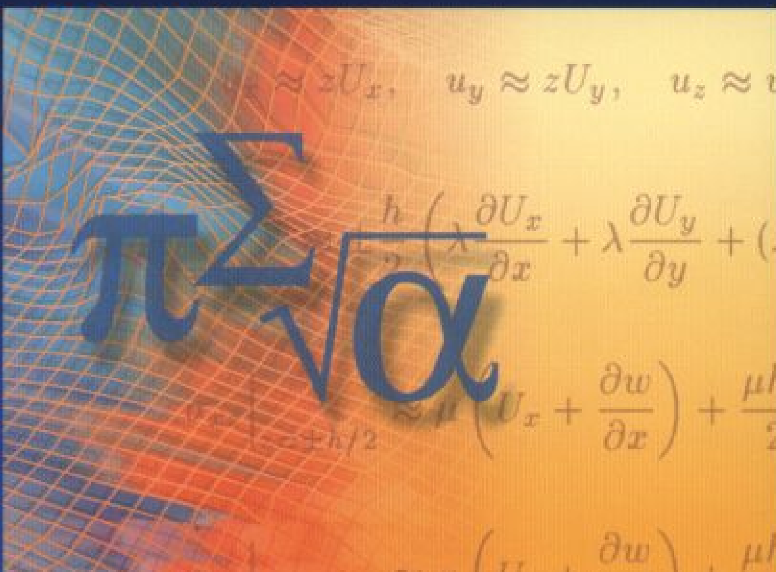
SERIES ON STABILITY, VIBRATION AND CONTROL OF SYSTEMS

Series A Volume 5

Generalized Point Models in Structural Mechanics

Ivan V. Andronov



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Preface

Most fields of human activity are influenced by phenomena of sound and vibration. Advances in scientific study of these phenomena have been driven by widespread occupance of technological processes in which interaction of sound and structural vibration is important. Examples abound in marine, aeronautical, mechanical and nuclear engineering, in physiological processes, geology, *etc.* Among thousands of works dealing with vibration of fluid loaded elastic plates and shells a noticeable place belongs to the analysis of specific physical effects simulated in simple models allowing exact analytical or almost analytical solution (up to algebraic equations and computation of integrals or series). Such are classical point models in hydroelasticity.

In recent years applications appeared which require higher accuracy of wave field representation both in fluid and in the structure than that achieved by the use of classical point models. With increasing accuracy it is desired to preserve simplicity of solution construction and analysis and not to violate mathematical correctness and rigorousity. All these can be achieved with the use of the technique of zero-range potentials. Zero-range potentials were first introduced by Fermi in 30-es for description of quantum mechanical phenomena. Later they came to mathematics as special selfadjoint perturbations of differential operators (see paper [29] by Beresin and Faddeev). At present applications of zero-range potentials are known not only in quantum mechanics, but also in diffraction by small slits in screens, analysis of resonators with small openings, simulation of scattering effects from small inclusions in electromagnetics and other fields.

This book introduces the idea of zero-range potentials to structural mechanics and allows generalized point models more accurate than classical ones to be constructed for obstacles presented both in the structure and in the fluid.

We discuss the zero-range potentials technique taking as an example one-sided fluid loaded thin elastic plate subject to flexural deformations described by Kirchhoff theory. Two and three dimensional problems of diffraction of stationary wave process are considered.

The ideas that form the basis of exposition combine specifics of boundary-value problems of hydro-elasticity and mathematically rigorous theory of operators and their extensions in Hilbert space. Detailed presentation of the theory of vibrations of thin-walled mechanical constructions was not in the scope of the exposition, believing that existing monographs on the theory of plates and shells can do that better. For the same reason the book does not present any complete list of literature. We cite only those directly related to the subject except some basic results with preference to Russian papers not much known to Western audience.

Nevertheless, the book contains some background material from the theory of flexure vibrations of thin elastic plates, it describes such important features of correctly set boundary-value problems as reciprocity principle and energy conservation law. The book contains a short introduction to the theory of operators in Hilbert space and describes particular spaces (L_2 and Sobolev spaces). Theory of supersingular integral equations is presented in the Appendix.

The first chapter presents some basic aspects of the theory of plates: it contains derivation of Kirchhoff model of flexural waves, which allows applicability of the approximation to be clarified; it describes general properties of scattering problems by thin elastic plates, conditions of correctness and uniqueness of solution; it discusses integral representation for the scattered field, used in the book for the analysis of particular problems of scattering, and presents important energetic identities such as optical theorem and reciprocity principle which are exploited for independent control of asymptotic and numerical results. Classical point models are subjected to more detailed analysis. Frequency and angular characteristics of scattering by clamped point, by stiffener of finite mass and momentum of inertia and by pointwise crack are presented for two examples of plate – fluid system. In one case the plate is heavily loaded by water, in the other it contacts light air. Peculiarities and general properties of scattered fields are discussed.

Chapter 2 gives a brief introduction to the theory of linear operators in

Hilbert space. It does not pretend to be complete, but may be used for getting acquainted to such objects as Hilbert space, symmetric and selfadjoint operators, operators extensions theory, generalized derivatives and Sobolev spaces. For more detailed and accurate presentation of these subjects the reader can refer to corresponding textbooks and recent developments in the perturbation theory of operators can be found in the book by S.Albeverio and P.Kurasov [2] and references listed there. Chapter 2 also formulates operator model adequate to the description of wave process in fluid loaded elastic plate and constructs zero-range potentials for this operator.

Analysis of the structure of the operator for fluid loaded plate, being two-component matrix one, permits the main hypothesis and basing on it procedure of generalized models construction to be proclaimed, which is done in Section 3.1. Other Sections of Chapter 3 deal with particular generalized models of inhomogeneities in fluid loaded thin elastic plates. Two-dimensional problem of diffraction by narrow crack is solved also in asymptotic approximation by integral equations method and allows the formulae written with the use of generalized model to be aposteriory justified. In three-dimensional case such justification is done for the generalized model of short crack. Solutions of diffraction problems by a round hole and by a narrow joint of two semi-infinite plates are considered in Chapter 3 with the use of generalized point models only. When examining auxiliary diffraction problems corresponding to isolated plates, Green's function method and method of Fourier transform is used to reduce the problems to integral equations of the convolution on an interval. For short crack the kernels of these equations are supersingular and for narrow joint these integral equations are solved in the class of nonintegrable functions. Theory of such integral equations and methods of their regularization are presented in the Appendix B.

In Chapter 4 the generalized models are analyzed from the point of view of accuracy, limitations and possible generalizations. The structure of generalized models and the reasons for the main hypothesis (of Section 3.1) to be true and the scheme of models construction to be successful are explained. An example of two-dimensional model of narrow crack generalization to the case of oblique incidence and to the analysis of edge waves is presented. Chapter 4 discusses also unsolved problems that may require further development of operator extensions theory.

We expect some mathematical background from the reader. When introducing a mathematical fact or formula for the first time a short explanation is included, and the index can help in finding that explanations in the book.

Appearance and development of the generalized models in structural mechanics based on operators extension theory began in late 80-s early 90-s in the time when after graduating St.Petersburg (at that time Leningrad) State University, I have caught excellent time for scientific research in the Department of Mathematical and Computational Physics of that University. My contacts with on one hand specialists in the field of application of mathematical physics to the theory of thin elastic plates such as B.P.Belinskiy and D.P.Kouzov and on the other hand with specialists in the theory of zero-range potentials, namely lectures of B.S.Pavlov and continuing discussions with P.B.Kurasov played invaluable role in the development of Generalized models theory in mechanics of fluid loaded elastic plates. Most of ideas were discussed at the seminars "On Wave Propagation" in St.Petersburg Branch of V.A.Steklov Mathematical Institute and "On Acoustics" held now in the Institute for Problems of in Mechanical Engineering.

I hope that dissemination of these ideas to a wider audience will be useful and bring to the use of the Generalized models in practical applications.

Pushkin, November 2001, *Ivan V. Andronov*

Contents

Chapter 1

Vibrations of Thin Elastic Plates and Classical Point Models

1.1 Kirchhoff model for flexural waves

1.1.1 *Fundamentals of elasticity*

The elastic properties of an isotropic body are described either by Lame coefficients λ and μ or by Young modulus E and Poison's ratio σ. These parameters are expressed via each other in the form

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \quad \mu = \frac{E}{2(1-\sigma)}.$$

The state of an elastic body under deformation is characterized by *strain tensor* $\boldsymbol{\varepsilon}$ and *stress tensor* $\boldsymbol{\sigma}$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{pmatrix}, \qquad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma xz \\ \sigma_{xy} & \sigma_{yy} & \sigma yz \\ \sigma_{xz} & \sigma_{yz} & \sigma zz \end{pmatrix}.$$

The diagonal elements of strain tensor describe relative elongations in the directions of x, y and z axes. The non-diagonal elements denote shear deformations in the corresponding planes. The volumetric strain or *dilatation* Θ is given by the trace of strain tensor

$$\Theta = \mathrm{Tr}\,\boldsymbol{\varepsilon} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}.$$

Let the displacements in an elastic body be given by vector function $\mathbf{u}(x, y, z) = (u_x, u_y, u_z)$, then the components of strain tensor can be expressed as follows

$$\varepsilon_{ii} = \frac{\partial u_i}{\partial i}, \qquad \varepsilon_{ij} = \frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}.$$

Here and below in this section subscripts i and j take the values x, y and z and j does not coincide with i.

The deformations ε cause stresses to appear. The diagonal components of *stress tensor* characterize normal stresses and non-diagonal components give shear stresses. In an isotropic material the stress and strain tensors are connected by Lame equations

$$\sigma_{ii} = \lambda\Theta + 2\mu\varepsilon_{ii}, \qquad \sigma_{ij} = \mu\varepsilon_{ij}. \tag{1.1}$$

The potential energy of an elastic body which undergoes deformations ε is given by the volume integral

$$P = \frac{1}{2}\iiint\Big(\varepsilon_{xx}\sigma_{xx} + \varepsilon_{yy}\sigma_{yy} + \varepsilon_{zz}\sigma_{zz} + \varepsilon_{xy}\sigma_{xy} + \varepsilon_{xz}\sigma_{xz} + \varepsilon_{yz}\sigma_{yz}\Big)\,dx\,dy\,dz.$$

Suppose that deformations ε are caused by external forces $\mathbf{f}(x, y, z)$. Then the energy becomes

$$P_f = P - \iiint (f_x u_x + f_y u_y + f_z u_z)\,dx\,dy\,dz. \tag{1.2}$$

According to the minimum energy principal the displacements $\mathbf{u}(x, y, z)$ in the elastic body are such that the total energy P_f is minimal. That is any problem of elasticity is equivalent to minimization of the functional (1.2) [52]. The class of functions is restricted by boundary conditions that should be satisfied on the surface of elastic body. The whole variety of boundary conditions can not be discussed here. Note only that on the fixed surface displacements are equal to zero and on the free surface stresses $\sigma_{nn}, \sigma_{nt_1}, \sigma_{nt_2}$ vanish (Here n stands for the normal to the surface and t_1 and t_2 are tangential directions).

1.1.2 *Flexural deformations of thin plates*

The problems of elasticity allow simple solutions to be found only in a small number of special geometries. In problems that contain small or large parameters asymptotic methods can be used.

Consider now a thin elastic layer and make asymptotic simplifications before solving the problem, that is at the stage of problem formulation. Let the Cartesian coordinates be chosen such that the midplane of the layer coincides with xy plane and the faces be at $z = \pm h/2$. Assuming h small compared to all other parameters of the problem allows the displacements, strains and stresses to be decomposed into series by z. It can be checked that potential energy splits into three parts corresponding to flexural, symmetric and shear deformations. The even terms in the series for u_z and odd terms in the series for u_x and u_y correspond to flexural deformations. Only these terms are considered below. To derive the principal order model for flexural waves it is sufficient only to keep terms up to quadratic in z in the series for displacements, that is take

$$u_x \approx zU_x, \quad u_y \approx zU_y, \quad u_z \approx w + \frac{z^2}{2}W.$$

Here U_x, U_y, w and W are functions of x and y only. Satisfying the free faces conditions at $z = \pm h/2$ allows all the functions to be expressed in terms of $w(x, y)$. For this substitute the above approximations for $\mathbf{u}$ into Lame equations, this yields

$$\sigma_{zz}\Big|_{z=\pm h/2} \approx \pm\frac{h}{2}\left(\lambda\frac{\partial U_x}{\partial x} + \lambda\frac{\partial U_y}{\partial y} + (\lambda + 2\mu)W\right) = 0,$$

$$\sigma_{xz}\Big|_{z=\pm h/2} \approx \mu\left(U_x + \frac{\partial w}{\partial x}\right) + \frac{\mu h^2}{2}\frac{\partial W}{\partial x} = 0,$$

$$\sigma_{yz}\Big|_{z=\pm h/2} \approx \mu\left(U_y + \frac{\partial w}{\partial y}\right) + \frac{\mu h^2}{2}\frac{\partial W}{\partial y} = 0.$$

Neglecting the terms containing h^2 in the last two equations, yields

$$U_x \approx -\frac{\partial w}{\partial x}, \quad U_y \approx -\frac{\partial w}{\partial y}, \quad W \approx -\frac{\lambda}{\lambda + 2\mu}\Delta w.$$

Here Δ denotes Laplace operator on the midplane of the layer. The applicability condition for the above relations can be written as

$$h|\nabla w| \ll |w|. \tag{1.3}$$

Computing the nonzero elements of the stress tensor and calculating integral by z allows the potential energy (1.2) to be written as the surface integral

$$P_f = \frac{D}{2}\iint \left((\triangle w)^2 + 2(1-\sigma)\left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \right) \right) dx\, dy \\ - \iint F w \, dx\, dy. \tag{1.4}$$

Here the Lame coefficients are expressed via Young modulus E and Poison's ratio σ and the *bending stiffness* of the plate D is introduced as

$$D = \frac{Eh^3}{12(1-\sigma^2)}.$$

The external force F in (1.4) is the integral of $f_z(x, y, z)$ from (1.2) by the thickness of the plate and the smaller order terms in the last integral are neglected. One can accept that $F(x, y)$ presents the difference of forces applied to the faces of the plate.

The formula (1.4) expresses the energy of the plate in the form of the functional of $w(x, y)$. This allows the z coordinate to be excluded and the problems of elasticity for thin plates to be reformulated in terms of midplane displacements only.

It is convenient also to rewrite the formula (1.4) in another form. The integral

$$I = \iint \left(\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \right) dx\, dy$$

can be rewritten as contour integral. Let the plate occupy the domain Ω on the midplane xy, and let $\partial\Omega$ be a smooth contour. Then integrating by parts in I yields

$$I = -\int_{\partial\Omega} \frac{\partial w}{\partial x}\frac{d}{ds}\left(\frac{\partial w}{\partial y}\right) ds.$$

Here s is the arc-length measured from some fixed point along the contour $\partial\Omega$. In the above contour integral one can integrate by parts. The smoothness of the contour $\partial\Omega$ yields absence of substitutes. Thus one finds the representation

$$I = \frac{1}{2}\int_{\partial\Omega} \left(\frac{\partial w}{\partial y}\frac{d}{ds}\left(\frac{\partial w}{\partial x}\right) - \frac{\partial w}{\partial x}\frac{d}{ds}\left(\frac{\partial w}{\partial y}\right) \right) ds.$$

The derivatives by x and y can be expressed in terms of derivatives by the normal ν and tangential s coordinates

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial \nu}\nu_x + \frac{\partial w}{\partial s}s_x, \quad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial \nu}\nu_y + \frac{\partial w}{\partial s}s_y.$$

Here (ν_x, ν_y) are the coordinates of unit normal and (s_x, s_y) are the coordinates of unit tangential vector. Differentiation by s in the above integral is applied both to the displacement w and to the unit vectors of local coordinates. Introducing radius of curvature $R(s)$ and using *Frenet formulae*

$$\frac{d\vec{\nu}}{ds} = \frac{1}{R}\vec{s}, \quad \frac{d\vec{s}}{ds} = \frac{1}{R}\vec{\nu}$$

the integral I can be written as

$$\begin{aligned} I=&\frac{1}{2}\int\limits_{\partial\Omega}\left(\frac{\partial w}{\partial \nu}\nu_y - \frac{\partial w}{\partial s}\nu_x\right)\left(\frac{\partial^2 w}{\partial \nu \partial s}\nu_x + \frac{\partial^2 w}{\partial s^2}\nu_y + \frac{1}{R}\frac{\partial w}{\partial \nu}\nu_y + \frac{1}{R}\frac{\partial w}{\partial s}\nu_x\right) ds \\ &-\frac{1}{2}\int\limits_{\partial\Omega}\left(\frac{\partial w}{\partial \nu}\nu_x + \frac{\partial w}{\partial s}\nu_y\right)\left(\frac{\partial^2 w}{\partial \nu \partial s}\nu_y - \frac{\partial^2 w}{\partial s^2}\nu_x - \frac{1}{R}\frac{\partial w}{\partial \nu}\nu_x + \frac{1}{R}\frac{\partial w}{\partial s}\nu_y\right) ds. \end{aligned}$$

In the above formulae the components of unit vector $\vec{s}$ are expressed via the components of the unit normal vector $s_x = \nu_y$, $s_y = -\nu_x$. Simplifying the integrand and noting that $\nu_x^2 + \nu_y^2 = 1$, yields

$$I = \frac{1}{2}\int\limits_{\partial\Omega}\left(\frac{\partial w}{\partial \nu}\frac{\partial^2 w}{\partial s^2} + \frac{1}{R}\left(\frac{\partial w}{\partial \nu}\right)^2 - \frac{\partial w}{\partial s}\frac{\partial^2 w}{\partial \nu \partial s} - \frac{1}{R}\left(\frac{\partial w}{\partial s}\right)^2\right) ds.$$

Finally integrating in the last two terms by parts one gets

$$\begin{aligned} P_f=&\iint\limits_{\Omega}\left(\frac{D}{2}(\triangle w)^2 - Fw\right) dx\, dy \\ &+\frac{D(1-\sigma)}{2}\int\limits_{\partial\Omega} w\left(\frac{\partial^3 w}{\partial \nu \partial s^2} + \frac{d}{ds}\left(\frac{1}{R}\frac{\partial w}{\partial s}\right)\right) ds \\ &+\frac{D(1-\sigma)}{2}\int\limits_{\partial\Omega}\frac{\partial w}{\partial \nu}\left(\frac{\partial^2 w}{\partial s^2} + \frac{1}{R}\frac{\partial w}{\partial \nu}\right) ds. \end{aligned}$$

If the contour $\partial\Omega$ is not smooth, then at every corner point the substitutes $F_c w$ appear when integration by parts is performed in the above derivations.

Here F_c denotes "corner" forces

$$F_c = \mathbb{F}_c w = D(1-\sigma)\left[\frac{\partial^2 w}{\partial\nu\partial s} - \frac{1}{R}\frac{\partial w}{\partial s}\right] \tag{1.5}$$

and by square brackets the jump of the enclosed expression is denoted.

1.1.3 *Differential operator and boundary conditions*

In real problems plates have edges, can be supported by stiffeners or joint to each other. In the problems of elasticity all these cases are described by some boundary conditions. To reformulate these conditions as boundary conditions for the displacement $w(x,y)$ one needs to know expressions for the angle of rotation of the plate, for bending momentum and force. It may be also convenient to formulate the problems as boundary value problems for a differential operator. The quadratic form P defines this operator. Applying Green's formula

$$\iint\limits_{\Omega}\Big(V\triangle U - U\triangle V\Big)d\Omega = \int_{\partial\Omega}\left(V\frac{\partial U}{\partial n} - U\frac{\partial V}{\partial n}\right)dS. \tag{1.6}$$

in the double integral of P_f one finds that displacements $w(x,y)$ satisfy the equilibrium equation

$$D\triangle^2 w = F, \quad (x,y)\in\Omega$$

and the boundary conditions should be such that the variation of the contour integral over $\partial\Omega$ vanishes, that is

$$\delta\int\limits_{\partial\Omega}\Big(w\,\mathbb{F}w + \frac{\partial w}{\partial\nu}\,\mathbb{M}w\Big)\,ds = 0. \tag{1.7}$$

Here the operators $\mathbb{F}$ and $\mathbb{M}$ on the contour $\partial\Omega$ are introduced by the formulae

$$\begin{aligned}\mathbb{F} &= -D\left(\frac{\partial}{\partial\nu}\triangle + (1-\sigma)\left(\frac{\partial^3}{\partial\nu\partial s^2} - \frac{d}{ds}\frac{1}{R}\frac{\partial^2}{\partial s^2}\right)\right),\\ \mathbb{M} &= D\left(\triangle - (1-\sigma)\left(\frac{\partial^2}{\partial s^2} + \frac{1}{R}\frac{\partial}{\partial\nu}\right)\right).\end{aligned} \tag{1.8}$$

To find the physics meaning of the boundary conditions one needs to identify displacement, angle of rotation, bending momentum and force in the integral (1.7). Generally speaking the formulae of plate theory do not allow continuation up to the edge, support or other inhomogeneity in the plate. Indeed the formulae of the previous section are derived in the supposition of infinite plate in x and y directions. If the layer is finite or if some body is attached to it, corner points appear. Near such corners smoothness of displacements is violated and the applicability condition (1.3) becomes not valid in a vicinity of order $O(h)$. Nevertheless, let the formulae of plate theory be extended up to the edge or line of support of the plate*. Then $w(x,y)$ stands for the displacement and $\partial w/\partial\nu$ specifies the angle of rotation of plate in the plane νz. Knowing that derivative of energy by displacement gives force and derivative of energy by angle of rotation gives bending momentum, one concludes that expressions $\mathbb{F}w$ and $\mathbb{M}w$ with the operators (1.8) express *force* and *bending momentum* at the edge of the plate.

Consider here some possible types of boundary conditions on the contour $\partial\Omega$. If the edge of the plate is *clamped*, its displacements and angles of rotation are equal to zero

$$w = 0, \quad \frac{\partial w}{\partial \nu} = 0.$$

On the contrary *free edge* is described by conditions

$$\mathbb{F}w = 0, \quad \mathbb{M}w = 0.$$

1.1.4 *Flexural waves*

The equations of free vibrations of thin elastic plate can be obtained from the equilibrium equation if the product of acceleration by surface mass is subtracted from the force F

$$D\Delta^2 w + \varrho h \frac{\partial^2 w}{\partial t^2} = F(x,y,t).$$

Here ϱ is the density of the plate material.

In the case of harmonic vibrations all the quantities are supposed dependent on time by the factor $\exp(-i\omega t)$ and due to phase shifts the function

*In some geometries this causes nonphysical results. For example near tips of cracks, sharp supports and inclusions stresses computed by plate theory have singularities. In reality such singularities are evidently not presented, but a kind of afterwards regularization allows *stress intensity coefficients* introduced as coefficients of singular terms, to characterize the probability of crack growth. See details in section 3.3.1.

w becomes complex[†]. Introducing the *wave number* of flexural waves k_0 in an isolated plate by the formula

$$k_0^4 = \frac{\omega^2 \varrho h}{D},$$

the equation for harmonic *flexural waves* can be written as

$$\Delta^2 w - k_0^4 w = \frac{1}{D} F(x, y). \tag{1.9}$$

The formula for the wavenumber k_0 shows that velocity c_f of flexural waves depends on frequency

$$c_f = \left(\frac{D}{\varrho h}\right)^{1/4} \omega^{1/2}.$$

Let $F = 0$. Multiplying the equation (1.9) by complex conjugate[‡] of w and integrating over the plate yields

$$\frac{D\omega}{4} \iint\limits_{\Omega} \left(|\Delta w|^2 + 2(1-\sigma)\left(\left|\frac{\partial^2 w}{\partial x \partial y}\right|^2 - \left|\frac{\partial^2 w}{\partial x^2}\right| \left|\frac{\partial^2 w}{\partial y^2}\right| \right) \right) dx\, dy$$
$$= \frac{D\omega}{4} k_0^4 \iint\limits_{\Omega} |w|^2\, dx\, dy$$

This formula expresses the well known equality of average potential and average kinetic energy of a vibrating body. Indeed, expressing displacements as the real part of w

$$\mathrm{Re} w = |w| \cos(\arg w - i\omega t)$$

and substituting into (1.4) causes w to be replaced by absolute values $|w|$ and the multiplier $\cos^2(\arg w - i\omega t)$ to appear in the integrand. Calculating the average by time t over the period $T = 1/\omega$ transforms this multiplier to $\omega/2$. Thus the left-hand side of the above identity gives the average potential energy of vibrating plate and its right-hand side gives average kinetic energy.

[†] Real part is assumed in final formulae for physical characteristics of the wave process.

[‡] Complex conjugation is denoted by overline.

The total energy is the sum of potential energy P_f given by (1.4) and kinetic energy

$$E = P_f + \frac{1}{2}\varrho h(\mathrm{Re}\dot{w})^2$$

Variation of the total energy in domain Ω is

$$\dot{E} = D\iint\limits_{\Omega}\left(\mathrm{Re}\,(\triangle w)\mathrm{Re}\,(\triangle \dot{w}) + (1-\sigma)\left(2\mathrm{Re}\,\left(\frac{\partial^2 w}{\partial x \partial y}\right)\mathrm{Re}\,\left(\frac{\partial^2 \dot{w}}{\partial x \partial y}\right)\right.\right.$$

$$\left.\left. - \mathrm{Re}\,\left(\frac{\partial^2 w}{\partial x^2}\right)\mathrm{Re}\,\left(\frac{\partial^2 \dot{w}}{\partial y^2}\right) - \mathrm{Re}\,\left(\frac{\partial^2 w}{\partial y^2}\right)\mathrm{Re}\,\left(\frac{\partial^2 \dot{w}}{\partial x^2}\right)\right)\right) dx\, dy$$

$$+ \varrho h \iint\limits_{\Omega} \mathrm{Re}\,\dot{w}\mathrm{Re}\,\ddot{w}\, dx\, dy.$$

Here dot denotes derivative by time. Acceleration $\ddot{w}$ can be expressed with the help of dynamics equation, after which integrating by parts yields

$$\dot{E} = \int\limits_{\partial\Omega}\left(\mathrm{Re}\,(\mathbb{F}w)\,\mathrm{Re}\,\dot{w} + \mathrm{Re}\,(\mathbb{M}w)\,\mathrm{Re}\,\left(\frac{\partial \dot{w}}{\partial \nu}\right)\right) ds.$$

Taking average by the period of oscillations one finds

$$\omega\int\limits_0^{1/\omega}\dot{E}dt = \frac{\omega}{2}\mathrm{Im}\int\limits_{\partial\Omega}\left(\mathbb{F}w\,\overline{w} + \mathbb{M}w\,\frac{\partial\overline{w}}{\partial\nu}\right) ds.$$

The integrand in the right-hand side of the above formula is the average *energy flux* density through the contour.

1.2 Fluid loaded plates

In the first order approximation for sufficiently rigid plates the influence of external media can be neglected. However more accurate considerations require the full system of plate and surrounding fluid to be studied. Differently from vibrations of elastic bodies, wave processes in fluids can be more conveniently described in terms of pressure, but not displacements. Under the assumption of the time factor $\exp(-i\omega t)$ same as in the previous section, the *acoustic pressure* satisfies Helmholtz equation

$$\triangle U(x, y, z) + k^2 U(x, y, z) = 0. \tag{1.10}$$

Here the wave number $k = \omega/c_a$ and c_a is the acoustic waves velocity in the fluid.

Acoustic pressure U causes external forces to be applied to the faces of the plate. If acoustic media is presented only on one side of the plate (let at $z > 0$), this results in the right-hand side in the equation (1.9)

$$\Delta^2 w - k_0^4 w = -\frac{1}{D} U(x, y, 0). \tag{1.11}$$

Note that due to small thickness of the plate $U(x, y, h/2)$ is replaced in (1.11) by $U(x, y, 0)$. Thus geometrically plate is considered infinitely thin and its thickness is presented in the equations only via bending stiffness D and wave number k_0. Evidently such simplification is possible if the wavelength in the acoustic media is small compared to h. That is besides the condition (1.3) the other applicability condition can be written as

$$kh \ll 1. \tag{1.12}$$

The plate displacements $w(x, y)$ and displacements in fluid in a vicinity of the plate coincide. This is described by *adhesion condition*

$$w(x, y) = \frac{1}{\varrho_0 \omega^2} \frac{\partial U(x, y, 0)}{\partial z}.$$

Here ϱ_0 is the density of fluid.

It is also convenient to rewrite the equations (1.11) and adhession condition as the *generalized impedance boundary condition* for acoustic pressure

$$\left(\Delta - k_0^4\right) \frac{\partial U(x, y, 0)}{\partial z} + N\, U(x, y, 0) = 0. \tag{1.13}$$

Here $N = \omega^2 \varrho_0 D^{-1}$. The boundary condition (1.13) allows the surface wave to propagate along the plate. Taking

$$U = A \exp(\pm i\kappa x - \sqrt{\kappa^2 - k^2} z),$$

with an arbitrary amplitude A, one gets the dispersion equation for the wave number κ

$$\left(\kappa^4 - k_0^4\right) \sqrt{\kappa^2 - k^2} - N = 0. \tag{1.14}$$

This dispersion equation can be reduced to the 5-th order polynomial equation with respect to κ^2. Two real solutions $\kappa_0 = -\kappa_5 = \kappa$ correspond to surface waves. Note that $\kappa > \max(k, k_0)$.

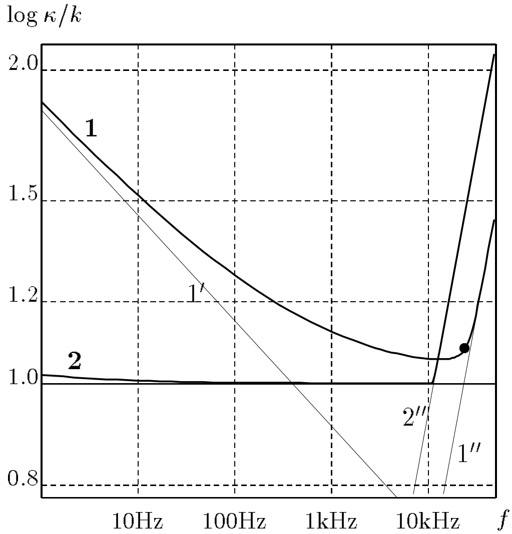


Fig. 1.1 Wave numbers of flexural waves in fluid loaded plates.

The characteristic dependence of the positive solution κ of the equation (1.14) as a function of frequency is illustrated on Fig. 1.1. The ratio κ/k_0 is plotted in logarithmic scale. Curve **1** corresponds to the steel plate of $h = 1$cm bounding water half-space. Curve **2** corresponds to 1mm steel plate being in one-side contact with air. Bullet on curve **1** characterizes the applicability of the model, it marks the frequency for which $kh = 1$. For the curve **2** this frequency exceeds the shown range.

In the low frequency approximation the solutions of the dispersion equation are (see Appendix A)

$$\kappa_j \approx N^{1/5} \exp\left(i\frac{\pi}{10}j\right), \quad j = 0, 1, \ldots 9. \tag{1.15}$$

That is the wavenumber κ appears proportional to $\omega^{2/5}$. Line $1'$ on Fig. 1.1 presents the asymptotics (1.15). Same asymptotics for the case of plate in air is valid only for very low frequencies and is not shown.

An important characteristics of the wave process in a fluid loaded plate is the *coincidence frequency*, that is such frequency f_c for which $c_f = c_a$. Lines $1''$ and $2''$ present the wavenumbers k in water and in air correspondingly. Their crossing points with the horizontal line mark coincidence frequencies.

Above coincidence frequency the wave number κ approaches k and the surface wave behaves in air as a "piston" wave. For water loaded plates coincidence frequency usually lies outside the frequency range where plate theory is applicable.

1.3 Scattering problems and general properties of solutions

1.3.1 *Problem formulation*

Consider problems of scattering in fluid loaded thin elastic plates. Such problems are formulated as boundary value problems for the equation (1.10) with the generalized impedance boundary condition (1.13) on the plate and some type of boundary and *contact conditions*[§] on the obstacle. Two types of obstacles are considered. The domain Ω can be bounded. In this case the scattering problem is three-dimensional. The other type of obstacles is infinite cylindrical obstacle $\Omega \times \mathbb{R}$ with a bounded cross-section Ω. If the incident field does not depend on y coordinate, the scattering problem is reduced to two-dimensional boundary value problem.

The domains for the equation (1.10) and for the operator impedance in (1.13) are unbounded, therefore radiation conditions should be formulated at infinity. Physically these conditions mean that except for the incident wave all other components of the elasto-acoustic field can carry energy only to infinity. Two types of waves are possible at infinity. These are spatial waves in fluid and surface waves appearing due to joint oscillations of plate and fluid. Thus the *radiation condition* can be written in the form of asymptotics [36] ($d = 2, 3$ is the dimension of the problem)

$$\begin{aligned} &\left(\frac{\partial}{\partial r} - ik\right) U = o\left(r^{\frac{1-d}{2}}\right), \quad r \to +\infty, \vartheta > \vartheta_0 > 0 \\ &\left(\frac{\partial}{\partial \rho} - i\kappa\right) w = o\left(\rho^{\frac{2-d}{2}}\right), \quad \rho \to +\infty \end{aligned} \tag{1.16}$$

Here $r = \sqrt{x^2 + y^2 + z^2}$ is the radius in spherical coordinate system and $\rho = \sqrt{x^2 + y^2}$ is its projection on the plate. The azimuthal angle ϑ is introduced such that $\rho/r = \cos\vartheta$. The first asymptotics in (1.16) is the usual radiation condition for acoustic waves. In the presence of infinite

[§]Contact conditions were first introduced by Krasilnikov in [47].

plate which allows surface waves to propagate this asymptotics becomes nonuniform and is valid at some distance from the plate. The second asymptotics is the radiation condition for flexural waves in fluid loaded plate. The wavenumber of these waves is defined by the dispersion equation (1.14).

The obstacle Ω and the boundary conditions on it can be of any type. It can be a rigid or elastic body joint to the plate or be separated from it, or it can be a set of holes in the plate. The only important restriction is that the boundary conditions on $\partial\Omega$ should be such that

$$E_\Omega = \mathrm{Im}\left(\frac{1}{2\varrho\omega} \iint\limits_{\partial\Omega} U \frac{\partial \overline{U}}{\partial n} dS + \frac{\omega}{2} \int\limits_{\partial\Omega_0} \left(\mathbb{F}w\,\overline{w} + \mathbb{M}w\,\frac{\partial \overline{w}}{\partial \nu} \right) ds + \frac{\omega}{2} \sum F_c \overline{w} \right) \geq 0. \tag{1.17}$$

Here n is the internal normal to the surface of the scatterer, Ω_0 is the domain where the obstacle is joint to the plate (in the case of holes, Ω_0 is union of all the holes), $\partial\Omega_0$ is the contour of Ω_0 on the plate and ν is the internal normal to this contour, overline stands for complex conjugation. Summation in (1.17) is carried on all corner points of $\partial\Omega_0$ and in the case of nonsmooth $\partial\Omega$ the integral is assumed as a limit from the exterior of Ω (see Meixner conditions below).

The inequality (1.17) sets restrictions on the boundary conditions satisfied on $\partial\Omega$ and $\partial\Omega_0$. If the boundaries have breaks or curves where boundary conditions change, in all such points the *Meixner conditions* should be specified [36]. That is, the integration is carried along spherical or cylindrical surfaces of small radius ε surrounding singularities of the boundary and then the limit for $\varepsilon \to 0$ is taken. The condition saying that the above limit is equal to zero is equivalent to the boundedness of the energy concentrated in any domain near the singular point. In particular for the acoustic pressure one assumes that

$$\nabla U = O\left(\varepsilon^{-\frac{1}{2}d+\delta}\right), \quad \delta > 0. \tag{1.18}$$

For flexural deformations the Meixner condition says

$$\Delta w = O\left(\varepsilon^{-1+\delta}\right), \quad \delta > 0. \tag{1.19}$$

We do not consider examples of correct conditions on the scatterer, note only that Neumann or Dirichlet boundary conditions for U cause the left-hand side of inequality (1.17) to be equal to zero.

From the point of view of physics E_Ω expresses the energy absorbed by the obstacle Ω. The first term in (1.17) gives the average energy flux carried by the acoustic field through the surface $\partial\Omega$ and the other terms give energy fluxes carried by flexural deformations (compare with the last formula in section 1.1.4).

1.3.2 *Green's function of unperturbed problem*

Green's function of unperturbed (without obstacle) problem is the universal tool that allows the scattering problem to be reduced to integral equations on the obstacle. Green's function represents the field of a point source. In the system plate–fluid two types of sources are possible, namely acoustic point source represented by Dirac's delta-function in the right-hand side of Helmholtz equation

$$\triangle U + k^2 U = -\delta(x - x_0)\delta(y - y_0)\delta(z - z_0),$$

and the other source is a concentrated force applied to the plate

$$\left(\triangle^2 - k_0^4\right) \frac{\partial U(x, y, 0)}{\partial z} + NU(x, y, 0) = \frac{1}{D}\delta(x - x_0)\delta(y - y_0).$$

Let the corresponding Green's functions be distinguished by the number of their arguments ($\mathbf{r}$ is three-dimensional vector and $\boldsymbol{\rho}$ is two-dimensional vector in the plane of the plate), $G(\mathbf{r}, \mathbf{r}_0)$ stands for the field of acoustic source and $G(\mathbf{r}, \boldsymbol{\rho}_0)$ denotes the field of the point force, and let $g(\boldsymbol{\rho}, \mathbf{r}_0)$ and $g(\boldsymbol{\rho}, \boldsymbol{\rho}_0)$ be the corresponding flexural displacements in the plate.

Both Green's functions satisfy the radiation conditions (1.16) at infinity.

The Green's functions can be constructed explicitly in the form of Fourier integrals. One can check the *reciprocity principle* and express all the functions via $G(x, y, z; x_0, y_0, z_0)$ by the formulae

$$G(\mathbf{r}, \boldsymbol{\rho}_0) = \frac{1}{\varrho_0 \omega^2} \frac{\partial G(\mathbf{r}, \boldsymbol{\rho}_0, 0)}{\partial z_0}, \qquad g = \frac{1}{\varrho_0 \omega^2} \frac{\partial G}{\partial z}.$$

The last formula is valid both in the case of acoustic and in the case flexural sources. Applying Fourier transform by x and y one finds

$$\begin{aligned} G(\mathbf{r},\mathbf{r}_0) =& \frac{1}{8\pi^2}\iint \exp\Big(i\lambda(x-x_0)+i\mu(y-y_0)\Big) \\ & \times\left(e^{-\gamma|z-z_0|}+\frac{L(\lambda,\mu)+2N}{L(\lambda,\mu)}e^{-\gamma(z+z_0)}\right)\frac{d\lambda d\mu}{\gamma}. \end{aligned} \tag{1.20}$$

For surface source this yields

$$G(\mathbf{r},\boldsymbol{\rho}_0) = -\frac{1}{4\pi^2 D}\iint \exp\Big(i\lambda(x-x_0)+i\mu(y-y_0)-\gamma z\Big)\frac{d\lambda d\mu}{L(\lambda,\mu)}. \tag{1.21}$$

Here

$$\gamma=\sqrt{\lambda^2+\mu^2-k^2}, \quad L(\lambda,\mu)=\left(\left(\lambda^2+\mu^2\right)^2-k_0^4\right)\gamma-N$$

The integration in the formulae (1.20) and (1.21) is carried along real axes of λ and μ avoiding singularities on positive semi-axes from below and on negative semi-axes from above. Such paths of integration accord to the radiation conditions (1.16), which is justified below.

Consider the asymptotics of the Green's functions at large distances from the source. For this use saddle point method [33]. This method is applied to integrals of the form

$$\int_C e^{ip\Phi(s)}f(s)ds,$$

where p is the large parameter and Φ and f are analytic functions of s. The path of integration C can be arbitrarily deformed into C'. One can achieve that $\operatorname{Re}\Phi =$ conts on C'. Then exponential factor does not oscillate. Then the main contribution gives the point on C' where $\operatorname{Im}\Phi$ is minimal. This is the *saddle point* defined by the equation

$$\frac{d\Phi(s)}{ds}=0.$$

Decomposing functions $\Phi(s)$ and $f(s)$ into Taylor series near the saddle point and computing integral of the principal order terms yields the saddle point asymptotics¶

$$\int e^{ip\Phi(s)} f(s)ds \sim \sqrt{\frac{2\pi}{p|\Phi''|}} f \exp\Big(ip\Phi + \tfrac{i}{4}\pi \text{sign}\,(\Phi'')\Big). \tag{1.22}$$

Here the phase function Φ, its second order derivative Φ'' and function f in the right-hand side are calculated in the saddle point.

Analogously one can derive the saddle point asymptotics in the case of double integral

$$\iint e^{ip\Phi(s_1,s_2)} f(s_1,s_2)ds_1 ds_2 \sim \frac{2\pi}{p\sqrt{|\Phi''|}} f \exp\Big(ip\Phi + \tfrac{i}{4}\pi \text{sgn}(\mathbf{\Phi}'')\Big). \tag{1.23}$$

Again functions Φ and f and the matrix $\mathbf{\Phi}''$ of second order derivatives

$$\mathbf{\Phi}'' = \begin{pmatrix} \partial^2\Phi/\partial s_1^2 & \partial^2\Phi/\partial s_1\partial s_2 \\ \partial^2\Phi/\partial s_1\partial s_2 & \partial^2\Phi/\partial s_2^2 \end{pmatrix}$$

are computed in the saddle point which is defined by the system of equations

$$\frac{\partial\Phi(s_1,s_2)}{\partial s_1} = 0, \qquad \frac{\partial\Phi(s_1,s_2)}{\partial s_2} = 0.$$

By $|\mathbf{\Phi}''|$ the determinant of the matrix $\mathbf{\Phi}''$ is denoted and sgn stands for the sign of matrix, which is equal to the difference of the number of positive and the number of negative eigenvalues.

The contribution of the two-dimensional saddle point (1.23) gives the outgoing spherical wave

$$G \sim \frac{2\pi}{kr} e^{ikr - i\pi/2} \Psi_g(\vartheta,\varphi), \qquad r \to +\infty.$$

The residue in the pole corresponding to $\lambda^2 + \mu^2 = \kappa^2$ gives the surface wave

$$G \sim \sqrt{\frac{2\pi}{\kappa\rho}} e^{i\kappa\rho - i\pi/4} \psi_g(\varphi) e^{-\sqrt{\kappa^2-k^2}z}, \qquad \rho \to +\infty.$$

¶Asymptotics is similar to the stationary phase asymptotics, though saddle point method is more general and allows contributions of poles of $f(s)$ to be calculated. When the path C is deformed to the steepest descent contour C' some poles of $f(s)$ may be crossed, then corresponding residues appear in the right-hand side of formula (1.22).

The above asymptotic decompositions show that the field G satisfies the radiation conditions (1.16). The far fields amplitudes in the above asymptotics are given by explicit expressions

$$\begin{aligned}\Psi_g(\vartheta,\varphi) = \frac{ik}{8\pi^2}\exp(-ikx_0\cos\vartheta\sin\varphi - iky_0\cos\vartheta\cos\varphi) \\ \times\left\{\exp(-ikz_0\sin\vartheta) + R(\vartheta)\exp(ikz_0\sin\vartheta)\right\},\end{aligned} \tag{1.24}$$

$$\psi_g(\varphi) = \frac{i}{2\pi\kappa}\exp\left(-i\kappa x_0\sin\varphi - i\kappa y_0\cos\varphi - \sqrt{\kappa^2-k^2}z_0\right)\times \\ \times\frac{N}{5\kappa^4 - 4k^2\kappa^2 - k_0^4},$$

where

$$R(\vartheta) = \frac{\mathcal{L}(\vartheta) - 2N}{\mathcal{L}(\vartheta)}, \qquad \mathcal{L}(\vartheta) = ik\sin\vartheta(k^4\cos^4\vartheta - k_0^4) + N. \tag{1.25}$$

is the reflection coefficient for a plane wave incident on the plate at angle ϑ. One can notice that functions $\Psi_g(\vartheta,\varphi)$ and $\psi_g(\varphi)$ are connected by the formula

$$\begin{aligned}\psi_g(\varphi) = -2\pi i\frac{\kappa}{k}\operatorname*{Res}_{\vartheta=\vartheta^*}\Psi_g(\vartheta,\varphi), \\ \vartheta^* = \cos^{-1}\left(\frac{\kappa}{k}\right) = i\ln\left(\frac{\kappa}{k} + \sqrt{\left(\frac{\kappa}{k}\right)^2 - 1}\right).\end{aligned} \tag{1.26}$$

It is convenient to change the variables λ,μ in (1.20) to α,β by the formulae $\lambda = k\cos\alpha\sin\beta$, $\mu = k\cos\alpha\cos\beta$, so that $\gamma = -ik\sin\alpha$. Then the integral (1.20) can be reduced to

$$\begin{aligned}G(\mathbf{r},\mathbf{r}_0) = -\iint\exp\Big(ik(x\cos\alpha\sin\beta + y\cos\alpha\cos\beta + z\sin\alpha)\Big) \\ \times\Psi_g(\alpha,\beta;\mathbf{r}_0)\cos\alpha\,d\alpha\,d\beta.\end{aligned} \tag{1.27}$$

This formula is valid for $z > z_0$ and allows two variants of the integration limits: 1) integration by α is carried along the *modified Sommerfeld contour* $(\pi - i\infty, +i\infty)$ with point $\vartheta = \pi - \vartheta^*$ avoided from the left and point $\vartheta = \vartheta^*$ avoided from the right (see Fig. 1.2), the integration by β takes place on the interval $[0,\pi]$; 2) integration by α is carried along the half of the modified

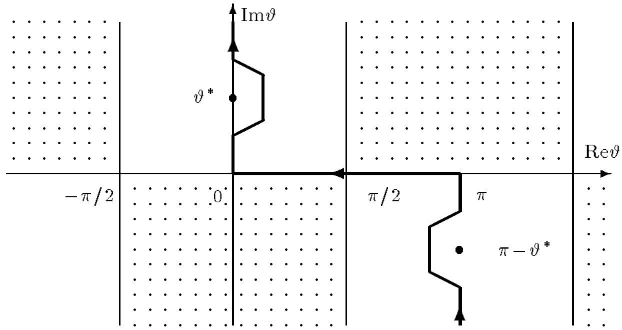


Fig. 1.2 Modified Sommerfeld contour

Sommerfeld contour $(\pi/2, +i\infty)$ with the point $\vartheta = \vartheta^*$ avoided from the right and integration by β is carried on the whole period $[0, 2\pi]$.

In the case of two-dimensional scattering problems the above formulae simplify. Only one Fourier transform is needed for the Green's function

$$G(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi} \int e^{i\lambda(x-x_0)} \left(e^{-\gamma|z-z_0|} + \frac{L(\lambda) + 2N}{L(\lambda)} e^{-\gamma(z+z_0)} \right) \frac{d\lambda}{\gamma}, \quad (1.28)$$

where $L(\lambda) = L(\lambda, 0)$. The far field asymptotics is derived by saddle point method formula (1.22). At large distances an outgoing cylindrical wave (the polar co-ordinates are introduced as $x = r\cos\vartheta$, $z = r\sin\vartheta$)

$$G \sim \sqrt{\frac{2\pi}{kr}} e^{ikr - i\pi/4} \Psi_g(\vartheta), \qquad r \to +\infty \quad (1.29)$$

and two surface waves

$$G \sim \psi_g^{\pm} e^{\pm i\kappa x} e^{-\sqrt{\kappa^2 - k^2} z}, \qquad x \to \pm\infty$$

are formed. The amplitude of the cylindrical wave is given by the expression

$$\Psi_g(\vartheta) = -\frac{i}{4\pi} e^{-ikx_0\cos\vartheta} \left(e^{-ikz_0\sin\vartheta} + R(\vartheta) e^{ikz_0\sin\vartheta} \right),$$

the amplitudes of the surface waves can be found from the formula similar to (1.26) which remains valid except the multiplier κ/k should be dropped out (This formula was first derived in [19])

$$\psi_g^{\pm} = -2\pi i \operatorname*{Res}_{\vartheta=\vartheta^{\pm}} \Psi_g(\vartheta), \qquad \vartheta^{\pm} = \cos^{-1}\left(\frac{\pm\kappa}{k}\right). \tag{1.30}$$

For the case of a surface source the Green's function $G(x, z; x_0)$ can be found as the derivative of (1.28) by z_0

$$\begin{aligned} G(x, z; x_0) &= \frac{1}{\varrho_0\omega^2}\frac{\partial G(x, z; x_0, 0)}{\partial z_0} \\ &= -\frac{1}{2\pi D}\int \exp\left(i\lambda(x - x_0) - \sqrt{\lambda^2 - k^2}z\right)\frac{d\lambda}{L(\lambda)}. \end{aligned} \tag{1.31}$$

It has similar as $G(x, z; x_0, z_0)$ asymptotics at large distances. The far field amplitude is given by the following formula

$$\Psi_g(\vartheta) = \frac{1}{2\pi D}\frac{k\sin\vartheta}{\mathcal{L}(\vartheta)}e^{-ikx_0\cos\vartheta}.$$

The formula (1.27) reduces in two dimensions to

$$G(x, z; x_0) = -\int \exp\Big(ik(x\cos\alpha + z\sin\alpha)\Big)\Psi_g(\alpha; x_0)d\alpha$$

with integration by α carried along the modified Sommerfeld contour shown on Fig. 1.2.

1.3.3 *Integral representation*

The introduced Green's functions allow the integral representation for the solution of the scattering problem to be derived. Let the incident field be a plane spatial wave

$$U^{(i)} = A\exp(ikx\cos\vartheta_0 - ikz\sin\vartheta_0). \tag{1.32}$$

Here $-\vartheta_0$ is the angle of incidence and A is arbitrary constant amplitude. It is natural to represent the solution of the scattering problem in the form

$$U = U^{(i)} + U^{(r)} + U^{(s)} = U^{(g)} + U^{(s)}.$$

The reflected field $U^{(r)}$ is the plane wave reflected from the plate in the absence of the scatterer

$$U^{(r)} = A\,R(\vartheta_0)\exp(ikx\cos\vartheta_0 + ikz\sin\vartheta_0),$$

where reflection coefficient $R(\vartheta_0)$ is given in (1.25). The sum of $U^{(i)}$ and $U^{(r)}$ is called the geometrical part of the field $U^{(g)}$. It satisfies all the conditions of the boundary value problem except the boundary (and contact) conditions on the obstacle. The boundary value problem for the scattered field $U^{(s)}$ remains the same as for the total field except the boundary and contact conditions on the scatterer become inhomogeneous.

Introduce a smooth surface Σ supported by a smooth contour Γ on the plate, such that the scatterer Ω be entirely inside Σ. In the domain bounded by the plate, surface Σ and the semi-sphere S_R of a large radius R one considers the Green's formula (1.6) for functions $G(\mathbf{r}, \mathbf{r}_0)$ and $U^{(s)}(\mathbf{r})$. The volume integral gives $-U^{(s)}(\mathbf{r}_0)$ and the surface integral is split into three parts: integral over Σ, integral over the plate and integral over the semi-sphere S_R. The integral over the semi-sphere S_R of large radius R vanishes as $R \to +\infty$ which is due to the radiation condition (1.16) for the solution $U^{(s)}$ and the asymptotics of the Green's function. In the integral over the plate the boundary conditions (1.13) can be used. This yields

$$-U^{(s)}(\mathbf{r}_0) = \iint\limits_{\Sigma} \left(U^{(s)} \frac{\partial G}{\partial n} - G \frac{\partial U^{(s)}}{\partial n} \right) d\Sigma$$
$$- \varrho_0 \omega^2 D \iint \left(w^{(s)} \left(\Delta^2 - k_0^4 \right) g - g \left(\Delta^2 - k_0^4 \right) w^{(s)} \right) dx\, dy$$

Integrating by parts in the last integral in a similar way as in the Section 1.1.2 allows this integral to be reduced to a contour integral[‖]. Due to the second radiation condition in (1.16) and asymptotics of the Green function $g(x, y; \mathbf{r}_0)$ the integral over the circle of large radius R tends to zero with $R \to +\infty$. Finally the required integral representation takes the form

$$-U^{(s)}(\mathbf{r}_0) = \iint\limits_{\Sigma} \left(U(\mathbf{r}) \frac{\partial G(\mathbf{r}_0, \mathbf{r})}{\partial n} - \frac{\partial U(\mathbf{r})}{\partial n} G(\mathbf{r}_0, \mathbf{r}) \right) d\Sigma$$
$$+ \varrho_0 \omega^2 \int\limits_{\Gamma} \Big(w(x, y) \mathbb{F} g(\mathbf{r}_0; x, y) - \mathbb{F} w(x, y) g(\mathbf{r}_0; x, y) \tag{1.33}$$
$$+ \frac{\partial w(x, y)}{\partial \nu} \mathbb{M} g(\mathbf{r}_0; x, y) - \mathbb{M} w(x, y) \frac{\partial g(\mathbf{r}_0; x, y)}{\partial \nu} \Big) ds.$$

The scattered field in the right-hand side of (1.33) can be replaced by the

‖See also [28] where the analog of Green's formula (1.6) is derived for an elastic plate.

total field. To justify this it is sufficient to show that the right-hand side of (1.33) with the geometrical part $U^{(g)}$ and $w^{(g)}$ substituted instead of U and w gives zero. This is due to the Green's formula (1.6) for $G(\mathbf{r},\mathbf{r}_0)$ and $U^{(g)}(\mathbf{r})$ inside Σ.

The identity (1.33) allows the field U to be computed in any point outside the surface Σ if the functions U, $\partial U/\partial n$, are known on Σ and w, $\partial\xi/\partial\nu$, $\mathbb{M}w$ and $\mathbb{F}w$ are known on Γ. In particular letting $r \to +\infty$ yields asymptotics for the field $U^{(s)}$

$$
\begin{aligned}
U^{(s)} &\sim \frac{2\pi}{kr}\, e^{ikr - i\pi/2}\Psi(\vartheta,\varphi), && r \to +\infty,\quad \vartheta > 0,\\
U^{(s)} &\sim \sqrt{\frac{2\pi}{\kappa\rho}}\, e^{i\kappa\rho - i\pi/4}\psi(\varphi)e^{-\sqrt{\kappa^2-k^2}z}, && \rho \to +\infty.
\end{aligned}
\tag{1.34}
$$

The far field amplitudes in these asymptotics can be expressed via the far field amplitude of point source by the formula

$$
\begin{aligned}
\Psi(\vartheta,\varphi) = &-\iint\limits_{\Sigma}\left(U(\mathbf{r}_0)\frac{\partial\Psi_g(\vartheta,\varphi;\mathbf{r}_0)}{\partial n} - \frac{\partial U(\mathbf{r}_0)}{\partial n_0}\Psi_g(\vartheta,\varphi;\mathbf{r}_0)\right) d\Sigma\\
&-\int\limits_{\Gamma}\Bigg(w(x,y)\mathbb{F}\Psi_g(\vartheta,\varphi;x,y) - \mathbb{F}w(x,y)\Psi_g(\vartheta,\varphi;x,y)\\
&\quad + \frac{\partial w(x,y)}{\partial\nu}\mathbb{M}\Psi_g(\vartheta,\varphi;x,y) - \mathbb{M}w(x,y)\frac{\partial\Psi_g(\vartheta,\varphi;x,y)}{\partial\nu}\Bigg) ds.
\end{aligned}
\tag{1.35}
$$

In the case of two-dimensional problem one integral disappears in the formulae (1.33) and (1.35). In particular integral over Γ reduces to substitutions in two points on the plate and expressions for the force and bending momentum simplify to

$$
\mathbb{F} = -D\frac{d^3}{dx^3}, \qquad \mathbb{M} = D\frac{d^2}{dx^2}. \tag{1.36}
$$

The formula (1.35) allows the analyticity of the far field amplitudes to be established. As the integration in (1.35) is performed in the bounded domain the analytic properties of the far field amplitude are inherited from functions $\Psi_g(\vartheta,\varphi)$ and $\psi_g(\varphi)$. That is the function $\Psi(\vartheta)$ is a meromorphic function of ϑ and has poles in the points corresponding to the solutions of the dispersion equation (1.14). The residue in the pole at $\vartheta = \cos^{-1}(\kappa/k)$ gives the far field amplitude of the surface wave and the formula (1.26) is valid for the field U.

Using the formula (1.27) for the Green's functions G and g and taking into account the relation (1.35) one can check the validity of similar formula for the field U

$$U(\mathbf{r}) = -\iint \exp\Big(ik(x\cos\vartheta\sin\varphi + y\cos\vartheta\cos\varphi + z\sin\vartheta)\Big) \times \Psi(\vartheta,\varphi)\cos\vartheta d\vartheta\, d\varphi. \tag{1.37}$$

The paths of integration here are the same as in (1.27).

The formula (1.37) is valid at some distance from the plate $z > z^* = \max_\Sigma z$. Under this condition the integral by α exponentially converges in the nonshaded in Fig. 1.2 semi-strips. For the derivation of similar formula in two dimensions see [17].

The integral representation (1.33) can be written for the case when Σ coincides with the surface of the obstacle. In this case part of functions in the right-hand side can be taken from the boundary conditions and integral equations on $\partial\Omega$ and $\partial\Omega_0$ can be derived for the other unknowns. Due to possible corner points of $\partial\Omega_0$ additional terms $\varrho_0\omega^2\sum(w\mathbb{F}_c g - g\mathbb{F}_c w)$ should be added to the right-hand side. Here $\mathbb{F}_c$ is the operator from (1.5). Representation (1.33) with $\Sigma = \partial\Omega$ is used in this book when solving particular problems of scattering.

1.3.4 *Optical theorem*

For any obstacle Ω the effective cross-section characterizes the portion of scattered energy. It is known [36] that the total effective cross-section on some compact obstacle is proportional to the real part of the scattering amplitude for the angle equal to zero. This identity follows from the *unitary property* of the *scattering operator* and is called "optical theorem". In the presence of infinite plate the optical theorem holds, but the expressions for the effective cross-section are different.

We derive here optical theorem in presence of infinite elastic plate being in one-side contact to fluid. Two cases of incident fields can be considered. The incident spatial plane wave (1.32) and the incident surface plane wave

$$U^{(i)} = Ae^{i\kappa x - \sqrt{\kappa^2 - k^2}z}. \tag{1.38}$$

Multiply the Helmholtz equation (1.10) by the complex conjugate solution $\overline{U}$ and integrate over the domain bounded by the semi-sphere S_R of large radius, by the plate and by the surface of the obstacle $\partial\Omega$. Apply then the Green's formula and take imaginary part

$$\frac{1}{2\varrho_0\omega}\mathrm{Im}\left(\iint \frac{\partial U}{\partial n}\overline{U}\,dS\right) = 0.$$

Here the multiplier $1/(2\varrho_0\omega)$ is introduced so that the left-hand side and expressions below have the meaning of the average energy fluxes carried through the surface.

All the function on the plate can be expressed via displacements with the help of boundary conditions

$$\mathrm{Im}\left(\frac{1}{2\varrho_0\omega}\iint\limits_{S_R+\partial\Omega} \frac{\partial U}{\partial n}\overline{U}\,dS + \frac{D\omega}{2}\iint\limits_{\mathbb{R}^2\setminus\Omega_0} w\Delta^2\overline{w}\,dx\,dy\right) = 0.$$

Integrating by parts in the last integral similarly to section 1.1.2 yields

$$\mathrm{Im}\left(\frac{1}{2\varrho_0\omega}\iint\limits_{S_R+\partial\Omega} \frac{\partial U}{\partial n}\overline{U}\,dS - \frac{\omega}{2}\int\limits_{C_R+\partial\Omega_0}\left(w\overline{\mathbb{F}w} + \frac{\partial w}{\partial \nu}\overline{\mathbb{M}w}\right)ds - \sum w\overline{\mathbb{F}_c w}\right) = 0.$$

Using (1.17) this equality can be rewritten as

$$\Pi_1(U,U) + \Pi_2(w,w) + E_\Omega = 0,$$

where

$$\Pi_1(P,Q) = \frac{1}{2\varrho_0\omega}\mathrm{Im}\left(R^2\int\limits_0^{\pi/2} d\vartheta \int\limits_0^{2\pi} d\varphi \frac{\partial P}{\partial r}\overline{Q}\cos\vartheta\right),$$

$$\Pi_2(p,q) = -\frac{\omega}{2}\mathrm{Im}\left(R\int\limits_0^{2\pi}\left(p\overline{\mathbb{F}q} + \frac{\partial p}{\partial \rho}\overline{\mathbb{M}q}\right)d\varphi\right).$$

The above identity expresses the energy conservation law. The first term is the energy carried through the semi-sphere S_R by the field U in fluid and the second term is the elastic energy carried through the circumference C_R by flexural deformations w. For noncoincident arguments Π_j give energies of interaction of two fields.

Representing the total field as the sum of geometrical part $U^{(g)}$ and scattered field $U^{(s)}$ yields

$$\Pi_1(U,U) = \Pi_1(U^{(g)},U^{(g)}) + \Pi_1(U^{(g)},U^{(s)}) + \Pi_1(U^{(s)},U^{(g)}) + \Pi_1(U^{(s)},U^{(s)})$$

and similar formula for Π_2. The geometrical part of the field satisfies the boundary value problem without an obstacle. Therefore,

$$\Pi_1(U^{(g)},U^{(g)}) + \Pi_2(w^{(g)},w^{(g)}) = 0.$$

The energy of interaction can be calculated by the saddle point method which gives exact result for $R \to \infty$. Consider the integral $\Pi_1(U^{(g)},U^{(s)})$. According to (1.34) the scattered field at large distances splits into the sum of spherical wave U_{sph} and cylindrical surface wave U_{surf}. Let the incident field be a plane wave (1.32). The geometrical part in this case is the sum of incident and reflected plane waves. Rewriting these waves in spherical coordinates and analysing the phase functions one finds out that saddle point exists only for the integral describing interaction of reflected wave with the spherical wave. The phase function in that integral is

$$\Phi = -1 + \cos\vartheta\cos\vartheta_0\sin\varphi + \sin\vartheta\sin\vartheta_0$$

and the saddle point is at $\vartheta = \vartheta_0$, $\varphi = \pi/2$.

Applying the formula (1.23) to the integrals Π_1 and letting $R \to +\infty$ yields

$$\Pi_1(U^{(i)},U_{\text{sph}}) \to 0, \quad \Pi_1(U^{(i)},U_{\text{surf}}) \to 0, \quad \Pi_1(U^{(r)},U_{\text{surf}}) \to 0$$

and

$$\Pi_1(U^{(r)},U_{\text{sph}}) \to \frac{2\pi^2}{\varrho_0\omega k}\mathrm{Re}\left(A\,R(\vartheta_0)\overline{\Psi(\vartheta_0,0)}\right).$$

Similar expressions can be found for the terms in $\Pi_1(U^{(s)},U^{(g)})$. Thus the total energy taken from the reflected plane wave can be written as

$$\begin{aligned} E &= -\Pi_1(U^{(g)},U^{(s)}) - \Pi_1(U^{(s)},U^{(g)}) \\ &= -\frac{4\pi^2}{\varrho_0\omega k}\mathrm{Re}\left(A\,R(\vartheta_0)\overline{\Psi(\vartheta_0,0)}\right). \end{aligned}$$

In the case of incident surface wave (1.38) the nonzero terms are only $\Pi_1(U^{(i)},U_{\text{surf}})$ and $\Pi_1(U_{\text{surf}},U^{(i)})$. It is more convenient to use cylindrical coordinates in the corresponding integrals and compute integrals by z

explicitly and integrals by φ by the saddle point method (1.22). One finds

$$\Pi_1(U^{(i)}, U_{\text{surf}}) \to \frac{\pi}{2\varrho_0\omega\sqrt{\kappa^2 - k^2}} \text{Re}\left(A\overline{\psi(0)}\right).$$

Consider now the terms Π_2. One can check that in the limit $R \to +\infty$ the nonvanishing term is possible only in the case when surface incident wave (1.38) interacts with the surface scattered field. Applying saddle point formula (1.22) one finds

$$\Pi_2(w^{(i)}, U_{\text{surf}}) \to \frac{2\left(\kappa^2 - k^2\right)}{\varrho_0\omega N} \text{Re}\left(A\overline{\psi(0)}\right).$$

Combining the above two expressions and taking into account the dispersion equation (1.14) one gets the expression for the energy taken from the incident surface wave in the form

$$E = -\frac{5\kappa^4 - 4\kappa^2k^2 - k_0^4}{2\varrho_0\omega N} \text{Re}\left(A\overline{\psi(0)}\right).$$

The scattered field carries energy flux to infinity

$$E^s = \Pi_1(U^{(s)}, U^{(s)}) + \Pi_2(w^{(s)}, w^{(s)}).$$

Consider the first term. It can be represented as the sum of contributions $\Pi_1(U_{\text{sph}}, U_{\text{sph}})$ and $\Pi_1(U_{\text{surf}}, U_{\text{surf}})$. The interaction of surface and spherical waves takes no place. This fact is similar to the absence of interaction between spatial components of $U^{(g)}$ and the surface scattered field and can be established with the help of saddle point method. Substituting the asymptotics (1.34) one finds

$$\Pi_1(U_{\text{sph}}, U_{\text{sph}}) \to \frac{2\pi^2}{\varrho_0\omega k} \iint |\Psi(\vartheta, \varphi)|^2 \cos\vartheta \, d\vartheta \, d\varphi.$$

The surface wave contribution can be easier computed in cylindrical coordinates

$$\Pi_1(U_{\text{surf}}, U_{\text{surf}}) \to \frac{\pi}{2\varrho_0\omega\sqrt{\kappa^2 - k^2}} \int |\psi(\varphi)|^2 d\varphi.$$

Consider now the second term in the formula for the scattered energy E^s. It can be checked that the contribution of spherical wave vanishes as $R \to +\infty$. In $\Pi_2(w_{\text{surf}}, w_{\text{surf}})$ only derivatives by ρ in the operators $\mathbb{F}$ and

$\mathbb{M}$ give nonvanishing contributions. One finds

$$\Pi_2(w_{\rm surf}, w_{\rm surf}) \to \frac{2\pi\kappa^2}{\varrho_0\omega N}\left(\kappa^2 - k^2\right)\int |\psi(\varphi)|^2 d\varphi.$$

Combining the three above expressions and taking into account that κ is the solution of the dispersion equation (1.14) yields

$$E^s = \frac{1}{2\varrho_0\omega}\left(\frac{4\pi^2}{k}\iint |\Psi(\vartheta,\varphi)|^2\cos\vartheta\, d\vartheta\, d\varphi \right.$$
$$\left. + \frac{\pi}{N}\left(5\kappa^4 - 4\kappa^2k^2 - k_0^4\right)\int |\psi(\varphi)|^2 d\varphi\right).$$

In terms of introduced above average energy fluxes the energy conservation law reads

$$E = E_\Omega + E^s. \tag{1.39}$$

That is, part of the energy taken from the geometrical part of the field is absorbed by the obstacle and the other part is scattered. In the case of nonabsorbing obstacle $E_\Omega = 0$ and the energy balance takes the form of *optical theorem*. We repeat here formulae for the energy E. In the case of spatial incident plane wave the energy E is taken from the reflected wave

$$E = -\frac{4\pi^2}{\varrho_0\omega k}\mathrm{Re}\left(A\, R(\vartheta_0)\overline{\Psi(\vartheta_0, 0)}\right).$$

For surface incident wave the energy is taken from the passed wave

$$E = -\frac{5\kappa^4 - 4\kappa^2k^2 - k_0^4}{2\varrho_0\omega N}\mathrm{Re}\left(A\overline{\psi(0)}\right).$$

An important characteristics of the scattering obstacle is the *effective cross-section* Σ defined as the ratio of scattered energy to the density of energy in the incident field. Calculate the density of energy flux carried by the incident wave. For the case of spatial plane incident wave one finds

$$E^i = \frac{1}{2\varrho_0\omega}\mathrm{Im}\left(\overline{U^{(i)}}\nabla U^{(i)}\right) = \frac{k|A|^2}{2\varrho_0\omega}. \tag{1.40}$$

Surface wave carries energy by acoustic pressure and by flexural deformations $E^i = E' + E''$. The flux in fluid is given by

$$E' = \frac{\kappa|A|^2}{4\varrho_0\omega\sqrt{\kappa^2 - k^2}}$$

and the energy flux in the plate is

$$E'' = -\frac{\omega}{2}\mathrm{Im}\left(\overline{\mathbb{F}w^{(i)}}\,w^{(i)} + \overline{\mathbb{M}w^{(i)}}\,\frac{\partial w^{(i)}}{\partial x}\right) = \frac{|A|^2}{\varrho_0\omega}\,\kappa^3\left(\kappa^2 - k^2\right).$$

Combining the above expressions and taking into account that $\sqrt{\kappa^2 - k^2} = N/(\kappa^4 - k_0^4)$ which follows from (1.14) yields finally the density of energy flux carried by the incident surface wave

$$E^i = \frac{\kappa|A|^2}{4\varrho_0\omega N}\left(5\kappa^4 - 4\kappa^2k^2 - k_0^4\right). \tag{1.41}$$

For the spatial incident plane wave the *effective cross-section* is

$$\Sigma = \frac{E^s}{E^i} = \frac{4\pi^2}{k^2}\int_0^{2\pi}\int_0^{\pi/2}\left|\frac{\Psi(\vartheta,\varphi)}{A}\right|^2\cos\vartheta\,d\vartheta\,d\varphi + \frac{\pi\left(5\kappa^4 - 4\kappa^2k^2 - k_0^4\right)}{Nk}\int_0^{2\pi}\left|\frac{\psi(\varphi)}{A}\right|^2 d\varphi.$$

This formula takes into account two channels of scattering, one presented by an outgoing spherical acoustic wave contributes to the first term, and the other channel of surface wave process contributes to the second term.

In terms of effective cross-sections optical theorem can be written in the form of two equalities: for spatial incident wave

$$\Sigma = -\frac{8\pi^2}{k^2}\mathrm{Re}\left(R(\vartheta_0)\frac{\overline{\Psi(\vartheta_0,0)}}{\overline{A}}\right) = \frac{4\pi^2}{k^2}\int_0^{2\pi}\int_0^{\pi/2}\left|\frac{\Psi(\vartheta,\varphi)}{A}\right|^2\cos\vartheta\,d\vartheta\,d\varphi + \frac{\pi\left(5\kappa^4 - 4\kappa^2k^2 - k_0^4\right)}{Nk}\int_0^{2\pi}\left|\frac{\psi(\varphi)}{A}\right|^2 d\varphi$$

and for surface incident wave

$$\Sigma = -\frac{2}{\kappa}\mathrm{Re}\left(\frac{\overline{\psi(0)}}{\overline{A}}\right)$$
$$= \frac{8N\pi^2}{\kappa k(5\kappa^4 - 4\kappa^2k^2 - k_0^4)}\int_0^{2\pi}\int_0^{\pi/2}\left|\frac{\Psi(\vartheta,\varphi)}{A}\right|^2\cos\vartheta\,d\vartheta\,d\varphi$$
$$+\frac{2\pi}{\kappa}\int_0^{2\pi}\left|\frac{\psi(\varphi)}{A}\right|^2 d\varphi.$$

Optical theorem for two-dimensional problems can be derived analogously (see details in [27]). In the case of spatial incident wave it reads

$$\Sigma = -\frac{4\pi}{k}\mathrm{Re}\left(R(\vartheta_0)\frac{\overline{\Psi(\vartheta_0)}}{\overline{A}}\right)$$
$$= \frac{2\pi}{k}\int_0^{\pi}\left|\frac{\Psi(\vartheta)}{A}\right|^2 d\vartheta + \frac{\kappa\left(5\kappa^4 - 4\kappa^2k^2 - k_0^4\right)}{2Nk}\left(\left|\frac{\psi^+}{A}\right|^2 + \left|\frac{\psi^-}{A}\right|^2\right). \tag{1.42}$$

For scattering by a compact obstacle in isolated plate optical theorem is presented in [23]. It reads

$$\begin{aligned}\Sigma_0 &= \frac{2\pi}{k_0|A|^2}\int_0^{2\pi}\left|\psi(\varphi)\right|^2 d\varphi \\ &= -\frac{4\pi}{k_0}\mathrm{Re}\left(A^{-1}\psi(\varphi_0)\right).\end{aligned} \tag{1.43}$$

Other variants of optical theorem dealing with diffraction by thin elastic plates can be found in [6], [23], [25], [48]. We present in Section 1.5.2 the optical theorem for an isolated plate with infinite crack in it.

Identities (1.39) and (1.42) allow additional independent control of numerical and analytical results to be performed.

