Gary Webb

Magnetohydrodynamics and Fluid Dynamics: Action Principles and Conservation Laws

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Gary Webb
CSPAR
The University of Alabama in Huntsville
Huntsville, Alabama
USA

Preface

The motivation for the present book originated in the quest to understand wave-wave interactions in magnetohydrodynamics (MHD) in a non-uniform background flow (this process is sometimes referred to as wave mixing in the solar wind and in cosmic ray modified shocks). The variational approach to WKB wave propagation in a non-uniform background plasma flow was developed by Dewar (1970). My initial aim was to understand linear, non-WKB wave propagation in the solar wind. The problem of wave mixing has also been identified as an important process in the evolution of turbulence and Alfvénic fluctuations in the solar wind (e.g. Zhou and Matthaeus 1990a,b; Zank et al. 2012). Waves in non-uniform flows also play an important role in Lagrangian averaged Euler-Poincaré equations (LAEP equations) of wave-mean flow interactions and the so-called alpha model of turbulence (e.g. Holm 2002).

Another motivation for the book was to understand the elegant non-canonical Hamiltonian formalism for MHD and fluids developed by Morrison and Greene (1980, 1982), Holm and Kupershmidt (1983a,b) and Marsden et al. (1984). The connection between a Clebsch variable action principle for MHD and the non-canonical Poisson bracket of Morrison and Greene (1980, 1982) and the Clebsch variational approach is developed by Zakharov and Kuznetsov (1997) (see also Zakharov and Kuznetsov (1971) for the canonical form of Hamilton's equations for MHD using Clebsch variables). In particular the work of Padhye and Morrison (1996a,b) shows the connection between Noether's second theorem and the conservation of potential vorticity in ideal fluid mechanics and MHD, due to the fluid relabelling symmetry of the equations (see also Salmon (1982, 1988) for an account of the fluid relabelling symmetry in ideal fluids). The fluid relabelling symmetries are due to the invariance of the action, in which the Lagrangian fluid labels can change (i.e. there are transformations or maps of the fluid labels onto new fluid labels that are diffeomorphisms) but the usual physical variables remain invariant. There are relationships between the fluid relabelling symmetries and the Casimirs of the non-canonical MHD Poisson bracket, which are explored in the present lecture notes.

Yet another motivation for the book is applications of topological methods in fluid dynamics and MHD. In the book we give examples of magnetic helicity conservation (e.g. Woltjer 1958; Kruskal and Kulsrud 1958; Berger and Field 1984; Finn and Antonsen 1985; Moffatt 1969, 1978; Moffatt and Ricca 1992) in solar physics and in solar wind physics. In Chap. 2, Sect. 2.5, we describe the investigation of Torok et al. (2010, 2014) on the evolution of the twist and writhe components of magnetic helicity in the evolution of the kink instability for solar magnetic flux ropes, and its role in coronal mass ejections (CMEs). Other applications to the magnetic helicity of the Parker interplanetary, Archimedean spiral magnetic field, to nonlinear Alfvén waves in the solar wind, and the MHD topological soliton solutions are described in Chap. 6.

Conservation laws obtained by Lie dragging advected invariants in magnetohydrodynamics (MHD) and gas dynamics or hydrodynamics (HD) were investigated by Moiseev et al. (1982), Sagdeev et al. (1990), Tur and Yanovsky (1993), Volkov et al. (1995), Kats (2001, 2003, 2004) and Webb et al. (2014a). The ten Galilean, Lie point symmetries of the action give rise to the energy conservation, momentum conservation, angular momentum and centre of mass conservation laws, via Noether's first theorem. The advected invariants are due to fluid relabelling symmetries, or diffeomorphisms associated with the Lagrangian map. There are different classes of geometrical quantities that are advected or Lie dragged with the flow. Examples are the entropy S (a 0-form) and the conservation of the magnetic flux ($\mathbf{B} \cdot d\mathbf{S}$ which is an invariant advected two-form), moving with the flow (i.e. Faraday's equation). Advected invariants are obtained by using the Euler-Poincaré approach to Noether's second theorem. Some of the invariants are important in topological fluid dynamics and MHD. We discuss different variants of helicity including kinetic helicity, cross helicity, magnetic helicity, Ertel's theorem and potential vorticity, the Hollman invariant and the Godbillon Vey invariant. Lie dragged invariants or Cauchy invariants play an important role in describing the dynamics of vortex and magnetic field lines in ideal hydrodynamics and MHD (e.g. Kuznetsov and Ruban 1998, 2000; Kuznetsov 2006; Besse and Frisch 2017).

The multi-symplectic and multi-momentum approach to Hamiltonian systems was originally developed by de Donder (1930) and Weyl (1935). They studied generalized Hamiltonian mechanics in which the Lagrangian $L = L(\mathbf{x}, \varphi^i, \partial \varphi^i / \partial x^\mu)$ where $x^\mu (1 \leq \mu \leq n)$ are the independent variables and φ^k $(1 \leq k \leq m)$ are the dependent variables. For the case where $n \geq 2$ one can define multi-momenta $\pi_i^\mu = \partial \varphi^i / \partial x^\mu$ corresponding to each x^μ (in the usual Hamiltonian formulation $x^0 = t$ is the evolution variable). The multi-symplectic approach has been developed in field theory in the search for a more covariant form of Hamiltonian mechanics (in the usual Hamiltonian formulation, there is only one evolution variable). Bridges et al. (2005, 2010), Marsden and Shkoller (1999), Hydon (2005) and Cotter et al. (2007) describe multi-symplectic systems. Our aim is to present both Eulerian and Lagrangian variational principles for ideal fluids and MHD obtained by, e.g. Newcomb (1962), Holm and Kupershmidt (1983a,b), Dewar (1970) and Webb et al. (2005a,b, 2014a,b). Both Eulerian and Lagrangian multi-symplectic forms of the equations can be obtained. In this book we concentrate on the Eulerian multi-

symplectic form of the equations (the Lagrangian, multi-symplectic ideal fluid equations are described by Webb (2015) and Webb and Anco (2016)). The multi-symplectic Noether's theorem and symplecticity and pullback conservation laws are obtained. Nonlocal conservation laws, for a non-barotropic equation of state for the fluid, in which the time integral of the temperature back along the fluid path plays an important memory role, are obtained (see also Mobbs (1981) for similar conservation laws for helicity in non-barotropic fluids). Yahalom (2016a, 2017a,b) explores the physical and topological meaning of the non-barotropic cross helicity and cross helicity per unit magnetic field flux, using a Clebsch potential formulation (see also Webb and Anco 2017). The connection of the multi-symplectic approach with Cartan's theory of differential equations using differential forms is developed. A potential vorticity type conservation law is derived for MHD using Noether's second theorem.

The motivation is to provide both local and nonlocal conservation laws of the fluid and MHD equations that give insight into the physics. Conservation laws are useful for the testing numerical codes and reveal new aspects of the physics (e.g. nonlocal conservation laws associated with potential symmetries and fluid relabelling symmetries, reveal the time history of the fluid elements can play an important role in understanding fluid vorticity). For example, the baroclinic effect leads to the creation of vorticity in fluids (e.g. in tornadoes), but the corresponding nonlocal conservation law for fluid helicity is not usually discussed. Casimirs (i.e. quantities with zero Poisson bracket with other functionals of the physical variables) are important in describing the stability of steady flows and equilibria. The knowledge of new conservation laws is important in fusion plasmas, space plasmas, fluid dynamics and atmospheric physics. New conservation laws are also important in mathematics in elucidating the symmetries responsible for the conservation laws (e.g. Lie pseudo groups are most likely related to fluid relabelling symmetries).

What Is Not Included in the Book

The abstract geometrical mechanics aspects of fluid mechanics and MHD are not developed in the present approach. Detailed descriptions of the geometrical mechanics approach to the theory are described in Marsden et al. (1984), Marsden and Ratiu (1994), Holm et al. (1998) and Holm (2008a,b). Holm and Kupershmidt (1983a,b), Marsden et al. (1984) and Holm et al. (1998) describe the role of semi-direct product Lie algebras and Lie groups inherent in the non-canonical Poisson bracket of Morrison and Greene (1980, 1982). Morrison (1982) gives a direct algebraic method to derive the Jacobi identity. Olver (1993) uses the variational complex to develop methods to check if a given co-symplectic differential operator used to define the Poisson bracket is a Hamiltonian operator (i.e. the bracket is skew symmetric and satisfies the Jacobi identity). Chandre et al. (2012, 2013) and Chandre (2013) derived Dirac brackets for MHD to obtain well-behaved brackets

that satisfy the Jacobi identity. Bridges et al. (2010) use the variational bi-complex to describe multi-symplectic systems. The analysis of Lie symmetries of differential equation systems using Lie's algorithm (e.g. Bluman and Kumei 1989; Olver 1993; Ovsjannikov 1962, 1982; Ibragimov 1985; Bluman et al. 2010) can be used to derive analytical solutions of the equations. We do not study conservation laws and symmetries for special and general relativistic MHD (see, e.g. Lichnerowicz 1967; Beckenstein and Oron 1978; Bekenstein 1987; Anile 1989; Achterberg 1983; D'Avignon et al. 2015). Pshenitsin (2016) has derived infinite classes of conservation laws for incompressible viscous MHD by using the so-called direct method developed by Anco and Bluman (see, e.g. Bluman et al. 2010). This method of determining conservation laws is illustrated for the case of the KdV equation in Chap. 4. However, we have not used this method to derive MHD conservation laws in the present book.

We discuss topological invariants in fluids and plasmas, using Lie dragged invariants in ideal fluids and MHD (see, e.g. Arnold and Khesin 1998; Berger and Field 1984; Berger 1999a,b; Moffatt and Ricca 1992; Besse and Frisch 2017 for detailed analysis). The papers by Kuznetsov and Ruban (1998, 2000) and Kuznetsov et al. (2004) give an account of vortex lines and magnetic field lines, using a mixed Eulerian and Lagrangian approach, which shows how one may resolve the degeneracy of the non-canonical Poisson brackets, by using Weber transformations and Lagrangian representations of the equations. They also show how the Hasimoto transformation arises from their analysis. Euler potential representations of the magnetic field and its use in fusion and space plasmas are another large area of research not covered in our treatment (see, e.g. Stern (1966) for applications in space plasmas, and Boozer (2004) in fusion plasmas).

Recent work by Webb (2015) and Webb and Anco (2016) on Lagrangian, multi-symplectic fluid mechanics and work on MHD gauge field theory by Webb and Anco (2017) are omitted from the present exposition. It is worth noting that Calkin (1963) developed a version of gauge field theory for a polarized version of MHD. Both Calkin (1963) and Webb and Anco (2017) identified the gauge symmetry responsible for the magnetic helicity conservation law in MHD. These developments lie beyond the scope of the present book.

Acknowledgements

Lie symmetries and Noether's theorems in MHD and fluid dynamics. I am indebted to G.P. Zank for suggesting to write a book on MHD and fluids and for discussions of the compressible, incompressible and reduced MHD equations and symmetries of the equations. I am indebted to Stephen Anco for discussions on some of the more obscure conservation laws of the MHD and fluid equations and the use of the direct method to obtain conservation laws for equation systems, which are not necessarily associated with variational principles (e.g. Cheviakov 2014; Cheviakov and Oberlack 2014; Pshenitsin 2016). I am grateful to Dr. Q. Hu (who drew many of the figures) for discussions on magnetic helicity and magnetic clouds, and to Dr. B. Dasgupta for his detailed knowledge of magnetic fields in plasma physics (e.g. the Kamchatnov MHD topological soliton and chaotic versus integrable magnetic field lines). I am indebted to E.D. Fackerell and C.B.G. McIntosh for their lectures on Lie symmetries and differential equations, whilst at Monash University in the period 1974–1977. I also acknowledge discussions on fluid relabelling symmetries with Nikhil Padhye and discussions with R.B. Sheldon on the importance of nonlocal conservation laws. The errors are mine.

Huntsville, AL, USA   Gary Webb
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Contents

Chapter 1
Introduction

Variational methods are widely used in physics, engineering and applied mathematics. Noether's theorems provide a route to deriving conservation laws for systems of differential equations governed by an action principle. Noether's theorem applies to systems of Euler-Lagrange equations that are in Kovalevskaya form (e.g. Olver 1993). For other Euler-Lagrange systems, each nontrivial variational symmetry leads to a conservation law, but there is no guarantee that it is non-trivial. Goldstein (1980) discusses action principles, Lagrangian and Hamiltonian equations and Noether's theorem in classical mechanics.

Noether's first theorem deals with conservation laws for equation systems, arising from variational principles, in which the action remains invariant with respect to infinitesimal transformations of finite dimensional Lie groups. Noether's second theorem applies for infinite dimensional pseudo Lie algebras, where the variational symmetries depend on arbitrary locally smooth functions of the independent variables. Noether's second theorem states that in this case, such symmetries only exist if there are differential relations between the Euler-Lagrange equations (see Brading (2002) for an historical overview). Hydon and Mansfield (2011) include constraint equations in their formulation of Noether's second theorem, by means of Lagrange multipliers. Rosenhaus (2002) determines the effects of boundary conditions on Noether's second theorem.

Another powerful method using a Lie dragging approach to derive conservation laws in MHD and fluid dynamics, was developed by Moiseev et al. (1982), Sagdeev et al. (1990), Tur and Yanovsky (1993), Volkov et al. (1995), and Besse and Frisch (2017). Tur and Yanovsky use the Calculus of exterior differential forms and Lie derivatives originally developed by Elie Cartan. These methods are used in general relativity (e.g. Misner et al. 1973). One of the advantages of this approach compared to classical work using Noether's theorem is that the invariance of geometrical quantities which are Lie dragged with the flow are naturally described by the formulas of the exterior differential Calculus. Cotter and Holm (2012) developed an Euler-Poincaré approach which takes into account the advection of geometrical

objects (e.g. tensors, vectors, differential forms) which are invariant under fluid relabeling symmetries.

We present the basic ideas used in deriving advected invariants in MHD and gas dynamics using the Calculus of exterior differential forms, and show how these conserved invariants and conservation laws are related to Clebsch variable formulations, Weber transformations, Noether's theorems and the Euler-Poincaré formulation of variational principles involving a symmetry group developed by Marsden and Ratiu (1994), Holm et al. (1998), Cotter et al. (2007) and others. Our aim is to study illustrative examples, and not to develop a complete discussion of all possibilities. Tur and Yanovsky (1993), Volkov et al. (1995) and Kats (2003, 2004) show that there are an infinite number of advected invariants in ideal fluid mechanics and MHD (Kats (2001) studies discontinuities in hydrodynamics using a variational approach). These invariants are related to fluid relabeling symmetries and gauge symmetries of the action. The magnetic helicity conservation equation is due to a gauge symmetry, which is not a fluid relabelling symmetry (e.g. Calkin 1963; Webb and Anco 2017). Tanehashi and Yoshida (2015) obtain gauge symmetries for Clebsch parameterized, barotropic MHD, by exploiting the known MHD Casimirs.

The MHD equations admit the ten-parameter Galilei Lie group. This includes the space and time translation symmetries, the space rotations and the Galilean boosts (e.g. Fuchs 1991; Grundland and Lalague 1995; Webb and Zank 2007). These symmetries are variational or divergence symmetries of the action, and give rise to conservation laws via Noether's first theorem, namely: (a) the energy conservation law due to the time translation symmetry (b) the momentum conservation laws (space translation symmetries), (c) angular momentum conservation laws (rotational symmetries) and (d) the center of mass conservation laws (Galilean boosts symmetries).

There is a class of infinite dimensional fluid relabelling symmetries that leave the MHD equations invariant under transformation of the Lagrangian fluid labels. The fluid relabelling symmetries conservation laws are associated with Noether's second theorem (e.g. Salmon 1982, 1988; Padhye and Morrison 1996a,b; Padhye 1998; Zakharov and Kuznetsov 1997; Kats 2003, 2004; Webb et al. 2005b; Webb and Zank 2007; Cotter and Holm 2012). Yahalom and Lynden-Bell (2008) developed simplified variational principles for barotropic MHD using Clebsch variables. In the Lagrangian fluid dynamics approach, one can search for Lie transformations of the form:

$$\mathbf{x}' = \mathbf{x} + \epsilon V^{\mathbf{x}}, \quad t' = t + \epsilon V^t, \quad \mathbf{x}_0' = \mathbf{x}_0 + \epsilon V^{\mathbf{x}_0}, \tag{1.1}$$

that leave the action invariant up to a divergence transformation, where $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ is the Lagrangian map between the Eulerian fluid particle position and its Lagrangian label $\mathbf{x}_0$. The Lagrangian map is the solution of the differential equation system $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$, where $\mathbf{u}$ is the fluid velocity, subject to the initial conditions $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$. The fluid relabelling symmetries correspond to the variational symmetries of the action of the form $\mathbf{x}' = \mathbf{x}$, $t' = t$ and $\mathbf{x}_0' = \mathbf{x}_0 + \epsilon V^{\mathbf{x}_0}$ in which $\mathbf{x}$ and t are fixed.

The inter-relationships between the fluid relabelling symmetries and the Lie point symmetries were investigated by Webb and Zank (2007). They converted the known Eulerian Lie point symmetries of the MHD equations to their corresponding Lagrangian form in which the Eulerian position vector $\mathbf{x}$ are the dependent variables and the Lagrange labels $\mathbf{x}_0$ and the time t are the independent variables. For polytropic equations of state with $p = p_0 \rho^\gamma \exp(S/C_v)$ there are three scaling symmetries of the equations, which can be judiciously combined to give a conservation law of the equations (Webb and Zank 2007; Webb et al. 2009). The form of the scaling symmetry Lie generators in Lagrange label space involve a modified form of the fluid relabelling symmetry equations. Sjöberg and Mahomed (2004) and Webb et al. (2009) (and references therein) consider potential symmetries of the 1D gas dynamic equations, which give rise to non-local conservation laws. Golovin (2011) derives the Lie symmetries and equivalence transformations for the Lagrangian MHD equations in which the Eulerian position of the fluid element $\mathbf{x} = \boldsymbol{\gamma}(t, \boldsymbol{\xi})$ leads to a vector wave equation for $\boldsymbol{\gamma}$ and (ξ^1, ξ^2, ξ^3) are appropriately chosen Lagrange labels (Webb et al. (2005b) used a similar formulation, but did not work out the Lie group of the equations).

Volkov et al. (1995) show the connection between advected invariants and the odd Buttin bracket and supersymmetry.

In Chap. 2, the MHD equations and the first law of thermodynamics for for the case of an ideal, non-barotropic gas in which $p = p(\rho, S)$ are introduced. An application of magnetic helicity conservation to the evolution of a kink unstable flux rope in the solar corona, in the solar atmosphere by Torok et al. (2010, 2014) is described in Sect. 2.5. We also indicate other applications in Chap. 6 to: (a) the magnetic helicity of the interplanetary Parker (1958) magnetic field, (b) the magnetic helicity of nonlinear shear and toroidal Alfvén waves in the Solar Wind, and to (c) topological solitons in MHD.

Chapter 3, gives an introduction to helicity in fluids and magnetohydrodynamics, including: helicity in barotropic fluids, magnetic helicity, cross helicity potential vorticity in MHD and fluids, and nonlocal conservation laws that apply for non-barotropic gas equation of state with $p = p(\rho, S)$. Both the differential and integrated forms of the helicity conservation laws are discussed. The helicity is defined as the integral of $\mathbf{u} \cdot \boldsymbol{\omega}$ over the volume V of the fluid of interest, where $\mathbf{u}$ is the fluid velocity and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the fluid vorticity (e.g. Moffatt 1969). Woltjer (1958) described the magnetic helicity in plasmas as the integral of $\mathbf{A} \cdot \mathbf{B}$ over a volume of the MHD fluid V, where $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic induction and $\mathbf{A}$ is the magnetic vector potential. It applies for the case where the normal magnetic field $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary ∂V of the volume of interest. Relative magnetic helicity (Berger and Field 1984; Finn and Antonsen 1985, 1988) is a gauge independent form of the helicity for cases where $\mathbf{B} \cdot \mathbf{n} \neq 0$ on the boundary ∂V of the plasma volume V. The magnetic helicity and cross helicity conservation laws for MHD are derived directly from the MHD equations.

Chapter 4, introduces basic tools of the Calculus of variations and the Euler Lagrange equations for a system of differential equations governed by a variational principle. We derive Noether's first theorem, and discuss Noether's second theorem.

We then discuss the direct method for deriving conservation laws developed by Anco and Bluman (1996, 1997, 2002a,b), which does not require that the differential equations are derived from a variational principle (see also Bluman et al. 2010). A simple example derives conservation laws for the Korteweg de Vries equation without using a variational principle. Cheviakov and Anco (2008) and Cheviakov (2014) applied this method to obtain conservation laws for fluid dynamics. Pshenitsin (2016) uses this method to obtain conservation laws of the incompressible MHD equations, including the effects of viscosity. Rosenhaus and Shankar (2016) investigate Noether's second theorem for quasi-Noether systems of equations, which they use to discuss the infinite number of conservation laws for the incompressible Euler equations of fluid dynamics. Rosenhaus and Shankar (2016, 2017a,b) investigate the role of sub-symmetries in the generation of infinite families of conservation laws, with application to the 2D and 3D Euler equations in the velocity and vorticity formulation. These analyses are related to the work of Cheviakov (2014) on infinite families of conservation laws for the Euler equations.

Chapter 5, derives advected invariants of the MHD and ideal fluid dynamics systems, by using the algebra of exterior differential forms:, i.e. exterior differentiation d, the Lie derivative $\mathcal{L}_{\mathbf{V}}$ with respect to a vector field $\mathbf{V}$, the formation of higher order forms by using the wedge product of forms, and the construction of lower order forms by contraction of vector fields with differential forms. Lie dragging of forms and vector fields are useful in obtaining geometric conservation laws for the equations (Tur and Yanovsky 1993). We discuss Faraday's equation the entropy advection equation and mass continuity equation in terms of advected invariant forms. Theorems, that are useful in combining known invariants, to obtain new invariants from old invariants are discussed.

Chapter 6 discusses topological invariants for fluids and plasmas. For cases with non-trivial magnetic field topology, there is no globally continuous form for the magnetic vector potential $\mathbf{A}$. However one can cover the whole manifold with two or more vector potentials that apply locally, and in which there is a jump in $\mathbf{A}$, between the vector potentials applicable in different regions. The prototypical example is that of the magnetic monopole field (e.g. Urbantke 2003; Webb et al. 2010a), where the sphere S^2 is covered by two separate regions S^+ and S^-, where S^+ is the sphere minus a small region about the north pole and S^- is a similar region of the surface of the sphere, excluding the south pole. Multi-valued magnetic vector potentials also occur in the MHD topological soliton (Kamchatnov 1982; Semenov et al. 2002). We discuss the Gauss link number formula and the Calugareanu invariant (Calugareanu 1959; Moffatt and Ricca 1992) and the linkage, twisting and writhing of magnetic flux tubes. Magnetic helicity, is an advected invariant. In Sect. 6.4 we discuss link numbers and signed crossing numbers for knots, and how these numbers may be used to calculate the magnetic helicity of knotted flux tubes. We discuss Dehn surgery in which knots are cut and reconnected, without change in the helicity. Taylor relaxation theory (Taylor 1974, 1986) is described in which the total magnetic helicity to lowest order is conserved in a high conductivity plasma during turbulent reconnection. In this theory the field evolves to a force-free magnetic field state satisfying the force-free magnetic field equation $\nabla \times \mathbf{B} = \Lambda \mathbf{B}$ where Λ is a

constant. Section 6.5 describes the Godbillon-Vey invariant in MHD, which arises if $\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} = 0$ or $\tilde{\mathbf{A}} \cdot \mathbf{B} = 0$ ($\tilde{\mathbf{A}} \cdot d\mathbf{x}$ is assumed to be advected with the background fluid flow). In this case there is a higher order topological invariant known as the Godbillon-Vey invariant which is advected with the flow (e.g. Tur and Yanovsky 1993; Webb et al. 2014a). Applications of magnetic helicity conservation to (a) the Parker interplanetary magnetic field, (b) toroidal and shear Alfvén waves in the solar wind, and (c) the MHD topological soliton are discussed in Sect. 6.6.

Chapter 7 discusses the Euler-Poincaré equation for MHD following the analysis of Holm et al. (1998) and Cotter and Holm (2012), followed by some applications of Noether's second theorem. The advection of the vorticity 2-form, as an advected invariant $\beta = \boldsymbol{\omega} \cdot dS$ in ideal, barotropic compressible gas dynamics is shown to be related to the mass conservation symmetry, via Noether's second theorem. Noether's second theorem, and the Euler-Poincaré formulation of fluid dynamics, are related to the advection of cross helicity, potential vorticity and Ertel's theorem. The differential conservation law for cross helicity for example, involves the the flux $h_c \mathbf{u} + \mathbf{B}(h + \Phi - |\mathbf{u}|^2/2)$ where h is the enthalpy, $\Phi(\mathbf{x})$ is the gravitational potential energy, and $h_c = \mathbf{u} \cdot \mathbf{B}$ is the cross helicity density.

Chapter 8 introduces the Hamiltonian formulation of MHD using Clebsch variables (e.g. Zakharov and Kuznetsov 1997). The Clebsch variable formulation involves a momentum map, in which the Lagrange multipliers in the constrained variational principle are the canonically conjugate momenta for the system. The canonical Poisson bracket using Clebsch variables is transformed to Eulerian physical variables to obtain the non-canonical Poisson bracket for MHD obtained by Morrison and Greene (1980, 1982) and Holm and Kupershmidt (1983a,b) (the Morrison and Greene formulation uses the magnetic field induction $\mathbf{B}$ as the basic variable describing the magnetic field, whereas Holm and Kupershmidt use the magnetic vector potential, for which the one-form $\mathbf{A} \cdot d\mathbf{x}$ is advected with the flow). The derivation of the noncanonical MHD Poisson bracket by Morrison and Greene (1980, 1982) was obtained directly from the Eulerian MHD equations, written in terms of the usual, noncanonical fluid variables, $\rho, \mathbf{u}, S$ and $\mathbf{B}$. Some of the subtleties of the MHD Poisson bracket discussed by Morrison and Greene (1982) associated with the Jacobi identity being satisfied for functionals with $\nabla \cdot \mathbf{B} \neq 0$ has been addressed by Chandre et al. (2012, 2013) and Chandre (2013), by using Dirac's theory of constraints and the Dirac bracket. We discuss the gauge case in which 1-form $\mathbf{A} \cdot d\mathbf{x}$ is advected with the flow (i.e. the advected $\mathbf{A}$ gauge). This approach circumvents the need to discuss the condition $\nabla \cdot \mathbf{B} = 0$ (see also Holm and Kupershmidt 1983a,b). Investigations of the Jacobi identity for non-canonical MHD Poisson brackets for the cases with (1) $\nabla \cdot \mathbf{B} \neq 0$ (Morrison and Greene 1982), (2) $\nabla \cdot \mathbf{B} = 0$ (Morrison and Greene 1980) and (3) for the advected $\mathbf{A}$ gauge (Holm and Kupershmidt 1983a) in which $\mathbf{B} = \nabla \times \mathbf{A}$, are carried out by using the functional multi-vectors approach of Olver (1993). We discuss methods to obtain the MHD Casimirs (see e.g. Morrison (1982), Holm et al. (1985), and Hameiri (1998, 2003, 2004) for discussions of Casimirs in studies of MHD stability for equilibria and steady flows).

Chapter 9 describes a multi-symplectic formulation of MHD based on the momentum map for the Clebsch variable action principle. We review the approach to multi-symplectic Hamiltonian systems developed by Bridges et al. (2005) and Hydon (2005). We use mainly the version of multi-symplectic Hamiltonian systems described by Hydon (2005). We extend the work of Cotter et al. (2007) to obtain a multi-symplectic formulation of MHD using Clebsch variables. We discuss the symplecticity conservation laws and Noether's theorem for multi-symplectic Hamiltonian systems. A conservation law obtained by using pull back of forms to the base manifold gives rise to the energy and momentum conservation laws for the MHD equations.

Chapter 10 introduces the Lagrangian map for MHD following the approach of Newcomb (1962). The Eulerian and Lagrangian variations of the plasma are defined. The Lagrangian and canonical Hamiltonian form of the equations are obtained. The reduction of the MHD equations using the Lagrangian map by Golovin (2011) is described in Sect. 10.4. Golovin (2010) used these ideas to describe steady state MHD flows which are pressure balance solutions. Analogous solutions for non-steady state pressure balance solutions were obtained by Golovin (2011). These solutions show knotted field line structures lying on the toroidal Maxwell surfaces (see Schief (2003) for similar solutions). We do not delve too deeply into these solutions. Our aim is to show how the solutions are related to the Lagrangian map.

Chapter 11 develops Noether's theorem based on the Lagrangian map, in which the Lagrangian fluid labels $\mathbf{x}_0$ are the independent coordinates and the Eulerian position coordinates $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ are the dependent variables (i.e. we use the Lagrangian map). This approach was developed by Padhye and Morrison (1996a,b), Webb et al. (2005a,b), and Webb and Zank (2007). The conservation laws are converted to their Eulerian form using the Lagrangian map. Conservation laws due to the Lie point symmetries of the MHD equations and the MHD action, and the fluid relabelling symmetries are discussed. We investigate the connection between the fluid relabelling symmetries and the Casimirs (see also Padhye and Morrison (1996a,b)). Padhye and Morrison used Lagrangian variations in their analysis. We use Eulerian variations to obtain similar determining equations for the Casimirs (there are some differences between the two approaches).

Chapter 12 discusses MHD stability methods using the Lagrangian fluid displacement $\boldsymbol{\xi}$. The Frieman and Rotenberg (1960) equations for the stability of steady MHD flows are obtained. These equations can be obtained by expanding the action as a power series in $\boldsymbol{\xi}$, and the Lagrangian variation ΔS of the entropy. This leads to the perturbed momentum equation for the fluid which is equivalent to the Frieman and Rotenberg equations. The characteristic manifolds for linear waves and their relationship to the magneto-acoustic, Alfvén and entropy waves are delineated (e.g. Webb et al. 2005a). The first and second variation of the action using Eulerian perturbations are developed. The second variation of the action is used to derive the Frieman and Rotenberg equations, and to write the equations in Hamiltonian form. We also show the connection between the Frieman and Rotenberg equations and accessible variations of the action using the non-canonical MHD Poisson bracket, as developed by Holm et al. (1985), Morrison and Eliezer (1986), and Hameiri (2003).

Chapter 13 concludes with an overview and discussion.

In Appendix A, we discuss the Lie derivatives of 0-forms, 1-forms and vector fields. Appendix B discusses Weber transformations and the Clebsch variable description of MHD using the approach of Zakharov and Kuznetsov (1997). Appendix C discusses the Cauchy invariant $\mathbf{b} = \mathbf{B}/\rho$. Appendix D describes magnetoacoustic N-waves of Webb et al. (1993). The phase and group velocities of the fast and slow magnetosonic waves, are discussed by using the wave eikonal formulation of the MHD dispersion equation, which is written as a first order, nonlinear partial differential equation for the wave phase $S(\mathbf{x}, t)$ or wave eikonal. The wave group velocity arises as an envelope solution of the wave eikonal equation, and the characteristics of the eikonal equation describe the group velocity surface via Hamilton's equations. These ideas (from Webb et al. 1993) are used to describe the magnetic field structure of the linear magneto-acoustic N-wave which corresponds to singular delta function initial data for the gas pressure. In Appendix E, we discuss Aharonov Bohm effects for the magnetic helicity H_M and the non-barotropic cross helicity H_{CNB} in MHD as developed by Yahalom (2013, 2017a,b) (see also Webb and Anco 2017). Appendix F gives a formal definition of equivalence transformations for a system of differential equations (see also Bluman et al. 2010). Golovin (2011) obtained the equivalence transformations for the Lagrangian MHD equations. Appendix G uses a modified form of the Lagrangian action principle to derive a covariant form of the MHD momentum equation from the action principle that uses generalized coordinates to specify the Eulerian position coordinates. Generalized coordinates are also used to describe the Lagrangian fluid labels.

Chapter 2
The Model

2.1 The MHD Equations

The magnetohydrodynamic equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u} + \left(p + \frac{B^2}{2\mu_0} \right) \mathbf{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0} \right] = -\rho \nabla \Phi, \tag{2.2}$$

$$\frac{\partial}{\partial t}(\rho S) + \nabla \cdot (\rho \mathbf{u} S) = 0, \tag{2.3}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u} \nabla \cdot \mathbf{B} = 0. \tag{2.4}$$

Here, ρ, $\mathbf{u}$, p, S and $\mathbf{B}$ are the gas density, fluid velocity, pressure, specific entropy, and magnetic induction $\mathbf{B}$ respectively, and $\mathbf{I}$ is the unit 3×3 dyadic. $p = p(\rho, S)$ is a function of the density ρ and entropy S, and μ_0 is the magnetic permeability. Equations (2.1)–(2.3) correspond to the mass, momentum and entropy conservation laws, and Faraday's equation in the MHD limit. In classical MHD, (2.1)–(2.4) are supplemented by Gauss' law:

$$\nabla \cdot \mathbf{B} = 0. \tag{2.5}$$

Ampere's law for non-relativistic MHD, which neglects the displacement currents for slow MHD phenomena, has the form

$$\mathbf{J} = \nabla \times \mathbf{B}/\mu_0. \tag{2.6}$$

On the right hand-side of (2.2) Φ is the gravitational potential of an external gravitational field. This term is important in stellar wind theory, where the gravity force $-\rho\nabla\Phi$ modifies the plasma flow, both for the case of stellar winds and accretion flows. The gravity force term is also important in static MHD models of solar magnetic structures such as prominences, in which the gravitational force is counter-balanced with the pressure gradient and magnetic forces (e.g. Low 1985).

There is an eigenmode of the MHD equations (2.1)–(2.4) with $\nabla\cdot\mathbf{B}\neq0$ known as the divergence mode, which is advected with the fluid, which is used in eight wave Riemann solvers in numerical MHD (e.g. Powell et al. 1999; Janhunen 2000; Webb et al. 2009). In physical applications it is necessary to set $\nabla\cdot\mathbf{B}=0$. However, for the sake of completeness we keep the $\nabla\cdot\mathbf{B}$ terms in the equations, in order to see the mathematical effects that result if $\nabla\cdot\mathbf{B}\neq0$. The Alfvén, fast and slow magnetoacoustic simple waves all apply to the case where $\nabla\cdot\mathbf{B}=0$.

If the equation of state for the gas is written in the form $S=f(p,\rho)$ the entropy conservation law (2.3) can alternatively be written in the form:

$$\frac{\partial p}{\partial t}+\mathbf{u}\cdot\nabla p+A(p,\rho)\nabla\cdot\mathbf{u}=0,\quad A(p,\rho)=a^2\rho, \tag{2.7}$$

where $a^2=\partial p/\partial\rho=-f_\rho/f_p$ is the square of the adiabatic sound speed of the gas. For the case of an ideal gas with entropy $S=C_v\ln[(p/p_1)/(\rho/\rho_1)^\gamma]$ where $\gamma=C_p/C_v$ is the ratio of specific heats at constant pressure and volume respectively, $A(p,\rho)=\gamma p$ in (2.7).

The above equations are supplemented by the first law of thermodynamics:

$$TdS=dQ=dU+pdV\quad\text{where}\quad V=\frac{1}{\rho}, \tag{2.8}$$

where U is the internal energy per unit mass and $V=1/\rho$ is the specific volume. If one uses the internal energy per unit volume $\varepsilon=\rho U$ instead of U (2.8) becomes:

$$TdS=\frac{1}{\rho}\left(d\varepsilon-hd\rho\right)\quad\text{where}\quad h=\frac{\varepsilon+p}{\rho}, \tag{2.9}$$

is the enthalpy of the gas. Since $\varepsilon=\varepsilon(\rho,S)$ (2.9) implies the relationships:

$$\rho T=\varepsilon_S,\quad h=\varepsilon_\rho,\quad p=\rho\varepsilon_\rho-\varepsilon, \tag{2.10}$$

between the temperature T, enthalpy h and pressure p to the internal energy density $\varepsilon(\rho,S)$. Equation (2.9) gives the equations:

$$TdS=dh-\frac{1}{\rho}dp\quad\text{and}\quad-\frac{1}{\rho}\nabla p=T\nabla S-\nabla h, \tag{2.11}$$

which gives an alternative expression for the pressure gradient force on the fluid.

2.2 Energy Conservation

The total energy equation for the system (2.1)–(2.4):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \varepsilon + \frac{B^2}{2\mu_0} + \rho\Phi\right) + \nabla\cdot\left[\rho\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2 + h + \Phi\right) + \frac{\mathbf{E}\times\mathbf{B}}{\mu}\right] = 0,$$

(2.12)

follows from adding together: (i) the electromagnetic energy equation:

$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right) = -\mathbf{J}\cdot\mathbf{E} - \mathbf{u}\cdot\mathbf{B}\frac{\nabla\cdot\mathbf{B}}{\mu_0},$$

(2.13)

where $\mathbf{S} = \mathbf{E}\times\mathbf{B}/\mu_0$ is the Poynting flux, and $\mathbf{E} = -\mathbf{u}\times\mathbf{B}$ is the motional electric field, and $\mathbf{J} = \nabla\times\mathbf{B}/\mu_0$ is Ampere's law for the electric current in the MHD limit; (ii) the co-moving gas energy equation:

$$\frac{\partial\varepsilon}{\partial t} + \nabla\cdot[\rho\mathbf{u}h] = \mathbf{u}\cdot\nabla p,$$

(2.14)

and (iii) the gas kinetic and gravitational energy equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho\Phi\right) + \nabla\cdot\left[\rho\mathbf{u}\left(\frac{1}{2}u^2 + \Phi\right)\right] = -\mathbf{u}\cdot\nabla p + \mathbf{J}\cdot\mathbf{E} + \mathbf{u}\cdot\mathbf{B}\frac{\nabla\cdot\mathbf{B}}{\mu_0},$$

(2.15)

Poynting's theorem (2.13) follows from using Faraday's equation (2.4) and Ampere's equation $\mathbf{J} = \nabla\times\mathbf{B}/\mu_0$ in the combination:

$$\frac{\mathbf{B}}{\mu_0}\cdot(\mathbf{B}_t + \nabla\times\mathbf{E} + \mathbf{u}\nabla\cdot\mathbf{B}) + \mathbf{E}\cdot(\mathbf{J} - \nabla\times\mathbf{B}/\mu_0) = 0,$$

(2.16)

and by using the identity

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = (\nabla\times\mathbf{E})\cdot\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{E},$$

(2.17)

The co-moving gas energy equation (2.14) follows from the second law of thermodynamics: $dQ = TdS = dU + pd\tau$, where $U = \varepsilon/\rho$ is the internal energy per unit mass of the gas, and $\tau = 1/\rho$ is the specific volume, and by noting that $dS/dt = 0$ for an adiabatic process where $d/dt = \partial/\partial t + \mathbf{u}\cdot\nabla$ is the time derivative following the flow.

The kinetic energy equation (2.15) for the gas follows from the momentum equation (2.2), which with the aid of the continuity equation (2.1) can be cast in the form:

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mathbf{J}\times\mathbf{B} - \rho\nabla\Phi + \mathbf{B}\frac{\nabla\cdot\mathbf{B}}{\mu_0},$$

(2.18)

where $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ is the current. Taking the scalar product of (2.18) with $\mathbf{u}$ and using the continuity equation (2.1) gives the gas kinetic energy equation (2.15). In the derivation of (2.15) it is useful to note that

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\mathbf{u} \times \boldsymbol{\omega} + \nabla \left(\frac{1}{2} u^2 \right), \qquad (2.19)$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the fluid vorticity.

In standard MHD, $\nabla \cdot \mathbf{B} = 0$. However, for MHD numerical simulations, where numerical generation of $\nabla \cdot \mathbf{B} \neq 0$ (e.g. Powell et al. 1999), it is useful to know the form of the equations for $\nabla \cdot \mathbf{B} \neq 0$.

Note that the total energy conservation Eq. (2.12) does not depend on whether $\nabla \cdot \mathbf{B} = 0$ or $\nabla \cdot \mathbf{B} \neq 0$. Morrison and Greene (1982), (e.g. Chandre et al. 2012), used the above form of the MHD equations with $\nabla \cdot \mathbf{B} \neq 0$, in their non-canonical Poisson bracket for MHD. They also study the case $\nabla \cdot \mathbf{B} = 0$.

2.3   Faraday's Equation and Flux Conservation

The relationship between Faraday's equation (2.4) and magnetic flux conservation for moving media is described by Panofsky and Phillips (1964) (ch. 9, p. 160 et seq., Parker (1979), p. 34, Chapter 4). Parker uses $\nabla \cdot \mathbf{B} = 0$ in his derivation, but Panofsky and Phillips allow for the possibility that $\nabla \cdot \mathbf{B} \neq 0$. The basic argument that $d\Phi/dt = 0$ where $\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$ for a surface S moving with the flow, is given below. Consider the volume V carved out by the fluid in the time interval $(t, t + \Delta t)$ in which $S(t) = S_1$ and $S(t + \Delta t) = S_2$ at time $t + \Delta t$. An area element on the side of the tube bounded by S_1 and S_2, has a surface element $d\boldsymbol{\ell} \times \mathbf{u} dt$ pointed out of the volume, where $d\boldsymbol{\ell}$ is an element of the curve C bounding $S(t)$ (see Fig. 2.1). At time t, Gauss's theorem gives:

$$\int_V (\nabla \cdot \mathbf{B}) \, d^3 x = \int \mathbf{B}(t) \cdot d\mathbf{S}_2 - \int \mathbf{B}(t) \cdot d\mathbf{S}_1 + \int \mathbf{B}(t) \cdot d\boldsymbol{\ell} \times \mathbf{u} \Delta t. \qquad (2.20)$$

Here we use the convention that dS_2 is pointed out of the volume, but dS_1 is pointed into the volume.

The change in magnetic flux through S in $(t, t + \Delta t)$ is:

$$\frac{\Delta}{\Delta t} \left(\int \mathbf{B} \cdot d\mathbf{S} \right) = \frac{1}{\Delta t} \left(\int \mathbf{B}(t + \Delta t) \cdot d\mathbf{S}_2 - \int \mathbf{B}(t) \cdot d\mathbf{S}_1 \right). \qquad (2.21)$$

Using the Taylor series expansion:

$$\mathbf{B}(t + \Delta t) = \mathbf{B}(t) + \frac{\partial \mathbf{B}}{\partial t} \Delta t + O (\Delta t)^2 + \dots, \qquad (2.22)$$

Fig. 2.1 Magnetic surface S
advected with the flow,
illustrating magnetic flux
conservation in MHD. The
flux surface $S(t)$ at time t is
advected to $S(t + \Delta t)$ at time
$t + \Delta t$. Input flux at time t
equals output flux at time
$t + \Delta t$. $C(t)$ is the contour
bounding $S(t)$. Flux out the
side of the tube accounted for
in the flux conservation
equation
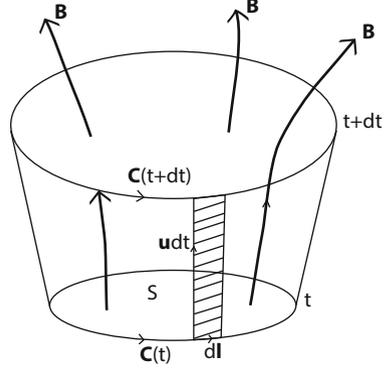
and using (2.22) in (2.21) gives, in the limit of small Δt, the equation:

$$\frac{d\Phi}{dt} = \frac{d}{dt}\left(\int \mathbf{B} \cdot d\mathbf{S}\right) = \int \frac{\partial \mathbf{B}(t)}{\partial t} \cdot d\mathbf{S} + \left(\int \mathbf{B} \cdot d\mathbf{S}_2 - \int \mathbf{B} \cdot d\mathbf{S}_1\right)/\Delta t, \quad (2.23)$$

where Φ is the magnetic flux through the surface S at time t. Using (2.20) in (2.23) to eliminate the surface integrals over $d\mathbf{S}_1$ and $d\mathbf{S}_2$, we obtain:

$$\frac{d\Phi}{dt} = \int \frac{\partial \mathbf{B}(t)}{\partial t} \cdot d\mathbf{S} + \int_S \nabla \cdot \mathbf{B} \frac{d^3 x}{\Delta t} - \int \mathbf{B}(t) \cdot (d\boldsymbol{\ell} \times \mathbf{u}). \quad (2.24)$$

Noting that $d^3 x / \Delta t = \mathbf{u} \cdot d\mathbf{S}$ in (2.24), we obtain:

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \int_S (\nabla \cdot \mathbf{B})\mathbf{u} \cdot d\mathbf{S} + \int_C (\mathbf{B} \times \mathbf{u}) \cdot d\boldsymbol{\ell}$$

$$= \int_S \left[\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}(\nabla \cdot \mathbf{B}) - \nabla \times (\mathbf{u} \times \mathbf{B})\right] \cdot d\mathbf{S}, \quad (2.25)$$

where Stokes theorem was used in the last step. Faraday's equation (2.4) follows by setting $d\Phi/dt = 0$ in (2.25). Thus, Faraday's equation implies the conservation of magnetic flux Φ moving with the flow. By taking the divergence of Faraday's equation (2.4) yields the continuity equation:

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) + \nabla \cdot [\mathbf{u}(\nabla \cdot \mathbf{B})] = 0, \quad (2.26)$$

which shows that numerically generated $\nabla \cdot \mathbf{B} \neq 0$ is advected with the flow out of the computational domain in a well designed code. The $\nabla \cdot \mathbf{B} \neq 0$ mode is known as the divergence mode in numerical MHD.

2.4 Field Line or Vortex Line Preservation

Definition A fluid motion is *line preserving* if two fluid particles initially on the same line remain on the same line at a later time throughout their motion

This definition is less general than a flux preserving motion studied in the previous section. Line preserving motions have been studied by many authors (e.g. Zorawski 1900; Truesdell 1954; Prim and Truesdell 1950; Truesdell and Toupin 1960; Newcomb 1958; Stern 1966; Parker 1979). Here we follow the development of Stern (1966) and Prim and Truesdell (1950). The motion of vortex lines and magnetic field lines has been studied by Kuznetsov and Ruban (1998, 2000), Kuznetsov et al. (2004), and Kuznetsov (2006), by using a combined Eulerian and Lagrangian approach involving non-canonical Poisson brackets for fluids and plasmas.

Consider the equation for position along the field line of the form: $\mathbf{x} = \mathbf{x}(\theta, t)$ where θ is the affine parameter, or distance along the field line from some fiducial point, describing the field quantity $\mathbf{Q}$ (in MHD we take $\mathbf{Q} = \mathbf{B}$, but in ideal fluid mechanics $\mathbf{Q} = \mathbf{\Omega} = \nabla \times \mathbf{u}$ is the fluid vorticity in the case of vortex lines). The parameter θ is assumed to be advected with the flow. The condition that $\mathbf{Q}$ and the tangent vector $\mathbf{x}_\theta$ are parallel implies:

$$\mathbf{x}_\theta \times \mathbf{Q} = 0. \tag{2.27}$$

The condition (2.27) implies that two particles a distance $\mathbf{x}_\theta \delta\theta$ apart on the same field line ($|\delta\theta| \ll 1$), forms a line segment parallel to $\mathbf{Q}$. The line preserving property asserts that the particles lie on the same field line at a later time t. The latter condition is equivalent to the requirement that:

$$\frac{d}{dt} (\mathbf{x}_\theta \times \mathbf{Q}) = 0. \tag{2.28}$$

The condition (2.27) implies that $\mathbf{x}_\theta$ and $\mathbf{Q}$ are parallel, and hence:

$$\mathbf{x}_\theta = \lambda \mathbf{Q}, \tag{2.29}$$

where λ is some scalar parameter.

From (2.28) and (2.29) we require:

$$\frac{d}{dt} (\mathbf{x}_\theta \times \mathbf{Q}) = \left(\frac{d\mathbf{x}_\theta}{dt} \right) \times \mathbf{Q} + \mathbf{x}_\theta \times \frac{d\mathbf{Q}}{dt} = \frac{\partial \mathbf{u}}{\partial \theta} \times \mathbf{Q} + \mathbf{x}_\theta \times \frac{d\mathbf{Q}}{dt}$$

$$= (\mathbf{x}_\theta \cdot \nabla \mathbf{u}) \times \mathbf{Q} + \mathbf{x}_\theta \times \frac{d\mathbf{Q}}{dt} = \lambda \mathbf{Q} \times \left(\frac{d\mathbf{Q}}{dt} - \mathbf{Q} \cdot \nabla \mathbf{u} \right) = 0, \tag{2.30}$$

where we used the result $\mathbf{u} = d\mathbf{x}/dt$ is the fluid velocity. The result (2.30) can also be written in the form:

$$\frac{d}{dt}(\mathbf{x}_\theta \times \mathbf{Q}) = \lambda \mathbf{Q} \times \left(\frac{\partial \mathbf{Q}}{\partial t} + [\mathbf{u}, \mathbf{Q}]\right) = 0, \qquad (2.31)$$

where

$$[\mathbf{u}, \mathbf{Q}]^i = (\mathbf{u} \cdot \nabla \mathbf{Q} - \mathbf{Q} \cdot \nabla \mathbf{u})^i, \qquad (2.32)$$

is the ith component of the commutator of the vector fields $\mathbf{u} \equiv \mathbf{u} \cdot \nabla$ and $\mathbf{Q} \equiv \mathbf{Q} \cdot \nabla$. Yet a further form of (2.30) is obtained by writing:

$$\frac{d}{dt}(\mathbf{x}_\theta \times \mathbf{Q}) = \lambda \mathbf{Q} \times \left(\frac{d\mathbf{Q}}{dt} + \mathbf{Q}\nabla \cdot \mathbf{u} - \mathbf{Q} \cdot \nabla \mathbf{u}\right)$$

$$= \lambda \mathbf{Q} \times \left[\frac{\partial \mathbf{Q}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{Q}) + \mathbf{u}(\nabla \cdot \mathbf{Q})\right] = 0. \qquad (2.33)$$

In (2.33) it is useful to note that:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_\mathbf{u}\right)(\mathbf{Q} \cdot d\mathbf{S}) = \left[\frac{\partial \mathbf{Q}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{Q}) + \mathbf{u}(\nabla \cdot \mathbf{Q})\right] \cdot d\mathbf{S} = 0, \qquad (2.34)$$

is equivalent to the condition that the flux $\mathbf{Q} \cdot d\mathbf{S}$ is Lie dragged with the flow, where $\mathcal{L}_\mathbf{u} = \mathbf{u} \cdot \nabla$ is the Lie derivative operator following the flow. For the case $\mathbf{Q} = \mathbf{B}$ (2.34) is equivalent to Faraday's equation (2.4). Similarly, (2.31) involves the formula:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_\mathbf{u}\right)(\mathbf{Q} \cdot \nabla) = \left(\frac{\partial \mathbf{Q}}{\partial t} + [\mathbf{u}, \mathbf{Q}]\right) \cdot \nabla, \qquad (2.35)$$

corresponding to Lie dragging the vector field $\mathbf{Q} \cdot \nabla$ with the flow.

To sum up, the field line preservation condition (2.30) or (2.31) requires that the component of the Lie dragged vector field $\mathbf{Q} \cdot \nabla$ perpendicular to the field line is conserved. Note that if the motion is flux preserving, so that (2.34) is satisfied (i.e. Faraday's equation in MHD is satisfied), then the motion of the fluid is field line preserving. However, the condition that the field line is preserved does not imply Faraday's equation is satisfied.

Figure 2.2 shows a simple example of a field line preserving flow in a fast magnetohydrodynamic shock. In the normal shock incidence frame, the shock is at rest, and the upstream fluid velocity $\mathbf{u}_1 = u_1\mathbf{e}_x$ is incident normal to the shock, which is located in the $x = 0$ plane. The magnetic field and fluid velocity are restricted to the xz plane, and the motional electric field $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ is directed down the y axis. The fluid is compressed and slows down in passing from the upstream to downstream region of the shock. The upstream magnetic field $\mathbf{B}_1$ lies in the xz-plane, and makes an angle ψ_1 with the x-axis. The transverse component

Fig. 2.2 Field line preserving flow upstream of a fast MHD shock. The figure depicts two plasma particles P_1 and P_2 on the same field line at time $t = t_0$, which move to the positions P_1' and P_2' at time $t = t_0 + dt$ respectively. For field line preserving flows, P_1' and P_2' also lie on the same field line at time $t_0 + dt$. P_1 and P_2 have been advected a distance $\mathbf{u}_1 dt$ toward the shock in the time interval $(t_0, t_0 + dt)$

of the field B_z increases across the shock, but the normal component of the field is conserved (i.e. $B_{x1} = B_{x2}$ where the subscripts 1 and 2 denote the regions upstream and downstream of the shock). The figure depicts two points P_1 and P_2 located on the same field line, upstream of the shock at time $t = t_0$ say. At a later time $t = t_0 + dt$, the field lines are advected further downstream toward the shock, to the positions P_1' and P_2' where $P_2 P_2' = \mathbf{u}_1 dt$ and $P_1 P_1' = \mathbf{u}_1 dt$. The perpendicular separation $P_1 Q_1$ of the position of the field line at $t_0 + dt$ is given by $P_1 Q_1 = (\mathbf{u} - \mathbf{u} \cdot \mathbf{e}_B \mathbf{e}_B) dt = \mathbf{V}_E dt$ where $\mathbf{V}_E = \mathbf{E} \times \mathbf{B}/|B|^2$ is the electric field drift velocity (see e.g. Webb et al. (1983) for an application of these ideas to cosmic ray acceleration at MHD shocks due to the drifts in the electric field at the shock). This example, illustrates the concept of fluid preserved field lines. The points P_1 and Q_1 on the same field line at time $t = t_0$, are advected to the points P_1' and Q_1', which are on the same advected field line at time $t_0 + dt$. Magnetic field line preservation is sometimes invoked in studies of magnetic reconnection in magnetospheric physics.

2.5   Incompressible Fluid and MHD Limit

The equations of ideal gas dynamics and MHD with an equation of state $p = p(\rho, S)$ implies the equation:

$$\frac{dp}{dt} = \frac{\partial p}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial p}{\partial S} \frac{dS}{dt}. \tag{2.36}$$

For isentropic flow $dS/dt = 0$. In this case (2.36) reduces to the equation:

$$\frac{dp}{dt} - a^2 \frac{d\rho}{dt} \equiv \frac{dp}{dt} + a^2 \rho \nabla \cdot \mathbf{u} = 0, \tag{2.37}$$

where $a^2 = (\partial p/\partial \rho)_S$ is the square of the adiabatic gas sound speed. In the derivation of (2.37) we used the mass continuity equation in the form:

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{u} = 0. \tag{2.38}$$

For an incompressible gas (i.e. in the incompressible gas limit) $a^2 \to \infty$ (i.e. for a highly subsonic flow for which the Mach number $M_s^2 = u^2/a^2 \sim \epsilon$ where $\epsilon << 1$) (2.37) requires $\nabla \cdot \mathbf{u} \to 0$ as $a^2 \to \infty$. Thus, in the incompressible flow limit balance of terms in (2.37) suggests:

$$\frac{dp}{dt} \sim O(1) \quad \text{and} \quad \nabla \cdot \mathbf{u} \to 0. \tag{2.39}$$

Note that $a^2 \rho \sim \rho u^2/\epsilon \sim O(1/\epsilon)$ and $\nabla \cdot \mathbf{u} \sim O(\epsilon)$, so that $dp/dt \sim O(1)$ in this limit (i.e. it is not necessarily true that $dp/dt \to 0$ in this limit). From the mass continuity equation (2.38) $d\rho/dt \to 0$ in this limit. Thus, for the incompressible limit for the gas dynamic equations ($M_s^2 \to 0$) leads to the basic equations:

$$\nabla \cdot \mathbf{u} = 0 \tag{2.40}$$

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p, \tag{2.41}$$

which are respectively, the mass continuity equation and the momentum equation for the fluid. In many applications it is assumed that $\rho = \rho_0 = constant$ for the density ρ. However, in the Boussinesq approximation used in describing gravity waves $\rho = \rho_0 + \delta\rho$ is not set equal to a constant, since variations of ρ in a gravitational field can give rise to buoyancy oscillations of the fluid (e.g. Whitham 1974). In the Boussinesq approximation ρ is set equal to ρ_0 in all equations, except in the momentum equation, where ρ is allowed to vary (in that case the gravitational force on the fluid element $-\rho\nabla\Phi$ should be included in the right hand side of the momentum equation (2.41)).

In vortex dynamics, one takes the curl of the momentum equation in (2.41) to get the vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = 0, \tag{2.42}$$

which is solved in conjunction with the mass continuity equation (2.40) to obtain the solution for $\mathbf{u}$ (subject to boundary and initial data). The gas pressure p is then obtained by taking the divergence of the momentum equation to obtain a Poisson equation for p (i.e. $\nabla^2 p = -\nabla \cdot (\rho_0 d\mathbf{u}/dt)$) which is solved for p.

2.5.1 Incompressible MHD

In the incompressible limit the MHD equations reduce to:

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho_0}\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0\rho_0},$$

$$\nabla\cdot\mathbf{u} = 0, \quad \nabla\cdot\mathbf{B} = 0,$$

$$\frac{\partial\mathbf{B}}{\partial t} - \nabla\times(\mathbf{u}\times\mathbf{B}) = 0. \tag{2.43}$$

The total pressure (magnetic plus gas pressure) is written as:

$$P = p + \frac{B^2}{2\mu_0}. \tag{2.44}$$

An application of incompressible MHD is in the description of the MHD topological soliton (e.g. Kamchatnov 1982; Semenov et al. 2002), which is described in Sect. 6.6.5. A more detailed description of incompressible MHD is given in Chandrasekhar (1961). Pshenitsin (2016) has analyzed the symmetry structure of the incompressible MHD model including the effects of viscosity. He uses the direct method of Anco and Bluman (1997) and Bluman et al. (2010) to obtain a large class of conservation laws of the equations.

We will not discuss this model in the present book. There are many applications of fluid dynamics and MHD in astrophysical and geophysical fluid dynamics. In particular in applications to meteorology, it is useful to write down the fluid equations in a frame rotating with the Earth. This leads to the addition of non-inertial force terms in the fluid momentum equation in the rotating frame (Pedlosky 1987), namely, the Coriolis force, the centrifugal force and the Darwin force (see e.g. Holm 2008a,b)

2.5.2 Reduced MHD

Kadomtsev and Pogutse (1974) and Strauss (1976) derived the so-called reduced MHD equations, in which the incompressible MHD system has a strong guide field $B = B_0\mathbf{e}_z$ along the z-axis or along the toroidal direction in a tokamak. The equations are derived using MHD perturbation theory, in which the perturbation parameter $\epsilon \sim B_\perp/B_z$ and in which spatial variations perpendicular to the guide field are much faster than parallel to the guide field. These equations are useful in describing plasma behaviour in tokamaks in fusion plasma physics. The equations have been to describe MHD turbulence in the solar wind, and in astrophysical plasmas (e.g. Zank and Matthaeus 1992; Oughton et al. 2017 and references therein). Morrison and

Hazeltine (1984) derived the Hamiltonian Poisson bracket form of the equations. The incompressible limit implies that magnetoacoustic waves do not occur at lowest order in these equations. We do not investigate the reduced MHD model in the present book.

2.6 Solar and Heliospheric Physics Applications

The main applications to solar and heliospheric plasma physics are centered around the use of magnetic helicity, to describe magnetic field structures. These applications are given mainly in Sect. 6.6 of the book, which includes (a) The magnetic helicity of the Parker, Archimedean spiral magnetic field, including a warped heliospheric current sheet across which the magnetic field polarity reverses (e.g. Bieber et al. 1987; Webb et al. 2010a); (b) the magnetic helicity of toroidal and shear Alfvén waves and more complicated versions of fully nonlinear Alfvén simple waves, in which the magnetic field **B** hodograph (i.e. (B_x, B_y, B_z) plot) lies on the sphere $B = const.$ (spacecraft data show $B \approx const.$, e.g. Bruno et al. 2001; Matteini et al. 2015; Gosling et al. 2009; Webb et al. 2010b). (c) MHD topological solitons derived by Kamchatnov (1982) which were subsequently investigated by Sagdeev et al. (1986), Semenov et al. (2002), and Thompson et al. (2014). These steady Alfvénic structures travel at the Alfvén velocity (i.e. $\mathbf{u} = \pm \mathbf{V}_A$). The total $p + B^2/(2\mu_0) = P$ is constant throughout the wave. However, unlike simple Alfvén waves, the magnetic pressure $B^2/(2\mu_0)$, is not constant. These hybrid MHD structures incorporate some of the features of both Alfvén waves and pressure balance structures; (d) linear magneto-acoustic N-waves arising from delta function initial data for the gas pressure ($p = A\delta(\mathbf{x})$ at time $t = 0$) from Webb et al. (1993) are described in Appendix D.

There are many applications of MHD in solar physics, and solar-heliospheric physics. An example of the type of phenomena of interest in heliospheric physics is the evolution of the writhe in unstable magnetic flux ropes and erupting filaments due to the kink instability (e.g. Torok et al. 2014). Figure 2.3, from Torok et al. (2014), shows examples of erupting and writhing solar magnetic filaments observed in the EUV (extreme ultraviolet) observations by the SOHO spacecraft (27 May, 2002) observed in 195 Å by the TRACE satellite (July, 2000) observed in 171 Å. Magnetic helicity can be decomposed into twist and writhe components via the formula *Link = Twist + Writhe* (e.g. Berger and Prior 2006) and the question arises how the writhe (out of plane distortion of the magnetic field) evolves in kink unstable flux ropes in CMEs. The writhe of a flux tube is sometimes referred to as the self-linkage of the flux tube with itself. Taylor's hypothesis that the magnetic helicity decays much slower than the magnetic energy density in a high conductivity, dissipative plasma (Taylor 1974, 1986) was developed to explain the magnetic field and current behavior of the reversed field pinch in toroidal fusion devices (e.g. the tokamak). Taylor's hypothesis also has important applications in solar physics in describing turbulent reconnection.
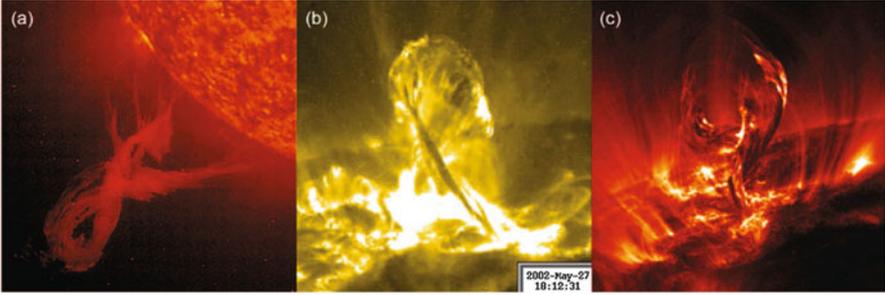
Fig. 2.3 Erupting and writhing solar filaments observed in extreme ultraviolet (EUV) wavelengths. (**a**) A full eruption (evolving into a CME on 18 January 2000, observed in 304 A° by the EIT telescope onboard the SOHO spacecraft. (**b**) A confined eruption (trapped in the low corona) on 27 May 2002, observed in 195 A° by the TRACE satellite. (**c**) An eruption, which most likely remained confined, on 19 July 2000, observed in 171 A° by TRACE (from Torok et al. 2014)

The main thrust of our analysis is to present a theoretical framework for fluid and MHD conservation laws (e.g. the magnetic helicity and cross helicity conservation laws). However, the main impetus for many space physicists is in the explanation of Earth based and satellite observations of solar, solar wind and magnetospheric phenomena. Kinetic plasma physics and numerical simulations are in many instances required to explain physical phenomena, which lie beyond the MHD fluid description.

Chapter 3
Helicity in Fluids and MHD

In this chapter we provide an overview of helicity and vorticity conservation laws in ideal fluid dynamics and MHD. For ideal barotropic fluids, in fluid mechanics, we derive the helicity conservation law for the helicity density $h_f = \mathbf{u} \cdot \boldsymbol{\omega}$, where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the fluid vorticity. The integral $H_f = \int_{V_m} h_f \, d^3x$ over a volume V_m moving with the fluid, is the fluid helicity. It is important in the description of the linkage of the vorticity streamlines (e.g. Moffatt 1969, Arnold and Khesin 1998). In MHD, the integral $H_M = \int_{V_m} \mathbf{A} \cdot \mathbf{B} \, d^3x$ is the magnetic helicity, where $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic induction and $\mathbf{A}$ is the magnetic vector potential. It is referred to as the Chern Simons term in field theory (the Chern Simons term in Yang-Mills theory has a totally different form). It describes the linkage and self linkage of the magnetic field lines (Woltjer 1958; Berger and Field 1984). The cross helicity $H_C = \int_{V_m} \mathbf{u} \cdot \mathbf{B} \, d^3x$ describes the linkage of the magnetic field flux tubes and the vorticity flux tubes. For the case of a barotropic gas with $p = p(\rho)$, H_C is conserved following the flow, i.e. $dH_C/dt = 0$. For non-barotopic flows, a modified form of the cross helicity, H_{CNB} is conserved following the flow. We derive topological invariants (topological charges) by determining invariants which are Lie dragged with the flow in Chap. 6 (e.g. Moiseev et al. 1982; Tur and Yanovsky 1993; Webb et al. 2014a).

The Aharonov-Bohm interpretation of: (a) magnetic helicity H_M (b) cross helicity H_C for barotropic flows and (c) non-barotropic cross-helicity H_{CNB} for non-barotropic flows was developed by Yahalom (2013, 2016a, 2017a,b) (see also Webb and Anco 2017). An account of these developments is given in Appendix E.

3.1 Helicity in Fluid Dynamics

For a barotropic, ideal fluid, in which $p = p(\rho)$, is independent of the entropy S, the helicity density

$$h_f = \mathbf{u} \cdot \boldsymbol{\omega} \quad \text{where} \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}, \tag{3.1}$$

satisfies the conservation law:

$$\frac{\partial h_f}{\partial t} + \nabla \cdot \left[\mathbf{u} h_f + \left(h + \Phi - \frac{1}{2} |\mathbf{u}|^2 \right) \boldsymbol{\omega} \right] = 0. \tag{3.2}$$

The net helicity for a fluid volume V_m moving with the fluid, in which there is no vorticity $\omega_n = \boldsymbol{\omega} \cdot \mathbf{n}$ normal to the boundary ∂V_m is conserved (e.g. Moffatt 1969). It satisfies the conservation law:

$$\frac{dH_f}{dt} = 0 \quad \text{where} \quad H_f = \int_{V_m} \mathbf{u} \cdot \nabla \times \mathbf{u} \, d^3 x. \tag{3.3}$$

Here $d/dt = \partial t + \mathbf{u} \cdot \nabla$ is the time derivative following the flow. The total helicity integral describes the linkage and knotting of the vorticity streamlines and is a key quantity in topological fluid dynamics (Moffatt 1969; Arnold and Khesin 1998).

To derive (3.2), note that for a ideal gas, the momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p - \nabla \Phi, \tag{3.4}$$

may be re-written in the form:

$$\frac{\partial \mathbf{u}}{\partial t} - \mathbf{u} \times \boldsymbol{\omega} = T \nabla S - \nabla \left(h + \Phi + \frac{1}{2} |\mathbf{u}|^2 \right). \tag{3.5}$$

For a barotropic gas, there is no $T \nabla S$ term in (3.5). To obtain (3.5) note that the first law of thermodynamics may be written in the form:

$$-\frac{1}{\rho} \nabla p = T \nabla S - \nabla h, \tag{3.6}$$

where

$$h = \varepsilon_\rho, \quad p = \rho \varepsilon_\rho - \varepsilon, \quad \rho T = \varepsilon_S \tag{3.7}$$

defines the enthalpy h, pressure p and temperature T in terms of the internal energy density $\varepsilon(\rho, S)$ per unit volume. For a barotropic gas set $T \nabla S = 0$ in (3.6). Using

the first law of thermodynamics (3.6) and the identity:

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\mathbf{u} \times \boldsymbol{\omega} + \frac{1}{2}\nabla|\mathbf{u}|^2, \tag{3.8}$$

in the momentum equation (3.4) gives the equivalent momentum equation (3.5).

The curl of the momentum equation (3.5) gives the vorticity equation:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = \nabla T \times \nabla S. \tag{3.9}$$

The scalar product of $\boldsymbol{\omega}$ with the momentum equation (3.5) plus the scalar product of $\mathbf{u}$ with the vorticity equation (3.9) gives the equation

$$\frac{\partial(\mathbf{u} \cdot \boldsymbol{\omega})}{\partial t} + \nabla \cdot \left[(\mathbf{u} \cdot \boldsymbol{\omega})\mathbf{u} + \left(h + \Phi - \frac{1}{2}|\mathbf{u}|^2\right)\boldsymbol{\omega}\right] = \boldsymbol{\omega} \cdot (T\nabla S) + \mathbf{u} \cdot \nabla T \times \nabla S. \tag{3.10}$$

For a barotropic fluid $p = p(\rho)$ and $\nabla S = 0$, and (3.10) reduces to the helicity conservation law (3.2).

To derive the integral fluid helicity conservation law (3.3), use the mass conservation law $\rho_t + \nabla \cdot (\rho\mathbf{u}) = 0$ in the form

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \mathbf{u}, \tag{3.11}$$

in (3.2) to obtain:

$$\frac{d}{dt}\left(\frac{\mathbf{u} \cdot \boldsymbol{\omega}}{\rho}\right) = -\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\left(h + \Phi - \frac{1}{2}|\mathbf{u}|^2\right). \tag{3.12}$$

The total helicity H_f in (3.3) can be expressed in the form:

$$H_f = \int_{V_m}\left(\frac{\boldsymbol{\omega} \cdot \mathbf{u}}{\rho}\right)\rho\,d^3x. \tag{3.13}$$

Noting that $d/dt(\rho d^3x) = 0$ (mass conservation equation), and using (3.12) results in the equation:

$$\begin{aligned}
\frac{dH_f}{dt} &= \int_{V_m}\left\{\frac{d}{dt}\left(\frac{\mathbf{u} \cdot \boldsymbol{\omega}}{\rho}\right)\rho d^3x + \left(\frac{\mathbf{u} \cdot \boldsymbol{\omega}}{\rho}\right)\frac{d}{dt}\left(\rho d^3x\right)\right\} \\
&= \int_{V_m}\left[-\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\left(h + \Phi - \frac{1}{2}|\mathbf{u}|^2\right)\right]\rho d^3x = -\int_{V_m}\boldsymbol{\omega} \cdot \nabla\left(h + \Phi - \frac{1}{2}|\mathbf{u}|^2\right)d^3x \\
&= -\int_{V_m}\nabla \cdot \left[\boldsymbol{\omega}\left(h + \Phi - \frac{1}{2}|\mathbf{u}|^2\right)\right]d^3x \\
&= -\int_{\partial V_m}(\boldsymbol{\omega} \cdot \mathbf{n})\left(h + \Phi - \frac{1}{2}|\mathbf{u}|^2\right)dS, \tag{3.14}
\end{aligned}$$

where Gauss's theorem was used to convert a volume integral to a surface integral over ∂V_m with outward normal $\mathbf{n}$. The assumption $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on ∂V_m, implies $dH_f/dt = 0$, which proves (3.3) (see also Moffatt 1969).

Kelvin's theorem implies that the circulation $\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{x}$ is conserved following the flow, for an ideal, barotropic fluid, where C is a closed path moving with the fluid, i.e. $d\Gamma/dt = 0$, where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative (this result also holds if there is a conservative, external gravitational field present). The circulation is not conserved if $\nabla S \neq 0$, in which case $d\Gamma/dt = \int_A (\nabla T \times \nabla S) \cdot \mathbf{n} dA$, where A is the area enclosing C with normal $\mathbf{n}$.

Theorem 3.1.1 (Ertel's Theorem) *Ertel's theorem for ideal fluids states that the potential vorticity $q = \boldsymbol{\omega} \cdot \nabla S/\rho$ is a scalar advected with the flow, i.e.,*

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega} \cdot \nabla S}{\rho}\right) = 0, \tag{3.15}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the fluid vorticity.

Proof The vorticity equation (3.9) may be written as:

$$\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega}\nabla \cdot \mathbf{u} - \boldsymbol{\omega} \cdot \nabla \mathbf{u} = \nabla T \times \nabla S. \tag{3.16}$$

Using the mass continuity equation (2.1), $\nabla \cdot \mathbf{u} = -(d\rho/dt)/\rho$ in (3.16) gives:

$$\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) - \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \mathbf{u} = \frac{\nabla T \times \nabla S}{\rho}. \tag{3.17}$$

The scalar product of (3.17) with ∇S gives the equation:

$$\nabla S \cdot \left[\frac{d}{dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) - \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \mathbf{u}\right] = 0. \tag{3.18}$$

From the entropy advection equation $dS/dt = 0$, we obtain:

$$\frac{d}{dt}\nabla S = \nabla\left(\frac{dS}{dt}\right) - (\nabla \mathbf{u})^T \cdot \nabla S \equiv -(\nabla \mathbf{u})^T \cdot \nabla S. \tag{3.19}$$

Taking the scalar product of (3.19) with $\boldsymbol{\omega}/\rho$ gives:

$$\frac{\boldsymbol{\omega}}{\rho} \cdot \left[\frac{d}{dt}\nabla S + (\nabla \mathbf{u})^T \cdot \nabla S\right] = 0. \tag{3.20}$$

Adding (3.18) and (3.20) gives the potential vorticity equation (3.15). $\square$

3.2 Helicity in MHD

Magnetic helicity in space and fusion plasmas is a key quantity describing the topology of magnetic fields (e.g. Moffatt 1969, 1978; Moffatt and Ricca 1992; Berger and Field 1984; Finn and Antonsen 1985, 1988; Rosner et al. 1989; Low 2006). The magnetic helicity H_M is given by the integral:

$$H_M = \int_V d^3 x \mathbf{A} \cdot \mathbf{B} \equiv \int_V \boldsymbol{\alpha} \wedge d\boldsymbol{\alpha}, \qquad (3.21)$$

where $\boldsymbol{\alpha} = \mathbf{A} \cdot d\mathbf{x}$ is the magnetic vector potential one-form, $d\boldsymbol{\alpha} = \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux 2-form; $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic induction, $\mathbf{A}$ is the magnetic vector potential and V is the volume in which the magnetic field of interest is located. The second form of the helicity in (3.21) is known as the Chern-Simons term, or the Hopf invariant. The magnetic helicity is an invariant of magnetohydrodynamics (MHD) (Elsässer 1956; Woltjer 1958; Moffatt 1969, 1978). In (3.21) it is assumed that $B_n = \mathbf{B} \cdot \mathbf{n}$ vanishes on the boundary ∂V of V.

