomial $f(Y_1, \ldots, Y_m)$ of degree q in $Y_1, \ldots, Y_m$ with coefficients in $k\}$, and let $H(q)$ be the vector space dimension of A_q over k. Then by a theorem of Hilbert [28: §12 of Chapter VII] there exist unique integers $c_0, \ldots, c_n$ such that

$$H(q) = c_n\tbinom{q}{n} + c_{n-1}\tbinom{q}{n-1} + \cdots + c_1\tbinom{q}{1} + c_0$$

for all sufficiently large q. Classically the arithmetic genus of X, relative to the representation (or "embedding") $X = \mathfrak{W}(k; y_1, \ldots,$

y_n), was defined to be $(-1)^n(c_0 - 1)$. Serre [22] has shown this to be independent of the said representation by proving that actually $(-1)^n(c_0 - 1) = p_a(X)$ (by the very definition, $p_a(X)$ is independent of any such representation).

Matsumura [17] has proved the following.

(9.6). *Assume that k is an algebraically closed field, and let X and X^* be any nonsingular projective models of K/k such that X^* dominates X. Then $h^i(X) \leqslant h^i(X^*)$ for all i. If, moreover, there exists a nonsingular irreducible closed subset T of X with $T \neq X$ such that X^* is the monoidal transform of X with center T then $h^i(X) = h^i(X^*)$ for all i.*

In view of (9.1.5), by (9.6) we get the following.

(9.7). *Assume that k is an algebraically closed field and* $\dim_k K \leqslant 3$. *Then for any two nonsingular projective models X and X^* of K/k we have that $h^i(X) = h^i(X^*)$ for all i, and hence in particular $p_a(X) = p_a(X^*)$.*

Once again note that in view of the alternative proof of (5.1) for the case when S_0 is of zero characteristic given in §5, the proof of (9.7) for the case when k is of zero characteristic has been made independent of the papers [5], [7], [8], and [9].

CHAPTER 3

Some Cases of Three-Dimensional Birational Resolution

For any ideal Q in any local ring R such that Q is primary for $M(R)$, by $\mathfrak{e}(Q)$ we denote the *multiplicity* of Q; also by $\mathfrak{e}(R)$ we denote the *multiplicity* of R, i.e., $\mathfrak{e}(R) = \mathfrak{e}(M(R))$; for definition see [28: page 294]; from the definition it follows that $\mathfrak{e}(QR^*) = \mathfrak{e}(Q) \geqslant \mathfrak{e}(R) = \mathfrak{e}(R^*)$ where R^* is the completion of R; note that by [28: Theorem 23 on page 296] we know that if R is regular then $\mathfrak{e}(R) = 1$.

§10. Uniformization of points of small multiplicity

The following result is due to Cohen [13: Theorem 8]; a proof is also given in [18: (30.6)].

(10.1). *For any complete local rings R and S such that S dominates R we have the following: If S is residually finite algebraic over R and $M(R)S$ is primary for $M(S)$ then S is a finite R-module. If S is residually rational over R and $M(R)S = M(S)$ then $R = S$.*

(10.2). *Let R and S be analytically irreducible local domains such that S dominates R. Assume that there exists a subring T of S with $R \subset T$ such that T is a finite R-module and $S = T_{T \cap M(S)}$. Also assume that R is a subspace of S. Let R^* and S^* be the completion of R and S respectively. Let K, L, K^*, and L^* be the quotient fields of R, S, R^*, and S^* respectively, where K^* is identified with a subfield of L^*. Then $S^* = R^*[T]$, S^* is a finite R^*-module, and $L^* = K^*(L)$.*

PROOF. Clearly $R^*[T]$ is a finite R^*-module and hence by [28: Theorem 15 on page 276 and Corollary 2 on page 283] we get that $R^*[T]$ is a complete local domain such that $R^*[T]$ dominates R^* and $M(R^*[T]) = \mathrm{rad}(M(R^*)R^*[T])$. It follows that S^* dominates $R^*[T]$. In particular $T \cap M(R^*[T]) = T \cap M(S^*) = T \cap M(S)$ and hence $R^*[T]$ dominates S. Consequently S^* is residually rational over $R^*[T]$ and $M(R^*[T])S^* = M(S^*)$. By (10.1) we now get that $S^* = R^*[T]$. Therefore S^* is a finite R^*-module and $L^* = K^*(L)$.

The following result is proved in [28: §7 of Chapter VII].

(10.3). *If R is any spot over a field and S is any spot over R such that S dominates R then* $\dim R + \mathrm{trdeg}_R S = \dim S + \mathrm{restrdeg}_R S$.

The following result is due to Zariski and Nagata; a proof is given in [18: (37.5)].

(10.4). *The completion of any normal spot over a field is a normal domain.*

The following result is a slight reformulation of [2: Lemma 13].

(10.5). *Let R be an analytically irreducible local domain, let R^* be the completion of R, let K and K^* be the quotient fields of R and R^* respectively, let V be a quasilocal domain with quotient field K such that V dominates R, and let H be the smallest subring of K^* such that H contains V and R^*. Then $M(V)H \neq H$ and there exists a valuation ring V^* of K^* such that V^* dominates V and R^*.*

PROOF. Suppose if possible that $M(V)H = H$. Then $1 = x_1y_1 + \cdots + x_ny_n$ where $x_1, \ldots, x_n$ are elements in $M(V)$ and $y_1, \ldots, y_n$ are elements in R^*. Since R and V have the same quotient field, we can write $x_i = z_i/z$ where $z, z_1, \ldots, z_n$ are elements in R with $z \neq 0$. Then $z = z_1y_1 + \cdots + z_ny_n \in R \cap (z_1, \ldots, z_n)R^* = (z_1, \ldots, z_n)R$, and hence $z = z_1r_1 + \cdots + z_nr_n$ with $r_1, \ldots, r_n$ in R. Consequently $1 = x_1r_1 + \cdots + x_nr_n \in M(V)$ which is a contradiction. Therefore $M(V)H \neq H$ and hence by the existence theorem of valuations there exists a valuation ring V^* of K^* such that $H \subset V^*$ and $M(V)H \subset M(V^*)$. Clearly then V^* dominates V and R^*.

(10.6). (A form of ZST (= Zariski's Subspace Theorem)). *Let R and S be local domains such that R is analytically irreducible, S is a spot over R, S dominates R, and the quotient fields of R and S coincide. Then R is a subspace of S.*

PROOF. Let R^* be the completion of R, and let K and K^* be the quotient fields of R and R^* respectively. Then $S \subset K \subset K^*$. By assumption there exists an affine ring A over R such that A is a subring of S and $S = A_{A \cap M(S)}$. Let $B = R^*[A]$. Then B is an affine ring over R^* and hence B is noetherian. Let H be the smallest subring of K^* such that H contains S and R^*. By (10.5) we know that $M(S)H \neq H$ and hence there exists a prime ideal Q in H such that $M(S)H \subset Q$. Now $B \subset H$ and hence $B \cap Q$ is a prime ideal in B. Let $T = B_{B \cap Q}$. Then T is a local domain which dominates S and R^*. In particular $\bigcap_{i=0}^{\infty} (R^* \cap M(T)^i) = \{0\}$ and hence by Chevalley's theorem [28: Theorem 13 on page 270] there exists a sequence of nonnegative integers $a(i)$ which tends to infinity with i such that $R^* \cap M(T)^i \subset M(R^*)^{a(i)}$ for all $i \geqslant 0$. Now for all $i \geqslant 0$ we have that $R \cap M(S)^i \subset R \cap M(T)^i \subset R \cap (R^* \cap M(T)^i) \subset R \cap M(R^*)^{a(i)} = M(R)^{a(i)}$, and hence R is a subspace of S.

(10.7). (A form of ZMT (= Zariski's Main Theorem)). *Let R and S be local domains such that R is analytically irreducible, S dominates R,* $\dim R = \dim S$, *S is residually finite algebraic over R, and $M(R)S$ is primary for $M(S)$. Let R^* and S^* be the completions of R and S respectively, and let f: $R^* \to S^*$ be the unique homomorphism such that $f(M(R^*)) \subset M(S^*)$ and $f(x) = x$ for all $x \in R$. Then f is a monomorphism (and hence R is a subspace of S by Chevalley's theorem* [28: Theorem 13 on page 270]). *If moreover R is normal and the quotient fields of R and S coincide then $R = S$.*

PROOF. Now $f(R^*)$ is a complete local ring (see [18: (17.9)]), S^* dominates $f(R^*)$, S^* is residually finite algebraic over $f(R^*)$, and $M(f(R^*))S^*$ is primary for $M(S^*)$. Therefore by (10.1) we get that S^* is integral over $f(R^*)$ and hence $\dim S^* = \dim f(R^*)$ (see [4: Lemmas 1.20, 1.22, and 1.24]). Consequently $\dim R^* =$

$\dim f(R^*)$. Since R^* is a domain, f must be a monomorphism. Now assume that R is normal and the quotient fields of R and S coincide. Let K and K^* be the quotient fields of R and R^* respectively. Since R is normal, by [28: Theorem 8 on page 17] we get that R is the intersection of all valuations rings of K dominating R. Therefore it suffices to show that if V is any valuation ring of K dominating R and z is any element in S then $z \in V$. Since K is the quotient field of R, we can write $z = x/y$ with $x \in R$ and $0 \neq y \in R$. Since $z \in S \subset S^*$ and S^* is integral over $f(R^*)$, there exist elements $z_1, \ldots, z_n$ in R^* such that $z^n + f(z_1)z^{n-1} + \cdots + f(z_n) = 0$. Now $f(x^n + z_1yx^{n-1} + \cdots + z_ny^n) = y^n(z^n + f(z_1)z^{n-1} + \cdots + f(z_n)) = 0$. Since f is a monomorphism, we get that $x^n + z_1yx^{n-1} + \cdots + z_ny^n = 0$, and hence $z^n + z_1z^{n-1} + \cdots + z_n = 0$. Therefore z is integral over R^*. By (10.5) there exists a valuation ring V^* of K^* such that V^* dominates V and R^*. Since V^* is normal, $R^* \subset V^*$, and z is integral over R^*, we get that $z \in V^*$. Now $V = V^* \cap K$ and hence $z \in V$.

(10.8). (A form of ZMT). *Let R be a normal spot over a field, let S be a local domain such that S dominates R,* $\dim R =$ *dim S, S is residually finite algebraic over R, $M(R)S$ is primary for R, and the quotient fields of R and S coincide. Then $R = S$.*

PROOF. Follows from (10.4) and (10.7).

(10.9). (A form of ZMT). *Let R and S be normal spots over a field k such that S dominates R,* $\operatorname{trdeg}_R S = 0 = \operatorname{restrdeg}_R S$, *and $M(R)S$ is primary for $M(S)$. Then $S = T_{T\cap M(S)}$ where T is the integral closure of R in the quotient field of S.*

PROOF. Let $S' = T_{T\cap M(S)}$. By (1.1.2) we know that T is a finite R-module, and hence in view of [4: Lemma 1.17] we get that S' is a normal spot over k. Also S dominates S', S is residually finite algebraic over S', $M(S')S$ is primary for $M(S)$, the quotient fields of S' and S coincide, and by (10.3) we get that $\dim S' = \dim S$. Therefore $S' = S$ by (10.8).

(10.10). (A form of ZST). *Let R be a normal spot over a field k, let T be the integral closure of R in a finite algebraic extension L*

of the quotient field K of R, let P be a prime ideal in T with $R \cap P = M(R)$, and let $S = T_P$. Then S is a normal spot over k, S dominates R, $\dim R = \dim S$, S is residually finite algebraic over R, $M(R)S$ is primary for $M(S)$, the completion R^* of R is a normal domain, the completion S^* of S is a normal domain, R is a subspace of S, and upon identifying the quotient field K^* of R^* with a subfield of the quotient field L^* of S^* we have that $S^* = R^*[T]$, S^* is a finite R^*-module, and $L^* = K^*(L)$.

PROOF. By (1.1.2) we know that T is a finite R-module, and hence in view of [4: Lemma 1.17] we get that S is a normal spot over k. Also S dominates R, and S is residually finite algebraic over R. By (10.3) we get that $\dim R = \dim S$. In view of [4: Lemma 1.19] we also have that $M(R)S$ is primary for $M(S)$. By (10.4) we get that R^* and S^* are normal domains. Since R^* is a domain, by (10.7) we get that R is a subspace of S. By (10.2) it now follows that $S^* = R^*[T]$, S^* is a finite R^*-module, and $L^* = K^*(L)$.

The following result was proved in [3: Theorem 1]; a proof is also given in [28: Appendix 2].

(10.11). *Let R be a local domain with quotient field K and let V be a valuation ring of K such that $V \neq K$, V dominates R, and* $\operatorname{restrdeg}_R V \geqslant (\dim R) - 1$. *Then* $\operatorname{restrdeg}_R V = (\dim R) - 1$, *and V is a one-dimensional regular local domain.*

The following result is due to Zariski [24: Theorem 4] for the case of spots over a field, and the general case is due to Sakuma [20: Proposition 1].

(10.12). *Let R be an n-dimensional local domain with $n > 0$, and let K be the quotient field of R. Then there exists a one-dimensional regular local domain V with quotient field K such that V dominates R and* $\operatorname{restrdeg}_R V = n - 1$.

PROOF. Now there exist elements $x_1, \ldots, x_n$ in R such that $(x_1, \ldots, x_n)R$ is primary for $M(R)$. Let $A = R[x_1/x_1, \ldots, x_n/x_1]$. Then by (1.3.3) we get that $M(R)A$ is a prime ideal in A and upon letting $S = A_{M(R)A}$ we have that S is a one-dimensional local domain, S dominates R, and $\operatorname{restrdeg}_R S = n - 1$. Let T be the integral closure of S in K, and let $V = T_P$ for some maximal ideal P in T. By a theorem of Krull [18: (33.2)], V is a one-dimensional regular local domain with quotient field K. Clearly V dominates S and $\operatorname{restrdeg}_S V = 0$. Therefore V dominates R and $\operatorname{restrdeg}_R V = n - 1$.

(10.13). (A form of ZST). *Let R and S be local domains such that R is analytically irreducible, S dominates R,* $\operatorname{trdeg}_R S < \infty$, *and* $\dim R + \operatorname{trdeg}_R S = \dim S + \operatorname{restrdeg}_R S$. *Then R is a subspace of S.*

PROOF. Let R^* be the completion of R. Let K, K^*, and L be the quotient fields of R, R^*, and S respectively. If $\dim R = 0$ then our assertion is trivial. So now assume that $\dim R > 0$. Then $\dim S > 0$ and hence by (10.12) there exists a one-dimensional regular local domain W with quotient field L such that W dominates S and $\operatorname{restrdeg}_S W = (\dim S) - 1$. Let $V = K \cap W$. Now W is the valuation ring of a valuation of L; consequently V is the valuation ring of a valuation of K, W dominates V, and V dominates R. In particular $R \cap M(V) = M(R) \neq \{0\}$ and hence $M(V) \neq \{0\}$; since W is the valuation ring of a discrete valuation of L, we get that V is the valuation ring of a discrete valuation of K and $M(V)W = M(W)^u$ where u is a positive integer, and then $K \cap M(W)^{ui} = M(V)^i$ for every nonnegative integer i. Since V is the valuation ring of a valuation of K, it follows that $\operatorname{restrdeg}_V W \leqslant \operatorname{trdeg}_V W$; now $\operatorname{trdeg}_V W = \operatorname{trdeg}_R S$, $\operatorname{restrdeg}_S W = (\dim S) - 1$, $\operatorname{restrdeg}_V W + \operatorname{restrdeg}_R V = \operatorname{restrdeg}_S W + \operatorname{restrdeg}_R S$, and by assumption $\dim R + \operatorname{trdeg}_R S = \dim S + \operatorname{restrdeg}_R S$; consequently we get that $\operatorname{restrdeg}_R V \geqslant (\dim R) - 1$. By (10.5) there exists a valuation ring V^* of K^* such that V^* dominates V and R^*. Since $\dim R^* = \dim R$ and R^* is residually rational over R, we get $\operatorname{restrdeg}_{R^*} V^* \geqslant (\dim R^*) - 1$. Therefore by (10.11) we get that V^* is a one-dimensional regular local domain. In particular $\bigcap_{i=0}^{\infty} M(V^*)^i = \{0\}$ and hence $\bigcap_{i=0}^{\infty} (R^* \cap M(V^*)^i) = \{0\}$. Consequently by Chevalley's theorem [28: Theorem 13 on page 270] there exists a sequence of nonnegative integers $a(i)$ which tends to infinity with i such that $R^* \cap M(V^*)^i \subset M(R^*)^{a(i)}$ for every nonnegative integer i. For every nonnegative integer i we now have: $R \cap M(S)^{ui} \subset R \cap M(W)^{ui} = R \cap M(V)^i \subset R \cap M(V^*)^i \subset R \cap (R^* \cap M(V^*)^i) \subset R \cap M(R^*)^{a(i)} = M(R)^{a(i)}$. Consequently there exists a sequence of nonnegative integers $b(i)$ which tends to infinity with i such that $R \cap M(S)^i \subset M(R)^{b(i)}$ for every nonnegative integer i. Therefore R is a subspace of S.

(10.14). (A form of ZST). *Let R and S be spots over a field k such that R is analytically irreducible and S dominates R. Then R is a subspace of S.*

PROOF. Follows from (10.3) and (10.13).

(10.15). REMARK. (10.6) to (10.10), (10.13), and (10.14) are variations of [6: Propositions 14 to 17] which in turn were variations of the results given by Zariski in [26]. (10.11) to (10.14) will not be used in this monograph. It may be noted that by a recent result of Ratliff [19: Corollary 2.9], if S is a spot over an analytically irreducible local domain R then $\dim R + \operatorname{trdeg}_R S = \dim S + \operatorname{restrdeg}_R S$; by using this result, (10.6) becomes a corollary of (10.13); this observation will not be used in this monograph.

(10.16). DEFINITION. Let R be a normal quasilocal domain with quotient field K, let T be the integral closure of R in a finite algebraic extension L of K, let $P_1, \ldots, P_m$ be the maximal ideals in T, and let S_i be the quotient ring of T with respect to P_i; then $S_1, \ldots, S_m$ are said to be the *extensions* of R to L. The following observation which follows from [4: §3] will be used tacitly in the rest of this section: (1) There exists at least one and at most finitely many maximal ideals in T, and for any prime ideal P in T we have that $R \cap P = M(R)$ if and only if P is a maximal ideal in T. (2) $S_1, \ldots, S_m$ are normal quasilocal domains with quotient field L and for $1 \leqslant i \leqslant m$ we have that S_i dominates R and $K \cap S_i = R$. (3) If L' is any finite algebraic extension of L and $S_{i,1}, \ldots, S_{i,u(i)}$ are the extensions of S_i to L' then the $u(1) + \cdots + u(m)$ quasilocal domains $S_{i,j}$ are all distinct and they are exactly all the extensions of R to L'. (4) If L is purely inseparable over K then $m = 1$ and S_1 is residually purely inseparable over R.

Let R be a normal quasilocal domain with quotient field K and let L be a finite algebraic extension of K. R is said to be ***unramified*** in L if for every extension S of R to L we have that S is residually separable algebraic over R and $M(R)S = M(S)$. By the ***inertial field*** of R in L we mean the compositum in K of all the subfields L_1 of L with $K \subset L_1$ such that R is unramified in L_1.

Let R be a one-dimensional regular local domain with quotient field K, let L be a finite algebraic extension of K, and let $S_1, ..., S_m$ be the extensions of R to L. The following result which follows from [27: §6, §7, and §8 of Chapter V] will be used tacitly in the rest of this section: (5) $S_1, ..., S_m$ are one-dimensional regular local domains and, upon letting w_i be the unique positive integer such that $M(R)S_i = M(S_i)^{w_i}$ and $q_i = [h_i(S_i) : h_i(R)]$ where $h_i : S_i \to S_i/M(S_i)$ is the canonical epimorphism, we have that $q_1w_1 + \cdots + q_mw_m \leqslant [L : K]$; moreover, if the integral closure of R in L is a finite R-module then $q_1w_1 + \cdots + q_mw_m = [L : K]$ (note that if L is separable over K then the integral closure of R in L is a finite R-module). The positive integer w_i is called the ***reduced ramification index*** of S_i over R. R is said to be ***tamely ramified*** in L if for $1 \leqslant i \leqslant m$ we have that S_i is residually separable algebraic over R and w_i is not divisible by the characteristic of $R/M(R)$.

The following observation, which follows from (1.1.1), [27: Theorem 15 on page 276 and Corollary 2 on page 283], and [4: Lemmas 1.20, 1.22, and 1.24], will be used tacitly in the rest of this section: (6) If R is any complete local domain and S is the integral closure of R in a finite algebraic extension of the quotient field of R, then S is a finite R-module, S is a complete local domain, S dominates R, $\dim R = \dim S$, and $M(R)S$ is primary for $M(S)$.

Let $f(Z)$ be a nonconstant monic polynomial in an indeterminate Z with coefficients in a field K. Take elements $z_1, ..., z_d$ in an overfield of K such that $f(Z) = (Z - z_1) \cdots (Z - z_d)$. Let $L = K(z_1, ..., z_d)$. Let $y_1, ..., y_e$ be the distinct elements among the elements $z_1, ..., z_d$. We *define:*

$$\mathfrak{d}_K^*(f(Z)) = \text{the norm of } \prod_{i \neq j} (y_i - y_j) \text{ relative to the field}$$
$$\text{extension } L \text{ of } K,$$

where the product is over $e(e - 1)$ term (note that by convention the product over an empty family is 1).

Note that then

$$0 \neq \mathfrak{d}_K^*(f(Z)) = \prod_{i \neq j} (y_i - y_j)^{[L:K]} \in K,$$

and $\mathfrak{d}_K^*(f(Z))$ depends only on $f(Z)$ and K, and not on the elements $y_1, \ldots, y_e$. The usual discriminant of $f(Z)$ is denoted by $\mathfrak{d}(f(Z))$, i.e.,

$$\mathfrak{d}(f(Z)) = \prod_{i \neq j} (z_i - z_j)$$

where the product is over $d(d - 1)$ terms. Note that then $\mathfrak{d}(f(Z) \in K$ and $\mathfrak{d}(f(Z))$ depends only on $f(Z)$, and not on K or the elements $z_1, \ldots, z_d$. Also note that for any normal quasilocal domain R' with quotient field K' such that K' is an overfield of K and $f(Z) \in R'[Z]$, we have the following: (7) $\mathfrak{d}_K^*(f(Z)) \in R'$ and $\mathfrak{d}(f(Z)) \in R'$; (8) if $f'(Z)$ is any nonconstant monic polynomial in Z with coefficients in K' such that $f'(Z)$ divides $f(Z)$ in $K'[Z]$ then $f'(Z) \in R'[Z]$, $f'(Z)$ divides $f(Z)$ in $R'[Z]$, and $\mathfrak{d}(f'(Z))$ divides $\mathfrak{d}(f(Z))$ in R'; and (9) if z is any element in an overfield of K' such that $f(z) = 0$ and z is separable over K' then upon letting $g(Z)$ be the minimal monic polynomial of z over K' we have that $g(Z) \in R'[Z]$ and $\mathfrak{d}(g(Z))$ divides $\mathfrak{d}_K^*(f(Z))$ in R'.

(10.17). *Let R be a normal quasilocal domain with quotient field K. Let $f_1(Z), \ldots, f_n(Z)$ be a finite number of nonconstant monic polynomials in an indeterminate Z with coefficients in R such that $\mathfrak{d}(f_i(Z)) \notin M(R)$ for $1 \leqslant i \leqslant n$. Let $r_1, \ldots, r_n$ be elements in an overfield of K such that $f_i(r_i) = 0$ for $1 \leqslant i \leqslant n$. Then R is unramified in $K(r_1, \ldots, r_n)$.*

PROOF. We shall make induction on n. The assertion is trivial for $n = 0$ because then $K(z_1, \ldots, z_n) = K$. So now let $n > 0$ and assume that the assertion is true for all values of n smaller than the given one. Let S be any extension of R to $K(z_1, \ldots, z_n)$ and let $R' = S \cap K(z_1, \ldots, z_{n-1})$. Then R' is an extension of R to $K(z_1, \ldots, z_{n-1})$, and S is an extension of R' to $K(z_1, \ldots, z_n)$. By the induction hypothesis, R is unramified in $K(z_1, \ldots, z_{n-1})$ and hence R' is residually separable algebraic over R and $M(R)R' = M(R')$. Let $g(Z)$ be the minimal monic polynomial of z_n over

$K(z_1, \ldots, z_{n-1})$. Then $\mathfrak{d}(g(Z))$ divides $\mathfrak{d}$ $(f_n(Z))$in R' and hence $\mathfrak{d}(g(Z)) \notin M(R')$; consequently by [4: Theorem 1.44] we get that S is residually separable algebraic over R' and $M(R')S = M(S)$. Therefore S is residually separable algebraic over R and $M(R)S = M(S)$. Since S was an arbitrary extension of R to $K(z_1, \ldots, z_n)$, we conclude that R is unramified in $K(z_1, \ldots, z_n)$.