1.3.5 *Uniqueness of the solution*

The question of uniqueness is studied for the case of nonabsorbing material of the plate and nonabsorbing fluid, that is $\mathrm{Im}\,k = 0$, $\mathrm{Im}\,k_0 = 0$ and $\mathrm{Im}\,N = 0$. If two different solutions (U_1, w_1) and (U_2, w_2) of the problem of scattering on an obstacle in presence of infinite elastic plate exist, then their difference $(U, w) = (U_1 - U_2,\, w_1 - w_2)$ solves the homogeneous problem, *i.e.* without any incident field. First, it can be shown that the far field amplitudes of spatial and surface waves in the solution (U, w) are equal to zero. Indeed, consider the expression (1.39). In this expression $E = 0$ as there is no incident wave and $E_\Omega \geq 0$ according to the supposition (1.17) of correct conditions on the scatterer. So, $E^s \leq 0$, that is

$$\frac{4\pi^2}{k}\int\limits_0^{2\pi}\int\limits_0^{\pi/2} |\Psi(\vartheta,\varphi)|^2 \sin\vartheta \, d\vartheta \, d\varphi$$

$$+ \pi\frac{D}{\varrho_0\omega^2}\left(5\kappa^4 - 4\kappa^2 k^2 - k_0^4\right)\int\limits_0^{2\pi} |\psi(\varphi)|^2 d\varphi \leq 0.$$

Noting that $\kappa > \max(k, k_0)$ (see dispersion equation (1.14) for κ) allows to conclude that $\Psi(\vartheta,\varphi) \equiv 0,\ \vartheta \in [0,\pi/2), \varphi \in [0, 2\pi]$ and $\psi(\varphi) \equiv 0,\ \varphi \in [0, 2\pi]$. Analyticity of the far field amplitude yields $\Psi(\vartheta,\varphi) \equiv 0$ for all complex arguments ϑ and φ. Then from (1.37) the function U is concluded identically zero at $z > z^*$. To the domain $z < z^*$ it is continued by zero as the solution of the Helmholtz equation. That is solutions U_1 and U_2 coincide. Due to adhession condition the corresponding displacements of the plate coincide too.

The above can be formulated as the uniqueness theorem:

Theorem 1 *The problem (1.10), (1.11) with adhession condition, radiation conditions (1.16) and any boundary and contact conditions on the obstacle that satisfy the inequality (1.17) has unique solution.*

Consider now the case of isolated plate. The problem of flexural wave scattering is the boundary value problem for the bi-harmonic equation (1.9), radiation condition given by the second formula in (1.16) and any pair of boundary conditions on $\partial\Omega_0$ such that the inequality

$$E_\Omega = \frac{\omega}{2}\mathrm{Im}\left(\int\limits_{\partial\Omega_0}\left((\mathbb{F}w)\,\overline{w} + (\mathbb{M}w)\,\frac{\partial\overline{w}}{\partial\nu}\right)ds + \sum F_c\overline{w}\right) \geq 0 \qquad (1.44)$$

holds. Note that inequality (1.44) coincides with (1.17) for $U \equiv 0$.

By the similar procedure it can be shown that the difference of two solutions $w = w_1 - w_2$ has the far field amplitude $\psi(\varphi) \equiv 0$. In the presence of fluid the fact that the far field amplitude is identically zero yields due to the Sommerfeld formula (1.37) the solution itself equal to zero. This is the property of the second order differential operator. For the bi-harmonic operator Sommerfeld formula is not true and absence of the far field amplitude does not yield $w = 0$. Nevertheless, to benefit from the Sommerfeld formula let the function $\zeta(x, y)$ be introduced as

$$\zeta(x, y) = \left(\Delta - k_0^2\right) w(x, y).$$

The Green's function for the isolated plate can be expressed in terms of Bessel functions of the third kind

$$g_0(\boldsymbol{\rho}, \boldsymbol{\rho}_0) = \frac{i}{8k_0^2 D}\left(H_0^{(1)}(k_0|\boldsymbol{\rho} - \boldsymbol{\rho}_0|) - H_0^{(1)}(ik_0|\boldsymbol{\rho} - \boldsymbol{\rho}_0|)\right). \qquad (1.45)$$

Analogously to section 1.3.3 the solution of the problem for isolated plate can be expressed in the form of convolution with the Green's function

$$w = -\int\limits_{\partial\Omega_0} \left(w\mathbb{F}g_0 + \frac{\partial w}{\partial \nu}\mathbb{M}g_0 - g_0\mathbb{F}w + \frac{\partial g_0}{\partial \nu}\mathbb{M}w\right) ds. \qquad (1.46)$$

This representation allows the radiation condition to be specified as

$$w \sim \sqrt{\frac{2\pi}{k_0\rho}} e^{ik_0\rho - i\pi/4}\psi_0(\varphi) \qquad (1.47)$$

Applying operator $\Delta - k_0^2$ to this formula yields the far field asymptotics for the function $\zeta(x, y)$. It has the same form as (1.47) with the far field amplitude $\psi_\zeta(\varphi) = -2k_0^2\psi_0(\varphi)$. Noting also that $\zeta(x, y)$ satisfies the two-dimensional Helmholtz equation with the wavenumber k_0 and thus Sommerfeld formula is valid for it one concludes that $\zeta \equiv 0$. That is the solution of the problem for flexural waves in isolated plate without sources of the field if exists satisfies the equation

$$\Delta w(x, y) - k_0^2 w(x, y) = 0.$$

Consider first the case of clamped or hinge-supported boundary. That is let on the whole contour $\partial\Omega_0$ be satisfied

$$w = 0, \quad \text{or} \quad \frac{\partial w}{\partial \nu} = 0,$$

or on part of $\partial\Omega_0$ the first of the above conditions is satisfied and on the rest of contour the second condition is satisfied (other two boundary conditions may be arbitrary). In this case the solution is unique. Indeed multiply the second order differential equation for w by the complex conjugate solution $\overline{w}$ and integrate by parts in the domain bounded by circumference C_R of large radius R and the obstacle Ω_0. Due to the fact that $\psi_0(\varphi) \equiv 0$ the integral over the circumference C_R vanishes in the limit $R \to 0$ which yields the integral identity

$$\iint \left(|\nabla w|^2 + k_0^2 |w|^2\right) dS = \int\limits_{\partial\Omega_0} \frac{\partial w}{\partial \nu} \overline{w} d\ell. \tag{1.48}$$

Due to the boundary conditions the contour integral in the right-hand side vanishes and, thus, $w \equiv 0$. That is, the following theorem holds

Theorem 2 *Solution of the boundary value problem for the equation (1.9) in the exterior of arbitrary bounded domain Ω_0 is unique if the boundary conditions are such that (1.44) holds and $w = 0$ on some part Γ_1 of the boundary and $\partial w / \partial \nu = 0$ on the remaining part $\Gamma_2 = \partial\Omega_0 \setminus \Gamma_1$ of the boundary.*

Consider now such problems of scattering that on some part of the boundary none of the above boundary conditions is satisfied. Let for example Ω_0 be a hole with a free edge, that is

$$\mathbb{F}w = 0, \qquad \mathbb{M}w = 0.$$

Or let in general case the boundary conditions be of impedance type

$$\mathbb{F}w = Z_f\, w, \qquad \mathbb{M}w = Z_m \frac{\partial w}{\partial \nu},$$

where Z_f and Z_m are the force and momentum impedances.

To prove the uniqueness theorem for the boundary value problem for the differential equation (1.9) with the impedance boundary conditions it is sufficient to show that for the solution of the problem without sources the following inequality holds

$$\int\limits_{\partial\Omega_0} \frac{\partial w}{\partial \nu} \overline{w} ds \leq 0.$$

Then the fact that $w \equiv 0$ follows from the integral identity (1.48). However it can be shown that for the general shape of the contour $\partial\Omega_0$ and natural assumptions for the impedances Z_f and Z_m the above inequality is not true. Moreover there exists an example of correctly formulated boundary conditions (such that (1.44) is satisfied) that allow *eigenfunction* concentrated near the boundary to exist. We discuss this example in the following Section.

1.3.6 Flexural wave concentrated near a circular hole

Consider stationary vibrations of infinite elastic plate with a round hole of radius ρ_0. Let the boundary conditions on the edge be

$$\mathbb{F}w = \omega^2 M w, \qquad \mathbb{M}w = \omega^2 I \frac{\partial w}{\partial \nu},$$

which correspond to a body of linear mass M and moment of inertia I attached to the edge.

The purpose of the analysis is to find such values of ρ_0, M and I for which there exists solution of a homogeneous problem. As it was shown in the previous section such wave should satisfy the equation $(\triangle - k_0^2)w = 0$, that is be of the form

$$w = H_j^{(1)}(ik_0\rho)e^{ij\varphi},$$

where j is arbitrary integer. Note that in general case the solution is decomposed in the form (see section 3.4.2)

$$\sum_j \left(\alpha_j H_j^{(1)}(k_0\rho) + \beta_j H_j^{(1)}(ik_0\rho)\right) e^{ij\varphi},$$

but the values of M and I will be chosen such that coefficient α_j be equal to zero. Computing force and momentum according to the formulae (1.8) and taking into account that functions $K_j(x) = H_j^{(1)}(ix)$ of real argument x take real values, one finds

$$\frac{\omega^2}{D} M = k_0^3 \left\{ \left[1 - (1-\sigma)\frac{j^2}{(k_0\rho_0)^2} \right] \frac{K_j'(k_0\rho_0)}{K_j(k_0\rho_0)} + (1-\sigma)\frac{j^2}{(k_0\rho_0)^3} \right\},$$

$$\frac{\omega^2}{D} I = k_0 \left\{ \frac{1-\sigma}{k_0\rho_0} - \left[1 + (1-\sigma)\frac{j^2}{(k_0\rho_0)^2} \right] \frac{K_j(k_0\rho_0)}{K_j'(k_0\rho_0)} \right\}.$$

That is if the values of M and I in the boundary conditions are taken according to the above formulae with any integer j, then the function $H_j^{(1)}(ik_0\rho)e^{ij\varphi}$ is the solution of the boundary-value problem of free flexural

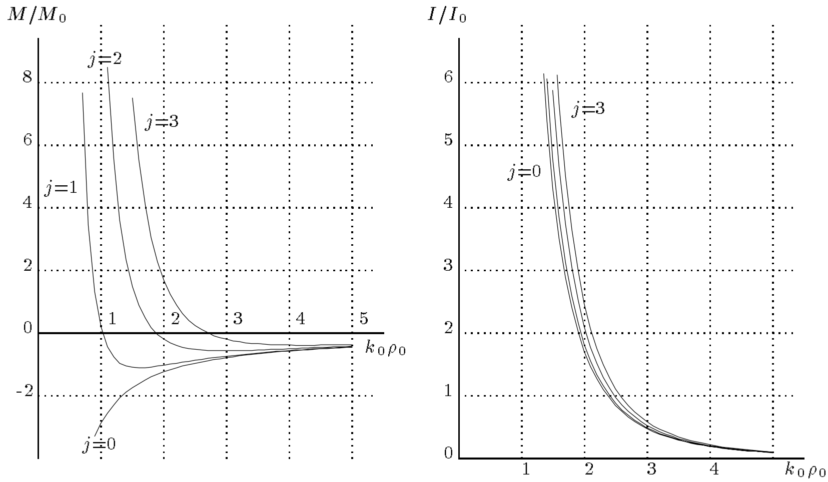


Fig. 1.3 Mass and moment of inertia for which eigenfunction exists ($\sigma = 1/3$).

vibrations of the plate. This function exponentially decreases with ρ and does not carry energy to infinity.

The values M and I can be conveniently compared with the mass M_0 and moment of inertia I_0 of the circular sector

$$M_0 = \frac{1}{2} h \varrho_0 \rho_0^2 \, d\varphi, \qquad I_0 = \frac{1}{12} h \varrho_0 \rho_0^4 \, d\varphi.$$

Taking into account that $h\varrho_0\omega^2/D = k_0^4$ one finds

$$\frac{M}{M_0} = \frac{2}{k_0\rho_0} \left\{ \left[1 - (1-\sigma)\frac{j^2}{(k_0\rho_0)^2} \right] \frac{K_j'(k_0\rho_0)}{K_j(k_0\rho_0)} + (1-\sigma)\frac{j^2}{(k_0\rho_0)^3} \right\},$$

$$\frac{I}{I_0} = \frac{12}{(k_0\rho_0)^3} \left\{ \frac{1-\sigma}{k_0\rho_0} - \left[1 + (1-\sigma)\frac{j^2}{(k_0\rho_0)^2} \right] \frac{K_j(k_0\rho_0)}{K_j'(k_0\rho_0)} \right\}.$$

Figure 1.3 presents above dependencies of mass and moment of inertia on the dimensionless parameter $k_0\rho_0$ for $j = 0, 1, 2$ and 3. For $j \neq 0$ force impedance can be either positive or negative (the latter corresponds to a spring attached to the edge). The momentum impedance Z_m should be positive.

Discussion of high quality resonances in Kirchhoff plate can be found in [30] and for Timoshenko-Mindlin plate in [14].

1.4 Classical point models

The use of approximate plate theory equations instead of Lame equations of elasticity simplifies the problems of diffraction in thin-walled mechanical constructions. In some cases this allows simple analytic solutions to be found in terms of integrals or series. In that case the analysis of the physical effects becomes the most simple one and many important phenomena were discovered and analyzed for such simple solutions. See e.g. [45] where resonances in a plate with two cracks were discovered, or [41], [46] where edge waves of Rayleigh type were discovered first in isolated and then in fluid loaded plates.

As it was shown in the previous section the correctly formulated boundary value problem should contain boundary-contact conditions for flexural displacements at all points (or lines) where the smoothness of the plate is violated (edges, joints, supports, *etc.*). Such boundary-contact conditions can be stated at any line or point which becomes the line or point of plate discontinuity. Classical point models use this possibility and formulate contact conditions in the midpoint of the obstacle.

1.4.1 *Point models in two dimensions*

Consider first the problems of scattering by obstacles that are infinitely long in one direction. We assume that neither geometry, nor the incident field depend on y co-ordinate. Such problems are reduced to two-dimensional boundary-value problems for the Helmholtz equation

$$\triangle U(x,z) + k^2 U(x,z) = 0, \quad z > 0$$

with the boundary condition (1.13) which is satisfied on the plate except the point $x = 0$. Such boundary condition can be written for all x with the δ-function and its derivatives in the right-hand side

$$\begin{aligned} \left(\frac{\partial^4}{\partial x^4} - k_0^4\right) \frac{\partial U(x,0)}{\partial z} &+ N\, U(x,0) \\ &= c_0\delta(x) + c_1\delta'(x) + c_2\delta''(x) + c_3\delta'''(x). \end{aligned} \tag{1.49}$$

The order of derivatives that may be presented in the right-hand side of the above equation is restricted by Meixner conditions (1.18) for the acoustic pressure U. The constants c_0, c_1, c_2 and c_3 are unknown and should be found from the boundary-contact conditions corresponding to particular point model.

Consider here some of classical point models. The simplest model is the model of fixed point which is formulated as

$$w(0) = 0, \quad w'(0) = 0. \tag{1.50}$$

Generally speaking, displacements corresponding to the boundary condition (1.49) are discontinuous and have discontinuous derivatives at the origin. So conditions (1.50) mean also that displacement $w(x)$ and angle $w'(x)$ are continuous functions.

The model of a crack with free edges is formulated as (see [44])

$$w''(\pm 0) = 0, \quad w'''(\pm 0) = 0, \tag{1.51}$$

which means absence of forces and bending momentums (formulae (1.8) reduce in the case of one-dimensional plate to (1.36)).

Models (1.50) and (1.51) take somewhat extreme positions in the scale of point models. Other possible model is the model of attached mass and moment of inertia. To formulate this model one can use the following approach. Suppose that the plate is cut along the line $x = 0$. Then the force, applied to the edge of the left semi-infinite part of the plate is

$$F^- = -D\frac{d^3}{d\nu^3}w = -D\frac{d^3w(-0)}{dx^3}$$

and the force applied to the edge of the right semi-infinite plate is

$$F^+ = -D\frac{d^3}{d\nu^3}w = D\frac{d^3w(+0)}{dx^3}.$$

The total force applied to the plate at point $x = 0$ is the sum of forces F^- and F^+. This force is due to the attached mass and is equal to the product of mass M and acceleration $\ddot{w}(0) = -\omega^2 w(0)$. Similarly computing the bending momentums applied to the edges of semi-infinite plates and equating the total momentum (difference of momentums applied to edges) to the product of angular acceleration of the stiffener $\ddot{w}'(0) = -\omega^2 w'(0)$ by moment of inertia I yields the second condition. Thus, the boundary-contact conditions of a point-wise stiffener are (see [43])

$$D[w'''] = \omega^2 M w(0), \quad D[w''] = -\omega^2 I w'(0). \tag{1.52}$$

Here again $w(x)$ and $w'(x)$ are assumed continuous at $x = 0$ and $w''(x)$ and $w'''(x)$ have jumps which are denoted by square brackets.

Note that in all the models the total number of conditions (including assumed smoothness) is 4 which coincides with the number of unknowns in the equation (1.49).

Usual procedure for constructing the solution of scattering by a point model consists of two steps. First so-called general solution $u(x, z)$ is introduced. This function satisfies all the conditions of the problem except the boundary-contact conditions formulated on the point scatterer. It contains arbitrary parameters c_j which play the role of amplitudes of passive sources applied to the plate. General solution is the linear combination of Green's function and its derivatives

$$u(x, z) = \sum_{\ell=0}^{3} c_\ell \left(\frac{\partial}{\partial x}\right)^\ell G(x, z; 0). \tag{1.53}$$

Here $G(x, z; 0)$ is the field of a point source applied to the plate. It is given by Fourier integral (1.31)

$$G(x, z; x_0) = -\frac{1}{2\pi D} \int \exp\left(i\lambda(x - x_0) - \sqrt{\lambda^2 - k^2} z\right) \frac{d\lambda}{L(\lambda)}. \tag{1.31}$$

Note that representation (1.53) is the reduced two dimensional integral representation (1.33) for $\Omega = \{0\}$. As it was mentioned the order of δ-function derivatives that may be written in the right-hand side of (1.49) is limited by the *Meixner conditions* (1.18) to which the acoustic pressure should satisfy. Analysis of the integrand in (1.31) shows that it decreases with $\lambda \to \pm\infty$ as $O(\lambda^{-5})$. Thus the Green's function and its derivatives of orders 1, 2 and 3 are bounded in the point of the source and the fourth-order derivative by x has logarithmic singularity. That is, general solution with arbitrary parameters c_ℓ satisfies the Meixner conditions (1.18), and no higher order derivatives can be allowed in the equation (1.49) and consequently in the representation (1.53).

The second step of the procedure is in finding the coefficients c_ℓ from the 4×4 linear system arising when general solution u is substituted into the boundary-contact conditions. In the general case it is not possible to substitute $x = \pm 0$ into the Green's function derivatives. The corresponding values should be considered as appropriate limits.

The solvability question of the system for coefficients c_ℓ in a particular point model can not be studied directly because the formulae for the matrix elements are too cumbersome. However solvability is directly connected with the uniqueness theorem. Indeed, following [24] suppose that the determinant of the system is zero. Then there exists nontrivial solution of a homogeneous problem. This solution has the far field amplitude

$$\Psi(\vartheta) = \sum_{\ell=0}^{3} c_\ell \Psi_g(\vartheta) \Big(ik\cos\vartheta\Big)^\ell$$

which according to the uniqueness theorem should be identically zero. The last fact due to the linear independence of trigonometric functions yields $c_\ell = 0$ contradicting our initial supposition. Thus the following theorem holds

Theorem 3 *The determinant of the system for boundary-contact constants is different from zero and the system is solvable uniquely for any right-hand side.*

Consider now the particular problems of scattering and construct solutions in explicit form according to the described above procedure. We shall also check the optical theorem and the reciprocity principle for incident spatial plane wave and for incident surface wave.

Scattering by fixed point

The boundary-contact conditions (1.50) yield the system

$$\begin{cases} c_1 D_1 + c_3 D_3 = 0, \\ c_0 D_1 + c_2 D_3 = 0, \\ c_0 D_0 + c_1 D_1 + c_2 D_2 + c_3 D_3 = D\dfrac{\partial U^{(g)}(0,0)}{\partial z}, \\ c_0 D_1 + c_1 D_2 + c_2 D_3 + c_3 D_4 = D\dfrac{\partial^2 U^{(g)}(0,0)}{\partial z \partial x}. \end{cases}$$

Here the integrals D_ℓ are introduced as limits (see Appendix A)

$$D_\ell = \frac{1}{2\pi}\int e^{+i0\lambda}(i\lambda)^\ell \frac{\sqrt{\lambda^2-k^2}}{L(\lambda)}d\lambda \equiv \lim_{s\to+0}\frac{1}{2\pi}\int e^{is\lambda}(i\lambda)^\ell \frac{\sqrt{\lambda^2-k^2}}{L(\lambda)}d\lambda \tag{1.54}$$

and symmetry properties of Green's function and its derivatives are taken into account.

Some properties of contact integrals D_ℓ can be immediately checked. The denominator $L(\lambda)$ behaves at infinity as $O(|\lambda|^5)$, thus integrals D_0, D_1 and D_2 absolutely converge. Noting that the integrand of D_1 is odd yields $D_1 = 0$. Thus from the first two equations of the above system, that express the continuity of $w(x)$ and $w'(x)$ at the origin, follows that $c_2 = 0$ and $c_3 = 0$. This simplifies the other two equations. One finds

$$c_0 = \frac{D}{D_0}\frac{\partial U^{(g)}(0,0)}{\partial z}, \qquad c_1 = \frac{D}{D_2}\frac{\partial^2 U^{(g)}(0,0)}{\partial z \partial x}.$$

If the incident wave is the plane acoustic wave

$$U^{(i)} = A \exp\Big(ikx\cos\vartheta_0 - ikz\sin\vartheta_0\Big)$$

then the geometrical part of the field is the sum of the incident and reflected plane waves (see (1.25) for the reflection coefficient)

$$U^{(g)} = U^{(i)} + A\,R(\vartheta_0)\exp\Big(ikx\cos\vartheta_0 + ikz\sin\vartheta_0\Big).$$

Differentiating the above expression and letting $x = z = 0$ yields

$$c_0 = -\frac{2\varrho_0\omega^2}{D_0}\frac{ik\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}, \qquad c_1 = \frac{2\varrho_0\omega^2}{D_2}\frac{k^2\sin\vartheta_0\cos\vartheta_0}{\mathcal{L}(\vartheta_0)}.$$

The far field amplitude of the scattered field can be expressed via the far field amplitude

$$\Psi_g(\vartheta) = \frac{1}{2\pi D}\frac{k\sin\vartheta}{\mathcal{L}(\vartheta)}e^{-ikx_0\cos\vartheta}$$

of the Green's function. One finds

$$\Psi(\vartheta;\vartheta_0) = A\frac{iN}{\pi}\frac{k^2\sin\vartheta\sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}\left(\frac{1}{D_0} - \frac{k^2\cos\vartheta\cos\vartheta_0}{D_2}\right). \tag{1.55}$$

The amplitudes of surface waves can be found as residues of $\Psi(\vartheta)$ by the formula (1.30)

$$\psi^{\pm}(\vartheta_0) = A\frac{2N}{\kappa}\frac{\sqrt{\kappa^2-k^2}}{5\kappa^4 - 4\kappa^4 k^4 - k_0^4}\frac{k\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}\left(\frac{1}{D_0} \mp \frac{\kappa k\cos\vartheta_0}{D_2}\right). \tag{1.56}$$

In the case of incident surface wave

$$U^{(i)} = A\exp\left(i\kappa x - \sqrt{\kappa^2-k^2}z\right)$$

analogous derivations yield

$$\Psi(\vartheta) = A\frac{1}{2\pi}\frac{k\sin\vartheta\sqrt{\kappa^2-k^2}}{\mathcal{L}(\vartheta)}\left(\frac{1}{D_0} - \frac{\kappa k\cos\vartheta}{D_2}\right), \tag{1.57}$$

$$\psi^{\pm} = A\frac{i}{\kappa}\frac{\kappa^2 - k^2}{5\kappa^4 - 4\kappa^2 k^4 - k_0^4}\left(\frac{1}{D_0} \mp \frac{\kappa^2}{D_2}\right).$$

Comparing the formulae (1.56) and (1.57) it is easy to note coincidence of angular factors of $\psi^+(\vartheta_0)$ and $\Psi(\vartheta)$ for $\vartheta = \vartheta_0$. This is the consequence of the *reciprocity principle* [51]. To find out the correspondence of the amplitudes one needs to consider energies.

The linear densities of energy fluxes carried by acoustic plane wave in the strip of wavelength size and by surface wave can be found from the formulae (1.40) and (1.41)

$$E^i_{\text{ac.}} = \frac{\pi|A_{\text{ac.}}|^2}{\varrho_0\omega}, \qquad E^i_{\text{surf.}} = \frac{\kappa|A_{\text{surf.}}|^2}{4\varrho_0\omega N}\left(5\kappa^4 - 4\kappa^2 k^2 - k_0^4\right).$$

The energy flux densities of the scattered waves are given by the two terms in the second formula in (1.42) expressing the optical theorem

$$E^s_{\text{ac.}} = \frac{\pi|\Psi(\vartheta)|^2}{\varrho_0\omega}, \qquad E^s_{\text{surf.}} = \frac{\kappa|\psi^+|^2}{4\varrho_0\omega N}\left(5\kappa^4 - 4\kappa^2 k^2 - k_0^4\right).$$

Here $E^s_{\text{ac.}}$ is the linear density of energy flux carried by the diverging cylindrical wave in a unit angle of ϑ and $E^s_{\text{surf.}}$ is the linear density of energy flux carried by the travelling to the right surface wave. Substituting the expressions (1.56) and (1.57) in the above formulae and letting $\vartheta = \vartheta_0$ it is easy to verify the following identity

$$\frac{E^s_{\text{ac.}}}{E^i_{\text{surf.}}} = \frac{E^s_{\text{surf.}}}{E^i_{\text{ac.}}}. \tag{1.58}$$

The results (1.55), (1.56) can be checked with the help of the optical theorem (1.42). Compute first the integral of the far field amplitude

$$\int_0^\pi |\Psi(\vartheta)|^2 d\vartheta = \frac{N^2}{\pi^2}\frac{k^2\sin^2\vartheta_0}{Z(\vartheta_0)}\left(\frac{1}{|D_0|^2}\int_0^\pi \frac{k^2\sin^2\vartheta}{Z(\vartheta)}d\vartheta \right.$$
$$\left. + \frac{k^2\cos^2\vartheta_0}{|D_2|^2}\int_0^\pi \frac{k^4\sin^2\vartheta\cos^2\vartheta}{Z(\vartheta)}d\vartheta\right).$$

Here the denominator Z is introduced as

$$Z(\vartheta) = \mathcal{L}(\vartheta)\overline{\mathcal{L}(\vartheta)} = N^2 + k^2\sin^2\vartheta\left(k^4\cos^4\vartheta - k_0^4\right)^2.$$

The cross terms with D_0 and D_2 contain $\cos\vartheta$ in the first power and do not contribute to the integral. Introducing new variable of integration $\lambda = k\cos\vartheta$ the above integrals can be written as

$$\int_0^\pi \frac{k^{2+\ell}\sin^2\vartheta\cos^\ell\vartheta}{Z(\vartheta)}d\vartheta = \int_{-k}^{k}\frac{\sqrt{k^2-\lambda^2}\,\lambda^\ell d\lambda}{N^2+(k^2-\lambda^2)(\lambda^4-k_0^4)^2}.$$

Computing $|\psi^+|^2+|\psi^-|^2$ and excluding complex values from denominators yields

$$|\psi^+|^2+|\psi^-|^2 = \frac{k^2\sin^2\vartheta_0}{Z(\vartheta_0)}\frac{8N^2(\kappa^2-k^2)}{\kappa^2(5\kappa^4-4k^2\kappa^2-k_0^4)}\left(\frac{1}{|D_0|^2}+\frac{\kappa^2k^2\cos^2\vartheta_0}{|D_2|^2}\right).$$

Combining the above expressions one finds the effective cross-section

$$\Sigma = \frac{4N}{k}\frac{k^2\sin^2\vartheta_0}{Z(\vartheta_0)}\sum_{\ell=0,2}\frac{(k\cos\vartheta_0)^\ell}{|D_\ell|^2}\times$$
$$\times\left(\frac{N}{2\pi}\int_{-k}^{k}\frac{\sqrt{k^2-\lambda^2}\lambda^\ell d\lambda}{N^2+(k^2-\lambda^2)(\lambda^4-k_0^4)^2}+\frac{1}{\kappa}\frac{(\kappa^2-k^2)\kappa^\ell}{5\kappa^4-4\kappa^2k^2-k_0^4}\right)$$

On the other hand the effective cross-section can be found as the energy taken from the reflected wave. Again excluding complex values from the denominators yields

$$\Sigma = \frac{4N}{k}\frac{k^2\sin^2\vartheta_0}{Z(\vartheta_0)}\left(\frac{1}{|D_0|^2}\mathrm{Im}\,D_0 - \frac{k^2\cos^2\vartheta_0}{|D_2|^2}\mathrm{Im}\,D_2\right). \tag{1.59}$$

Compute now the imaginary parts of the boundary-contact integrals D_0 and D_2. The path of integration for D_0 and D_2 can coincide with the real axis except the small neighborhoods of points $\lambda = \pm\kappa$ where the denominator $L(\lambda)$ has zeros. The point $\lambda = \kappa$ is avoided from below and the point $\lambda = -\kappa$ is avoided from above. The integrals can be also rewritten as the principal value integrals along real axis plus the sum of semi-residues in that points

$$D_\ell = \frac{1}{2\pi}\mathrm{v.p.}\int \ldots d\lambda + \frac{i}{2}\operatorname*{Res}_{\lambda=\kappa}\ldots - \frac{i}{2}\operatorname*{Res}_{\lambda=-\kappa}\ldots$$

For $|\lambda| > k$ the integrated functions in the above integrals are real and imaginary parts of D_ℓ are only due to the integrals form $-k$ to k and residues. Excluding complex values from the denominator of the integrated functions and computing residues yields

$$\operatorname{Im} D_\ell = i^\ell \left(\frac{N}{2\pi} \int\limits_{-k}^{k} \frac{\sqrt{k^2-\lambda^2}\lambda^\ell d\lambda}{N^2+(k^2-\lambda^2)(\lambda^4-k_0^4)^2} + \frac{1}{\kappa} \frac{(\kappa^2-k^2)\kappa^\ell}{5\kappa^4-4\kappa^2k^2-k_0^4} \right).$$

This formula shows that the two expressions derived above for the effective cross-section coincide.

Analogously one can check the optical theorem for the case of incident surface wave.

Numerical characteristics of scattering by fixed point are presented on Fig. 1.4 (thick line). Discussion of the characteristics see in the end of this section.

For low frequencies ($kh \ll 1$) substituting the asymptotics (A.11) of the integrals D_ℓ and retaining only principal order terms yields

$$\Sigma \sim 10\Big(1+\cos(\pi/5)\Big) kN^{-2/5} \sin^2 \vartheta_0.$$

Scattering by a pointwise crack

Analysis of this problem requires regularization of the integrals (1.54). The solution is expressed in terms of integrals D_4 and D_6. These integrals are defined by (1.54) as limits. First the exponential factor $\exp(i\varepsilon\lambda)$ is assumed. Then for positive ε the ends of the integration path are shifted to the upper half-plane of λ and convergence of the integrals is achieved due to exponential decay of the above factor. Finally the limit for $\varepsilon \to +0$ is taken. Details of regularization and explicit expressions for the integrals D_ℓ in terms of solutions κ_j of dispersion equation (1.14) are presented in Appendix A. In particular the values of odd integrals are

$$D_1 = 0, \quad D_3 = \frac{1}{2}, \quad D_5 = 0.$$

The boundary-contact conditions (1.51) on the crack yield

$$c_0 = 0, \quad c_1 = 0, \quad c_2 = \frac{D}{D_4} \frac{\partial^3 U^{(g)}(0,0)}{\partial z \partial x^2}, \quad c_3 = \frac{D}{D_6} \frac{\partial^4 U^{(g)}(0,0)}{\partial z \partial x^3}.$$

Computing displacements corresponding to the geometrical part of the field and substituting the above constants into the representation (1.53) gives explicit formula for the scattered field. Its far field amplitude is given by the formula

$$\Psi(\vartheta;\vartheta_0) = A\frac{iN}{\pi}\frac{k^2\sin\vartheta\sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}\left(\frac{k^4\cos^2\vartheta\cos^2\vartheta_0}{D_4} - \frac{k^6\cos^3\vartheta\cos^3\vartheta_0}{D_6}\right). \tag{1.60}$$

The optical theorem (1.42) can be checked in the same way as it was done above for the scattering by a fixed point. The effective cross-section of a pointwise crack is

$$\Sigma = \frac{4Nk^5\sin^2\vartheta_0\cos^4\vartheta_0}{Z(\vartheta_0)}\left(\frac{\operatorname{Im} D_4}{|D_4|^2} - \frac{k^2\cos^2\vartheta_0 \operatorname{Im} D_6}{|D_6|^2}\right). \tag{1.61}$$

For low frequencies the first term dominates and taking into account (A.11) one finds

$$\Sigma \sim 10\Big(1-\cos(2\pi/5)\Big)k^5N^{-6/5}\sin^2\vartheta_0\cos^4\vartheta_0.$$

In low frequency limit the main contribution to the effective cross-section is due to the surface wave process. For incident flexural wave one can introduce reflection and transition coefficients of the crack. The connection formula (1.30) allows the amplitudes of surface waves to be found

$$\psi^{\pm} = iA\frac{\kappa^3(\kappa^2-k^2)}{5\kappa^4-4\kappa^2k^2-k_0^4}\left(\frac{1}{D_4}\mp\frac{\kappa^2}{D_6}\right).$$

Substituting the asymptotics (A.11) one finds the amplitude of passed wave

$$A+\psi^+ \sim A - \frac{A}{2}\Big(\Big(1-e^{2\pi i/5}\Big)+\Big(1+e^{\pi i/5}\Big)\Big) = A\frac{e^{2\pi i/5}-e^{\pi i/5}}{2}$$

and the amplitude of reflected wave

$$\psi^- \sim \frac{A}{2}\Big(-\Big(1-e^{2\pi i/5}\Big)+\Big(1+e^{\pi i/5}\Big)\Big) = A\frac{e^{2\pi i/5}+e^{\pi i/5}}{2}.$$

One can check the energy conservation law in the form

$$|A+\psi^+|^2+|\psi^-|^2 = 1.$$

Numerical characteristics computed by the formula (1.61) are presented on Fig. 1.4 by thin lines.

Scattering by stiffener

Analogously to the case of fixed point continuity of plate at the support yields $c_2 = 0$, $c_3 = 0$. The system arising from the other two conditions splits and one finds the far field amplitude

$$\Psi(\vartheta;\vartheta_0) = A\frac{iN}{\pi}\frac{k^2\sin\vartheta\sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}\left(\frac{1}{D_0 - D/(M\omega^2)} - \frac{k^2\cos\vartheta\cos\vartheta_0}{D_2 + D/(I\omega^2)}\right) \tag{1.62}$$

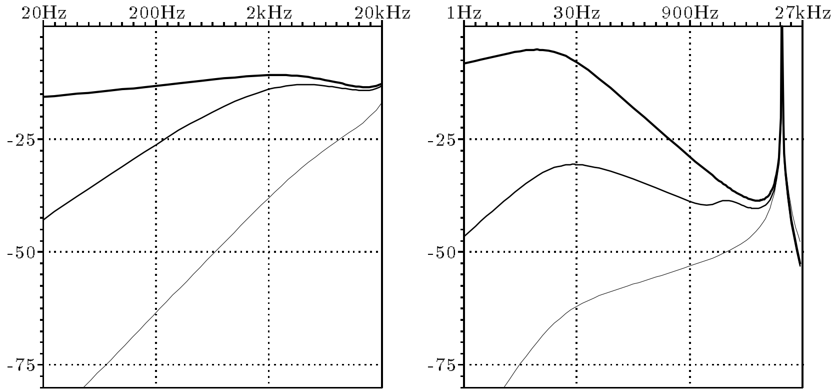
and the effective cross-section

$$\Sigma = \frac{4Nk\sin^2\vartheta_0}{Z(\vartheta_0)}\left(\frac{\operatorname{Im} D_0}{|D_0 - D/(M\omega^2)|^2} - \frac{k^2\cos^2\vartheta_0 \operatorname{Im} D_2}{|D_2 - D/(I\omega^2)|^2}\right).$$

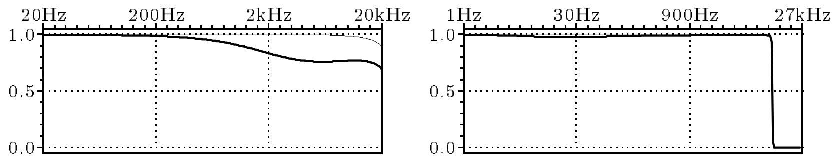
Numerical characteristics are presented on the upper graphs of Fig. 1.4 by medium lines.

The two systems plate–water (left-side graphs) and plate–air (right-side graphs) are taken for illustration of the scattering process by point models. Parameters of the systems and characteristic frequencies see in Appendix C. The frequency characteristics are presented on the upper graphs in Fig. 1.4. Lower graphs show the portion of energy carried by the surface waves, that is the ratio of the second term in the last expression of (1.42) to Σ. At the bottom graphs of Fig. 1.4 the angular dependencies of effective cross-sections are plotted for different frequencies.

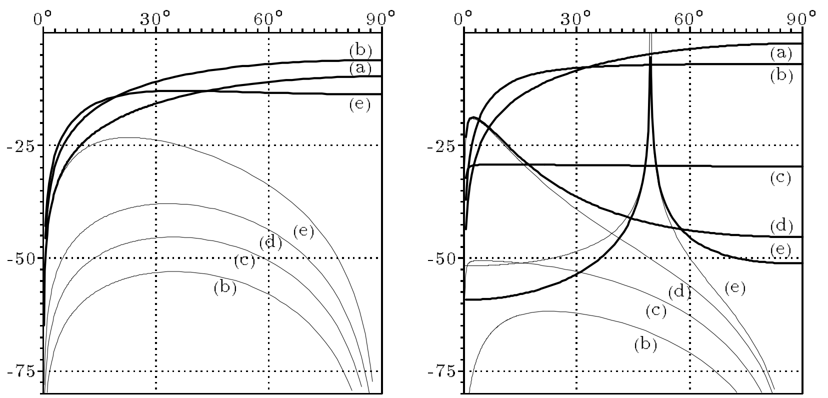
Comparing the above three point models one can notice some general properties of the scattering process. At low frequencies (below the coincidence frequency f_c) all models distinguish significantly. In that range of frequencies most of energy is concentrated near the plate and it is natural that different conditions for displacements cause large difference in the scattering characteristics. The scattering by a point crack is minimal and the scattering by a fixed point is maximal. The stiffener occupies intermediate position. For the system 1 of steel plate in water the inequality $\Sigma_{\text{crack}} < \Sigma_{\text{stiffener}} < \Sigma_{\text{fixed}}$ holds in the whole range of frequencies for which the model of thin plate is valid. For air loaded steel plate (system 2) one can consider frequencies above f_c, where the character of scattering process changes significantly. At a small interval near coincidence frequency f_c energy redistributes from the surface channel of scattering to acoustic channel and at frequencies above f_c almost all the energy is carried by diverging cylindrical wave. For weakly loaded plates redistribution of energy appears very sharp. For the system 1 surface energy begins to decrease already for $f \approx 0.5 f_c$. At frequencies above coincidence frequency all the models have



Frequency characteristics. Angle of incidence is $\vartheta_0 = 30°$.



Portion of scattered energy carried by surface waves.



Frequencies: (a) – 20Hz, (b) – 500Hz, (c) – 1kHz, (d) – 2kHz, (e) – 10kHz

Frequencies: (a) – 1Hz, (b) – 30Hz, (c) – 1kHz, (d) – 11kHz, (e) – 27kHz

Angular characteristics of scattering for fixed point and point crack.

Fig. 1.4 Effective cross-sections (in dB) of models: fixed point (bold), crack (thin), stiffener (medium) in 1cm steel plate in water (left) and in 1mm steel plate in air (right).

close scattering characteristics with a sharp maximum of effective cross-section (upper right graph of Fig. 1.4). This maximum corresponds to the critical frequency f^* for which the trace of the incident wave front on the plate moves with the velocity of flexural waves c_f. That is $k\cos\vartheta_0 = k_0$ and thus $f^* = f_c/\cos^2\vartheta_0$. The same sharp maximum is seen on the angular characteristics at lower-right graph of Fig. 1.4. Angular dependencies of the effective cross-section of fixed point and of point stiffener (not shown) are similar. The angular characteristics of the point crack differs significantly. This difference is due to the factor $\cos^4\vartheta_0$ presented in the formula (1.61).

1.4.2 *Scattering by crack at oblique incidence*

The explicitly solvable models discussed above deal with incident waves running orthogonally to the obstacle. These models allow generalization to the case of oblique incidence, that is the incident wave can run at some angle φ_0 to the fixed line, crack, stiffener or any other linear inhomogeneity. The factor $e^{i\mu_0 y}$, $\mu_0 = k\cos\vartheta_0\cos\varphi_0$ of the incident plane wave can be separated from all the components of the field and the problems of scattering can be reformulated as two-dimensional. These two-dimensional problems differ from the discussed above by replacement of the Laplace operator by $\Delta - \mu_0^2$ and by expressions for force and bending momentum, which after computing derivatives by y in (1.8) become (prime denotes derivative by x)

$$\mathbb{F}w = -D\Big(w''' - (2-\sigma)\mu_0^2 w'\Big), \qquad \mathbb{M}w = D\Big(w'' - \sigma\mu_0^2 w\Big).$$

We consider here only the model of a point crack. Let the field scattered by the infinite straight crack be denoted by $U^{(c)}$. This field can be found by the same approach as in the previous section. However let the other approach be demonstrated. This approach is based on Fourier transform method. Let the solution of the problem be searched in the form of the integral

$$U^{(c)}(x,y,z) = -\frac{e^{i\mu_0 y}}{2\pi D}\int \exp\left(i\lambda x - \sqrt{\lambda^2+\mu_0^2-k^2}\,z\right)\times \\ \times\left(\sum_{\ell=0}^{3} c_\ell(\mu_0)(i\lambda)^\ell\right)\frac{d\lambda}{L(\lambda,\mu_0)}, \tag{1.63}$$

where c_ℓ are unknown. The path of integration is chosen the same as in the Fourier representation of the Green's function $G(x,z;0,0)$, the singularities on the negative semi-axis are avoided from above and the singularities on the positive semi-axis are avoided from below. The dependence of $U^{(c)}$ on y co-ordinate is inherited from the incident wave.

The representation (1.63) satisfies the Helmholtz equation (1.10) and the boundary condition (1.13) on the whole plane $\{z = 0\}$. The boundary-contact conditions on the crack (at $x = 0$) yield the system of algebraic equations for the coefficients c_ℓ. It is convenient to rewrite the boundary-contact conditions in the form

$$\mathbb{F}w(+0, y) - \mathbb{F}w(-0, y) = 0, \quad \mathbb{M}w(+0, y) - \mathbb{M}w(-0, y) = 0$$

and

$$\mathbb{F}w(+0, y) = 0, \quad \mathbb{M}w(+0, y) = 0.$$

The first two conditions remain homogeneous when the geometrical part of the field is separated. Inverting the Fourier integral in these conditions yields

$$c_0 = -\sigma\mu_0^2 c_2, \qquad c_1 = -(2-\sigma)\mu_0^2 c_3.$$

The other two conditions yield the system

$$\begin{cases} c_1\Big(D_4(\mu_0) - (2-\sigma)\mu_0^2 D_2\Big) + c_3\Big(D_6(\mu_0) - (2-\sigma)\mu_0^2 D_4\Big) \\ \qquad\qquad\qquad\qquad = \varrho_0\omega^2\mathbb{F}w^{(g)}(0), \\ c_0\Big(D_2(\mu_0) - \sigma\mu_0^2 D_0\Big) + c_2\Big(D_4(\mu_0) - \sigma\mu_0^2 D_2\Big) = -\varrho_0\omega^2\mathbb{M}w^{(g)}(0). \end{cases}$$

Here the integrals $D_\ell(\mu_0)$ are introduced as

$$D_\ell(\mu_0) = \frac{1}{2\pi}\int e^{+i0\lambda}\sqrt{\lambda^2 + \mu_0^2 - k^2}\,\frac{(i\lambda)^\ell\, d\lambda}{L(\lambda, \mu_0)}. \tag{1.64}$$

Note that integrals D_ℓ from the previous section are the values of integrals $D_\ell(\mu_0)$ for $\mu_0 = 0$. When computing limits of $w^(s)$ and its derivatives at $x = \pm 0$ which are involved in the above system the parity properties of different terms in the representation (1.63) are taken into account.

Computing displacements of the geometrical part $U^{(g)}$

$$w^{(g)} = -\frac{2i}{D}\frac{k\sin\vartheta_0}{L(\vartheta_0)}e^{ikx\cos\vartheta_0\sin\varphi_0}$$

and substituting expressions for the coefficients c_0 and c_1 in the above system yields

$$c_2 = -\frac{2\varrho_0\omega^2 ik^3 \sin\vartheta_0 \cos^2\vartheta_0 \left(1 - (1-\sigma)\cos^2\varphi_0\right)}{\mathcal{L}(\vartheta_0)\Delta_e(\mu_0)},$$

$$c_3 = \frac{2\varrho_0\omega^2 k^4 \sin\vartheta_0 \cos^3\vartheta_0 \sin\varphi_0 \left(1 + (1-\sigma)\cos^2\varphi_0\right)}{\mathcal{L}(\vartheta_0)\Delta_o(\mu_0)}.$$

Here the denominators Δ_e and Δ_o are introduced as

$$\Delta_e(\mu_0) = D_4(\mu_0) - 2\sigma\mu_0^2 D_2(\mu_0) + \sigma^2\mu_0^4 D_0(\mu),$$

$$\Delta_o(\mu_0) = D_6(\mu_0) - 2(2-\sigma)\mu_0^2 D_4(\mu_0) + (2-\sigma)^2\mu_0^4 D_2(\mu_0).$$

Note that for some values of μ_0 the denominators $\Delta_o(\mu_0)$ or $\Delta_e(\mu_0)$ can be equal to zero. Real solutions of dispersion equations

$$\Delta_e(\varkappa_e) = 0, \quad \Delta_o(\varkappa_o) = 0 \tag{1.65}$$

define wave numbers $\varkappa_e$ of symmetric and $\varkappa_o$ of anti-symmetric edge waves that may run along the crack. Analysis of the above dispersion equations is carried out in section 4.2. The edge waves are defined by the formula (1.63) where one lets for symmetric wave $c_0 = \sigma\mu_0^2 c_2$, $c_1 = 0$, $c_3 = 0$ and for anti-symmetric wave $c_0 = 0$, $c_1 = (2-\sigma)\mu_0^2 c_3$, $c_2 = 0$. It can be shown that both $\varkappa_e$ and $\varkappa_o$ (if exist) are greater than k. So for any angles ϑ_0 and φ_0 coefficients c_2 and c_3 are finite.

Applying saddle point method (1.22) to the integral (1.63) allows the asymptotics of the scattered field to be found. Let the cylindric co-ordinate system (d, α, y) be introduced by the formulae

$$x = d\cos\alpha, \quad z = d\sin\alpha.$$

Then the phase function in (1.63) becomes

$$\Phi(\lambda) = \lambda\cos\alpha + \sqrt{k^2 - \mu_0^2 - \lambda^2}\,\sin\alpha$$

with the saddle point at

$$\lambda = k\xi\cos\alpha, \qquad \xi = \sqrt{1 - \cos^2\vartheta_0\cos^2\varphi_0}.$$

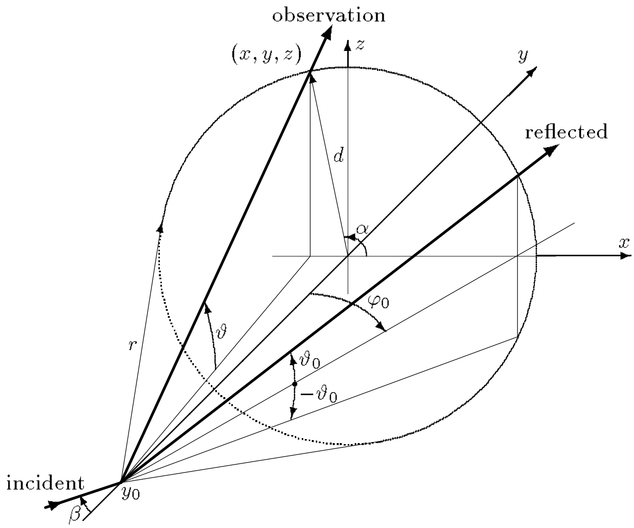


Fig. 1.5 Diffraction by straight crack at oblique incidence.

The contribution of the saddle point to the asymptotics of the integral for $d \gg 1$ gives diverging cylindrical wave

$$U^{(c)}(x,z) \sim \sqrt{\frac{2\pi}{k\xi d}} e^{ik\xi d - i\pi/4} \Psi(\alpha)$$

with the far field amplitude

$$\begin{aligned}
\Psi(\alpha) = & \frac{iNk^6\xi}{\pi} \frac{\sin\alpha \sin\vartheta_0 \cos^2\vartheta_0}{-L(k\xi\cos\alpha, k\cos\vartheta_0\sin\vartheta_0)\mathcal{L}(\vartheta_0)} \times \\
\times & \left(\frac{1-(1-\sigma)\cos^2\varphi_0}{\Delta_e(k\cos\vartheta_0\cos\varphi_0)} \left(\cos^2\alpha - \cos^2\vartheta_0\cos^2\varphi_0(\cos^2\alpha-\sigma)\right) \right. \\
& - \frac{1+(1-\sigma)\cos^2\varphi_0}{\Delta_o(k\cos\vartheta_0\cos\varphi_0)} k^2\xi\cos\alpha\cos\vartheta_0\cos\varphi_0 \times \\
& \left. \times \left(\cos^2\alpha - \cos^2\vartheta_0\cos^2\varphi_0(\cos^2\alpha - 2 + \sigma)\right)\right).
\end{aligned}$$

The residues in the zeros of the denominator $L(\lambda, \mu_0)$ which are crossed when the contour of integration is shifted to the steepest descent path give surface waves.

The oblique incident wave diffracts in a *diffraction cone*. The ray geometry of the diffraction phenomenon is presented on Fig. 1.5. Consider the incident ray that intersects the crack at point $(0, y_0, 0)$. The diffracted rays originated from this incident ray form the cone which in Keller terminology is called diffraction cone [36]. The opening angle of this cone is equal to the angle β between the incident ray and the crack. In the case of orthogonal incidence diffraction cone transforms to diffraction plane and one deals with two-dimensional problem. The angle of the cone opening β is given by the formula

$$\beta = \cos^{-1}\Big(\cos\vartheta_0 \cos\varphi_0\Big).$$

In the case of plane incident wave all the cones with opening β and revolution axis y are equivalent. The point of observation (x, y, z) defines the particular diffraction cone. The y co-ordinate of its vertex is

$$y_0 = y - d\cot\beta.$$

The coordinates (d, α) specify the point on the cone. In order to rewrite the asymptotics of diverging cylindrical wave in symmetric form let the angle of elevation ϑ (see Fig. 1.5) be introduced

$$\vartheta = \sin^{-1}\Big(\sin\beta \sin\alpha\Big).$$

Now the asymptotics can be written as

$$U^{(c)}(x, y, z) \sim \frac{e^{iky_0\cos\beta}}{\sqrt{\sin\beta}} \sqrt{\frac{2\pi}{kr}}\, e^{ikr - i\pi/4}\Psi(\vartheta),$$

$$\Psi(\vartheta) = \frac{iNk^6}{\pi} \frac{\sin\vartheta}{\mathcal{L}(\vartheta)} \frac{\sin\vartheta_0}{\mathcal{L}(\vartheta_0)} \left(\frac{(\cos^2\vartheta - p)(\cos^2\vartheta_0 - p)}{\Delta_e(k\cos\beta)} \right.$$
$$\left. - \frac{(\cos^2\vartheta + p)(\cos^2\vartheta_0 + p)}{\Delta_o(k\cos\beta)} k^2 \sqrt{\cos^2\vartheta - \cos^2\beta}\sqrt{\cos^2\vartheta_0 - \cos^2\beta} \right).$$

Here $r = \sqrt{x^2 + z^2 + (y - y_0)^2}$ is the distance from the cone vertex to the observation point and the parameter p is introduced as $p = (1 - \sigma)\cos^2\beta$.

1.4.3 *Point models in three dimensions*

For the problems of scattering by compact obstacles in thin elastic plates the set of classical point models shrinks to such models in which the boundary-contact conditions are formulated for displacements w only. In infinite plate this is the model of a fixed point, which is formulated as

$$w(0,0) = 0. \tag{1.66}$$

The solution is searched again in two steps. First the general solution is constructed. This solution satisfies the boundary-value problem with the inhomogeneous generalized boundary condition (1.13) on the plate. The right-hand side of this condition contains arbitrary generalized function concentrated in the point $(x = 0, y = 0)$. Evidently this is a linear combination of delta-function and its derivatives. The order of admissible derivatives is regulated by the *Meixner conditions* (1.18), (1.19) which are formulated in that case both for acoustic pressure U and flexural displacements w. It can be shown that only delta-function itself can be presented and no its derivatives are admissible

$$\left(\Delta^2 - k_0^4\right) w(x,y) + \frac{1}{D} U(x,y,0) = \frac{c}{D}\delta(x)\delta(y).$$

Therefore the general solution is simply the Green's function $G(\mathbf{r}; 0, 0)$ multiplied by an unknown constant c. This constant is found from the condition (1.66).

In order to formulate the point model of an attached to the plate mass M, the expression for the force F in a separate point is needed. Note that the first formula (1.8) gives force on arbitrary line. Let a small circle of radius ε be cut from the plate. The total force with which that circle acts on the plate is given by the integral of $\mathbb{F}w$ over the circumference. Letting $\varepsilon \to 0$ yields

$$F = \lim_{\varepsilon \to 0} \int_0^{2\pi} \mathbb{F} w \varepsilon d\varphi = c.$$

Thus the point model of an attached mass M can be written as the condition

$$c = \omega^2 M w(0,0). \tag{1.67}$$

Scattering by a fixed point

Consider first the case of isolated plate. That is the boundary value problem for the homogeneous bi-harmonic equation (1.9)

$$\Delta^2 w - k_0^4 w = 0, \quad \mathbb{R}^2 \setminus \{0\}$$

with the condition (1.66) and radiation condition (1.47).

The general solution** satisfies the equation

$$\triangle^2 w^{(s)} - k_0^4 w^{(s)} = \frac{c}{D}\delta(x)\delta(y)$$

with unknown amplitude c in the right-hand side. This amplitude of passive source should be found from the condition (1.66), which becomes inhomogeneous with the flexural displacement of the incident wave in the right-hand side

$$w^{(s)}(0,0) = -w^{(g)}(0,0).$$

The general solution is the Green's function (1.45) multiplied by the coefficient c

$$w^{(s)}(x,y) = cg_0(x,y) = \frac{ic}{8k_0^2 D}\left(H_0^{(1)}(k_0\rho) - H_0^{(1)}(ik_0\rho)\right).$$

Satisfying the condition in the fixed point, yields

$$c = -\frac{w^{(g)}(0,0)}{g_0(0,0)}.$$

The two Bessel functions in the formula for g_0 have logarithmic singularities, but their difference is bounded and the value of Green's function at the point of the source is defined. Using the representation of Bessel functions $H_0^{(1)}(\xi) = J_0(\xi) + iY_0(\xi)$ in the form of series [1]

$$J_0(\xi) = \sum_{j=0}^{\infty} \frac{(-\xi^2/4)^j}{(j!)^2} = 1 - \frac{\xi^2}{4} + \ldots,$$
$$Y_0(\xi) = \frac{2}{\pi}\left(\ln(\xi/2) + C_E\right) J_0(\xi) + \frac{2}{\pi}\left(\frac{\xi^2}{4} - \frac{3\xi^4}{64} + \ldots\right)$$

yields

$$g_0(0,0) = \frac{i}{8k_0^2 D}.$$

There is evidently rotational symmetry in the problem of scattering, so let $\varphi_0 = 0$ and the incident wave be

$$w^{(i)} = w^{(g)} = Ae^{ik_0 x}.$$

**General solution is denoted here by $w^{(s)}$ as well as the scattered field to which it transforms when constant c is found.

Then the scattered field is

$$w^{(s)} = -A\left(H_0^{(1)}(k_0\rho) - H_0^{(1)}(ik_0\rho)\right).$$

At large distances it has the asymptotics of diverging circular wave (1.47) with the far field amplitude

$$\psi_0(\varphi) = -\frac{A}{\pi}. \tag{1.68}$$

The above result satisfies the optical theorem (1.43).

Consider now the case of fluid loaded plate. In the above derivations the Green's function $g_0(x, y)$ should be replaced by the Green's function (1.21) corresponding to a surface source in a fluid loaded plate

$$U^{(s)} = c\varrho_0\omega^2 G(x, y, z; 0, 0).$$

For the incident spatial wave at angle ϑ_0 one finds

$$w^{(g)} = \frac{2A}{D}\frac{ik\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}e^{ikx\cos\vartheta_0}.$$

The value of the Green's function $g(x, y; 0, 0)$ in the point of the surface source is given by the boundary-contact integral

$$g(0,0) = \frac{1}{D\varrho_0\omega^2}D_{00}, \qquad D_{00} = \frac{1}{4\pi^2}\iint\frac{\sqrt{\lambda^2+\mu^2-k^2}}{L(\lambda,\mu)}d\lambda\, d\mu.$$

The analysis of the above integral see in Appendix A.

The far field amplitude of the scattered field differs by a multiplier $-w^{(g)}(0,0)D/D_{00}$ from the far field amplitude of the point source field $G(\mathbf{r}; 0, 0)$ and can be easily found by differentiating the formula (1.24)

$$\Psi(\vartheta,\varphi) = -\frac{iANk^3}{2\pi^2 D_{00}}\frac{\sin\vartheta\sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}. \tag{1.69}$$

In a similar way one can consider the surface wave scattering by the fixed point, check reciprocity principle (1.58) and optical theorem (1.39).

In the low frequency approximation one can use the asymptotics (A.14) of the integral D_{00}. This yields

$$\Psi(\vartheta,\varphi) \approx \frac{20Ak^3}{\pi^2 N^{3/5}}\frac{i\cot(2\pi/5)-2}{4+\cot^2(2\pi/5)}\sin\vartheta\sin\vartheta_0.$$

Scattering by a pointwise mass

In the case of attached pointwise mass the derivations are analogous. The solution differs by a multiplier from the corresponding Green's function for a point source in isolated or fluid loaded plate. Further one substitutes $w(0,0) = w^{(g)}(0,0) + w^{(s)}(0,0)$ into the condition (1.67) and finds the amplitude c. Finally this yields

$$\psi_0(\varphi) = -\frac{A}{\pi}\left(1 + i\frac{8k_0^2 D}{\omega^2 M}\right)^{-1}$$

in the case of isolated plate and

$$\Psi(\vartheta,\varphi) = -\frac{iANk^3}{2\pi^2}\left(D_{00} - \frac{D}{M\omega^2}\right)^{-1}\frac{\sin\vartheta\,\sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}$$

for acoustic wave diffraction by fluid loaded plate.

1.5 Scattering problems for plates with infinite crack

1.5.1 *General properties of boundary value problems*

The geometry in the problems of scattering discussed in the previous sections can be considered as a perturbation of some basic geometry by a compact or pointwise obstacle. This basic geometry is the geometry of fluid half-space bounded by the plate. For such problems both in two and in three dimensions the Green's functions can be written in explicit form. This fact is the keynote reason why the point models allow explicit solution to be constructed in the form of integrals.

Consider now a more complicated basic geometry. Namely let the plate, bounding fluid half-space, be cut along the line $\{x = 0\}$. Such geometry is already considered in section 1.4.2 when dealing with oblique incidence at a straight crack. The Green's function for that basic geometry can be found explicitly by methods similar to that used in section 1.4.2. Indeed one can decompose the field of a point source by plane waves which results in additional Fourier transform by the parameter μ_0 in (1.63). In this chapter we shall not use that Green's function, however in Chapter 3 it will be needed and it is derived there (see section 3.5.4). The Green's function in the case of a plate with a crack is essentially more complicated. This is one technical difficulty that appears due to the crack. The other circumstance is that infinite crack causes additional channel(s) of scattering to appear.

This channel is associated with edge waves of Rayleigh type. Edge waves that run along the free edge and exponentially decrease far from it were first discovered in the model of isolated plate [41]. Then the influence of fluid was taken into account as a small perturbation [46]. Further analysis of edge waves that propagate along a straight crack are presented in [10]. Results of [41] for edge waves in isolated plate are briefly presented in the section 1.5.2 and summary of results concerning edge waves in fluid loaded plate can be found in Chapter 4. Presence of edge waves that propagate along the crack requires additional radiation conditions to be included in the formulation of the scattering problem. Note also that additional terms appear in the expression for the effective cross-section. We discuss optical theorem only in the isolated plate with the crack (see section 1.5.2).

We present only one point model for plates cut by an infinite crack. This is the model for a joint of two semi-infinite plates. It is formulated in the form of boundary-contact conditions in separate points of plates edges. As it was discussed before, such boundary-contact conditions can be formulated for the displacements only (no derivatives are allowed). The pointwise joint of two semi-infinite plates is formulated as the continuity condition for displacements and absence of forces

$$w(+0,0) - w(-0,0) = 0, \quad F(0,0) = 0. \tag{1.70}$$

Actually there are two functions, one $w_1(x,y)$ defined for $x \geq 0$ gives displacements of semi-infinite plate $P^+ = \{x > 0\}$, the other $w_2(x,y)$ defined for $x \leq 0$ expresses displacements in the semi-infinite plate $P^- = \{x < 0\}$. These two functions have coincident values at the point of joint which is assumed at the origin. The force F in that point is defined by the same limiting procedure as in the section 1.4.3.

1.5.2 *Scattering problems in isolated plates*

Consider first the simpler case of isolated plates. Let an obstacle Ω_0 be of arbitrary nature. We assume that its boundary $\partial\Omega_0$ crosses the crack in two points $y = a$ and $y = b$, so that the segment $[a,b]$ belongs to Ω_0. The boundary value problem of diffraction by an obstacle Ω_0 reads

$$\begin{aligned} &\Delta^2 w - k_0^4 w = 0, \qquad (x,y) \in \mathbb{R}^2_+ \cup \mathbb{R}^2_- \setminus \Omega_0, \\ &\mathbb{F}w(\pm 0, y) = 0, \quad \mathbb{M}w(\pm 0, y) = 0, \qquad y < a \;\text{ or }\; y > b. \end{aligned}$$

Besides, the boundary conditions on $\partial\Omega_0$ and radiation conditions at infinity should be specified for the solution $w(x, y)$. The radiation conditions combine (1.47) which is valid at some distance from the crack and radiation conditions for edge waves

$$\frac{\partial w(\pm 0, y)}{\partial y} - i\varkappa w(\pm 0, y) = o(1), \quad y \to \pm\infty.$$

Here $\varkappa$ is the wave number of edge waves (see below).

Note that in the absence of the obstacle flexural waves in two semi-infinite plates P^+ and P^- are completely independent. The incident plane wave

$$w^{(i)} = A \exp\left(ik_0(-x\sin\varphi_0 + y\cos\varphi_0)\right)$$

in the plate P^+ causes two waves to appear. These are the reflected plane wave and the wave that is concentrated near the edge

$$\begin{aligned} w^{(r)} =& Ar(\varphi_0)\exp\left(ik_0(x\sin\varphi_0 + y\cos\varphi_0)\right) \\ &+ At(\varphi_0)\exp\left(-k_0 x\sqrt{1+\cos^2\varphi_0} + ik_0 y\cos\varphi_0\right). \end{aligned} \tag{1.71}$$

Here the coefficients $r(\varphi)$ and $t(\varphi)$ are given by the formulae

$$r(\varphi) = -\frac{\overline{l(\varphi)}}{l(\varphi)}, \qquad t(\varphi) = \frac{-2i\sin\varphi A_+(\varphi)A_-(\varphi)}{l(\varphi)},$$

where

$$l(\varphi) = i\sin\varphi A_+^2(\varphi) + \sqrt{1+\cos^2\varphi}A_-^2(\varphi), \quad A_\pm(\varphi) = (1-\sigma)\cos^2\varphi \pm 1.$$

One can easily check that the field $w^{(g)} = w^{(i)} + w^{(r)}$ satisfies the boundary conditions on the free edge.

The denominator $l(\varphi)$ of the above coefficients equals to zero for some value of the parameter $\varkappa = k_0\cos\varphi^+$ which corresponds to complex angle. The solution of the dispersion equation $l(\varphi^+) = 0$ can be found explicitly

$$\varkappa = k_0\Big((1-\sigma)\big(3\sigma - 1 + 2\sqrt{1 - 2\sigma + 2\sigma^2}\big)\Big)^{-1/4}. \tag{1.72}$$

The boundary conditions on the obstacle Ω_0 should satisfy the correctness condition similar to (1.44). The only difference is that corner forces contributions given by the last terms in (1.44) should be presented at points

$(0, a)$ and $(0, b)$ even if $\partial\Omega_0$ is smooth, but $\partial\Omega_0$ intersects the crack (line $x = 0$) at nonzero angle.

The scattered field can be shown subject to the integral representation similar to (1.46) where the Green's function $g_0(\boldsymbol{\rho}, \boldsymbol{\rho}_0)$ should be replaced by the Green's function for the semi-infinite plate with free edge. The Green's function $g^+(\boldsymbol{\rho}, \boldsymbol{\rho}_0)$ for the semi-infinite plate P^+ with free edge can be constructed by Fourier method. It is convenient to represent it as the sum of Green's function $g_0(\boldsymbol{\rho}, \boldsymbol{\rho}_0)$ for the infinite plate given in (1.45) and a correction $g^{(c)}(\boldsymbol{\rho})$ which causes force and momentum on the edge to vanish

$$g^+(\boldsymbol{\rho}, \boldsymbol{\rho}_0) = g_0(\boldsymbol{\rho}, \boldsymbol{\rho}_0) + g^{(c)}(\boldsymbol{\rho}).$$

The correction is searched in the form of Fourier integral and it is convenient to represent the Green's function g_0 by a similar Fourier integral

$$g_0(\boldsymbol{\rho}, \boldsymbol{\rho}_0) = \frac{1}{8\pi k_0^2 D} \int e^{i\mu(y-y_0)} \left(\frac{e^{-\gamma_-|x-x_0|}}{\gamma_-} - \frac{e^{-\gamma_+|x-x_0|}}{\gamma_+} \right) d\mu,$$
$$g^{(c)}(x, y) = \frac{1}{8\pi k_0^2 D} \int e^{i\mu(y-y_0)} \left(\alpha(\mu; x_0) e^{-\gamma_- x} + \beta(\mu; x_0) e^{-\gamma_+ x} \right) d\mu.$$

The functions $\gamma_\pm$ in the above representations and other functions which are used in this section are introduced below

$$\gamma_\pm = \sqrt{\mu^2 \pm k_0^2}, \quad a_\pm = (1-\sigma)\mu^2 \pm k_0^2, \quad \ell(\mu) = \gamma_- a_+^2 - \gamma_+ a_-^2.$$

Note that $\ell(\mu)$ is connected to $l(\varphi)$ by the formula

$$\ell(\mu) = -k_0^3 l\left(\cos^{-1}(k_0^{-1}\mu)\right).$$

Computing force and momentum on the edge at $x = 0$ yields algebraic system for functions $\alpha(\mu; x_0)$ and $\beta(\mu; x_0)$

$$\begin{cases} a_- \alpha + a_+ \beta = -\dfrac{a_-}{\gamma_-} e^{-\gamma_- x_0} + \dfrac{a_+}{\gamma_+} e^{-\gamma_+ x_0}, \\ \gamma_- a_+ \alpha + \gamma_+ a_- \beta = a_+ e^{-\gamma_- x_0} - a_- e^{-\gamma_+ x_0}. \end{cases}$$

Solving this system yields

$$g^{(c)} = \frac{1}{8\pi k_0^2 D} \int \frac{e^{i\mu(y-y_0)}}{\ell(\mu)} \left(\left(\frac{\gamma_+}{\gamma_-} a_-^2 + a_+^2 \right) e^{-\gamma_-(x+x_0)} \right.$$
$$-2a_+ a_- \left(e^{-\gamma_- x - \gamma_+ x_0} + e^{-\gamma_+ x - \gamma_- x_0} \right) \tag{1.73}$$
$$\left. + \left(\frac{\gamma_-}{\gamma_+} a_+^2 + a_-^2 \right) e^{-\gamma_+(x+x_0)} \right) d\mu.$$

Numerical study of the Green's function (1.73) is undertaken in [35].

At far distances it has the asymptotics of diverging circular wave and two edge waves that run in positive and in negative directions of y axis. Only the first term in the above Fourier integral contributes to the saddle point asymptotics

$$g^{+} \sim \sqrt{\frac{2\pi}{k_0\rho}}\, e^{ik_0\rho - i\pi/4}\psi_g(\varphi), \qquad \rho \to \infty. \tag{1.74}$$

The edge waves running along the edge $\{x = +0\}$ are due to the residues in the zeros of the denominator $\ell(\mu)$. These zeros coincide with the wavenumbers of edge waves $\mu = \pm\varkappa$ defined by (1.72)

$$g^{+} \sim \psi_g^{\pm} e^{\pm i\varkappa y} \left(e^{-\sqrt{\varkappa^2 - k_0^2}|x|} - \frac{(1-\sigma)\varkappa^2 - k_0^2}{(1-\sigma)\varkappa^2 + k_0^2} e^{-\sqrt{\varkappa^2 + k_0^2}|x|} \right), \quad y \to \pm\infty. \tag{1.75}$$

We do not present here the expression for the far field amplitude $\psi(\varphi)$ of the diverging circular wave. It can be easily derived from the integrand of the representation (1.73) in which integration variable μ should be replaced by its value $k_0 \cos\varphi$ in the stationary point and a multiplier $k_0 \sin\varphi$ should be included which appears from the second order derivative of the phase function. Similarly we do not present the formulae for the amplitudes $\psi^{\pm}$ of edge waves.

For $x_0 = y_0 = 0$ the formulae for the Green's function simplify. Combining both integrals yields

$$g^{+}(\boldsymbol{\rho}; 0, 0) = \frac{1}{2\pi D} \int e^{i\mu y} \frac{a_+ e^{-\gamma_- x} - a_- e^{-\gamma_+ x}}{\ell(\mu)}\, d\mu. \tag{1.76}$$

The far field amplitude in the case of the source on the edge is given by a simple formula

$$\psi_g(\varphi) = -\frac{1}{2\pi D} \frac{k_0 \sin\varphi A_+(\varphi)}{l(\varphi)}. \tag{1.77}$$

The amplitudes of edge waves can be found the following

$$\psi_g^{\pm} = \frac{i}{D} \frac{(1-\sigma)\varkappa^2 + k_0^2}{\ell'(\varkappa)},$$

$$\ell'(\varkappa) = \frac{1}{\sqrt{\varkappa^2 - k_0^2}} \left(5(1-\sigma)^2\varkappa^4 + 2\varkappa^2 k_0^2(1-\sigma)(1+2\sigma) - (3-4\sigma)k_0^4\right)$$
$$- \frac{1}{\sqrt{\varkappa^2 - k_0^2}} \left(5(1-\sigma)^2\varkappa^4 - 2\varkappa^2 k_0^2(1-\sigma)(1+2\sigma) - (3-4\sigma)k_0^4\right).$$

One can check that the formula similar to (1.30) is valid

$$\psi_g^{\pm} = -2\pi i \operatorname*{Res}_{\varphi=\varphi^{\pm}} \psi_g(\varphi), \quad \varphi^{\pm} = \cos^{-1}(\pm\varkappa/k_0). \tag{1.78}$$

The connection formula (1.78) is valid not only for the field of a point source applied to the edge, but also for the asymptotics of $g^+(\boldsymbol{\rho}, \boldsymbol{\rho}_0)$ with arbitrary coordinates of the point source. Using the same idea as in Section 1.3.3 one can derive the integral representation for the scattered field in the form of convolution of some unknown functions and constants with the Green's function $g^+(\boldsymbol{\rho}, \boldsymbol{\rho}_0)$. If the obstacle is finite the integration in that formula is carried over a finite contour and the properties of $\psi_g(\varphi)$ are inherited by the far field amplitude of the scattered field. In particular that yields the formula (1.78) for the amplitudes $\psi(\varphi)$ and $\psi^{\pm}$ of the field scattered by arbitrary compact obstacle.