For magnetic fields, in which $\mathbf{B} \cdot \mathbf{n} \neq 0$ on the boundary surface ∂V, a gauge independent definition of relative helicity (Finn and Antonsen 1985, 1988) is:

$$H_r = \int_V d^3 x \, (\mathbf{A}_1 + \mathbf{A}_2) \cdot (\mathbf{B}_1 - \mathbf{B}_2), \qquad (3.22)$$

(see also Berger and Field (1984) for an equivalent definition) where $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$ describes the magnetic field of interest and $\mathbf{B}_2 = \nabla \times \mathbf{A}_2$ is a reference magnetic field with the same normal flux as $\mathbf{B}_1$ (in many cases $\mathbf{B}_2$ a potential magnetic field, i.e. $\nabla \times \mathbf{B}_2 = 0$). Relative helicity is used to model solar magnetic structures (Longcope and Malanushenko 2008; Low 2006). Bieber et al. (1987) and Webb et al. (2010a) studied the relative helicity of the Parker interplanetary spiral magnetic field. Berger and Ruzmaikin (2000) determined the injection of magnetic helicity into the solar wind from the photospheric base using normal magnetic field observations and taking into account the differential rotation of the Sun.

3.2.1 Magnetic Field Line Flow

Cary and Littlejohn (1983) describe a variational principle for magnetic field line flow, using non-canonical Hamiltonian mechanics. This description of magnetic field lines has been used by Yeates and Hornig (2013) and by Prior and Yeates (2014) to characterize the magnetic field lines (see also Berger (1988) who defined fieldline helicity). Berger (1991) studies third order braid invariants. In this section, we give an overview of the Cary and Littlejohn (1983) variational principle for magnetic field line flow. We indicate its connection to the magnetic helicity density $\mathbf{A} \cdot \mathbf{B}$.

Proposition 3.2.1 *The magnetic field line equations:*

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} \quad or \quad \mathbf{dx} \times \mathbf{B} = 0, \tag{3.23}$$

may be obtained by requiring that the action:

$$J = \int \mathbf{A}[(\mathbf{x}(\lambda)] \cdot \frac{d\mathbf{x}}{d\lambda} d\lambda \equiv \int \mathbf{A}[(\mathbf{x})] \cdot d\mathbf{x}, \tag{3.24}$$

is stationary, where $\mathbf{x} = \mathbf{x}(\lambda)$ *describes the field line.*

Proof First note that the Lagrangian for the action (3.24) is:

$$L = \mathbf{A}[\mathbf{x}(\lambda)]^s \frac{dx^s}{d\lambda}. \tag{3.25}$$

The condition for J to be stationary is given by the variational equations:

$$\frac{\delta J}{\delta x^i} = \frac{\partial L}{\partial x^i} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial x^i_\lambda}\right) = \frac{\partial A^s}{\partial x^i}\frac{dx^s}{d\lambda} - \frac{dA^i}{d\lambda} = \frac{dx^s}{d\lambda}\left(\frac{\partial A^s}{\partial x^i} - \frac{\partial A^i}{\partial x^s}\right) = 0. \tag{3.26}$$

Writing

$$\Omega_{si} = \frac{\partial A^s}{\partial x^i} - \frac{\partial A^i}{\partial x^s}, \quad B^i = (\nabla \times \mathbf{A})^i = \epsilon_{ijk}\frac{\partial A^k}{\partial x^j}, \tag{3.27}$$

we obtain:

$$B^p = \frac{1}{2}\epsilon_{psi}\Omega_{si}, \quad \Omega_{ab} = \epsilon_{abp}B^p, \tag{3.28}$$

(this is referred to as the hat map by Holm (2008a)). In effect, Ω_{ab} is the magnetic part of the Faraday tensor, which is dual to **B**.

Using (3.27) and (3.28), variational equation (3.26) becomes:

$$\frac{\delta J}{\delta x^i} = \frac{dx^s}{d\lambda}\Omega_{si} = \frac{dx^s}{d\lambda}\left(\epsilon_{sip}B^p\right) = \left(\mathbf{B} \times \frac{d\mathbf{x}}{d\lambda}\right)^i = 0. \tag{3.29}$$

This proves the proposition because (3.29) is equivalent to (3.23).   □

Cary and Littlejohn (1983) study both canonical and non-canonical Hamiltonian field line variational principles. Berger (1988) introduced the notion of field line magnetic helicity, which was used by Yeates and Hornig (2013) and by Prior and Yeates (2014) to describe magnetic braids. Following Yeates and Hornig (2013), consider braided magnetic fields confined to a cylinder $V = \{(r, \phi, z) : 0 \leq r \leq R, 0 \leq z \leq 1\}$, satisfying $B_z > 0$ everywhere in V, and impose boundary conditions $B = \mathbf{e}_z$ and $\mathbf{u} = 0$ on ∂V, where $\mathbf{u}$ is the fluid velocity. In

general, two magnetic field braids have the same topology if they have the same field line mapping from $z = 0$ to $z = 1$. The map $x^i(\mathbf{x}_0, z) = f^i(\mathbf{x}_0, z)$ represents a point on a field line passing through the point $\mathbf{x}_0 = (r_0, \phi_0, 0)$, where (r, ϕ, z) are cylindrical polar coordinates, and $\mathbf{z} = \mathbf{x}_0$ at $z = 0$. The field line equations can be written as:

$$\frac{d\mathbf{x}}{dz} = \mathbf{f}(\mathbf{x}_0, z) = \frac{\mathbf{B}\left[\mathbf{f}(\mathbf{x}_0, z)\right]}{B_z\left[\mathbf{f}(\mathbf{x}_0, z)\right]}. \tag{3.30}$$

The action integral J in (3.24) can be written as:

$$J = \int_0^1 \mathbf{A}\left[\mathbf{x}(z)\right] \cdot \frac{d\mathbf{x}}{dz} \, dz, \tag{3.31}$$

where the affine parameter $\lambda \to z$ in (3.31). Using (3.30) in (3.31) we obtain:

$$J = \int_0^1 \frac{\mathbf{A} \cdot \mathbf{B}}{B_z} \, dz. \tag{3.32}$$

The integral J depends on the magnetic helicity density $h_m = \mathbf{A} \cdot \mathbf{B}$. The field line action principle (3.23)–(3.24) is related to symplectic field line maps, in fusion plasma devices (Morrison 2000) or for line tied magnetic equilibria in the solar atmosphere.

3.2.2 Magnetic Helicity Conservation Law

In ideal MHD, $h_m = \mathbf{A} \cdot \mathbf{B}$ satisfies the conservation law:

$$\frac{\partial h_m}{\partial t} + \nabla \cdot \left[\mathbf{u} h_m + \mathbf{B}(\phi_E - \mathbf{A} \cdot \mathbf{u})\right] = 0, \tag{3.33}$$

where

$$\mathbf{E} = -\nabla \phi_E - \frac{\partial \mathbf{A}}{\partial t} = -\mathbf{u} \times \mathbf{B}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{3.34}$$

To derive (3.33) we use Faraday's law $\mathbf{B}_t + \nabla \times \mathbf{E} = 0$ and the formula $\mathbf{B} = \nabla \times \mathbf{A}$ relating $\mathbf{B}$ to the magnetic vector potential $\mathbf{A}$ to obtain the equations:

$$\mathbf{B}_t + \nabla \times \mathbf{E} = 0, \tag{3.35}$$

$$\mathbf{A}_t + \mathbf{E} + \nabla \phi_E = 0, \tag{3.36}$$

where ϕ_E is the electric field potential. The curl of (3.36) gives Faraday's law (3.35). From (3.35)–(3.36) we obtain:

$$\mathbf{A} \cdot (\mathbf{B}_t + \nabla \times \mathbf{E}) + \mathbf{B} \cdot (\mathbf{A}_t + \mathbf{E} + \nabla \phi_E) = 0. \tag{3.37}$$

Using the identity:

$$\nabla \cdot (\mathbf{E} \times \mathbf{A}) = \mathbf{A} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}, \tag{3.38}$$

in (3.37) gives the equation:

$$\frac{\partial}{\partial t}(\mathbf{A} \cdot \mathbf{B}) + \nabla \cdot (\mathbf{E} \times \mathbf{A} + \phi_E \mathbf{B}) = -2\mathbf{E} \cdot \mathbf{B}, \tag{3.39}$$

Since the electric field $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ for ideal MHD, and setting $h_m = \mathbf{A} \cdot \mathbf{B}$ in (3.39) gives helicity conservation equation (3.33) for ideal MHD.

For non-ideal MHD with a finite conductivity σ, the simplest form of Ohm's law has the form:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{\mathbf{J}}{\sigma} \quad \text{or} \quad \mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{\mathbf{J}}{\sigma}, \tag{3.40}$$

(e.g. Boyd and Sanderson 1969, Eq. (3.61)). In this case the magnetic helicity equation (3.39) reduces to:

$$\frac{\partial}{\partial t}(\mathbf{A} \cdot \mathbf{B}) + \nabla \cdot \left[\mathbf{A} \cdot \mathbf{B}\mathbf{u} + (\phi_E - \mathbf{A} \cdot \mathbf{u}) \mathbf{B} + \frac{\mathbf{J} \times \mathbf{A}}{\sigma} \right] = -\frac{2\mathbf{J} \cdot \mathbf{B}}{\sigma}. \tag{3.41}$$

Thus, a finite conductivity σ results in dissipation of magnetic helicity. The dissipation term on the right hand side of (3.41) depends on $\mathbf{J} \cdot \mathbf{B} = \mathbf{B} \cdot \nabla \times \mathbf{B}/\mu_0$ which is the current helicity of the plasma (there is also a modification of the magnetic helicity flux in (3.41) of $\mathbf{J} \times \mathbf{A}/\sigma$ for a finite plasma conductivity). More complicated forms of the generalized Ohm's law including Hall current effects and plasma pressure and current effects (Boyd and Sanderson 1969), lie beyond the scope of the present analysis. In the limit as $\sigma \to \infty$ one recovers the ideal MHD helicity conservation equation (3.33). The dissipative helicity transport equation can be written in terms of the resistivity $\eta = 1/(\mu_0\sigma)$ (e.g. Berger and Field 1984).

Below we prove that the total magnetic helicity $H_M = \int_{V_m} \mathbf{A} \cdot \mathbf{B} \, d^3x$ moving with the flow is invariant, i.e. $dH_M/dt = 0$ provided $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary surface ∂V_m of the volume V_m. Using the continuity equation $(1/\rho)d\rho/dt = -\nabla \cdot \mathbf{u}$, the helicity conservation law may be written as:

$$\frac{d}{dt}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho}\right) = \frac{\mathbf{B}}{\rho}\nabla(\mathbf{A} \cdot \mathbf{u} - \phi_E). \tag{3.42}$$

The total helicity H_m is given by

$$H_M = \int_{V_m} \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho} \right) \rho \, d^3 x. \tag{3.43}$$

Using the continuity equation in the form: $d/dt(\rho d^3 x) = 0$ and taking the total Lagrangian time derivative of (3.43), we obtain:

$$\begin{aligned}
\frac{dH_M}{dt} &= \int_{V_m} \frac{d}{dt} \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho} \right) \rho \, d^3 x \\
&= \int_{V_m} \frac{\mathbf{B}}{\rho} \cdot \nabla (\mathbf{A} \cdot \mathbf{u} - \phi_E) \rho \, d^3 x = \int_{V_m} \mathbf{B} \cdot \nabla (\mathbf{A} \cdot \mathbf{u} - \phi_E) \, d^3 x \\
&= \int_{V_m} \nabla \cdot [\mathbf{B}(\mathbf{A} \cdot \mathbf{u} - \phi_E)] \, d^3 x \\
&= \int_{\partial V_m} \mathbf{B} \cdot \mathbf{n}(\mathbf{A} \cdot \mathbf{u} - \phi_E) \, d\mathbf{S}.
\end{aligned} \tag{3.44}$$

Assuming $\mathbf{B} \cdot \mathbf{n} = 0$ on ∂V_m then implies $dH_M/dt = 0$. Thus, the magnetic helicity (3.43) is conserved following the flow provided $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary ∂V_m of the moving volume V_m.

Consider the gauge for $\mathbf{A}$. By setting $\mathbf{B} = \nabla \times \mathbf{A}$, (3.34) may be written as:

$$\frac{d\mathbf{A}}{dt} = \nabla (\mathbf{A} \cdot \mathbf{u} - \phi_E) - (\nabla \mathbf{u})^T \cdot \mathbf{A}, \tag{3.45}$$

where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$. Introducing the gauge transformation: $\tilde{\mathbf{A}} = \mathbf{A} + \nabla \Lambda$ in (3.45) gives the evolution equation for $\tilde{A}$ as:

$$\frac{d\tilde{\mathbf{A}}}{dt} + (\nabla \mathbf{u})^T \cdot \tilde{\mathbf{A}} = \nabla \left(\frac{d\Lambda}{dt} + \mathbf{A} \cdot \mathbf{u} - \phi_E \right). \tag{3.46}$$

By choosing the gauge potential Λ such that:

$$\frac{d\Lambda}{dt} + \mathbf{A} \cdot \mathbf{u} - \phi_E = 0, \tag{3.47}$$

results in the formula:

$$\Lambda = \int^t (\phi_E - \mathbf{A} \cdot \mathbf{u}) \, dt', \tag{3.48}$$

for Λ, where the integration in (3.48) is with respect to the Lagrangian time variable t', in which the labels $\mathbf{x}_0$ are kept constant. Faraday's equation, for $\tilde{\mathbf{A}}$ from

(3.46) becomes:

$$\frac{d\tilde{\mathbf{A}}}{dt} + (\nabla\mathbf{u})^T \cdot \tilde{\mathbf{A}} = 0, \tag{3.49}$$

which can also be written in the form:

$$\frac{\partial\tilde{\mathbf{A}}}{\partial t} - \mathbf{u} \times (\nabla \times \tilde{\mathbf{A}}) + \nabla(\mathbf{u} \cdot \tilde{\mathbf{A}}) = 0, \tag{3.50}$$

This equation can be written as $d/dt(\tilde{\mathbf{A}} \cdot d\mathbf{x}) = 0$ which shows that $\tilde{\mathbf{A}} \cdot d\mathbf{x}$ is Lie dragged with the flow (see Chap. 5). The latter equation is equivalent to (3.34) for $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ in the form:

$$\mathbf{E} = -\nabla(\mathbf{u} \cdot \tilde{\mathbf{A}}) - \frac{\partial\tilde{\mathbf{A}}}{\partial t}, \tag{3.51}$$

Thus, in the new gauge is $\tilde{\phi}_E = \mathbf{u} \cdot \tilde{\mathbf{A}}$. The Cauchy solution of (3.49) for $\tilde{\mathbf{A}}$ is:

$$\tilde{A}_k = \tilde{A}_j^0 \frac{\partial x_0^j}{\partial x^k} \quad \text{where} \quad \frac{d\tilde{\mathbf{A}}_0}{dt} = 0 \tag{3.52}$$

(e.g. Parker 1979; Holm and Kupershmidt 1983a,b). Combining (3.49) with Faraday's equation for $\mathbf{B}$ gives the helicity transport equation:

$$\frac{\partial\tilde{h}}{\partial t} + \nabla \cdot (\tilde{h}\mathbf{u}) = 0, \tag{3.53}$$

where

$$\tilde{h} = \tilde{\mathbf{A}} \cdot \mathbf{B} = \frac{\tilde{\mathbf{A}}_0 \cdot \mathbf{B}_0}{J}, \quad B^i = \frac{x_{ij}}{J}B_0^j, \tag{3.54}$$

is the magnetic helicity density in this special gauge. Here $x_{ij} = \partial x^i/\partial x_0^j$ and $J = \det(x_{ij})$.

The gauge choice (3.48) appears to be the best choice of the gauge potential Λ since it fits in with the idea that $\tilde{\mathbf{A}} \cdot d\mathbf{x}$ is an invariant, Lie dragged one form, and gives the simplest continuity equation for the helicity conservation law (3.53).

3.2.3   Cross Helicity

The cross helicity in MHD (for $p = p(\rho)$) is defined as the integral:

$$H_C = C[u, B] = \int_{V_m} d^3x \, \mathbf{u} \cdot \mathbf{B}, \tag{3.55}$$

where it is assumed that $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary ∂V_m of the volume V_m. It is a Casimir of barotropic MHD ($p = p(\rho)$). C is a Casimir if $\{F, C\} = 0$, for any functional F where $\{., .\}$ is MHD Poisson brackets (Padhye and Morrison 1996a,b). It is referred to as a rugged invariant in MHD turbulence theory (Matthaeus and Goldstein 1982). For a barotropic gas, the cross helicity (3.55) is conserved following the flow:

$$dH_C/dt = 0 \qquad (3.56)$$

The cross helicity density conservation law (for $p = p(\rho)$) is:

$$\frac{\partial h_c}{\partial t} + \nabla \cdot \left[\mathbf{u} h_c + \mathbf{B} \left(h + \Phi - \frac{1}{2}|\mathbf{u}|^2 \right) \right] = 0 \quad \text{where} \quad h_c = \mathbf{u} \cdot \mathbf{B}, \qquad (3.57)$$

and $h = (p + \varepsilon)/\rho$ is the gas enthalpy. Equation (3.57) also holds if $p = p(\rho, S)$ and $\mathbf{B} \cdot \nabla S = 0$.

To derive the cross helicity conservation law (3.57) we use the Faraday and momentum equations:

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0, \quad \frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p + \frac{\mathbf{J} \times \mathbf{B}}{\rho} - \nabla \Phi, \qquad (3.58)$$

where $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ and $d\mathbf{u}/dt = (\partial_t + \mathbf{u} \cdot \nabla)\mathbf{u}$ and we assume $\nabla \cdot \mathbf{B} = 0$. Using the thermodynamic Eq. (2.11), the MHD momentum equation reduces to:

$$\mathbf{u}_t - \mathbf{u} \times \boldsymbol{\omega} + \nabla \left(h + \Phi + \frac{1}{2}|\mathbf{u}|^2 \right) - \frac{\mathbf{J} \times \mathbf{B}}{\rho} - T\nabla S = 0, \qquad (3.59)$$

where $\mathbf{u}_t = \partial \mathbf{u}/\partial t$. The scalar product of Faraday's equation with $\mathbf{u}$ plus the scalar product of the momentum equation (3.59) with $\mathbf{B}$ gives the equation:

$$\mathbf{u} \cdot (\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}))$$
$$+\mathbf{B} \cdot \left(\mathbf{u}_t - \mathbf{u} \times \boldsymbol{\omega} + \nabla \left(h + \Phi + \frac{1}{2}|\mathbf{u}|^2 \right) - \frac{\mathbf{J} \times \mathbf{B}}{\rho} - T\nabla S \right) = 0. \quad (3.60)$$

Using the identity:

$$\nabla \cdot (\mathbf{E} \times \mathbf{u}) = \mathbf{u} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{u} \quad \text{where} \quad \mathbf{E} = -\mathbf{u} \times \mathbf{B}, \qquad (3.61)$$

in (3.60) gives the cross helicity equation:

$$\frac{\partial}{\partial t}(\mathbf{u} \cdot \mathbf{B}) + \nabla \cdot \left[\mathbf{E} \times \mathbf{u} + \left(h + \Phi + (1/2)|\mathbf{u}|^2 \right) \mathbf{B} \right] = T\mathbf{B} \cdot \nabla S. \qquad (3.62)$$

If $\mathbf{B} \cdot \nabla S = 0$, Eq. (3.62) reduces to the cross-helicity conservation law (3.57). The cross helicity conservation law (3.56) follows by integrating (3.57) over a volume V_m moving with the flow, and by assuming $\mathbf{B} \cdot \mathbf{n} = 0$ on ∂V_m.

Magnetic helicity and cross helicity are widely recognized as important quantities in topological MHD. However, there are other invariants, such as the magnetized version of potential vorticity (e.g. Kats 2003), and other advected invariants (e.g. Tur and Yanovsky 1993), which are also important. We discuss these invariants in the following analysis.

3.3   Nonlocal Conservation Laws

We introduce a nonlocal potential $r(\mathbf{x}, t)$ satisfying the equation:

$$\frac{\partial r}{\partial t} + \mathbf{u} \cdot \nabla r = -T(\mathbf{x}, t), \tag{3.63}$$

where T is the temperature of the gas. This allows one to obtain nonlocal helicity and cross helicity conservation laws that generalize the fluid helicity law (3.2) and the cross helicity law (3.57) for the case of a non-barotropic equation of state for the gas. The variable r is related to the Clebsch potential $\beta = r\rho$ of Zakharov and Kuznetsov (1997) in their Clebsch potential variational principle for fluids and plasmas. Note that:

$$r = -\int_0^t T(\mathbf{x}(\mathbf{x}_0, t'), t') \, dt' + r_0(\mathbf{x}_0), \tag{3.64}$$

is the integral of the temperature from time $t = 0$ to time t in the fluid frame ($r_0(\mathbf{x}_0)$ is integration 'constant'), where $\mathbf{x}_0$ is a Lagrange label advected with the flow.

Proposition 3.3.1 *For a non-barotropic gas, the fluid helicity law (3.2) generalizes to the nonlocal conservation law:*

$$\frac{\partial}{\partial t} \left[\boldsymbol{\Omega} \cdot (\mathbf{u} + r\nabla S) \right] + \nabla \cdot \left\{ \mathbf{u} \left[\boldsymbol{\Omega} \cdot (\mathbf{u} + r\nabla S) \right] + \boldsymbol{\Omega} \left(h + \Phi - \frac{1}{2} |\mathbf{u}|^2 \right) \right\} = 0. \tag{3.65}$$

(Webb et al. 2014a,b), where

$$\boldsymbol{\Omega} = \boldsymbol{\omega} + \nabla r \times \nabla S, \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}, \tag{3.66}$$

and $r(\mathbf{x}, t)$ is the nonlocal potential (3.63).

Proposition 3.3.2 *For a nonbarotropic gas, the MHD cross-helicity law (3.57) generalizes to the nonlocal conservation law:*

$$\frac{\partial}{\partial t} \left[\mathbf{B} \cdot (\mathbf{u} + r\nabla S) \right] + \nabla \cdot \left\{ \mathbf{u} [\mathbf{B} \cdot (\mathbf{u} + r\nabla S)] + \mathbf{B} \left(h + \Phi - \frac{1}{2} |\mathbf{u}|^2 \right) \right\} = 0, \tag{3.67}$$

(Webb et al. 2014a,b) where $\nabla \cdot \mathbf{B} = 0$ (Gauss's law) and $r(\mathbf{x}, t)$ satisfies (3.63). The non-barotropic cross helicity:

$$H_{CNB} = \int_{V_m} \mathbf{B} \cdot (\mathbf{u} + r\nabla S) \, d^3x, \tag{3.68}$$

is conserved following the flow, i.e.

$$\frac{dH_{CNB}}{dt} = 0, \tag{3.69}$$

The generalized vorticity $\mathbf{\Omega}$ in (3.66) satisfies the analog of Faraday's equation for MHD:

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{\Omega}) = 0. \tag{3.70}$$

Note that $\nabla \cdot \mathbf{\Omega} = 0$, and that (3.67) has the same form as (3.65) in which $\mathbf{\Omega}$ is replaced by $\mathbf{B}$. For the isentropic case, $S = const.$, $\nabla S = 0$, and (3.65) reduces to the fluid helicity conservation law (3.2). Similarly, (3.67) reduces to (3.57) if $\nabla S = 0$.

3.3.1 Example: 1D Gas Dynamics

Webb (2015) describes a multi-symplectic formulation of Lagrangian, 1D gas dynamics (multi-symplectic here refers to a Hamiltonian form of an equation system, in which all the independent variables can be thought of as evolution variables). The equations were shown to admit a nonlocal conservation law involving the nonlocal variable $r(x, t)$ of (3.63). The Eulerian form of the equations of 1D gas dynamics are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{3.71}$$

$$u_t + uu_x = -\frac{1}{\rho}p_x, \tag{3.72}$$

$$S_t + uS_x = 0, \tag{3.73}$$

where $p = p(\rho, S)$. Using (2.11), the pressure gradient force may be written as:

$$-\frac{1}{\rho}p_x = TS_x - h_x, \tag{3.74}$$

where h and T are the enthalpy and temperature of the gas. Using the nonlocal variable r, (3.71)–(3.74) imply the nonlocal conservation law:

$$\frac{\partial}{\partial t}\left(u + rS_x\right) + \frac{\partial}{\partial x}\left(h + \frac{1}{2}u^2 + urS_x\right) = 0. \tag{3.75}$$

Of course, one could include an extra term $-\Phi_x$ to the momentum equation in (3.72) due the an external gravitational field. In that case $h \rightarrow h + \Phi$ in (3.75).

Remark This example shows that the nonlocal conservation law associated with the nonlocal variable $r(x,t)$ also applies in flows, in which there is no vorticity.

3.4 Potential Vorticity in MHD

Cheviakov (2014) derived new conservation laws for fluid systems involving vorticity and vorticity related equations (potential type systems) including MHD and Maxwell's equations. Webb and Mace (2015) derived a potential vorticity type equation for MHD by using a non-field aligned fluid relabeling symmetry of the equations, in conjunction with Noether's second theorem, as developed by Hydon and Mansfield (2011). The Webb and Mace (2015) conservation law is a special case of the conservation law obtained by Cheviakov (2014).

Cheviakov (2014) studied the system of equations:

$$\nabla \cdot \mathbf{N} = 0, \tag{3.76}$$

$$\frac{\partial \mathbf{N}}{\partial t} + \nabla \times \mathbf{M} = 0. \tag{3.77}$$

Equations (3.76) and (3.77) in electromagnetic theory are Gauss's law (3.76) and Faraday's magnetic induction equation, in which

$$\mathbf{N} \rightarrow \mathbf{B} \quad \text{and} \quad \mathbf{M} \rightarrow \mathbf{E}, \tag{3.78}$$

where $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields respectively.

Proposition 3.4.1 *The equation system (3.76)–(3.77) admits the conservation law:*

$$\frac{\partial}{\partial t}\left(\mathbf{N} \cdot \nabla F\right) + \nabla \cdot \left(\mathbf{M} \times \nabla F - \mathbf{N}F_t\right) = 0, \tag{3.79}$$

where $F(\mathbf{x}, t)$ is an arbitrary function of $\mathbf{x}$ and t.

Remark The conservation law (3.79) might be thought to be trivial, since there is no constraint on $F(\mathbf{x}, t)$, except that it is differentiable. However, if $F(\mathbf{x}, t)$ is a potential

or function, which is part of a larger equation system, then (3.79) can have physical significance.

Proof Using the identity:

$$\nabla \cdot (\mathbf{A} \times \mathbf{C}) = \nabla \times \mathbf{A} \cdot \mathbf{C} - \nabla \times \mathbf{C} \cdot \mathbf{A}, \tag{3.80}$$

it follows that:

$$
\begin{aligned}
\nabla \cdot (\mathbf{M} \times \nabla F) &= \nabla \times \mathbf{M} \cdot \nabla F - (\nabla \times \nabla F) \cdot \mathbf{M} \equiv \nabla \times \mathbf{M} \cdot \nabla F \\
&= -\frac{\partial \mathbf{N}}{\partial t} \cdot \nabla F = -\left\{ \frac{\partial}{\partial t} (\mathbf{N} \cdot \nabla F) - \mathbf{N} \cdot \nabla F_t \right\} \\
&= -\left\{ \frac{\partial}{\partial t} (\mathbf{N} \cdot \nabla F) - \nabla \cdot (\mathbf{N} \cdot F_t) \right\}.
\end{aligned}
\tag{3.81}
$$

Equation (3.81) is equivalent to (3.79). □

Remark Equations (3.76) and (3.77) are satisfied by writing $\mathbf{N} = \nabla \times \mathbf{A}$ and by uncurling (3.77) to obtain the equations:

$$\mathbf{N} = \nabla \times \mathbf{A}, \quad \mathbf{M} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \tag{3.82}$$

where $\mathbf{A}$ and ϕ are potentials describing $\mathbf{N}$ and $\mathbf{M}$. This is in fact, the standard approach used in classical electrodynamics (e.g. Jackson 1975), for expressing $\mathbf{B}$ and $\mathbf{E}$ in terms the magnetic vector potential $\mathbf{A}$ and electrostatic potential ϕ when the identifications (3.78) are used.

Proposition 3.4.2 *The MHD equations,including gravity (2.1)–(2.5), satisfy the MHD potential vorticity type conservation equation:*

$$\frac{\partial}{\partial t} (\boldsymbol{\omega} \cdot \nabla \psi) + \nabla \cdot \left[(\boldsymbol{\omega} \cdot \nabla \psi) \mathbf{u} - \left(T \nabla S + \frac{\mathbf{J} \times \mathbf{B}}{\rho} \right) \times \nabla \psi \right] = 0, \tag{3.83}$$

where $\psi(\mathbf{x}, t)$ is a scalar function advected with the background plasma flow, i.e.

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \mathbf{u} \cdot \nabla \psi = 0. \tag{3.84}$$

This conservation law was derived by Webb and Mace (2015) by other methods.

Proof First note that the MHD momentum equation (2.2) can be written in the form:

$$\mathbf{u}_t - \mathbf{u} \times \boldsymbol{\omega} = -\nabla \left(\frac{1}{2} u^2 + h + \Phi \right) + \mathbf{F}, \tag{3.85}$$

where

$$\mathbf{F} = T\nabla S + \frac{\mathbf{J} \times \mathbf{B}}{\rho}, \quad \mathbf{J} = \frac{\nabla \times \mathbf{B}}{\mu_0}, \quad \omega = \nabla \times \mathbf{u}, \tag{3.86}$$

in which $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ is the electric current density and ω is the fluid vorticity. Taking the curl of (3.85) gives the fluid vorticity equation in the form:

$$\frac{\partial \mathbf{N}}{\partial t} + \nabla \times \mathbf{M} = 0, \tag{3.87}$$

where

$$\mathbf{M} = -\mathbf{u} \times \omega - \mathbf{F}, \quad \mathbf{N} = \omega, \tag{3.88}$$

Since $\omega = \nabla \times \mathbf{u}$, then $\nabla \cdot \omega = 0$. Using (3.85)–(3.88) with $F \to \psi(\mathbf{x}, t)$, the conservation law (3.79) reduces to:

$$\frac{\partial}{\partial t}(\omega \cdot \nabla \psi) + \nabla \cdot \left[(\omega \cdot \nabla \psi)\mathbf{u} - (\psi_t + \mathbf{u} \cdot \nabla \psi)\,\omega - \left(T\nabla S + \frac{\mathbf{J} \times \mathbf{B}}{\rho} \right) \times \nabla \psi \right] = 0, \tag{3.89}$$

where ψ at this point in the proof, is an arbitrary function of $\mathbf{x}$ and t. If in addition, we require that $\psi(\mathbf{x}, t)$ is advected with the background plasma flow, and satisfies the scalar advection equation (3.84), then (3.89) reduces to (3.83).   □

Chapter 4
Noether's Theorems and the Direct Method

In this chapter we give a general discussion of Noether's theorems and the Calculus of Variations, for systems of differential equations governed by a variational principle. Noether's theorems are discussed by Gelfand and Fomin (1963), Ibragimov (1985), Marsden and Ratiu (1994), Holm (2008a,b), Bluman and Kumei (1989), Olver (1993), Anco and Bluman (1996, 1997, 2002a,b), and Bluman et al. (2010) and others. We use the analysis of Bluman and Kumei (1989) and Ibragimov (1985) as summarized by Webb et al. (2005b). Hydon and Mansfield (2011) give a clear presentation of Noether's second theorem. The main aim is to briefly present Noether's theorems and methods to derive conservation laws. Noether's theorems link the symmetries of the action and conservation laws, for systems of differential equations governed by a variational principle. Kara and Mahomed (2000, 2002) describe how to generate new conservation laws from known conservation laws by using Lie symmetries of the equations. Bluman et al. (2010) also describe these methods and nonlocal conservation laws which can arise from potential symmetries and higher order symmetries. They also discuss recursion operators for symmetries and maps between diiferent equation systems using symmetries.

We also discuss the so-called direct method to derive conservation laws for systems of differential equations that are not necessarily derivable from a variational principle, as developed by Anco and Bluman (1997, 2002a,b) and Bluman et al. (2010). In the direct method, integrating factors or multipliers of each of the equations in the system are sought, which result in a pure divergence expression, encapsulating the conservation law. This method exploits the mathematical properties of the Euler operator E_u for the dependent variables u which commutes with the total derivative operators $D_i = \partial/\partial x^i$ of the system. The Euler operator (or variational derivative operator) also plays a central role in Noether's theorems. However, the Euler operator is defined independent of a variational principle, and thus can be used to derive conservation laws for systems not described by a variational principle. Section 4.1 describes some basic results in the Calculus of variations, namely the derivation of the Euler Lagrange equations from a variational

principle. Section 4.2 describes the variational equations used to derive Noether's two theorems. We also list the properties of canonical or evolutionary symmetry operators used in the analysis. Noether's first theorem is stated in Sect. 4.2.1. Section 4.2.2 describes Noether's second theorem, and Sect. 4.3 describes the direct method of Anco and Bluman (1997, 2002a,b) for deriving conservation laws with a simple example.

4.1   Euler Lagrange Equations and the Calculus of Variations

Consider a system of differential equations in the dependent variables u^α ($1 \le \alpha \le m$) and independent variables x^i ($1 \le i \le n$) of the form:

$$R^s(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots) = 0, \quad 1 \le s \le m, \tag{4.1}$$

(the subscripts in the u^α_i and u^α_{ij} denote partial derivatives with respect to the independent variables x^i ($1 \le i \le n$)), which are the Euler-Lagrange equations of the action functional:

$$\mathscr{A}[u] = \int_R d\mathbf{x} L(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij}, \ldots). \tag{4.2}$$

At a critical point, the action is stationary, i.e.,

$$\delta\mathscr{A} = \lim_{\epsilon \to 0} \frac{\mathscr{A}[u + \epsilon v] - \mathscr{A}[u]}{\epsilon} = \lim_{\epsilon \to 0} \int_R d\mathbf{x}\, \frac{\delta L}{\epsilon} = 0. \tag{4.3}$$

Expanding δL as a power series in ϵ gives:

$$\delta L \equiv L[u + \epsilon v] - L[u] = \epsilon \left(v^\gamma E_\gamma(L) + D_i W^i[u, v] \right) + O(\epsilon^2). \tag{4.4}$$

Here, D_i denotes the total partial derivative with respect to x^i (Bluman and Kumei 1989). The critical point requirement $\delta\mathscr{A} = 0$ is satisfied if the u^α satisfy the Euler-Lagrange equations:

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - D_i\left(\frac{\partial L}{\partial u^\alpha_i}\right) + D_i D_j\left(\frac{\partial L}{\partial u^\alpha_{ij}}\right) - D_i D_j D_k\left(\frac{\partial L}{\partial u^\alpha_{ijk}}\right) + \ldots = 0. \tag{4.5}$$

An abbreviated notation is to use multi-index notation where $u^\alpha_{,J} = \partial u^\alpha / \partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_n}$, i.e. $J = j_1 j_2 j_3 \ldots j_n$ for some arbitrary given n. Using the multi-index notation, the Euler-Lagrange equations (4.5) can be written in the form:

$$E_\alpha(L) = (-D)_J\left(\frac{\partial L}{\partial u^\alpha_J}\right) = 0, \qquad \alpha = 1, 2, \ldots, q. \tag{4.6}$$

Here, $(-D)_J$ is defined as:

$$(-D)_J = (-1)^k D_J = (-D_{j_1})(-D_{j_2})\ldots(-D_{j_k}), \qquad (4.7)$$

(see e.g. Olver 1993, p. 245). The surface term $\mathbf{W} \cdot \mathbf{n}$ that arises from using Gauss's theorem is assumed to vanish on the boundary ∂R. Here $\mathbf{n}$ is the outward unit normal of the integration region R for $\mathscr{A}[u]$. The boundary vector $W^i[u, v]$ is given by

$$
\begin{aligned}
W^i[\mathbf{u}, \mathbf{v}] = {} & v^\gamma \left[\frac{\partial L}{\partial u_i^\gamma} - D_j \left(\frac{\partial L}{\partial u_{ij}^\gamma} \right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}^\gamma} \right) - \ldots \right] \\
& + v_j^\gamma \left[\frac{\partial L}{\partial u_{ij}^\gamma} - D_k \left(\frac{\partial L}{\partial u_{ijk}^\gamma} \right) + D_\ell D_k \left(\frac{\partial L}{\partial u_{ijk\ell}^\gamma} \right) - \ldots \right] \\
& + v_{jk}^\gamma \left[\frac{\partial L}{\partial u_{ijk}^\gamma} - D_s \left(\frac{\partial L}{\partial u_{ijks}^\gamma} \right) + \ldots \right] + \ldots . \qquad (4.8)
\end{aligned}
$$

The surface term $W[\mathbf{u}, \mathbf{v}]$ can be written more succinctly in the form:

$$W^i[\mathbf{u}, \mathbf{v}] = v^\gamma \frac{\delta L}{\delta u_i^\gamma} + v_j^\gamma \frac{\delta L}{\delta u_{ij}^\gamma} + v_{jk}^\gamma \frac{\delta L}{\delta u_{ijk}^\gamma} + \ldots, \qquad (4.9)$$

where

$$\frac{\delta L}{\delta \psi} = E_\psi(L) = \frac{\partial L}{\partial \psi} - D_i \left(\frac{\partial L}{\partial \psi_i} \right) + D_i D_j \left(\frac{\partial L}{\partial \psi_{ij}} \right) + \ldots, \qquad (4.10)$$

(e.g. Ibragimov (1994): note that this definition of $\delta L/\delta \psi$ is used for convenience; it is not the usual meaning associated with a variational derivative). The vector $W^i[u, v]$ will vanish on ∂R if $\delta u^\alpha = \epsilon v^\alpha$ and its normal derivatives all vanish on the boundary. In the above equations, $E_\alpha[L]$ defines the Euler operator E_α for the system. The Euler-Lagrange equations (4.5) are the differential equations $R^s = 0$ $(1 \leq s \leq m)$ for the system in this case. The surface vector W^i is important in Noether's theorem.

4.2 Noether's Theorems

In the proof of Noether's theorem an important result is that two Lagrangian densities L_1 and L_2 that differ by a pure divergence have the same Euler-Lagrange equations (4.5). This property follows from the result that $E_\gamma[D_i F] = 0$ for any sufficiently smooth functional $F[u]$. Thus if $L_2 - L_1 = D_i A^i$, then $E_\alpha[L_1] = E_\alpha[L_2]$ (e.g. Bluman and Kumei 1989).

Consider the variation:

$$\delta \mathscr{A} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\int_{R'} d\mathbf{x}' L'(x'^i, u'^\alpha, u_i'^\alpha, u_{ij}'^\alpha, \ldots) - \int_R d\mathbf{x} L(x^i, u^\alpha, u_i^\alpha, u_{ij}^\alpha, \ldots) \right),$$
(4.11)

due to the infinitesimal Lie transformations:

$$x'^i = x^i + \epsilon V^i(\mathbf{x}, \mathbf{u}, \mathbf{u}_i, \mathbf{u}_{ij}, \ldots) + O(\epsilon^2), \quad 1 \le i \le n,$$
(4.12)

$$u'^\alpha = u^\alpha + \epsilon \eta^\alpha(\mathbf{x}, \mathbf{u}, \mathbf{u}_i, \mathbf{u}_{ij}, \ldots) + O(\epsilon^2), \quad 1 \le \alpha \le m,$$
(4.13)

and the divergence transformation:

$$L' = L + \epsilon D_i \Lambda^i + O(\epsilon^2).$$
(4.14)

The Euler-Lagrange equations $E_\alpha(L') = E_\alpha(L) = 0$ are invariant under the divergence transformation (4.14). From (4.12)–(4.14), we obtain:

$$\delta \mathscr{A} = \int_R d\mathbf{x} \left[\tilde{X} L + L D_i V^i + D_i \Lambda^i \right],$$
(4.15)

$$\equiv \int_R d\mathbf{x} \left[\hat{X} L + D_i \left(L V^i + \Lambda^i \right) \right].$$
(4.16)

Here, the prolonged symmetry operator $\tilde{X}$ is related to the prolonged, canonical symmetry operator $\hat{X}$ by the equation

$$\tilde{X} = \hat{X} + V^i D_i.$$
(4.17)

The canonical symmetry operator (or evolutionary symmetry operator $\hat{X}$ with characteristic $\hat{\eta}$) is the infinitesimal Lie symmetry transformation for which

$$x'^i = x^i, \quad u'^\alpha = u^\alpha + \epsilon \hat{\eta}^\alpha, \quad \hat{\eta}^\alpha = \eta^\alpha - V^i u_i^\alpha,$$
(4.18)

(see e.g. Ibragimov (1985), Bluman and Kumei (1989) and Olver (1993) for further discussion of canonical or characteristic symmetries). The prolonged symmetry operator $\hat{X}[\hat{\eta}]$ is given by

$$\hat{X}[\hat{\eta}] = \hat{\eta}^\alpha \frac{\partial}{\partial u^\alpha} + D_i \hat{\eta}^\alpha \frac{\partial}{\partial u_i^\alpha} + D_i D_j \hat{\eta}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \ldots$$

$$+ D_{i_1} D_{i_2} \ldots D_{i_s} \hat{\eta}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha} + \ldots.$$
(4.19)

The prolonged symmetry operator $\tilde{X}$ gives the transformation of x^i, u^α and the derivatives of the u^α under the Lie transformation (4.12)–(4.13). $\hat{X}$ is the extended symmetry operator corresponding to the canonical symmetry (4.18). The terms in (4.15) consist of: the change in L due to the Lie transformation (4.12)–(4.13); the

change in the volume element $d\mathbf{x}$ due to Lie transformation; and the change in $\mathscr{A}[u]$ due to the divergence transformation (4.14).

Since the integration region R is arbitrary in (4.15), it follows that a Lie group, with infinitesimal generators (4.12)–(4.13) is a *divergence symmetry* of the variational functional (4.2) if

$$\tilde{X}L + LD_iV^i = -D_i\Lambda^i, \tag{4.20}$$

for some vector field Λ^i. The condition (4.20) ensures that $\delta\mathscr{A} = 0$ in (4.15). If $\Lambda^i = 0$, the symmetry is called a *variational symmetry*. From (4.16), (4.17), (4.20) is equivalent to:

$$\hat{X}[\hat{\eta}]L + D_i(LV^i) = -D_i\Lambda^i, \tag{4.21}$$

where $\hat{X}[\hat{\eta}]$ is the prolonged canonical symmetry operator (4.19).

The invariance condition (4.20) implies that (4.12)–(4.13) is a Lie symmetry of the Euler-Lagrange equations $E_\gamma(L) = 0$, i.e. $\tilde{X}E_\gamma(L) = 0$ when $E_\gamma(L) = 0$. One can show that $\hat{X}E_\gamma(L) = 0$ and hence from (4.17) $\tilde{X}E_\gamma(L) = V^iD_i[E_\gamma(L)]$. The proof that $\hat{X}E_\gamma(L) = 0$ is given in Bluman and Kumei (1989). The proof depends on the facts: $L(\mathbf{x}, \mathbf{u}', \mathbf{u}'_i, \ldots) = \exp(\epsilon\hat{X}[\hat{\eta}])L(\mathbf{x}, \mathbf{u}, \mathbf{u}_i, \ldots)$, for a finite Lie transformation; that the commutator $[\hat{X}[\hat{\eta}], D_i] = 0$ and the result (4.21).

The Lie bracket of two evolutionary symmetry operators with characteristics $\hat{\eta}_1$ and $\hat{\eta}_2$ is given by

$$[\hat{X}[\hat{\eta}_1], \hat{X}[\hat{\eta}_2]] = \hat{X}[\hat{\eta}_3] \quad \text{where} \quad \hat{\eta}_3 = \hat{X}[\hat{\eta}_1]\hat{\eta}_2 - \hat{X}[\hat{\eta}_2]\hat{\eta}_1. \tag{4.22}$$

One can also show that $[\hat{X}[\hat{\eta}], D_i] = 0$ and $[D_i, D_j] = 0$ (Ibragimov 1985). The canonical symmetry operators form a Lie algebra, $\hat{L}$, which is infinite dimensional if there is an infinite number of distinct $\hat{\eta}_j$. From Ibragimov (1985), the algebra $\hat{L}$ is a subalgebra of $\tilde{L}$, the symmetry algebra of the vector fields $\tilde{X}$. One can show, that $\hat{L}$ is isomorphic to the factor algebra $\tilde{L}/L_*$ where $L_* = \{X_* \in \tilde{L} : X_* = \xi^j_* D_j\}$ is a closed ideal in $\tilde{L}$ (i.e. $[\tilde{X}, X_*] \in L_*$ for all $X_* \in L_*$ and $\tilde{X} \in \tilde{L}$).

The main idea behind Noether's theorem is given below. We first note from (4.4) with $v^\gamma = \hat{\eta}^\gamma$ that

$$\hat{X}[\hat{\eta}]L = \hat{\eta}^\gamma E_\gamma(L) + D_iW^i[u, \hat{\eta}], \tag{4.23}$$

Using (4.23) in (4.16) now gives:

$$\delta\mathscr{A} = \int_R d\mathbf{x} \left[\hat{\eta}^\gamma E_\gamma(L) + D_i\left(W^i[u, \hat{\eta}] + LV^i + \Lambda^i\right)\right] = 0, \tag{4.24}$$

Because the integration region R is arbitrary in (4.24), then $\delta\mathscr{A} = 0$ if

$$\hat{\eta}^\gamma E_\gamma(L) + D_i\left(W^i[u, \hat{\eta}] + LV^i + \Lambda^i\right) = 0. \tag{4.25}$$

Equations (4.24)–(4.25) are the basis of Noether's first and second theorems.

4.2.1   Noether's First Theorem

If the Euler-Lagrange equations $E_\gamma(L) = 0$ are *non-degenerate*, then to each *divergence symmetry* (4.12)–(4.14) of the variational functional (4.2), there is a corresponding conservation law:

$$D_i\left(W^i[u, \hat{\eta}] + LV^i + \Lambda^i\right) = 0. \tag{4.26}$$

More precisely, there is a one-to-one correspondence between the symmetries and the class of conservation laws equivalent to (4.26). In the usual case where there is a finite Lie group of transformations that leave the action invariant (up to a divergence transformation), then each independent Lie symmetry of the group gives rise to a conservation law. The theorem follows from (4.25) if we assume that the Euler-Lagrange equations $E_\gamma(L) = 0$ are satisfied and are independent, i.e., the Euler Lagrange equations and their differential consequences are independent. A system of differential equations is called *non-degenerate* if it is *locally solvable* and satisfies the *maximum rank condition* (see e.g. Olver (1993), Ch. 2, p. 158 et seq. and Ch. 4 and 5 for a detailed discussion of these notions).

Noether's first theorem can be used in the derivation of conservation laws associated with Lie point symmetries, or with multi-parameter Lie groups with a finite number of parameters. For example, the momentum and energy conservation laws correspond to space-translation and time translation variational symmetries of a system of differential equations governed by a variational principle.

4.2.2   Noether's Second Theorem

Noether's second theorem describes so-called, under-determined systems of Euler-Lagrange equations, and relates infinite dimensional Lie groups of variational and divergence symmetries to dependencies among the Euler-Lagrange equations. In this case, one can obtain solutions of the variational equations (4.24), i.e. $\delta\mathscr{A} = 0$, for the independent variables u^α, that do not satisfy the Euler Lagrange equations, but nevertheless correspond to a divergence symmetry (or possibly a variational symmetry) satisfying the infinitesimal Lie symmetry equations (4.20) and (4.21). These solutions satisfy so-called generalized Bianchi identities obtained by integrating (4.25) over all the independent variables x^i over some region R, or by integrating only a subset of the independent variables over a hypersurface embedded within R. A detailed description of Noether's second theorem may be found in Olver (1993) (Ch. 5).

A typical application of the use of Noether's second theorem is in the description of fluid re-labeling symmetries in ideal MHD and fluid mechanics (e.g. Padhye and Morrison 1996a,b; Padhye 1998). In this case one obtains generalized Bianchi identities by integrating the analogue of (4.25) over the Lagrangian fluid labels

x_0^i, but not over the time variable t. The Eulerian fluid coordinates $x^i = x^i(\mathbf{x}_0, t)$ ($i = 1, 2, 3$) in this development are the independent variables, and the Lagrangian fluid labels x_0^i and the time t are the dependent variables. For the case of gas dynamics, there is an infinite dimensional Lie symmetry involving transformations of the fluid labels that corresponds to Ertel's theorem (i.e., the conservation of potential vorticity following the flow). Ertel's theorem results if one assumes that the Euler Lagrange equations of the fluid (i.e. the momentum equation for the fluid) are satisfied. This conservation law is known as a weak conservation law. However, there are also other solutions of the generalized Bianchi identity, which do not satisfy the individual Euler Lagrange equations, and are associated with so-called strong conservation laws.

Noether's second theorem (Olver 1993; Hydon and Mansfield 2011) expresses the idea that there must exist a relation between the Euler-Lagrange equations if the symmetry operator $\hat{\eta}^\alpha(\mathbf{x}, [\mathbf{u}, \mathbf{g}])$ depends on an arbitrary function $\mathbf{g}(\mathbf{x})$.

Proposition 4.2.1 (Olver 1993) *The variational problem (4.24)–(4.25) admits an infinite dimensional group of variational symmetries whose characteristics $\hat{\eta}^\alpha(\mathbf{x}, [\mathbf{u}, \mathbf{g}])$ depend on an arbitrary function $\mathbf{g}(\mathbf{x})$ (and its derivatives) if and only if there exist differential operators $\mathcal{D}^1, \mathcal{D}^2, \dots \mathcal{D}^q$, not all zero, such that:*

$$\mathcal{D}^1 E_1(L) + \mathcal{D}^2 E_2(L) + \dots + \mathcal{D}^q E_q(L) = 0, \tag{4.27}$$

for all $\mathbf{x}$ and $\mathbf{u}$.

A proof and discussion of the theorem is given by Olver (1993). Hydon and Mansfield (2011) give a simpler proof of the *only if* part of the proof, (based on Noether (1918)) and identify the operators $\{\mathcal{D}^s : 1 \le s \le q\}$.

In Hydon and Mansfield (2011), an explicit form of the relationship (4.27) between the Euler-Lagrange equations is obtained by applying the operator E_g to (4.25) to obtain:

$$E_g \{\hat{\eta}^\alpha(\mathbf{x}, [\mathbf{u}; g]) E_\alpha(L)\} = 0, \tag{4.28}$$

as the required differential relation (4.27) between the Euler-Lagrange equations. The theorem extends immediately to variational symmetries whose characteristics $\hat{\eta}^\alpha$ depend on R independent arbitrary functions $\mathbf{g} = (g^1(\mathbf{x}), \dots, g^R(\mathbf{x}))$ and their derivatives. This gives R differential relations between the Euler Lagrange equations:

$$E_{g^r} \{\hat{\eta}^\alpha(\mathbf{x}, [\mathbf{u}; \mathbf{g}]) E_\alpha(L)\} = (-D)_J \left(\frac{\partial \hat{\eta}^\alpha(\mathbf{x}, [\mathbf{u}; \mathbf{g}])}{\partial g^r_{,J}} E_\alpha(L) \right) = 0, \quad r = 1, \dots, R. \tag{4.29}$$

It is useful to consider (4.28) as Euler Lagrange equations for the action

$$\hat{J}[\mathbf{u}; \mathbf{g}] = \int \hat{L}(\mathbf{x}; [\mathbf{u}; \mathbf{g}]) \, d\mathbf{x}, \tag{4.30}$$

where

$$\hat{L}(\mathbf{x}, [\mathbf{u}; \mathbf{g}]) = \hat{\eta}^\alpha(\mathbf{x}, [\mathbf{u}; g]) E_\alpha(L(\mathbf{x}, [\mathbf{u}])). \tag{4.31}$$

In the case, where the functions $\mathbf{g} = (g^1, g^2, \ldots, g^R)$ are subject to S constraints, of the form:

$$\mathcal{D}_{sr}(g^r) = 0, \quad s = 1, \ldots, S, \tag{4.32}$$

where the $\mathcal{D}_{sr}$ are differential operators, the constraints can be incorporated in the Lagrangian $\hat{L}$, by using the modified Lagrangian:

$$\hat{L}(\mathbf{x}, [\mathbf{u}; \mathbf{g}]) = \hat{\eta}^\alpha(\mathbf{x}, [\mathbf{u}; \mathbf{g}]) E_\alpha(L(\mathbf{x}, [\mathbf{u}])) - v^s \mathcal{D}_{sr}(g^r). \tag{4.33}$$

Here the v^s are Lagrange multipliers which ensure that the constraint equations (4.32) are satisfied.

By varying the action (4.30) and (4.33) with respect to g^r, we note that

$$\delta\left(\int v^s \mathcal{D}_{sr}(g^r)\, d\mathbf{x}\right) = \langle v^s, \mathcal{D}_{sr}(\delta g^r)\rangle = \langle \mathcal{D}^\dagger_{sr}(v^s), \delta g^r\rangle, \tag{4.34}$$

where we have dropped the surface integral terms. The angle brackets define the usual inner product for functions. Taking variations of (4.30) and (4.33) with respect to g^r yields

$$\frac{\delta \hat{J}}{\delta g^r} = (-D)_J\left(\frac{\partial \hat{\eta}^\alpha(\mathbf{x}, [\mathbf{u}, \mathbf{g}])}{\partial g^r_{,J}} E_\alpha(L)\right) - \mathcal{D}^\dagger_{sr}(v^s) = 0, \quad r = 1, \ldots, R. \tag{4.35}$$

Thus, if $S < R$, it may be possible to eliminate the Lagrange multipliers in (4.35). In any event, (4.35) relates the Lagrange multipliers v^s to the solutions of the original Euler Lagrange equations (4.5).

We use the above form of Noether's second theorem by Hydon and Mansfield (2011) was used by Webb and Mace (2015) to derive potential vorticity conservation laws for MHD using fluid relabelling symmetries of the equations.

4.3   The Direct Method

The direct method for obtaining conservation laws of a differential equation system was developed by Anco and Bluman (1996, 1997, 2002a,b). This approach is described in Bluman et al. (2010). Cheviakov (2007, 2014) and Cheviakov and Anco (2008) solved the determining equations for the multipliers $\Lambda_\sigma[\mathbf{u}]$ for each of the equations $R^\sigma[u] = 0$ in (4.1) to obtain non-trivial conservation laws. Before

describing the direct method, it is useful to have at hand some definitions of what actually constitutes a non-trivial conservation law (e.g. Bluman et al. 2010). A trivial conservation law arises in two cases, and contains no information about the partial differential equation system of interest.

Definition A local conservation law of a system of partial differential equations:

$$R^\sigma[\mathbf{u}] = R^\sigma[\mathbf{x}, \mathbf{u}, \partial\mathbf{u}, \partial^2[\mathbf{u}], \ldots, \partial^k\mathbf{u}] = 0, \quad 1 \leq \sigma \leq m, \tag{4.36}$$

is a divergence expression:

$$D_i\Phi^i[\mathbf{u}] = D_1\Phi^1 + D_2\Phi^2 + \ldots D_n\Phi^n = 0, \tag{4.37}$$

which holds for all differentiable solutions of (4.36).

Definition A local conservation law of (4.36) is trivial if its fluxes have the form:

$$\Phi^i[\mathbf{u}] = P^i[\mathbf{u}] + Q^i[\mathbf{u}], \tag{4.38}$$

where $P^i[\mathbf{u}]$ and $Q^i[\mathbf{u}]$ are such that (i) $P^i[\mathbf{u}] = 0$ on solutions of (4.36) and (ii) $D_iQ^i[\mathbf{u}] = 0$ is valid for arbitrary fluxes, not necessarily related to (4.36).

An example of a trivial conservation law of type (ii) for $n = 3$, is the identity $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ which is an identity for arbitrary differentiable and continuous $\mathbf{F}$.

Definition Two conservation laws $D_i\Phi^i[\mathbf{u}] = 0$ and $D_i\Psi^i[\mathbf{u}] = 0$ are equivalent if $D_i(\Phi^i[\mathbf{u}] - \Psi^i[\mathbf{u}]) = 0$ is a trivial conservation law.

Thus, conservation laws split up into *equivalence classes*, of non-trivial conservation laws.

Definition A set of ℓ conservation laws $D_i\Phi^i_{(j)} = 0$, where $i \leq j \leq \ell$ is linearly independent if there exist a set of constants $\{a^j : 1 \leq j \leq \ell\}$, not all zero, such that the linear combination $D_i\left(a^j\Phi^i_{(j)}[\mathbf{u}]\right) = 0$ is a trivial conservation law.

In general, one is only interested in linearly independent conserved currents and conservation laws.

The Euler operator with respect to U^α is defined by the equation:

$$E_{U^\alpha} = \frac{\partial}{\partial U^\alpha} - D_i\left(\frac{\partial}{\partial U_i^\alpha}\right) + D_iD_j\left(\frac{\partial}{\partial U_{ij}^\alpha}\right) + \ldots (-1)^s D_{i_1}D_{i_2}\ldots D_{i_s}\left(\frac{\partial}{\partial U_{i_1 i_2 \ldots i_s}^\alpha}\right) + \ldots. \tag{4.39}$$

It can be shown (e.g. Bluman and Kumei 1989; Ibragimov 1985) that

$$E_{U^\alpha}(D_iF) = 0, \tag{4.40}$$

for $F = F(\mathbf{x}, \mathbf{U}, \partial\mathbf{U}, \ldots \partial^k\mathbf{U})$ an arbitrary differentiable and continuous function of its arguments.

Definition A set of conservation law multipliers $\Lambda_\sigma[U]$, $1 \le \sigma \le N$, for the partial differential system (4.36) , i.e. $R^\sigma[u] = 0$, has the property

$$\Lambda_\sigma[U]R^\sigma[U] = D_i\Phi^i[U], \tag{4.41}$$

where the $\{\Phi^i[U]\}$ are the conserved fluxes or currents for the conservation law, where the $D_i\Phi^i$ are non-trivial divergence expressions. On the solution manifold $\mathbf{U} = \mathbf{u}$ and $R^\sigma[u] = 0$. Thus $D_i\Phi^i[\mathbf{u}] = 0$ on the solution manifold.

Proposition 4.3.1 *The equations*

$$E_{U^j}\left(F[\mathbf{x}, \mathbf{U}, \partial\mathbf{U}, \partial^2\mathbf{U}, \dots \partial^k\mathbf{U}]\right) = 0, \tag{4.42}$$

($1 \le j \le m$) holds for arbitrary $\mathbf{U}(\mathbf{x})$ $\Longleftrightarrow$

$$F[\mathbf{x}, \mathbf{U}, \partial\mathbf{U}, \partial^2\mathbf{U}, \dots \partial^k\mathbf{U}] = D_i\Psi^i\left(\mathbf{x}, \mathbf{U}, \partial\mathbf{U}, \partial^2\mathbf{U}, \dots \partial^{k-1}\mathbf{U}\right), \tag{4.43}$$

holds for some functions $\Psi^i[\mathbf{x}, \mathbf{U}]$ where $1 \le i \le n$.

Remark The above proposition is described in Bluman et al. (2010) (see also Bluman and Kumei (1989) and Ibragimov (1985) for related results). The result does not depend on the existence of a variational functional formulation of the equations $R^\sigma[\mathbf{u}] = 0$. It is related to the property that the variational derivative of a pure divergence term is zero in Noether's theorem, and hence does not change the Euler Lagrange equations $E_{u^\alpha}[L] = 0$ in cases where the equation system is described by a variational principle.

To derive non-trivial conservation laws of the system (4.36) it is necessary to determine non-singular multipliers (integrating factors) $\Lambda_\sigma[\mathbf{U}]$ satisfying (4.41). by applying the Euler operator E_{U^j} to both sides of (4.41) gives the determining equations for the multipliers $\Lambda_\sigma[\mathbf{U}]$ as:

$$E_{U^j}\left(\Lambda_\sigma[\mathbf{U}]R^\sigma[\mathbf{U}]\right) = E_{U^j}\left(D_i\Phi^i\right) = 0, \quad 1 \le j \le m. \tag{4.44}$$

Equation (4.44) applies for general $\mathbf{U}(\mathbf{x})$, and not just for solutions $\mathbf{U} = \mathbf{u}$ that lie on the solution manifold $R^\sigma[\mathbf{u}] = 0$. It is also required that the differential consequences of (4.44) are taken into account. Notice that the right-hand side of (4.44) vanishes by virtue of (4.42)–(4.43). The net upshot is that the integrating factors $\Lambda_\sigma[\mathbf{U}]$ must satisfy the determining equations (4.44).

Once the determining equations (4.44) are solved for the multipliers $\Lambda_\sigma[\mathbf{U}]$, one needs to determine the flux functions Φ^i satisfying (4.41). This can be achieved, for example, by using the homotopy formula given by Olver (1993) (see e.g. Bluman et al. 2010; Anco and Bluman 1996, 1997, 2002a,b).

A question not addressed by the above analysis is: under what conditions do non-trivial conservation laws arise? Conversely, under what conditions does a set of local multipliers Λ_σ give non-trivial conservation laws? To answer these questions,

it is necessary to write the equation system in a *solved form* (i.e. in Cauchy-Kovalevskaya form), with respect to some leading order derivatives.

Our aim is to give an outline of the direct method for deriving the integrating factors or multipliers Λ_σ and the conserved currents $\{\Phi^i : \ 1 \leq i \leq n\}$ for the system of partial differential equations $R^\sigma[\mathbf{u}] = 0$ ($1 \leq \sigma \leq N$), in m dependent variables $\{u^j : \ 1 \leq j \leq m\}$ and n independent variables $\{x^i : \ 1 \leq i \leq n\}$. Below we use an example from Anco and Bluman (1997, 2002a,b) to illustrate the direct method.

4.3.1 Example: KdV Equation

Our aim is to find some of the lower order conservation laws for the KdV equation:

$$R[u] = u_t + uu_x + u_{xxx} = 0. \tag{4.45}$$

by using the determining equations (4.44) for multipliers or integrating factors of the form $\Lambda(x, t, U, U_x, U_{xx}, \ldots)$, i.e. we search for multipliers Λ such that

$$\Lambda R[U] = D_t \Phi^t + D_x \Phi^x, \tag{4.46}$$

is a pure divergence expression. From (4.44) the integrating factors Λ must satisfy the determining equations:

$$E_U (\Lambda R[U]) = 0, \tag{4.47}$$

for general $U(x, t)$, i.e. $U(x, t)$ is not in general a solution $U = u$ of the KdV equation $R[u] = 0$. Using the formula (4.39) for the Euler operator E_U, we obtain:

$$E_U (\Lambda U_t) = - D_t \Lambda + U_t \Lambda_U - D_x \left(\frac{\partial \Lambda}{\partial U_x} U_t \right) + D_x^2 \left(\frac{\partial \Lambda}{\partial U_{xx}} U_t \right) + \ldots,$$

$$E_U (\Lambda UU_x) = - UD_x \Lambda + UU_x \Lambda_U - D_x \left(\frac{\partial \Lambda}{\partial U_x} UU_x \right) + D_x^2 \left(\frac{\partial \Lambda}{\partial U_{xx}} UU_x \right) + \ldots,$$

$$E_U (\Lambda U_{xxx}) = - D_x^3 \Lambda + U_{xxx} \Lambda_U - D_x \left(\frac{\partial \Lambda}{\partial U_x} U_{xxx} \right) + D_x^2 \left(\frac{\partial \Lambda}{\partial U_{xx}} U_{xxx} \right) + \ldots,$$
$$\tag{4.48}$$

where we have assumed (without loss of generality) that $\Lambda = \Lambda(x, t, U, U_x, U_{xx}, \ldots)$. Adding the terms in (4.48) gives:

$$E_U [\Lambda (U_t + UU_x + U_{xxx})]$$
$$= -D_t \Lambda - UD_x \Lambda - D_x^3 \Lambda + \Lambda_U (U_t + UU_x + U_{xxx})$$

$$- D_x \left(\frac{\partial \Lambda}{\partial U_x} (U_t + U U_x + U_{xxx}) \right) + D_x^2 \left(\frac{\partial \Lambda}{\partial U_{xx}} (U_t + U U_x + U_{xxx}) \right) + \dots$$
$$= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.49)$$

for the determining equations.

Consider the special case where $\Lambda = \Lambda(x, t, U)$. In that case:

$$D_t \Lambda = \Lambda_t + \Lambda_U U_t, \quad D_x \Lambda = \Lambda_x + \Lambda_U U_x,$$
$$D_x^3 \Lambda = \Lambda_{xxx} + 3 U_{xx} \Lambda_{xU} + 3 U_x U_{xx} \Lambda_{UU} + 3 U_x \Lambda_{xxU} + 3 U_x^2 \Lambda_{UUx}$$
$$+ U_{xxx} \Lambda_U + U_x^3 \Lambda_{UUU}, \qquad\qquad\qquad (4.50)$$

and (4.49) reduces to:

$$E_U (\Lambda R[U]) = - \Bigg(\Lambda_t + \Lambda_{xxx} + U \Lambda_x + 3 \Lambda_{xxU} U_x + 3 \Lambda_{xUU} U_x^2$$
$$+ 3 \Lambda_{xU} U_{xx} + 3 \Lambda_{UU} U_x U_{xx} + \Lambda_{UUU} U_x^3 \Bigg) = 0. \qquad (4.51)$$

Setting the powers of U_x and U_{xx} equal to zero in (4.51), the determining equations (4.51) splits up into the equations:

$$\Lambda_t + \Lambda_{xxx} + U \Lambda_x = 0, \qquad\qquad\qquad\qquad (4.52)$$
$$\Lambda_{xxU} = \Lambda_{xUU} = \Lambda_{xU} = \Lambda_{UU} = \Lambda_{UUU} = 0. \qquad (4.53)$$

The determining equations (4.53) reduce to:

$$\Lambda_{UU} = \Lambda_{xU} = 0. \qquad\qquad\qquad\qquad\qquad (4.54)$$

Equations (4.52) and (4.54) have solutions:

$$\Lambda = \beta (tU - x) + \gamma U + \delta. \qquad\qquad\qquad (4.55)$$

where β, γ and δ are arbitrary constants. From (4.55) there are three independent solutions for Λ, namely:

$$\text{(i) } \Lambda = 1, \qquad \text{(ii) } \Lambda = U, \qquad \text{(iii) } \Lambda = tU - x. \qquad (4.56)$$

corresponding to $(\delta, \gamma, \beta) = (1, 0, 0); (0, 1, 0);$ and $(0, 0, 1)$ respectively.

Using the multipliers (4.56) we obtain three conservation laws of the form:

$$D_t \Phi_i^t + D_x \Phi_i^x = 0, \quad 1 \le i \le 3, \qquad\qquad (4.57)$$

where

$$\Phi_1^t = U, \quad \Phi_1^x = \frac{1}{2}U^2 + U_{xx}, \tag{4.58}$$

$$\Phi_2^t = \frac{1}{2}U^2, \quad \Phi_2^x = \frac{1}{3}U^3 + UU_{xx} - \frac{1}{2}U_x^2, \tag{4.59}$$

$$\Phi_3^t = -xU + \frac{1}{2}tU^2, \quad \Phi_3^x = \frac{t}{3}U^3 - \frac{x}{2}U^2 + t\left(UU_{xx} - \frac{1}{2}U_x^2\right) + U_x - xU_{xx}, \tag{4.60}$$

where $U = u$ satisfies the KdV equation.

The integrating factors (multipliers) Λ_i and the conservation laws and conserved densities Φ_i^t and Φ_i^x are related to the linearized KdV equation:

$$\hat{X}(\eta)R[U] = R'[U]\eta = \left(D_t + U_x + UD_x + D_x^3\right)\eta = 0, \tag{4.61}$$

where $\hat{X}(\eta)$ is the canonical or characteristic (evolutionary) Lie symmetry operator in which the independent variables are frozen, i.e., $x' = x$, $t' = t$ and $U' = U + \epsilon\eta$ are the infinitesimal Lie transformation forms. The Green's function identity:

$$\Lambda R'[U]Y = YR'^\dagger[U]\Lambda + D_t S^0 + D_x S^1, \tag{4.62}$$

relates $R'[U]$ to its adjoint operator $R'^\dagger[U]$, where

$$R'^\dagger[U]\Lambda = -\left(D_t + D_x^3 + UD_x\right)\Lambda. \tag{4.63}$$

Note that $R'^\dagger[U]\Lambda = 0$ by (4.52) where Λ is an integration multiplier of the KdV equation. In (4.62) S^0 and S^1 in the divergence term, are given by:

$$S^0 = \Lambda Y, \quad S^1 = Y\Lambda_{xx} + \Lambda Y_{xx} - Y_x\Lambda_x + UY\Lambda. \tag{4.64}$$

The divergence term $D_t S^0 + D_x S^1$ is analogous to the bilinear concomitant in the theory of ordinary differential equations (e.g. Morse and Feshbach 1953, Vol. 1, p. 528).

In the case where $Y = U$, (4.62) reduces to the equation:

$$\Lambda R'[U]U = UR'^\dagger[U]\Lambda + D_t S^0 + D_x S^1, \tag{4.65}$$

where

$$S^0 = \Lambda U, \quad S^1 = U\Lambda_{xx} + \Lambda U_{xx} - U_x\Lambda_x + U^2\Lambda. \tag{4.66}$$

Equation (4.66) can be written in the form:

$$\frac{d}{d\epsilon} (\Lambda R[U + \epsilon Y])_{Y=U} = D_t S^0 + D_x S^1, \qquad (4.67)$$

where $d/d\epsilon$ corresponds to a Lie derivative symmetry operator at an arbitrary point $\epsilon = \epsilon_0$ say on the Lie trajectory ($\epsilon_0 \neq 0$ in general) corresponding to the scaling symmetry transformation:

$$x' = x, \quad t' = t, \quad U' = \lambda U, \quad \lambda = \exp(\epsilon), \qquad (4.68)$$

where $\Lambda[U]R[U] = 0$ and $R'^\dagger[U]\Lambda = 0$ on the KdV solution manifold where $U = u$. By integrating (4.67) with respect to ϵ (or equivalently with respect to λ) we obtain the conservation law:

$$D_t \Phi^t + D_x \Phi^x = \Lambda R[U] = 0, \qquad (4.69)$$

where Φ^t and Φ^x are given by the homotopy formulas:

$$\Phi^t = \int_0^1 \frac{d\lambda}{\lambda} S^0[U]_{|U'=\lambda U} + \Phi^t(\lambda = 0),$$

$$\Phi^x = \int_0^1 \frac{d\lambda}{\lambda} S^1[U]_{|U'=\lambda U} + \Phi^x(\lambda = 0). \qquad (4.70)$$

In (4.70) we have assumed that the integrals over λ (or ϵ) are well defined (this is the case in our applications). In some cases it is useful integrate λ over a different range than in (4.70) if the integrals do not converge (e.g. from $\lambda = 1$ to $\lambda = \infty$ in some cases: e.g. Webb et al. 2010a). By setting the initial value terms equal to zero in (4.70), and using the three multipliers (4.56) for the Λ's, and carrying out the integrals (4.70) we obtain the conserved densities for the KdV equation Φ_i^t and Φ_i^x for $1 \leq i \leq 3$ listed in (4.58)–(4.60) where $U = u$ satisfies the KdV equation.

Remark By using the recursion operator for the symmetries associated with Φ_1^t and Φ_3^t one can generate two infinite sequences of conservation laws by application of the recursion operator (e.g. Anco and Bluman 1997).

Remark One can search for multipliers $\Lambda = \Lambda(x, t, U, U_x, U_{xx}, \ldots)$ which depend on the derivatives of U_x, U_{xx} as well as on (x, t, U). This possibility is discussed by Anco and Bluman (1997, 2002a,b) and Bluman et al. (2010) .

There are substantial computational algebra investigations of conservation laws for nonlinear partial differential and discrete nonlinear equations (e.g. Hereman et al. 2006; Cheviakov and Anco 2008). Cheviakov (2007, 2014) has obtained large classes of conservation laws for fluid and potential type systems of pdes using the direct approach of Anco and Bluman (1997, 2002a,b). Kelbin et al. (2013) use the direct method to obtain conservation laws for helically symmetric, plane and

rotationally symmetric viscous and inviscid flows. These methods are important in deriving conservation laws for systems of pdes, such as fluid equations and the MHD equations. However, these methods will not be the focus of the present book, because our emphasis is on the use of action principles in MHD and fluid mechanics, in deriving conservation laws. Pshenitsin (2016) has used these methods to derive conservation laws for the viscous and finite conductivity, incompressible MHD equations.