(10.18). *Let R be a normal complete local domain with quotient field K, let S be the integral closure of R in a finite algebraic extension L of K, let h: $S \to S/M(S)$ be the canonical epimorphism, and let K' be the inertial field of R in L. Then R is unramified in K', $h(K' \cap S)$ is the maximal separable algebraic extension of $h(R)$ in $h(S)$, $[K' : K] = [h(K' \cap S) : h(R)]$, and there exists a primitive element s of K' over K such that $K' \cap S = R[s]$ and, upon letting $g(Z)$ be the minimal monic polynomial of s over K where Z is an indeterminate, we have that $g(Z) \in R[Z]$ and $\mathfrak{d}(g(Z)) \notin M(R)$.*

PROOF. We can take $s^* \in h(S)$ such that $h(R)(s^*)$ is the maximal separable algebraic extension of $h(R)$ in $h(S)$, and then we can take a monic polynomial $g(Z)$ in Z with coefficients in R such that upon applying h to the coefficients of $g(Z)$ we get the minimal monic polynomial of s^* over $h(R)$. By Hensel's lemma [28: Theorem 17 on page 279] there exists $s \in S$ such that $g(s) = 0$ and $h(s) = s^*$. Then $[K(s) : K] = [h(R)(s^*) : h(R)]$, $h(R[s]) = h(R)(s^*)$, $g(Z)$ is the minimal monic polynomial of s over K, and $\mathfrak{d}(g(Z)) \notin M(R)$. Since $\mathfrak{d}(g(Z)) \notin M(R)$, by [4: Proposition 1.43 and Theorem 1.44] we get that R is unramified in $K(s)$, and $R[s]$ is the integral closure of R in $K(s)$. Let L_1 be any subfield of L with $K \subset L_1$ such that R is unramified in L_1. We shall show that then $L_1 \subset K(s)$ and this will complete the proof. Let S_1 be the integral closure of R in L_1. Upon replacing S by S_1 in the above argument we find $s_1 \in S_1$ such that $R[s_1]$ is the integral closure of R in $K(s_1)$, $h(R[s_1])$ is the maximal separable algebraic extension of $h(R)$ in $h(S_1)$, and $\mathfrak{d}(g_1(Z)) \notin M(R)$ where $g_1(Z)$ is the minimal monic polynomial of s_1 over K. Since R is unramified in L_1, we get that S_1 is residually rational over $R[s_1]$ and $M(R[s_1])S_1 = M(S_1)$. Consequently by (10.1) we get that $R[s_1] = S_1$ and hence $L_1 = K(s_1)$. Let S^* be the integral closure of $R[s]$ in $K(s, s_1)$. Now $\mathfrak{d}(g_1(Z)) \notin M(R[s])$ and hence by (10.17) we get that S^* is residually separable algebraic over $R[s]$ and $M(R[s])S^* = M(S^*)$. Since $h(R[s])$ is the maximal separable algebraic extension of $h(R)$ in $h(S)$, we get that S^* is residually rational over $R[s]$. Consequently by (10.1) we must have $R[s] = S^*$ and hence $L_1 \subset K(s)$.

(10.19). *Let R be a complete local domain with quotient field K and let L be a finite algebraic extension of K. Assume that there exists a one-dimensional regular local domain H in* $\mathfrak{B}(R)$ *such that H is tamely ramified in L. Then L is separable over K.*

PROOF. Let K' be the maximal separable algebraic extension of K in L and let H' be an extension of H to K'. Then H' is tamely ramified in L. Let H^* be the integral closure of H' in L. Since L is purely inseparable over K', we get that H^* is a one-dimensional regular local domain and H^* is residually purely inseparable over H'. Since H' is tamely ramified in L, we conclude that H^* is residually rational over H' and the reduced ramification index w of H^* over H' is not divisible by the characteristic of K'. By (1.1.1) and (1.1.2) we know that H^* is a finite H'-module, and hence $[L : K'] = w$. Therefore $L = K'$.

The following two results ((10.20) and (10.21)) are slight variations of some results given in [1: §2].

(10.20). *Let R be a d-dimensional regular local domain with* $d > 0$, *let K be the quotient field of R, let* $(y_1, \ldots, y_d)$ *be a basis of* $M(R)$, *let* $z_1, \ldots, z_d$ *be elements in an overfield of K such that* $z_i^{n(i)} = y_i$ *for* $1 \leqslant i \leqslant d$ *where* $n(i)$ *is a positive integer, let* $L = K(z_1, \ldots, z_d)$, *and let S be the integral closure of R in L. Then we have the following.*

(10.20.1). *S is a finite R-module, S is a d-dimensional regular local domain,* $S = R[z_1, \ldots, z_d]$, $M(S) = (z_1, \ldots, z_d)S$, *S is residually rational over R, and* $[L : K] = n(1) \cdots n(d)$.

(10.20.2). *Let n be the least common multiple of* $n(1), \ldots, n(d)$. *Assume that n is not divisible by the characteristic of K, and K contains a primitive nth root z of n. Let L′ be any subfield of L with* $K \subset L'$. *Then there exists a finite number of elements* $r_1, \ldots, r_m$

in L' *such that* $L' = K(r_1, \ldots, r_m)$ *and* $r_j^n = y_1^{a(1,j)} \ldots y_d^{a(d,j)}$ *for* $1 \leqslant j \leqslant m$ *where* $a(1,j), \ldots, a(d,j)$ *are nonnegative integers.*

PROOF OF (10.20.1). By induction, the general case follows from the case when $n(2) = \cdots = n(d) = 1$. So assume that $n(2) = \cdots = n(d) = 1$. Let $R' = R[z_1]$. Then R' is noetherian, R' is a finite R-module, and R' is integral over R. Since R' is integral over R, by [4: Lemmas 1.19 and 1.20] we get that there exists a maximal ideal in R' and if P is any maximal ideal in R' then $P \cap R = M(R)$; now $z_1 \in \operatorname{rad}(M(R)R') \subset P$ and hence $P = (M(R) \cup \{z_1\})R' = (z_1, y_2, \ldots, y_d)R'$. It follows that R' is a local domain, $M(R') = (z_1, y_2, \ldots, y_d)R'$, and R' is residually rational over R. Since R' is integral over R, by [4: Lemmas 1.20, 1.22, and 1.24] we get that $\dim R' = \dim R = d$ and hence R' is regular. Consequently R' is normal and hence $R' = S$. Let H be the quotient ring of R with respect to y_1R, let H^* be an extension of H to L, and let w be the reduced ramification index of H^* over H. Since $L = K(z_1)$ and $z_1^{n(1)} = y_1$, we get that $[L : K] \leqslant n(1) \leqslant w$ and hence $[L : K] = n(1)$.

PROOF OF (10.20.2). For $1 \leqslant i \leqslant d$, by (10.20.1) we have that $[K(z_i) : K] = n(i)$ and hence we get that: $K(z_i)$ is a Galois extension of K, the Galois group G_i of $K(z_i)$ over K is a cyclic group of order $n(i)$, and G_i has a generator g_i' such that $g_i'(z_i) = z^{n/n(i)}z_i$. Now L is the compositum of $K(z_1), \ldots, K(z_d)$ in L, and by (10.20.1) we have that $[L : K] = n(1) \cdots n(d)$; consequently L is a Galois extension of K, the Galois group G of L over K is abelian, and there exist elements $g_1, \ldots, g_d$ in G such that: $g_i(z_i) = z^{n/n(i)}z_i$ and $g_i(z_j) = z_j$ whenever $1 \leqslant i \leqslant d$, $1 \leqslant j \leqslant d$, and $i \neq j$; and every element in G can be uniquely expressed as $g_1^{b(1)} \cdots g_d^{b(d)}$ with $1 \leqslant b(1) \leqslant n(1), \ldots, 1 \leqslant b(d) \leqslant n(d)$. Since G is abelian, it follows L' is the compositum in L of a finite number of Galois extensions of K with cyclic Galois groups. Consequently, without loss of generality, we may assume that L' is a Galois extension of K and the Galois group G' of L' over K is cyclic. Since G' is isomorphic to a factor group of G, it follows that the order of G' divides n. Consequently there exists a nonzero primitive element x of L' over K such that $x^n \in K$, and then $g(x)/x \in K$ for all $g \in G$. Since $[L : K] = n(1) \cdots n(d)$, upon letting $Q = \{z_1^{v(1)} \cdots z_d^{v(d)}$:

$0 \leqslant v(1) < n(1), ..., 0 \leqslant v(d) < n(d)\}$, we get that Q is a free K-basis of L. In particular

$$x = \sum x(v(1), ..., v(d)) z_1^{v(1)} \cdots z_d^{v(d)}$$

with $x(v(1), ..., v(d)) \in K$, where the sum is over $0 \leqslant v(1) < n(1)$, ..., $0 \leqslant v(d) < n(d)$. Suppose if possible that there exist two d-tuples $(u(1), ..., u(d))$ and $(v(1), ..., v(d))$ such that $u(j) \neq v(j)$ for some j and $x(u(1), ..., u(d)) \neq 0 \neq x(v(1), ..., v(d))$; since Q is a free K-basis of L, we get that $g_j(x)/x \notin K$ which is a contradiction. Therefore $x = x' z_1^{a(1)} \cdots z_d^{a(d)}$ where $0 \neq x' \in K$ and $a(1), ..., a(d)$ are integers with $0 \leqslant a(1) < n(1), ..., 0 \leqslant a(d) < n(d)$. Let $r = x/x'$. Then $L' = K(r)$ and $r^n = y_1^{a(1)} \cdots y_d^{a(d)}$.

(10.21). *Let R be a d-dimensional complete regular local domain with $d > 0$, let p be the characteristic of $R/M(R)$, let $(y_1, ..., y_d)$ be a basis of $M(R)$, let H_i be the quotient ring of R with respect to $y_i R$, let K be the quotient field of R, let L be a finite algebraic extension of K, let $H_{i,1}, ..., H_{i,u(i)}$ be the extensions of H_i to L, and let $w(i, j)$ be the reduced ramification index of $H_{i,j}$ over H_i. Assume that every one-dimensional element in $\mathfrak{B}(R) - \{H_1, ..., H_d\}$ is unramified in L, and for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant u(i)$ we have that $H_{i,j}$ is residually separable algebraic over H_i and $w(i, j) \not\equiv 0 \bmod p$. Then we have the following.*

(10.21.1). *There exists a finite number of elements $s_0, ..., s_m$ in an algebraic extension of L, a positive integer n with $n \not\equiv 0 \bmod p$, and nonnegative integers $a(1, j), ..., a(d, j)$ for $1 \leqslant j \leqslant m$, such that: $K(s_0, ..., s_m) = L(s_0)$, $s_j^n = y_1^{a(1,j)} \cdots y_d^{a(d,j)}$ for $1 \leqslant j \leqslant m$, $K(s_0)$ contains a primitive* n*th root of* 1, *$K(s_0)$ is the inertial field of R in $L(s_0)$, and, upon letting $g(Z)$ be the minimal monic polynomial of s_0 over K where Z is an indeterminate, we have that $g(Z) \in R[Z]$ and $\mathfrak{d}(g(Z)) \notin M(R)$.*

(10.21.2). *Let K' be the inertial field of R in L and assume that $L \neq K'$. Then there exist elements s_0 and s in an algebraic extension of L, a prime number q with $q \not\equiv 0 \bmod p$, and nonnegative integer $a(1), ..., a(d)$, such that: $s \in L(s_0)$, $[K(s_0, s) : K(s_0)] = q$, $[L : K'] = q[L(s_0) : K(s_0, s)]$, $s^q = y_1^{a(1)} \cdots y_d^{a(d)}$, $K(s_0)$ contains a* *primitive* qth *root of* 1, $K(s_0)$ *is the inertial field of* R *in* $L(s_0)$, *and, upon letting* $g(Z)$ *be the minimal monic polynomial of* s_0 *over* K *where* Z *is an indeterminate, we have that* $g(Z) \in R[Z]$ *and* $\mathfrak{d}(g(Z)) \notin M(R)$.

PROOF OF (10.21.1). We can take a positive integer n such that $n \not\equiv 0 \bmod p$ and $n \equiv 0 \bmod w(i, j)$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant u(i)$. Then we can take elements $z_0, \ldots, z_d$ in an overfield of L such that z_0 is a primitive nth root of 1, and $z_i^n = y_i$ for $1 \leqslant i \leqslant d$. Let Z be an indeterminate and let $f_0(Z) = Z^n - 1$ and $f_i(Z) = Z^n - y_i$ for $1 \leqslant i \leqslant d$. Then $f_i(Z) \in R[Z]$ and $f_i(z_i) = 0$ for $0 \leqslant i \leqslant d$. Let $L^* = L(z_0, \ldots, z_d)$. By (10.18) there exists $s_0 \in L^*$ such that upon letting $K' = K(s_0)$, $R' = R[s_0]$, and $g(Z) =$ the minimal monic polynomial of s_0 over K, we have that K' is the inertial field of R in L^*, R' is the integral closure of R in K', $g(Z) \in R[Z]$, and $\mathfrak{d}(g(Z)) \notin M(R)$. Now $\mathfrak{d}(f_0(Z)) \notin M(R)$ and hence by (10.17) we get that R is unramified in $K(z_0)$; consequently $z_0 \in K'$ and hence upon letting $K^* = K'(z_1, \ldots, z_d)$ we have that $L^* = L(s_0, z_1, \ldots, z_d) = K^*(L)$. Now H_1 is tamely ramified in L and hence by (10.19) we get that L is separable over K; therefore L^* is separable over K^*. Let R^* and S^* be the integral closures of R in K^* and L^* respectively. Then R^* and S^* are complete local domains and S^* dominates R^*; by (10.18) we know that S^* is residually purely inseparable over R', and hence S^* is residually purely inseparable over R^*. By (10.18) we know that R is unramified in K', and hence R' is a d-dimensional regular local domain and $M(R') = (y_1, \ldots, y_d)R'$. Now by (10.20.1) we get that R^* is a d-dimensional regular local domain and $M(R^*) = (z_1, \ldots, z_d)R^*$. Let H_i^* be the quotient ring of R^* with respect to z_iR^*.

Let V^* be any one-dimensional element in $\mathfrak{V}(R^*)$ and let W^* be any extension of V^* to L^*. We claim that then: (1) W^* is residually separable algebraic over V^* and $M(V^*)W^* = M(W^*)$. To prove this let $V = K \cap V^*$ and $W = L \cap W^*$; note that then V^*, W^*, V, W are one-dimensional regular local domains with quotient fields K^*, L^*, K, L respectively, V^* is an extension of V to K^*, W^* is an extension of W to L^*, and W is an extension of V to L; in view of [4: Proposition 1.24B] we also have that $V \in \mathfrak{V}(R)$. First suppose that $V^* \neq H_i^*$ for $1 \leqslant i \leqslant d$; then $V \neq H_i$ for $1 \leqslant i \leqslant d$ and hence by assumption V is unramified in L; consequently W is residually separable algebraic over V and $M(V)W = M(W)$; also $\mathfrak{d}(f_{i'}(Z)) \notin M(W)$ for $0 \leqslant i' \leqslant d$ and hence by (10.17) we get that W^* is residually separable algebraic over W and $M(W)W^* = M(W^*)$; therefore W^* is residually separable algebraic over V^* and $M(V^*)W^* = M(W^*)$. Next suppose that $V^* = H_i^*$ for some i with $1 \leqslant i \leqslant d$; then $V = H_i$ and hence $W = H_{i,j}$ for some j with $1 \leqslant j \leqslant u(i)$; we can take $x \in W$ such that $xW = M(W)$, and then $y_i = x'x^{w(i,j)}$ where x' is a unit in W; let $z = z_i^{n/w(i,j)}x^{-1}$ and $f(Z) = Z^{w(i,j)} - x'$; then $z \in L^*$, $f(Z) = 0$, and $\mathfrak{d}(f(Z)) \notin M(W)$; also $\mathfrak{d}(f_{i'}(Z)) \notin M(W)$ whenever $0 \leqslant i' \leqslant d$ and $i' \neq i$; consequently, upon letting $L' = L(z_0, \ldots, z_{i-1}, z, z_{i+1}, \ldots, z_d)$, by (10.17) we get that W is unramified in L'; therefore, upon letting $W' = L' \cap W^*$ we get that W' is an extension of W to L', W' is residually separable algebraic over W, $M(W') = xW'$, and W^* is an extension of W' to L^*; now $L^* = L'(z_i)$, $z_i^{n/w(i,j)} = zx$, and $M(W') = (zx)W'$; consequently by (10.20.1) we get that W^* is residually rational over W' and $M(W^*) = z_i W^*$; it follows that W^* is residually separable algebraic over V^* and $M(V^*)W^* = M(W^*)$.

This completes the proof of (1). Thus we have shown that every one-dimensional element in $\mathfrak{B}(R^*)$ is unramified in L^*. Therefore by the Zariski-Nagata Purity Theorem [18: (41.1)], R^* is unramified in L^*. Since S^* is residually purely inseparable over R^*, it follows that S^* is residually rational over R^* and $M(R^*)S^* = M(S^*)$. Consequently by (10.1) we get that $R^* = S^*$ and hence $K^* = L^*$. Thus, $K' \subset L(s_0) \subset L^* = K'(z_1, \ldots, z_d)$, and hence by (10.20.2) there exist elements $s_1, \ldots, s_m$ in $L(s_0)$ such that $L(s_0) = K(s_0, \ldots, s_m)$ and $s_j^n = y_1^{a(1,j)} \cdots y_d^{a(d,j)}$ for $1 \leqslant j \leqslant m$ where $a(1,j), \ldots, a(d,j)$ are nonnegative integers. Since $K(s_0)$ is the inertial field of R in L^*, it follows that $K(s_0)$ is the inertial field of R in $L(s_0)$.

PROOF OF (10.21.2). Let $s_0, \ldots, s_m, n, a(1, j), \ldots, a(d, j)$, and $g(Z)$ be as in (10.21.1). By (10.18) we know that R is unramified in K' and hence $K' \subset K(s_0)$. Let R', S, R^*, and S^* be the integral closures of R in K', L, $K(s_0)$, and $L(s_0)$ respectively. Let h: $S^* \to S^*/M(S^*)$ be the canonical epimorphism. By (10.17) we have that R' and S are unramified in $K(s_0)$ and $L(s_0)$ respectively, and hence $h(R^*)$ and $h(S^*)$ are finite separable algebraic extensions of

$h(R')$ and $h(S)$ respectively; by (10.18) we also have that $h(S)$ and $h(S^*)$ are finite purely inseparable extensions of $h(R')$ and $h(R^*)$ respectively; therefore by [27: Corollary 2 on page 79] we get that $[h(R^*) : h(R')] = [h(S^*) : h(S)]$. Since R' and S are unramified in $K(s_0)$ and $L(s_0)$ respectively, by (10.18) we get that $[K(s_0) : K'] = [h(R^*) : h(R')]$ and $[L(s_0) : L] = [h(S^*) : h(S)]$. Therefore $[K(s_0) : K'] = [L(s_0) : L]$, and hence $[L(s_0) : K(s_0)] = [L : K']$. By assumption $L \neq K'$ and hence $L(s_0) \neq K(s_0)$. Consequently $s_j \notin K(s_0)$ for some j with $1 \leqslant j \leqslant m$. Let n' be the smallest positive integer such that $s_j^{n'} \in K(s_0)$ and $n \equiv 0 \bmod n'$. Then $n' > 1$ and hence there exists a prime number q such that $n' \equiv 0 \bmod q$. Note that then $q \not\equiv 0 \bmod p$. Since $K(s_0)$ contains a primitive nth root z_0 of 1, it follows that $K(s_0)$ contains a primitive qth root of 1. Let $s' = s_j^{n'/q}$, $u = n/n'$, and $z = z_0^{n'}$. Then $u \not\equiv 0 \bmod p$, z is a primitive uth root of 1 in $K(s_0)$, $s' \in L(s_0)$, $[K(s_0, s') : K(s_0)] = q$, $s'^q \in K(s_0)$, and $(s'^q)^u = y_1^{a(1,j)} \cdots y_d^{a(d,j)}$. By (10.17) we know that R is unramified in $K(s_0)$, and hence R^* is a d-dimensional regular local domain with quotient field $K(s_0)$ and $M(R^*) = (y_1, \dots, y_d)R^*$. It follows that $a(i, j) \equiv 0 \bmod u$ for $1 \leqslant i \leqslant d$. Let $y = y_1^{a(1)} \cdots y_d^{a(d)}$ where $a(i) = a(i, j)/u$ for $1 \leqslant i \leqslant d$. Then $(s'^q y^{-1})^u = 1$ and hence $s'^q y^{-1} = z^v$ for some integer v. Let $s = s'(z_0^{n'/q})^{-v}$. Then $s \in L(s_0)$, $s^q = y$, $[K(s_0, s): K(s_0)] = q$, and $[L : K'] = q[L(s_0) : K(s_0, s)]$.

(10.22). *Let R be a d-dimensional regular local domain with $d \geqslant 2$. Let $(y_1, \dots, y_d)$ be a basis of $M(R)$. Let V be a valuation ring of the quotient field K of R such that V dominates R and there do not exist any positive integers n and n' such that $y_2^n/y_1^{n'}$ is a unit in V. Let q be a positive integer. Then we have the following.*

(10.22.1). *Assume that $y_2/y_1 \in V$. Then there exists a sequence of d-dimensional regular local domains $R_0, \dots, R_q$ with quotient field K, elements y_1^* and y_2^* in R_q, and a two-dimensional element S_i in $\mathfrak{B}(R_i)$ having a simple point at R_i for $0 \leqslant i < q$, such that: $R_0 = R$; R_{i+1} is a monoidal transform of (R_i, S_i) for $0 \leqslant i < q$; V dominates R_q; $M(R_q) = (y_1^*, y_2^*, y_3, \dots, y_d)R_q$; and $y_1 = y_1^{*u} y_2^{*v}$ and $y_2 = y_1^{*u'} y_2^{*v'}$ where u, v, u', v' are nonnegative integers such that $u' \geqslant q$.*

(10.22.2). *Let a and b be nonnegative integers. Then there exists a sequence of d-dimensional regular local domains $R_0, \ldots, R_e$ with quotient field K, elements y_1' and y_2' in R_e, and a two-dimensional element S_i in $\mathfrak{B}(R_i)$ having a simple point at R_i for $0 \leqslant i < e$, such that: $0 \leqslant e < q$; $R_0 = R$; R_{i+1} is a monoidal transform of (R_i, S_i) for $0 \leqslant i < e$; V dominates R_e; $M(R_e) = (y_1', y_2', y_3, \ldots, y_d)R_e$; and $y_1^a y_2^b = (y_1'^{a^*} y_2'^{b^*})^q (y_1'^{a'} y_2'^{b'})$ where a^*, b^*, a', b' are nonnegative integers such that $a' + b' < q$.*

PROOF. Clearly there exists a unique infinite sequence of pairs of nonzero elements $(x_i, z_i)_{0 \leqslant i < \infty}$ in $M(V)$ with $(x_0, z_0) = (y_1, y_2)$ such that for $0 < i < \infty$ we have that: if $z_{i-1}/x_{i-1} \in V$ then $x_{i-1} = x_i$ and $z_{i-1} = x_i z_i$; and if $z_{i-1}/x_{i-1} \notin V$ then $x_{i-1} = x_i z_i$ and $z_{i-1} = x_i$. Let R_i be the quotient ring $R[x_i, z_i]$ with respect to $M(V) \cap R[x_i, z_i]$. Then $R_0 = R$, and for $0 \leqslant i < \infty$ we have that: R_i is a d-dimensional regular local domain with quotient field K; V dominates R_i; $M(R_i) = (x_i, z_i, y_3, \ldots, y_d)R_i$; and R_{i+1} is a monoidal transform of (R_i, S_i) where S_i is the quotient ring of R_i with respect to $(x_i, z_i)R_i$.

To prove (10.22.1) assume that $y_2/y_1 \in V$. Then $y_1 = x_1^{u(1)} z_1^{v(1)}$ and $y_2 = x_1^{u'(1)} z_1^{v'(1)}$ where $u(1) = 1$, $v(1) = 0$, $u'(1) = 1$, $v'(1) = 1$. By induction on i we see that for $0 < i < \infty$: $y_1 = x_i^{u(i)} z_i^{v(i)}$ and $y_2 = x_i^{u'(i)} z_i^{v'(i)}$ where $u(i)$, $v(i)$, $u'(i)$, $v'(i)$ are nonnegative integers such that $u'(i) \geqslant i$ and $v'(i) \geqslant 1$. It now suffices to take $y_1^* = x_q$, $y_2^* = z_q$, $u = u(q)$, $v = v(q)$, $u' = u'(q)$, $v' = v'(q)$.