The far field asymptotics allow the optical theorem to be derived. We derive the optical theorem for the case of plane flexural wave incident on a bounded obstacle in the plate cut by the infinite straight crack. For the average energy fluxes carried by the incident and by the diverging circular flexural waves the results of section 1.3.4 can be used. The energy flux carried by the flexural wave w is given by the quadratic form $\Pi_2(w, w)$. The energy taken form the geometrical part of the field E, that is the energy of interaction of geometrical and scattered parts of the field, is only due to the interaction of diverging circular wave with the plane reflected wave given by the first term in (1.71). Applying saddle point method one finds

$$E = -\Pi_2(w^{(g)}, w^{(s)}) - \Pi_2(w^{(s)}, w^{(g)}) = 2\pi\omega D k_0^2 \mathrm{Re}\left(Ar(\varphi_0)\overline{\psi(\varphi_0)}\right).$$

The energy flux carried by the circular wave can be found as the integral

$$\Pi_2(w_{\text{circ.}}, w_{\text{circ.}}) \to 2\pi\omega D k_0^2 \int_0^{2\pi} |\psi(\varphi)|^2 d\varphi.$$

The energy flux carried by the edge waves consists of two parts. One part expresses the flux in the plate and coincides with $\Pi_2(w_{\text{edge}}, w_{\text{edge}})$, the other is due to the contribution of corner forces (1.5) and is given by the expression

$$\Pi_3(w,w) = -\omega \mathrm{Im}\left(w(0,y)\overline{\mathbb{F}_c w(0,y)} \right).$$

Substituting from (1.75) into the above expressions and combining this with the energy flux carried by circular wave one finds

$$E^s = 2\pi\omega D k_0^2 \int\limits_0^{2\pi} |\psi(\varphi)|^2 d\varphi + \omega D k_0^2 Q \left(|\psi_+^+|^2 + |\psi_+^-|^2 + |\psi_-^+|^2 + |\psi_-^-|^2 \right).$$

Here the amplitudes of edge waves are marked with superscript $\pm$ specifying the direction of edge wave propagation (positive or negative direction along y-axis) and by subscript indicating the plate P^+ of P^- in which the edge wave propagates. The quantity Q is introduced as follows

$$Q = \frac{\varkappa}{2\sqrt{\varkappa^2 - k_0^2}} \left(1 - \frac{\left((1-\sigma)\varkappa^2 - k_0^2\right)^4}{\left((1-\sigma)\varkappa^2 + k_0^2\right)^4} - 8(1-\sigma)\frac{k_0^2(\varkappa^2 - k_0^2)}{(1-\sigma)^2\varkappa^4 - k_0^4} \right).$$

Dividing the above energy flux by the energy flux carried on a unit front of the incident wave

$$E^i = |A|^2 \omega D k_0^3$$

gives the formulation of the optical theorem for the isolated plate with infinite straight crack in the form of two expressions for the effective cross-section

$$\begin{aligned} \Sigma =& \frac{2\pi}{k_0} \int\limits_0^{2\pi} |\psi(\varphi)|^2 d\varphi + \frac{Q}{k_0}\left(|\psi_+^+|^2 + |\psi_+^-|^2 + |\psi_-^+|^2 + |\psi_-^-|^2 \right) \\ =& -\frac{4\pi}{k_0} \mathrm{Re}\left(r(\varphi_0)\overline{\psi(\varphi_0)} \right). \end{aligned} \tag{1.79}$$

The formulae presented in this section show the similarity of diffraction by an obstacle in isolated semi-infinite plate with the two-dimensional problem of diffraction by an obstacle in presence of infinite elastic plate. In both cases the two channels of scattering are presented. One is due to the diverging wave and the other is due to waves concentrated near the boundary. In the two-dimensional problems for fluid loaded plate this is the surface wave and in the problems of scattering in semi-infinite isolated plate with a free edge this is the Rayleigh type edge wave. In both types of problems the scattered field can be represented in the form of convolution with the appropriate Green's function. The scattering characteristics in the two channels are connected by the formulae (1.30) and (1.78) respectively. These formulae exactly coincide though the edge wave is given by a more complicated formula (1.75) combining two exponentials. As it was noted in section 1.3.5 the Sommerfeld formula is not valid for the solution of biharmonic operator. The uniqueness theorem can not be proved by the same reason as in the absence of the crack, however no example of concentrated solution is known as well.

1.5.3 *Scattering by pointwise joint*

Return now back to the problem of scattering by a pointwise joint. That is let the obstacle Ω_0 be the point $(0,0)$ and the conditions on the obstacle be the conditions (1.70). Excluding $w^{(g)}$ reduces the boundary value problem, to the problem for the scattered field $w^{(s)}$ which differs by the first contact condition in (1.70) that becomes inhomogeneous with the right hand side

$$-[w^{(g)}] = -A\frac{4i\,A_+(\varphi_0)\sin\varphi_0}{l(\varphi_0)}.$$

In order the second condition in (1.70) be satisfied the passive source in the plate P^+ should be combined with the passive source of the same amplitude, but opposite phase in the plate P^-. That is the general solution of the problem is

$$w^{(s)} = \begin{cases} cg^+(x,y), & x>0, \\ -cg^+(-x,y), & x<0 \end{cases}$$

with some amplitude c. Substituting this formula into the first condition in (1.70) allows that amplitude to be found

$$c = -\frac{[w^{(g)}]}{2g^+(0,0)}.$$

The limiting value of the Green's function in the point of the source can be obtained by direct substitution of $x=0$ and $y=0$ in the formula (1.76)

$$g^+(0,0) = \frac{1}{k_0^2 D}d_0, \quad d_0 = \frac{k_0^4}{\pi}\int\frac{d\mu}{\ell(\mu)}.$$

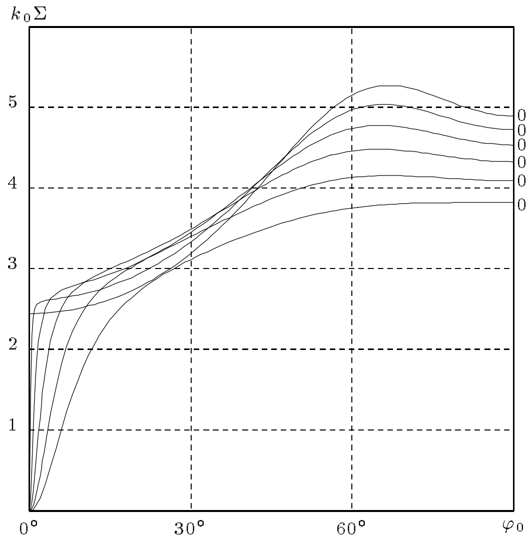


Fig. 1.6 Effective cross-section of the point joint for different values of Poisson's ratio σ (values of σ are indicated at the right margin).

The integrand can be checked to behave at infinity as $O(\mu^{-3})$. It is also convenient to rewrite this integral in dimensionless variable ($\tau = k_0^{-1}\mu$)

$$d_0 = \frac{1}{\pi}\int \frac{d\tau}{\left((1-\sigma)\tau^2+1\right)^2\sqrt{\tau^2-1} - \left((1-\sigma)\tau^2-1\right)^2\sqrt{\tau^2+1}}.$$

This integral plays the role of contact integral and is studied in Appendix A.

Using the far field asymptotics of Green's function g^+ allows the far field amplitude of the scattered circular wave to be found

$$\psi(\varphi;\varphi_0) = \frac{iA}{\pi}\frac{\sin\varphi A_+(\varphi)\sin\varphi_0 A_+(\varphi_0)}{d_0\, l(\varphi)\, l(\varphi_0)}\mathrm{sign}(x). \tag{1.80}$$

The optical theorem (1.79) can be checked in the same way as it was done in section 1.4.1. The integral of the far field amplitude can be reduced to the imaginary part of the integral of $1/\ell(\tau)$ over $[-1,1]$. When excluding complex values from the denominator only the terms with $\sqrt{\tau^2-1}$ contribute to the imaginary part of the integral. The residues of $1/\ell(\tau)$ in the points $\tau = \pm k_0^{-1}\varkappa$ coincide with the contribution of edge waves to the effective cross-section. Finally the formula for the effective cross-section of the point-wise joint takes the form

$$\Sigma = \frac{4}{k_0} \frac{\sin^2 \varphi_0 A_+^2(\varphi_0)}{\sin^2 \varphi_0 A_+^4(\varphi_0) + (1 + \cos^2 \varphi_0) A_-^4(\varphi_0)} \frac{\operatorname{Im} d_0}{|d_0|^2}.$$

The only parameter that is involved in nontrivial way in the above formulae is the Poisson's ratio σ. Angular characteristics $k_0\Sigma(\varphi_0)$ computed numerically by the above formula are presented on Fig. 1.6 for different values of σ. One can note that the case of $\sigma = 0$ is somewhat degenerated. For that value of Poison's ratio no edge waves exist (see formula (1.72) which gives $\varkappa = k_0$) and the effective-cross section does not vanish at grazing incidence (for $\varphi_0 = 0$) which is the case for all other values of Poison's ratio.

Chapter 2

Operator methods in diffraction

2.1 Abstract operator theory

2.1.1 *Hilbert space*

Let L be a *Hilbert space*. That means that some operations are defined on the elements f of the set L and that the set L is supplied with some structure. To define Hilbert space we need some mathematical objects and properties to be introduced.

We start with the property of the space to be *linear*. That property means that two operations are defined on elements of L. Namely, these are the summation of elements $f_1 + f_2$ and the multiplication of an element by a complex number αf, $\alpha \in \mathbb{C}$. These two operations satisfy the usual properties: the order of terms in the sum does not influence the result

$$f_1 + f_2 = f_2 + f_1$$

and there exists unique zero element 0 for which

$$f + 0 = f;$$

the multiplication by complex constants satisfies the following rules

$$\begin{aligned} \alpha(f_1 + f_2) &= \alpha f_1 + \alpha f_2, \\ (\alpha_1 + \alpha_2) f &= \alpha_1 f + \alpha_2 f, \\ \alpha_1(\alpha_2 f) &= (\alpha_1 \alpha_2) f, \\ 1 \cdot f &= f. \end{aligned}$$

An important characteristics of a linear set is its dimension. They say that the elements $\{f_j\}_{j=1}^N$ of a linear set L are *linearly independent* if the equality

$$\sum_{j=1}^{N} \alpha_j f_j = 0$$

is possible only if all coefficients are equal to zero, that is $\alpha_j = 0$ for $j = 1, 2, \ldots, N$. In the opposite case the elements $\{f_j\}_{j=1}^{N}$ are linear dependent. Evidently that if one of the elements of $\{f_j\}$ is equal to zero, then these elements are linear dependent.

The *dimension* of a set is defined as the maximal number of linear independent elements in this set. The dimension of a linear set can be finite or infinite, i.e. in the latter case the set contains infinitely many linear independent elements.

The following structure is introduced in Hilbert space. A complex number $\langle f_1, f_2 \rangle$ is associated to every pair of elements f_1, f_2 from the set L. This number is the *scalar product* and it is introduced in such a way that:

(1) $$\langle f_1, f_2 \rangle = \overline{\langle f_2, f_1 \rangle}$$

(2) $$\langle \alpha_1 f_1 + \alpha_2 f_2, f_3 \rangle = \alpha_1 \langle f_1, f_3 \rangle + \alpha_2 \langle f_2, f_3 \rangle$$

(3) $$\langle f, f \rangle \geq 0 \text{ and } \langle f, f \rangle = 0 \text{ only if } f = 0.$$

The property (2) expresses linearity of scalar product by its first argument. Analogously combining (1) and (2) the linearity by the second argument can be written as

(2') $$\langle f_1, \alpha_2 f_2 + \alpha_3 f_3 \rangle = \overline{\alpha_2} \langle f_1, f_2 \rangle + \overline{\alpha_3} \langle f_1, f_3 \rangle .$$

The quantity $||f|| \equiv \sqrt{\langle f, f \rangle}$ is the *norm* of the element f. The property (3) of scalar product guarantees that norm is non-negative and $||f|| = 0$ only if $f = 0$.

The properties of scalar product and norm are described by the following

Theorem 1 *The scalar product and the norm possess the following properties:*

(1) Scaling property: $||\alpha f|| = |\alpha| \cdot ||f||$.

(2) Schwartz inequality: $|\langle f_1, f_2 \rangle| \leq ||f_1|| \cdot ||f_2||$, *and equal sign is possible only if the elements* f_1 *and* f_2 *are linear dependent.*

(3) Triangle inequality: $||f_1 + f_2|| \leq ||f_1|| + ||f_2||$, *and equal sign takes place only if* f_1 *and* f_2 *are linear dependent.*

The scaling property follows directly from the properties (2) and (2') of the scalar product

$$\langle \alpha f, \alpha f \rangle = \alpha \langle f, \alpha f \rangle = \alpha \overline{\alpha} \langle f, f \rangle = |\alpha|^2 \langle f, f \rangle .$$

To prove Schwartz inequality one can compute the norm of the element $f^\circ \equiv \alpha | \langle f_1, f_2 \rangle | f_1 + \langle f_1, f_2 \rangle f_2$, where $\alpha \in \mathbb{R}$. Using the linearity of scalar product one finds that

$$\langle f^\circ, f^\circ \rangle = | \langle f_1, f_2 \rangle |^2 \left(\alpha^2 \langle f_1, f_1 \rangle + 2\alpha | \langle f_1, f_2 \rangle | + \langle f_2, f_2 \rangle \right) \geq 0$$

for any real constant α. Thus the discriminant of the quadratic polynomial in α in the above formula should be non-positive, which yields Schwartz inequality.

Finally, the triangle inequality follows form Schwartz inequality. Indeed,

$$||f_1 + f_2||^2 = \langle f_1, f_1 \rangle + 2\mathrm{Re} \left(\langle f_1, f_2 \rangle \right) + \langle f_2, f_2 \rangle \leq ||f_1||^2 + 2||f_1|| \cdot ||f_2|| + ||f_2||^2.$$

Introduction of the scalar product and the norm makes the space *metrized*. The distance from one element f_1 to another element f_2 can be defined as the norm of their difference $f_1 - f_2$. Thus, Hilbert space is a *metric space*.

The other important property of Hilbert space is its *completeness*. Consider an infinite sequence of elements $\{f_n\}_{n=1}^{\infty}$. If for any $\varepsilon > 0$ there exists such number N that $||f_n - f_m|| < \varepsilon$ for any $n, m > N$, then the sequence is called *fundamental sequence*. The limiting element of infinite sequence is such an element f° that

$$\lim_{n \to \infty} ||f_n - f^\circ|| = 0.$$

Generally speaking, not every sequence has its limiting element, moreover even not every fundamental sequence has its limiting element. However if the sequence has the limiting element then it is fundamental. In a Hilbert space all fundamental sequences have their limiting elements, or in another words the limiting elements of all fundamental sequences belong to the space. This property expresses the *completeness* of the space.

Definition 1 An infinite dimensional linear metrized set L which is a complete space in the metrics generated by the scalar product is called *Hilbert space.*

Besides the Hilbert space L one needs to deal with its linear parts. Remind that subset L' of L is linear if for any elements $f_1, f_2 \in L'$ all the elements $\alpha_1 f_1 + \alpha_2 f_2$ also belong to L'. Such subsets can be also characterized by dimension which is defined analogously to the definition of the dimension of the total space L. The dimension of subset can be finite or infinite as well. One can construct a linear subset by choosing some number of linear independent elements f_j, $j = 1, 2, \ldots, n$ of the space L and defining L' as the set of all elements

$$\sum_{j=1}^{n} \alpha_j f_j$$

with arbitrary complex coefficients α_j.

The limiting point of a set $L' \subset L$ is such a point $f^\circ \in L$ that for any $\varepsilon > 0$ there exists a point $f \in L'$ such that $\|f - f^\circ\| < \varepsilon$. Adding all limiting points to the set L' performs the *closure* of the set. Below closure is denoted by overline, that is closure of L' is $\overline{L'}$. The property of the linear subset to be a *closed set* is analogous to the completeness of the space. The closed linear set in a Hilbert space is a Hilbert space itself.

One defines the sum of Hilbert spaces L' and L'' as the set of elements $f' + f''$, where elements f' and f'' are arbitrary elements from the corresponding spaces $f' \in L'$ and $f'' \in L''$. If such representation is unique, then the sum is direct.

Two elements f_1 and f_2 are *orthogonal* to each other if $\langle f_1, f_2 \rangle = 0$. An element f is orthogonal to the set L' if it is orthogonal to all elements of L'. All the elements of L that are orthogonal to L' compose *orthogonal compliment* $(L')^\perp$ of L' in L

$$L = L' \oplus (L')^\perp.$$

Any element f of L can be represented in the form

$$f = f' + f'', \qquad f' \in L', \quad f'' \in (L')^\perp$$

and such representation is unique. The element f' is the *orthogonal projection* of element f on the set L'.

A linear set L' is *dense* in L if for any element f° of L and any positive number ε there exists an element f of L' such that

$$\|f^\circ - f\| \leq \varepsilon.$$

Dense sets play an important role which is described by the following

Theorem 2 *If L' is dense in L and every element $f' \in L'$ is orthogonal to some element f°, then $f^\circ = 0$.*

Indeed, let f be an element from L' such that $||f^\circ - f|| \leq \varepsilon$. Then the element $f'' \equiv f^\circ - f$ has the norm $||f''|| \leq \varepsilon$. Further f° is orthogonal to any element of L', that is to f as well. Therefore

$$0 = \langle f^\circ, f\rangle = \langle f, f\rangle + \langle f'', f\rangle$$

that is

$$||f||^2 = |\langle f'', f\rangle| \leq ||f''|| \cdot ||f|| \leq \varepsilon ||f||$$

which is true only if $||f|| \leq \varepsilon$. Using arbitrariness of ε one concludes that $||f|| = 0$, that is $f = 0$.

2.1.2 *Operators*

Now one can consider functionals and operators in Hilbert space. The functionals and operators are defined on elements of some set D which is part of the space L. Functionals map the elements from D to complex numbers and operators transform elements form D to another elements from some other subset R of the space L.

An example of a functional is the scalar product by some fixed element f° of the space

$$a(f) = \langle f, f^\circ\rangle$$

and an example of an operator is multiplication by some fixed complex constant α°

$$A(f) = \alpha^\circ f.$$

In this particular case the functional $a()$ and the operator $A()$ can be applied to any element of the space L. However we consider the domains of definition $D(a)$ and $D(A)$ to be an inseparable part of the functional and operator. That is, for example, the functional $a_1(f) = ||f||$ defined on $D_1 \subset L$ and the functional $a_2(f) = ||f||$ defined on $D_2 \subset L$ where $D_1 \neq D_2$ are different functionals. Both are restrictions of the functional of norm to subsets D_1 and D_2 correspondingly. Similarly operators expressed by the same formula, but defined on different domains are different operators.

Definition 2 If the domain of an operator A_2 contains the domain of operator A_1, that is $D_1 \subset D_2$, and $A_2(f) = A_1(f)$, when $f \in D_1$, then the operator A_2 is the *extension* of the operator A_1, $(A_1 \subset A_2)$.

Oppositely the operator A_1 is the restriction of the operator A_2 to the domain D_1.

The domain of an operator A is denoted below as $\mathrm{Dom}(A)$ and the set of all elements $A(f)$, where $f \in \mathrm{Dom}(A)$, is denoted as $\mathrm{Res}(A)$. That is $A: \mathrm{Dom}(A) \to \mathrm{Res}(A)$.

If for any different elements $f_1, f_2 \in \mathrm{Dom}(A)$ the elements $A(f_1)$ and $A(f_2)$ do not coincide there exists an *inverse operator* A^{-1} defined on $\mathrm{Dom}(A^{-1}) = \mathrm{Res}(A)$ such that

$$A^{-1}(A(f)) = f, \quad f \in \mathrm{Dom}(A)$$

and

$$A(A^{-1}(f)) = f, \quad f \in \mathrm{Res}(A).$$

Below only linear operators are considered, that is such operators for which

$$A(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 A(f_1) + \alpha_2 A(f_2), \qquad \alpha_{1,2} \in \mathbb{C}.$$

For linear operators $A(f)$ is denoted below as Af.

If there exists such positive constant C that for any $f \in \mathrm{Dom}(A)$

$$||Af|| \leq C||f||$$

then the operator A is *bounded* and the minimal possible constant in the above estimate is the *norm* $||A||$ of the operator A.

Any number λ for which there exists an element $f \neq 0$ such that

$$Af = \lambda f$$

is the *eigen number* of the operator A. The element f in that case is the *eigen element* (eigen function) and the set of all eigen elements if the *eigen subspace* of the operator. The number of linear independent eigen elements corresponding to an eigen number λ is the *multiplicity of an eigen number.*

2.1.3 *Adjoint, symmetric and selfadjoint operators*

Consider the scalar product $\langle Af, g\rangle$ where f is arbitrary element from $\mathrm{Dom}(A)$. Let g and $g^\star$ be such elements from L that

$$\langle Af, g\rangle = \langle f, g^\star\rangle \tag{2.1}$$

for any $f \in \mathrm{Dom}(A)$. Evidently such pairs of elements g and $g^\star$ exist, at least one can take $g = g^\star = 0$. We are going to introduce an operator that defines the element $g^\star$ for every element g. Such operator exists if the domain $\mathrm{Dom}(A)$ is dense in L. Indeed, if $\mathrm{Dom}(A)$ is not dense in L, then there exists an element h which is orthogonal to all elements of $\mathrm{Dom}(A)$ and besides (2.1) the equality

$$\langle Af, g\rangle = \langle f, g^\star + h\rangle$$

holds. On the contrary if $\mathrm{Dom}(A)$ is dense in L, then if for any $f \in \mathrm{Dom}(A)$

$$\langle Af, g\rangle = \langle f, g_1^\star\rangle \quad \text{and} \quad \langle Af, g\rangle = \langle f, g_2^\star\rangle\,,$$

then the element $g_1^\star - g_2^\star$ is orthogonal to $\mathrm{Dom}(A)$, which yields $g_1^\star = g_2^\star$.

Definition 3 If $\mathrm{Dom}(A)$ is dense in L, then the operator A has its adjoint operator $A^\star$. The domain of $A^\star$ is the set of all elements g for which $g^\star$ exists such that (2.1) is satisfied for every $f \in \mathrm{Dom}(A)$ and

$$A^\star g = g^\star.$$

It is easy to check that the adjoint operator is linear. Let $\mathrm{Ker}(A)$ be the set of elements from $\mathrm{Dom}(A)$ such that

$$Af = 0, \quad \text{if} \quad f \in \mathrm{Ker}(A).$$

Theorem 3 *The following decomposition of the space holds*

$$\mathrm{Ker}(A^\star) \oplus \overline{\mathrm{Res}(A)} = L.$$

Indeed, let f be orthogonal to $\mathrm{Res}(A)$, that is for any $x \in \mathrm{Dom}(A)$

$$0 = \langle Ax, f\rangle = \langle x, A^\star f\rangle$$

Due to the supposition that $\mathrm{Dom}(A)$ is dense in L the above identity means that $A^\star f = 0$, that is

$$\mathrm{Res}(A)^\perp \subset \mathrm{Ker}(A^\star). \tag{2.2}$$

Similarly if $f \in \mathrm{Ker}(A^\star)$, then for any $x \in \mathrm{Dom}(A)$

$$\langle Ax, f\rangle = \langle x, A^\star f\rangle = 0,$$

that is f is orthogonal to $\mathrm{Res}(A)$ and the opposite to (2.2) inclusion holds.

Definition 4 The linear operator A is symmetric if $\mathrm{Dom}(A)$ is dense in L and for any two elements $f_1, f_2 \in \mathrm{Dom}(A)$ the following equality holds

$$\langle Af_1, f_2\rangle = \langle f_1, Af_2\rangle .$$

This definition yields that for any $f \in \mathrm{Dom}(A)$ the scalar product $\langle Af, f\rangle$ is real. If besides

$$\langle Af, f\rangle \geq c\langle f, f\rangle , \quad c \in \mathbb{R}$$

for any $f \in \mathrm{Dom}(A)$ then the symmetric operator A is semibounded from below. Similarly if the opposite inequality holds, the operator is semibounded from above. In particular if

$$\langle Af, f\rangle \geq 0, \qquad f \in \mathrm{Dom}(A)$$

then A is *positive operator.*

Definition 5 The *boundary form* of an operator A is defined by the formula

$$\mathcal{I}[f, g] \equiv \langle Af, g\rangle - \langle f, Ag\rangle .$$

The boundary form of a symmetric operator is equal to zero and oppositely if the boundary form is equal to zero on the domain of an operator, then this operator is symmetric.

Theorem 4 *If A is symmetric operator, then its adjoint $A^\star$ is its extension, that is $A \subset A^\star$.*

Indeed, let f_1, $f_2 \in \mathrm{Dom}(A)$, then

$$\langle Af_1, f_2\rangle = \langle f_1, Af_2\rangle = \langle f_1, A^\star f_2\rangle .$$

That is $Af_2 = A^\star f_2$ for any $f_2 \in \mathrm{Dom}(A)$, which proves the theorem (see Definition 2).

In the procedure of constructing selfadjoint extensions of symmetric operators this theorem plays an important role.

If $\widetilde{A}$ is a symmetric extension of a symmetric operator A, then $\mathrm{Dom}(\widetilde{A}^\star)$ $\subset \mathrm{Dom}(A^\star)$. Indeed, for the definition of adjoint operator $\widetilde{A}^\star$ one needs to check existence of g and $g^\star$ such that (2.1) is satisfied for all $f \in \mathrm{Dom}(A)$. This defines $\mathrm{Dom}(A^\star)$. Besides the equality (2.1) should be checked for such f' that belong to $\mathrm{Dom}(\widetilde{A})$, but do not belong to $\mathrm{Dom}(A)$. As $\mathrm{Dom}(A) \subset$

$\mathrm{Dom}(\widetilde{A})$ such elements $f' \neq 0$ exist and the condition defining $\mathrm{Dom}(\widetilde{A})$ appears more restrictive than that for $\mathrm{Dom}(A)$.

Combining the above two statements yields

$$A \subset \widetilde{A} \subseteq \widetilde{A}^{\star} \subset A^{\star}. \tag{2.3}$$

Definition 6 If $A = A^{\star}$ (domains $\mathrm{Dom}(A)$ and $\mathrm{Dom}(A^{\star})$ coincide) the operator A is *selfadjoint*.

2.1.4 *Extension theory*

We start with some auxiliary tools that allow the extensions of symmetric operators to be described.

Definition 7 A point λ is the *point of regular type* of an operator A if

$$||(A-\lambda)f|| \geq c(\lambda)||f||, \quad c(\lambda) > 0$$

for any $f \in \mathrm{Dom}(A)$.

If λ_0 is a point of regular type, then for all points λ such that $|\lambda - \lambda_0| \leq \frac{1}{2}c(\lambda_0)$ and all $f \in \mathrm{Dom}(A)$ the following estimate holds

$$\begin{aligned} ||(A-\lambda)f|| &\geq ||(A-\lambda_0)f|| - |\lambda - \lambda_0|||f|| \\ &\geq c(\lambda_0)||f|| - \frac{1}{2}c(\lambda_0)||f|| = \frac{1}{2}c(\lambda_0)||f||. \end{aligned}$$

That is, the domain of regular type points is open (not closed).

Note also that points of regular type can not be eigen numbers.

Definition 8 The *deficiency number* of an operator A is the dimension of orthogonal compliment of $\mathrm{Res}(A)$

$$\mathrm{def}(A) = \dim\left(\mathrm{Res}(A)^{\perp}\right).$$

The set $\mathrm{Res}(A)^{\perp}$ is the *deficiency subspace* of the operator A and any element of deficiency subspace is a *deficiency element.*

The deficiency number is stable with respect to small perturbations of an operator. Namely,

Theorem 5 *Let operator A' be semibounded, that is for any $f \in \mathrm{Dom}(A')$ $||A'f|| \geq C||f||$, $C > 0$, and let operator A'' be such that* $\mathrm{Dom}(A' + A'') =$

$\mathrm{Dom}(A')$ *and* $||A''f|| \leq \alpha ||A'f||$, $\alpha \leq 1$ *for any* $f \in \mathrm{Dom}(A')$. *Then* $\mathrm{def}(A' + A'') = \mathrm{def}(A')$.

In order to prove this theorem assume the opposite. First, let $\mathrm{def}(A' + A'') > \mathrm{def}(A')$. That is, assume the dimension of orthogonal compliment of $\mathrm{Res}(A')$ be less than the dimension of the orthogonal compliment of $\mathrm{Res}(A'+A'')$. Then there exists a non zero element f° from $\mathrm{Res}(A'+A'')^\perp$ that does not belong to $\mathrm{Res}(A')^\perp$. That is, it belongs to $\mathrm{Res}(A')$ and can be represented as $f^\circ = A'x$, where $x \in \mathrm{Dom}(A')$. Further orthogonality of f° to $\mathrm{Res}(A' + A'')$ yields

$$0 = \langle A'x, (A' + A'')x \rangle = ||A'x||^2 + \langle A'x, A''x \rangle .$$

That is

$$||A'x||^2 = - \langle A'x, A''x \rangle . \tag{2.4}$$

Using Schwartz inequality one finds

$$||A'x||^2 = | \langle A'x, A''x \rangle | \leq ||A'x|| \cdot ||A''x|| \leq \alpha ||A'x||^2.$$

Due to $\alpha < 1$ the above is possible only if $||A'x|| = 0$, that is $A'x = 0$ which contradicts with $f^\circ \equiv A'x \neq 0$.

Assume now that $\mathrm{def}(A' + A'') < \mathrm{def}(A')$. Then by similar arguments there exists nonzero element $f^\circ \in \mathrm{Res}(A')^\perp$ that belongs to $\mathrm{Res}(A' + A'')$ and therefore can be represented as $f^\circ = (A' + A'')x$. Again its orthogonality to $\mathrm{Res}(A')$ yields

$$\langle f^\circ, A'x \rangle = \langle (A' + A'')x, A'x \rangle = ||A'x||^2 + \langle A''x, A'x \rangle ,$$

which again brings to the equality (2.4) and one concludes $A'x = 0$. As the operator A' is assumed semibounded from below, $A'x = 0$ yields $x = 0$ and $f^\circ \equiv (A' + A'')x = 0$ which again contradicts with our supposition.

Thus, $\mathrm{def}(A' + A'')$ can not be less and can not be greater than $\mathrm{def}(A')$. The theorem is proved.

Definition 9 The *deficiency number in the point* λ of an operator A is the deficiency number of $A - \lambda$,

$$\mathrm{def}_\lambda(A) = \mathrm{def}(A - \lambda).$$

Theorem 6 *If Λ is a connected domain of regular type points of an operator A, then the deficiency number* $\mathrm{def}_\lambda(A)$ *remains the same in all points $\lambda \in \Lambda$.*

Prove first that deficiency number does not change if the points are sufficiently close to each other. Indeed, introduce the operators A' and A'' by the formula

$$A - \lambda = (A - \lambda_0) + (\lambda_0 - \lambda) = A' + A''.$$

Theorem 5 says that $\mathrm{def}(A) = \mathrm{def}(A')$ if for any $f \in \mathrm{Dom}(A)$ holds $||A''f|| \leq \alpha ||A'f||$. This estimate is true if $|\lambda - \lambda_0| \leq c(\lambda_0)$, indeed, due to λ_0 being regular type point of A the estimate

$$||(A - \lambda_0)f|| \geq c(\lambda_0)||f||$$

holds and one finds

$$||A''f|| = |\lambda_0 - \lambda| \cdot ||f|| \leq \frac{|\lambda_0 - \lambda|}{c(\lambda_0)} ||A'f||.$$

Now let λ_1, λ_2 be two arbitrary points in Λ. As Λ is connected, the path from λ_1 to λ_2 can be covered by circles of appropriate radii such that the above estimate holds in every circle and therefore in every circle the deficiency number does not change. Walking along this path allows the theorem to be proved.

Consider now symmetric operators. If A is symmetric, then

$$\begin{aligned} ||(A - \lambda)f||^2 &= \langle Af - \lambda f, Af - \lambda f \rangle \\ &= ||(A - \mathrm{Re}\,\lambda)f||^2 + |\mathrm{Im}\,\lambda|^2 ||f||^2 \geq |\mathrm{Im}\,\lambda|^2 ||f||^2. \end{aligned}$$

The above estimate means that the upper $\mathbb{C}^+$ and the lower $\mathbb{C}^-$ half-planes of λ are the domains of regular type points. Thus any symmetric operator has two deficiency numbers, one in $\mathbb{C}^+$ and the other in $\mathbb{C}^-$.

Definition 10 The deficiency numbers of a symmetric operator A are the *deficiency indices* $n_\pm(A)$.

If a symmetric operator A has a real point of regular type then $n_+(A) = n_-(A)$. If no real regular type points exist then the deficiency indices can be different. Below we deal mainly with semi-bounded operators for which there exist real points of regular type and therefore the deficiency indices coincide. Moreover the deficiency indices are finite.

Theorem 7 *If A is a symmetric operator, then any non-real number λ is the eigen number of the adjoint operator $A^\star$. The multiplicity of this eigen number is equal to n_+ if $\operatorname{Im}\lambda > 0$ and is equal to n_- if $\operatorname{Im}\lambda < 0$.*

Indeed, if f is arbitrary element of $\operatorname{Dom}(A)$ and g is any element of $\operatorname{Res}(A-\overline{\lambda})^\perp$, then

$$0 = \left\langle g, (A-\overline{\lambda})f \right\rangle = \langle A^\star g - \lambda g, f\rangle .$$

As $\operatorname{Dom}(A)$ is dense in L the latter means that

$$A^\star g = \lambda g.$$

That is any $g \in \operatorname{Res}(A-\overline{\lambda})^\perp$ is the eigen element of $A^\star$ and therefore the multiplicity of the eigen number is equal to the dimension of $\operatorname{Res}(A-\overline{\lambda})^\perp$.

Examine the structure of domains of adjoint operators.

Theorem 8 ***(First Neumann formula)***
Let A be arbitrary symmetric operator and $R_+ = \operatorname{Res}(A-\lambda)^\perp$, $R_- = \operatorname{Res}(A-\overline{\lambda})^\perp$, ($\operatorname{Im}\lambda > 0$) be any pair of its deficiency subspaces. Then the domain of the adjoint operator $A^\star$ can be decomposed into three parts

$$\operatorname{Dom}(A^\star) = \operatorname{Dom}(A) + \operatorname{Res}(A-\lambda)^\perp + \operatorname{Res}(A-\overline{\lambda})^\perp$$

and the decomposition of any element

$$f = f_0 + f_+ + f_-, \quad f \in \operatorname{Dom}(A),\ f_+ \in R_+,\ f_- \in R_-$$

is unique. Besides

$$A^\star f = Af_0 + \lambda f_+ + \overline{\lambda} f_- .$$

Proof. Let $f \in \operatorname{Dom}(A^\star)$. Decompose $A^\star f - \lambda f$ into components in orthogonal subspaces $\operatorname{Res}(A-\lambda)$ and R_+

$$A^\star f - \lambda f = (Af_0 - \lambda f_0) + g_+,$$

where $f_0 \in \operatorname{Dom}(A)$ and $g_+ \in R_+$. Note that g_+ as an element of R_+ is an eigen element of $A^\star$ corresponding to an eigen number $\overline{\lambda}$. Introducing $f_+ \equiv (\overline{\lambda}-\lambda)^{-1} g_+$ and $f_- = f - f_0 - f_+$ and computing $A^\star f_-$, one finds

$$\begin{aligned} A^\star f_- &= A^\star f - A^\star f_0 - \overline{\lambda} g_+ = (A^\star f - \lambda f) + \lambda f - Af_0 - \overline{\lambda} f_+ \\ &= \lambda f - \lambda f_0 + \left(1 - \frac{\overline{\lambda}}{\overline{\lambda}-\lambda}\right) g_+ = \lambda f_- . \end{aligned}$$

That is, $f_- \in \mathrm{Ker}(A^\star - \lambda)$ and according to Theorem 3 the element $f_- \in R_-$. Thus

$$f = f_0 + f_+ + f_-, \quad f_0 \in \mathrm{Dom}(A),\ f_\pm \in R_\pm.$$

Finally, the uniqueness of such representation should be proved. For this let us check that in the decomposition of the zero element both f_0, f_+ and f_- are zeros. Suppose that

$$0 = f_0 + f_+ + f_-. \tag{2.5}$$

Apply operator $A^\star - \lambda$ to the above equality

$$0 = (Af_0 - \lambda f) + \left(\overline{\lambda} - \lambda\right) f_+.$$

The two terms in the above sum are orthogonal, therefore $f_+ = 0$. Analogously applying $A^\star - \overline{\lambda}$ to (2.5) allows to conclude that $f_- = 0$. Thus from equality (2.5) follows that

$$f_0 = f_+ = f_- = 0,$$

which concludes the proof.

Introduce a linear operator V defined on $\mathrm{Ker}(A^\star - \lambda)$ and taking values in $\mathrm{Ker}(A^\star - \overline{\lambda})$. In our applications the sets $\mathrm{Ker}(A^\star - \lambda)$ and $\mathrm{Ker}(A^\star - \overline{\lambda})$ will be of finite and equal dimensions. In that case operator V is a matrix with complex coefficients.

Consider now an extension $\widetilde{A}$ of the operator A. Let this extension be defined on elements $\widetilde{f}$ that can be represented in the form

$$\widetilde{f} = f_0 + (I + V)f_+.$$

Comparing this with the first Neumann formula yields

$$\mathrm{Dom}(\widetilde{A}) \subset \mathrm{Dom}(A^\star).$$

Let $\widetilde{A} \subset A^\star$. Then

$$\widetilde{A}f = Af_0 + \overline{\lambda}f_+ + \lambda V f_+.$$

We want $\widetilde{A}$ to be symmetric, that is

$$\left\langle \widetilde{A}f, f\right\rangle - \left\langle f, \widetilde{A}f\right\rangle = 0.$$

This property after cumbersome, but straight forward derivations yields

$$\|f_+\| = \|Vf_+\|.$$

That is the operator $\widetilde{A}$ is symmetric if the operator V preserves the norm*.

Generally speaking, the operator V can be defined on some subset $\mathrm{Dom}(V)$ of $\mathrm{Ker}(A^\star - \lambda)$ and can have $\mathrm{Res}(V) \subset \mathrm{Ker}(A^\star - \overline{\lambda})$.

All the above can be formulated as

Theorem 9 ***(Second Neumann formula)***
If $\widetilde{A}$ is a symmetric extension of operator A, then

$$\mathrm{Dom}(\widetilde{A}) = \mathrm{Dom}(A) \oplus (I + V)\mathrm{Dom}(V),$$

where $\mathrm{Dom}(V) \subset \mathrm{Ker}(A^\star - \lambda)$ *and* $\mathrm{Res}(V) \subset \mathrm{Ker}(A^\star - \overline{\lambda})$. *The representation*

$$f = f_0 + f_+ + Vf_+ \tag{2.6}$$

is unique and

$$\widetilde{A}f = Af_0 + \overline{\lambda}f_+ + \lambda V f_+. \tag{2.7}$$

The second Neumann formula by the two equalities (2.6) and (2.7) describes all symmetric extensions $\widetilde{A}$ of a given symmetric operator A. In particular, if the deficiency indices of the operators A are equal ($n_+ = n_-$) there exists a selfadjoint extension $\widetilde{A}$ for which $\mathrm{Dom}(V) = \mathrm{Ker}(A^\star - \lambda)$ and $\mathrm{Res}(V) = \mathrm{Ker}(A^\star - \overline{\lambda})$. And vice versa, if $n_+ = n_-$ and operator V is such that $\mathrm{Dom}(V) = \mathrm{Ker}(A^\star - \lambda)$ and $\mathrm{Res}(V) = \mathrm{Ker}(A^\star - \overline{\lambda})$, then $\widetilde{A} = \widetilde{A}^\star$.

2.2 Space L_2 and differential operators

2.2.1 *Hilbert space L_2*

An important example of Hilbert space is the space of square integrable functions L_2. Let Ω be a bounded or unbounded domain in $\mathbb{R}^n$. Define by $L_2(\Omega)$ the set of functions $f(\mathbf{x})$ such that $|f(\mathbf{x})|^2$ is integrable in Lebesgue sense [66] in the domain Ω. The functions that differ only on a domain of zero measure are not distinguished as elements of $L_2(\Omega)$.

---

*The operators that preserve norm are the *unitary* operators

The set $L_2(\Omega)$ is linear set. Indeed, the estimate

$$|f_1 + f_2|^2 \leq 2|f_1|^2 + 2|f_2|^2$$

yields that if $f_1 \in L_2(\Omega)$ and $f_2 \in L_2(\Omega)$, then $f_1 + f_2 \in L_2(\Omega)$. The possibility to multiply any function by a complex constant α is evident.

The metrics in $L_2(\Omega)$ is introduced by the formula

$$\langle f_1, f_2 \rangle = \int_\Omega f_1(\mathbf{x})\overline{f_2(\mathbf{x})}d\mathbf{x}. \tag{2.8}$$

The existence of the integral in the right-hand side of the above definition follows from the estimate

$$|f_1 f_2| \leq \frac{1}{2}|f_1|^2 + \frac{1}{2}|f_2|^2.$$

Using the properties of Lebesgue integral allows the conditions (1), (2) and (3) of the scalar product (see page 64) to be checked. The norm of a function is defined as

$$\|f\| = \sqrt{\int_\Omega |f(\mathbf{x})|^2 d\mathbf{x}}.$$

The Riesz–Ficher theorem [65] establishes the completeness of the space $L_2(\Omega)$.

Theorem 10 ***(Riesz–Ficher)***
Let a sequence of functions $\{f_j\}_{j=1}^{\infty}$, $f_j \in L_2(\Omega)$ be such that for any $\varepsilon > 0$ there exists number $N(\varepsilon)$ such that $\|f_i - f_j\| \leq \varepsilon$ for any $i, j > N(\varepsilon)$. Then there exists such function $f^\star \in L_2(\Omega)$ that $\|f_j - f^\star\| \to 0$.

Thus the space $L_2(\Omega)$ is a Hilbert space and the theory briefly presented in Section 2.1 can be applied. Further the following fact is needed.

Theorem 11 *For any function $f \in L_2(\Omega)$ and any $\varepsilon > 0$ there exists infinitely smooth function φ, such that $\|f - \varphi\| \leq \varepsilon$.*

The proof of this theorem requires some auxiliary tools to be introduced. For any function $f \in L_2(\Omega)$ consider the mean function

$$\varphi_h(\mathbf{x}) = \frac{1}{h^n}\int \omega\left(\frac{|\mathbf{x} - \mathbf{x}'|}{h}\right) f(\mathbf{x}')d\mathbf{x}', \tag{2.9}$$

where $\omega(t)$ is the kernel possessing the following properties

- the function $\omega \in C^\infty(\mathbb{R}^n)$;
- if $|t| > 1$, then $\omega(t)$ and all its derivatives are equal to zero;
- the function is normalized by the condition

$$\int \omega(|\mathbf{x}|)d\mathbf{x} = 1. \tag{2.10}$$

The particular formula for the function $\omega(t)$ is not important. One can take for example

$$\omega(t) = \begin{cases} C\exp\left(\dfrac{1}{t^2-1}\right), & |t| < 1, \\ 0, & |t| \geq 1. \end{cases}$$

The constant C depends on the dimension n of the domain Ω and is chosen such that the normalization (2.10) holds.

One can easily check that the mean functions $\varphi_h(\mathbf{x})$ defined by the formula (2.9) are infinitely smooth ($\varphi_h \in C^\infty$). Indeed, one can change the order of differentiation and integration and apply differentiation to the kernel $\omega(|\mathbf{x}-\mathbf{x}'|/h)$. This is approved because the integral converges absolutely (see [64]).

Consider the difference $f(\mathbf{x}) - \varphi_h(\mathbf{x})$. Evidently this difference can be written in the form of integral

$$f(\mathbf{x}) - \varphi_h(\mathbf{x}) = \frac{1}{h^n}\int \omega\left(\frac{|\mathbf{x}-\mathbf{x}'|}{h}\right)\Big(f(\mathbf{x}) - f(\mathbf{x}')\Big)d\mathbf{x}'.$$

The integration in the above formula takes place on the whole space $\mathbb{R}^n$, but can be restricted to the ball $|\mathbf{x}-\mathbf{x}'| < h$ because outside this ball the kernel is identically equal to zero. Then, using the Schwartz inequality (in the case of L_2 this inequality belongs to Bounyakovskii), one finds

$$|f - \varphi_h|^2 \leq \frac{1}{h^{2n}} \int\limits_{|\mathbf{x}'-\mathbf{x}|<h} |f(\mathbf{x}') - f(\mathbf{x})|^2 d\mathbf{x}' \int \omega^2\left(\frac{|\mathbf{x}-\mathbf{x}'|}{h}\right)d\mathbf{x}'.$$

Introducing new integration variable $\mathbf{y} = h^{-1}(\mathbf{x}-\mathbf{x}')$ in the second integral yields

$$|f - \varphi_h|^2 \leq \frac{C_1}{h^n} \int\limits_{|\mathbf{x}'-\mathbf{x}|<h} |f(\mathbf{x}') - f(\mathbf{x})|^2 d\mathbf{x}',$$

where

$$C_1 = \int \omega^2(|\mathbf{y}|)d\mathbf{y}.$$

Integrating the last inequality and changing the order of integration yields (here $\mathbf{x}'' = \mathbf{x}' - \mathbf{x}$)

$$\|f - \varphi_h\|^2 \leq \frac{C_1}{h^n} \int\limits_{|\mathbf{x}''|<h} d\mathbf{x}'' \int\limits_{\Omega} |f(\mathbf{x} + \mathbf{x}'') - f(\mathbf{x})|^2 d\mathbf{x}.$$

Further, we base on the following theorem [66], [62]

Theorem 12 *If $f \in L_2(\Omega)$, then for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$\int_{\Omega} |f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x})|^2 d\mathbf{x} < \varepsilon$$

when $|\mathbf{y}| < \delta(\varepsilon)$.

By appropriate choice of $h = \delta(\varepsilon)$ the inner integral can be made less than any given ε. Therefore, introducing constant C_2 as the volume of a unit ball in $\mathbb{R}^n$, one finds

$$\|f - \varphi_h\|^2 \leq \frac{C_1 \varepsilon}{h^n} \int\limits_{|\mathbf{x}''|<h} d\mathbf{x}'' = C_1\, C_2 \varepsilon,$$

Due to arbitrariness of ε this estimate proves the theorem 11.

The theorem 11 allows many properties to be checked only on functions from C^∞ which compose a dense set in $L_2(\Omega)$. Moreover it is sufficient to use only polynomials and even polynomials with rational coefficients. Below we shall use other dense sets C_0^∞ and $C_0^\infty(\Omega)$. These sets are sets of functions from C^∞ that have finite support S and in the case of $C_0^\infty(\Omega)$ the support S is entirely inside the domain Ω. To prove the fact that such sets are dense in L_2 requires only to check that these sets are dense in C^∞ in the metrics (2.8) which can be easily done.

2.2.2 *Generalized derivatives*

Let $f \in L_2(\Omega)$. Consider function φ continuously differentiable and identically equal to zero in a vicinity of the boundary $\partial\Omega$. Introduce notation

$$D^\ell \equiv \frac{\partial^\ell}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \dots \partial x_n^{\ell_n}}, \qquad (\ell_1 + \ell_2 + \dots + \ell_n = \ell).$$

If f is differentiable, then the formula of integration by parts in Ω reads

$$\int_\Omega f \, D^\ell \varphi \, d\mathbf{x} = (-1)^\ell \int_\Omega \varphi \, D^\ell f \, d\mathbf{x}.$$

The integral over the boundary $\partial\Omega$ is not presented in the above formula because φ vanishes in a vicinity of the boundary.

The above formula defines the generalized derivative, namely

Definition 11 Let f and $f' \in L_2(\Omega)$ and

$$\int_\Omega f \, D^\ell \varphi \, d\mathbf{x} = (-1)^\ell \int_\Omega f' \, \varphi \, d\mathbf{x} \tag{2.11}$$

for any $\varphi \in C_0^\infty(\Omega)$. Then f' is the *generalized derivative* of order $\{\ell_j\}$ of function f ($f' = D^\ell f$).

One can check that the generalized derivative is unique. Indeed, supposing that (2.11) holds for f_1' and f_2' and subtracting the two formulae yields

$$\int (f_1' - f_2') \, \varphi \, d\mathbf{x} = 0.$$

This identity expresses orthogonality of $f_1' - f_2'$ to any φ from a dense set $C_0^\infty(\Omega)$ in $L_2(\Omega)$. Therefore theorem 2 of section 2.1 concludes that $f_1' - f_2' = 0$, That is, $f_1' = f_2'$. If function f is differentiable in a usual sense then the usual derivative and the generalized derivative coincide up to a function equivalent to zero.

The usual rules of differentiation can be checked applicable to calculation of generalized derivatives. In particular the operator of generalized derivative is linear. Combining generalized derivatives one can define any generalized differential operator with constant coefficients. Below we consider harmonic $\triangle$ and bi-harmonic $\triangle^2$ operators.

2.2.3 *Sobolev spaces and embedding theorems*

Define now special subsets of the space $L_2(\Omega)$. Let $W_2^\ell(\Omega)$ be composed of all functions $f \in L_2(\Omega)$ that have all generalized derivatives of orders up to ℓ and all these derivatives belong to $L_2(\Omega)$. The sets $W_2^\ell(\Omega)$ are Hilbert spaces with the scalar products defined by the following formula [49]

$$\langle f_1, f_2\rangle_\ell \equiv \int\limits_\Omega \sum_{k=0}^{\ell} \sum_{k_1,\ldots,k_n} \frac{\partial^k f_1}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \frac{\partial^k \overline{f_2}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}} \, d\mathbf{x}.$$

The Hilbert spaces $W_2^\ell(\Omega)$ form the sequence

$$L_2(\Omega) \supset W_2^1(\Omega) \supset W_2^2(\Omega) \supset \ldots$$

Each of these spaces contains $C_0^\infty(\Omega)$ and thus each $W_2^\ell(\Omega)$ is dense in $W_2^k(\Omega)$, $k < \ell$.

The spaces $W_2^\ell(\Omega)$ are the maximal domains where differential operators of order ℓ can be defined. In particular the harmonic (Laplace) operator can be defined on function from $W_2^2(\Omega)$ while bi-harmonic operator $\triangle^2$ can be defined on $W_2^4(\Omega)$. However having in mind that domain of operator is its inseparable part, below we consider operators $\triangle$ and $\triangle^2$ defined on different subsets of corresponding spaces. In particular additional conditions that define these subsets can be the boundary conditions on $\partial\Omega$. The types of boundary conditions that can be used are described by *embedding theorems* [66], [62], [49]. We remind that functions that differ only on sets of measure zero are not distinguished as elements of $W_2^\ell(\Omega)$.

We present only two embedding theorems. The domains Ω are assumed in $\mathbb{R}^n$.

Theorem 13 *If $f \in W_2^\ell(\Omega)$ and $\ell > n/2$, then there exists an equivalent function $\widetilde{f} \in C(\Omega)$ and*

$$\max_\Omega |\widetilde{f}| \le C \|f\|_{W_2^\ell(\Omega)},$$

where constant C does not depend on function f.

Theorem 14 *Let $f \in W_2^\ell(\Omega)$, $n \ge 2\ell$ and S is any smooth section of Ω having dimension s with $s > n - 2\ell$. Then $f \in L_2(S)$ and*

$$\|f\|_{L_2(S)} \le C \|f\|_{W_2^\ell(\Omega)},$$

where constant C does not depend on function f.

The section S in the above theorem can consist of a finite set of domains $S = \sum_j S_j$ such that every domain S_j can be transformed to flat domain in $\mathbb{R}^s$ and the coordinate transform is expressed by $C^\ell(S_j)$ functions. This section can be the boundary of Ω as well.

The last theorem in particular confirms that in two or three dimensional boundary value problem for a differential operator of order ℓ in a domain with not too bad boundary the boundary conditions can be specified for any derivative of order up to $\ell - 1$.

2.3 Problems of scattering

Here from the general point of view of operator theory in Hilbert space we examine the problems of scattering by bounded domains formulated in Chapter 1†. These problems deal with harmonic and bi-harmonic operators.

2.3.1 *Harmonic operator*

One can start with the operator that corresponds to the problem without a scatterer. In the case of acoustic waves in a half-space bounded by an absolutely rigid screen such operator is a harmonic operator. In the case of three-dimensional scattering problem it is

$$A_3 = \Delta, \quad \mathrm{Dom}(A) = \left\{ U \in W_2^2(\mathbb{R}_+^3),\ \left.\frac{\partial U}{\partial z}\right|_{z=0} = 0 \right\}.$$

In the case of two-dimensional problem (that is when neither the incident field nor the obstacle depend on y co-ordinate) the operator is

$$A_2 = \Delta, \quad \mathrm{Dom}(A) = \left\{ U \in W_2^2(\mathbb{R}_+^2),\ \left.\frac{dU}{dz}\right|_{z=0} = 0 \right\}.$$

One can easily check that the above operators are selfadjoint in the corresponding spaces $L_2(\mathbb{R}_+^3)$ and $L_2(\mathbb{R}_+^2)$. Indeed, first check that A_3 and

†Generally speaking, the connection of the problems of scattering and operators in Hilbert space is not that simple. Rigorous mathematical explanation of how radiation conditions appear instead of square integrability lies outside the scope of this book. Interested reader can be addressed, for example, to monographs [50] and [54], see also Section 3.1

A_2 are symmetric, that is the *boundary form* of these operators vanishes

$$\mathcal{I}_3[U,V] \equiv \langle A_3 U, V\rangle - \langle U, A_3 V\rangle = 0$$

and

$$\mathcal{I}_2[U,V] \equiv \langle A_2 U, V\rangle - \langle U, A_2 V\rangle = 0.$$

This evidently follows from integration by parts. Secondly, one can calculate the deficiency indices (see Definition 10). This requires finding square integrable solutions of the equations

$$\triangle f_\pm = \pm i f_\pm$$

which evidently are only zeros $(f\pm \equiv 0)^{\ddagger}$. Thus deficiency indices are equal to zero and the operators are selfadjoint.

Consider now the case when an obstacle is presented. Let this obstacle occupy domain Ω with a piecewise smooth boundary. This domain can consist of a set of connected pieces attached to the rigid screen or separated from it. Let $\partial\Omega$ be the part of the boundary of Ω which lies outside the screen. The formulation of the scattering problem requires some boundary conditions to be fixed on $\partial\Omega$. As it was shown in Chapter 1, physically correct boundary conditions should bring to the inequality

$$\operatorname{Im}\left(\int_{\partial\Omega} U \frac{\partial \overline{U}}{\partial n} dS\right) \geq 0. \tag{2.12}$$

This inequality is a particular case of (1.17) when $w = 0$.

We consider further only such obstacles that do not absorb energy, that is (see Section 1.3.4) the equal sign is assumed in (2.12). One can check that the operators $\triangle$ defined on functions $U \in W_2^2(\mathbb{R}_+^{2,3}\backslash\Omega)$ that satisfy the correctly set boundary conditions on $\partial\Omega$ are selfadjoint. Again the boundary form of the corresponding operators vanishes due to the integration by parts and thus the symmetry property is established. Further, due to the uniqueness theorem proved in Chapter 1 the kernels of the operators are empty.

---

‡Solutions of these equations are any combinations of functions

$$\exp\left(e^{\pm i\pi/4} x \cos\vartheta\right) \cos\left(e^{\pm i\pi/4} z \sin\vartheta\right)$$

with arbitrary complex angle ϑ. However these functions do not belong to $L_2(\mathbb{R}_+^d)$.

2.3.2 *Bi-harmonic operator*

Consider now the bi-harmonic operators. The operator that describes the wave process in the absence of an obstacle is the selfadjoint operator

$$B_2 = \triangle^2, \quad \mathrm{Dom}(B) = W_2^4(\mathbb{R}^2).$$

This follows from integration by parts similarly as in the previous section.

In the case of an obstacle Ω_0 presented in the plate two boundary conditions should be specified on the surface $\partial\Omega_0$. These conditions make the operator selfadjoint if satisfy the condition (1.44) with equal sign. Indeed, the boundary form of the bi-harmonic operator defined on functions that do not satisfy any conditions on the surface of the obstacle can be reduced to

$$\mathcal{I}[w_1, w_2] = \int\limits_{\partial\Omega_0} \left(\mathbb{F}w_1 \overline{w_2} + \mathbb{M}w_1 \frac{\partial \overline{w_2}}{\partial \nu} - w_1 \overline{\mathbb{F}w_2} - \frac{\partial w_1}{\partial \nu} \overline{\mathbb{M}w_2} \right) ds$$
$$+ \sum \left(\mathbb{F}_c w_1 \overline{w_2} - w_1 \overline{\mathbb{F}_c w_2} \right)$$

If the boundary conditions are such that $E_\Omega = 0$, where E_Ω is defined in (1.44), then $\mathcal{I}[w_1, w_2] = 0$ and the operator is symmetric.

Now we check that in fact this operator is selfadjoint. For that we calculate deficiency elements and find that they are trivial. In Section 1.3.5 the uniqueness of the solution was studied for the case of real values of k_0^4. However for complex wavenumbers proof of uniqueness is a simple matter. One assumes that two different solutions exist and considers the equation for their difference $w = w_1 - w_2$. Then the equation is multiplied by $\overline{w}$ and is integrated over the domain in $\mathbb{R}^2$ bounded by the surface of the obstacle and the circumference C_R of large radius. Applying Green's formula in this domain and using the boundary conditions on $\partial\Omega_0$ yield

$$\iint |\triangle w|^2 dx\, dy - k_0^4 \iint |w|^2 dx\, dy = 0.$$

When computing deficiency elements one takes k_0^4 with nonzero imaginary part. Then the above equality is possible only if $w \equiv 0$. That is, homogeneous equation has only trivial solution and therefore deficiency indices of the operator are equal to zero. That proves that the operator is selfadjoint.

The example of configuration presented in Section 1.3.6 that allows existence of solution of homogeneous problem corresponds to an eigennumber on the continuous spectrum of the operator.

Consider the case of a plate cut along the straight line $\{x = 0\}$. The bi-harmonic operator of the problem without an obstacle is defined in the following domain

$$\mathrm{Dom}(B_c) = \left\{ w \in W_2^4(\mathbb{R}_+^2) \oplus W_2^4(\mathbb{R}_-^2) : \ \mathbb{F}w|_{x=\pm 0} = 0, \ \mathbb{M}w|_{x=\pm 0} = 0 \right\}.$$

Note that second and third order derivatives presented in $\mathbb{F}w$ and $\mathbb{M}w$ belong to L_2 space on the sections $x = \pm 0$ which follows from the fact that if $w \in W_2^4$m then $\mathbf{F}w \in W_2^1$ and $\mathbb{M}w \in W_2^2$ and by embedding theorem 14 these functions belong to L_2 on any one-dimensional section.

Actually the operator B_c is the sum of two operators B_c^+ and B_c^-. The first acts in $L_2(\mathbb{R}_+^2)$, the second in $L_2(\mathbb{R}_-^2)$. Each of these operators is selfadjoint. In the presence of an obstacle Ω_0 the boundary conditions on $\partial\Omega_0$ can be set in such a way that interaction of the operators B_c^+ and B_c^- appears. One can check that selfadjoint operator appears if the boundary conditions satisfy the equality

$$\mathrm{Im} \left(\int\limits_{\partial\Omega_0} \left(\mathbb{F}w\overline{w} + \mathbb{M}w \frac{\partial \overline{w}}{\partial \nu} \right) ds + \sum F_c \overline{w} \right) = 0.$$

The corner points where $\partial\Omega_0$ intersects the crack $\{x = 0\}$ should be included in the sum in the left-hand side of the above equality.

2.3.3 *Operator of fluid loaded plate*

Consider first the case of two-dimensional model. The behavior of the mechanical system consisting of acoustic medium occupying half-plain $\mathbb{R}_+^2$ and elastic plate on its boundary $\mathbb{R}$ is characterized by two functions: acoustic pressure $U(x, z)$ and displacement of the plate $w(x)$. Therefore the operator should act upon pairs of functions $\mathcal{U} = \big(U(x, y), u(x)\big)$. The first component $U(x, z)$ is a function from the Hilbert space $L_2(\mathbb{R}_+^2)$. The second component $u(x)$ is from $L_2(\mathbb{R})$. The total Hilbert space is

$$\mathcal{L} = L_2(\mathbb{R}_+^2) \oplus L_2(\mathbb{R})$$

with the scalar product

$$\langle \mathcal{U}, \mathcal{V} \rangle = \langle U, V \rangle + \Theta \langle u, v \rangle.$$

(We use the same notations for the scalar product in the total space and in its components. These scalar products are distinguished by their arguments.) Here Θ is a positive parameter. It is convenient to choose (see page 102)

$$\Theta = \frac{\varrho h c_a^2}{D}. \tag{2.13}$$

For convenience of physical interpretation the functions $U(x,z)$ and $u(x)$ can be chosen such that they represent natural physical characteristics. For the first component we use acoustic pressure $U(x,z)$ and the second component $u(x)$ is taken proportional to flexural displacement $w(x)$.

First, consider problems that describe wave processes in acoustic medium and elastic plate separately. The operators that act upon $U(x,z)$ and $u(x)$ are correspondingly harmonic operator $A_2 = -\Delta$ and bi-harmonic operator $B_1 = d^4/dx^4$. The operator A_2 is defined on functions from $W_2^2(\mathbb{R}^2_+)$ that satisfy the Neumann boundary condition, that is

$$\mathrm{Dom}(A_2) = \left\{ U \in W_2^2(\mathbb{R}^2_+),\ \frac{\partial U(x,0)}{\partial z} = 0 \right\}.$$

The domain of the operator B_1 is $\mathrm{Dom}(B_1) = W_2^4(\mathbb{R})$. As it was shown in the previous section the operator A_2 is selfadjoint. The operator B_1 is ordinary differential operator and it can be easily checked to be selfadjoint, too.

Combining these two operators yet without interaction yields composite operator

$$\mathcal{H}_0 = \begin{pmatrix} A_2 & 0 \\ 0 & B_1 \end{pmatrix}$$

that is defined on pairs of functions from $\mathrm{Dom}(A_2) \oplus \mathrm{Dom}(B_1)$. In order the equations in the components of the spectral problem coincide with (1.10) and (1.11) describing wave processes in acoustic media and in the plate respectively, one composes the matrix operator pencil

$$\mathcal{H}_0 \mathcal{U} = \omega^2 \Lambda_0 \mathcal{U}\ , \qquad \Lambda_0 = \mathrm{diag}\left(\frac{1}{c_a^2}, \frac{\varrho h}{D} \right).$$

Note that it is possible to invert the operator Λ_0 and rewrite the spectral problem for the operator pencil as ordinary spectral problem for symmetric operator

$$\mathrm{diag}\left(c_a^2 A_2, \frac{D}{\varrho h} B_1 \right).$$

To introduce interaction of wave processes in acoustic medium and in the plate one restricts the operator A_2 to functions $\mathcal{U}$ whose first component satisfies also the Dirichlet condition on $\{z = 0\}$. The resulting operator is denoted below as $\mathring{A}_2$. The domain of the adjoint operator $(\mathring{A}_2)^*$ is composed of functions from $\mathrm{Dom}((\mathring{A}_2)^*) = W_2^2(\mathbb{R}_+^2)$ with no boundary condition imposed on $\{z = 0\}$. According to the Neumann formula the selfadjoint extensions of the matrix operator $\mathring{\mathcal{H}}_0 = \mathring{A}_2 \oplus B_1$ are defined on pairs of functions from $\mathrm{Dom}((\mathring{A}_2)^*) \oplus \mathrm{Dom}(B_1)$ that satisfy an additional linear restriction. The noninteracting operator appears, if one introduces restriction involving only the first component $U(x,z)$. If both components are involved into the linear restriction, then interaction of wave processes in the plate and in the acoustic medium appears.

Let the operator Π that takes the trace of function defined in $\mathbb{R}_+^2$ on the boundary $\mathbb{R}$ be introduced

$$\Pi U(x,z) = U(x,0).$$

The embedding theorems say that the operator Π is bounded as an operator from $W_2^1(\mathbb{R}_+^2)$ to $L_2(\mathbb{R})$. Having in mind the adhesion condition let the following restriction on the elements $\mathcal{U}$ be set

$$u(x) = \alpha \Pi \frac{\partial U(x,z)}{\partial z} \ .$$

The domain of the operator that describes interacting acoustic medium and elastic plate is the following

$$\mathrm{Dom}(\mathcal{H}) = \left\{ \mathcal{U} : \ U \in W_2^2(\mathbb{R}_+^2), \ u \in W_2^4(\mathbb{R}), \ \alpha \Pi \frac{\partial U}{\partial z} = u \right\}.$$

The interaction in the operator is given by nondiagonal terms. Let this interaction be introduced in the operator Λ_0, namely let

$$\Lambda = \begin{pmatrix} \dfrac{1}{c_a^2} & 0 \\ \widehat{\beta}\Pi & \dfrac{\varrho h}{D} \end{pmatrix}.$$

The parameters α and $\widehat{\beta}$ should be chosen in such a way that the operator $\mathcal{H} - \omega^2 \Lambda$ is selfadjoint in $\mathcal{L}$ and that the last equation coincides with the generalized boundary condition (1.13) on the plate. However it is more convenient to invert the matric Λ and consider the operator

$$\mathcal{H}_2 = \begin{pmatrix} -c_a^2\Delta & 0 \\ \beta\Pi & \dfrac{D}{\varrho h}\dfrac{d^4}{dx^4} \end{pmatrix}$$

with the same domain $\mathrm{Dom}(\mathcal{H}_2) = \mathrm{Dom}(\mathcal{H})$. The spectral problem for the operator $\mathcal{H}_2$ can coincide with Helmholtz equation in the first component and with the boundary condition (1.13) in the second component only if frequency ω is involved into the parameters α and β. Namely, one finds

$$\frac{\beta}{\alpha} = \frac{\varrho_0\omega^2}{\varrho h}.$$

The other equation that together with the above one fixes the parameters α and β in a unique way appears form equating to zero the boundary form of the operator $\mathcal{H}_2$. By integrating by parts one finds

$$\mathcal{I}_{\mathcal{H}_2}[\mathcal{U},\mathcal{V}] = \left(c_a^2 - \Theta\alpha\beta\right)\int\limits_{\mathbb{R}}\left(U\frac{\partial\overline{V}}{\partial z} - \frac{\partial U}{\partial z}\overline{V}\right)dx.$$

The boundary form is zero for any $\mathcal{U}$ and $\mathcal{V}$, if the parameters satisfy the condition

$$\beta\alpha = \frac{c_a^2}{\Theta} = \frac{D}{\varrho h}.$$

Finally

$$\alpha = \sqrt{\frac{D}{\varrho_0}}\,\omega^{-1}, \qquad \beta = \frac{\sqrt{D\varrho_0}}{\varrho h}\,\omega.$$

That is the operator describing vibrations of fluid loaded plate takes the form

$$\mathcal{H}_2 = \begin{pmatrix} -c_a^2\Delta & 0 \\ \dfrac{\sqrt{D\varrho_0}\omega}{\varrho h}\Pi & \dfrac{D}{\varrho h}\dfrac{d^4}{dx^4} \end{pmatrix} \tag{2.14}$$

and is defined on

$$\mathrm{Dom}(\mathcal{H}_2) = \left\{\mathcal{U} : \ U \in W_2^2(\mathbb{R}^2_+),\ u \in W_2^4(\mathbb{R}),\ \sqrt{\frac{D}{\varrho_0}}\frac{1}{\omega}\Pi\frac{\partial U}{\partial z} = u\right\}.$$

Note that the parameter ω^2 which is taken usually as the spectral parameter appears involved in the definition of the operator. This can be corrected if the scheme of [21] is used. We describe this scheme briefly in Section 2.3.4. However this scheme introduces nonphysical objects and the structure of the operator becomes more complicated. Besides, it is difficult to be generalized to the case of three-dimensional problems of scattering.

Consider now problems of scattering that do not contain symmetry and can not be reduced to two-dimensional problems. In that case the functions introduced above get dependence on y co-ordinate. That is

$$\mathcal{L} = L_2(\mathbb{R}^3_+) \oplus L_2(\mathbb{R}^2).$$

All the derivations performed above for two-dimensional problems can be repeated. Finally, the selfadjoint operator describing vibrations of fluid loaded thin elastic plate can be written in the matrix form

$$\mathcal{H}_3 = \begin{pmatrix} -c_a^2 \Delta & 0 \\ \dfrac{\sqrt{D\varrho_0}\omega}{\varrho h}\Pi & \dfrac{D}{\varrho h}\Delta^2 \end{pmatrix} \tag{2.15}$$

with the domain

$$\mathrm{Dom}(\mathcal{H}_3) = \left\{ \mathcal{U} : \ U \in W_2^2(\mathbb{R}^3_+), \ u \in W_2^4(\mathbb{R}^2), \ \sqrt{\frac{D}{\varrho_0}} \frac{1}{\omega} \Pi \frac{\partial U}{\partial z} = u \right\}.$$

The operators $\mathcal{H}$ for two and three-dimensional problems are composed of two differential operators that describe wave processes in the acoustic medium and in the plate separately. This block structure is inherited by the selfadjoint perturbations of these operators constructed below.

2.3.4 *Another operator model of fluid loaded plate*

The approach first suggested in [21] and then developed in [12] uses more complicated two component function, but yields the usual spectral problem for the operator rather than for operator pencil. It is applicable only in two-dimensional case. One introduces $\mathcal{U} = (U, v)$ and considers the operator

$$\mathcal{H}\mathcal{U} = \begin{pmatrix} -c_a^2 \Delta U \\ \dfrac{\varrho_0}{\varrho h} \dfrac{d^2}{dx^2} \Pi U + \dfrac{D}{\varrho h} \dfrac{d^4}{dx^4} v \end{pmatrix} \tag{2.16}$$

with domain

$$\begin{aligned} \mathrm{Dom}(\mathcal{H}) = \Big\{ & U \in W_2^2(\mathbb{R}^2), v \in W_2^2(\mathbb{R}), \\ & \frac{\varrho_0}{\varrho_h}\Pi U + \frac{D}{\varrho h} v'' = \Pi \frac{\partial U}{\partial z} \in W_2^2(\mathbb{R}) \Big\}. \end{aligned} \tag{2.17}$$

In [21] the operator $\mathcal{H}$ is proved to be selfadjoint.

The boundary-value problem of acoustic wave diffraction on a homogeneous elastic plate can be rewritten as the spectral problem

$$\mathcal{H}\mathcal{U} = \omega^2 \mathcal{U}.$$

It is essential to note that the frequency ω is not contained in the formulation of the operator and the above problem is a classical spectral problem.

Nevertheless the models of Section 2.3.3 which deal with physical objects in the channels of scattering are mainly used in the book. Afterwards reformulation can be made using the formulae

$$v = \frac{1}{\omega^2}\, w'', \qquad w = \frac{D}{\varrho h}\, v'' + \frac{\varrho_0}{\varrho h}\, \Pi U.$$

The operator (2.16) is used in Section 4.4.

2.4 Extensions theory for differential operators

Operator extensions are described by Neumann's second formula (Theorem 9). However when dealing with differential operators it is more convenient to characterize the domain of operator extensions by fixing some kind of "boundary" conditions. We consider such selfadjoint perturbations that deal with "boundary" conditions fixed in one point [§]. They say that such conditions correspond to the *zero-range potential* and the point where these conditions are formulated is the force center of the zero-range potential. (For more general cases see [2] and the list of references presented there). The general scheme for the construction of such selfadjoint perturbations consists of two steps [58]. First the original selfadjoint operator is restricted to such functions that vanish in a vicinity of some point M which then becomes the potential center. Closure of such restricted operator is a symmetric operator with equal and finite deficiency indices. The second

[§]A more complicated example is presented in Chapter 4.

step of the procedure is in the description of all selfadjoint extensions of the restricted operator. One can perform this extension according to the Neumann's formulae, but then the domain can be described in terms of local asymptotic decompositions. In the following sections we present this scheme for harmonic operator in half-spaces $\mathbb{R}^2_+$ and $\mathbb{R}^3_+$, for bi-harmonic operator in $\mathbb{R}^2$, for the operator describing vibrations of fluid loaded plate and for the operator corresponding to a plate cut by an infinite straight crack.

2.4.1 *Zero-range potentials for harmonic operator*

Zero-range potentials were first introduced by Fermi in early 30's. He fixed the value of the logarithmic derivative of wave function in a chosen point (potential center). Rigorous mathematical analysis of zero-range potentials in [29] shows that the "boundary" conditions formulated for the logarithmic derivative of the wave function correspond to some selfadjoint perturbation of the initial harmonic operator. Further development of the zero-range potentials technique was carried out both by physicists and mathematicians. For the current state of the theory see [2]. The zero-range potentials for the harmonic operator are constructed and used for modeling narrow slots in [59], [69], etc.

Consider first two-dimensional problems. One starts with the harmonic operator A_2 in a half-space $(x, z) \in \mathbb{R}^2_+$. As noted in Section 2.3.1 this operator is selfadjoint. Restrict now this operator to such smooth functions that together with all derivatives vanish near the origin $x = z = 0$. The closure of this restricted operator domain in the metrics of W^2_2 gives

$$\mathrm{Dom}(A^\circ_2) = \left\{ U \in W^2_2(\mathbb{R}^2_+),\ \left.\frac{\partial U}{\partial z}\right|_{z=0} = 0,\ U(0,0) = 0 \right\}. \tag{2.18}$$

Note that there is embedding of $W^2_2(\mathbb{R}^2)$ to the space of continuous functions and therefore the last condition is sensible.

The operator $A^\circ_2 = -\Delta$ on $\mathrm{Dom}(A^\circ_2)$ is symmetric. Its deficiency indices are equal to $(1, 1)$. Indeed, deficiency elements $G_\pm$ are such elements that are orthogonal to $\mathrm{Res}(A^\circ_2 \pm i)$, that is

$$\iint\limits_{\mathbb{R}^2_+} \overline{G_\pm(x,z)}\,(-\Delta \pm i)\,U(x,z)dx\,dz = 0$$

for any $U \in \mathrm{Dom}(A_2^\circ)$. Integrating by parts yields

$$\int_{\mathbb{R}} U(x,0)\frac{\partial \overline{G_\pm(x,0)}}{\partial z}dx + \iint_{\mathbb{R}^2_+} U(x,z)\,(-\Delta \pm i)\,\overline{G_\pm(x,z)}dx\,dz = 0.$$

Due to arbitrariness of the function $U(x,z)$ one concludes that

$$\frac{\partial G_\pm(x,0)}{\partial z} = 0, \qquad x \neq 0$$

and

$$(\Delta \pm i)\,G_\pm = 0, \qquad (x,z) \neq (0,0).$$

The above equations are satisfied for the functions

$$G_+(x,z) = \frac{i}{2}H_0^{(1)}\left(e^{i\pi/4}\sqrt{x^2+z^2}\right),$$
$$G_-(x,z) = \frac{i}{2}H_0^{(1)}\left(e^{3i\pi/4}\sqrt{x^2+z^2}\right).$$

These are the only nonzero elements that satisfy to the above equations and belong to the space $L_2(\mathbb{R}^2_+)$.