Chapter 5
Advected Invariants

Tur and Yanovsky ([1993](#)) developed a formalism for Lie dragging of geometrical objects **G** (tensors, p-forms and vectors) that are advected with the flow in ideal gas dynamics and MHD. The basic requirement for **G** to be advected or Lie dragged with the flow **u** is that

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \mathbf{G} \equiv \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{G} = 0, \tag{5.1}$$

where $\mathcal{L}_{\mathbf{u}}$ is the Lie derivative with respect to the vector field **u**. As in the Calculus of exterior differential forms and in differential geometry (e.g. Harrison and Estabrook [1971](#); Misner et al. [1973](#); Fecko [2006](#)), vector fields **V** and one-forms $\alpha = A_i dx^i \equiv \mathbf{A} \cdot d\mathbf{x}$ are dual.

5.1 Exterior Differential Forms and Vector Fields

Discussions of the algebra of exterior differential forms are given in the books by Frankel ([1997](#)), Bott and Tu ([1982](#)), Misner et al. ([1973](#)), Marsden and Ratiu ([1994](#)), Holm ([2008a](#),[b](#)), and Flanders ([1963](#)). A short summary is given by Harrison and Estabrook ([1971](#)), who develop a geometric approach to invariance groups and solutions of partial differential systems using Cartan's geometric formulation of partial differential equations in terms of exterior differential forms and vector fields.

The vector field **V** in 3D Cartesian geometry is a directional derivative operator:

$$\mathbf{V} = V^x \frac{\partial}{\partial x} + V^y \frac{\partial}{\partial y} + V^z \frac{\partial}{\partial z}, \tag{5.2}$$

and the one-form $\mathbf{A} \cdot d\mathbf{x}$ has the form:

$$\boldsymbol{\omega} = \mathbf{A} \cdot d\mathbf{x} = A_x dx + A_y dy + A_z dz. \tag{5.3}$$

The inner product of vector $\mathbf{u}$ and 1-form ω is the scalar or dot product:

$$\langle \mathbf{u}, \omega \rangle = \left\langle u^x \frac{\partial}{\partial x} + u^y \frac{\partial}{\partial y} + u^z \frac{\partial}{\partial z}, A_x dx + A_y dy + A_z dz \right\rangle$$

$$= u^x A_x + u^y A_y + u^z A_z \equiv \mathbf{u} \cdot \mathbf{A}. \tag{5.4}$$

Equivalent forms for the inner product are:

$$\langle \mathbf{u}, \omega \rangle \equiv \mathbf{u} \lrcorner \omega \equiv \mathbf{i_u} \omega, \tag{5.5}$$

where $\mathbf{i_u}$ denotes inner product of contravariant field $\mathbf{u}$ with a covariant field ω. In particular,

$$\left\langle \frac{\partial}{\partial x^i}, dx^j \right\rangle = \delta_{ij}, \quad \text{e.g.} \quad \left\langle \frac{\partial}{\partial x}, dx \right\rangle = 1, \quad \left\langle \frac{\partial}{\partial x}, dy \right\rangle = 0. \tag{5.6}$$

In an n-dimensional, differentiable manifold, a p-form $\boldsymbol{\omega}$ is a completely anti-symmetric covariant pth rank tensor, described by its anti-symmetric components $\omega_{\mu_1 \dots \mu_p}$. In general, a p-form can be expressed in the form:

$$\boldsymbol{\omega} = \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p}. \tag{5.7}$$

One forms are elements of the cotangent space at a point of a manifold, involving the mapping of the vector fields in the tangent space onto the reals. In (5.7) $\wedge$ denotes a non-commutative anti-symmetrized multiplication. At each point of the manifold, the forms have a Grassmann algebra defined by the properties of the wedge product operator $\wedge$. On the manifold, one may use the three operations of of exterior differentiation, d, of contraction with a vector field $\mathbf{V}$ (a contravariant vector V^μ), $\mathbf{V} \lrcorner \boldsymbol{\omega}$, and of Lie derivative with respect to $\mathbf{V}$, $\mathcal{L}_{\mathbf{V}} \boldsymbol{\omega}$, to give rise to new geometric quantities from given geometrical objects. These operations give forms of rank $p + 1$, $p - 1$ and p respectively.

An application of the Grassmann algebra is the integration of tensors over surface and volumes and to Stokes and Gauss's theorem in vector Calculus where the oriented volume element is $dx \wedge dy \wedge dz$, and the integral of a vector field $\mathbf{V}$ over a surface element $d\mathbf{S}$ can be written in terms of the two form: $\alpha = \mathbf{V} \cdot d\mathbf{S} = V_x dy \wedge dz + V_y dz \wedge dx + V_x dx \wedge dy$ and $d\alpha = \nabla \cdot \mathbf{V} dx \wedge dy \wedge dz \equiv \nabla \cdot \mathbf{V} dV$. Thus, the differential form of Gauss's divergence theorem may be written using the exterior derivative in the form:

$$d(\mathbf{V} \cdot d\mathbf{S}) = \nabla \cdot \mathbf{V} dV. \tag{5.8}$$

Elementary derivations of the Lie derivatives: $\mathcal{L}_{\mathbf{u}}f$, $\mathcal{L}_{\mathbf{u}}\alpha$ and $\mathcal{L}_{\mathbf{u}}\mathbf{w}$ where f is a 0-form (function), $\alpha = \mathbf{A} \cdot d\mathbf{x}$ is a 1-form and $\mathbf{w} = w^x \partial/\partial x + w^y \partial/\partial y + w^z \partial/\partial z$ is a vector field, are given in Appendix A.

Misner et al. (1973) and Schutz (1980) discuss the geometrical picture of the commutator of two vector fields in terms of the closure of the quadrilateral associated with the two vector fields. The Faraday two form is visualized in terms of an egg-crate like structure by Misner et al. (1973). The magnetic field two-form $\mathbf{B} \cdot d\mathbf{S}$, has a geometric structure associated with the oriented surface element $d\mathbf{S}$ and the vector field $\mathbf{B}$, which describes magnetic flux tubes. Lie dragging of vectors, differential forms and tensors involves parallel transport, in which the change in the geometric quantity at a point along a Lie orbit or trajectory, is pulled back to the initial point. It is useful in applications to use the dual of vector fields and forms using the hodge star formalism (e.g. Flanders 1963; Frankel 1997; Fecko 2006).

Basic properties of the wedge product, $\wedge$, of the exterior derivative d and the Lie derivative $\mathcal{L}_{\mathbf{V}}$ are given below.

If ω be a p-form, σ a q-form, f a 0-form, c a constant, $\mathbf{V}$ and $\mathbf{W}$ be vector fields, then:

$$\omega \wedge \sigma = (-1)^{pq}\sigma \wedge \omega,$$

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^p \omega \wedge d\sigma,$$

$$dd\omega = 0, \quad dc = 0, \tag{5.9}$$

$$(\mathbf{V} + \mathbf{W}) \lrcorner \omega = \mathbf{V} \lrcorner \omega + \mathbf{W} \lrcorner \omega, \quad (f\mathbf{V}) \lrcorner \omega = f(\mathbf{V} \lrcorner \omega),$$

$$\mathbf{V} \lrcorner (\omega \wedge \sigma) = (\mathbf{V} \lrcorner \omega) \wedge \sigma + (-1)^p \omega \wedge (\mathbf{V} \lrcorner \sigma). \tag{5.10}$$

Cartan's magic formula for the Lie derivative of the p form ω:

$$\mathcal{L}_{\mathbf{V}}\omega = \mathbf{V} \lrcorner d\omega + d(\mathbf{V} \lrcorner \omega), \tag{5.11}$$

is useful in applications. Further Lie derivative formulae are:

$$\mathcal{L}_{\mathbf{V}}f = \mathbf{V} \lrcorner df, \quad \mathcal{L}_{\mathbf{V}}d\omega = d(\mathcal{L}_{\mathbf{V}}\omega),$$

$$\mathcal{L}_{\mathbf{V}}(\omega \wedge \sigma) = (\mathcal{L}_{\mathbf{V}}\omega) \wedge \sigma + \omega \wedge (\mathcal{L}_{\mathbf{V}}\sigma),$$

$$\mathcal{L}_{\mathbf{V}}(\mathbf{W} \lrcorner \omega) = [\mathbf{V}, \mathbf{W}] \lrcorner \omega + \mathbf{W} \lrcorner (\mathcal{L}_{\mathbf{V}}\omega). \tag{5.12}$$

5.1.1   Exterior Derivative Formula Relations (Vector Notation)

Let $\mathbf{X}$ and $\mathbf{V}$ be vector fields. Then:

$$df = \nabla f \cdot d\mathbf{x},$$

$$d(\mathbf{V} \cdot d\mathbf{x}) = (\nabla \times \mathbf{V}) \cdot d\mathbf{S} \quad \text{(Stokes thm)},$$

$$d(\mathbf{A} \cdot d\mathbf{S}) = (\nabla \cdot \mathbf{A})dV \quad \text{(Gauss thm)},$$

$$d^2f = d(\nabla f \cdot d\mathbf{x}) = (\nabla \times \nabla f) \cdot d\mathbf{S} = 0 \quad \text{(Poincaré Lemma)},$$

$$d^2(\mathbf{V} \cdot d\mathbf{x}) = d[(\nabla \times \mathbf{V}) \cdot d\mathbf{S}] = \nabla \cdot (\nabla \times \mathbf{V})dV = 0 \quad \text{(Poincaré Lemma)}$$

$$\mathbf{X}_\lrcorner(\mathbf{V} \cdot d\mathbf{x}) = \mathbf{V} \cdot \mathbf{X},$$

$$\mathbf{X}_\lrcorner(\mathbf{B} \cdot d\mathbf{S}) = -(\mathbf{X} \times \mathbf{B}) \cdot d\mathbf{x},$$

$$\mathbf{X}_\lrcorner dV = \mathbf{X} \cdot d\mathbf{S},$$

$$d(\mathbf{X}_\lrcorner dV) = d(\mathbf{X} \cdot d\mathbf{S}) = (\nabla \cdot \mathbf{X})dV. \tag{5.13}$$

Let $\boldsymbol{\omega}$ be a p-form, then $dd\omega = 0$. It implies the equality of mixed second order partial derivatives. If $\omega = d\alpha$ is a p-form, where α is a global form of order $p - 1$, then ω is exact. A form ω with $d\omega = 0$ is closed. Not all closed forms are exact. Exactness means 'globally integrable'. Inexact forms, imply in many cases, that the geometric structure has a non-trivial topological structure (e.g. the MHD topological soliton: Kamchatnov 1982; Semenov et al. 2002).

5.1.2 Lie Derivative Relations

$$\mathcal{L}_\mathbf{X}f = \mathbf{X}_\lrcorner df = \mathbf{X} \cdot \nabla f,$$

$$\mathcal{L}_\mathbf{X}(\mathbf{V} \cdot d\mathbf{x}) = (-\mathbf{X} \times (\nabla \times \mathbf{V}) + \nabla(\mathbf{X} \cdot \mathbf{V})) \cdot d\mathbf{x},$$

$$\mathcal{L}_\mathbf{X}(\mathbf{B} \cdot d\mathbf{S}) = (-\nabla \times (\mathbf{X} \times \mathbf{B}) + \mathbf{X}(\nabla \cdot \mathbf{B})) \cdot d\mathbf{S},$$

$$\mathcal{L}_\mathbf{X}(fdV) = \nabla \cdot (\mathbf{X}f)dV. \tag{5.14}$$

For vector fields $\mathbf{X}$ and $\mathbf{Y}$

$$\mathcal{L}_\mathbf{X}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}] = (\mathbf{X} \cdot \mathbf{Y} - \mathbf{Y} \cdot \mathbf{X}) \cdot \nabla \equiv \mathrm{ad}_\mathbf{X}(\mathbf{Y}). \tag{5.15}$$

Here $[\mathbf{X}, \mathbf{Y}]$ is the left Lie bracket.

For a 1-form density $\mathsf{m} = \mathbf{m} \cdot d\mathbf{x} \otimes dV$, the Lie derivative of m with respect to the vector field $\mathbf{x}$ is given by:

$$\mathcal{L}_\mathbf{X}\mathsf{m} = \left(\nabla \cdot (\mathbf{X} \otimes \mathbf{m}) + (\nabla\mathbf{X})^T \cdot \mathbf{m}\right) \cdot d\mathbf{x} \otimes dV =: \mathsf{ad}_\mathbf{X}^*\mathsf{m}. \tag{5.16}$$

The pairing between the one form density m and the vector field $\mathbf{u}$ is defined by the inner product:

$$\langle \mathsf{m}, \mathbf{u} \rangle = \int_\Omega \mathbf{u}_\lrcorner \mathbf{m} \, dV. \tag{5.17}$$

Vector fields are either left or right invariant vector fields. Associated with the group transformation $\mathbf{x} = g\mathbf{x}_0$, the right invariant vector field $\mathbf{u} = \dot{g}\mathbf{x}_0 = \dot{g}g^{-1}\mathbf{x}$ defines the right invariant vector field $\mathbf{u} = \dot{g}g^{-1}$. The left invariant version of the same vector field is $\mathbf{v} = g^{-1}\dot{g}$. The right and left Lie brackets are related by: $[\mathbf{U}, \mathbf{V}]_R = -[\mathbf{U}, \mathbf{V}]_L$. The left Lie bracket is used in (5.15). The right Lie bracket in (5.16) is given by:

$$ad_{\mathbf{U}}(\mathbf{V}) = [\mathbf{U}, \mathbf{V}]_R = (\mathbf{V} \cdot \nabla \mathbf{U} - \mathbf{U} \cdot \nabla \mathbf{V}) \cdot \nabla. \tag{5.18}$$

Further discussion of the difference between right and left vector fields of a Lie algebra are given by Marsden and Ratiu (1994), Holm et al. (1998), Holm (2008a,b) and Fecko (2006).

5.1.3   Lie Dragging of Forms and Vector Fields

Formulas for the Lie dragging of 0-forms, 1-forms, 2-forms, 3-forms and vector fields are given below. These formulae are very useful in describing advected invariants.

For 0-forms or functions I:

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \mathbf{u} \cdot \nabla I = 0. \tag{5.19}$$

For 1-forms: $\mathbf{S} \cdot d\mathbf{x}$

$$\frac{d}{dt}(\mathbf{S} \cdot d\mathbf{x}) = \left(\frac{\partial \mathbf{S}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{S}) + \nabla(\mathbf{u} \cdot \mathbf{S}) \right) \cdot d\mathbf{x} = 0. \tag{5.20}$$

For 2-forms $\mathbf{B} \cdot d\mathbf{S}$:

$$\frac{d}{dt}(\mathbf{B} \cdot d\mathbf{S}) = \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}(\nabla \cdot \mathbf{B}) \right) \cdot d\mathbf{S} = 0. \tag{5.21}$$

For 3-forms $\rho dx \wedge dy \wedge dz$:

$$\frac{d}{dt}(\rho dx \wedge dy \wedge dz) = \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) \right) dx \wedge dy \wedge dz = 0. \tag{5.22}$$

For vector fields (the dual of one-forms): $\mathbf{J} = J^i \nabla_i$:

$$\frac{d\mathbf{J}}{dt} = \frac{\partial \mathbf{J}}{\partial t} + [\mathbf{u}, \mathbf{J}] = 0, \quad \text{where} \quad [\mathbf{u}, \mathbf{J}] = \left(\mathbf{u} \cdot \nabla J^i - \mathbf{J} \cdot \nabla u^i \right) \nabla_i, \tag{5.23}$$

is the Lie bracket of $\mathbf{u}$ and $\mathbf{J}$.

5.2 Applications

Tur and Yanovsky (1993) derive a variety of forms and vector fields that are Lie dragged with the flow. Some of these invariants are obtained by combining old invariants or known invariants, using the wedge product, Lie derivative and contraction of vector fields and forms. The inner product of an invariant vector field and an invariant 1-form is an invariant scalar. Similarly, the wedge product of an invariant p-form and an invariant q-form is an invariant $p + q$ form. The exterior derivative of an invariant p-form is an invariant $p + 1$ form. The Lie derivative of an invariant p-form, with respect to an invariant vector field $\mathbf{V}$ is an invariant p-form. These theorems were given in Tur and Yanovsky (1993) (see also the next subsection).

Below we list some invariants based on invariant 0-forms, 1-forms, 2-forms, 3-forms and vector fields. We use the results (5.19)–(5.23) and Cartan's magic formula (5.28) to discuss Faraday's equation described by the advection of the magnetic flux 2-form, the magnetic potential 1-form and the Lie dragged vector field $\mathbf{b} = \mathbf{B}/\rho$. We describe the entropy S (a 0-form) and the mass 3-form in terms of advection of invariant forms.

5.2.1 *Invariants Related to* $\mathbf{A}$ *and* $\mathbf{B}$ *in MHD*

There are many invariants in MHD involving $\mathbf{A}$ and $\mathbf{B}$ (Tur and Yanovsky 1993). However, for the magnetic vector potential 1-form $\mathbf{A} \cdot d\mathbf{x}$ to be Lie dragged by the flow, it is necessary to choose the gauge of $\mathbf{A}$ so that $\mathbf{u} \cdot \mathbf{A} = \phi_E$ where ϕ_E is the electric field potential that arises from uncurling Faraday's equation.

Some of these invariants are:

$$\mathbf{S}' = \nabla S(\mathbf{x}, t), \quad I' = \frac{\mathbf{A} \cdot \mathbf{B}}{\rho}, \quad \rho' = \mathbf{A} \cdot \mathbf{B}. \tag{5.24}$$

Here $\mathbf{S}' \cdot d\mathbf{x}$ is 1-form, I' is a scalar, and $\rho' dx \wedge dy \wedge dz$ is a three-form, which are Lie dragged by the the flow.

Another class of invariants (second generation) are:

$$I'' = \frac{\mathbf{B}}{\rho} \cdot \nabla S, \quad \mathbf{S}'' = \nabla \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho} \right), \quad \mathbf{J}'' = \frac{\mathbf{B}}{\mathbf{A} \cdot \mathbf{B}},$$

$$\rho'' = \mathbf{B} \cdot \nabla S, \quad \mathbf{J}_1 = \frac{1}{\rho} \left(\nabla S \times \mathbf{A} \right). \tag{5.25}$$

Here I'' is a 0-form, $\mathbf{S}'' \cdot d\mathbf{x}$ is a 1-form, $\mathbf{J}''$ is a vector field, $\rho'' dx \wedge dy \wedge dz$ is a 3-form and $\mathbf{J}_1$ is a vector field, that are invariant under Lie dragging with the flow.

5.2.2   Some Integral Invariants

$$\Gamma_1^1 = \oint_{\gamma(t)} \Phi \mathbf{A} \cdot d\mathbf{l}, \quad \Gamma_1^2 = \int_{\mathbf{S(t)}} \Phi \mathbf{B} \cdot d\mathbf{S}',$$

$$I_2^3 = \int_{\Omega(t)} \Phi(\mathbf{A} \cdot \mathbf{B}) \, d^3x,$$

$$I_3^4 = \int_{\Omega(t)} \Phi \mathbf{A} \cdot \left[\nabla S \times \nabla \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho} \right) \right] d^3x,$$

$$\Phi = \Phi \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho}, S, \frac{\mathbf{B}}{\mathbf{A} \cdot \mathbf{B}} \cdot \nabla \left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho} \right), \frac{\mathbf{B}}{\rho} \cdot \nabla \left(\frac{\mathbf{B} \cdot \nabla S}{\rho} \right) \dots \right). \quad (5.26)$$

There are many more examples.

- If $\Phi = 1$, Γ_1^1 is circulation of $\mathbf{A}$ (note $\phi_E = \mathbf{u} \cdot \mathbf{A}$).
- If $\Phi = 1$, Γ_1^2 is magnetic flux; I_2^3 is magnetic helicity.

5.3   Lie Dragging

In this section we use Faraday's equation (2.4) to illustrate the Lie dragging of 2-forms, 1-forms and vector fields discussed in (5.19)–(5.23). A key equation in the analysis is Cartan's magic formula (5.11).

Example 1 Consider Lie dragging the magnetic flux 2-form:

$$\beta = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \equiv \mathbf{B} \cdot d\mathbf{S}, \quad (5.27)$$

by the fluid velocity vector field $\mathbf{u} = u^x \partial_x + u^y \partial_y + u^z \partial_z = \mathbf{u} \cdot \nabla$. We determine $\mathcal{L}_{\mathbf{u}}(\beta)$ by using Cartan's magic formula (5.11):

$$\mathcal{L}_{\mathbf{u}}(\beta) = \mathbf{u} \lrcorner d\beta + d(\mathbf{u} \lrcorner \beta). \quad (5.28)$$

We obtain:

$$d\beta = \nabla \cdot \mathbf{B} \, dx \wedge dy \wedge dz,$$

$$\mathbf{u} \lrcorner d\beta = \nabla \cdot \mathbf{B} \, (\mathbf{u} \cdot d\mathbf{S}),$$

$$\mathbf{u} \lrcorner \beta = -(\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{x},$$

$$d(\mathbf{u} \lrcorner \beta) = -\nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}. \quad (5.29)$$

Use (5.29) in Cartan's formula (5.28) gives the equation:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\beta = \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}\nabla \cdot \mathbf{B}\right) \cdot d\mathbf{S} = 0, \qquad (5.30)$$

expressing the Lie dragging (5.21) of the magnetic flux $\mathbf{B} \cdot d\mathbf{S}$ with the flow. Equation (5.30) is Faraday's equation (2.4). By taking the divergence of (5.30) we obtain the continuity equation

$$\frac{\partial(\nabla \cdot \mathbf{B})}{\partial t} + \nabla \cdot (\mathbf{u}\nabla \cdot \mathbf{B}) = 0, \qquad (5.31)$$

which implies a non-zero $\nabla \cdot \mathbf{B}$ is advected with the flow. This result is used in numerical MHD in which numerically generated $\nabla \cdot \mathbf{B} \neq 0$ is advected with the flow (e.g. Powell et al. 1999; Janhunen 2000; Webb et al. 2009; Balsara and Kim 2004; Dedner et al. 2002; Evans and Hawley 1988).

Example 2 Lie drag the one form:

$$\alpha = A_x dx + A_y dy + A_z dz \equiv \mathbf{A} \cdot d\mathbf{x}. \qquad (5.32)$$

Use Cartan's magic formula: $\mathcal{L}_{\mathbf{u}}(\alpha) = \mathbf{u} \lrcorner d\alpha + d(\mathbf{u} \lrcorner \alpha)$ and the results:

$$d\alpha = (\nabla \times \mathbf{A}) \cdot d\mathbf{S}, \quad \mathbf{u} \lrcorner d\alpha = -[\mathbf{u} \times (\nabla \times \mathbf{A})] \cdot d\mathbf{x},$$
$$\mathbf{u} \lrcorner \alpha = (\mathbf{u} \cdot \mathbf{A}), \quad d(\mathbf{u} \lrcorner \alpha) = \nabla(\mathbf{u} \cdot \mathbf{A}) \cdot d\mathbf{x}, \qquad (5.33)$$

to obtain the Lie dragged 1-form equation for α:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\alpha = \left(\frac{\partial \mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla(\mathbf{u} \cdot \mathbf{A})\right) \cdot d\mathbf{x} = 0. \qquad (5.34)$$

Equation (5.34) is the same as (5.20) for the advection of a one-form, but with $\mathbf{S}$ replaced by $\mathbf{A}$. From (3.36) the uncurled form of Faraday's equation for $\mathbf{A}$ where $\mathbf{B} = \nabla \times \mathbf{A}$, is:

$$\frac{\partial \mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla\phi_E = 0. \qquad (5.35)$$

Here $\mathbf{E} = -\nabla\phi_E - \partial\mathbf{A}/\partial t$ is equivalent to Faraday's equation and ϕ_E is the electric field potential. Note that (5.35) is equivalent to the Lie dragged 1-form equation (5.34) if the gauge for $\mathbf{A}$ is chosen so that $\phi_E = \mathbf{u} \cdot \mathbf{A}$. For this particular gauge the magnetic helicity conservation equation (3.33) assumes the continuity equation form (3.42).

Example 3 Faraday's equation (2.4) can be written in the form:

$$\frac{d\mathbf{b}}{dt} - \mathbf{b} \cdot \nabla\mathbf{u} = 0, \qquad (5.36)$$

where $\mathbf{b} = \mathbf{B}/\rho$ and $d\mathbf{b}/dt = \partial\mathbf{b}/\partial t + \mathbf{u}\cdot\nabla\mathbf{b}$. Equation (5.36) comes from combining Faraday's equation (2.4) with the mass continuity equation (2.1). It can also be written in the form:

$$\frac{\partial\mathbf{b}}{\partial t} + [\mathbf{u}, \mathbf{b}] \equiv \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\mathbf{b} = 0. \tag{5.37}$$

Equation (5.37) is equivalent to (5.23) for the advection of the vector field $\mathbf{b} = b^i\partial/\partial x^i$, but with $\mathbf{J} \rightarrow \mathbf{b}$. Note that $\mathcal{L}_{\mathbf{u}}(\mathbf{b}) = [\mathbf{u}, \mathbf{b}]$ where $[\mathbf{u}, \mathbf{b}]$ is the Lie bracket of the vector fields $\mathbf{u}$ and $\mathbf{b}$ (see Appendix A).

The Lie dragged vector field equation (5.37) may be directly integrated to give the solution for $\mathbf{b}$. First note that (5.37) implies that

$$b^i\frac{\partial}{\partial x^i} = b_0^j\frac{\partial}{\partial x_0^j} \quad \text{or} \quad b^i = x_{ij}b_0^j, \tag{5.38}$$

where $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ is the Lagrangian map and $x_{ij} = \partial x^i/\partial x_0^j$. This is the Cauchy solution for the magnetic field (e.g. Newcomb 1962). To see this explicitly, it is necessary to use the mass conservation equation in the form

$$\rho d^3x = \rho_0 d^3x_0 \quad \text{which implies} \quad \rho = \frac{\rho_0}{J}, \tag{5.39}$$

where $J = \det(x_{ij})$ is the Jacobian of the Lagrangian map. Combining (5.38) and (5.39) gives:

$$B^i = \frac{x_{ij}B_0^j}{J}, \tag{5.40}$$

which is the Cauchy solution for B^i given by Newcomb (1962).

5.3.1 Entropy and Mass Advection

The entropy $S = S(x_0)$, a 0-form (i.e. a function) which is Lie dragged with the fluid, i.e.

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)S \equiv \frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S = 0, \tag{5.41}$$

which is (5.19) for the advection of a 0-form I, but with $I \rightarrow S$. The integral of (5.41) is $S = S_0(\mathbf{x}_0)$, where $\mathbf{x}_0$ is the Lagrange fluid label for which $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$.

Consider the mass 3-form:

$$\beta = \rho \, dx \wedge dy \wedge dz. \tag{5.42}$$

Using Cartan's formula (5.28) we find $d\beta = 0$ as β is a 3-form in 3D xyz-space, and $\mathbf{u} \lrcorner \beta = \rho \mathbf{u} \cdot d\mathbf{S}$, which implies:

$$\mathcal{L}_{\mathbf{u}}(\beta) = 0 + d(\mathbf{u} \lrcorner \beta) = \nabla \cdot (\rho \mathbf{u}) dx \wedge dy \wedge dz, \tag{5.43}$$

and

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) \beta = \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) d^3 x = 0. \tag{5.44}$$

Equation (5.44) is the same as (5.22) for an advected 3-form $\rho dx \wedge dy \wedge dz$. The integral of (5.44) is:

$$\rho d^3 x = \rho_0 d^3 x_0, \quad \text{where} \quad \rho = \rho_0(\mathbf{x}_0)/J, \quad J = \det(x_{ij}). \tag{5.45}$$

Thus the mass continuity, entropy advection and Faraday's equation can all be expressed in terms of the Lie dragging of forms by the vector field $\mathbf{u}$.

5.4 Theorems for Advected Invariants

Theorem 5.4.1 *If ω^p is an invariant, then $\omega^{p+1} = d\omega^p$ is an invariant $(p+1)$-form.*

Proof ω^p is invariant implies:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) \omega^p = 0. \tag{5.46}$$

Take d of (5.46). Use $d\partial_t = \partial_t d$, and $d\mathcal{L}_{\mathbf{u}} = \mathcal{L}_{\mathbf{u}} d$ gives (5.46) but with $\omega^p \rightarrow \omega^{p+1}$.
                                                                                                                               □

Example The entropy S is a scalar invariant implies $\alpha = dS = \nabla S \cdot d\mathbf{x}$ is a conserved 1-form. Note that

$$\frac{\partial \nabla S}{\partial t} - \mathbf{u} \times (\nabla \times \nabla S) + \nabla(\mathbf{u} \cdot \nabla S) = 0, \tag{5.47}$$

shows that ∇S satisfies (5.20). From (5.47) and noting $dS/dt \equiv \partial_t S + \mathbf{u} \cdot \nabla S = 0$, implies α is Lie dragged with the flow:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) \alpha = \nabla \left(\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S \right) = 0. \tag{5.48}$$

Similarly, $dI_e = \nabla I_e \cdot d\mathbf{x}$ is conserved 1-form which is Lie dragged with the flow, where $I_e = \boldsymbol{\omega} \cdot \nabla S / \rho$ is the Ertel invariant.

Theorem 5.4.2 *Let* ω_1^k *and* ω_2^l *be advected k and l-form invariants, then* $\omega^{k+l} = \omega_1^k \wedge \omega_2^l$ *is an advected $(k + l)$-form invariant.*

Proof Use

$$\frac{\partial}{\partial t} (\omega_1 \wedge \omega_2) = \frac{\partial \omega_1}{\partial t} \wedge \omega_2 + \omega_1 \wedge \frac{\partial \omega_2}{\partial t}$$

$$\mathcal{L}_{\mathbf{u}} (\omega_1 \wedge \omega_2) = \mathcal{L}_{\mathbf{u}} (\omega_1) \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_{\mathbf{u}} (\omega_2), \tag{5.49}$$

to get

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (\omega_1 \wedge \omega_2) = 0. \tag{5.50}$$

$\square$

Example $\omega_1 = \mathbf{S}_1 \cdot d\mathbf{x}$ and $\omega_2 = \mathbf{S}_2 \cdot d\mathbf{x}$ are advected one-forms, then

$$\omega_1 \wedge \omega_2 = (\mathbf{S}_1 \times \mathbf{S}_2) \cdot d\mathbf{S}, \tag{5.51}$$

is an advected 2-form, and $(\omega_1 \wedge \omega_2)/\rho$ is an advected invariant vector field.

Theorem 5.4.3 *If ω is a conserved p-form, and $\mathbf{J}$ is a conserved vector, then* $\omega^{(p-1)} = \mathbf{J} \lrcorner \omega$ *is a conserved $(p - 1)$ form.*

Proof Use

$$\mathcal{L}_{\mathbf{u}} (\mathbf{J} \lrcorner \omega) = [\mathbf{u}, \mathbf{J}] \lrcorner \omega + \mathbf{J} \lrcorner (\mathcal{L}_{\mathbf{u}} \omega), \quad (\partial_t + \mathcal{L}_{\mathbf{u}}) \omega = 0,$$

$$[\partial_t, \mathbf{J}] = \mathbf{J}_t, \quad \partial_t (\mathbf{J} \lrcorner \omega) = [\partial_t, \mathbf{J}] \lrcorner \omega + \mathbf{J} \lrcorner (\omega_t),$$

$$(\partial_t + \mathcal{L}_{\mathbf{u}}) (\mathbf{J} \lrcorner \omega) = [\partial_t + \mathbf{u}, \mathbf{J}] \lrcorner \omega + \mathbf{J} \lrcorner [(\partial_t + \mathcal{L}_{\mathbf{u}}) \omega]. \tag{5.52}$$

But since $\mathbf{J}$ is an invariant vector field, then

$$[\partial_t + \mathbf{u}, \mathbf{J}] = [\partial_t, \mathbf{J}] + [\mathbf{u}, \mathbf{J}] = \mathbf{J}_t + [\mathbf{u}, \mathbf{J}] = 0. \tag{5.53}$$

and the last equation in (5.52) reduces to:

$$(\partial_t + \mathcal{L}_{\mathbf{u}}) (\mathbf{J} \lrcorner \omega) = 0, \tag{5.54}$$

which proves the theorem. Note that $[\partial_t, \mathbf{J}] = \mathbf{J}_t$ and $\mathcal{L}_{\mathbf{u}} \mathbf{J} = [\mathbf{u}, \mathbf{J}]$. $\square$

Theorem 5.4.4 *If ω is an invariant p-form, and $\mathbf{J}$ is an invariant vector field, then* $\omega' = \mathcal{L}_{\mathbf{J}} \omega$ *is an invariant p-form.*

Proof Use Cartan's magic formula:

$$\mathcal{L}_{\mathbf{J}}\omega = \mathbf{J}\lrcorner d\omega + d\left(\mathbf{J}\lrcorner\omega\right). \tag{5.55}$$

Note $d\omega$ is invariant (Theorem 5.4.1), and $\mathbf{J}\lrcorner d\omega$ is invariant (Theorem 5.4.3); also $\mathbf{J}\lrcorner\omega$ is invariant (Theorem 5.4.3) and $d\left(\mathbf{J}\lrcorner\omega\right)$ is invariant (Theorem 5.4.1). Net result: $\omega' = \mathcal{L}_{\mathbf{J}}\omega$ is an invariant p-form. □

Theorem 5.4.5 *If $\mathbf{J}_1$ and $\mathbf{J}_2$ are invariant vector fields then so is $[\mathbf{J}_1, \mathbf{J}_2]$ iff $\{\mathbf{J}_1, \mathbf{J}_2, \mathbf{u}\}$ are elements of a Lie algebra.*

Proof It is based on the equations

$$\frac{\partial \mathbf{J}_1}{\partial t} + [\mathbf{u}, \mathbf{J}_1] = 0, \quad \frac{\partial \mathbf{J}_2}{\partial t} + [\mathbf{u}, \mathbf{J}_2] = 0. \tag{5.56}$$

Using (5.56) we obtain:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)[\mathbf{J}_1, \mathbf{J}_2] &= \frac{\partial}{\partial t}[\mathbf{J}_1, \mathbf{J}_2] + \mathcal{L}_{\mathbf{u}}[\mathbf{J}_1, \mathbf{J}_2] \\
&= [\mathbf{J}_{1,t}, \mathbf{J}_2] + [\mathbf{J}_1, \mathbf{J}_{2,t}] + [\mathbf{u}, [\mathbf{J}_1, \mathbf{J}_2]] \\
&= [-[\mathbf{u}, \mathbf{J}_1], \mathbf{J}_2] + [\mathbf{J}_1, -[\mathbf{u}, \mathbf{J}_2] + [\mathbf{u}, [\mathbf{J}_1, \mathbf{J}_2] \\
&= [\mathbf{u}, [\mathbf{J}_1, \mathbf{J}_2]] + [\mathbf{J}_1, [\mathbf{J}_2, \mathbf{u}]] + [\mathbf{J}_2, [\mathbf{u}, \mathbf{J}_1]]. \tag{5.57}
\end{aligned}$$

If $\mathbf{u}, \mathbf{J}_1, \mathbf{J}_2$ are vector fields that are elements of a Lie algebra, then the right handside of (5.57) is zero by the Jacobi identity for $\mathbf{J}_1, \mathbf{J}_2$ and $\mathbf{u}$. This proves the theorem. □

5.4.1 Comment

The question of Lie algebraic structures for fluid relabeling symmetries has been addressed by Volkov et al. (1995). Their work shows that there is a hidden supersymmetry in hydrodynamical systems (i.e. ideal MHD and hydrodynamics), with respect to the odd Buttin bracket.

5.5 Magnetic Helicity

$\alpha = \mathbf{A} \cdot d\mathbf{x}$ is advected one form for the magnetic vector potential, provided the gauge for $\mathbf{A}$ is chosen so that $\phi_E = \mathbf{u} \cdot \mathbf{A}$. The Lie dragging condition for $\mathbf{A}$ is:

$$\frac{\partial \mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla(\mathbf{u} \cdot \mathbf{A}) = 0. \tag{5.58}$$

This equation can be written as $d\mathbf{A}/dt + (\nabla\mathbf{u})^T \cdot \mathbf{A} = 0$. The magnetic flux 2-form $\beta = \mathbf{B} \cdot d\mathbf{S}$ and the vector field $\mathbf{b} = \mathbf{B}/\rho$ are Lie dragged with the flow. Thus, $\mathbf{b} \; \lrcorner \; (\mathbf{A} \cdot d\mathbf{x}) \equiv \mathbf{A} \cdot \mathbf{B}/\rho$ is a Lie dragged scalar invariant. Thus, we obtain the magnetic helicity conservation law:

$$\frac{d}{dt}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho}\right) = 0 \quad \text{or} \quad \frac{\partial h_m}{\partial t} + \nabla \cdot (h_m \mathbf{u}) = 0, \tag{5.59}$$

where $h_m = \mathbf{A} \cdot \mathbf{B}$ is the magnetic helicity in the gauge $\phi_E = \mathbf{u} \cdot \mathbf{A}$. Gordin and Petviashvili (1987, 1989) obtained these results using the advected $\mathbf{A}$ gauge where $\mathbf{A}$ satisfies (5.58).

5.6 The Ertel Invariant and Related Invariants

In this section we discuss Ertel's theorem in gas dynamics, and the generalization of Ertel's equation to MHD (e.g. Kats 2003). The MHD generalization of Ertel's theorem uses the Clebsch variable representation of the fluid velocity, that arises from using Lagrangian constraints in the variational principle for MHD discussed by Zakharov and Kuznetsov (1997) (see Appendix B). We also discuss the Hollmann (1964) invariant, which is related to the Ertel invariant (e.g. Tur and Yanovsky 1993). The Ertel invariant is:

$$I_e = \frac{\omega \cdot \nabla S}{\rho} \quad \text{where} \quad \omega = \nabla \times \mathbf{u}. \tag{5.60}$$

To derive the Ertel invariant we use the Clebsch representation for $\mathbf{u}$:

$$\mathbf{u} = \nabla\phi - r\nabla S - \lambda\nabla\mu,$$

$$\phi = \int_0^t \left(\frac{1}{2}|\mathbf{u}|^2 - h\right)(\mathbf{x}_0, t')\,dt', \quad r = -\int_0^t T_0(\mathbf{x}_0, t')\,dt', \tag{5.61}$$

where $h = (p + \varepsilon)/\rho =$ is the enthalpy, S is the entropy, ϕ is the velocity potential, and $T_0(\mathbf{x}_0, t) = T(\mathbf{x}, t)$ is the temperature. λ and μ are related to the Lin constraints associated with vorticity in a Lagrangian variational principle with constraints (e.g. Zakharov and Kuznetsov (1997), see also Appendix B). The Clebsch variable representation for $\mathbf{u}$ is related to Weber transformations (Appendix C).

Let

$$\mathbf{w} = \mathbf{u} - \nabla\phi + r\nabla S \equiv -\lambda\nabla\mu, \tag{5.62}$$

$\nabla \times \mathbf{w} = -\nabla\lambda \times \nabla\mu$ represents the component of the vorticity of the fluid that is not generated by entropy gradients, i.e. it does not depend on ∇S. The one-form $\alpha = \mathbf{w} \cdot d\mathbf{x}$ is Lie dragged with the fluid. Thus $\mathbf{w}$ satisfies Eq. (5.20):

$$\frac{\partial \mathbf{w}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{w}) + \nabla(\mathbf{u} \cdot \mathbf{w}) = 0. \tag{5.63}$$

Using (5.37) it follows that $\mathbf{b} = (\nabla \times \mathbf{w})/\rho$ is a Lie dragged vector field. By Theorem 5.4.1, $\nabla S \cdot d\mathbf{x}$ is a conserved 1-form. Thus, $\mathbf{b} \lrcorner (\nabla S \cdot d\mathbf{x}) = \mathbf{b} \cdot \nabla S$ is a conserved scalar. Inspection of $\mathbf{b} \cdot \nabla S$ reveals that:

$$I_e \equiv \mathbf{b} \cdot \nabla S = \frac{\nabla \times (\mathbf{u} + r\nabla S - \nabla\phi)}{\rho} \cdot \nabla S = \frac{\nabla \times \mathbf{u}}{\rho} \cdot \nabla S, \tag{5.64}$$

is the Ertel invariant.

Theorem 5.6.1 *The generalization for the Ertel invariant in MHD, given by Kats (2003) is:*

$$I_e^{(m)} = \frac{\nabla \times (\mathbf{u} - \mathbf{u}_M)}{\rho} \cdot \nabla S, \tag{5.65}$$

where

$$\mathbf{u}_M = -\frac{(\nabla \times \boldsymbol{\Gamma}) \times \mathbf{B}}{\rho} - \boldsymbol{\Gamma}\frac{(\nabla \cdot \mathbf{B})}{\rho}, \tag{5.66}$$

$$\frac{\partial \boldsymbol{\Gamma}}{\partial t} - \mathbf{u} \times (\nabla \times \boldsymbol{\Gamma}) + \nabla(\boldsymbol{\Gamma} \cdot \mathbf{u}) = -\frac{\mathbf{B}}{\mu_0}, \tag{5.67}$$

and μ_0 is the magnetic permeability. We can also write (5.67) as:

$$\frac{d}{dt}(\boldsymbol{\Gamma} \cdot d\mathbf{x}) = -\frac{\mathbf{B} \cdot d\mathbf{x}}{\mu_0}. \tag{5.68}$$

Proof Use the Clebsch representation for $\mathbf{u}$:

$$\mathbf{u} = \nabla\phi - r\nabla S - \tilde{\lambda}\nabla\mu + \mathbf{u}_M. \tag{5.69}$$

Inspection shows that $\mathbf{w}$ satisfies the equation:

$$\mathbf{w} = \mathbf{u} - \nabla\phi + r\nabla S - \mathbf{u}_M \equiv -\tilde{\lambda}\nabla\mu, \tag{5.70}$$

and hence $\alpha = \mathbf{w} \cdot d\mathbf{x}$ is an invariant 1-form. It follows that $\mathbf{b} = \nabla \times \mathbf{w}/\rho$ is a Lie advected vector field. By Theorem 5.4.1, $dS = \nabla S \cdot d\mathbf{x}$ is an invariant advected 1-form; Thus, $I_e^m = \mathbf{b} \cdot \nabla S$ is an invariant scalar, given by:

$$\begin{aligned} I_e^m &= \nabla \times (\mathbf{u} - \nabla\phi + r\nabla S - \mathbf{u}_M) \cdot \nabla S/\rho \\ &= [\nabla \times (\mathbf{u} - \mathbf{u}_M) + \nabla r \times \nabla S] \cdot \nabla S/\rho \\ &\equiv \nabla \times (\mathbf{u} - \mathbf{u}_M) \cdot \nabla S/\rho. \end{aligned}$$

$$\tag{5.71}$$

The quantity I_e^m is the MHD analogue of the Ertel invariant. It reduces to the Ertel invariant in the case where $\mathbf{b}$ and $\mathbf{u}_M$ are zero. $\square$

Theorem 5.6.2 *The Hollmann invariant is:*

$$I_h = (\mathbf{u} - \nabla\phi) \cdot \frac{\nabla S \times \nabla I_e}{\rho} \quad where \quad I_e = \frac{(\nabla \times \mathbf{u})\cdot\nabla S}{\rho}, \tag{5.72}$$

is the Ertel invariant. Here ϕ is the Clebsch potential in (5.61) associated with potential flow. The Hollmann invariant I_h is Lie dragged with the flow.

Proof $\omega_1 = \nabla S \cdot d\mathbf{x}$ and $\omega_2 = \nabla I_e \cdot d\mathbf{x}$ are conserved one-forms. Thus, $\omega = \omega_1 \wedge \omega_2 = (\nabla S \times \nabla I_e) \cdot d\mathbf{S}$ is a conserved two form, and

$$\mathbf{b} = \nabla S \times \nabla I_e/\rho, \tag{5.73}$$

is a conserved vector. $\alpha = \mathbf{w} \cdot d\mathbf{x}$ is a conserved one-form, where

$$\mathbf{w} = \mathbf{u} - \nabla\phi + r\nabla S, \tag{5.74}$$

and $\mathbf{w}$ satisfies the equation:

$$\frac{\partial \mathbf{w}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{w}) + \nabla(\mathbf{u} \cdot \mathbf{w}) = 0. \tag{5.75}$$

Using (5.73) and (5.74) it follows that

$$I_h = \mathbf{w} \cdot \mathbf{b} \equiv (\mathbf{u} - \nabla\phi) \cdot \frac{\nabla S \times \nabla I_e}{\rho}, \tag{5.76}$$

is a scalar invariant (i.e. the Hollmann invariant). $\square$

Similarly, the MHD version of the Hollmann invariant is:

$$I_h^m = \mathbf{w}_m \cdot \mathbf{b}_m = (\mathbf{u} - \mathbf{u}_M - \nabla\phi) \cdot \left(\frac{\nabla S \times \nabla I_e^m}{\rho}\right), \tag{5.77}$$

where

$$\mathbf{w}_m = \mathbf{u} - \nabla\phi + r\nabla S - \mathbf{u}_M, \quad \mathbf{b}_m = \frac{\nabla S \times \nabla I_e^m}{\rho}. \tag{5.78}$$

Chapter 6
Topological Invariants

In this chapter we discuss topological invariants of MHD and gas dynamics. Topological invariants and integrals of differential forms over a volume V that are non-zero are sometimes referred to as topological charges. A more complete discussion is given by Tur and Yanovsky (1993). Topological fluid dynamics and invariants are discussed in more detail in Arnold (1974), Arnold and Khesin (1998), Berger and Field (1984), and in many other works. Tur and Yanovsky (2017) present examples of vortices in two fluid plasmas with nontrivial topology in which the streamlines and magnetic field lines are linked. They also discuss MHD topological solitons.

Below we first recall the definitions of closed and exact differential forms. This is followed a discussion of the magnetic monopole and non-global **A**, advected invariants, the Hopf invariant, the Calugareanu invariant, the *Link = Twist + Writhe* formula for the linkage of magnetic flux tubes, link numbers and signed crossing numbers in knot theory, Dehn surgery and magnetic reconnection, and the Godbillon-Vey invariant. This is followed by a sequence of examples of magnetic helicity in space plasmas and in MHD (the Parker (1958) interplanetary magnetic field, Alfvén simple waves, MHD topological solitons and the Hopf fibration). Appendix E discusses the Aharonov-Bohm interpretation of magnetic helicity, cross helicity and fluid helicity for both barotropic and non-barotropic fluids developed by Yahalom (2013, 2016a,b, 2017a,b) (see also Webb and Anco 2017).

6.1 Closed and Exact Differential Forms

Definition 1 A p-form ω^p is closed if its exterior derivative $d\omega^p = 0$.

Definition 2 A p-form ω^p is exact if it can be expressed as the exterior derivative of a $(p-1)$-form ω^{p-1}, i.e., $\omega^p = d\omega^{p-1}$. It is assumed that ω^p and ω^{p-1} are

sufficiently smooth and differentiable on a star-shaped region of the manifold on which the forms are defined.

Lemma 6.1.1 (Poincaré) *The Poincaré Lemma states that if X is a contractible open set of R^n, then any closed p-form defined on X is exact, for any integer $0 < p \leq n$.*

Definition Contractibility means that there is a homotopy $F_t : X \times [0, 1] \to X$ that continuously deforms X to a point. Thus every cycle c in X is the boundary of some cone. One can take the cone to be the image of c under the homotopy. A dual version of this result gives the Poincaré Lemma.

From the above definitions, it follows that an exact p-form is closed, but a closed p-form is not necessarily exact. To verify these statements, note that if ω^p is exact, then $\omega^p = d\omega^{p-1}$ for some $p - 1$ form ω^{p-1}. By the Poincaré Lemma, $d\omega^p = dd\omega^{p-1} = 0$ (i.e. the Poincaré Lemma states that $dd\alpha = 0$ for a differential form α, where α is sufficiently differentiable, i.e. at least twice differentiable on the star shaped region of the manifold M on which the form is defined). However, a closed form ω^p with $d\omega^p = 0$ is not necessarily exact, i.e. there might not exist a $(p - 1)$ form such that $\omega^p = d\omega^{p-1}$. The word exact is synonymous with the notion of global integrability.

6.1.1   The Magnetic Monopole and Non-global A

The notion of exactness means that given a p-form ω^p, there exists a *global* $(p - 1)$-form ω^{p-1} such that $\omega^p = d\omega^{p-1}$. In that case $d\omega^p = dd\omega^{p-1} = 0$ by the Poincaré Lemma, and hence the exact form ω^p is closed. A counter example to this situation is that of a magnetic monopole field (e.g. Urbantke 2003; Webb et al. 2010a). Webb et al. (2010a) point out that the split monopole solution is applicable to the description of the Parker, interplanetary spiral magnetic field, where the radial component of the field is outward (inward) above the interplanetary current sheet and inward (outward) below the sheet. In addition there is an azimuthal component of the field due to solar rotation. The radial component of the field is represented by the split monopole solution in which the polarity of the field changes across the current sheet.

For the monopole field $\mathbf{B} = I/r^2 \mathbf{e}_{\hat{r}}$ where $\mathbf{e}_{\hat{r}} = \mathbf{r}/r$ is the radial unit vector. In this case $\omega^2 = \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux 2-form, i.e. $\omega^2 = I \sin\theta d\theta d\phi$. It is not possible to define a global, continuous magnetic vector potential $\mathbf{A}$, because the solutions for the magnetic vector potential have singularities at the poles at $\theta = 0$ (north pole) and at $\theta = \pi$ (south pole). To circumvent these problems, define two open sets on the sphere S^2 of radius r: $U^+ := S^2 \backslash \{\text{south pole}\}$ and $U^- := S^2 \backslash \{\text{north pole}\}$. On the region $S_m^2 = U^+ \cap U^-$, excluding the poles the vector potentials

$$\mathbf{A}^+ = \frac{I(1 - \cos\theta)}{r \sin\theta} \mathbf{e}_{\hat{\phi}}, \quad \mathbf{A}^- = \frac{I(-1 - \cos\theta)}{r \sin\theta} \mathbf{e}_{\hat{\phi}}, \quad (6.1)$$

are analytic and bounded (i.e. excluding small regions about the poles). Thus, for the region U^+, $\mathbf{A}^+$ is analytic and $\mathbf{B} = \nabla \times \mathbf{A}^+$, whereas on U^-, $\mathbf{B} = \nabla \times \mathbf{A}^-$. It turns out that

$$\mathbf{A}^+ - \mathbf{A}^- = \frac{2I}{r \sin \theta} \mathbf{e}_{\hat{\phi}} = \nabla \Lambda, \tag{6.2}$$

where

$$\Lambda = 2I\phi[H(\phi) - H(\phi - 2\pi)], \tag{6.3}$$

where $H(x)$ is the Heaviside step function. Here Λ is discontinuous and has jumps of $4\pi I$ each period of 2π in ϕ.

- The main point of this example is that there is not a global $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$ which is valid over the whole sphere S^2. Note also that there are discontinuous jumps in the difference of the two magnetic vector potentials involved. The existence of jumps in the magnetic vector potentials is indicative of non-trivial magnetic topology (see Urbantke 2003; Webb et al. 2010a for further discussion).
- There does not exist a global 1-form $\alpha = \mathbf{A} \cdot d\mathbf{x}$ such that $\beta = d\alpha = \mathbf{B} \cdot d\mathbf{S}$. However, $d\beta = \nabla \cdot \mathbf{B} d^3x = 0$ everywhere, except in the immediate neighbourhood of $\mathbf{r} = 0$. Thus, β is closed meaning $d\beta = 0$ but β is not exact as there does not exist a global 1-form α with $\beta = d\alpha$.
- A similar jump in the magnetic vector potentials used to describe the MHD topological soliton occurs in the Euler potential representation for $\mathbf{A} = \alpha \nabla \beta + \nabla \Lambda$, where there exists jumps in the potential Λ (e.g. Kamchatnov 1982; Semenov et al. 2002).

6.2 Advected Invariants: Closed and Non-closed Forms

It is clear that invariant p-forms advected with the flow split into two classes: the forms are either closed, or they are not closed, i.e. either $d\omega^p = 0$ or $d\omega^p \neq 0$. The so-called $\mathbf{S}$ invariants (Tur and Yanovsky 1993), are invariant 1-forms of the form: $\alpha = \mathbf{A} \cdot d\mathbf{x}$, are in general not closed as the 2-form $d\alpha = \nabla \times \mathbf{A} \cdot d\mathbf{S} \neq 0$ if $\nabla \times \mathbf{A} \neq 0$. However, the exterior derivative of a 0-form I: $\beta = dI = \nabla I \cdot d\mathbf{x}$ is an advected invariant 1-form but $d\beta = ddI = (\nabla \times \nabla I) \cdot d\mathbf{S} = 0$. Thus $\beta = dI$ is a closed 1-form. Thus, there are both closed and non-closed invariant 1-forms advected by the flow (e.g. the entropy S is an advected 0-form and dS is an invariant, closed, advected 1-form).

Invariant 2-forms ω^2 can also be either *closed* or *not closed*. An example of a closed two form is the magnetic flux $\beta = \mathbf{B} \cdot d\mathbf{S}$ where $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{B}$ is the magnetic induction and $\mathbf{A}$ is the magnetic vector potential. The exterior derivative $d\beta = \nabla \cdot \mathbf{B} \, d^3x = 0$ as $\nabla \cdot \mathbf{B} = 0$. Alternatively, note that $d\beta = dd\alpha = 0$ where $\alpha = \mathbf{A} \cdot d\mathbf{x}$.

6.2.1   Topological Charge

If $\beta = \omega \cdot d\mathbf{S}$ is an advected invariant 2-form, then $\mathbf{J} = \boldsymbol{\omega}/\rho \equiv (\omega^i/\rho)\partial/\partial x^i$ is an invariant advected vector field, and $d\beta = \nabla \cdot \boldsymbol{\omega}\, d^3x \equiv \nabla \cdot (\rho\mathbf{J})d^3x \neq 0$ if $\nabla \cdot (\rho\mathbf{J}) \neq 0$. If $\nabla \cdot (\rho\mathbf{J}) \neq 0$, the integral $I^q = \int d\beta$ has non-zero *topological charge*. Examples of two-forms with non-zero topological charge can be constructed from the wedge product of two invariant 1-forms. For example, if

$$\omega^1_{S_1} = \mathbf{S}_1 \cdot d\mathbf{x}, \quad \omega^1_{S_2} = \mathbf{S}_2 \cdot d\mathbf{x}, \tag{6.4}$$

are invariant 1-forms, then

$$\omega^2 = \omega^1_{S_1} \wedge \omega^1_{S_2} = \mathbf{S}_1 \cdot d\mathbf{x} \wedge \mathbf{S}_2 \cdot d\mathbf{x} = (\mathbf{S}_1 \times \mathbf{S}_2) \cdot d\mathbf{S}, \tag{6.5}$$

is an invariant 2-form. Taking the exterior derivative of ω^2 gives

$$d\omega^2 = \nabla \cdot (\mathbf{S}_1 \times \mathbf{S}_2)\, d^3x. \tag{6.6}$$

In general $\nabla \cdot (\mathbf{S}_1 \times \mathbf{S}_2) \neq 0$, and hence the 3-form $d\omega^2$ has non-zero topological charge. More precisely, the topological charge for a volume $V = D_3(t)$ is given by the equivalent expressions:

$$I^q = \int_{D_3(t)} d\omega^2 = \int_{\partial D_3(t)} \omega^2 = \int_{\partial D_3(t)} (\mathbf{S}_1 \times \mathbf{S}_2) \cdot d\mathbf{S}. \tag{6.7}$$

Thus I^2 is zero if the normal component of $\mathbf{S}_1 \times \mathbf{S}_2$ is zero on the boundary $\partial D_3(t)$ of the volume $D_3(t)$ of the region of interest.

Example 1 For compressible ideal fluid flows:

$$\omega^1_1 = \nabla S \cdot d\mathbf{x}, \quad \omega^1_2 = (\mathbf{u} - \nabla\phi - r\nabla S) \cdot d\mathbf{x} \equiv \mathbf{w} \cdot d\mathbf{x}, \tag{6.8}$$

are invariant 1-forms advected with the flow. We show that $\mathbf{w} \cdot d\mathbf{x}$ is an invariant 1-form in Appendix B, where $\mathbf{u} = \nabla\phi + r\nabla S + \lambda\nabla\mu$ is a Clebsch representation for the fluid velocity $\mathbf{u}$. The two-form ω^2 with properties:

$$\omega^2 = \omega^1_1 \wedge \omega^1_2 = \nabla S \times (\mathbf{u} - \nabla\phi) \cdot d\mathbf{S},$$
$$d\omega^2 = \nabla \cdot [\nabla S \times (\mathbf{u} - \nabla\phi)]d^3x, \tag{6.9}$$

is an advected invariant 2-form. Using the identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{A}) = \mathbf{A} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{A}, \tag{6.10}$$

with $\mathbf{E} = \nabla S$ and $\mathbf{A} = \mathbf{u} - \nabla\phi$ in (6.9) we obtain:

$$d\omega^2 = -\nabla S \cdot \nabla \times \mathbf{u}\, d^3x = -\rho I_e\, d^3x, \tag{6.11}$$

where I_e is the Ertel invariant. In this case, in general $d\omega^2 = \nabla \cdot (\rho \mathbf{J})\, d^3x \neq 0$ where $\rho \mathbf{J} = \nabla S \times (\mathbf{u} - \nabla\phi)$.

Example 2   A second example in MHD is to set

$$\omega_1^1 = \mathbf{A} \cdot d\mathbf{x}, \quad \omega_2^1 = \nabla(\mathbf{b} \cdot \nabla S) \cdot d\mathbf{x} \quad \text{where} \quad \mathbf{b} = \frac{\mathbf{B}}{\rho}. \tag{6.12}$$

In this case,

$$I_2^q = \int \omega_1^1 \wedge \omega_2^1 \equiv \int \mathbf{A} \times \nabla(\mathbf{b} \cdot \nabla S) \cdot d\mathbf{S}. \tag{6.13}$$

Using Gauss's theorem we obtain:

$$I_2^q = \int \nabla \cdot [\mathbf{A} \times \nabla(\mathbf{b} \cdot \nabla S)]\, d^3x \equiv \int \mathbf{b} \cdot \nabla(\mathbf{b} \cdot \nabla S)\rho\, d^3x, \tag{6.14}$$

where $\mathbf{b} = \mathbf{B}/\rho$ is an invariant vector field advected with the flow and $\mathbf{B} = \nabla \times \mathbf{A}$, and we used the identity (3.38). In this case $\rho \mathbf{J} = \mathbf{A} \times \nabla(\mathbf{b} \cdot \nabla S)$ and $\nabla \cdot (\rho \mathbf{J}) = \rho \mathbf{b} \cdot \nabla(\mathbf{b} \cdot \nabla S)$ is in general non-zero.

It is interesting to note that the invariant I_2^q above is in general non-zero if $\mathbf{B} \cdot \nabla S \neq 0$. On the other hand, the cross helicity conservation law (3.57), only applies in the opposite case for which $\mathbf{B} \cdot \nabla S = 0$.

6.3   The Hopf Invariant

Arnold (1974), Tur and Yanovsky (1993) and Arnold and Khesin (1998) discuss the Hopf invariant (see also Berger and Field 1984; Moffatt and Ricca 1992; Finn and Antonsen 1985, 1988). Arnold and Khesin (1998) discuss the Hopf invariant in 3D and higher dimensions. Below we give a discussion of the magnetic helicity, which is the Hopf invariant for the magnetic field (e.g. Moffatt and Ricca 1992). We address in particular, the form of the magnetic helicity when the magnetic vector potential 1-form $\alpha = \tilde{\mathbf{A}} \cdot d\mathbf{x}$ is Lie dragged with the flow. Consider the 2-form:

$$\omega_b^2 = \mathbf{b} \lrcorner \omega^3, \quad \omega^3 = f\, d^3x, \tag{6.15}$$

where f is a scalar function and $\mathbf{b}$ is a Lie dragged vector field that is advected with the flow. The most obvious choice for f is $f = \rho$, but other choices are possible. We

require $\boldsymbol{\omega}_b^2$ to be closed. Note that

$$\boldsymbol{\omega}_b^2 = \mathbf{b} \lrcorner (f \, d^3x) = f\mathbf{b} \cdot d\mathbf{S}. \tag{6.16}$$

Thus, $\boldsymbol{\omega}_b^2$ is closed if

$$d\boldsymbol{\omega}_b^2 = \nabla \cdot (\mathbf{b}f) \, d^3x = 0. \tag{6.17}$$

By the Poincaré Lemma, for a smooth manifold, the closed form condition $d\boldsymbol{\omega}_b^2 = 0$ implies that locally, there exists a 1-form $\boldsymbol{\omega}^1$ such that

$$\boldsymbol{\omega}_b^2 = d\boldsymbol{\omega}^1 \quad \text{and} \quad \boldsymbol{\omega}^1 = \mathbf{A} \cdot d\mathbf{x},$$
$$\boldsymbol{\omega}_b^2 = d(\mathbf{A} \cdot d\mathbf{x}) = (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = f\mathbf{b} \cdot d\mathbf{S}. \tag{6.18}$$

Thus,

$$f\mathbf{b} = \nabla \times \mathbf{A} \quad \text{or} \quad \mathbf{b} = \frac{\nabla \times \mathbf{A}}{f}. \tag{6.19}$$

The one form $\boldsymbol{\omega}^1 = \mathbf{A} \cdot d\mathbf{x}$ might not be an invariant 1-form, that is Lie dragged with the flow. However, it is possible to introduce a 1-form $\tilde{\boldsymbol{\omega}}^1$ that is Lie dragged with the flow, such that:

$$\tilde{\boldsymbol{\omega}}^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x} \quad \text{where} \quad \tilde{\mathbf{A}} = \mathbf{A} + \nabla\Lambda, \tag{6.20}$$

and Λ is a gauge potential. Note that $\tilde{\boldsymbol{\omega}}^1 = \boldsymbol{\omega}^1 + d\Lambda$. For the sake of simplicity we omit a discussion of the gauge potential in the magnetostatic limit in which $|\mathbf{u}| \to 0$ (see (3.50) et seq. for a discussion of this limit).

From (6.20) we obtain:

$$\frac{d}{dt}\left(\tilde{\mathbf{A}} \cdot d\mathbf{x}\right) = \frac{d}{dt}\left(\mathbf{A} \cdot d\mathbf{x}\right) + \nabla\left(\frac{d\Lambda}{dt}\right) \cdot d\mathbf{x}, \tag{6.21}$$

where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative following the flow. Writing $d\Lambda/dt = g$ gives:

$$\Lambda = \int^t \tilde{g}(t', \mathbf{x}_0) \, dt', \tag{6.22}$$

where $\tilde{g}(t', \mathbf{x}_0) := g(t, \mathbf{x})$. Equation (6.21) then reduces to:

$$\frac{d\tilde{\boldsymbol{\omega}}^1}{dt} = \frac{d\boldsymbol{\omega}^1}{dt} + \nabla g \cdot d\mathbf{x}. \tag{6.23}$$

We can choose the gauge potential Λ and gauge function g so that $d\tilde{\boldsymbol{\omega}}^1/dt = 0$ which ensures that $\boldsymbol{\omega}^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x}$ is Lie dragged with the flow. The gauge transformation (6.20)–(6.23) is similar to that used in (3.45)–(3.51) in the discussion of magnetic helicity using vector Calculus in Chap. 5. Because $\tilde{\boldsymbol{\omega}}^1$ is Lie dragged with the flow, $\tilde{\mathbf{A}}$ satisfies (5.20) but with $\mathbf{S} \to \tilde{\mathbf{A}}$, i.e.

$$\frac{d\tilde{\boldsymbol{\omega}}^1}{dt} = \left[\frac{\partial \tilde{\mathbf{A}}}{\partial t} - \mathbf{u} \times (\nabla \times \tilde{\mathbf{A}}) + \nabla(\mathbf{u} \cdot \tilde{\mathbf{A}}) \right] \cdot d\mathbf{x} = 0. \tag{6.24}$$

Because $\tilde{\boldsymbol{\omega}}^1$ is a Lie dragged 1-form, then $d\tilde{\boldsymbol{\omega}}^1$ and $\tilde{\boldsymbol{\omega}}^1 \wedge d\tilde{\boldsymbol{\omega}}^1$ are also invariant, Lie dragged forms. Noting that $d\tilde{\boldsymbol{\omega}}^1 = (\nabla \times \tilde{\mathbf{A}}) \cdot d\mathbf{S}$ we obtain the Hopf invariant:

$$I_h = \int_V \tilde{\boldsymbol{\omega}}^1 \wedge d\tilde{\boldsymbol{\omega}}^1 = \int_V (\tilde{\mathbf{A}} \cdot d\mathbf{x}) \wedge \left[(\nabla \times \tilde{\mathbf{A}}) \cdot d\mathbf{S} \right] = \int_V \tilde{\mathbf{A}} \cdot (\nabla \times \tilde{\mathbf{A}}) \, d^3x. \tag{6.25}$$

Thus, the Hopf invariant can be written in the form:

$$I_h = \int_V \tilde{\mathbf{A}} \cdot \mathbf{B} \, d^3x \quad \text{where} \quad \mathbf{B} = \nabla \times \tilde{\mathbf{A}}. \tag{6.26}$$

The formula (6.26) is the usual formula for magnetic helicity but it is given for the special case for which $\tilde{\boldsymbol{\omega}}^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x}$ is advected with the flow. The magnetic vector potential $\mathbf{A}$ satisfies the equation:

$$\frac{\partial \mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla(\mathbf{u} \cdot \mathbf{A} + g) = 0. \tag{6.27}$$

In the analysis in Chap. 5, Eq. (5.35), the function $g = \phi_E - \mathbf{u} \cdot \mathbf{A}$ in (6.27), where ϕ_E is the electric field potential. Note that in general $\mathbf{A} \cdot d\mathbf{x}$ is not Lie dragged with the flow for $g \neq 0$.

Tur and Yanovsky (1993) make the important point that the helicity based on the Lie dragged invariants $\tilde{\boldsymbol{\omega}}^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x}$ and $d\tilde{\boldsymbol{\omega}}^1 = (\nabla \times \tilde{\mathbf{A}}) \cdot d\mathbf{S}$ in (6.26) and (6.27) does not require $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary $\partial V(t)$ of the region $V(t)$ moving with the fluid, as is the case with the Moffatt (1969) analysis of the magnetic helicity discussed in Chap. 3. The Hopf invariant (6.26) is sometimes written in the form:

$$I_h = \int_{V(t)} \mathbf{B} \cdot \text{curl}^{-1}(\mathbf{B}) \, d^3x, \tag{6.28}$$

where $\tilde{\mathbf{A}} = \text{curl}^{-1}(\mathbf{B})$ is given in terms of $\mathbf{B}$ using the Biot Savart formula for $\tilde{\mathbf{A}}$.

6.3.1 The Calugareanu Invariant

Moffatt and Ricca (1992) show that the magnetic helicity I_h of two isolated magnetic flux tubes, can be expressed in the form:

$$I_h = 2n\Phi_1\Phi_2 \quad \text{where} \quad n = \text{Lk}(C_1, C_2), \tag{6.29}$$

is an integer known as the link number or Calugareanu invariant, or Gauss link number of the two curves $C_1(t)$ and $C_2(t)$ representing (in the present case) the central field lines of the two flux tubes (labelled 1 and 2) with magnetic fluxes $\Phi_1 = \mathbf{B}_1 \cdot d\mathbf{S}_1$ and $\Phi_2 = \mathbf{B}_2 \cdot d\mathbf{S}_2$ with cross-sectional areas $d\mathbf{S}_1$ and $d\mathbf{S}_2$ respectively.

As a simple example of the formula (6.29) consider the case of two flux tubes T_1 and T_2 that are coupled by a single link, as illustrated in Fig. 6.1. The magnetic helicity integrals for the two separate flux tubes (assumed not to be twisted) are:

$$I_m = \int_{V_m} (\mathbf{A} \cdot \mathbf{B}) \, d^3x \quad \text{where} \quad \mathbf{B} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial V_m, \quad (m = 1, 2), \tag{6.30}$$

where $\mathbf{n}$ is the outward normal to the flux tube. In Fig. 6.1, the area S_1 is the area bounded by the closed curve C_1 and S_2 is the corresponding area enclosed by the curve C_2. The integral of $\mathbf{A} \cdot \mathbf{B}$ over flux tube 1 is:

$$\begin{aligned} I_1 &= \int_{V_1} (\mathbf{A} \cdot \mathbf{B}) \, d^3x = \int_{C_1} \mathbf{A} \cdot (B_1 \sigma_1 d\mathbf{x}) \\ &= \Phi_1 \oint_{C_1} \mathbf{A} \cdot d\mathbf{x} = \Phi_1 \int_{S_1} (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\sigma} = \Phi_1 \int_{S_1} \mathbf{B} \cdot d\boldsymbol{\sigma} = \Phi_1 \Phi_2. \end{aligned} \tag{6.31}$$

Similarly, the integral of $\mathbf{A} \cdot \mathbf{B}$ over the flux tube 2 is $I_2 = \Phi_2 \Phi_1$. The total integral of $\mathbf{A} \cdot \mathbf{B}$ over both flux tubes is:

$$I_\infty = I_1 + I_2 = 2n\Phi_1\Phi_2 = 2\Phi_1\Phi_2, \tag{6.32}$$

Fig. 6.1 Two untwisted magnetic flux tubes T_1 and T_2, coupled by a single link. The magnetic helicity integral for the two flux tubes reduces to $2n\Phi_1\Phi_2$, where $n = 1$ is the number of links
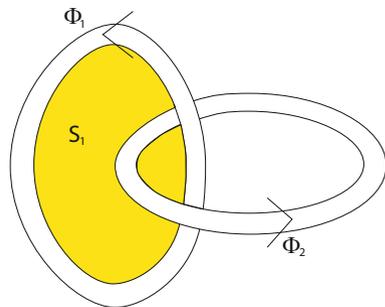
Fig. 6.2 Two untwisted
magnetic flux tubes T_1 and
T_2, coupled by two links. The
magnetic helicity integral for
the two flux tubes reduces to
$2n\Phi_1\Phi_2$, where $n = 2$ is the
number of links
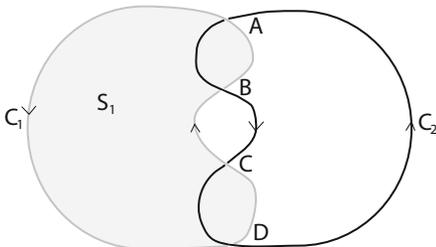
where $n = 1$ is the link number of the two curves C_1 and C_2.

As a second example, consider two flux tubes T_1 and T_2, which have a link
number of $n = 2$ as illustrated in Fig. 6.2 (again the tubes are assumed not be
twisted). The helicity integral for flux tube 1 is:

$$I_1 = \int_{V_1} (\mathbf{A} \cdot \mathbf{B})\, d^3x = \Phi_1 \oint_{C_1} \mathbf{A} \cdot d\mathbf{x} = \Phi_1 \int_{S_1} (\nabla \times \mathbf{A}) \cdot d\boldsymbol{\sigma}$$

$$= \Phi_1 \int_{S_1} \mathbf{B} \cdot d\boldsymbol{\sigma} = \Phi_1 \,(\text{flux } \Phi_2 \text{ at A } + \text{ flux } \Phi_2 \text{ at C}) = 2\Phi_1\Phi_2. \qquad (6.33)$$

Similarly, for flux tube 2, $I_2 = 2\Phi_2\Phi_1$. The total integral of $\mathbf{A} \cdot \mathbf{B}$ in this case is:

$$I_\infty = I_1 + I_2 = 4\Phi_1\Phi_2 = 2n\Phi_1\Phi_2 \quad \text{where} \quad n = 2. \qquad (6.34)$$

In this case the link number $n = 2$.

More generally, the Gauss link number or Calugareanu invariant $Lk(C_1, C_2)$ for
the two closed interlinked curves $C_1(t)$ and $C_2(t)$ is given by the formula:

$$\text{Lk}(C_1, C_2) = \frac{1}{4\pi} \oint_{C_1} \oint_{C_2} \frac{[\mathbf{x}_1(t) - \mathbf{x}_2(t')] \cdot d\mathbf{x}_1(t) \times d\mathbf{x}_2(t')}{|\mathbf{x}_1(t) - \mathbf{x}_2(t')|^3}, \qquad (6.35)$$

(Calugareanu 1959; Aldinger et al. 1995; Moffatt and Ricca 1992). In general the
link number of the two curves $C_1(t)$ and $C_2(t)$ can be split up into twist (Tw) and
writhe (Wr) components:

$$\text{Lk}(C_1, C_2) = \text{Tw}(C_1, C_2) + \text{Wr}(C_1), \qquad (6.36)$$

where the writhe is the self-linking number of the curve C_1 with itself, which is
given by the formula

$$\text{Wr}(C_1) = \frac{1}{4\pi} \oint_{C_1} \oint_{C_1} \frac{[\mathbf{x}_1(t) - \mathbf{x}_2(t')] \cdot d\mathbf{x}_1(t) \times d\mathbf{x}_2(t')}{|\mathbf{x}_1(t) - \mathbf{x}_2(t')|^3}, \qquad (6.37)$$

(e.g. Aldinger et al. 1995). In the above formulation, the curves $C_1(t)$ and $C_2(t)$ can
be thought of as a ribbon, with edges $C_1(t)$ and $C_2(t)$, in which one of the curves

$C_2(t)$ is conceived as winding about the axis curve $C_1(t)$. The writhe $\mathrm{Wr}(C_1)$ is the out of the plane buckling that occurs in a tangled telephone cord due to the stresses on the cord. Note that a single curve $C_1(t)$ can have a non-zero link number due to its writhe. Note also that there is an interplay between the twist (Tw) and the writhe (Wr) in such a way that their sum is a constant (i.e. twist can be converted into writhe and vice versa).