To prove (10.22.2) let nonnegative integers a and b be given. Let $a(0) = a$ and $b(0) = b$. Define integers $a(i)$ and $b(i)$ for $0 < i < \infty$ by the following recurrence relations for $0 < i < \infty$: $a(i) = a(i-1) + b(i-1)$; if $z_{i-1}/x_{i-1} \in V$ then $b(i) = b(i-1)$; and if $z_{i-1}/x_{i-1} \notin V$ then $b(i) = a(i-1)$. Let $a^*(i)$, $b^*(i)$, $a'(i)$, $b'(i)$ be the integers such that $a(i) = a^*(i)q + a'(i)$, $b(i) = b^*(i)q + b'(i)$, $0 \leqslant a'(i) < q$, $0 \leqslant b'(i) < q$. By induction on i we see that for $0 \leqslant i < \infty$: $y_1^a y_2^b = (x_i^{a^*(i)} z_i^{b^*(i)})^q (x_i^{a'(i)} z_i^{b'(i)})$, and if $a'(i) + b'(i) \geqslant q$ then $a'(i+1) + b'(i+1) < a'(i) + b'(i)$. Since $a'(0) + b'(0) \leqslant 2q - 2$, there exists an integer e with $0 \leqslant e < q$ such that $a'(e) + b'(e) < q$. It now suffices to take $y_1' = x_e$, $y_2' = z_e$, $a^* = a^*(e)$, $b^* = b(e)$, $a' = a'(e)$, $b' = b'(e)$.

(10.23). *Let R be a d-dimensional regular local domain with $d \geqslant 2$. Let $(y_1, \ldots, y_d)$ be a basis of $M(R)$. Let V be a valuation ring* *of the quotient field K of R such that V dominates R and there exist positive integers n and n' such that $y_2^n/y_1^{n'}$ is a unit in V. Then there exists a sequence of regular local domains $R_0, \dots, R_e$ with quotient field K, an element y_2' in R_e, and a two-dimensional element S_i in $\mathfrak{B}(R_i)$ having a simple point at R_i for $0 \leqslant i < e$, such that: e is a positive integer; $R_0 = R$; $d - 1 \leqslant \dim R_e \leqslant d = \dim R_i$ for $0 \leqslant i < e$; R_{i+1} is a monoidal transform of (R_i, S_i) for $0 \leqslant i < e$; V dominates R_e; $y_1 = y_1^* y_2'^u$ and $y_2 = y_2^* y_2'^v$ where u and v are nonnegative integers and y_1^* and y_2^* are units in R_e; if $\dim R_e = d - 1$ then $M(R_e) = (y_2', y_3, \dots, y_d)R_e$; and if $\dim R_e = d$ then $M(R_e) = (y_1', y_2', y_3, \dots, y_d)R_e$ for some $y_1' \in R_e$.*

PROOF. Now there exists a unique finite sequence of pairs of nonzero elements $(x_i, z_i)_{0 \leqslant i < e}$ in $M(V)$ such that $(x_0, z_0) = (y_1, y_2)$, e is a positive integer, x_{e-1}/y_{e-1} is a unit in V, and for $0 < i < e$ we have that: if $z_{i-1}/x_{i-1} \in V$ then $x_{i-1} = x_i$ and $z_{i-1} = x_i z_i$; and if $z_{i-1}/x_{i-1} \notin V$ then $x_{i-1} = x_i z_i$ and $z_{i-1} = x_i$. For $0 \leqslant i < e$ let R_i be the quotient ring of $R[x_i, z_i]$ with respect to $M(V) \cap R[x_i, y_i]$. Then $R_0 = R$ and for $0 < i < e$ we have that: R_i is a d-dimensional regular local domain with quotient field K; V dominates R_i; $M(R_i) = (x_i, z_i, y_3, \dots, y_d)R_i$; R_i is a monoidal transform of (R_{i-1}, S_{i-1}) where S_{i-1} is the quotient ring of R_{i-1} with respect to $(x_{i-1}, z_{i-1})R_{i-1}$; and $y_1 = x_i^{a(i)} z_i^{b(i)}$ and $y_2 = x_i^{a'(i)} z_i^{b'(i)}$ where $a(i)$, $b(i)$, $a'(i)$, $b'(i)$ are nonnegative integers. Let $y_2' = x_{e-1}$, $u = a(e-1) + b(e-1)$, $v = a'(e-1) + b'(e-1)$, $y_1^* = (z_{e-1}/x_{e-1})^{b(e-1)}$, and $y_2^* = (z_{e-1}/x_{e-1})^{b'(e-1)}$. Let S_{e-1} be the quotient ring of R_{e-1} with respect to $(x_{e-1}, z_{e-1})R_{e-1}$, and let R_e be the quotient ring of $R_{e-1}[z_{e-1}/x_{e-1}]$ with respect to $M(V) \cap R_{e-1}[z_{e-1}/x_{e-1}]$. Then: R_e is a regular local domain with quotient field K; $d - 1 \leqslant \dim R_e \leqslant d$; V dominates R_e; R_e is a monoidal transform of (R_{e-1}, S_{e-1}); $y_1 = y_1^* y_2'^u$, $y_2 = y_2^* y_2'^v$, u and v are nonnegative integers, y_1^* and y_2^* are units in R_e; if $\dim R_e = d - 1$ then $M(R_e) = (y_2', y_3, \dots, y_d)R_e$; and if $\dim R_e = d$ then $M(R_e) = (y_1', y_2', y_3, \dots, y_d)R_e$ for some $y_1' \in R_e$.

(10.24). *Let R be a d-dimensional regular local domain with $d > 0$. Let $(y_1, \dots, y_d)$ be a basis of $M(R)$, let V be a valuation ring of the quotient field K of R such that V dominates R, let q be a prime number, let d' be an integer with $0 < d' \leqslant d$, and let $a(1), \dots, a(d')$*

be nonnegative integers. Then there exists a sequence $R_0, \ldots, R_e$ *of regular local domains with quotient field* K, *an integer* d^* *with* $0 < d^* \leqslant d'$, *elements* $z_1, \ldots, z_{d^*}$ *in* R_e, *and a two-dimensional element* S_i *in* $\mathfrak{V}(R_i)$ *having a simple point at* R_i *for* $0 \leqslant i < e$, *such that*: e *is a nonnegative integer;* $R_0 = R$; R_{i+1} *is a monoidal transform of* (R_i, S_i) *for* $0 \leqslant i < e$; V *dominates* R_e; $\dim R_e = d - d' + d^*$; $M(R_e) = (z_1, \ldots, z_{d^*}, y_{d'+1}, \ldots, y_d)R_e$; *and*

$$y_1^{a(1)} \cdots y_{d'}^{a(d')} = y^*(z_1^{m(1)} \cdots z_{d^*}^{m(d^*)})^q(z_1^{a'(1)} \cdots z_{d^*}^{a'(d^*)})$$

where y^* *is a unit in* R_e *and* $m(1), \ldots, m(d^*)$, $a'(1), \ldots, a'(d^*)$ *are nonnegative integers such that* $a'(1) + \cdots + a'(d^*) < q$ *(note that if* V *is residually algebraic over* R *then* $\dim R_e = d$, *i.e.,* $d^* = d'$).

PROOF. Let w be the number of distinct integers i with $0 < i \leqslant d'$ such that $a(i) \not\equiv 0 \bmod q$. We shall make induction on w. For $w \leqslant 1$ our assertion is trivial, and for $w = 2$ it follows from (10.22.2) and (10.23). So now let $w > 2$ and assume that the assertion is true for all values of w smaller than the given one. Upon relabeling $y_1, \ldots, y_{d'}$ we may assume that $a(i) \not\equiv 0 \bmod q$ for $i = 1, 2, 3$, and $y_3/y_2 \in V$ and $y_2/y_1 \in V$. If there exist positive integers n and n' such that $y_2^n/y_1^{n'}$ is a unit in V then by (10.23) we get a reduction in w. So now suppose that there do not exist any positive integers n and n' such that $y_2^n/y_1^{n'}$ is a unit in V. Then by (10.22.1) there exists a sequence of d-dimensional regular local domains $R_0, \ldots, R_q$ with quotient field K, elements y_1^* and y_2^* in R_q, and a two-dimensional element S_i in $\mathfrak{V}(R_i)$ having a simple point at R_i for $0 \leqslant i < q$, such that: $R_0 = R$; R_{i+1} is a monoidal transform of R_i for $0 \leqslant i < q$; V dominates R_q; $M(R_q) = (y_1^*, y_2^*, y_3, \ldots, y_d)R_q$; and $y_1 = y_1^{*u}y_2^{*v}$ and $y_2 = y_1^{*u'}y_2^{*v'}$ where u, v, u', v' are nonnegative integers such that $u' \geqslant q$. It follows that then $y_3/y_1^{*q} \in V$ and

$$y_1^{a(1)} \cdots y_{d'}^{a(d')} = y_1^{*a}y_2^{*b}y_3^{a(3)} \cdots y_{d'}^{a(d')}$$

where $a = a(1)u + a(2)u'$ and $b = a(1)v + a(2)v'$. Since q is a prime number, there exists an integer e' with $q \leqslant e' < 2q$ such that $a + (e' - q)a(3) = 0 \bmod q$. For $q \leqslant i \leqslant e'$ let $x_i = y_3/y_1^{*^{i-q}}$. Then for $q < i \leqslant e'$, upon letting R_i be the quotient ring of

$R_q[x_i]$ with respect to $M(V) \cap R_q[x_i]$, we get that: R_i is a d-dimensional regular local domain with quotient field K; V dominates R_i; $M(R_i) = (y_1^*, y_2^*, x_i, y_4, ..., y_d)R_i$; and R_i is a monoidal transform of (R_{i-1}, S_{i-1}) where S_{i-1} is the quotient ring of R_{i-1} with respect to $(y_1^*, x_{i-1})R_{i-1}$. Let $a' = a + (e' - q)a(3)$ and $y_3^* = x_{e'}$. Then $M(R_{e'}) = (y_1^*, y_2^*, y_3^*, y_4, ..., y_d)R_{e'}$, a' is a nonnegative integer with $a' \equiv 0 \bmod q$, and

$$y_1^{a(1)} \cdots y_{d'}^{a(d')} = y_1^{*a'} y_2^{*b} y_3^{*a(3)} y_4^{a(4)} \cdots y_{d'}^{a(d')}.$$

Thus we have again obtained a reduction in w.

(10.25). *Let L be a function field over an infinite perfect field k of characteristic p, let $d = \mathrm{trdeg}_k L$, and let V be a valuation ring of L with $k \subset V$ such that V is residually algebraic over k. Assume that $d > 0$, and the following two conditions are satisfied.*

(*) *If $x_1, ..., x_d$ are any elements in $M(V)$ which constitute a transcendence basis of L over k, and t is any nonzero element in R where R is the quotient ring of $k[x_1, ..., x_d]$ with respect to $M(V) \cap k[x_1, ..., x_d]$, then there exists a regular spot R_0 over R and a basis $(y_1, ..., y_d)$ of $M(R_0)$ such that: V dominates R_0, the quotient field of R_0 is $k(x_1, ..., x_d)$, and $t = t' y_1^{b(1)} \cdots y_d^{b(d)}$ where t' is a unit R_0 and $b(1), ..., b(d)$ are nonnegative integers* (note that by (10.3) we have that $\dim R = d = \dim R_0$; also $M(R) = (x_1, ..., x_d)R$ and hence R is regular).

(**) *There exists a spot S over k with quotient field L such that V dominates S and $\mathfrak{e}(S)! \not\equiv 0 \bmod p$.*

Then there exists a regular spot S_1 over S such that V dominates S_1.

PROOF. By [18: (25.9), (25.10), (25.12), (34.9), (40.6)] we have that if S is any spot over a field such that $\mathfrak{e}(S) = 1$ then S is regular. Therefore, assuming that condition (*) is satisfied, it suffices to show that if S is any spot over k with quotient field L such that V dominates S, $\mathfrak{e}(S) > 1$, and $\mathfrak{e}(S)! \not\equiv 0 \bmod p$, then there exists a spot S_1 over S such that V dominates S_1 and $\mathfrak{e}(S_1) < \mathfrak{e}(S)$. Let A be the integral closure of S in L, and let $B = A_{A \cap M(V)}$. By (1.1.2) we know that A is a finite S-module, and hence B is a spot over S such that V dominates B. In view of (10.3), by [28:

Corollary 1 on page 299] we get that $\mathfrak{e}(B) \leqslant \mathfrak{e}(S)$. Therefore, without loss of generality we may assume that S is normal. Note that by (10.3) we have that $\dim S = d$. By [28: Theorem 22 on page 294] there exist elements $x_1, ..., x_d$ in S such that $(x_1, ..., x_d)S$ is primary for $M(S)$ and $\mathfrak{e}((x_1, ..., x_d)S) = \mathfrak{e}(S)$; by [28: Corollary 1 on page 293] we know that then $(x_1, ..., x_d)$ is a transcendence basis of L over k. Let R be the quotient ring of $k[x_1, ..., x_d]$ with respect to $M(V) \cap k[x_1, ..., x_d]$, and let $K = k(x_1, ..., x_d)$. Then R is a regular local domain with quotient field K, $\dim R = d$, $M(R) = (x_1, ..., x_d)R$, S dominates R, S is residually finite algebraic over R, $M(R)S$ is primary for S, and $\mathfrak{e}(M(R)S) = \mathfrak{e}(S)$. By (10.9) we get that $S = T_{T \cap M(S)}$ where T is the integral closure of R in L. We can take elements $r_1, ..., r_n$ in T such that $L = K(r_1, ..., r_n)$, and then we can take nonconstant monic polynomials $f_1(Z), ..., f_n(Z)$ in an indeterminate Z with coefficients in R such that $f_j(r_j) = 0$ for $1 \leqslant j \leqslant n$. Let $t = \mathfrak{d}_K^*(f_1(Z)) \cdots \mathfrak{d}_K^*(f_n(Z))$. Then $0 \neq t \in R$ and hence by assumption there exists a d-dimensional regular local domain R_0 with quotient field K and a basis $(y_1, ..., y_d)$ of $M(R_0)$ such that R_0 is a spot over R, V dominates R_0, and $t = t' y_1^{b(1)} \cdots y_d^{b(d)}$ where t' is a unit in R_0 and $b(1), ..., b(d)$ are nonnegative integers. Let T_0 be the integral closure of R_0 in L and let S_0 be the quotient ring of T_0 with respect to $T_0 \cap M(V)$. By (1.1.2) we know that T_0 is a finite R_0-module, and hence S_0 is a spot over R_0. It follows that S_0 is a normal spot over k, S_0 is a spot over S, and S_0 dominates S. Let R^*, R_0^*, S^*, and S_0^* be the completions of R, R_0, S, and S_0 respectively. Then R^* and R_0^* are normal domains, and by (10.4) so are S^* and S_0^*. Let K^*, K_0^*, L^*, and L_0^* be the quotient fields of R^*, R_0^*, S^*, and S_0^* respectively. By (10.6) and (10.10) we know that R, R_0, and S are subspaces of S^*, and hence we may identify K^*, K_0^*, and L^* with subfields of L_0^*. Then in view of (10.10) we get that $L^* = K^*(L)$, $L_0^* = K_0^*(L)$, and S^* and S_0^* are the integral closures of R^* and R_0^* in L^* and L_0^* respectively. Also note that R_0^* and S_0^* dominate R^* and S^* respectively.

Let K' and K_0' be the inertial fields of R^* and R_0^* in L^* and L_0^* respectively. By (10.18) there exist primitive elements s' and s_0' of K' and K_0' over K^* and K_0^* respectively, such that, upon letting $g'(Z)$ and $g_0'(Z)$ be the minimal monic polynomials of s' and s_0' over K^* and K_0^* respectively, we have $g'(Z) \in R^*[Z]$, $\mathfrak{d}(g'(Z)) \notin$

$M(R^*)$, $g_0'(Z) \in R_0^*[Z]$, and $\mathfrak{d}(g_0'(Z)) \notin M(R_0^*)$. Since $\mathfrak{d}(g'(Z)) \notin M(R^*)$, we have $\mathfrak{d}(g'(Z)) \notin M(R_0^*)$ and so by (10.17) we get that R_0^* is unramified in $K_0^*(s')$; consequently $K_0^*(s') \subset K_0'$ and hence $K' \subset K_0'$; since $L_0^* = K_0^*(L)$, $K_0^* \subset K_0' \subset L_0^*$, and $L \subset L^* \subset L_0^*$, we get that $L_0^* = K_0'(L^*)$; therefore $[L_0^* : K_0'] \leqslant [L^* : K']$. Let $h\colon S^* \to S^*/M(S^*)$ be the canonical epimorphism. By (10.18) we know that $h(S^* \cap K')$ is the maximal separable algebraic extension of $h(R^*)$ in $h(S^*)$ and $[K' : K^*] = [h(S^* \cap K') : h(R^*)]$; since by assumption k is perfect, we conclude that $[K' : K^*] = [h(S^*) : h(R^*)]$; by [28: Corollary 1 on page 299] we know that $\mathfrak{e}(M(R^*))[L^* : K^*] = \mathfrak{e}(M(R^*)S^*)[h(S^*) : h(R^*)]$; since $\mathfrak{e}(M(R^*)) = 1$ and $\mathfrak{e}(M(R^*)S^*) = \mathfrak{e}(M(R)S) = \mathfrak{e}(S)$, we get that $\mathfrak{e}(S) = [L^* : K']$. Therefore $[L_0^* : K_0'] \leqslant \mathfrak{e}(S)$, and hence $[L_0^* : K_0']! \not\equiv 0 \bmod p$. Let H_i be the quotient ring of R_0^* with respect to $y_i R_0^*$ for $1 \leqslant i \leqslant d$. Since $\mathfrak{d}(g_0'(Z)) \notin M(R_0^*)$, we get that $\mathfrak{d}(g_0'(Z)) \neq 0$ and hence by (10.17) we get that K_0' is separable over K_0^*; since $[L_0^* : K_0']! \not\equiv 0 \bmod p$, we conclude that L_0^* is separable over K_0^*; consequently, upon letting $f_j'(Z)$ be the minimal monic polynomial of r_j over K_0^*, we get that $\mathfrak{d}(f_j'(Z)) \in R_0^*$ and $\mathfrak{d}(f_j'(Z))$ divides $\mathfrak{d}_K^*(f_j(Z))$ in R_0^* for $1 \leqslant j \leqslant n$; since $\mathfrak{d}_K^*(f_1(Z)) \cdots \mathfrak{d}_K^*(f_n(Z)) = t' y_1^{b(1)} \cdots y_d^{b(d)}$, we conclude that $\mathfrak{d}(f_j'(Z)) = t_j y_1^{b(1,j)} \cdots y_d^{b(d,j)}$ for $1 \leqslant j \leqslant n$ where t_j is a unit in R_0^* and $b(1,j), \dots, b(d,j)$ are non-negative integers; since $L_0^* = K_0^*(L)$ and $L = K(r_1, \dots, r_n)$, we get that $L_0^* = K_0^*(r_1, \dots, r_n)$ and hence by (10.17) it follows that every one-dimensional element in $\mathfrak{V}(R_0^*) - \{H_1, \dots, H_d\}$ is unramified in L_0^*. Let $H_{i,j}, \dots, H_{i,u(i)}$ be the extensions of H_i to L_0^*, and let $w(i,j)$ be the reduced ramification index of $H_{i,j}$ over H_i. Now $\mathfrak{d}(g_0'(Z)) \notin M(H_i)$ and hence by (10.17) we get that H_i is unramified in K_0' for $1 \leqslant i \leqslant d$; since $[L_0^* : K_0']! \not\equiv 0 \bmod p$, we conclude that $H_{i,j}$ is residually separable algebraic over H_i and $w(i,j) \not\equiv 0 \bmod p$ for $1 \leqslant i \leqslant d$ and $1 \leqslant j \leqslant u(i)$. Let R_0' be the integral closure of R_0^* in K_0'; by (10.18) we know that R_0^* is unramified in K_0'; consequently R_0' is regular and hence $\mathfrak{e}(M(R_0')) = 1$; by [28: Corollary 1 on page 299] we know that $\mathfrak{e}(M(R_0')S_0^*) \leqslant \mathfrak{e}(M(R_0'))[L_0^* : K_0']$; now $\mathfrak{e}(S_0) = \mathfrak{e}(S_0^*) \leqslant \mathfrak{e}(M(R_0')S_0^*)$ and hence $\mathfrak{e}(S_0) \leqslant [L_0^* : K_0']$. Therefore if $L_0^* = K_0'$ then it suffices to take $S_1 = S_0$. So assume that $L_0^* \neq K_0'$.

Now by (10.21.2) there exist elements s_0 and s in an algebraic closure L_0^{**} of L_0^*, a prime number q with $q \not\equiv 0 \bmod p$, and nonnegative integers $a(1), ..., a(d)$, such that: $s \in L_0^*(s_0)$, $[L_0^* : K_0'] = q[L_0^*(s_0) : K_0^*(s_0, s)]$, $s^q = y_1^{a(1)} \cdots y_d^{a(d)}$, and upon letting $g(Z)$ be the minimal monic polynomial of s_0 over K_0^* we have that $g(Z) \in R_0^*[Z]$ and $\mathfrak{d}(g(Z)) \notin M(R_0^*)$. Since $[L_0^* : K_0'] \leqslant \mathfrak{e}(S)$, we get that $[L_0^*(s_0) : K_0^*(s_0, s)] \leqslant \mathfrak{e}(S)/q$. By (10.24) there exists a d-dimensional regular spot R_1 over R_0 and a basis $(z_1, ..., z_d)$ of $M(R_1)$ such that V dominates R_1 and

$$y_1^{a(1)} \cdots y_d^{a(d)} = y^*(z_1^{m(1)} \cdots z_d^{m(d)})^q (z_1^{a'(1)} \cdots z_d^{a'(d)})$$

where y^* is a unit in R_1, and $m(1), ..., m(d)$, $a'(1), ..., a'(d)$ are nonnegative integers such that $a'(1) + \cdots + a'(d) < q$. Let T_1 be the integral closure of R_1 in L and let S_1 be the quotient ring of T_1 with respect to $T_1 \cap M(V)$. By (1.1.2) we know that T_1 is a finite R_1-module, and hence S_1 is a spot over R_1. It follows that S_1 is a normal spot over k, S_1 is a spot over S_0, and S_1 dominate S_0. Let R_1^* and S_1^* be the completions of R_1 and S_1 respectively. Then R_1^* is a normal domain, and by (10.4) so is S_1^*. Let K_1^* and L_1^* be the quotient fields of R_1^* and S_1^* respectively. By (10.6) and (10.10) we know that R_0, R_1, and S_0 are subspaces of S_1, and hence we may identify K_1^* and L_0^* with subfields of L_1^*. Then R_1^* dominates R_0^*, and in view of (10.10) we get that $L_1^* = K_1^*(L_0^*)$ and S_1^* is the integral closure of R_1^* in L_1^*. Let L_1^{**} be an algebraic closure of L_1^*. We can then identify L_0^{**} with a subfield of L_1^{**}. Now $[L_1^*(s_0) : K_1^*(s_0, s)] \leqslant [L_0^*(s_0) : K_0^*(s_0, s)]$ and hence $[L_1^*(s_0) : K_1^*(s_0, s)] \leqslant \mathfrak{e}(S)/q$. Let E, F, and G be the integral closures of R_1^* in $K_1^*(s_0)$, $K_1^*(s_0, s)$, and $L_1^*(s_0)$ respectively. Since $\mathfrak{d}(g(Z)) \notin M(R_0^*)$ and R_1^* dominates R_0^*, we get that $\mathfrak{d}(g(Z)) \notin M(R_1^*)$; consequently by (10.17) we have that R_1^* is unramified in $K_1^*(s_0)$; therefore E is a d-dimensional regular local domain and $M(E) = (z_1, ..., z_d)E$. Let $x = s(z_1^{m(1)} \cdots z_d^{m(d)})^{-1}$. Then $x^q \in E$ and $\operatorname{ord}_E x^q < q$. Also $K_1^*(x) = K_1^*(s)$ and hence $[L_1^*(s_0) : K_1^*(s_0, x)] \leqslant \mathfrak{e}(S)/q$. Since $\mathfrak{d}(g(Z)) \notin M(R_0^*)$, we get that $\mathfrak{d}(g(Z)) \notin M(S_1^*)$; consequently by (10.17) we know that S_1^* is unramified in $L_1^*(s_0)$; hence $M(S_1^*)G = M(G)$ and in view of (10.18) we get that $[L_1^*(s_0) : L_1^*] = [h^*(G) : h^*(S_1^*)]$ where $h^* : G \to G/M(G)$ is the canonical epimorphism; by [28: Corollary 1 on page 299] we know that $\mathfrak{e}(M(S_1^*))[L_1^*(s_0) : L_1^*] = \mathfrak{e}(M(S_1^*)G)[h^*(G) : h^*(S_1^*)]$; therefore $\mathfrak{e}(S_1^*) = \mathfrak{e}(G)$ and hence $\mathfrak{e}(S_1) = \mathfrak{e}(G)$. By [28: Corollary 1 on page 299] we also have that $\mathfrak{e}(G) \leqslant \mathfrak{e}(F)[L_1^*(s_0) : K_1^*(s_0, x)]$; since $[L_1^*(s_0) : K_1^*(s_0, x)] \leqslant \mathfrak{e}(S)/q$, we get that $\mathfrak{e}(G) \leqslant (\mathfrak{e}(F)/q)\mathfrak{e}(S)$. Thus $\mathfrak{e}(S_1) \leqslant (\mathfrak{e}(F)/q)\mathfrak{e}(S)$; we shall show that $\mathfrak{e}(F) < q$ and this will complete the proof.