Now the adjoint operator can be constructed. According to the first Neumann's formula (see Theorem 8) the domain of this operator can be decomposed into the sum of $\mathrm{Dom}(A_2^\circ)$ and two one-dimensional lineals $R_+ = \{\alpha G_+,\ \alpha \in \mathbb{C}\}$ and $R_- = \{\beta G_-,\ \beta \in \mathbb{C}\}$.

The second step of the procedure is in setting appropriate restrictions on the domain $\mathrm{Dom}((A_2^\circ)^*)$ involving elements of the lineals $R_\pm$, that is, in defining unitary operator V from the second Neumann's formula (see Theorem 9) that acts on R_+ and maps it on R_-. Evidently that in the considered case of one-dimensional lineals $R_\pm$ the operator V is simply multiplication by $e^{i\theta}$, $\mathrm{Im}\,\theta = 0$, that is $\beta = e^{i\theta}\alpha$. Therefore the domain of the selfadjoint perturbation $\mathcal{A}_2^\theta$ is specified as follows

$$\mathrm{Dom}(\mathcal{A}_2^\theta) = \mathrm{Dom}(A_2^\circ) + \alpha\left(G_+ + e^{i\theta}G_-\right).$$

For differential operators the elements from $\mathrm{Dom}(\mathcal{A}_2^\theta)$ can be more conveniently pointed out by their local asymptotics in a vicinity of the potential center. The asymptotics of Bessel functions of the third kind (see [1]) for small argument yield (here C_E is the Euler constant)

$$\left(G_+ + e^{i\theta}G_-\right) \sim -c\frac{1}{\pi}\ln(r) + b + \ldots, r = \sqrt{x^2+z^2} \to 0,$$
$$c = \left(1+e^{i\theta}\right), \quad b = \left(1+e^{i\theta}\right)\frac{1}{\pi}\left(\ln 2 - C_E\right) - \left(1-e^{i\theta}\right)\frac{i}{4}.$$

Direct calculation shows that for any real parameter θ the coefficients c and b are proportional to each other with real coefficient

$$b = Sc, \quad \operatorname{Im} S = 0.$$

There is one to one correspondence between parameters θ and S

$$S = \frac{1}{\pi}\left(\ln 2 - C_E\right) - \frac{1}{4}\frac{\sin\theta}{1+\cos\theta},$$
$$\theta = \arg\left(\frac{i-X}{i+X}\right), \quad X = 4S - \frac{4}{\pi}\left(\ln 2 - C_E\right).$$

Therefore real parameter S can be used for the parameterization of selfadjoint perturbations of operator A_2. Namely, domain $\mathrm{Dom}(\mathcal{A}_2^S)$ consists of functions belonging to $W_2^2(\mathbb{R}_+^2 \setminus \{0\})$ that have the asymptotics

$$U \sim c\left(-\frac{1}{\pi}\ln r + S\right) + o(1), \quad r \to 0. \tag{2.19}$$

the operator $\mathcal{A}_2^S$ acts on smooth functions as Laplace operator and on singular functions $g_\pm$ as the multiplication by $\pm i$. That is

$$\mathcal{A}_2^S U = \Delta U' - icg_+ = \Delta U'' + icg_-,$$

where

$$U' = U - cg_+ \in \mathrm{Dom}(A_2), \qquad U'' = U - cg_- \in \mathrm{Dom}(A_2)$$

and c is the coefficient from (2.19).

Note that one can come to the above parameterization directly by examining the boundary form of the adjoint operator. This approach is used below for the construction of zero-range potentials for harmonic operator A_3 and for bi-harmonic operator B_2.

Consider the case of three-dimensional problem. Again the initial operator A_3 is restricted to functions that vanish at the origin. The deficiency indices are again equal to $(1,1)$ and the deficiency elements satisfy the equations

$$\frac{\partial G_{\pm}(x,y,0)}{\partial z} = 0, \qquad (x,y) \neq (0,0)$$

and

$$(\Delta \pm i)\, G_{\pm} = 0, \qquad (x,y,z) \neq (0,0,0).$$

The asymptotics of functions from $\mathrm{Dom}((A_3^{\circ})^*)$ can be found directly from these equations

$$U \sim c\frac{1}{2\pi}\frac{1}{r} + b + o(1), \qquad r = \sqrt{x^2+y^2+z^2} \to 0.$$

Consider the boundary form

$$\langle (A_3^{\circ})^* U_1, U_2\rangle - \langle U_1, (A_3^{\circ})^* U_2\rangle = c_1\overline{b_2} - b_1\overline{c_2}.$$

Here c_1 and b_1 are the coefficients of the asymptotics of function U_1 and c_2, b_2 are the coefficients of U_2. Now restrict A_3° to selfadjoint operator $\mathcal{A}_3^S$ which is a perturbation of A_3 with zero-range potential. The boundary form is equal to zero if the functions U_1 and U_2 satisfy the asymptotics

$$U \sim c\left(\frac{1}{2\pi}\frac{1}{r} + S\right) + o(1), \quad r \to 0 \tag{2.20}$$

where S is any real parameter (including infinity, which corresponds to $c = 0$ and cS arbitrary).

Thus, the selfadjoint operator $\mathcal{A}_3^S$ is defined on functions that satisfy the asymptotics (2.20). On smooth functions it acts as Laplacian and singular functions $g_{\pm}$ it multiplies by $\pm i$.

2.4.2 *Zero-range potentials for bi-harmonic operator*

Zero-range potentials for bi-harmonic operator were first considered in [39]. Their construction can be performed according to the general scheme described in the previous section.

Consider first the case of one-dimensional isolated plate. In that case one deals with an ordinary differential operator and its perturbations are parameterized by classical boundary conditions. However, for convenience of further constructing zero-range potentials for the operator $\mathcal{H}_2$, introduced in Section 2.3.2, we present here the analysis of selfadjoint perturbations of the operator d^4/dx^4 according to the general scheme.

One restricts the operator to functions that vanish near $\{x = 0\}$ and after closure in metrics of W_2^4 gets

$$\mathrm{Dom}(B_1^\circ) = \left\{ u \in W_2^4(\mathbb{R}) : \quad \frac{d^j}{dx^j} u(0) = 0,\ j = 0,1,2,3 \right\}. \tag{2.21}$$

The adjoint operator is defined on functions from $W_2^4(\mathbb{R}_+) \oplus W_2^4(\mathbb{R}_-)$. For these functions boundary values at $x = \pm 0$ can be considered and boundary values for the derivatives up to the third order are defined (see embedding theorem 13). The particular zero-range potential is defined by setting linear constraints for those values. Consider the asymptotic decomposition of function from $W_2^4(\mathbb{R} \backslash \{0\})$ when $x \to \pm 0$

$$u \sim c_3 \frac{\mathrm{sign}(x)}{2} + c_2 \frac{|x|}{2} + c_1 \frac{x|x|}{4} + c_0 \frac{|x|^3}{12} + b_0 + b_1 x + b_2 \frac{x^2}{2} + b_3 \frac{x^3}{6} + \ldots \tag{2.22}$$

The formula (2.22) is simply the Taylor decomposition for two functions defined on positive and negative semi-axes. The terms with the coefficients c_j form singular part u^s of the field, the other terms are regular. The forth order derivative of functions having the asymptotics (2.22) is a combination of the generalized functions

$$c_3 \delta'''(x) + c_2 \delta''(x) + c_1 \delta'(x) + c_0 \delta(x)$$

and regular terms.

The zero-range potential can be characterized by the condition for the terms of the asymptotics (2.22). To find the class of conditions one examines the boundary form of the operator $(B_1^\circ)^*$

$$\mathcal{I}[u,v] = \sum_{j=0}^{3} (-1)^j \left(c_j(u) \overline{b_j(v)} - b_j(u) \overline{c_j(v)} \right). \tag{2.23}$$

Let $\mathbf{c} = (c_0, -c_1, c_2, -c_3)^T$ and $\mathbf{b} = (b_0, b_1, b_2, b_3)^T$ (Here and below ^{T} stands for the transposition of vector). Then the above boundary form is zero for any u and v and the operator is selfadjoint provided vectors $\mathbf{c}$ and $\mathbf{b}$ are related by some Hermitian matrix $\mathbf{S}$

$$\mathbf{b} = \mathbf{S}\mathbf{c}. \tag{2.24}$$

We denote this selfadjoint perturbation of operator B_1 as $\mathcal{B}_1^S$. Evidently the representation (2.24) exists not for all classical boundary conditions. One should permit infinitely large values for the elements of the matrix. For example classical point model of crack in an elastic plate is characterized by the boundary conditions

$$w''(\pm 0) = w'''(\pm 0) = 0 \ .$$

For the coefficients of the asymptotic expansion (2.22) these conditions reduce to

$$u''(\pm 0) = \pm\frac{c_1}{2} + b_2 = 0, \qquad u'''(\pm 0) = \pm\frac{c_0}{2} + b_3 = 0,$$

that is $c_0 = c_1 = b_2 = b_3 = 0$. If one formally accepts that $\infty \cdot 0$ is any finite number then the boundary conditions on the crack are equivalent to (2.24) with

$$\mathbf{S} = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ * & * & \infty & * \\ * & * & * & \infty \end{pmatrix},$$

where asterisks denote arbitrary numbers.

Description of the domain in the form (2.24) permits to calculate the number of parameters that uniquely define the zero-range potential. The number of parameters is equal to the dimension of the space of Hermitian matrices in $\mathbb{C}^4$, that is to 16.

Consider the operator B_2 and construct its selfadjoint perturbations of the zero-range potential type. Again restrict this operator to smooth functions that vanish near the origin. Closure yields

$$\mathrm{Dom}(B_2^\circ) = \Big\{ u \in W_2^4(\mathbb{R}^2),\ u(0,0) = 0,\ \frac{\partial u(0,0)}{\partial x} = 0,\ \frac{\partial u(0,0)}{\partial y} = 0, \\ \frac{\partial^2 u(0,0)}{\partial x^2} = 0,\ \frac{\partial^2 u(0,0)}{\partial x \partial y} = 0,\ \frac{\partial^2 u(0,0)}{\partial y^2} = 0 \Big\}.$$

Embedding theorems confirm that the above conditions are sensible for functions from $W_2^4(\mathbb{R}_+^2)$.

The adjoint operator $(B_2^\circ)^*$ is defined on such functions that may have singularities in the point $(0,0)$. These singularities are due to the deficiency elements of the restricted operator B_2°. In order to find the deficiency elements one considers the scalar product $\langle B_2^{\circ} u, g_\pm \rangle$ which is equal to zero for any $u \in \mathrm{Dom}(B_2^{\circ})$. Integrating by parts in this equality yields

$$(\triangle^2 \pm i)\, g_\pm = 0, \qquad (x, y) \neq (0, 0).$$

These equations have the following solutions

$$g_+ = \frac{e^{i\pi/4}}{8}\left(H_0^{(1)}\left(e^{i\pi/8}\rho\right) - H_0^{(1)}\left(e^{5i\pi/8}\rho\right)\right),$$
$$\frac{\partial g_+}{\partial x}, \ \frac{\partial g_+}{\partial y}, \ \frac{\partial^2 g_+}{\partial x^2}, \ \frac{\partial^2 g_+}{\partial x \partial y}, \ \frac{\partial^2 g_+}{\partial y^2}$$

and

$$g_- = \frac{e^{3i\pi/4}}{8}\left(H_0^{(1)}\left(e^{3i\pi/8}\rho\right) - H_0^{(1)}\left(e^{7i\pi/8}\rho\right)\right),$$
$$\frac{\partial g_-}{\partial x}, \ \frac{\partial g_-}{\partial y}, \ \frac{\partial^2 g_-}{\partial x^2}, \ \frac{\partial^2 g_-}{\partial x \partial y}, \ \frac{\partial^2 g_-}{\partial y^2}.$$

Higher order differentiation causes non square integrable singularities to appear.

That is, there are six deficiency elements corresponding to the spectral parameter i and six deficiency elements corresponding to the spectral parameter $-i$. The deficiency indices are $(6, 6)$ and the domain of the adjoint operator is composed of functions that can be represented in the form

$$u = u_0 + \sum_{i,j=0}^{i+j=2} \alpha_{ij} \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j g_+ + \sum_{i,j=0}^{i+j=2} \beta_{ij} \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j g_-,$$

where $u_0 \in \mathrm{Dom}(B_2^{\circ})$ and α_{ij}, β_{ij} are arbitrary complex constants. As previously let the functions from $\mathrm{Dom}((B_2^{\circ})^*)$ be specified by their asymptotics as $\rho \to 0$. Using the asymptotics [1] of function $H_0^{(1)}$ it is not difficult to find the general form of the asymptotic decomposition of a function from $\mathrm{Dom}((B_2^{\circ})^*)$, namely

$$\begin{aligned} u \sim {} & \frac{c_{00}}{8\pi}\rho^2 \ln\rho + \frac{c_{10}}{4\pi}\rho\ln\rho\cos\varphi + \frac{c_{01}}{4\pi}\rho\ln\rho\sin\varphi \\ & + \frac{c_{20}}{4\pi}\left(\ln\rho + \cos^2\varphi\right) + \frac{c_{11}}{4\pi}\cos\varphi\sin\varphi \\ & + \frac{c_{02}}{4\pi}\left(\ln\rho + \sin^2\varphi\right) + b_{00} + b_{10}\rho\cos\varphi \\ & + b_{01}\rho\sin\varphi + \frac{b_{20}}{2}\rho^2\cos^2\varphi \end{aligned}$$

$$+ b_{11}\rho^2 \cos\varphi \sin\varphi + \frac{b_{02}}{2}\rho^2 \sin^2\varphi + o(r^2) \tag{2.25}$$

and c_{ij}, b_{ij} are arbitrary complex constants.

The selfadjoint perturbation of the original operator B_2 can be constructed by introducing a unitary operator V that maps the lineal of elements $\partial^{i+j}g_+/\partial x^i\partial y^j$, $i+j \le 2$ onto the lineal of elements $\partial^{i+j}g_-/\partial x^i\partial y^j$, $i+j \le 2$. In the basis of the pointed out elements this operator is represented by unitary 6×6 matrices. Another parameterization involves the coefficients c_{ij} and b_{ij} of the asymptotics (2.25). Let such conditions be fixed in the form

$$(b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})^T = \mathbf{S}\,(c_{00}, -c_{10}, -c_{01}, c_{20}, c_{11}, c_{02})^T . \tag{2.26}$$

The class of possible matrices $\mathbf{S}$ can be found directly by equating the boundary form to zero

$$\mathcal{I}(u,v) = \sum_{i,j} (-1)^{i+j}\left(c_{ij}(u)\overline{b_{ij}(v)} - b_{ij}(u)\overline{c_{ij}(v)}\right) = 0.$$

Therefore matrix $\mathbf{S}$ should be Hermitian.

When studying perturbations of harmonic operator, we allow parameter S to be equal to infinity which corresponds to $c = 0$, that is to the original operator A_2 or A_3. Similarly for bi-harmonic operator perturbations the matrix $\mathbf{S}$ can contain infinitely large elements. That is, one can apply arbitrary permutations of c_{ij} and b_{ij} and form two vectors $\widehat{c}$ and $\widehat{b}$ such that

$$\widehat{c}_{ij} = c_{ij},\ \ \widehat{b}_{ij} = b_{ij} \qquad \text{or} \qquad \widehat{c}_{ij} = b_{ij},\ \ \widehat{b}_{ij} = c_{ij}.$$

Then the selfadjoint perturbation is fixed by Hermitian matrix

$$\widehat{b} = \widehat{\mathbf{S}}\widehat{c}. \tag{2.27}$$

In the nondegenerated case the parameterizations in the form (2.26) and in the form (2.27) are equivalent, but if the matrix $\widehat{\mathbf{S}}$ is degenerated then infinitely large elements appear formally in the equation (2.26).

2.4.3 *Zero-range potentials for fluid loaded plates*

The zero-range potentials for the components of the matrix operator $\mathcal{H}$ are defined in sections 2.4.1 and 2.4.2. Now the zero-range potentials for the matrix operator itself can be constructed. The idea is in replacing the operators A_2 and d^4/dx^4 or A_3 and B_2 by the corresponding operators with zero-range potentials in the matrices (2.14) or (2.15). However the procedure used in Section 2.3.3 does not preserve the zero-range potentials. Indeed restricting the operator $\mathcal{A}_2^S$ (or $\mathcal{A}_3^S$) to functions that satisfy the Dirichlet boundary condition on $\{z = 0\}$ brings to the same operator A_2° (or A_3°) as in section 2.3.3. Therefore we apply the restriction-extension procedure directly to the operators $\mathcal{H}_2$ and $\mathcal{H}_3$. However, it is useful to notice objects that appear in sections 2.4.1 and 2.4.2 for the zero-range potentials of operators in components of $\mathcal{H}_2$ and $\mathcal{H}_3$.

Consider the restricted matrix operator $\mathcal{H}_2^\circ$ with the domain

$$\mathrm{Dom}(\mathcal{H}_2^\circ) = \Big\{ \mathcal{U} : U \in W_2^2(\mathbb{R}_+^2), \quad u \in W_2^4(\mathbb{R}), \quad \alpha\Pi \frac{\partial U}{\partial z} = u, \\ U(0,0) = 0, \quad \frac{d^j}{dx^j} u(0) = 0, j = 0, 1, 2, 3 \Big\}.$$

This domain is obtained by setting constraints to the domain $\mathrm{Dom}(\mathcal{H})$ in the manner similar to (2.18) and (2.21). By direct calculation the deficiency indices of the operator $\mathcal{H}_2^\circ$ can be found equal to $(5, 5)$, that is to the sums of the deficiency indices for the components of the operator. The domain of the adjoint operator $(\mathcal{H}_2^\circ)^*$ is the sum of $\mathrm{Dom}(\mathcal{H}_2^\circ)$ and linear span of the deficiency elements. One can find that these deficiency elements (as generalized functions) satisfy the equations

$$\begin{pmatrix} -c_a^2 \Delta & 0 \\ \frac{\varrho_0}{\varrho h}\Pi & \frac{D}{\varrho h}\frac{d^4}{dx^4} \end{pmatrix} \mathcal{G} - \omega^2 \mathcal{G} = \begin{pmatrix} c_a^2\, c\delta(x)\delta(z) \\ \frac{D}{\varrho h}\sum_{j=0}^{3} c_j \frac{d^j}{dx^j}\delta(x) \end{pmatrix}$$

with arbitrary constants c and c_j in the right-hand side. Consider first the deficiency element that corresponds to $c = 1$, $c_0 = c_1 = c_2 = c_3 = 0$. This deficiency element $\mathcal{G} = (G, g)$ has singular component G and regular component g. The singularity of G can be shown the same as in the deficiency elements $G_\pm$ of the harmonic operator A_2° studied in section 2.4.1. Namely,

$$G \sim -\frac{1}{\pi} \ln r + b + \ldots, \qquad r \to 0.$$

The deficiency element $\mathcal{G}_0 = (G_0, g_0)$ which corresponds to $c = c_1 = c_2 = c_3 = 0$, $c_0 = 1$ has regular component G_0 and singular component g_0.

The singularity of g_0 appears the same as the singularity of the deficiency elements of the operator B_1°, namely

$$g_0 \sim \frac{1}{12}|x|^3 + b_{01}x + b_{02}\frac{x^3}{6} + \ldots, \qquad x \to \pm 0. \tag{2.28}$$

Other deficiency elements corresponding to nonzero c_1, c_2 or c_3 can be obtained by differentiation of $\mathcal{G}_0$. All these deficiency elements have regular components G_j and the singularities in the components g_j are the same as in the deficiency elements of the operator B_1°. All the mentioned above properties of the deficiency elements follow from the well known fact that singularities of solution are defined by the principal part of differential operator. However more precise asymptotics of the deficiency elements will be needed in Chapter 3, so exact expressions for these elements are presented below.

$$\begin{aligned} G(x,z;\omega) &= \frac{1}{2\pi}\int e^{i\lambda x-\sqrt{\lambda^2-k^2}z}\,\frac{\lambda^4-k_0^4}{L(\lambda)}d\lambda, \\ g(x;\omega) &= -\frac{\omega}{2\pi}\sqrt{\frac{\varrho_0}{D}}\int e^{i\lambda x}\frac{d\lambda}{L(\lambda)}, \end{aligned} \tag{2.29}$$

$$\begin{aligned} G_0(x,z;\omega) &= -\frac{\omega}{2\pi}\sqrt{\frac{\varrho_0}{D}}\int e^{i\lambda x-\sqrt{\lambda^2-k^2}z}\,\frac{d\lambda}{L(\lambda)}, \\ g_0(x;\omega) &= \frac{1}{2\pi}\int e^{i\lambda x}\frac{\sqrt{\lambda^2-k^2}}{L(\lambda)}d\lambda. \end{aligned} \tag{2.30}$$

Here $k = \omega/c_a$, $k_0^4 = \omega^2\varrho h/D$, $N = \omega^2\varrho_0/D$ and the symbol $L(\lambda)$ of the boundary condition is

$$L(\lambda) = \left(\lambda^4 - k_0^4\right)\sqrt{\lambda^2-k^2} - N.$$

When ω is complex (one takes $\omega = \exp(\pm i\pi/4)$ for defining the deficiency elements) no singularities are presented on the real axis along which the integration in (2.29) and (2.30) is carried.

The noted above similarity of the deficiency elements with that of harmonic operator and operator of ordinary fourth order derivative may be confirmed by the following decompositions

$$\frac{\lambda^4-k_0^4}{L(\lambda)} = \frac{1}{\sqrt{\lambda^2-k^2}} + \frac{N}{L(\lambda)\sqrt{\lambda^2-k^2}}$$

$$\frac{\sqrt{\lambda^2 - k^2}}{L(\lambda)} = \frac{1}{\lambda^4 - k_0^4} + \frac{N}{L(\lambda)(\lambda^4 - k_0^4)}.$$

Consider the first formula. When substituted into the Fourier integral for $G(x, z; \omega)$ the first term gives deficiency element of A_2°. The second term decreases at infinity as $O(\lambda^{-6})$ and the corresponding Fourier integral has no singularity. When substituting the second formula into the Fourier integral for $g_0(x; \omega)$ the integral of the first term can be computed by residues theorem. This gives the first singular term in the asymptotics (2.28). The second term decreases as $O(\lambda^{-9})$, therefore it does not contribute to singularities even after being three times differentiated.

Using the first Neumann's formula and combining the asymptotic decompositions of the deficiency elements yield representation for functions from the domain of the adjoint operator $(\mathcal{H}_2^\circ)^*$, namely

$$\begin{aligned} U(x, z) &= \left(-c\frac{1}{\pi}\ln r + b\right)\chi(r) + U_0(x, z), \\ u(x) &= \left(c_3\frac{\operatorname{sign}(x)}{2} + c_2\frac{|x|}{2} + c_1\frac{x|x|}{4} + c_0\frac{|x|^3}{12}\right. \\ &\quad \left. + b_0 + b_1 x + b_2\frac{x^2}{2} + b_3\frac{x^3}{6}\right)\chi(|x|) + u_0(x). \end{aligned} \tag{2.31}$$

Here χ is infinitely smooth function with finite support, that is identically equal to unity when its argument is less than one. The functions $U_0(x, z)$ and $u_0(x)$ are functions from the domain of the restricted operator.

To define the selfadjoint operator some constraints should be imposed on elements from $\mathrm{Dom}((\mathcal{H}_2^\circ)^*)$. These constraints can be written for the coefficients of the asymptotic decomposition (2.31) in a vicinity of the zero-range potential center. The components of the elements from $\mathrm{Dom}((\mathcal{H}_2^\circ)^*)$ have the same asymptotics as the elements from $\mathrm{Dom}((A_2^\circ)^*)$ and $\mathrm{Dom}((B_1^\circ)^*)$. Therefore one can use the constraints (2.19) and (2.24). Nominally that corresponds to substituting the operators with zero-range potentials instead of A_2 and d^4/dx^4 into matrix operator $\mathcal{H}_2$. The number of real parameters that define the zero-range potential in this case is equal to 17. One parameter (constant S) characterizes the zero-range potential for the harmonic operator A_2^S involved in the composite operator $\mathcal{H}_2^S$ and 16 parameters (matrix $\mathbf{S}$) fix the zero-range potential in the component B_1^S.

Note that the conditions (2.19) and (2.24) are sufficient conditions for the composite operator to be selfadjoint, but are not necessary conditions.

Indeed, examining the boundary form $\mathcal{I}[\mathcal{U},\mathcal{V}]$ of the operator $(\mathcal{H}_2^\circ)^*$ and equating it to zero yields the following equation

$$\begin{aligned} & c_a^2\left(c(U)\overline{b(V)} - b(U)\overline{c(V)}\right) \\ & +\Theta\frac{D}{\varrho h}\sum_{j=0}^{3}(-1)^j\left(c_j(u)\overline{b_j(v)} - b_j(u)\overline{c_j(v)}\right) = 0. \end{aligned} \tag{2.32}$$

Here c, b, c_j and b_j are the coefficients of the asymptotics (2.31). The chosen in (2.13) value of Θ makes the above equation the most simple, in that case it reduces to

$$c(U)\overline{b(V)} - b(U)\overline{c(V)} + \sum_{j=0}^{3}(-1)^j\left(c_j(u)\overline{b_j(v)} - b_j(u)\overline{c_j(v)}\right) = 0.$$

The more general conditions that guarantee (2.32) to be satisfied are

$$\begin{pmatrix} b \\ \mathbf{b} \end{pmatrix} = \mathcal{S}\begin{pmatrix} c \\ \mathbf{c} \end{pmatrix}, \tag{2.33}$$

where $\mathbf{c} = (c_0, -c_1, c_2, -c_3)^T$, $\mathbf{b} = (b_0, b_1, b_2, b_3)^T$ and matrix $\mathcal{S}$ is arbitrary Hermitian matrix in $\mathbb{C}^5$. Thus, the number of parameters characterizing the zero-range potential increases up to 25. The additional 8 parameters that are not involved in the conditions (2.19) and (2.24) and appear in the condition (2.33) describe interaction of the components U and u in the zero-range potential of the operator $\mathcal{H}_2^S$. Therefore matrix $\mathcal{S}$ has block structure

$$\mathcal{S} = \begin{pmatrix} S & s \\ s^* & \mathbf{S} \end{pmatrix} \tag{2.34}$$

which is exploited in the procedure of choosing parameters of selfadjoint perturbations in Chapter 3.

The condition (2.33) describes not all the zero-range potentials. Again one should allow arbitrary interchange of elements of two vectors (see page 98).

Consider now the case of three-dimensional scattering problem. The scheme of zero-range potential construction for the operator $\mathcal{H}_3$ is exactly the same as above. The restricted matrix operator $\mathcal{H}_3^\circ$ is defined on the domain

$$\mathrm{Dom}(\mathcal{H}_3^\circ) = \Big\{\mathcal{U} : U \in W_2^2(\mathbb{R}_+^3), \quad u \in W_2^4(\mathbb{R}^2), \quad \alpha\Pi\frac{\partial U}{\partial z} = u,$$
$$U(0,0,0) = 0, \quad \frac{\partial^{i+j}}{\partial x^i \partial y^j} u(0,0) = 0, i+j \le 2\Big\}.$$

The deficiency elements $\mathcal{G} = (G, g)$ can be found in the form of Fourier integrals

$$G(x,y,z;\omega) = \frac{1}{4\pi^2}\iint e^{i\lambda x + i\mu y - \sqrt{\lambda^2+\mu^2-k^2}z}\,\frac{(\lambda^2+\mu^2)^2 - k_0^4}{L(\lambda,\mu)}\,d\lambda\,d\mu,$$

$$g(x,y;\omega) = -\frac{\omega}{4\pi^2}\sqrt{\frac{\varrho_0}{D}}\iint e^{i\lambda x + i\mu y}\,\frac{d\lambda\,d\mu}{L(\lambda,\mu)}, \tag{2.35}$$

$$\begin{aligned} G_{00}(x,y,z;\omega) &= -\frac{\omega}{4\pi^2}\sqrt{\frac{\varrho_0}{D}}\iint e^{i\lambda x + i\mu y - \sqrt{\lambda^2+\mu^2-k^2}z}\,\frac{d\lambda\,d\mu}{L(\lambda,\mu)}, \\ g_{00}(x,y;\omega) &= \frac{1}{4\pi^2}\frac{\varrho h}{D}\iint e^{i\lambda x + i\mu y}\,\frac{\sqrt{\lambda^2+\mu^2-k^2}}{L(\lambda,\mu)}\,d\lambda\,d\mu. \end{aligned} \tag{2.36}$$

Here the spectral parameter ω^2 is taken complex and the notations of Chapter 1 are used, namely $k = \omega/c_a$, $k_0^4 = \omega^2\varrho h/D$, $N = \omega^2\varrho_0/D$ and

$$L(\lambda,\mu) = \left((\lambda^2+\mu^2)^2 - k_0^4\right)\sqrt{\lambda^2+\mu^2-k^2} - N.$$

The functions defined by the formulae (2.36) can be differentiated by x and by y up to 2 times, that is, $\mathcal{G}_{10} = \partial\mathcal{G}_{00}/\partial x$, $\mathcal{G}_{01} = \partial\mathcal{G}_{00}/\partial y$, $\mathcal{G}_{20} = \partial^2\mathcal{G}_{00}/\partial x^2$, $\mathcal{G}_{11} = \partial^2\mathcal{G}_{00}/\partial x\partial y$ and $\mathcal{G}_{02} = \partial^2\mathcal{G}_{00}/\partial y^2$ are also deficiency elements. The deficiency indices of the operator $\mathcal{H}_3^\circ$ are $(7,7)$ and the domain of the adjoint operator $(\mathcal{H}_3^\circ)^*$ is composed of functions

$$\begin{aligned} U(x,y,z) &= \left(c\frac{c_a^2}{2\pi}\frac{1}{r} + b\right)\chi(r) + U_0(x,y,z), \\ u(x,y) &= \Big(\frac{c_{00}}{8\pi}\rho^2\ln\rho + \frac{c_{10}}{4\pi}\rho\ln\rho\cos\varphi + \frac{c_{01}}{4\pi}\rho\ln\rho\sin\varphi \\ &+ \frac{c_{20}}{4\pi}\left(\ln\rho + \cos^2\varphi\right) + \frac{c_{11}}{4\pi}\cos\varphi\sin\varphi + \frac{c_{02}}{4\pi}\left(\ln\rho + \sin^2\varphi\right) \\ &+ b_{00} + b_{10}\rho\cos\varphi + b_{01}\rho\sin\varphi + b_{20}\frac{1}{2}\rho^2\cos^2\varphi \\ &+ b_{11}\rho^2\cos\varphi\sin\varphi + b_{02}\frac{1}{2}\rho^2\sin^2\varphi\Big)\chi(\rho) + u_0(x,y). \end{aligned} \tag{2.37}$$

Here χ is infinitely smooth cut-off function as in (2.31) and $U_0(x,y,z)$ and $u_0(x,y)$ are functions from the domain of the restricted operator.

The constraints on elements from $\mathrm{Dom}((\mathcal{H}_3^\circ)^*)$ that make the operator selfadjoint can be written for the coefficients of the asymptotic decomposition of (2.37) in a vicinity of the zero-range potential center

$$\begin{pmatrix} b \\ \mathbf{b} \end{pmatrix} = \mathcal{S} \begin{pmatrix} c \\ \mathbf{c} \end{pmatrix}, \tag{2.38}$$

Here $\mathbf{c} = (c_{00}, -c_{10}, -c_{01}, c_{20}, c_{11}, c_{02})^T$, $\mathbf{b} = (b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})^T$ and the matrix $\mathcal{S}$ is arbitrary Hermitian matrix in $\mathbb{C}^7$. Thus, the number of parameters characterizing the zero-range potential for the operator $\mathcal{H}_3^S$ is 49. Similarly to the case of $\mathcal{H}_2^S$ matrix $\mathcal{S}$ has block structure (2.34).

2.4.4 *Zero-range potentials for the plate with infinite crack*

Consider the zero-range potentials for the operator B_c which describes vibrations of thin elastic plate cut along the line $\{x = 0\}$. For the operators A_2, A_3, B_1 and B_3 the most wide class of selfadjoint perturbations was considered. For this the initial operator was restricted to such functions that vanish in a chosen point together with all derivatives which are defined. However it is possible to deal with partial restrictions and obtain a subset of selfadjoint perturbations. For the operator B_c only six-dimensional set of perturbations is constructed below. The restricted operator is defined on

$$\mathrm{Dom}(B_c^\circ) = \Big\{u \in W_2^4(\mathbb{R}_+^2) \oplus W_2^4(\mathbb{R}_-^2) \;:\; \mathbb{F}u|_{x=\pm 0} = 0,$$
$$\mathbb{M}u|_{x=\pm 0} = 0,\ u(\pm 0, 0) = 0,\ \frac{\partial u(\pm 0,0)}{\partial x} = 0,\ \frac{\partial u(\pm 0,0)}{\partial y} = 0\Big\}.$$

The deficiency indices of B_c° are $(6,6)$ and functions from $\mathrm{Dom}((B_c^\circ)^*)$ have the asymptotics for $\rho \to 0$ of the form

$$\begin{aligned} u \sim{}& c_0^\pm \frac{\rho^2}{4\pi}\left(\ln\rho + \frac{1+\sigma}{1-\sigma}\Big(\ln\rho\cos 2\varphi - (\varphi \mp \pi/2)\sin 2\varphi\Big)\right) \\ &+ c_1^\pm \frac{\rho}{2\pi}\left(\frac{2\ln\rho}{1-\sigma}\cos\varphi - \frac{1+\sigma}{1-\sigma}(\varphi \mp \pi/2)\sin\varphi\right) \\ &+ c_2^\pm \frac{\rho}{2\pi}\left(\frac{2\ln\rho}{1-\sigma}\sin\varphi + \frac{1+\sigma}{1-\sigma}(\varphi \mp \pi/2)\cos\varphi\right) \\ &\quad + b_0^\pm + b_1^\pm \rho\cos\varphi + b_2^\pm \rho\sin\varphi + \ldots, \qquad \pm\pi \ge \pm\varphi \ge 0. \end{aligned} \tag{2.39}$$

Here the terms with coefficients $c_j^{\pm}$ are singular terms due to six deficiency elements and the terms denoted by dots belong to the domain of the restricted operator. The boundary form of $(B_c^\circ)^*$ can be found equal to the following expression involving the coefficients of the asymptotics (2.39)

$$\mathcal{I}[u,v] = \sum_{j=0}^{2}(-1)^j \left(c_j^+(u)\overline{b_j^+(v)} + c_j^-(u)\overline{b_j^-(v)} \right.$$
$$\left. - b_j^+(u)\overline{c_j^+(v)} - b_j^-(u)\overline{c_j^-(v)}\right).$$

The restriction of $(B_c^\circ)^*$ that makes the operator selfadjoint can be defined by setting system of equations for the coefficients of the asymptotics (2.39). All such perturbations are parameterized by Hermitian matrices from $\mathbb{C}^6$ in the form

$$(b_0^+, b_1^+, b_2^+, b_0^-, b_1^-, b_2^-)^T = \mathbf{S}(c_0^+, -c_1^+, c_2^+, c_0^-, -c_1^-, c_2^-)^T. \tag{2.40}$$

Similarly to the above arbitrary interchange of vectors components is allowed in (2.40).

The operator of the problem for fluid loaded elastic plate can be constructed similarly to Section 2.4.3. The initial operator of fluid loaded plate with infinite crack can be written in matrix form

$$\mathcal{H}_c = \begin{pmatrix} -c_a^2\Delta & 0 \\ \dfrac{\varrho_0}{\varrho h}\Pi & \dfrac{D}{\varrho h}B_c \end{pmatrix},$$

$$\mathrm{Dom}(\mathcal{H}_c) = \left\{ U \in W_2^2(\mathbf{R}_+^3),\ u \in \mathrm{Dom}(B_c),\ \Pi\,\frac{\partial U}{\partial z} = u \right\}.$$

The zero-range potentials for the operator $\mathcal{H}_c$ can be constructed according to the same scheme of [58] as it was done in the previous sections for other differential operators. Among these zero-range potentials there is a subset of zero-range potentials that are presented only in the elastic component. Only such zero-range potentials are used below. The selfadjoint operator $\mathcal{H}_c^S$ is defined on functions $\mathcal{U} = (U, u)$ such that U is bounded at the origin and u has the asymptotics (2.39) with coefficients satisfying the system (2.40) with the given matrix $\mathbf{S}$.

Chapter 3

Generalized point models

3.1 Shortages of classical point models and the general procedure of generalized models construction

The point models discussed in Chapter 1 allow an explicit form for the representation of the field. This simplifies analysis of many physical effects. The obstacle in these models is specified by boundary-contact conditions formulated in its central point. That is one deals with the same basic idea that allowed the Lame equations of elasticity in thin layer to be reduced to the biharmonic equation of Kirchhoff model for flexural displacements. Evidently that point models can be used only if the size of real obstacle is sufficiently small. However problems of diffraction for fluid loaded thin elastic plates contain many parameters and the applicability question for a particular point model should be studied. The analysis of applicability of the point model of stiffener is performed in [22] where the stiffener is assumed of finite height and the correction to the point model is found (see also Section 4.4). The applicability of the pointwise crack model was studied only recently [20]. It was discovered that the classical point model correctly represents the far field amplitude only under very restrictive conditions (when the width of the crack is exponentially smaller than the thickness of the plate and at nonorthogonal direction of observation and incidence). Therefore the point model of crack studied in Chapter 1 appears hardly applicable to real problems of scattering and should be corrected.

We noticed also in Chapter 1 that the set of point models in three dimensional problems of scattering by fluid loaded plates is rather small. Namely it is restricted to such models for which contact conditions only contain displacement and possibly the force in a point. These are the model of attached mass and its limiting case for infinitely large mass corresponding to fixed point. The cracks or holes with free edges are described by other types of contact conditions and can not be represented by classical point models. This lack of models takes place in the case of isolated plates as well.

The technique of zero-range potentials allows the set of solvable models to be enlarged and more precise models of obstacles to be formulated both in the case of isolated and fluid loaded plates. To apply the technique of zero-range potentials first notice the connection between the problems of scattering by compact obstacles in the presence of thin elastic plate and selfadjoint operators in appropriate Hilbert spaces. Consider the problem for an operator A in $L_2(\mathbb{R}^d)$

$$AU = V. \tag{3.1}$$

Let the operator A be a differential operator and the boundary conditions be involved as the conditions imposed on the domain of the operator. This problem differs from the boundary-value problem of diffraction theory by replacement of conditions $U, V \in L_2(\mathbb{R}^d)$ by radiation conditions for U. In spite of this difference, however, there is much in common. Namely as was noted in Chapter 2 if the boundary-value problem is such that conditions on the obstacle yield $E_\Omega = 0$ (see (1.17) for definition of E_Ω), then the operator A is selfadjoint. Oppositely if the operator A is selfadjoint, then the boundary-value problem that corresponds to it is mathematically correct and its solution satisfies all basic physical properties such as reciprocity principle and balance of energy (or energy conservation law) expressed by optical theorem.

Intuitively it is natural that if the conditions specified at finite distances (boundary and contact conditions) are mathematically correct, then changing conditions specified at infinitely large distances from one mathematically correct (square integrability) to another mathematically correct condition (radiation condition) can not spoil correctness of the entire problem. In another words one can consider all objects locally in L_2 on any bounded domain. Rigorous mathematical justification of that connection uses spectral theory of operators (eigenfunctions of continuous spectrum) and scattering theory. We address interested readers to classical books [50] and [54].

We present here some nonrigorous explanations of why the operator A should be selfadjoint in order the boundary-value problem be correctly set and its solution satisfies such basic physical properties as reciprocity principle and energy balance expressed by optical theorem. Consider the problem (3.1). Function V is the source of the field and function U is the solution to be found. Further, let W be the characteristics of the observer, that is let the measured value of the field be $\langle U, \overline{W} \rangle$. Perform some transformations in this scalar product using symmetric property of operator A

$$\langle U, \overline{W} \rangle = \langle A^{-1}V, \overline{W} \rangle = \langle V, A^{-1}\overline{W} \rangle = \langle A^{-1}W, \overline{V} \rangle .$$

This equality expresses the reciprocity principle. One can interchange the source and the observer. Letting W_n and V_n be delta-type sequences and taking the limit in the above equality states the symmetry of Green's function.

Now we try to explain why the operator A should be not only symmetric, but selfadjoint*. If the operator A is symmetric, but not selfadjoint, then $\mathrm{Res}(A)^{\perp}$ is not empty and A^{-1} (which is defined on $\mathrm{Res}(A)$) is defined on not dense set. Therefore, for example, if the source field $V \in \mathrm{Res}(A)^{\perp}$, then no solution U exists.

Thus in order to get mathematically correct boundary-value problem one should start with selfadjoint operator in an appropriate Hilbert space.

Classical point models appear as boundary conditions formulated in a separate point. In an isolated one-dimensional plate these conditions describe all possible zero-range potentials B_1^S. The zero-range potentials for the operator $\mathcal{H}_2$ of fluid loaded plate are parameterized by Hermitian matrices in $\mathbb{C}^5$ in the form (2.33). The boundary-contact conditions of classical point models appear to be a subset of conditions (2.33). In the case of two-dimensional plate the zero-range potentials for the operator B_2 are parameterized by Hermitian matrices from $\mathbb{C}^6$ in the form of conditions (2.27). The variety of such conditions is certainly more reach that the only model of attached mass (1.67). For fluid loaded plates in three dimensions the generalized point models are parameterized by Hermitian matrices from $\mathbb{C}^7$ in the form (2.38). Again the conditions of classical point models form a subset of conditions (2.38).

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*Often in techniques and physics these two types of operators are not distinguished as different because the domain is not included in the definition of the operator.

All the above shows that the discussed in Chapter 2 zero-range potentials are the generalizations of classical point models of hydroelasticity. The main idea of the generalized models construction both in two and three dimensions for fluid loaded and for isolated plates is in the choice of such zero-range potential that the field scattered by it approximately coincides with the field scattered by the real obstacle, and this coincidence is desirable as close as possible. Generally speaking one can hope to achieve better coincidence than in the case of classical point models. The main question that appears in the procedure of generalized models construction is the choice of the Hermitian matrices that define the particular zero-range potential. The block structure of the matrix $\mathcal{S}$ presented in (2.34) and physical interpretation of the parameters allow the following HYPOTHESIS to be suggested:

Hypothesis

Let the zero-range potential of the operator A_d^S ($d = 2, 3$ is the dimension of the problem) approximate the scattering process on an obstacle Ω in acoustic media bounded by infinitely rigid screen and let the zero-range potential B_{d-1}^S approximate the scattering process on the obstacle Ω in isolated elastic plate. Then the parameters of the zero-range potential that approximates scattering process on the obstacle Ω in fluid loaded elastic plate are given by the matrix

$$\begin{pmatrix} S & s \\ s^* & \mathbf{S} \end{pmatrix},$$

where S is the parameter of A_d^S, $\mathbf{S}$ is the matrix characterizing B_{d-1}^S and vectors s and s^* are zero vectors.

We can not prove this HYPOTHESIS for the general case, it will be checked for some particular obstacles. In Section 3.2 the problem of scattering by a crack of small, but finite width in fluid loaded plate is considered. This is a two-dimensional problem which is studied in [20] and by zero-range potential approach in [9]. The generalized model of narrow crack is suggested and examined. In order to choose the parameters two auxiliary problems are considered. These are the diffraction by a narrow slit in absolutely rigid screen and the problem of scattering by the crack in isolated plate. (These problems are the limiting cases of the original boundary-value problem for $D \to \infty$ and for $\varrho_0 \to 0$ correspondingly). Analysis of the far field amplitudes of scattered waves in these two problems allows the zero-range potentials A_2^S and B_1^S to be chosen. According to the HYPOTHESIS this determines the element $\mathcal{S}_{11} = S$ and the block $\{\mathcal{S}_{ij}\}_{i,j\geq 2} = \mathbf{S}$ of the complete zero-range potential $\mathcal{H}_2^S$. The problem of scattering by this zero-range potential for fluid loaded plate is solved in closed form. In Section 3.2.6 the problem of diffraction by a narrow crack is examined by the classical approach based on integral equations. The derived asymptotics for the far field amplitude is compared to the results of Section 3.2.5 which allows the applicability of the above HYPOTHESIS in the case of narrow crack to be justified.

Section 3.3 deals with the model of short crack in three-dimensional problem. First (Section 3.3.1) the case of isolated plate is considered. By means of Fourier transform the problem is reduced to integral equations which are then analyzed numerically and asymptotically. Comparing the far field amplitude asymptotics with the fields scattered by the zero-range potentials B_2^S allows the generalized model of short crack in an isolated plate to be suggested (Section 3.3.2). Further this model is used in the model of short crack in fluid loaded plate. Evidently that for absolutely rigid plate an infinitely thin crack is not noticeable. Indeed, such crack is characterized by contact conditions only. That is, the conditions on the crack are written for the displacements which are identically equal to zero in absolutely rigid plate. Thus the parameter S in the model of short crack should be taken equal to zero. According to the HYPOTHESIS the other parameters are taken the same as in the case of isolated plate. The field scattered by the generalized model of short crack is found in explicit form. It is compared to the asymptotics of the far field amplitude derived in Section 3.3.4 by means of integral equations approach. Again the approach suggested by the above HYPOTHESIS is justified by this comparison.

For further models (in Sections 3.4 and 3.5) of a small round hole and of a short joint of semi-infinite plates the HYPOTHESIS is not checked, but similarity of these obstacles with that considered in Sections 3.2 and 3.3 allows one to believe in its applicability.

It is useful to note that generalized models reproduce only some of the characteristics of the real scattering process. They give far field amplitudes and some of the spectral properties, namely those associated with far fields. The generalized models cause nonphysical singularities to appear at the potential canter. Therefore these models give wrong results for near fields. Generally speaking classical point models though not bringing to singularities of the field do not describe near fields as well. This is seen for example from the analysis of the applicability conditions for Kirchhoff model of flexure waves in thin elastic plates and unjustified extension of the formulae for the force and bending momentum up to the edge or support of the plate.

3.2 Model of narrow crack

3.2.1 *Introduction*

Consider the problem of scattering by a straight crack in a fluid loaded plate. The plate is supposed to be in one-side contact with fluid and the incident field is assumed independent on the coordinate y measured along the crack. Therefore the problem appears two-dimensional and the zero-range potentials for such problems are constructed in Section 2.4.3. These zero-range potentials are parameterized by matrices from $\mathbb{C}^5$ in the form (2.33). The matrices of parameters have block diagonal structure and according to the main HYPOTHESIS proclaimed in the previous section the blocks can be found separately. The first block consisting of a single element $\mathcal{S}_{11} = S$ can be found if the solution of the problem of scattering by a slit in absolutely rigid screen is constructed and analyzed. This problem is well known (see e.g. [36]). Nevertheless its analysis is presented in Section 3.2.2. The other diagonal block, consisting of elements $\mathcal{S}_{ij}$, $i, j = 2, 3, 4, 5$ is the matrix $\mathbf{S}$ that can be found if the problem of scattering in the isolated plate is considered. This problem is very simple as it deals with ordinary differential operator. It is studied in Section 3.2.3. When all the parameters of the matrix $\mathcal{S}$ from (2.33) are defined, one can consider the problem of scattering by the zero-range potential that models the narrow crack. The corresponding problem is studied in Section 3.2.4 and the explicit solution is constructed. Section 3.2.5 deals with the exact formulation of the scattering problem on the crack of finite width. The asymptotics of the far field amplitude constructed in this section allows the main HYPOTHESIS to be checked.

3.2.2 *The case of absolutely rigid plate*

Consider the acoustic media $\{-\infty < x < \infty,\ z > 0\}$ bounded by the absolutely rigid screen $\{-\infty < x < \infty,\ z = 0\}$ with the slit at $-a < x < a$. Let the source of the field be the plane wave

$$U^{(i)} = A \exp\big(ikx\cos\vartheta_0 - ikz\sin\vartheta_0\big).$$

Then the total field occurs independent of co-ordinate y and the problem of scattering is actually two-dimensional. It is formulated as follows. The acoustic pressure U satisfies the Helmholtz equation

$$\left(\Delta + k^2\right) U(x,z) = 0, \qquad z > 0$$

with the boundary conditions

$$U(x,0) = 0, \qquad |x| < a,$$
$$\frac{\partial U(x,0)}{\partial z} = 0, \qquad |x| > a.$$

Except for the incident plane wave all other components of the field carry energy only to infinity, that is, satisfy the radiation condition

$$\left(\frac{\partial U}{\partial r} - ikU\right) = o\left(r^{-1/2}\right), \quad r \to \infty$$

The field satisfies the Meixner conditions (1.18) at points $x = \pm a,\ z = 0$ where the boundary condition changes.

The above problem is well known. It can be solved by variables separation method in elliptic coordinates which allows the solution to be represented in the form of decomposition by Mathieu functions. Though such representation is exact, it may appear inconvenient for computations. Below following [36] the asymptotic solution for small width $ka \ll 1$ is constructed.

The problem can be reduced to Rayleigh integral equation. Separating geometrical part of the field $U^{(g)}$ consisting of the incident field and the field reflected from a homogeneous rigid screen allows the problem to be formulated for the scattered field $U^{(s)} = U - U^{(g)}$. Noting that the Green's function for the case of Neumann boundary condition on the plate is the Bessel function of the third kind

$$G_N(x,z;x_0,0) = \frac{i}{2} H_0^{(1)}\left(k\sqrt{(x-x_0)^2 + z^2}\right), \tag{3.2}$$

one obtains the integral representation for the scattered field in the form similar to (1.33)

$$U^{(s)}(x,z) = -\frac{i}{2}\int_{-a}^{a}\phi(x_0)H_0^{(1)}\left(k\sqrt{(x-x_0)^2+z^2}\right)dx_0.$$

The function $\phi(x_0)$ is defined from the integral equation

$$\int_{-a}^{a}\phi(x_0)H_0^{(1)}(k|x-x_0|)dx_0 = -2iU^{(g)}(x), \qquad |x|<a,$$

which appears when the above representation is substituted into the Dirichlet boundary condition on the slit. The function $\phi(x_0)$ coincides with the normal derivative of the total field in the slit. Near the edges of semi-infinite screens it can have weak singularities which are allowed by Meixner conditions (1.18).

It is convenient to perform scaling of co-ordinates in such a way that the crack becomes of unit semi-width. That is let co-ordinate t be introduced as $t = x/a$. In new co-ordinates the integral equation takes the form

$$\int_{-1}^{1}\phi(at_0)H_0^{(1)}(ka|t-t_0|)dt_0 = -2ia^{-1}U^{(g)}(at), \qquad |t|<1.$$

The kernel of the above integral equation has logarithmic singularity at $t = t_0$ which can be concluded from the asymptotics [1]

$$H_0^{(1)}(t) \sim \frac{2i}{\pi}\left(\ln(t/2) - i\frac{\pi}{2} + C_E\right),$$

where $C_E = 0.5772\ldots$ is the Euler constant. Replacing the kernel by this asymptotics and using the formula (B.3) allows the solution $\phi(x_0)$ to be found in the principal order

$$\phi(x_0) = \frac{1}{\sqrt{a^2-x_0^2}}\frac{1}{\ln(ka/4)+C_E-i\pi/2}.$$

Substituting this expression into the integral representation and applying the saddle point method (1.22) yields the far field amplitude

$$\Psi^{\circ}(\vartheta) = \frac{i}{\ln(ka/4)+C_E-i\pi/2} + \ldots \qquad (3.3)$$

3.2.3 *The case of isolated plate*

The other limiting case of diffraction by a narrow crack in an isolated plate is described by ordinary differential equations. The two semi-infinite plates at $x < -a$ and at $x > a$ are completely independent. Let an incident wave be running from $x = -\infty$

$$w^{(i)} = e^{ik_0 x}$$

and be reflected from the free edge at $x = -a$. It is easy to find that the total field is the sum of the incident and reflected propagating waves and the reflected exponentially decreasing wave (compare with formula (1.71) valid for the case of oblique incidence)

$$w = e^{ik_0 x} + r_a e^{-ik_0 x} + t_a e^{+k_0 x}, \qquad x < -a.$$

The coefficients r_a and t_a in the above formula can be found from the boundary conditions

$$w''(-a) = 0, \qquad w'''(-a) = 0.$$

The exact expressions are

$$r_a = -\frac{1}{2}(1-i)^2 e^{-2ik_0 a}, \qquad t_a = (1+i)e^{k_0 a(1-i)}.$$

Assuming $k_0 a$ to be small the reflection coefficient reduces to the reflection coefficient (1.71) from the point crack at $x = 0$, that is

$$r_a \approx r(\pi/2) = i.$$

In particular the above formula means that contact conditions in the model of narrow crack in isolated plate can be taken the same as for pointwise crack.

3.2.4 *Generalized point model of narrow crack*

The formulae of the above sections allow the parameters of the generalized point model to be defined. Following the HYPOTHESIS declared in Section 3.1 the matrix $\mathcal{S}$ that specifies the zero-range potential in the composite operator $\mathcal{H}_2^S$ is composed of blocks corresponding to the zero-range potentials of acoustic and elastic components. These parameters are $\mathcal{S}_{11} = S$ that sets proportionality of the coefficients in the asymptotic decomposition of acoustic pressure and the contact conditions that are specified for flexure displacement. The latter can be taken the same as in the case of pointwise crack. Thus, to determine the operator $\mathcal{H}_2^S$ describing scattering by narrow crack one needs to find the only parameter S. For that consider the problem of scattering for the operator A_2^S. This problem is in finding such solution of the equation

$$\Delta U^{(s)} + k^2 U^{(s)} = 0, \qquad (x, y) \neq (0, 0), \tag{3.4}$$

that satisfies the Neumann boundary condition on the screen and being combined with the incident $U^{(i)}$ and reflected $U^{(r)}$ fields has the asymptotics (2.19)

$$U = U^{(i)} + U^{(r)} + U^{(s)} \sim c\left(-\frac{1}{\pi}\ln r + S\right) + \ldots, \qquad r \to 0.$$

Let $U^{(i)}$ be the incident plane wave

$$U^{(i)} = \exp(ikx\cos\vartheta_0 - ikz\sin\vartheta_0),$$

then the reflected field $U^{(r)}$ is

$$U^{(r)} = \exp(ikx\cos\vartheta_0 + ikz\sin\vartheta_0).$$

The solution of the Helmholtz equation (3.4) that satisfies the radiation condition and has square integrable singularity is defined up to a multiplier

$$U^{(s)} = c\frac{i}{2}H_0^{(1)}(kr).$$

The asymptotics of the total field is

$$U = -c\frac{1}{\pi}\ln r - c\left(\frac{C_E + \ln(k/2)}{\pi} - \frac{i}{2}\right) + 2 + o(1), \quad r \to 0.$$

Comparing it with (2.19) yields

$$-c\left(\frac{C_E + \ln(k/2)}{\pi} - \frac{i}{2}\right) + 2 = Sc. \tag{3.5}$$

The parameter S in this equation should be chosen such that the far field amplitude of U^s approximately coincides with (3.3), that is

$$\frac{ic}{2\pi} \approx \frac{i}{\ln(ka/4) + C_E - i\pi/2}$$

Therefore

$$c \approx \frac{2\pi}{\ln(ka/4) + C_E - i\pi/2}$$

and the equation (3.5) defines the parameter S. We neglect smaller order terms by ka and define S uniquely. Namely

$$S = \frac{1}{\pi}\ln(a/2).$$

In this particular case the requirement $\operatorname{Im} S = 0$ is satisfied automatically. It is important to note that the parameters of the incident field (frequency, angle of incidence, amplitude) and of the media (sound velocity c_a) are not involved in the parameter S. This observation allows one to hope that the same parameter S can be used in other problems of scattering by a soft segment of length $2a$. In particular according to the HYPOTHESIS we use this parameter in the generalized model of narrow crack in fluid loaded elastic plate.

Conditions that fix the *generalized model of narrow crack* are formulated as the asymptotics of acoustic field

$$U \sim -\frac{c}{\pi}\ln(2r/a) + o(1), \qquad r \to 0 \tag{3.6}$$

with arbitrary c and the contact conditions

$$w''(\pm 0) = 0, \qquad w'''(\pm 0) = 0 \tag{3.7}$$

for displacements of the plate.

3.2.5 *Scattering by point model of narrow crack*

Now consider the problem of scattering by the zero-range potential in the fluid-loaded plate which is parameterized by conditions (3.6) and (3.7). Find the scattered field $U^{(s)}$ that is generated by an incident plane wave. The field is searched for in the form

$$U^{(s)} = cG(x, z; \omega) + \sum_{\ell=0}^{3} c_\ell \frac{\partial^\ell}{\partial x_0^\ell} G_0(x, z; \omega), \tag{3.8}$$

where $G(x, z; \omega)$ and $G_0(x, z; \omega)$ are defined by the formulae (2.29) and (2.30). These functions are the limiting values of the deficiency elements for $\operatorname{Im}\omega \to +0$ and up to a multiplier coincide with the Green's functions

$G(x, z; x_0, z_0)$ and $G(x, z; x_0)$ presented by formulae (1.28) and (1.31). The representation (3.8) is the general solution (see page 36) that satisfies the Helmholtz equation and the boundary condition on the homogeneous plate. Additionally to representation (1.53) a point source of acoustic pressure appears in (3.8). The parameters c and c_ℓ in (3.8) should be chosen such that the conditions (3.6) and (3.7) are satisfied for the sum of the fields $U^{(g)}$ and $U^{(s)}$.

The coefficients of the representation (3.8) can be found from the algebraic system that appears when (3.8) is substituted into the conditions (3.6), (3.7). Singular solutions $\mathcal{G}$ and $\mathcal{G}_\ell$ are normalized in such a way that the coefficients c and c_ℓ in the representation (3.8) and in the asymptotic decompositions (2.31) coincide. Singular solutions manifest themselves also in smooth terms. To calculate the coefficients b and b_ℓ asymptotic decompositions of the integrals (2.29) and (2.30) should be examined. The asymptotic decompositions of the Fourier integrals for $r \to 0$ and $x \to 0$ are defined by the behavior of the transform at the infinity. Consider first the asymptotics of the component $G(\mathbf{r}; \omega)$. It is convenient to exclude the Green's function $G_N(x, y; 0, 0)$ corresponding to the case of absolutely rigid screen (with Neumann boundary condition on it). This function given by formula (3.2) can be rewritten in the form of Fourier integral

$$G_N = \frac{1}{2\pi} \int e^{i\lambda x - \sqrt{\lambda^2 - k^2} z} \frac{d\lambda}{\sqrt{\lambda^2 - k^2}}.$$

It is a simple matter to find that the correction $G - G_N$ is a smooth function. This yields asymptotic decomposition

$$G(x, y; \omega) = -\frac{1}{\pi} \ln(r) - \frac{1}{\pi} (\ln(k/2) + C_E) + \frac{i}{2} + N\, J + \dots$$

Here the following integral is introduced

$$J = \frac{1}{2\pi} \int \frac{d\lambda}{L(\lambda)\sqrt{\lambda^2 - k^2}}. \tag{3.9}$$

The component $g(x; \omega)$ of that singular solution can be decomposed in Taylor series. Its derivatives up to the third order are bounded. Therefore

$$g(x; \omega) = -\sqrt{N} \left(J_0 + J_2 \frac{x^2}{2} + \dots \right),$$

where

$$J_\ell = \frac{1}{2\pi}\int \frac{(i\lambda)^\ell}{L(\lambda)}d\lambda, \quad \ell = 0, 2. \tag{3.10}$$

Consider now the asymptotics of the singular solutions $\mathcal{G}_\ell$. The components $G_\ell(\mathbf{r};\omega)$ are smooth functions and one finds

$$G_0(0,0) = -\sqrt{N}\, J_0, \quad G_1(0,0) = 0, \quad G_2(0,0) = -\sqrt{N}\, J_2, \quad G_3(0,0) = 0.$$

The components $g_\ell(x;\omega)$ were already studied in Section 1.4.1 These are discontinuous functions. By direct substitution the asymptotics of $g_\ell(x;\omega)$ is expressed in terms of contact integrals D_ℓ introduced in (1.54) as

$$D_\ell = \frac{1}{2\pi}\int e^{+i0\lambda}(i\lambda)^\ell \frac{\sqrt{\lambda^2-k^2}}{L(\lambda)}\, d\lambda.$$

Using properties of integrals D_ℓ described in the Appendix A, one finds the asymptotics

$$\begin{aligned}
g_0(x;\omega) &= D_0 + \frac{x^2}{2}D_2 + \frac{x^3}{12}\mathrm{sign}(x) + O(x^4),\\
g_1(x;\omega) &= xD_2 + \frac{x^2}{4}\mathrm{sign}(x) + \frac{x^3}{6}D_4 + O(x^4),\\
g_2(x;\omega) &= D_2 + \frac{|x|}{2} + \frac{x^2}{2}D_4 + O(x^4),\\
g_3(x;\omega) &= \frac{\mathrm{sign}(x)}{2} + xD_4 + \frac{x^3}{6}D_6 + O(x^4).
\end{aligned}$$

With the help of the above decompositions for the scattered field one can find that the total field has the asymptotic behaviour (2.31) near the zero-range potential center. The coefficients of singular terms coincide with that in the representation (3.8) and the coefficients of regular terms are the following

$$\begin{aligned}
b &= -c\left\{\frac{\ln(k/2)+C_E}{\pi} - \frac{i}{2} - N\,J\right\} - \sqrt{N}\Big(c_0J_0 + c_2J_2\Big) + b^{(g)},\\
b_0 &= -\sqrt{N}\,c\,J_0 + c_0D_0 + c_2D_2 + b_0^{(g)}, \quad b_1 = c_1D_2 + c_3D_4 + b_1^{(g)},\\
b_2 &= -\sqrt{N}\,c\,J_2 + c_0D_2 + c_2D_4 + b_2^{(g)}, \quad b_3 = c_1D_4 + c_3D_6 + b_3^{(g)}.
\end{aligned}$$

Here $b^{(g)}$ and $b_\ell^{(g)}$ are the geometrical part contributions to the corresponding coefficients of representation (2.33).

Substitution of these formulae into conditions (2.33) that fix the zero-range potential yields algebraic system for coefficients c and c_ℓ. Introducing matrix

$$\mathcal{Z}(\omega) = \begin{pmatrix} -Z & -\sqrt{N}\,J_0 & 0 & -\sqrt{N}\,J_2 & 0 \\ -\sqrt{N}\,J_0 & D_0 & 0 & D_2 & 0 \\ 0 & 0 & D_2 & 0 & D_4 \\ -\sqrt{N}J_2 & D_2 & 0 & D_4 & 0 \\ 0 & 0 & D_4 & 0 & D_6 \end{pmatrix},$$

where

$$Z = \frac{1}{\pi}\Big(\ln(k/2) + C_E\Big) - \frac{i}{2} - N\,J,$$

this system can be written in the form

$$\Big(\mathcal{S} - \mathcal{Z}(\omega)\Big)\begin{pmatrix} c \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} b^{(g)} \\ \mathbf{b}^{(g)} \end{pmatrix}. \tag{3.11}$$

It is important to note the structure of the system (3.11). The matrix $\mathcal{S}$ defines the scattering object. It does not depend upon the incident field and the parameters that characterize properties of acoustic medium and elastic plate far from the scattering object. All the information on the incident field is concentrated in vector $(b^{(g)}, \mathbf{b}^{(g)})^T$. The mechanical system characteristics are involved only in matrix $\mathcal{Z}(\omega)$. Therefore it is natural to suggest the main HYPOTHESIS . Indeed taking the limit for $D \to \infty$ that corresponds to the case of absolutely rigid screen affects only matrix $\mathcal{Z}(\omega)$ and matrix $\mathcal{S}$ remains unchanged. Another limit for $\varrho_0 \to 0$ also transforms only matrix $\mathcal{Z}(\omega)$. Note also that for $\omega = i$ the integrals J, J_ℓ and D_ℓ are real and matrix $\mathcal{Z}$ becomes Hermitian. That Hermitian matrix $\mathcal{Z}(i)$ could be added to matrix $\mathcal{S}$, but in this case the structure of the system (3.11) would be spoiled and the HYPOTHESIS would be wrong.

The far field asymptotics of the scattered field (3.8) can be found by applying saddle point method to the integrals (2.29) and (2.30). The contribution of the saddle point gives diverging cylindrical wave (1.29) with the far field amplitude

$$\Psi(\vartheta) = \frac{\sqrt{N}}{2\pi}\frac{k\sin\vartheta}{\mathcal{L}(\vartheta)}\Big(\frac{c}{\sqrt{N}}(k^4\cos^4\vartheta - k_0^4) \\ + c_0 + c_1 ik\cos\vartheta - c_2k^2\cos^2\vartheta - c_3ik^3\cos^3\vartheta\Big).$$

The residues in the zeros of the denominator $L(\lambda)$ correspond to surface waves. The amplitudes of these surface waves can be computed by the general formula (1.30).

In the case of the generalized point model of narrow crack the system (3.11) splits. Coefficients c_0 and c_1 in (3.8) are zeros similarly to the case of pointwise crack. Coefficients c, c_2 can be found from the system

$$\begin{cases} \left(Z + \frac{1}{\pi}\ln(a/2)\right) c + \sqrt{N}\, J_2\, c_2 = b^{(g)} \\ \sqrt{N}\, J_2\, c + D_4\, c_2 = -b_2^{(g)} \end{cases}.$$

The first equation follows directly from condition (3.6) and the second equation expresses the condition $u''(-0) + u''(+0) = 0$ which follows from (3.7). One finds

$$c = -\pi \frac{D_4 b^{(g)} + \sqrt{N} J_2 b_2^{(g)}}{D_4(\pi Z + \ln(a/2)) + N J_2^2},$$
$$c_2 = -\frac{b_2^{(g)}}{D_4} + \pi \frac{\sqrt{N} J_2 b^{(g)} + N J_2^2 D_4^{-1}\, b_2^{(g)}}{D_4(\pi Z + \ln(a/2)) + N J_2^2}.$$

The coefficient c_3 is defined from the last condition in (3.7)

$$c_3 = -\frac{1}{D_6} b_3^{(g)}.$$

The above formulae express the amplitudes of passive sources via coefficients $b^{(g)}$, $b_2^{(g)}$ and $b_3^{(g)}$ of the geometrical part of the field. For the case of incident plane wave one finds

$$b^{(g)} = 2A \frac{ik \sin\vartheta_0}{\mathcal{L}(\vartheta_0)}(k^4 \cos\vartheta_0 - k_0^4), \quad b_\ell^{(g)} = -2A\sqrt{N} \frac{ik\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}(ik\cos\vartheta_0)^\ell.$$

This yields

$$\Psi(\vartheta) = A \frac{iN}{\pi} \frac{k^2 \sin\vartheta \sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)} \left\{ \frac{k^4 \cos^2\vartheta \cos^2\vartheta_0}{D_4} - \frac{k^6 \cos^3\vartheta \cos^3\vartheta_0}{D_6} \right.$$
$$\left. - \frac{\pi}{N} \frac{M(\vartheta)\, M(\vartheta_0)}{\ln(ka/4) + C_E - i\pi/2 - \pi N\, J + \pi N J_2^2 D_4^{-1}} \right\}, \tag{3.12}$$

where

$$M(\vartheta) = k^4 \cos^4\vartheta - k_0^4 + \frac{k^2\, N\, J_2}{D_4} \cos^2\vartheta. \tag{3.13}$$

Using formula (1.30) one computes amplitudes of surface waves

$$\psi^{\pm} = A\,\frac{2N\sqrt{\kappa^2-k^2}}{5\kappa^4-4k^2\kappa^2-k_0^4}\,\frac{k\sin\vartheta_0}{\kappa\mathcal{L}(\vartheta_0)}\left\{\frac{\kappa^2k^2\cos^2\vartheta_0}{D_4}\mp\frac{\kappa^3k^3\cos^3\vartheta_0}{D_6}\right. \\ \left. -\frac{\pi}{N}\frac{\left(\kappa^4-k_0^4+NJ_2\kappa^2D_4^{-1}\right)M(\vartheta_0)}{\ln(ka/4)+C_E-i\pi/2-\pi N\,J+\pi NJ_2^2D_4^{-1}}\right\}. \tag{3.14}$$

In the case of incident surface wave

$$U^i = A\exp\left(i\kappa x-\sqrt{\kappa^2-k^2}z\right)$$

only the right-hand sides of the equations for c, c_2 and c_3 change. One finds

$$b^{(g)}=1,\quad b_2^{(g)}=\sqrt{N}\sqrt{\kappa^2-k^2}\kappa^2,\quad b_3^{(g)}=i\sqrt{N}\sqrt{\kappa^2-k^2}\kappa^3,$$

which yields

$$\Psi(\vartheta) = A\,\frac{1}{2\pi}\,\frac{k\sin\vartheta\sqrt{\kappa^2-k^2}}{\mathcal{L}(\vartheta)}\left\{\frac{\kappa^2k^2\cos^2\vartheta}{D_4}-\frac{\kappa^3k^3\cos^3\vartheta}{D_6}\right. \\ \left. -\frac{\pi}{N}\frac{\left(\kappa^4-k_0^4+NJ_2\kappa^2D_4^{-1}\right)M(\vartheta)}{\ln(ka/4)+C_E-i\pi/2-\pi N\,J+\pi NJ_2^2D_4^{-1}}\right\} \tag{3.15}$$

and

$$\psi^{\pm} = A\,\frac{i}{\kappa}\,\frac{\kappa^2-k^2}{5\kappa^4-4k^2\kappa^2-k_0^4}\left\{\frac{\kappa^4}{D_4}\mp\frac{\kappa^6}{D_6}\right. \\ \left. -\frac{\pi}{N}\frac{\left(\kappa^4-k_0^4+NJ_2\kappa^2D_4^{-1}\right)^2}{\ln(ka/4)+C_E-i\pi/2-\pi N\,J+\pi NJ_2^2D_4^{-1}}\right\}. \tag{3.16}$$

The far field amplitudes and the amplitudes of scattered surface waves given by the formulae (3.12), (3.14), (3.15) and (3.16) are reciprocal and exactly satisfy the optical theorem (1.42). (For the discussion of reciprocity of the formulae (3.14) and (3.15) see page 39). Two terms in these formulae coincide with the corresponding solution of scattering problem by pointwise crack. Detailed discussion of the above formulae and numerical results are presented in Section 3.2.7.

3.2.6 *Diffraction by a crack of finite width in fluid loaded elastic plate*

In this section the same problem of plane wave scattering by a submerged plate weakened by a crack of small width is examined by classical asymptotic approach. With the use of integral representation (1.33) the problem is reduced to a system of integro-algebraic equations. Then this system is analysed asymptotically with respect to small parameter $ka \ll 1$. The asymptotics of the far field amplitude is derived. The results of this section allow the main HYPOTHESIS to be checked.

Formulation of the problem

Let the acoustic system consist of the homogeneous liquid half space $\{z > 0\}$ bounded by thin elastic plate $\{z = 0\}$ with the crack $\{|x| < a, z = 0\}$. The field in the system satisfies Helmholtz equation in the halfspace

$$\triangle U + k^2 U = 0, \quad z > 0$$

and the boundary condition on the plate

$$\left(D\frac{d^4}{dx^4} - \varrho\omega^2 h \right) w + U = 0, \quad |x| > a,\, z = 0,$$

where $w = \varrho_0^{-1}\omega^{-2}\partial U/\partial z|_{z=0}$ is flexural displacement of the plate. The edges of semi-infinite plates are supposed to be free of forces and bending momentums. That is expressed by the boundary-contact conditions

$$w''(\pm a) = 0, \quad w'''(\pm a) = 0. \tag{3.17}$$

Inside the crack the Dirichlet boundary condition

$$U = 0, \quad |x| < a,\, z = 0,$$

is assumed. This condition describes the free surface of fluid. Meixner conditions (1.18) near the points $\{x = \pm a,\ z = 0\}$, where the boundary condition changes, guarantee the finiteness of energy in any compact domain in acoustic medium.

Note that letting $a = 0$ yields classical model of pointwise crack.

Let the wave field be generated by the incident plane wave

$$U^{(i)} = A \exp\left(ik\left(x\cos\vartheta_0 - z\sin\vartheta_0\right)\right),$$

The field $U^{(g)}$ that would be generated if the plate has no crack is known

$$U^{(g)} = U^{(i)} + A\,R(\vartheta_0)\exp\left(ik\left(x\cos\vartheta_0 + z\sin\vartheta_0\right)\right).$$

The reflection coefficient $R(\vartheta_0)$ is given by formula (1.25). The scattered field $U^{(s)} = U - U^{(g)}$ satisfies radiation condition (1.16).

Reducing the problem to integral equations

The boundary value contact problem for the scattered field $U^{(s)}$ differs from that presented above for the total field by introducing inhomogeneous boundary condition

$$U^{(s)} = -U^{(g)}, \quad -a < x < a,\ z = 0, \tag{3.18}$$

and inhomogeneous contact conditions

$$\frac{d^n w^{(s)}}{dx^n} = -\frac{d^n w^{(g)}}{dx^n}, \quad x = \pm a, n = 2, 3. \tag{3.19}$$

Here the values $w^{(s)}$ and $w^{(g)}$ denote displacements corresponding to the fields $U^{(s)}$ and $U^{(g)}$.