Self and Mutual Helicity of Two Flux Tubes

The total helicity of two flux tubes in a volume V consists in general of a contribution from the helicities of the separate flux tubes, plus the mutual helicity of the two tubes due to their winding around each other (e.g. Berger and Prior 2006; Campbell and Berger 2014). Let T_1 and T_2 be the self helicities of tubes 1 and 2, and let w_{12} be the winding number of the two tubes about each other. The self helicity of tube 1 say, is due to the twisting of the field lines inside the tube about its central axis, as well as the writhe Wr_1 (out of plane buckling) of the axis itself, i.e.

$$T_1 = T_{w1} + W_{r1}, \tag{6.38}$$

(Berger and Prior 2006; Campbell and Berger 2014). If the fluxes of the two tubes are Φ_1 and Φ_2 then the total helicity is given by:

$$H = T_1\Phi_1^2 + T_2\Phi_2^2 + 2w_{12}\Phi_1\Phi_2, \tag{6.39}$$

(e.g. Berger 1986; Ruzmaikin and Akhmetiev 1994). This generalizes the previous formula (6.29) where the self helicities of the two tubes were neglected, i.e. (6.29) has $w_{12} = n$ and $T_1 = T_2 = 0$).

The formulas (6.29)–(6.37) are derived using formulas from differential geometry, and have a wider significance in topological problems in physics and mathematics (i.e. (6.29)–(6.37) also apply to the knotting and linking of DNA (Summners 1992)). The Hopf, fibration describes the stereographic map of the three sphere in a 4-dimensional manifold onto the two sphere in 3D space. The map is used in the construction of the MHD topological soliton (e.g. Kamchatnov 1982; Semenov et al. 2002). Kamchatnov (1982) uses the work of Nicole (1978) to derive the MHD topological soliton.

Proof (of (6.29)) Here we use the work of Moffatt and Ricca (1992) to derive (6.29)–(6.35). Using Ampéré law $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$, using $\mathbf{B} = \nabla \times \mathbf{A}$ and the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ gives Poisson's equation for $\mathbf{A} \equiv \mathbf{A}_c$ as:

$$\nabla^2\mathbf{A}_c = -\mu_0\mathbf{J}, \tag{6.40}$$

with solution:

$$\mathbf{A}_c = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} \, d^3x. \tag{6.41}$$

Expressing $\mathbf{J} = \nabla \times \mathbf{B}/\mu_0$ in (6.41) integrating by parts, and using a generalized version of Stokes' theorem we obtain:

$$\mathbf{A}_c = -\frac{1}{4\pi} \int_V \frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{B}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3x' - \int_{\partial V} \frac{\mathbf{B}(\mathbf{x}') \times \mathbf{n}(\mathbf{x}')}{4\pi |\mathbf{x} - \mathbf{x}'|} \, d\mathbf{S}', \tag{6.42}$$

where $\mathbf{n}(\mathbf{x}')$ is the outward normal to the surface ∂V. Assuming the integral over the boundary ∂V vanishes (e.g. the boundary is at infinity or $\mathbf{B}$ is of compact support), we obtain the Biot Savart form for $\mathbf{A}_c$:

$$\mathbf{A}_c = -\frac{1}{4\pi} \int_V \frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{B}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \, d^3x'. \tag{6.43}$$

Multiplying (6.43) by $\mathbf{B}(\mathbf{x})$ and integrating over the volume V gives the magnetic helicity for the volume V as:

$$H_M = \frac{1}{4\pi} \int_V d^3x \int_V d^3x' \frac{(\mathbf{x} - \mathbf{x}') \cdot \mathbf{B}(\mathbf{x}) \times \mathbf{B}'(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}. \tag{6.44}$$

For the case of two linked flux tubes:

$$\mathbf{B}(\mathbf{x_1})d^3x_1 = \mathbf{B}(\mathbf{x_1})d\mathbf{x}_1 \cdot d\mathbf{S}_1 = \Phi_1 d\mathbf{x}_1,$$
$$\mathbf{B}(\mathbf{x_2})d^3x_2 = \mathbf{B}(\mathbf{x_2})d\mathbf{x}_2 \cdot d\mathbf{S}_2 = \Phi_2 d\mathbf{x}_2, \tag{6.45}$$

where $\mathbf{x}_1 \equiv \mathbf{x}(t)$ is the position vector on curve C_1 and $\mathbf{x}_2(t')$ is the position vector on C_2 used in the integration (6.44). Using (6.45) in (6.44) gives:

$$H_M = \frac{\Phi_1 \Phi_2}{4\pi} \oint_{C1} \oint_{C_2} \frac{(\mathbf{x}_1 - \mathbf{x}_2) \cdot d\mathbf{x}_1 \times d\mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} + (1 \leftrightarrow 2), \tag{6.46}$$

where $1 \leftrightarrow 2$ corresponds to replacing $\mathbf{x}_1$ by $\mathbf{x}_2$ and vice-versa. Equation (6.46) gives the flux in tube 1, that links with the flux in tube 2 plus the flux in tube 2 that links to the flux in tube 1. The net result from (6.46) is

$$H_M = 2n\Phi_1 \Phi_2, \tag{6.47}$$

where $n = \mathrm{Lk}(C_1, C_2)$ is the link number of curves C_1 and C_2 given by (6.35). This completes the proof. $\square$

6.4 Link Numbers and Signed Crossing Numbers

In general, the linkage of one or more curves in knot theory is not intrinsically associated with magnetic fields or fluid vortices (e.g. DNA strands can be knotted and linked). There is a vast literature on knot theory. The books by Kauffman (1987) and Gilbert and Porter (1994) are sufficient for our purposes. As a simple example, consider the linkage of two curves depicted in Fig. 6.3. The link number of the two curves C_1 and C_2 in Fig. 6.3 is given by:

$$Lk(1, 2) = \frac{1}{2} (\sigma_A + \sigma_B), \tag{6.48}$$

where σ_A and σ_B are the signed crossing numbers of the two strands at A and B, which is illustrated by the sub-diagrams b and c in Fig. 6.3. The signed crossing number of two strands in which the over-strand is right-hand related to the under-strand has a crossing number $\sigma = 1$. Similarly, if the overstrand is not right hand related to the under-strand (i.e. it is left hand related), then $\sigma = -1$. If the knot is represented by a closed curve in the xy-plane, then the crossing number $\sigma = 1$ if the cross product of the overstrand with the understrand is directed along the positive z-axis, and $\sigma = -1$ if the above cross product is along the negative z-axis. In Fig. 6.3b and c, $\sigma_A = \sigma_B = 1$ and the total link number using (6.48) is given by:

$$Lk(1, 2) = \frac{1}{2} (1 + 1) = 1. \tag{6.49}$$

If the two curves are replaced by magnetic flux tubes, then the total helicity integral is given by:

$$I_\infty = 2Lk(1, 2)\Phi_1\Phi_2 = 2\Phi_1\Phi_2, \tag{6.50}$$

in this case.

Figure 6.4 shows a variant of the Whitehead link, involving two curves C_1 and C_2 (e.g. Kauffman 1987, Ch. 2, p. 15). A calculation of the link number using the
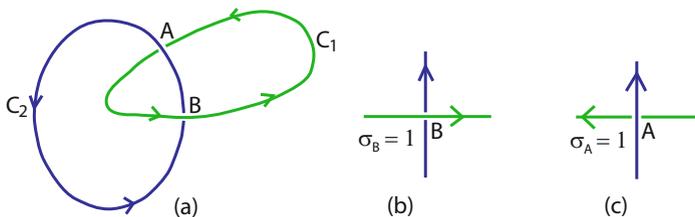
Fig. 6.3 (**a**) Two linked curves C_1 and C_2, (**b**) the signed crossing number of the curves at B, (**c**) the signed crossing number at A

Fig. 6.4 A variant of the
Whitehead link. By adding up
the signed crossing numbers
of the two curves at A, B, C,
D, E, F, the total link number
of the two curves turns out to
be zero, as there are as many
positive crossing numbers as
negative crossing numbers
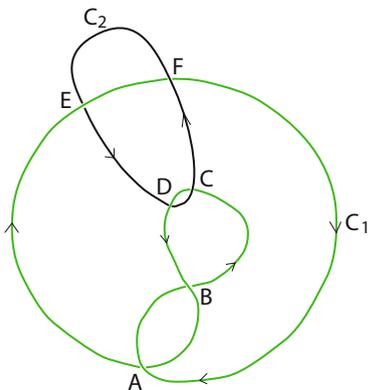
signed crossing numbers gives:

$$Lk(1,2) = \frac{1}{2} \left(\sigma_A + \sigma_B + \sigma_C + \sigma_D + \sigma_E + \sigma_F \right)$$

$$= \frac{1}{2} \left(1 + 1 + 1 - 1 - 1 - 1 \right) = 0. \tag{6.51}$$

Thus the link number $n = 0$. In general, knots can be linked, even when their link
number is zero.

6.4.1   Dehn Surgery and Reconnection

In Dehn surgery of flux tubes, one cuts and reconnects flux tubes without altering
the total helicity of the flux tube structure as a whole. At crossing points where the
knot or tube crosses itself, one can cut the knot with pairs of backward and forward
cuts in such a way that after reconnection, H_M is conserved (e.g. Berger and Field
1984; Ruzmaikin and Akhmetiev 1994).

Consider a planar trefoil knot, that does not kink out of the plane (Fig. 6.5), with
crossing numbers $\sigma_A = \sigma_B = \sigma_C = -1$ at the cross-over points A, B and C of the
knot. The knot (flux tube) is a single tube and the helicity of the tube is

$$H_M = (\sigma_A + \sigma_B + \sigma_C) \, \Phi^2 = -3\Phi^2, \tag{6.52}$$

Figure 6.6 shows the same trefoil knot as in Fig. 6.5 (left panel). At each cross-
over point, a backward and forward cut of the flux tube is made in the manner
illustrated in Fig. 6.6. The rules for insertion of the cuts and the re-connections are:
(a) the forward and backward cuts occur in pairs, so that no helicity is injected into
the knot by the cuts and the net helicity in the reconnected configuration (right panel)
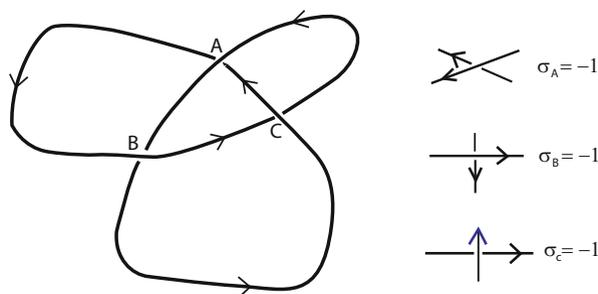
Fig. 6.5 Trefoil knot and signed crossing numbers $\sigma_A = \sigma_B = \sigma_C = -1$
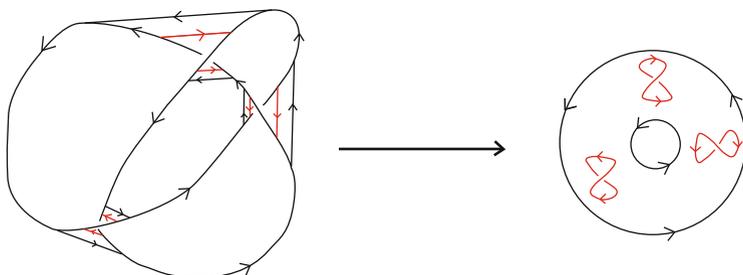
Fig. 6.6 Dehn surgery for the trefoil knot

is the same as the original trefoil knot; (b) the cuts and reconnections do not violate the direction of the knot. The net result of the surgery is shown on the right. The large outer circle comes from reconnecting the outer parts of the trefoil knot (left panel). The inner small circle is a deformed version of the inner part of the knot on the left. Similarly, the three figure eight knots on the right are equivalent to the figure 8 structures created by cutting and reconnecting the field about the cross-over points in the left figure. The net upshot results in no change in $H_M = -3\Phi^2$ after the surgery.

If one twists a figure 8 curve to obtain a circle, then the twist of the resultant circle must convert the twist into writhe (i.e. the circle is under torsional stress), so that Link = Twist + Writhe is conserved.

In Dehn surgery, one can omit some of the backward and forward cuts (e.g. one could omit the outermost cuts leading to the large circle in the right panel of Fig. 6.6), but still obtain interesting knot configurations that have the same helicity as the original trefoil knot of Fig. 6.5. This leads to the knot configuration in Fig. 6.7.

One can also presumably, eliminate the inner circle loop on the right of Fig. 6.7 by merging it with the outer loop. This process is analogous to island merging in magnetic reconnection. In merging two magnetic islands, with the same sense of rotation, reconnection occurs where the islands collide with each other, in such a way that the total magnetic flux of the two separate islands equals the net magnetic flux of the single magnetic island that results from the merging process.
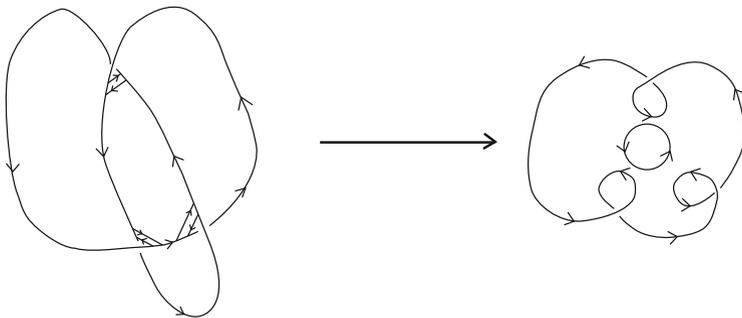
Fig. 6.7   Partial Dehn surgery for the trefoil knot

Not all of the possible knot configurations obtained by Dehn surgery are necessarily stable, and correspond to a minimum magnetic field energy state (i.e Dehn surgery may need to be supplemented by other energetic criteria to obtain an acceptable configurations of the re-connected field).

The above excursion into knot theory is only the tip of the iceberg. For example, the linkage and topological structure of more complicated knots can be described by knot polynomials: (e.g. the Kauffman, Jones, Homfly and Alexander polynomials). One can use Dehn surgery to calculate the linkage of knots. Knots can be defined by using Seifert surfaces in which the knot is embedded.

6.4.2   Taylor's Hypothesis and Magnetic Reconnection

Taylor's relaxation hypothesis (Taylor 1974, 1986) is that in a high conductivity plasma, the total magnetic helicity to lowest order is conserved during turbulent magnetic reconnection. The helicity of individual flux tubes is not conserved. Because of the high magnetic Reynolds number the plasma undergoes turbulent reconnection. The application of Taylor's hypothesis was initially developed to describe the plasma evolution and relaxation in fusion devices, such as the tokamak and the spheromak. Later work applied the same idea to astrophysical plasmas. In most cases of interest, the magnetic energy is dissipated or converted into heat energy (e.g. by possibly accelerating particles in the electric and magnetic fields) and also into flow kinetic energy (e.g. as in a CME). Magnetic reconnection is widely thought to be one of the basic physical processes at work in heating the solar corona, and in driving the solar wind (e.g. Parker 1979, 1994; Low 2015). Dewar et al. (2015, 2017) have developed the theory of Taylor relaxation for multi-region relaxed MHD with application in fusion plasma devices.

Taylor (1974, 1986) argued that the action principle for the relaxed turbulent magnetic reconnection state in low β, high conductivity plasmas reduces to $\delta \mathcal{A} = 0$

where the action $\mathcal{A}$ is given by:

$$\mathcal{A} = \int_V \left(\frac{B^2}{2\mu_0} - \lambda \mathbf{A} \cdot \mathbf{B} \right) d^3 x. \tag{6.53}$$

The magnetic helicity for individual flux tubes is not conserved in the turbulent relaxation of the plasma, but the magnetic helicity for the turbulent plasma region V as a whole is conserved for a weakly dissipative plasma (magnetic Reynolds number $R_M \gg 1$). In that case, the Lagrange multiplier λ in (6.53) is taken to be constant, which implies the total magnetic helicity is conserved at lowest order. Using integration by parts, the variation of the action $\delta\mathcal{A}$ is given by:

$$\delta\mathcal{A} = \int_V \delta\mathbf{A} \cdot \left(\frac{\nabla \times \mathbf{B}}{\mu_0} - 2\lambda\mathbf{B} \right) + \nabla \cdot \left(\frac{\delta\mathbf{A} \times \mathbf{B}}{\mu_0} - \lambda\delta\mathbf{A} \times \mathbf{A} \right) d^3 x$$

$$= \int_V \delta\mathbf{A} \cdot \left(\frac{\nabla \times \mathbf{B}}{\mu_0} - 2\lambda\mathbf{B} \right) + \int_{\partial V} \left(\frac{\delta\mathbf{A} \times \mathbf{B}}{\mu_0} - \lambda\delta\mathbf{A} \times \mathbf{A} \right) \cdot \mathbf{n} \, dS, \tag{6.54}$$

where $\mathbf{n}$ is the outward normal to the boundary ∂V. Assuming that $\mathbf{n} \times \delta\mathbf{A} = 0$ on the boundary ∂V, the stationary point conditions $\delta\mathcal{A}/\delta\mathbf{A} = 0$ imply:

$$\nabla \times \mathbf{B} = \Lambda\mathbf{B} \quad \text{where} \quad \Lambda = 2\lambda\mu_0. \tag{6.55}$$

Thus, the required field is a constant Λ force-free magnetic field. The application of the solutions of (6.55), in general requires a superposition of solutions with different Λ in order to fit the boundary conditions (Taylor 1986).

Taylor (1986) shows that the magnetic helicity decays much slower than the magnetic energy in a highly conducting plasma. The gist of his argument is given below. Let

$$w_B = \frac{B^2}{2\mu_0}, \quad h_m = \mathbf{A} \cdot \mathbf{B}. \tag{6.56}$$

If the plasma has conductivity σ, then use of Ohm's law in its simplest form: $\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \mathbf{J}/\sigma$ coupled with Maxwell's equations gives Poynting's theorem:

$$\frac{\partial w_B}{\partial t} + \nabla \cdot \left(\frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) = \mathbf{J} \cdot \mathbf{E} = \mathbf{J} \cdot \left(-\mathbf{u} \times \mathbf{B} + \frac{\mathbf{J}}{\sigma} \right), \tag{6.57}$$

From (3.41), the magnetic helicity evolution equation for a dissipative plasma obeying the above form of Ohm's law, satisfies the evolution equation:

$$\frac{\partial h_m}{\partial t} + \nabla \cdot \left(\mathbf{u} h_m + (\phi_E - \mathbf{u} \cdot \mathbf{A}) \mathbf{B} + \frac{\mathbf{J} \times \mathbf{A}}{\sigma} \right) = -\frac{2\mathbf{J} \cdot \mathbf{B}}{\sigma}. \tag{6.58}$$

Integrating (6.57) and (6.58) over the plasma volume V, and dropping surface terms, gives the estimates:

$$\frac{dH_M}{dt} \sim -2 \int \frac{\mathbf{J} \cdot \mathbf{B}}{\sigma} \, d^3x = -2\mu_0 \eta \int_V \mathbf{J} \cdot \mathbf{B} \, d^3x,$$

$$\frac{dW_B}{dt} \sim - \int \frac{J^2}{\sigma} \, d^3x = -\mu_0 \eta \int_V J^2 \, d^3x, \tag{6.59}$$

where $\eta = 1/(\mu\sigma)$ is the plasma resistivity. Using the formulae:

$$H_M = \int_V h_m \, d^3x = \sum H_k = \sum k B_k^2,$$

$$W_B = \int_V w_B \, d^3x = \sum W_k = \sum \frac{B_k^2}{\mu_0}, \tag{6.60}$$

for the Fourier decomposition of H_M and W_B we obtain the estimates:

$$\frac{dH_k}{dt} \sim -2\eta k B_k^2, \quad \frac{dW_k}{dt} \sim -\eta k^2 B_k^2/\mu_0, \tag{6.61}$$

Ohmic dissipation with $B_k \sim \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$ gives $\omega \sim \eta k^2$. Thus, $\ell = 1/k = \sqrt{\eta/\omega}$ is the dissipation scale and $k = 1/\ell = \sqrt{\omega/\eta}$ is the corresponding wave number. From (6.61) we obtain the decay estimates:

$$\dot{W}_k \sim -\frac{\omega B_k^2}{\mu_0}, \quad \dot{H}_k \sim -2\sqrt{\eta\omega}B_k^2, \quad \frac{\dot{H}_k}{\dot{W}_k} \sim 2\sqrt{\eta/\omega}\,\mu_0. \tag{6.62}$$

Thus, the helicity decay rate is $\sim O(\sqrt{\eta})$ the magnetic energy decay rate, and the decay rate for the magnetic helicity is much smaller than that of the magnetic energy density in the high conductivity limit in which the plasma diffusivity $\eta = 1/(\mu_0\sigma) \to 0$ (cf. Taylor 1986).

6.5   The Godbillon Vey Invariant

Consider the Pfaffian differential form (1-form) $\tilde{\boldsymbol{\omega}}_A^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x}$, for which $d\tilde{\boldsymbol{\omega}}_A^1 = (\nabla \times \tilde{\mathbf{A}}) \cdot d\mathbf{S}$ and

$$\tilde{\boldsymbol{\omega}}_A^1 \wedge d\tilde{\boldsymbol{\omega}}_A^1 = \mathbf{A} \cdot d\mathbf{x} \wedge (\nabla \times \tilde{\mathbf{A}}) \cdot d\mathbf{S} = (\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}}) \, d^3x. \tag{6.63}$$

The Pfaffian differential equation:

$$\tilde{\boldsymbol{\omega}}_A^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x} = 0, \tag{6.64}$$

determines planes perpendicular to the vector field $\tilde{\mathbf{A}}$ at each point. For these planes to exist, i.e. for the Pfaffian equation (6.64) to have a solution requires that the integrability conditions

$$\tilde{\boldsymbol{\omega}}_A^1 \wedge d\tilde{\boldsymbol{\omega}}_A^1 = (\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}}) \, d^3x = 0. \tag{6.65}$$

are satisfied. If

$$\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} = 0, \tag{6.66}$$

the Pfaffian equation (6.64) is integrable (e.g. Sneddon 1957).

Tur and Yanovsky (1993) discuss the geometric obstruction to integrability when $\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} \neq 0$ in terms of non-closure of the integral paths. Note that the helicity or Hopf invariant

$$I^\tau = \int_V \tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} d^3x, \tag{6.67}$$

is non-zero only if $\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} \neq 0$ in some region in the volume V (i.e. $\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} = 0$ throughout the whole of V is not possible). Thus $I^\tau \neq 0$ implies $\alpha = \tilde{\mathbf{A}} \cdot d\mathbf{x}$ is non-integrable in sub-regions of V where α does not change sign.

A natural question (e.g. Tur and Yanovsky 1993), is: given that the differential form $\tilde{\boldsymbol{\omega}}^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x} = 0$ is integrable, and satisfies the integrability condition (6.65), are there then higher order topological invariants that have non-zero topological charge? The answer to this question is yes, there is a higher order topological quantity that can be non-zero in this case called the Godbillon Vey invariant. It is defined by the equation:

$$I^g = \int_{D^3(t)} \boldsymbol{\eta} \cdot \nabla \times \boldsymbol{\eta} \, d^3x \quad \text{where} \quad \boldsymbol{\eta} = \frac{\tilde{\mathbf{A}} \times \mathbf{B}}{|\tilde{\mathbf{A}}|^2}. \tag{6.68}$$

where $\mathbf{B} = \nabla \times \tilde{\mathbf{A}}$, and $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary $\partial D^3(t)$ of the region $D^3(t)$ with outward normal $\mathbf{n}$. I^g is a topological invariant that is advected with the flow, i.e.,

$$\frac{dI^g}{dt} = 0, \tag{6.69}$$

where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative moving with the flow. It is important to note that the Godbillon Vey invariant (6.68) only applies to zero helicity flows for which $\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} = 0$.

In (6.68) $\boldsymbol{\eta}$ is defined by the integrability equation:

$$d\tilde{\boldsymbol{\omega}}_A^1 = \boldsymbol{\omega}_\eta^1 \wedge \tilde{\boldsymbol{\omega}}_A^1, \tag{6.70}$$

where

$$\tilde{\omega}_A^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x}, \quad \text{and} \quad \omega_\eta^1 = \eta \cdot d\mathbf{x}, \tag{6.71}$$

are 1-forms. Taking the exterior derivative of $\tilde{\omega}_A^1$ and using it in (6.70) we obtain the equivalent flux equation:

$$(\nabla \times \tilde{\mathbf{A}}) \cdot d\mathbf{S} = (\eta \times \tilde{\mathbf{A}}) \cdot d\mathbf{x} \quad \text{or} \quad \nabla \times \tilde{\mathbf{A}} = \eta \times \tilde{\mathbf{A}}. \tag{6.72}$$

From (6.72) we obtain:

$$\tilde{\mathbf{A}} \times (\nabla \times \tilde{\mathbf{A}}) = \tilde{\mathbf{A}} \times (\eta \times \tilde{\mathbf{A}}) = (\tilde{\mathbf{A}} \cdot \tilde{\mathbf{A}})\eta - (\tilde{\mathbf{A}} \cdot \eta)\tilde{\mathbf{A}}. \tag{6.73}$$

The general solution of (6.73) for η is:

$$\eta = \frac{1}{|\tilde{\mathbf{A}}|^2}\left(\tilde{\mathbf{A}} \times \mathbf{B} + \eta \cdot \tilde{\mathbf{A}}\tilde{\mathbf{A}}\right) \tag{6.74}$$

By dropping the arbitrary component of η parallel to $\tilde{\mathbf{A}}$ we obtain the solution (6.68) for η.

A derivation of the Godbillon Vey invariant (6.68) and and the invariance equation (6.69) for I^g (see also Tur and Yanovsky 1993) is outlined below.

Proof (of Godbillon Vey Formula (6.69)) The Frobenius integrability condition (6.65) is satisfied if there exists a 1-form ω_η^1 such that

$$d\tilde{\omega}_A^1 = \omega_\eta^1 \wedge \tilde{\omega}_A^1, \tag{6.75}$$

Note that

$$\tilde{\omega}_A^1 \wedge d\tilde{\omega}_A^1 = \tilde{\omega}_A^1 \wedge (\omega_\eta^1 \wedge \tilde{\omega}_A^1) = -\tilde{\omega}_A^1 \wedge \tilde{\omega}_A^1 \wedge \omega_\eta^1 = 0, \tag{6.76}$$

where we used the associative and anti-symmetry properties of the $\wedge$ operation. Equation (6.75) ensures $d\tilde{\omega}_A^1 = 0$ whenever $\tilde{\omega}_A^1 = 0$. The condition $d\tilde{\omega}_A^1 = 0$ implies by the Poincaré Lemma that there exist a 0-form Φ such that $\tilde{\omega}_A^1 = d\Phi$. The Pfaffian equation $\tilde{\omega}_A^1 = \tilde{\mathbf{A}} \cdot d\mathbf{x} = 0$ is then satisfied by $\Phi(x, y, z) = const$. Equation (6.75) implies that the set of forms $\{\tilde{\omega}_A^1, d\tilde{\omega}_A^1\}$ is a closed ideal of differential forms which are in involution according to Cartan's theory of differential equations (e.g. Harrison and Estabrook 1971), i.e. the equations $\tilde{\omega}_A^1 = 0$ are integrable and satisfy the integrability conditions (6.65)). Equations (6.75) are similar to the Maurer Cartan equations, which are differentiability conditions in differential geometry.

We require that $d\tilde{\boldsymbol{\omega}}_A^1$ is advected with the flow, i.e.

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) d\tilde{\boldsymbol{\omega}}_A^1 \equiv \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\left(\boldsymbol{\omega}_\eta^1 \wedge \tilde{\boldsymbol{\omega}}_A^1\right) = 0. \tag{6.77}$$

Expanding (6.77) using the properties of the Lie derivative $\mathcal{L}_{\mathbf{u}}$ gives:

$$\left[\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\tilde{\boldsymbol{\omega}}_\eta^1\right] \wedge \tilde{\boldsymbol{\omega}}_A^1 + \boldsymbol{\omega}_\eta^1 \wedge \left[\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\tilde{\boldsymbol{\omega}}_A^1\right] = 0. \tag{6.78}$$

Using (6.78) and the condition that $\tilde{\boldsymbol{\omega}}_A^1$ is Lie dragged with the flow (6.78) simplifies to:

$$\left[\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\boldsymbol{\omega}_\eta^1\right] \wedge \tilde{\boldsymbol{\omega}}_A^1 = 0, \tag{6.79}$$

Equation (6.79) is satisfied if

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right)\boldsymbol{\omega}_\eta^1 = \alpha\tilde{\boldsymbol{\omega}}_A^1, \tag{6.80}$$

Equation (6.80) can also be written in the form:

$$\frac{\partial \boldsymbol{\eta}}{\partial t} - \mathbf{u} \times (\nabla \times \boldsymbol{\eta}) + \nabla(\mathbf{u} \cdot \boldsymbol{\eta}) = \alpha\tilde{\mathbf{A}}. \tag{6.81}$$

Taking the scalar product of (6.81) with $\tilde{\mathbf{A}}$ gives:

$$\alpha|\tilde{\mathbf{A}}|^2 = \tilde{\mathbf{A}} \cdot \left[\frac{\partial \boldsymbol{\eta}}{\partial t} - \mathbf{u} \times (\nabla \times \boldsymbol{\eta}) + \nabla(\mathbf{u} \cdot \boldsymbol{\eta})\right]. \tag{6.82}$$

An alternative expression for α can be obtained by noting that $\tilde{\mathbf{A}} \cdot d\mathbf{x}$ is Lie dragged with the flow. Thus, $\tilde{\mathbf{A}}$ satisfies (3.50), and hence:

$$0 = \boldsymbol{\eta} \cdot \left[\frac{\partial \tilde{\mathbf{A}}}{\partial t} - \mathbf{u} \times (\nabla \times \tilde{\mathbf{A}}) + \nabla(\mathbf{u} \cdot \tilde{\mathbf{A}})\right]. \tag{6.83}$$

Noting that $\tilde{\mathbf{A}} \cdot \boldsymbol{\eta} = \tilde{\mathbf{A}} \cdot (\tilde{\mathbf{A}} \times \mathbf{B}/|\tilde{\mathbf{A}}|^2) = 0$ and adding (6.82) and (6.83) we obtain:

$$\alpha = \frac{1}{|\tilde{\mathbf{A}}|^2}\left\{\tilde{\mathbf{A}} \cdot [\mathbf{u} \cdot \nabla\boldsymbol{\eta} + (\nabla\mathbf{u})^T \cdot \boldsymbol{\eta}] + \boldsymbol{\eta} \cdot [\mathbf{u} \cdot \nabla\tilde{\mathbf{A}} + (\nabla\mathbf{u})^T \cdot \tilde{\mathbf{A}}]\right\}. \tag{6.84}$$

Next we investigate if the 3-form:

$$\boldsymbol{\omega}_\eta^3 = \boldsymbol{\omega}_\eta^1 \wedge d\boldsymbol{\omega}_\eta^1, \tag{6.85}$$

is an advected (Lie dragged) 3-form. We find:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_\mathbf{u}\right)\boldsymbol{\omega}_\eta^3 = -d\left(\alpha d\tilde{\boldsymbol{\omega}}_A^1\right). \tag{6.86}$$

In the derivation of (6.86) we use the fact that $d\boldsymbol{\omega}_\eta^1 \wedge \tilde{\boldsymbol{\omega}}_A^1 = 0$ which follows by noting

$$d(d\tilde{\boldsymbol{\omega}}_A^1) = 0 \equiv d\left(\boldsymbol{\omega}_\eta^1 \wedge \tilde{\boldsymbol{\omega}}_A^1\right) = d\boldsymbol{\omega}_\eta^1 \wedge \tilde{\boldsymbol{\omega}}_A^1 - \boldsymbol{\omega}_\eta^1 \wedge d\tilde{\boldsymbol{\omega}}_A^1, \tag{6.87}$$

and that $\boldsymbol{\omega}_\eta^1 \wedge d\tilde{\boldsymbol{\omega}}_A^1 = 0$ by (6.75).

Next consider the integral

$$I^g = \int \boldsymbol{\omega}_\eta^3 = \int \boldsymbol{\omega}_\eta^1 \wedge d\boldsymbol{\omega}_\eta^1 \equiv \int_{D^3(t)} \boldsymbol{\eta} \cdot \nabla \times \boldsymbol{\eta}\, d^3 x. \tag{6.88}$$

Using (6.86) gives:

$$\frac{\partial I^g}{\partial t} = \int \frac{\partial \boldsymbol{\omega}_\eta^3}{\partial t} = \int \left[-\mathcal{L}_\mathbf{u}\left(\boldsymbol{\omega}_\eta^3\right) - d\left(\alpha d\tilde{\boldsymbol{\omega}}_A^1\right)\right]. \tag{6.89}$$

However, using Cartan's magic formula gives

$$\mathcal{L}_\mathbf{u}\left(\boldsymbol{\omega}_\eta^3\right) = d\left(\mathbf{u} \lrcorner \boldsymbol{\omega}_\eta^3\right) + \mathbf{u} \lrcorner d\boldsymbol{\omega}_\eta^3 = d\left(\mathbf{u} \lrcorner \boldsymbol{\omega}_\eta^3\right), \tag{6.90}$$

(note $\boldsymbol{\omega}_\eta^3$ is a 3-form and hence $d\boldsymbol{\omega}_\eta^3 = 0$). From (6.90) and (6.89) we obtain:

$$\frac{\partial I^g}{\partial t} = \int_{D^3(t)} -d\left(\mathbf{u} \lrcorner \boldsymbol{\omega}_\eta^3 + \alpha d\tilde{\boldsymbol{\omega}}_A^1\right) = -\int_{\partial D^3(t)} \left(\mathbf{u} \lrcorner \boldsymbol{\omega}_\eta^3 + \alpha d\tilde{\boldsymbol{\omega}}_A^1\right), \tag{6.91}$$

Writing

$$\psi = \boldsymbol{\eta} \cdot \nabla \times \boldsymbol{\eta}, \tag{6.92}$$

(6.91) can be written in the form:

$$\int_{D^3(t)} \frac{\partial \psi}{\partial t} \, d^3x = -\int \left\{ \mathbf{u} \lrcorner \left(\boldsymbol{\omega}_\eta^1 \wedge d\boldsymbol{\omega}_\eta^1 \right) + \alpha d(\tilde{\mathbf{A}} \cdot d\mathbf{x}) \right\}$$

$$= -\int \left\{ \mathbf{u} \lrcorner \left[(\boldsymbol{\eta} \cdot d\mathbf{x}) \wedge (\nabla \times \boldsymbol{\eta}) \cdot d\mathbf{S} \right] + \alpha (\nabla \times \tilde{\mathbf{A}}) \cdot d\mathbf{S} \right\}$$

$$= -\int \left[\mathbf{u} \lrcorner (\boldsymbol{\eta} \cdot \nabla \times \boldsymbol{\eta}) d^3x + \alpha \mathbf{B} \cdot d\mathbf{S} \right]$$

$$= -\int_{\partial D^3(t)} \left[\psi \mathbf{u} \cdot d\mathbf{S} + \alpha \mathbf{B} \cdot d\mathbf{S} \right]$$

$$= -\int_{D^3(t)} \nabla \cdot (\mathbf{u}\psi + \alpha \mathbf{B}) \, d^3x. \tag{6.93}$$

Equation (6.93) implies the conservation law:

$$\frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{u}\psi + \alpha \mathbf{B}) = 0. \tag{6.94}$$

where α is given in (6.84).

Integrating the continuity equation (6.94) for ψ over the volume $D^3(t)$, and using the results

$$\psi d^3x = \psi(\mathbf{x}_0) d^3x_0, \quad d^3x = J d^3x_0, \quad \psi J = \psi_0(x_0), \quad \frac{d \ln J}{dt} = \nabla \cdot \mathbf{u}, \tag{6.95}$$

from Lagrangian fluid mechanics where $J = \det(x_{ij})$ is the Jacobian determinant of $x_{ij} = \partial x^i / \partial x_0^j$ of the Lagrangian map relating the Eulerian position coordinate $\mathbf{x}$ and the Lagrangian label $\mathbf{x}_0$ where $\mathbf{x} = \mathbf{x}_0$ at $t = 0$, we obtain:

$$0 = \int_{D^3(t)} \left[\frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{u}\psi + \alpha \mathbf{B}) \right] d^3x$$

$$= \int_{D^3(t)} \left[\frac{\partial \psi}{\partial t} + \left(\psi \frac{d \ln J}{dt} + \mathbf{u} \cdot \nabla \psi \right) \right] J d^3x_0$$

$$= \int_{D^3(t)} \left[J \frac{d\psi}{dt} + \psi \frac{dJ}{dt} \right] d^3x_0$$

$$= \int_{D^3(t)} \left[\frac{d\psi}{dt} d^3x + \psi \frac{d}{dt}(d^3x) \right]. \tag{6.96}$$

In the second line in (6.96) there is no contribution from the $\alpha \mathbf{B}$ term, because if we apply Gauss's theorem $\nabla \cdot (\alpha \mathbf{B}) d^3x \to \alpha \mathbf{B} \cdot d\mathbf{S} = \alpha \mathbf{B} \cdot \tilde{\mathbf{A}} dS / |\tilde{\mathbf{A}}| = 0$ and because $\mathbf{B} \cdot \tilde{\mathbf{A}} = 0$ is the integrability condition for $\tilde{\mathbf{A}} \cdot d\mathbf{x} = 0$. The last integral in (6.96) can

be recognized as dI^g/dt. Thus, (6.96) implies the Lagrangian conservation law:

$$\frac{dI^g}{dt} = 0. \tag{6.97}$$

Thus I^g is a constant moving with the flow. This completes the proof of (6.69). □

6.6 Magnetic Helicity Examples

In this section provide examples of the application of magnetic helicity (3.21) and the relative helicity (3.22). In particular we study the magnetic helicity of the Parker, Archimedean spiral magnetic field derived by Parker (1958, 1963). The relative magnetic helicity of the Parker field was derived by Bieber et al. (1987) and later by Webb et al. (2010a). The approach to the magnetic helicity of the Parker field of Webb et al. (2010a) is described below. Berger and Ruzmaikin (2000) study the related issue of the injection of magnetic helicity into the solar wind, based in part on observations of the Sun's photospheric magnetic field. After describing the relative helicity of the Parker field, We then discuss the helicity (relative helicity) of fully nonlinear shear and toroidal Alfven waves which is of interest in solar and heliospheric physics (e.g. Alfvén waves in the solar wind), based on the work of Webb et al. (2010b, 2011).

6.6.1 The Parker Archimedean Spiral Field

The Parker Archimedean spiral interplanetary magnetic field field beyond a few solar radii, with a flat current sheet in the helio-equatorial plane has the form:

$$\mathbf{B} = \frac{af(\theta)}{r^2}\left[\mathbf{e}_r - \frac{\Omega r \sin\theta}{u}\mathbf{e}_\phi\right], \tag{6.98}$$

where

$$a = \sigma B_0 r_0^2, \quad f(\theta) = 1 - 2H(\theta - \pi/2) \equiv \text{sgn}(cos\theta). \tag{6.99}$$

Here $H(\theta)$ is the Heaviside step function, Ω is the angular speed of rotation of the Sun, u is the radial solar wind speed (both Ω and u are assumed to be constant) and θ is the helio-colatitude. Parameter $\sigma = 1$ if the field is radially outward above the current sheet, and $\sigma = -1$ corresponds to the opposite polarity case where the field is inward above the current sheet.

The Parker magnetic field is sometimes referred to as the garden hose spiral because the pattern produced by a rotating sprinkler, consists of radially moving

water droplets, which form an Archimedean spiral pattern due to the rotation of the sprinkler at the center of the pattern. Similarly, the frozen in magnetic field blobs (elements) released from the rotating Sun move radially outward with the plasma. However as the blobs are emitted from the rotating solar source surface the net pattern of the frozen in field elements forms the Archimedean spiral. It is implicitly assumed that the magnetic field is sufficiently weak, that it does not modify the background flow (this assumption is questionable near to the solar surface where the plasma β is small). Gleeson and Webb (1980) provides a derivation of the Parker Archimedean spiral field, by directly solving Faraday's equation which implies the magnetic flux is frozen in to the flow, subject to the constraint $\nabla \cdot \mathbf{B} = 0$. The solar wind velocity is assumed to be a constant radial outflow from the Sun (this latter condition that u is constant is not necessary), in which the field does not modify the background flow.

An illustration of the Parker spiral magnetic field in the heliosphere is shown schematically in Fig. 6.8 (from Balogh et al. 2008). The field lines for the Parker spiral are obtained by integrating the field line equations:

$$\frac{dr}{B_r} = \frac{r d\theta}{B_\theta} = \frac{r \sin \theta}{B_\phi}, \tag{6.100}$$

Fig. 6.8 Parker spiral field in the heliosphere, including the Sun, the heliospheric termination shock, the heliopause and the heliotail (Balogh et al. 2008)

with integrals:

$$\theta = c_1 = const., \quad \phi + \frac{\Omega r}{u} = c_2, \tag{6.101}$$

where c_1 and c_2 are integration constants. The field lines (6.101) consist of Archimedean spirals $r = -(u/\Omega)(\phi - \phi_0) + r_0$ that lie on the surface of a cone $\theta = c_1$. The Parker field for the case of a flat current sheet in the helio-equatorial plane is illustrated in Fig. 6.8 (Balogh et al. 2008: figure made by Steve Suess, MSFC, in March 1999 (private communication)). It shows the Parker magnetic field north of the flat current sheet located in the helio-equatorial plane. Also shown is the (assumed) spherical termination shock, where the solar wind undergoes a supersonic-subsonic transition at the spherical termination shock. The helio-pause is the contact surface beyond the termination shock, where the gas of solar origin meets the interstellar gas. The heliotail streamlines of the flow inside the helio-pause are illustrated on the top right-handside of the figure. A more realistic form of the magnetic field, involves a warped current sheet in the vicinity of the helio-equator is illustrated in Fig. 6.9 (e.g. Jokipii and Thomas 1981). The magnetic field south of the current sheet (not shown) is also an Archimedean spiral, but the field has opposite polarity to the field north of the current sheet.

In many applications, it is necessary to define a gauge invariant form of the magnetic helicity (3.21), for cases, where $\mathbf{B} \cdot \mathbf{n} \neq 0$ on the boundary ∂V of the volume V bounding the plasma by using the relative helicity (3.22), i.e.

$$H_r = \int_V d^3x \, (\mathbf{A}_1 + \mathbf{A}_2) \cdot (\mathbf{B}_1 - \mathbf{B}_2), \tag{6.102}$$

where $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$ is the magnetic field of interest, and $\mathbf{B}_2 = \nabla \times \mathbf{A}_2$ is a reference magnetic field, with the same normal flux as $\mathbf{B}_1$ ($\mathbf{B}_2$ is sometimes taken as a potential magnetic field where $\mathbf{B}_2 \cdot \mathbf{n} = \mathbf{B}_1 \cdot \mathbf{n}$ on ∂V). The relative helicity (6.102)

Fig. 6.9 Parker spiral field and heliospheric current sheet (Jokipii and Thomas 1981; Wikipedia)
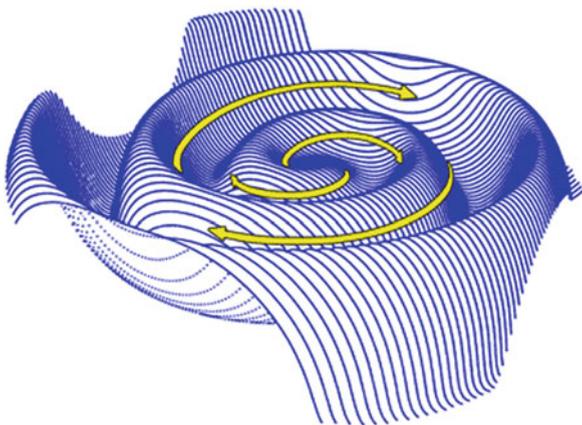
is independent of the gauges chosen for the magnetic vector potentials $\mathbf{A}_1$ and $\mathbf{A}_2$. The usual magnetic helicity integral is recovered if $\mathbf{B}_2 = \mathbf{A}_2 = 0$.

Two methods to determine the magnetic vector potential $\mathbf{A}$ for a given $\mathbf{B}$-field, is to either use the Biot Savart formula or Coulomb gauge form for $\mathbf{A}$, or to use the homotopy formula (Webb et al. 2010a). The Biot-Savart formula:

$$\mathbf{A}_c = -\frac{1}{4\pi} \int_V d^3x' \frac{(\mathbf{x} - \mathbf{x}') \times \mathbf{B}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3}, \tag{6.103}$$

gives the vector potential for the Coulomb gauge. For magnetic fields that are well behaved near $r = 0$ (i.e. $\mathbf{B} \sim \mathbf{C}r^s$ with $s > -2$) the homotopy formula for $\mathbf{A}$ is

$$\mathbf{A}_h = \int_0^1 \mathbf{B}(\lambda\mathbf{r}) \times \lambda\mathbf{r} \, d\lambda, \tag{6.104}$$

(an alternative homotopy formula for $\mathbf{A}_h$ applies if $s < -2$: Webb et al. 2010a). Formula (6.104) was used by Webb et al. (2010a) to determine the relative helicity of magnetic flux ropes observed by the wind spacecraft. In general, $\mathbf{A}_c = \mathbf{A}_h + \nabla\Phi_{ch}$ where Φ_{ch} is a gauge potential.

6.6.2   Magnetic Field Representations

The magnetic helicity of the Parker field (6.98) was studied by Bieber et al. (1987). The related problem of magnetic helicity injection into the solar wind was also investigated by Berger and Ruzmaikin (2000) (see also Webb et al. 2010a). An overview of helicity injection into the solar wind is given by Berger (1999b). The role of helicity injection in coronal mass ejections was studied by Low (1994) and Rust (1994).

Bieber et al. (1987) showed that the magnetic vector potential $\mathbf{A}_c$ for the Parker spiral magnetic field (6.98)–(6.99) using the Coulomb gauge ($\nabla\cdot\mathbf{A}_c = 0$) is given by

$$
\begin{aligned}
A_c^r &= \frac{2a\Omega}{3u} f(\theta) \left(1 - \frac{3}{2}x - x\ln(1 + x) \right), \\
A_c^\theta &= \frac{2a\Omega}{3u} \sin\theta f(\theta) \left(\frac{x}{1 + x} + \ln(1 + x) \right), \\
A_c^\phi &= \frac{a}{r\sin\theta}(1 - x), \quad x = |\cos\theta|,
\end{aligned}
\tag{6.105}
$$

The magnetic vector potential for the split monopole magnetic field for $\Omega = 0$ is obtained if $\mathbf{A} = A_c^\phi \mathbf{e}_\phi$.

Webb et al. (2010a) used the homotopy formula for $\mathbf{A}$ to obtain the magnetic vector potential $\mathbf{A} = \mathbf{A}_h$ of the form:

$$\mathbf{A}_h = a\left(\frac{1 - |\cos\theta|}{r\sin\theta}\mathbf{e}_\phi - \frac{f(\theta)\Omega\sin\theta}{u}\mathbf{e}_\theta\right), \tag{6.106}$$

which gives the Parker field (6.98) (i.e. $\mathbf{B} = \nabla \times \mathbf{A}_h$).

By using the holonomic base vectors ∇r, $\nabla\theta$, $\nabla\phi$, and using $\mathbf{B} = \nabla \times \mathbf{A}_h$ we obtain:

$$\mathbf{B} = \nabla\alpha \times \nabla\beta = \nabla \times \mathbf{A}_e,$$

$$\mathbf{A}_e = \alpha\nabla\beta, \quad \alpha = -a|\cos\theta|, \quad \beta = \phi + \frac{\Omega r}{u} - \Omega t, \tag{6.107}$$

as another representation of the Parker magnetic field in terms of the Euler potentials α and β, which are Lagrangian variables which are advected with the flow. In this representation the field lines are given by the intersection of the surfaces $\alpha = const.$ and $\beta = const.$ The Euler potential representation for magnetic fields are discussed by Parker (1979, Ch. 4), and by Low (2006).

The Parker magnetic field $\mathbf{B}$ can also be decomposed as $\mathbf{B} = \mathbf{B}_P + \mathbf{B}_T$ where $\mathbf{B}_P$ and $\mathbf{B}_T$ are the toroidal and poloidal components of $\mathbf{B}$, where $\mathbf{B}_P$ is the split monopole field, and the toroidal field is the azimuthal component of the field. We obtain:

$$\mathbf{B}_P = \frac{a}{r^2}f(\theta)\mathbf{e}_r = \nabla \times \mathbf{A}_P, \quad \mathbf{A}_P = \nabla \times (\mathbf{r}P) = \frac{a(1 - |\cos\theta|)}{r\sin\theta}\mathbf{e}_\phi,$$

$$\mathbf{B}_T = -\frac{a}{r^2}f(\theta)\frac{\Omega r\sin\theta}{u}\mathbf{e}_\phi = \nabla \times \mathbf{A}_T, \quad \mathbf{A}_T = \mathbf{r}T = -\frac{a\Omega|\cos\theta|}{u}\mathbf{e}_r, \tag{6.108}$$

where explicit formulae for P and T are given in Webb et al. (2010a). The magnetic vector potentials $\mathbf{A}_P$ and $\mathbf{A}_T$ can be expressed in the Euler potential forms:

$$\mathbf{A}_P = a(1 - |\cos\theta|)\nabla\phi \equiv \alpha_1\nabla\beta_1,$$

$$\mathbf{A}_T = -\frac{a\Omega|\cos\theta|}{u}\nabla(r - ut) \equiv \alpha_2\nabla\beta_2, \tag{6.109}$$

where the α_i and β_i surfaces enclose elemental volumes in spherical polar coordinates. Thus, the magnetic helicity analysis of Low (2006) in which $\mathbf{B}$ is split up into a poloidal and a toroidal field applies in this case (see also Kruskal and Kulsrud 1958).

The poloidal and toroidal decomposition (6.108) can be used to describe the helicity in terms of the linkage of the poloidal and toroidal flux (e.g. Kruskal and Kulsrud 1958; Berger and Field 1984; Finn and Antonsen 1985, 1988; Low 2006). The poloidal flux of the split monopole field passes through the closed loops of

the toroidal field. One can visualise in a more general model, the poloidal flux as consisting of closed field loops that return at large distances from the Sun at the symmetric point south of the current sheet at $\theta = \pi - \theta_1$ where θ_1 corresponds to the outward poloidal field, with toroidal field loops of the opposite polarity south of the current sheet.

6.6.3   Magnetic Helicity of the Parker Field

The gauge independent relative helicity density in the spherical shell $R_1 < r < R_2$ (see (6.102)) is:

$$h_r = (\mathbf{A}_1 + \mathbf{A}_2) \cdot (\mathbf{B}_1 - \mathbf{B}_2) = -\frac{2a^2 f(\theta)}{r^2} \frac{\Omega(1 - |\cos\theta|)}{u}, \qquad (6.110)$$

where $\mathbf{A}_2$ and $\mathbf{B}_2$ are the split monopole contributions to the Parker spiral field, and $\mathbf{B}_1$ and $\mathbf{A}_1$ are the total fields associated with the Parker spiral field (i.e. $\mathbf{A}_1$ and $\mathbf{B}_1$ are given by (6.106) and (6.98) and $\mathbf{B}_2$ is the radial component of $\mathbf{B}_1$ and $\mathbf{A}_2$ is the azimuthal component of $\mathbf{A}_h$ in (6.106)).

Integrating h_r over the northern hemispherical shell with $R_1 < r < R_2$:

$$H_r^N = \int_{R_1}^{R_2} dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \, r^2 \sin\theta \, h_r = -\frac{2\pi a^2 \Omega(R_2 - R_1)}{u}, \qquad (6.111)$$

we obtain the net relative helicity above the current sheet, in the region $R_1 < r < R_2$. The total relative helicity over the shell $R_1 < r < R_2$ is $H_r = H_r^N + H_r^S = 0$ where H_r^S is the total relative helicity below the current sheet. Note that $H_r^N < 0$ above the current sheet, and $H_r^S > 0$ below the current sheet.

The formula (6.110) can be written in the form:

$$H_r^N = -\frac{\Omega(R_2 - R_1)}{u} \frac{\Phi_N^2}{2\pi}, \qquad (6.112)$$

where

$$\Phi_N = 2\pi r_0^2 B_{r0} = 2\pi |a|, \qquad (6.113)$$

is the magnetic flux in the northern hemisphere above the current sheet.

Below, we calculate the magnetic helicity for the region $R_1 < r < R_2$ using the poloidal-toroidal decomposition of the field. The net result is the same as the result (6.112). Using the toroidal and poloidal decomposition (6.108), we obtain the

helicity density

$$h_{PT}^N = (\mathbf{A}_p + \mathbf{A}_T) \cdot \mathbf{B} = -\frac{a^2 \Omega}{r^2 u}, \tag{6.114}$$

Note that $h_{PT} = \mathbf{A}_P \cdot \mathbf{B}_T + \mathbf{A}_T \cdot \mathbf{B}_P$ and that the helicity H_{PT}^N obtained by integrating h_{PT}^N over a spherical shell north of the current sheet is given by H_r^N in (6.111). This example illustrates the theory of Low (2006) in which it is not necessary to worry about the gauge of $\mathbf{A}$ in order to calculate a gauge free magnetic helicity if the field admits a poloidal-toroidal decomposition.

The helicity integral H_{PT}^N can also be written in the form (Webb et al. 2010a, Appendix E):

$$H_{PT}^N = \int_{V_N} h_{PT}\, d^3x = \int_0^{\pi/2} d\theta \left(F_T^N \frac{dF_P^N}{d\theta} - F_P^N \frac{dF_T^N}{d\theta} \right), \tag{6.115}$$

where V_N is the volume north of the current sheet and

$$F_N^T = \int B_T dS_T = -\frac{a\Omega(R_2 - R_1)}{u} \cos\theta, \quad F_P^N = \int B_P\, dS_P = 2\pi a(1 - \cos\theta), \tag{6.116}$$

are the flux integrals of the toroidal and poloidal magnetic field components for the Parker spiral magnetic field. We find $H_{PT}^N = H_r^N$ where H_r^N is given by (6.112)–(6.113). In (6.116) $dS_T = rdr \wedge d\theta$ and $dS_P = r^2 \sin\theta\, d\theta \wedge d\phi$ are the toroidal and poloidal surface elements. In this formulation, the $\theta = const.$ surfaces are magnetic flux surfaces (i.e. $\mathbf{B} \cdot \nabla\theta = 0$ and $\mathbf{B}$ has no component normal to the surface, i.e., $B_\theta = 0$). The toroidal flux is located in the helio-equatorial band $\theta < \theta' < \pi/2$, $R_1 < r < R_2$ and $0 < \phi' < 2\pi$ where θ' and ϕ' are integration variables corresponding to θ and ϕ. The poloidal (i.e. radial) flux is in the polar heliolatitude band $0 < \theta' < \theta$, $0 < \phi' < 2\pi$ and $R_1 < r < R_2$. Thus, the helicity integral H_{PT}^N represents the linkage of the poloidal and toroidal fluxes (e.g. Kruskal and Kulsrud 1958; Low 2006). Note that H_{PT}^N in (6.115) depends only on the poloidal and toroidal flux integrals, which are gauge independent.

The result (6.112) for the helicity H_r^N north of the current sheet also applies for a model Parker field, with a warped current sheet as in Fig. 6.9 (Webb et al. 2010a). This is expected as H_r^N describes a topological invariant, which does not depend on the detailed geometry of the current sheet.

Using the magnetic helicity transport equation (Webb et al. 2010a; Berger and Field 1984; Berger and Ruzmaikin 2000; Finn and Antonsen 1985) results in the helicity injection rate north of the current sheet as:

$$\frac{\partial H_r^N}{\partial t} = \int_{r=R_1} 2(\mathbf{A}_2 \times \mathbf{E}_1) \cdot \mathbf{n} dS = -\frac{\Phi_N^2}{T}, \quad T = \frac{2\pi}{\Omega}, \tag{6.117}$$

where $\mathbf{E}_1 = -\mathbf{u} \times \mathbf{B}_1$ is the motional electric field and $T = 2\pi/\Omega$ is the solar rotation period (see e.g. Berger and Ruzmaikin 2000; Webb et al. 2010a) for estimates of the helicity injection rate.

6.6.4 Alfvén Simple Waves

Consider the relative helicity of multi-dimensional Alfvén simple waves. Webb et al. (2010b) identified two basic Alfvén modes: (a) the planar 1D simple Alfvén wave (the shear mode), in which the wave propagates along the z-axis, with wave normal $\mathbf{n} = (0, 0, 1)$ and phase $\varphi = k_0(z - \lambda t)$, where k_0 is the wave number and $\lambda \mathbf{e}_z$ is the group velocity of the wave, and (b) the generalized Barnes (1976) simple wave with wave normal $\mathbf{n} = (\cos\varphi, \sin\varphi, 0)$ in the xy-plane, and the mean field $B_z = const.$ (possibly $B_z = 0$). More complex Alfvén simple waves are given by Webb et al. (2010b, 2011), in which all physical variables depend on a single wave phase $\varphi(x, y, z, t)$, where φ satisfies an implicit equation of the form: $f(\varphi) = \mathbf{x} \cdot \mathbf{n} - \lambda(\varphi)t$. $\mathbf{k} = \nabla\varphi$ and $\omega = -\varphi_t$ are the local wavenumber and frequency of the wave. $\mathbf{n}(\varphi) = \mathbf{k}/k$ is the wave normal and ω/k is an eigenvalue of the MHD equations ($\lambda(\varphi)$ is the normal speed of the wave front). Simple Alfvén waves admit the six integrals:

$$\mathbf{u} \pm \frac{\mathbf{B}}{\sqrt{\mu\rho}} = \mathbf{u} \pm \mathbf{V}_A = (V_1, V_2, V_3) = \mathbf{V} = const.,$$

$$p = c_4, \quad \rho = c_5, \quad B^2 = c_6. \tag{6.118}$$

where the $\{V_i : 1 \leq i \leq 3\}$ and the $\{c_i : 4 \leq i \leq 6\}$ are integration constants. Here $(\rho, \mathbf{u}, \mathbf{B}, p)$ are the gas density, fluid velocity, magnetic field induction, and gas pressure and $\mathbf{V}_A = \mathbf{B}/\sqrt{\mu\rho}$ is the Alfvén velocity. We consider only the forward wave for which $\mathbf{u} + \mathbf{V}_A = \mathbf{V} = const.$

The Shear Mode

For the shear Alfvén wave

$$\mathbf{B} = B_\perp(\cos\varphi, \sin\varphi, 0) + B_\parallel(0, 0, 1),$$

$$\varphi = k_0(z - \lambda t), \quad \lambda = u_z + V_{Az}, \quad \mathbf{n} = (0, 0, 1). \tag{6.119}$$

where $B_\parallel = B_0 \cos\alpha_0$ and $B_\perp = B_0 \sin\alpha_0$. The magnetic field lines for the wave are described by the helix $\tilde{\mathbf{r}} = (\tan\alpha_0 \sin\varphi, -\tan\alpha_0 \cos\varphi, \varphi)/k_0$ in the traveling wave frame ($\tilde{\mathbf{r}} = \mathbf{r} - \lambda t \mathbf{e}_z$). An illustration of a magnetic field line for the shear Alfvén wave (6.119) is shown in Fig. 6.10. The hodograph of $\mathbf{B}(\varphi) = (B_x, B_y, B_z)$ is a circle at the co-latitude $\theta = \alpha_0$ of radius $B_\perp = B_0 \sin\alpha_0$ and center $(0, 0, B_0 \cos\alpha_0)$ on the sphere $|\mathbf{B}| = B_0 = const.$
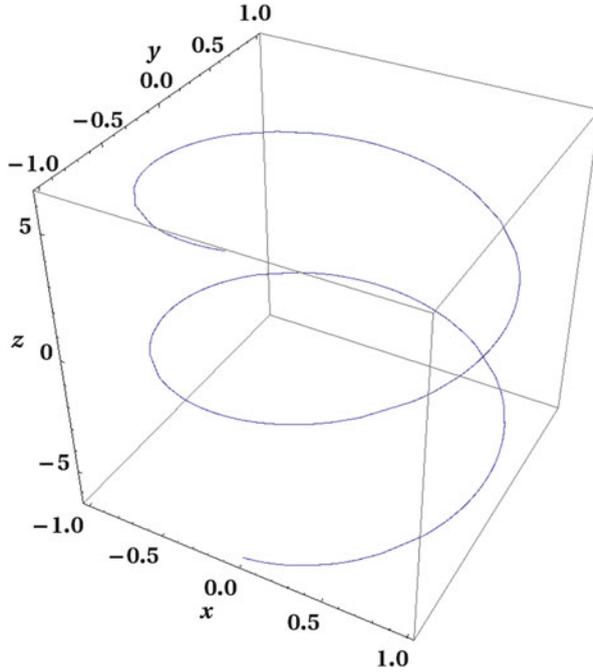
Fig. 6.10 Magnetic field line for the 1D, shear Alfvén wave (6.119), with $\alpha_0 = \pi/4$, $k_0 = 1$. The wave is a traveling, non-centered simple wave, propagating along the z-axis. The wave normal $\mathbf{n} = (0, 0, 1)^T$. The current is finite and azimuthal about the z-axis

The relative helicity H_r of the shear Alfvén wave for a volume $V = \{(x, y, z) : 0 \le x \le a, 0 \le y \le b, 0 \le z \le c\}$ where $c = 2\pi m/k_0$ is an integral number of wave lengths may be written in the form:

$$H_r = abc\langle h_r \rangle, \quad \langle h_r \rangle = -\sigma \frac{B_\perp^2}{|k_0|}(1 - \delta), \quad \sigma = \pm 1,$$

$$\delta = \frac{32c^2}{\pi^4 ab} \frac{\cosh(k_0 a) - 1}{\sinh(|k_0|a)} \sum_{n=1}^{\infty} \frac{n_1[1 - (-1)^n]}{(n^2 - n_1^2)^2 n} \frac{\cosh(n\pi b/c) - 1}{\sinh(n\pi b/c)}. \tag{6.120}$$

where $n_1 = 2m$. Effectively, the sum (6.120) is over odd values of n. Note that $n_1 = 2m$ is an even integer, so the sum in (6.120) is well defined. The result (6.120) applies for a fixed volume V in the traveling wave frame. For $c/a \ll 1$ and $c/b \ll 1$ the relative helicity density $\langle h_r \rangle$ simplifies to:

$$\langle h_r \rangle \approx -\sigma \frac{B_\perp^2}{k_0}, \tag{6.121}$$

which is the helicity density for a shear Alfvén wave packet with wavelength much less than its transverse dimensions given by Berger and Field (1984).

Torsional 2D Mode Solutions

A relatively simple 2D mode or generalized (Barnes 1976) solution, has magnetic field **B** and wave normal **n** given by Webb et al. (2010b):

$$\mathbf{B} = B_\perp(\cos\varphi, \sin\varphi, 0) + B_\parallel(0, 0, 1), \quad \mathbf{n} = (\cos\varphi, \sin\varphi, 0), \qquad (6.122)$$

where the wave phase satisfies a wavefront equation of the form $f(\varphi) = \tilde{\mathbf{r}} \cdot \mathbf{n}$, and $\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{V}t$ is the position in the traveling wave frame. Choosing $f(\varphi) = 0$ (centered simple wave case), the wavefront equation reduces to $\tilde{\mathbf{r}}.\mathbf{n} = \tilde{x}\cos\varphi + \tilde{y}\sin\varphi = 0$, which may be written as:

$$\varphi = \theta + \sigma\pi/2, \quad \sigma = \pm 1, \qquad (6.123)$$

where (ϖ, θ, z) are cylindrical coordinates (θ is the azimuthal angle and ϖ is cylindrical radius). Thus, the magnetic field in the wave is $\mathbf{B} = \sigma B_\perp \mathbf{e}_\theta + B_\parallel \mathbf{e}_z$ and σ determines the sense of rotation of the transverse field about the z axis. A schematic of the solution (6.122) for **B** and wave normal **n**, for the case $B_\parallel = 0$ is shown in Fig. 6.11. and the wave normal **n** for the case $B_\parallel = 0$ is given in figure. The relative helicity of the wave can be determined either from the relative helicity formula (6.102) using $\mathbf{B} = B_\parallel \mathbf{e}_z$ for the comparison magnetic field, or by expressing the helicity integral in terms of toroidal and poloidal fluxes. Note that the cylindrical surface $\varpi = const.$ are flux surfaces for the field. The poloidal-toroidal decomposition of the field, and the fluxes for a cylindrical volume $V = \{(\varpi, \theta, z) : 0 < \varpi < R, \ 0 < \theta \leq 2\pi, \ 0 < z < L\}$ are:

$$\mathbf{B} = \mathbf{B}_P + \mathbf{B}_T, \quad \mathbf{B}_P = B_\parallel \mathbf{e}_z, \quad \mathbf{B}_T = \sigma B_\perp \mathbf{e}_\theta,$$
$$\mathbf{A}_P = \frac{F_P(\varpi)}{2\pi}\nabla\theta, \quad \mathbf{A}_T = \frac{F_T(\varpi)}{L}\nabla z,$$
$$F_P = \pi B_\parallel \varpi^2, \quad F_T = L(R - \varpi)\sigma B_\perp. \qquad (6.124)$$

Using these results we obtain:

$$H_{PT} = \frac{2\pi\sigma B_\perp B_\parallel L R^3}{3}, \qquad (6.125)$$

for the relative helicity $H_r \equiv H_{PT}$. This corresponds to a mean relative helicity density of

$$\langle h_r \rangle \equiv \langle h_{PT} \rangle = \frac{H_{PT}}{V} = \frac{4}{3}\sigma\frac{B_\perp B_\parallel}{\langle k_\perp \rangle}, \quad \text{here} \quad \langle k_\perp \rangle = \frac{2}{R}. \qquad (6.126)$$
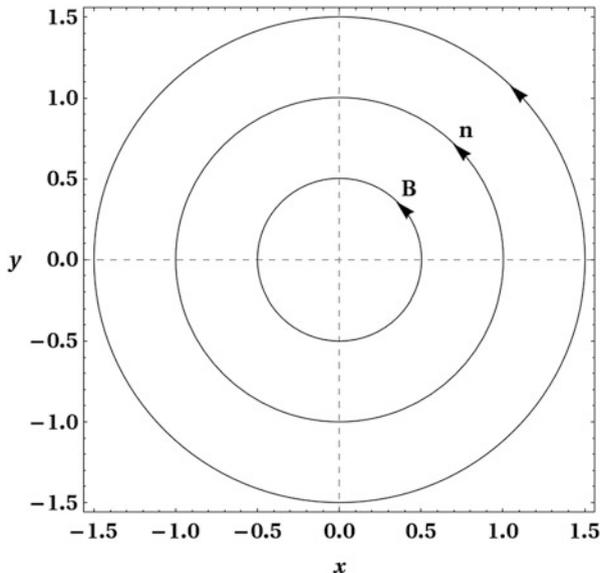
Fig. 6.11 The magnetic field lines and wave normal for the Barnes (1976) simple Alfvén wave solution (6.122) with $B_{\parallel} = 0$. The wave normal **n** and magnetic induction **B** are parallel and are directed in the azimuthal direction ($\mathbf{n} = (\cos\varphi, \sin\varphi, 0)^T$ where φ is the wave phase). The wave phase fronts are planes $\varphi = const.$ perpendicular to **B**, passing through the origin of the xy-plane

Here $\langle k_{\perp} \rangle = \int_V (1/\varpi) d^3x/V$ is the mean wave number of the wave in the volume V.

The torsional simple Alfvén wave (6.122) has current density $\mathbf{J} = B_{\perp}/(\mu_0 \varpi)\mathbf{e}_z$ directed along the z-axis. For the shear wave (6.119) $\mathbf{J} = -k_0 B_{\perp}(\cos\varphi, \sin\varphi, 0)/\mu_0$ is non-singular and has no z component and $\varphi = k_0(z - \lambda t)$. Both waves have the same field lines (i.e. a helix about the z-axis). The helicity of the shear mode in (6.120) and (6.121) depends on $B_{\perp}^2$ whereas the 2D mode helicity in (6.126) depends on $B_{\perp} B_{\parallel}$.

Wave Breaking for Alfvén Simple Waves

For simple waves, the wave normal $\mathbf{n}(\varphi)$ and wave speed $\lambda(\varphi)$ satisfy the equations:

$$\mathbf{n}(\varphi) = \frac{\nabla\varphi}{|\nabla\varphi|}, \quad \lambda(\varphi) == \frac{\omega}{k} = -\frac{\varphi_t}{|\nabla\varphi|}. \tag{6.127}$$

Thus, φ must satisfy the first order partial differential equations:

$$\nabla\varphi - \mathbf{n}(\varphi)|\nabla\varphi| = 0, \quad \varphi_t + \lambda(\varphi)|\nabla\varphi| = 0, \tag{6.128}$$

where $\mathbf{n}(\varphi)$ and $\lambda(\varphi)$ are given functions of φ. Boillat (1970) showed that the partial differential equations (6.128) have general, implicit solutions for $\varphi(x, y, z, t)$ of the form:

$$f(\varphi) = \mathbf{r} \cdot \mathbf{n}(\varphi) - \lambda(\varphi)t, \tag{6.129}$$

where $\mathbf{r} = (x, y, z)$, and $f(\varphi)$ is an arbitrary differentiable function of φ. One can verify the implicit solution (6.129) by implicit differentiation of (6.129) to obtain the equations:

$$\varphi_t = -\frac{\lambda}{F}, \quad \nabla\varphi = \frac{\mathbf{n}(\varphi)}{F}, \tag{6.130}$$

where

$$F = f'(\varphi) + \frac{d\lambda}{d\varphi}t - \mathbf{r} \cdot \frac{d\mathbf{n}}{d\varphi} = \frac{1}{|\nabla\varphi|}. \tag{6.131}$$

For a consistent solution F must be positive. At points where $F \to 0$, $k = |\nabla\varphi| \to \infty$ and wave breaking occurs.

Equation (6.129), for a fixed parameter φ, consists of a family of planes in (t, x, y, z) space, i.e.

$$G = f(\varphi) + \lambda(\varphi)t - xn^x(\varphi) - yn^y(\varphi) - zn^z(\varphi) = 0. \tag{6.132}$$

A *characteristic curve* of the family of planes (6.132) is obtained by solving the equations:

$$G(x, y, z, t, \varphi) = 0 \quad \text{and} \quad G_\varphi(x, y, z, t, \varphi) = 0, \tag{6.133}$$

simultaneously for a fixed φ (e.g. Sneddon 1957, Appendix). Calculating G_φ in (6.132) we obtain:

$$G_\varphi = F = f'(\varphi) + \frac{d\lambda}{d\varphi}t - \mathbf{r} \cdot \frac{d\mathbf{n}}{d\varphi} = \frac{1}{|\nabla\varphi|}. \tag{6.134}$$

The simultaneous solution of (6.133) defines the *envelope* of the family of planes (6.132). From (6.133)–(6.134) we note that the wave breaks on the envelope of the family of planes (6.132) (see also Appendix of Sneddon 1957). Note that current:

$$\mathbf{J} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \frac{|\nabla\varphi|}{\mu_0}\mathbf{n} \times \frac{d\mathbf{B}}{d\varphi} \to \infty \quad \text{as} \quad F \to 0, \tag{6.135}$$

on the wave envelope.