For a moment suppose that x^q is a unit in E; then $\mathfrak{d}(Z^q - x^q) \notin M(E)$ and hence by (10.17) we get that E is unramified in $K_1^*(s_0, x)$; consequently F is regular and hence $\mathfrak{e}(F) = 1$; since q is a prime number, we thus get that $\mathfrak{e}(F) < q$. So now assume that x^q is a nonunit in E. Then $0 < \operatorname{ord}_E x^q < q$. Since $\mathfrak{e}(M(E)) = 1$, by [28: Corollary 1 on page 299] we get that $\mathfrak{e}(F) \leqslant [K_1^*(s_0, x) : K_1^*(s_0)]$. Hence if $[K_1^*(s_0, x) : K_1^*(s_0)] < q$ then we have nothing more to show. So also assume that $[K_1^*(s_0, x) : K_1^*(s_0)] \geqslant q$. Then $Z^q - x^q$ is the minimal monic polynomial of x over $K_1^*(s_0)$. Let $F' = E[x]$. Then F' is a local domain with quotient field $K_1^*(s_0, x)$, and F is a finite F'-module; hence once again by [28: Corollary 1 on page 299] we get that $\mathfrak{e}(F) \leqslant \mathfrak{e}(F')$. Therefore it suffices to show that $\mathfrak{e}(F') < q$. In view of Cohen's structure theorem [28: Corollary on page 307] we may identify E with the formal power series ring $k'[[Z_1, ..., Z_d]]$ in indeterminates $Z_1, ..., Z_d$ over a field k'. Let $E' = k'[[Z_1, ..., Z_d, Z]]$ and $E^* = k'[[Z_1, ..., Z_d]][Z]$. Since $Z^q - x^q$ is the minimal monic polynomial of x over $K_1^*(s_0)$, we get that F' is isomorphic to $E^*/(Z^q - x^q)E^*$; by Weirstrass Preparation Theorem [28: Theorem 5 on page 139] we also know that $E^*/(Z^q - x^q)E^*$ is isomorphic to $E'/(Z^q - x^q)E'$; now E' is a regular local domain and hence by [18: (40.2)] we get that $\mathfrak{e}(E'/(Z^q - x^q)E') = \operatorname{ord}_{E'}(Z^q - x^q)$; since $\operatorname{ord}_E x^q < q$, we also get that $\operatorname{ord}_{E'}(Z^q - x^q) = \operatorname{ord}_E x^q < q$. Therefore $\mathfrak{e}(F') < q$.

§11. Three-dimensional birational resolution over a ground field of characteristic zero

By (5.2.3), (9.1.7), and (10.25) we get the following.

(11.1). *Let k be an infinite perfect field of characteristic p and let K be a function field over k with* $\operatorname{trdeg}_k K \leqslant 3$. *Assume that given any valuation ring V of K with $k \subset V$ such that V is residually algebraic over k, there exists a spot S over k with quotient field K*

such that V dominates S and $\mathfrak{e}(S)! \not\equiv 0 \bmod p$. *Then there exists a nonsingular projective model of K/k.*

By (11.1) we get the following.

(11.2). *Let k be a field of characteristic zero and let K be a function field over k with* $\mathrm{trdeg}_k K \leqslant 3$. *Then there exists a nonsingular projective model of K/k.*

Note that, in view of the alternative proof of (5.1) for the case when S_0 is of zero characteristic given in §5, the proof of (11.2) has been made independent of the papers [5], [7], [8], and [9].

§12. Existence of projective models having only points of small multiplicity

In accordance with the usual notation for field extensions, the dimension of a vector space L over a field K is denoted by $[L : K]$.

(12.1). *Homogeneous domains.* By a *homogeneous domain* A we mean a domain A together with a family $(A_n)_{0 \leqslant n < \infty}$ of additive subgroups of A such that: the underlying additive group of A is the direct sum of the family $(A_n)_{0 \leqslant n < \infty}$; $\{xy: x \in A_m, y \in A_n\} \subset A_{m+n}$ for all m and n; A_0 is a field; $A = A_0[A_1]$; and $0 < [A_1 : A_0] < \infty$. Note that then A is noetherian and $0 < [A_n : A_0] < \infty$ for all n. We *define*: $\mathfrak{r}(A) = [A_1 : A_0]$.

An ideal Q in A is said to be *homogeneous* if the following two equivalent conditions are satisfied: (1) $\sum_{0 \leqslant n < \infty} r_n \in Q$; with $r_n \in A_n$ for all n (where $r_n = 0$ for all sufficiently large n) $\Rightarrow$ $r_n \in Q$ for all n; (2) $Q = IA$ for some $I \subset \bigcup_{0 \leqslant n < \infty} A_n$. Note that $A_1 A$ is the only homogeneous maximal ideal in A and it contains every nonunit homogeneous ideal in A. Also note that for any homogeneous ideal Q in A with $Q \neq A$ we have that: $\mathrm{rad}\, Q = A_1 A \Leftrightarrow A_n \subset Q$ for some $n > 0 \Leftrightarrow A_n \subset Q$ for all sufficiently large n. Recall that by a *minimal prime ideal* of any ideal Q in any ring E we mean a prime ideal P in E with $Q \subset P$ such that there does not exist any prime ideal P' in E for which $Q \subset P' \subset P$ and $P' \neq P$. Note that by [28: Corollary on page 154] we know that if Q is any homogeneous ideal in A then all the associated prime ideals of Q in A are homogeneous; in particular, all the minimal prime ideals of Q in A are homogeneous and hence $\operatorname{rad} Q$ is homogeneous. For any homogeneous ideals I and Q in A we *define:* $\mathfrak{u}(I, Q) =$ the number of minimal prime ideals P of Q in A such that $P \neq A_1 A$ and $I \not\subset P$; note that then $\mathfrak{u}(A, Q) = 0 \Leftrightarrow \operatorname{rad} Q = A$ or $A_1 A$.

For any nonmaximal homogeneous prime ideal P in A we *set:* $(A/P)_n = f(A_n)$ for all n where $f: A \to A/P$ is the canonical epimorphism; note that then A/P becomes a homogeneous domain. By a ***homogeneous subdomain*** of A we mean a homogeneous domain B such that B is a subring of A, $B_0 = A_0$, and $B_n = B \cap A_n$ for all $n > 0$. For any subring B of A of the form $B = A_0[L]$ for some nonzero A_0-subspace L of A_1 we *set:* $B_n = B \cap A_n$ for all n; note that then B becomes a homogeneous subdomain of A, and this gives a one-to-one correspondence between homogeneous subdomains of A and nonzero A_0-subspaces of A_1. For any positive integer d we *define:* $A_n^{(d)} = A_{dn}$ for all n, and $A^{(d)} =$ the direct sum of $(A_n^{(d)})_{0 \leqslant n < \infty}$; note that then $A^{(d)}$ is a homogeneous domain and it is a subring of A (but not a homogeneous subdomain of A unless $d = 1$, and then $A^{(1)} = A$).

We *define*: $\mathfrak{K}(A) = \bigcup_{0 \leqslant n < \infty} \{x/y\colon\ x \in A_n,\ 0 \neq y \in A_n\}$. Note that then: $\mathfrak{K}(A)$ is a subfield of the quotient field of A; $\mathfrak{K}(A) = A_0(\{x/y\colon x \in A_n\})$ for all $n > 0$ and $0 \neq y \in A_n$; $\mathfrak{K}(A)$ is a function field over A_0; and $\mathfrak{K}(A) = \mathfrak{K}(A^{(d)})$ for all $d > 0$. We *define:* $\mathfrak{t}(A) =$ the transcendence degree of $\mathfrak{K}(A)$ over A_0. Note that then $\mathfrak{t}(A) + 1 =$ the transcendence degree of the quotient field of A over A_0. Also note that for any homogeneous subdomain B of A we have that $\mathfrak{K}(A)$ is a function field over $\mathfrak{K}(B)$ and $0 \leqslant \mathfrak{t}(A) - \mathfrak{t}(B) \leqslant \mathfrak{r}(A) - \mathfrak{r}(B)$. We *define:* $\mathfrak{W}(A) = \mathfrak{W}(A_0, A_1)$. Note that then $\mathfrak{W}(A)$ is a projective model of $\mathfrak{K}(A)/A_0$, and $\mathfrak{W}(A) = \mathfrak{W}(A_0, A_n) = \mathfrak{W}(A^{(d)})$ for all $n > 0$ and $d > 0$. For any nonmaximal homogeneous prime ideal P in A we *define*: $\mathfrak{R}(A, P) = \bigcup_{0 \leqslant n < \infty} \{x/y\colon x \in A_n,\ y \in A_n - P\}$. Note that then $\mathfrak{R}(A, \{0\}) = \mathfrak{K}(A)$. Concerning these notions we have the following.

(12.1.1). *Let P be any nonmaximal homogeneous prime ideal in A. Then $\mathfrak{K}(A, P)$ is a quasilocal domain with quotient field $\mathfrak{K}(A)$, and $M(\mathfrak{R}(A, P)) = \bigcup_{0 \leqslant n < \infty} \{x/y\colon x \in A_n \cap P, y \in A_n - P\}$.*

(12.1.2). *Given any nonmaximal homogeneous prime ideal P in A and $z \in A_1 - P$, let $P' = \bigcup_{0 \leqslant n < \infty} \{x/z^n \colon x \in A_n \cap P\}$. Then P' is a prime ideal in $A_0[A_1 z^{-1}]$, $\mathfrak{R}(A, P) = (A_0[A_1 z^{-1}])_{P'}$, and $A_n \cap P = \{x \in A_n \colon x/z^n \in M(\mathfrak{R}(A, P))\}$ for all n. If I is any subset of $\bigcup_{0 \leqslant n < \infty} A_n$ such that $P = IA$ then $\bigcup_{0 \leqslant n < \infty} \{x/z^n \colon x \in A_n \cap I\}$ is a basis of P'.*

(12.1.3). *Given any $0 \neq z \in A_1$ and $R \in \mathfrak{V}(A_0[A_1 z^{-1}])$, let $P = \{\sum_{0 \leqslant n < \infty} r_n \colon r_n \in A_n$ and $r_n/z^n \in M(R)$ for all n, and $r_n = 0$ for all sufficiently large $n\}$. Then P is a nonmaximal homogeneous prime ideal in A, $z \notin P$, and $\mathfrak{R}(A, P) = R$.*

(12.1.4). *$P \to \mathfrak{R}(A, P)$ is a one-to-one inclusion-reversing map of the set of all nonmaximal homogeneous prime ideals in A onto $\mathfrak{W}(A)$.*

The proofs of (12.1.1), (12.1.2), and (12.1.3) are straightforward. (12.1.4) follows from (12.1.2) and (12.1.3).

(12.1.5). *For any A_0-subspace L of A_1 with $[L : A_0] = \mathfrak{r}(A) - 1$ and any $0 \neq z \in A_1 - L$, the following three conditions are equivalent:* (1) rad$(LA) = A_1 A$; (2) *there exists a positive integer q and $r_n \in A_n \cap (A_0[L])$ for $1 \leqslant n \leqslant q$ such that $z^q + r_1 z^{q-1} + \cdots + r_q = 0$;* (3) *there exists a positive integer q and $s_n \in A_0$ for $1 \leqslant n \leqslant q$ such that $z^q + s_1 z^{q-1} + \cdots + s^q \in LA$.*

PROOF. First suppose that (1) holds; then there exists a positive integer q such that $z^q \in LA$; since $z^q \in LA$, we can write

$$z^q = \sum_{i=1}^{d} t_i \left(\sum_{j=0}^{e} y_{i,j} \right) \qquad \text{with} \quad t_i \in L \quad \text{and} \quad y_{i,j} \in A_j\,,$$

where d and e are integers with $d \geqslant 1$ and $e \geqslant q - 1$; since $z^q \in A_q$ and A is the direct sum of $(A_n)_{0 \leqslant n < \infty}$, we get that

$$z^q = \sum_{i=1}^{d} t_i y_i \qquad \text{where} \quad y_i = y_{i,q-1} \in A_{q-1}\,;$$

since $[L : A_0] = \mathfrak{r}(A) - 1$, we get that

$$y_i = \sum_{n=1}^{q} x_{i,n} z^{q-n} \qquad \text{with} \quad x_{i,n} \in A_{n-1} \cap (A_0[\mathrm{L}]) ;$$

now $z^q + r_1 z^{q-1} + \cdots + r_q = 0$ where

$$r_n = - \sum_{i=1}^{d} t_i x_{i,n} \in A_n \cap (A_0[L]) \qquad \text{for} \quad 1 \leqslant n \leqslant q ;$$

thus (1) implies (2). Next suppose that (2) holds; since $r_n \in A_0[L]$, there exists $s_n \in A_0$ such that $r_n - s_n \in LA$ for $1 \leqslant n \leqslant q$, and then $z^q + s_1 z^{q-1} + \cdots + s_q \in LA$; thus (2) implies (3). Finally suppose that (3) holds; now $z^q \in A_q$ and $s_n z^{q-n} \in A_{q-n}$ for $1 \leqslant n \leqslant q$; since LA is a homogeneous ideal in A, we must have $z^q \in LA$; since $[L : A_0] = \mathfrak{r}(A) - 1$, we conclude that $\mathrm{rad}(LA) = A_1 A$; thus (3) implies (1).

(12.1.6). *Let W_1 be the set of all A_0-subspaces L of A_1 such that $[L : A_0] = \mathfrak{r}(A) - 1$. Let W be the set of all $L \in W_1$ such that LA is a nonmaximal homogeneous prime ideal in A. Let W' be the set of all $L \in W$ such that LA is not contained in any homogeneous prime ideal in A other than LA and A_1A. Let W'' be the set of all $L \in W_1$ such that* $\mathrm{rad}(LA) \neq A_1A$. *Let W^* be the set of all $R \in \mathfrak{W}(A)$ such that R is residually rational over A_0. Then $W' = W = W''$, and $L \to \mathfrak{R}(A, LA)$ is a one-to-one map of W onto W^*.*

PROOF. Clearly $W' \subset W \subset W''$. Conversely let any $L \in W''$ be given; since $L \neq A_1$, we can take $0 \neq z \in A_1 - L$; let $h: A \to A/(LA)$ be the canonical epimorphism; since $[L : A_0] = \mathfrak{r}(A) - 1$, we get that $h(A) = h(A_0)[h(z)]$; also $A_0 \cap (LA) = \{0\}$ and hence $h(A_0)$ is isomorphic to the field A_0; by (12.1.5) we know that if $s_0 z^q + s_1 z^{q-1} + \cdots + s_q \in LA$ for some positive integer q and some elements $s_0, \ldots, s_q$ in A_0 then necessarily $s_0 = 0$; consequently $h(A)$ is isomorphic to the ring of polynomials in one indeterminate with coefficients in the field $h(A_0)$; it follows that $L \in W'$. Thus $W' = W = W''$. By (12.1.2) we get that $\mathfrak{R}(A, LA) \in W^*$ for all $L \in W$. If L_1 and L_2 are any two elements in W such that $\mathfrak{R}(A, L_1A) = \mathfrak{R}(A, L_2A)$ then by (12.1.4) we would get that $L_1A = L_2A$ and hence $L_1 = L_2$. Finally, let any $R \in W^*$ be given; we can then take a free A_0-basis $(x_1, \ldots, x_v, z)$ of A_1, where $v = \mathfrak{r}(A) - 1$, such that $R \in \mathfrak{V}(A_0[A_1 z^{-1}])$; clearly $A_0[A_1 z^{-1}] = A_0[x_1/z, \ldots, x_v/z]$; since R is residually rational over A_0, there exists $t_i \in A_0$ such that $(x_i/z) - t_i \in M(R)$ for $1 \leqslant i \leqslant v$; let $y_i = x_i - zt_i$ and let L be the A_0-subspace of A_1 generated by $y_1, \ldots, y_v$; then $[L : A_0] = \mathfrak{r}(A) - 1$, $A_0[A_1 z^{-1}] = A_0[y_1/z, \ldots, y_v/z]$, and $y_i/z \in M(R)$ for $1 \leqslant i \leqslant v$; let $P = \{ \sum_{0 \leqslant n < \infty} r_n \colon r_n \in A_n$ and $r_n/z^n \in M(R)$ for all n, and $r_n = 0$ for all sufficiently large $n\}$; then by (12.1.3) we know that P is a nonmaximal homogeneous prime ideal in A and $\mathfrak{R}(A, P) = P$; clearly $LA \subset P$ and hence $L \in W''$; since $W' = W = W''$, we conclude that $LA = P$ and $L \in W$.

(12.1.7). *Given any positive integer d and any homogeneous ideal Q in $A^{(d)}$, let $W' = \{R \in \mathfrak{W}(A^{(d)})$: $R = \mathfrak{R}(A^{(d)}, P')$ for some nonmaximal homogeneous prime ideal P' in $A^{(d)}$ with $Q \subset P'\}$, and let $W^* = \{R \in \mathfrak{W}(A)$: $R = \mathfrak{R}(A, P^*)$ for some nonmaximal homogeneous prime ideal P^* in A with $QA \subset P^*\}$. Then $W' = W^*$, and $\mathfrak{u}(A^{(d)}, Q) = \mathfrak{u}(A, QA)$.*

PROOF. Given any $R \in \mathfrak{W}(A^{(d)}) = \mathfrak{W}(A)$, let P' and P^* be the unique (see (12.1.4)) nonmaximal homogeneous prime ideals in $A^{(d)}$ and A respectively such that $\mathfrak{R}(A^{(d)}, P') = R = \mathfrak{R}(A, P^*)$; we can take $0 \neq z \in A_1$ such that $R \in \mathfrak{V}(A_0[A_1 z^{-1}])$; then $0 \neq z^d \in A_1^{(d)}$ and $A_0[A_1 z^{-1}] = A_0^{(d)}[A_1^{(d)}(z^d)^{-1}]$; consequently by (12.1.3) we get that $P' \cap A_n^{(d)} = \{r_n \in A_n^{(d)} \colon r_n/z^{dn} \in M(R)\}$ and $P^* \cap A_n = \{s_n \in A_n : s_n/z^n \in M(R)\}$ for all n; it follows that $Q \subset P' \Leftrightarrow QA \subset P^*$. This shows that $W' = W^*$. Let $W_1' = \{R \in W' \colon R \not\subset S$ for all $S \in W'$ with $S \neq R\}$ and $W_1^* = \{R \in W^*$: $R \not\subset S$ for all $S \in W^*$ with $S \neq R\}$. By (12.1.4) we get that $\mathfrak{u}(A^{(d)}, Q) =$ the number of elements in W_1', and $\mathfrak{u}(A, QA) =$ the number of elements in W_1^*. Since $W' = W^*$, we get that $W_1' = W_1^*$ and hence $\mathfrak{u}(A^{(d)}, Q) = \mathfrak{u}(A, QA)$.

(12.1.8). *Let E be any noetherian domain, let D be any affine ring over R, and let I be any nonzero ideal in D. Then there exists a nonzero ideal J in E such that for every ideal Q in E, upon letting v be the number of minimal prime ideals P of Q in E with $J \not\subset P$ and u be the number of minimal prime ideals P' of QD in D with $I \not\subset P'$, we have that $u \geqslant v$.*

PROOF. By a straightforward induction, the general case follows from the case when $D = E[x]$ for some $x \in D$. So assume that $D = E[x]$ for some $x \in D$.

First suppose that x is transcendental over the quotient field of D. Since I is nonzero, there exists a nonnegative integer a and elements $r_0, \dots, r_a$ in E with $r_a \neq 0$ such that $r_0 + r_1x + \cdots + r_ax^a \in I$. Take $J = r_aE$. Given any ideal Q in E, let $P_1, \dots, P_v$ $(v \geqslant 0)$ be those minimal prime ideals of Q in E which do not contain J. Let $P_i' = P_iD$ for $1 \leqslant i \leqslant v$. Let $h_i: E \to E/P_i$ be the canonical epimorphism; since x is transcendental over the quotient field of E, there exists a unique epimorphism $h_i' : D \to (E/P_i)[Z]$, where Z is an indeterminate, such that $h_i'(y) = h_i(y)$ for all $y \in E$ and $h_i'(x) = Z$; now $(E/P_i)[Z]$ is a domain and clearly $h_i'^{-1}(0) = P_i'$; therefore P_i' is a prime ideal in D and $E \cap P_i' = P_i$; now $h_i'(r_0 + r_1x + \cdots + r_ax^a) = h_i(r_0) + h_i(r_1)Z + \cdots + h_i(r_a)Z^a$, and, since $r_aE = J \not\subset P_i$, we get that $h_i(r_a) \neq 0$; consequently $h_i'(r_0 + r_1x + \cdots + r_ax^a) \neq 0$ and hence $r_0 + r_1x + \cdots + r_ax^a \notin P_i'$; therefore $I \not\subset P_i'$. If P^* is any prime ideal in D with $QD \subset P^* \subset P_i'$ then $E \cap P^*$ is a prime ideal in E and $Q \subset (E \cap (QD)) \subset (E \cap P^*) \subset (E \cap P_i') = P_i$, and hence $E \cap P^* = P_i$; consequently $P_i' \subset P^*$ and hence $P^* = P_i'$. Thus $P_1', \dots, P_v'$ are minimal prime ideals of QD in D, and $I \not\subset P_i'$ for $1 \leqslant i \leqslant v$; since $E \cap P_i' = P_i$, we also get that $P_1', \dots, P_v'$ are distinct.

Next suppose that x is algebraic over the quotient field of E. Then there exists a positive integer q and elements $z, z_1, \dots, z_q$ in E with $z \neq 0$ such that $zx^q + z_1x^{q-1} + \cdots + z_q = 0$. Let $S = E[1/z]$ and $R = D[1/z]$. Then R is integral over S and IR is a nonzero ideal in R; consequently $S \cap (IR) \neq 0$ (see [4: Lemma 1.23]) and hence there exists $0 \neq s \in S \cap (IR)$; we can take a nonnegative integer b such that upon letting $t = z^bs$ we have that $0 \neq t \in E \cap I$ (alternatively, take $0 \neq y \in I$; since x is algebraic over the quotient field L of E, we have that y is algebraic over L; let $t_0Z^e + t_1Z^{e-1} + \cdots + t_e$ be the minimal monic polynomial of y over L where Z is an indeterminate and $t_0 = 1$; since $y \neq 0$, we must have $t_e \neq 0$; since L is the quotient field of E, there exists $0 \neq t' \in E$ such that $t't_j \in E$ for $0 \leqslant j \leqslant e$; upon letting $t = t't_e$ we now get that $0 \neq t = -(t't_0y^{e-1} + \cdots + t't_{e-1})y \in E \cap I$). Take $J = (zt)E$. Given any ideal Q in E, let $P_1, \dots, P_v$ $(v \geqslant 0)$ be those minimal prime ideals of Q in E which do not contain J. Note that then $t \notin P_i$ and $z \notin P_i$ for $1 \leqslant i \leqslant v$. Let $H = \{z^j\colon 0 \leqslant j < \infty\}$. Then S is the quotient ring of E with respect to H, and $H \cap P_i = \varnothing$ for $1 \leqslant i \leqslant v$; consequently $P_1S, \ldots, P_vS$ are distinct minimal prime ideals of QS in S; also $E \cap (P_iS) = P_i$, $t \in E$, $t \notin P_i$, and $t \in E \cap I \subset S \cap (IR)$; consequently $S \cap (IR) \not\subset P_iS$ for $1 \leqslant i \leqslant v$. Since R is integral over S, by [4: Lemma 1.20] there exist prime ideals $P_1'', \ldots, P_v''$ in R such that $S \cap P_i'' = P_iS$ for $1 \leqslant i \leqslant v$. Since $P_1S, \ldots, P_vS$ are distinct and $S \cap (IR) \not\subset P_iS$ for $1 \leqslant i \leqslant v$, we get that $P_1'', \ldots, P_v''$ are distinct and $I \not\subset P_i''$ for $1 \leqslant i \leqslant v$. If P^* is any prime ideal in R with $QR \subset P^* \subset P_i''$, then $S \cap P^*$ is a prime ideal in S and $QS \subset (S \cap (QR)) \subset (S \cap P^*) \subset (S \cap P_i'') = P_iS$, and hence $S \cap P^* = S \cap P_i''$; since R is integral over S, by [4: Lemma 1.24] we then get that $P^* = P_i''$. Thus $P_1'', \ldots, P_v''$ are distinct minimal prime ideals of QR in R and $I \not\subset P_i''$ for $1 \leqslant i \leqslant v$; let $P_i' = D \cap P_i''$ for $1 \leqslant i \leqslant v$; now R is the quotient ring of D with respect to H, and hence we conclude that $P_1', \ldots, P_v'$ are distinct minimal prime ideals of QD in D, and $I \not\subset P_i'$ for $1 \leqslant i \leqslant v$.