Further derivations use the Green's function $G(x, z; x_0, z_0)$ for homogeneous plate (without crack) defined by (1.28). The integral representation for the scattered field can be derived with the help of Green's formula written for functions $U^{(s)}$ and $G(x, z; x_0, z_0)$ in the semicircle of large radius. In the considered case this representation takes the form

$$\begin{aligned} U^{(s)}(x_0, z_0) = &\int_{-a}^{a} \frac{\partial U(x,0)}{\partial z} G(x, 0; x_0, z_0)dx \\ &+ \varrho_0\omega^2 D\Big(w(a)\frac{\partial^3 g(a; x_0, z_0)}{\partial x^3} - w(-a)\frac{\partial^3 g(-a; x_0, z_0)}{\partial x^3} \\ &- w'(a)\frac{\partial^2 g(a; x_0, z_0)}{\partial x^2} + w'(-a)\frac{\partial^2 g(-a; x_0, z_0)}{\partial x^2}\Big). \end{aligned} \tag{3.20}$$

This integral representation is the particular form of the representation (1.33) in the case when fields do not depend on the third coordinate y and the obstacle that causes diffraction occupies the segment $[-a, a]$ on the plate. Besides the substitutes like $w''(a)g(a, 0; x_0, z_0)$ are dropped out due to absence of forces and bending momentums on the edges of the plates specified by conditions (3.17). The reciprocity principle for the Green's functions reads $G(x, z; x_0, z_0) = G(x_0, z_0; x, z)$ and $g(x; x_0, z_0) =$

$G(x_0, z_0; x)$. We remind that the fields of point acoustic source and of point source applied to the plate are distinguished by the number of arguments $G(x, z; x_0, z_0)$ and $G(x, z; x_0)$ correspondingly. This allows the representation (3.20) to be interpreted as the sum of fields generated by acoustic sources distributed along the segment (first term) and by point sources applied to the plate in points $x = \pm a$. If the amplitudes of these sources are known, that is the normal derivative of the total field on the crack $\phi(x) \equiv \partial U(x, 0)/\partial z$, the total displacements $w(\pm a)$, and angles $w'(\pm a)$ are known, then the scattered field $U^{(s)}$ can be computed by formula (3.20). The field $U^{(s)}$ given by (3.20) satisfies the Helmholtz equation, the boundary condition on the plate and the radiation condition for any integrable function $\phi(x)$ and any constants $w(\pm a)$, $w'(\pm a)$. That is the formula (3.20) defines general solution in the terminology of section 1.4.1.

The scattered field at large distances from the crack is now examined. The Green's function and its derivatives involved in the representation (3.20), can be calculated by the saddle point method (1.22). The field at large distances from the crack can be represented as the sum of cylindrical diverging wave (1.29) and two surface waves running to the right and to the left from the crack. The far field amplitude $\Psi(\vartheta)$ of the scattered cylindrical wave is given by exact formula (see also (1.35) for the general representation of the far field amplitude)

$$\begin{aligned}\Psi(\vartheta) = \frac{\varrho_0\omega^2}{\pi}\frac{k\sin\vartheta}{\mathcal{L}(\vartheta)}\Bigg\{ & \frac{k^4\cos^4\vartheta - k_0^4}{2\varrho_0\omega^2}\int_{-a}^{a} e^{-ikx\cos\vartheta}\phi(x)dx \\ & + ik^2\cos^2\vartheta\sin(ka\cos\vartheta)\{w'\} - k^2\cos^2\vartheta\cos(ka\cos\vartheta)[w'] \\ & - k^3\cos^3\vartheta\sin(ka\cos\vartheta)\{w\} - ik^3\cos^3\vartheta\cos(ka\cos\vartheta)[w]\Bigg\}.\end{aligned} \tag{3.21}$$

The amplitudes of surface waves can be found by taking residue of $\Psi(\vartheta)$ according to the formula (1.30).

The amplitudes of passive sources $\phi(x)$, $w(a)$, $w(-a)$, $w'(a)$ and $w'(-a)$ should be chosen such, that the field satisfies boundary condition (3.18) on the crack and contact conditions at free edges. Letting $z_0 \longrightarrow 0$ yields the integral equation

$$\int_{-a}^{a} G(x_0, 0; x, 0)\phi(x)dx + \varrho_0\omega^2 D\Big(w(a)\,\frac{\partial^3 G(x_0, 0; a)}{\partial x^3}$$

$$
\begin{aligned}
&- w(-a)\,\frac{\partial^3 G(x_0,0;-a)}{\partial x^3} - w'(a)\,\frac{\partial^2 G(x_0,0;a)}{\partial x^2} \\
&+ w'(-a)\,\frac{\partial^2 G(x_0,0;-a)}{\partial x^2}\Big) = U^{(g)}(x_0,0), \quad -a < x_0 < a.
\end{aligned}
\tag{3.22}
$$

The Meixner conditions (1.18) are used to choose the appropriate form for the solution $\phi(x)$. Namely $\phi(x)$ can have weak singularities near the ends of the interval

$$\phi(x) \sim (x \mp a)^{\delta-1}, \qquad x \to \pm a, \quad \delta > 0. \tag{3.23}$$

Substituting from (3.20) into contact conditions (3.19) yields four equations $(n = 2, 3)$

$$
\begin{aligned}
&\int\limits_{-a}^{a} \frac{\partial^n g(a;x,0)}{\partial x_0^n}\phi(x)dx + \varrho_0\omega^2 D\Big(w(a)\frac{\partial^{3+n} g(a,a)}{\partial x^3 \partial x_0^n} - w(-a)\frac{\partial^{3+n} g(a,-a)}{\partial x^3 \partial x_0^n} \\
&\qquad - w'(a)\frac{\partial^{2+n} g(a,a)}{\partial x^2 \partial x_0^n} + w'(-a)\frac{\partial^{2+n} g(a,-a)}{\partial x^2 \partial x_0^n}\Big) = \frac{d^n}{dx^n} w^{(g)}(a),
\end{aligned}
$$

$$
\begin{aligned}
&\int\limits_{-a}^{a} \frac{\partial^n g(-a;x,0)}{\partial x_0^n}\phi(x)dx + \varrho_0\omega^2 D\Big(w(a)\frac{\partial^{3+n} g(-a,a)}{\partial x^3 \partial x_0^n} \\
&\qquad -w(-a)\frac{\partial^{3+n} g(-a,-a)}{\partial x^3 \partial x_0^n} - w'(a)\frac{\partial^{2+n} g(-a,a)}{\partial x^2 \partial x_0^n} \\
&\qquad\qquad +w'(-a)\frac{\partial^{2+n} g(-a,-a)}{\partial x^2 \partial x_0^n} = \frac{d^n}{dx^n} w^{(g)}(-a).
\end{aligned}
$$

Together with the integral equation (3.22) these forms the integro-algebraic system for the unknown function $\phi(x)$ and four constants $w(a)$, $w(-a)$, $w'(a)$ and $w'(-a)$.

The above integro-algebraic system is rather cumbersome and it is convenient to write it in "matrix" form. Examine first the Green's functions involved in that system. The Green's function $G(x,z;x_0,z_0)$ represents pressure $U(x,z)$ produced by an acoustic source placed in the point (x_0,z_0). It is given by Fourier integral (1.28). Other Green's functions can be expressed in terms of derivatives of $G(x,z;x_0,z_0)$ by z and z_0. We present here only expressions for the traces of Green's functions on the plate (at

$z = z_0 = 0$)

$$G(x,0;x_0,0) = \frac{1}{2\pi}\text{v.p.}\int e^{i\lambda(x-x_0)}\frac{\lambda^4 - k_0^4}{L(\lambda)}d\lambda, \tag{3.24}$$

$$G(x_0,0;x) = g(x;x_0,0) = -\frac{1}{2\pi D}\int e^{i\lambda(x-x_0)}\frac{d\lambda}{L(\lambda)},$$

$$g(x,x_0) = \frac{1}{2\pi\varrho_0\omega^2 D}\int e^{i\lambda(x-x_0)}\frac{\sqrt{\lambda^2-k^2}}{L(\lambda)}d\lambda.$$

The denominator in the above integrals is the Fourier symbol of the generalized boundary condition (1.13)

$$L(\lambda) = \sqrt{\lambda^2-k^2}\left(\lambda^4 - k_0^4\right) - N$$

and the path of integration coincides with the real axis of λ except for small neighborhoods of points $\lambda = \pm\kappa$, where symbol $L(\lambda)$ has zeros. These points are avoided in such a way that $\lambda = -\kappa$ appears below and $\lambda = \kappa$ appears above the path of integration. The first integral is understood as the principal value, that is the semi-infinite ends of the path of integration are assumed shifted to the upper for $x > x_0$ or lower for $x < x_0$ half-planes of complex parameter λ.

Analysis of the above Fourier integrals shows that the kernel $G(x,0;x_0,0)$ has logarithmic singularity as $x \to x_0$. Indeed, the denominator $L(\lambda)$ behaves at infinity as $O(\lambda^5)$ and therefore the integrand of (3.24) decreases only as $O(\lambda^{-1})$. The derivatives of the Green's function $G(x,0;x_0,z_0)$ that are presented in the integro-algebraic system can not be computed by formal differentiation of the above integrated functions. Accurate analysis of these integrals requires the following limits to be introduced. First the points x and x_0 are supposed noncoincident. In that case before letting $z_0 \to 0$ the ends of the path of integration can be shifted to the upper (for $x - x_0 > 0$) or to the lower (for $x - x_0 < 0$) half plane of complex variable λ. Then z_0 can be let equal to zero. Manipulation with the above limits allows the following formulae to be established

$$\left(\frac{d^4}{dx^4} - k_0^4\right)g(x;x_0,0) = G(x_0,0;x,0),$$
$$\left(\frac{d^4}{dx^4} - k_0^4\right)g(x;x_0) = G(x_0,0;x).$$

Therefore the integro-algebraic system can be written in compact form

$$\mathcal{L}W\Phi = \mathcal{L}w^{(g)}, \tag{3.25}$$

where $\Phi = \big(\phi(x),\ w(a),\ w(-a),\ w'(a),\ w'(-a)\big)^T$ is the vector of unknown amplitudes of passive sources,

$$W\Phi(x_0) = \int_{-a}^{a} G(x_0, 0; x)\phi(x)dx + \varrho_0\omega^2 D\Big(g'''(x_0, a)w(a) - g'''(x_0, -a)w(-a) - g''(x_0, a)w'(a) + g''(x_0, -a)w'(-a)\Big)$$

and $\mathcal{L}$ is the differential operator

$$\mathcal{L} = \left(\frac{d^4}{dx^4} - k_0^4,\ \frac{d^2}{dx^2}\bigg|_{x_0=a},\ \frac{d^2}{dx^2}\bigg|_{x_0=-a},\ \frac{d^3}{dx^3}\bigg|_{x_0=a},\ \frac{d^3}{dx^3}\bigg|_{x_0=-a}\right)^T.$$

When rewriting the integral equation in the form (3.25) the following property of the geometrical part of the field is used

$$-U^{(g)}(x_0, 0) = -D\left(\frac{d^4}{dx^4} - k_0^4\right) w^{(g)}(x_0),$$

which follows immediately from the generalized boundary condition on the plate.

The integro-algebraic system (3.25) is evidently satisfied if

$$W\Phi(x_0) = w^{(g)}(x_0). \tag{3.26}$$

This integral equation can be derived directly if one uses the integral representation for the displacements of the plate that appears when applying Green's formula to $U^{(s)}$ and $g(x; x_0, z_0)$.

The amplitude of acoustic sources $\phi(x)$ is searched in the class of functions defined by condition (3.23), that is in the space S_0 (see Appendix B). The integral equation (3.26) has smooth kernel and one can hope for the solution to exist only due to presence of arbitrary constants $w(a)$, $w(-a)$, $w'(a)$ and $w'(-a)$. That is, this integral equation determines amplitudes of all passive sources.

As it was already mentioned the kernel of the integral equation in (3.25) has logarithmic singularity. Appendix B presents some results on the solvability of such integral equations. In particular it follows that such integral equations are subject to Fredholm theory. Therefore the total integro-algebraic system (3.25) is subject to it, too. The following theorems describe the solvability properties of the integro-algebraic system (3.25) and integral equation (3.26).

Theorem 1 *Any solution of the integral equation (3.26) satisfies the system (3.25). Any solution of the integro-algebraic system (3.25) satisfies the equation*

$$W\Phi = w^{(g)} + w^{\circ}$$

with some function w° such that $\mathcal{L}w^{\circ} = 0$.

Theorem 2 *If the frequency is different from critical frequencies of operator $\mathcal{L}$ (that is $\mathcal{L}w = 0$ has only trivial solution), then solution Φ of the integro-algebraic system (3.25) exists and is unique for any right-hand side. If the frequency is critical, the solution Φ exists if the right-hand side satisfies the orthogonality condition*

$$-\int\limits_{-a}^{a} U^{(g)}(x,0)\phi_0(x)dx + \frac{d^2 w^{(g)}(a)}{dx^2} w_0(a) + \frac{d^2 w^{(g)}(-a)}{d} x^2 w_0(-a)$$
$$+ \frac{d^3 w^{(g)}(a)}{dx^3} w_0'(a) + \frac{d^3 w^{(g)}(-a)}{dx^3} w_0'(-a) = 0,$$

where $\Phi_0 = (\phi_0(x), w_0(a), w_0(-a), w_0'(a), w_0'(-a))$ is the solution of the homogeneous system (3.25). In that case Φ is defined up to Φ_0.

Theorem 3 *Solution of integral equation (3.26) exists and is unique for any right-hand side.*

Asymptotics of the field for $ka \ll 1$

The integro-algebraic system appears more suitable for asymptotic analysis. It is more convenient to rearrange the unknowns as follows. Let the following quantities be introduced

$$[w] = w(a) - w(-a), \quad [w'] = w'(a) - w'(-a),$$

$$\{w\} = \frac{w(a) + w(-a)}{2}, \quad \{w'\} = \frac{w'(a) + w'(-a)}{2}.$$

When the width of the crack tends to zero two of the introduces constants become the jump of displacement on the crack and the jump of bending angle. The integro-algebraic system takes the form

$$\int_{-a}^{a}\Big(P_4(x_0-x)-k_0^4P_0(x_0-x)\Big)\phi(x)dx+\varrho_0\omega^2\Big(\{w'\}[P_2](x_0)$$
$$+[w']\{P_2\}(x_0)-\{w\}[P_3](x_0)-[w']\{P_3\}(x_0)\Big)=U^{(g)}(x_0,0),$$

$$\frac{1}{D}\int_{-a}^{a}\{P_2\}(x)\phi(x)dx+[w']\frac{D_4+E_4(2a)}{2}-\{w\}E_5(2a)=-\left\{\frac{d^2w^{(g)}}{dx^2}\right\},$$

$$\frac{1}{D}\int_{-a}^{a}[P_3](x)\phi(x)dx+[w']E_5(2a)+\{w\}2\big(D_6-E_6(2a)\big)=-\left[\frac{d^3w^{(g)}}{dx^3}\right],$$

$$-\frac{1}{D}\int_{-a}^{a}[P_2](x)\phi(x)dx+\{w'\}2\big(D_4-E_4(2a)\big)+[w]E_5(2a)=-\left[\frac{d^2w^{(g)}}{dx^2}\right],$$

$$\frac{1}{D}\int_{-a}^{a}\{P_3\}(x)\phi(x)dx+\{w'\}E_5(2a)-[w]\frac{D_6+E_6(2a)}{2}=\left\{\frac{d^3w^{(g)}}{dx^3}\right\}.$$

Here

$$P_\ell(x)=\frac{1}{2\pi}\int e^{i\lambda x}\frac{(i\lambda)^\ell d\lambda}{L(\lambda)},\qquad E_\ell(x)=\frac{1}{2\pi}\int e^{i\lambda x}\frac{(i\lambda)^\ell\sqrt{\lambda^2-k^2}d\lambda}{L(\lambda)}.$$

Note that $P_\ell(0)=J_\ell$ where J_ℓ are introduced in (3.10) and $E_\ell(+0)=D_\ell$ where D_ℓ are introduced in (1.54). Also we introduce

$$\{f\}(x)=\frac{f(x-a)+f(x+a)}{2},\quad [f](x)=f(x-a)-f(x+a).$$

Consider the case of narrow crack, that is let $ka\ll 1$. In that case one can accept that $\phi(x)\approx\phi_0/\sqrt{a^2-x^2}$, $\phi_0=const.$ Then one substitutes this representation into the integro-algebraic system and decomposes the first equation into Taylor series near $x_0=0$. The kernel $P_4(x_0-x)-k_0^4P_0(x_0-x)$ can be replaced by its asymptotics when $x\to x_0$ (compare with page 118)

$$P_4(x-x_0)-k_0^4P_0(x-x_0)\sim\frac{1}{\pi}\ln|x-x_0|+\frac{1}{\pi}\left(\ln(k/2)+C_E\right)-\frac{i}{2}-NJ+\dots$$

and other functions are decomposed into Taylor series. After calculating the integrals the symmeterized integro-algebraic system takes in the principal by $ka \ll 1$ order the following form

$$\begin{aligned} &Q\,\phi_0 + \varrho_\omega^2 D J_2[w'] \approx U^{(g)}(0,0), \\ &J_2\phi_0 + \varrho_0\omega^2 D_4[w'] \approx -D\frac{d^2 w^{(g)}(0)}{dx^2} \end{aligned} \tag{3.27}$$

and

$$\varrho_0\omega^2 D_6[w] \approx -\frac{d^3 w^{(g)}(0)}{dx^3}. \tag{3.28}$$

The coefficient Q in (3.27) is given by the formula

$$Q = \ln(ka/4) + C_E - i\,\pi/2 - \pi N\,J$$

and the integrals J and J_2 are introduced in (3.9) and (3.10). The quantities $\{w'\}$ and $\{w\}$ disappear from the leading order terms in the integro-algebraic equation. Analysis of equations in the next order of ka allows these constants to be proved bounded as $ka \to 0$. Therefore their contribution to the far field asymptotics appears of order $O\left((ka)^2\right)$ and is neglected below.

The right-hand sides in the equations (3.27), (3.28) are functions of the incidence angle ϑ_0

$$\begin{aligned} U^{(g)}(0,0) &= \frac{ik\sin\vartheta_0\left(k^4\cos^4\vartheta_0 - k_0^4\right)}{\mathcal{L}(\vartheta_0)}, \\ w^{(g)} &= \frac{1}{\varrho_0\omega^2}\frac{ik\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}e^{ikx\cos\vartheta_0}. \end{aligned}$$

The unknowns are found from the system (3.27) and equation (3.28) as

$$\phi_0 = \frac{2ik\sin\vartheta_0}{\mathcal{L}(\vartheta_0)Q}\left\{\pi M(\vartheta_0) + O\left(\frac{1}{\ln^2(ka)}\right)\right\},$$

$$[w'] = -\frac{2iNk\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}\left\{\frac{k^2\cos^2\vartheta_0}{D_4} - \frac{J_2}{D_4 Q}\pi M(\vartheta_0) + O\left(\frac{1}{\ln^2(ka)}\right)\right\},$$

$$[w] = \frac{2Nk^4\sin\vartheta_0\cos^3\vartheta_0}{D_6\mathcal{L}(\vartheta_0)} + o(1),$$

where $\mathcal{L}(\vartheta_0)$ is defined in (1.25) and $M(\vartheta)$ is defined in (3.13).

The far field amplitude is given by the formula (3.21). In two principal orders by ka this yields

$$\Psi(\vartheta;\vartheta_0) = \frac{iN}{\pi}\frac{k^2\sin\vartheta\sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}\left\{\frac{k^4}{D_4}\cos^2\vartheta\cos^2\vartheta_0 - \frac{k^6}{D_6}\cos^3\vartheta\cos^3\vartheta_0 - \frac{\pi^2 M(\vartheta)M(\vartheta_0)}{N\,Q} + \dots\right\}. \tag{3.29}$$

The first two terms coincide with the far field amplitude of scattering by classical model of pointwise crack. The last term is the logarithmically smaller correction with respect to ka.

3.2.7 *Discussion and numerical results*

The above asymptotic formula coincides with the exact expression (3.12) for the far field amplitudes in the problem of scattering by the generalized point model of narrow crack up to terms of order $o\big(1/\ln(ka)\big)$. The amplitudes of surface waves are obtained from the formulae (3.29) and (3.12) by taking residues as specified in (1.30). Therefore asymptotic coincidence of the amplitudes of surface waves in the two problems follows from such coincidence of far field amplitudes of diverging cylindrical waves. Analogously one could check that the generalized model of narrow crack reproduces principal order terms and logarithmic corrections of the far field amplitude and amplitudes of surface waves in the case of surface wave scattering. This fact allows the main HYPOTHESIS to be justified for the problem of scattering by a narrow crack in fluid loaded elastic plate.

Below we discuss the formulae (3.12) and (3.14) of scattering by the generalized point model of narrow crack. Note that the two leading terms, for $\varepsilon = ka \ll 1$, of the asymptotics (3.12) coincide with those presented in Chapter 1, see formula (1.60) where scattering by the plate with the pointwise crack is examined. The correction given by the last term in (3.12) depends on the width $2a$ of the crack. Under the assumption that the problem contains only one small parameter ka the characteristics of scattering on a narrow crack can be concluded close to those for classical model. At the same time, if the incidence is orthogonal, then the first two terms in (3.12) and (3.14) vanish and the main contribution is of order $1/\ln(ka)$. That is, classical model of pointwise crack is not applicable if the incidence angle ϑ_0 or the angle of observation ϑ are close to $\pi/2$, (namely, if their declination is less than $1/\sqrt{|\ln(ka)|}$). By similar reason in the case of surface wave scattering the classical model of pointwise crack is not applicable if the direction of observation is close to orthogonal.

Formulae (3.12) and (3.14) contain special integrals D_4, D_6 J and J_2. These integrals depend on parameters of the plate and fluid. In the case of thin plate, ($kh \ll 1$), the integrals can be simplified and hence simpler formulae can be derived. The asymptotic expressions for contact integrals are derived in Appendix A. Comparison of the coefficients of trigonometric functions presented in the asymptotics of the far field amplitude allows the range of applicability of classical model of pointwise crack to be established. Substituting from (A.11), (A.12) and (A.13) into the asymptotics (3.12) it is immediately apparent that the last term giving correction with respect to small ka appears to be leading in parameter $\varepsilon \equiv kh$. Indeed, the ratio of the scattering pattern for the pointwise crack to that given in (3.12) is of order $\varepsilon^{8/5}\ln(ka)$. For the amplitudes of surface waves the last term is again the leading one with respect to ε, though analogous ratio is larger, namely it is of order $\varepsilon^{4/5}\ln(ka)$. This means that even for nonorthogonal incidence terms corresponding to the classical model of pointwise crack prevail only for very small ka, namely in the formula for the far field amplitude if $ka \ll \exp(-\varepsilon^{-8/5})$, and in the formula for the amplitudes of surface waves if $ka \ll \exp(-\varepsilon^{-4/5})$. The applicability of the model examined here for such narrow cracks is of doubt and cannot be used for justification of classical model. However, if $ka > kh$ generalized model of narrow crack is valid and classical model of pointwise crack is not. The doubts concerning applicability of generalized model of narrow crack for $ka \ll kh$ are similar to those for the applicability of contact conditions briefly discussed in Section 1.1.3.

As for not exponentially small widths a the last terms in (3.12) and (3.14) give main effect of scattering, it is of interest to describe it by the asymptotic formulae. For that consider first the denominator of the last term in the asymptotics (3.12)

$$Q \approx \ln(ka/4) + C_E - \frac{\pi i}{2} + \frac{3\pi i}{10} + \frac{2\pi i}{5}\frac{1 - e^{2\pi i/5}}{\left(1 + e^{\pi i/5}\right)^2}$$
$$= \ln(ka/4) + C - \frac{\pi i}{5},$$

$$C = C_E + \frac{2\pi}{5}\tan\left(\frac{\pi}{10}\right).$$

Excluding complex values from the denominator one finds

$$\Psi(\vartheta;\vartheta_0) \sim 5\varepsilon^2 d^2 \frac{\pi - 5i\left(\ln(ka/4) + C\right)}{\pi^2 + 25\left(\ln(ka/4)+C\right)^2} \sin\vartheta \sin\vartheta_0,$$

$$\psi_\pm = 2\pi\varepsilon d e^{\pi i/5} \frac{\pi i + 5\left(\ln(ka/4)+C\right)}{\pi^2 + 25\left(\ln(ka/4)+C\right)^2} \sin\vartheta_0.$$

The above asymptotics can be verified to satisfy the optical theorem (1.42). Computations show that the main portion of energy scattered by narrow crack is carried away by surface waves. The scattering cross-section has the asymptotics

$$\Sigma(\vartheta_0) = \frac{20\pi^2}{25\left(\ln(ka/4)+C\right)^2 + \pi^2} \varepsilon d^2 h \sin^2\vartheta_0.$$

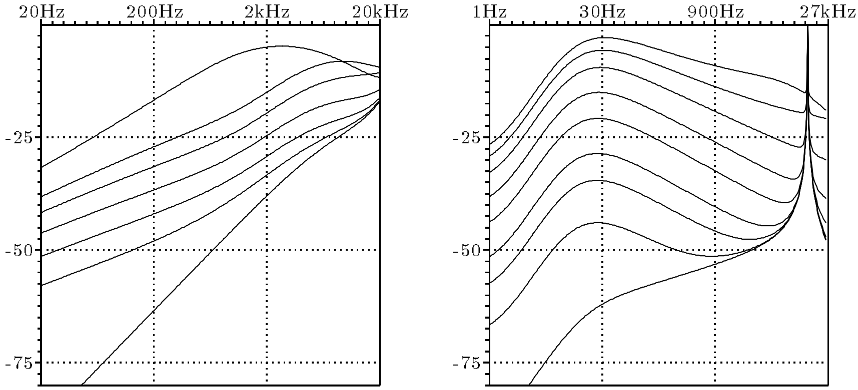
We analysed validity of classical model of pointwise crack for plane wave scattering. Now the case of surface wave scattering is analysed briefly. The asymptotics (3.16) leads to the expression

$$\psi_\pm = \frac{1}{2}\left(1 - e^{2\pi i/5}\right) \pm \frac{1}{2}\left(1 - e^{6\pi i/5}\right)$$
$$-\pi e^{9\pi i/10} \frac{5\left(\ln(ka/4)+C\right) + \pi i}{25\left(\ln(ka/4)+C\right)^2 + \pi^2}.$$

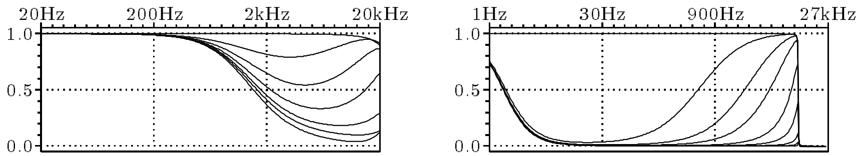
The first two terms correspond to classical model and are the leading terms. The correction is $1/|\ln(ka)|$ times smaller. Hence in the case of surface wave scattering classical model of pointwise crack gives correct asymptotics.

The above analysis is made with the assumptions of thin plate when the asymptotics (1.15) is valid. In other cases numerical calculation of integrals D_0, D_2, J_2 and J should be performed. Numerical results are presented on Figs. 3.1–3.3. Two examples of plate–fluid systems are examined. For 1cm steel plate in water the semi-widths a of the crack are assumed 0, 10^{-10}, 10^{-5}, 0.001, 0.01, 0.03 and 0.1m †. The effective cross-section Σ (in dB) and the portion of energy carried by surface waves in that system are presented on the left-hand side graphs of Fig. 3.1. The right-hand side graphs correspond to the system of 1mm steel plate in air. For this system classical and generalized models show larger difference and to follow the

†The small values of a are taken formally (without respect to the molecular structure) just to illustrate the limit of $a \to 0$ in the generalized model of narrow crack.



Effective cross-section (in dB) as a function of frequency for $\vartheta_0 = 30°$.



Portion of energy carried by surface waves.

Fig. 3.1 Scattering by narrow crack: 1cm steel plate in water (left) and 1mm steel plate in air (right). On the left graphs $a = 0$, 10^{-10}, 10^{-5}, 10^{-3}, 0.01 and 0.1m on curves from bottom to top on the upper graph and from top to bottom on the lower graph. On the right graphs $a = 0$, 10^{-300}, 10^{-100}, 50^{-50}, 10^{-20}, 10^{-10}, 10^{-5}, 10^{-3} and 0.01m.

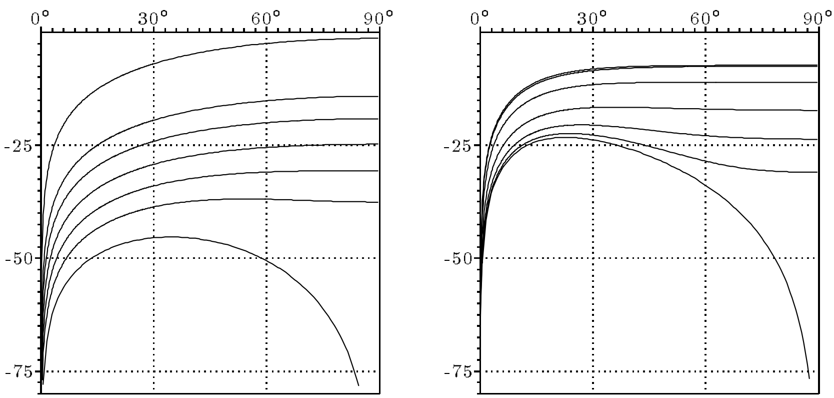


Fig. 3.2 Angular characteristics of scattering by a narrow crack in 1cm steel plate in water at 1kHz (left) and at 10kHz (right).

limit of $a \to 0$ the characteristics for $a = 0$, 10^{-300}, 10^{-100}, 10^{-50}, 10^{-20}, 10^{-10}, 10^{-5}, 0.001 and 0.01m are presented. The order of curves on the graphs is the following. On the graphs for the effective cross-section Σ the lower curves correspond to the smaller widths. On the graphs for the portion of surface energy the order of curves is the reverse.

One can see that even for very narrow cracks the low frequency limits in the case of classical point model and in the case of generalized model of narrow crack are different. At larger frequencies the curves approach to each

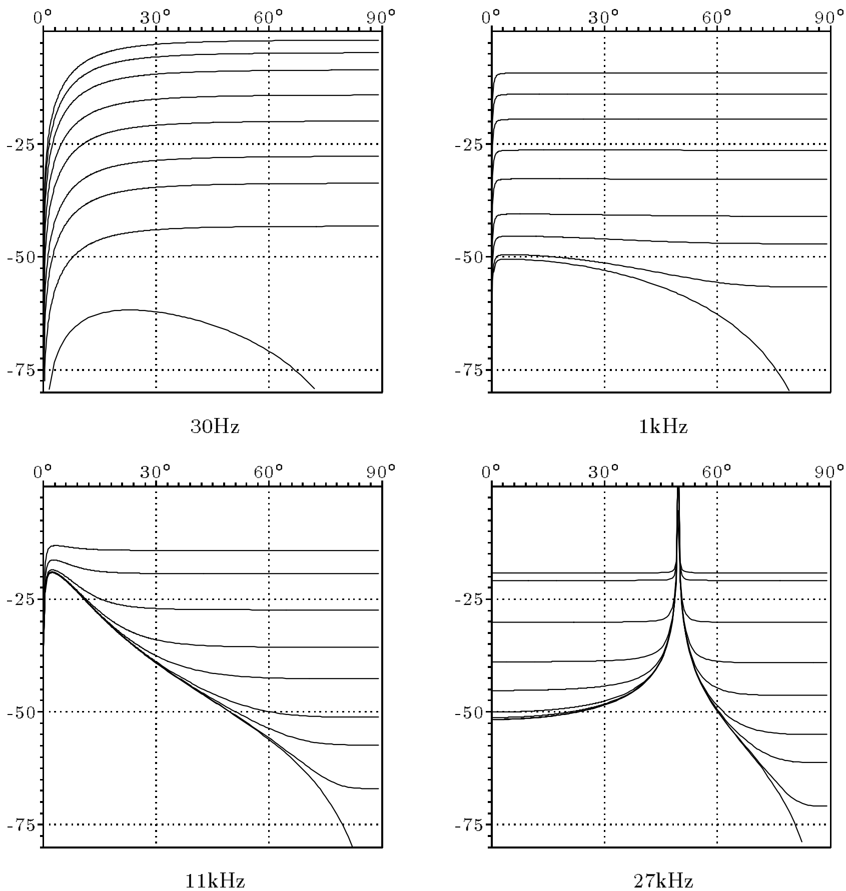


Fig. 3.3 Angular characteristics of scattering by crack in 1mm steel plate in air.

other and in a vicinity of the critical frequency f^* both models give similar results as the semi-width a does not manifest itself. The lower graph shows that the balance of energy for $a \neq 0$ is different from that in the case of pointwise crack. In the low frequency range almost all the energy is carried by surface waves, but in the intermediate domain a significant portion of energy is scattered by cylindrical wave. This increase of cylindrical wave contribution starts at about coincidence frequency f_c. In the model of pointwise crack such rearrangement of energy fluxes is not presented.

Figure 3.2 presents angular characteristics for the system 1cm plate — water for frequencies 1kHz and 10kHz. The order of curves and the parameters a are the same as on the upper left graph of Fig. 3.1. One can see that the model of pointwise crack gives smaller values of the effective cross-section. This Figure also illustrates the inapplicability of classical point model in a vicinity of orthogonal direction which is established above.

Same characteristics for the system 1mm plate — air are presented on Fig. 3.3 for frequencies 30Hz, 1kHz, 11kHz and 27kHz. The semi-widths of the crack are accepted the same as on Fig. 3.1 (right), that is the curves from bottom to top correspond to $a = 0$, 10^{-300}, 10^{-100}, 10^{-50}, 10^{-20}, 10^{-10}, 10^{-5}, 0.001 and 0.01 m. One can see that even for very narrow cracks classical and generalized models give different characteristics. Coincidence is noticed only in a vicinity of critical frequency f^*.

3.3 Model of a short crack

Consider the problem of scattering by a short infinitely narrow crack in elastic plate. Let the crack be oriented along the x axis, the edges of the crack be free and fluid occupy the halfspace $\{z > 0\}$ on one side of the plate. We assume the length $2a$ of the crack be small compared to the wavelength of the incident acoustic or surface wave. In that case to obtain the principal order terms in the asymptotics of scattered field the obstacle (crack) can be replaced by a generalized point model in the form of appropriate zero-range potential for operator $\mathcal{H}_3$. Such zero-range potentials are parameterized by Hermitian matrices $\mathcal{S}$ according to formula (2.38). The matrix $\mathcal{S}$ has block structure as described in Section 2.3.3 and its blocks can be determined as suggested by the HYPOTHESIS of Section 3.1. That is, the element $\mathcal{S}_{11} = S$ is determined from the analysis of scattering by absolutely rigid plate, the block $\{\mathcal{S}_{ij}\}_{i,j=2}^{7} = \mathbf{S}$ is the same as in the case of isolated plate and other elements of matrix $\mathcal{S}$ are zeros.

The diffraction process is first considered in isolated plate by means of integral equations method. This allows the far field amplitude of the scattered flexure wave to be derived and the parameters $\mathbf{S}$ of the zero-range potential to be chosen. Then the generalized model for the crack in fluid loaded plate is constructed according to the main HYPOTHESIS as described above. Further scattering by this zero-range potential is studied. Finally considerations of Section 3.3.4, where the problem of diffraction by a short crack in fluid loaded plate is solved in classical formulation, justify the HYPOTHESIS .

3.3.1 *Diffraction by a short crack in isolated plate*

Problem formulation

The problem of scattering of an incident field of flexural displacements $w^{(i)}(x,y)$ by the crack $\Lambda = \{|y| < a,\ x = 0\}$ in an infinite plate is in finding such scattered field $w^{(s)}(x,y)$, that satisfies the equation

$$\triangle^2 w^{(s)} - k_0^4 w^{(s)} = 0, \qquad (x,y) \notin \Lambda, \tag{3.30}$$

and such that the field $w = w^{(i)} + w^{(s)}$ satisfies the boundary conditions on both sides of the crack

$$\mathbb{M}^{\pm} w^{(s)} = -\mathbb{M}^{\pm} w^{(i)}, \quad \mathbb{F}^{\pm} w^{(s)} = -\mathbb{F}^{\pm} w^{(i)}$$

denoting that edges are free of forces and bending momentums. Here

$$\mathbb{M}^{\pm} w \equiv D \lim_{x\to\pm 0} (w_{xx} + \sigma w_{yy}) = 0, \qquad |y| < a,$$

$$\mathbb{F}^{\pm} w \equiv -D \lim_{x\to\pm 0} (w_{xxx} + (2-\sigma) w_{xyy}) = 0, \qquad |y| < a.$$

The problem formulation includes the radiation condition and Meixner conditions (1.19) at the ends of the crack. The latter guarantee the energy finiteness in any bounded domain and may be written in the form of asymptotic expansion [16]

$$w^{(s)} = w_0 + r w_2(\varphi) + r^{3/2} w_3(\varphi) + O(r^2), \quad r \to 0. \tag{3.31}$$

Condition (3.31) means that the term $r^{1/2} w_1(\varphi)$ is absent in the expansions of $w^{(s)}$ in both polar systems with the poles at each end of the crack $(0,-a)$ and $(0,a)$.

For further derivations it is convenient to use dimensionless coordinates x', y' and wave number k_0':

$$x' = x/a, \quad y' = y/a, \quad k_0' = k_0 a.$$

Henceforth the primes are dropped.

Governing integral equations

The scattered field $w^{(s)}$ may be searched in the form of Fourier transform (compare with (1.73))

$$w^{(s)} = \sum_{j=0}^{3} \operatorname{sign}^j(x) \int e^{i\mu y} p_j(\mu) \left(\gamma_-^{j-1} e^{-|x|\gamma_-} - \gamma_+^{j-1} e^{-|x|\gamma_+} \right) d\mu. \quad (3.32)$$

Here $\gamma_\pm = \sqrt{\mu^2 \pm k_0^2}$. The integration path in (3.32) goes along the real axis of μ in such a way that the branch point $\mu = -k_0$ is below and the branch point $\mu = k_0$ is above it. This form of contour is fixed by the radiation conditions. The integrand in each term satisfies the equation (3.30) and the number of terms in (3.32) is defined by the order of the differential operator. The terms with numbers $j = 1$ and 3 form the odd part of the field and the other two terms form its even part. Unknown functions p_j should behave at infinity as prescribed by the asymptotics (3.31), namely

$$p_j(\mu) = O\big(\mu^{1/2-j}\big), \qquad \mu \to \pm\infty. \quad (3.33)$$

Therefore integrals in (3.32) converge for all values of (x, y), but when substituted into equation (3.30) or into boundary conditions the integrals should be treated as Hadamard integrals or should be regularized (see Appendix B).

The representation (3.32) automatically satisfies the equation (3.30) when $x \neq 0$. The Fourier transform in the formula (3.32) is written in such a way that $4k_0^2 p_j$ is the Fourier transform of the jump of the derivative by x of order $3 - j$ at $x = 0$. That is, the following formula takes place

$$\left(\Delta^2 - k_0^4\right) w^{(s)} = -4k_0^4 \sum_{j=0}^{3} \int e^{i\mu y} p_j(\mu) d\mu \, \left(-\frac{d}{dx}\right)^j \delta(x).$$

To satisfy the equation (3.30) everywhere except at the crack it is necessary to fix conditions for p_j $(j = 0, 1, 2, 3)$

$$\int e^{i\mu y} p_j(\mu) d\mu = 0, \quad |y| > 1. \tag{3.34}$$

Consider the boundary conditions. It is convenient to rewrite these conditions as

$$\left(\mathbb{M}^+ - \mathbb{M}^-\right) w^{(s)} = 0, \quad \left(\mathbb{F}^+ - \mathbb{F}^-\right) w^{(s)} = 0, \tag{3.35}$$

$$\mathbb{F}^+ \left(w^{(s)} + w^{(i)}\right) = 0, \quad \mathbb{M}^+ \left(w^{(s)} + w^{(i)}\right) = 0, \qquad |y| < a. \tag{3.36}$$

The conditions (3.35) are satisfied on the whole axis and do not contain the incident field which is continuous. Substituting representation (3.32) into conditions (3.35) yields integral equations of convolution on the whole axis. Solving these equations one finds

$$p_0(\mu) = -\sigma\mu^2 p_2(\mu), \qquad p_1(\mu) = -(2-\sigma)\mu^2 p_3(\mu). \tag{3.37}$$

The boundary conditions (3.36) yield integral equations

$$\begin{aligned} \int e^{i\mu y} h_2(\mu) p_2(\mu) d\mu &= f_2(y), \qquad |y| < 1, \\ \int e^{i\mu y} h_3(\mu) p_3(\mu) d\mu &= f_3(y), \qquad |y| < 1, \end{aligned} \tag{3.38}$$

where

$$h_2(\mu) = \frac{a_-^2}{\gamma_-} - \frac{a_+^2}{\gamma_+}, \quad h_3(\mu) = a_+^2\gamma_- - a_-^2\gamma_+, \quad a_\pm = (1-\sigma)\mu^2 \pm k_0^2,$$

$$f_2(y) = -\frac{1}{D}\mathbb{M}^+ w^{(i)}(y), \qquad f_3(y) = -\frac{1}{D}\mathbb{F}^+ w^{(i)}(y).$$

Superscripts $^+$ of operators $\mathbb{M}$ and $\mathbb{F}$ can be suppressed because the incident field $w^{(i)}$ is continuous.

Equations (3.38) together with equations (3.34) form two systems of dual integral equations. Note that equations of this kind often appear in diffraction problems. For sufficiently large values of k_0 the Wiener-Hopf method [55] can be applied to find the asymptotics of the scattered field as the sum of waves many times rescattered by the ends of the crack. The other case is considered here. The wave number k_0 is supposed to be small.

To satisfy the homogeneous equations (3.34) let p_n be Fourier transforms of functions q_n different from zero only on the interval $[-1, 1]$

$$p_n(\mu) = \int_{-1}^{1} q_n(t) e^{-it\mu} dt. \tag{3.39}$$

The index n here and below takes values 2 and 3. Integrating over μ in (3.38) gives integral equations for new unknown functions $q_n(t)$:

$$\mathbb{H}_n q_n \equiv \int_{-1}^{1} H_n(y-t) q_n(t) dt = k_0^n f_n(y), \qquad |y| < 1. \tag{3.40}$$

Here kernels $H_n(s)$ are Fourier transforms of $h_n(\tau)$ (the factor k_0^{-2} is introduced to exclude frequency from the principal singular parts of the integral operators):

$$H_n(s) = k_0^{-2} \int e^{i\mu s} h_n(\mu) d\mu.$$

The kernels can be expressed by Bessel functions of the third kind. Using representation of Bessel functions in the form of series [1] allows the kernels to be concluded representable in the form

$$H_n(s) = \frac{d^{2n-2}}{ds^{2n-2}} \left(\ln|s| + a_n(s^2) \ln|s| + b_n(s^2) \right) \tag{3.41}$$

where $a(s^2)$ and $b(s^2)$ are smooth functions from $C^\infty[-1, 1]$ and $a_n(0) = 0$. Indeed, singularities of kernels are defined by the asymptotics of $h_n(\mu)$ when $\mu \to \pm\infty$

$$h_2(\mu) = -\chi k_0^2 |\mu| + h_2'(\mu), \quad h_2'(\mu) = O(\mu^{-3}),$$

$$h_3(\mu) = \chi k_0^2 |\mu|^3 + h_3'(\mu), \quad h_3'(\mu) = O(\mu^{-1}).$$

Here $\chi = (1-\sigma)(3+\sigma)$. Therefore the singular terms are

$$H_2(s) = -2\chi \frac{d^2}{ds^2} \ln|s| + H_2'(s),$$

$$H_3(s) = -2\chi \frac{d^4}{ds^4} \ln|s| + H_3'(s).$$

The kernels H_n' are functions integrable on the interval. Therefore to regularize the equations (3.40) one changes the order of integration and differentiation in singular terms. That gives

$$-2\chi \frac{d^2}{dy^2} \int_{-1}^{1} \ln|y-t| q_2(t)dt + \int_{-1}^{1} H_2'(y-t) q_2(t)dt = k_0^2 f_2(y), \qquad (3.42)$$

$$-2\chi \frac{d^4}{dy^4} \int_{-1}^{1} \ln|y-t| q_3(t)dt + \int_{-1}^{1} H_3'(y-t) q_3(t)dt = k_0^3 f_3(y). \qquad (3.43)$$

The solutions $q_n(t)$ of these equations should belong to spaces defined by conditions (3.33). Let $\mathbb{S}_m$ denote spaces of functions $u \in C^\infty(-1,1)$ satisfying inequalities (see Appendix B, page 245)

$$|u(t)| < \text{const} \left(1-t^2\right)^{m-1+\delta}, \qquad \delta > 0. \qquad (3.44)$$

Then $q_2(t) \in S_1$ and $q_3(t) \in S_2$ as follows from the properties of Fourier integrals‡.

To examine existence and uniqueness of solutions of integral equations (3.42) and (3.43) in spaces defined by condition (3.44), one can check that the symbols $h_n(\mu)$ have positive imaginary part when μ is real. That is, the symbols of the integral equations are sectorial. The analysis of supersingular integral equations of the convolution on a finite interval can be found in Appendix B. It uses the representation (3.41) for the kernels $H_n(s)$ and sectorial property of $h_n(\mu)$ and allows the following result to be formulated

Theorem 4 *Solutions of equations (3.42) and (3.43) in the classes $\mathbb{S}_0$ and $\mathbb{S}_1$ respectively exist for every right-hand side from $C^\infty[-1,1]$ and such solutions are unique.*

‡One can integrate by parts in (3.39) and check that Fourier transform of q_2 decreases at infinity as $o(\mu^{-1})$ and Fourier transform of q_3 decreases as $o(\mu^{-2})$. Exact behaviour can be studied with the use of integral representation for Bessel function

$$J_0(\mu) = \frac{1}{\pi} \int_{-1}^{1} \frac{e^{i\mu t} dt}{\sqrt{1-t^2}}$$

and the estimate $J_0(\mu) = O(\mu^{-1/2})$, for $\mu \to infty$.

Numerical analysis

For numerical analysis of integral equations (3.42), (3.43) it is convenient to use Galerkin method with orthogonal polynomials as basic functions. The same method was previously used in [60] when examining static problems for plates with cuts, ribs, fixtures and other thin inhomogeneities. Solutions $q_n(t)$ are represented in the form of infinite series

$$\begin{aligned} q_2(t) &= \sqrt{1-t^2}\sum_{j=0}^{\infty}\alpha_j U_j(t), \\ q_3(t) &= \left(1-t^2\right)^{3/2}\sum_{j=0}^{\infty}\beta_j C_j^{(2)}(t). \end{aligned} \tag{3.45}$$

Here $U_j(t)$ are Chebyshev polynomials of the second kind and $C_j^{(2)}(t)$ are ultraspherical polynomials. These polynomials are used in representation (3.45) because they are the eigen functions of the singular parts of the integral operators of equations (3.42) and (3.43) respectively [60]

$$\frac{d^2}{dy^2}\int_{-1}^{1}\ln|y-t|\sqrt{1-t^2}U_j(t)dt = \pi(j+1)U_j(y),$$

$$\frac{d^4}{dy^4}\int_{-1}^{1}\ln|y-t|\left(1-t^2\right)^{3/2}C_j^{(2)}(t)dt = \pi(j+1)(j+2)(j+3)C_j^{(2)}(y).$$

Substituting representations (3.45) into the integral equations gives infinite systems of linear algebraic equations for the coefficients α_j and β_j:

$$\begin{aligned} -\chi(j+1)\alpha_j + \sum_{m=0}^{\infty} A_j^m \alpha_m &= \frac{k_0^2}{\pi} f_j^{(2)}, \\ \frac{\chi}{4}(j+1)^2(j+2)(j+3)^2\beta_j + \sum_{m=0}^{\infty} B_j^m \beta_m &= \frac{k_0^3}{\pi} f_j^{(3)}. \end{aligned} \tag{3.46}$$

The functions on the right-hand sides of equations (3.42) and (3.43) are expanded into series in orthogonal polynomials, and $f_j^{(n)}$ are the coefficients of these series

$$f_j^{(2)} = \frac{1}{\pi}\int_{-1}^{1} f_2(y)\sqrt{1-y^2}U_j(y)dy,$$

$$f_j^{(3)} = \frac{1}{\pi}\int\limits_{-1}^{1} f_3(y)\left(1-y^2\right)^{3/2} C_j^{(2)}(y)dy.$$

The elements of matrices A_j^m and B_j^m are given by integrals

$$A_j^m = \frac{1}{\pi^2}\int\limits_{-1}^{1}\int\limits_{-1}^{1} H_2'(y-t)\sqrt{1-t^2}U_j(t)\sqrt{1-y^2}U_m(y)\,dt\,dy,$$
$$B_j^m = \frac{1}{\pi^2}\int\limits_{-1}^{1}\int\limits_{-1}^{1} H_3'(y-t)\left(1-t^2\right)^{3/2} C_j^{(2)}(t)\left(1-y^2\right)^{3/2} C_m^{(2)}(y)\,dt\,dy. \tag{3.47}$$

The elements A_j^m and B_j^m having indices of different parity are equal to zero. This corresponds to the symmetry of the problem. Therefore each system (3.46) separates into two independent systems for the coefficients of odd and even with respect to x parts of scattered field.

To avoid integrating twice in expressions (3.47), it is convenient to use Fourier transforms of kernels H_n'. After that, integrals over y and t may be rewritten in terms of Bessel functions. For that one uses the formulae [1]

$$(1-t^2)U_\ell(t) = \frac{1}{2}\Big(T_\ell(t) - T_{\ell+2}(t)\Big),$$

$$(1-t^2)C_\ell^{(2)}(t) = \frac{1}{4}\Big((\ell+3)U_\ell(t) - (\ell+1)U_{\ell+2}(t)\Big),$$

$$J_\ell(\mu) = \frac{i^{-\ell}}{\pi}\int\limits_0^{\pi} e^{i\mu\theta}\cos(\ell\theta)d\theta == \frac{i^{-\ell}}{\pi}\int\limits_{-1}^{1} e^{it\mu}T_\ell(t)\frac{dt}{1-t^2}$$

and

$$J_{\ell-1}(\mu) + J_{\ell+1}(\mu) = \frac{2\ell}{\mu}J_\ell(\mu).$$

Finally representations (3.47) are reduced to

$$A_j^m = -2k_0^{-2}(j+1)(m+1)i^{j+m}\int\limits_0^{\infty} h_2'(\mu)J_{j+1}(\mu)J_{m+1}(\mu)\frac{d\mu}{\mu^2}, \tag{3.48}$$

$$B_j^m = \frac{1}{2} k_0^{-2} \frac{(j+3)!}{j!} \frac{(m+3)!}{m!} i^{j+m} \int_0^\infty h_3'(\mu) J_{j+2}(\mu) J_{m+2}(\mu) \frac{d\mu}{\mu^4}. \quad (3.49)$$

The integrated functions in expressions (3.48) and (3.49) decrease at infinity rather slowly (only as $O(\mu^{-6})$). To improve convergence singular terms $(y-t)^2 \ln|y-t|$ and $\ln|y-t|$ can be extracted from kernels H_2' and H_3' respectively. These terms give five diagonal matrices and increase convergence of the integrals up to $O(\mu^{-10})$. Convergence can be made as rapid as desired by extracting further terms from kernels.

Systems (3.46) can be truncated. To prove this, it is necessary to estimate the behavior of A_j^m and B_j^m with respect to indices. Consider representation (3.48) for A_j^m and divide the interval of integration into two parts by some point $\mu = T$. For the integral from 0 to T the estimate $|J_m(z)| \leq (z/2)^m/m!$ for Bessel functions and the integrability of the symbol $h_2'(\mu)$ leads to

$$(j+1)(m+1) \left| \int_0^T h_2'(\mu) J_{j+1}(\mu) J_{m+1}(\mu) \frac{d\mu}{\mu^2} \right|$$

$$< C_1 \frac{(T/2)^{j+m}}{j!\, m!} \left(1 + \frac{(T/2)^2}{(j+1)(j+2)}\right) \left(1 + \frac{(T/2)^2}{(m+1)(m+2)}\right).$$

For the integral from T to infinity the Bessel functions are limited by 1 and the expression $|h_2'(\mu)\mu^{-2}|$ decreases as $C_2\mu^{-5}$. Therefore this gives

$$(j+1)(m+1) \left| \int_T^\infty h_2'(\mu) J_{j+1}(\mu) J_{m+1}(\mu) \frac{d\mu}{\mu^2} \right| \leq \frac{C_2(j+1)(m+1)}{4T^4}.$$

The value T is chosen such as to minimize the sum of the right-hand sides in the above estimates. More rough, but sufficient estimate appears if one chooses value of T simply by equating the right-hand sides in the two above estimates. This corresponds to

$$T \approx \Big((j+1)!(m+1)!\Big)^{1/(j+m+4)}$$

and finally gives

$$|A_j^m| \leq C_3(j+1)(m+1)\big((j+1)!(m+1)!\big)^{-4/(j+m)}.$$

Using Stirling formula [1] for the factorials allows to conclude that

$$|A_j^m| = \begin{cases} O\left((m+j)^{-2}\right), & j \approx m, \\ O(m^{-3}), & j \ll m. \end{cases}$$

An analogous procedure for B_j^m leads to insufficiently accurate estimates. Therefore it is necessary to extract singular term $C_4 \ln|y-t|$ out of h_3'. This term may be explicitly integrated in the representation (3.47):

$$B_j^m = \widehat{B}_j^m - \frac{C_4}{8}\left\{\delta_j^m \left(\eta_j (j+3)^2 + 4(j+2) + \frac{(j+1)^2}{j+4}\right) - 4\delta_j^{m+1}(j+1) + \delta_j^{m+4}\eta_j(j^2-9)\right\},$$

where

$$\eta_j = \begin{cases} \ln 2, & j = 0 \\ (4j)^{-1}, & j > 0. \end{cases}$$

The described above procedure gives for $\widehat{B}_j^m$ the following estimate

$$|\widehat{B}_j^m| \le C_5 (k_0/2)^5 j^3 m^3 (j!\, m!)^{-8/(j+m)}.$$

Therefore

$$|B_j^m| = \begin{cases} O(m+j), & j \approx m, \\ O(m^{-5}), & j \ll m. \end{cases}$$

It is shown in [37] that

Theorem 5 *Truncation method is applicable to systems*

$$\gamma_m + \sum_{j \ne m} C_m^j \gamma_j = r_m$$

satisfying the following two conditions:
(i) for any m the sum of nondiagonal terms is finite

$$S_m \equiv \sum_{j \ne m} |C_m^j| < \infty;$$

(ii) there exists some number N such that for any $m \ge N$

$$S_m < 1 - \varepsilon, \quad |r_m| < 1 - \varepsilon, \quad \varepsilon > 0.$$

The estimates for A_j^m and B_j^m mean that systems (3.46) satisfy conditions (i) and (ii) for any finite k_0. Therefore solutions of truncated systems converge to solutions of (3.46) if the systems of N equations (N is the same as in (ii)) are not singular. Calculations show that for large values of k_0 matrices of truncated systems become nearly singular. Therefore, the method is applicable when k_0 is not large.

Numerical and asymptotic results

When systems (3.46) are solved the scattered field may be calculated according to formulae (3.32), (3.37), (3.39) and (3.45). This gives

$$\begin{aligned} w^{(s)} = &\int e^{i\mu y} p_2(\mu) \left(\frac{a_-}{\gamma_-} e^{-|x|\gamma_-} - \frac{a_+}{\gamma_+} e^{-|x|\gamma_+} \right) d\mu \\ &- \operatorname{sign}(x) \int e^{i\mu y} p_3(\mu) \Big(a_+ e^{-|x|\gamma_-} - a_- e^{-|x|\gamma_+} \Big) d\mu, \end{aligned} \tag{3.50}$$

$$\begin{aligned} p_2(\mu) &= \pi \sum_{j=0}^{\infty} \alpha_j (-i)^j (j+1) \frac{J_{j+1}(\mu)}{\mu}, \\ p_3(\mu) &= \pi \sum_{j=0}^{\infty} \beta_j (-i)^j \frac{(j+3)!}{j!} \frac{J_{j+2}(\mu)}{\mu^2}. \end{aligned} \tag{3.51}$$

The integrals over t in equation (3.39) are rewritten in terms of Bessel functions. Properties of the integral equations lead to superpower decrease of coefficients α_j, β_j.

The main characteristics of scattered field at infinity is its far field amplitude. The saddle point method (1.22) for the integral (3.50), when $k_0 r \to \infty$ is applicable. The factors $e^{-|x|\gamma_+}$ in each integral form exponentially small contributions, the exponentials $e^{-|x|\gamma_-}$ give circular wave

$$w^{(s)} \sim \sqrt{\frac{2\pi}{k_0\rho}} e^{ik_0\rho - i\pi/4} \psi_0(\varphi).$$

The far field amplitude $\psi_0(\varphi)$ is expressed by the values of functions p_2 and p_3 in the stationary point, that is for $\mu = k_0 \cos\varphi$

$$\psi_0(\varphi) = A_-(\varphi) p_2(k_0 \cos\varphi) + A_+(\varphi) \sin\varphi \, p_3(k_0 \cos\varphi), \tag{3.52}$$

where

$$A_\pm(\varphi) = (1-\sigma)\cos^2\varphi \pm 1.$$

The formula (3.52) is valid for any incident field which characteristics are implicitly presented in functions $p_n(\mu)$ defined by the solutions α_j, β_j of the infinite algebraic systems (3.46). Let the incident field be the plane wave

$$w^{(i)} = A\exp\Big(ik_0(x\sin\varphi_0 + y\cos\varphi_0)\Big).$$

If the crack is short that is if $k_0 a \ll 1$ it is possible to derive the asymptotics of the far field amplitude $\psi(\varphi)$ from the algebraic systems (3.46). For this purpose the coefficients A_j^m and B_j^m can be replaced by their asymptotics. To obtain the leading terms of the asymptotic expansion for $\psi_0(\varphi)$ it is sufficient to notice that $B_j^m = o(k_0^3)$ and to replace all B_j^m by zeros. Indeed, changing the integration variable from μ to $\tau = \mu/k_0$ one finds that $H_3'(s) = O\left(k_0^4\ln(k_0 s)\right)$. Therefore the second formula in (3.47) yields $B_j^m = O(k_0^4\ln k_0)$. For A_j^m it is necessary to obtain a more precise estimate

$$A_0^0 = \frac{i\pi}{32}k_0^2(3\sigma^2 + 2\sigma + 3) + O(k_0^4), \qquad A_j^m = O(k_0^4), \quad j + m > 0.$$

This estimate may be derived by examining expansion of kernel $H_2'(y-t)$ in the neighborhood of $t = y$. Again changing integration variable to $\tau = \mu/k_0$ yields

$$H_2' \approx k_0^2 \int \left(\frac{\left((1-\sigma)\tau^2 - 1\right)^2}{\sqrt{\tau^2 - 1}} - \frac{\left((1-\sigma)\tau^2 + 1\right)^2}{\sqrt{\tau^2+1}} + \chi|\tau| \right) d\tau + o(1). \tag{3.53}$$

The above integral can be calculated in closed form. In particular its imaginary part is equal to

$$2k_0^2 \int_0^1 \frac{\left((1-\sigma)\tau^2 - 1\right)^2}{\sqrt{1-\tau^2}} d\tau = \frac{\pi}{8}(3\sigma^2 + 2\sigma + 3)k_0^2$$

and its real part after cumbersome derivations appears equal to zero. Substituting from (3.53) into the first formula (3.47) gives the above asymptotics of A_j^m.

The right-hand sides $f_j^{(2)}$ and $f_j^{(3)}$ for the incident plane wave $w^{(i)}$ are expressed via Bessel functions. That gives the asymptotics

$$f_j^{(2)} = -A\, i^j (j+1) A_-(\varphi_0) \frac{J_{j+1}(k_0\cos\varphi_0)}{k_0\cos\varphi_0}$$

$$\sim -A\, i^j \frac{j+1}{2} A_-(\varphi_0)\left(k_0\cos\varphi_0\right)^j\,,$$

$$f_j^{(3)} = A\, i^{j+1}\frac{(j+3)!}{2j!}A_+(\varphi_0)\sin\varphi_0\frac{J_{j+2}(k_0\cos\varphi_0)}{(k_0\cos\varphi_0)^2}$$

$$\sim A\, i^{j+1}\frac{(j+3)!}{8j!}A_+(\varphi_0)\sin\varphi_0\left(\tfrac{1}{2}k_0\cos\varphi_0\right)^j\,.$$

The Bessel functions in expressions (3.51) may be replaced by the first terms of their Taylor series. All this gives

$$\begin{aligned}\psi_0(\varphi) = A\, i\left(\frac{k_0 a}{2}\right)^2\frac{A_-(\varphi)A_-(\varphi_0)}{(1-\sigma)(3+\sigma)}\Bigg\{1+\left(\frac{k_0 a}{2}\right)^2\cos\varphi\cos\varphi_0 \\ +i\frac{\pi}{8}\left(\frac{k_0 a}{2}\right)^2\frac{3\sigma^2+2\sigma+3}{(1-\sigma)(3+\sigma)}+O\big((k_0a)^4\big)\Bigg\} \\ +2i\left(\frac{k_0 a}{2}\right)^4\frac{A_+(\varphi)\sin\varphi A_+(\varphi_0)\sin\varphi_0}{(1-\sigma)(3+\sigma)}\Big\{1+O\big((k_0a)^2\big)\Big\}.\end{aligned} \tag{3.54}$$

In the asymptotics (3.54) the former dimensioned variables are used.

The effective cross section is defined as the portion of energy scattered by the inhomogeneity and is expressed via the far field amplitude in two ways (1.43). This equality allows the result (3.54) and all numerical results presented below to be checked. The asymptotics of the effective cross-section of a short crack is as follows:

$$\Sigma = \frac{\pi}{32}k_0^3a^4\frac{3\sigma^2+2\sigma+3}{(1-\sigma)^2(3+\sigma)^2}A_-^2(\varphi_0)+O(k_0^5a^6).$$

Numerical calculations show that, for not long cracks with $k_0a \le 5$ in order to have an error in optical theorem less than 1%, it is possible to truncate the systems (3.46) at the 10-th equation. That is, it is necessary to solve four systems of not more than five equations each. The integrals (3.48), (3.49) representing matrices **A** and **B** were calculated applying basic integrals subtraction method to improve convergence at infinity.

The obtained results permit to discover some particular features of the field diffracted by a crack. Figure 3.4 presents the effective cross-section Σ against the length of the crack for incidence angles $\varphi_0 = 0°$, $5°$, $45°$ and $90°$. The effective cross-section is normalized by the "visible" cross-section of the crack $\Sigma_0 = a\sin\varphi_0$ when $\varphi_0 \neq 0$ and by $0.1a$ otherwise. These curves are characterized by presence of maximums approximately at $k_0a = 2$–2.5. Similar maximum exists for diffraction of acoustics waves on a segment with

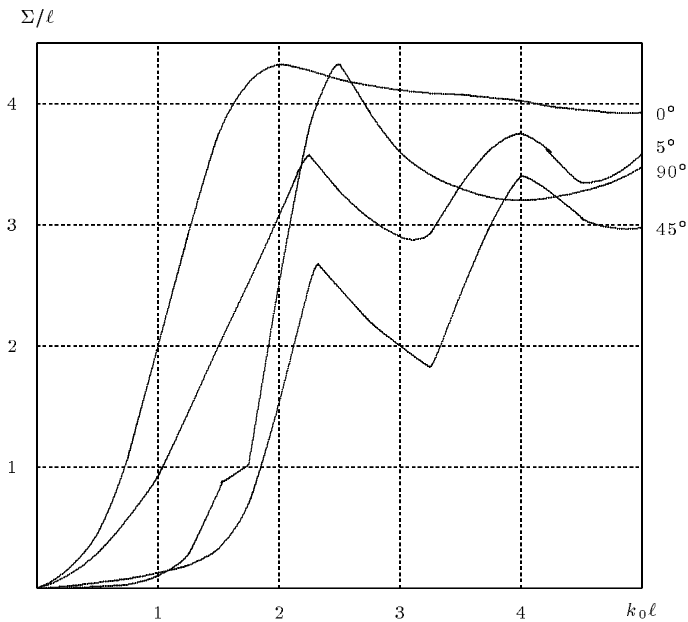


Fig. 3.4 Normalized effective cross-section

Neumann boundary condition [36] and can be explained by the interference of waves scattered by the ends of the segment. This observation means that the applied in this investigation long wave approach leads to formulae that are valid at the beginning of the short wave region.

Note that the edge waves of Rayleigh type (see formulae (1.75), (1.72) in Section 1.5.2) do not cause any resonances. This result may be established either by examining Fig. 3.4 or concluded from the fact that the integral equations are solvable for any complex values of parameter k_0.

Apart from the noted similarity of diffraction processes of flexural and acoustic waves scattering there are some differences. When an acoustic incident wave goes along the obstacle the scattered field vanishes, but when flexural wave goes along the crack the effective cross section is not equal to zero. This is due to the propriety of elastic deformations. The fact is that deformation of elastic body in one direction causes normal stresses in all other directions to appear, see (1.1). This effect is characterized by Poisson's ratio σ and exists for any $\sigma \neq 0$.

The formula for scattered field makes it possible to calculate the *stress*

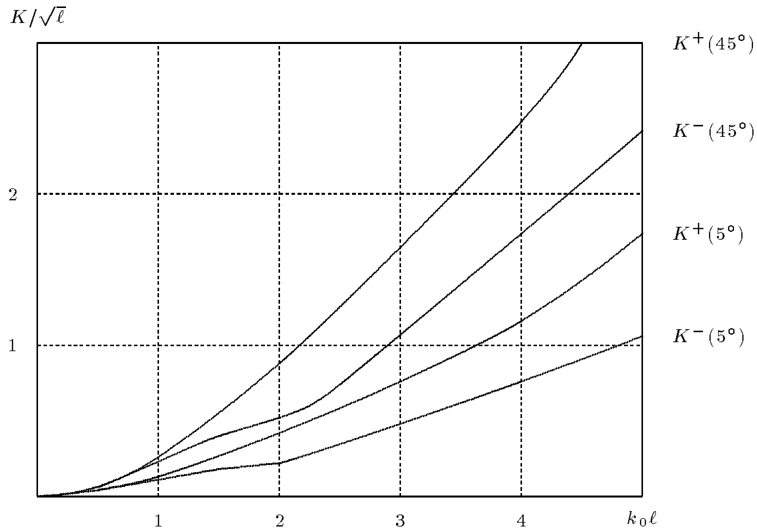


Fig. 3.5 Stress intensity coefficients

intensity coefficients. These coefficients are important for the theory of material failure and characterize singularity of stress tensor (see Section 1.1):

$$K^{\pm} = \lim_{y \to \pm(a+0)} \sqrt{2\pi(y \mp a)}\, \sigma_{xx}(y, 0)$$

By using the following proprieties of orthogonal polynomials [60]

$$-\frac{1}{\pi}\frac{d^2}{dy^2}\int_{-1}^{1} \ln|y-t| U_j(t)\sqrt{1-t^2}dt = |y|\frac{U_j(y)}{\sqrt{y^2-1}} + \operatorname{sign}(y)\sqrt{y^2-1}\frac{dU_j(y)}{dy} - \frac{j+1}{2}U_j(y), \qquad |y| > 1,$$

it is not difficult to derive

$$K^{\pm} = 2\pi^{3/2} iEh\sqrt{a}\frac{1-\sigma}{1+\sigma}\sum_{j=0}^{\infty}\alpha_j(\pm 1)^j(j+1).$$

Here bending stiffness D is expressed via Young modulus E and thickness of the plate h. The above representations allow the asymptotics for $a \to 0$

to be found

$$K^{\pm} = -A\sqrt{\pi}k_0^2 a^{5/2} Eh \frac{A_-(\varphi_0)}{(1-\sigma)(3+\sigma)} + O\left(k_0^4 a^{9/2}\right).$$

The numerically calculated stress intensity coefficients are presented on Fig. 3.5 for two angles of incidence at $\varphi_0 = 5°$ and $\varphi_0 = 45°$.

3.3.2 *Generalized point model of short crack*

Isolated plate

The scattering process by a short crack in an isolated plate can be simulated by means of appropriate zero-range potential of operator B_2^S. Parameters that define the operator B_2^S, namely Hermitian matrix $\mathbf{S}$ from (2.26) can be chosen with the help of the far field amplitude asymptotics constructed in the previous section.

The problem of scattering by a zero-range potential of operator B_2^S is in finding such scattered field that satisfies the equation

$$\triangle^2 w^{(s)} - k_0^2 w^{(s)} = 0, \qquad (x,y) \neq (0,0)$$

and combined with the incident wave $w = w^{(i)} + w^{(s)}$ has the asymptotics (2.25) with coefficients satisfying condition (2.26) or (2.27) with a given matrix $\mathbf{S}$.