As an example of wave breaking for Alfvén simple waves, consider the Alfvén wave family with:

$$\mathbf{n} = (\sin \Theta_0 \cos \varphi, \sin \theta_0 \sin \varphi, \cos \Theta_0),$$

$$f(\varphi) = \sin \theta_0 (x \cos \varphi + y \sin \varphi) + z \cos \Theta_0 - \lambda t,$$

$$\mathbf{B} = (\sin \alpha_0 \cos \varphi, \sin \alpha_0 \sin \varphi, \cos \alpha_0),$$

$$\lambda = (\mathbf{u} + \mathbf{v}_A) \cdot \mathbf{n}, \tag{6.136}$$

and the wave possesses the six integrals (6.118)

Figure 6.12 shows two examples of Alfvén simple waves in which there is a current sheet singularity in the wave as $|\nabla \varphi| \to \infty$ and $F \to 0$. The left panel of Fig. 6.12 corresponds to a centered simple wave in which $f(\varphi) = 0$. In this case (6.132) reduces to the equation:

$$\tilde{\mathbf{x}} \cdot \mathbf{n}(\varphi) = \tilde{x} \cos \varphi + \tilde{y} \sin \varphi + \tilde{z} \cot \Theta_0 = 0, \tag{6.137}$$

where

$$\tilde{\mathbf{x}} = \mathbf{x} - (\mathbf{u} + \mathbf{V}_A)t, \tag{6.138}$$
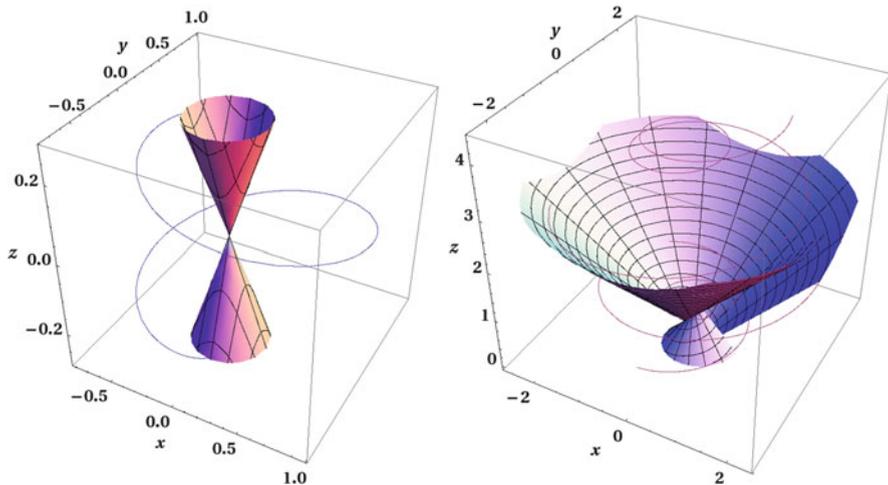
Fig. 6.12 *Left panel*: Magnetic field line for centered simple Alfvén wave, (6.136) which generalizes the (Barnes 1976) simple wave. The current is singular on the cone $\tilde{z} = \tilde{r} \tan \Theta_0$ where $\tilde{r} = (\tilde{x}^2 + \tilde{y}^2)^{1/2}$ is radial distance from the $\tilde{z}$-axes. $\Theta_0 = 45°$ and $\alpha_0 = 85°$. $\varphi = 0$ at the top of the figure, and $\varphi = \pi/c_0$ at the bottom, where the field line intersects the cone. *Right panel*: Field line and current sheet for the non-centered simple Alfvén wave (6.136), for $k_0 = 10$, $r_0 = 1$ and $0 < \varphi < 3\pi/c_0$. The field line in this case does not hit the singular current surface

is position in the wave frame. In this example F reduces to:

$$F = \sin \Theta_0 \left(\tilde{r}^2 - \tilde{z}^2 \tan^2 \Theta_0 \right)^{1/2}, \tag{6.139}$$

where $\tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2$. Thus, $F \to 0$ and $|\nabla \varphi| \to \infty$ on the conical surface $\tilde{z} = \pm \tilde{r} \tan \Theta_0$ and $|\mathbf{J}| \to \infty$. The example in the right hand panel of Fig. 6.12 is for a non-centered simple Alfvén wave in which $f(\varphi) = \varphi/k_0$ and hence

$$G = \frac{\varphi}{k_0} - \sin \Theta_0 \left(\tilde{x} \cos \varphi + \tilde{y} \sin \varphi \right) - \cos \theta_0 \tilde{z} = 0, \tag{6.140}$$

which implicitly defines $\varphi(\mathbf{x}, t)$. The figure shows a very complicated current sheet structure, and a typical example of a field line that does not intersect the current sheet. In the limit as $\Theta_0 \to 0$ the left panel solution approaches a modified form of the (Barnes 1976) solution, in which there is a current singularity on the $\tilde{z}$ axis as $\tilde{r} \to 0$.

Observations and Alfvén Wave Examples

There are other more complex 2D simple Alfvén waves discussed in Webb et al. (2010b), which show a complicated hodograph of $\mathbf{B}$ in which $|\mathbf{B}| = const.$ similar to the hodographs of $\mathbf{B}$ given in spacecraft data (e.g. Bruno et al. 2001; Roberts and Goldstein 2006).

Bruno et al. (2001, 2005) observed Alfvénic structures in the solar wind shown in the left panel of Fig. 6.13. The tip of the $\mathbf{B}$ vector moves over the sphere
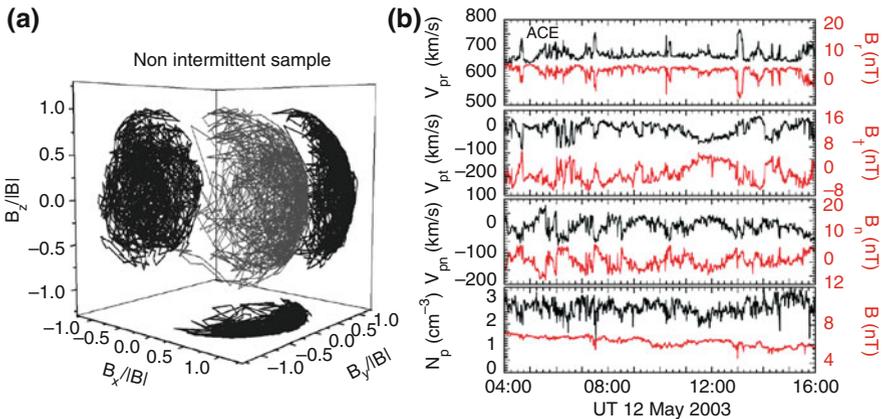

Fig. 6.13 Evidence for Alfvén waves in the solar wind: (**a**) left panel shows the tip of the magnetic field vector in minimum variance reference system in a time period without intermittency (from Bruno et al. 2001), and (**b**) right panel shows ACE data for the fluid velocity, $\mathbf{V}$, magnetic field $\mathbf{B}$ and density N on May 12, 2003 (from Gosling et al. 2009)

$|\mathbf{B}| = const.$ as expected for fully nonlinear Alfvén waves. During Alfvénic periods, the minimum variance direction tends to align with the mean magnetic field (see Barnes 1981 for a statistical model). This alignment is not expected from turbulence which decreases the alignment. The Alfvénicity of the fluctuations decreases with increasing distance from the Sun, due to wave mixing. The right panel of Fig. 6.13 shows data from Gosling et al. (2009) of nonlinear Alfvénic structures in the solar wind on May 12, 2003.

Matteini et al. (2015) (Fig. 6.14) provide evidence of large amplitude Alfvénic fluctuations in the solar wind obtained from an analysis of the Helios data of protons and alpha particles and magnetic field fluctuations. They show that the protons velocity distribution in the wave frame is roughly spherical implying kinetic energy of the protons is conserved in the wave frame. The wave frame is identified as the mean frame of the alpha particles, which move approximately with the velocity $\mathbf{u} + \mathbf{V}_A$, where $\mathbf{u}$ is the mean fluid velocity and $\mathbf{V}_A = \mathbf{B}/\sqrt{\mu\rho}$ is the Alfvén velocity. This observation is consistent to a first approximation, with fully nonlinear outward propagating, Alfvén waves in which $\mathbf{u} + \mathbf{V}_A = \mathbf{V}$ is constant for which

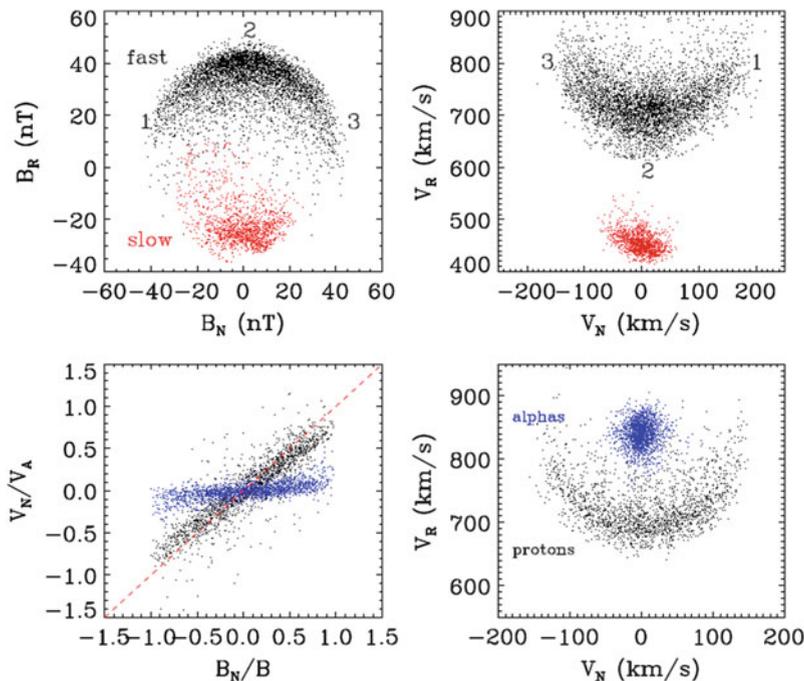**Fig. 6.14** Top: Scatterplot of normal and radial magnetic field (left) and proton velocity (right) red points: from earlier low speed solar wind; Bottom: (left) scatterplot of normal Alfvénic components for protons and alphas (blue); red dashed straight line corresponds to Alfvén speed; (right) protons and alphas (blue) in the (V_N, V_R) plane for same time as left panel (from Matteini et al. 2015)

$\delta\mathbf{u} = -\delta\mathbf{V}_A$ (the relative alpha-proton velocity $V_{\alpha p} = \langle V_\alpha - V_p\rangle \approx 0.85V_A$ is always approximately aligned with the magnetic field (Marsch 1982)).

Figure 6.15 shows the magnetic field lines for an Alfvén simple wave that has both a fast evolution of the wave phase φ superimposed on a longer scale periodic evolution of the phase (the fast evolution is due to the change in the wave normal $\mathbf{n}(\varphi)$ and the longer scale evolution comes from the evolution of the components of $\mathbf{B}$ in the moving trihedron frame in which $\mathbf{n}(\varphi)$ is the tangent vector to some curve $\mathbf{X}(\varphi)$ (Webb et al. 2010b, 2011). The main point is that the hodograph of $\mathbf{B}$ in the (B_x, B_y, B_z) coordinates moves on the $B = const.$ sphere, which is similar, at lowest order to the spacecraft observations of $\mathbf{B}$ depicted in Figs. 6.13 and 6.14. Figure 6.16 shows a similar nonlinear simple Alfvén wave with $\mathbf{n} = \mathbf{n}(\varphi)$. The left panel shows the field line and the right panel shows the hodograph of $\mathbf{B}$, which traces out a path on the $B = const.$ sphere. The hodograph of $\mathbf{B}$ is a re-scaled version by a factor of $B = const.$ of the tangent indicatrix (tantrix) of the field line.

Webb et al. (2010b) determine the relative helicity of simple Alfvén waves similar to the examples in Figs. 6.15 and 6.16. It was found that magnetic helicity decreases monotonically as $n = R_X/R_B$ increases. This suggests that magnetic helicity increases as the large scale component of curved field increases, which in turn is possibly related to an inverse cascade or dynamo action that increases the large scale field.
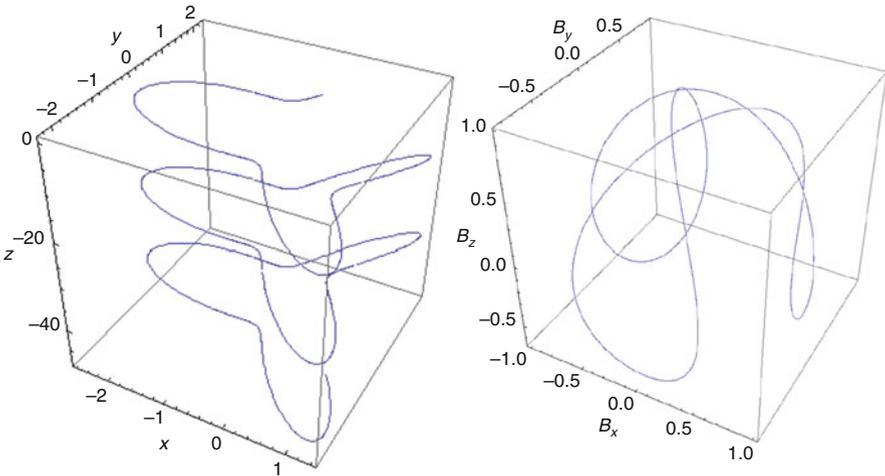
Fig. 6.15 Left panel: A field line for a centered simple Alfvén wave which generalizes Barnes (1976) solution (Webb et al. 2010b, 2011). The wave normal $\mathbf{n}(\varphi)$ evolves with the phase $\varphi_X = \varphi/R_X$ with scale R_X and the magnetic field in the Frenet frame evolves with phase $\varphi_B = \varphi/R_B$. Right panel: hodograph of $\mathbf{B}$ for the wave, lies on the $B = const.$ sphere. There are three main lobes to the field line because the parameter $n = R_X/R_B = 3$
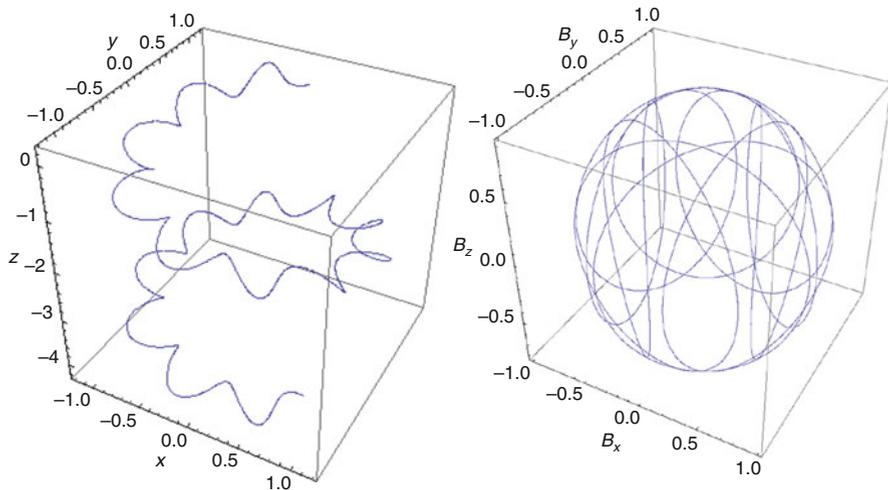
Fig. 6.16 Left panel shows a field line for a centered simple Alfvén wave. There are $n = 10 = R_X/R_B$ small curls for each 2π-period in θ. The right panel shows the hodograph of **B**, in which **B** moves on the $B = const.$ sphere (Webb et al. 2010b, 2011)

6.6.5 MHD Topological Solitons

Topological solitons in electromagnetic theory have been reviewed by Arrayás et al. (2017). The MHD topological soliton was derived by Kamchatnov (1982) by using the Hopf fibration which is a map from the 3-sphere S^3 to the two-sphere S^2 in which each point of the 2-sphere is associated with a circle S^1 on the 3-sphere. Formally, the Hopf fibration is a map $p : S^3 \to S^2 \times S^1$ where the circles S^1 are referred to as the fibers of the map or as Villarceaux circles. The Villarceaux circles in S^3 are linked to each other, and lead to linked and knotted electromagnetic field structures, by mapping tangent vectors on the 3-sphere to tangent vectors on the 2-sphere, where the tangent vectors are described by 1-forms $\tilde{A}_\mu dq^\mu$ ($1 \le \mu \le 4$) which are mapped onto tangent vectors in R^3 described by 1-forms $A_i dx^i$ ($1 \le i \le 3$) where **A** is the magnetic field vector potential and $\mathbf{B} = \nabla \times \mathbf{A}$ is the magnetic field induction. For the Kamchatnov topological soliton, the magnetic helicity is an invariant of the Hopf fibration map which allows one to construct linked magnetic fields, with specific values for the magnetic helicity.

The steady MHD equations:

$$\nabla \cdot (\rho \mathbf{u}) = 0, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times (\mathbf{u} \times \mathbf{B}) = 0, \tag{6.141}$$

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0}, \tag{6.142}$$

admits solutions for which:

$$\mathbf{u} = \pm \frac{\mathbf{B}}{\sqrt{\mu_0 \rho}} = \pm \mathbf{V}_A, \quad p + \frac{B^2}{2\mu_0} = P = const., \quad \rho = const, \qquad (6.143)$$

where $\mathbf{V}_A = \mathbf{B}/\sqrt{\mu_0 \rho}$ is the Alfvén velocity. In particular, the magnetic force balance equation (6.142) splits into the equation:

$$\rho \sigma \mathbf{V}_A \cdot \nabla \sigma \mathbf{v}_A = \rho \frac{\mathbf{B}}{\sqrt{\mu_0 \rho}} \cdot \frac{\mathbf{B}}{\sqrt{\mu_0 \rho}} = \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0}, \qquad (6.144)$$

where $\sigma = \pm 1$, and the pressure balance equation $p + B^2/(2\mu_0) = P = const.$ listed in (6.143). Equation (6.144) implies that the magnetic tension force is balanced by the inertial force. This force balance also applies to the simple Alfvén wave solutions (Webb et al. 2010b). This solution of the MHD equations was noted by Chandrasekhar (1961) and by Kamchatnov (1982) for the case of incompressible MHD. For non-barotropic, compressible MHD, with equation of state $p = p(\rho, S)$, the pressure balance equation is a consistent solution provided there is a variation in the entropy through the wave. It is clearly different than the simple Alfvén waves investigated in the previous section for which the magnetic pressure $B^2/(2\mu_0) = const$. The above solution used by Kamchatnov (1982) to construct the MHD topological soliton (see also Sagdeev et al. 1986; Semenov et al. 2002; Thompson et al. 2014). Chanteur (1999) discusses constraints on localized Alfvénic solutions for compressible MHD. Tsinganos (1981) discusses the analog of Hill's spherical vortex for steady, Alfvénic MHD flows with an ignorable coordinate.

Below we give a short account of the MHD topological soliton based on the work of Semenov et al. (2002) and Kamchatnov (1982).

In Semenov et al. (2002), the magnetic vector potential $\mathbf{A}$ and magnetic field induction $\mathbf{B}$ have the form:

$$\mathbf{A} = -\beta \nabla \alpha + \nabla \psi, \quad \mathbf{B} = \nabla \alpha \times \nabla \beta. \qquad (6.145)$$

If $\mathbf{A}$, α, β, and ψ are smooth, single valued functions of space and time, then the magnetic helicity integral:

$$H_M = \int_V \mathbf{A} \cdot \mathbf{B} \, d^3x = \int_V \nabla \psi \cdot \nabla \alpha \times \nabla \beta \, d^3x = \int_V \nabla \psi \cdot \mathbf{B} \, d^3x$$

$$= \int_V \nabla \cdot (\mathbf{B} \psi) \, d^3x = \int_{\partial V} \psi \mathbf{B} \cdot \mathbf{n} \, dS = 0, \qquad (6.146)$$

because it is assumed that $\mathbf{B} \cdot \mathbf{n} = 0$ on the boundary ∂V.

In order to obtain a non-zero magnetic helicity H_M, it is necessary that the potential ψ is discontinuous at a set of surfaces $\{\Sigma_j : 1 \leq j \leq n - 1\}$ that splits the volume V into sub-volumes $V_1, V_2, \ldots V_n$. Consider the case where $V = V_1 \cup V_2$

is split up into two disjoint regions V_1 and V_2 in which the adjoining surface Σ is a surface of discontinuity, across which ψ jumps by $[\psi] = \psi_2 - \psi_1$. In this case:

$$H_M = \int_{V_1} \mathbf{A} \cdot \mathbf{B} \, d^3x + \int_{V_2} \mathbf{A} \cdot \mathbf{B} \, d^3x = \int_{V_1} \nabla \cdot (\mathbf{B}\psi) \, d^3x + \int_{V_2} \nabla \cdot (\mathbf{B}\psi) \, d^3x$$

$$= \int_{S_1} \mathbf{B} \cdot \mathbf{n}\psi \, dS + \int_{\Sigma} \mathbf{B} \cdot \mathbf{n}_1 \psi_1 + \mathbf{B} \cdot \mathbf{n}_2 \psi_2 \, dS + \int_{S_2} \mathbf{B} \cdot \mathbf{n}\psi \, dS$$

$$= \int_{S_1 \cup S_2} \mathbf{B} \cdot \mathbf{n}\psi \, dS + \int_{\Sigma} \mathbf{B} \cdot \mathbf{n}_2[\psi] \, dS \equiv \int_{\Sigma} \mathbf{B} \cdot \mathbf{n}_2[\psi] \, dS. \tag{6.147}$$

Thus, $H_M \neq 0$ due to the jump in $[\psi] = \psi_2 - \psi_1$ across Σ. This proves the assertion that a jump in ψ across Σ results in a non-zero H_M.

Hopf Fibration

The Hopf fibration in our application is a map between S^3 and S^2. The three sphere S^3 is defined as the set of points $(q_1, q_2, q_3, q_4)^T \in \mathbb{R}^4$ such that $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$. One can also define S^3 by introducing two complex numbers:

$$Z_1 = q_1 + iq_2 \quad \text{and} \quad Z_2 = q_3 + iq_4. \tag{6.148}$$

In terms of Z_1 and Z_2, S^3 is described by the equation

$$|Z_1|^2 + |Z_2|^2 = 1. \tag{6.149}$$

The Hopf fibration from the 3-sphere S^3 to the 2-sphere S^2 is defined by the map:

$$p(Z_1, Z_2) = \left(2Z_1 Z_2^*, |Z_1|^2 - |Z_2|^2\right), \tag{6.150}$$

where we think of the 2-sphere as $\mathbb{C} \times \mathbb{R}$. From (6.150) we note:

$$|2Z_1 Z_2^*|^2 + \left(|Z_1|^2 - |Z_2|^2\right)^2 = 4|Z_1|^2|Z_2|^2 + \left(|Z_1|^2 - |Z_2|^2\right)^2 = \left(|Z_1|^2 + |Z_2|^2\right)^2 = 1, \tag{6.151}$$

which shows that $p(Z_1, Z_2)$ lies on the two sphere S^2. Because $p(\lambda Z_1, \lambda Z_2) = |\lambda|^2 p(Z_1, Z_2) = p(Z_1, Z_2)$ if $|\lambda|^2 = 1$, the circle $|\lambda|^2 = 1$ on S^3 is mapped onto a single point on S^2, which is identified as a Villarceaux circle on S^3. In addition the inverse image $p^{-1}(m)$ of a point on the 2-sphere S^2 is a Villarceaux circle on S^3. The Hopf fibration map (6.151) can be thought of as a stereographic projection from S^3 onto S^2 in which

$$p(Z_1, Z_2) = (x, y, z) = \left(\frac{2\,\mathrm{Re}(\zeta)}{1 + |\zeta|^2}, \frac{2\,\mathrm{Im}(\zeta)}{1 + |\zeta|^2}, \frac{1 - |\zeta|^2}{1 + |\zeta|^2}\right), \tag{6.152}$$

where

$$\zeta = \frac{Z_2}{Z_1}, \tag{6.153}$$

Note that $x^2 + y^2 + z^2 = 1$, i.e. (x, y, z) lies on S^2. The projective definition (6.152) is equivalent to the complex number definition of the Hopf map in (6.150).

The Hopf fibration, can be described using the $SU(2)$ and $SO(3)$ Lie groups. $SU(2)$ can be described by the Pauli spin matrices, or in terms of quaternions, and by using angular coordinates (θ, φ, ψ) in S^3 and in terms of the angular variables (θ, φ) on S^2.

Hopf Map and Topological Soliton

A circle on S^3 can be represented by the equation:

$$I(t) = (Z_1 \exp(i\omega_1 t), Z_2 \exp(i\omega_2 t)) . \tag{6.154}$$

Two circles which correspond to different initial points Z_1 and Z_2, (where ω_1 and ω_2 are integers) link each other $\omega_1 \omega_2$ times (Semenov et al. 2002). The topological soliton obtained by Kamchatnov (1982) corresponds to the case $\omega_1 = 1$ and $\omega_2 = -1$.

A tangential vector field on S^3 is obtained by differentiating the curve (6.154) with respect to t to obtain the curve:

$$\tilde{A}_\mu = \mathbf{Y}(\omega_1, \omega_2) = \frac{dI(t)}{dt} = (-\omega_1 q_2, \omega_1 q_1, -\omega_2 q_4, \omega_2 q_3) , \tag{6.155}$$

where $\omega_A = \tilde{A}_\mu dq^\mu$ is the magnetic vector potential one form on S^3. Note that $q_\mu \tilde{A}_\mu = 0$ implies $\tilde{A}^\mu$ is tangent to S^3. Here

$$q_1 = Z_1 \cos \omega_1 t, \quad q_2 = Z_1 \sin \omega_1 t, \quad q_3 = Z_2 \cos \omega_2 t, \quad q_4 = Z_2 \sin \omega_2 t. \tag{6.156}$$

Curves with different Z_1 and Z_2 initial data also have link number of $\omega_1 \omega_2$. From Kamchatnov (1982) the magnetic vector potential one-form ω_A is such that

$$\omega_A = \tilde{A}_\mu dq^\mu = A_i dx^i, \tag{6.157}$$

where $\omega_A = A_i \, dx^i$ is the form in the range R^3. From (6.157)

$$A_i = \tilde{A}_\mu \frac{\partial q^\mu}{\partial x^i}, \tag{6.158}$$

To obtain the topological soliton solutions, Kamchatnov (1982) and Semenov et al. (2002) use the stereographic projection from $S^3 \to R^3$ from the south pole

$(0, 0, 0, -1)$. This transformation preserves the Hopf invariant. They obtained the transformations:

$$x_i = \frac{q_i}{1 + q_4}, \quad i = 1, 2, 3, \tag{6.159}$$

$$q_4 = \frac{1 - x^2}{1 + x^2}, \quad q_i = \frac{2x_i}{1 + x^2}, \quad x^2 = x_1^2 + x_2^2 + x_3^2, \quad 1 = 1, 2, 3. \tag{6.160}$$

Using (6.157)–(6.160) one obtains the formulae:

$$A_i = (1 + q_4)\tilde{A}_i - q_i\tilde{A}_4, \tag{6.161}$$

$$\tilde{A}_\mu = \frac{1}{2}(1 + x^2)A_i - x_i x_j A_j, \quad \tilde{A}_4 = -x_i A_i, \tag{6.162}$$

as the transformations between the A_i and $\tilde{A}_\mu$ (cf. Kamchatnov 1982).

Using (6.155) and (6.161) we obtain the magnetic vector potential $\mathbf{A}$ in R^3 of the form:

$$\mathbf{A} = \frac{4}{(1 + x^2)^2}\left[-(\omega_1 x_2 + \omega_2 x_1 x_3), \omega_1 x_1 - \omega_2 x_2 x_3, -\frac{1}{2}\omega_2\left(1 + 2x_3^2 - x^2\right)\right]. \tag{6.163}$$

Taking the curl of (6.163), gives the magnetic field induction $\mathbf{B} = \nabla \times \mathbf{A}$ as:

$$\mathbf{B} = \frac{16}{(1 + x^2)^3}\left[\omega_1 x_1 x_3 + \omega_2 x_2, \omega_1 x_2 x_3 - \omega_2 x_1, \frac{1}{2}\omega_1\left(1 + 2x_3^2 - x^2\right)\right]. \tag{6.164}$$

The magnetic vector potential solution (6.163) for $\mathbf{A}$ and the solution (6.164) for $\mathbf{B}$ at first sight, appear to be different than the solutions given by Semenov et al. (2002). However, if we use the notation $\mathbf{A}^W(\omega_1, \omega_2)$ and $\mathbf{B}^W(\omega_1, \omega_2)$ to denote the solutions (6.163) and (6.164) for $\mathbf{A}$ and $\mathbf{B}$ (here the superscript W denotes Webb), we obtain:

$$\mathbf{A}^S(\omega_1, \omega_2) = \frac{1}{4}\mathbf{A}^W(-\omega_2, -\omega_1), \quad \mathbf{B}^S(\omega_1, \omega_2) = \frac{1}{4}\mathbf{B}^W(-\omega_2, -\omega_1), \tag{6.165}$$

where the superscript S refers to the forms for $\mathbf{A}$ and $\mathbf{B}$ given in equations (26) and (16) of Semenov et al. (2002). Note that the magnetic helicity density h_M in Semenov et al. (2002) depends only on the link number combination $\omega_1\omega_2$. Thus, the helicity integral H_M^W is related to H_M^S by the equation:

$$H_M^W = 16H_M^S = -4\pi^2\omega_1\omega_2, \tag{6.166}$$

where $H_M^S \equiv K$ and K is given by (31) in Semenov et al. (2002).

Figure 6.17 from Semenov et al. (2002) shows magnetic flux tubes for the case $\omega_1 = \omega_2 = 1$ in (6.165). The magnetic field lines are linked with a link number

Fig. 6.17 A magnetic flux
tube for the topological
soliton (6.165) in the form of
a Mobius strip, for the case
$\omega_1 = \omega_2 = 1$ (Figure 1 of
Semenov et al. 2002)
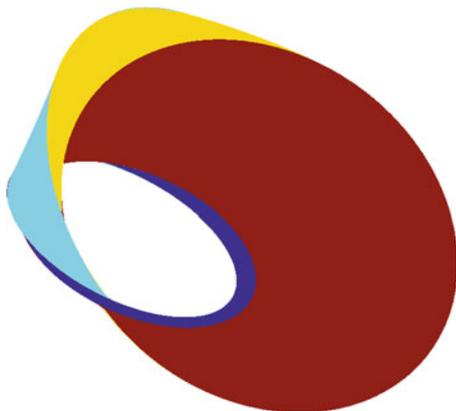
Fig. 6.18 Two linked
magnetic flux tubes for the
topological soliton (6.165) for
the case $\omega_1 = 2$ and $\omega_2 = 1$
(Figure 8 of Semenov et al.
2002)
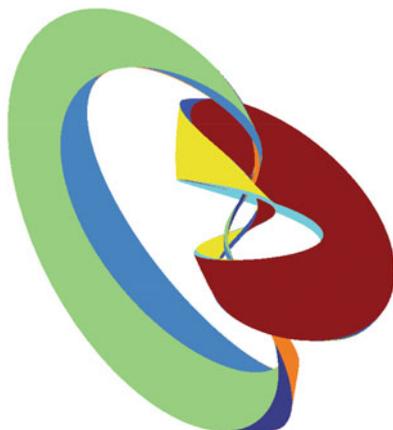
of $Lk(\omega_1, \omega_2) = \omega_1 \omega_2$ as expected for a single flux tube in which the edges of the
flux ribbon are linked field lines, with link number $Lk = 1$. Figure 6.18 shows two
linked magnetic flux tubes for the MHD topological soliton described by (6.165) for
the case $\omega_1 = 2$ and $\omega_2 = 1$ (from Semenov et al. 2002). Other more complicated
examples of magnetic flux surfaces and flux tubes for other values of ω_1 and ω_2 are
given by Semenov et al. (2002) and by Thompson et al. (2014).

Case $\omega_2 = -\omega_1$

An inspection of (6.163) and (6.164) reveals that if $\omega_2 = -\omega_1$, then:

$$
\begin{aligned}
\mathbf{A} &= \frac{4\omega_1}{(1 + x^2)^2} \left[x_1 x_3 - x_2, x_1 + x_2 x_3, \frac{1}{2}(1 + 2x_3^2 - x^2) \right], \\
\mathbf{B} &= \frac{16\omega_1}{(1 + x^2)^3} \left[x_1 x_3 - x_2, x_1 + x_2 x_3, \frac{1}{2}(1 + 2x_3^2 - x^2) \right].
\end{aligned}
\tag{6.167}
$$

The solution (6.167) for **A** and **B** has the property:

$$\mathbf{B} = \nabla \times \mathbf{A} = \lambda \mathbf{A} \quad \text{where} \quad \lambda = \frac{4}{1 + x^2}. \tag{6.168}$$

A straightforward calculation gives

$$|\mathbf{A}| = \frac{N}{2(1 + x^2)} \quad \text{where} \quad N = 4|\omega_1| \tag{6.169}$$

Equations (6.168)–(6.169) imply (6.167) is a special solution of the equation:

$$\nabla \times \mathbf{A} = k|\mathbf{A}|\mathbf{A} \quad \text{where} \quad k = \frac{8 \, \text{sgn}(\omega_1)}{N} = \frac{2}{|\omega_1|}. \tag{6.170}$$

One can clearly rescale the coordinates in (6.170) so that $k = 1$. The Lie point symmetry group of (6.170) was determined by Bila (1999). Bila (1999), refers to Blair's solution for **A**, which is essentially the same as the Kamchatnov (1982) solution for **A**, given in (6.167). Blair was interested in the Riemannian geometry of contact metric manifolds.

Chapter 7
Euler-Poincaré Equation Approach

Poincaré (1901) wrote down the Euler equations for a rigid body on so(3) in a matrix commutator form (see also Holm 2008b, Volume 2, p. 46). Arnold (1966) showed that the equations for ideal, incompressible fluid dynamics could be derived from a variational principle in which the Lagrangian consists of the fluid kinetic energy, subject to an infinite Lie group (pseudo-Lie group) constraint, associated with the Lagrangian map (the constraint is that the Lagrangian map $\mathbf{x} = \varphi(\mathbf{x}_0, t)$ for fixed t, is a differentiable (smooth) and measure preserving diffeomorphism). The group G, is known as $Sdiff(\mathbf{R}^3)$. The variational formulation showed that when the Lagrangian l is a metric on the tangent space TG, the resultant variational equations (the Euler-Poincaré equations) are geodesic spray equations for geodesic motion on the group G with respect to the metric l. For the case of rigid body dynamics the group involved is the semi-direct product Lie group $SE(3) = SO(3)\circledS\mathbf{R}^3$. Euler-Poincaré variational principles have been developed by a number of authors (e.g. Marsden et al. 1984; Holm and Kupershmidt 1983a,b; Holm et al. 1998; Cendra et al. 2003; Arnold and Khesin 1998). The geodesic spray equations for MHD were obtained by Ono (1995a,b). These equations are sometimes referred to as the Euler-Arnold equations. Araki (2015, 2017) determine the geodesic spray equations for incompressible Hall plasmas, known as XMHD (i.e. extended MHD). The curvature associated with the geodesic metric is negative for unstable flows. Holm et al. (1985) describes the use of Casimirs in stability analyses. Squire et al. (2013) derive the Hamiltonian structure and Euler-Poincaré formulations of the Vlasov-Maxwell and gyro-kinetic systems.

Our analysis in this chapter is based in part, on the analysis of Holm et al. (1998) and Cotter and Holm (2012). In action principles in MHD and gas dynamics, it is useful to use both Lagrangian and Eulerian variations. The Euler-Poincaré approach uses Eulerian variations in which $\mathbf{x}$ is held constant.

The solution of $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$ with $\mathbf{x} = \mathbf{x}_0$ at $t = 0$ is written as $\mathbf{x} = g\mathbf{x}_0 = \mathbf{X}(\mathbf{x}_0, t)$. The inverse map $\mathbf{x}_0 = g^{-1}\mathbf{x}$ defines $\mathbf{x}_0 = \mathbf{x}_0(\mathbf{x}, t)$. The Lagrange label $\mathbf{x}_0$

is advected with the flow:

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \mathbf{x}_0 = \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) \mathbf{x}_0 = 0, \tag{7.1}$$

Write $\mathbf{x} = g\mathbf{x}_0$, $\mathbf{x}_0 = g^{-1}\mathbf{x}$. Notice that $\dot{\mathbf{x}}_0 = (g^{-1})\dot{\mathbf{x}} = -g^{-1}\dot{g}g^{-1}\mathbf{x}_0$ (use $g^{-1}g = e$ where e is the identity). Here $\dot{\mathbf{x}}_0 = \partial\mathbf{x}_0/\partial t$ where $\mathbf{x}$ is held constant. Thus,

$$\dot{\mathbf{x}}_0 = -g^{-1}\dot{g}g^{-1}g\mathbf{x}_0 = -g^{-1}\dot{g}\mathbf{x}_0 = -\mathcal{L}_{\mathbf{u}}\mathbf{x}_0. \tag{7.2}$$

We identify

$$\boldsymbol{\xi} = \mathcal{L}_{\mathbf{u}} = \mathbf{u} \cdot \nabla \equiv g^{-1}\dot{g}, \tag{7.3}$$

with the fluid velocity $\mathbf{u}$. Note $\boldsymbol{\xi} = g^{-1}\dot{g}$ is left invariant vector field. Similarly, for a geometrical object Lie dragged with the flow:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}}\right) a = 0. \tag{7.4}$$

Let $a_0 = ga$ then $a = g^{-1}a_0$ and

$$\delta a = \delta\left(g^{-1}\right) a_0 = -g^{-1}\delta g g^{-1}a_0 = -g^{-1}\delta g\, a = -\mathcal{L}_{\boldsymbol{\eta}}(a). \tag{7.5}$$

We write

$$\boldsymbol{\eta} = g^{-1}\delta g. \tag{7.6}$$

as the vector field associated with the variations. Note $\boldsymbol{\eta}$ is a left invariant vector field (i.e. $(hg)^{-1}\delta(hg) = g^{-1}\delta g$, assuming that $\delta h = 0$).

To compute $\delta\boldsymbol{\xi}$ where $\boldsymbol{\xi} = g^{-1}\dot{g}$ we note:

$$\delta\boldsymbol{\xi} = \delta g^{-1}\dot{g} + g^{-1}\delta\dot{g} = -(g^{-1}\delta g g^{-1})\dot{g} + g^{-1}\delta\dot{g}, \tag{7.7}$$

which gives:

$$\delta\boldsymbol{\xi} = -\boldsymbol{\eta}\boldsymbol{\xi} + g^{-1}\delta\dot{g}, \tag{7.8}$$

Similarly, for $\boldsymbol{\eta} = g^{-1}\delta g$ we find

$$\dot{\boldsymbol{\eta}} = (g^{-1})\dot{}\delta g + g^{-1}\delta\dot{g} = -g^{-1}\dot{g}g^{-1}\delta g + g^{-1}\delta\dot{g}, \tag{7.9}$$

which gives:

$$\dot{\boldsymbol{\eta}} = -\boldsymbol{\xi}\boldsymbol{\eta} + g^{-1}\delta\dot{g}. \tag{7.10}$$

Subtract (7.10) from (7.8) gives:

$$\delta\boldsymbol{\xi} = \dot{\boldsymbol{\eta}} + \boldsymbol{\xi}\boldsymbol{\eta} - \boldsymbol{\eta}\boldsymbol{\xi} \equiv \dot{\boldsymbol{\eta}} + [\boldsymbol{\xi}, \boldsymbol{\eta}]_L. \tag{7.11}$$

where $[\boldsymbol{\xi}, \boldsymbol{\eta}]_L = ad_{\boldsymbol{\xi}}(\boldsymbol{\eta})_L$ is the left Lie bracket. The right Lie bracket $[\boldsymbol{\xi}, \boldsymbol{\eta}]_R = -[\boldsymbol{\xi}, \boldsymbol{\eta}]_L$.

7.1 The Euler-Poincaré Equation

Consider the Variational Principle (Holm et al. 1998; Cotter and Holm 2012) in which the action:

$$J = \int \ell(\mathbf{u}, a) \, d^3x \, dt, \tag{7.12}$$

is stationary, i.e.

$$\delta J = \int \left(\frac{\delta\ell}{\delta\mathbf{u}} \cdot \delta\mathbf{u} + \frac{\delta\ell}{\delta a}\delta a \right) d^3x \, dt \equiv \int \left\langle \frac{\delta\ell}{\delta\mathbf{u}}, \delta\mathbf{u} \right\rangle + \left\langle \frac{\delta\ell}{\delta a}, \delta a \right\rangle dt = 0. \tag{7.13}$$

However from (7.11) with $\boldsymbol{\xi} = \mathbf{u}$, and (7.5),

$$\delta\mathbf{u} = \dot{\boldsymbol{\eta}} + [\mathbf{u}, \boldsymbol{\eta}], \quad \delta a = -\mathcal{L}_{\boldsymbol{\eta}}(a). \tag{7.14}$$

Thus

$$\delta J = \int \left\langle \frac{\delta\ell}{\delta\mathbf{u}}, \dot{\boldsymbol{\eta}} + [\mathbf{u}, \boldsymbol{\eta}] \right\rangle + \left\langle \frac{\delta\ell}{\delta a}, -\mathcal{L}_{\boldsymbol{\eta}}(a) \right\rangle dt. \tag{7.15}$$

Integrate (7.15) by parts, and use $ad_{\mathbf{u}}(\boldsymbol{\eta}) = [\mathbf{u}, \boldsymbol{\eta}]$ to obtain:

$$\delta J = \int \left\{ \left(\frac{d}{dt} \left\langle \frac{\delta\ell}{\delta\mathbf{u}}, \boldsymbol{\eta} \right\rangle - \left\langle \boldsymbol{\eta}, \frac{d}{dt} \left(\frac{\delta\ell}{\delta\mathbf{u}} \right) \right\rangle \right) + \left\langle \frac{\delta\ell}{\delta\mathbf{u}}, ad_{\mathbf{u}}(\boldsymbol{\eta}) \right\rangle \right.$$
$$\left. - \left\langle \frac{\delta\ell}{\delta a}, \mathcal{L}_{\boldsymbol{\eta}}(a) \right\rangle \right\} dt. \tag{7.16}$$

for δJ.

In the further analysis of (7.16) it is useful to introduce the diamond operator. The diamond operator $\diamond$ in the present application allows one to take the adjoint of the $\langle \delta\ell/\delta\mathbf{u}, ad_{\mathbf{u}}(\boldsymbol{\eta}) \rangle$ term in (7.16) and thereby isolate its $\boldsymbol{\eta}$ component, by using the formula

$$\left\langle \frac{\delta\ell}{\delta a} \diamond a, \eta \right\rangle = -\left\langle \frac{\delta\ell}{\delta a}, \mathcal{L}_{\eta}(a) \right\rangle, \tag{7.17}$$

A more formal definition of the diamond operator is given below.

Definition 7.1.1 The diamond operator $\diamond$ is defined as minus the dual of the Lie derivative, with respect to the pairing induced by the variational derivative $\mathbf{p} = \delta\ell/\delta\mathbf{q}$, namely:

$$\langle \mathbf{p} \diamond \mathbf{q}, \boldsymbol{\xi} \rangle = \langle \mathbf{p}, -\mathcal{L}_{\boldsymbol{\xi}}(\mathbf{q}) \rangle. \tag{7.18}$$

Using (7.17) and the definition of $ad_{\mathbf{u}}^*$:

$$\left\langle ad_{\mathbf{u}}^* \left(\frac{\delta\ell}{\delta\mathbf{u}} \right), \eta \right\rangle = \left\langle \frac{\delta\ell}{\delta\mathbf{u}}, ad_{\mathbf{u}}(\eta) \right\rangle, \tag{7.19}$$

in (7.16) where $\diamond$ is the diamond operator (this involves integration by parts, and dropping surface terms). We obtain:

$$\delta J = \int \left\langle \eta, -\frac{d}{dt}\left(\frac{\delta\ell}{\delta\mathbf{u}}\right) + ad_{\mathbf{u}}^*\left(\frac{\delta\ell}{\delta\mathbf{u}}\right) + \frac{\delta\ell}{\delta a} \diamond a \right\rangle dt + \left[\left\langle \frac{\delta\ell}{\delta\mathbf{u}}, \eta \right\rangle\right]_{t_0}^{t_1}. \tag{7.20}$$

Assuming the surface term vanishes in (7.20), and η is arbitrary, then $\delta J = 0$ implies the Euler-Poincaré equation:

$$\frac{d}{dt}\left(\frac{\delta\ell}{\delta\mathbf{u}}\right) + ad_{\mathbf{u}}^*\left(\frac{\delta\ell}{\delta\mathbf{u}}\right)_R = \frac{\delta\ell}{\delta a} \diamond a, \tag{7.21}$$

where

$$ad_{\mathbf{u}}^*\left(\frac{\delta\ell}{\delta\mathbf{u}}\right)_R = -ad_{\mathbf{u}}^*\left(\frac{\delta\ell}{\delta\mathbf{u}}\right)_L. \tag{7.22}$$

Here, (7.21) is the Euler-Poincaré equation associated with the variational principle $\delta J = 0$ (Holm et al. 1998). In (7.21), $d/dt \equiv \partial/\partial t$ keeping $\mathbf{x}$ constant. Below, we show that:

$$ad_{\mathbf{u}}^*\left(\frac{\delta\ell}{\delta\mathbf{u}}\right)_R = \nabla \cdot \left(\mathbf{u} \otimes \frac{\delta\ell}{\delta\mathbf{u}}\right) + (\nabla\mathbf{u})^T \cdot \left(\frac{\delta\ell}{\delta\mathbf{u}}\right). \tag{7.23}$$

The proof of (7.23) is given below.

Proof Let $\mathbf{m} = \delta\ell/\delta\mathbf{u}$. We obtain:

$$\langle \eta, ad_{\mathbf{u}}^*(\mathbf{m})_R \rangle = \langle ad_{\mathbf{u}}(\eta), \mathbf{m} \rangle = \langle -[\mathbf{u}, \eta]_L, \mathbf{m} \rangle$$

$$= -\int [(\mathbf{u} \cdot \nabla\eta - \eta \cdot \nabla\mathbf{u})\nabla]\lrcorner\mathbf{m} \cdot dx\, d^3x$$

$$= \int -\nabla \cdot (\mathbf{u}(\mathbf{m} \cdot \boldsymbol{\eta})) + \boldsymbol{\eta} \cdot \left(\nabla \cdot (\mathbf{u} \otimes \mathbf{m}) + (\nabla \mathbf{u})^T \cdot \mathbf{m} \right) \, d^3x,$$

$$= \langle \boldsymbol{\eta}, \nabla \cdot (\mathbf{u} \otimes \mathbf{m}) + (\nabla \mathbf{u})^T \cdot \mathbf{m} \rangle, \tag{7.24}$$

where we dropped the surface term. This proves (7.23). □

It can be shown that:

$$\mathcal{L}_{\mathbf{u}}(\mathbf{m} \cdot d\mathbf{x} \otimes dV) = \left(\nabla \cdot (\mathbf{u} \otimes \mathbf{m}) + (\nabla \mathbf{u})^T \cdot \mathbf{m} \right) \cdot d\mathbf{x} \otimes dV. \tag{7.25}$$

For MHD the Lagrange density ℓ is given by:

$$\ell = \frac{1}{2} \rho u^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu_0}. \tag{7.26}$$

We now determine the different terms in the Euler-Poincaré equation (7.21). From (7.13), the variation of the action $\delta J = \delta J_{\mathbf{u}} + \delta J_a$ where:

$$\delta J_{\mathbf{u}} = \int \frac{\delta \ell}{\delta \mathbf{u}} \cdot \delta \mathbf{u} \, d^3x \, dt,$$

$$\delta J_a = \int \left(\frac{\delta \ell}{\delta \rho} \delta \rho + \frac{\delta \ell}{\delta S} \delta S + \frac{\delta \ell}{\delta \mathbf{B}} \cdot \delta \mathbf{B} \right) d^3x \, dt. \tag{7.27}$$

From (7.26) we obtain:

$$\frac{\delta \ell}{\delta \rho} = \frac{1}{2} u^2 - \varepsilon_\rho = \frac{1}{2} u^2 - h, \quad \frac{\delta \ell}{\delta \mathbf{u}} \equiv \mathbf{m} = \rho \mathbf{u}$$

$$\frac{\delta \ell}{\delta S} = -\varepsilon_S = -\rho T, \quad \frac{\delta \ell}{\delta \mathbf{B}} = -\frac{\mathbf{B}}{\mu_0}, \tag{7.28}$$

where T is the temperature and h is the enthalpy of the gas.

Using the formulae:

$$\delta \left(\rho d^3x \right) = -\mathcal{L}_{\mathbf{u}} \left(\rho d^3x \right) = -\nabla \cdot (\rho \mathbf{u}) \, d^3x,$$

$$\delta S = -\mathcal{L}_{\mathbf{u}}(S) = -\mathbf{u} \cdot \nabla S,$$

$$\delta(\mathbf{B} \cdot d\mathbf{S}) = -\mathcal{L}_{\mathbf{u}}(\mathbf{B} \cdot d\mathbf{S})$$

$$= [\nabla \times (\mathbf{u} \times \mathbf{B}) - \mathbf{u}(\nabla \cdot \mathbf{B})] \cdot d\mathbf{S}, \tag{7.29}$$

we obtain:

$$\delta \rho = -\nabla \cdot (\rho \mathbf{u}), \quad \delta S = -\mathbf{u} \cdot \nabla S,$$

$$\delta \mathbf{B} = [\nabla \times (\mathbf{u} \times \mathbf{B}) - \mathbf{u}(\nabla \cdot \mathbf{B})]. \tag{7.30}$$

Note that $\delta \rho$, δS and $\delta \mathbf{B}$ are Eulerian variations in which $\Delta x^i = -x_{ij} \delta x_0^j$ is replaced by u^i, where $\Delta \mathbf{x}$ is the Lagrangian variation of $\mathbf{x}$, and $x_{ij} = \partial x^i / \partial x_0^j$ (e.g. Webb et al.

2005a,b; Newcomb 1962). Using $\delta\ell/\delta\mathbf{u} = \rho\mathbf{u} = \mathbf{m}$ in (7.23) gives:

$$ad_{\mathbf{u}}^* \left(\frac{\delta\ell}{\delta\mathbf{u}} \right)_R = \nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u}) + \rho\nabla \left(\frac{1}{2}|\mathbf{u}|^2 \right). \tag{7.31}$$

for the advected term on the left hand side of the Euler-Poincaré equation (7.21).

Next we find the $(\delta\ell/\delta a) \diamond a$ term on right hand side of (7.21). We obtain:

$$\begin{aligned}
\frac{\delta\ell}{\delta a}\delta a &= \frac{\delta\ell}{\delta\rho}\delta\rho + \frac{\delta\ell}{\delta S}\delta S + \frac{\delta\ell}{\delta\mathbf{B}}\delta\mathbf{B} \\
&= \frac{\delta\ell}{\delta\rho}(-\nabla \cdot (\rho\mathbf{u})) + \frac{\delta\ell}{\delta S}(-\mathbf{u} \cdot \nabla S) \\
&\quad + \frac{\delta\ell}{\delta\mathbf{B}} \cdot [\nabla \times (\mathbf{u} \times \mathbf{B}) - \mathbf{u}\nabla \cdot \mathbf{B}]
\end{aligned} \tag{7.32}$$

Thus

$$\begin{aligned}
\frac{\delta\ell}{\delta a}\delta a &= -\nabla \cdot \left(\rho\mathbf{u}\frac{\delta\ell}{\delta\rho} \right) + \nabla \cdot \left[(\mathbf{u} \times \mathbf{B}) \times \frac{\delta\ell}{\delta\mathbf{B}} \right] \\
&\quad + \mathbf{u} \cdot \left\{ \rho\nabla \left(\frac{\delta\ell}{\delta\rho} \right) - \frac{\delta\ell}{\delta S}\nabla S + \mathbf{B} \times \left(\nabla \times \left(\frac{\delta\ell}{\delta\mathbf{B}} \right) \right) - \frac{\delta\ell}{\delta\mathbf{B}}\nabla \cdot \mathbf{B} \right\}
\end{aligned} \tag{7.33}$$

From (7.33) we find:

$$\frac{\delta\ell}{\delta a} \diamond a = \rho\nabla \left(\frac{\delta\ell}{\delta\rho} \right) - \frac{\delta\ell}{\delta S}\nabla S + \mathbf{B} \times \left(\nabla \times \left(\frac{\delta\ell}{\delta\mathbf{B}} \right) \right) - \frac{\delta\ell}{\delta\mathbf{B}}\nabla \cdot \mathbf{B}. \tag{7.34}$$

Integrate (7.33) over d^3x over the volume, V, drop surface terms, and set $\boldsymbol{\eta} \to \mathbf{u}$ in (7.20) gives the result (7.34) for $\delta\ell/\delta a \diamond a$.

Using the first law of thermodynamics in the form: $T\nabla S - \nabla h = -\nabla p/\rho$ and the expressions (7.28) for $\delta\ell/\delta\rho$, $\delta\ell/\delta S$, $\delta\ell/\delta\mathbf{B}$ in (7.34) gives:

$$\frac{\delta\ell}{\delta a} \diamond a = \left(-\nabla p + \mathbf{J} \times \mathbf{B} + \frac{\mathbf{B}}{\mu_0}\nabla \cdot \mathbf{B} \right) + \rho\nabla \left(\frac{1}{2}|\mathbf{u}|^2 \right). \tag{7.35}$$

Using $ad_{\mathbf{u}}^*(\delta\ell/\delta\mathbf{u})_R$ (7.31) and $\delta\ell/\delta a \diamond a$ (7.35) in the Euler-Poincaré equation (7.21) gives the MHD momentum equation in the form:

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla \cdot (\rho\mathbf{u} \otimes \mathbf{u}) = -\nabla p + \mathbf{J} \times \mathbf{B} + \frac{\mathbf{B}}{\mu_0}\nabla \cdot \mathbf{B}. \tag{7.36}$$

the momentum equation (7.36) can also be written in the conservative form:

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla \cdot \left(\rho\mathbf{u} \otimes \mathbf{u} + \left(p + \frac{B^2}{2\mu_0} \right)\mathsf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right) = 0, \tag{7.37}$$

where the magnetic terms involve the Maxwell stress energy tensor. The above derivation of the Euler-Poincaré equation is essentially that of Holm et al. (1998). It is also discussed by Cotter and Holm (2012) in their analysis of symmetries and conservation laws associated with advection of physical quantities i.e., the Tur and Yanovsky (1993) conservation laws.

7.2 Noether's Second Theorem

Consider the application of the above ideas to obtain a version of Noether's second theorem associated with the symmetries $\boldsymbol{\eta}$. In the derivation of Noether's theorem, it is useful to keep track of all the surface or divergence terms that arise when integrating by parts. These terms are assumed to vanish in the derivation of the Euler-Poincaré equation (7.35) or (7.36). The variation of the action δJ is again given by (7.13), which reduces to the result (7.15), i.e.

$$\delta J = \int \left(\left\langle \frac{\delta \ell}{\delta \mathbf{u}}, \dot{\boldsymbol{\eta}} + [\mathbf{u}, \boldsymbol{\eta}] \right\rangle + \left\langle \frac{\delta \ell}{\delta a}, -\mathcal{L}_{\boldsymbol{\eta}}(a) \right\rangle \right) dt \equiv \delta J_u + \delta J_a, \tag{7.38}$$

where δJ_u and δJ_a are given by (7.27). Using integration by parts, the first term δJ_u in (7.38) reduces to:

$$\delta J_u = -\int \left\langle \boldsymbol{\eta}, \frac{d}{dt}\left(\frac{\delta \ell}{\delta \mathbf{u}}\right) + ad_{\mathbf{u}}^*\left(\frac{\delta \ell}{\delta \mathbf{u}}\right)_R \right\rangle dt$$
$$+ \int \frac{\partial}{\partial t}\left(\boldsymbol{\eta} \cdot \frac{\delta \ell}{\delta \mathbf{u}}\right) + \nabla \cdot \left[\left(\boldsymbol{\eta} \cdot \frac{\delta \ell}{\delta \mathbf{u}}\right)\mathbf{u}\right] d^3x dt. \tag{7.39}$$

The variations of the a variables is given by (7.5), i.e. $\delta a = -\mathcal{L}_{\boldsymbol{\eta}}(a)$. Thus, we compute the variations $\delta(\rho d^3 x)$, δS and $\delta(\mathbf{B} \cdot d\mathbf{S})$ as in (7.29) but with $\mathbf{u}$ replaced by $\boldsymbol{\eta}$. The net result from (7.30) is:

$$\delta \rho = -\nabla \cdot (\rho \boldsymbol{\eta}), \quad \delta S = -\boldsymbol{\eta} \cdot \nabla S,$$
$$\delta \mathbf{B} = [\nabla \times (\boldsymbol{\eta} \times \mathbf{B}) - \boldsymbol{\eta}(\nabla \cdot \mathbf{B})]. \tag{7.40}$$

Using the results (7.28) and (7.40) we obtain Eq. (7.32) but with $\mathbf{u}$ replaced by $\boldsymbol{\eta}$. The net upshot is the result (7.33) but with $\mathbf{u}$ replaced by $\boldsymbol{\eta}$, i.e.,

$$\frac{\delta \ell}{\delta a}\delta a = -\nabla \cdot \left(\rho \boldsymbol{\eta} \frac{\delta \ell}{\delta \rho}\right) + \nabla \cdot \left[(\boldsymbol{\eta} \times \mathbf{B}) \times \frac{\delta \ell}{\delta \mathbf{B}}\right]$$
$$+ \boldsymbol{\eta} \cdot \left\{\rho \nabla \left(\frac{\delta \ell}{\delta \rho}\right) - \frac{\delta \ell}{\delta S}\nabla S + \mathbf{B} \times \left(\nabla \times \left(\frac{\delta \ell}{\delta \mathbf{B}}\right)\right) - \frac{\delta \ell}{\delta \mathbf{B}}\nabla \cdot \mathbf{B}\right\}$$
$$\tag{7.41}$$

Using (7.41) we obtain:

$$
\delta J_a = \int \frac{\delta \ell}{\delta a} \delta a \, d^3x \, dt
$$

$$
\int \left\langle \boldsymbol{\eta}, \frac{\delta \ell}{\delta a} \diamond a \right\rangle dt + \int \nabla \cdot \left(-\rho \boldsymbol{\eta} \frac{\delta \ell}{\delta \rho} + (\boldsymbol{\eta} \times \mathbf{B}) \times \frac{\delta \ell}{\delta \mathbf{B}} \right) d^3x \, dt,
$$

$$(7.42)$$

where $\delta \ell / \delta a \diamond a$ is given by (7.34), or the coefficient of $\boldsymbol{\eta}$ in (7.41). Adding (7.39) and (7.42) for δJ_u and δJ_a we obtain:

$$
\delta J = \delta J_u + \delta J_a = -\int \left\langle \boldsymbol{\eta}, \frac{d}{dt}\left(\frac{\delta \ell}{\delta \mathbf{u}}\right) + ad_{\mathbf{u}}^*\left(\frac{\delta \ell}{\delta \mathbf{u}}\right)_R - \frac{\delta \ell}{\delta a} \diamond a \right\rangle dt
$$

$$
+ \int \int \left[\frac{\partial}{\partial t}\left(\boldsymbol{\eta} \cdot \frac{\delta \ell}{\delta \mathbf{u}}\right) + \nabla \cdot \left(\boldsymbol{\eta} \cdot \frac{\delta \ell}{\delta \mathbf{u}} \mathbf{u} - \rho \boldsymbol{\eta} \frac{\delta \ell}{\delta \rho} + (\boldsymbol{\eta} \times \mathbf{B}) \times \frac{\delta \ell}{\delta \mathbf{B}} \right) \right] d^3x \, dt.
$$

$$(7.43)$$

We require $\delta J = 0$ in (7.43) in order for $\boldsymbol{\eta}$ to be a variational symmetry of the action. Because there are an infinite number of fluid relabeling symmetries $\boldsymbol{\eta}$ one cannot automatically assume that the Euler Lagrange equations (7.21) are satisfied. We can write (7.43) in the form:

$$
\delta J = \int \left\langle \boldsymbol{\eta}, E_{\{\mathbf{u},a\}}(\ell) \right\rangle dt + \int \int \left(\frac{\partial D}{\partial t} + \nabla \cdot \mathbf{F} \right) d^3x \, dt, \qquad (7.44)
$$

where

$$
E_{\{\mathbf{u},a\}}(\ell) = -\left\{ \frac{d}{dt}\left(\frac{\delta \ell}{\delta \mathbf{u}}\right) + ad_{\mathbf{u}}^*\left(\frac{\delta \ell}{\delta \mathbf{u}}\right)_R - \frac{\delta \ell}{\delta a} \diamond a \right\}, \qquad (7.45)
$$

is the Euler operator and

$$
D = \boldsymbol{\eta} \cdot \frac{\delta \ell}{\delta \mathbf{u}},
$$

$$
\mathbf{F} = \boldsymbol{\eta} \cdot \frac{\delta \ell}{\delta \mathbf{u}} \mathbf{u} - \rho \boldsymbol{\eta} \frac{\delta \ell}{\delta \rho} + (\boldsymbol{\eta} \times \mathbf{B}) \times \frac{\delta \ell}{\delta \mathbf{B}}, \qquad (7.46)
$$

are the density D and flux $\mathbf{F}$ surface terms. Further analysis of (7.44) involving integration by parts is necessary before one can arrive at a conservation law for particular Lie symmetries (which involve arbitrary function(s)). In particular, Padhye and Morrison (1996a,b) and Padhye (1998) describe how this procedure results in Ertel's theorem, which is associated with a fluid relabeling symmetry.

The variational equation (7.44) can be written in the form:

$$\delta J = \int \langle \boldsymbol{\eta}, \mathbf{E}\,(\ell) \rangle \; dt + C(t) + \int \int \nabla \cdot \mathbf{F} \, d^3x \, dt, \qquad (7.47)$$

where

$$C(t) = \int \int \frac{\partial D}{\partial t} \, d^3x \, dt \equiv \left[\left\langle \boldsymbol{\eta}, \frac{\delta\ell}{\delta\mathbf{u}} \right\rangle \right]_{t_0}^{t},$$

$$\langle \boldsymbol{\eta}, \frac{\delta\ell}{\delta\mathbf{u}} \rangle = \int_V d^3x \left(\boldsymbol{\eta} \cdot \frac{\delta\ell}{\delta\mathbf{u}} \right) \qquad (7.48)$$

and D and $\mathbf{F}$ are given by (7.46).

For the case of MHD, use of the formulae (7.28) for $\delta\ell/\delta\rho$, $\delta\ell/\delta\mathbf{u}$, $\delta\ell/\delta S$ and $\delta\ell/\delta\mathbf{B}$ gives:

$$D = \hat{V}^{\mathbf{x}} \cdot \rho\mathbf{u},$$

$$\mathbf{F} = \hat{V}^{\mathbf{x}} \cdot \left(\rho\mathbf{u} \otimes \mathbf{u} + \left(\varepsilon + p + \frac{B^2}{\mu_0} - \frac{1}{2}\rho|\mathbf{u}|^2 \right) \mathsf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right), \qquad (7.49)$$

where use the notation:

$$\hat{V}^{\mathbf{x}} = \boldsymbol{\eta}. \qquad (7.50)$$

Here $\hat{V}^{\mathbf{x}}$ is the canonical symmetry generator associated with fluid relabeling symmetries, in which $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ is the Lagrangian map, in which the x^i are the dependent variables and Lagrange labels $\mathbf{x}_0$ are the independent variables (e.g. Webb et al. 2005a,b; Webb and Zank 2007). From Ibragimov (1985) and Webb et al. (2005a,b)

$$\hat{V}^{x^i} = V^{x^i} - V^{x_0^s} D_{x_0^s} x^i \equiv -V^{x_0^s} x_{is}, \qquad (7.51)$$

gives the formula for the canonical symmetry generator $\hat{V}^{\mathbf{x}}$ in terms of the Lagrange label symmetry generator $V^{x_0^s}$ where $x_{is} = \partial x^i / \partial x_0^s$.

An alternative approach to Noether's second theorem is based on the Lagrangian variational approach of Padhye and Morrison (1996a,b) and Padhye (1998). Webb and Mace (2015) use the formulation of Noether's second theorem of Hydon and Mansfield (2012), to determine a generalized potential vorticity conservation law for MHD.

7.2.1 Fluid Relabeling Determining Equations

For fluid relabeling symmetries, Eulerian physical variables do not change (e.g. Webb and Zank 2007). Advected quantities a satisfy:

$$\delta a = -\mathcal{L}_\eta(a) = 0, \tag{7.52}$$

where η is the vector field generator of the relabeling symmetry.

The Eulerian fluid velocity $\mathbf{u}$ does not change under fluid relabeling symmetry. Thus,

$$\delta\mathbf{u} = \dot{\boldsymbol{\eta}} + [\mathbf{u}, \boldsymbol{\eta}] = 0. \tag{7.53}$$

Equation (7.53) is condition for the vector field η to be Lie dragged by the fluid, i.e. $d\boldsymbol{\eta}/dt = 0$ moving with the flow.

The conditions (7.52) are equivalent in the case of MHD of setting $\delta\rho$, δS and $\delta\mathbf{B}$ equal to zero. Using the notation $\hat{V}^{\mathbf{x}} \equiv \eta$, (7.40) reduce to:

$$\nabla \cdot (\rho\hat{V}^{\mathbf{x}}) = 0, \quad \hat{V}^{\mathbf{x}} \cdot \nabla S = 0,$$
$$\nabla \times \left(\hat{V}^{\mathbf{x}} \times \mathbf{B}\right) = 0, \tag{7.54}$$

where we used Gauss's law $\nabla \cdot \mathbf{B} = 0$. Setting $\delta\mathbf{u} = 0$ in (7.53) gives the equation:

$$\frac{d\hat{V}^{\mathbf{x}}}{dt} - \hat{V}^{\mathbf{x}} \cdot \nabla\mathbf{u} = 0, \tag{7.55}$$

where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative moving with the flow. The condition (7.55) shows that the vector field $\hat{V}^{\mathbf{x}}$ is Lie dragged with the flow.

Comment

One can derive similar Lie determining equations for fluid relabeling symmetries by requiring that the Lagrangian form of the action is invariant under the Lie transformations: $\mathbf{x}' = \mathbf{x}$, $t' = t$, and $\mathbf{x}'_0 = \mathbf{x}_0 + \epsilon\hat{V}^{\mathbf{x}_0}$ to $O(\epsilon)$. The Lie determining equations in this case reduce to (Padhye 1998; Webb et al. 2005b):

$$\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) = 0, \quad V^{\mathbf{x}_0} \cdot \nabla_0 S = 0, \quad u^i x_{ij} D_t\left(\rho_0 V^{x_0^j}\right) = 0,$$
$$\frac{x_{ia}x_{ib}}{J}B_0^b\left[\nabla_0 \times (V^{\mathbf{x}_0} \times \mathbf{B}_0)\right]^a = 0, \quad \nabla_0 \cdot \mathbf{B}_0 = 0, \tag{7.56}$$

where $D_t = \partial_t + \mathbf{u} \cdot \nabla$ (i.e. ∂_t keeping $\mathbf{x}_0$ constant), $x_{ij} = \partial x^i / \partial x_0^j$, $J = \det(x_{ij})$. The relation between $\hat{V}^{x^i}$ and $V^{x_0^s}$ is given by (7.51).

Equations (7.56) are slightly more general than (7.54)–(7.55) and may be written in terms of $\hat{V}^{\mathbf{x}}$ and $\mathbf{B}$ as:

$$\nabla \cdot (\rho \hat{V}^{\mathbf{x}}) = 0, \quad \hat{V}^{\mathbf{x}} \cdot \nabla S = 0,$$

$$\mathbf{u} \cdot \left(\frac{d\hat{V}^{\mathbf{x}}}{dt} - \hat{V}^{\mathbf{x}} \cdot \nabla \mathbf{u} \right) = 0, \quad \mathbf{B} \cdot \left[\nabla \times \left(\hat{V}^{\mathbf{x}} \times \mathbf{B} \right) \right] = 0, \quad \nabla \cdot \mathbf{B} = 0.$$

$$(7.57)$$

Equation (7.57) has more solutions than the relabeling symmetry determining equations (7.54)–(7.55), but there is a class of solutions of (7.57) that satisfy the relabeling symmetry equations (7.54)–(7.55).

7.2.2   Noether's Second Theorem: Mass Conservation Symmetry

In this section we consider the conservation law associated with the mass conservation equation for the case of an ideal, isobaric fluid, with equation of state $p = p(\rho)$ (see also Cotter and Holm 2012). For Noether's second theorem the variation of J, δJ, is given by (7.47), i.e. we require:

$$\delta J = \int d^3x \int dt \left[\boldsymbol{\eta} \cdot \mathbf{E}(\ell) + \frac{\partial D}{\partial t} + \nabla \cdot \mathbf{F} \right] = 0, \tag{7.58}$$

where $\mathbf{E}(\ell)$ is the Euler operator given by (7.45). For the fluid relabeling symmetry associated with mass conservation, the variation δa of $a = \rho d^3x$ is set equal to zero, i.e.,

$$\delta a = -\mathcal{L}_{\boldsymbol{\eta}}(\rho d^3x) = 0, \tag{7.59}$$

Use Cartan's magic formula:

$$\mathcal{L}_{\boldsymbol{\eta}}(a) = d(\boldsymbol{\eta} \lrcorner a) + \boldsymbol{\eta} \lrcorner da, \tag{7.60}$$

$da = 0$ (as a is a three-form in 3D-space).
Also $\boldsymbol{\eta} \lrcorner a = \rho \boldsymbol{\eta} \cdot d\mathbf{S}$. Thus

$$\mathcal{L}_{\boldsymbol{\eta}}(\rho d^3x) = d[\rho \boldsymbol{\eta} \cdot d\mathbf{S}] = 0. \tag{7.61}$$

By the Poincaré Lemma, there exists a 1-form $\boldsymbol{\psi} \cdot d\mathbf{x}$ such that

$$\boldsymbol{\eta}_{\lrcorner} a = \rho \boldsymbol{\eta} \cdot d\mathbf{S} = d(\boldsymbol{\psi} \cdot d\mathbf{x}) \equiv \nabla \times \boldsymbol{\psi} \cdot d\mathbf{S}. \tag{7.62}$$

Since $\boldsymbol{\eta}_{\lrcorner} a$ is a conserved advected 2-form, then

$$\boldsymbol{\eta} = \frac{\nabla \times \boldsymbol{\psi}}{\rho} \quad \text{is a conserved (Lie dragged) vector field.} \tag{7.63}$$

A simpler derivation of (7.63) is to note that $\boldsymbol{\eta} \equiv \hat{V}^{\mathbf{x}}$ satisfies the first Lie determining equation in (7.54), i.e. $\nabla \cdot (\rho \boldsymbol{\eta}) = 0$.

The first term in (7.58) containing the Euler operator : $\mathbf{E}(\ell)$ is:

$$\begin{aligned}
T_1 &= \int d^3x \int dt \, \boldsymbol{\eta} \cdot \mathbf{E}(\ell) = \int d^3x \int dt \, \frac{\nabla \times \boldsymbol{\psi}}{\rho} \cdot \mathbf{E}(\ell) \\
&= \int d^3x \int dt \, \{ \nabla \cdot [\boldsymbol{\psi} \times \mathbf{E}(\ell)/\rho] + \boldsymbol{\psi} \cdot \nabla \times (\mathbf{E}(\ell)/\rho) \} \\
&= \int d^3x \int dt \, \boldsymbol{\psi} \cdot \nabla \times (\mathbf{E}(\ell)/\rho),
\end{aligned} \tag{7.64}$$

where the surface term due to $\nabla \cdot [\boldsymbol{\psi} \times \mathbf{E}(\ell)/\rho]$ is assumed to vanish on the boundary ∂V of the volume V of integration.

The remaining integrals in δJ in (7.58):

$$T_2 = \int d^3x \int dt \, \left(\frac{\partial D}{\partial t} + \nabla \cdot \mathbf{F} \right) = C(t) + \int d^3x \int dt \, \nabla \cdot \mathbf{F}, \tag{7.65}$$

can be reduced to the form:

$$T_2 = \int d^3x \int dt \, \left\{ \boldsymbol{\psi} \cdot \left[\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) \right] + \nabla \cdot \mathbf{W} \right\}, \tag{7.66}$$

where

$$\mathbf{W} = \nabla \times \left[\left(h + \frac{1}{2} |\mathbf{u}|^2 \right) \boldsymbol{\psi} - (\boldsymbol{\psi} \cdot \mathbf{u}) \mathbf{u} \right]. \tag{7.67}$$

and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity of the fluid. Note that $\nabla \cdot \mathbf{W} = 0$, because $\mathbf{W}$ may be written in the form of a 'curl': $\mathbf{W} = \nabla \times \mathbf{M}$. Put another way

$$\int_V \nabla \cdot \mathbf{W} \, d^3x = \int_{\partial V} \nabla \times \mathbf{M} \cdot d\mathbf{S} = \int_{\partial \partial V} \mathbf{M} \cdot d\mathbf{x}, \tag{7.68}$$

which is zero since $\partial \partial V$ does not exist (i.e. the boundary of a boundary is zero for a simply connected region: e.g. Misner et al. 1973). Combining (7.64) and (7.66) we

obtain:

$$\delta J = \int d^3x \int dt \left\{ \boldsymbol{\psi} \cdot \left[\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nabla \times \left(\frac{\mathbf{E}(\ell)}{\rho} \right) \right] + \nabla \cdot \mathbf{W} \right\} . \tag{7.69}$$

Thus, invoking the du-Bois Reymond lemma of the Calculus of variations and noting that $\nabla \cdot \mathbf{W} = 0$, (7.69) yields the generalized Bianchi identity:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \nabla \times \left(\frac{\mathbf{E}(\ell)}{\rho} \right) = 0. \tag{7.70}$$

Equation (7.70) is the basic result of Noether's second theorem, which shows that there are differential relations between the Euler-Lagrange variational derivatives $E_i(\ell)$ $(1 \leq i \leq 3)$ in this case. Note that (7.70) does not necessarily imply that the Euler Lagrange equations $E_i(\ell) = 0$ $(1 \leq i \leq 3)$ are satisfied. In the case where $E_i(\ell) = 0$ $(1 \leq i \leq 3)$, (7.70) implies the vorticity conservation law:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (\boldsymbol{\omega} \cdot d\mathbf{S}) = \left(\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) + \mathbf{u} \nabla \cdot \boldsymbol{\omega} \right) \cdot d\mathbf{S} = 0. \tag{7.71}$$

Note that $\nabla \cdot \boldsymbol{\omega} = 0$ as $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity. Equation (7.71) shows that the vorticity 2-form $\boldsymbol{\omega} \cdot d\mathbf{S}$ is advected with the flow.

The generalized Bianchi identity could also be derived using the method of Lagrange multipliers for Noether's second theorem developed by Hydon and Mansfield (2011). The proof of (7.65)–(7.66) is given below.