(12.1.9). *Let I be any nonzero homogeneous ideal in A and let B be any homogeneous subdomain of A. Then there exists a nonzero homogeneous ideal J in B such that for every homogeneous ideal Q in B we have that* $\mathrm{u}(I, QA) \geqslant \mathrm{u}(J, Q)$.

PROOF. Take $0 \neq x \in B_1$. Let $D = A_0[A_1z^{-1}]$ and $E = B_0[B_1z^{-1}]$. For every homogeneous ideal Q in A let $f(Q) = \bigcup_{0 \leqslant n < \infty} \{x/z^n\colon x \in A_n \cap Q\}$; then $f(Q)$ is an ideal in D. For every ideal Q' in D let $f'(Q') = \{\sum_{0 \leqslant n < \infty} r_n\colon r_n \in A_n$ and $r_n/z^n \in Q'$ for all n, and $r_n = 0$ for all sufficiently large $n\}$; then $f'(Q')$ is a homogeneous ideal in A. Concerning the maps f and f' we note the following: if Q is any homogeneous ideal in A then: $Q = \{0\} \Leftrightarrow f(Q) = \{0\}$; if $Q^* \subset Q$ are any homogeneous ideals in A then $f(Q^*) \subset f(Q)$; if Q is any homogeneous ideal in A then $Q \subset f'(f(Q))$; if Q is any homogeneous prime ideal in A with $z \notin Q$ then $f(Q)$ is a prime ideal in D and $Q = f'(f(Q))$; if Q' is any ideal in D then $Q' = f(f'(Q'))$; if $Q'^* \subset Q'$ are any ideals in D then $f'(Q'^*) \subset f'(Q')$; if Q' is any prime ideal in D then $f'(Q')$ is a prime ideal in A and $z \notin f'(Q')$. Let g and g' be the maps which correspond to f and f'

when in the above definitions we replace A and D by B and E respectively; then g and g' enjoy the corresponding properties; also note that for every homogeneous ideal Q in B we have that $g(Q)D = f(QA)$.

Now $f(I)$ is a nonzero ideal in D, and hence by (12.1.8) there exists a nonzero ideal J' in E such that for every ideal Q' in E, upon letting v be the number of minimal prime ideals of Q' in E which do not contain J' and u be the number of minimal prime ideals of $Q'D$ in D which do not contain $f(I)$, we have that $u \geqslant v$. Let $J^* = g'(J')$. Then J^* is a nonzero homogeneous ideal in B. Take $J = zJ^*$. Then J is a nonzero homogeneous ideal in B. Given any homogeneous ideal Q in B, let $v = \mathfrak{u}(J, Q)$ and let $Q_1, ..., Q_v$ be the minimal prime ideals of Q in B such that $Q_i \neq B_1B$ and $J \not\subset Q_i$ for $1 \leqslant i \leqslant v$. Since $J \not\subset Q_i$ we get that $z \notin Q_i$ for $1 \leqslant i \leqslant v$. It follows that $g(Q_1), ..., g(Q_v)$ are distinct minimal prime ideals of $g(Q)$ in E, and $J' \not\subset g(Q_i)$ for $1 \leqslant i \leqslant v$. Therefore there exist distinct minimal prime ideals $P_1', ..., P_v'$ of $g(Q)D$ in D such that $f(I) \not\subset P_i'$ for $1 \leqslant i \leqslant v$. Let $P_i = f'(P_i')$ for $1 \leqslant i \leqslant v$. Then $P_1, ..., P_v$ are distinct minimal prime ideals of QA in A, and $z \notin P_i$ and $I \not\subset P_i$ for $1 \leqslant i \leqslant v$. Since $z \notin P_i$ we get that $P_i \neq A_1A$ for $1 \leqslant i \leqslant v$. It follows that $\mathfrak{u}(I, QA) \geqslant v$, i.e., $\mathfrak{u}(I, QA) \geqslant \mathfrak{u}(J, Q)$.

Finally we note the following.

(12.1.10). *If* $\mathfrak{r}(A) = 1$ *then* $\mathfrak{t}(A) = 0$. *Conversely, if* $\mathfrak{t}(A) = 0$ *and* A_0 *is algebraically closed then* $\mathfrak{r}(A) = 1$.

PROOF. Obviously, if $\mathfrak{r}(A) = 1$ then $\mathfrak{t}(A) = 0$. Now assume that $\mathfrak{t}(A) = 0$ and A_0 is algebraically closed. Take $0 \neq y \in A_1$. Then $A_0 = \mathfrak{K}(A) = A_0(\{x/y\colon\ x \in A_1\})$ and hence $x/y \in A_0$ for all $x \in A_0$. Therefore $[A_1 : A_0] = 1$, i.e., $\mathfrak{r}(A) = 1$.

(12.2). *Hilbert polynomial of a local ring.* By [28: §8, §9, and §10 of Chapter VIII] we know that: given any local ring R, there exists a unique polynomial, to be denoted by $\mathfrak{H}(R, Z)$, in an indeterminate Z with rational coefficients such that

$$\mathfrak{H}(R, n) = \sum_{j=0}^{n-1} [M(R)^j/M(R)^{j+1} : R/M(R)]$$

for all sufficiently large n; moreover, upon letting $t = \dim R$, we have that $t =$ degree of $\mathfrak{H}(R, Z)$ in Z, and $\mathfrak{e}(R)/(t!) =$ coefficient of Z^t in $\mathfrak{H}(R, Z)$; note that if R contains a subfield k such that R is residually rational over k then clearly

$$\sum_{j=0}^{n-1} [M(R)^j/M(R)^{j+1} : R/M(R)] = \sum_{j=0}^{n-1} [M(R)^j/M(R)^{j+1} : k]$$
$$= [R/M(R)^n : k]$$

for all n, and hence $\mathfrak{H}(R, n) = [R/M(R)^n : k]$ for all sufficiently large n.

(12.2.1). *Let A be a homogeneous domain, and let B be a homogeneous subdomain of A such that* $\mathfrak{r}(B) = \mathfrak{r}(A) - 1$ *and* $\operatorname{rad}(B_1A) \neq A_1A$. *Let $0 \neq z \in A_1 - B_1$, and for each nonnegative integer j let $B'_j = \{x \in B_j$: there exists a nonnegative integer w and $x_{j+i} \in B_{j+i}$ for $1 \leqslant i \leqslant w$ such that $xz^w + x_{j+1}z^{w-1} + \cdots + x_{j+w} = 0\}$. Let* $R = \mathfrak{R}(A, B_1A)$ (note that by (12.1.6) we know that B_1A is a nonmaximal homogeneous prime ideal in A). *Then R is a local domain,* $\dim R = \mathfrak{t}(A)$, *R is residually rational over A_0, and* $[B_j/B'_j : A_0] = [M(R)^j/M(R)^{j+1} : A_0]$ *for all j.*

Proof. By (12.1.4) and (12.1.6) we know that R is a local domain with quotient field $\mathfrak{K}(A)$, R is a spot over A_0, and R is residually rational over A_0; in view of [28: §7 of Chapter VII] we also get that $\dim R = \mathfrak{t}(A)$. Let $D = A_0[B_1z^{-1}]$ and $P = (\{x/z\colon x \in B_1\})D$. By (12.1.2) we get that P is a prime ideal in D and $R = D_P$. It follows that $M(R)^j = (\{x/z^j\colon x \in B_j\})R$ for all j. Given any nonnegative integer j, we get an A_0-homomorphism $f\colon B_j \to M(R)^j$ by taking $f(x) = x/z^j$ for all $x \in B_j$. Given any $y \in M(R)^j$ we can write $y = r_1f(y_1) + \cdots + r_qf(y_q)$ with $r_1, \ldots, r_q$ in R and $y_1, \ldots, y_q$ in B_j; since R is residually rational over A_0, there exists $r'_i \in A_0$ such that $r_i - r'_i \in M(R)$ for $1 \leqslant i \leqslant q$, and then $r'_1y_1 + \cdots + r'_qy_q \in B_j$ and $y - f(r'_1y_1 + \cdots + r'_qy_q) \in M(R)^{j+1}$. Thus, upon letting $h\colon M(R)^j \to M(R)^j/M(R)^{j+1}$ be the canonical epimorphism, we get that $h(M(R)^j) = h(f(B_j))$. Let any $x \in B'_j$ be given; then there exists a nonnegative integer w and $x_{j+i} \in B_{j+i}$ for $1 \leqslant i \leqslant w$ such $xz^w + x_{j+1}z^{w-1} + \cdots + x_{j+w} = 0$; now $f(x) =$

$x/z^j = -((x_{j+1}/z^{j+1}) + \cdots + (x_{j+w}/z^{j+w})) \in M(R)^{j+1}$. Conversely, let any $x \in B_j$ be given such that $f(x) \in M(R)^{j+1}$; then $x/z^j = (s_1t_1 + \cdots + s_dt_d)/z^{j+1}$ with $s_1, \ldots, s_d$ in R and $t_1, \ldots, t_d$ in B_{j+1}; since $R = D_P$, we can write $s_a = s'_a/s'_0$ for $1 \leqslant a \leqslant d$ where $s'_0, \ldots, s'_a$ are elements in D and $s'_0 \notin P$; since $D = A_0[B_1z^{-1}]$, we can write

$$s'_a = \sum_{u=0}^{w-1} s'_{a,u}/z^u \qquad \text{for} \quad 0 \leqslant a \leqslant d, \quad \text{with} \quad s'_{a,u} \in B_u,$$

where w is a positive integer; since $s'_0 \notin P$, we must have $s'_{0,0} \neq 0$; now

$$xzs'_0 = \sum_{a=1}^{d} s'_a t_a$$

and hence, upon letting $s'_{0,w} = 0$, we get that

$$xz^w + \sum_{i=1}^{w} x_{j+i}z^{w-i} = 0$$

where

$$x_{j+i} = x(s'_{0,i}/s'_{0,0}) - \sum_{a=1}^{d} (s'_{a,i-1}/s'_{0,0})t_a \in B_{j+i}$$

for $1 \leqslant i \leqslant w$; consequently $x \in B'_j$. Thus $f^{-1}(M(R)^{j+1}) = B'_j$, and hence $[B_j/B'_j : A_0] = [M(R)^j/M(R)^{j+1} : A_0]$.

(12.3). *Hilbert polynomial of a homogeneous domain.* Let A be a homogeneous domain. By [28: §12, Chapter VII] we know that: given any homogeneous ideal Q in A, there exists a unique polynomial, to be denoted by $\mathfrak{h}(A, Q, Z)$, in an indeterminate Z with rational coefficients such that $\mathfrak{h}(A, Q, n) = [A_n/(A_n \cap Q) : A_0]$ for all sufficiently large n; we *define*: $\mathfrak{t}(A, Q)$ = degree of $\mathfrak{h}(A, Q, Z)$ in Z in case $\mathfrak{h}(A, Q, Z)$ is a nonzero polynomial, and $\mathfrak{t}(A, Q) = -1$ in case $\mathfrak{h}(A, Q, Z)$ is the zero polynomial; upon letting $t = \mathfrak{t}(A, Q)$ we *define:* $\mathfrak{g}(A, Q) = t!$ times the coefficient of Z^t in $\mathfrak{h}(A, Q, Z)$ in case $t \geqslant 0$, and $\mathfrak{g}(A, Q) = 0$ in case $t = -1$. For any homogeneous ideal Q in A, by [28: §12 of Chapter VII] we have that: $\mathfrak{t}(A, Q) = -1 \Leftrightarrow \operatorname{rad} Q = A$ or A_1A; and if $\mathfrak{t}(A, Q) \neq -1$ then $\mathfrak{t}(A, Q) =$

$\max\{\mathfrak{t}(A/P)\colon P \in$ (the set of all minimal prime ideals of Q in A other than $A_1A)\}$, and $\mathfrak{g}(A, Q)$ is a positive integer. Note that in particular $\mathfrak{t}(A, \{0\}) = \mathfrak{t}(A)$, and $\mathfrak{t}(A, Q) < \mathfrak{t}(A)$ for every nonzero homogeneous ideal Q in A. We *define:* $\mathfrak{g}(A) = \mathfrak{g}(A, \{0\})$.

(12.3.1). (A weak form of Bezout's theorem). *Let m be a nonnegative integer, and for $1 \leqslant i \leqslant m$ let $f_i \in A_{n(i)}$ where $n(i)$ is a positive integer. Let $P_1, \ldots, P_u$ $(u \geqslant 0)$ be the minimal prime ideals of $(f_1, \ldots, f_m)A$ in A different from A_1A. Then $\mathfrak{g}(A, P_1) + \cdots + \mathfrak{g}(A, P_u) \leqslant \mathfrak{g}(A)n(1) \cdots n(m)$ (and hence in particular $\mathfrak{u}(A, Q) \leqslant \mathfrak{g}(A)n(1) \cdots n(m)$).*

PROOF. First suppose that $m = 0$; then $(f_1, \ldots, f_m)A = \{0\}$, $u = 1$, and $P_1 = \{0\}$; hence $\mathfrak{g}(A, P_1) + \cdots + \mathfrak{g}(A, P_u) = \mathfrak{g}(A) = \mathfrak{g}(A)n(1) \cdots n(m)$.

Next suppose that $m = 1$. If $u = 0$ then $\mathfrak{g}(A, P_1) + \cdots + \mathfrak{g}(A, P_u) = 0 \leqslant \mathfrak{g}(A)n(1)$; if $f_1 = 0$ then $u = 1$ and $\mathfrak{g}(A, P_1) + \cdots + \mathfrak{g}(A, P_u) = \mathfrak{g}(A) \leqslant \mathfrak{g}(A)n(1)$. So assume that $u \geqslant 1$ and $f_1 \neq 0$. Let $P_1', \ldots, P_v'$ $(v \geqslant 0)$ be the associated prime ideals of f_1A in A different from $P_1, \ldots, P_u$. Let $t = \mathfrak{t}(A)$. Then in view of Krull's principal ideal theorem [27: Theorem 29 on page 238], by [28: §7 of Chapter VII] we get that $t > 0$, $\mathfrak{t}(A, P_i) = t - 1$ for $1 \leqslant i \leqslant u$, and $\mathfrak{t}(A, P_j') < t - 1$ for $1 \leqslant j \leqslant v$. By [28: Theorem 9 on page 153] there exist homogeneous primary ideals $Q_1, \ldots, Q_u, Q_1', \ldots, Q_v'$ in A such that $f_1A = Q_1 \cap \cdots \cap Q_u \cap Q_1' \cap \cdots \cap Q_v'$, $\operatorname{rad} Q_i = P_i$ for $1 \leqslant i \leqslant u$, and $\operatorname{rad} Q_j' = P_j'$ for $1 \leqslant j \leqslant v$. For $1 \leqslant i < u$ let $Q_i^* = Q_{i+1} \cap \cdots \cap Q_u \cap Q_1' \cap \cdots \cap Q_v'$; let $Q_u^* = Q_1' \cap \cdots \cap Q_v'$ in case $v \neq 0$, and $Q_u^* = A$ in case $v = 0$. For $1 \leqslant i \leqslant u$, by the standard formula for finite-dimensional vector spaces, we have that $[A_n/(A_n \cap Q_i \cap Q_i^*) : A_0] = [A_n/(A_n \cap Q_i) : A_0] + [A_n/(A_n \cap Q_i^*) : A_0] - [A_n/(A_n \cap (Q_i + Q_i^*)) : A_0]$ for all n, and hence $\mathfrak{h}(A, Q_i \cap Q_i^*, Z) = \mathfrak{h}(A, Q_i, Z) + \mathfrak{h}(A, Q_i^*, Z) - \mathfrak{h}(A, Q_i + Q_i^*, Z)$; now $\mathfrak{t}(A, Q_i + Q_i^*) < t - 1 = \mathfrak{t}(A, Q_i) = \mathfrak{t}(A, Q_i \cap Q_i^*)$; also $\mathfrak{t}(A, Q_i^*) = t - 1$ in case $i < u$, and $\mathfrak{t}(A, Q_i^*) < t - 1$ in case $i = u$; therefore we get that: $\mathfrak{g}(A, Q_i \cap Q_i^*) = \mathfrak{g}(A, Q_i) + \mathfrak{g}(A, Q_i^*)$ in case $i < u$, and $\mathfrak{g}(A, Q_i \cap Q_i^*) = \mathfrak{g}(A, Q_i)$ in case $i = u$. It follows that $\mathfrak{t}(A, f_1A) = t - 1$ and $\mathfrak{g}(A, f_1A) = \mathfrak{g}(A, Q_1) + \cdots + \mathfrak{g}(A, Q_u)$. For $1 \leqslant i \leqslant u$, since $Q_i \subset P_i$, we get that $[A_n/(A_n \cap P_i) : A_0] \leqslant [A_n/(A_n \cap Q_i) :$

$A_0]$ for all n; since $\mathfrak{t}(A, P_i) = t - 1 = \mathfrak{t}(A, Q_i)$, we deduce that $\mathfrak{g}(A, P_i) \leqslant \mathfrak{g}(A, Q_i)$. Therefore $\mathfrak{g}(A, P_1) + \cdots + \mathfrak{g}(A, P_u) \leqslant \mathfrak{g}(A, f_1 A)$. Clearly $[A_n \cap (f_1 A) : A_0] = [A_{n-n(1)} : A_0]$ for all $n \geqslant n(1)$, and hence $[A_n/(A_n \cap (f_1 A)) : A_0] = [A_n : A_0] - [A_{n-n(1)} : A_0]$ for all $n \geqslant n(1)$; consequently $\mathfrak{h}(A, f_1 A, Z) = \mathfrak{h}(A, \{0\}, Z) - \mathfrak{h}(A, \{0\}, Z - n(1))$, and hence $\mathfrak{g}(A, f_1 A) = \mathfrak{g}(A) n(1)$. Therefore $\mathfrak{g}(A, P_1) + \cdots + \mathfrak{g}(A, P_u) \leqslant \mathfrak{g}(A) n(1)$.

Now we shall prove the general case by induction on m. So let $m > 1$ and assume that the assertion is true for all values of m smaller than the given one. If $u = 0$ then we have nothing to show. So also assume that $u \neq 0$. Then $\operatorname{rad}(f_1, \ldots, f_m)A = P_1 \cap \cdots \cap P_u$, and, upon letting $P_i^*, \ldots, P_w^*$ be the minimal prime ideals of $(f_2, \ldots, f_m)A$ in A different from $A_1 A$, we get that $w \neq 0$ and $\operatorname{rad}(f_2, \ldots, f_m)A = P_i^* \cap \cdots \cap P_w^*$. Let $P_{i,1}^*, \ldots, P_{i,v(i)}^*$ $(v(i) \geqslant 0)$ be the minimal prime ideals of $P_i^* + (f_1 A)$ in A. Then by the induction hypothesis we get that $\mathfrak{g}(A, P_1^*) + \cdots + \mathfrak{g}(A, P_w^*) \leqslant g(A) n(2) \cdots n(m)$, and by applying the case of $m = 1$ proved above to A/P_i^* we get that $\mathfrak{g}(A, P_{i,1}^*) + \cdots + \mathfrak{g}(A, P_{i,v(i)}^*) \leqslant \mathfrak{g}(A, P_i^*) n(1)$ for $1 \leqslant i \leqslant w$. Therefore

$$\sum_{i=1}^{w} \sum_{j=1}^{v(i)} \mathfrak{g}(A, P_{i,j}^*) \leqslant \mathfrak{g}(A) n(1) \cdots n(m).$$

Now

$$\begin{aligned} \bigcap_{i=1}^{w} \bigcap_{j=1}^{v(i)} P_{i,j}^* &= \bigcap_{i=1}^{w} \operatorname{rad}(P_i^* + (f_1 A)) \\ &= \operatorname{rad} \bigcap_{i=1}^{w} (P_i^* + (f_1 A)) \\ &= \operatorname{rad} \prod_{i=1}^{w} (P_i^* + (f_1 A)), \end{aligned}$$

and

$$\begin{aligned} \prod_{i=1}^{w} (P_i^* + (f_1 A)) &\subset (P_1^* \cap \cdots \cap P_w^*) + (f_1 A) \\ &\subset (P_1^* \cap \cdots \cap P_w^*) + (\operatorname{rad}(f_1 A)) \\ &= (\operatorname{rad}(f_2, \ldots, f_m)A) + (\operatorname{rad}(f_1 A)) \\ &\subset \operatorname{rad}(f_1, \ldots, f_m)A \\ &= P_1 \cap \cdots \cap P_u; \end{aligned}$$

consequently

$$\bigcap_{i=1}^{w} \bigcap_{j=1}^{v(i)} P_{i,j}^{*} \subset P_1 \cap \cdots \cap P_u .$$

Also

$$(f_1, \ldots, f_m)A \subset \bigcap_{i=1}^{w} (P_i^{*} + (f_1 A))$$

and hence

$$\begin{aligned} P_1 \cap \cdots \cap P_u &= \operatorname{rad}(f_1, \ldots, f_m)A \\ &\subset \operatorname{rad} \bigcap_{i=1}^{w} (P_i^{*} + (f_1 A)) \\ &= \bigcap_{i=1}^{w} \bigcap_{j=1}^{v(i)} P_{i,j}^{*} . \end{aligned}$$

Thus

$$P_1 \cap \cdots \cap P_u = \bigcap_{i=1}^{w} \bigcap_{j=1}^{v(i)} P_{i,j}^{*}$$

and hence for each q with $1 \leqslant q \leqslant u$ we have that $P_q = P_{i,j}^{*}$ for some (i, j). Therefore

$$\mathfrak{g}(A, P_1) + \cdots + \mathfrak{g}(A, P_u) \leqslant \sum_{i=1}^{w} \sum_{j=1}^{v(i)} \mathfrak{g}(A, P_{i,j}^{*})$$

and hence $\mathfrak{g}(A, P_1) + \cdots + \mathfrak{g}(A, P_u) \leqslant g(A)n(1) \cdots n(m)$.

(12.3.2). *Let d be any positive integer and let L be any A_0-subspace of $A_1^{(d)}$. Then* $\mathfrak{u}(A^{(d)}, LA^{(d)}) \leqslant \mathfrak{g}(A)d^q$ *where* $q = [L : A_0]$.

PROOF. By (12.3.1) we get that $\mathfrak{u}(A, LA) \leqslant g(A)d^q$, and by (12.1.7) we get that $\mathfrak{u}(A^{(d)}, LA^{(d)}) = \mathfrak{u}(A, LA)$.

(12.3.3). *Let* $t = \mathfrak{t}(A)$ *and let d be any positive integer. Then* $\mathfrak{g}(A^{(d)}) = \mathfrak{g}(A)d^t$.

PROOF. Now $[A_n^{(d)} : A_0^{(d)}] = [A_{dn} : A_0]$ for all n, and hence $\mathfrak{h}(A^{(d)}, \{0\}, Z) = \mathfrak{h}(A, \{0\}, dZ)$. Therefore $\mathfrak{g}(A^{(d)}) = \mathfrak{g}(A)d^{\mathfrak{t}}$.

(12.3.4). *For any homogeneous subdomain B of A with* $\mathfrak{r}(B) = \mathfrak{r}(A) - 1$, *we have the following:* (1) *if* $\operatorname{rad}(B_1A) = A_1A$ *then* $\mathfrak{t}(B) = \mathfrak{t}(A)$ *and* $\mathfrak{g}(A) = \mathfrak{g}(B)[\mathfrak{K}(A) : \mathfrak{K}(B)]$; (2) *if* $\operatorname{rad}(B_1A) \neq A_1A$ *and* $\mathfrak{t}(B) = \mathfrak{t}(A)$ *then* $\mathfrak{g}(A) = \mathfrak{g}(B)[\mathfrak{K}(A) : \mathfrak{K}(B)] + \mathfrak{e}(\mathfrak{R}(A, B_1A))$; *and* (3) *if* $\operatorname{rad}(B_1A) \neq A_1A$ *and* $\mathfrak{t}(B) \neq \mathfrak{t}(A)$ *then* $\mathfrak{g}(A) = \mathfrak{g}(B) = \mathfrak{e}(\mathfrak{R}(A, B_1A))$ (note that by (12.1.6) we know that if $\operatorname{rad}(B_1A) \neq A_1A$ then $\mathfrak{R}(A, B_1A)$ is a local domain).