The general solution of the above equation that locally belongs to $L_2(\mathbb{R}^2)$ has the form of multipole decomposition

$$w^{(s)} = \sum_{i,j=0}^{i+j\le 2} c_{ij} \frac{\partial^{i+j}}{\partial x^i \partial y^j} g_0(x,y;0,0),$$

where $g_0(\boldsymbol{\rho},\boldsymbol{\rho}_0)$ is the Green's function defined by (1.45). The far field amplitude (3.54) allows coefficients of this decomposition to be found. Replacing the Green's function in the above multipole decomposition by its asymptotics for large $\rho = \sqrt{x^2+y^2}$ yields expression for the far field amplitude

$$\begin{aligned}\psi(\varphi) = \frac{i}{8\pi k_0^2 D}\Big(&c_{00} + ik_0 c_{10}\sin\varphi + ik_0 c_{01}\cos\varphi \\ &- k_0^2 c_{20}\sin^2\varphi - k_0^2 c_{11}\sin\varphi\cos\varphi - k_0^2 c_{02}\cos^2\varphi\Big).\end{aligned} \tag{3.55}$$

Comparing it with asymptotics (3.54), firstly from the point of view of symmetry, one concludes that

$$c_{10} = c_{01} = c_{11} = 0.$$

Then, analysis of angular factors shows that (3.55) can not reproduce some terms of (3.54) that have the order $O\big((k_0 a)^4\big)$. Therefore let no terms of this order be used for further comparison. Finally the coefficients c_{20} and c_{02} are expressed via c_{00} up to smaller order terms

$$c_{20} = k_0^{-2} c_{00} + \sigma Q + o\big((k_0 a)^2\big), \quad c_{02} = k_0^{-2} c_{00} + Q + o\big((k_0 a)^2\big), \quad (3.56)$$

where Q is defined by the incident plane flexural wave

$$Q = -2\pi D (k_0 a)^2 \frac{\sin^2 \varphi_0 + \sigma \cos^2 \varphi_0}{(1-\sigma)(3+\sigma)}.$$

Consider now the system (2.26). Excluding zero amplitudes c_{10}, c_{01} and c_{11} it can be written in the form

$$\begin{pmatrix} c_{00} \\ c_{20} \\ c_{02} \end{pmatrix} = \begin{pmatrix} R_{11} & R_{14} & R_{16} \\ R_{41} & R_{44} & R_{46} \\ R_{61} & R_{64} & R_{66} \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{20} \\ b_{02} \end{pmatrix}. \quad (3.57)$$

Here matrix $\mathbf{R}$ is the inversion of $\mathbf{S}$ from (2.26). The coefficients of regular part of solution appear not only due to the incident wave, but also from the scattered field. Using asymptotics of Bessel functions [1] (see page 63)

$$\begin{aligned} J_0(\xi) &= \sum_{j=0}^{\infty} \frac{(-\xi^2/4)^j}{(j!)^2} = 1 - \frac{\xi^2}{4} + \ldots, \\ Y_0(\xi) &= \frac{2}{\pi}\Big(\ln(\xi/2) + \gamma\Big) J_0(\xi) + \frac{2}{\pi}\left(\frac{\xi^2}{4} - \frac{3\xi^4}{64} + \ldots\right) \end{aligned}$$

one finds

$$\begin{aligned} b_{00} &= 1 + \frac{i}{8k_0^2 D} c_{00} + \left(Z + \frac{3}{8\pi D}\right) c_{20} + \left(Z + \frac{3}{8\pi D}\right) c_{02}, \\ b_{20} &= -k_0^2 \cos^2 \varphi_0 + Z c_{00} + \frac{3i}{64D} k_0^2 c_{20} + \frac{i}{64D} k_0^2 c_{02}, \\ b_{02} &= -k_0^2 \sin^2 \varphi_0 + Z c_{00} + \frac{i}{64D} k_0^2 c_{20} + \frac{3i}{64D} k_0^2 c_{02}. \end{aligned}$$

Here

$$Z = \frac{1}{4\pi D}\Big(\ln(k_0/2) + C_E - 1 - i\pi/4\Big).$$

With the help of formula (3.56) one expresses all the coefficients of regular part of the field $w(x, y)$ via the monopole amplitude c_{00}

$$\begin{aligned}
b_{00} &= 1 + \left(Z + \frac{3}{8\pi D}\right)(1+\sigma)Q + 2Z' k_0^{-2} c_{00},\\
b_{20} &= -k_0^2 \cos^2\varphi_0 + \frac{i}{64D} k_0^2 (3+\sigma)Q + Z c_{00},\\
b_{02} &= -k_0^2 \sin^2\varphi_0 + \frac{i}{64D} k_0^2 (1+3\sigma)Q + Z c_{00}.
\end{aligned}$$

Here

$$Z' = Z + \frac{3}{8\pi D} + \frac{i}{16D}.$$

Substituting these values into the system (3.57) and expressing c_{20} and c_{02} via c_{00} yields

$$\begin{aligned}
c_{00} = \left(R_{11}\frac{2Z'}{k_0^2} + R_{14}Z + R_{16}Z\right) c_{00} + R_{11} - R_{14}k_0^2\cos^2\varphi_0 \\
- R_{16}k_0^2\sin^2\varphi_0 + R_{11}\left(Z + \frac{3}{8\pi D}\right)(1+\sigma)Q \\
+ R_{14}\frac{ik_0^2}{64}(3\sigma+1)Q + R_{16}\frac{ik_0^2}{64}(\sigma+3)Q,
\end{aligned}$$

$$\begin{aligned}
\frac{c_{00}}{k_0^2} + \sigma Q = \left(R_{41}\frac{2Z'}{k_0^2} + R_{44}Z + R_{46}Z\right) c_{00} + R_{41} - R_{44}k_0^2\cos^2\varphi_0 \\
- R_{46}k_0^2\sin^2\varphi_0 + R_{41}\left(Z + \frac{3}{8\pi D}\right)(1+\sigma)Q \\
+ R_{44}\frac{ik_0^2}{64}(3\sigma+1)Q + R_{46}\frac{ik_0^2}{64}(\sigma+3)Q,
\end{aligned}$$

$$\begin{aligned}
\frac{c_{00}}{k_0^2} + Q = Z\left(R_{61}\frac{2Z'}{k_0^2} + R_{64}Z + R_{66}Z\right) c_{00} + R_{61} - R_{64}k_0^2\cos^2\varphi_0 \\
- R_{66}k_0^2\sin^2\varphi_0 + R_{61}\left(Z + \frac{3}{8\pi D}\right)(1+\sigma)Q \\
+ R_{64}\frac{ik_0^2}{64}(3\sigma+1)Q + R_{66}\frac{ik_0^2}{64}(\sigma+3)Q.
\end{aligned}$$

The matrix **S** and consequently **R** of the generalized model should not depend on the incident field. In particular on wave number k_0 and angle of incidence φ_0. Assuming R_{ij} not depending on k_0 allows the above equations to be split into equations for the coefficients of different powers of k_0. It is easy to conclude from the above system that c_{00} is decomposed into series in even powers of k_0,

$$c_{00} = c_2 k_0^2 + c_4 k_0^4 + \ldots,$$

with coefficients c_2, c_4 possibly containing only logarithms of k_0. Neglecting presence of $\ln(k_0)$ in the coefficient Z one finds in the principal order

$$\begin{cases} 0 = R_{11} 2Z' c_2 + R_{11}, \\ c_2 = R_{41} 2Z' c_2 + R_{41}, \\ c_2 = R_{61} 2Z' c_2 + R_{61}. \end{cases}$$

This is only possible if

$$R_{11} = 0, \qquad R_{41} = R_{61}, \qquad c_2 = \frac{R_{41}}{1 - 2Z' R_{41}}. \tag{3.58}$$

Consider the next order equations. Substituting from (3.58) into the first equation and accounting for the symmetry $R_{14} = \overline{R_{41}}$, $R_{16} = \overline{R_{61}}$ yields

$$\text{Re}\,(R_{41}) = 2\left(Z + \frac{3}{8\pi D} + \frac{i}{16D}\right)|R_{41}|^2.$$

Remember now that quantity Z depends on k_0 and R_{41} does not. Therefore the above equation is possible only if $R_{41} = 0$. The other two equations of the same order k_0^2 take the form

$$\sigma Q = -R_{44} k_0^2 \cos^2\varphi_0 - R_{46} k_0^2 \sin^2\varphi_0,$$
$$Q = -R_{64} k_0^2 \cos^2\varphi_0 - R_{66} k_0^2 \sin^2\varphi_0.$$

Equating coefficients of linear independent trigonometric functions $\cos^2\varphi_0$ and $\sin^2\varphi_0$ allows the remaining coefficients of the matrix **R** to be determined. One finds

$$\begin{pmatrix} R_{44} & R_{46} \\ R_{64} & R_{66} \end{pmatrix} = \frac{2\pi a^2}{(1-\sigma)(3+\sigma)} \begin{pmatrix} \sigma^2 & \sigma \\ \sigma & 1 \end{pmatrix}. \tag{3.59}$$

The far field amplitude in the problem of flexural wave scattering by the zero-range potential B_2^R corresponding to matrix **R** with the only nonzero elements given in (3.59) can be easily found. For that one should solve the system of two equations

$$\begin{pmatrix} c_{20} \\ c_{02} \end{pmatrix} = \begin{pmatrix} R_{44} & R_{46} \\ R_{64} & R_{66} \end{pmatrix} \begin{pmatrix} b_{20} \\ b_{02} \end{pmatrix}$$

and substitute coefficients c_{20} and c_{02} into the formula (3.55) for the far field amplitude. This yields exact expression

$$\psi(\varphi;\varphi_0) = i\left(\frac{k_0 a}{2}\right)^2 \frac{(\sin^2\varphi_0 + \sigma\cos^2\varphi_0)(\sin^2\varphi + \sigma\cos^2\varphi)}{(1-\sigma)(3+\sigma) - i\pi/32\,(k_0 a)^2(3+2\sigma+3\sigma^2)}.$$

This formula is reciprocal (see page 39) and exactly satisfies the optical theorem. Both these properties automatically follow from the selfadjointness of the operator B_2^R.

Fluid loaded plate

The generalized point model of short crack constructed in the previous section can be easily transformed to the case of fluid loaded plate. According to the proclaimed HYPOTHESIS the zero-range potential of operator $\mathcal{H}_3^S$ has block structure. These blocks correspond to the zero-range potentials in the components. The first component characterizes the obstacle in absolutely rigid screen. Infinitely narrow crack in a rigid screen does not manifest itself. Indeed, in classical formulation of the boundary conditions on such crack only displacements are used and the latter are identically zeros if the plate is absolutely rigid. Therefore acoustic passive source is not presented in the model. Therefore $S = 0$. The block $\mathbf{S}$ is inherited from the model for isolated plate. All this allows the generalized model of short crack in fluid loaded elastic plate to be suggested. This model specifies the asymptotics of total displacements in the form (2.37) with

$$c_{00} = c_{10} = c_{01} = c_{11} = 0,$$

coefficients b_{00}, b_{10}, b_{01} and b_{11} arbitrary and

$$c_{20} = \sigma c_{02} = \frac{2\pi a^2}{(1-\sigma)(3+\sigma)}\Big(\sigma b_{20} + b_{02}\Big).$$

The problem of scattering by the generalized model of short crack in fluid loaded elastic plate is formulated as follows. The scattered field satisfies Helmholtz equation

$$\left(\Delta + k^2\right)U^{(s)}(x,y,z) = 0, \qquad z > 0,$$

boundary condition on the plate

$$\left(\Delta^2 - k_0^4\right) w^{(s)}(x,y) + U^{(s)}(x,y,0) = 0, \qquad (x,y) \neq (0,0),$$

where $w^{(s)} = \varrho_0^{-1}\omega^{-2}\partial U^{(s)}(x,y,0)/\partial z$, radiation conditions (1.16). The conditions on the obstacle are formulated for the total flexure displacements, namely $w = w^{(g)} + w^{(s)}$ should have the asymptotics

$$\begin{aligned} w \sim \frac{\sigma b_{20} + b_{02}}{(1-\sigma)(3+\sigma)} \frac{a^2}{2} \left((1+\sigma)\ln \rho/a + \sigma \cos^2 \varphi + \sin^2 \varphi\right) \\ + b_{00} + b_{10}x + b_{01}y + b_{20}\frac{x^2}{2} + b_{11}xy + b_{02}\frac{y^2}{2} + \ldots, \quad \rho \to 0 \end{aligned} \tag{3.60}$$

with arbitrary coefficients b_{ij}.

3.3.3 *Scattering by the generalized point model of short crack*

Consider the problem of scattering by the generalized model of short crack. Let the length of the crack be $2a$. This dimension is not taken into account by the geometry of the problem, but is involved in the condition (3.60) specified at the potential center. Consider the case of incident acoustic wave

$$U^{(i)} = A \exp\Big(ikx \cos\vartheta_0 \cos\varphi_0 + iky \cos\vartheta_0 \sin\varphi_0 - ikz \sin\vartheta_0\Big).$$

The scattered field is searched as usually in two steps. First the general solution is introduced. It satisfies all the equations of the problem except the asymptotics (3.60) and has the form of multipole decomposition. In the general case of scattering by a zero-range potential in fluid loaded plate there could be 7 passive sources. One is a center of extension in acoustic media and the others are sourced applied to the plate. The amplitudes of these sources are presented directly in the asymptotics, specified in a vicinity of potential center. In the generalized model of short crack only two passive sources are presented. Therefore the general solution is given by the following multipole decomposition

$$U^{(s)}(\mathbf{r}) = c_{20}\frac{\partial^2}{\partial x^2}G(\mathbf{r};0,0) + c_{02}\frac{\partial^2}{\partial y^2}G(\mathbf{r};0,0). \tag{3.61}$$

Here $G(\mathbf{r}; x_0, y_0)$ is the Green's function (1.21) corresponding to the point source on the plate.

To find the amplitudes of passive sources the asymptotics of displacements $w^{(s)}(\boldsymbol{\rho})$ in the general solution (3.61) as $\rho \to 0$ is required. Comparing coefficients in that asymptotics with that prescribed by the generalized model yields the system of algebraic equations for amplitudes c_{20} and c_{02}.

Consider displacements $g(\boldsymbol{\rho}; 0, 0)$ caused by the point source applied to the plate

$$g(\boldsymbol{\rho}; 0, 0) = \frac{1}{D\varrho_0\omega^2} \frac{1}{4\pi^2} \iint e^{i\lambda x + i\mu y} \frac{\sqrt{\lambda^2 + \mu^2 - k^2}}{L(\lambda, \mu)} \, d\lambda \, d\mu.$$

One can also introduce new integration variables τ and α by the formulae

$$\lambda = \tau \cos\alpha, \qquad \mu = \tau \sin\alpha$$

and discover that integral by α gives Bessel function

$$\begin{aligned} g &= \frac{1}{D\varrho_0\omega^2} \frac{1}{4\pi^2} \iint e^{i\lambda x + i\mu y} \frac{\sqrt{\lambda^2 + \mu^2 - k^2}}{L(\lambda, \mu)} d\lambda \, d\mu \\ &= \frac{1}{D\varrho_0\omega^2} \frac{1}{2\pi} \int\limits_0^\infty J_0(\tau\rho) \frac{\sqrt{\tau^2 - k^2}}{L(\tau)} \tau \, d\tau. \end{aligned}$$

In order to find the asymptotics of $g(\boldsymbol{\rho}; 0, 0)$ as $\rho \to 0$ it is convenient to extract the Green's function $g_0(\boldsymbol{\rho}; 0, 0)$ for isolated plate given in (1.45). For this represent the fraction in the above integral as

$$\frac{1}{(\lambda^2 + \mu^2)^2 - k_0^4} + \frac{N}{L(\lambda, \mu)\Big((\lambda^2 + \mu^2)^2 - k_0^4\Big)}.$$

Using the property of Bessel functions

$$H_0^{(1)}(t) - H_0^{(1)}(e^{i\pi} t) = 2J_0(t),$$

the integral with the first fraction can be rewritten as

$$\frac{1}{D\varrho_0\omega^2} g_0 = \frac{1}{D\varrho_0\omega^2} \frac{1}{2\pi} \int H_0^{(1)}(\tau\rho) \frac{\tau \, d\tau}{\tau^4 - k_0^4}.$$

and calculated as the sum of residues in the poles $\tau = k_0$ and $\tau = ik_0$, which gives formula (1.45) for $g_0(\boldsymbol{\rho}; 0, 0)$ (the branch cut of function $H_0^{(1)}$

is below the path of integration). Therefore, one finds

$$g(\boldsymbol{\rho};0,0)=\frac{1}{D\varrho_0\omega^2}g_0(\boldsymbol{\rho};0,0)+\frac{1}{D^2}\frac{1}{2\pi}\int\limits_0^\infty J_0(\tau\rho)\frac{\tau\,d\tau}{L(\tau)\,(\tau^4-k_0^4)}.$$

The asymptotics of $g_0(\boldsymbol{\rho};0,0)$ was already used when deriving expressions for coefficients b_{00}, b_{20} and b_{02} on page 153. It reads

$$g_0(\boldsymbol{\rho};0,0)=\frac{1}{8\pi D}\rho^2\ln\rho+\frac{i}{8k_0^2D}+Z\frac{\rho^2}{2}+\frac{k_0^2\rho^4}{256D}+o(\rho^4).$$

In order to derive the asymptotics of the correction one can decompose Bessel function into Taylor series. In the terms up to the 4-th the integrals converge. This finally yields

$$\begin{aligned}g(\boldsymbol{\rho};0,0)&=\frac{1}{8\pi D\varrho_0\omega^2}\rho^2\ln\rho+\frac{i}{8k_0^2D\varrho_0\omega^2}+\frac{D_0'}{D^2}\\&+\left(\frac{Z}{2\varrho_0\omega^2}+\frac{D_2'}{4D^2}\right)\rho^2+\left(\frac{k_0^2}{256D\varrho_0\omega^2}+\frac{D_4'}{64D^2}\right)\rho^4+o(\rho^4),\end{aligned}\tag{3.62}$$

where

$$D_\ell'=\frac{1}{2\pi}\int\limits_0^\infty\frac{(i\tau)^\ell\tau\,d\tau}{L(\tau)(\tau^4-k_0^4)}.$$

Note that the value $g(0,0;0,0)=(D\varrho_0\omega^2)^{-1}D_{00}$ (see page 52) can be computed by setting $x=y=0$ in the double Fourier integral. That is,

$$D_{00}=\frac{i}{8k_0^2}+ND_0'.$$

Using formula (3.62) it is a simple matter to find asymptotics of displacements $w^{(s)}(\boldsymbol{\rho})$ in the general solution. Comparing it with the asymptotics (3.60) specified by generalized model of short crack, one finds equations for amplitudes c_{20}, c_{02} and coefficients b_{20} and b_{02}. Namely equating coefficients of singular terms yields

$$\begin{aligned}\frac{c_{20}}{4\pi D\varrho_0\omega^2}&=\frac{\sigma b_{20}+b_{02}}{(1-\sigma)(3+\sigma)}\frac{a^2}{2}\sigma,\\\frac{c_{02}}{4\pi D\varrho_0\omega^2}&=\frac{\sigma b_{20}+b_{02}}{(1-\sigma)(3+\sigma)}\frac{a^2}{2},\end{aligned}$$

wherefrom in particular follows

$$c_{20} = \sigma c_{02}.$$

Equating coefficients of x^2 and y^2 in the asymptotics of the total field yields

$$b_{20} = 8\left(\frac{k_0^2}{256 D\varrho_0\omega^2} + \frac{D_4'}{64D^2}\right)\Big(3c_{20} + c_{02}\Big) + \frac{\partial^2 w^{(g)}(0,0)}{\partial x^2},$$
$$b_{02} = 8\left(\frac{k_0^2}{256 D\varrho_0\omega^2} + \frac{D_4'}{64D^2}\right)\Big(c_{20} + 3c_{02}\Big) + \frac{\partial^2 w^{(g)}(0,0)}{\partial y^2}.$$

Solving that system and taking into account that

$$w^{(g)} = -\frac{2A}{D}\frac{ik\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}\exp\Big(ikx\cos\vartheta_0\cos\varphi_0 + iky\cos\vartheta_0\sin\varphi_0\Big)$$

allows the amplitudes of passive sources to be found

$$c_{02} = \left(1 - \frac{\pi}{4}\frac{3\sigma^2+2\sigma+3}{(1-\sigma)(3+\sigma)}\left(\frac{k_0^2}{4} + N\,D_4'\right)a^2\right)^{-1}\times$$
$$\times\frac{4i\pi\varrho_0\omega^2 a^2}{(1-\sigma)(3+\sigma)}\frac{k^3\sin\vartheta_0\cos^2\vartheta_0}{\mathcal{L}(\vartheta_0)}\Big(\sigma\cos^2\varphi_0 + \sin^2\varphi_0\Big).$$

The scattered field forms an outgoing spherical wave with the far field amplitude $\Psi(\vartheta,\varphi)$. Using the formula (1.24) one finds

$$\Psi(\vartheta,\varphi) = A\frac{ikNa^2}{\pi(1-\sigma)(3+\sigma)}\frac{k^3\sin\vartheta_0\cos^2\vartheta_0}{\mathcal{L}(\vartheta_0)}\frac{k^3\sin\vartheta\cos^2\vartheta}{\mathcal{L}(\vartheta)}\times$$
$$\times\Big(\sigma\cos^2\varphi_0+\sin^2\varphi_0\Big)\Big(\sigma\cos^2\varphi+\sin^2\varphi\Big)\times \tag{3.63}$$
$$\times\left(1 - \frac{\pi}{4}\frac{3\sigma^2+2\sigma+3}{(1-\sigma)(3+\sigma)}\left(\frac{k_0^2}{4} + N\,D_4'\right)a^2\right)^{-1}.$$

The far field amplitude of surface circular wave can be found by computing residue of the above expression in the point $\vartheta = \kappa/k$ according to formula (1.26). Note that expression (3.63) for the far field amplitude is symmetric and can be checked to satisfy the optical theorem. These properties follow directly from the selfadjointness of the operator in the formulation of the generalized model of short crack.

In the following section by means of asymptotic analysis of diffraction by a short crack in fluid loaded elastic plate we check the formula (3.63), that is check applicability of the HYPOTHESIS .

3.3.4 *Diffraction by a short crack in fluid loaded plate*

Consider the problem of acoustic wave diffraction by an elastic plate with the short crack $\Lambda = \{x = 0, |y| < a\}$. The boundary-value problem consists of Helmholtz equation

$$\triangle U + k^2 U = 0, \qquad z > 0$$

with the boundary condition on the plate

$$\triangle^2 w - k_0^4 w + \frac{1}{D} U|_{z=0} = 0, \quad (x, y) \notin \Lambda, \tag{3.64}$$

where flexural displacement is defined by adhesion condition

$$w = \frac{1}{\varrho_0 \omega^2} \left. \frac{\partial U}{\partial z} \right|_{z=0}.$$

The radiation conditions (1.16) are specified for the scattered field and the conditions on the crack are formulated for displacement w in the same way as in the case of isolated plate

$$\mathbb{F} w(\pm x, y) = 0, \quad \mathbb{M} w(\pm x, y) = 0, \qquad |y| < a.$$

The Meixner conditions are specified for the acoustic pressure in a vicinity of the crack Λ and for displacements w near the ends of the crack, see (1.18) and (1.19).

By co-ordinates scaling the length a can be made equal to 1, which is accepted below. The scattered field $U^{(s)}$ can be expressed as a convolution of Green's function (1.20) with unknown displacements and bending angles on the edges of the crack as explained in Section 1.3.1. However this requires regularization of integral equations and another approach is used here. Let the field $U^{(s)}$ be searched in the form of double Fourier transform

$$U^{(s)}(\mathbf{r}) = \frac{1}{4\pi^2} \iint e^{i\lambda x + i\mu y} e^{-\sqrt{\lambda^2+\mu^2-k^2} z} \frac{P(\lambda, \mu)}{L(\lambda, \mu)} d\lambda \, d\mu. \tag{3.65}$$

Here $P(\lambda, \mu)$ is unknown and the exponential is chosen as the solution of Helmholtz equation corresponding to outgoing waves. The displacement $w^{(s)}$ is given then by Fourier integral

$$w^{(s)}(x, y) = -\frac{1}{\varrho_0 \omega^2} \frac{1}{4\pi^2} \iint e^{i\lambda x + i\mu y} \frac{\sqrt{\lambda^2 + \mu^2 - k^2}}{L(\lambda, \mu)} P(\lambda, \mu) \, d\lambda \, d\mu.$$

By substituting this representation into the boundary condition on the plate at $y \neq 0$ and taking into account that the order of differential operator by x in that condition is equal to 4, allows the function $P(\lambda, \mu)$ to be concluded polynomial by λ of degree 3. That is

$$P(\lambda, \mu) = p_0(\mu) + p_1(\mu)\lambda + p_2(\mu)\lambda^2 + p_3(\mu)\lambda^3.$$

For such $P(\lambda, \mu)$ with arbitrary $p_j(\mu)$ the boundary condition (3.64) is satisfied everywhere except the line $x = 0$. Requirement of condition (3.64) at $\{x = 0,\ |y| > 1\}$ (note that the co-ordinates are scaled to have $a = 1$) yields

$$\int e^{i\mu y} p_j(\mu)\, d\mu = 0, \qquad |y| > 1. \tag{3.66}$$

Similarly to the case of isolated plate conditions on edges of the crack can be rewritten for the jumps of force and momentum and for mean values of these quantities. The jumps are equal to zero on the whole line $x = 0$ and the integral equations expressing these conditions for jumps can be solved explicitly (inversion of Fourier transform), which yields

$$p_0(\mu) = \sigma\mu^2 p_2(\mu), \qquad p_1(\mu) = (2-\sigma)\mu^2 p_3(\mu).$$

The remaining contact conditions reduce to equations

$$\int e^{i\mu y} h_n(\mu) p_n(\mu) d\mu = f_n(y), \quad |y| < 1. \tag{3.67}$$

Here and below $n = 2, 3$ and the following functions are introduced

$$h_2(\mu) = \frac{2k_0^2}{\pi} \int \left(\mu^2 + \sigma\lambda^2\right)^2 \frac{\sqrt{\lambda^2 + \mu^2 - k^2}}{L(\lambda, \mu)}\, d\lambda,$$

$$h_3(\mu) = -\frac{2k_0^2}{\pi} \int \left(\mu^2 + (2-\sigma)\lambda^2\right)^2 \frac{\sqrt{\lambda^2 + \mu^2 - k^2}}{L(\lambda, \mu)}\, \lambda^2\, d\lambda,$$

$$f_2(y) = 8\pi N D k_0^2 \left(\frac{\partial^2 w^{(g)}(0, y)}{\partial x^2} + \sigma \frac{\partial^2 w^{(g)}(0, y)}{\partial y^2} \right),$$

$$f_3(y) = 8i\pi N D k_0^2 \left(\frac{\partial^3 w^{(g)}(0, y)}{\partial x^3} + (2-\sigma) \frac{\partial^3 w^{(g)}(0, y)}{\partial x \partial^2 y} \right).$$

Convergence of Fourier integrals and Meixner conditions (1.18) for acoustic pressure and for displacements eliminate possible behaviour of functions $p_j(\mu)$ at infinity, namely

$$p_n(\mu) = O(\mu^{1-n-\delta}), \qquad \delta > 0.$$

Further derivations follow the scheme of Section 3.3.1. To satisfy equations (3.66) one represents unknown functions $p_n(\mu)$ as Fourier transforms

$$p_n(\mu) = \int\limits_{-1}^{1} e^{-i\mu t} q_n(t)\, dt$$

and computes integrals by μ in (3.67). This results in integral equations of convolution on the interval $[-1, 1]$. The kernels of these integral equations are supersingular and regularization is needed. However using the formula

$$\frac{\sqrt{\lambda^2+\mu^2-k^2}}{L(\lambda,\mu)} = \frac{1}{(\lambda^2+\mu^2)^2 - k_0^4} + \frac{N}{L(\lambda,\mu)\left((\lambda^2+\mu^2)^2 - k_0^4\right)}$$

allows the singularities to be concluded the same as in the equations (3.42), (3.43) for the case of isolated plate. Indeed the above formula allows the functions $h_n(\mu)$ to be represented as

$$h_n(\mu) = h_n^{(0)}(\mu) + h_n^{(1)}(\mu),$$

where $h_n^{(0)}(\mu)$ are the symbols (Fourier transforms of kernels) of integral equations in the case of isolated plate and $h_n^{(1)}(\mu)$ rapidly decreasing at infinity corrections. These corrections give bounded contributions to the kernels $H_n(y-t)$. Therefore integral equations for $q_n(t)$ can be written as previously in the form of integro-differential equations

$$-2\chi \frac{d^2}{dy^2} \int\limits_{-1}^{1} \ln|y-t| q_2(t) dt + \int\limits_{-1}^{1} H_2'(y-t) q_2(t) dt = k_0^2 f_2(y), \tag{3.68}$$

$$-2\chi \frac{d^4}{dy^4} \int\limits_{-1}^{1} \ln|y-t| q_3(t) dt + \int\limits_{-1}^{1} H_3'(y-t) q_3(t) dt = k_0^3 f_3(y). \tag{3.69}$$

Here $\chi = (1-\sigma)(3+\sigma)$ and kernels $H_n'(s)$ are given by cumbersome expressions involving Fourier integrals

$$H_2'(s) = 2k_0^{-2}\int_0^\infty e^{i\mu s}\left(h_2(\mu) + \chi k_0^2\mu\right)d\mu,$$

$$H_3'(s) = 2k_0^{-2}\int_0^\infty e^{i\mu s}\left(h_3(\mu) - \chi k_0^2\mu^3\right)d\mu.$$

Solutions $q_2(t)$ and $q_3(t)$ of integral equations (3.68) and (3.69) are searched in the classes $\mathbb{S}_1$ and $\mathbb{S}_2$ (see Appendix B) which follows from the asymptotic behavior of $p_2(\mu)$ and $p_3(\mu)$ at infinity. Smooth components that differ the kernels of these equations from that of Section 3.3.1 do not violate the theorem 4 of existence and uniqueness of solutions. The numerical procedure may remain the same as in section 3.3.1. One can also check that the principal order contribution to the far field asymptotics is due to terms containing $q_2(t)$ and solution of integral equation (3.69) gives smaller order correction by ka. For checking the HYPOTHESIS it is sufficient to derive only the principal order term in the asymptotics of the far field amplitude. Therefore the leading order approximation for $q_2(t)$ is needed only. This approximation is

$$q_2(t) \approx \sqrt{1-t^2}\alpha_0,$$

where α_0 is the zero-th term in the decomposition (3.45) for $q_2(t)$. Substituting this approximation into the integral equation and calculating the left and the right-hand sides at $y = 0$ yields

$$\alpha_0\left(-2\chi\pi + \int_{-1}^{1} H_2'(t)\sqrt{1-t^2}dt\right) \approx k_0^2 f_2(0).$$

The correction to the kernel $H_2'(t)$ is a smooth function of ta and for small a it can be replaced by $H_2'(0)$. That yields

$$\alpha_0 \approx \frac{2}{\pi}\frac{k_0^2 f_2(0)}{H_2'(0) - 4\chi}.$$

Analysis of the integral for $H_2'(s)$ allows it to be concluded asymptotically small. Therefore it is further neglected.

Applying saddle point formula (1.23) to (3.65) one finds

$$\Psi(\vartheta,\varphi) \approx -\frac{k^4}{4\pi}\frac{\sin\vartheta\cos^2\vartheta}{\mathcal{L}(\vartheta)}\Big(\sigma\cos^2\varphi + \sin^2\varphi\Big)p_2(k\cos\vartheta\cos\varphi).$$

Calculating Fourier transform of the derived approximation for $q_2(t)$ and substituting it into the above formula yields the leading term of the far field amplitude asymptotics. In dimensioned variables it reads

$$\Psi(\vartheta,\varphi) \approx A\frac{ikNa^2}{\pi(1-\sigma)(3+\sigma)}\frac{k^3\sin\vartheta_0\cos^2\vartheta_0}{\mathcal{L}(\vartheta_0)}\frac{k^3\sin\vartheta\cos^2\vartheta}{\mathcal{L}(\vartheta)}\times$$
$$\times\Big(\sigma\cos^2\varphi_0 + \sin^2\varphi_0\Big)\Big(\sigma\cos^2\varphi + \sin^2\varphi\Big).$$

3.3.5 *Discussion*

One can note that the above approximate formula coincides with the far field amplitude of the field scattered by the generalized model of short crack up to terms of order $O\big((ka)^4\big)$. Thus one can conclude that the procedure of generalized models construction based on the main HYPOTHESIS is applicable in the case of short crack in fluid loaded elastic plate. The validity of the HYPOTHESIS was checked in the case of two-dimensional problems taking as an example scattering by narrow crack and in three-dimensional problem in the case of short crack. The following observation can explain the reason for the HYPOTHESIS to be valid. The parameterization of the operator extension in the form of operator with zero-range potential was chosen such that the parameters (real constant S and Hermitian matrix $\mathbf{S}$) do not depend on the characteristics of plate, fluid, incident field and objects located outside a small vicinity of the obstacle. These parameters contain only the size of the obstacle, depend on its type (boundary or contact conditions on its surface) and shape. Therefore it appears natural that these parameters remain the same in problems of scattering in two limiting cases of absolutely rigid plate and of isolated plate.

The other suggestion of the HYPOTHESIS sets the parameters s and s^* in (2.34) equal to zero. Physically these parameters describe additional interaction between acoustic pressure and flexure displacements that is caused by the obstacle. However in both problems considered in this chapter such interaction is not presented. Indeed the boundary conditions on the narrow crack can be separated into conditions specified for acoustic pressure only and conditions that involve flexure displacements only. No conditions on the crack contain both acoustic pressure and displacements simultaneously. This is the case of three-dimensional problem of scattering by a short crack as well. Therefore approximations of these conditions formulated as asymptotic decompositions near the zero-range potential center naturally contain no terms corresponding to additional interaction of acoustic pressure and flexure displacements.

Further in this chapter when considering scattering by a circular hole and by a short joint of two semi-infinite plates no checking of the HYPOTHESIS validity is performed. However the structure of boundary and contact conditions on the obstacles allows one to believe in the validity of the HYPOTHESIS and therefore in the applicability of the procedure of generalized models construction.

3.4 Model of small circular hole

Consider the problem of scattering by a circular hole in elastic plate. Let the edges of the hole be free and fluid occupy the halfspace $\{z > 0\}$ on one side of the plate. The radius a of the hole is supposed to be small. The problem is in finding the principal order terms in the asymptotics of scattered field by small ka. For that the hole can be replaced by an appropriate zero-range potential. The matrix $\mathcal{S}$ that parameterizes zero-range potentials has block structure. According to the HYPOTHESIS of Section 3.1 the element $\mathcal{S}_{11} = S$ is the same as in the model of the hole in absolutely rigid plate, the block $\{\mathcal{S}_{ij}\}_{i,j=2}^{7} = \mathbf{S}$ is the same as in the model of the hole in isolated plate and other elements of matrix $\mathcal{S}$ are equal to zero. By considering the problem of diffraction by the hole in rigid screen (Section 3.4.1) and the problem of flexural wave diffraction by the hole in isolated plate (Section 3.4.2) the blocks of matrix $\mathcal{S}$ can be defined. Then in Section 3.4.3 the problem of scattering by the generalized model is studied.

3.4.1 *The case of absolutely rigid plate*

The problem of diffraction by a circular hole in a rigid screen is well known. It is formulated as follows

$$\left(\Delta + k^2\right) U^{(s)}(x, y, z) = 0, \qquad z > 0,$$

$$\frac{\partial U^{(s)}(x,y,0)}{\partial z} = 0, \qquad x^2 + y^2 > a^2,$$
$$U^{(s)}(x,y,0) = -f(x,y), \qquad x^2 + y^2 < a^2.$$

Here $f(x,y)$ is the trace of the geometrical part of the field composed of incident and reflected waves from the plane $\{z = 0\}$ without the hole. Meixner conditions (1.18) near the edge of the hole $x^2 + y^2 = a^2$ and radiation condition conclude the problem formulation.

This problem can be solved by variables separation method in ellipsoidal coordinates [40]. On the other hand there are works that calculate approximate solutions by reducing the problem to integral equation (see e.g. [36]). The derivations of [36] show in particular that at large distances from the hole the scattered field forms an outgoing spherical wave

$$U^{(s)} \sim -A\frac{2a}{\pi r}e^{ikr}, \qquad r \to \infty. \tag{3.70}$$

Here A is the amplitude of the incident plane wave and r is the radius in spherical coordinates. Formula (3.70) corresponds to the far field amplitude

$$\Psi(\vartheta,\varphi) \sim -A\frac{i}{\pi^2}ka.$$

Consider now the problem of scattering by a zero-range potential A_3^S formulated in Section 2.4.1. That is the scattered field satisfies Helmholtz equation and Neumann boundary condition on the screen everywhere except the origin. In a vicinity of the origin the field should have the asymptotics (2.20) with a given parameter S. Evidently that the scattered field in that case is proportional to Green's function

$$U^{(s)} = c\frac{e^{ikr}}{2\pi r}.$$

Therefore

$$cS = c\frac{ik}{2\pi} + U^{(i)}(0,0,0) + U^{(r)}(0,0,0).$$

In the case of an incident plane wave

$$U^{(i)} = A\exp\Big(ikx\cos\vartheta_0\cos\varphi_0 + iky\cos\vartheta_0\sin\varphi_0 - ikz\sin\vartheta_0\Big)$$

one finds

$$c = A\frac{4\pi}{2\pi S - ik}.$$

On the other hand solution (3.70) corresponds to

$$c \approx -4aA.$$

The above two equations allow the parameter S to be found

$$S \approx -\frac{1}{2a} + \frac{ik}{2\pi}.$$

However, parameter S should be real to fix selfadjoint operator. Therefore the second term in the above expression, which is ka times smaller than the first one, should be neglected. That defines the S parameter of the zero-range potential

$$S = -\frac{1}{2a}.$$

3.4.2 *The case of isolated plate*

Consider now the other limiting case, namely the case of isolated plate. That is, the problem is in finding solution of biharmonic equation

$$\left(\Delta^2 - k_0^4\right) w^{(s)}(x, y) = 0, \qquad x^2 + y^2 > a$$

with the boundary conditions

$$\mathbb{F} w^{(s)}|_{\rho=a} = -\mathbb{F} w^{(i)}|_{\rho=a},$$

$$\mathbb{M} w^{(s)}|_{\rho=a} = -\mathbb{M} w^{(i)}|_{\rho=a}$$

and radiation condition. Here the operators of force $\mathbb{F}$ and of bending momentum $\mathbb{M}$ are given by the formulae (1.8). In the case of circumference these operators are reduced to

$$\mathbb{F} = -D\left(\frac{\partial}{\partial\rho}\Delta + (1-\sigma)\frac{1}{a^2}\left(\frac{\partial^3}{\partial\rho\partial\varphi^2} - \frac{1}{a}\frac{\partial^2}{\partial\varphi^2}\right)\right),$$
$$\mathbb{M} = D\left(\Delta - (1-\sigma)\frac{1}{a}\left(\frac{\partial}{\partial\rho} + \frac{1}{a}\frac{\partial^2}{\partial\varphi^2}\right)\right).$$

The above problem is solved in [42] by variables separation method in polar coordinates. We present these results briefly. Represent the incident plane wave in polar coordinates

$$w^{(i)} = Ae^{ik_0\rho\sin\varphi}.$$

Search the scattered field in the form

$$w^{(r)} = \sum_{j=-\infty}^{+\infty} \left(\alpha_j H_j^{(1)}(k_0\rho) + \beta_j H_j^{(1)}(ik_0\rho)\right) e^{ij\varphi}, \tag{3.71}$$

where $H_j^{(1)}$ are Bessel functions of the third kind. Representation (3.71) satisfies the radiation condition. The terms containing coefficients α_j represent outgoing waves and satisfy the equation

$$\left(\triangle + k_0^2\right) H_j^{(1)}(k_0\rho)e^{ij\varphi} = 0.$$

The terms that contain coefficients β_j represent quickly decaying waves and satisfy an analogous equation with minus before k_0^2 (compare to section 1.3.6).

Coefficients α_j and β_j are found from the boundary conditions on the edge of the hole. Substituting representation (3.71) into the boundary conditions and introducing $\xi = k_0 a$ yields

$$\begin{aligned}
&\sum_{j=-\infty}^{+\infty} \left(\alpha_j M_j(\xi) + \beta_j M_j(i\xi)\right) e^{ij\varphi} = A\, M(\xi, i\sin\varphi)e^{i\xi\sin\varphi}, \\
&\sum_{j=-\infty}^{+\infty} \left(\alpha_j F_j(\xi) + \beta_j F_j(i\xi)\right) e^{ij\varphi} = A\, F(\xi, i\sin\varphi)e^{i\xi\sin\varphi},
\end{aligned} \tag{3.72}$$

where

$$M_j(\xi) = \left((1-\sigma)j^2 - \xi^2\right) H_j^{(1)}(\xi) - (1-\sigma)\xi \dot{H}_j^{(1)}(\xi),$$

$$F_j(\xi) = (1-\sigma)j^2 H_j^{(1)}(\xi) - \left(\xi^3 + (1-\sigma)j^2\xi^2\right) \dot{H}_j^{(1)}(\xi),$$

$$M(\xi, s) = \xi^2\left(\sigma - (1-\sigma)s^2\right),$$

$$F(\xi, s) = (1-\sigma)\xi^3 s^3 + 2(1-\sigma)\xi^2 s^2 + (2-\sigma)\xi^3 s + (1-\sigma)\xi^2.$$

Equations (3.72) mean that quantities $\alpha_j M_j(\xi) + \beta_j M_j(i\xi)$ and $\alpha_j F_j(\xi) + \beta_j F_j(i\xi)$ are the Fourier coefficients for $A\, M(\xi, i\sin\varphi)e^{i\xi\sin\varphi}$ and

$A\,F(\xi, i\sin\varphi)e^{i\xi\sin\varphi}$ correspondingly. Inverse Fourier transform yields linear algebraic systems for each harmonics

$$\begin{cases} \alpha_j M_j(\xi) + \beta_j M_j(i\xi) = A\,M(\xi, j), \\ \alpha_j F_j(\xi) + \beta_j F_j(i\xi) = A\,F(\xi, j). \end{cases} \tag{3.73}$$

Here the following integrals are introduced

$$M(\xi, j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} M(\xi, i\sin\varphi)\exp\Big(i\xi\sin\varphi - ij\varphi\Big)\,d\varphi,$$

$$F(\xi, j) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\xi, i\sin\varphi)\exp\Big(i\xi\sin\varphi - ij\varphi\Big)\,d\varphi.$$

These integrals can be expressed via Bessel functions (see [1], dot denotes derivative)

$$M(\xi, j) = \Big(\xi^2 - j^2(1-\sigma)\Big)J_j(\xi) + \xi(1-\sigma)\dot{J}_j(\xi),$$

$$F(\xi, j) = -j^2(1-\sigma)J_j(\xi) + \Big(\xi^3 + (1-\sigma)j^2\xi\Big)\dot{J}_j(\xi).$$

Solving systems (3.73) one finds

$$\alpha_j = A\frac{M(\xi, j)F_j(i\xi) - F(\xi, j)M_j(i\xi)}{M_j(\xi)F_j(i\xi) - F_j(\xi)M_j(i\xi)},$$

$$\beta_j = A\frac{F(\xi, j)M_j(\xi) - M(\xi, j)F_j(\xi)}{F_j(i\xi)M_j(\xi) - M_j(i\xi)F_j(\xi)}.$$

Replacing the Bessel functions $H_j^{(1)}(k_0\rho)$ by their asymptotics for large ρ allows the far field amplitude of diverging circular flexural wave to be found

$$\psi_0(\varphi) = -\frac{i}{\pi}\sum_{j=-\infty}^{+\infty}\alpha_j e^{ij\varphi}.$$

In the case of small ξ the above formula can be decomposed in series by ξ and $\ln\xi$. In the principal order one finds (next order terms can be found in [42])

$$\psi_0(\varphi) = A\frac{i\sigma}{4(1-\sigma)}(k_0 a)^2 + O\big((k_0 a)^4\ln(k_0 a)\big). \tag{3.74}$$

Now reproduce the same asymptotics in the problem of scattering by the zero-range potential of operator B_2^S and choose parameters in its matrix $\mathbf{S}$. The problem of scattering by the zero-range potential is formulated as the boundary-value problem for biharmonic operator

$$\left(\triangle^2 - k_0^4\right) w^{(s)}(x, y) = 0, \qquad (x, y) \neq (0, 0)$$

with a specific "boundary" condition in the point $x = y = 0$. This condition is formulated for the coefficients b_{ij} and c_{ij} of the near field asymptotics (2.25). We use the parameterization of the zero-range potential in the form

$$(c_{00}, -c_{10}, -c_{01}, c_{20}, c_{11}, c_{02})^T = \mathbf{S}(b_{00}, b_{10}, b_{01}, b_{20}, b_{11}, b_{02})^T.$$

The solution is given by multipole decomposition

$$w^{(s)} = \sum_{i+j\leq 2} c_{ij} \frac{\partial^{i+j}}{\partial x^i \partial y^j} g_0(x, y; 0, 0),$$

where $g_0(x, y; 0, 0)$ is the Green's function (1.45) in isolated plate. The Green's function is circular symmetric and such should be the principal order term in the asymptotics of the scattered field according to the formula (3.74). Therefore the coefficients c_{10}, c_{01} and c_{11} should be equal to zero and c_{20} should be equal to c_{02}. That is the matrix $\mathbf{S}$ should be such that

$$w^{(s)} = c_{00} g_0(x, y; 0, 0) + c_{20} \triangle g_0(x, y; 0, 0).$$

Therefore the elements S_{ij} with $i = 2, 3, 5$ are equal to zero, and the rows $S_{4,j}$ and $S_{6,j}$ coincide.

Replacing $g_0(x, y; 0, 0)$ by its asymptotics at large distances (for asymptotics of $H_0^{(1)}$ see [1]) yields the formula for the far field amplitude

$$\psi_0(\varphi) = -\frac{i}{8\pi D}\left(\frac{1}{k_0^2} c_{00} - c_{20}\right).$$

To reproduce the result (3.74) one should have

$$\frac{1}{k_0^2} c_{00} - c_{20} = -A \frac{2\pi D\sigma}{1-\sigma}(k_0 a)^2. \tag{3.75}$$

Replacing $g_0(x, y; 0, 0)$ by its asymptotics near the source (see page 153) allows the coefficients b_{ij} to be expressed via c_{00} and c_{20}. Due to the symmetry of matrix $\mathbf{S}$ only coefficients b_{00}, b_{20} and b_{02} are involved in the "boundary" condition in the zero-range potential center. Introducing

$$Z = \frac{1}{4\pi D}\left(\ln(k_0/2) + C_E - 1\right),$$

these coefficients can be written as

$$b_{00} = A + \frac{i}{8k_0^2 D}c_{00} + 2\left(Z + \frac{3}{8\pi D}\right)c_{20},$$
$$b_{20} + b_{02} = -Ak_0^2 + Zc_{00} + \frac{ik_0^2}{8D}c_{20}.$$

The equations for the amplitudes of passive sources reduce to

$$\left\{\begin{aligned} c_{00} &= S_{11}\left(A + \frac{i}{8k_0^2 D}\,c_{00} + 2\left(Z + \frac{3}{8\pi D}\right)c_{20}\right) \\ &\quad + S_{14}\left(-Ak_0^2 + Zc_{00} + \frac{ik_0^2}{8D}\,c_{20}\right), \\ c_{20} &= \overline{S_{14}}\left(A + \frac{i}{8k_0^2 D}\,c_{00} + 2\left(Z + \frac{3}{8\pi D}\right)c_{20}\right) \\ &\quad + S_{44}\left(-Ak_0^2 + Zc_{00} + \frac{ik_0^2}{8D}\,c_{20}\right). \end{aligned}\right. \tag{3.76}$$

Combining it with (3.75) and assuming that matrix $\mathbf{S}$ does not depend on k_0 allows the elements S_{11}, S_{14} and S_{44} to be found uniquely. Indeed, let $c_{00} = ck_0^2$ and let c and c_{20} be decomposed in series by powers of k_0. One can check that system (3.76) yields decomposition

$$c = c_0 + c_1 k_0^2 + \ldots, \qquad c_{20} = c_0 + c_2 k_0^2 + \ldots$$

where c_j can depend on logarithm of k_0. The principal order terms in the above decompositions coincide which follows form (3.75). Now substitute decompositions for c and c_{20} into equations (3.76) and separate terms of order $O(k_0^0)$

$$\begin{aligned} 0 &= S_{11}\Big(A + Z'c_0\Big), \\ c_0 &= \overline{S_{14}}\Big(A + Z'c_0\Big), \end{aligned} \tag{3.77}$$

where

$$Z' = Z + \frac{3}{8\pi D} + \frac{i}{16D}.$$

It is easy to find that such system can be satisfied only if $S_{11} = 0$.

In the order $O(k_0^2)$ the first equation in (3.76) yields

$$c_0 = S_{14}\Big(-A + \left(Z + \frac{i}{8D}\right) c_0\Big).$$

Combining it with the second equation in (3.77) and asking S_{14} to be not dependent on k_0 yields $S_{14} = 0$. Finally, from (3.75) and the second equation of (3.76) one finds

$$S_{44} = 2\pi \frac{\sigma}{1-\sigma} a^2.$$

Taking into account the noted above symmetry of matrix $\mathbf{S}$ allows it to be written as

$$\mathbf{S} = 2\pi \frac{\sigma}{1-\sigma} a^2 \begin{pmatrix} \mathbb{O} & & \mathbb{O} & \\ & 1 & 0 & 1 \\ \mathbb{O} & 0 & 0 & 0 \\ & 1 & 0 & 1 \end{pmatrix}.$$

3.4.3 *Generalized point model*

Models of the small hole in a rigid screen and in isolated elastic plate are constructed in the previous sections. According to the HYPOTHESIS the generalized model of a small hole in fluid loaded elastic plate combines the above two models and is formulated in the form of two asymptotics:

$$U = c\left(\frac{1}{2\pi r} - \frac{1}{2a}\right) + o(1), \qquad r \to 0 \tag{3.78}$$

and

$$\begin{aligned} w = &-\frac{\sigma a^2}{1-\sigma}(b_{20} + b_{02}) \ln \rho/a + b_{00} + b_{10}\rho\cos\varphi + b_{01}\rho\sin\varphi \\ &+ b_{11}\rho^2 \sin 2\varphi + \frac{b_{20}}{2}\rho^2\cos^2\varphi + \frac{b_{20}}{2}\rho^2 \sin^2\varphi + o(\rho^2), \quad \rho \to 0. \end{aligned} \tag{3.79}$$

Here c and b_{ij} are arbitrary coefficients.

The scattered field can be found by the similar procedure as used in Section 3.3.3. The general solution is the field of two passive sources

$$U^{(s)} = cG(\mathbf{r}; 0, 0, 0) + c'\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(\mathbf{r}; 0, 0).$$

Here $G(\mathbf{r};0,0,0)$ and $G(\mathbf{r};0,0)$ are the Green's functions introduced in Section 1.3.2. The amplitudes c and c' are defined by two linear constrains hidden in the asymptotics of U and w, namely by proportionality of coefficients of singular and regular terms in (3.78) and by proportionality of the coefficients of singular term and terms $\rho^2\cos^2\varphi$ and $\rho^2\sin^2\varphi$ in (3.79).

One needs the asymptotics of Green's functions for small r. By extracting Green's function of the problem with absolutely rigid screen it is not difficult to find

$$G(\mathbf{r};0,0,0) = \frac{1}{2\pi r} + \frac{ik}{2\pi} + N\,J' + o(1),$$

where

$$J' = \frac{1}{4\pi^2}\iint \frac{d\lambda\,d\mu}{L(\lambda,\mu)\sqrt{\lambda^2+\mu^2-k^2}} = \frac{1}{2\pi}\int\limits_0^\infty \frac{\tau\,d\tau}{L(\tau)\sqrt{\tau^2-k^2}}.$$

Using formula (1.21) for Green's function $G(\mathbf{r};0,0)$ one finds

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)G(\mathbf{r};0,0) = -\frac{2}{D}J_2' + o(1),$$

where

$$J_\ell' = \frac{1}{4\pi^2}\iint \frac{(i\lambda)^\ell\,d\lambda\,d\mu}{L(\lambda,\mu)}.$$

Asymptotics of displacements $g(\boldsymbol{\rho};0,0,0)$ corresponding to Green's function $G(\mathbf{r};0,0,0)$ can be found by direct decomposition into Taylor series. One finds

$$g(\boldsymbol{\rho};0,0,0) = -\frac{1}{D}J_0' - \frac{1}{2D}J_2'\rho^2 + o(\rho^2).$$

Asymptotics of displacements $g(\boldsymbol{\rho};0,0)$ corresponding to $G(\mathbf{r};0,0)$ was found before and is given by (3.62). It yields

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)g(\boldsymbol{\rho};0,0) = \frac{1}{2\pi D^2N}\ln\rho + \frac{1}{2\pi D^2N} + \frac{16DZ-i}{8D^2N} + \frac{D_2'}{D^2} + \left(\frac{k_0^2}{16D^2N}+\frac{D_4'}{4D^2}\right)\rho^2 + o(\rho^2).$$

The above formulae for Green's functions allow the asymptotics (3.78) and (3.79) to be written in terms of amplitudes c and c'

$$U = \frac{c}{2\pi r} + c\left(\frac{ik}{2\pi} + NJ'\right) - c'\frac{2}{D}J_2' + 2A\frac{ik\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}M(\vartheta_0) + o(1),$$

$$\begin{aligned} w = &\frac{c'}{2\pi D^2 N}\ln\rho - \frac{c}{D}J_0' + \frac{c'}{D^2N}\left(\frac{1}{2\pi} + 2DZ - \frac{i}{8} + ND_2'\right) \\ &- \frac{2A}{D}\frac{k\sin\vartheta_0}{\mathcal{L}(\vartheta_0)} - \frac{2Ai}{D}\frac{k^2\sin\vartheta_0\cos\vartheta_0}{\mathcal{L}(\vartheta_0)}\rho\cos(\varphi-\varphi_0) - \frac{c}{2D}J_2'\rho^2 \\ &+ \frac{c'}{4D^2N}\left(\frac{k_0^2}{4} + ND_4'\right)\rho^2 + \frac{A}{D}\frac{k^3\sin\vartheta_0\cos^2\vartheta_0}{\mathcal{L}(\vartheta_0)}\rho^2\cos^2(\varphi-\varphi_0) + o(\rho^2), \end{aligned}$$

where

$$M(\vartheta) = k^4\cos^4\vartheta - k_0^4.$$

Comparing these formulae with (3.78) and (3.79) yields algebraic system

$$\begin{cases} \left(\dfrac{1}{2a} + \dfrac{ik}{2\pi} + NJ'\right)c - \dfrac{2}{D}J_2'c' = -2A\dfrac{ik\sin\vartheta_0(k^4\cos^4\vartheta_0 - k_0^4)}{\mathcal{L}(\vartheta_0)}, \\ 2J_2'c - \left(\dfrac{1-\sigma}{2\pi DNa^2} + \dfrac{k_0^2}{4DN} + \dfrac{D_4'}{D}\right)c' = 2A\dfrac{ik^3\sin\vartheta_0\cos^2\vartheta_0}{\mathcal{L}(\vartheta_0)}. \end{cases}$$

This system allows the amplitudes c and c' of passive sources to be found

$$c = -2iA\frac{k\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}\frac{1}{Q}\left\{\left(\frac{1-\sigma}{2\pi N\sigma a^2} + \frac{k_0^2}{4N} + D_4'\right)M(\vartheta_0) + 2J_0'k^2\cos^2\vartheta_0\right\},$$

$$c' = -2iA\frac{k\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}\frac{1}{DQ}\left\{2J_2'M(\vartheta_0) + \left(\frac{1}{2a} + \frac{ik}{2\pi} + NJ'\right)k^2\cos^2\vartheta_0\right\},$$

where

$$Q = \left(\frac{1-\sigma}{2\pi N\sigma a^2} + \frac{k_0^2}{4N} + D_4'\right)\left(\frac{1}{2a} + \frac{ik}{2\pi} + NJ'\right) - 4(J_2')^2.$$

Finally one gets exact expression for the far field amplitude of scattered field

$$\Psi(\vartheta,\varphi) = \frac{iAk}{2\pi^2 Q}\frac{k\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}\frac{k\sin\vartheta}{\mathcal{L}(\vartheta)}\left\{\left(\frac{1-\sigma}{2\pi N\sigma a^2}+\frac{k_0^2}{4N}+D_4'\right)M(\vartheta_0)M(\vartheta)\right.$$
$$+2k^2J_2'\left(\cos^2\vartheta_0 M(\vartheta)+M(\vartheta)\cos^2\vartheta_0\right)$$
$$\left.+\left(\frac{1}{2a}+\frac{ik}{2\pi}+NJ'\right)k^4\cos^2\vartheta_0\cos^2\vartheta\right\}.$$

Two principal order terms by small a are

$$\Psi(\vartheta,\varphi) = \frac{iA}{\pi^2}\frac{k\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}\frac{k\sin\vartheta}{\mathcal{L}(\vartheta)}\times$$
$$\times\left\{\left(ka-\left(\frac{i}{\pi}+\frac{2NJ'}{k}\right)(ka)^2+\ldots\right)M(\vartheta_0)M(\vartheta)\right.$$
$$\left.+\pi Nk^3\frac{\sigma}{1-\sigma}(ka)^2\cos^2\vartheta_0\cos^2\vartheta+\ldots\right\}.$$

3.4.4 *Other models of circular holes*

Consider the case of clamped edge. The problem of scattering in isolated plate can be solved similarly to the above case of free edge by variables separation method. Substituting representation (3.71) into the boundary conditions

$$w=0,\quad \frac{\partial w}{\partial\rho}=0,\qquad \rho=a$$

yields algebraic equations for coefficients α_j and β_j of Fourier series. The far field amplitude is expressed via the coefficients α_j and can be written in explicit form [42]

$$\psi_0(\varphi) = -\frac{i}{\pi}\sum_{j=-\infty}^{+\infty}\frac{J_j(k_0a)i\dot{H}_j^{(1)}(ik_0a)-\dot{J}_j(k_0a)H_j^{(1)}(ik_0a)}{\dot{H}_j^{(1)}(k_0a)H_j^{(1)}(ik_0a)-H_j^{(1)}(k_0a)i\dot{H}_j^{(1)}(ik_0a)}e^{ij\varphi}.$$

Asymptotic analysis of this expression for $k_0a \ll 1$ yields

$$\psi_0(\varphi) = -\frac{1}{\pi}\left(1-\frac{i\pi}{2}\frac{\cos\varphi}{\ln(k_0a/2)+C_E-i\pi/4}+\ldots\right).$$

One can check that this asymptotics corresponds to the model

$$\begin{pmatrix} b_{00} \\ b_{10} \\ b_{01} \end{pmatrix} = \frac{\ln(a)+1/2}{4\pi} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{00} \\ -c_{10} \\ -c_{01} \end{pmatrix},$$

which is a particular case of zero-range potential for bi-harmonic operator. Combining this model with the zero-range potential in the acoustic channel yields composite generalized model for a circular hole with clamped edges in fluid loaded elastic plate.

Similarly one can consider other boundary conditions on the edge of the hole (see [42]) and derive other generalized models. According to the HYPOTHESIS these models then can be reformulated for fluid loaded plates.

3.5 Model of narrow joint of two semi-infinite plates

In this section we consider an acoustic wave diffraction by two semi-infinite plates joint along a short segment. Considering the problem for the isolated structure, we derive the generalized point model that reproduces not only the principal order term in the asymptotics by the width of the joint (compare to the model of Chapter 1), but also the logarithmically smaller corrections. Then the model is reformulated for fluid loaded structure. The asymptotics of the far field amplitude of the diverging from the joint spherical wave is found.

3.5.1 *Problem formulation*

Consider mechanical system, consisting of two semi-infinite thin elastic plates $\Pi_\pm = \{z=0, \pm x > 0\}$. Let fluid be on one side of these plates at $\{z > 0\}$ and let the plates be joined along the short segment $\Lambda = \{z = x = 0, |y| < a\}$. The contact is assumed ideal, that is the displacements, angles, momentums and forces are continuous on Λ.

Acoustic pressure $U(x,y,z)$ in the fluid and flexure displacements $w(x,y)$ in the plate satisfy the following boundary value problem

$$\left(\Delta + k^2\right) U = 0, \qquad z > 0, \tag{3.80}$$

$$\left(\Delta^2 - k_0^4\right) w + NU\big|_{z=0} = 0, \qquad x \neq 0, \tag{3.81}$$

$$w(x,y) = \frac{\partial U(x,y,0)}{\partial z}, \tag{3.82}$$

$$\mathbb{F}w = 0, \quad \mathbb{M}w = 0, \qquad |y| > a, \; x = \pm 0,$$
$$[w] = 0, \; [w_x] = 0, \; [\mathbb{F}w] = 0, \; [\mathbb{M}w] = 0, \qquad |y| < a. \qquad (3.83)$$

(Here $[f]$ denotes the jump $f|_{x=+0} - f_{x=-0}$.) Besides, acoustic field $U(x, y, z)$ and displacements $w(x, y)$ satisfy Meixner conditions (1.18) and (1.19) in all points where the boundary conditions have discontinuities. Namely, Meixner conditions (1.18) for $U(x, y, z)$ are assumed along the rays $\{z = x = 0, y > a\}$ and $\{z = x = 0, y < -a\}$, Meixner conditions (1.19) for $w(x, y)$ are assumed at the tips of semi-infinite cracks, that is in the points $(0, a)$ and $(0, -a)$.

The operators of force $\mathbb{F}$ and bending momentum $\mathbb{M}$ on the line $x = 0$ are given by the formulae (1.8), namely

$$\mathbb{F}w \equiv w_{xxx} + (2 - \sigma) w_{xyy}, \qquad \mathbb{M}w \equiv w_{xx} + \sigma w_{yy}.$$

Let the field in the system be excited by the incident plane wave

$$U^{(i)}(x, y, z) = A \exp\big(ik(x \cos\vartheta_0 \sin\varphi_0 + y \cos\vartheta_0 \cos\varphi_0 - z \sin\vartheta_0)\big).$$

All the other components of the total field are subject to radiation condition, that is represent waves that propagate to infinity. Mathematical formulation of radiation conditions for fluid loaded elastic plate with infinite crack is given below in the form of asymptotics (3.84), (3.85) and (3.86).

It is natural to separate from the total field the incident wave and the wave that would be reflected from a homogeneous plate, i.e.

$$U^{(r)}(x, y, z) = AR(\vartheta_0) \exp\big(ik(x \cos\vartheta_0 \sin\varphi_0 + y \cos\vartheta_0 \cos\varphi_0 + z \sin\vartheta_0)\big),$$

Presence of infinite crack causes an additional component of the field $U^{(c)}$ to appear. This field is found in Section 1.4.2. The field $U^{(i)} + U^{(r)} + U^{(c)}$ satisfies all the conditions of the boundary value problem except the continuity conditions on the joint. The boundary value problem (3.80) – (3.83) can be rewritten for the remaining part of the field $U^{(s)}$. The function $U^{(s)}(x, y, z)$ satisfies the same boundary value problem as the total field except that the conditions (3.83) on the joint become inhomogeneous. Namely

$$[w^{(s)}] = -[w^{(c)}], \quad \left[w_x^{(s)}\right] = -\left[w_x^{(c)}\right], \quad [\mathbb{M}w^{(s)}] = 0, \quad [\mathbb{F}w^{(s)}] = 0.$$

One can write the radiation conditions for the scattered field $U^{(s)}$ in the form of asymptotic decompositions at large distances from the joint. Far from the plate this field behaves as a diverging spherical wave

$$U^{(s)} \sim \frac{2\pi}{kr} e^{ikr - i\pi/2} \Psi(\varphi, \vartheta), \quad r \to +\infty. \tag{3.84}$$

In a vicinity of the plate, but far from the edges it forms cylindrical wave that exponentially decreases with z

$$U^{(s)} \sim \sqrt{\frac{2\pi}{\kappa\rho}} e^{i\kappa\rho - i\pi/4} \psi(\varphi) e^{-\sqrt{\kappa^2 - k^2} z}, \quad \rho \to +\infty. \tag{3.85}$$

Finally in a vicinity of the semi-infinite cracks specific edge waves (see Section 4.2) are generated. It is natural to represent edge wave process as the sum of symmetric with respect to x co-ordinate wave $U^{(e)}$ and anti-symmetric wave $U^{(o)}$ with amplitudes $\psi^e_\pm$ and $\psi^o_\pm$

$$U^{(s)} \sim \psi^e_\pm e^{\pm \varkappa_e y} V_e(x, z) + \psi^o_\pm e^{\pm \varkappa_o y} V_o(x, z), \qquad y \to \pm\infty. \tag{3.86}$$

The wavenumbers $\varkappa_e$ and $\varkappa_o$ are the positive solutions of dispersion equations (1.65)[§]

$$\Delta_e(\varkappa_e) = 0 \tag{3.87}$$

and

$$\Delta_o(\varkappa_o) = 0. \tag{3.88}$$

Functions V_e and V_o are given by Fourier integrals similar to (1.63)

$$V_e(x, z) = \int_{-\infty}^{+\infty} \exp\left(i\lambda x - \sqrt{\varkappa_e^2 + \lambda^2 - k^2}\, z\right) \frac{\lambda^2 + \sigma \varkappa_e^2}{L(\lambda, \varkappa_e)}\, d\lambda,$$

$$V_o(x, z) = \int_{-\infty}^{+\infty} \exp\left(i\lambda x - \sqrt{\varkappa_o^2 + \lambda^2 - k^2}\, z\right) \frac{\lambda^2 + (2 - \sigma)\varkappa_o^2}{L(\lambda, \varkappa_o)}\, \lambda\, d\lambda.$$

Radiation conditions specified in the form of asymptotics (3.84), (3.85) and (3.86) allow asymptotics of displacements $w^{(s)}(x, y)$ to be written. The

[§]We examine these dispersion equations in Section 4.2.2 and discover that $\varkappa_e$ always exists and $\varkappa_o$ may exist or not depending on the parameters of the problem.

field of displacements at large distances from the joint is represented as diverging circular wave

$$w^{(s)} \sim -\sqrt{\frac{2\pi}{\kappa\rho}} e^{i\kappa\rho - i\pi/4} \psi(\varphi) \sqrt{\kappa^2 - k^2}, \quad \rho \to +\infty. \tag{3.89}$$

and edge waves running along the crack

$$w^{(s)} \sim \psi^e_\pm e^{\pm\varkappa_e y} \frac{\partial V_e(x,0)}{\partial z} + \psi^o_\pm e^{\pm\varkappa_o y} \frac{\partial V_o(x,0)}{\partial z}, \qquad y \to \pm\infty. \tag{3.90}$$

3.5.2 *Isolated plate*

Boundary value problem

Consider first the case of isolated plate. Evidently the boundary value problem (3.80–3.83) in this case reduces to the boundary value problem for flexure displacements $w(x,y)$ in the plates

$$\begin{cases} \left(\Delta^2 - k_0^2\right) w(x,y) = 0, \\ \mathbb{F}w = 0, \quad \mathbb{M}w = 0, \qquad |y| > a, \ x = 0, \\ [w] = 0, \ [w_x] = 0, \ [\mathbb{F}w] = 0, \ [\mathbb{M}w] = 0, \qquad |y| < a. \end{cases} \tag{3.91}$$

Besides, w satisfies Meixner conditions (1.19) at the tips of the cracks and the scattered field $w^{(s)}(x,y)$ is subject to radiation conditions (3.89) and (3.90). For the case of isolated plate surface waves propagate with the wavenumber k_0 (instead of κ) and the dispersion equations (3.87) and (3.88) for the wavenumbers of symmetric and anti-symmetric waves simplify and reduce to one equation

$$\left((1-\sigma)\varkappa^2 - k_0^2\right)^2 \sqrt{\varkappa^2 + k_0^2} - \left((1-\sigma)\varkappa^2 + k_0^2\right)^2 \sqrt{\varkappa^2 - k_0^2} = 0.$$

The solution of this equation is known (1.72)

$$\varkappa = k_0 \Big((1-\sigma)\big(3\sigma - 1 + 2\sqrt{1 - 2\sigma + 2\sigma^2}\big)\Big)^{-1/4}.$$

The functions V_e and V_o simplify as well and finally the edge waves are given by the formula (1.75)

$$w^{(s)} \sim \psi^{\pm}_{\pm} e^{\pm\varkappa y} \left(e^{-\sqrt{\varkappa^2 - k_0^2}|x|} - \frac{(1-\sigma)\kappa^2 - k_0^2}{(1-\sigma)\kappa^2 + k_0^2} e^{-\sqrt{\varkappa^2 + k_0^2}|x|} \right).$$

This formula shows that in the absence of fluid edge waves propagate in semi-infinite plates independently with the same wave number $\varkappa$. This is natural as in the absence of fluid no interactions take place across the crack.

Geometrical part of the field

Let the incident plane wave

$$w^{(i)} = A \exp\bigl(\bigl(ik(x \sin\varphi_0 - y\cos\varphi_0)\bigr) \tag{3.92}$$

excite the wave process in the plates. Define the geometrical part of displacements field as

$$w^{(g)}(x,y) = \begin{cases} w^{(i)} + w^{(r)}, & x > 0 \\ 0, & x < 0 \end{cases}. \tag{3.93}$$

Here the reflected field $w^{(r)}$ is the sum of plane reflected wave and inhomogeneous wave concentrated near the edge (1.71)

$$w^{(r)} = A\, r(\varphi_0) \exp\bigl(ik_0(x\sin\varphi_0 + y\cos\varphi_0)\bigr)$$
$$+ A\, t(\varphi_0) \exp\Bigl(-k_0 x\sqrt{1+\sin^2\varphi_0} + ik_0 y\cos\varphi_0\Bigr)$$

with reflection coefficients

$$r(\varphi) = -\frac{\overline{l(\varphi)}}{l(\varphi)}, \qquad t(\varphi) = \frac{-2i\sin\varphi A_+(\varphi)A_-(\varphi)}{l(\varphi)},$$

$$l(\varphi) = i\sin\varphi A_+^2(\varphi) + \sqrt{1+\cos^2\varphi}A_-^2(\varphi), \qquad A_\pm(\varphi) = (1-\sigma)\cos^2\varphi \pm 1.$$

Excluding $w^{(g)}$ reduces the boundary value problem (3.91) to the boundary value problem for scattered field $w^{(s)}$ which differs from (3.91) by the boundary conditions on the joint that become inhomogeneous with the right-hand sides

$$[w^{(g)}] = \frac{4i\ A_+(\varphi_0)\sin\varphi_0}{l(\varphi_0)} e^{ik_0 y\cos\varphi_0},$$

$$[w_x^{(g)}] = \frac{4ik_0\ A_-(\varphi_0)\sqrt{1+\cos^2\varphi_0}\sin\varphi_0}{l(\varphi_0)} e^{ik_0 y\cos\varphi_0}.$$

Integral equations

With the use of Green's function $g^+(\boldsymbol{\rho}; \boldsymbol{\rho}_0)$ for semi-infinite plate with free edge derived in Section 1.5.2 one can reduce the problem (3.91) to two integral equations of the convolution on the interval $-a < y < a$ [15]. However this yields the first kind integral equation with smooth kernel. Solution of this equation, which physically corresponds to the force on the joint, belongs to the class of functions with strong singularities at the ends of the interval. Another approach is based on Fourier transform. The solution $w^{(s)}(x, y)$ is searched in the form

$$\begin{aligned} w^{(s)}(x,y) = \operatorname{sign}(y) \times \\ \times \int e^{i\mu y} \left(\alpha^-(\mu) e^{-\sqrt{\mu^2 - k_0^2}|x|} + \alpha^+(\mu) e^{-\sqrt{\mu^2 + k_0^2}|x|} \right) \frac{d\mu}{\ell(\mu)} \\ + \int e^{i\mu y} \left(\beta^-(\mu) e^{-\sqrt{\mu^2 - k_0^2}|x|} + \beta^+(\mu) e^{-\sqrt{\mu^2 + k_0^2}|x|} \right) \frac{d\mu}{\ell(\mu)}. \end{aligned} \tag{3.94}$$

Here

$$\ell(\mu) = a_+^2(\mu)\sqrt{\mu^2 - k_0^2} - a_-^2(\mu)\sqrt{\mu^2 + k_0^2}, \qquad a_\pm(\mu) = (1-\sigma)\mu^2 \pm k_0^2$$

and functions $\alpha^\pm(\mu)$ and $\beta^\pm(\mu)$ are defined from the boundary conditions. The first integral in representation (3.94) gives the odd with respect to x part of the field and the second integral corresponds to the even part of the field. Noting the symmetry of the two integrals one can eliminate two unknown functions. For the first integral the condition $\mathbb{M}w = 0$ is satisfied on the whole axis. It yields integral equation that can be easily solved by Fourier transform. This gives

$$\alpha^\pm(\mu) = \mp a^\mp(\mu)\alpha(\mu),$$

where $\alpha(\mu)$ is a new unknown.

Analogously for the second integral the condition $\mathbb{F}w = 0$ can be inverted and gives

$$\beta^\pm(\mu) = \mp a_\pm(\mu)\sqrt{\mu^2 \pm k_0^2}\beta(\mu).$$

The other conditions give dual integral equations for functions $\alpha(\mu)$ and $\beta(\mu)$. Consider first integral equations for the odd part of the field

$$\int e^{i\mu y}\alpha(\mu)\, d\mu = 0, \quad |y| > a. \tag{3.95}$$

$$2k_0^2 \int e^{i\mu y} \frac{\alpha(\mu)}{\ell(\mu)}\, d\mu = -\frac{1}{2}[w^{(g)}](y), \quad |y| < a. \tag{3.96}$$

Equations (3.95) and (3.96) uniquely determine the function $\alpha(\mu)$ in the class of functions with the allowed growth at infinity as $O(\mu)$.

Note that formal inversion of equation (3.95) yields the first kind integral equation for the Fourier transform of $\alpha(\mu)$. The kernel of this equation is bounded, but the solution due to the allowed growth of $\alpha(\mu)$ at infinity can have nonintegrable singularities. (Analysis of [16] allows to find that these singularities are of order $(y \pm a)^{-3/2}$).

Instead of applying Fourier transform directly to $\alpha(\mu)$ let it be searched in the form

$$\alpha(\mu) = \mu^2 \widehat{\alpha}(\mu) + \alpha_0 \rho_0(\mu) + \alpha_1 \rho_1(\mu). \tag{3.97}$$

Here function $\widehat{\alpha}(\mu)$ decreases at infinity not slower than $O(\mu^{-1-\delta})$, $\delta > 0$ and can be represented by its Fourier transform $p(y)$. The two remaining in (3.97) terms compensate the second order zero at $\mu = 0$ introduced in the first term. Functions $\rho_0(\mu)$ and $\rho_1(\mu)$ are fixed functions that do not violate asymptotics of $\alpha(\mu)$ at infinity and are such that vectors $\big(\rho_0(0), \rho_0'(0)\big)$ and $\big(\rho_1(0), \rho_1'(0)\big)$ are linear independent. Besides, it is convenient to choose functions $\rho_0(\mu)$ and $\rho_1(\mu)$ with Fourier transform support on to the interval $-a < y < a$.

Substituting representation (3.97) into (3.96) and accounting the mentioned above properties of functions $\rho_0(\mu)$ and $\rho_1(\mu)$ yields integro-algebraic equation

$$\begin{aligned} \int_{-a}^{a} p(y) K_o(y - y_0)\, dy - k_0^2 \alpha_0 \int e^{i\mu y_0} \frac{\rho_0(\mu)}{\ell(\mu)} \mu^2 d\mu \\ - k_0^2 \alpha_1 \int e^{i\mu y_0} \frac{\rho_1(\mu)}{\ell(\mu)} \mu^2 d\mu = -\frac{1}{4}[w^{(g)}](y_0). \end{aligned} \tag{3.98}$$

The kernel $K_o(y - y_0)$ coincides with the second order derivative of Green's function

$$K_o(y - y_0) = 2\pi(3 + \sigma)^{-1} g^+_{yy}(0, y, 0, y_0) = k_0^2 \int e^{i\mu(y - y_0)} \frac{(i\mu)^2 d\mu}{\ell(\mu)}. \tag{3.99}$$

Equation (3.98) defines function $p(y)$ and constants α_0 and α_1. Due to the behavior $\widehat{\alpha}(\mu) = O(\mu^{-1-\delta})$, $\delta > 0$, function $p(y)$ should be searched in the class S_1. That is, it can be represented in the form

$$p(y) = \sqrt{a^2 - y^2}\, P(y), \tag{3.100}$$

where $P(y)$ is continuous on $[-a, a]$. Afterwards function $P(y)$ can be found smooth.