Proof We use the analysis of Cotter and Holm (2012) to calculate $C(t)$. Using (7.48) and (7.66) $C(t)$ is given by:

$$\begin{aligned} C(t) &= \left\langle \frac{\delta \ell}{\delta \mathbf{u}}, \boldsymbol{\eta} \right\rangle = \int_D \left(\frac{\delta \ell}{\delta \mathbf{u}} \cdot \boldsymbol{\eta} \right) d^3x \\ &= \int \left(\frac{1}{\rho} \frac{\delta \ell}{\delta u_j} \right) \rho \eta_j \, d^3x = \int \frac{1}{\rho} \frac{\delta \ell}{\delta u_j} (\nabla \times \boldsymbol{\psi})_j dS_j dx_j \\ &= \int \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge d(\boldsymbol{\psi} \cdot d\mathbf{x}). \end{aligned} \tag{7.72}$$

From (7.72)

$$\frac{dC}{dt} = \int \left\{ \frac{\partial}{\partial t} \left(\frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge d(\boldsymbol{\psi} \cdot d\mathbf{x}) + \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge \frac{\partial}{\partial t} [d(\boldsymbol{\psi} \cdot d\mathbf{x})] \right\} . \tag{7.73}$$

Write $dC/dt = t_1 + t_2$ where t_1 is first term and t_2 second term in (7.73). Note that a, $\boldsymbol{\eta}$ and $(\boldsymbol{\eta} \lrcorner a)$, where $a = \rho \, d^3x$ are advected with the flow. Thus,

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (\boldsymbol{\eta} \lrcorner a) \equiv \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) [d(\boldsymbol{\psi} \cdot d\mathbf{x})] = 0. \tag{7.74}$$

At this point it is useful to introduce the notation:

$$\alpha = \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot d\mathbf{x}, \quad \beta = \mathcal{L}_{\mathbf{u}}(\boldsymbol{\psi} \cdot d\mathbf{x}), \quad \gamma = \boldsymbol{\psi} \cdot d\mathbf{x}. \tag{7.75}$$

Using (7.74) in (7.73), the second term in (7.73) reduces to:

$$t_2 = - \int \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge \mathcal{L}_{\mathbf{u}} d(\boldsymbol{\psi} \cdot d\mathbf{x}) = - \int \frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot d\mathbf{x} \wedge d\mathcal{L}_{\mathbf{u}}(\boldsymbol{\psi} \cdot d\mathbf{x})$$

$$\equiv \int \{ \mathcal{L}_{\mathbf{u}}(d\alpha) \wedge \gamma + d[\alpha \wedge \beta - \mathbf{u} \lrcorner (d\alpha \wedge \gamma)] \}. \tag{7.76}$$

Similarly, we can write t_1 in the form:

$$t_1 = \int \frac{\partial}{\partial t} \left(\frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \cdot d\mathbf{x} \right) \wedge d(\boldsymbol{\psi} \cdot d\mathbf{x})$$

$$= \int \left\{ \frac{\partial}{\partial t}(d\alpha) \wedge \gamma - d(\alpha_t \wedge \gamma) \right\}. \tag{7.77}$$

In the derivation of (7.76) we used the results:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta,$$

$$\mathcal{L}_{\mathbf{u}}(d\alpha \wedge \gamma) = \mathcal{L}_{\mathbf{u}}(d\alpha) \wedge \gamma + d\alpha \wedge \mathcal{L}_{\mathbf{u}}(\gamma),$$

$$\mathcal{L}_{\mathbf{u}}(d\alpha \wedge \gamma) = \mathbf{u} \lrcorner d(d\alpha \wedge \gamma) + d[\mathbf{u} \lrcorner (d\alpha \wedge \gamma)]. \tag{7.78}$$

Note that $d(d\alpha \wedge \gamma) = 0$ since $d\alpha \wedge \gamma$ is a 3-form in 3D space. In (7.77), we used the result:

$$\alpha_t \wedge d\gamma = d\alpha_t \wedge \gamma - d(\alpha_t \wedge \gamma). \tag{7.79}$$

Adding (7.76) and (7.77) gives:

$$\frac{dC}{dt} = \int \left\{ \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (d\alpha) \wedge \gamma + d[\alpha \wedge \beta - \mathbf{u} \lrcorner (d\alpha \wedge \gamma) - \alpha_t \wedge \gamma] \right\}. \tag{7.80}$$

Using (7.80) for dC/dt in (7.58) for δJ gives:

$$\delta J = \int dt \left\{ \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (d\alpha) \wedge \gamma + d \left[\boldsymbol{\psi} \times \frac{\mathbf{E}(\ell)}{\rho} \cdot d\mathbf{S} + \mathbf{F} \cdot d\mathbf{S} \right. \right.$$
$$\left. \left. + \alpha \wedge \beta - \mathbf{u} \lrcorner (d\alpha \wedge \gamma) - \alpha_t \wedge \gamma \right] \right\} + \int d^3x \int dt \, \boldsymbol{\psi} \cdot \nabla \times (\mathbf{E}(\ell)/\rho)$$

$$(7.81)$$

Next we note that the surface term:

$$d \left[\mathbf{F} \cdot d\mathbf{S} + \alpha \wedge \beta - \mathbf{u} \lrcorner (d\alpha \wedge \gamma) - \alpha_t \wedge \gamma \right]$$
$$= d(\mathbf{W} \cdot d\mathbf{S}) = \nabla \cdot \mathbf{W} d^3x, \tag{7.82}$$

where

$$\mathbf{W} = \nabla \times \left[\left(h + \frac{1}{2}|\mathbf{u}|^2 \right) \boldsymbol{\psi} - (\boldsymbol{\psi} \cdot \mathbf{u})\mathbf{u} \right]. \tag{7.83}$$

Note that $\nabla \cdot \mathbf{W} = 0$. In (7.83) we assumed a barotropic equation of state, with $p = p(\rho)$, and used the momentum equation (3.5) to determine α_t. Also note that

$$\int \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (d\alpha) \wedge \gamma = \int \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (\boldsymbol{\omega} \cdot d\mathbf{S}) \wedge (\boldsymbol{\psi} \cdot d\mathbf{x})$$
$$= \int \boldsymbol{\psi} \cdot [\boldsymbol{\omega}_t - \nabla \times (\mathbf{u} \times \boldsymbol{\omega})] \, d^3x. \tag{7.84}$$

Substituting (7.82)–(7.84) into (7.81), and assuming the surface term due to $\boldsymbol{\psi} \times \mathbf{E}(\ell)/\rho$ is zero, we obtain the result (7.69) for δJ. This completes the proof. $\qquad\square$

Comment

In the derivation of (7.82)–(7.83) we used the results:

$$\mathbf{F} \cdot d\mathbf{S} = \left[(\nabla \times \boldsymbol{\psi}) \cdot \mathbf{u}\mathbf{u} + \left(h - \frac{1}{2}|\mathbf{u}|^2 \right) \nabla \times \boldsymbol{\psi} \right] \cdot d\mathbf{S},$$
$$\alpha \wedge \beta = \mathbf{u} \times [\nabla(\mathbf{u} \cdot \boldsymbol{\psi}) - \mathbf{u} \times (\nabla \times \boldsymbol{\psi})] \cdot d\mathbf{S}$$
$$\mathbf{u} \lrcorner (d\alpha \wedge \gamma) = (\boldsymbol{\psi} \cdot \boldsymbol{\omega})\mathbf{u} \cdot d\mathbf{S},$$
$$\alpha_t \wedge \gamma = \left[\mathbf{u} \times \boldsymbol{\omega} + \delta \, T\nabla S - \nabla \left(h + \frac{1}{2}|\mathbf{u}|^2 \right) \right] \times \boldsymbol{\psi} \cdot d\mathbf{S}, \tag{7.85}$$

where for an isobaric equation of state $p = p(\rho)$, the parameter $\delta = 0$, but for a non-isobaric equation of state with $p = p(\rho, S)$, $\delta = 1$.

Alternative Proof of Bianchi Identity (7.70)

Proof An alternative, equivalent approach to obtain the vorticity conservation law (7.71) and the Bianchi identity (7.70), using vector Calculus is given below.

First note that for the mass conservation symmetry $\eta = \nabla \times \boldsymbol{\psi} / \rho$, that $\boldsymbol{\psi} \cdot d\mathbf{x}$ is a conserved 1-form that is advected with the flow, and hence $\boldsymbol{\psi}$ satisfies (5.20), but with $\mathbf{S} \to \boldsymbol{\psi}$, i.e.

$$\frac{\partial \boldsymbol{\psi}}{\partial t} - \mathbf{u} \times (\nabla \times \boldsymbol{\psi}) + \nabla(\mathbf{u} \cdot \boldsymbol{\psi}) = 0. \tag{7.86}$$

Noting that

$$D = \rho \mathbf{u} \cdot \boldsymbol{\eta} = \mathbf{u} \cdot \nabla \times \boldsymbol{\psi},$$

$$\mathbf{F} = (\nabla \times \boldsymbol{\psi}) \cdot \mathbf{u}\mathbf{u} + \left(h - \frac{1}{2}|\mathbf{u}|^2\right) \nabla \times \boldsymbol{\psi}, \tag{7.87}$$

(7.58) for δJ reduces to:

$$\delta J = \int_V d^3x \int dt \left\{\frac{\partial}{\partial t}(\boldsymbol{\psi} \cdot \boldsymbol{\omega}) + \nabla \cdot \left[\frac{\partial}{\partial t}(\boldsymbol{\psi} \times \mathbf{u})\right] + \nabla \cdot \mathbf{F} + \frac{\nabla \times \boldsymbol{\psi}}{\rho} \cdot \mathbf{E}(\ell)\right\} \tag{7.88}$$

Using (7.86) to eliminate $\boldsymbol{\psi}_t$ in the first term in (7.88) we obtain:

$$\delta J = \int_V d^3x \int dt \left\{\boldsymbol{\psi} \cdot \left[\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega})\right] + \nabla \cdot \mathbf{P} + \boldsymbol{\psi} \cdot \nabla \times (\mathbf{E}(\ell)/\rho)\right\} \tag{7.89}$$

where

$$\mathbf{P} = \frac{\partial}{\partial t}(\boldsymbol{\psi} \times \mathbf{u}) + \boldsymbol{\psi} \times (\boldsymbol{\omega} \times \mathbf{u}) - \boldsymbol{\omega} \cdot \nabla(\mathbf{u} \cdot \boldsymbol{\psi}) + \mathbf{F}. \tag{7.90}$$

Using (7.86) for $\boldsymbol{\psi}_t$ and (3.5) for $\mathbf{u}_t$, and $\mathbf{F}$ from (7.87), we find

$$\mathbf{P} \equiv \mathbf{W} = \nabla \times \left[\left(h + \frac{1}{2}|\mathbf{u}|^2\right) \boldsymbol{\psi} - (\boldsymbol{\psi} \cdot \mathbf{u})\mathbf{u}\right]. \tag{7.91}$$

(see also (7.83)). Use of (7.91) in (7.89) gives the same expression for δJ obtained previously in (7.66)–(7.69). This completes the proof. $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\square$

Comment

Note that $\mathbf{W}$ is of the form: $\mathbf{W} = \nabla \times \mathbf{M}$. Thus, $\mathbf{W} \cdot d\mathbf{S} = d(\mathbf{M} \cdot d\mathbf{x})$ and $\mathbf{W} \cdot d\mathbf{S}$ is exact. Also $d(\mathbf{W} \cdot d\mathbf{S}) = dd(\mathbf{M} \cdot d\mathbf{x}) = 0$ by the Poincaré lemma.

Symmetry Operators

Consider the relabelling symmetries:

$$X_1 = \left(\frac{\nabla \times \boldsymbol{\psi}_1}{\rho} \right) \cdot \nabla = \frac{\mathbf{A}}{\rho} \cdot \nabla \quad \text{where} \quad \mathbf{A} = \nabla \times \boldsymbol{\psi}_1,$$

$$X_2 = \left(\frac{\nabla \times \boldsymbol{\psi}_2}{\rho} \right) \cdot \nabla = \frac{\mathbf{B}}{\rho} \cdot \nabla \quad \text{where} \quad \mathbf{B} = \nabla \times \boldsymbol{\psi}_2, \quad\quad (7.92)$$

Note that $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{B} = 0$. The Lie bracket of X_1 and X_2 is:

$$[X_1, X_2] = X_1 X_2 - X_2 X_1 = \left[\frac{\mathbf{A}}{\rho} \cdot \nabla \left(\frac{\mathbf{B}}{\rho} \right) - \frac{\mathbf{B}}{\rho} \cdot \nabla \left(\frac{\mathbf{A}}{\rho} \right) \right] \cdot \nabla$$

$$= -\frac{1}{\rho} \nabla \times \left(\frac{\mathbf{A} \times \mathbf{B}}{\rho} \right) \cdot \nabla. \quad\quad (7.93)$$

Write:

$$[X(\boldsymbol{\psi}_1), X(\boldsymbol{\psi}_2)] = X(\boldsymbol{\psi}_{12}) \quad \text{where} \quad X(\boldsymbol{\psi}) = \frac{\nabla \times \boldsymbol{\psi}}{\rho} \cdot \nabla,$$

$$\boldsymbol{\psi}_{12} = \frac{(\nabla \times \boldsymbol{\psi}_2) \times (\nabla \times \boldsymbol{\psi}_1)}{\rho}. \quad\quad (7.94)$$

Jacobi Identity

One can verify that the Lie bracket Jacobi identity is satisfied for the symmetries $X(\boldsymbol{\psi})$. Write

$$X(\boldsymbol{\psi}_1) = \frac{\mathbf{A}}{\rho} \cdot \nabla, \quad X(\boldsymbol{\psi}_2) = \frac{\mathbf{B}}{\rho} \cdot \nabla, \quad X(\boldsymbol{\psi}_3) = \frac{\mathbf{C}}{\rho} \cdot \nabla$$

$$\mathbf{A} = \nabla \times \boldsymbol{\psi}_1, \quad \mathbf{B} = \nabla \times \boldsymbol{\psi}_2, \quad \mathbf{C} = \nabla \times \boldsymbol{\psi}_3, \quad\quad (7.95)$$

$$S_{ABC} = [[X(\boldsymbol{\psi}_1), X(\boldsymbol{\psi}_2)], X(\boldsymbol{\psi}_3)]. \quad\quad (7.96)$$

$$S_{ABC} = \left[\frac{\mathbf{W}}{\rho} \cdot \nabla \left(\frac{\mathbf{C}}{\rho} \right) - \frac{\mathbf{C}}{\rho} \cdot \nabla \left(\frac{\mathbf{W}}{\rho} \right) \right] \cdot \nabla, \quad [X_1, X_2] = \frac{\mathbf{W}}{\rho} \cdot \nabla,$$

$$\mathbf{W} = \mathbf{A} \cdot \nabla \left(\frac{\mathbf{B}}{\rho} \right) - \mathbf{B} \cdot \nabla \left(\frac{\mathbf{A}}{\rho} \right). \tag{7.97}$$

Using (7.95)–(7.97) we obtain:

$$S_{ABC} + S_{BCA} + S_{CAB} = 0, \tag{7.98}$$

which verifies the Jacobi identity.

7.2.3 Cross Helicity

To obtain the cross helicity conservation law (3.57) using Noether's theorem, it is necessary to obtain the appropriate solution of (7.52)–(7.55) for the fluid relabeling symmetries. The condition that the mass 3-form $\alpha = \rho d^3 x$ is a fluid relabeling symmetry using Cartan's magic formula requires that:

$$\mathcal{L}_{\boldsymbol{\eta}}(\rho d^3 x) = d\boldsymbol{\eta} \lrcorner (\rho d^3 x) = d(\rho \boldsymbol{\eta} \cdot d\mathbf{S}) = \nabla \cdot (\rho \boldsymbol{\eta}) d^3 x = 0. \tag{7.99}$$

The entropy variation $\delta S = -\boldsymbol{\eta} \cdot \nabla S = 0$, and the magnetic field variation $\delta \mathbf{B} = \nabla \times (\boldsymbol{\eta} \times \mathbf{B}) = 0$ and the fluid velocity variation $\delta \mathbf{u} = \dot{\boldsymbol{\eta}} + [\mathbf{u}, \boldsymbol{\eta}] = 0$ are all satisfied by the choice:

$$\boldsymbol{\eta} \equiv \hat{V}^{\mathbf{x}} = \zeta(\mathbf{x}_0)\mathbf{b} \quad \text{where} \quad \mathbf{b} = \frac{\mathbf{B}}{\rho} \quad \text{and} \quad \mathbf{B} \cdot \nabla S = 0. \tag{7.100}$$

Note that $\mathbf{b} = \mathbf{B}/\rho$ is an invariant vector field that is Lie dragged with the flow (see (5.36) and (5.37)). From (7.49) the surface term D in the variational principle (7.44) is given by:

$$D = \rho \mathbf{u} \cdot \boldsymbol{\eta} = \rho \mathbf{u} \cdot \zeta(\mathbf{x}_0)\mathbf{b} \equiv \zeta(\mathbf{x}_0)\mathbf{u} \cdot \mathbf{B}. \tag{7.101}$$

Similarly, the flux $\mathbf{F}$ surface term in (7.49) is given by:

$$\mathbf{F} = \zeta(\mathbf{x}_0) \frac{\mathbf{B}}{\rho} \cdot \left[\rho \mathbf{u} \otimes \mathbf{u} + \rho \left(h - \frac{1}{2}|\mathbf{u}|^2 \right) \mathsf{I} + \frac{B^2}{\mu_0} \mathsf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right]$$

$$= \zeta(\mathbf{x}_0) \left[(\mathbf{u} \cdot \mathbf{B})\mathbf{u} + \left(h - \frac{1}{2}|\mathbf{u}|^2 \right) \mathbf{B} \right]. \tag{7.102}$$

In the variational principle (7.44) δJ reduces to:

$$\delta J = \int d^3x \int dt \left\{ \zeta(\mathbf{x}_0)\frac{\mathbf{B}}{\rho} \cdot \mathbf{E}(\ell) + \frac{\partial}{\partial t}\left(\zeta(\mathbf{x}_0)\mathbf{u}\cdot\mathbf{B}\right)\right.$$
$$\left. + \nabla\cdot\left[\zeta(\mathbf{x}_0)\left((\mathbf{u}\cdot\mathbf{B})\mathbf{u} + \left(h - \frac{1}{2}|\mathbf{u}|^2\right)\mathbf{B}\right)\right]\right\}$$
$$= \int d^3x \int dt \left\{\zeta(\mathbf{x}_0)\left[\frac{\mathbf{B}\cdot\mathbf{E}(\ell)}{\rho} + \frac{\partial h_c}{\partial t}\right.\right.$$
$$\left.\left. + \nabla\cdot\left[\mathbf{u}h_c + \left(h - \frac{1}{2}|\mathbf{u}|^2\right)\mathbf{B}\right]\right] + R\right\}, \tag{7.103}$$

where $h_c = \mathbf{u}\cdot\mathbf{B}$ is the cross helicity, and

$$R = h_c\left(\frac{\partial\zeta}{\partial t} + \mathbf{u}\cdot\nabla\zeta\right) + \left(h - \frac{1}{2}|\mathbf{u}|^2\right)\mathbf{B}\cdot\nabla\zeta. \tag{7.104}$$

The fluid relabeling symmetry must satisfy $\mathbf{B}\cdot\nabla S(\mathbf{x}_0) = \mathbf{B}\cdot\nabla\zeta(\mathbf{x}_0) = 0$. and $d\zeta/dt = 0$. Thus, the remainder term in (7.103) and (7.104) $R = 0$. The net upshot from (7.103) is the generalized Bianchi identity:

$$\frac{\mathbf{B}\cdot\mathbf{E}(\ell)}{\rho} + \frac{\partial h_c}{\partial t} + \nabla\cdot\left[\mathbf{u}h_c + \left(h - \frac{1}{2}|\mathbf{u}|^2\right)\mathbf{B}\right] = 0. \tag{7.105}$$

Thus, if the Euler Lagrange equations $\mathbf{E}(\ell) = 0$ are satisfied, then (7.105) reduces to the cross helicity conservation equation (3.57), i.e.

$$\frac{\partial h_c}{\partial t} + \nabla\cdot\left[\mathbf{u}h_c + \left(h - \frac{1}{2}|\mathbf{u}|^2\right)\mathbf{B}\right] = 0. \tag{7.106}$$

The only constraint on (7.106) is that we require $\mathbf{B}\cdot\nabla S = 0$. If $\mathbf{B}\cdot\mathbf{n} = 0$ on the boundary ∂V_m of the volume V_m of interest, then the integral form of the (7.106) reduces to $dH_c/dt = 0$.

7.2.4 Helicity in Fluids

In a barotropic, ideal fluid in which the pressure $p = p(\rho)$ is independent of the entropy S, the helicity density:

$$h_f = \mathbf{u}\cdot\boldsymbol{\omega} \quad \text{where} \quad \boldsymbol{\omega} = \nabla\times\mathbf{u}, \tag{7.107}$$

satisfies the conservation law:

$$\frac{\partial h_f}{\partial t} + \nabla \cdot \left[\mathbf{u} h_f + \left(h - \frac{1}{2} |\mathbf{u}|^2 \right) \boldsymbol{\omega} \right] = 0. \tag{7.108}$$

This conservation law is the analogue of the cross helicity conservation law (7.106) where

$$\mathbf{B} \to \boldsymbol{\omega} \quad \text{and} \quad h_c \to h_f, \tag{7.109}$$

The helicity

$$H_f = \int_V \mathbf{u} \cdot \nabla \times \mathbf{u} \, d^3 x, \tag{7.110}$$

is a topological invariant describing the linkage of the vortex tubes (Moffatt 1969, 1978), where it is assumed that $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on the boundary of the region V with outward normal $\mathbf{n}$. This is equivalent to assuming that the volume V consists of vortex tubes with $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on the boundary ∂V. H_f is the fluid dynamical analogue of the magnetic helicity $H_m = \int_V \mathbf{A} \cdot \mathbf{B} \, d^3 x$ which gives the linkage of the magnetic flux tubes (e.g. Moffatt and Ricca 1992; Kruskal and Kulsrud 1958; Woltjer 1958). H_m describes the linkage of the poloidal and toroidal magnetic flux (e.g. Kruskal and Kulsrud 1958). The integral form of the helicity conservation equation (7.108) is $dH_f/dt = 0$ for a volume V moving with the flow (see Section 3 for more detail) where it is assumed $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ on the boundary ∂V_m.

The Lie symmetry associated with the helicity (kinetic helicity) conservation equation (7.108) is:

$$\boldsymbol{\eta} \equiv \hat{V}^{\mathbf{x}} = \frac{\zeta(\mathbf{x}_0) \boldsymbol{\omega}}{\rho} \quad \text{where} \quad \boldsymbol{\omega} \cdot \nabla \zeta(\mathbf{x}_0) = 0. \tag{7.111}$$

One can verify that the solution (7.111) satisfies the fluid relabelling Lie determining equations (7.53)–(7.55) with $\mathbf{B} = 0$. In particular (7.55) reduces to the vorticity equation:

$$\frac{d}{dt} \left(\frac{\boldsymbol{\omega}}{\rho} \right) = \frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \mathbf{u} \quad \text{or} \quad \frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) = 0, \tag{7.112}$$

which applies for a barotropic equation of state with $p = p(\rho)$. The derivation of the helicity conservation law (7.108) using Noether's theorem is analogous to the derivation of the cross helicity conservation law (7.106) except that $\mathbf{B} \to \boldsymbol{\omega}$ and $h_c \to h_f$.

7.2.5 Potential Vorticity and Ertel's Theorem

The Lie determining equations (7.54)–(7.55) admit the symmetry:

$$\boldsymbol{\eta} \equiv \hat{V}^{\mathbf{x}} = \frac{\nabla \times (\Phi \nabla S)}{\rho} = \frac{\nabla \times \boldsymbol{\psi}}{\rho}, \quad \text{where} \quad \boldsymbol{\psi} = \Phi \nabla S, \tag{7.113}$$

and $\Phi = \Phi(\mathbf{x}_0)$ depends only on the Lagrange labels $\mathbf{x}_0$, i.e. Φ is a 0-form Lie dragged by the flow:

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \mathbf{u} \cdot \nabla \Phi = 0. \tag{7.114}$$

Note that

$$\boldsymbol{\eta} \lrcorner \rho \, d^3 x = \rho \boldsymbol{\eta} \cdot d\mathbf{S} = \nabla \times \boldsymbol{\psi} \cdot d\mathbf{S} = d(\boldsymbol{\psi} \cdot d\mathbf{x}) = d(\Phi dS). \tag{7.115}$$

The condition (7.55) implies $\hat{V}^{\mathbf{x}} \equiv \boldsymbol{\eta}$ is a Lie dragged vector field which satisfies (7.53). Similarly, the 1-form $\alpha = \boldsymbol{\psi} \cdot d\mathbf{x}$ is Lie dragged with the flow, i.e. $\boldsymbol{\psi}$ satisfies the equation:

$$\frac{\partial \boldsymbol{\psi}}{\partial t} - \mathbf{u} \times (\nabla \times \boldsymbol{\psi}) + \nabla(\mathbf{u} \cdot \boldsymbol{\psi}) = 0. \tag{7.116}$$

Using $\boldsymbol{\psi} = \Phi \nabla S$, (7.116) reduces to:

$$\Phi \nabla \left(\frac{dS}{dt} \right) + \nabla S \frac{d\Phi}{dt} = 0. \tag{7.117}$$

Equation (7.55) is equivalent to the curl of (7.117). Since $dS/dt = 0$, (7.117) implies $d\Phi/dt = 0$. Note that $\boldsymbol{\psi} \cdot d\mathbf{x} = \Phi \nabla S \cdot d\mathbf{x}$ are Lie dragged 1-forms and hence Φ is necessarily an advected invariant 0-form or function.

Proof (Ertel's Theorem) To derive Ertel's theorem from Noether's theorem, we require $\delta J = 0$ in (7.58). From (7.81):

$$\delta J = \int dt \int_V \left[\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (d\alpha) \wedge \gamma + d(\mathbf{W} \cdot d\mathbf{S}) \right] + \int dt \int_V d^3 x \, \boldsymbol{\psi} \cdot \nabla \times (\mathbf{E}(\ell)/\rho), \tag{7.118}$$

where $\mathbf{W}$ is given by (7.83). Note that $\mathbf{W}$ is a solenoidal vector field, i.e. $\nabla \cdot \mathbf{W} = 0$. In (7.118) $\boldsymbol{\psi} = \Phi \nabla S$ and $d\Phi/dt = 0$. We introduce the notation:

$$I = \int_V \left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) (d\alpha) \wedge \gamma, \tag{7.119}$$

for the first integral in (7.118), where α, β and γ are the differential 1-forms given in (7.75). From (7.119) and (7.75) we obtain:

$$
\begin{aligned}
I &= \int_V \frac{d}{dt} \left[\nabla \times \left(\frac{1}{\rho} \frac{\delta \ell}{\delta \mathbf{u}} \right) \cdot d\mathbf{S} \right] \wedge \Phi \nabla S \cdot d\mathbf{x} \\
&= \int_V \frac{d}{dt} (\boldsymbol{\omega} \cdot d\mathbf{S}) \wedge \Phi \nabla S \cdot d\mathbf{x} \\
&= \int_V \frac{d}{dt} (\boldsymbol{\omega} \cdot d\mathbf{S} \wedge \Phi \nabla S \cdot d\mathbf{x}).
\end{aligned}
\tag{7.120}
$$

In (7.120) we use the fact that Φ is a 0-form and $\nabla S \cdot d\mathbf{x}$ is a 1-form, which are Lie dragged with the flow. The integral I in (7.120) can be further reduced to:

$$
I = \int_V \frac{d}{dt} \left(\frac{\boldsymbol{\omega} \cdot \nabla S}{\rho} \Phi \rho \, d^3 x \right) = \int_V \frac{d}{dt} \left(\frac{\boldsymbol{\omega} \cdot \nabla S}{\rho} \right) \Phi \rho \, d^3 x.
\tag{7.121}
$$

Note that $d/dt(\Phi \rho d^3 x) = 0$ as $\rho d^3 x$ is an invariant 3-form and Φ is an invariant 0-form.

Using (7.121) in (7.118) gives:

$$
\delta J = \int dt \int_V d^3 x \left\{ \Phi \left[\rho \frac{d}{dt} \left(\frac{\boldsymbol{\omega} \cdot \nabla S}{\rho} \right) + \nabla S \cdot \nabla \times \left(\frac{\mathbf{E}(\ell)}{\rho} \right) \right] + \nabla \cdot \mathbf{W} \right\}.
\tag{7.122}
$$

Because $\nabla \cdot \mathbf{W} = 0$, and using the du-Bois Reymond lemma in (7.122), we obtain the generalized Bianchi identity:

$$
\rho \frac{d}{dt} \left(\frac{\boldsymbol{\omega} \cdot \nabla S}{\rho} \right) + \nabla S \cdot \nabla \times \left(\frac{\mathbf{E}(\ell)}{\rho} \right) = 0.
\tag{7.123}
$$

If the Euler-Lagrange equations $\mathbf{E}(\ell) = 0$ are satisfied, then (7.123) implies Ertel's theorem:

$$
\frac{d}{dt} \left(\frac{\boldsymbol{\omega} \cdot \nabla S}{\rho} \right) = 0.
\tag{7.124}
$$

This completes the proof. $\square$

Chapter 8
Hamiltonian Approach

This chapter describes the Hamiltonian approach to MHD and gas dynamics. In Sect. 8.1 we describe a constrained MHD variational principle by using Lagrange multipliers to enforce the constraints of mass conservation; the entropy advection equation; Faraday's equation and the so-called Lin constraint describing in part, the vorticity of the flow (i.e. Kelvin's theorem). This leads to Hamilton's canonical equations in terms of Clebsch potentials. The Lagrange multipliers define the Clebsch variables, which gives a Clebsch representation for the fluid velocity **u** (Zakharov and Kuznetsov 1997). In Sect. 8.2 we transform the canonical, Clebsch variable, Poisson bracket to different non-canonical forms that use Eulerian physical variables (see e.g. Morrison and Greene 1980, 1982; Morrison 1982; Holm and Kupershmidt 1983a,b). The different MHD brackets are described in Sect. 8.3. Section 8.4 verifies the Jacobi identity for the bracket of Morrison and Greene (1982) in which $\nabla \cdot \mathbf{B}$ can be non-zero. We discuss how the Morrison and Greene (1980) bracket, with $\nabla \cdot \mathbf{B} = 0$, has been placed on a more rigorous footing by the use of the Dirac bracket and projectors by Chandre et al. (2012, 2013) and Chandre (2013) (see also Banerjee and Kumar 2016). We use the functional multi-vector approach of Olver (1993) to investigate and check the Jacobi identity for: (1) the Morrison and Greene (1982) bracket, (2) the advected **A** bracket in which $\mathbf{A} \cdot d\mathbf{x}$ is Lie dragged with the flow used by Holm and Kupershmidt (1983a,b) and (3) the Morrison and Greene (1980) bracket. The non-canonical Poisson brackets are used to determine the MHD Casimirs in Sect. 8.5 (e.g. Hameiri 2004). The Casimirs are related to the advected invariants.

8.1 Clebsch Variables and Hamilton's Equations

Consider the MHD action (modified by constraints):

$$J = \int d^3x \, dt L, \tag{8.1}$$

where

$$L = \left\{ \frac{1}{2}\rho u^2 - \epsilon(\rho, S) - \frac{B^2}{2\mu_0} \right\} + \phi \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right)$$

$$+ \beta \left(\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S \right) + \lambda \left(\frac{\partial \mu}{\partial t} + \mathbf{u} \cdot \nabla \mu \right)$$

$$+ \mathbf{\Gamma} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}(\nabla \cdot \mathbf{B}) \right). \tag{8.2}$$

The Lagrangian in curly brackets equals the kinetic minus the potential energy (internal thermodynamic energy plus magnetic energy). The Lagrange multipliers ϕ, β, λ, and $\mathbf{\Gamma}$ ensure that the mass, entropy, Lin constraint, Faraday equations are satisfied. We do not enforce $\nabla \cdot \mathbf{B} = 0$, since we are interested in the effect of $\nabla \cdot \mathbf{B} \neq 0$ (which is useful for numerical MHD where $\nabla \cdot \mathbf{B} \neq 0$). It is straightforward to impose $\nabla \cdot \mathbf{B} = 0$ if desired, although some care is required in the formulation of the Poisson bracket, to ensure that the Jacobi identity is satisfied (e.g. Morrison 1982; Chandre et al. 2012, 2013; Chandre 2013).

Stationary point conditions for the action are $\delta J = 0$. $\delta J / \delta \mathbf{u} = 0$ gives Clebsch representation for $\mathbf{u}$:

$$\mathbf{u} = \nabla \phi - \frac{\beta}{\rho} \nabla S - \frac{\lambda}{\rho} \nabla \mu + \mathbf{u}_M \tag{8.3}$$

where

$$\mathbf{u}_M = -\frac{(\nabla \times \mathbf{\Gamma}) \times \mathbf{B}}{\rho} - \mathbf{\Gamma} \frac{\nabla \cdot \mathbf{B}}{\rho}, \tag{8.4}$$

is magnetic contribution to $\mathbf{u}$. Setting $\delta J / \delta \phi$, $\delta J / \delta \beta$, $\delta J / \delta \lambda$, $\delta J / \delta \mathbf{\Gamma}$ consecutively equal to zero gives the mass, entropy advection, Lin constraint, and Faraday (magnetic flux conservation) constraint equations:

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$S_t + \mathbf{u} \cdot \nabla S = 0,$$

$$\mu_t + \mathbf{u} \cdot \nabla \mu = 0,$$

$$\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}(\nabla \cdot \mathbf{B}) = 0. \tag{8.5}$$

Setting $\delta J / \delta \rho$, $\delta J / \delta S$, $\delta J / \delta \mu$, $\delta J / \delta \mathbf{B}$ equal to zero gives evolution equations for the Clebsch potentials ϕ, β λ and $\mathbf{\Gamma}$ as:

$$-\left(\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi \right) + \frac{1}{2}u^2 - h = 0, \tag{8.6}$$

$$\frac{\partial \beta}{\partial t} + \nabla \cdot (\beta \mathbf{u}) + \rho T = 0, \tag{8.7}$$

$$\frac{\partial \lambda}{\partial t} + \nabla \cdot (\lambda \mathbf{u}) = 0, \tag{8.8}$$

$$\frac{\partial \mathbf{\Gamma}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{\Gamma}) + \nabla(\mathbf{\Gamma} \cdot \mathbf{u}) + \frac{\mathbf{B}}{\mu_0} = 0. \tag{8.9}$$

Equation (8.6) is related to Bernoulli's equation for potential flow. The $\nabla(\mathbf{\Gamma} \cdot \mathbf{u})$ term in (8.9) is associated with $\nabla \cdot \mathbf{B} \neq 0$. Taking the curl of (8.9) gives:

$$\frac{\partial \tilde{\mathbf{\Gamma}}}{\partial t} - \nabla \times (\mathbf{u} \times \tilde{\mathbf{\Gamma}}) = -\frac{\nabla \times \mathbf{B}}{\mu_0} \quad \text{where} \quad \tilde{\mathbf{\Gamma}} = \nabla \times \mathbf{\Gamma}. \tag{8.10}$$

Equations (8.6)–(8.10) can be written in the form:

$$\frac{d\phi}{dt} = \frac{1}{2}u^2 - h, \quad \frac{d}{dt}\left(\frac{\beta}{\rho}\right) = -T,$$

$$\frac{d}{dt}\left(\lambda d^3 x\right) = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{\lambda}{\rho}\right) = 0,$$

$$\frac{d}{dt}(\mathbf{\Gamma} \cdot d\mathbf{x}) = -\frac{\mathbf{B} \cdot d\mathbf{x}}{\mu_0}, \quad \frac{d}{dt}(\tilde{\mathbf{\Gamma}} \cdot d\mathbf{S}) = -\mathbf{J} \cdot d\mathbf{S}. \tag{8.11}$$

where $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$, is the Lagrangian time derivative following the flow.

Introduce the Hamiltonian functional:

$$\mathcal{H} = \int H d^3 x \quad \text{where} \quad H = \frac{1}{2}\rho u^2 + \epsilon(\rho, S) + \frac{B^2}{2\mu_0}. \tag{8.12}$$

Substitute the Clebsch expansion (8.3)–(8.4) for $\mathbf{u}$ in (8.12). Evaluating the variational derivatives of $\mathcal{H}$ gives Hamilton's equations:

$$\frac{\partial \rho}{\partial t} = \frac{\delta \mathcal{H}}{\delta \phi}, \quad \frac{\partial \phi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \rho}, \quad \frac{\partial S}{\partial t} = \frac{\delta \mathcal{H}}{\delta \beta}, \quad \frac{\partial \beta}{\partial t} = -\frac{\delta \mathcal{H}}{\delta S},$$

$$\frac{\partial \mu}{\partial t} = \frac{\delta \mathcal{H}}{\delta \lambda}, \quad \frac{\partial \lambda}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \mu}, \quad \frac{\partial \mathbf{B}}{\partial t} = \frac{\delta \mathcal{H}}{\delta \mathbf{\Gamma}}, \quad \frac{\partial \mathbf{\Gamma}}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \mathbf{B}}. \tag{8.13}$$

Here $\{\rho, \phi\}, \{S, \beta\}, \{\mu, \lambda\}, \{\mathbf{B}, \mathbf{\Gamma}\}$ are canonically conjugate variables.

The canonical Poisson bracket is:

$$\{F, G\} = \int d^3 x \left(\frac{\delta F}{\delta \rho}\frac{\delta G}{\delta \phi} - \frac{\delta F}{\delta \phi}\frac{\delta G}{\delta \rho} + \frac{\delta F}{\delta \mathbf{B}} \cdot \frac{\delta G}{\delta \mathbf{\Gamma}} - \frac{\delta F}{\delta \mathbf{\Gamma}} \cdot \frac{\delta G}{\delta \mathbf{B}}\right.$$

$$\left. + \frac{\delta F}{\delta S}\frac{\delta G}{\delta \beta} - \frac{\delta F}{\delta \beta}\frac{\delta G}{\delta S} + \frac{\delta F}{\delta \mu}\frac{\delta G}{\delta \lambda} - \frac{\delta G}{\delta \mu}\frac{\delta F}{\delta \lambda}\right). \tag{8.14}$$

It is straightforward to verify that the canonical Poisson bracket (8.14) satisfies the linearity, skew symmetry and Jacobi identity necessary for a Hamiltonian (i.e. the Poisson bracket defines a Lie algebra).

8.2 Non-canonical Poisson Brackets

Morrison and Greene (1980, 1982) and Holm and Kupershmidt (1983a,b) introduced non-canonical Poisson brackets for MHD.

Introduce the new variables:

$$\mathbf{M} = \rho \mathbf{u}, \quad \sigma = \rho S, \tag{8.15}$$

The formulae for the transformation of variational derivatives in the old variables $(\rho, \phi, S, \beta, \mu, \lambda, \mathbf{B}, \mathbf{\Gamma})$ in terms of the new variables $(\rho, \sigma, \mathbf{B}, \mathbf{M})$ are:

$$\frac{\delta F}{\delta \rho} = \frac{\delta F}{\delta \rho} + S\frac{\delta F}{\delta \sigma} + \frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \phi, \quad \frac{\delta F}{\delta \phi} = -\nabla \cdot \left(\rho \frac{\delta F}{\delta \mathbf{M}} \right),$$

$$\frac{\delta F}{\delta S} = \rho \frac{\delta F}{\delta \sigma} + \nabla \cdot \left(\beta \frac{\delta F}{\delta \mathbf{M}} \right), \quad \frac{\delta F}{\delta \beta} = -\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla S,$$

$$\frac{\delta F}{\delta \mu} = \nabla \cdot \left(\lambda \frac{\delta F}{\delta \mathbf{M}} \right), \quad \frac{\delta F}{\delta \lambda} = -\nabla \mu \cdot \frac{\delta F}{\delta \mathbf{M}},$$

$$\frac{\delta F}{\delta \mathbf{B}} = \left[\frac{\delta F}{\delta \mathbf{B}} + \nabla \left(\frac{\delta F}{\delta \mathbf{M}} \right) \cdot \mathbf{\Gamma} + \frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \mathbf{\Gamma} \right] \equiv \frac{\delta F}{\delta \mathbf{B}} + \nabla \left(\mathbf{\Gamma} \cdot \frac{\delta F}{\delta \mathbf{M}} \right) + (\nabla \times \mathbf{\Gamma}) \times \frac{\delta F}{\delta \mathbf{M}},$$

$$\frac{\delta F}{\delta \mathbf{\Gamma}} = \left[\mathbf{B} \cdot \nabla \left(\frac{\delta F}{\delta \mathbf{M}} \right) - \nabla \cdot \left(\frac{\delta F}{\delta \mathbf{M}} \right) \mathbf{B} - \frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \mathbf{B} \right] \equiv \nabla \times \left(\frac{\delta F}{\delta \mathbf{M}} \times \mathbf{B} \right) - \frac{\delta F}{\delta \mathbf{M}} (\nabla \cdot \mathbf{B}).$$
$$\tag{8.16}$$

Note that

$$\mathbf{M} = \rho \mathbf{u} = \rho \nabla \phi - \beta \nabla S - \lambda \nabla \mu + \mathbf{B} \cdot (\nabla \mathbf{\Gamma})^T - \mathbf{B} \cdot \nabla \mathbf{\Gamma} - \mathbf{\Gamma} (\nabla \cdot \mathbf{B}). \tag{8.17}$$

Using the transformations (8.16) in the canonical Poisson bracket (8.14) we obtain the Morrison and Greene (1982) non-canonical Poisson bracket:

$$\{F, G\} = -\int d^3 x \left\{ \rho \left[\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \left(\frac{\delta G}{\delta \rho} \right) - \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \left(\frac{\delta F}{\delta \rho} \right) \right] \right.$$

$$+ \sigma \left[\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \left(\frac{\delta G}{\delta \sigma} \right) - \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \left(\frac{\delta F}{\delta \sigma} \right) \right]$$

$$+ \mathbf{M} \cdot \left[\left(\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \right) \frac{\delta G}{\delta \mathbf{M}} - \left(\frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \right) \frac{\delta F}{\delta \mathbf{M}} \right]$$

$$+ \, \mathbf{B} \cdot \left[\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \left(\frac{\delta G}{\delta \mathbf{B}} \right) - \frac{\delta G}{\delta \mathbf{M}} \cdot \nabla \left(\frac{\delta F}{\delta \mathbf{B}} \right) \right]$$

$$+ \, \mathbf{B} \cdot \left[\left(\nabla \frac{\delta F}{\delta \mathbf{M}} \right) \cdot \frac{\delta G}{\delta \mathbf{B}} - \left(\nabla \frac{\delta G}{\delta \mathbf{M}} \right) \cdot \frac{\delta F}{\delta \mathbf{B}} \right] \Bigg\}. \tag{8.18}$$

The bracket (8.18) has the Lie-Poisson form and satisfies the Jacobi identity for all functionals F, G and H of the physical variables, and in general applies both for $\nabla \cdot \mathbf{B} \neq 0$ and in the limit as $\nabla \cdot \mathbf{B} = 0$. (Marsden and Ratiu 1994, Chapter 13), discuss the Lie Poisson bracket and the third term in the Poisson bracket (8.18) and how it is related to the Lie-Poisson reduction theorem, and to the mass preserving diffeomorphism group (for incompressible fluids, the group is that of volume preserving diffeomorphisms). The condition $\nabla \cdot \mathbf{B} = 0$ needs to be accounted for in the evaluation of the Jacobi identity (Chandre et al. 2012, 2013; Chandre 2013). Banerjee and Kumar (2016) carry out further analysis of MHD using the Dirac bracket. We verify the Jacobi identity for the bracket (8.18) for $\nabla \cdot \mathbf{B} \neq 0$ in the next section, using Olver (1993)'s method. We also use the Olver functional multi-vector method to verify the Jacobi identity for the Holm and Kupershmidt (1983a,b) bracket based on the magnetic vector potential $\mathbf{A}$. Holm and Kupershmidt (1983a,b) and Holm et al. (1998) use the semi-direct product and Lie Poisson form of their bracket to deduce the Jacobi identity.

By using the transformation $\sigma = \rho S$, and the variational derivative transformations:

$$\frac{\delta \hat{F}}{\delta \sigma} = \frac{1}{\rho} \frac{\delta \tilde{F}}{\delta S}, \quad \frac{\delta \hat{F}}{\delta \rho} = \frac{\delta \tilde{F}}{\delta \rho} - \frac{S}{\rho} \frac{\delta \tilde{F}}{\delta S}, \tag{8.19}$$

the bracket (8.18) may be written in the form:

$$\{F, G\} = \int_V d^3 x \, \Bigg\{ (G_\mathbf{M} \cdot \nabla F_\mathbf{M} - F_\mathbf{M} \cdot \nabla G_\mathbf{M}) \cdot \mathbf{M}$$

$$+ (G_\mathbf{M} \cdot \nabla (F_\mathbf{B}) - F_\mathbf{M} \cdot (\nabla G_\mathbf{B}) \cdot \mathbf{B}$$

$$+ F_\mathbf{B} \cdot [(\mathbf{B} \cdot \nabla) G_\mathbf{M}] - G_\mathbf{B} \cdot [(\mathbf{B} \cdot \nabla) F_\mathbf{M}]$$

$$+ \rho [G_\mathbf{M} \cdot \nabla (F_\rho) - F_\mathbf{M} \cdot \nabla (G_\rho)]$$

$$+ S \nabla \cdot [G_\mathbf{M} F_S - F_\mathbf{M} G_S] \Bigg\}. \tag{8.20}$$

In (8.19) $\hat{F}$ refers to the functional in (8.18) in the old variables, and $F = \tilde{F}$ refers to the same functional in the new variables used in (8.20). The Poisson bracket (8.20) was used by Hameiri (2004) in a paper on the MHD Casimirs.

Another useful form of the Poisson bracket is obtained by changing variables from $(\mathbf{M}, \rho)$ to the new variables $(\mathbf{u}, \rho)$ in (8.20). The transformation of variational

derivatives from the old to the new variables are given by:

$$\frac{\delta \hat{F}}{\delta \mathbf{M}} = \frac{1}{\rho} \frac{\delta \tilde{F}}{\delta \mathbf{u}}, \quad \frac{\delta \hat{F}}{\delta \rho} = \frac{\delta \tilde{F}}{\delta \rho} - \frac{1}{\rho} \mathbf{u} \cdot \frac{\delta \tilde{F}}{\delta \mathbf{u}}, \tag{8.21}$$

in the bracket (8.20), where $\hat{F}$ refers to the old variables functional used in (8.20) and $\tilde{F}$ to the new variables $\mathbf{u}$ and ρ. The bracket in the new variables is:

$$\begin{aligned}
\{F, G\} = -\int_V d^3x \Bigg\{ & F_\rho \nabla \cdot (G_\mathbf{u}) + F_\mathbf{u} \cdot \nabla (G_\rho) \\
& + \frac{\nabla \times \mathbf{u}}{\rho} \cdot (G_\mathbf{u} \times F_\mathbf{u}) + \frac{\nabla S}{\rho} (F_S G_\mathbf{u} - G_S F_\mathbf{u}) \\
& + \mathbf{B} \cdot \left[\frac{1}{\rho} F_\mathbf{u} \cdot \nabla (G_\mathbf{B}) - \frac{1}{\rho} G_\mathbf{u} \cdot \nabla (F_\mathbf{B}) \right] \\
& + \mathbf{B} \cdot \left[\nabla \left(\frac{1}{\rho} F_\mathbf{u} \right) \cdot G_\mathbf{B} - \nabla \left(\frac{1}{\rho} G_\mathbf{u} \right) \cdot F_\mathbf{B} \right] \Bigg\}. \tag{8.22}
\end{aligned}$$

This bracket was discussed by Morrison and Greene (1982) and Morrison (1982). We use this form of the Poisson bracket later to discuss the MHD Casimirs.

8.2.1 Advected **A** Formulation

Below we formulate the MHD variational principle using the magnetic vector potential **A** instead of using **B** (see e.g. Holm and Kupershmidt 1983a,b for a similar formulation using **A**). The condition that the magnetic flux $\mathbf{B} \cdot d\mathbf{S}$ is Lie dragged with the flow (i.e. Faraday's equation) as a constraint equation, is replaced by the constraint that the magnetic vector potential 1-form $\alpha = \mathbf{A} \cdot d\mathbf{x}$ is Lie dragged by the flow. This implies Faraday's equation, where $\mathbf{B} = \nabla \times \mathbf{A}$ is a solenoidal vector field with $\nabla \cdot \mathbf{B} = 0$. This approach is tantamount to a gauge choice for **A**, which simplifies conservation laws associated with **A** and **B**, since both $\alpha = \mathbf{A} \cdot d\mathbf{x}$ and $\beta = \mathbf{B} \cdot d\mathbf{S}$ are Lie dragged with the flow. The condition that $\mathbf{A} \cdot d\mathbf{x}$ is Lie dragged with the flow is equivalent to:

$$\frac{\partial \mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla (\mathbf{u} \cdot \mathbf{A}) = 0 \tag{8.23}$$

(see Chap. 4). The condition that $\alpha = \mathbf{A} \cdot d\mathbf{x}$ is Lie dragged with the flow implies $\mathbf{A} \cdot d\mathbf{x} = \mathbf{A}_0(\mathbf{x}_0) \cdot d\mathbf{x}_0$, where $\mathbf{A}_0(\mathbf{x}_0)$ is the magnetic vector potential in Lagrange label space ($\mathbf{x}_0$), i.e. $d\mathbf{A}_0(\mathbf{x}_0)/dt = 0$.

We use the variational principle $\delta\mathcal{A} = 0$ where the action $\mathcal{A}$ is given by:

$$\mathcal{A} = \int_V d^3x \int dt \left\{ \left[\frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \frac{|\nabla \times \mathbf{A}|^2}{2\mu} \right] \right.$$
$$+ \phi \left(\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) \right) + \beta \left(\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S \right) + \lambda \left(\frac{\partial\mu}{\partial t} + \mathbf{u} \cdot \nabla\mu \right)$$
$$\left. + \boldsymbol{\gamma} \cdot \left[\frac{\partial\mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla(\mathbf{u} \cdot \mathbf{A}) \right] \right\}. \tag{8.24}$$

By setting the variational derivative $\delta\mathcal{A}/\delta\mathbf{u} = 0$ gives the Clebsch variable expansion:

$$\mathbf{u} = \nabla\phi - \frac{\beta}{\rho}\nabla S - \frac{\lambda}{\rho}\nabla\mu - \frac{\boldsymbol{\gamma} \times (\nabla \times \mathbf{A})}{\rho} + \frac{\nabla \cdot \boldsymbol{\gamma}}{\rho}\mathbf{A}, \tag{8.25}$$

for the fluid velocity $\mathbf{u}$.

Setting the variational derivatives $\delta\mathcal{A}/\delta\phi$, $\delta\mathcal{A}/\delta\beta$, $\delta\mathcal{A}/\delta\lambda$, and $\delta\mathcal{A}/\delta\boldsymbol{\gamma}$ equal to zero gives the constraint equations:

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0, \quad \frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S = 0,$$

$$\frac{\partial\mu}{\partial t} + \mathbf{u} \cdot \nabla\mu = 0,$$

$$\frac{\partial\mathbf{A}}{\partial t} - \mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla(\mathbf{u} \cdot \mathbf{A}) = 0. \tag{8.26}$$

Similarly setting $\delta\mathcal{A}/\delta\rho$, $\delta\mathcal{A}/\delta S$, $\delta\mathcal{A}/\delta\mu$ and $\delta\mathcal{A}/\delta\mathbf{A}$ equal to zero gives the equations:

$$\frac{\partial\phi}{\partial t} + \mathbf{u} \cdot \nabla\phi + h - \frac{1}{2}|\mathbf{u}|^2 = 0,$$

$$\frac{\partial\beta}{\partial t} + \nabla \cdot (\beta\mathbf{u}) + \rho T = 0,$$

$$\frac{\partial\lambda}{\partial t} + \nabla \cdot (\lambda\mathbf{u}) = 0,$$

$$\frac{\partial\boldsymbol{\gamma}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\gamma}) + \mathbf{u}(\nabla \cdot \boldsymbol{\gamma}) + \frac{\nabla \times \mathbf{B}}{\mu} = 0. \tag{8.27}$$

The Euler-Lagrange equations (8.25)–(8.27) together imply Hamilton's equations:

$$\frac{\partial \rho}{\partial t} = \frac{\delta \mathcal{H}}{\delta \phi}, \quad \frac{\partial \phi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \rho}, \quad \frac{\partial S}{\partial t} = \frac{\delta \mathcal{H}}{\delta \beta}, \quad \frac{\partial \beta}{\partial t} = -\frac{\delta \mathcal{H}}{\delta S},$$

$$\frac{\partial \mu}{\partial t} = \frac{\delta \mathcal{H}}{\delta \lambda}, \quad \frac{\partial \lambda}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \mu}, \quad \frac{\partial \mathbf{A}}{\partial t} = \frac{\delta \mathcal{H}}{\delta \boldsymbol{\gamma}}, \quad \frac{\partial \boldsymbol{\gamma}}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \mathbf{A}}. \tag{8.28}$$

Here $\{\rho, \phi\}$, $\{S, \beta\}$, $\{\mu, \lambda\}$, and $\{\mathbf{A}, \boldsymbol{\gamma}\}$ are canonically conjugate variables. The Hamiltonian functional $\mathcal{H}$ is given by (8.12), and $\mathbf{u}$ is given by the Clebsch expansion (8.25). The canonical Poisson bracket is:

$$\{F, G\} = \int d^3x \left(\frac{\delta F}{\delta \rho} \frac{\delta G}{\delta \phi} - \frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \rho} + \frac{\delta F}{\delta \mathbf{A}} \cdot \frac{\delta G}{\delta \boldsymbol{\gamma}} - \frac{\delta F}{\delta \boldsymbol{\gamma}} \cdot \frac{\delta G}{\delta \mathbf{A}} \right.$$

$$\left. + \frac{\delta F}{\delta S} \frac{\delta G}{\delta \beta} - \frac{\delta F}{\delta \beta} \frac{\delta G}{\delta S} + \frac{\delta F}{\delta \mu} \frac{\delta G}{\delta \lambda} - \frac{\delta F}{\delta \lambda} \frac{\delta G}{\delta \mu} \right). \tag{8.29}$$

The transformations of the variational derivatives from canonical Clebsch variables $(\rho, \phi, S, \beta, \mathbf{A}, \boldsymbol{\gamma})$ in terms of the non-canonical new variables $(\rho, \sigma, \mathbf{A}, \mathbf{M})$ are:

$$\frac{\delta F}{\delta \rho} = \frac{\delta F}{\delta \rho} + S \frac{\delta F}{\delta \sigma} + \frac{\delta F}{\delta \mathbf{M}} \cdot \nabla \phi, \quad \frac{\delta F}{\delta \phi} = -\nabla \cdot \left(\rho \frac{\delta F}{\delta \mathbf{M}} \right),$$

$$\frac{\delta F}{\delta S} = \rho \frac{\delta F}{\delta \sigma} + \nabla \cdot \left(\beta \frac{\delta F}{\delta \mathbf{M}} \right), \quad \frac{\delta F}{\delta \beta} = -\frac{\delta F}{\delta \mathbf{M}} \cdot \nabla S,$$

$$\frac{\delta F}{\delta \mu} = \nabla \cdot \left(\lambda \frac{\delta F}{\delta \mathbf{M}} \right), \quad \frac{\delta F}{\delta \lambda} = -\nabla \mu \cdot \frac{\delta F}{\delta \mathbf{M}},$$

$$\frac{\delta F}{\delta \mathbf{A}} = \frac{\delta F}{\delta \mathbf{A}} + \nabla \cdot \boldsymbol{\gamma} \frac{\delta F}{\delta \mathbf{M}} - \nabla \times \left(\frac{\delta F}{\delta \mathbf{M}} \times \boldsymbol{\gamma} \right),$$

$$\frac{\delta F}{\delta \boldsymbol{\gamma}} = -\mathbf{B} \times \frac{\delta F}{\delta \mathbf{M}} - \nabla \left[\mathbf{A} \cdot \left(\frac{\delta F}{\delta \mathbf{M}} \right) \right], \tag{8.30}$$

we obtain the non-canonical Poisson bracket:

$$\{F, G\} = -\int d^3x \left\{ [F_{\mathbf{M}} \cdot \nabla(G_{\mathbf{M}}) - G_{\mathbf{M}} \cdot \nabla(F_{\mathbf{M}})] \cdot \mathbf{M} \right.$$

$$+ \rho \left[F_{\mathbf{M}} \cdot \nabla(G_\rho) - G_{\mathbf{M}} \cdot \nabla(F_\rho) \right]$$

$$+ \sigma \left[F_{\mathbf{M}} \cdot \nabla(G_\sigma) - G_{\mathbf{M}} \cdot \nabla(F_\sigma) \right]$$

$$+ \mathbf{A} \cdot \left[F_{\mathbf{M}} \nabla \cdot (G_{\mathbf{A}}) - G_{\mathbf{M}} \nabla \cdot (F_{\mathbf{A}}) \right]$$

$$\left. + \nabla \times \mathbf{A} \cdot \left[G_{\mathbf{A}} \times F_{\mathbf{M}} - F_{\mathbf{A}} \times G_{\mathbf{M}} \right] \right\}, \tag{8.31}$$

where $F_\mathbf{M} \equiv \delta F/\delta \mathbf{M}$ and similarly for the other variational derivatives in (8.31). The non-canonical bracket (8.31) was obtained by Holm and Kupershmidt (1983a,b). It is a skew symmetric bracket and satisfies the Jacobi identity.

8.3 Differences Between the MHD Brackets

The differences between MHD brackets are subtle. Both Holm and Kupershmidt (1983a,b) and Marsden et al. (1984) discussed the form of the brackets using semi-direct product Lie algebraic approaches using the Lie-Poisson bracket which implies that the Jacobi bracket is satisfied.

Chandre et al. (2012) developed the theory of non-canonical Poisson brackets in incompressible fluids and MHD by using Dirac brackets. They use projectors $\mathcal{P}$ which split vector fields in 3D space up into a solenoidal and an irrotational part (e.g. Panofsky and Phillips 1964, Ch. 1). The vector field $\mathbf{V}$ is written in the form $\mathbf{V} = \nabla \times \mathbf{a} + \nabla \phi$ where $\mathcal{P}(\mathbf{V}) = \nabla \times \mathbf{a}$. Chandre (2013) and Chandre et al. (2013) study the relationship between projectors and Dirac brackets, for both incompressible and compressible MHD. Below, we discuss in more detail the different MHD brackets.

For brackets with $\nabla \cdot \mathbf{B} = 0$,

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla \cdot \mathbf{B} = 0. \tag{8.32}$$

We show how (8.32) and the variational derivatives $F_\mathbf{A}$ and $F_\mathbf{B}$ can be described by using the projector formalism of Chandre et al. (2012, 2013) and Chandre (2013). Taking the curl of the first equation (8.32) gives:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \tag{8.33}$$

where $\mathbf{J}$ is the electric current. Equation (8.33) can be written in the form:

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}, \tag{8.34}$$

or as:

$$\mathcal{P}\mathbf{A} = -\nabla^{-2}(\mu_0 \mathbf{J}) \equiv \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, d\mathbf{x}, \tag{8.35}$$

where ∇^{-2} is the inverse Laplacian, used in solving Poisson's equation. Here:

$$\mathcal{P} = \mathsf{I} - \nabla^{-2}\nabla[\nabla(\cdot)] \equiv \mathsf{I} - \nabla[\nabla^{-2}\nabla \cdot (\cdot)], \tag{8.36}$$

is the projection tensor that maps a 3D vector field $\mathbf{V}$ onto its solenoidal part, and I is the unit 3×3 dyadic. The kernel of the operator $\mathcal{P}$. is defined as

$$\mathrm{Ker}(\mathcal{P}) = \{\mathbf{V} : \mathcal{P}\mathbf{V} = 0\}. \tag{8.37}$$

For vector fields $\mathbf{V}$ that vanish sufficiently fast as $|\mathbf{x}| \rightarrow \infty$, with specified divergence Q and curl $\boldsymbol{\gamma}$ sources:

$$\nabla \cdot \mathbf{V} = Q \quad \text{and} \quad \nabla \times \mathbf{V} = \boldsymbol{\gamma}, \tag{8.38}$$

the vector field $\mathbf{V}$ has the form:

$$\mathbf{V} = \nabla \times \mathbf{a} + \nabla\phi. \tag{8.39}$$

From (8.39)

$$\begin{aligned}
\nabla \times \mathbf{V} &= \nabla \times (\nabla \times \mathbf{a}) = -\nabla^2(\mathcal{P}\mathbf{a}) = \boldsymbol{\gamma}, \\
\nabla \cdot \mathbf{V} &= \nabla^2\phi = Q,
\end{aligned} \tag{8.40}$$

give the curl and divergence of $\mathbf{V}$ respectively. From (8.40)

$$\mathcal{P}\mathbf{a} = -\nabla^{-2}(\boldsymbol{\gamma}), \quad \phi = \nabla^{-2}(Q), \tag{8.41}$$

are the solutions for $\mathcal{P}\mathbf{a}$ and ϕ. Note that the Laplacian operator $\Delta = \nabla^2$ and $\Delta^{-1} = \nabla^{-2}$.

Chandre et al. (2012) list some of the properties of $\mathcal{P}$, namely:

$$\begin{aligned}
\mathcal{P}^2\mathbf{V} &= \mathcal{P}\mathbf{V}, \quad \mathcal{P} \cdot \nabla\mathbf{V} = 0, \quad \mathcal{P} \cdot \nabla \times \mathbf{V} = \nabla \times \mathbf{V}, \\
(\nabla \times \mathcal{P})\mathbf{V} &= \nabla \times \mathbf{V}, \quad (\nabla \cdot \mathcal{P})\mathbf{V} = \nabla \cdot (\mathcal{P}\mathbf{V}) = 0.
\end{aligned} \tag{8.42}$$

The projection operator $\mathcal{P}$ is self adjoint:

$$\int d^3x \, \mathbf{a} \cdot \mathcal{P} \cdot \mathbf{b} = \int d^3x \, \mathbf{b} \cdot \mathcal{P} \cdot \mathbf{a}. \tag{8.43}$$

By noting that

$$\int F_{\mathbf{A}} \cdot \delta\mathbf{A} \, d\mathbf{x} = \int F_{\mathbf{B}} \cdot \delta\mathbf{B} \, d\mathbf{x}, \quad \delta\mathbf{B} = \nabla \times \delta\mathbf{A}, \tag{8.44}$$

integrating by parts, and dropping surface terms gives,

$$\int F_{\mathbf{A}} \cdot \delta\mathbf{A} \, d\mathbf{x} = \int \nabla \times F_{\mathbf{B}} \cdot \delta\mathbf{A} \, d\mathbf{x}, \tag{8.45}$$

which suggests:

$$F_{\mathbf{A}} = \nabla \times F_{\mathbf{B}}. \tag{8.46}$$

However, if $\nabla \cdot F_{\mathbf{A}} \neq 0$, then (8.46) cannot be true. For general $F_{\mathbf{A}}$, (8.45) is satisfied if:

$$\mathcal{P} F_{\mathbf{A}} = \nabla \times F_{\mathbf{B}}. \tag{8.47}$$

Chandre et al. (2012) derived (8.47) by setting $F_{\mathbf{A}} \to F_{\mathbf{A}} + \mathbf{\Upsilon}$ and then determined $\mathbf{\Upsilon}$ by requiring $\nabla \cdot (F_{\mathbf{A}} + \mathbf{\Upsilon}) = 0$. Note that $\nabla \cdot (\mathcal{P} F_{\mathbf{A}}) = 0$. By noting that

$$\nabla \times (\mathcal{P} F_{\mathbf{B}}) = \nabla \times \left\{ F_{\mathbf{B}} - \nabla \left[\nabla^{-2} (\nabla \cdot F_{\mathbf{B}}) \right] \right\} = \nabla \times F_{\mathbf{B}}, \tag{8.48}$$

(8.47) can also be written in the form:

$$\mathcal{P} F_{\mathbf{A}} = \nabla \times (\mathcal{P} F_{\mathbf{B}}). \tag{8.49}$$

Starting from the Holm and Kupershmidt (1983a,b) bracket (8.31), and naively using the transformations (8.46) and $\mathbf{B} = \nabla \times \mathbf{A}$ we obtain the Morrison and Greene (1980) bracket for $\nabla \cdot \mathbf{B} = 0$ in the form:

$$
\begin{aligned}
\{F, G\}_{80} = -\int d^3x \bigg\{ & [F_{\mathbf{M}} \cdot \nabla(G_{\mathbf{M}}) - G_{\mathbf{M}} \cdot \nabla(F_{\mathbf{M}})] \cdot \mathbf{M} \\
& + \rho \left[F_{\mathbf{M}} \cdot \nabla G_\rho - G_{\mathbf{M}} \cdot \nabla F_\rho \right] \\
& + \sigma \left[F_{\mathbf{M}} \cdot \nabla(G_\sigma) - G_{\mathbf{M}} \cdot \nabla(F_\sigma) \right] \\
& + \mathbf{B} \cdot \left[(F_{\mathbf{M}} \cdot \nabla) G_{\mathbf{B}} - (G_{\mathbf{M}} \cdot \nabla) F_{\mathbf{B}} \right] \\
& + \mathbf{B} \cdot \left[\nabla(F_{\mathbf{B}}) \cdot G_{\mathbf{M}} - \nabla(G_{\mathbf{B}}) \cdot F_{\mathbf{M}} \right] \bigg\}.
\end{aligned}
\tag{8.50}
$$

The magnetic part of the Poisson bracket (8.50) can be written in the form:

$$\{F, G\}_{80}^B = -\int d^3x \bigg\{ F_{\mathbf{M}} \cdot \mathbf{B} \times (\nabla \times G_{\mathbf{B}}) - G_{\mathbf{M}} \cdot \mathbf{B} \times (\nabla \times F_{\mathbf{B}}) \bigg\} \, d^3x, \tag{8.51}$$

The Morrison and Greene (1980) bracket (8.50) is tainted because the Jacobi identity is satisfied only if $\nabla \cdot \mathbf{B} = 0$ (see later in this chapter). Chandre (2013) and Chandre et al. (2013) use the modified bracket for $\mathbf{B}$ in which

$$\mathbf{B} \to \tilde{\mathbf{B}} = \mathbf{B} - \nabla \Delta^{-1} \nabla \cdot \mathbf{B} = \mathcal{P} \mathbf{B}, \tag{8.52}$$

in $\{F, G\}_{80}^{B}$ in (8.51). This results in the bracket:

$$\{F, G\}_{80}^{\tilde{B}} = -\int d^3x \left\{ F_{\mathbf{M}} \cdot \tilde{\mathbf{B}} \times (\nabla \times G_{\mathbf{B}}) - G_{\mathbf{M}} \cdot \tilde{\mathbf{B}} \times (\nabla \times F_{\mathbf{B}}) \right\} d^3x, \quad (8.53)$$

Bracket (8.53) gives a modified version of the Morrison and Greene (1980) bracket which satisfies the Jacobi identity unconditionally. This latter bracket also has $\nabla \cdot \mathbf{B}$ as a Casimir. The magnetic component of the Poisson bracket (8.53) can also be written in the form:

$$\{F, G\}_{80}^{\tilde{B}} = \{F, G\}_{80}^{B} + \int d^3x \, \nabla \cdot \mathbf{B} \Delta^{-1}(\nabla \cdot \mathbf{C}), \quad (8.54)$$

where

$$\mathbf{C} = F_{\mathbf{M}} \times (\nabla \times G_{\mathbf{B}}) - G_{\mathbf{M}} \times (\nabla \times F_{\mathbf{B}}). \quad (8.55)$$

Thus, the effect of using the transformation $\mathbf{B} \rightarrow \mathcal{P}\mathbf{B}$ is to add an integrand term proportional to $\nabla \cdot \mathbf{B}$ to the Poisson bracket (8.50) (see Chandre et al. 2013, Equation (19)).

The corresponding Poisson bracket of Morrison and Greene (1982) with $\nabla \cdot \mathbf{B} \neq 0$ is given by (8.18) (hereafter denoted by $\{F, G\}_{82}$). The difference between these two brackets is given by:

$$\{F, G\}_{82} - \{F, G\}_{80} = \int d^3x \, \nabla \cdot \mathbf{B} \left[F_{\mathbf{M}} \cdot G_{\mathbf{B}} - F_{\mathbf{B}} \cdot G_{\mathbf{M}} \right]. \quad (8.56)$$

For the $\{F, G\}_{82}$ bracket, with $\nabla \cdot \mathbf{B} \neq 0$, the MHD momentum equation can be written in the conservative form:

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot \left(\rho \mathbf{u} \otimes \mathbf{u} + \left(p + \frac{B^2}{2\mu} \right) \mathsf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu} \right) = 0, \quad (8.57)$$

Equation (8.57) can also be written in the form:

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mathbf{J} \times \mathbf{B} + \frac{\mathbf{B}(\nabla \cdot \mathbf{B})}{\mu}, \quad (8.58)$$

where $\mathbf{J} = \nabla \times \mathbf{B}/\mu$ is the current.

Faraday's equation for the Morrison and Greene (1982) bracket for $\nabla \cdot \mathbf{B} \neq 0$ may be written in the form:

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}(\nabla \cdot \mathbf{B}) = 0. \quad (8.59)$$

It corresponds to Lie dragging the magnetic flux $\int \mathbf{B} \cdot d\mathbf{S}$ with the flow. Taking the divergence of (8.59) gives the magnetic flux conservation law:

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) + \nabla \cdot [\mathbf{u}(\nabla \cdot \mathbf{B})] = 0, \tag{8.60}$$

which shows that $\nabla \cdot \mathbf{B}$ is advected with the flow (this is sometimes referred to as the divergence wave eigenmode, e.g. Webb et al. 2009; Powell et al. 1999). For the Morrison and Greene (1980) bracket the momentum equation has the form (8.58) and Faraday's equation has the form (8.59), but with $\nabla \cdot \mathbf{B} = 0$.

8.4 The Jacobi Identity

In this section, we derive the Jacobi identity for the Morrison and Greene (1982) bracket for $\nabla \cdot \mathbf{B} \neq 0$. The Jacobi identity for $\nabla \cdot \mathbf{B} = 0$ can then be obtained by setting $\nabla \cdot \mathbf{B} = 0$ after all the calculations are done (one can also take the limit as $\nabla \cdot \mathbf{B} \to 0$ if desired after the calculation is done). We prove the Jacobi identity for the bracket (8.18) for the general case where $\nabla \cdot \mathbf{B} \neq 0$ by using the functional multi-vector approach of Olver (1993, Chapter 7, Theorem 7.18). We use the same method to verify the Jacobi identity for the advected $\mathbf{A}$ bracket, and then convert the advected $\mathbf{A}$ bracket to the bracket (8.18). Morrison (1982) gives a direct approach to the derivation of the Jacobi identity that is applicable for a restricted range of Hamiltonian operators or co-symplectic forms. However, this class of co-symplectic forms is sufficient for a wide class of fluid and plasma models.

8.4.1 The Morrison and Greene (1982) Bracket

In the functional multi-vector approach to the Jacobi identity of Olver (1993), one first writes the bracket in the standard, co-symplectic form:

$$\{F, G\} = -\int F_\alpha \mathcal{D}_{\alpha\beta} G_\beta \, d\mathbf{x}, \tag{8.61}$$

where $\mathcal{D}_{\alpha\beta}$ is the co-symplectic form or Hamiltonian operator and $F_\alpha = \delta F/\delta u^\alpha$ is the variational derivative of the functional F with respect to the dependent field variable u^α.

The Poisson bracket (8.61) is required to satisfy the conditions of skew-symmetry:

$$\{F, G\} = -\{G, F\}, \tag{8.62}$$

and to satisfy the Jacobi identity:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0, \tag{8.63}$$

for all admissible functionals F, G and H. These conditions are required for $\mathcal{D}_{\alpha\beta}$ to define a Hamiltonian operator (co-symplectic form) and Poisson bracket (8.61). From (8.61) these conditions imply that the bracket (8.61) is bilinear in both arguments:

$$\begin{aligned}
\{c_1 F + c_2 G, H\} &= c_1 \{F, H\} + c_2 \{G, H\}, \\
\{F, d_1 G + d_2 H\} &= d_1 \{F, G\} + d_2 \{F, H\}.
\end{aligned} \tag{8.64}$$

The Leibniz rule for a finite dimensional Hamiltonian system (e.g. Olver 1993, Ch. 6), is not required for the infinite dimensional Poisson bracket (8.61).

Using the non-canonical, MHD Poisson bracket (8.18) where $u^\alpha = (\rho, \mathbf{M}^T, \sigma, \mathbf{B}^T)^T$ are the dependent variables, we obtain:

$$\begin{aligned}
\{F, G\} = -\int \Bigg\{ & \left(F_\rho \mathcal{D}_{\rho M^j} G_{M^j} + F_{M^j} \mathcal{D}_{M^j \rho} G_\rho \right) \\
& + \left(F_\sigma \mathcal{D}_{\sigma M^j} G_{M^j} + F_{M^j} \mathcal{D}_{M^j \sigma} G_\sigma \right) \\
& + \left(F_{M^i} \mathcal{D}_{M^i M^j} G_{M^j} + F_{M^i} \mathcal{D}_{M^i B^j} G_{B^j} + F_{B^i} \mathcal{D}_{B^i M^j} G_{M^j} \right) \Bigg\} d\mathbf{x},
\end{aligned} \tag{8.65}$$

where $\mathcal{D}_{\alpha\beta}$ in (8.65) are given by:

$$\begin{aligned}
\mathcal{D}_{\rho M^j} &= D_j(\rho \cdot), \quad \mathcal{D}_{M^j \rho} = \rho D_j, \\
\mathcal{D}_{\sigma M^j} &= D_j(\sigma \cdot), \quad \mathcal{D}_{M^j \sigma} = \sigma D_j, \\
\mathcal{D}_{M^i M^j} &= D_j \left(M^i \cdot \right) + M^j D_i(\cdot), \\
\mathcal{D}_{M^i B^j} &= B^j D_i(\cdot) - \delta_{ij} \left[\nabla \cdot \mathbf{B} + B^s D_s \right] (\cdot), \\
\mathcal{D}_{B^i M^j} &= D_j(B^i \cdot) - \delta_{ij} B^s D_s(\cdot).
\end{aligned} \tag{8.66}$$

Here we use the notation $D_i \equiv \nabla_i$, which is the total partial derivative with respect to $\mathbf{x}^i$ (i.e. we use Olver's notation).