PROOF. Take $z \in A_1 - B_1$. Let $D_j^{[w]} = \{x \in B_j :$ there exists $x_{j+i} \in B_{j+i}$ for $1 \leqslant i \leqslant w$ such that $xz^w + x_{j+1}z^{w-1} + \cdots + x_{j+w} = 0\}$, $B_j' = \bigcup_{0 \leqslant w < \infty} D_j^{[w]}$, $D^{[w]} = \{\sum_{0 \leqslant j < \infty} r_j : r_j \in D_j^{[w]}$ for all j, and $r_j = 0$ for all sufficiently large $j\}$, and $B' = \bigcup_{0 \leqslant w < \infty} D^{[w]}$. Then B' and $D^{[0]} \subset D^{[1]} \subset D^{[2]} \subset \cdots$ are homogeneous ideals in B, $D^{[0]} = \{0\}$, and $B_j \cap B' = B_j'$ and $B_j \cap D^{[w]} = D_j^{[w]}$ for all j and w. Since B is noetherian, there exists a unique nonnegative integer u such that $D^{[w]} = B'$ for all $w \geqslant u$. Let $E_n^{[w]} = \{y_n + y_{n-1}z + \cdots + y_{n-w}z^w$: $y_i \in B_i$ for $n - w \leqslant i \leqslant n\}$, and let $E_n^{[-1]} = \{0\}$. Then $E_n^{[-1]} \subset E_n^{[0]} \subset E_n^{[1]} \subset \ldots \subset E_n^{[n]} = A_n$ are A_0-subspaces of A_n, and hence

$$[A_n : A_0] = \sum_{w=0}^{n} [E_n^{[w]}/E_n^{[w-1]} : A_0].$$

Given any integers w and n with $0 \leqslant w \leqslant n$, upon letting $T = \{y_n + y_{n-1} + \cdots + y_{n-w} : y_i \in B_i$ for $n - w \leqslant i \leqslant n\}$, we get A_0-epimorphisms f: $T \to E_n^{[w]}$ and g: $T \to B_{n-w}$ by taking $f(y_n + y_{n-1} + \cdots + y_{n-w}) = y_n + y_{n-1}z + \cdots + y_{n-w}z^w$ and $g(y_n + y_{n-1} + \cdots + y_{n-w}) = y_{n-w}$; clearly $f^{-1}(E_n^{[n-w]}) = g^{-1}(D_{n-w}^{[w]})$ and hence $E_n^{[w]}/E_n^{[w-1]}$ and $B_{n-w}/D_{n-w}^{[w]}$ are A_0-isomorphic. Therefore

$$(*) \qquad [A_n : A_0] = \sum_{w=0}^{n} [B_{n-w}/D_{n-w}^{[w]} : B_0] \qquad \text{for all } n.$$

If $D^{[w]} \neq \{0\}$ for some w then let q be the smallest integer such that $D^{[q]} \neq \{0\}$, and if $D^{[w]} = \{0\}$ for all w then let $q = \infty$. Then q is either a positive integer, or $q = \infty$. Clearly $\mathfrak{t}(B) \neq \mathfrak{t}(A) \Leftrightarrow q = \infty$; and if $\mathfrak{t}(B) = \mathfrak{t}(A)$ then $[\mathfrak{K}(A) : \mathfrak{K}(B)] = q$. First suppose that $\mathrm{rad}(B_1A) = A_1A$; then by (12.1.5) we get that $B' = B$; consequently $q \neq \infty$, $\mathfrak{t}(B) = \mathfrak{t}(A)$, and $1 \leqslant q \leqslant u$; by $(*)$ we now get that

$$[A_n : A_0] = \sum_{w=0}^{q-1} [B_{n-w} : B_0] + \sum_{w=q}^{n} [B_{n-w}/D_{n-w}^{[w]} : B_0]$$

for all $n \geqslant u$; consequently

$$\mathfrak{h}(A, \{0\}, Z) = \sum_{w=0}^{q-1} \mathfrak{h}(B, \{0\}, Z - w) + \sum_{w=q}^{u-1} \mathfrak{h}(B, D^{[w]}, Z - w);$$

now $\mathfrak{t}(A, \{0\}) = \mathfrak{t}(B, \{0\}) > \mathfrak{t}(B, D^{[w]})$ for $q \leqslant w \leqslant u - 1$, and hence $\mathfrak{g}(A) = \mathfrak{g}(B)q = \mathfrak{g}(B)[\mathfrak{K}(A) : \mathfrak{K}(B)]$. Next suppose that $\mathrm{rad}(B_1A) \neq A_1A$ and $\mathfrak{t}(B) = t(A)$; then $q \neq \infty$ and $1 \leqslant q \leqslant u$; by $(*)$ we now get that

$$[A_n : A_0] = \sum_{w=0}^{q-1} [B_{n-w} : B_0] + \sum_{w=q}^{u-1} [B_{n-w}/D_{n-w}^{[w]} : B_0] + \sum_{w=u}^{n} [B_{n-w}/B'_{n-w} : B_0]$$

for all $n \geqslant u$; in view of (12.2.1) we get that

$$\begin{aligned}\sum_{w=u}^{n} [B_{n-w}/B'_{n-w} : B_0] &= \sum_{j=0}^{n-u} [B_j/B'_j : A_0] \\ &= [\mathfrak{N}(A, B_1A)/M(\mathfrak{N}(A, B_1A))^{n-u+1} : A_0]\end{aligned}$$

for all $n \geqslant u$, and hence

$$\begin{aligned}\mathfrak{h}(A, \{0\}, Z) = \sum_{w=0}^{q-1} \mathfrak{h}(B, \{0\}, Z - w) &+ \sum_{w=q}^{u-1} \mathfrak{h}(B, D^{[w]}, Z - w) \\ &+ \mathfrak{H}(\mathfrak{N}(A, B_1A), Z - u + 1);\end{aligned}$$

now $\mathfrak{t}(A, \{0\}) = \mathfrak{t}(B, \{0\}) > \mathfrak{t}(B, D^{[w]})$ for $q \leqslant w \leqslant u - 1$, and by (12.2) we have that $\mathfrak{H}(\mathfrak{N}(A, B_1A), Z)$ is a polynomial of degree t in

Z, where $t = \mathfrak{t}(A, \{0\})$, and the coefficient of Z^t in $\mathfrak{H}(\mathfrak{R}(A, B_1A), Z)$ is $\mathfrak{e}(\mathfrak{R}(A, B_1A))/(t!)$; therefore $\mathfrak{g}(A) = \mathfrak{g}(B)q + \mathfrak{e}(\mathfrak{R}(A, B_1A)) = \mathfrak{g}(B)[\mathfrak{K}(A) : \mathfrak{K}(B)] + \mathfrak{e}(\mathfrak{R}(A, B_1A))$. Finally suppose that $\mathrm{rad}(B_1A) \neq A_1A$ and $\mathfrak{t}(A) \neq \mathfrak{t}(B)$; then $q = \infty$, and hence $u = 0$ and $D_j^{[w]} = B_j' = \{0\}$ for all j and w; since $D_j^{[w]} = \{0\}$ for all j and w, by $(*)$ we get that

$$[A_n : A_0] = \sum_{j=0}^{n} [B_j : B_0]$$

for all n; now $\mathfrak{t}(B, \{0\}) = \mathfrak{t}(A, \{0\}) - 1$ and hence $\mathfrak{g}(A) = \mathfrak{g}(B)$; since $D_j^{[w]} = B_j'$ for all j and w, by $(*)$ we get that

$$[A_n : A_0] = \sum_{j=0}^{n} [B_j/B_j' : B_0]$$

for all n, and hence by (12.2.1) we get that $\mathfrak{h}(A, \{0\}, Z) = \mathfrak{H}(\mathfrak{R}(A, B_1A), Z + 1)$; therefore $\mathfrak{g}(A) = \mathfrak{e}(\mathfrak{R}(A, B_1A))$.

(12.3.5). *Assume that A_0 is algebraically closed. Then* $\mathfrak{g}(A) + \mathfrak{t}(A) \geqslant \mathfrak{r}(A)$.

PROOF. We shall make induction on $\mathfrak{r}(A)$. If $\mathfrak{r}(A) = 1$ then clearly $\mathfrak{g}(A) = 1$ and $\mathfrak{t}(A) = 0$. So now let $\mathfrak{r}(A) > 1$ and assume that the assertion is true for all values of $\mathfrak{r}(A)$ smaller than the given one. Since A_0 is algebraically closed, by the Hilbert Nullstellensatz [28: Lemma on page 165] there exists $R \in \mathfrak{W}(A)$ such that R is residually rational over A_0. Now by (12.1.6) there exists a homogeneous subdomain B of A such that $\mathfrak{r}(B) = \mathfrak{r}(A) - 1$, B_1A is a nonmaximal homogeneous prime ideal in A, and $\mathfrak{R}(A, B_1A) = R$. Since $\mathrm{rad}(B_1A) \neq A_1A$, by (12.3.4) we get that $\mathfrak{g}(A) + \mathfrak{t}(A) \geqslant \mathfrak{g}(B) + \mathfrak{t}(B) + 1$. By the induction hypothesis we have that $\mathfrak{g}(B) + \mathfrak{t}(B) \geqslant \mathfrak{r}(B)$. It follows that $\mathfrak{g}(A) + \mathfrak{t}(A) \geqslant \mathfrak{r}(A)$.

(12.3.6). *Let $t = \mathfrak{t}(A)$ and $g = \mathfrak{g}(A)$. Assume that A_0 is algebraically closed, and let I be any nonzero homogeneous ideal in A. Then there exist A_0-subspaces $L, L_1, \ldots, L_g$ of A_1 such that: $L \subset L_1 \cap \cdots \cap L_g$; $[L : A_0] = t$; $\{L_1A, \ldots, L_gA\}$ is the set of all minimal prime ideals of LA in A; $L_1A, \ldots, L_gA$ are distinct; $LA = (L_1A) \cap$*

$\cdots \cap (L_g A) \cap T$ where either $T = A$ or T is a homogeneous ideal in A which is primary for $A_1 A$; and for $1 \leqslant i \leqslant g$ we have that $[L_i : A_0] = \mathfrak{r}(A) - 1$, $I \not\subset L_i A \neq A_1 A$, $L_i A$ is not contained in any homogeneous prime ideal in A other than $L_i A$ and $A_1 A$, $\mathfrak{N}(A, L_i A)$ is a t-dimensional regular local domain which is residually rational over A_0, and if $(s_1, \ldots, s_t)$ is any A_0-basis of L and s'_i is any element in $A_1 - L_i$ then $(s_1/s'_i, \ldots, s_t/s'_i)\mathfrak{N}(A, L_i A) = M(\mathfrak{N}(A, L_i A))$.

PROOF. By [27: Theorem 31 on page 105 and Theorem 8 on page 266] we can find a free A_0-basis $(x, y_1, \ldots, y_t, z_1, \ldots, z_q)$ of A_1 such that $(x, y_1, \ldots, y_t)$ is a separating transcendence basis of the quotient field of A over A_0, and A is integral over $A_0[x, y_1, \ldots, y_t]$. Let $B = A_0[x, y_1, \ldots, y_t]$. Then B is a homogeneous subdomain of A, and clearly $\mathfrak{t}(B) = t$ and $\mathfrak{g}(B) = 1$. Let $E = B_0[B_1 x^{-1}]$ and $D = A_0[A_1 x^{-1}]$. Then $E = A_0[y_1/x, \ldots, y_t/x]$, $D = E[z_1/x, \ldots, z_q/x]$, and $\mathfrak{K}(B)$ and $\mathfrak{K}(A)$ are the quotient fields of E and D respectively.

Given any integer u with $1 \leqslant u \leqslant q$, let $F(Z) = Z^d + F_1 Z^{d-1} + \cdots + F_d$ be the minimal monic polynomial of z_u over the quotient field of B, where Z is a indeterminate; since B is normal, we get that $F_v \in B$ for $1 \leqslant v \leqslant d$, and then, since A is a homogeneous domain, we get that $F_v \in B_v$ for $1 \leqslant v \leqslant d$; it follows that $z_u \in \mathrm{rad}(B_1 B[z_u])$; let $G(Z) = Z^d + (F_1/x)Z^{d-1} + \cdots + (F_d/x^d)$; then $G(z_u/x) = 0$ and $F_v/x^v \in E$ for $1 \leqslant v \leqslant d$, and hence z_u/x is integral over E; let $F'(Z)$ and $G'(Z)$ be the Z-derivatives of $F(Z)$ and $G(Z)$ respectively; since z_u is separable over the quotient field of B, we must have $F'(z_u) \neq 0$; now $G'(z_u/x) = F'(z_u)/x^{d-1}$ and hence $G'(z_u/x) \neq 0$; therefore z_u/x is separable over $\mathfrak{K}(B)$. This being so for $1 \leqslant u \leqslant q$, we get that D is integral over E, and $\mathfrak{K}(A)$ is separable over $\mathfrak{K}(B)$; also we have shown that $z_u \in \mathrm{rad}(B_1 B[z_u])$ for $1 \leqslant u \leqslant q$. Let $B^{[u]} = B[z_1, \ldots, z_u]$ for $0 \leqslant u \leqslant q$; then $B^{[u]}$ is a homogeneous subdomain of A and $\mathfrak{t}(B^{[u]}) = t$ for $0 \leqslant u \leqslant q$; also $B^{[0]} = B$ and $B^{[q]} = A$. For any integer u with $1 \leqslant u \leqslant q$, we now have that $B^{[u-1]}$ is a homogeneous subdomain of $B^{[u]}$ and $\mathfrak{r}(B^{[u-1]}) = \mathfrak{r}(B^{[u]}) - 1$; since $z_u \in \mathrm{rad}(B_1 B[z_u])$, we also have that $\mathrm{rad}(B_1^{[u-1]} B^{[u]}) = B_1^{[u]} B^{[u]}$; consequently by (12.3.4) we get that $\mathfrak{g}(B^{[u]}) = \mathfrak{g}(B^{[u-1]})[\mathfrak{K}(B^{[u]}) : \mathfrak{K}(B^{[u-1]})]$. This being so for $1 \leqslant u \leqslant q$, we conclude that $\mathfrak{g}(A) = \mathfrak{g}(B)[\mathfrak{K}(A) : \mathfrak{K}(B)]$, and hence $[\mathfrak{K}(A) : \mathfrak{K}(B)] = g$.

Thus we have shown that $\mathfrak{K}(A)$ is separable over $\mathfrak{K}(B)$, D is integral over E, and $[\mathfrak{K}(A) : \mathfrak{K}(B)] = g$. We can now take $z \in D$ such that $\mathfrak{K}(A) = \mathfrak{K}(B)(z)$. Let $H(Z) = Z^g + H_1 Z^{g-1} + \cdots + H_g$ be the minimal monic polynomial of z over $\mathfrak{K}(B)$, where Z is an indeterminate. Since E is normal, we must have $H_v \in E$ for $1 \leqslant v \leqslant g$. Let J be the Z-discriminant of $H(Z)$. Then $J \in E$; since $\mathfrak{K}(A)$ is separable over $\mathfrak{K}(B)$, we must have $J \neq 0$. Now $I \cap B$ is a homogeneous ideal in B; since $I \neq \{0\}$ and A is integral over B, by [4: Lemma 1.23] we have that $I \cap B \neq \{0\}$; consequently we can take $0 \neq J' \in I \cap B_w$ for some nonnegative integer w. Let $J^* = J(J'/x^w)$. Then $0 \neq J^* \in E$. Since A_0 is infinite, there exist elements $a_1, \ldots, a_t$ in A_0 such that $J^* \notin ((y_1/x) - a_1, \ldots, (y_t/x) - a_t)E$. Let $x_v = y_v - a_v x$ for $1 \leqslant v \leqslant t$. Let L be the A_0-subspace of B_1 generated by $(x_1, \ldots, x_t)$. Then $[L : A_0] = t$, $B = A_0[x, x_1, \ldots, x_t]$, and $E = A_0[x_1/x, \ldots, x_t/x]$. Let $P = (x_1/x, \ldots, x_t/x)E$. Then P is a maximal ideal in E, E/P is algebraically closed, $J'/x^w \notin P$, and $J \notin P$; now by [4: Lemma 1.19, Theorem 1.42, and Theorem 1.44] we get that $PD = P_1 \cap \cdots \cap P_g$ where $P_1, \ldots, P_g$ are distinct maximal ideals in D, and, upon letting R_i be the quotient ring of D with respect to P_i, we have that R_i is residually rational over A_0 and $(x_1/x, \ldots, x_t/x)R_i = M(R_i)$ for $1 \leqslant i \leqslant g$. Now $P_i \cap E = P$ and hence $J'/x^w \notin P_i$ for $1 \leqslant i \leqslant g$. Since R_i is residually rational over A_0, by [28: §7 of Chapter VII] we get that $\dim R_i = t$ and hence R_i is regular for $1 \leqslant i \leqslant g$. Since R_i is residually rational over A_0, there exist elements $a_{i1}, \ldots, a_{iq}$ in A_0 such that $P_i = (x_1/x, \ldots, x_t/x, (z_1/x) - a_{i1}, \ldots, (z_q/x) - a_{iq})D$. Let L_i be the A_0-subspace of A_1 generated by $(x_1, \ldots, x_t, z_1 - a_{i1}x, \ldots, z_q - a_{iq}x)$. Then $[L_i : A_0] = \mathfrak{r}(A) - 1$.

For every homogeneous ideal Q in A let $f(Q) = \bigcup_{0 \leqslant n < \infty} \{r/x^n : r \in Q \cap A_n\}$; then $f(Q)$ is an ideal in D. It is easily seen that for any homogeneous ideals Q and Q' in A we have the following: (1) if $Q' \subset Q$ then $f(Q') \subset f(Q)$; (2) if Q is prime, Q' is primary for Q, and $x \notin Q$, then $f(Q)$ is prime, $f(Q')$ is primary for $f(Q)$, and: $Q' = Q \Leftrightarrow f(Q') = f(Q)$; (3) if Q and Q' are prime, $x \notin Q$, and $x \notin Q'$, then: $Q' = Q \Leftrightarrow f(Q') = f(Q)$. Clearly $f(LA) = PD$; also $LA \subset L_i A$ and $f(L_i A) = P_i$ for $1 \leqslant i \leqslant g$. Since $f(L_i A) = P_i$ for $1 \leqslant i \leqslant g$, we get that $L_1 A, \ldots, L_g A$ are distinct. Since $J'/x^w \notin P_i = f(L_i A)$, we get that $J' \notin L_i A$ and hence $I \not\subset L_i A$. Since

$D \neq P_i = f(L_iA)$, we get that $x^n \notin L_iA$ for all $n > 0$, and hence $\mathrm{rad}(L_iA) \neq A_1A$. By (12.1.6) we now get that L_iA is a nonmaximal homogeneous prime ideal in A, and L_iA is not contained in any homogeneous prime ideal in A other than L_iA and A_1A. Since $z_u \in \mathrm{rad}(B_1B[z_u])$ for $1 \leqslant u \leqslant q$, we get that A_1A is the only homogeneous prime ideal in A which contains x and LA. In view of [28: Theorem 9 on page 153 and Corollary on page 154] we now conclude that $\{L_1A, ..., L_gA\}$ is the set of all minimal prime ideals of LA in A, and $LA = L_1A \cap \cdots \cap L_gA \cap T$ where either $T = A$ or T is a homogeneous ideal in A which is primary for A_1A.

Since $x \notin L_iA$ and $f(L_iA) = P_i$, by (12.1.2) we get that $\mathfrak{R}(A, L_iA) = R_i$ and hence $M(\mathfrak{R}(A, L_iA)) = (x_1/x, ..., x_t/x)\mathfrak{R}(A, L_iA)$. If s_i' is any element in $A_1 - L_iA$ then by (12.1.2) we get that s_i'/x is a unit in $\mathfrak{R}(A, L_iA)$; it follows that if $(s_1, ..., s_t)$ is any A_0-basis of L then $(s_1/s_i', ..., s_t/s_i')\mathfrak{R}(A, L_iA) = M(\mathfrak{R}(A, L_iA))$.

(12.3.7). *Assume that A_0 is algebraically closed. Then given any nonzero homogeneous ideal I in A and any positive real number m, we have that either: there exists a homogeneous subdomain A' of A such that $\mathfrak{t}(A') = \mathfrak{t}(A)$ and $\mathfrak{e}(R')[\mathfrak{K}(A) : \mathfrak{K}(A')] < m$ for every element R' in $\mathfrak{W}(A')$ which is residually rational over A_0; or: there exists an A_0-subspace L of A_1 such that $[L : A_0] = \max(0, \mathfrak{t}(A) - 1)$ and $\mathfrak{u}(I, LA) \geqslant \mathfrak{r}(A) - \mathfrak{t}(A) - (\mathfrak{g}(A)/m)$.*

PROOF. We shall make induction on $\mathfrak{r}(A)$. If $\mathfrak{r}(A) = 1$ then $\mathfrak{t}(A) = 0$ and $\mathfrak{g}(A) = 1$, and upon taking $L = \{0\}$ we get that $[L : A_0] = 0 = \max(0, \mathfrak{t}(A) - 1)$ and $\mathfrak{u}(I, LA) = 1 \geqslant \mathfrak{r}(A) - \mathfrak{t}(A) - (\mathfrak{g}(A)/m)$. So now let $\mathfrak{r}(A) > 1$ and assume that the assertion is true for all values of $\mathfrak{r}(A)$ smaller than the given one. If $\mathfrak{e}(R) < m$ for every element R in $\mathfrak{W}(A)$ which is residually rational over A_0 then it suffices to take $A' = A$. So now assume that there exists an element R in $\mathfrak{W}(A)$ such that R is residually rational over A_0 and $\mathfrak{e}(R) \geqslant m$. By (12.1.6) there exists a homogeneous subdomain B of A such that $\mathfrak{r}(B) = \mathfrak{r}(A) - 1$, $\mathrm{rad}(B_1A) \neq A_1A$, B_1A is a nonmaximal homogeneous prime ideal in A, and $\mathfrak{R}(A, B_1A) = R$. By (12.1.9) there exists a nonzero homogeneous ideal J in B such that (1): for every homogeneous ideal Q in B we have that $\mathfrak{u}(I, QA) \geqslant \mathfrak{u}(J, Q)$. For a moment suppose that $\mathfrak{t}(B) \neq \mathfrak{t}(A)$; then $\mathfrak{t}(B) = \mathfrak{t}(A) - 1$; by (12.3.6) there exists an A_0-subspace L of B_1

such that $[L : A_0] = \mathfrak{t}(B)$ and $\mathfrak{u}(J, LB) = \mathfrak{g}(B)$; by (12.3.5) we also have that $\mathfrak{g}(B) \geqslant \mathfrak{r}(B) - \mathfrak{t}(B)$; thus $[L : A_0] = \mathfrak{t}(A) - 1 = \max(0, \mathfrak{t}(A) - 1)$, and $\mathfrak{u}(I, LA) \geqslant \mathfrak{u}(J, LB) = \mathfrak{g}(B) \geqslant \mathfrak{r}(B) - \mathfrak{t}(B) = \mathfrak{r}(A) - \mathfrak{t}(A) \geqslant \mathfrak{r}(A) - \mathfrak{t}(A) - (\mathfrak{g}(A)/m)$. So now assume that $\mathfrak{t}(B) = \mathfrak{t}(A)$. Let $q = [\mathfrak{K}(A) : \mathfrak{K}(B)]$. Then by (12.3.4) we get that $\mathfrak{g}(B) = (\mathfrak{g}(A) - \mathfrak{e}(R))/q$; since $\mathfrak{e}(R) \geqslant m$, we get that $\mathfrak{g}(B) \leqslant (\mathfrak{g}(A) - m)/q$, and hence (2): $\mathfrak{g}(B)/(m/q) \leqslant (\mathfrak{g}(A)/m) - 1$. Upon applying the induction hypothesis to $(B, J, m/q)$ we get that either (3): there exists a homogeneous subdomain A' of B such that $\mathfrak{t}(A') = \mathfrak{t}(B)$ and $\mathfrak{e}(R')[\mathfrak{K}(B) : \mathfrak{K}(A')] < m/q$ for every element R' in $\mathfrak{W}(A')$ which is residually rational over A_0, or (4) : there exists an A_0-subspace L of B_1 such that $[L : A_0] = \max(0, \mathfrak{t}(B) - 1)$ and $\mathfrak{u}(J, LB) \geqslant \mathfrak{r}(B) - \mathfrak{t}(B) - (\mathfrak{g}(B)/(m/q))$. If condition (3) prevails then $\mathfrak{t}(A') = \mathfrak{t}(A)$ and $\mathfrak{e}(R')[\mathfrak{K}(A) : \mathfrak{K}(A')] < m$ for every element R' in $\mathfrak{W}(A')$ which is residually rational over A_0. If condition (4) prevails then $[L : A_0] = \max(0, \mathfrak{t}(A) - 1)$, and by (1) and (2) we get that $\mathfrak{u}(I, LA) \geqslant \mathfrak{u}(J, LB) \geqslant \mathfrak{r}(A) - \mathfrak{t}(A) - (\mathfrak{g}(A)/m)$.