For the even part function $\beta(\mu)$ decreases at infinity and standard procedure yields integral equation

$$\int_{-a}^{a} q(y) K_e(y - y_0)\, dy = -\frac{1}{4}[w_x^{(g)}](y_0) \tag{3.101}$$

for its Fourier transform $q(y)$ in the class

$$q(y) = \frac{Q(y)}{\sqrt{a^2 - y^2}}, \qquad Q \in C([-a, a]). \tag{3.102}$$

The kernel is again related to Green's function g^+, namely

$$\begin{aligned} K_e(y - y_0) &= -\frac{2\pi}{3 + \sigma} g^+_{xx_0}(0, y, 0, y_0) \\ &= k_0^2 \int e^{i\mu(y - y_0)} \frac{\sqrt{\mu^4 - k_0^4}\, d\mu}{\ell(\mu)}. \end{aligned} \tag{3.103}$$

Analysis of kernels and asymptotic solution

The singularities of kernels K_o and K_e of integral equations (3.98) and (3.101) are defined by the asymptotics of the integrated functions of (3.99), (3.103) at infinity. It is easy to find that

$$\ell(\lambda) = k_0^2 \chi |\lambda|^3 + \ldots, \qquad \chi = (1 - \sigma)(3 + \sigma).$$

Therefore function $g^+(0, y, 0, y_0)$ is bounded at $y = y_0$ and its second order derivatives $g^+_{yy}(0, y, 0, y_0)$ and $g^+_{xx_0}(0, y, 0, y_0)$ have logarithmic singularities. More accurate analysis shows that kernels have the following asymptotic decompositions by small $\delta = y - y_0$

$$\begin{aligned} g^+(0, y, 0, y_0) = k_0^{-2}\pi d_0 + \frac{\delta^2 \ln|\delta|}{\chi} \\ + \frac{1}{\chi}\left(\ln(k_0/2) + C_E + i\frac{\pi}{4} + \frac{1 - \sigma^2}{2} d_1 - \frac{1}{2} d_2 - \frac{3}{2}\right)\delta^2 + O(\delta^4), \end{aligned}$$

$$K_o(\delta) = \frac{2}{\chi}\ln|\delta| + \frac{2}{\chi}\left(\ln(k_0/2) + C_E + i\frac{\pi}{4} + \frac{1-\sigma^2}{2}d_1 - \frac{1}{2}d_2\right) + O(\delta^2),$$

$$K_e(\delta) = -\frac{2}{\chi}\ln|\delta| - \frac{2}{\chi}\left(\ln(k_0/2) + C_E + i\frac{\pi}{4} - (1-\sigma)d_1 + \left(1 - \sigma - \frac{\sigma^2}{2}\right)d_2\right) + O(\delta^2).$$

Here the following integrals are introduced (for d_0 see Section 1.5.3)

$$d_1 = \int \frac{l^2 - \sqrt{l^4-1}}{\mathcal{L}(l)\sqrt{l^4-1}}\, l^2\, dl, \quad d_2 = \int \frac{dl}{\mathcal{L}(l)\sqrt{l^4-1}}, \tag{3.104}$$

where

$$\mathcal{L}(l) = k_0^{-5}\ell(k_0 l) = \mathcal{A}_+^2(l)\sqrt{l^2-1} - \mathcal{A}_-^2(l)\sqrt{l^2+1},$$

$$\mathcal{A}_\pm(l) = k_0^{-2}a_\pm(k_0 l) = (1-\sigma)l^2 \pm 1.$$

Note that integrals (3.104) depend only on Poisson's ratio σ and do not depend on $k_0 a$. These integrals can be reduced to sums of residues of denominator $\mathcal{L}(l)$ on both sheets of Riemann surface of complex variable l as described in Appendix A.

In order to find the asymptotics of solution (p, α_0, α_1) of equation (3.98) it is convenient to choose functions $\rho_0(\lambda)$ and $\rho_1(\lambda)$ in the form

$$\rho_0(\lambda) = \int_{-a}^{a} \frac{e^{-i\lambda s}ds}{\sqrt{a^2-s^2}}, \qquad \rho_1(\lambda) = \int_{-a}^{a} \frac{e^{-i\lambda s}s\, ds}{\sqrt{a^2-s^2}}.$$

Let function $p(y)$ be represented in the form (3.100) and function $P(y)$ from that representation be decomposed in Taylor series. Substituting this representation into integro-algebraic equation (3.98) and equating coefficients at powers of y_0 yields a linear algebraic system for the constants α_0 and α_1 and for the Taylor coefficients of $P(y)$. Similarly, solution q of equation (3.101) is searched in the form (3.102) and function $Q(y)$ is decomposed in Taylor series. We do not present here cumbersome formulae for coefficients α_0 and α_1 and for the Taylor coefficients of $P(y)$ and $Q(y)$. These formulae are given in [15].

Calculating integrals in representation (3.94) according to the saddle point method yields the far field asymptotics of the scattered field. The

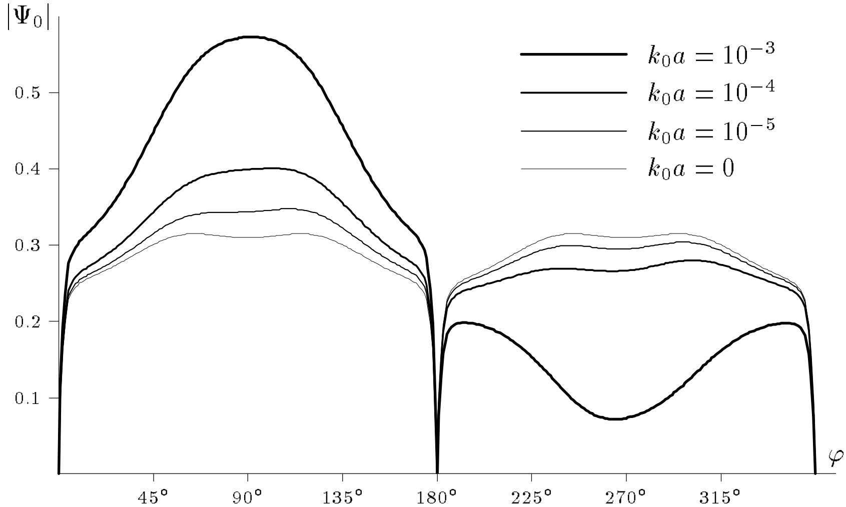


Fig. 3.6 The far field amplitudes $|\psi(\varphi)|$ for scattering on joints of different widths $k_0 a$. Poisson's ratio is assumed $\sigma = 1/3$.

contribution of the saddle point gives diverging circular wave with the far field amplitude

$$\psi(\varphi;\varphi_0) = i\frac{\sin\varphi}{L(\varphi)}\frac{\sin\varphi_0}{L(\varphi_0)}\Big\{\mathrm{sign}(x)A_+(\varphi)A_+(\varphi_0)\Big[\frac{1}{\pi d_0} - \frac{\chi\cos\varphi\cos\varphi_0}{\ln(k_0a/4) + C_E + (1-\sigma^2)d_1 - d_2 + i\pi/2} + \ldots\Big]$$
$$+ \frac{\chi\, A_-(\varphi)A_-(\varphi_0)\sqrt{1+\cos^2\varphi}\sqrt{1+\cos^2\varphi_0}}{\ln(k_0a/4) + C_E - 2(1-\sigma)d_1 + (2-2\sigma-\sigma^2)d_2 + i\pi/2} + \ldots\Big\}. \quad (3.105)$$

The asymptotics of edge waves can be found with the help of formula (1.78).

Numerical analysis of the far field amplitude

Formula (3.105) shows that the principal order term in the asymptotics does not depend on the width $2a$ of the joint. This term coincides with the exact solution (1.80) for diffraction by a pointwise joint of two semi-infinite plates. The corrections in (3.105) are only logarithmically smaller. It is natural to expect that validity of the principal term approximation is restricted to very narrow joints. Numerical results presented on Fig. 3.6 confirm this expectation. Noticeable difference is seen already for $k_0 a = 10^{-5}$. Therefore classical point model studied in Section 1.5.3, should be corrected. This is done in the following section.

3.5.3 *Generalized model*

Generalized models of obstacles in the plate with infinite crack are derived in Section 2.4.4. The parameters of the zero-range potential can be chosen according to the HYPOTHESIS from analysis of the problem for isolated plates. That is, matrix $\mathbf{S}$ in (2.40) should be chosen such that the field scattered by the zero-range potential of operator B_c^S approximately coincides with the asymptotics (3.105).

Analysis of the symmetry of the scattered field (3.105) with respect to x co-ordinate yields

$$c_j^+ = -c_j^- \equiv c_j, \qquad j = 0, 1, 2.$$

Such zero-range potentials of B_c are parameterized by 3×3 Hermitian matrices $\mathbf{S}'$ in the form

$$\mathbf{S}' \begin{pmatrix} c_0 \\ -c_1 \\ -c_2 \end{pmatrix} = \begin{pmatrix} b_0^+ - b_0^- \\ b_1^+ - b_1^- \\ b_2^+ - b_2^- \end{pmatrix}. \tag{3.106}$$

Consider the problem of scattering by a zero-range potential in an isolated mechanical structure. Let the incident plane wave be given by the formula (3.92). Excluding geometrical part $w^{(g)}(x, y)$ defined in (3.93) from the total field $w(x, y)$ one considers the problem

$$\begin{aligned} \triangle^2 w^{(s)} &= k_0^4 w^{(s)}, \quad && x \neq 0, \\ \mathbb{F} w^{(s)}(\pm 0, y) &= 0, && y \neq 0, \\ \mathbb{M} w^{(s)}(\pm 0, y) &= 0, && y \neq 0 \end{aligned}$$

in the class of functions that have asymptotics (2.39) with coefficients satisfying equations (3.106).

Evidently solution of this problem is the sum of Green's functions

$g^{\pm}(x, y, 0, 0)$ and their first derivatives over x and y

$$w^{(s)}(x, y) = \operatorname{sign}(x)\left(c_0 + c_1 \frac{\partial}{\partial y}\right) g^{+}(|x|, y, 0, 0) + c_2 \frac{\partial}{\partial x} g^{+}(|x|, y, 0, 0). \tag{3.107}$$

Here the symmetry $g^{-}(x, y, 0, 0) = g^{+}(-x, y, 0, 0)$ of Green's functions for the plates $\Pi^{\pm}$ is used. The coefficients c_j in the above representation are the same as in asymptotics (2.39).

Comparison of representation (3.107) with asymptotics (3.105) of the far field amplitude allows coefficients c_j to be found. For this consider the far field asymptotics (1.74), (1.77) of Green's function $g^{+}(x, y, 0, 0)$. Formula (1.74) allows differentiation by y to be performed. The principal order term is due to differentiation of the exponential and results in additional multiplier $ik_0 \cos\varphi$ in the far field amplitude. The asymptotics of $g^{+}_{y_0}(x, y, 0, 0)$ is similar to (1.74) with the far field amplitude

$$\psi_{g_y}(\varphi) = \frac{3+\sigma}{4\pi k_0} \frac{A_{-}(\varphi)\sqrt{1+\cos^2\varphi}\,\sin\varphi}{L(\varphi)}. \tag{3.108}$$

Comparing expressions (1.77) and (3.108) with asymptotics (3.105) it is easy to find that coefficients c_j in representation (3.107) should be the following

$$c_0 \approx -i\frac{4}{3+\sigma}\,\frac{A_{+}(\varphi_0)\sin\varphi_0}{L(\varphi_0)}\,\frac{k_0^2}{d_0},$$

$$c_1 \approx \frac{A_{+}(\varphi_0)\sin\varphi_0\cos\varphi_0}{L(\varphi_0)} \frac{4\pi k_0(1-\sigma)}{\ln(k_0 a/4) + C_E + (1-\sigma^2)d_1 - d_2 + i\pi/2},$$

$$c_2 \approx \frac{A_{-}(\varphi_0)\sqrt{1+\cos^2\varphi_0}\,\sin\varphi_0}{L(\varphi_0)} \times \frac{4i\pi k_0(1-\sigma)}{\ln(k_0 a/4) + C_E - 2(1-\sigma)d_1 + (2-2\sigma-\sigma^2)d_2 + i\pi/2}.$$

Now it is easy to find coefficients $b_j^{\pm}$. Regular terms in (2.39) are partly due to the geometrical part of the field and partly due to regular terms in the asymptotics of Green's function. The contributions of the geometrical part are evidently equal to the following values

$$b_0^+(w^{(g)}) = \frac{4i\, A_+(\varphi_0)\sin\varphi_0}{L(\varphi_0)}, \quad b_0^-(w^{(g)}) = 0,$$

$$b_1^+(w^{(g)}) = -\frac{4k_0\, A_+(\varphi_0)\sin\varphi_0\cos\varphi_0}{L(\varphi_0)}, \quad b_1^-(w^{(g)}) = 0,$$

$$b_2^+(w^{(g)}) = \frac{4ik_0\, A_-(\varphi_0)\sqrt{1+\cos^2\varphi_0}\sin\varphi_0}{L(\varphi_0)}, \quad b_2^-(w^{(g)}) = 0.$$

Regular terms in the asymptotics of Green's function and its derivatives give

$$b_0^+(w^{(s)}) = -b_0^-(w^{(s)}) = -\frac{2i\, A_+(\varphi_0)\sin\varphi_0}{L(\varphi_0)},$$

$$b_1^+(w^{(s)}) = -b_1^-(w^{(s)}) = \frac{2k_0\, A_+(\varphi_0)\sin\varphi_0\cos\varphi_0}{L(\varphi_0)} \times$$
$$\times \frac{\ln(k_0/2) + C_E + (1-\sigma^2)d_1 - d_2 - 3/2 + i\pi/2}{\ln(k_0 a/4) + C_E + (1-\sigma^2)d_1 - d_2 + i\pi/2},$$

$$b_2^+(w^{(s)}) = -b_2^-(w^{(s)}) = \frac{4ik_0\, A_-(\varphi_0)\sqrt{1+\cos^2\varphi_0}\sin\varphi_0}{L(\varphi_0)} \times$$
$$\times \frac{\ln(k_0/2) + C_E - 2(1-\sigma)d_1 + (2-2\sigma-\sigma^2)d_2 + i\pi/2}{\ln(k_0 a/4) + C_E - 2(1-\sigma)d_1 + (2-2\sigma-\sigma^2)d_2 + i\pi/2}.$$

Substituting the values of $c_j^\pm$ and $b_j^\pm$ into condition (3.106) yields the system of equations for matrix $\mathbf{S}'$. This system should be completed with equations expressing Hermitian property of $\mathbf{S}'$. Besides, one takes into account that $\mathbf{S}'$ does not depend on the angle of incidence φ_0 and wavenumber k_0. All this allows matrix $\mathbf{S}'$ to be determined uniquely

$$\mathbf{S}' = \mathrm{diag}\left(0, \frac{2\ln(a/2)+3}{4\pi(1-\sigma)}, -\frac{\ln(a/2)}{2\pi(1-\sigma)}\right). \tag{3.109}$$

The generalized model of narrow joint of width $2a$ is formulated as the condition (3.106) for the coefficients of the asymptotics (2.39) of displacements $w(x,y)$. The matrix $\mathbf{S}'$ in that condition is given by (3.109). According to the HYPOTHESIS this condition remains the same in the case of fluid loaded plates.

In terms of parameterization of Section 2.4.4 this corresponds to

$$\mathbf{S}^{-1} = \begin{pmatrix} \mathbf{S}'' & -\mathbf{S}'' \\ -\mathbf{S}'' & \mathbf{S}'' \end{pmatrix},$$

$$\mathbf{S}'' = (\mathbf{S}')^{-1} = \operatorname{diag}\left(\infty, \frac{4\pi(1-\sigma)}{2\ln(a/2)+3}, -\frac{2\pi(1-\sigma)}{\ln(a/2)}\right).$$

3.5.4 *Scattering by the generalized model of narrow joint*

Problem formulation

The problem of scattering by the constructed zero-range potential is formulated as the spectral problem for the operator $\mathcal{H}_c^S$ with matrix $\mathcal{S} = \operatorname{diag}(0, \mathbf{S})$. This spectral problem for acoustic pressure $U^{(s)}(x, y, z)$ and flexure displacement $w^{(s)}(x, y)$ in the scattered field can be written as the boundary-value problem for equations (3.80), (3.81), (3.82) with the boundary conditions

$$\mathbb{F}w^{(s)}(x, \pm 0) = 0, \mathbb{M}w^{(s)}(x, \pm 0) = 0, \quad y \neq 0$$

on edges of the plate and the "boundary" condition in the point $\{x = y = z = 0\}$ which is specified for coefficients $c_j^{\pm}$, $b_j^{\pm}$ in asymptotics (2.39) of $w^{(s)}$. This "boundary" condition is formulated in the form of algebraic system

$$\frac{1}{4\pi(1-\sigma)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\ln(a/2)-3 & 0 \\ 0 & 0 & -2\ln(a/2) \end{pmatrix} \begin{pmatrix} c_0^+ \\ -c_1^+ \\ -c_2^+ \end{pmatrix}$$

$$= \begin{pmatrix} b_0^+ - b_0^- \\ b_1^+ - b_1^- \\ b_2^+ - b_2^- \end{pmatrix} + \begin{pmatrix} w^{(c)}(0,+0) - w^{(c)}(0,-0) \\ w_x^{(c)}(0,+0) - w_x^{(c)}(0,-0) \\ w_y^{(c)}(0,+0) - w_y^{(c)}(0,-0) \end{pmatrix}, \quad (3.110)$$

$$\begin{pmatrix} c_0^- \\ -c_1^- \\ -c_2^- \end{pmatrix} + \begin{pmatrix} c_0^+ \\ -c_1^+ \\ -c_2^+ \end{pmatrix} = 0.$$

Green's function

Similarly to the case of isolated plates, the scattered by the zero-range potential field is expressed in terms of Green's function for fluid loaded plate with infinite crack. This Green's function is the field of a point source applied to the plate. It satisfies the following boundary value problem

$$\triangle\mathcal{G} + k^2\mathcal{G} = 0, \quad z > 0,$$

$$\left(\triangle^2 - k_0^4\right) \left.\frac{\partial\mathcal{G}}{\partial z}\right|_{z=0} + N\,\mathcal{G}|_{z=0} = \frac{1}{D}\delta(x - x_0)\delta(y - y_0), \quad x \neq 0,$$

$$\mathbb{F}\left.\frac{\partial\mathcal{G}}{\partial z}\right|_{z=0} = 0, \quad \mathbb{M}\left.\frac{\partial\mathcal{G}}{\partial z}\right|_{z=0} = 0. \tag{3.111}$$

Besides, Green's function satisfies the radiation conditions specified in the form of asymptotics (3.84), (3.85), (3.86) and Meixner conditions (1.18) near the edges of semi-infinite plates.

Evidently that Green's function depends on the difference of y and y_0. Below y_0 is taken equal to zero. Let $\mathcal{G}$ be searched in the form

$$\mathcal{G}(x, y, z; x_0) = G(x - x_0, y, z) + \mathcal{G}'(x, y, z; x_0),$$

where $G(x - x_0, y, z) = G(\mathbf{r}; x_0, 0)$ is Green's function for the plate without crack and $\mathcal{G}'(x, y, z; x_0)$ is a more smooth correction. Green's function $G(x, y, z)$ is given by (1.21) in the form of double Fourier transform

$$G(x, y, z) = -\frac{1}{4\pi^2 D}\iint e^{i\lambda x + i\mu y - \sqrt{\lambda^2+\mu^2-k^2}\,z}\,\frac{d\lambda\, d\mu}{L(\lambda, \mu)}. \tag{1.21}$$

The correction can be also searched in the form of double Fourier transform

$$\mathcal{G}'(x, y, z; y_0) = \frac{1}{4\pi^2 D}\int\int e^{i\lambda x + i\mu y - \sqrt{\lambda^2+\mu^2-k^2}\,z} \times \\ \times \left(\sum_{j=0}^{3}\lambda^j \alpha_j(\mu; x_0)\right)\frac{d\lambda\, d\mu}{L(\lambda, \mu)}. \tag{3.112}$$

Expression (3.112) satisfies Helmholtz equation and the boundary condition on the plate. Functions $\alpha_j(\mu; x_0)$ are chosen such that contact conditions on the crack are satisfied for the total field $\mathcal{G}$. Substituting from (1.21) and

(3.112) into contact conditions (3.111) yields the system of equations

$$\frac{i}{2}\alpha_0(\mu;x_0) - \left(E_4(0;\mu) - (2-\sigma)\mu^2 E_2(0;\mu)\right)\alpha_1(\mu;x_0)$$
$$- \frac{i}{2}\sigma\mu^2\alpha_2(\mu;x_0) + \left(E_6(0;\mu) - (2-\sigma)\mu^2 E_4(0;\mu)\right)\alpha_3(\mu;x_0)$$
$$= E_3(x_0;\mu) - (2-\sigma)\mu^2 E_1(x_0;\mu),$$
$$-\frac{i}{2}\alpha_0(\mu;x_0) - \left(E_4(0;\mu) - (2-\sigma)\mu^2 E_2(0;\mu)\right)\alpha_1(\mu;x_0)$$
$$+ \frac{i}{2}\sigma\mu^2\alpha_2(\mu;x_0) + \left(E_6(0;\mu) - (2-\sigma)\mu^2 E_4(0;\mu)\right)\alpha_3(\mu;x_0)$$
$$= E_3(x_0;\mu) - (2-\sigma)\mu^2 E_1(x_0;\mu),$$
$$\left(E_2(0;\mu) - \sigma\mu^2 E_0(0;\mu)\right)\alpha_0(\mu;x_0) + \frac{1}{2}\alpha_1(\mu;x_0)$$
$$- \left(E_4(0;\mu) - \sigma\mu^2 E_2(0;\mu)\right)\alpha_2(\mu;x_0) + \frac{1}{2}(2-\sigma)\mu^2\alpha_3(\mu;x_0)$$
$$= -E_2(x_0;\mu) + \sigma\mu^2 E_0(x_0;\mu),$$
$$\left(E_2(0;\mu) - \sigma\mu^2 E_0(0;\mu)\right)\alpha_0(\mu;x_0) - \frac{1}{2}\alpha_1(\mu;x_0)$$
$$- \left(E_4(0;\mu) - \sigma\mu^2 E_2(0;\mu)\right)\alpha_2(\mu;x_0) - \frac{1}{2}(2-\sigma)\mu^2\alpha_3(\mu;x_0)$$
$$= -E_2(x_0;\mu) + \sigma\mu^2 E_0(x_0;\mu).$$

Here

$$E_\ell(x;\mu) = \frac{1}{2\pi}\int e^{i\lambda x}\,\frac{(i\lambda)^\ell\sqrt{\lambda^2+\mu^2-k^2}}{L(\lambda,\mu)}\,d\lambda$$

generalize the integrals $E_\ell(x)$ and $D_\ell(\mu)$, Namely, $E_\ell(x;0) = E_\ell(x)$ and $E_\ell(0;\mu) = D_\ell(\mu)$. When deriving the above system the following properties of integrals $E_j(\pm 0;\mu)$ are used (see (A.6) in Appendix A)

$$E_1(\pm 0;\mu) = 0, \quad E_3(\pm 0;\mu) = \pm\frac{1}{2}, \quad E_5(\pm 0;\mu) = \pm\mu^2.$$

From the above system one finds

$$\alpha_0(\mu;x_0) = \sigma\mu^2\alpha_2(\mu;x_0), \quad \alpha_1(\mu;x_0) = (2-\sigma)\mu^2\alpha_3(\mu;x_0),$$

$$\alpha_2(\mu; x_0) = \frac{E_2(x_0; \mu) - \sigma\mu^2 E_0(x_0; \mu)}{\Delta_e(\mu)},$$

$$\alpha_3(\mu; x_0) = i\frac{E_3(\mu; x_0) - (2-\sigma)\mu^2 E_1(\mu; x_0)}{\Delta_o(\mu)}.$$

Note that denominators in the above expressions coincide with the left-hand sides of dispersion equations (1.65) for symmetric and anti-symmetric edge waves

$$\Delta_o(\mu) = D_6(\mu) - 2(2-\sigma)\mu^2 D_4(\mu) + (2-\sigma)^2 D_2(\mu),$$

$$\Delta_e(\mu) = D_4(\mu) - 2\sigma\mu^2 D_2(\mu) + \sigma^2 D_0(\mu).$$

Solution

Taking into account the last three equations of (3.110), which say that elementary passive sources in the plates Π^+ and Π^- have opposite amplitudes, the scattered field $U^s(x, y, z)$ is searched in the form

$$U^s(x, y, z) = \sum_{j=0}^{2} c_j \mathcal{F}_j(x, y, z) \ , \tag{3.113}$$

where

$$\mathcal{F}_0 = \mathcal{G}(x, y, z; +0) - \mathcal{G}(x, y, z; -0)$$
$$= \frac{i}{4\pi^2 D}\iint \frac{e^{i\lambda x + i\mu y - \sqrt{\lambda^2+\mu^2-k^2}z}\left(\lambda^2 + (2-\sigma)\mu^2\right)\lambda d\mu d\lambda}{L(\lambda,\mu)\,\Delta_o(\mu)},$$

$$\mathcal{F}_1 = \mathcal{G}_y(x, y, z; +0) - \mathcal{G}_y(x, y, z; -0)$$
$$= -\frac{1}{4\pi^2 D}\iint \frac{e^{i\lambda x + i\mu y - \sqrt{\lambda^2+\mu^2-k^2}z}\left(\lambda^2 + (2-\sigma)\mu^2\right)\mu\lambda d\mu d\lambda}{L(\lambda,\mu)\,\Delta_o(\mu)},$$

$$\mathcal{F}_2 = \mathcal{G}_{x_0}(x, y, z; +0) - \mathcal{G}_{x_0}(x, y, z; -0)$$
$$= \frac{1}{4\pi^2 D}\iint \frac{e^{i\lambda x + i\mu y - \sqrt{\lambda^2+\mu^2-k^2}z}\left(\lambda^2 + \sigma\mu^2\right) d\mu d\lambda}{L(\lambda,\mu)\,\Delta_e(\mu)}.$$

Representation (3.113) satisfies all the equations and boundary conditions of the scattering problem except the first three equations of (3.110).

In order to find amplitudes c_j it is necessary to consider near field asymptotics for displacements $w^s(x, y)$, that is, as $r \to 0$. One can show that $w^s(x, y)$ has asymptotics (2.39). Regular terms in that asymptotics can be represented as the sum of contributions of $U^{(c)}(x, y, z)$ and $U^{(s)}(x, y, z)$. Incident $U^{(i)}(x, y, z)$ and reflected $U^{(r)}(x, y, z)$ fields are continuous and do not contribute to system (3.110).

From (1.63) one finds

$$w^{(c)}(0, +0) - w^{(c)}(0, -0) = \frac{2k^4 \sin\vartheta_0 \cos^3\vartheta_0 A_+(\varphi_0) \sin\varphi_0}{D\,\mathcal{L}(\vartheta_0)\Delta_o(k\cos\vartheta_0\cos\varphi_0)},$$

$$w_y^{(c)}(0, +0) - w_y^{(c)}(0, -0) = \frac{2ik^5 \sin\vartheta_0 \cos^4\vartheta_0 A_+(\varphi_0) \sin\varphi_0 \cos\varphi_0}{D\,\mathcal{L}(\vartheta_0)\Delta_o(k\cos\vartheta_0\cos\varphi_0)},$$

$$w_{x_0}^{(c)}(0, +0) - w_{x_0}^{(c)}(0, -0) = \frac{2ik^3 \sin\vartheta_0 \cos^2\vartheta_0 A_-(\varphi_0)}{D\,\mathcal{L}(\vartheta_0)\Delta_e(k\cos\vartheta_0\cos\varphi_0)}. \qquad (3.114)$$

Coefficients $b_j^{\pm}$ are proportional to the corresponding amplitudes of passive sources c_j

$$b_j^+ - b_j^- = c_j \frac{\mathcal{D}_j}{4D^2N}. \qquad (3.115)$$

Here $\mathcal{D}_0$ can be found by direct differentiation of $\mathcal{F}_0$ by z and substitution of $x = 0$, $y = 0$ and $z = 0$

$$\mathcal{D}_0 = \int \frac{d\mu}{\Delta_o(\mu)}.$$

The denominator $\Delta_o(\mu)$ as follows from the asymptotics (A.8) behaves at infinity as

$$\Delta_o(\mu) = \frac{(\sigma - 1)(3 + \sigma)}{4}\mu^3 + \ldots$$

Differentiating $\mathcal{F}_1$ by z and y and excluding logarithmic singularity one finds

$$\mathcal{D}_1 = \frac{1}{\chi}\left(2\gamma + 2\ln(k/2) - i\pi - \widehat{\mathcal{D}}_1\right), \quad \widehat{\mathcal{D}}_1 = \int \left(\frac{\chi\mu^2}{\Delta_o(\mu)} + \frac{1}{\sqrt{\mu^2 - 1}}\right) d\mu.$$

Analogously differentiating $\mathcal{F}_2$ one finds

$$\mathcal{D}_2 = \frac{1}{\chi}\left(2\gamma + 2\ln(k/2) - i\pi - \widehat{\mathcal{D}}_2\right), \quad \widehat{\mathcal{D}}_2 = \int \left(\frac{\chi}{\Delta_e(\mu)} + \frac{1}{\sqrt{\mu^2-1}}\right) d\mu.$$

Substituting from (3.114) and (3.115) into (3.110) yields three separate equations for the amplitudes c_j. One finds

$$c_0 = -\frac{8\pi N k^4 \sin\vartheta_0 \cos^3\vartheta_0 A_+(\varphi_0)\sin\varphi_0}{\mathcal{D}_0\,\mathcal{L}(\vartheta_0)\Delta_o(k\cos\vartheta_0\cos\varphi_0)},$$

$$c_1 = -\frac{8i\pi N k^5 \sin\vartheta_0 \cos^4\vartheta_0 A_+(\varphi_0)\sin\varphi_0\cos\varphi_0}{\mathcal{L}(\vartheta_0)\Delta_o(k\cos\vartheta_0\cos\varphi_0)}\times$$
$$\times\frac{(1-\sigma)(3+\sigma)}{2\ln(ka/4) + 2\gamma + 3 - i\pi - \widehat{\mathcal{D}}_1},$$

$$c_2 = -\frac{8i\pi N k^3 \sin\vartheta_0 \cos^2\vartheta_0 A_-(\varphi_0)}{\mathcal{L}(\vartheta_0)\Delta_e(k\cos\vartheta_0\cos\varphi_0)}\,\frac{(1-\sigma)(3+\sigma)}{2\ln(ka/4) + 2\gamma - i\pi - \widehat{\mathcal{D}}_2}.$$

The far field amplitude

The far field amplitude of the scattered field $U^{(s)}$ can be found as the sum of the far field amplitudes of point sources

$$\Psi(\vartheta,\varphi) = c_0\Psi_{\mathcal{F}_0}(\vartheta,\varphi) + c_1\Psi_{\mathcal{F}_1}(\vartheta,\varphi) + c_2\Psi_{\mathcal{F}_2}(\vartheta,\varphi).$$

Applying saddle point method to the integrals $\mathcal{F}_j$ with the saddle point at $\lambda = k\cos\vartheta\cos\varphi$, $\mu = k\cos\vartheta\sin\varphi$ one finds

$$\Psi_{\mathcal{F}_0}(\vartheta,\varphi) = -\frac{ik^4}{4\pi^2 D}\frac{\cos^3\vartheta\sin\vartheta A_+(\varphi)\sin\varphi}{\mathcal{L}(\vartheta)\Delta_o(k\cos\vartheta\cos\varphi)},$$

$$\Psi_{\mathcal{F}_1}(\vartheta,\varphi) = \frac{k^5}{4\pi^2 D}\frac{\cos^4\vartheta\sin\vartheta A_+(\varphi)\sin\varphi\cos\varphi}{\mathcal{L}(\vartheta)\Delta_o(k\cos\vartheta\cos\varphi)},$$

$$\Psi_{\mathcal{F}_2}(\vartheta,\varphi) = \frac{k^3}{4\pi^2 D}\frac{\cos^2\vartheta\sin\vartheta A_-(\varphi)}{\mathcal{L}(\vartheta)\Delta_e(k\cos\vartheta\cos\varphi)}.$$

Combining these formulae and the values of amplitudes c_j it is a simple matter to write the far field amplitude for the field, scattered by the zero-range potential.

According to the main HYPOTHESIS this formula gives the principal order terms in the far field amplitude asymptotics by small width of the joint. The first term does not depend on the width of the joint and gives the far field amplitude in the problem of scattering by the classical model (1.70) of pointwise joint. This field is anti-symmetric by x and is symmetric by y co-ordinates. The other two terms introduce logarithmically smaller corrections that discard the symmetry of the total field. Next order terms that are not reproduced by the suggested generalized model are of order $O\left((ka)^2\right)$ [15].

Chapter 4

Discussions and recommendations for future research

4.1 General properties of models

Generalized point models of small obstacles in fluid loaded and isolated thin elastic plates simulate the scattered field by means of appropriate passive sources placed in the midpoint of the obstacle. The amplitudes of these passive sources are defined via the local field asymptotics in a vicinity of the potential center. However this is not a direct proportionality of the incident field and the amplitudes of passive sources, a kind of "self-action" is included in the models which can be illustrated by Fig. 4.1. The incident field (both acoustic and/or flexural) creates near the scatterer some regular field which is given by its Taylor decomposition. The model directly connects amplitudes of passive sources with this regular field and this connection as declared by the hypothesis of Section 3.2 is defined by the obstacle only. However passive sources produce field which passes through the media, reflects by distinct objects and contributes to regular field. This "self-action" depends on the global characteristics of the acoustic media and the plate, it knows the global geometry of the problem and it does not know anything of the obstacle. Such separation of parameters that characterize the obstacle from those describing surrounding media appear crucial.

Analysis of classical point models shows these models to be particular examples of generalized point models which involve only such passive sources that do not bring to singular fields. The structure shown on Fig. 4.1 can be found in classical point models as well.

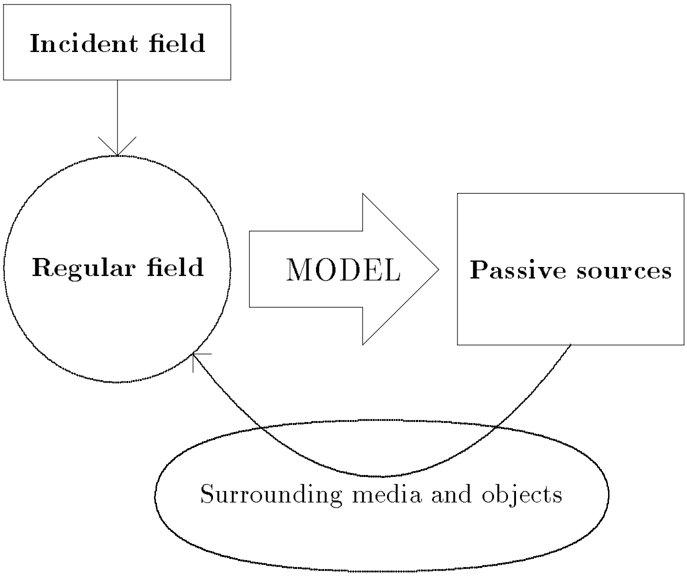


Fig. 4.1 Scheme of generalized models

4.1.1 *Generalized models in two dimensions*

The generalized model of a narrow crack constructed in Section 3.3 generalizes the classical point model of Section 1.4.1 by adding passive acoustic source in the acoustic media near the plate. The "boundary" condition that regulates the amplitude of this source is written in the form of asymptotic decomposition

$$U = -c\frac{1}{\pi}\ln(r/r_0) + o(1), \qquad r \to 0,$$

where c is arbitrary and r_0 is fixed. Combining such acoustic passive source with passive sources specified by usual contact conditions on the plate yields the following two-dimensional models:

(1) Classical point model of crack

$$c = 0\ , \qquad w''(\pm 0) = 0\ , \qquad w'''(\pm 0) = 0\ .$$

(2) Model of narrow crack (with free edges)

$$U = -\frac{c}{\pi}\ln(2r/a) + o(1), \quad r \to 0, \qquad w''(\pm 0) = 0\ , \quad w'''(\pm 0) = 0\ .$$

(3) Fixed point

$$c = 0 \ , \qquad w(\pm 0) = 0 \ , \qquad w'(\pm 0) = 0 \ .$$

(4) Model of a narrow slit with clamped edges ($2a$ is the width of the slit)

$$U = -\frac{c}{\pi}\ln(2r/a) + o(1), \quad r \to 0, \qquad w(\pm 0) = 0 \ , \qquad w'(\pm 0) = 0 \ .$$

(5) Model of a bubble (a is the radius of the bubble)

$$U = -\frac{c}{\pi}\ln(2r/a) + o(1), \quad r \to 0, \qquad w \in C^3.$$

Note that for $a \to 0$ the models (2) and (4) are close to classical models (1) and (3) correspondingly. The bubble presented by model (5) is not described by contact conditions and has no limiting classical model.

Figure 4.2 presents frequency characteristics of effective cross-section for the models (1)–(5). The plane wave is incident at $\vartheta_0 = 30°$ and two models of fluid loaded plates are accepted, namely 1cm steel plate in water (left) and 1mm steel plate in air (right). The obstacle is taken of width $2a =$ 5cm. Consider first the model of water loaded steel plate. Schematically one can accept that the above models cause scattering by different types of passive sources separately. The above models deal with three sources, namely source of model (1), source of model (3) and acoustic source of model (5). Other models can be thought as combinations. Narrow crack is

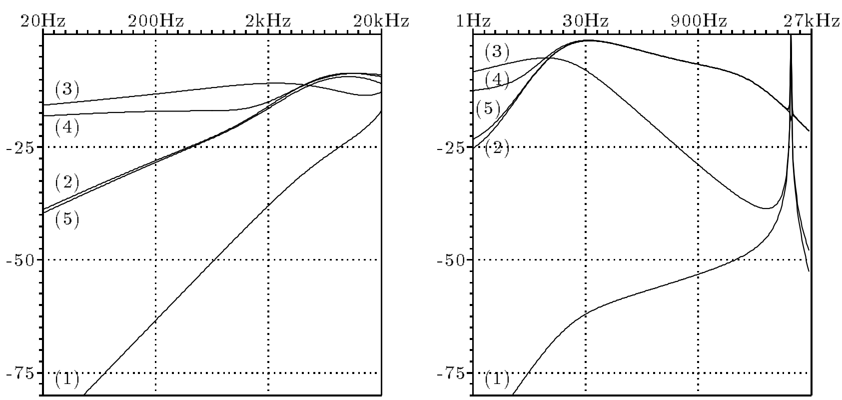


Fig. 4.2 Frequency characteristics for the models (1)–(5)

a combination of pointwise crack and the bubble, narrow slit with clamped edges is a combination of fixed point and the bubble. We compare scattering characteristics for the above models and analyse effects caused by different elementary sources.

Figure 4.2 shows that classical point crack presents the “weakest” obstacle. The energy scattered by the point source corresponding to free edges of the crack rapidly increases with frequency though remains the smallest in the whole range of applicability of thin plate theory. The acoustic source presented in the models (2), (4) and (5) is stronger. Its amplitude depends on the size of the obstacle and is proportional to the inverse of the logarithm of effective radius. For intermediate sizes a the acoustic source appears much stronger than the source of model (1). In particular scattering characteristics do now allow the narrow crack and the bubble to be distinguished. That is the acoustic source completely hides weather the plate is cut or not. The strongest source in the low frequency domain is the source corresponding to a fixed point (or clamped edges of semi-infinite plates). At larger frequencies both flexural sources behave similarly and the acoustic source begins to dominate.

Similar conclusions can be made from the analysis of frequency characteristics in the model of air loaded steel plate. Again at low frequencies the bubble and the narrow crack have close effective cross-sections and other models distinguish more significantly, with the increase of frequency acoustic source begins to dominate. However for steel plate in air there is a large range of intermediate frequencies where only three models can be distinguished, classical point crack, fixed point and the bubble. That is, in the case of light fluid no interference of sources occurs. If the model possesses acoustic source then this source defines the scattering characteristics. If acoustic source is not presented in the model, then the scattering characteristics are defined by one of the sources of flexure vibrations applied to the plate. The source corresponding to free edges appear weaker than that of fixed point. Close to critical frequency one can notice rearrangements in the scattering characteristics. Scaled graphs in a vicinity of critical frequency are shown on Fig. 4.3. In that domain both sources in the plate behave similarly and can not be distinguished. Therefore frequency characteristics of models (1) and (3) almost coincide. Similarly there is little difference between models (2) and (4) that combine correspondingly passive sources of models (1) and (3) with the passive source of model (5). However in a vicinity of critical frequency acoustic source becomes less strong and one

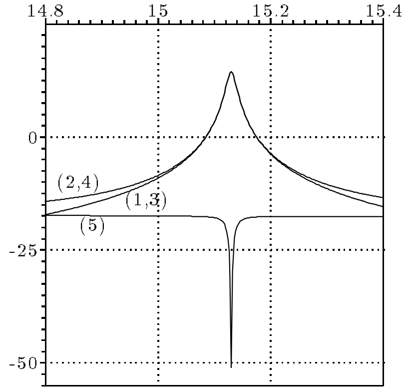


Fig. 4.3 Frequency characteristics for the models (1)–(5) near critical frequency from 14.8kHz to 15.4kHz

can see interference of sources. In particular models (2) and (4) those scattering characteristics at lower frequencies are hardly distinguishable from scattering characteristics of model (5) can be separated at critical frequencies. For all the models except the bubble effective cross-sections behave similarly with very sharp maximum. On the contrary, critical frequency brings minimum to the effective cross-section of the bubble.

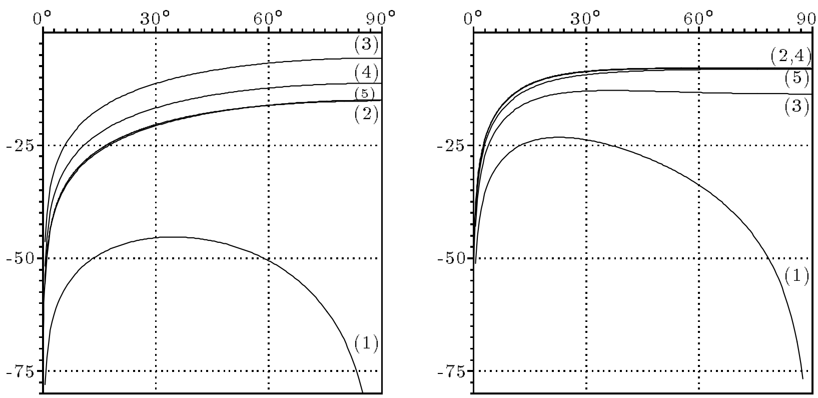


Fig. 4.4 Angular characteristics of scattering by models (1) – (5) in water loaded 1cm steel plate at frequencies 1kHz (left) and 10kHz (right)

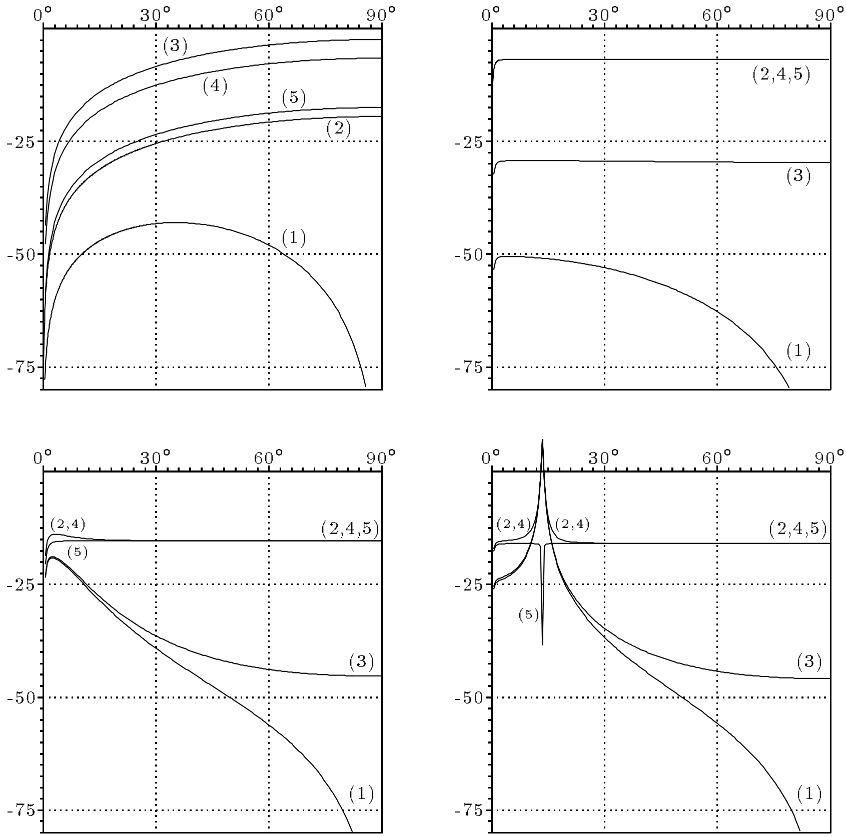


Fig. 4.5 Angular characteristics of scattering by models (1) – (5) in air loaded 1mm steel plate at frequencies 1Hz (top left, curve (1) is shifted up by 50 dB), 1kHz (top right), 11kHz (bottom left) and 12kHz (bottom right)

Figures 4.4 and 4.5 present angular characteristics for scattering by models (1) – (5). One can conclude that at frequencies below the critical frequency significant difference is seen only for the model (1) of pointwise crack. This obstacle scatters no energy in the orthogonal to the plate direction. Other obstacles produce scattered fields with similar angular distributions of energy fluxes. At very low frequencies these obstacles can be distinguished by the amplitudes of scattered fields. At intermediate frequencies scattering characteristics of models (2), (4) and (5) coincide and field scattered by model (3) appears with the same uniform angular distribution, but it has much smaller amplitude. When approaching critical frequency model (5) begins to be separated from models (2) and (4) which remain indistinguishable. It is known that fields scattered by classical point models have far field amplitudes with very sharp lobes oriented at critical angles $\vartheta^* = \cos^{-1}\left(\sqrt{f_c/f}\right)$ (see page 45). Similar lobes can be found on the angular characteristics of scattering by models (2) and (4). However these lobes are superimposed on stronger than in the case of models (1) and (3) uniformly directed field. The scattering characteristics of the bubble on the contrary has very sharp minima at critical angles.

4.1.2 *Structure of generalized models in three dimensions*

Generalized models in three dimensional problems combine 6 sources in the channel of flexural waves and 1 acoustic passive source. In two dimensional models contributions of all the sources applied to the plate do not depend on the size of the obstacle and the far field amplitudes in models (1)–(4) appear of order $O(a^0)$. In three dimensional models passive sources become dependent on the size of the obstacle. Analysis of these effect allows the structure of parameters of the model to be understood better.

Consider first the case of isolated plate. Similarly as in the case of two dimensions classical point models of Chapter 1 can be formulated as generalized models with restricted set of passive sources. Combining all these models and models studied in Chapter 3 gives the following list*:

(6) Fixed point

$$b_{00} = 0, \quad c_{01} = c_{10} = c_{20} = c_{11} = c_{02} = 0,$$

(7) Attached point mass (compare to page 53)

$$b_{00} = \frac{D}{\omega^2 M} c_{00}, \quad c_{01} = c_{10} = c_{20} = c_{11} = c_{02} = 0,$$

(8) Short crack,
(9) Circular hole with free edge,
(10) Circular hole with clamped edge .

*We use continuous enumeration of models.

All these models can be written in the form of condition (2.26) or (2.27). Analyse first flexural components of the generalized models. Matrices $\mathbf{S}$ in the corresponding conditions (2.26) and (2.27) possess specific structure

$$\mathbf{S} = \begin{pmatrix} S_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & S_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & S_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_3 & 0 & S_6 \\ 0 & 0 & 0 & 0 & S_4 & 0 \\ 0 & 0 & 0 & S_6 & 0 & S_5 \end{pmatrix} \qquad (4.1)$$

which expresses the symmetry of passive sources with respect to x and y co-ordinates. The element S_0 specifies the mass M of the obstacle

$$S_0 = \frac{D}{\omega^2 M}.$$

This parameter does not depend on the geometry of the obstacle, its size and shape. For models (8) and (9) this parameter is equal to infinity, which means that the corresponding passive source is not presented.

The parameters S_1 and S_2 characterize dipole source. They are proportional to the logarithm of effective radius and are responsible for logarithmically smaller terms in the far field asymptotics. Such sources are presented in model (10).

The parameters S_3. S_4, S_5 and S_6 are proportional to the square of the effective radius. These parameters characterize quadruple sources and correspond to contact conditions expressing absence of bending momentum on some curve. In models (8) and (9) these are the only sources. To determine the parameters S_3, S_4, S_5 and S_6 in the model of a particular obstacle one needs to know second order terms in the far field asymptotics.

Note here that rotation of the obstacle causes the following transformation in the parameters. Let the obstacle be rotated by angle α, then its matrix $\mathbf{S}$ is changed to $\mathbf{S}'$

$$\mathbf{S}' = \mathbf{R}(\alpha)\mathbf{S}\mathbf{R}(-\alpha),$$

where

$$\mathbf{R}(\alpha) = \begin{pmatrix} 1 & \mathbf{0} & 0 \\ 0 & \mathbf{R}_1 & 0 \\ 0 & 0 & \mathbf{R}_2 \end{pmatrix}, \quad \mathbf{R}_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

$$\mathbf{R}_2 = \begin{pmatrix} cos^2\alpha & \sin 2\alpha & \sin^2\alpha \\ -\frac{1}{2}\sin 2\alpha & \cos 2\alpha & \frac{1}{2}\sin 2\alpha \\ \sin^2\alpha & -\sin 2\alpha & \cos^2\alpha \end{pmatrix}.$$

Generalized models for fluid loaded plates differ by the presence of additional acoustic source. The parameter $\mathcal{S}_{11}$ that characterizes this source appears proportional to the effective radius. According to the hypothesis the parameters in the composite model are inherited from the models in acoustic and flexural components. Therefore the structure (4.1) is preserved in the composite model as well.

4.1.3 *Generalized models in the plate with infinite crack*

In this book we studied only a subset of zero-range potentials in the problems dealing with elastic plates cut by straight infinitely narrow crack. The sources that are included are of monopole and dipole types. The structure of the matrix **S** for such models resembles the 3×3 upper left block of (4.1). One can check that the parameter S_0 describes the mass of the attached obstacle and the parameters S_1 and S_2 depend on the logarithm of the size of the obstacle and characterize its shape.

The generalized point models presented in this book were constructed on the basis of analysis of auxiliary problems of diffraction in the case of absolutely rigid plate and isolated plate. However after having sufficiently large set of generalized models and knowing the structure of the matrices **S** that is described above, one can distinguish some elementary blocks which can be used for constructing other generalized models.

4.2 Extending the model of narrow crack to oblique incidence and edge waves analysis

4.2.1 *Reformulation of the model*

The two dimensional generalized point models can be extended to the analysis of scattering effects at oblique incidence and in particular to the study of edge waves running along a narrow straight crack in fluid loaded elastic plate.

Consider first the formulation of the diffraction problem on a narrow straight crack in elastic plate. The crack is described by boundary condition

$$U(x, y, 0) = 0, \qquad |x| < a \tag{4.2}$$

and contact conditions

$$\mathbb{F}w(\pm a, y) = \mathbb{M}w(\pm a, y) = 0.$$

If the incident wave does not depend on y co-ordinate the problem of diffraction is reduced to two-dimensional problem studied in Section 3.2.6. Let the incident wave depend on y co-ordinate by the factor $e^{i\mu_0 y}$. Then all the components of the field contain this factor and it can be separated. Again the problem is reduced to two-dimensional one. Namely the above boundary and contact conditions are reduced to†

$$U(x, 0) = 0, \qquad |x| < a \tag{4.3}$$

$$\begin{aligned} w'''(\pm a) - (2-\sigma)\mu_0^2 w'(\pm a) = 0, \\ w''(\pm a) - \sigma\mu_0^2 w(\pm a) = 0. \end{aligned} \tag{4.4}$$

The contact conditions (4.4) are easily transformed into the corresponding conditions of generalized model. One simply lets $a = 0$ in (4.4). The boundary condition (4.3) exactly coincides with the corresponding condition in two-dimensional problem. That is, the assumption of factor $e^{i\mu_0 y}$ changes only the *surrounding media* in the scheme of generalized models presented on Fig. 4.1 . The parameterization of the zero-range potentials in the generalized models is organized in such a way that the parameters are completely defined by the obstacle and do not depend on the incident field, its frequency and direction of incidence. All this allows the zero-range potential in the cross-section $y = \text{const}$ to be concluded the same as defined by conditions (3.6), that is the same as in the case of $\mu_0 = 0$. Therefore in the initial three-dimensional formulation the generalized model of straight narrow crack is defined by fixing asymptotics

$$U = -\frac{c(y)}{\pi} \ln\left(2\sqrt{x^2+z^2}/a\right) + o(1), \qquad \sqrt{x^2+z^2} \to 0 \tag{4.5}$$

and contact conditions

$$\mathbb{F}w(\pm 0, y) = \mathbb{M}w(\pm 0, y) = 0. \tag{4.6}$$

†We preserve notations U and w for the pressure and displacement in the cross-section $y = \text{const}$.

Here $c(y)$ is arbitrary function and the asymptotics is valid for any fixed co-ordinate y.

4.2.2 *Edge waves propagating along a narrow crack*

Edge waves that propagate along infinitely narrow cracks are studied in [46] by means of perturbation technique assuming weak loading. In the case when perturbation technique is not applicable numerical analysis of dispersion equations (1.65) is required. We modify these dispersion equations by taking into account the width of the crack and study edge waves that can propagate along such cracks. The generalized model (4.5), (4.6) is accepted.

Consider solutions in the form of waves propagating along y co-ordinate

$$U = e^{i\mu y} v(x,z).$$

Separating exponential factor yields the eigen problem for parameter μ and function $v(x,z)$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) v + \left(k^2 - \mu^2\right) v = 0, \qquad z > 0, \tag{4.7}$$

$$\left[\left(\frac{\partial^2}{\partial x^2} - \mu^2\right)^2 - k_0^4\right] \frac{\partial v}{\partial z} - N\, v = 0, \qquad z = 0, x \neq 0, \tag{4.8}$$

$$v = \text{const}\,\log(2\sqrt{x^2+z^2}/a) + o(1), \qquad \sqrt{x^2+z^2} \to 0, \tag{4.9}$$

$$\begin{aligned} &\left(\frac{d^3}{dx^3} - (2-\sigma)\mu^2 \frac{d}{dx}\right) \frac{\partial v(x,0)}{\partial z}\bigg|_{x=\pm 0} = 0, \\ &\left(\frac{d^2}{dx^2} - \sigma\mu^2\right) \frac{\partial v(x,0)}{\partial z}\bigg|_{x=\pm 0} = 0. \end{aligned} \tag{4.10}$$

If μ is known, then the general solution v that satisfies the equation (4.7) and the boundary condition (4.8), is given by Fourier integral

$$v = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \left\{ c\left[\left(\lambda^2+\mu^2\right)^2 - k_0^4\right] + \sum_{\ell=0}^{3} c_\ell (i\lambda)^\ell \right\} \times$$
$$\times \exp\left(i\lambda x - \sqrt{\lambda^2+\mu^2-k^2}\, z\right) \frac{d\lambda}{L(\lambda,\mu)}.$$

Substitution of this expression into the conditions (4.9) and (4.10), yields the linear algebraic system for the constants c_0, c_1, c_2, c_3 and c. The symmetry of the problem allows this system to be split. One subsystem corresponds to symmetric by x part

$$\begin{cases} c_0 + \sigma\mu^2 c_2 = 0, \\ \big(D_2(\mu) - \sigma\mu^2 D_0(\mu)\big)c_0 + \big(D_4(\mu) - \sigma\mu^2 D_2(\mu)\big)c_2 \\ \qquad\qquad + N\big(J_2(\mu) - \sigma\mu^2 J_0(\mu)\big)c = 0, \\ J_0(\mu)c_0 + J_2(\mu)c_2 \\ \qquad + \dfrac{1}{\pi}\left(\ln\left(\dfrac{a}{2}\sqrt{\mu^2 - k^2}\right) + C_E + \pi N J(\mu)\right) c = 0. \end{cases} \tag{4.11}$$

The integrals $D_\ell(\mu)$ are introduced in (1.64) and integrals $J_\ell(\mu)$ and $J(\mu)$ generalize the integrals J_ℓ and J introduced in (3.10) and (3.9) for the case of oblique incidence, namely

$$J_\ell(\mu) = \frac{1}{2\pi}\int (i\lambda)^\ell \frac{d\lambda}{L(\lambda,\mu)}, \quad J(\mu) = \frac{1}{2\pi}\int \frac{d\lambda}{L(\lambda,\mu)\sqrt{\lambda^2 + \mu^2 - k^2}}.$$

Equating determinant of the system (4.11) to zero yields dispersion equation

$$\Delta_e(\mu) - \frac{\pi N\ \big(J_2(\mu) - \sigma\mu^2 J_0(\mu)\big)^2}{\ln\left(\sqrt{\mu^2 - k^2}a/2\right) + C_E + J(\mu)} = 0, \tag{4.12}$$

$$\Delta_e(\mu) \equiv D_4(\mu) - 2\sigma\mu^2 D_2(\mu) + \sigma^2\mu^4 D_0(\mu).$$

For classical model one formally lets $a = 0$ and drops out the last term which yields the first equation form (1.65).

The subsystem for the anti-symmetric part of the field does not contain the width of the crack a and coincides with the second equation from (1.65)

$$\Delta_o(\mu) \equiv D_6(\mu) - 2(2-\sigma)\mu^2 D_4(\mu) + (2-\sigma)^2\mu^4 D_2(\mu) = 0. \tag{4.13}$$

We denote positive solutions of the above dispersion equations as $\varkappa_e$ and $\varkappa_o$ correspondingly. If these solutions are known the symmetric and anti-symmetric edge waves are given by the formulae

$$U_e = e^{i\mu_e y}\int \exp\left(i\lambda x - \sqrt{\lambda^2+\mu_o^2-k^2}\,z\right)\left(\lambda^2+\sigma\mu_e^2 - \pi\frac{\left(J_2(\mu_e)-\sigma\mu_o^2 J_0(\mu_e)\right)\left(\left(\lambda^2+\mu_e^2\right)^2-k_0^4\right)}{\ln\left(\frac{1}{2}a\sqrt{\mu_e^2-k^2}\right)+C_E+\pi N\,J(\mu_e)}\right)\frac{d\lambda}{L(\lambda,\mu_o)},$$

$$U_o = e^{i\mu_o y}\int \exp\left(i\lambda x - \sqrt{\lambda^2+\mu_o^2-k^2}\,z\right)\frac{\lambda^2+(2-\sigma)\mu_o^2}{L(\lambda,\mu_o)}\lambda\,d\lambda.$$

Before proceeding to solving dispersion equations (4.12) and (4.13), investigate the left-hand sides as functions of complex variable μ. One notes that if μ is real and is greater than the wavenumber κ of surface waves, then the paths of integration in all the contact integrals $D_\ell(\mu)$, $J_\ell(\mu)$ and $J(\mu)$ can be brought in coincidence with the real axis of λ. Therefore for $\mu > \kappa$ all these integrals are real. All these integrals can be reduced to sums of residues in the zeros κ_s of the denominator (see (A.7)). The explicit formulae for these integrals contain logarithms and it is easy to verify that all $\mu = \kappa_s$ are branch points. Moreover in a vicinity of $\mu = \kappa_s$ the function $\Delta_e(\mu)$ infinitely increases while $\Delta_o(\mu)$ remains bounded. By calculating the residue in $\lambda = \sqrt{\mu^2-\kappa^2}$ it can be shown that

$$\lim_{\mu\to\kappa+0}\Delta_e(\mu) = +\infty.$$

It is also easy to determine the sign of the function $\Delta_e(\mu)$ for large positive μ. For this one should take into account that the integrals $D_4(\mu)$ and $D_6(\mu)$ require regularization which is equivalent to subtraction of quantities 1 and λ^2 from the respective integrated function. One finds

$$\lim_{\mu\to+\infty}\Delta_e(\mu) < 0.$$

Analysis of the dispersion equation (4.12) concludes that dependently on the parameters of the plate – fluid system two cases are possible. Either two real solutions exist on the ray $\mu > \kappa$, or no real solutions exist. For the interpretation of this result one should take into account that the model of narrow crack assumes smallness of a compared to the characteristic wavelength. That means that the dispersion equation (4.12) is physically meaningful only in a bounded domain of the complex plane of μ, namely

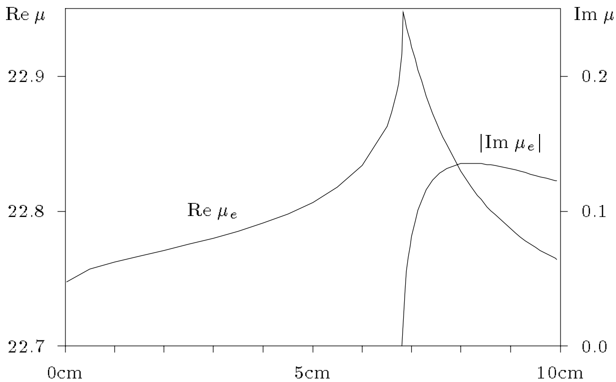


Fig. 4.6 Even wave wavenumber dependence on the crack width in 1sm steel plate in water at 5kHz.

when $\sqrt{\mu^2 - k^2}a \ll 1$. Therefore in the case when two solutions exist on the ray $\mu > \kappa$ the smaller solution corresponds to the wavenumber of symmetric edge wave and the larger lies outside of the applicability domain of generalized model. In the case when no real solutions exist generalized model of narrow crack is inapplicable.

Figure 4.6 shows the dependence of the solution $\varkappa_e$ of dispersion equation (4.12) for symmetric edge waves on the crack width a. The system of 1cm steel plate on the surface of water is chosen. If $a\mu$ is small correction due to the opening of the crack is negligible. For larger values of $a\mu$ a parasitic nonphysical solution approaches from infinity, two solutions collide and move to the complex plane. In this case the generalized model becomes inapplicable. Similar behaviour of solutions remains for other combinations of plate and fluid parameters. Therefore further numerical results are presented for the classical point model of crack.

The wave numbers of symmetric and anti-symmetric edge waves appear very close to κ. To distinguish these values Fig. 4.7 presents quantities $s_e = \varkappa_e^2 - \kappa^2$ and $s_o = \varkappa_o^2 - \kappa^2$ for edge waves in 1cm steel plate on the surface of water (left) and in 1mm steel plate in air (right). The symmetric wave exists in both models in the whole range of frequencies. Below coincidence frequency ($f < f_c$) its wavenumber κ_e noticeable differs from the wave number κ of surface waves. Above coincidence frequency ($f > f_c$) $\varkappa_e$

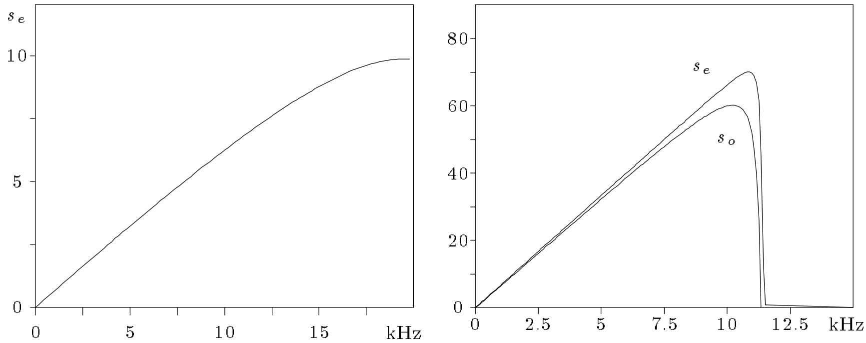


Fig. 4.7 Quantities $s_e = \mu_e^2 - \kappa^2$ and $s_o = \mu_o^2 - \kappa^2$ for edge waves in water (left) and in air (right) loaded steel plates

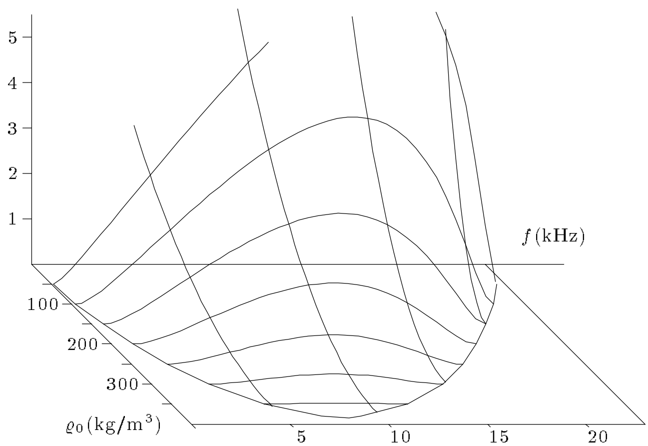


Fig. 4.8 Dependence of s_o on frequency f and density of fluid ϱ_0

approaches to κ and symmetric edge wave becomes weakly localized. Anti-symmetric edge wave exists only in the case of lightly loaded plate in some range of frequencies $[f_1, f_2]$. For air loaded steel plate $f_1 \approx 0$ and $f_2 \approx f_c$. For heavier fluids f_1 increases and f_2 decreases. In particular for steel plate in water the interval $[f_1, f_2]$ is degenerated and no anti-symmetric edge wave exist. Figure 4.8 presents the wavenumber $\varkappa_o$ of anti-symmetric edge wave in 1cm steel plate loaded by some hypothetical fluids with $c_a = 1500$m/c and variable densities.

Analysis of the profiles of displacements in symmetric and anti-symmetric edge waves shows that both waves are exponentially decreasing far from the crack. Details see in [13] where energy flows carried by edge waves and their distribution between acoustic and elastic channels are studied. See also [56] and [57] where edge waves are studied in a different geometry for orthotropic plate and for plate described by Timoshenko–Mindlin plate theory.

4.3 Further generalizations and unsolved problems

4.3.1 *Models with internal structure*

Generalized models of Chapter 3 describe scattering by such obstacles that are described by ideal boundary and contact conditions. These are absolutely rigid bodies, or holes with free edges. If one needs to take into account deformations that may be presented in the obstacle, then more complicated generalized models should be used. Such models are formulated as zero-range potentials with internal structure [58]. To formulate such models one adds the third component to the space ($d = 2, 3$ is the dimension)

$$\mathcal{L} \oplus L = \Big(L_2(\mathbb{R}^d_+) \oplus L_2(\mathbb{R}^{d-1})\Big) \oplus L.$$

The two component space $\mathcal{L}$ is the usual space for fluid loaded plate. The second component L is the space that should be chosen adequately to the wave process in the obstacle. For example if one deals with a beam of length H which only allows compressional waves, then the component L can be taken as $L_2([0, H])$.

Let the wave process in detached obstacle be described by selfadjoint differential operator C. Then one starts with the operator

$$\operatorname{diag}(\mathcal{H}_d, C)$$

which describes noninteracting fluid loaded elastic plate and the obstacle. Further one restricts this operator, that is restricts both components $\mathcal{H}_d$ and C. Restriction of the operator $\mathcal{H}_d$ are performed to functions that vanish in a vicinity of some chosen point (see Section 2.4.3). Restriction of operator C are performed analogously. The deficiency indices of the restricted operator are equal to the sums of deficiency indices of its restricted components. If for example the operator C is a second order differential operator, then in the case of $d = 2$ the indices are (6,6) and in the case of $d = 3$ the deficiency indices are $(8, 8)$. Selfadjoint extensions are separated into two types. Some extensions are equivalent to the sum of selfadjoint operators $\mathcal{H}_d^S$ and perturbation C^ς of operator C. These are models of nondeformable obstacle attached to the plate. The other selfadjoint extensions contain interaction of functions from $\mathrm{Dom}(\mathcal{H}_d)$ and $\mathrm{Dom}(C)$. The eigen oscillations of the body (eigen numbers of operator C) become resonances of the plate with the obstacle. Therefore generalized models with internal structure describe resonance character of scattering phenomena.

4.3.2 *Restrictions of accuracy*

Two dimensional classical point models reproduce only the principal order terms in the asymptotics of the scattered fields. These terms do not depend on the size of the obstacle. The generalized models are designed to give also the first order correction which is logarithmically smaller by the size of the obstacle. One may need a more precise model that would reproduce terms of smaller orders.

Consider for example the problem of scattering by a stiffener of mass M and moment of inertia I which protrudes by small height H into fluid (formulation see in Section 4.4). Asymptotics of diffracted field is found in [22] by means of integral equations method. The far field amplitude is

$$\Psi(\vartheta) = \frac{iNA}{\pi}\frac{k^2 \sin\vartheta \sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}\left\{\frac{1}{D_0 - D/(M\omega^2)} - \frac{k^2\cos\vartheta\cos\vartheta_0}{D_2 + D/(I\omega^2)}\right.$$
$$\left. - \frac{\pi}{2}N(kH)^2\cos\vartheta\cos\vartheta_0 M(\vartheta)M(\vartheta_0) + o\Big((kH)^2\Big)\right\}, \qquad (4.14)$$
$$M(\vartheta) = \frac{k^4\cos^4\vartheta - k_0^4}{N} - \frac{J_2}{D_2 + D/(I\omega^2)}.$$

In the principal order the stiffener can be replaced by the point mass and momentum of inertia attached to the plate. The far field amplitude for this model is given in (1.62). The next order terms which correct the asymptotics of the scattered field taking into account the height of the stiffener are proportional to $(kH)^2$. Terms of such order are not reproduced by the generalized models presented in this book.

Similarly in diffraction by a stiffener of small, but nonzero width a

considered in [18] the correction to the classical point model appears of order $O\left((ka)^2\right)$. It is not reproduced by generalized models, too.

Analogous restrictions are presented in three dimensional generalized models. As is established in Section 4.1.2 the generalized models in elastic component take into account boundary-contact conditions specified for displacements, angles and bending momentums. The boundary-contact conditions written only for forces are neglected. In particular in the problem of diffraction by a short crack (see Section 3.3) one may replace the condition

$$\mathbb{F}w(\pm 0, y) = 0, \qquad |y| < a$$

by the condition

$$[w] = 0, \quad [\mathbb{F}w] = 0, \qquad |y| < a$$

which will be not noticed by the generalized model. Therefore to be able to reproduce scattering effects associated with forces one needs to increase accuracy of generalized point models.

For this one needs to add other passive sources in both channels of scattering. For example the smaller order corrections in the asymptotics (4.14) of the far field amplitude in the problem of diffraction by protruding stiffener correspond to dipole acoustic source $\delta'(x)\delta(z)$. The field generated by such source is $\partial G(x, z; 0, 0)/\partial x$. It has the asymptotics

$$\frac{\partial G(x, z; 0, 0)}{\partial x} \sim -\frac{x}{\pi\, r^2}, \qquad r \to 0,$$

which shows that such derivative does not belong to $L_2(\mathbb{R}^2_+)$.

From the point of view of operator extensions necessity to add passive sources of multipole type means necessity to increase the dimension of the deficiency subspace.

In the case of the generalized model of narrow joint such enlargement of deficiency subspace can be achieved. Indeed, when constructing zero-range potentials for the basic geometry of a plate cut by straight crack, the initial operator was restricted to functions that vanish at the origin and those first order derivatives vanish, too. However embedding theorems allow the second order derivatives to be equated to zero, which causes further restriction of the operator and increases the dimension of the deficiency subspace up to 12. The domain of adjoint operator is extended then by adding second order derivatives of singular solution $\mathcal{G}(x, y, z; x_0)$ defined in (3.112).

For all other generalized models the dimension of deficiency subspace can not be increased. Indeed, the restricted operator can not be further restricted by adding conditions in the potential center. Besides one can not add higher order derivatives of singular solutions to the domain of the operator because such derivatives do not belong to the main space L_2 used for the model construction.

This shows that in order to increase accuracy of the model one needs to use some other space instead of L_2. Two approaches are known. One deals with spaces $L_2(\mu)$ of square integrable functions with appropriately chosen weight μ which vanishes near the origin and therefore higher order singularities are allowed. This approach is developed for example in [53] for arbitrary second order differential operator. Generalization to the case of matrix operator of hydroelasticity may be also possible. However the approach based on weighted spaces causes spectral parameter to appear in the matrix that parameterizes zero-range potentials. This property spoils the structure of the model and the main hypothesis may appear wrong.

Another approach uses Pontryagin spaces [61], [32]. These spaces have no metrics, but are supplied with the metric-like function which possesses all the properties of usual metric except it can be negative. Roughly speaking such constructions are equivalent to adding to L_2 a span of fixed functions ϕ_j that do not belong to L_2. Then the "scalar product" is set as

$$\langle U, V\rangle = \langle u, v\rangle + \sum_{j=1}^{n} u_j \overline{v}_j,$$

where

$$U(\mathbf{r}) = u(\mathbf{r}) + \sum_{j=1}^{n} u_j \phi_j(\mathbf{r}), \qquad V(\mathbf{r}) = v(\mathbf{r}) + \sum_{j=1}^{n} v_j \phi_j(\mathbf{r}). \tag{4.15}$$

The set of all functions U that are decomposed in the form (4.15) with finite complex u_j and $u \in L_2$ form the Pontryagin space with n negatives squares. In [61] such spaces are used for constructing models of narrow slits in rigid screens. No generalizations of the approach based on Pontryagin spaces to the case of fluid loaded plates are known.

Another attempt is undertaken recently in [11] where the generalized model of protruding stiffener in absolutely rigid plate is suggested. The construction is carried out in a specially designed Hilbert space. Unfortunately this construction becomes very cumbersome if quadruple and higher order sources are involved. Besides it is difficult to be extended for the case of fluid loaded elastic plate.

Finally, one can replace the obstacle by a set of passive point sources. The model of protruding stiffener in the form of zero-range potential with two potential centers in acoustic component is presented in Section 4.4.

4.3.3 *Other basic geometry*

The generalized models described in this book deal with elastic plates being in one side contact to acoustic media. In a similar manner one can consider plates plunged into acoustic media. Moreover one can have different fluids separated by the plate. In that cases the zero-range potentials are considered in the space

$$\mathcal{L} = L_2(\mathbb{R}^d_+) \oplus L_2(\mathbb{R}^d_-) \oplus L_2(\mathbb{R}^{d-1})$$

and the operator of fluid loaded plate is a three component operator. The additional component increases the deficiency indices by 1.