Below we verify the Jacobi identity (8.63) for the bracket (8.61) where the $\mathcal{D}_{\alpha\beta}$ are given by (8.66). Olver (1993) shows that the Jacobi identity (8.63) is equivalent to the functional bi-vector Lie derivative condition:

$$\mathrm{pr} V_{\mathcal{D}\theta}(\Theta) \equiv \frac{1}{2} \int V_{\mathcal{D}\theta} \wedge \left(\theta^\alpha \wedge \mathcal{D}_{\alpha\beta} \theta^\beta \right) d\mathbf{x} = 0, \tag{8.67}$$

where

$$\theta^\alpha = \left(\theta^\rho, \theta^{\mathbf{M}}, \theta^\sigma, \theta^{\mathbf{B}}\right), \tag{8.68}$$

are 1-form, dual basis elements of the cotangent space, T_Q^* dual to the tangent vector space T_Q. T_Q is spanned by the vector fields $\mathbf{e}_\alpha = \partial/\partial u^\alpha$, and $V_Q = V_{Q^\alpha} \mathbf{e}_\alpha$ live in the tangent space T_Q. The vector valued 2-form in (8.67) is defined as:

$$\Theta = \frac{1}{2} \int \theta^\alpha \wedge \mathcal{D}_{\alpha\beta} \theta^\beta \, d\mathbf{x}, \tag{8.69}$$

where $\wedge$ is the usual skew symmetric operator acting on forms. The symbol prV_Q denotes the prolonged vector field V_Q or extended evolutionary symmetry operator, defined by the equation:

$$\mathrm{pr}\left(V_Q\right) = Q^\alpha \frac{\partial}{\partial u^\alpha} + D_{x^s}\left(Q^\alpha\right) \frac{\partial}{\partial u^\alpha_{,s}} + D_{x^s} D_{x^p}\left(Q^\alpha\right) \frac{\partial}{\partial u^\alpha_{,sp}} + \ldots, \tag{8.70}$$

where

$$Q^\alpha = \mathcal{D}_{\alpha\beta} \theta^\beta, \tag{8.71}$$

is the vector-field valued one form, associated with the co-symplectic form $\mathcal{D}_{\alpha\beta}$. In (8.70)

$$u^\alpha_{,s} = D_{x^s}(u^\alpha), \quad u^\alpha_{,sp} = D_{x^s} D_{x^p}(u^\alpha) \equiv \frac{\partial u^\alpha}{\partial x^s \partial x^p}, \ldots, \tag{8.72}$$

denote the partial derivatives of the u^α with respect to the x^k.

It is straightforward to verify the skew-symmetry of the co-symplectic form $\mathcal{D}_{\alpha\beta}$ by using integration by parts, i.e.

$$(\mathcal{D}_{\alpha\beta})^\dagger = -\mathcal{D}_{\alpha\beta}, \tag{8.73}$$

where $\mathcal{D}^\dagger_{\alpha\beta}$ is the adjoint of the operator $\mathcal{D}_{\alpha\beta}$. As a consequence, the bracket (8.61) is skew-symmetric, i.e. $\{F, G\} = -\{G, F\}$. The derivation of (8.67) is given in Olver (1993, Chapter 7 (Proposition 7.7 and Theorem 7.18, pp. 443–444)). The one-forms θ^α are dual to the vector field $\partial/\partial u^\alpha$ i.e.,

$$\left\langle \frac{\partial}{\partial u^\alpha}, \theta^\beta \right\rangle = \delta^\beta_\alpha. \tag{8.74}$$

where $\langle \, , \, \rangle$ denotes the pairing (inner product) between the basis vector fields $\partial/\partial u^\alpha$ and 1-forms θ^β. In (8.67) it is implicitly assumed that the integrand is a sum of perfect derivatives that vanish on the boundary ∂R of the integration region R. The

boundaries ∂R could be at $|\mathbf{x}| \to \infty$, or periodic boundary conditions are used in the case of a periodic domain R.

In the above translation of the Jacobi identity into the form (8.67), one needs to keep in mind the map between the tangent space of vector fields T_M to the manifold M and the co-tangent space or dual vector space T_M^* of differential forms. If ω^1, $\omega^2 \dots \omega^n$ are elements of T_M^* and V_1, $V_2 \dots V_n$ are vector fields in T_M then the inner product map:

$$\langle \omega^1 \wedge \omega^2, \dots \wedge \omega^n; V_1, V_2, \dots V_n \rangle = \det(a_{ij}) \equiv \det(\langle \omega^i; V_j \rangle) \tag{8.75}$$

maps the n-form $\omega^1 \wedge \omega^2 \dots \wedge \omega^n$ onto the reals. Furthermore, the Poisson bracket $\{\mathcal{P}, \mathcal{R}\}$ is defined by the inner product:

$$
\begin{aligned}
\langle P, R; \Theta \rangle &= \frac{1}{2} \int \frac{\delta \mathcal{P}}{\delta u^\alpha} \wedge \mathcal{D}_{\alpha\beta} \frac{\delta \mathcal{R}}{\delta u^\beta} \, d\mathbf{x} \\
&= \frac{1}{2} \int \left(P^\alpha \mathcal{D}_{\alpha\beta} R^\beta - R^\alpha \mathcal{D}_{\alpha\beta} P^\beta \right) \, d\mathbf{x} \\
&= \int \frac{\delta \mathcal{P}}{\delta u^\alpha} \mathcal{D}_{\alpha\beta} \frac{\delta \mathcal{R}}{\delta u^\beta} d\mathbf{x} \\
&\equiv \{\mathcal{P}, \mathcal{R}\},
\end{aligned}
\tag{8.76}
$$

where

$$P^\alpha = \frac{\delta \mathcal{P}}{\delta u^\alpha} = E_{u^\alpha}(\mathcal{P}), \quad R^\alpha = \frac{\delta \mathcal{R}}{\delta u^\alpha} = E_{u^\alpha}(\mathcal{R}), \tag{8.77}$$

are the variational derivatives of $\mathcal{P}$ and $\mathcal{R}$, and E_{u^α} is the Euler operator.

Effectively, (8.67) is the Jacobi identity in which direct reference to the functionals $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ have been stripped away from the identity. One can of course, include the variational derivatives of the functionals $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ back into the formulation, but this is best done at the end of the calculation. We illustrate this process later in our investigation of the Jacobi identity for the Morrison and Greene (1980) bracket in (8.113) et seq.

To evaluate the integral (8.67) modulo perfect derivative terms, we first note that:

$$
\begin{aligned}
\mathrm{pr} V_{\mathcal{D}\theta}(\Theta) &\equiv \frac{1}{2} \int V_{\mathcal{D}\theta} \wedge \left(\theta^\alpha \wedge \mathcal{D}_{\alpha\beta} \theta^\beta \right) d\mathbf{x} \\
&= \frac{1}{2} \int \left\{ Q^\rho \frac{\partial}{\partial \rho} + Q^\sigma \frac{\partial}{\partial \sigma} + Q^{M^k} \frac{\partial}{\partial M^k} + Q^{B^k} \frac{\partial}{\partial B^k} \right. \\
&\quad + D_{x^s} \left(Q^\rho \right) \frac{\partial}{\partial \rho_{,s}} + D_{x^s} \left(Q^\sigma \right) \frac{\partial}{\partial \sigma_{,s}}
\end{aligned}
$$

$$+ D_{x^s} \left(Q^{M^k} \right) \frac{\partial}{\partial M_{,s}^k} + D_{x^s} \left(Q^{B^k} \right) \frac{\partial}{\partial B_{,s}^k} + \cdots \Big\}$$

$$\times \left(\theta^\rho \wedge Q^\rho + \theta^\sigma \wedge Q^\sigma + \theta^{M^P} \wedge Q^{M^P} + \theta^{B^P} \wedge Q^{B^P} \right) \, d\mathbf{x}, \quad (8.78)$$

where

$$Q^\rho = \mathcal{D}_{\rho M^k} \left(\theta^{M^k} \right) = \nabla \cdot \left(\rho \theta^{\mathbf{M}} \right),$$

$$Q^\sigma = \mathcal{D}_{\sigma M^k} \left(\theta^{M^k} \right) = \nabla \cdot \left(\sigma \theta^{\mathbf{M}} \right),$$

$$Q^{M^i} = \mathcal{D}_{M^i \rho} \theta^\rho + \mathcal{D}_{M^i \sigma} \theta^\sigma + \mathcal{D}_{M^i M^j} \theta^{M^j} + \mathcal{D}_{M^i B^j} \theta^{B^j}$$

$$\equiv \rho D_i \theta^\rho + \sigma D_i \theta^\sigma + D_j \left(M^i \theta^{M^j} \right) + M^j D_i \left(\theta^{M^j} \right)$$

$$+ B^j D_i \theta^{B^j} - [\nabla \cdot \mathbf{B} + B_s D_s] \, \theta^{B^i},$$

$$Q^{B^i} = D_j \left(B^i \theta^{M^j} \right) - B^s D_s \theta^{M^i}, \quad (8.79)$$

are the components of the Q^α. Here we use the abbreviated notation $D_s \equiv D_{x^s}$. Note that we need up to first order derivatives in pr(V_Q) in (8.78) as Q depends at most on first order derivatives and the function values of the u^α. The Q^α are obtained from (8.71).

From (8.78) to (8.79) we obtain:

$$\mathrm{pr} \left[V_{\mathcal{D}\theta}(\Theta) \right] = T_\rho + T_M + T_\sigma + T_B, \quad (8.80)$$

where T_ψ consists of terms linear in ψ and $\nabla \psi$ (here $\psi = \rho, \mathbf{M}, \sigma$, or $\mathbf{B}$). Thus, the task of verifying the Jacobi identity reduces to the evaluation of the different terms (8.80). This objective is achieved by evaluating the integrals in (8.78), integrating by parts, and dropping surface terms (assumed to vanish).

The term T_ρ has the form:

$$T_\rho = \frac{1}{2} \int \Big\{ \nabla \cdot \left(\rho \theta^{\mathbf{M}} \right) \wedge \left(\nabla \cdot \theta^{\mathbf{M}} \wedge \theta^\rho + \theta_{,p}^\rho \wedge \theta^{M^P} \right)$$

$$+ \rho \theta_{,k}^\rho \wedge \left(\nabla \cdot \theta^{\mathbf{M}} \wedge \theta^{M^k} + \theta_{,p}^{M^k} \wedge \theta^{M^P} \right)$$

$$+ D_j [\nabla \cdot \left(\rho \theta^{\mathbf{M}} \right)] \wedge \theta^{M^j} \wedge \theta^\rho \Big\} \, d\mathbf{x}. \quad (8.81)$$

After some integration by parts and dropping surface terms we obtain:

$$T_\rho = -\frac{1}{2} \int \theta^\rho_{,k} \wedge D_j \left(\rho \theta^{M^j} \wedge \theta^{M^k} \right) d\mathbf{x}$$

$$\equiv \frac{1}{4} \int \rho \theta^\rho_{,kj} \wedge \left(\theta^{M^j} \wedge \theta^{M^k} + \theta^{M^k} \wedge \theta^{M^j} \right) d\mathbf{x} = 0. \tag{8.82}$$

T_ρ in the second line of (8.82) is zero by the antisymmetry of the terms in braces, whereas the $\theta^\rho_{,jk}$ derivative is symmetric with respect to x^j and x^k. T_σ is also given by (8.82) except that ρ is replaced by σ in (8.82), i.e. $T_\sigma = 0$.

A similar calculation for T_M gives:

$$T_M = \int \theta^{M^k}_{,p} \wedge D_j \left(M^k \theta^{M^j} \wedge \theta^{M^p} \right) d\mathbf{x}$$

$$\equiv -\frac{1}{2} \int M^k \theta^{M^k}_{,pj} \wedge \left(\theta^{M^j} \wedge \theta^{M^p} + \theta^{M^p} \wedge \theta^{M^j} \right) d\mathbf{x} = 0. \tag{8.83}$$

Note that T_M is (8.83) has a similar form to T_ρ in (8.82).

Both Q^ρ and Q^σ do not contribute to T_B. Only linear terms in $\mathbf{B}$ and first order derivatives of $\mathbf{B}$ contribute to T_B. We find:

$$T_B = \frac{1}{2} \int \left\{ \theta^{M^p} \wedge \tilde{X}_B \left(Q^{M^p} \right) + \theta^{B^p} \wedge \tilde{X}_B \left(Q^{B^p} \right) \right\} d\mathbf{x}, \tag{8.84}$$

where

$$\tilde{X}_B = \hat{Q}^{M^k} \frac{\partial}{\partial M^k} + Q^{B^k} \frac{\partial}{\partial B^k} + D_{x^s} \left(\hat{Q}^{M^k} \right) \frac{\partial}{\partial M^k_{,s}} + D_{x^s} \left(Q^{B^k} \right) \frac{\partial}{\partial B^k_{,s}}. \tag{8.85}$$

In (8.85):

$$\hat{Q}^{M^i} = B^j \theta^{B^j}_{,i} - (\nabla \cdot \mathbf{B} + B^s D_s) \theta^{B^i}, \tag{8.86}$$

is the component of Q^{M^i} that is independent of ρ, σ and M^k. After some algebra, (i.e. integrating by parts and dropping surface terms, and using the anti-symmetry of the wedge product), we obtain:

$$T_B = \int \left\{ Q^{B^s} \wedge \left(\theta^{B^p} \wedge \theta^{M^p}_{,s} - \theta^{M^p} \wedge \theta^{B^s}_{,p} \right) - \hat{Q}^{M^p} \wedge \theta^{M^j} \wedge \theta^{M^p}_j \right\} d\mathbf{x}. \tag{8.87}$$

Using (8.86) for $\hat{Q}^{M^i}$ and (8.79) for Q^{B^i} in (8.87) and integrating by parts, and dropping surface terms, gives:

$$T_B = T^1_B + T^2_B = 0, \tag{8.88}$$

where

$$T_B^1 = \int \left\{ B^a \theta^{Bp} \wedge \left(\theta_{,a}^{Ms} \wedge \theta_{,s}^{Mp} + \theta_{,s}^{Mp} \wedge \theta_{,a}^{Ms} \right) \right.$$

$$\left. + B^a \theta^{Mp} \wedge \left(\theta_{,p}^{Ms} \wedge \theta_{,s}^{Ba} + \theta_{,s}^{Ba} \wedge \theta_{,p}^{Ms} \right) \right\} d\mathbf{x}$$

$$\equiv 0, \tag{8.89}$$

depends on the first derivatives of θ^α. By using the anti-symmetry of the wedge product it follows that $T_B^1 = 0$. The term T_B^2 is given by:

$$T_B^2 = \int \left\{ - B^s \theta^{Ma} \wedge \theta^{Mp} \wedge \theta_{,pa}^{Bs} \right.$$

$$\left. - B^s \left(\theta^{Ma} \wedge \theta^{Bp} + \theta^{Bp} \wedge \theta^{Ma} \right) \wedge \theta_{,as}^{Mp} \right\} d\mathbf{x} \equiv 0. \tag{8.90}$$

T_B^2 depends on second order derivatives of θ^α. By the anti-symmetry of the wedge product, and noting that $\theta_{,ab}^\alpha = \theta_{,ba}^\alpha$, it follows that $T_B^2 = 0$. Thus, $T_B = T_B^1 + T_B^2 = 0$ which verifies (8.88).

To sum up, $\mathrm{pr}[V_{\mathcal{D}\theta}(\Theta)] = 0$ because T_ρ, T_M, T_σ and T_B are all zero. It follows from Olver (1993, Proposition 7.7 and Theorem 7.18), that the Jacobi identity for the Morrison and Greene (1982) bracket (8.18) with $\nabla \cdot \mathbf{B} \neq 0$ is satisfied.

8.4.2 The Advected A Bracket

In this section, we show that the advected **A** bracket (8.31) of Holm and Kupershmidt (1983a,b) satisfies the Jacobi identity. Note that for this bracket $\nabla \cdot \mathbf{B} = 0$, because $\mathbf{B} = \nabla \times \mathbf{A}$. The $\nabla \cdot \mathbf{B} = 0$ bracket of Morrison and Greene (1980) is more complicated to describe, because of the constraint $\nabla \cdot \mathbf{B} = 0$ imposed on the variations. This bracket has been investigated by Chandre et al. (2012, 2013) and Chandre (2013) using the Dirac bracket to define more precisely the conditions on the functionals allowed for this bracket. They also show that the bracket can be obtained by using a projection operator applied to the variational derivatives to ensure that the functionals F_ψ are divergence free.

We prove the Jacobi identity using Theorem 7.18 and Proposition 7.7 (Chapter 7) of Olver (1993) to simplify the analysis. The non-canonical Poisson bracket (8.31) with field variables $u^\alpha = (\rho, \mathbf{M}^T, \sigma, \mathbf{A}^T)^T$, where **A** is the advected magnetic vector

potential, can be written in the co-symplectic form:

$$
\begin{aligned}
\{F, G\}_A = -\int \Bigg\{ & \left(F_\rho \mathcal{D}_{\rho M^j} G_{M^j} + F_{M^j} \mathcal{D}_{M^j \rho} G_\rho \right) \\
& + \left(F_\sigma \mathcal{D}_{\sigma M^j} G_{M^j} + F_{M^j} \mathcal{D}_{M^j \sigma} G_\sigma \right) \\
& + \left(F_{M^i} \mathcal{D}_{M^i M^j} G_{M^j} + F_{M^i} \mathcal{D}_{M^i A^j} G_{A^j} + F_{A^i} \mathcal{D}_{A^i M^j} G_{M^j} \right) \Bigg\} d\mathbf{x},
\end{aligned}
\tag{8.91}
$$

where

$$
\begin{aligned}
\mathcal{D}_{\rho M^j} &= D_j(\rho \cdot), \quad \mathcal{D}_{M^j \rho} = \rho D_j, \\
\mathcal{D}_{\sigma M^j} &= D_j(\sigma \cdot), \quad \mathcal{D}_{M^j \sigma} = \sigma D_j, \\
\mathcal{D}_{M^i M^j} &= D_j \left(M^i \cdot \right) + M^j D_i(\cdot), \\
\mathcal{D}_{M^i A^j} &= D_j(A^i \cdot) - A^j_{,i}, \\
\mathcal{D}_{A^i M^j} &= A^j D_i(\cdot) + A^i_{,j} \cdot.
\end{aligned}
\tag{8.92}
$$

The advected **A** bracket (8.91)–(8.92) is similar to the Morrison and Greene (1982) bracket (8.65)–(8.66) except that **A** is used, rather than **B** and the skew symmetric operators $\mathcal{D}_{M^i A^j}$ and $\mathcal{D}_{A^i M^j}$ are different operators describing the magnetic vector potential **A**. The operators $\mathcal{D}_{\alpha\beta}$ in (8.92) are skew adjoint, so that $\{F, G\} = -\{G, F\}$ for the bracket (8.91).

From Olver (1993, Proposition 7.7 and Theorem 7.18 (pp. 443–444)), the proof of the Jacobi identity for the bracket (8.91)–(8.92) reduces to proving

$$
\mathrm{pr}V_{\mathcal{D}\theta}(\Theta) \equiv \frac{1}{2} \int V_{\mathcal{D}\theta} \wedge \left(\theta^\alpha \wedge \mathcal{D}_{\alpha\beta} \theta^\beta \right) \, d\mathbf{x} = 0,
\tag{8.93}
$$

where $\mathrm{pr}(V_Q)$ is defined in (8.70)–(8.71) and $Q^\alpha = \mathcal{D}_{\alpha\beta} \theta^\beta$.

Following the approach of (8.61) et seq., we obtain essentially formula (8.78) for $\mathrm{pr}V_{\mathcal{D}\theta}(\Theta)$ but with **B** replaced by **A** in (8.78). Thus, the canonical symmetry generator **Q** in the present application has components:

$$
\begin{aligned}
Q^\rho &= \nabla \cdot (\rho \theta^\mathbf{M}), \quad Q^\sigma = \nabla \cdot (\sigma \theta^\mathbf{M}), \\
Q^{M^i} &= \rho D_i \theta^\rho + \sigma D_i \theta^\sigma + D_j \left(M^i \theta^{M^j} \right) + M^j D_i \theta^{M^j} + \hat{Q}^{M^i}, \\
\hat{Q}^{M^i} &= \left(A^i_{,j} - A^j_{,i} \right) \theta^{A^j} + A^i \nabla \cdot \theta^\mathbf{A}, \\
Q^{A^i} &= A^j \theta^{M^j}_{,i} + A^i_{,j} \theta^{M^j}.
\end{aligned}
\tag{8.94}
$$

In (8.94) we have split off the **A**-dependent part of Q^{M^i}, which is denoted by $\hat{Q}^{M^i}$.

As in the previous proof of the Jacobi identity in (8.67) et seq., we split $\mathrm{pr}V_{\mathcal{D}\theta}(\Theta)$ up into ρ, M, σ and A components as:

$$\mathrm{pr}V_{\mathcal{D}\theta}(\Theta) = T_\rho + T_M + T_\sigma + T_A. \tag{8.95}$$

The terms T_ρ, T_M, T_σ are the same as in (8.81) et seq. The term T_A is linear in $\mathbf{A}$ and the first order derivatives $A^i_{,j}$. The term T_A has the form:

$$T_A = \frac{1}{2} \int \left[\theta^{M^p} \wedge \tilde{X}_A \left(Q^{M^p} \right) + \theta^{A^p} \wedge \tilde{X}_A \left(Q^{A^p} \right) \right] \, d\mathbf{x}, \tag{8.96}$$

where

$$\tilde{X}_A = \hat{Q}^{M^k} \frac{\partial}{\partial M^k} + Q^{A^k} \frac{\partial}{\partial A^k} + D_{x^s} \left(\hat{Q}^{M^k} \right) \frac{\partial}{\partial M^k_{,s}} + D_{x^s} \left(Q^{A^k} \right) \frac{\partial}{\partial A^k_{,s}}. \tag{8.97}$$

Equations (8.96)–(8.97) are analogous to (8.84)–(8.85) in the $\nabla \cdot \mathbf{B}$ bracket analysis.

$$T_A = \int \left\{ Q^{A^k} \wedge \left[\theta^{A^k} \wedge \left(\nabla \cdot \theta^{\mathbf{M}} \right) - \theta^{A^p} \wedge \theta^{M^k}_{,p} - \theta^{M^p} \wedge \theta^{A^k}_{,p} \right] \right.$$
$$\left. - \hat{Q}^{M^k} \wedge \left(\theta^{M^p} \wedge \theta^{M^k}_{,p} \right) \right\} \, d\mathbf{x}. \tag{8.98}$$

Using (8.94) for Q^{A^k} and $\hat{Q}^{M^k}$ in (8.98) and integrating by parts gives:

$$T_A = \int A^k \left\{ - D_j \left(\theta^{A^p} \wedge \theta^{M^j} + \theta^{M^p} \wedge \theta^{A^j} \right) \wedge \theta^{M^k}_{,p} \right.$$
$$+ \theta^{M^k}_{,j} \wedge \left[\theta^{A^j} \wedge \nabla \cdot \theta^{\mathbf{M}} - \theta^{A^p} \wedge \theta^{M^j}_{,p} - \theta^{M^p} \wedge \theta^{A^j}_{,p} \right]$$
$$\left. - \nabla \cdot \theta^{\mathbf{A}} \wedge \theta^{M^p} \wedge \theta^{M^k}_{,p} \right\} \, d\mathbf{x}. \tag{8.99}$$

The term T_A in (8.99) can be reduced to the form:

$$T_A = \int \left(t_{A1} + t_{A2} + t_{A3} + t_{A4} \right) \, d\mathbf{x}, \tag{8.100}$$

where

$$t_{A1} = - A^k \left(\theta^{A^p} \wedge \theta^{M^j} + \theta^{M^j} \wedge \theta^{A^p} \right) \wedge \theta^{M^k}_{,pj} = 0,$$
$$t_{A2} = A^k \theta^{M^j} \wedge \left[\theta^{A^p}_{,j} \wedge \theta^{M^k}_{,p} + \theta^{M^k}_{,p} \wedge \theta^{A^p}_{,j} \right] = 0,$$

$$t_{A3} = A^k \theta^{Aj} \wedge \left[\theta_{,j}^{Mp} \wedge \theta_{,p}^{Mk} + \theta_{,p}^{Mk} \wedge \theta_{,j}^{Mp} \right] = 0,$$

$$t_{A4} = A^k \nabla \cdot \theta^{\mathbf{A}} \wedge \theta^{Mp} \wedge \theta_{,p}^{Mk} (1 - 1) = 0. \tag{8.101}$$

Thus, $T_A = 0$. As in the previous analysis in (8.78) et seq. for the $\nabla \cdot \mathbf{B} \neq 0$ case, $T_\rho = T_\sigma = T_M = 0$, which implies from (8.95) that $\mathrm{pr} V_{\mathcal{D}\theta}(\Theta) = 0$ and that the Jacobi identity is satisfied by the advected $\mathbf{A}$ bracket (8.91).

8.4.3   The Morrison and Greene (1980) Bracket

In this section we look at the Jacobi identity for the Morrison and Greene (1980) bracket for MHD, by using Olver's version of the Jacobi identity (8.67), (here-after referred to as the MG80 bracket). The MG80 bracket has the form (8.65) except that the components of the co-symplectic form (8.66) are different for the $\mathcal{D}_{M^iB^j}$ and $\mathcal{D}_{B^iM^j}$ terms. For the MG80 bracket,

$$\mathcal{D}_{M^iB^j} = B^j D_i - \delta_{ij} B^s D_s,$$

$$\mathcal{D}_{B^iM^j} = D_j \left(B^i \cdot \right) - \delta_{ij} \left(\nabla \cdot \mathbf{B} + B^s D_s \right). \tag{8.102}$$

The other components of the co-symplectic form $\mathcal{D}_{\alpha\beta}$ are the same as in (8.66). The quantity $\mathrm{pr}[V_{\mathcal{D}\theta}(\Theta)]$ in the Jacobi identity analog $\mathrm{pr}[V_{\mathcal{D}\theta}(\Theta)] = 0$, in the present case reduces to the form (8.67), i.e.,

$$\mathrm{pr}[V_{\mathcal{D}\theta}(\Theta)] = T_\rho + T_M + T_\sigma + T_B, \tag{8.103}$$

where T_ρ, T_M and T_σ are zero as in (8.81) et seq. Thus, we find:

$$\mathrm{pr}[V_{\mathcal{D}\theta}(\Theta)] = T_B, \tag{8.104}$$

where T_B is formally given by (8.84)–(8.85), except that now

$$Q^{M^i} = \rho D_i \theta^\rho + \sigma D_i \theta^\sigma + D_j \left(M^i \theta^{Mj} \right) + M^j D_i \left(\theta^{Mj} \right) + \hat{Q}^{M^i},$$

$$\hat{Q}^{M^i} = B^j D_i \theta^{Bj} - B^s D_s \theta^{Bi},$$

$$Q^{B^i} = D_j \left(B^i \theta^{Mj} \right) - D_s \left(B^s \theta^{Mi} \right) \equiv D_s \left(B^i \theta^{Ms} - B^s \theta^{Mi} \right). \tag{8.105}$$

Note that $\hat{Q}^{M^i}$ and Q^{B^i} differ from the MG82 bracket by terms proportional to $\nabla \cdot \mathbf{B}$, namely:

$$\hat{Q}_{80}^{M^i} = \hat{Q}_{82}^{M^i} + \nabla \cdot \mathbf{B} \, \theta^{B^i}, \quad Q_{80}^{B^i} = Q_{82}^{B^i} - \nabla \cdot \mathbf{B} \, \theta^{M^i}, \tag{8.106}$$

where the subscripts 80 and 82 refer to the MG80 and MG82 brackets respectively.

The expression for T_B in the present case is obtained by using (8.84)–(8.85) where $\hat{Q}^{M^i}$ and Q^{B^i} are given by (8.105). Integrating (8.84) by parts, and dropping surface terms gives:

$$T_B = \int \left[\hat{Q}^{M^p} \wedge \theta^{M^p}_{,s} \wedge \theta^{M^s} + Q^{B^p} \wedge \theta^{M^k} \wedge \left(\theta^{B^k}_{,p} - \theta^{B^p}_{,k} \right) \right] d\mathbf{x}. \qquad (8.107)$$

Then using the expressions (8.105) for $\hat{Q}^{M^i}$ and Q^{B^i} in (8.107) we obtain:

$$\begin{aligned} T_B = \int \Bigg\{ & \left(B^a \theta^{M^p} - B^p \theta^{M^a} \right) \wedge \theta^{M^k}_{,a} \wedge \left(\theta^{B^k}_{,p} - \theta^{B^p}_{,k} \right) \\ &+ \left(B^a \theta^{M^p} - B^p \theta^{M^a} \right) \wedge \theta^{M^k} \wedge \left(\theta^{B^k}_{,pa} - \theta^{B^p}_{,ka} \right) \\ &+ B^a \left(\theta^{B^a}_{,p} - \theta^{B^p}_{,a} \right) \wedge \theta^{M^p}_{,s} \wedge \theta^{M^s} \Bigg\} \ d\mathbf{x}. \end{aligned} \qquad (8.108)$$

The terms in (8.108) can be split up into first order derivatives terms $T_B^{(1)}$ and second order derivative terms $T_B^{(2)}$ as:

$$T_B = T_B^{(1)} + T_B^{(2)}. \qquad (8.109)$$

Using integration by parts in (8.108) we obtain:

$$\begin{aligned} T_B^{(2)} = & -\int B^a \theta^{M^p} \wedge \theta^{M^k} \wedge \theta^{B^p}_{,ka} \ d\mathbf{x} \\ \equiv & \int \Bigg\{ \nabla \cdot \mathbf{B} \ \theta^{M^p} \wedge \theta^{M^k} \wedge \theta^{B^p}_{,k} \\ &+ B^a \left[\theta^{M^p}_{,a} \wedge \theta^{M^k} + \theta^{M^p} \wedge \theta^{M^k}_{,a} \right] \wedge \theta^{B^p}_{,k} \Bigg\} \ d\mathbf{x}, \end{aligned} \qquad (8.110)$$

and

$$\begin{aligned} T_B^{(1)} = \int \Bigg\{ & B^a \theta^{M^p} \wedge \theta^{M^k}_{,a} \wedge \left(\theta^{B^k}_{,p} - \theta^{B^p}_{,k} \right) \\ &+ B^a \theta^{M^p} \wedge \theta^{M^k}_{,a} \wedge \left(\theta^{B^a}_{,k} - \theta^{B^k}_{,a} \right) + B^a \theta^{M^p}_{,s} \wedge \theta^{M^s} \wedge \left(\theta^{B^a}_{,p} - \theta^{B^p}_{,a} \right) \Bigg\} \ d\mathbf{x}. \end{aligned} \qquad (8.111)$$

Substituting $T_B^{(1)}$ from (8.111) and $T_B^{(2)}$ from (8.110) in (8.109) we obtain:

$$\mathrm{pr}\left[V_{\mathcal{D}\theta} \left(\Theta \right) \right] = T_B = \int \nabla \cdot \mathbf{B} \ \theta^{M^p} \wedge \theta^{M^k} \wedge \theta^{B^p}_{,k} \ d\mathbf{x}, \qquad (8.112)$$

for pr $[V_{\mathcal{D}\theta}(\Theta)]$. The result (8.112) is the most important result of this sub-section. It shows that the Jacobi identity condition pr $[V_{\mathcal{D}\theta}(\Theta)] = 0$ requires that $\nabla \cdot \mathbf{B} = 0$. However, in the derivation of (8.112), we did not assume $\nabla \cdot \mathbf{B} = 0$. Assuming $\nabla \cdot \mathbf{B} = 0$ at the outset, can be implemented by using the Dirac bracket, where the constraint $\nabla \cdot \mathbf{B} = 0$ is properly taken account of, at the outset (e.g. Chandre et al. 2012, 2013; Chandre 2013).

Using the inner product formalism (8.75)–(8.76), we find:

$$\langle P, Q, R; \ \mathrm{pr}\,[V_{\mathcal{D}\theta}(\Theta)]\rangle = \int \nabla \cdot \mathbf{B}\,\det(\mathsf{A})\,d\mathbf{x}, \tag{8.113}$$

where $\mathsf{A}_{pk} = a_{pk}$ is given by the equations:

$$a_{pk} = \frac{\delta \mathcal{P}}{\delta M^p}\frac{\delta \mathcal{Q}}{\delta M^k}D_{x^k}\left(\frac{\delta \mathcal{R}}{\delta B^p}\right). \tag{8.114}$$

Olver (1993) argues that if $\langle P, Q, R; \ \mathrm{pr}\,[V_{\mathcal{D}\theta}(\Theta)]\rangle = 0$, for arbitrary functionals $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$, then the cyclic sum of such terms will be zero, which implies the Jacobi identity. In general $\det(a_{pk}) \neq 0$. Thus, if $\nabla \cdot \mathbf{B} = 0$, the Jacobi identity for the MG80 bracket can be satisfied, but if $\nabla \cdot \mathbf{B} \neq 0$ then the Jacobi identity will not be satisfied.

8.5 The MHD Casimirs

The Casimirs in Hamiltonian mechanics, are defined as functionals that have zero Poisson bracket with any functional K defined on the phase space. The functional K is usually thought of as a Hamiltonian (not necessarily the MHD Hamiltonian). The condition for a Casimir is:

$$\{C, K\} = 0, \tag{8.115}$$

for arbitrary functionals K. The Casimirs typically reveal underlying symmetries of the phase space, implying dependence among the variables used to describe the system. The reduced Hamiltonian dynamics, taking into account the Casimir constants of motion (note $C_t = 0$) is said to take place on the symplectic leaves foliating the phase space (e.g. Marsden and Ratiu 1994; Morrison 1998; Holm et al. 1998).

To obtain the Casimir determining equations, we introduce the vector:

$$\zeta = \left(K_{\mathbf{u}}, K_{\mathbf{B}}, K_\rho, K_S\right) = \left(\boldsymbol{\xi}, \boldsymbol{\chi}, \lambda, \sigma\right), \tag{8.116}$$

where $K_{\mathbf{u}} \equiv \delta K / \delta \mathbf{u}$, and similarly for the other variational derivatives in (8.116). The MHD Poisson bracket $\{C, K\}$ can be written in the form:

$$\{C, K\} = \int \frac{\delta C}{\delta \psi^a} \mathcal{A}^{ab} \frac{\delta K}{\delta \psi^b} \, d^3x = \int \frac{\delta C}{\delta \psi^a} \mathcal{A}^{ab} \zeta_b \, d^3x$$

$$= -\int \zeta_a \mathcal{A}^{ab} \frac{\delta C}{\delta \psi^b} \, d^3x, \tag{8.117}$$

where $\boldsymbol{\psi}$ is the state vector of the system (in the MHD case we take $\boldsymbol{\psi} = (\mathbf{u}, \mathbf{B}, \rho, S)$). The matrix differential operator in (8.117) is skew-symmetric, since the Poisson bracket is skew symmetric, i.e. $\{C, K\} = -\{K, C\}$. From (8.117) it follows that for arbitrary $\zeta_b = \delta K / \delta \psi^b$, the Casimirs must satisfy the equations:

$$\mathcal{A}^{ab} \frac{\delta C}{\delta \psi^b} = 0. \tag{8.118}$$

In the present analysis we use the MHD variables $\boldsymbol{\psi} = (\mathbf{u}, \mathbf{B}, \rho, S)$ and the non-canonical Poisson bracket (8.22). Hameiri (2004) carried out a similar analysis to that developed here, except that he used the variables $(\mathbf{M}, \mathbf{B}, \rho, S)$ where $\mathbf{M} = \rho\mathbf{u}$ is the mass flux.

The gas dynamic terms in the Poisson bracket $\{C, K\}$ are:

$$F_\rho \nabla \cdot K_{\mathbf{u}} + F_{\mathbf{u}} \cdot \nabla K_\rho = F_\rho \nabla \cdot \boldsymbol{\xi} + F_{\mathbf{u}} \cdot \nabla \lambda \equiv -\boldsymbol{\xi} \cdot \nabla C_\rho - \lambda \nabla \cdot (C_{\mathbf{u}}),$$

$$\frac{\nabla \times \mathbf{u}}{\rho} \cdot K_{\mathbf{u}} \times F_{\mathbf{u}} = -\frac{\boldsymbol{\xi}}{\rho} \cdot (\boldsymbol{\omega} \times C_{\mathbf{u}}),$$

$$\frac{\nabla S}{\rho} \cdot (F_S K_{\mathbf{u}} - K_S F_{\mathbf{u}}) = \frac{1}{\rho} [\boldsymbol{\xi} \cdot \nabla S C_s - \sigma \nabla S \cdot C_{\mathbf{u}}]. \tag{8.119}$$

In (8.119) we have dropped pure divergence terms which give rise to surface terms when integrated over the volume V involved (i.e. we assume the surface terms vanish). Similarly, the magnetic field terms in $\{C, K\}$ are:

$$\mathbf{B} \cdot \left[\frac{1}{\rho} F_{\mathbf{u}} \cdot \nabla K_{\mathbf{B}} - \frac{1}{\rho} K_{\mathbf{u}} \cdot \nabla (F_{\mathbf{B}}) \right] + \mathbf{B} \cdot \left[\nabla \left(\frac{1}{\rho} F_{\mathbf{u}} \right) \cdot K_{\mathbf{B}} - \nabla \left(\frac{1}{\rho} K_{\mathbf{u}} \right) \cdot F_{\mathbf{B}} \right]$$

$$= \frac{\boldsymbol{\xi}}{\rho} \cdot \mathbf{B} \times (\nabla \times C_{\mathbf{B}}) + \boldsymbol{\chi} \cdot \nabla \times \left(\frac{1}{\rho} C_{\mathbf{u}} \times \mathbf{B} \right). \tag{8.120}$$

Using the results (8.119) and (8.120) in the Casimir equation $\{C, K\} = 0$, and setting the coefficients of λ, σ and $\boldsymbol{\chi}$ to zero gives the equations:

$$\nabla \cdot (C_{\mathbf{u}}) = 0, \quad \frac{1}{\rho} C_{\mathbf{u}} \cdot \nabla S = 0, \quad \nabla \times \left(\frac{1}{\rho} C_{\mathbf{u}} \times \mathbf{B} \right) = 0. \tag{8.121}$$

Similarly, by setting the coefficient of $\boldsymbol{\xi}$ equal to zero in the equation $\{C, K\} = 0$ gives the vector equation:

$$\mathbf{B} \times [\nabla \times (C_{\mathbf{B}})] + \rho \nabla (C_\rho) - C_S \nabla S + \boldsymbol{\omega} \times C_{\mathbf{u}} = 0, \tag{8.122}$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the fluid vorticity. The form (8.122) is useful in the case of ordinary fluid mechanics in deriving conservation laws associated with the fluid vorticity.

An alternative, equivalent form of the Casimir determining equations (8.121) and (8.122) was given by Padhye and Morrison (1996a,b) and Hameiri (2004), who used the variables $\tilde{\boldsymbol{\psi}} = (\mathbf{M}, \mathbf{B}, \rho, \sigma)$ where $\mathbf{M} = \rho \mathbf{u}$ and $\sigma = \rho S$ instead of the variables $\boldsymbol{\psi} = (\mathbf{u}, \mathbf{B}, \rho, S)$. Noting that

$$\int d^3 x \, C_{\psi^k} \delta \psi^k = \int d^3 x \, \tilde{C}_{\tilde{\psi}^k} \delta \tilde{\psi}^k, \tag{8.123}$$

we obtain the transformations:

$$C_\rho = \tilde{C}_\rho + \frac{\mathbf{M}}{\rho} \cdot \tilde{C}_{\mathbf{M}} + \frac{\sigma}{\rho} \tilde{C}_\sigma,$$
$$C_S = \rho \tilde{C}_\sigma, \quad C_{\mathbf{u}} = \rho \tilde{C}_{\mathbf{M}}, \quad C_{\mathbf{B}} = \tilde{C}_{\mathbf{B}}. \tag{8.124}$$

Using (8.124) the Casimir determining equations (8.121) and (8.122) become:

$$\nabla \cdot (\rho \tilde{C}_{\mathbf{M}}) = 0, \quad \tilde{C}_{\mathbf{M}} \cdot \nabla (\sigma/\rho) = 0, \quad \nabla \times (\tilde{C}_{\mathbf{M}} \times \mathbf{B}) = 0, \tag{8.125}$$

$$M_j \nabla \tilde{C}_{M_j} + \rho \tilde{C}_{\mathbf{M}} \cdot \nabla (\mathbf{M}/\rho) + \rho \nabla \tilde{C}_\rho + \sigma \nabla \tilde{C}_\sigma + \mathbf{B} \times (\nabla \times \tilde{C}_{\mathbf{B}}) = 0. \tag{8.126}$$

Equations (8.125) and (8.126) are equivalent to the Casimir determining equations of Padhye and Morrison (1996a,b) and Hameiri (2004) (the term $\rho \tilde{C}_{\mathbf{M}} \cdot \nabla (\mathbf{M}/\rho)$ in (8.126) in Padhye and Morrison (1996a) is missing the first ρ factor; they also omit magnetic field terms in their equation (20), Phys. Lett. A. paper). Solutions of (8.125) and (8.126) are given by Padhye and Morrison (1996a,b) in terms of advected scalar invariants that are Lie dragged by the flow (see also below).

Writing

$$C_{\mathbf{u}} = \rho \hat{V}^{\mathbf{x}} = \rho \boldsymbol{\eta}, \tag{8.127}$$

(8.121) can be recognized as a subset of the fluid relabeling symmetry determining equations:

$$\nabla \cdot (\rho \boldsymbol{\eta}) = 0, \quad \boldsymbol{\eta} \cdot \nabla S = 0, \quad \nabla \times (\boldsymbol{\eta} \times \mathbf{B}) = 0. \tag{8.128}$$

However, the Casimir determining equations do not appear to require

$$\delta \mathbf{u} = \boldsymbol{\eta}_t + [\mathbf{u}, \boldsymbol{\eta}] = 0, \tag{8.129}$$

which is required for a fluid relabeling symmetry.

Solutions of the Casimir equations (8.122)–(8.127) or the equivalent Casimir equations (8.125) and (8.126) are given by Padhye and Morrison (1996a,b). The Casimirs turn out to be combinations of Lie dragged invariants. Thus, for example the functionals:

$$C_1[\rho, S] = \int \rho G(S) \, d^3x, \quad C_2[\rho, S, \mathbf{B}] = \int \rho G\left(S, \frac{\mathbf{B} \cdot \nabla S}{\rho}\right) d^3x,$$

$$C_3[\rho, S, \mathbf{u}] = \int \rho G\left(S, \frac{\boldsymbol{\omega} \cdot \nabla S}{\rho}\right) d^3x, \tag{8.130}$$

are Casimirs. The main point to note is that the Casimirs (8.130) are composed of advected invariants. Thus, for example $C_1[\rho, S]$ depends on the advected invariants ρd^3x and S; $C_2[\rho, S, \mathbf{B}]$ depends on the invariants ρd^3x, S and $\mathbf{B} \cdot \nabla S/\rho$; and $C_3[\rho, S, \mathbf{u}]$ depends on ρd^3x, S, and the potential vorticity $\boldsymbol{\omega} \cdot \nabla S/\rho$. One can verify that the Casimirs (8.130) satisfy (8.121)–(8.122) or (8.125)–(8.126).

Examples

As an example, consider the Casimir C_2 which can be written in the form:

$$C_2 = \int d^3x \, \rho G(S, \theta) \quad \text{where} \quad \theta = \frac{\mathbf{B} \cdot \nabla S}{\rho}. \tag{8.131}$$

To check the determining equations requires the variational derivatives:

$$\frac{\delta C_2}{\delta \rho} = G(S, \theta) - \theta G_\theta(S, \theta), \quad \frac{\delta C_2}{\delta \mathbf{u}} = 0,$$

$$\frac{\delta C_2}{\delta S} = \rho G_S - \mathbf{B} \cdot \nabla G_\theta \equiv \rho G_S - \mathbf{B} \cdot (G_{\theta\theta} \nabla \theta + G_{\theta S} \nabla S),$$

$$\frac{\delta C_2}{\delta S} \equiv \rho(G_S - \theta G_{\theta S}) - (\mathbf{B} \cdot \nabla \theta)G_{\theta\theta}, \quad \frac{\delta C_2}{\delta \mathbf{B}} = G_\theta \nabla S. \tag{8.132}$$

Using (8.132) one can verify that (8.121) and (8.122) are satisfied. Since $\delta C_2/\delta \mathbf{u} = 0$, it is not possible (at least in the present analysis) to associate the conservation law

$$\frac{d}{dt} C_2[\rho, S, \mathbf{B}] = 0, \tag{8.133}$$

with a fluid relabelling symmetry. This does not necessarily mean that there is not a fluid relabelling symmetry association, since the work of Volkov et al. (1995) associates the Lie dragged invariants with hidden supersymmetries.

The Casimir $C_3[\rho, S, \mathbf{u}]$ can directly be related to a fluid relabelling symmetry, as it depends explicitly on $\mathbf{u}$. This can be verified directly by calculating the variational derivatives of C_3. Writing

$$C_3[\rho, S, \mathbf{u}] = \int \rho G(S, \phi)\, d^3x \quad \text{where} \quad \phi = \frac{\boldsymbol{\omega} \cdot \nabla S}{\rho}, \tag{8.134}$$

we find:

$$\frac{\delta C_3}{\delta \rho} = G - \phi G_\phi, \quad \frac{\delta C_3}{\delta \mathbf{u}} = \nabla \times (G_\phi \nabla S) = G_{\phi\phi} \nabla \phi \times \nabla S,$$

$$\frac{\delta C_3}{\delta S} = \rho G_\phi - \boldsymbol{\omega} \cdot \nabla G_\phi. \tag{8.135}$$

The determining equations (8.121)–(8.122) are satisfied. The invariant C_3 is associated with the fluid relabelling symmetry with generator

$$\hat{V}^{\mathbf{x}} = \frac{1}{\rho} \frac{\delta C_3}{\delta \mathbf{u}} = \frac{\nabla \times (G_\phi \nabla S)}{\rho}. \tag{8.136}$$

8.5.1   Casimir Equations for Advected A

The determining equations for the Casimirs $C(\mathbf{M}, \mathbf{A}, \rho, \sigma)$ where $\mathbf{B} = \nabla \times \mathbf{A}$ for the advected $\mathbf{A}$ Poisson bracket (8.31) are derived below. The Casimirs are defined by the equations $\{C, K\} = 0$. Following the approach in (8.116) et seq., we introduce the variational derivative vector:

$$\boldsymbol{\zeta} = \left(K_{\mathbf{M}}, K_{\mathbf{A}}, K_\rho, K_\sigma\right) = (\boldsymbol{\xi}, \boldsymbol{\chi}, \lambda, \nu). \tag{8.137}$$

The Casimirs C satisfy equations of the form (8.118) where in the present case $\boldsymbol{\psi} = (\mathbf{M}, \mathbf{A}, \rho, \sigma)$.

Using the notation (8.137), the gas dynamic terms in the bracket (8.31) are given by:

$$[F_{\mathbf{M}} \cdot \nabla(G_{\mathbf{M}}) - G_{\mathbf{M}} \cdot (\nabla F_{\mathbf{M}})] \cdot \mathbf{M} = [(C_{\mathbf{M}} \cdot \nabla)\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla(C_{\mathbf{M}})] \cdot \mathbf{M},$$

$$\rho\left(F_{\mathbf{M}} \cdot \nabla G_\rho - G_{\mathbf{M}} \cdot \nabla F_\rho\right) = \rho\left(C_{\mathbf{M}} \cdot \nabla \lambda - \boldsymbol{\xi} \cdot \nabla C_\rho\right),$$

$$\sigma\left(F_{\mathbf{M}} \cdot \nabla G_\sigma - G_{\mathbf{M}} \cdot \nabla F_\sigma\right) = \sigma\left(C_{\mathbf{M}} \cdot \nabla \nu - \boldsymbol{\xi} \cdot \nabla C_\sigma\right). \tag{8.138}$$

Similarly, the magnetic vector potential terms in the Poisson bracket (8.31) are:

$$(\mathbf{A} \cdot F_{\mathbf{M}})\nabla \cdot G_{\mathbf{A}} - (\mathbf{A} \cdot G_{\mathbf{M}})\nabla \cdot F_{\mathbf{A}} = (\mathbf{A} \cdot C_{\mathbf{M}})\nabla \cdot \boldsymbol{\chi} - (\mathbf{A} \cdot \boldsymbol{\xi})\nabla \cdot C_{\mathbf{A}},$$

$$\mathbf{B} \cdot [G_{\mathbf{A}} \times F_{\mathbf{M}} - F_{\mathbf{A}} \times G_{\mathbf{M}}] = \boldsymbol{\chi} \cdot (C_{\mathbf{M}} \times \mathbf{B}) - \boldsymbol{\xi} \cdot (\mathbf{B} \times C_{\mathbf{A}}). \tag{8.139}$$

In (8.139) $\mathbf{B} = \nabla \times \mathbf{A}$ and we make the identifications $F = C$ and $G = K$.

Substituting (8.138)–(8.139) in the Poisson bracket (8.31) and integrating the derivative terms by parts, and dropping the surface terms gives:

$$\{C, K\} = \int \left\{ -\boldsymbol{\xi} \cdot \left[(\nabla \cdot C_{\mathbf{M}})\mathbf{M} + (C_{\mathbf{M}} \cdot \nabla)\mathbf{M} + \mathbf{M} \cdot (\nabla C_{\mathbf{M}})^T \right] \right.$$

$$- \left[\lambda \nabla \cdot (\rho C_{\mathbf{M}}) + \rho \boldsymbol{\xi} \cdot \nabla C_\rho \right] - \left[\nu \nabla \cdot (\sigma C_{\mathbf{M}}) + \sigma \boldsymbol{\xi} \cdot \nabla C_\sigma \right]$$

$$\left. - \left[\boldsymbol{\chi} \cdot \nabla(\mathbf{A} \cdot C_{\mathbf{M}}) + (\boldsymbol{\xi} \cdot \mathbf{A})\nabla \cdot C_{\mathbf{A}} \right] + \boldsymbol{\chi} C_{\mathbf{M}} \times \mathbf{B} - \boldsymbol{\xi} \cdot (\mathbf{B} \times C_{\mathbf{A}}) \right\} \, d^3 x$$

$$= 0. \tag{8.140}$$

Setting the coefficients of λ and ν equal to zero in (8.140) gives the equations:

$$\nabla \cdot (\rho C_{\mathbf{M}}) = 0, \quad \nabla \cdot (\sigma C_{\mathbf{M}}) = 0. \tag{8.141}$$

which are analogous to the steady state mass continuity equation and entropy conservation equation with advection velocity

$$\hat{V}^{\mathbf{x}} = C_{\mathbf{M}}. \tag{8.142}$$

Setting the coefficient of $\boldsymbol{\chi}$ equal to zero in (8.140) gives the steady state advection equation:

$$- C_{\mathbf{M}} \times (\nabla \times \mathbf{A}) + \nabla(\mathbf{A} \cdot C_{\mathbf{M}}) = 0, \tag{8.143}$$

associated with Lie dragging the magnetic vector potential 1-form $\alpha = \mathbf{A} \cdot d\mathbf{x}$ with velocity $\hat{V}^{\mathbf{x}} = C_{\mathbf{M}}$. Taking the curl of (8.143) we obtain $\nabla \times (C_{\mathbf{M}} \times \mathbf{B}) = 0$, which is the steady state Faraday equation for $\mathbf{B}$ with advection velocity $C_{\mathbf{M}}$, obtained previously in (8.125) in the analysis of the Casimirs for the Morrison and Greene bracket (8.20). Noting that $\mathbf{M} = \rho \mathbf{u}$ and setting the coefficient of $\boldsymbol{\xi}$ equal to zero in (8.140) we obtain the equation:

$$M^k \nabla C_{M^k} + \rho C_{\mathbf{M}} \cdot \nabla(\mathbf{M}/\rho) + \rho C_\rho + \sigma \nabla C_\sigma + \mathbf{A}(\nabla \cdot C_{\mathbf{A}}) + \mathbf{B} \times C_{\mathbf{A}} = 0. \tag{8.144}$$

By noting that for $\mathbf{B} = \nabla \times \mathbf{A}$, that

$$C_{\mathbf{A}} = \nabla \times C_{\mathbf{B}}, \quad \nabla \cdot C_{\mathbf{A}} = 0, \tag{8.145}$$

(8.144) reduces to:

$$M^k \nabla C_{M^k} + \rho C_{\mathbf{M}} \cdot \nabla(\mathbf{M}/\rho) + \rho C_\rho + \sigma \nabla C_\sigma + \mathbf{B} \times (\nabla \times C_{\mathbf{B}}) = 0, \qquad (8.146)$$

which is (8.126) obtained for the Morrison and Greene bracket previously. Note that this latter result depends on Gauss's law $\nabla \cdot \mathbf{B} = 0$ for which $\mathbf{B} = \nabla \times \mathbf{A}$.

Padhye and Morrison (1996a) give the Casimir solutions:

$$C[\rho, S, \mathbf{A}] = \int_V \rho G\left(S, \frac{\mathbf{A} \cdot \mathbf{B}}{\rho}, \frac{\mathbf{B} \cdot \nabla S}{\rho}, \frac{\mathbf{B} \cdot \nabla}{\rho}\left(\frac{\mathbf{B} \cdot \nabla S}{\rho}\right)\right.$$
$$\left. + \frac{\mathbf{B} \cdot \nabla}{\rho}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{\rho}\right), \dots\right) d^3 x, \qquad (8.147)$$

It is clear that this family of Casimirs has $C_{\mathbf{M}} = 0$ and hence the gauge dependent condition (8.143) does not affect the solution of the Casimir determining equations (8.141) and (8.143). This illustrates that the Casimirs (8.147) are not in general Lie dragged with the flow. Only for cases where $\alpha = \mathbf{A} \cdot d\mathbf{x}$ is lie dragged with the flow (i.e. α satisfies (8.26)) does $d/dt(\mathbf{A} \cdot \mathbf{B}/\rho) = 0$ (see also (3.42)). Since $C_{\mathbf{M}} = 0$, it is not clear how one can identify the symmetry associated with the Casimirs (8.147) (possibly the Casimirs (8.147) are related to the supersymmetries discussed by Volkov et al. (1995)). More general forms of the MHD Casimirs are clearly possible (see e. g. Tur and Yanovsky 1993 and (5.26) of the present paper).

To sum up, the Casimir equations (8.141)–(8.146) obtained by using the Holm and Kupershmidt (1983a,b) bracket (8.31) are the same as for the Morrison and Greene bracket (8.20), except Faraday's equation (8.125) is replaced by the more restrictive (8.143). However, use of the Holm and Kupershmidt (1983a,b) bracket (8.31) leads more naturally to conservation laws involving the magnetic vector potential $\mathbf{A}$, which are Lie dragged with the flow.

Chapter 9
Multi-Symplectic Clebsch Approach

Multi-symplectic formulations of Hamiltonian systems with two or more independent variables x^α have been developed as a useful extension of Hamiltonian systems with one evolution variable t. This development has connections with dual variational formulations of traveling wave problems (e.g. Bridges 1992; Webb et al. 2014d), and is useful in numerical schemes for multisymplectic systems. Bridges and co-workers used the multi-symplectic approach to study linear and nonlinear wave propagation, generalizations of wave action, wave modulation theory, and wave stability problems (Bridges 1997a,b). Reich (2000) and Bridges (2006) develop difference schemes. Multi-symplectic Hamiltonian systems have been studied by Marsden and Shkoller (1999), Kanatchikov (1993, 1997, 1998), Gotay (1991), Gotay et al. (2004a,b), Forger et al. (2003), Carenina et al. (1991) and Bridges et al. (2005). Bridges et al. (2010) shows the connection between multi-symplectic systems and the variational bi-complex. Marsden et al. (2001) discuss multisymplectic geometry and continuum mechanics.

Cotter et al. (2007) developed a multi-symplectic, Euler-Poincaré formulation of fluid mechanics. They showed that multi-symplectic ideal fluid mechanics type systems are related to the Clebsch variable formulations of Hamiltonian fluid type systems, in which the Lagrange multipliers play the role of canonically conjugate momenta. Thus, Clebsch variables in ideal MHD and fluid type systems, involves a momentum map. In the next section we give an introduction to multi-symplectic systems, based on the work of Hydon (2005) (see also Brio et al. (2010) for a similar discussion). We obtain multi-symplectic equations for ideal gas dynamics and MHD, based on the Clebsch variables formulation. The energy and momentum equations for the gas dynamics and MHD are obtained from the symplecticity conservation law for these systems.

We consider the MHD equations for the case of no external gravitational potential. The analysis is based on the work of Webb et al. (2014c, 2015). Webb et al. (2015) is an addendum and erratum of Webb et al. (2014c) (Proposition 4.3 of Webb et al. (2014c) contains errors, which are corrected in Webb et al. (2015)).

9.1 Overview of Multi-Symplectic Systems

Hamiltonian systems, with one evolution variable t, can in general be written in the form:

$$K_{ij}(z)\frac{dz^j}{dt} = \nabla_{z^i}H(z), \tag{9.1}$$

where the invariant phase space volume element:

$$\kappa = \frac{1}{2}K_{ij}(z)dz^i \wedge dz^j, \tag{9.2}$$

is a closed two-form, i.e. $d\kappa = 0$. Here d denotes the exterior derivative and $\wedge$ denotes the anti-symmetric wedge product used in the exterior Calculus. The condition that κ be a closed 2-form, implies $\kappa = dg$ where $g = L_j dz^j$ is a one-form (note that $d\kappa = ddg = 0$ by antisymmetry of the wedge product). It turns out, that the condition that κ be a closed 2-form implies that $K_{ij} = -K_{ji}$ is a skew symmetric operator (see Zakharov and Kuznetsov 1997; Hydon 2005). Taking the exterior derivative of the 2-form (9.2) and setting the result equal to zero, we obtain the identity:

$$K_{ij,k} + K_{jk,i} + K_{ki,j} = 0, \tag{9.3}$$

which in some cases is related to the Jacobi identity for the Poisson bracket. If the system (9.1) has an even dimension, and if K_{ij} has non-zero determinant , then (9.1) can be written in the form:

$$\frac{dz^i}{dt} = R_{ij}\nabla_{z^j}H(z), \tag{9.4}$$

where R_{ij} is the inverse of the matrix K_{ij}. Here $R_{ij} = -R_{ji}$ is a skew-symmetric matrix. The closure relation (9.3) then are equivalent to the relations:

$$R_{im}\frac{\partial R_{jk}}{\partial z^m} + R_{km}\frac{\partial R_{ij}}{\partial z^m} + R_{jm}\frac{\partial R_{ki}}{\partial z^m} = 0, \tag{9.5}$$

(see e.g. Zakharov and Kuznetsov 1997). The Poisson bracket for the system in the finite dimensional case is given by

$$\{A, B\} = \sum R_{ij}\frac{\partial A}{\partial z^i}\frac{\partial B}{\partial z^j}. \tag{9.6}$$

Using the Poisson bracket description (9.6) the Jacobi identity reduces to (9.5). Casimir functionals have zero Poisson bracket with respect to any other functional of the variables describing the system. For finite dimensional systems Casimirs always occur for odd dimensional systems.

Consider a finite dimensional Hamiltonian system of dimension $2n$ with canonical variables $z = (q^1, q^2, \ldots q^n, p_1, p_2, \ldots p_n)^t$ which can be written in the form (9.1), where

$$\mathbf{K} = \mathbf{J}^t = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{9.7}$$

Here the matrix $\mathbf{K}$ is the inverse of the symplectic matrix $\mathbf{J}$ and I_n is the unit $n \times n$ matrix. The invariant phase space element form (9.2) is:

$$\kappa = dp_j \wedge dq^j = d(p_j dq^j). \tag{9.8}$$

Hamiltonian, multi-symplectic systems with n independent variables x^α can be written in the form:

$$K_{ij}^\alpha z_{,\alpha}^j = \nabla_{z^i} H(z), \tag{9.9}$$

where $z_\alpha^j = \partial z^j / \partial x^\alpha$. The fundamental invariant 2-forms are:

$$\kappa^\alpha = \frac{1}{2} K_{ij}^\alpha dz^i \wedge dz^j, \quad \alpha = 1(1)n, \tag{9.10}$$

Invariance of the phase space element $D_t(dp_j \wedge dq^j) = 0$ for the standard canonical Hamiltonian formulation with evolution variable t is replaced by the symplectic, or structural conservation law:

$$\kappa_{,\alpha}^\alpha = 0, \tag{9.11}$$

which is referred to as the symplecticity conservation law.

The closure of the 2-forms κ^α implies that the exterior derivative of $\kappa^\alpha = 0$. By the Poincaré Lemma κ^α is the exterior derivative of a 1-form, i.e.,

$$\kappa^\alpha = d(L_j^\alpha dz^j) = d\omega^\alpha \quad \text{where} \quad \omega^\alpha = L_j^\alpha dz^j. \tag{9.12}$$

Note that $d\kappa^\alpha = dd\omega^\alpha = 0$. Taking the exterior derivative of ω^α in (9.12) and using the anti-symmetry of the wedge product we obtain:

$$\kappa^\alpha = \frac{1}{2} \left(\frac{\partial L_k^\alpha}{\partial z^j} - \frac{\partial L_j^\alpha}{\partial z^k} \right) dz^j \wedge dz^k. \tag{9.13}$$

From (9.10) and (9.13) we obtain:

$$K_{jk}^{\alpha} = \frac{\partial L_k^{\alpha}}{\partial z^j} - \frac{\partial L_j^{\alpha}}{\partial z^k}. \tag{9.14}$$

Thus, the matrices K_{ij}^{α} are skew-symmetric, i.e. $K_{ij}^{\alpha} = -K_{ji}^{\alpha}$.

Proposition 9.1.1 *The Legendre transformation for multi-symplectic systems is the identity*

$$\left(L_j^{\alpha} dz^j\right)_{,\alpha} = d\left\{L_j^{\alpha}(z)z_{,\alpha}^j - H(z)\right\} \equiv dL, \tag{9.15}$$

where

$$L = L_j^{\alpha}(z)z_{,\alpha}^j - H(z), \tag{9.16}$$

is the Lagrangian density and $H(z)$ is the multi-symplectic Hamiltonian.

Proof The proof of (9.15) proceeds by noting

$$\begin{aligned}
\left(L_j^{\alpha} dz^j\right)_{,\alpha} &= \frac{\partial L_j^{\alpha}}{\partial z^i} z_{,\alpha}^i dz^j + L_j^{\alpha}(z) D_{\alpha} dz^j \\
&= \frac{\partial L_j^{\alpha}}{\partial z^i} z_{,\alpha}^i dz^j + L_j^{\alpha}(z) d\left(z_{,\alpha}^j\right).
\end{aligned} \tag{9.17}$$

Here we used the fact that the operators d and D_{α} commute. Equation (9.17) can be further reduced to:

$$\left(L_j^{\alpha} dz^j\right)_{,\alpha} = -K_{ji}^{\alpha} z_{,\alpha}^i dz^j + d\left(L_j^{\alpha}(z)z_{,\alpha}^j\right). \tag{9.18}$$

The identity (9.15) then follows by using the Hamiltonian evolution equations (9.9).
□

The symplecticity or structural conservation law (9.11) now follows by taking the exterior derivative of (9.15) and using the results $ddL = 0$ and $dD_{\alpha} = D_{\alpha}d$, i.e.,

$$D_{\alpha}\kappa^{\alpha} = D_{\alpha}[d(L_j^{\alpha} dz^j)] = dD_{\alpha}(L_j^{\alpha} dz^j) = ddL = 0, \tag{9.19}$$

which is (9.11). Other conservation laws are obtained by sectioning the forms in (9.15) (i.e. we impose the requirement that $z^j = z^j(\mathbf{x})$, which is also referred to as the pull-back to the base manifold). The pullback, applied to (9.15) gives

$$\left(L_j^{\alpha} dz^j\right)_{,\alpha} = \left(L_j^{\alpha} z_{,\beta}^j dx^{\beta}\right)_{,\alpha} = \left(L_j^{\alpha} z_{,\beta}^j\right)_{,\alpha} dx^{\beta} = dL = \frac{\partial L}{\partial x^{\beta}} dx^{\beta}. \tag{9.20}$$

Thus, (9.20) gives the conservation law:

$$D_\alpha \left(L_j^\alpha(z) z_{,\beta}^j - L\delta_\beta^\alpha \right) = 0. \tag{9.21}$$

This conservation law is in fact, the conservation law obtained due to the invariance of the action $A = \int L dx$ under translations in x^β which follows from Noether's first theorem (i.e. $x'^\alpha = x^\alpha + \epsilon \delta_\beta^\alpha$).

A further set of $n(n-1)/2$ conservation laws is obtained from pull-back of the structural conservation law (9.11) to the base manifold, namely:

$$D_\alpha \left(K_{ij}^\alpha z_{,\beta}^i z_{,\gamma}^j \right) = 0, \quad \beta < \gamma. \tag{9.22}$$

The conservation laws (9.22) can be obtained by cross-differentiation of the conservation laws (9.19), i.e. they are a consequence of the equations:

$$D_\gamma \left\{ D_\alpha \left(L_j^\alpha(z) z_{,\beta}^j \right) - D_\beta(L) \right\} - D_\beta \left\{ D_\alpha \left(L_j^\alpha(z) z_{,\gamma}^j \right) - D_\gamma(L) \right\} = 0. \tag{9.23}$$

A multi-symplectic version of Noether's theorem (discussed by Hydon (2005)) for the multi-symplectic system (9.9) is described below:

Proposition 9.1.2 *If the action:*

$$J = \int L \, d^3x dt \tag{9.24}$$

is invariant to $O(\epsilon)$ under the infinitesimal Lie transformation:

$$z'^s = z^s + \epsilon V^{z^s}, \quad x'^\alpha = x^\alpha + \epsilon V^{x^\alpha}, \quad (0 \le \alpha \le 3, \quad 1 \le s \le N), \tag{9.25}$$

and under the divergence transformation:

$$L' = L + \epsilon D_\alpha \Lambda^\alpha + O(\epsilon^2), \tag{9.26}$$

where L has the multi-symplectic form (9.16):

$$L = L_j^\alpha(z) z_{,\alpha}^j - H(z) \quad where \quad \mathcal{H} = \int H(z) d^3x dt, \tag{9.27}$$

is the Hamiltonian functional, then the Euler Lagrange equations for the action:

$$E_{z^s}(L) = \frac{\partial L}{\partial z^s} - \frac{\partial}{\partial x^\alpha} \left(\frac{\partial L}{\partial z_{,\alpha}^s} \right) \equiv -\frac{\partial H}{\partial z^s} + K_{sj}^\alpha z_{,\alpha}^j = 0, \tag{9.28}$$

admit the conservation law

$$D_\alpha \left\{ V^{x^\alpha} L + W^\alpha[\mathbf{z}, \hat{V}^{\mathbf{z}}] + \Lambda^\alpha \right\} = 0, \tag{9.29}$$

where

$$W^\alpha[\mathbf{z}, \hat{V}^{\mathbf{z}}] = \hat{V}^{z^s} \frac{\partial L}{\partial z^s_{,\alpha}} \equiv \hat{V}^{z^s} L^\alpha_s(z), \tag{9.30}$$

and

$$\hat{V}^{z^s} = V^{z^s} - V^{x^\alpha} z^s_{,\alpha}, \tag{9.31}$$

is the canonical or characteristic Lie symmetry generator (i.e., the infinitesimal Lie symmetry transformation $z'^s = z^s + \epsilon \hat{V}^{z^s}$, $x'^\alpha = x^\alpha$ which is equivalent to Lie transformation (9.25)). Thus, the conservation law (9.29) reduces to:

$$D_\alpha \left\{ V^{x^\alpha} \left[L^\mu_s(\mathbf{z}) z^s_{,\mu} - H(\mathbf{z}) \right] + \hat{V}^{z^s} L^\alpha_s(\mathbf{z}) + \Lambda^\alpha \right\} = 0. \tag{9.32}$$

or alternatively:

$$D_\alpha \left\{ V^{x^\alpha} L + \hat{V}^{z^s} L^\alpha_s(\mathbf{z}) + \Lambda^\alpha \right\} = 0. \tag{9.33}$$

This is the multi-symplectic form of Noether's first theorem for the system (9.9).

The condition for the Lie symmetry (9.25)–(9.26) to be a divergence symmetry of the action is:

$$\tilde{X} L + V^{x^\alpha} D_\alpha L + D_\alpha \Lambda^\alpha = 0, \tag{9.34}$$

where

$$\tilde{X} = V^{x^\alpha} \frac{\partial}{\partial x^\alpha} + V^{z^s} \frac{\partial}{\partial z^s} + V^{z^s_{,\alpha}} \frac{\partial}{\partial z^s_{,\alpha}} + \dots, \tag{9.35}$$

is the extended Lie symmetry operator. The extended Lie symmetry operator $\tilde{X}$ can be expressed in terms of the characteristic symmetry operator $\hat{X}$ by the formula

$$\tilde{X} = \hat{X} + V^{x^\alpha} D_\alpha, \quad where \quad \hat{X} = \hat{V}^{z^s} \frac{\partial}{\partial z^s} + D_\alpha \left(\hat{V}^{z^s} \right) \frac{\partial}{\partial z^s_{,\alpha}} + \dots. \tag{9.36}$$

The Lie invariance condition (9.34) written in terms of $\hat{X}$ is:

$$\hat{X} L + D_\alpha \left(V^{x^\alpha} L + \Lambda^\alpha \right) = 0. \tag{9.37}$$

Example As an example of Noether's theorem, consider the invariance of the action J under the Lie symmetry:

$$V^{x^{\alpha}} = \delta^{\alpha}_{\beta}, \quad V^{z^s} = 0, \quad \Lambda^{\alpha} = 0, \tag{9.38}$$

corresponding to translation invariance with respect to x^{β}. The canonical Lie symmetry generator $\hat{V}^{z^s}$ is given by:

$$\hat{V}^{z^s} = -z^s_{,\beta}, \tag{9.39}$$

The Lie invariance condition (9.37) is satisfied for $\Lambda^{\alpha} = 0$, i.e. the action is invariant under a variational symmetry (one can show $\hat{X}L = -D_{\beta}L$). The conservation law (9.32) or (9.33) reduces to the symplectic conservation law (9.21). Thus we have shown that the symplectic conservation law is due to invariance of the action under translations in x^{β}.

9.2 Multi-Symplectic MHD

In the Clebsch variables approach, the fluid velocity is given by the expression:

$$\rho\mathbf{u} = \rho\nabla\phi - \beta\nabla S - \lambda\nabla\mu - (\nabla \times \mathbf{\Gamma}) \times \mathbf{B} - \mathbf{\Gamma}(\nabla \cdot \mathbf{B}), \tag{9.40}$$

In the standard Clebsch variable formulation (Sect. 8.1), in which t is the evolution variable, the canonical coordinates are the physical variables $(\rho, S, \mu, \mathbf{B}^{\mathbf{T}})$ and the Lagrange multipliers $(\phi, \beta, \lambda, \mathbf{\Gamma}^T)$ are the corresponding canonical momenta (the role of the canonical momenta and coordinates can be interchanged, simply by changing the sign of the Hamiltonian). In the multi-symplectic formulation both space and time can be thought of as evolution variables. The Lagrangian L is equal to the kinetic minus the potential energy of the system, subject to the constraints of mass, entropy and magnetic flux conservation (Faraday's law), plus the Lin constraints. Thus, the multi-symplectic Lagrangian L is given by (8.2). It is worth noting that there may be up to three Lin constraints that need to be imposed for three dimensional flow in some cases (e.g. Yoshida 2009; Webb and Anco 2016).

In the multi-symplectic approach used in the present analysis, the Clebsch variable expansion for the fluid velocity $\mathbf{u}$ in (9.40) is re-written in the form:

$$\beta\nabla S + \lambda\nabla\mu + \mathbf{\Gamma}(\nabla \cdot \mathbf{B}) + \mathbf{B} \cdot \nabla\mathbf{\Gamma} - \mathbf{B} \cdot (\nabla\mathbf{\Gamma})^T - \rho\nabla\phi = -\rho\mathbf{u} \equiv -\frac{\delta\ell}{\delta\mathbf{u}}, \tag{9.41}$$

where

$$\ell = \int_V \left(\frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu_0}\right) d^3x, \tag{9.42}$$

is the MHD Lagrangian without constraints.