(12.3.8). *Assume that A_0 is algebraically closed. Then there exists a positive integer d and a homogeneous subdomain A' of $A^{(d)}$ such that $\mathfrak{t}(A') = \mathfrak{t}(A)$ and $\mathfrak{e}(R')[\mathfrak{K}(A) : \mathfrak{K}(A')] \leqslant \mathfrak{t}(A)!$ for every element R' in $\mathfrak{W}(A')$ which is residually rational over A_0.*

PROOF. If $\mathfrak{t}(A) = 0$ then $\mathfrak{t}(A)! = 1$, $\mathfrak{W}(A) = \{A_0\}$, and $\mathfrak{e}(A_0) = 1$, and hence it suffices to take $d = 1$ and $A' = A$. So now assume that $\mathfrak{t}(A) > 0$. Let $t = \mathfrak{t}(A)$. Now $\mathfrak{r}(A^{(n)}) = [A_n : A_0]$ for all $n > 0$, $[A_n : A_0] = \mathfrak{h}(A, \{0\}, n)$ for all sufficiently large n, $\mathfrak{h}(A, \{0\}, Z)$ is a polynomial of degree t in an indeterminate Z with rational coefficients, the coefficient of Z^t in $\mathfrak{h}(A, \{0\}, Z)$ is $\mathfrak{g}(A)/(t!)$, and $\mathfrak{g}(A)$ is a positive integer. Therefore there exists a positive integer d' such that (1): for every integer d with $d \geqslant d'$ we have that $\mathfrak{g}(A)d^{t-1} < \mathfrak{r}(A^{(d)}) - t - ((\mathfrak{g}(A)d^t)/(t! + 1))$. Let d be any integer with $d \geqslant d'$. Then, in view (1), by (12.3.2) and (12.3.3) we get that (2): if L is any A_0-subspace of $A_1^{(d)}$ with $[L : A_0] = t - 1$ then $\mathfrak{u}(A^{(d)}, LA^{(d)}) < \mathfrak{r}(A^{(d)}) - t - (\mathfrak{g}(A^{(d)})/(t! + 1))$. In view of (2), upon taking $(A^{(d)}, A^{(d)}, t! + 1)$ for (A, I, m) in (12.3.7) we get that: there exists a homogeneous subdomain A' of $A^{(d)}$ such that $\mathfrak{t}(A') = t$ and $\mathfrak{e}(R')[\mathfrak{K}(A) : \mathfrak{K}(A')] < (t! + 1)$ for every element R' in $\mathfrak{W}(A')$ which is residually rational over A_0; now $\mathfrak{e}(R')[\mathfrak{K}(A) :$

$\mathfrak{K}(A')]$ is an integer and hence we must have $\mathfrak{e}(R')[\mathfrak{K}(A):\mathfrak{K}(A')] \leqslant t!$.

(12.4). *Projective models.*

(12.4.1). *Let k be any field, let K be any function field over k, and let X be any projective model of K/k. Then there exists a homogeneous domain A such that $A_0 = k$, $\mathfrak{K}(A) = K$, and $\mathfrak{W}(A) = X$.*

PROOF. By definition there exist elements $y_1, ..., y_q$ in K such that $y_1 \neq 0$ and $X = \mathfrak{W}(k; y_1, ..., y_q)$. Take an element z in an overfield of K such that z is transcendental over K. Let $z_i = (y_i z)/y_1$ for $1 \leqslant i \leqslant q$. Then $X = \mathfrak{W}(k; z_1, ..., z_q)$. Let $A = k[z_1, ..., z_q]$, and for each nonnegative integer n let $A_n = \{0\} \cup \{u \in A\colon u = f(z_1, ..., z_q)$ for some nonzero homogeneous polynomial $f(Z_1, ..., Z_q)$ of degree n in indeterminates $Z_1, ..., Z_q$ with coefficients in $k\}$. It is easily seen that then A is a homogeneous domain, $A_0 = k$, $\mathfrak{K}(A) = K$, and $\mathfrak{W}(A) = X$.

(12.4.2). *Let k be any algebraically closed field and let K be any function field over k. Then there exists a function field K' over k and a projective model X' of K'/k such that K' is a subfield of K, $[K : K'] < \infty$, and upon letting $X = \bigcup_{R' \in X'} X(R')$, where $X(R')$ is the set of all quotient rings of the integral closure T of R' in K with respect to the various maximal ideals in T, we have that X is a normal complete model of K/k and $\mathfrak{e}(R) \leqslant (\mathrm{trdeg}_k K)!$ for every element R in X which is residually algebraic over k.*

PROOF. By (12.3.8) and (12.4.1) there exists a function field K' over k and a projective model X' of K'/k such that K' is a subfield of K, $[K : K'] < \infty$, and $\mathfrak{e}(R')[K : K'] \leqslant (\mathrm{trdeg}_k K)!$ for every element R' in X' which is residually algebraic over k. Let $X(R')$ and X be as defined above. Now $X' = \mathfrak{V}(E_1) \cup \cdots \cup \mathfrak{V}(E_q)$ where $E_1, ..., E_q$ are affine rings over k. Let D_i be the integral closure of E_i in K. Then by (1.1.2) we know that D_i is a finite E_i-module for $1 \leqslant i \leqslant q$, and by [4: Lemmas 1.17, 1.19, and 1.28] we see that every element in $\mathfrak{V}(D_1) \cup \cdots \cup \mathfrak{V}(D_q)$ is normal and $X = \mathfrak{V}(D_1) \cup \cdots \cup \mathfrak{V}(D_q)$; it follows that X is a normal complete model of K/k, and clearly X dominates X'. Given any element R in X such that R is residually algebraic over k, let R' be the center of R on X'; then clearly $R \in X(R')$ and R' is residually algebraic over k; by (1.1.2) we know that the integral closure of R' in K is a finite R'-module, and by [28: §7 of Chapter VII] we know that $\dim R^* = \dim R'$ for all $R^* \in X(R')$; consequently by [28: Corollary 1 on page 299] we get that $\mathfrak{e}(R) \leqslant \mathfrak{e}(R')[K : K']$, and hence $\mathfrak{e}(R) \leqslant (\mathrm{trdeg}_k K)!$.

(12.4.3). *Let k be any algebraically closed field, let K be any function field over k, and let V be any valuation ring of K such that $k \subset V$ and V is residually algebraic over k. Then there exists a normal spot R over k with quotient field K such that V dominates R and $\mathfrak{e}(R) \leqslant (\mathrm{trdeg}_k K)!$.*

PROOF. Follows from (12.4.2).

The following consequence of (12.4.2) will not be used in this monograph.

(12.4.4). *Let k be any algebraically closed field and let K be any function field over k. Then there exists a normal projective model X of K/k such that for every element R in X which is residually algebraic over k we have that $\mathfrak{e}(R) \leqslant (\mathrm{trdeg}_k K)!$.*

PROOF. Let X be as in (12.4.2). By [28: §18 of Chapter VI] we get that X is a projective model of K/k.

§13. Three-dimensional birational resolution over an algebraically closed ground field of characteristic $\neq 2, 3, 5$

Let K be a function field over a field k, let p be the characteristic of k, and let $n = \mathrm{trdeg}_k K$.

By (11.1) and (12.4.3) we get the following.

(13.1). *Assume that k is algebraically closed, and either: $n \leqslant 1$; or: $n = 2$ and $p \neq 2$; or: $n = 3$ and $p \neq 2, 3, 5$. Then there exists a nonsingular projective model of K/k.*

Note that we have proved the existence of a nonsingular projective model of K/k: in (9.4) when $n \leqslant 1$; in (9.3) when $n \leqslant 2$ and k is perfect; and in (11.2) when $n \leqslant 3$ and $p = 0$. Finally note that, as remarked in the Introduction, Hironaka [15] has proved the existence of a nonsingular projective model of K/k when $p = 0$ (any n).

Appendix on Analytic Desingularization in Characteristic Zero

Text of a talk given by Shreeram S. Abhyankar
at various places in 1996-97

ABSTRACT. As witnessed by the famous works of Zariski and Hironaka, desingularization proofs tend to be very long and difficult. Here I shall present a very short and simple proof of analytic desingularization in characteristic zero for any dimension which I have recently found. It is hoped that this will remove the fear of desingularization from young minds and embolden them to study it further. The said proof is extracted from my previous work on good points. It was inspired by discussions with the Control Theorist Hector Sussmann, the Subanalytic Geometer Adam Parusiński, and the Algebraic Geometer Wolfgang Seiler. Once again this illustrates the fundamental unity of all Mathematics from Control Theory to Complex Analysis to Algebra and Algebraic Geometry.

Section 1. Introduction

In this talk I shall give a very short and simple proof of the Analytic Desingularization Theorem for a hypersurface f in the neighborhood of a point P in the local space V of any dimension n.[1] This could serve

[1]In effect this proof is a slight variation of the good point proof for surfaces which I gave in the Purdue Seminar of 1966 as reported by Lipman in the 1973 Arcata Conference; see page 218 of his paper [A12] in the Proceedings of that Conference; for an exhaustive general theory of good points of hypersurfaces see my 1988 paper [A6] in Advances in Mathematics. The trick which converts the good point surface proof to the present proof for any dimension can be found in (10.24) on pages 255-257 of my 1966 Resolution Book which is being reprinted here. The Reduction Lemma and the Principalization Lemma of Section 3 are explicit avatars of this trick. As a telegraphic preview of the present proof: By an inductive procedure incorporating the Principalization Lemma, the hypersurface f is approximated by a binomial hypersurface, i.e., a hypersurface of the form $X_1^e + X_2^{b_{e2}} \dots X_n^{b_{en}} = 0$ where e is a positive integer and $b_{e2}, \dots, b_{en}$ are nonnegative integers. The Reduction Lemma enables us to further arrange matters so that

as an introduction to the great works of Zariski [23, 25] and Hironaka [15]. The precise statement of the theorem will be given in the next section. Here in the introduction let us give a quick geometric overview of the theorem and its proof.

In brief, inductively, the singularity of the hypersurface f is converted into a "good point" which is then dealt with by an easy and explicit algorithm. In effect f has a "good point" at P means that the equimultiplicity locus[2] of f at P is a pure $(n-2)$-dimensional subvariety having a normal crossing[3] at P and by iteratively blowing up its components the multiplicity gets reduced.[4] Equationally speaking, we can choose coordinates $X_1, \ldots, X_n$ at P such that the line $X_2 = \cdots = X_n = 0$ is not tangent to f, and then by the Weierstrass[5] Preparation Theorem we can write $f = h_0 g$ where h_0 is a unit in the power series ring R in the variables $X_1, \ldots, X_n$ and

$$g = X_1^m + \sum_{1 \le j \le m} G_j X_1^{m-j} \quad \text{with} \quad G_j \in M(R^*)^j \text{ for } 1 \le j \le m$$

where m is the multiplicity of f at P and $M(R^*)$ is the maximal ideal in the power series ring R^* in the variables $X_2, \ldots, X_n$. Next we

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$\bar{b}_{e2} + \cdots + \bar{b}_{en} < e$ where $\bar{b}_{e2}, \ldots, \bar{b}_{en}$ are the residues of $b_{e2}, \ldots, b_{en}$ modulo e. This then is a prototype of a good point of a hypersurface. The Good Point Lemma of Section 3 shows how to reduce the multiplicity of a good point.

[2]The equimultiplicity locus of f at P consists of those points near P where f has the same multiplicity as it has at P.

[3]A variety has a normal crossing at P means, with respect to some analytic coordinate system, all its components are linear subspaces through P. For example a variety consisting of some coordinate axes and some coordinate hyperplanes has a normal crossing at the origin.

[4]These blow ups are called monoidal transformations. The elementary monoidal transformation of the $(X_1, \ldots, X_n)$-space with center $X_1 = \cdots = X_t = 0$ (to be blown up) amounts to making a substitution of the form $X_1 = X_1'$, $X_j = X_1'(X_j' + \lambda_j)$ with a constant λ_j for $2 \le j \le t$, and $X_k = X_k'$ for $t < k \le n$. If $t = n$ then we call it a quadratic transformation. It was Max Noether [A13] who first proved that the singularities of a plane curve can be resolved by a finite sequence of quadratic transformations of the plane. Monoidal transformations were introduced by Zariski [24].

[5]See Weierstrass [A16]. To quote from the Preface of my book [A1]: Impressed by the power of the Preparation Theorem – indeed it prepares us so well! – I considered "Weierstrass Preparation Theorem and its immediate consequences" as a possible title for the entire book.

make the Shreedharacharya[6] Transformation[7] $X_1 \to X_1 + (G_1/m)$, i.e., by "completing the m-th power" we arrange matters so that $G_1 = 0$. If $G_j = 0$ for all j then we have nothing to show. So assume that $G_j \neq 0$ for some j, and let G be the product of all nonzero G_j. Now inductively transforming the local $(X_2, \ldots, X_n)$-space V^* by an iterated elementary monoidal transformation we arrange matters so that, at the origin P^* of V^*, the hypersurface G has a normal crossing and the ideal $(G_2^{m!/2}, \ldots, G_m^{m!/m})$ is principal. Then for some integer e with $2 \leq e \leq m$ we have $G_e \neq 0$ and $G_j^{m!/j}/G_e^{m!/e} \in R^*$ for $2 \leq j \leq m$, and for some nonnegative integers $b_{j2}, \ldots, b_{jn}$ we have $G_j/(X_2^{b_{j2}} \ldots X_n^{b_{jn}}) \in R^* \setminus M(R^*)$ for all nonzero G_j. In the Reduction Lemma we shall show that after further transforming V^* by an iterated elementary monoidal transformation it can be arranged that $\bar{b}_{e2} + \cdots + \bar{b}_{en} < e$ where, for every nonnegative integer b, we are letting $\bar{b}$ denote the residue of b modulo e. In the Good Point Lemma we shall show that f now has a "good point" at P; indeed, upon letting $q_2, \ldots, q_n$ to be the nonnegative integers with $b_{e2} = eq_2 + \bar{b}_{e2}, \ldots, b_{en} = eq_n + \bar{b}_{en}$, we shall see that the multiplicity is reduced when, for $2 \leq i \leq n$, we make q_i elementary monoidal transformations with center $X_1 = X_i = 0$, i.e., after making a total of $q_2 + \cdots + q_n$ elementary monoidal transformations with $(n-2)$-dimensional centers. The said reduction of multiplicity refers to the "proper transform" of f, i.e., to $f/(X_2^{mq_2} \ldots X_n^{mq_n})$. The "total transform" of f, i.e., f itself, has the "exceptional divisor" $X_2^{mq_2} \ldots X_n^{mq_n}$ as a factor. For keeping a record of the exceptional divisor, it is best to relax the assumption of h_0 being a unit in R by only assuming that to start with $h_0/(X_2^{s_2} \ldots X_n^{s_n}) \in R \setminus M(R)$ for some nonnegative integers $s_2, \ldots, s_n$. But now after a reduction of the multiplicity of the proper transform we may need to change the direction of projection. This may have the effect that the exceptional divisor may cease to have normal crossings, but every component of it will continue to be

[6] To quote from the Preface of my book [A5]: Our method (of desingularization) may be termed the method of Shreedharacharya, the fifth century Indian mathematician, to whom Bhaskaracharya ascribes the device of solving quadratic equations by completing the square. The said device is given in verse number 116 of Bhaskaracharya's Beejaganit [A10] of 1150 A.D.

[7] For a previous use of a Shreedharacharya Transformation see my 1955 joint paper with Zariski [10].

nonsingular. Thus in the end, when the proper transform of f has only simple points, all the components of the total transform of f will be nonsingular but they may meet each other tangentially. One more application of the Good Point Lemma will finally make the total transform of f have only normal crossings. Formally this will be achieved by introducing the biorder of f as a lexicographically ordered pair of nonnegative integers; the first entry being the multiplicity of the proper transform while the second entry will keep track of the number of "historically older" components of the exceptional divisor. By using this biorder, formally we have to invoke the Good Point Lemma only once rather than twice. Finally, the inductive passage from the total transform having only normal crossings to the ideal becoming principal is taken care of by the Principalization Lemma.[8] Collectively, the Reduction Lemma, the Principalization Lemma, and the Good Point Lemma comprise the auxiliary lemmas.

Section 2. Statement of the Theorem

Let $f = f_0 \dots f_l$ be the product of a finite number of nonzero elements $f_0, \dots, f_l$ in the power series ring $R = K[\{X_1, \dots, X_n\}]$ where $l \geq 0$ and $n \geq 1$ are integers, and K is an algebraically closed field of characteristic zero. Here $K[\{X_1, \dots, X_n\}]$ either stands for the formal power series ring $K[[X_1, \dots, X_n]]$ or the convergent power series ring $K[\langle X_1, \dots, X_n \rangle]$; in the latter case K is assumed to be equipped with an absolute value, such as the field of complex numbers. Let V be the "local space" K^n near the origin $P = (0, \dots, 0)$. In this paper we shall present a simple proof of the following version of the

Analytic Desingularization Theorem. *We can construct an IAMT (= Iterated Analytic Monoidal Transform) W of V such that, at every point Q of W, the hypersurface f has an NC (= Normal Crossing),*

[8]In the language of models this Lemma corresponds to "domination" (see the main part of the present book), whereas in the language of schemes it corresponds to "trivialization of a coherent sheaf of ideals" (see [15]). It also corresponds to "removal of points of indeterminacy of a birational correspondence" (again see [15]). It may be noted that "Proper Resolution for dimension $n \Rightarrow$ Total Resolution (= Monomialization) for dimension $n \Rightarrow$ Principalization for dimension $n \Rightarrow$ Proper Resolution for dimension $n+1$" is a common feature of many resolution proofs; for example see [15] and the main part of the present book.

and, for every subsequence $0 \leq u(0) < \cdots < u(v) \leq l$, *the ideal* $(f_{u(0)}, \ldots, f_{u(v)})$, *is principal.*

Our monoidal transformations are always assumed to have nonsingular centers. In brief, to apply an analytic monoidal transformation to V means to take an analytic coordinate system $\widehat{X}_1, \ldots, \widehat{X}_n$ on V at P, i.e., to take $\widehat{X}_1, \ldots, \widehat{X}_n$ in the maximal ideal $M(R)$ of R such that $R = K[\{\widehat{X}_1, \ldots, \widehat{X}_n\}]$, and then take an integer t with $1 \leq t \leq n$, and finally, for $1 \leq i \leq t$, to make the substitutions: $\widehat{X}_i = Y_{ii}, \widehat{X}_j = Y_{ii}(Y_{ij} + \lambda_{ij})$ where $\lambda_{ij} \in K$ for $1 \leq j \leq t$ with $j \neq i$, and $\widehat{X}_k = Y_{ik}$ for $t < k \leq n$. Now the AMT (=Analytic Monoidal Transform) V_1 of V with center $\widehat{X}_1 = \cdots = \widehat{X}_t = 0$ is covered by the t charts $(\lambda_{i1}, \ldots, \lambda_{it}) \in K^{t-1}$ with $\lambda_{ii} = 0$ and local coordinates $(Y_{i1}, \ldots, Y_{in})$ for $i = 1, \ldots, t$. We may now apply an analytic monoidal transformation to V_1 to get V_2. Then we may apply an analytic monoidal transformation to V_2 to get V_3. And so on. Thus we get a finite sequence of analytic spaces $V = V_0, V_1, V_2, \ldots, V_\tau$. An IAMT of V is an analytic space $W = V_\tau$ obtained in this manner for some integer $\tau \geq 0$.

At any rate, if S is the local ring of a point Q of an IAMT W of V, then S dominates R, i.e., R is a subring of S and $M(R) = R \cap M(S)$. Moreover, for any basis $Y = (Y_1, \ldots, Y_n)$ of $M(S)$ we have $S = K[\{Y\}]$ which makes S abstractly isomorphic to $R = K[\{X\}]$ with $X = (X_1, \ldots, X_n)$. The hypersurface f has an NC at Q means that for some basis Y of $M(S)$ we have $f = \delta(Y)Y_1^{b_1} \ldots Y_n^{b_n}$ where $b_1, \ldots, b_n$ are nonnegative integers, and $\delta(Y)$ is a unit in S, i.e., $\delta(Y) \in K[\{Y\}]$ with $\delta(0) \neq 0$; if Y is such a basis of $M(S)$ then we may say that f has an NC at Q relative to Y. It follows that then for $0 \leq w \leq l$ we have $f_w = \delta_w(Y)Y_1^{b_{w1}} \ldots Y_n^{b_{wn}}$ where $b_{w1}, \ldots, b_{wn}$ are nonnegative integers, and $\delta_w(Y)$ is a unit in S. Note that now, for a subsequence $0 \leq u(0) < \cdots < u(v) \leq l$ with $v \geq 1$, the ideal $(f_{u(0)}, \ldots, f_{u(v)})$ is principal at Q means that for some i with $0 \leq i \leq v$ (where i depends on Q) we have $b_{u(i)1} \leq b_{u(j)1}, \ldots, b_{u(i)n} \leq b_{u(j)n}$ for $0 \leq j \leq v$.

In the above description of AMT, if $X = \widehat{X}$ where $\widehat{X} = (\widehat{X}_1, \ldots, \widehat{X}_n)$, then we may call V_1 an EMT (=Elementary Monoidal Transform) of V relative to X, and we may say that the basis $(Y_{i1}, \ldots, Y_{in})$ of the maximal ideal of the local ring of the relevant point of V_1 is elementarily related to X. By constructing V_2 in terms of such a basis

$(Y_{i1}, \ldots, Y_{in})$ and so on, we get the idea of an IEMT (= Iterated Elementary Monoidal Transform) $W = V_\tau$ of V relative to X, and we get the idea of a basis of the maximal ideal $M(S)$ of the local ring S of a point Q of W to be elementarily related to X. It is clear that if f has an NC at P relative to X, and if Y is a basis of $M(S)$ elementarily related to X then f has an NC at Q relative to Y; let us call this the *invariance of NC under IEMT.*

The construction of the IAMT to be described in this paper was culled from my old desingularization papers [A4,A6]; for background material see the main part of the present book as well as my book [A7]. This culling was inspired by the stimulating conversations which I had with Parusiński [A14], Seiler and Sussmann [A15], while we were attending the conferences at Kraków and Warsaw in September-October 1996 to celebrate the 70th birthday of Łojasiewicz.

In the next section we shall prove some auxiliary lemmas, and then in the last section we shall construct the IAMT asserted in the above theorem by induction on n. The construction of the IAMT can be carried out, and hence the theorem can be proved, without assuming K to be algebraically closed. But for simplicity of exposition we shall continue to assume K to be algebraically closed.

Section 3. Auxiliary Lemmas

In this section we shall prove the three auxiliary lemmas.

Reduction Lemma. *Assume that f has an NC at P relative to $X = (X_1, \ldots, X_n)$, i.e., for $0 \le w \le l$ we have $f_w = \epsilon_w(X) X_1^{a_{w1}} \ldots X_n^{a_{wn}}$ where ϵ_w is a unit in $R = K[\{X\}]$, and $a_{w1}, \ldots, a_{wn}$ are nonnegative integers. Let e be a positive integer. For every nonnegative integer b, let $\overline{b}$ be the residue of b modulo e, i.e., let $\overline{b}$ be the unique integer with $0 \le \overline{b} < e$ such that $b - \overline{b}$ is divisible by e. Then there exists an IEMT W of V relative to X such that, for a basis $Y = (Y_1, \ldots, Y_n)$ of the maximal ideal $M(S)$ of the local ring S of any point Q of W which is elementarily related to X, for $0 \le w \le l$ we have $f_w = \delta_w(Y) Y_1^{b_{w1}} \ldots Y_n^{b_{wn}}$ where $\delta_w(Y)$ is a unit in $S = K[\{Y\}]$, and $b_{w1}, \ldots, b_{wn}$ are nonnegative integers such that for $w = 0$ we have $\overline{b}_{01} + \cdots + \overline{b}_{0n} < e$.*

We shall prove this by induction on $\overline{a}_{01} + \cdots + \overline{a}_{0n}$. If $\overline{a}_{01} + \cdots + \overline{a}_{0n} < e$ then we have nothing to show. So let $\overline{a}_{01} + \cdots + \overline{a}_{0n} \ge e$ and assume for all values of $\overline{a}_{01} + \cdots + \overline{a}_{0n}$ smaller than the given value. Now there is a unique integer t with $2 \le t \le n$ such that for every subsequence $1 \le u(1) < \cdots < u(t-1) \le n$ we have $\overline{a}_{0u(1)} + \cdots + \overline{a}_{0u(t-1)} < e$, but for some subsequence $1 \le u(1) < \cdots < u(t) \le n$ we have $\overline{a}_{0u(1)} + \cdots + \overline{a}_{0u(t)} \ge e$. Upon relabelling $X_1, \ldots, X_n$ we can arrange matters so that $\overline{a}_{01} + \cdots + \overline{a}_{0t} \ge e$ Let R' be the local ring of any point P' of the AMT V' of V with center $X_1 = \cdots = X_t = 0$. We can take i with $1 \le i \le t$ such that $X_j/X_i \in S$ for $1 \le j \le t$. Let $X' = (X'_1, \ldots, X'_n)$ where $X'_i = X_i$, $X'_j = (X_j/X_i) - \lambda_j$ with $\lambda_j \in K$ for $1 \le j \le t$ with $j \ne i$, and $X'_k = X_k$ for $t < k \le n$. Then X' is a basis of $M(R')$ which is elementarily related to X, and for $0 \le w \le l$ we have $f_w = \epsilon'_w(X') {X'_1}^{a'_{w1}} \ldots {X'_n}^{a'_{wn}}$ where $\epsilon'_w(X')$ is a unit in $R' = K[\{X'\}]$, $a'_{wi} = a_{w1} + \cdots + a_{wt}, a'_{wj} = a_{wj}$ for $1 \le j \le t$ with $j \ne i$ and $\lambda_j = 0, a'_{wj} = 0$ for $1 \le j \le t$ with $j \ne i$ and $\lambda_j \ne 0$, and $a'_{wk} = a_{wk}$ for $t < k \le n$. Since $\overline{a}_{01} + \cdots + \overline{a}_{0t} - \overline{a}_{0i} < e$ and $\overline{a}_{0i} < e$ and $\overline{a}_{01} + \cdots + \overline{a}_{0t} \ge e$, we see that $\overline{a}'_{0i} = \overline{a}_{01} + \cdots + \overline{a}_{0t} - e < \overline{a}_{0i}$ and hence $\overline{a}'_{01} + \cdots + \overline{a}'_{0n} < \overline{a}_{01} + \cdots + \overline{a}_{0n}$. Therefore we are done by induction.