More essential changes appear if one considers semi-infinite objects. Only one model is known. Zero-range potentials for isolated plate with semi-infinite crack are constructed in [38].

4.3.4 *Other approximate theories of vibrations*

This book only deals with flexural deformations in the plates described by Kirchhoff theory. This theory is valid for sufficiently thin plates, or for sufficiently low frequencies. At higher frequencies shear deformations become important and Timoshenko–Mindlin equations [67] are used for the description of plate vibrations. One can reformulate generalized models of this book for the case of Timoshenko–Mindlin theory.

Symmetric deformations exist in thin plates independently of flexural deformations and are almost not influenced by fluid loading. However at wedge-shaped junctions of semi-infinite plates interaction of flexural and symmetric waves may appear. Such junctions can be also approximated by generalized models using zero-range potentials. For this the channel of symmetric waves (additional component of the space) should be added.

One can also try to extend the zero-range potentials approach to models of obstacles in thin elastic shells.

4.4 Model of protruding stiffener in elastic plate

4.4.1 *Introduction*

In this section a simple model of protruding stiffener of small height H is formulated. This model reproduces not only the principal order terms in the far field amplitude, but also takes into account corrections proportional to the height H. We use the operator model of fluid loaded plate formulated in Section 2.3.4. That approach treats the square of frequency as the spectral parameter and yields a usual spectral problem rather than a spectral problem for an operator pencil as in models of Chapter 3.

The terms in the asymptotics of the far field amplitude that are proportional to the height of the stiffener require the zero-range potential of dipole type. As described in previous sections such zero-range potential can not be constructed in L_2. Instead we simulate dipole source by zero-range potential with two potential centers.

The parameters of the zero-range potential are organized in Hermitian matrix and its block structure is similar to noted in Section 2.4.3. It is essential that the matrix only depends on the parameters of the stiffener (its height, mass and momentum of inertia) and does not depend on the global characteristics of the mechanical system (such as thickness of the plate, its rigidity, densities of the plate and fluid, *etc.*) and on the incident field. The two diagonal blocks characterize the scattering properties of the obstacle in the isolated channels of scattering. One of these problems of scattering can be thought of as the limit of the initial scattering problem for infinitely rigid plate, the other as the limit for infinitely light fluid (vacuum). The nondiagonal blocks are assumed zero. Afterwards comparison with the asymptotics (4.14) found in [22] allows the applicability of the model to be justified for stiffeners of small height, that is for $kH \ll 1$.

4.4.2 *Classical formulation*

Let the acoustic system consist of a homogeneous liquid half-space $\{z > 0\}$ bounded by a thin elastic plate $\{z = 0\}$ supported by protruding stiffener $\{x = 0, 0 < z < H\}$. The wave field is formed by an incident harmonic plane wave

$$U^{(i)}(x,z) = A\exp\big(ikx\cos\vartheta_0 - ikz\sin\vartheta_0\big).$$

The acoustic pressure $U(x,z)$ satisfies Helmholtz equation and the generalized impedance boundary condition (1.13) on the plate. The stiffener is assumed absolutely rigid, that is the following conditions (same as in [22]) are satisfied

$$\frac{\partial U(0,z)}{\partial x} = -z\frac{\partial^2 U(0,0)}{\partial x \partial z} \qquad 0 < z < H,$$

$$Z_f w(0) = w'''(+0) - w'''(-0), \tag{4.16}$$

$$\begin{aligned} -Z_m w'(\pm 0) &= w''(+0) - w''(-0) \\ &+ \frac{1}{D}\int_0^H z\Big(U(+0,z) - U(-0,z)\Big)dz. \end{aligned} \tag{4.17}$$

Here Z_f and Z_m are the force and the momentum impedances

$$Z_f = \frac{\varrho_1 h_1 H\omega^2}{D}, \qquad Z_m = \frac{\varrho_1 h_1 H^3\omega^2}{3D},$$

ϱ_1 and h_1 are the density and the thickness of the stiffener. Besides, in the singular points of the boundary, that is in the points $(0,0)$ and $(0,H)$, the Meixner conditions (1.18) for the acoustic pressure $U(x,z)$ are satisfied. Scattered field is subject to radiation conditions (1.16).

The use of Green's function (1.28) and integral representation for the scattered field allows the above boundary value problem to be reduced to an integro-algebraic system of equations. Asymptotic analysis of this system allows approximate formula (4.14) for the far field amplitude to be found. Detailed derivations can be found in [22].

4.4.3 *Zero-range potentials*

Let the generalized model that reproduces terms of two orders in kH in the far field amplitude $\Psi(\vartheta;\vartheta_0)$ for $kH \to 0$ be constructed in the form of zero-range potential for the operator $\mathcal{H}$ of Section 2.3.4. This approach treats frequency as the spectral parameter.

Zero-range potential in the acoustic component

Consider first zero-range potentials in the acoustic channel. Restrict Laplacian to functions that vanish in a vicinity of the points $(\epsilon, 0)$ and $(-\epsilon, 0)$. The closure of this operator is the symmetric operator with the deficiency indices equal to $(2, 2)$. The adjoint operator A^* is defined on such functions from $L_2(\mathbb{R}^2_+)$ that can be represented in the form

$$\begin{aligned} U(x, z) = U_0(x, z) + \chi(x, z) \left(-\frac{c^+}{2\pi} \ln\left((x-\epsilon)^2 + z^2\right) \right. \\ \left. + b^+ \frac{x+\epsilon}{2\epsilon} - \frac{c^-}{2\pi} \ln\left((x+\epsilon)^2 + z^2\right) + b^- \frac{x-\epsilon}{2\epsilon} \right). \end{aligned} \tag{4.18}$$

Here U_0 belongs to the domain of the restricted operator, χ is the cut-off function and $c^\pm, b^\pm$ are arbitrary complex constants. The expression (4.18) is in fact a simple combination of the two asymptotics of the type (2.19).

Following the technique of [58] the selfadjoint operator appears if appropriate constrains are set to the constants $c^\pm, b^\pm$. It is easy to find out that the boundary form reduces to the sum of the boundary forms corresponding to each center, namely

$$\langle A^* U_1, U_2 \rangle - \langle U_1, A^* U_2 \rangle = b_1^+ \overline{c_2^+} + b_1^- \overline{c_2^-} - c_1^+ \overline{b_2^+} - c_1^- \overline{b_2^-}.$$

In the case of non-interacting centers linear constrains are taken for constants c^+, b^+ and c^-, b^- independently. This is a simple combination of two zero-range potentials. The interaction between the centers appears if the vectors $\mathbf{c} = (c^+, c^-)$ and $\mathbf{b} = (b^+, b^-)$ are connected by a Hermitian non-diagonal matrix.

It is convenient to rewrite these conditions in another form. Instead of amplitudes $c^\pm$ of the passive monopole sources one can take the amplitude of an equivalent monopole and dipole sources. These amplitudes are

$$c = c^+ + c^-, \qquad c' = \epsilon\left(c^+ - c^-\right).$$

The quantities $b^\pm$ are replaced by mean value and by finite difference

$$b = \frac{b^+ + b^-}{2}, \qquad b' = \frac{b^+ - b^-}{2\epsilon}.$$

If ϵ approaches zero, the parameter b' tends to the derivative of the regular part of the field and the parameter c' tends to the amplitude of the passive dipole source. In terms of these new coefficients the operator is selfadjoint provided

$$\begin{pmatrix} c' \\ c \end{pmatrix} = S \begin{pmatrix} b' \\ b \end{pmatrix} \tag{4.19}$$

and matrix S is Hermitian.

Zero-range potentials in the elastic component

For the component v (see Section 2.3.4) the zero-range potential with one potential center is sufficient to model effects of diffraction by the stiffener. These are the zero-range potentials for ordinary differential operator and can be set by usual boundary conditions formulated in the point of the potential center. To write these conditions in the form convenient for combining the zero-range potential for fluid loaded plate from the acoustic and elastic components one introduces function

$$\Upsilon(x) \equiv \frac{\varrho_0}{D} U(x,0) + v''(x).$$

and finds

$$w''(x) = \omega^2 v(x), \qquad w(x) = \frac{D}{\varrho h}\Upsilon(x). \tag{4.20}$$

Formulae (4.20) remain valid both in the case of fluid loaded plate and in the case of isolated plate (when $\varrho_0 = 0$).

The parameterization of the zero-range potentials in the elastic component can be taken in the form

$$\begin{pmatrix} [\Upsilon] \\ -[\Upsilon'] \\ [v] \\ -[v'] \end{pmatrix} = \mathbf{S} \begin{pmatrix} \{v'\} \\ \{v\} \\ \{\Upsilon'\} \\ \{\Upsilon\} \end{pmatrix}. \tag{4.21}$$

Remind that square brackets denote jumps and curly braces stand for mean values.

Zero-range potentials for fluid loaded plate

The conditions that fix selfadjoint operator can be written in matrix form

$$\mathbf{c} = \mathcal{S}\mathbf{b}. \tag{4.22}$$

Here

$$\mathbf{c} = \left(\frac{\varrho h}{D}c',\ \frac{\varrho h}{D}c,\ [\Upsilon],\ -[\Upsilon'],\ [v],\ -[v']\right)^T,$$
$$\mathbf{b} = \left(\frac{\varrho_0}{D}b',\ \frac{\varrho_0}{D}b,\ \{v'\},\ \{v\},\ \{\Upsilon'\},\ \{\Upsilon\}\right)$$

and $\mathcal{S}$ is Hermitian matrix having the block structure similar to (2.34)

$$\mathcal{S} = \begin{pmatrix} \frac{\varrho h}{\varrho_0}S & \mathbf{s} \\ \mathbf{s}^* & \mathbf{S} \end{pmatrix}.$$

The block S is the 2×2 matrix that parameterizes the zero-range potential in acoustic channel, the block $\mathbf{S}$ is the 4×4 matrix that parameterizes the zero-range potential in elastic channel and 2×4 matrix $\mathbf{s}$ characterizes interaction of the channels that appears due to the stiffener.

Deficiency elements

Deficiency elements are solutions of boundary-value problem with delta-functions in the right-hand sides of all equations. Introduce elements $\mathcal{G}_j \equiv \big(G_j(x,z;x_0), \gamma_j(x;x_0)\big)$ $(j=1,2,3)$ as solutions of the following boundary-value problems

$$\left\{\begin{aligned} & -c_a^2\Delta G_j(x,z;x_0) = \omega^2 G_j(x,z;x_0) + c_a^2\delta_1^j\delta(x-x_0)\delta(z-0), \\ & \frac{\varrho_0}{\varrho h}\frac{\partial^2 G_j(x,0;x_0)}{\partial x^2} + \frac{D}{\varrho h}\frac{\partial^4\gamma_j(x;x_0)}{\partial x^4} = \omega^2\gamma_j(x;x_0) \\ & \qquad\qquad + \frac{D}{\varrho h}\delta_2^j\delta(x-x_0), \\ & \frac{\varrho_0}{\varrho h}G_j(x,0;x_0) + \frac{D}{\varrho h}\frac{\partial^2\gamma_j(x;x_0)}{\partial x^2} = \frac{\partial G_j(x,0;x_0)}{\partial z} \\ & \qquad\qquad + \frac{D}{\varrho h}\delta_3^j\delta(x-x_0). \end{aligned}\right. \tag{4.23}$$

Here $\delta_k^k = 1$ and $\delta_k^j = 0$ if $k \neq j$. The normalization of the field corresponding to the delta-function in the Helmholtz equation is taken such that the coefficient c in the representation (2.19) for $G_1(x,z;0)$ be equal to 1. The fields produced by other sources are normalized in such a way that the jumps $[\Gamma_2']$ and $[\gamma_3']$ are equal to 1. Here

$$\Gamma_j(x;x_0) = \frac{\varrho_0}{D} G_j(x;x_0) + \gamma_j''(x;x_0).$$

The solutions $\mathcal{G}_j$ of the boundary value problem (4.23) can be found in the form of Fourier integrals

$$\mathcal{G}_1(x,y;x_0) = \begin{pmatrix} \dfrac{1}{2\pi} \displaystyle\int e^{i\lambda(x-x_0)-\sqrt{\lambda^2-k^2}z} \frac{\lambda^4 - k_0^4}{L(\lambda)}\, d\lambda \\ \dfrac{\varrho_0}{D} \dfrac{1}{2\pi} \displaystyle\int e^{i\lambda(x-x_0)} \frac{\lambda^2}{L(\lambda)}\, d\lambda, \end{pmatrix},$$

$$\mathcal{G}_2(x,z;x_0) = \begin{pmatrix} -\omega^2 \dfrac{1}{2\pi} \displaystyle\int e^{i\lambda(x-x_0)-\sqrt{\lambda^2-k^2}z} \frac{1}{L(\lambda)}\, d\lambda \\ -\dfrac{1}{2\pi} \displaystyle\int e^{i\lambda(x-x_0)} \frac{\sqrt{\lambda^2-k^2}\lambda^2}{L(\lambda)}\, d\lambda \end{pmatrix},$$

$$\mathcal{G}_3(x,z;x_0) = \begin{pmatrix} \dfrac{D}{\varrho h} \dfrac{1}{2\pi} \displaystyle\int e^{i\lambda(x-x_0)-\sqrt{\lambda^2-k^2}z} \frac{\lambda^2}{L(\lambda)}\, d\lambda \\ \dfrac{1}{2\pi} \displaystyle\int e^{i\lambda(x-x_0)} \frac{\sqrt{\lambda^2-k^2} + \rho_0(\rho h)^{-1}}{L(\lambda)}\, d\lambda \end{pmatrix}.$$

For the deficiency elements of the restricted operator the point source in the Helmholtz equation can be placed in any of the points $(\epsilon, +0)$ and $(-\epsilon, +0)$ and in the elastic channel both $\delta(x)$ and $\delta'(x)$ can be presented. Therefore totally 6 deficiency elements exist

$$\mathcal{G}_1(x,z;\epsilon), \quad \mathcal{G}_1(x,z;-\epsilon), \quad \mathcal{G}_2(x,z;0), \quad \frac{\partial \mathcal{G}_2(x,z;0)}{\partial x},$$
$$\mathcal{G}_3(x,z;0) \quad \text{and} \quad \frac{\partial \mathcal{G}_3(x,z;0)}{\partial x}.$$

4.4.4 *Scattering by the zero-range potential*

Algebraic system for the amplitudes

The field scattered by the zero-range potential can be represented as the sum of fields radiated by elementary passive sources, namely

$$U^{(s)} = c^+ G_1(x,z;\epsilon) + c^- G_1(x,z;-\epsilon) + [\Upsilon] \frac{\partial G_2(x,z;0)}{\partial x} + [\Upsilon'] G_2(x,z;0)$$
$$+ [v] \frac{\partial G_3(x,z;0)}{\partial x} + [v'] G_3(x,z;0).$$

Computing asymptotics of singular solutions in a vicinity of potential centers and decomposing exponentials $\exp(ik\epsilon)$ and $\exp(2ik\epsilon)$ into Taylor series by ϵ allows the system (4.22) to be rewritten for vector $\mathbf{b}^{(i)}$ defined only by the geometrical part of the field and vector $\mathbf{c}$

$$\Big(I + \mathcal{S}\boldsymbol{\mathcal{G}} + O(\epsilon^2)\Big)\mathbf{c} = \mathcal{S}\mathbf{b}^{(i)}. \tag{4.24}$$

Here

$$\boldsymbol{\mathcal{G}} = \begin{pmatrix} \frac{\varrho_0}{\varrho h} Z' & 0 & \frac{\varrho_0}{\varrho h} J_4 & 0 & NJ_2 & 0 \\ 0 & \frac{\varrho_0}{\varrho h} Z & 0 & -\frac{\varrho_0}{\varrho h} J_2 & 0 & -NJ_0 \\ \frac{\varrho_0}{\varrho h} J_4 & 0 & -D_2 - \frac{\varrho_0}{\varrho h} J_2 & 0 & -D_4 & 0 \\ 0 & -\frac{\varrho_0}{\varrho h} J_2 & 0 & D_0 + \frac{\varrho_0}{\varrho h} J_0 & 0 & D_2 \\ NJ_2 & 0 & -D_4 & 0 & -k_0^4 D_2 & 0 \\ 0 & -NJ_0 & 0 & D_2 & 0 & k_0^4 D_0 \end{pmatrix},$$

$$Z' = \frac{k^2}{2\pi}\Big(\ln(r/2) + C_e - 1\Big) - \frac{i}{4}k^2 - \frac{N}{2}\Big(D_0 + k^2 J\Big),$$

$$Z = \frac{1}{\pi}\Big(\ln(k/2) + C_E\Big) - \frac{i}{4} - NJ.$$

Decomposing exponential factors of small argument $ik\epsilon\cos\vartheta_0$ in Taylor series and preserving only the principal order terms gives the following approximate expression for vector $\mathbf{b}^{(i)}$

$$\mathbf{b}^{(i)} = A\frac{2\varrho_0}{D}\frac{ik\sin\vartheta_0}{\mathcal{L}(\vartheta_0)}\boldsymbol{\Phi}(\vartheta_0) + O(\epsilon^2), \tag{4.25}$$

where

$$\boldsymbol{\Phi}(\vartheta) = \begin{pmatrix} ik\cos\vartheta\Big(k^4\cos^4\vartheta - k_0^4\Big) \\ k^4\cos^4\vartheta - k_0^4 \\ ik^3\cos^3\vartheta \\ k^2\cos^2\vartheta \\ -ik_0^4\cos\vartheta \\ -k_0^4 \end{pmatrix}.$$

Far field amplitude

The asymptotics of the scattered field at large distances can be calculated if the saddle point method is applied to the Fourier expressing the first components of $\mathcal{G}_j$. The approximate expression for the far field amplitude $\Psi(\vartheta)$ neglecting terms proportional to ϵ^2 can be written as the scalar product

$$\Psi(\vartheta) = \frac{A}{2\pi}\frac{D}{\varrho h}\frac{k \sin\vartheta}{\mathcal{L}(\vartheta)}\Big\langle \mathbf{c}, \mathbf{\Phi}(\vartheta)\Big\rangle + O(\epsilon^2). \tag{4.26}$$

(Note that complex conjugation is contained in the scalar product.)

Inverting matrix in the equation (4.24) yields

$$\Psi(\vartheta;\vartheta_0) = \frac{iA}{\pi}\frac{\varrho_0}{\varrho h}\frac{k^2 \sin\vartheta \sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}\Big\langle (\mathbf{I} - \mathcal{S}\mathcal{G})^{-1}\mathcal{S}\mathbf{\Phi}(\vartheta_0), \mathbf{\Phi}(\vartheta)\Big\rangle. \tag{4.27}$$

It is important to note that the neglected terms of order $O(\epsilon^2)$ in the formulae (4.24), (4.25) and (4.26) do not spoil the symmetry of the far field amplitude.

4.4.5 *Choice of parameters in the model*

According to the experience of Chapter 3 let the matrices S and $\mathbf{S}$ be taken the same as in the models of stiffener in rigid and in isolated plates. The matrix $\mathbf{s}$ from (2.34) that describes interaction of acoustic and elastic channels in all the models of Chapter 3 is equal to zero as no such interaction is presented in the classical formulation of diffraction problems. For the protruding stiffener with finite impedances Z_f and Z_m the interaction of two channels is presented. It is due to contact condition (4.17) which contains both acoustic pressure U on the stiffener and flexure displacement w. Nevertheless, let $\mathbf{s} = 0$. This supposition is *a posteriori* justified by comparing the far field amplitude with the asymptotics (4.14).

Auxiliary diffraction problems

Consider first two auxiliary problems of diffraction. Assume that the plate is rigid. In that case the boundary condition (1.13) is replaced by the Neumann boundary condition and the contact conditions (4.16), (4.17) disappear. Solution of such boundary value problem can be obtained from results on diffraction by rigid strip (see [36]) by taking even part of the field. In the two orders of small parameter kH the far field amplitude $\Psi^\circ(\vartheta;\vartheta_0)$ is given by the formula

$$\Psi^{\circ}(\vartheta;\vartheta_0)=\frac{i}{2}A(kH)^2\cos\vartheta\cos\vartheta_0\times$$
$$\times\left\{1-\frac{(KH)^2}{4}\ln(kH/4)\right. \tag{4.28}$$
$$\left.+\frac{(kH)^2}{8}\left(\pi i+\cos^2\vartheta+\cos^2\vartheta_0-\frac{1}{2}-2C_E\right)+\dots\right\}.$$

Instead of considering boundary-value problem for isolated plate, let the results of Chapter 1 be used. Diffraction by nonprotruding stiffener is studied in Section 1.4.1 and the far field amplitude is given by formula (1.62).

Choice of parameters in acoustic channel

The 2×2 matrix S can be chosen with the help of the asymptotics (4.28). Assuming for a moment that matrix S is known, the solution for the problem of plane wave scattering is searched in the form

$$U(x,z)=2A\exp(ikx\cos\vartheta_0)\cos(kz\sin\vartheta_0)$$
$$+\frac{c^+}{2i}H_0^{(1)}(kr_+)+\frac{c^-}{2i}H_0^{(1)}(kr_-).$$

The amplitude of the outgoing cylindrical wave can be easily calculated if Bessel functions $H_0^{(1)}(kr_\pm)$ of large argument are replaced by asymptotics and the radii $r_\pm$ are approximated as $r\mp\epsilon\cos\vartheta$. This yields

$$\Psi_0(\vartheta;\vartheta_0)=-\frac{i}{2\pi}\left(c^+\ e^{-ik\epsilon\cos\vartheta}+c^-\ e^{ik\epsilon\cos\vartheta}\right)$$
$$\approx-\frac{i}{2\pi}\ c-\frac{k\cos\vartheta}{2\pi}\ c'.$$

Comparing this formula with the asymptotics (4.28) for diffraction by protruding stiffener in a rigid screen allows the amplitudes c and c' to be found. Namely,

$$c=0,\qquad c'=-A\pi i\ H^2k\cos\vartheta_0. \tag{4.29}$$

Coefficients $b^{\pm}$ are given by the formulae

$$b^{\pm} = -c^{\mp}\frac{i}{2}H_0^{(1)}(2k\epsilon) + c^{\pm}\frac{1}{\pi}\left(\ln(k\epsilon) + C_E - i\frac{\pi}{2}\right) + 2A\exp\left(\pm ik\epsilon\,\cos\vartheta_0\right).$$

If $k\epsilon \ll 1$ Bessel function of small argument $k\epsilon$ can be replaced by asymptotics. This gives approximate expressions

$$b = 2A + O\big((k\epsilon)^2\ln(k\epsilon)\big),$$

$$b' = iAk\cos\vartheta_0\left(1 + \frac{(kH)^2}{2}\left(\ln(k\epsilon) + C_E - 1 - i\frac{\pi}{2}\right)\right) + O\big((k\epsilon)^2\ln(k\epsilon)\big). \tag{4.30}$$

Using expressions (4.29) and (4.30) it is easy to find matrix S. This matrix should not depend on angle ϑ_0, therefore the second row can be found identically zero, that is $S_{21} = S_{22} = 0$. Hermitian property yields $S_{12} = 0$. Taking into account that S does not depend on the wave number k, the remaining component S_{11} is defined uniquely as $-\pi H^2/2$. Thus protruding stiffener in rigid screen is modeled by the two centers zero-range potential with the matrix

$$S = \begin{pmatrix} -\frac{1}{2}\pi H^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is important to note that matrix S fixing the zero-range potential does not depend on parameter ϵ (which allows the limit $\epsilon \to 0$ to be considered) and is defined only by the geometrical characteristics of the protruding stiffener. This was achieved by introducing coefficients c, c', b and b' instead of $c^{\pm}$ and $b^{\pm}$ in the parameterization formula (4.19).

Choice of parameters in elastic component

Comparing (4.21) with contact conditions (1.52) yields the matrix

$$\mathbf{S} = \mathrm{diag}\Big(0, 0, z_m, z_f\Big).$$

Here

$$z_f = \frac{D}{\omega^2\varrho h}Z_f = \frac{\varrho_1 h_1}{\varrho h}H, \qquad z_m = \frac{D}{\omega^2\varrho h}Z_m = \frac{\varrho_1 h_1}{\varrho h}\frac{H^3}{3}.$$

These quantities do not contain spectral parameter ω^2 and depend only on characteristics of the stiffener, its relative density ϱ_1/ϱ, relative thickness h_1/h and height H.

4.4.6 Generalized model of protruding stiffener in fluid loaded plate

It was noted above that as both the acoustic pressure and the displacement of the plate are presented in the contact condition (4.17), the scatterer adds additional interaction of the scattering channels. Thus, generally speaking, matrices $\mathbf{s}$ and $\mathbf{s}^*$ in the model of the protruding stiffener can be different from zero. However, let $\mathbf{s} = 0$ be accepted. That is let the matrix $\mathcal{S}$ be

$$\mathcal{S} = \operatorname{diag}\left(-\frac{\rho h}{\rho_0}\frac{\pi}{2}H^2, 0, 0, 0, z_m, z_f\right).$$

Performing simple though cumbersome calculations by the formula (4.27) yields expression for the scattering amplitude

$$\Psi(\vartheta;\vartheta_0) = \frac{iNA}{\pi}\frac{k^2\sin\vartheta\sin\vartheta_0}{\mathcal{L}(\vartheta)\mathcal{L}(\vartheta_0)}\left\{\frac{1}{D_0 - 1/Z_f} - \frac{k^2\cos\vartheta\cos\vartheta_0}{D_2 + 1/Z_m}\right.$$
$$\left. - \frac{\pi}{2}N(kH)^2\cos\vartheta\cos\vartheta_0 M(\vartheta)M(\vartheta_0)\right\}. \qquad (4.31)$$

Here

$$M(\vartheta) = \frac{k^4\cos^4\vartheta - k_0^4}{N} - \frac{J_2}{D_2 + 1/Z_m}.$$

The two first terms in the formula (4.31) coincide with the far field amplitude (1.62) for non-protruding stiffener. The last term gives correction that appears due to the protruding part. One can compare the asymptotics (4.31) with (4.14). This comparison shows that up to terms of order $O\big((kH)^2\big)$ both expressions coincide. That is the additional interaction of the channels which is described by the condition (4.17) does not influence upon the terms of order $O\big((kH)^2\big)$ and can manifest itself only in smaller order corrections. These corrections are of the form of a field radiated by a quadruple passive source and are not reproduced by the suggested model with two potential centers in the acoustic component. To include quadrupole sources one can use zero-range potentials with 4 potential centers. The problem of choosing appropriate parameters $\mathbf{s}$ for these zero-range potentials should be studied.

Appendix A

Regularization and analysis of boundary-contact integrals

A.1 Boundary-contact integrals in two dimensional problems

Consider the boundary-contact integrals D_ℓ

$$D_\ell = \frac{1}{2\pi}\int e^{i0\lambda}\frac{(i\lambda)^\ell\sqrt{\lambda^2-k^2}}{L(\lambda)}d\lambda \equiv \lim_{x\to+0}\frac{1}{2\pi}\int e^{ix\lambda}\frac{(i\lambda)^\ell\sqrt{\lambda^2-k^2}}{L(\lambda)}d\lambda.$$

The integration path coincides with the real axis of λ except for the small neighborhoods of the poles $\lambda = \pm\kappa$ which are avoided according to the limiting absorption principle, that is $\lambda = \kappa$ from below and $\lambda = -\kappa$ from above (see Fig. A.1). The denominator behaves as $O(\lambda^5)$ at infinity, therefore integrals with $\ell > 2$ are understood as limits. First the exponential factor $e^{i\lambda x}$ is introduced and the ends of the integration path are shifted for positive x to the upper complex half-plane of λ. Then the limit as $x \to +0$ is taken in the resulting integral. This limit is symbolically indicated by the exponential factor $e^{i0\lambda}$ in the above formula.

The deformation of the integration path into the loop around the upper branch-cut Γ_+ (see Fig. A.1) performs regularization of the integrals. Taking into account the residues in the poles, which are crossed, one finds

$$D_\ell = -\sum_{\substack{\kappa=\kappa_0,\\ \kappa=-\kappa_4}}\frac{(i\kappa)^{\ell-1}(\kappa^2-k^2)}{5\kappa^4-4\kappa^2k^2-k_0^4} + \frac{1}{2\pi}\int_{\Gamma_+} e^{i0\lambda}\frac{(i\lambda)^\ell\sqrt{\lambda^2-k^2}}{L(\lambda)}d\lambda.$$

Here κ_s are solutions of the dispersion equation

$$L(\kappa_s) \equiv (\kappa_s^4 - k_0^4)\sqrt{\kappa_s^2-k^2} - N = 0. \tag{A.1}$$

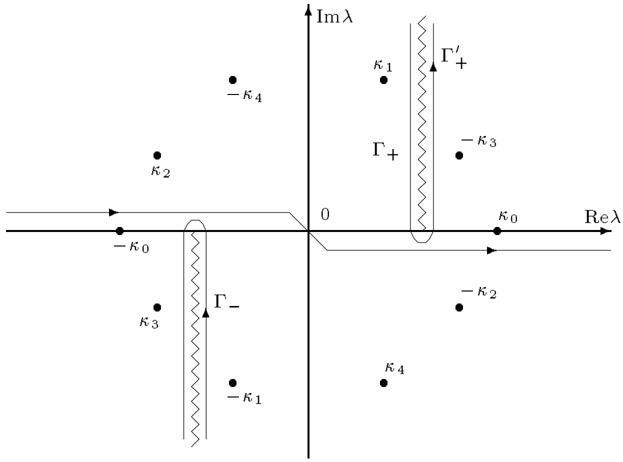


Fig. A.1 Integration path and singularities of the integrand

The integral over Γ_+ can be reduced to the integral over the right-hand side shore of the branch-cut Γ'_+. This excludes growing terms from the integrand and one can drop out the exponential factor

$$D_\ell = -\sum_{\substack{\kappa=\kappa_0,\\ \kappa=-\kappa_4}} \frac{(i\kappa)^{\ell-1}(\kappa^2-k^2)}{5\kappa^4-4\kappa^2k^2-k_0^4} + \frac{N}{\pi}\int\limits_{\Gamma'_+} \frac{(i\lambda)^\ell\sqrt{\lambda^2-k^2}}{(\lambda^4-k_0^4)^2(\lambda^2-k^2)-N^2}d\lambda.$$

Further the integrals by Γ'_+ can be reduced to integrals over the union of two loops Γ_+ and Γ_- with the help of the formulae

$$\int\limits_{\Gamma'_+} \lambda f(\lambda^2)\sqrt{\lambda^2-k^2}d\lambda = \frac{1}{4}\int\limits_{\Gamma_+\cup\Gamma_-} \lambda f(\lambda^2)\sqrt{\lambda^2-k^2}d\lambda,$$

$$\int\limits_{\Gamma'_+} f(\lambda^2)\sqrt{\lambda^2-k^2}d\lambda$$
$$= \frac{1}{2\pi i}\int\limits_{\Gamma_+\cup\Gamma_-} f(\lambda^2)\sqrt{\lambda^2-k^2}\ln\left(\frac{\lambda+\sqrt{\lambda^2-k^2}}{k}\right)d\lambda.$$

In the above formula logarithm is taken positive when $\lambda > k$.

Finally integrals by the union of loops can be evidently reduced to sums of residues. The poles of the denominator coincide with the solutions of the dispersion equation taken on both sheets of the Riemann surface of square root $\sqrt{\lambda^2 - k^2}$. Exploiting the symmetry one expresses the residues in the poles $\lambda = -\kappa_s$ in terms of residues in the poles $\lambda = \kappa_s$. This yields exact formulae

$$D_{2\ell+1} = -\frac{1}{2}\sum_{s=0}^{4} \frac{(i\kappa_s)^{2\ell}(\kappa_s^2 - k^2)}{5\kappa_s^4 - 4\kappa_s^2 k^2 - k_0^4}, \tag{A.2}$$

$$D_{2\ell} = -\sum_{s=0}^{4} \frac{(i\kappa_s)^{2\ell-1}(\kappa_s^2 - k^2)}{5\kappa_s^4 - 4\kappa_s^2 k^2 - k_0^4} \left(\frac{i}{\pi}\ln\left(\frac{\kappa_s + \sqrt{\kappa_s^2 - k^2}}{k}\right) + \frac{1}{2}\right). \tag{A.3}$$

Analogous derivations for integrals

$$J = \frac{1}{2\pi}\int \frac{d\lambda}{L(\lambda)\sqrt{\lambda^2 - k^2}}$$

and

$$J_\ell = \frac{1}{2\pi}\int \frac{(i\lambda)^\ell}{L(\lambda)}\,d\lambda, \qquad \ell = 0, 1, 2, 3$$

yields exact formulae

$$J = \frac{1}{\pi}\sum_{s=0}^{4} \frac{\ln\left(\left(\kappa_s + \sqrt{\kappa_s^2 - k^2}\right)/k\right) - i\pi/2}{5\kappa_s^4 - 4\kappa_s^2 k^2 - k_0^4},$$

$$J_{2\ell} = \sum_{s=0}^{4} (-1)^\ell \frac{\kappa_s^{2\ell-1}\sqrt{\kappa_s^2 - k^2}}{5\kappa_s^4 - 4\kappa_s^2 k^2 - k_0^4}\left(\frac{1}{\pi}\ln\left(\frac{\kappa_s + \sqrt{\kappa_s^2 - k^2}}{k}\right) - \frac{i}{2}\right),$$

$$J_{2\ell+1} = -\frac{i}{2}\sum_{s=0}^{4} (-1)^\ell \frac{\kappa_s^{2\ell}\sqrt{\kappa_s^2 - k^2}}{5\kappa_s^4 - 4\kappa_s^2 k^2 - k_0^4}. \tag{A.4}$$

On the other hand integrals with odd indices can be expressed by simpler formulae. The limit $x \to +0$ for the integral D_1 can be taken in the integrand. Therefore integrals D_1, J_1 and J_3 absolutely converge and as integrals of odd functions over symmetric path these integrals are equal to zero

$$D_1 = 0, \quad J_1 = 0, \quad J_3 = 0.$$

For D_3 and D_5 one can use the representation

$$\frac{\sqrt{\lambda^2 - k^2}}{L(\lambda)} = \frac{1}{\lambda^4 - k_0^4} + \frac{N}{L(\lambda)(\lambda^4 - k_0^4)} \tag{A.5}$$

The integrals with the second term are equal to zero by the same reason of symmetry and computing integrals with the first terms by the residue theorem (poles are at $\lambda = k_0$ and $\lambda = ik_0$) yields

$$D_3 = \frac{1}{2}, \qquad D_5 = 0.$$

Computations by the formulae (A.2) and (A.4) can be checked with the exact values of the integrals which allows accuracy of numerical results to be controlled.

A.2 Boundary-contact integrals for oblique incidence

Consider the integrals

$$D_\ell(\mu) = \frac{1}{2\pi} \int e^{i0\lambda} \frac{(i\lambda)^\ell \sqrt{\lambda^2 + \mu^2 - k^2}}{L(\lambda, \mu)} d\lambda.$$

Integrals with odd indices are again easily computed. For that change the integration variable to $\tau = \sqrt{\lambda^2 + \mu^2}$ and apply the formula (A.5). This yields

$$D_1(\mu) = 0, \quad D_3(\mu) = \frac{1}{2}, \quad D_5(\mu) = \mu^2. \tag{A.6}$$

In the integrals with even indices the path deformation to the upper loop around the branch-cut and then reducing the integrals to the sums of residues gives formulae similar to (A.3)

$$\begin{aligned} D_{2\ell}(\mu) = i(-1)^\ell \sum_{s=0}^{4} \frac{(\kappa_s^2 - \mu^2)^{\ell - 1/2} (\kappa_s^2 - k^2)}{5\kappa_s^4 - 4\kappa_s^2 k^2 - k_0^4} \times \\ \times \left(\frac{i}{\pi} \ln \left(\frac{\sqrt{\kappa_s^2 - \mu^2} + \sqrt{\kappa_s^2 - k^2}}{\sqrt{k^2 - \mu^2}} \right) + \frac{1}{2} \right). \end{aligned} \tag{A.7}$$

In particular if μ is large this yields

$$D_{2\ell}(\mu) \sim \frac{2\ell-1}{4}\mu^{2\ell-3} + \ldots, \qquad |\mu| \to \infty. \tag{A.8}$$

When deriving the above asymptotics the factor $(\tau^2-\mu^2)^{\ell-1/2}$ was decomposed in Taylor series. Then the terms without logarithm give

$$\mu^{2\ell-1}D_1 + \frac{2\ell-1}{2}\mu^{2\ell-3}D_3 + \ldots \tag{A.9}$$

The logarithm can be replaced by its asymptotics

$$\ln\left(\frac{\sqrt{\kappa_s^2-\mu^2}+\sqrt{\kappa_s^2-k^2}}{\sqrt{k^2-\mu^2}}\right) \sim \sqrt{\frac{\kappa_s^2-k^2}{\kappa_2^2-\mu^2}} + O(\mu^{-3}).$$

When substituting the principal order approximation into the expressions (A.7) one gets in the leading order by μ

$$\frac{(-1)^{\ell+1}}{\pi}\mu^{2\ell-2}\sum_{s=0}^{4}\frac{(\kappa_s^2-k^2)^{3/2}}{5\kappa_s^4-4\kappa_s^2k^2-k_0^4} + O(\mu^{2\ell-4})$$

which equals up to a constant to $\mu^{2\ell-2}(J_3+k^2J_1)$. Therefore this term is equal to zero and the asymptotics (A.8) is due to the second term in (A.9).

A.3 Low frequency asymptotics

The boundary-contact integrals are expressed in terms of zeros of the dispersion equation (A.1). This equation is equivalent to the fifth power equation and can not be solved in closed form.

Examine the asymptotics of the integrals $D_{2\ell}$, J and $J_{2\ell}$ when the plate is thin or the frequency is low. For this introduce dimensionless parameters

$$p = 12(1-\sigma^2)\frac{\varrho_0 c_a^2}{E}, \qquad d = \frac{\varrho}{\varrho_0}, \qquad \varepsilon = kh,$$

where E and σ are the Young modulus and the Poisson's ratio in the material of the plate c_a is the wave velocity in fluid, ϱ, ϱ_0 are the densities of the material of the plate and of fluid and h is the thickness of the plate. Then

$$Nh^5 = p\varepsilon^2, \quad k_0^4h^4 = pd\varepsilon^2.$$

The dispersion equation (A.1) can be written as

$$L(\kappa)h^5 = \left((\kappa h)^4 - pd\varepsilon^2\right)\sqrt{(\kappa h)^2 - k^2} - p\varepsilon^2 = 0.$$

In the principal order one finds the asymptotics

$$\kappa_s h \approx p^{1/5}\varepsilon^{2/5}e^{2\pi i s/5}, \qquad s = 0, 1, 2, 3, 4. \tag{A.10}$$

The corrections can be found in the form of decomposition by powers of $\varepsilon^{2/5}$, in particular

$$\kappa_s h = p^{1/5}\varepsilon^{2/5}e^{2\pi i s/5} + \frac{1}{5}p^{2/5}d\varepsilon^{4/5}e^{4\pi i s/5} - \frac{1}{25}p^{3/5}d^2\varepsilon^{6/5}e^{6\pi i s/5}$$
$$+ \left(\frac{1}{125}p^{4/5}d^3 + \frac{1}{10}p^{-1/5}\right)\varepsilon^{8/5}e^{-2\pi i s/5} + \ldots$$

Therefore the asymptotics (A.10) is valid if

$$\varepsilon \ll d^{-5/2}p^{-1/2}, \quad \varepsilon \ll p^{1/3}.$$

Under this assumptions one can neglect all k and k_0 in the expressions for the boundary-contact integrals. This allows simple asymptotic formulae for the integrals $D_{2\ell}$, J and $J_{2\ell}$ to be found

$$D_{2\ell} \sim \frac{2i}{5}(-1)^\ell N^{\frac{2\ell-3}{2}}\left(1 - \exp\left((2\ell-3)\frac{2\pi i}{5}\right)\right)^{-1}, \tag{A.11}$$

$$J \sim \frac{3i}{10}N^{-1}, \tag{A.12}$$

$$J_0 \sim \frac{2i}{5}\frac{N^{-4/5}}{1 - \exp\left(\frac{2\pi i}{5}\right)}, \quad J_2 \sim -\frac{2i}{5}\frac{N^{-2/5}}{1 - \exp\left(\frac{6\pi i}{5}\right)}. \tag{A.13}$$

A.4 Boundary-contact integrals in three dimensions

Consider the three dimensional boundary-contact integrals. The integral

$$D_{00} = \frac{1}{4\pi^2}\iint \frac{\sqrt{\lambda^2 + \mu^2 - k^2}}{L(\lambda, \mu)}\,d\lambda\,d\mu$$

can be simplified if variables of integration are changed to τ, α by the formulae $\lambda = \sqrt{\tau}\cos\alpha$, $\mu = \sqrt{\tau}\sin\alpha$. This yields

$$D_{00} = \frac{1}{4\pi} \int_0^{+\infty} \frac{\sqrt{\tau - k^2}}{(\tau^2 - k_0^4)\sqrt{\tau - k^2} - N}\, d\tau.$$

Further, excluding irrationality from the denominator reduces this integral to the sum of two integrals. Integrand in one is a fraction of two polynomials and the other contains $\sqrt{\tau - k^2}$. Both integrals can be expressed in elementary functions with coefficients depending on zeros of the denominator $\tau_s = \kappa_s^2$. These formulae are rather cumbersome and we present here only the asymptotics for low frequencies when the zeros are given by the asymptotics (A.10).

In the case of low frequency, neglecting smaller order terms in the integrand one finds

$$D_{00} \sim \frac{1}{4\pi} \int_0^{+\infty} \frac{\tau^2\, d\tau}{\tau^5 - N} = \frac{N^{-2/5}}{4\pi} \int_0^{+\infty} \frac{t^2\, dt}{t^5 - 1}.$$

The integration path in the above integral avoids the pole at $\tau = N^{1/5}$, $t = 1$ from below. Excluding semi-residue one gets the principal value integral. Further, in the integral over $t \in [1, \infty)$ one changes the integration variable to its inverse and combines the result with the integral over $t \in [0, 1]$. This yields

$$D_{00} \sim \frac{i}{20} N^{-2/5} + \frac{N^{-2/5}}{4\pi} \int_0^1 \frac{t\, dt}{t^4 + t^3 + t^2 + t + 1}$$

Finally

$$D_{00} \sim -\frac{\cot(2\pi/5)}{40} N^{-2/5} + \frac{i}{20} N^{-2/5}. \tag{A.14}$$

Similar expressions can be found for integrals D'_ℓ.

Another approach to computing the asymptotics of the boundary contact integrals in three dimensions see in [7].

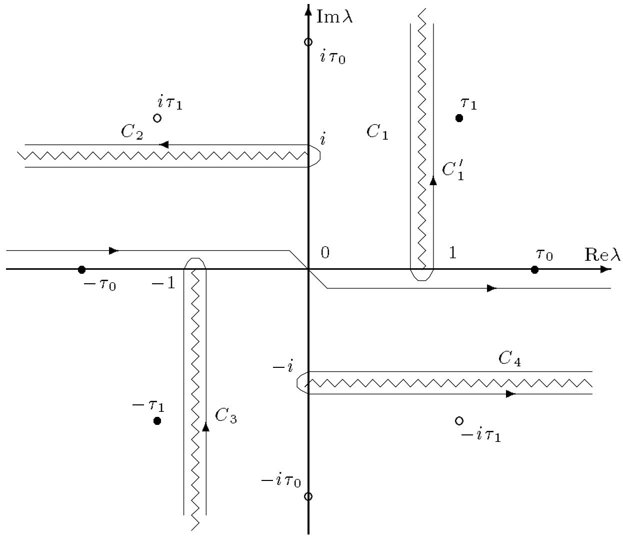


Fig. A.2 Integration path and singularities of the integrated functions

A.5 Boundary-contact integrals for the plate with infinite crack

Consider the integrals of the problem of diffraction by a joint of isolated semi-infinite plates. These are the integrals d_ℓ

$$d_0 = \frac{1}{\pi}\int \frac{d\tau}{\mathcal{L}(\tau)}, \qquad d_1 = \int \frac{\tau^2 - \sqrt{\tau^4-1}}{\mathcal{L}(\tau)\sqrt{\tau^4-1}}\tau^2 d\tau$$

and

$$d_2 = \int \frac{d\tau}{\mathcal{L}(\tau)\sqrt{\tau^4-1}}.$$

These integrals can be reduced to sums of residues in the zeros of the denominator $\mathcal{L}(\tau)$ on the Riemann surface of the square root $\sqrt{\tau^4-1}$. Let the values of square roots be fixed by choosing branch-cuts on the complex plane of τ along the rays $L_1 = [1, 1+i\infty)$, $L_2 = [i, -\infty + i)$, $L_3 = [-1, -1-i\infty)$ and $L_4 = [-i, +\infty - i)$ (see Fig. A.2).

Examine the zeros of the denominator

$$\mathcal{L}(\tau) \equiv \Big((1-\sigma)\tau^2+1\Big)^2\sqrt{\tau^2-1} - \Big((1-\sigma)\tau^2-1\Big)^2\sqrt{\tau^2+1}.$$

The zeros are located in pairs on the 4-sheet Riemann surface. There are four zeros on the "physical" sheet, namely $\tau = \tau_0 = \varkappa/k_0$, $\tau = -\tau_0$, $\tau = \tau_1$ and $\tau = -\tau_1$, where

$$\tau_1 = \Big((1-\sigma)\big(-3\sigma+1+2\sqrt{1-2\sigma+2\sigma^2}\big)\Big)^{-1/4} e^{i\pi/4}.$$

On the other sheets of Riemann surface the zeros are in the points $\tau = i\tau_0$, $\tau = -i\tau_0$, $\tau = i\tau_1$ and $\tau = -i\tau_1$.

Consider first the integral d_0. Deform the integration path to the upper complex half-plane to the loops around the branch-cuts L_1 and L_2. The residues in the poles at τ_0 and τ_1 are extracted. The integrals over the loops C_1 and C_2 around the rays L_1 and L_2 reduce to integrals along the right-hand side shores C_1' and C_2'. In the integral over C_2' change the integration variable $\tau = it$ which maps the ray L_2 to the ray L_1. Then combine both integrals. One can find that the integrals cancel each other. Therefore

$$d_0 = 2i\left(\frac{1}{\mathcal{L}'(\tau_0)} + \frac{1}{\mathcal{L}'(\tau_1)}\right).$$

The above expression is rather cumbersome, however it can be shown that $\operatorname{Re} d_0 = 0$.

Perform now similar derivations with the integrals d_1 and d_2. The contributions of the integrals over the loops do not cancel for that integrals. However integrals over the semi-loop C_1' can be reduced to the sums of residues on all the sheets of Riemann surface analogously to section A.1. For this introduce the function

$$f(\tau) = \ln\left(\tau - \sqrt{\tau^2-1}\right).$$

One can easily find that $f(-\tau) = i\pi - f(\tau)$ which allows the following transformation of the integrals to be performed

$$\int\limits_1^{1+i\infty} F(\tau^2)d\tau = \frac{1}{2\pi i}\int\limits_{C_1} F(\tau^2)\big(f(\tau)-f(-\tau)\big)\,d\tau = \frac{1}{2\pi i}\int\limits_{C_1\cup C_3} F(\tau^2)f(\tau)d\tau.$$

Now the integrals over the union of loops C_1 and C_3 can be computed as sums of residues. Finally one finds

$$d_1 = 2\pi i \sum_{j=0,1} \frac{\tau_j^2\left(\tau_j^2 - \sqrt{\tau_j^4 - 1}\right)}{\mathcal{L}'(\tau_j)\sqrt{\tau_j^4 - 1}} + \frac{1}{4}\sum_{j=0}^{7} \frac{\tau_j^4\left((1-\sigma)\tau_j^2 - 1\right)^2 - \tau_j^2(\tau_j^2 - 1)\left((1-\sigma)\tau_j^2 + 1\right)^2}{(1-\sigma)\left((1-\sigma)^2(3+\sigma)\tau^4 - 1 + 3\sigma\right)\tau^3\sqrt{\tau_j^2 - 1}} f(\tau_j),$$

$$d_2 = 2\pi i \sum_{j=0,1} \frac{1}{\mathcal{L}'(\tau_j)\sqrt{\tau_j^4 - 1}} + \frac{1}{4}\sum_{j=0}^{7} \frac{\left((1-\sigma)\tau_j^2 - 1\right)^2 f(\tau_j)}{(1-\sigma)\left((1-\sigma)^2(3+\sigma)\tau^4 - 1 + 3\sigma\right)\tau^3\sqrt{\tau_j^2 - 1}}.$$

In the above formulae $\tau_2 = i\tau_0$, $\tau_3 = i\tau_1$, $\tau_4 = -\tau_0$, $\tau_5 = -\tau_1$, $\tau_6 = -i\tau_0$ and $\tau_7 = -i\tau_1$.

Appendix B

Integral equations of convolution on a finite interval

Problems of diffraction by a segment like inhomogeneity in fluid loaded or isolated thin elastic plate can be reduced to integral equations of the convolution on a finite interval. In classical problems of diffraction such equations usually have kernels with logarithmic singularity. For the case of diffraction by thin elastic plate the class of possible kernel singularities enlarges. Theorems of existence and uniqueness of solutions of these equations are presented here.

B.1 Integral equations of convolution

Consider the integral equation

$$\int_{-a}^{a} K(x-t)p(t)dt = f(x), \qquad |x| < a \tag{B.1}$$

with the kernel $K(x-t)$. By introducing scaled coordinates the equations of convolution can be rewritten for the interval $[-1,1]$. This is assumed below. The following types of kernels are studied:

(1) Kernels having logarithmic singularity

$$K(s) = \ln|s| + a(s^2)\ln|s| + b(s^2).$$

(2) Supersingular kernels

$$K(s) = \frac{d^{2m}}{ds^{2m}}\left(\ln|s| + a(s^2)\ln|s| + b(s^2)\right), \quad m = 1, 2, \ldots$$

(3) Smooth kernels

$$K(s) = s^2 \ln |s| + s^2 a(s^2) \ln |s| + b(s^2).$$

Here $a(s^2)$ and $b(s^2)$ are smooth functions from C^∞ and $a(0) = 0$. Further the Fourier symbol of the kernel will be used

$$k(\lambda) = \int e^{i\lambda s} K(s) ds.$$

An important class of integral equations that appear in problems of diffraction have *sectorial symbol* $k(\lambda)$. That is the domain of values $k(\lambda)$, when $\operatorname{Im}\lambda = 0$ belongs to a half-plane of complex plane.

B.2 Logarithmic singularity of the kernel

Consider first the simplest case of logarithmic kernel

$$\int_{-1}^{1} \Big(\ln |t - x| + C\Big) p_0(t) dt = f(x). \tag{B.2}$$

The solution of this equation can be written in explicit form as the singular integral [34]

$$\begin{gathered} p_0(t) = \frac{1}{\pi^2} \frac{1}{\sqrt{1 - t^2}} \left(\int_{-1}^{1} \frac{\sqrt{1 - s^2} f'(s)}{s - t} ds + C_1 \right) \\ C_1 = \frac{1}{C - \ln 2} \int_{-1}^{1} \frac{f(s)}{\sqrt{1 - s^2}} ds. \end{gathered} \tag{B.3}$$

Here the denominator $C - \ln 2$ is assumed not equal to zero. In the case when $C - \ln 2 = 0$ the solution of (B.2) exists only if

$$\int_{-1}^{1} \frac{f(s)}{\sqrt{1 - s^2}} ds = 0$$

and is defined by the first formula in (B.3) with arbitrary constant C_1.

Analysis of the formula (B.3) shows that

Theorem 1 *For $f \in C^{n+2}[-1,1]$ the solution of the integral equation (B.2) has the form*

$$p_0(t) = \frac{\varphi(t)}{\sqrt{1-t^2}}$$

with $\varphi \in C^n[-1,1]$.

For the proof it is sufficient to show that the n-th order derivative of the singular integral in (B.3) is bounded. Rewrite the singular integral in the form

$$SI = \int_{-1}^{1} \frac{(f'(s) - f'(t))\sqrt{1-s^2}}{s-t} ds - \pi t f'(t)$$

and apply formally n-th order differentiation by t

$$\frac{d^n}{dt^n} SI = n! \int_{-1}^{1} \frac{\sqrt{1-s^2}}{s-t} \left| f'(s) - \sum_{j=1}^{n+1} \frac{d^j f(t)}{dt^j} \frac{(s-t)^{j-1}}{(j-1)!} \right| ds - \pi \frac{d^n}{dt^n} t f'(t).$$

Noting that the sum in the last multiplier represents the Taylor formula for $f'(s)$ and using the error estimate [63]

$$\left| f'(s) - \sum_{j=1}^{n+1} \frac{d^j f(t)}{dt^j} \frac{(s-t)^{j-1}}{(j-1)!} \right| \le \frac{f^{(n+2)}(\xi)}{(n+1)!} (s-t)^{n+1}$$

allows uniform convergence of the integral to be established. This fact justifies the change of order of integration and differentiation performed above. Thus the n-th order derivative of the singular integral in (B.3) is bounded which proves the theorem.

Let now the kernel be of the form

$$K(x,t) = \ln|t-x| + N(x,t).$$

And let the function $N(x,t)$ be sufficiently smooth by its arguments. With the help of the formula (B.3) the integral equation can be semi-inverted and rewritten in the form

$$p(t) + \int_{-1}^{1} \widehat{K}(x,t) p(t) dt = \widehat{f}(x)$$

Careful analysis shows that the kernel $\widehat{K}(x,t)$ is bounded and the above integral equation is of Fredholm type [34]. That is the two cases are possible

(1) The integral equation is uniquely solvable for any smooth right-hand side (smoothness is required for the formula (B.3) to be applicable).
(2) There exists a nonzero solution $q(t)$ of the homogeneous adjoint equation and for the solvability of the initial integral equation the right-hand side should be orthogonal to $q(t)$, that is

$$\int_{-1}^{1} q(t)f(t)dt = 0.$$

In that case the solution $p(t)$ contains one-parameter arbitrariness.

Theorem 2 *For the integral equation with difference kernel $K(x-t)$ and sectorial symbol $k(\lambda)$ only the case (1) is possible.*

Indeed, as the kernel depends on the difference $t - x$ only, the adjoint equation coincides with the initial one

$$\int_{-1}^{1} K(x-t)q(t)dt = 0, \quad |x| < 1.$$

Multiplying this equation by $\overline{q(x)}$ and integrating over the interval $x \in [-1,1]$ yields

$$\int_{-1}^{1}\int_{-1}^{1} K(x-t)q(t)\overline{q(x)}\, dt\, dx = 0.$$

Substituting the kernel in the form of Fourier transform and introducing Fourier transform $\eta(\lambda)$ of the solution $q(t)$ yields

$$\int_{0}^{+\infty} k(\lambda)|\eta(\lambda)|^2 d\lambda = 0.$$

Sectorial property of the symbol $k(\lambda)$ allows $\eta(\lambda)$ to be concluded identically equal to zero. Thus $q(t) \equiv 0$ and the case (2) is not possible.

Examine now the smoothness properties of the solution. First, let the kernel be of the form

$$K(s) = \ln |s| + b(s^2).$$

Such kernels appear in static problems of diffraction [68]. With the help of formula (B.3) the integral equation can be rewritten in the form

$$\int_{-1}^{1} \ln |t - x| p_1(t) dt = f_1(x) - \int_{-1}^{1} b\left((t-x)^2\right) p_1(t) dt, \quad |x| < 1.$$

Here the solution of the integral equation is represented as

$$p(t) = p_0(t) + p_1(t),$$

where $p_0(t)$ is defined by the formula (B.3). The function $f_1(x)$ in the right-hand side of the above equation is given by the formula

$$f_1(x) = \int_{-1}^{1} \left(b\left((t-x)^2\right) - C\right) p_0(t) dt.$$

The function $b(s^2)$ is assumed infinitely differentiable. Thus the right-hand side of the above equation for any integrable $p_0(t)$ and $p_1(t)$ belongs to $C^\infty[-1, 1]$. Therefore theorem 1 yields

$$p_1(t) = \frac{\varphi_1(t)}{\sqrt{1-t^2}}, \qquad \varphi_1 \in C^\infty[-1, 1].$$

Combining this fact and theorem 1 allows the following structure of the solution to be discovered [68]

Theorem 3 *The solution $P(t)$ of the integral equation*

$$\int_{-1}^{1} \left(\ln |t-x| + b\left((x-t)^2\right)\right) p(t)\, dt = f(x), \quad |x| < 1$$

for any $b \in C^\infty$ and $f \in C^{n+2}[-1, 1]$ is representable in the form

$$p(t) = \frac{\varphi(t)}{\sqrt{1-t^2}}, \tag{B.4}$$

with $\varphi \in C^n[-1, 1]$

Actually one can ask less smoothness from the right-hand side, namely $f \in C^{n+1}[-1,1]$ and f^{n+1} from Hölder class of any positive degree. For details see [68]. However in the problems of diffraction considered in Chapter 3 the right-hand sides are infinitely smooth and theorem 3 is sufficient for further analysis.

The result of theorem 3 can be generalized to the case of more general kernels of type (1) with arbitrary smooth $a(s^2)$. First, let $a(s)$ be polynomial.

Theorem 4 *If the integral equation*

$$\int_{-1}^{1} \ln|t-x| \left(\sum_{\ell=0}^{N} \mu_\ell (t-x)^\ell \right) p(t)\, dt = f(x),$$

is solvable and $f \in C^{n+2}[-1,1]$, then its solution $p(t)$ is representable in the form (B.4) with $\varphi \in C^n[-1,1]$.

For the proof the kernel can be rewritten as

$$\sum_{\ell=0}^{N} \mu'_\ell \frac{d^{N-\ell}}{dx^{N-\ell}} \left(\ln|t-x|(t-x)^N \right) + R(t-x),$$

where $R(t-x)$ is some polynomial and coefficients μ'_ℓ are defined by μ_ℓ. Changing the order of integration and differentiation yields the differential equation

$$\sum_{\ell=0}^{N} \mu'_\ell \frac{d^{N-\ell}}{dx^{N-\ell}} r(x) = f(x) - \int_{-1}^{1} R(t-x) p(t)\, dt$$

for the function

$$r(x) = \int_{-1}^{1} (t-x)^N \ln|t-x| p(t)\, dt.$$

The right-hand side of this equation belongs to $C^{n+2}[-1,1]$, therefore the solution $r(x)$ is from $C^{N+n+2}[-1,1]$ and its N-th order derivative is from $C^{n+2}[-1,1]$

$$\frac{d^N r(x)}{dx^N} = \int_{-1}^{1} (\ln|t-x| + C)\, p(t)\, dt \in C^{n+2}[-1,1].$$

Finally theorem 3 states the structure (B.4) of the solution $p(t)$.

Now let the function $a(s)$ be arbitrary function from C^∞. Then

Theorem 5 *Solution of the integral equation with the kernel* $K(s) = \ln|s| + a(s)\ln|s| + b(s)$ *is representable in the form (B.4) with* $\varphi \in C^n[-1,1]$ *if* $f \in C^{n+1}[-1,1]$ *and* $a, b \in C^\infty$.

For the proof assume the opposite. Let the N-th order derivative of $\varphi(t)$ be discontinuous. Then decompose $a(t-x)$ according to Taylor formula. The terms up to $N+1$ keep in the left-hand side and the remainder move to the right-hand side together with the convolution of $p(t)$ and $b(t-x)$. Performing differentiation it is not difficult to show that the right-hand side is a function from $C^{N+2}[-1,1]$. Thus theorem 4 establishes that $\varphi \in C^N[-1,1]$ which contradicts our supposition. Due to the uniqueness of the solution this proves the theorem.

B.3 Supersingular kernels

Consider not the case of supersingular kernels. Such integral equations can be considered in the sense of Hadamard integrals or can be regularized. The regularization is in the change of order of integration and differentiation. That is integral equations with supersingular kernels are considered as integro-differential equations of the form

$$\begin{gathered} \frac{d^{2m}}{dx^{2m}} \int\limits_{-1} K'(t-x)p(t)\,dt = f(x), \quad |x| < 1, \\ K'(s) = \ln|s| + a\left(s^2\right)\ln|s| + b\left(s^2\right). \end{gathered} \tag{B.5}$$

Solutions of equations (B.5) are considered in the classes of functions with special behavior near the ends of the integration interval.

Definition 1 Denote by $\mathbb{S}_m$ the class of functions $p(t)$ that are representable in the form

$$p(t) = (1-t^2)^{m-1+\delta}\varphi(t) \quad \text{with any} \quad \delta > 0,\ \varphi \in C^1[-1,1]. \tag{B.6}$$

Theorem 6 *The integral equation (B.5) with* $a, b \in C^\infty$ *and sectorial symbol* $k(\lambda)$ *has unique solution in* $\mathbb{S}_m$ *for any* $f \in C[-1,1]$. *In the representation (B.6) of this solution* $\delta = 1/2$ *and* $\varphi \in C^{m+n}[-1,1]$ *for* $f \in C^{n+2}[-1,1]$.

For $m = 0$ the statement of the theorem combines the theorems 1 and 5. For $m > 0$ the proof can be achieved by induction.

Let the theorem be true for some order m'. Then the supersingular integral equation of order $m = m' + 1$ can be integrated twice by x. This yields

$$\frac{d^{2m'}}{dx^{2m'}} \int_{-1}^{1} K'(t-x)p(t)\,dt = F(x) + c_0 + c_1 x.$$

Here $F(x)$ is the double integral of $f(x)$ and c_0, c_1 are arbitrary constants of integration. According to our assumption that the theorem is true for m' the solution $p(t)$ is representable in the form

$$p(t) = (1-t^2)^{m'-1/2}\Phi(t), \quad \Phi \in C^{n+2+m'}[-1,1]. \tag{B.7}$$

For the proof of the statement for $m = m' + 1$ one need to show that the constants c_0 and c_1 can be chosen such that

$$\Phi(\pm 1) = 0. \tag{B.8}$$

Introducing solutions $p_*(t)$, $p_0(t)$ and $p_1(t)$ of the integral equations

$$\frac{d^{2m'}}{dx^{2m'}} \int_{-1}^{1} K'(t-x) \begin{pmatrix} p_*(t) \\ p_0(t) \\ p_1(t) \end{pmatrix} dt = \begin{pmatrix} F(x) \\ 1 \\ x \end{pmatrix},$$

the condition (B.8) can be rewritten in the form of linear algebraic system for constants c_0 and c_1

$$\begin{cases} c_0\Phi_0(1) + c_1\Phi_1(1) = -\Phi_*(1), \\ c_0\Phi_0(-1) + c_1\Phi_1(-1) = -\Phi_*(-1). \end{cases}$$

Here functions $\Phi_*(t)$, $\Phi_0(t)$ and $\Phi_1(t)$ are the functions from the representations (B.7) for $p_*(t)$, $p_0(t)$ and $p_1(t)$ correspondingly

$$\begin{pmatrix} p_*(t) \\ p_0(t) \\ p_1(t) \end{pmatrix} = (1-t^2)^{m'-1/2} \begin{pmatrix} \Phi_*(t) \\ \Phi_0(t) \\ \Phi_1(t) \end{pmatrix}.$$

The function $P_0(t)$ is even and the function $P_1(t)$ is odd. Therefore the determinant of the above system is not equal to zero if $\Phi_0(1) \neq 0$ and $\Phi_1(1) \neq 0$.

To prove this fact let integrate by parts in the integral equations for $p_0(t)$ and $p_1(t)$. Then differentiation by x yields

$$\frac{d^{2m'}}{dx^{2m'}} \int_{-1}^{1} K'(t-x) \frac{dp_j(t)}{dt} dt = j, \quad |x| < 1.$$

Assuming that $p_0 \in \mathbb{S}_{m'+1}$ yields $dp_0/dt \in \mathbb{S}_{m'}$ and due to the uniqueness of the solution in $\mathbb{S}_{m'}$ one concludes that $dp_0(t)/dt = 0$. Further using the structure of the solution given in (B.7) yields $p_0(t) \equiv 0$ which contradicts the definition of $p_0(t)$.

For $p_1(t)$ the above procedure yields that

$$p_1(t) = \int_{-1}^{t} p_0(t') \; dt'. \tag{B.9}$$

Multiplying the initial equation for $p_0(t)$ by $\overline{p_0(x)}$ and integrating over the interval $[-1, 1]$ yields

$$\int_{-1}^{1} \overline{p_0(x)} \frac{d^{2m'}}{dx^{2m'}} \int_{-1}^{1} K'(t-x) p_0(t) \; dt \; dx = \int_{-1}^{1} \overline{p_0(x)} \; dx.$$

Due to the formula (B.9) and our assumption that $p_1(1) = 0$ the last integral is equal to zero. Performing m' times integration by parts and expressing the kernel in the form of Fourier integral yields

$$\int_{0}^{+\infty} k(\lambda) |\eta(\lambda)|^2 d\lambda = 0.$$

Here substitutions at $x = \pm 1$ disappear due to the supposition that $p_0 \in S_{m'}$ and

$$\eta(\lambda) = \int_{-1}^{1} e^{-i\lambda t} \frac{d^{m'}}{dt^{m'}} p_0(t) \; dt.$$

Sectoriality of the symbol yields $d^{m'}p_0(t)/dt^{m'} \equiv 0$ which again due to the structure of the solution contradicts the definition of $p_0(t)$. Thus both $p_0(t)$ and $p_1(t)$ are not equal to zero at $t = \pm 1$ and the algebraic system for the constants c_0 and c_1 is solvable for any right-hand side. Therefore the solution of the supersingular integral equation of order $m' + 1$ in the class $\mathbb{S}_{m'+1}$ exists for any $f(x)$ and is unique.

Check the smoothness of the function $\varphi(t)$ in the representation (B.6). Evidently that for $f \in C^{n+2}[-1,1]$ its second order integral $F \in C^{n+4}[-1,1]$. Therefore according to the supposition that the theorem is true for m' one concludes that $\Phi \in C^{n+2+m'}$. After division by $(1-t^2)$ its smoothness can decrease no more than by one. Thus $\varphi \in C^{n+1+m'}[-1,1]$. The theorem is proved for $m' + 1$ and by induction for any positive m.

B.4 Smooth kernels

Consider now the third case, namely integral equations with smooth kernels. Solutions of such equations are searched in the class of functions with nonintegrable singularities near the ends of the interval. More precisely the solution belongs to $\mathbb{S}_{-1}$.

The first problem that appears is the necessity of regularization. For this one returns back to the procedure of the integral equation derivation. Integral equations in the problems of diffraction are obtained with the help of Green's formula. First the integral representation for the scattered field is derived at some distance from the obstacle surface. The limit when the point of observation tends to the surface is taken formally and may yield integral equations for nonintegrable functions. Such singularities appear near the corner points of the obstacle surface. In particular in the simplest case these are the end points of the interval. By extracting those singularities the integral equation may be regularized [60].

Another approach is based on Fourier transform. Consider the system of dual integral equations‡

$$\begin{cases} \displaystyle\int e^{i\lambda x}\eta(\lambda)\, d\lambda = 0, & |x| > 1, \\ \displaystyle\int e^{i\lambda x}k(\lambda)\eta(\lambda)\, d\lambda = f(x), & |x| < 1. \end{cases}$$

‡Here we repeat derivations of Section 3.5.2 on page 182.

Here Fourier transform $k(\lambda)$ of the kernel $K(t-x)$ decreases at infinity as $|\lambda|^{-3}$ and growth of solution at infinity is allowed as $\eta(\lambda) = O(\lambda)$. The first integral is understood in the sense of distributions. By inverting it one formally gets the integral equation of convolution with smooth kernel.

To avoid nonintegrable solutions, let the function $\eta(\lambda)$ be represented as

$$\eta(\lambda) = \lambda^2 \widehat{\eta}(\lambda) + \eta_0 \rho_0(\lambda) + \eta_1 \rho_1(\lambda). \tag{B.10}$$

Here the function $\widehat{\eta}(\lambda)$ decreases at infinity not slower than $O(\lambda^{-1})$ and can be represented by its Fourier transform $\widehat{p}(x)$. The two remaining in (B.10) terms compensate the second order zero at $\lambda = 0$ introduced in the first term. The functions $\rho_0(\lambda)$ and $\rho_1(\lambda)$ are fixed functions that do not violate the asymptotics of $\alpha(\lambda)$ at infinity and are such that the vectors $\big(\rho_0(0), \rho_0'(0)\big)$ and $\big(\rho_1(0), \rho_1'(0)\big)$ are linear independent. Besides it is convenient to choose functions $\rho_0(\lambda)$ and $\rho_1(\lambda)$ such that their Fourier transform support belongs to the segment $-1 < x < 1$.

Substituting the representation (B.10) into the dual integral equations and accounting the mentioned above properties of functions $\rho_0(\lambda)$ and $\rho_1(\lambda)$ yields the integro-algebraic equation

$$\int_{-1}^{1} \widehat{p}(t) K''(t-x)dt + \sum_{\ell=0,1} \eta_\ell \int e^{i\lambda x} \rho_\ell(\lambda) k(\lambda) d\lambda = f(x). \tag{B.11}$$

The kernel $K''(t-x)$ of this equation has logarithmic singularity. Thus the integral equation of convolution with smooth kernel of the type

$$K(s) = s^2 \ln|s| + s^2 a(s^2) \ln|s| + b(s^2), \quad a(0) = 0$$

on an interval is regularized in the form of integro-algebraic equation (B.11). The unknowns in that system are the function $\widehat{p}$ which is searched in the class $\mathbb{S}_1$ and two constants η_0 and η_1.

The theory presented in section B.2 for the integral equations with logarithmic singularity in the kernel yields that for any η_0 and η_1 there exists unique solution $\widehat{p}(t)$ in the class $\mathbb{S}_0$. Therefore the question of solvability of the integro-algebraic equation (B.11) can be reduced to the analysis of solvability of the algebraic system for the constants η_0 and η_1. By introducing solutions of the integral equations

$$\int_{-1}^{1} K''(t-x)\frac{P_*(t)}{\sqrt{1-t^1}}dt = f(x),$$
$$\int_{-1}^{1} K''(t-x)\frac{P_\ell(t)}{\sqrt{1-t^1}}dt = \int e^{i\lambda x}k(\lambda)\rho_\ell(\lambda)d\lambda, \quad \ell = 0,1$$

this system can be written as

$$\begin{cases} \eta_0 P_0(1) + \eta_1 P_1(1) = P_*(1), \\ \eta_0 P_0(-1) + \eta_1 P_1(-1) = P_*(-1). \end{cases}$$

The solvability question is studied in [4]. Unfortunately there is no elegant proof of solvability. One may use arbitrariness of functions $\rho_\ell(\lambda)$ and show that at least for some $\rho_\ell(\lambda)$ determinant of the above algebraic system is different from zero and therefore solution of integro-algebraic equation (B.11) with $\widehat{p} \in \mathbb{S}_1$ exists.

Appendix C

Models used for numerical analysis

Numerical results presented in this book are computed for the following two plate–fluid systems. The first corresponds to the case of heavily loaded plate, the second to weakly loaded plate.

Model 1

This is a 1cm thick steel plate being in one-side contact to water. The parameters of the model are:

Young modulus E of steel taken equal to $200 \cdot 10^9$kg/c^{2}m,
Poison's ratio σ is 0.29,
density ϱ of steel is 7800kg/m^3,
thickness of the plate h is 0.01m,
sound velocity c_a in water is taken equal to 1500m/c,
density ϱ_0 of water is 1000kg/m^3.

Computing bending stiffness one finds

$$D = \frac{Eh^3}{12(1-\sigma^2)} = 18\ 197\text{kg m}^2/\text{c}^2.$$

The coincidence frequency f_c can be found equal to

$$f_c = 23\ 445\text{Hz}$$

and limiting frequency of Kirchhoff model applicability for which $kh = 1$ is

$$f_\infty = 23\ 873\text{Hz}.$$

The stiffener in such plate is considered of height $H = 20$cm and thickness $h_1 = 1$cm. Its linear mass M can be computed by the formula

$$M = \varrho h_1 H.$$

Momentum of inertia I is given by the integral§

$$I = \varrho \int_{-W/2}^{W/2} dx \int_0^H dz(x^2 + z^2) = \frac{\varrho h_1 H}{12}\left(h_1^2 + 4H^2\right).$$

One finds

$$M = 15.6\text{kg/m}, \qquad I = 0.2\text{kg m}.$$

Model 2

This is a 1mm thick steel plate being in one-side contact to air. The parameters of the model are:

Young modulus E is $200 \cdot 10^9$kg/c^{2}m,
Poison's ratio σ is 0.29,
density ϱ of steel is 7800kg/m^3,
thickness h of the plate is 0.001m,
sound velocity c_a in air is taken equal to 330m/c,
density ϱ_0 of air is 1.29kg/m^3.

The bending stiffness of such plate is

$$D = 18.197\text{kg m}^2/\text{c}^2,$$

coincidence frequency is

$$f_c = 11\,347\text{Hz}$$

and limit of applicability is

$$f_\infty = 52\,521\text{Hz}.$$

The stiffener is taken with $h_1 = 1$mm, $H = 20$mm. One finds

$$M = 0.156\text{kg/m}, \qquad I = 2 \cdot 10^{-5}\text{kg m}.$$

§One can neglect h_1^2 compared to H^2 which yields

$$I \approx \frac{\varrho h_1 H^3}{3}$$

accepted for example in [22] and in Section 4.4.

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