Principalization Lemma. *Assume that f has an NC at P relative to $X = (X_1, \ldots, X_n)$. Then there exists an IEMT W of V relative to X such that, at every point Q of W, the hypersurface f has an NC relative to a basis $Y = (Y_1, \ldots, Y_n)$ of the maximal ideal $M(S)$ of the local ring S of Q which is elementarily related to X, and, for every subsequence $0 \le u(0) < \cdots < u(v) \le l$, the ideal $(f_{u(0)}, \ldots, f_{u(v)})$ is principal.*

To prove this, in view of the invariance of NC under IEMT, it suffices to show that, assuming $l = 1$ there exists an IEMT of W of V relative to X such that, at every point Q of W, the ideal (f_0, f_1) is principal. Since f has an NC at P relative to X, for $0 \le w \le 1$ we can write $f_w = \epsilon_w(X) X_1^{a_{w1}} \ldots X_n^{a_{wn}}$ where ϵ_w is a unit in $R = K[\{X\}]$, and $a_{w1}, \ldots, a_{wn}$ are nonnegative integers. Let us make induction on n. If $n = 1$ then every ideal in R is principal and hence we have nothing to show. So let $n > 1$ and assume for $n - 1$. Let us make a second induction on a nonnegative integer N such that $|a_{0i} - a_{1i}| \le N$ for some i with $1 \le i \le n$. If N is zero then we are done by the $n - 1$ case. So let $|a_{0i} - a_{1i}| = N > 0$ for some i with $1 \le i \le n$, and assume for all values of N smaller than the given value. Relabelling $X_1, \ldots, X_n$ and relabelling f_0, f_1 we may assume that $a_{01} - a_{11} = N$.

By the $n-1$ case we may assume that either $a_{02} \le a_{12}, \dots, a_{0n} \le a_{1n}$ or $a_{02} \ge a_{12}, \dots, a_{0n} \ge a_{1n}$. In the latter case $f_0/f_1 \in R$ and hence we have nothing more to show. So assume that $a_{02} \le a_{12}, \dots, a_{0n} \le a_{1n}$. Under the assumption that $a_{01} - a_{11} = N$ and $a_{02} \le a_{12}, \dots, a_{0n} \le a_{1n}$, we shall make a third induction on the sum $|a_{12} - a_{02}| + \dots + |a_{1n} - a_{0n}|$. If this sum is zero then $f_0/f_1 \in R$ and hence we have nothing to show. So let the sum be positive and assume true for all values of the sum smaller than the given value. If $|a_{12} - a_{02}| < N$ then we are done by the second induction hypothesis. So assume that $|a_{12} - a_{02}| \ge N$. Let R' be the local ring of any point P' of the AMT V' of V with center $X_1 = X_2 = 0$. If $X_2/X_1 \in R'$, then $f_1/f_0 \in R'$ and hence we have nothing more to do. If $X_2/X_1 \notin R'$, then $X' = (X_1', \dots, X_n) = (X_1/X_2, X_2, \dots, X_n)$ is a basis of $M(R')$ which is elementarily related to X, and for $0 \le w \le 1$ upon letting $a'_{w1} = a_{w1}, a'_{w2} = a_{w1} + a_{w2}, a'_{w3} = a_{w3}, \dots, a'_{wn} = a_{wn}$ we have $f_w = \epsilon'(X') X_1'^{a'_{w1}} \dots X_n'^{a'_{wn}}$ where $\epsilon'(X') = \epsilon(X) =$ a unit in R', and clearly $a'_{01} - a'_{11} = N$ and $a'_{02} \le a'_{12}, \dots, a'_{0n} \le a'_{1n}$ and $|a'_{12} - a'_{02}| + \dots + |a'_{1n} - a'_{0n}| = |a_{12} - a_{02}| + \dots + |a_{1n} - a_{0n}| - N$, and hence we are done by the third induction hypothesis.

Good Point Lemma. *Let $R^* = K[\{X_2, \dots, X_n\}]$ and assume that there exist nonnegative integers $d, D, s_2, \dots, s_n$, together with nonzero elements $h_0, \dots, h_{D+1}$ in R, such that*

$$f = h_0 \dots h_{D+1} \quad \textit{with} \quad h_0/(X_2^{s_2} \dots X_n^{s_n}) \in R \setminus M(R)$$

and

$$h_i = X_1 + H_i \quad \textit{with} \quad H_i \in R^* \textit{ for } 1 \le i \le D$$

and

$$h_{D+1} = X_1^d + \sum_{1 \le j \le d} L_j X_1^{d-j} \quad \textit{with} \quad L_j \in R^* \textit{ for } 1 \le j \le d.$$

Let

$$g = h_1 \dots h_{D+1} \quad \textit{and} \quad m = D + d$$

and assume that $m \ge 2$ and

$$g = X_1^m + \sum_{2 \le j \le m} G_j X_1^{m-j} \quad \textit{with} \quad G_j \in M(R^*)^j \textit{ for } 2 \le j \le m.$$

Also assume that for some e with $2 \le e \le m$ we have

$$G_e \neq 0 \quad \text{and} \quad G_j^{m!/j}/G_e^{m!/e} \in R^* \text{ for } 2 \le j \le m.$$

Finally assume that there are nonnegative integers $b_{j2}, \dots, b_{jn}$ such that

$$G_j/(X_2^{b_{j2}} \dots X_n^{b_{jn}}) \in R^* \backslash M(R^*) \text{ for all } j \text{ with } 2 \le j \le m \text{ and } G_j \neq 0$$

and

$$\overline{b}_{e2} + \dots + \overline{b}_{en} < e$$

where, for every nonnegative integer b, we are letting $\overline{b}$ denote the residue of b modulo e. Then there exists an IEMT W of V relative to $(X_1, \dots, X_n)$ such that the maximal ideal $M(S)$ of the local ring S of any point Q of W has a basis $(Y_1, \dots, Y_n)$ which is elementarily related to $(X_1, \dots, X_n)$ and for which, either f has an NC at Q relative to $(Y_1, \dots, Y_n)$, or we have

$$X_i = Y_i \text{ for } 2 \le i \le n \quad \text{and} \quad X_1 = Y_2^{t_2} \dots Y_n^{t_n}(Y_1 + \lambda) \text{ with } \lambda \in K$$

where $t_2, \dots, t_n$ are nonnegative integers, and

$$f = h_0^\sharp \dots h_{D+1}^\sharp \quad \text{with} \quad h_0^\sharp/(Y_2^{s_2+mt_2} \dots Y_n^{s_n+mt_n}) \in S \setminus M(S)$$

and

$$h_i^\sharp = h_i/(Y_2^{t_2} \dots Y_n^{t_n}) = Y_1 + H_i^\sharp \quad \text{with} \quad H_i^\sharp \in R^* \text{ for } 1 \le i \le D$$

and

$$h_{D+1}^\sharp = h_{D+1}/(Y_2^{dt_2} \dots Y_n^{dt_n}) = Y_1^d + \sum_{1 \le j \le d} L_j^\sharp Y_1^{d-j}$$

with

$$L_j^\sharp \in R^* \text{ for } 1 \le j \le d$$

and

$$\begin{cases} \text{either } D \ge 1 \text{ and } H_i^\sharp \notin M(R^*) \text{ for some } i \text{ with } 1 \le i \le D \\ \text{or } d \ge 1 \text{ and } L_j^\sharp \notin M(R^*)^j \text{ for some } j \text{ with } 1 \le j \le d. \end{cases}$$

We shall prove this by induction on $q_2 + \cdots + q_n$ where $q_2, \ldots, q_n$ are the nonnegative integers with $b_{e2} = eq_2 + \overline{b}_{e2}, \ldots, b_{en} = eq_n + \overline{b}_{en}$. Clearly $q_2 + \cdots + q_n \geq 1$. We shall simultaneously treat the initial case of $q_2 + \cdots + q_n = 1$ and the inductive step which says that the general case follows from the case of smaller values of $q_2 + \cdots + q_n$. By suitably relabelling $X_2, \ldots, X_n$ we may assume that $q_2 \geq 1$. Let R' be the local ring of any point P' of the AMT V' of V with center $X_1 = X_2 = 0$. If $X_1/X_2 \notin R'$ then $(X_1', \ldots, X_n') = (X_1, X_2/X_1, X_3, \ldots, X_n)$ is a basis of $M(R')$ which is elementarily related to $(X_1, \ldots, X_n)$ and we have

$$f = h_0 g$$

with

$$h_0/(X_1'^{s_2} X_2'^{s_2} X_3'^{s_3} \ldots X_n'^{s_n}) = h_0/(X_2^{s_2} \ldots X_n^{s_n}) \in R' \setminus M(R')$$

and

$$g/X_1'^m = 1 + \sum_{2 \leq j \leq m} (G_j/X_2^j) X_2'^j$$

with

$$G_j/X_2^j \in R^* \subset R' \text{ for } 2 \leq j \leq m$$

and hence $g/X_1'^m \in R' \setminus M(R')$, and therefore f has an NC at P' relative to $(X_1', \ldots, X_n')$.

So henceforth assume that $X_1/X_2 \in R'$. Then $(X_1', \ldots, X_n') = ((X_1/X_2) - \lambda, X_2, \ldots, X_n)$, with a unique $\lambda \in K$, is a basis of $M(R')$ which is elementarily related to $(X_1, \ldots, X_n)$ and upon letting

$$s_2' = m + s_2, s_3' = s_3, \ldots, s_n' = s_n \quad \text{and} \quad h_0' = h_0 X_1'^m$$

and

$$g' = g/X_1'^m$$

we see that

$$f = h_0' g' \quad \text{with} \quad h_0'/(X_2'^{s_2'} \ldots X_n'^{s_n'}) \in R' \setminus M(R')$$

and

$$g' = (X_1' + \lambda)^m + \sum_{2 \leq j \leq m} (G_j/X_2^j)(X_1' + \lambda)^{m-j}$$

with

$$G_j/X_2^j \in R^* \text{ for } 2 \le j \le m$$

where we note that

$$R^* = K[\{X_2, \dots, X_n\}] = K[\{X_2', \dots, X_n'\}] \subset K[\{X_1', \dots, X_n'\}] = R'.$$

Since $g = h_1 \dots h_{D+1}$ and $q_2 \ge 1$, it follows that

$$g' = h_1' \dots h_{D+1}'$$

where

$$h_i' = h_i/X_2' = (X_1'+\lambda)+(H_i/X_2) \quad \text{with} \quad H_i/X_2 \in R^* \text{ for } 1 \le i \le D$$

and

$$h_{D+1}' = h_{D+1}/X_2'^d = (X_1' + \lambda)^d + \sum_{1 \le j \le d} (L_j/X_2^j)(X_1' + \lambda)^{d-j}$$

with

$$L_j/X_2^j \in R^* \text{ for } 1 \le j \le d.$$

Note that now

$$g' = X_1'^m + \sum_{1 \le j \le m} G_j' X_1'^{m-j}$$

where

$$G_j' \in R^* \text{ for } 1 \le j \le m \text{ with } G_1' = m\lambda$$

and

$$h_i' = X_1' + H_i' \quad \text{with} \quad H_i' \in R^* \text{ for } 1 \le i \le D$$

and

$$h_{D+1}' = X_1'^d + \sum_{1 \le j \le d} L_j' X_1'^{d-j} \quad \text{with} \quad L_j' \in R^* \text{ for } 1 \le j \le d.$$

Since $g' = h_1' \dots h_{D+1}'$, by the above three displayed equations we get the implication saying that:

$$G_j' \notin M(R^*)^j \text{ for some } j \text{ with } 1 \le j \le m$$
$$\Leftrightarrow \begin{cases} \text{either } D \ge 1 \text{ and } H_i' \notin M(R^*) \text{ for some } i \text{ with } 1 \le i \le D \\ \text{or } d \ge 1 \text{ and } L_j' \notin M(R^*)^j \text{ for some } j \text{ with } 1 \le j \le d. \end{cases}$$

If $\lambda \neq 0$ then $G'_1 = m\lambda \notin M(R^*)$ and hence we are done by the above implication. If $\lambda = 0$ then $G'_1 = 0$ and $G'_j = G_j/X_2^j$ for $2 \leq j \leq m$. Therefore if $\lambda = 0$ and $q_2 + \cdots + q_n = 1$ then, because of the inequality $\overline{b}_{e2} + \cdots + \overline{b}_{en} < e$, we see that $G'_e \notin M(R^*)^e$ and hence again we are done by the above implication. Finally, if $\lambda = 0$ and $q_2 + \cdots + q_n > 1$ then

$$G'_1 = 0 \quad \text{and} \quad G'_j \in M(R^*)^j \text{ for } 2 \leq j \leq m$$

and

$$G'_e \neq 0 \quad \text{and} \quad {G'_j}^{m!/j}/{G'_e}^{m!/e} \in R^* \text{ for } 2 \leq j \leq m$$

and upon letting $b'_{j2} = b_{j2} - j, b'_{j3} = b_{j3}, \ldots, b'_{jn} = b_{jn}$ we see that $b'_{j2}, \ldots, b'_{jn}$ are nonnegative integers such that

$$G'_j/({X'_2}^{b'_{j2}} \ldots {X'_n}^{b'_{jn}}) \in R^* \backslash M(R^*) \text{ for all } j \text{ with } 2 \leq j \leq m \text{ and } G'_j \neq 0$$

and

$$\overline{b}'_{e2} + \cdots + \overline{b}'_{en} < e$$

and upon letting $q'_2 = q_2 - 1, q'_3 = q_3, \ldots, q'_n = q_n$ we see that $q'_2, \ldots, q'_n$ are nonnegative integers with $b'_{e2} = eq'_2 + \overline{b}'_{e2}, \ldots, b'_{en} = eq'_n + \overline{b}'_{en}$ and $q'_2 + \cdots + q'_n < q_2 + \cdots + q_n$ and hence we are done by induction.

Section 4. Proof of the Theorem

We shall now prove the Analytic Desingularization Theorem by induction on n. For $n = 1$ we have nothing to show; just take $W = V$. So let $n > 1$ and assume the Theorem to be true for $n - 1$. In view of the Principalization Lemma, it suffices to show that (*) there exists an IAMT W of V such that f has an NC at every point Q of W. Recall that the R-order of f is defined by putting $\text{ord}_R f = c$ where c is the unique nonnegative integer such that $f \in M(R)^c \setminus M(R)^{c+1}$. Also recall that for any $I \subset M(R)$, the (R/I)-order of f is defined by putting $\text{ord}_{R/I} f = c'$ where c' is the unique nonnegative integer such that $f \in (IR + M(R)^{c'}) \backslash (IR + M(R)^{c'+1})$ if $f \notin IR$, and $\text{ord}_{R/I} f = \infty$ if $f \in IR$. Let us consider pairs (d, D) of nonnegative integers and order them lexicographically, i.e., $(d, D) < (d', D') \Leftrightarrow$ either $d = d'$ and $D < D'$ or $d < d'$.

Since K is infinite, we can find a basis $(X_1^*, \dots, X_n^*)$ of $M(R)$ such that upon letting $I^* = (X_2^*, \dots, X_n^*)$ we have $\mathrm{ord}_R f = \mathrm{ord}_{R/I^*} f$. Therefore it makes sense to define the R-biorder of f to be the smallest pair (d, D) for which there is a basis $(\widehat{X}_1, \dots, \widehat{X}_n)$ of $M(R)$ together with nonnegative integers $r_2, \dots, r_n$ and nonzero elements $h_0, \dots, h_{D+1}$ in R such that

$$f = h_0 \dots h_{D+1} \quad \text{with} \quad h_0/(\widehat{X}_2^{r_2} \dots \widehat{X}_n^{r_n}) \in R \setminus M(R)$$

and upon letting $\widehat{I} = (\widehat{X}_2, \dots, \widehat{X}_n)$ we have

$$\mathrm{ord}_R h_i = \mathrm{ord}_{R/\widehat{I}} h_i = 1 \text{ for } 1 \leq i \leq D$$

and

$$\mathrm{ord}_R h_{D+1} = \mathrm{ord}_{R/\widehat{I}} h_{D+1} = d.$$

We shall prove (*) by induction on (d, D). If $(d, D) = 0$ then there is nothing to show; just take $W = V$. So let $(d, D) > (0, 0)$ and assume true or all values of (d, D) smaller than the given value. Upon letting $\widehat{R} = K[\{\widehat{X}_2, \dots, \widehat{X}_n\}]$ and absorbing suitable unit factors of $h_1, \dots, h_{D+1}$ into h_0, by WPT (=Weierstrass Preparation Theorem) we may assume that

$$h_i = \widehat{X}_1 + H_i \quad \text{with} \quad H_i \in M(\widehat{R}) \text{ for } 1 \leq i \leq D$$

and

$$h_{D+1} = \widehat{X}_1^d + \sum_{1 \leq j \leq d} L_j \widehat{X}_1^{d-j} \quad \text{with} \quad L_j \in M(\widehat{R})^j \text{ for } 1 \leq j \leq d.$$

Let

$$g = h_1 \dots h_{D+1} \quad \text{and} \quad m = D + d.$$

Then $m \geq 1$ and

$$g = \widehat{X}_1^m + \sum_{1 \leq j \leq m} G_j \widehat{X}_1^{m-j} \quad \text{with} \quad G_j \in M(\widehat{R})^j \text{ for } 1 \leq j \leq m.$$

After making an SDT (= Shreedharacharya Transformation), i.e., writing $\widehat{X}_1$ for $\widehat{X}_1 + (G_1/m)$, we may assume that $G_1 = 0$. If $G_j = 0$ for

$2 \le j \le m$ then f has an NC at P relative to $(\widehat{X}_1, \ldots, \widehat{X}_n)$ and hence we have nothing more to show. So also assume that $m \ge 2$ and $G_j \ne 0$ for some j with $2 \le j \le m$.

Let $\widehat{V}$ be the local space K^{n-1} near the origin $\widehat{P} : \widehat{X}_2 = \cdots = \widehat{X}_n = 0$. By the basic induction hypothesis, i.e., by the $n-1$ case of the theorem, we can find an IAMT $\widehat{W}$ of $\widehat{V}$ such that, at every point $\widehat{Q}$ of $\widehat{W}$, the hypersurface

$$\widehat{X}_2^{r_2} \ldots \widehat{X}_n^{r_n} \left(\prod_{2 \le j \le m \text{ with } G_j \ne 0} G_j \right)$$

has an NC and the ideal

$$(G_2^{m!/2}, \ldots, G_m^{m!/m})$$

is principal. It follows that for some e with $2 \le e \le m$ we have

$$G_e \ne 0 \quad \text{and} \quad G_j^{m!/j} / G_e^{m!/e} \in \widehat{S} \text{ for } 2 \le j \le m$$

where $\widehat{S}$ is the local ring of $\widehat{Q}$. In view of the Reduction Lemma we may assume that the maximal ideal $M(\widehat{S})$ of $\widehat{S}$ has a basis $(\widetilde{Y}_2, \ldots, \widetilde{Y}_n)$ for which there are nonnegative integers $s_2, \ldots, s_n$ and $b_{j2}, \ldots, b_{jn}$ such that

$$(\widehat{X}_2^{r_2} \ldots \widehat{X}_n^{r_n}) / (\widetilde{Y}_2^{s_2} \ldots \widetilde{Y}_n^{s_n}) \in \widehat{S} \setminus M(\widehat{S})$$

and

$$G_j / (\widetilde{Y}_2^{b_{j2}} \ldots \widetilde{Y}_n^{b_{jn}}) \in \widehat{S} \setminus M(\widehat{S}) \text{ for all } j \text{ with } 2 \le j \le m \text{ and } G_j \ne 0$$

and

$$\bar{b}_{e2} + \cdots + \bar{b}_{en} < e$$

where, for every nonnegative integer b, we letting $\bar{b}$ denote the residue of b modulo e. Let $\widetilde{W}$ be the corresponding IAMT of V, let $\widetilde{Q}$ be the corresponding point of $\widetilde{W}$, and let $\widetilde{Y}_1 = \widehat{X}_1$. Then $(\widetilde{Y}_1, \ldots, \widetilde{Y}_n)$ is a basis of the maximal ideal $M(\widetilde{S})$ of the local ring $\widetilde{S}$ of $\widetilde{Q}$ on $\widetilde{W}$, and we have

$$f = h_0 \ldots h_{D+1} \quad \text{with} \quad h_0 / (\widetilde{Y}_2^{s_2} \ldots \widetilde{Y}_n^{s_n}) \in \widetilde{S} \setminus M(\widetilde{S})$$

and

$$h_i = \widetilde{Y}_1 + H_i \quad \text{with} \quad H_i \in M(\widehat{S}) \text{ for } 1 \leq i \leq D$$

and

$$h_{D+1} = \widetilde{Y}_1^d + \sum_{1 \leq j \leq d} L_j \widetilde{Y}_1^{d-j} \quad \text{with} \quad L_j \in M(\widehat{S})^j \text{ for } 1 \leq j \leq d$$

and

$$g = h_1 \dots h_{D+1} \quad \text{and} \quad m = D + d \geq 2$$

and

$$g = \widetilde{X}_1^m + \sum_{2 \leq j \leq m} G_j \widetilde{X}_1^{m-j} \quad \text{with} \quad G_j \in M(\widehat{S})^j \text{ for } 2 \leq j \leq m.$$

Now by the Good Point Lemma we can achieve a reduction in (d, D). This completes all the inductions, and hence proves the theorem.

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Index of Notation

Index of Definitions

List of Corrections

Here: $n.i$ = line i on page n, and $n.j^*$ = line j from the bottom on page n.

where	current material	corrected material or remark
ix.4*	*Bibliography* 283	*Bibliography* 285
24.7*	*be a pseudogeometric regular*	*be a regular*
53.1*	$(z/x')^{c'}R'$.	$(z/x')^{c}R'$.
62.15	hence $z \in R_{u(u(0))}$. Thus	hence $z \in R_{u(0)}$. Thus
63.17*	*every element in E has*	*every element in E' has*
64.4*	*every element in E has*	*every element in E' has*
126.14	Since $w/x^{*e} \in$ ... whenever $a \neq 0$,	delete the entire line
126.15	and hence t^*_{abc} ... $= x^*R'$ and	delete the entire line
126.16	hence t^*_{abc} ... Let $h' : R' \to$	Let $h' : R' \to$
126.19	$h'((z^*/x^*)^e R') \not\in h'((z^*/x^*)^{e+1}R')$	$h'((z^*/x^*)^{e+1}R')$
126.20	hence $h'((w/x^{*e})R') =$	hence $h'((w/x^{*e})R')$
126.21	$h'((z^*/x^*)^e R')$; since	$\not\in h'((z^*/x^*)^{e+1}R')$; since
168.14	$I(Z_n, X)$	$\mathfrak{I}(Z_n, X)$
176.22	$Z^* \cup \cdots \cup Z_n^*$	$\{\cup_{i \neq j}(Z_i \cap Z_j)\} \cup (Z_1^* \cup \cdots \cup Z_n^*)$
195.11	$R_0 \mathfrak{S}\mathfrak{S}^*(J_0)$	$R_0 \in \mathfrak{S}^*(J_0)$
220.4*	$\mathfrak{S}^*(J'') = \mathfrak{S}^*(\mathfrak{Z}(J''))$	$\mathfrak{S}^*(J'') = \mathfrak{S}(\mathfrak{Z}(J''))$
263.2*	*Then $\mathfrak{K}(A, P)$ is*	*Then $\mathfrak{R}(A, P)$ is*