Proposition 9.2.1 *The evolution equations (8.5)–(8.9) and the Clebsch variable equation (9.41) for $-\delta\ell/\delta\mathbf{u}$ can be written in the multi-symplectic form:*

$$\mathsf{A}\mathbf{z} \equiv \left(\mathsf{K}^0 \frac{\partial}{\partial t} + \mathsf{K}^1 \frac{\partial}{\partial x} + \mathsf{K}^2 \frac{\partial}{\partial y} + \mathsf{K}^3 \frac{\partial}{\partial z} \right) \mathbf{z} = \frac{\delta\mathcal{H}}{\delta\mathbf{z}}, \tag{9.43}$$

where A *is a* 15×15 *matrix differential operator. In (9.43)*

$$\mathbf{z} = \left(\mathbf{u}^T, \rho, S, \mu, \mathbf{B}^T, \mathbf{\Gamma}^T, \lambda, \beta, \phi \right)^T, \tag{9.44}$$

is a 15-dimensional state vector for the system and the K^α *($\alpha = 0, 1, 2, 3$) are skew-symmetric* 15×15 *matrices, and*

$$\mathcal{H} = -\ell \equiv -\int_V \left(\frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu_0} \right) d^3x = \int_V H(\mathbf{z})d^3x, \tag{9.45}$$

is the multi-symplectic Hamiltonian functional for the system. The functional or variational derivative $\delta\mathcal{H}/\delta z^s = \partial H/\partial z^s$ *in the present case. The skew-symmetric matrices* K^α *satisfy equations of the form:*

$$d\omega^\alpha = \frac{1}{2}\mathsf{K}^\alpha_{ij}dz^i \wedge dz^j \quad where \quad \omega^\alpha = L^\alpha_j dz^j, \tag{9.46}$$

are symplectic one-forms. For the MHD system, the one-forms ω^α *are given by (up the exterior derivative of a scalar function):*

$$\omega^0 = \phi d\rho + \beta dS + \lambda d\mu + \mathbf{\Gamma} \cdot d\mathbf{B}, \tag{9.47}$$

$$\omega^i = [\mathbf{u}(\beta dS + \lambda d\mu + \phi d\rho) + \rho\phi d\mathbf{u} + (\mathbf{\Gamma} \cdot \mathbf{B})d\mathbf{u} - \mathbf{B}(\mathbf{\Gamma} \cdot d\mathbf{u}) + \mathbf{u}(\mathbf{\Gamma} \cdot d\mathbf{B})]^i, \tag{9.48}$$

$$\equiv [\mathbf{u}(\beta dS + \lambda d\mu - \rho d\phi) + d(\rho\phi\mathbf{u}) + (\mathbf{\Gamma} \cdot \mathbf{u})d\mathbf{B} - (\mathbf{\Gamma} \times d\mathbf{E})]^i, \tag{9.49}$$

where $1 \le i \le 3$ *and*

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}, \tag{9.50}$$

is the electric field in ideal MHD. The adjoint $\mathsf{A}^\dagger$ *of the matrix differential operator* A *satisfies the equation:*

$$\boldsymbol{\psi}^T \cdot \mathsf{A}\mathbf{z} = \frac{\partial}{\partial x^\alpha} \left(\boldsymbol{\psi}^T \cdot \mathsf{K}^\alpha \mathbf{z} \right) + \mathbf{z}^T \cdot \mathsf{A}^\dagger \boldsymbol{\psi}, \tag{9.51}$$

where

$$A^\dagger \psi = \frac{\partial}{\partial x^\alpha} (K^\alpha \psi). \tag{9.52}$$

Note that $\langle \psi, Az \rangle = \langle z, A^\dagger \psi \rangle$, *where* $\langle, \rangle$ *is the usual inner product.*

Proof Below we give a straightforward proof of the theorem. Later, in (9.65)–(9.75) we give a simpler, more elegant derivation of the matrices K_{ij}^α based on the differential forms ω^0 and ω^i given in (9.47)–(9.48).

To derive (9.43)–(9.48) first note that the Clebsch variable equation (9.41) for $\rho \mathbf{u}$ and the evolution equations (8.5)–(8.9) can be written in the form:

$$\beta \nabla S + \lambda \nabla \mu + \Gamma(\nabla \cdot \mathbf{B}) + \mathbf{B} \cdot \nabla \Gamma - \mathbf{B} \cdot (\nabla \Gamma)^T - \rho \nabla \phi = H_\mathbf{u},$$

$$- D_t \phi = H_\rho, \quad -\beta \nabla \cdot \mathbf{u} - D_t \beta = H_S, \quad -\lambda \nabla \cdot \mathbf{u} - D_t \lambda = H_\mu,$$

$$- \Gamma \cdot (\nabla \mathbf{u})^T - D_t \Gamma = H_\mathbf{B}, \quad \mathbf{B}(\nabla \cdot \mathbf{u}) - \mathbf{B} \cdot \nabla \mathbf{u} + D_t \mathbf{B} = H_\Gamma,$$

$$D_t \mu = H_\lambda, \quad D_t S = H_\beta, \quad \rho \nabla \cdot \mathbf{u} + D_t \rho = H_\phi, \tag{9.53}$$

where $D_t = \partial_t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative and the multi-symplectic Hamiltonian is given by (9.45). In (9.53) we use the notation $H_\psi \equiv \partial H/\partial \psi$.

To obtain the matrices K^α in (9.43) write (9.53) in the matrix form:

$$Az = H_z \quad \text{where} \quad A = K^\alpha \frac{\partial}{\partial x^\alpha}, \tag{9.54}$$

and $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$. Note that the equations involving $H_\mathbf{u}$, $H_\mathbf{B}$ and H_Γ each consist of three equations, but the other equations involving H_ρ, H_S, H_μ, H_λ, H_β and H_ϕ are single equations. The matrix differential operator A in (9.54) has the form:

$$A = \begin{pmatrix} O_{3\times3} & 0 & \beta\nabla & \lambda\nabla & \Gamma\nabla\cdot & V_\mathbf{B} & 0 & 0 & -\rho\nabla \\ O_{1\times3} & 0 & 0 & 0 & O_{1\times3} & O_{1\times3} & 0 & 0 & -D_t \\ -\beta\nabla\cdot & 0 & 0 & 0 & O_{1\times3} & O_{1\times3} & 0 & -D_t & 0 \\ -\lambda\nabla\cdot & 0 & 0 & 0 & O_{1\times3} & O_{1\times3} & -D_t & 0 & 0 \\ -\Gamma \cdot (\nabla\circ)^T & 0 & 0 & 0 & O_{3\times3} & -D_t & 0 & 0 & 0 \\ -V_\mathbf{B}^\dagger & 0 & 0 & 0 & D_t & O_{3\times3} & 0 & 0 & 0 \\ O_{1\times3} & 0 & 0 & D_t & O_{1\times3} & O_{1\times3} & 0 & 0 & 0 \\ O_{1\times3} & 0 & D_t & 0 & O_{1\times3} & O_{1\times3} & 0 & 0 & 0 \\ \rho\nabla\cdot & D_t & 0 & 0 & O_{1\times3} & O_{1\times3} & 0 & 0 & 0 \end{pmatrix}, \tag{9.55}$$

where

$$V_\mathbf{B} = \mathbf{B} \cdot \nabla \circ -\mathbf{B} \cdot (\nabla\circ)^T, \quad V_\mathbf{B}^\dagger = \mathbf{B} \cdot (\nabla\circ)^T - \mathbf{B}\nabla \cdot \circ. \tag{9.56}$$

In (9.55) $\mathsf{O}_{3\times3}$ is the zero 3×3 matrix, and $\mathsf{O}_{1\times3}$ is the 1×3 zero matrix. The operator:

$$D_t = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla, \tag{9.57}$$

is the Lagrangian time derivative following the flow. Note that $V_\mathbf{B}^\dagger$ is the adjoint of the operator $V_\mathbf{B}$ with respect to the usual inner product $\langle f,g\rangle = \int fg\,d^3x$ for real functions.

Using (9.55) the skew symmetric matrices K_{ij}^α have the form:

$$\mathsf{K}_{ij}^\alpha = \mathsf{k}_{[i,j]}^\alpha = \mathsf{k}_{ij}^\alpha - \mathsf{k}_{ji}^\alpha. \tag{9.58}$$

In particular:

$$\mathsf{k}_{ij}^0 = \delta_{15}^i\delta_4^j + \delta_{14}^i\delta_5^j + \delta_{13}^i\delta_6^j + \delta_{10}^i\delta_7^j + \delta_{11}^i\delta_8^j + \delta_{12}^i\delta_9^j. \tag{9.59}$$

Similarly:

$$
\begin{aligned}
\mathsf{k}_{ij}^1 &= \Gamma^x\delta_1^i\delta_7^j + \Gamma^y\delta_2^i\delta_7^j + \Gamma^z\delta_3^i\delta_7^j \\
&\quad + B^x\left(\delta_2^i\delta_{11}^j + \delta_3^i\delta_{12}^j\right) + B^y\delta_{11}^i\delta_1^j + B^z\delta_{12}^i\delta_1^j + u^x\left(\delta_{10}^i\delta_7^j + \delta_{11}^i\delta_8^j + \delta_{12}^i\delta_9^j\right) \\
&\quad + \left\{u^x\left(\delta_{14}^i\delta_5^j + \delta_{13}^i\delta_6^j + \delta_{15}^i\delta_4^j\right) + \beta\delta_1^i\delta_5^j + \lambda\delta_1^i\delta_6^j - \rho\delta_1^i\delta_{15}^j\right\},
\end{aligned}\tag{9.60}
$$

$$
\begin{aligned}
\mathsf{k}_{ij}^2 &= \Gamma^x\delta_1^i\delta_8^j + \Gamma^y\delta_2^i\delta_8^j + \Gamma^z\delta_3^i\delta_8^j \\
&\quad + B^x\delta_{10}^i\delta_2^j + B^y\left(\delta_1^i\delta_{10}^j + \delta_3^i\delta_{12}^j\right) + B^z\delta_{12}^i\delta_2^j + u^y\left(\delta_{10}^i\delta_7^j + \delta_{11}^i\delta_8^j + \delta_{12}^i\delta_9^j\right) \\
&\quad + \left\{u^y\left(\delta_{13}^i\delta_6^j + \delta_{14}^i\delta_5^j + \delta_{15}^i\delta_4^j\right) + \beta\delta_2^i\delta_5^j + \lambda\delta_2^i\delta_6^j - \rho\delta_2^i\delta_{15}^j\right\},
\end{aligned}\tag{9.61}
$$

$$
\begin{aligned}
\mathsf{k}_{ij}^3 &= \Gamma^x\delta_1^i\delta_9^j + \Gamma^y\delta_2^i\delta_9^j + \Gamma^z\delta_3^i\delta_9^j \\
&\quad + B^x\delta_{10}^i\delta_1^j + B^y\delta_{11}^i\delta_2^j + B^z\left(\delta_1^i\delta_{10}^j + \delta_2^i\delta_{11}^j\right) + u^z\left(\delta_{10}^i\delta_7^j + \delta_{11}^i\delta_8^j + \delta_{12}^i\delta_9^j\right) \\
&\quad + \left\{u^z\left(\delta_{13}^i\delta_6^j + \delta_{14}^i\delta_5^j + \delta_{15}^i\delta_4^j\right) + \beta\delta_3^i\delta_5^j + \lambda\delta_3^i\delta_6^j - \rho\delta_3^i\delta_{15}^j\right\}.
\end{aligned}\tag{9.62}
$$

The one-form solutions for $\omega^\alpha = L_j^\alpha dz^j$ in (9.47)–(9.48) are related to the K_{jk}^α by (9.14), i.e.

$$\mathsf{K}_{jk}^\alpha = \frac{\partial L_k^\alpha}{\partial z^j} - \frac{\partial L_j^\alpha}{\partial z^k}. \tag{9.63}$$

Note that the solution of (9.63) for the L_j^α are not unique because $\omega^\alpha = L_j^\alpha dz^j + d\Phi(\mathbf{z})$ will also give the same K_{jk}^α.

As an example we find $\omega^0 = L_j^0 dz^j$ is given by:

$$\omega^0 = \left(z^{15}dz^4 + z^{14}dz^5 + z^{13}dz^6\right) + \left\{z^{10}dz^7 + z^{11}dz^8 + z^{12}dz^9\right\}$$
$$\equiv \phi d\rho + \beta dS + \lambda d\mu + \mathbf{\Gamma} \cdot d\mathbf{B}. \tag{9.64}$$

Similarly, $\omega^i = L_j^i dz^j$ gives (9.48) for ω^i. This completes the proof. $\qquad\square$

9.2.1 Exterior Differential Forms

Perhaps, the most elegant way to derive the above results of Proposition 9.2.1 is to use differential forms to deduce the skew symmetric matrices K^α and the one forms ω^α ($\alpha = 0, 1, 2, 3$) describing the system. This approach is described below. From (8.2) the MHD Lagrangian may be written in the form:

$$L = \frac{1}{2}\rho u^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu} + L_{z^s}^\alpha \frac{\partial z^s}{\partial x^\alpha}, \tag{9.65}$$

where

$$L_{z^s}^\alpha \frac{\partial z^s}{\partial x^\alpha} = \phi \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{u})\right) + \beta \left(\frac{\partial S}{\partial t} + \mathbf{u} \cdot \nabla S\right) + \lambda \left(\frac{\partial \mu}{\partial t} + \mathbf{u} \cdot \nabla \mu\right)$$
$$+ \mathbf{\Gamma} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) + \mathbf{u}(\nabla \cdot \mathbf{B})\right). \tag{9.66}$$

In particular:

$$L_{z^s}^0 \frac{\partial z^s}{\partial x^0} = \phi\rho_t + \beta S_t + \lambda\mu_t + \mathbf{\Gamma} \cdot \mathbf{B}_t, \tag{9.67}$$

and hence

$$\omega^0 = L_\rho^0 d\rho + L_S^0 dS + L_\mu^0 d\mu + L_\mathbf{B}^0 \cdot d\mathbf{B},$$
$$\equiv \phi d\rho + \beta dS + \lambda d\mu + \mathbf{\Gamma} \cdot d\mathbf{B}, \tag{9.68}$$

(note (9.68) define the non-zero $L_{z^s}^0$). The result (9.68) for ω^0 is the same as (9.47). Taking the exterior derivative of (9.68) gives:

$$d\omega^0 = d\phi \wedge d\rho + d\beta \wedge dS + d\lambda \wedge d\mu + d\Gamma_s \wedge dB^s \equiv \frac{1}{2}K_{z^s,z^p}^0 dz^s \wedge dz^p. \tag{9.69}$$

Hence

$$\mathsf{K}^0_{\phi,\rho} = \mathsf{K}^0_{\beta,S} = \mathsf{K}^0_{\lambda,\mu} = \mathsf{K}^0_{\Gamma_s,B^s} = 1 \quad (s = 1, 2, 3). \tag{9.70}$$

Thus we obtain the skew symmetric matrix K^0_{ij} given by (9.58) and (9.59).

A similar calculation gives:

$$\begin{aligned}
L^k_{z^s} \frac{\partial z^s}{\partial x^k} &= \phi \left(\rho \nabla_k u^k + u^k \nabla_k \rho \right) + \beta \left(u^k \nabla_k S \right) + \lambda u^k \nabla_k \mu \\
&\quad + \Gamma_s \left(u^k \nabla_k B^s + B^s \nabla_k u^k - B^k \nabla_k u^s \right),
\end{aligned} \tag{9.71}$$

from which we read off:

$$\begin{aligned}
L^k_\rho &= \phi u^k, \quad L^k_{u^i} = (\rho \phi + \mathbf{\Gamma} \cdot \mathbf{B}) \delta^k_i - \Gamma_i B^k, \\
L^k_S &= \beta u^k, \quad L^k_\mu = \lambda u^k, \quad L^k_{B^i} = \Gamma_i u^k,
\end{aligned} \tag{9.72}$$

Using (9.72) we obtain:

$$\omega^k = L^k_{z^s} dz^s = \{\mathbf{u}[\phi\, d\rho + \beta dS + \lambda d\mu] + \rho \phi d\mathbf{u} + (\mathbf{\Gamma} \cdot \mathbf{B}) d\mathbf{u} - \mathbf{B}(\mathbf{\Gamma} \cdot d\mathbf{u}) + \mathbf{u}(\mathbf{\Gamma} \cdot d\mathbf{B})\}^k, \tag{9.73}$$

which is the result (9.48) for ω^k. Taking the exterior derivative of (9.73) gives:

$$\begin{aligned}
d\omega^k &= du^k \wedge (\beta dS + \lambda d\mu - \rho d\phi - B^s d\Gamma_s) \\
&\quad + u^k (d\phi \wedge d\rho + d\beta \wedge dS + d\lambda \wedge d\mu + d\Gamma_s \wedge dB^s) \\
&\quad - \Gamma_s dB^k \wedge du^s - B^k d\Gamma_s \wedge du^s.
\end{aligned} \tag{9.74}$$

From (9.74) we obtain:

$$\begin{aligned}
\mathsf{K}^k_{u^k,S} &= \beta, \quad \mathsf{K}^k_{u^k,\mu} = \lambda, \quad \mathsf{K}^k_{u^k,\phi} = -\rho, \\
\mathsf{K}^k_{\Gamma_s,B^s} &= \mathsf{K}^k_{\phi,\rho} = \mathsf{K}^k_{\beta,S} = \mathsf{K}^k_{\lambda,\mu} = u^k, \quad \mathsf{K}^k_{u^s,B^k} = \Gamma_s, \\
\mathsf{K}^k_{u^k,\Gamma_s} &= -B^s, \quad \mathsf{K}^k_{u^s,\Gamma_s} = B^k \quad (k \neq s).
\end{aligned} \tag{9.75}$$

By using the state vector $\mathbf{z} = \left(\mathbf{u}^T, \rho, S, \mu, \mathbf{B}^T, \mathbf{\Gamma}^T, \lambda, \beta, \phi \right)^T$ and (9.75) gives the results (9.60)–(9.62) for the K^k_{ij} ($k = 1, 2, 3$).

Proposition 9.2.2 *The multi-symplectic conservation law (9.21) for $\beta = 0$ gives the conservation law:*

$$\frac{\partial D}{\partial t} + \nabla \cdot \mathbf{F} = 0, \tag{9.76}$$

where

$$D = \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \varepsilon(\rho, S) + \frac{B^2}{2\mu_0} \right) - \nabla \cdot (\mathbf{E} \times \mathbf{\Gamma}),$$

$$\mathbf{F} = \mathbf{u} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \varepsilon(\rho, S) + p \right) + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} + \frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{\Gamma}) - \nabla \times [(\mathbf{\Gamma} \cdot \mathbf{u})\mathbf{E}].$$
(9.77)

Because of null divergence terms in (9.77), the conservation law (9.76) reduces to the MHD energy conservation equation:

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \varepsilon(\rho, S) + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\mathbf{u} \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \varepsilon(\rho, S) + p \right) + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) = 0.$$
(9.78)

Similarly, the multi-symplectic conservation law (9.21) for $\beta = k$ gives a conservation law of the form (9.76) but with

$$D \equiv D^k = -\rho u^k + \nabla_k(\rho\phi + \mathbf{\Gamma} \cdot \mathbf{B}) - \nabla \cdot (\mathbf{\Gamma}^k \mathbf{B}),$$

$$F^i \equiv F^{ik} = - \left\{ \rho u^i u^k + \left(p + \frac{B^2}{2\mu_0} \right) \delta^{ik} - \frac{B^i B^k}{\mu_0} \right\}$$

$$+ \left[-\frac{\partial}{\partial t}(\rho\phi + \mathbf{\Gamma} \cdot \mathbf{B})\delta^{ik} + \frac{\partial}{\partial t} \left(\Gamma^k B^i \right) \right]$$

$$+ \nabla \times \left(\Gamma^k \mathbf{E} \right)^i + \nabla_k[\mathbf{\Gamma} \cdot \mathbf{B}u^i] - \nabla \cdot [\mathbf{\Gamma} \cdot \mathbf{B}u]\,\delta^{ik}.$$
(9.79)

The conservation law (9.76) reduces to:

$$-\left\{ \frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla \cdot \left[\rho\mathbf{u} \otimes \mathbf{u} + \left(p + \frac{B^2}{2\mu_0} \right)\mathsf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right] \right\}^k = 0,$$
(9.80)

i.e., the conservation law reduces to the MHD momentum conservation equation in the x^k- direction.

Proof The multi-symplectic Hamiltonian density H, and the 1-forms $\omega^\alpha = \Gamma_j^\alpha dz^j$ from (9.47) and (9.48) give:

$$L_j^0 z_{,\alpha}^j = \phi\rho_{,\alpha} + \beta S_{,\alpha} + \lambda\mu_{,\alpha} + \Gamma^s B_{,\alpha}^s,$$

$$L_j^i z_{,\alpha}^j = u^i [\beta S_{,\alpha} + \lambda\mu_{,\alpha} - \rho\phi_{,\alpha}] + \mathbf{\Gamma} \cdot \mathbf{u}B_{,\alpha}^i - \epsilon_{ijk}\Gamma^j E_{,\alpha}^k \quad (\alpha = 0, 1, 2, 3).$$
(9.81)

Using (9.81) we obtain:

$$L = L_j^\alpha z_{,\alpha}^j - H = p - \frac{B^2}{2\mu_0} + \frac{\partial}{\partial t}(\rho\phi),$$

$$H = -\left(\frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu_0}\right). \tag{9.82}$$

Using the results (9.81)–(9.82) in the symplectic conservation law (9.21) for $\beta = 0$ and $\beta = k$ gives the energy and momentum conservation laws (9.78) and (9.80).

□

The pullback conservation laws (9.21)

$$G_\beta = D_\alpha \left(L_j^\alpha(z)z_{,\beta}^j - L\delta_\beta^\alpha\right) = 0, \tag{9.83}$$

follow from the pullback of the identities:

$$\left(L_j^\alpha dz^j\right)_{,\alpha} = d\left\{L_j^\alpha(z)z_{,\alpha}^j - H(z)\right\} = dL,$$

$$\left(L_j^\alpha dz^j\right)_{,\alpha} = \left(L_j^\alpha z_{,\beta}^j dx^\beta\right)_{,\alpha} = \left(L_j^\alpha z_{,\beta}^j\right)_{,\alpha} dx^\beta = dL = \frac{\partial L^\alpha}{\partial x^\beta}dx^\beta. \tag{9.84}$$

The pullback equation $\kappa_{,\alpha}^\alpha = 0$ where $\kappa^\alpha = d\omega^\alpha$ gives rise to the symplecticity or phase-space conservation laws (structural conservation laws):

$$D_\alpha \left(\mathsf{K}_{ij}^\alpha z_{,\beta}^i z_{,\gamma}^j\right) = 0, \quad \beta < \gamma. \tag{9.85}$$

These conservation laws can also be written in the form:

$$D_\beta G_\gamma - D_\gamma G_\beta = D_\alpha \left(\mathsf{K}_{ij}^\alpha z_{,\beta}^i z_{,\gamma}^j\right) = 0, \tag{9.86}$$

i.e., the symplecticity conservation laws (9.86) are compatibility conditions for the pullback conservation laws (9.83).

The pullback conservation law for $\beta = 0$ in (9.83) reduces to the energy conservation law:

$$G_0 = -\left\{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \varepsilon(\rho, S) + \frac{B^2}{2\mu_0}\right) + \nabla\cdot\left(\rho\mathbf{u}\left(\frac{1}{2}u^2 + h\right) + \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right)\right\} = 0, \tag{9.87}$$

where $h = (\varepsilon + p)/\rho$ is the enthalpy of the gas, $\mathbf{E} = -\mathbf{u}\times\mathbf{B}$ is the electric field, and $\mathbf{E}\times\mathbf{B}/\mu_0$ is the Poynting flux. Similarly, the pullback conservation laws (9.83) for $\beta = i$ ($1 \leq i \leq 3$) give rise to the MHD momentum conservation equation:

$$G^i = -\left\{\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot\left[\rho\mathbf{u}\otimes\mathbf{u} + \left(p + \frac{B^2}{2\mu_0}\right)\mathbf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0}\right]\right\}^i = 0. \tag{9.88}$$

9.3   Symplecticity Conservation Laws Interpretation

The symplecticity conservation laws (9.85)–(9.86) have a generalized curl form. Consider the symplecticity laws (9.86) for $1 \leq i, k \leq 3$, namely:

$$\Omega_{ik} = D_i G_k - D_k G_i = 0. \tag{9.89}$$

Introduce the dual of the tensor Ω_{ik} defined as:

$$V^p = -\frac{1}{2}\varepsilon_{pik}\Omega_{ik} = -(\nabla \times \mathbf{G})^p, \tag{9.90}$$

where $\nabla \times \mathbf{G}$ is the spatial curl of $\mathbf{G}$. Taking into account the momentum conservation law (9.88) for $\mathbf{G}$, (9.90) reduces to:

$$\nabla \times \mathbf{G} = -\left\{\frac{\partial}{\partial t}\nabla \times \mathbf{M} + \nabla \times \left[\nabla \cdot \left(\mathbf{M} \otimes \mathbf{u} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0}\right)\right]\right\} = 0, \tag{9.91}$$

where

$$\mathbf{M} = \rho\mathbf{u}, \tag{9.92}$$

is the momentum density or mass flux $\mathbf{M}$ of the MHD fluid. Note there is no contribution from the magnetic pressure $(B^2/(2\mu_0))$ and gas pressure (p) gradient force terms in (9.91) because $\nabla \times \nabla(p + B^2/(2\mu_0)) = 0$ when one takes the curl of the momentum equation (9.88). The evolution of $\nabla \times \mathbf{M}$ in (9.91) is thus determined by the inertia and magnetic tension components:

$$\mathbf{M} \otimes \mathbf{u} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0}, \tag{9.93}$$

of the MHD stress-energy tensor. This suggests that (9.91) describes Alfvénic type disturbances, in which both fluid spin and magnetic tension forces are part of the dynamics. Equation (9.91) can also be expressed in the conservation law form:

$$\frac{\partial}{\partial t}\nabla \times \mathbf{M} + \nabla_s\left[\nabla \times \left(\mathbf{M}u^s - \frac{\mathbf{B}B^s}{\mu_0}\right)\right] = 0. \tag{9.94}$$

Pressure gradient forces play no role in the vorticity-like conservation laws (9.91) and (9.94).

Example For simple and double Alfvén waves (e.g. Webb et al. 1996, 2010b, 2011, 2012a), $p + B^2/2\mu_0 = const.$ so that the momentum conservation equation (9.88)

reduces to the form:

$$\frac{\partial \mathbf{M}}{\partial t} + \nabla \cdot \left(\mathbf{M} \otimes \mathbf{u} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right) = 0, \tag{9.95}$$

Simple and double phase Alfvén waves admit the integrals:

$$\mathbf{u} \pm \mathbf{V}_A = \mathbf{u} \pm \frac{\mathbf{B}}{\sqrt{\mu_0 \rho}} = (V_1, V_2, V_3) \equiv \mathbf{V} = const.,$$

$$p = c_4, \quad \rho = c_5, \quad p_B = \frac{B^2}{2\mu_0} = c_6, \tag{9.96}$$

where $V_1, V_2, V_3, c_4, c_5, c_6$ are constants. In a simple MHD wave, the physical variables depend on a single phase function $\varphi(\mathbf{x}, t)$ where $\lambda(\mathbf{n}(\varphi)) = \omega/k$ is the phase speed of the wave, $\mathbf{n}(\varphi) = \nabla\varphi/|\nabla\varphi|$ is the wave normal, which is a given function of φ such that $|\mathbf{n}| = 1$, $\mathbf{k} = \nabla\varphi$ is the wave vector of the wave and $\omega = -\varphi_t$ is the local frequency of the wave (note $\lambda(\mathbf{n}) = \omega/k$) (see e.g. Boillat 1970, Webb et al. 1996, 2011). A double MHD wave depends on two independent phase functions $\varphi_1(\mathbf{x}, t)$ and $\varphi_2(\mathbf{x}, t)$ (e.g. Grundland and Picard 2004; Webb et al. 2012a). The main point to note is that for simple and double Alfvén waves $p + B^2/2\mu_0 = const.$ and $\nabla(p + B^2/2\mu_0) = 0$, which explains the absence of the $\nabla(p + B^2/2\mu_0)$ in (9.95).

In the frame moving with velocity $\mathbf{V} = (V_1, V_2, V_3)$ (the group velocity frame), the velocity of the fluid is:

$$\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{V} = \mp \mathbf{V}_A = \sigma \mathbf{V}_A, \quad \sigma = \mp 1. \tag{9.97}$$

The momentum equation (9.95) in this frame reduces to:

$$\rho \tilde{\mathbf{u}} \cdot \nabla \tilde{\mathbf{u}} = \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0}, \tag{9.98}$$

which is a balance between the inertial force $\rho\tilde{\mathbf{u}} \cdot \nabla\tilde{\mathbf{u}}$ and the magnetic tension force $\mathbf{B} \cdot \nabla\mathbf{B}/\mu_0$. In general the magnetic field and fluid velocity rotate as one progresses through the wave. It is straightforward to verify (9.98) using the expression for $\tilde{\mathbf{u}}$ from (9.97). The generalized vorticity equation (9.91) clearly applies for general MHD flows. The above example illustrates (9.91) for the case of simple and double Alfvén waves.

The symplecticity conservation law (9.91) is different than that obtained by taking the curl of Euler momentum equation in the form $d\mathbf{u}/dt = \mathbf{F}$ where $\mathbf{F}$ is the net force on the fluid element, to obtain an equation for the evolution of the fluid vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Webb et al. (2014a,b) and Webb and Anco (2016) obtained a conservation law in ideal fluid mechanics (i.e. for $\mathbf{B} = 0$) for the generalized

vorticity

$$\boldsymbol{\Omega} = \boldsymbol{\omega} + \nabla r \times \nabla S \quad \text{where} \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}, \tag{9.99}$$

is the fluid vorticity and r satisfies the equation:

$$\frac{dr}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) r = -T. \tag{9.100}$$

Here $r = \beta/\rho$ where β is the Clebsch potential that ensures $dS/dt = 0$ in the Eulerian, Clebsch variational approach (e.g. Zakharov and Kuznetsov 1997) and $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative following the flow. Webb et al. (2014a,b) and Webb and Anco (2016) show that for fluid dynamics ($\mathbf{B} = 0$) the modified vorticity flux $\boldsymbol{\Omega} \cdot dS$ is advected or Lie dragged with the flow, i.e.,

$$\frac{d}{dt}(\boldsymbol{\Omega} \cdot dS) = \left[\frac{\partial \boldsymbol{\Omega}}{\partial t} - \nabla \times (\mathbf{u} \times \boldsymbol{\Omega}) + \mathbf{u}(\nabla \cdot \boldsymbol{\Omega})\right] \cdot dS = 0. \tag{9.101}$$

The conservation law (9.101) and the associated conservation law for the modified fluid helicity $\mathbf{u} \cdot \boldsymbol{\Omega}$ are nonlocal conservation laws that depend on the nonlocal variable $r = -\int_0^t T(\mathbf{x}, t)dt$ where the integration is with respect to the Lagrangian time t (e.g. Webb et al. 2014a,b). Conservation law (9.101) in fluid dynamics is analogous to Faraday's equation in MHD. Note that $\nabla \cdot \boldsymbol{\Omega} = 0$ in (9.101).

The generalized fluid helicity conservation equation obtained by using (9.101) has the form:

$$\frac{\partial}{\partial t}\left[\boldsymbol{\Omega} \cdot (\mathbf{u} + r\nabla S)\right] + \nabla \cdot \left[\mathbf{u}[\boldsymbol{\Omega} \cdot (\mathbf{u} + r\nabla S)] + \boldsymbol{\Omega}\left(h - \frac{1}{2}|\mathbf{u}|^2\right)\right] = 0. \tag{9.102}$$

where h is the enthalpy of the gas. This is a nonlocal conservation law as it depends on the nonlocal variable r (r satisfies (9.100)). In the case where $\nabla S = 0$, the conservation law reduces to the usual local fluid helicity conservation law. There is an analogous nonlocal conservation law for the generalized magnetic cross helicity for flows in which $\nabla S \neq 0$, namely:

$$\frac{\partial}{\partial t}\left[\mathbf{B} \cdot (\mathbf{u} + r\nabla S)\right] + \nabla \cdot \left[\mathbf{u}[\mathbf{B} \cdot (\mathbf{u} + r\nabla S)] + \mathbf{B}\left(h - \frac{1}{2}|\mathbf{u}|^2\right)\right] = 0. \tag{9.103}$$

Since the variable r is obtained by integrating the temperature back along the fluid element trajectory with respect to the Lagrangian time t, then the conservation laws (9.102) and (9.103) involve the memory of the past history of the fluid element.

There is another symplecticity conservation law obtained from (9.86) for the case $\beta = 0$ and $\gamma = 1, 2, 3$. In that case (9.86) reduces to:

$$\frac{\partial}{\partial t}\mathbf{G} - \nabla G_0 = 0, \tag{9.104}$$

where $\mathbf{G} = 0$ is the momentum equation (9.88) and $G_0 = 0$ is the energy conservation equation (9.87).

The general form of the symplecticity equations for MHD using Eulerian Clebsch potentials were given in Webb et al. (2014c) [equations (5.44) et seq. of that paper]. There were some typographical errors in the flux F_{ab}^k, indicated below. The general symplecticity conservation laws obtained by Webb et al. (2014c) have the form:

$$\frac{\partial}{\partial t}\left(F_{ab}^0\right) + \frac{\partial}{\partial x^k}\left(F_{ab}^k\right) = 0, \tag{9.105}$$

where

$$F_{ab}^0 = \frac{\partial(\phi, \rho)}{\partial(x^a, x^b)} + \frac{\partial(\beta, S)}{\partial(x^a, x^b)} + \frac{\partial(\lambda, \mu)}{\partial(x^a, x^b)} + \frac{\partial(\Gamma_s, B^s)}{\partial(x^a, x^b)}, \tag{9.106}$$

and

$$\begin{aligned}
F_{ab}^k = {} & -\frac{\partial(\rho u^k, \phi)}{\partial(x^a, x^b)} + \frac{\partial(\beta u^k, S)}{\partial(x^a, x^b)} + \frac{\partial(\lambda u^k, \mu)}{\partial(x^a, x^b)} \\
& + \frac{\partial(\Gamma_s B^s, u^k)}{\partial(x^a, x^b)} + \frac{\partial(\Gamma_s u^k, B^s)}{\partial(x^a, x^b)} - \frac{\partial(\Gamma_s B^k, u^s)}{\partial(x^a, x^b)}.
\end{aligned} \tag{9.107}$$

In (9.107) $1 \leq k \leq 3$. The fourth term on the right-hand side of (9.107) was missed in Eq. (5.48) in Webb et al. (2014c). Also in (9.107) we used the identity:

$$\frac{\partial(\phi u^k, \rho)}{\partial(x^a, x^b)} + \frac{\partial(\rho\phi, u^k)}{\partial(x^a, x^b)} = -\frac{\partial(\rho u^k, \phi)}{\partial(x^a, x^b)}, \tag{9.108}$$

to simplify (5.48) of Webb et al. (2014c). The derivation of the symplecticity conservation laws (9.91) and (9.104) using the general symplecticity laws (9.105) and using (8.3)–(8.9) is a non-trivial algebraic exercise.

9.4   Differential Forms Approach

Proposition 4.3 in Webb et al. (2014c) contains flaws. These flaws are corrected in Webb et al. (2015). The first statement in Proposition 4.3 of Webb et al. (2014c) is correct, but it is not a variational statement. Equation (4.46) of Webb et al. (2014c)

is incorrect, i.e. $\Omega \neq dz^\mu \wedge \beta_\mu$. Propositions 4.1 and 4.2 given below replace Proposition 4.3 of Webb et al. (2014c). A consistent approach to the multi-symplectic equations using differential forms for 1D Lagrangian gas dynamics was given by Webb (2015). Webb and Anco (2016) have given the corresponding theory for multi-dimensional, ideal, compressible, Lagrangian gas dynamics. Below we use differential forms to describe the Eulerian, Clebsch variable MHD variational principle of Webb et al. (2014c, 2015).

Proposition 9.4.1 *The multi-symplectic system (9.43) is a stationary point of the action:*

$$J = \int \psi^*(\Theta) = \int L dV, \qquad (9.109)$$

where $\psi^*(\Theta)$ is the pullback of the differential form Θ given below, namely:

$$\Theta = \omega^\alpha \wedge d\tilde{x}^\alpha - H dV, \quad \omega^\alpha = L_j^\alpha dz^j,$$

$$dV = dt \wedge dx \wedge dy \wedge dz, \quad d\tilde{x}^\alpha = \partial_\alpha \lrcorner dV \equiv (-1)^\alpha dx_0 \wedge \ldots dx^{\alpha-1} \wedge dx^{\alpha+1} \ldots \wedge dx^n, \qquad (9.110)$$

where we use the notation $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, and L is the constrained Lagrangian (8.2).

Proof The pullback of the form Θ is given by:

$$\psi^*(\Theta) = \psi^* \left(L_j^\alpha dz^j \wedge d\tilde{x}_\alpha - H dV \right)$$

$$= L_j^\alpha \frac{\partial z^j}{\partial x^s} dx^s \wedge d\tilde{x}^\alpha - H dV. \qquad (9.111)$$

However,

$$dx^s \wedge d\tilde{x}_\alpha = dx^s \wedge (-1)^\alpha dx^0 \ldots \wedge dx^{\alpha-1} \wedge dx^{\alpha+1} \ldots \wedge dx^n \equiv (-1)^{2\alpha} \delta_\alpha^s dV. \qquad (9.112)$$

Thus,

$$\psi^*(\Theta) = \left(L_j^\alpha \frac{\partial z^j}{\partial x^\alpha} - H \right) dV \equiv L dV, \qquad (9.113)$$

where L is the multi-symplectic Lagrangian (8.2).

The stationary point conditions, $\delta J/\delta z^i = 0$, give the Euler-Lagrange equations:

$$\frac{\delta J}{\delta z^i} = \frac{\partial L}{\partial z^i} - \frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial z^i_{,j}} \right) = \mathsf{K}_{ij}^\alpha \frac{\partial z^j}{\partial x^\alpha} - \frac{\partial H}{\partial z^i} = 0, \qquad (9.114)$$

which is the multi-symplectic system (9.43) (see also Hydon 2005). $\qquad \square$

Proposition 9.4.2 *Consider the variational functional:*

$$G[\Omega] = \int_M \Omega, \tag{9.115}$$

where

$$\Omega = d\Theta = d\omega^\alpha \wedge d\tilde{x}_\alpha - dH \wedge dV. \tag{9.116}$$

and M is a region of the jet space (fiber bundle space) with boundary ∂M, in which the z^s are taken as independent of the base variables x^α ($\alpha = 0, 1, 2, 3$). Consider the variational principle:

$$\delta G[\Omega] = \int_M \mathcal{L}_{\mathbf{V}}(\Omega) = 0, \tag{9.117}$$

where

$$\mathcal{L}_{\mathbf{V}} = \frac{d}{d\epsilon} = V^i \frac{\partial}{\partial z^i}, \tag{9.118}$$

is the Lie derivative with respect to the arbitrary, but smooth vector field $\mathbf{V}$. The variational equation $\delta G[\Omega] = 0$ reduces to:

$$\delta G[\Omega] = \int_{\partial M} V^p \beta_p = 0, \tag{9.119}$$

where the forms β_p are given by the formulae:

$$\beta_p = \frac{\partial}{\partial z^p} \lrcorner \Omega = \mathsf{K}^\alpha_{pj} dz^j \wedge d\tilde{x}_\alpha - \frac{\partial H}{\partial z^p} dV. \tag{9.120}$$

($1 \leq p \leq N$). Because the V^p are arbitrary smooth functions of the z^s, the variational principle $\delta G[\Omega] = 0$ implies:

$$\beta_p = 0, \quad 1 \leq p \leq N. \tag{9.121}$$

The pullback of the forms $\{\beta_p\}$ to the base manifold gives the equations:

$$\tilde{\beta}_p = \left(\mathsf{K}^\alpha_{pj} \frac{\partial z^j}{\partial x^\alpha} - \frac{\partial H}{\partial z^p} \right) dV = 0. \tag{9.122}$$

Thus, the sectioned forms $\tilde{\beta}_p$ vanish on the solution manifold of the multi-symplectic system (9.43), and the $\{\beta_p\}$ can be used as a basis of Cartan forms describing the system (9.43).

Proof The proof is essentially the same as that given by Webb (2015) for the case of 1D gas dynamics (see also Webb et al. 2015). A critical component of the proof is the use of Cartan's magic formula:

$$\mathcal{L}_{\mathbf{V}}(\Omega) = \mathbf{V} \lrcorner d\Omega + d(\mathbf{V} \lrcorner \Omega) = d(\mathbf{V} \lrcorner \Omega),\qquad (9.123)$$

where we used the facts $\Omega = d\Theta$ and $d\Omega = dd\Theta = 0$. Note that the Lie derivative with respect to a vector field $\mathbf{V}$ means the derivative along a curve, with tangent vector $\mathbf{V}$ in which ϵ refers to the parameter along the curve. Usually one has in mind a continuous group of transformations, in which $\epsilon = 0$ corresponds to the identity transformation. We use the notation $\mathcal{L}_{\mathbf{V}} \equiv d/d\epsilon$ to denote the directional derivative along the curve with tangent vector $\mathbf{V}$. Using (9.123) and Stokes theorem, (9.117) reduces to:

$$\delta G[\Omega] = \int_M d(\mathbf{V} \lrcorner \Omega) = \int_{\partial M} \mathbf{V} \lrcorner \Omega = \int_{\partial M} V^p \left(\frac{\partial}{\partial z^p} \lrcorner \Omega \right) = \int_{\partial M} V^p \beta_p = 0,$$
$$(9.124)$$

which verifies (9.119). The formula (9.120) for β_p is obtained by using (9.46) for $d\omega^{\alpha}$ and (9.116) for Ω, to obtain:

$$\beta_p = \frac{\partial}{\partial z^p} \lrcorner \Omega = \frac{\partial}{\partial z^p} \lrcorner \left(\frac{1}{2} \mathsf{K}_{ij}^{\alpha} dz^i \wedge dz^j \wedge d\tilde{x}_{\alpha} - \frac{\partial H}{\partial z^a} dz^a \wedge dV \right),\qquad (9.125)$$

Using the skew symmetry of K_{ij}^{α} and $dz^i \wedge dz^j$, (9.125) reduces to the expression (9.120) for β_p. This completes the proof. $\qquad\square$

Remark The integrability conditions $dd\omega^{\alpha} = 0$ imply the identities:

$$\mathsf{K}_{ij,k}^{\alpha} + \mathsf{K}_{jk,i}^{\alpha} + \mathsf{K}_{ki,j}^{\alpha} = 0,\qquad (9.126)$$

where $\mathsf{K}_{ij,k}^{\alpha} \equiv \partial \mathsf{K}_{ij}^{\alpha}/\partial z^k$, $i \neq j \neq k$, $1 \leq i,j,k \leq N$ and $0 \leq \alpha \leq 3$. These identities are equivalent to the Jacobi identity in the case of finite dimensional, Hamiltonian systems in which there is only one evolution variable in which K^0 is invertible (Zakharov and Kuznetsov 1997). They are equivalent to the requirement that the L_i^{α} have continuous second order partial derivatives with respect to z^j and z^k, i.e. $\left(L_i^{\alpha} \right)_{,j,k} = \left(L_i^{\alpha} \right)_{,k,j}$.

9.5 The Differential Forms β_p

The differential forms $\beta_p = \partial_{z^p} \lrcorner \Omega$ in (9.120) may be used to represent the MHD system described by the Clebsch variable variational principle. The dependent variables $\mathbf{z}$ are listed in (9.44). In the time evolution variational principle (e.g. Zakharov and Kuznetsov 1997), the fluid velocity $\mathbf{u}$ is expressed in terms of the Clebsch potentials, and is eliminated from the Hamiltonian density $H = (1/2)\rho u^2 +$

$\varepsilon(\rho, S) + B^2/(2\mu_0)$ and (ρ, ϕ), (S, β), (μ, λ), $(\mathbf{B}, \boldsymbol{\Gamma})$ are canonically conjugate pairs in the canonical Poisson bracket. We use the notation:

$$\beta^{z^i} = \partial_{z^i} \lrcorner \Omega, \tag{9.127}$$

where the Cartan Poincaré form Ω in (9.116) has the form:

$$\Omega = d\omega^0 \wedge d\tilde{x}_0 + d\omega^k \wedge d\tilde{x}_k - \frac{\partial H}{\partial z^p} dz^p \wedge dV, \tag{9.128}$$

and the differential forms ω^0 and ω^k are listed in (9.47)–(9.48). From (9.127) and (9.128)

$$\beta^{z^i} = \partial_{z^i} \lrcorner \left\{ d\omega^0 \wedge d\tilde{x}_0 + d\omega^k \wedge d\tilde{x}_k - \frac{\partial H}{\partial z^p} dz^p \wedge dV \right\}. \tag{9.129}$$

Using (9.47)–(9.48) and (9.129) we obtain:

$$\beta^{u^i} = (\beta dS + \lambda d\mu - \rho d\phi - B^s d\Gamma_s) \wedge d\tilde{x}_i + \left(\Gamma_i dB^k + B^k d\Gamma_i\right) \wedge d\tilde{x}_k + \rho u^i dV, \tag{9.130}$$

for the differential forms associated with $\mathbf{u}$. Using the identity

$$dx^a \wedge d\tilde{x}_i = \delta_i^a dV, \tag{9.131}$$

the sectioned form equation $\tilde{\beta}^{u^i} = 0$ yields the expression:

$$\rho \mathbf{u} = \rho \nabla \phi - \beta \nabla S - \lambda \nabla \mu + \mathbf{B} \cdot (\nabla \boldsymbol{\Gamma})^T - \mathbf{B} \cdot \nabla \boldsymbol{\Gamma} - \boldsymbol{\Gamma} \nabla \cdot \mathbf{B}, \tag{9.132}$$

which is equivalent to the Clebsch expansion for the mass flux $\rho \mathbf{u}$ given in (8.3).

The differential form β^ρ is given by:

$$\beta^\rho = \partial_\rho \lrcorner \Omega = (\partial_\rho \lrcorner d\omega^0) \wedge d\tilde{x}_0 + (\partial_\rho \lrcorner d\omega^k) \wedge d\tilde{x}_k - \frac{\partial H}{\partial \rho} dV. \tag{9.133}$$

Using (9.82) for H, we obtain:

$$\frac{\partial H}{\partial \rho} = -\left(\frac{1}{2}u^2 - \varepsilon_\rho\right) = -\left(\frac{1}{2}u^2 - h\right), \tag{9.134}$$

where $h = (\varepsilon + p)/\rho$ is the gas enthalpy. Substituting (9.134) in (9.133) gives

$$\beta^\rho = -\left(d\phi \wedge d\tilde{x}_0 + u^k d\phi \wedge d\tilde{x}_k\right) + \left(\frac{1}{2}u^2 - h\right) dV, \tag{9.135}$$

$$\tilde{\beta}^\rho = -\left[\frac{d\phi}{dt} - \left(\frac{1}{2}u^2 - h\right)\right] dV, \tag{9.136}$$

for the differential form β^ρ and for the sectioned form $\tilde{\beta}^\rho$. Note that $\tilde{\beta}^\rho = 0$ is equivalent to Bernoulli's equation (8.6). Similarly, we obtain:

$$\beta^S = \partial_S \lrcorner \Omega = -d\beta \wedge d\tilde{x}_0 - d(\beta u^k) \wedge d\tilde{x}_k - \rho T dV, \tag{9.137}$$

$$\tilde{\beta}^S = -\left(\frac{\partial \beta}{\partial t} + \nabla \cdot (\beta \mathbf{u}) + \rho T \right) dV. \tag{9.138}$$

The equation $\tilde{\beta}^S = 0$ corresponds to (8.7) for β.

Following the above procedure we obtain the equations:

$$\beta^\mu = \partial_\mu \lrcorner \Omega = -\left[d\lambda \wedge d\tilde{x}_0 + d(\lambda u^k) \wedge d\tilde{x}_k \right],$$

$$\beta^{B^i} = -\left[d\Gamma_i \wedge d\tilde{x}_0 + u^k d\Gamma_i \wedge d\tilde{x}_k + \Gamma_s du^s \wedge d\tilde{x}_i + \frac{B^i}{\mu_0} dV \right],$$

$$\beta^{\Gamma_i} = dB^i \wedge d\tilde{x}_0 + \left(B^i du^k + u^k dB^i - B^k du^i \right) \wedge d\tilde{x}_k,$$

$$\beta^\lambda = \partial_\lambda \lrcorner \Omega = \left(d\mu \wedge d\tilde{x}_0 + u^k d\mu \wedge d\tilde{x}_k \right),$$

$$\beta^\beta = \partial_\beta \lrcorner \Omega = dS \wedge d\tilde{x}_0 + u^k dS \wedge d\tilde{x}_k,$$

$$\beta^\phi = \partial_\phi \lrcorner \Omega = d\rho \wedge d\tilde{x}_0 + \left(u^k d\rho + \rho du^k \right) \wedge d\tilde{x}_k. \tag{9.139}$$

The pullback of the above equations, i.e. $\tilde{\beta}^{z^p} = 0$, gives the evolution equations for $(\lambda, \mathbf{\Gamma}, \mathbf{B}, \mu, S, \rho)$ listed in (8.5)–(8.9). The differential form equations $\beta^{z^p} = 0$ thus represent the partial differential equation system (9.43).

It is not obvious that the system of forms $\{\beta^{z^p}\}$ above is a closed ideal. A check on the closure of the forms for the case of non-barotropic, 1D gas dynamics (i.e. $\mathbf{B} = 0$) indicates that the ideal of forms $\mathcal{I} = \{\beta^u, \beta^\rho, \beta^S, \beta^\beta, \beta^\phi\}$ can be closed by adjoining the form $d\beta^u$. The ideal $\mathcal{I}$ is closed for the case of a barotropic gas. The Cartan approach to Lie symmetries requires that $\mathcal{I}$ is a closed ideal (e.g. Harrison and Estabrook 1971). The ideal is closed if $d\beta_i = c_{ij} \wedge \beta^{z^j}$, where the c_{ij} are forms. It should be noted that the ideal of forms obtained by Webb (2015) for 1D, Lagrangian, multi-symplectic gas dynamics is closed. The ideal of forms for multi-dimensional, Lagrangian, compressible gas dynamics obtained from the Cartan-Poincaré form is a closed ideal (Webb and Anco 2016). The ideal of forms using the Clebsch variable description has a more complicated structure than the set of forms that arise in the Lagrangian variational approach. The exact relationship between the Clebsch and Lagrangian approaches remains to be elucidated.

Chapter 10
The Lagrangian Map

In this chapter we give a synopsis of Lagrangian MHD, as initially developed by Newcomb (1962). The analysis is also based on the work of Webb et al. (2005a,b), Webb and Zank (2007) and Golovin (2011) where the MHD, Lie point symmetries and the fluid relabelling symmetries were investigated using the Lagrangian map. Golovin (2011) converted the MHD equations to Lagrangian form, to obtain a vector wave equation form for the Lagrangian momentum equation, that takes into account the symmetries of the equation associated with Faraday's equation (see also e.g. Schief 2003).

Golovin (2011) also obtains equivalence transformations for both the incompressible and compressible MHD equations. Equivalence transformations preserve the functional form of the equations, but may change the arguments and the scaling of the different forms in the equations (e.g. the equation of state can change under an equivalence transformation). A simple example of equivalence transformations are:

$$\tilde{t} = t, \quad \tilde{x} = ax, \quad \tilde{u} = au, \quad \tilde{\kappa} = a^2\kappa, \tag{10.1}$$

which maps the heat equation onto a modified heat equation:

$$u_t - \kappa u_{xx} = 0 \quad \rightarrow \quad \tilde{u}_{\tilde{t}} - \tilde{\kappa}\tilde{u}_{\tilde{x}\tilde{x}} = 0. \tag{10.2}$$

A more formal definition of an equivalence transformation is given in Bluman et al. (2010, p. 21) (see also Appendix F). Webb and Zank (2007) investigated both the ten parameter Galilei Lie group of Lie point symmetries, and also a class of scaling, Lie point symmetries. The scaling symmetries, are interesting in the sense, that the fluid relabeling symmetry determining equations become modified when the scaling symmetries are transformed from Eulerian form, to their Lagrange label space form.

10.1 Lagrangian MHD

The Lagrangian map: $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ is obtained by integrating the fluid velocity equation $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$, subject to the initial condition $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$. In this approach, the mass continuity equation and entropy advection equation are replaced by the equivalent algebraic equations:

$$\rho = \frac{\rho_0(\mathbf{x}_0)}{J}, \quad S = S(\mathbf{x}_0), \tag{10.3}$$

where

$$J = \det(x_{ij}) \quad \text{and} \quad x_{ij} = \frac{\partial x^i(\mathbf{x}_0, t)}{\partial x_0^j}. \tag{10.4}$$

Similarly, Faraday's equation (2.4) has the formal solution for the magnetic field induction $\mathbf{B}$ of the form:

$$B^i = \frac{x_{ij}B_0^j}{J}, \quad \nabla_0 \cdot \mathbf{B}_0 = 0. \tag{10.5}$$

The solution (10.5) for B^i is equivalent to the frozen in field theorem in MHD (e.g. Stern 1966; Parker 1979), and the initial condition $\nabla_0 \cdot \mathbf{B}_0 = 0$ is imposed in order to ensure that Gauss's law $\nabla \cdot \mathbf{B} = 0$ is satisfied.

The Lagrangian map $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ and its inverse $\mathbf{x}_0 = \mathbf{X}_0(\mathbf{x}, t)$ are characterized by the relations:

$$x_{is}y_{sp} = \delta_p^i, \quad \frac{\partial x^i}{\partial t} + x_{is}\frac{\partial x_0^s}{\partial t} = 0, \tag{10.6}$$

where

$$x_{is} = \frac{\partial x^i}{\partial x_0^s} \quad \text{and} \quad y_{sp} = \frac{\partial x_0^s}{\partial x^p}. \tag{10.7}$$

From (10.6) and (10.7) we obtain:

$$\frac{\partial x_0^i}{\partial t} + u^s\frac{\partial x_0^i}{\partial x^s} = 0, \tag{10.8}$$

showing that the Lagrange label $\mathbf{x}_0$ is advected with the background flow with velocity $\mathbf{u} = \partial\mathbf{x}(\mathbf{x}_0, t)/\partial t$.

From Cramer's rule:

$$y_{ij} = \frac{A_{ji}}{J}, \quad x_{ij} = JB_{ji}, \tag{10.9}$$

where $A_{ij} = \text{cofac}(x_{ij})$ and $B_{ij} = \text{cofac}(y_{ij})$ are the co-factor matrices associated with x_{ij} and y_{ij} (note A_{ij} and B_{ij} are inverse matrices). One can show:

$$A_{ij} = \frac{1}{2}\epsilon_{ipq}\epsilon_{jmn}x_{pm}x_{qn}, \quad B_{ij} = \frac{1}{2}\epsilon_{ipq}\epsilon_{jmn}y_{pm}y_{qn}, \tag{10.10}$$

where ϵ_{ijk} is the anti-symmetric permutation tensor density [see e.g. Newcomb (1962)]. From (10.10) it follows that $\partial A_{ij}/\partial x_0^j = 0$ and $\partial B_{ij}/\partial x^j = 0$.

The action for the MHD system is:

$$A = \int\int \mathcal{L}\, d^3x dt \equiv \int\int \mathcal{L}^0\, d^3x_0 dt, \tag{10.11}$$

where

$$\mathcal{L} = \frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu} - \rho\Phi, \quad \mathcal{L}^0 = \mathcal{L}J, \tag{10.12}$$

are the Eulerian ($\mathcal{L}$) and Lagrangian ($\mathcal{L}^0$) Lagrange densities respectively. Using (10.3)–(10.5), and (10.12) we obtain:

$$\mathcal{L}^0 = \frac{1}{2}\rho_0|\mathbf{x}_t|^2 - J\varepsilon\left(\frac{\rho_0}{J}, S\right) - \frac{x_{ij}x_{is}B_0^j B_0^s}{2\mu J} - \rho_0\Phi, \tag{10.13}$$

for $\mathcal{L}^0$. Note that in the Lagrange density $\mathcal{L}^0 = \mathcal{L}^0(\mathbf{x}_0, t; \mathbf{x}, \mathbf{x}_t, x_{ij})$, $\mathbf{x}_0$ and t are the independent variables, and $\mathbf{x}$ and its derivatives with respect to $\mathbf{x}_0$ and t are dependent variables.

Extremization of the action in (10.11) gives the Euler-Lagrange equations:

$$\frac{\delta A}{\delta x^i} = \frac{\partial \mathcal{L}^0}{\partial x^i} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}^0}{\partial x_t^i}\right) - \frac{\partial}{\partial x_0^s}\left(\frac{\partial \mathcal{L}^0}{\partial x_{ij}}\right) = 0, \tag{10.14}$$

where $x_{ij} \equiv \partial x^i/\partial x_0^j$. Evaluation of the variational derivative (10.14) gives the Lagrangian momentum equation for the system in the form (Newcomb 1962):

$$\rho_0\left(\frac{\partial^2 x^i}{\partial t^2} + \frac{\partial \Phi}{\partial x^i}\right) + \frac{\partial}{\partial x_0^j}\left\{A_{kj}\left[\left(p + \frac{B^2}{2\mu}\right)\delta^{ik} - \frac{B^i B^k}{\mu_0}\right]\right\} = 0, \tag{10.15}$$

where $A_{kj} = \text{cofac}(x_{kj})$. Dividing (10.15) by J, and using the fact that $\partial A_{kj}/\partial x_0^j = 0$, gives the Eulerian form of the momentum equation (2.2).

The above analysis uses Lagrangian variations of the action in which $\mathbf{x}_0$ is fixed. The Lagrangian variation of $\mathbf{x}(\mathbf{x}_0, t; \epsilon)$ is defined as $\Delta\mathbf{x} = \partial\mathbf{x}/\partial\epsilon$ evaluated at $\epsilon = 0$ and keeping $\mathbf{x}_0$ fixed. The Eulerian variation $\delta\psi$ and Lagrangian variation $\Delta\psi$ of a

physical variable ψ are related by the equations:

$$\Delta\psi = \delta\psi + \Delta\mathbf{x}\cdot\nabla\psi, \quad \delta\psi = \Delta\psi + \delta\mathbf{x}_0\cdot\nabla_0\psi, \tag{10.16}$$

where $\delta\mathbf{x}_0$ is the Eulerian variation of $\mathbf{x}_0$. In (10.16) the Lagrangian variation $\Delta\psi = (\psi_\epsilon)_{\mathbf{x}_0}$ is evaluated with $\mathbf{x}_0$ held constant and $\partial\psi/\partial\epsilon$ is evaluated at $\epsilon = 0$, whereas the Eulerian variation $\delta\psi = (\psi_\epsilon)_{\mathbf{x}}$ is evaluated at $\epsilon = 0$ with $\mathbf{x}$ held constant. In (10.14) the action is extremized using Lagrangian variations in which $\mathbf{x}_0$ is held constant. It is also possible to extremize the first form of the action in (10.11) using Eulerian variations in which $\mathbf{x}$ is held constant, leading to the Eulerian form of the momentum equation (2.2).

10.2   Hamiltonian Formulation

For the above Lagrangian description of MHD, the equations for the system can be written in terms of the canonical Poisson bracket:

$$\{F, G\} = \int \sum_{k=1}^{3} \left(\frac{\delta F}{\delta q^k} \frac{\delta G}{\delta \Pi_k} - \frac{\delta G}{\delta q^k} \frac{\delta F}{\delta \Pi_k} \right) d^3x_0. \tag{10.17}$$

where

$$q^k = x^k(\mathbf{x}_0, t) \quad \text{and} \quad \Pi_k = \frac{\partial \mathcal{L}^0}{\partial \dot{x}^k} = \rho_0 \dot{x}^k \equiv \rho_0 u^k. \tag{10.18}$$

The Lagrangian density $\mathcal{L}^0$ is given by (10.13). The Hamiltonian functional $\mathcal{H}$ is given by

$$\mathcal{H} = \int H \, d^3x_0, \tag{10.19}$$

and the Hamiltonian density H is given by the Legendre transformation, i.e.,

$$H = \sum_{k=1}^{3} \Pi_k \dot{q}^k - \mathcal{L}^0 \equiv \frac{\Pi_k^2}{2\rho_0} + J\varepsilon\left(\frac{\rho_0}{J}, S\right) + \rho_0\Phi(\mathbf{x}) + \frac{x_{ij}x_{is}B_0^j B_0^s}{2\mu J}, \tag{10.20}$$

is the total energy density. Here $\dot{x}^k = dx^k/dt$ and $\dot{\Pi}_k = d\Pi_k/dt$ are Lagrangian time derivatives, keeping $\mathbf{x}_0$ constant. The equations of motion can be expressed in the canonical Hamiltonian form:

$$\dot{x}^k = \frac{\delta\mathcal{H}}{\delta\Pi_k} = \frac{\Pi_k}{\rho_0}, \quad \dot{\Pi}_k = -\frac{\delta\mathcal{H}}{\delta x_k}. \tag{10.21}$$

Hamilton's equations (10.21) can also be written in terms of the Poisson bracket (10.17) in the form:

$$\dot{x}^k = \{x^k, \mathcal{H}\} \quad \text{and} \quad \dot{\Pi}_k = \{\Pi_k, \mathcal{H}\}. \tag{10.22}$$

The canonical form of the MHD equations in (10.17)–(10.22) is useful in the discussion of the MHD Casimirs and their relation to the fluid relabelling symmetries (e.g. Padhye and Morrison 1996a,b). The nonlinear stability of fluids and plasmas (e.g. Holm et al. 1985; Arnold and Khesin 1998) uses the Casimir constraints and the Lagrangian displacement $\boldsymbol{\xi}$ to describe MHD instabilities.

10.3 Lagrangian Wave Equations

In this section we reduce the Lagrangian momentum equation (10.15) to a system of three coupled nonlinear MHD wave equations for $x^i(\mathbf{x}_0, t)$ ($1 \le i \le 3$). We also discuss the characteristic manifold of the system of coupled pdes for the $x^i(\mathbf{x}_0, t)$.

Using the Lagrangian map equations (10.3)–(10.5) for ρ and $\mathbf{B}$ and noting that $p = p(\rho, S)$, (10.15) reduces the coupled wave system:

$$A_j^{i\alpha\beta} x_{\alpha\beta}^j + R^i = 0, \tag{10.23}$$

where $x_{\alpha\beta}^j = \partial^2 x^j / \partial x_0^\alpha \partial x_0^\beta$ ($\alpha, \beta = 0, 1, 2, 3$, $1 \le j \le 3$), i.e. Greek indices assume the values $0, 1, 2, 3$ and Latin indices assume the values $1, 2, 3$) and $(t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$. In (10.23),

$$
\begin{aligned}
A_j^{i\alpha\beta} &= \delta^{ij} \left[\delta^{\alpha 0} \delta^{\beta 0} - \delta^{\alpha p} \delta^{\beta q} b^s b^k y_{pk} y_{qs} \right] \\
&\quad + \delta^{\alpha p} \delta^{\beta q} \left[-(a^2 + b^2) y_{qi} y_{pj} + b^j b^s y_{qi} y_{ps} + b^s b^i y_{pj} y_{qs} \right], \tag{10.24}
\end{aligned}
$$

$$
\begin{aligned}
R^i &= \frac{B^r}{\mu \rho_0} \frac{\partial B_0^s}{\partial x_0^p} \left(y_{pi} x_{rs} - x_{is} y_{pr} \right) \\
&\quad + \frac{A_{ij}}{\rho_0} \left(\frac{a^2}{J} \frac{\partial \rho_0}{\partial x_0^j} + \frac{\partial p}{\partial S} \frac{\partial S}{\partial x_0^j} \right) + \frac{\partial \Phi}{\partial x^i}. \tag{10.25}
\end{aligned}
$$

Here $\mathbf{b} = \mathbf{B}/\sqrt{\mu\rho}$ is the local Alfvén velocity and $a = (\partial p/\partial \rho)^{1/2}$ is the adiabatic sound speed of the gas. The characteristic manifolds of the partial differential equation system (10.23) are defined as manifolds $\phi(\mathbf{x}, t) = const.$ on which the Cauchy, initial value problem does not have a unique solution. The characteristic manifolds of (10.23) are given by the solutions of the determinantal equation:

$$\det(\tilde{\mathbf{A}}) = 0 \quad \text{where} \quad \tilde{A}_j^i = A_j^{i\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \tag{10.26}$$

and $\phi_{,\alpha} \equiv \partial\phi/\partial x_0^\alpha$. The matrix $\tilde{\mathbf{A}}$ can be written in the form:

$$\tilde{A}_j^i = \left[\omega'^2 - (\mathbf{b}\cdot\mathbf{k})^2\right]\delta^{ij} - (a^2 + b^2)k^i k^j + (\mathbf{b}\cdot\mathbf{k})\left(b^i k^j + b^j k^i\right), \qquad (10.27)$$

where

$$\omega' = -\frac{\partial\phi(\mathbf{x}_0, t)}{\partial t} \equiv -(\phi_t + \mathbf{u}\cdot\nabla\phi), \quad \mathbf{k} = \nabla\phi, \qquad (10.28)$$

(ϕ_t denotes the time derivative of ϕ keeping $\mathbf{x}$ constant) define the wave frequency ω' in the fluid frame and wave number $\mathbf{k}$. The determinant of $\tilde{\mathbf{A}}$ is:

$$\det(\tilde{\mathbf{A}}) = \left[\omega'^2 - (\mathbf{b}\cdot\mathbf{k})^2\right]\left[\omega'^4 - (a^2 + b^2)\omega'^2 k^2 + a^2 k^2(\mathbf{b}\cdot\mathbf{k})^2\right] = 0, \qquad (10.29)$$

(cf. Webb et al. 2005a). The first factor in (10.29) corresponds to the Alfvén wave modes, and the second factor in square brackets corresponds to the fast and slow magnetosonic modes respectively.

10.3.1 One Dimensional Gas Dynamics

As a simple example of the Lagrangian wave equation (10.23) consider the case of one dimensional gas dynamics in one Cartesian space dimension x, in which the fluid velocity u is along the x-axis. In the Eulerian description the physical quantities ρ, u, p and S depend only on (x, t). The Lagrangian map $x = x(x_0, t)$ is the solution of the ordinary differential equation $dx/dt = u(x, t)$ where $u(x, t)$ is assumed to be known and $x = x_0$ at time $t = 0$. Using (10.23)–(10.25) (or alternatively using (10.15)), we obtain the correspondences:

$$A_j^{i\alpha\beta}x_{\alpha\beta}^j \to \frac{\partial^2 x}{\partial t^2} - \frac{a^2}{J^2}\frac{\partial^2 x}{\partial x_0^2},$$

$$R^1 \to \frac{1}{\rho_0}\left(\frac{\partial p}{\partial\bar{S}}\frac{\partial\bar{S}}{\partial x_0} + \frac{a^2}{J}\frac{\partial\rho_0}{\partial x_0}\right), \qquad (10.30)$$

where

$$\rho J = \rho_0 \quad \text{and} \quad J = \frac{\partial x}{\partial x_0}, \qquad (10.31)$$

is the Jacobian of the Lagrangian map, and $a = (\gamma p/\rho)^{1/2}$ is the adiabatic gas sound speed. Here we use the adiabatic gas law:

$$p = A\rho^\gamma \exp(\bar{S}) \quad \text{where} \quad \bar{S} = \frac{S}{C_v}, \tag{10.32}$$

is a normalized form of the gas entropy S and C_v is the specific heat of the gas at constant volume. Note that p_0, ρ_0 and $S = S(x_0)$ are related by the equation:

$$p_0 = p(x_0) = A\rho_0^\gamma \exp(\bar{S}). \tag{10.33}$$

Using (10.32)–(10.33) in (10.30), the Lagrangian wave equation (10.23) reduces to:

$$\frac{\partial^2 x}{\partial t^2} - a_0^2 \left(\frac{\partial x}{\partial x_0}\right)^{-\gamma-1} \left[\frac{\partial^2 x}{\partial x_0^2} - \frac{1}{\gamma p_0}\frac{\partial p_0}{\partial x_0}\frac{\partial x}{\partial x_0}\right] = 0. \tag{10.34}$$

Here p_0, ρ_0 and S are functions of x_0 only, and $a_0 = (\gamma p_0/\rho_0)^{1/2}$ is the adiabatic gas sound speed at time $t = 0$.

If one chooses the Lagrangian mass coordinate $m = \int^x \rho(x', t)dx'$ to replace x_0 then for compact support with $\rho_0 = 1$ one obtains $m = x_0 + L$ where L is a constant. For this choice, the Lagrangian wave equation (10.34) reduces to the form:

$$x_{tt} - a_0^2 x_m^{-\gamma-1}\left[x_{mm} - \frac{1}{\gamma}\bar{S}_m x_m\right] = 0, \tag{10.35}$$

which is the Lagrangian wave equation for 1D gas dynamics given by Webb and Zank (2009) and Webb (2015).

The nonlinear wave equation (10.34) can be written as a first order partial differential equation system by setting

$$u = \frac{\partial x}{\partial t}, \quad v = \frac{\partial x}{\partial x_0}, \tag{10.36}$$

in (10.34) to obtain the matrix equation system:

$$\left(I_2\frac{\partial}{\partial t} + A\frac{\partial}{\partial x_0}\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} R^1 \\ 0 \end{pmatrix}, \tag{10.37}$$

where

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -c^2 \\ -1 & 0 \end{pmatrix}. \tag{10.38}$$

Here

$$c^2 = \frac{a^2}{J^2} = a_0^2 \left(\frac{\partial x}{\partial x_0} \right)^{-\gamma-1}, \quad R^1 = -\frac{c^2}{\gamma p_0} \left(\frac{\partial p_0}{\partial x_0} \right) \frac{\partial x}{\partial x_0}. \tag{10.39}$$

On a characteristic manifold, $\phi(\mathbf{x}_0, t) = const.$, the Cauchy initial value problem does not have a unique solution. Searching for solutions of (10.37) of the form $u = u(\phi)$ and $v = v(\phi)$, requires:

$$(\mathsf{A}\phi_{x_0} + \mathsf{I}_2\phi_t) \begin{pmatrix} u'(\phi) \\ v'(\phi) \end{pmatrix} = \begin{pmatrix} R^1 \\ 0 \end{pmatrix}. \tag{10.40}$$

This system does not have a unique solution for $(u'(\phi), v'(\phi))^T$ if the determinant of the matrix $\mathsf{A}\phi_{x_0} + \mathsf{I}_2\phi_t$ is zero. This condition can also be written as:

$$\det(\mathsf{A} - \lambda \mathsf{I}_2) = \lambda^2 - c^2 = 0 \quad \text{where} \quad \lambda = -\frac{\phi_t}{\phi_{x_0}}. \tag{10.41}$$

The first order partial differential equation for $\phi(x_0, t) \equiv \Phi(x, t)$ in (10.41) can be written in the forms:

$$\frac{\partial \phi}{\partial t} \pm c \frac{\partial \phi}{\partial x_0} \equiv \frac{d\Phi}{dt} \pm a \frac{\partial \Phi}{\partial x} = 0, \tag{10.42}$$

The bicharacteristics are defined as the characteristics of the first order partial differential equations (10.42), i.e.:

$$\frac{dx_0}{dt} = \pm c \quad \text{or} \quad \frac{dx}{dt} = u \pm a. \tag{10.43}$$

The first version of the bicharacteristics in (10.43) is the Lagrangian version, whereas the second form of the bi-characteristics in (10.43) are the Eulerian version of the equations (note $d\Phi/dt = \partial\Phi/\partial t + \mathbf{u} \cdot \nabla\Phi$ in (10.42)).

An alternative, and instructive approach is given by Courant and Hilbert (1989), who use inner derivatives of the characteristic surfaces, which are directional derivatives confined within the characteristic surface. Once the initial data is specified on a characteristic surface, then it is not possible to get off the surface by using inner derivatives (non-uniqueness of the solution then occurs). One searches for left eigenvectors of the matrix equation (10.37) which we write in the form:

$$M_{ij}^{\alpha} \frac{\partial \psi^j}{\partial x_0^{\alpha}} = Q_i, \tag{10.44}$$

where

$$M_{ij}^0 = \delta_{ij}, \quad M_{ij}^1 = A_{ij}, \quad \boldsymbol{\psi} = (u, v)^T, \quad \mathbf{Q} = (R^1, 0)^T, \tag{10.45}$$

where we use the notation $(x_0^0, x_0^1) = (t, x_0)$ for 1D gas dynamics. Next consider the scalar equation:

$$\ell_i M_{ij}^\alpha \frac{\partial \psi^j}{\partial x_0^\alpha} = \ell_i Q^i, \tag{10.46}$$

obtained by taking a linear combination of Eq. (10.44) with left multipliers ℓ_i. Equation (10.46) may be written in the form:

$$D_j \psi^j = M_j^\alpha \frac{\partial \psi^j}{\partial x_0^\alpha} = \boldsymbol{\ell} \cdot \mathbf{Q}, \tag{10.47}$$

where

$$M_j^\alpha = \ell_i M_{ij}^\alpha \quad \text{and} \quad D_j = M_j^\alpha \frac{\partial}{\partial x_0^\alpha}. \tag{10.48}$$

The condition for the directional derivative D_j to be an inner derivative for the characteristic surface $\phi(x_0, t) = const.$ is for the vector M_j^α to be perpendicular to the normal $N_\alpha = \partial \phi / \partial x_0^\alpha$, i.e.

$$M_j^\alpha N_\alpha = M_j^\alpha \frac{\partial \phi}{\partial x_0^\alpha} \equiv \ell_i \left(M_{ij}^\alpha \frac{\partial \phi}{\partial x_0^\alpha} \right) = 0. \tag{10.49}$$

Equation (10.49) will have non-trivial solutions for the ℓ_i if

$$\det \left(M_{ij}^\alpha \phi_{x_0^\alpha} \right) = \det \left(\mathsf{I}_2 \phi_t + \mathsf{A} \phi_{x_0} \right) = 0. \tag{10.50}$$

which is equivalent to the determinantal equation (10.41) obtained previously.

   To determine the inner derivative operators D_j ($j = 1, 2$) it is necessary to determine the left eigenvectors in (10.49), i.e. $\boldsymbol{\ell}$ satisfies the eigenvalue equation:

$$\boldsymbol{\ell} \cdot (\mathsf{A} - \lambda \mathsf{I}) = 0. \tag{10.51}$$

   The left eigenvectors of (10.51) have the form:

$$\boldsymbol{\ell} = \ell_1(1, -\lambda) \quad \text{where} \quad \lambda = \pm c. \tag{10.52}$$

On the backward characteristic:

$$\lambda = -c, \quad \frac{dx_0}{dt} = -c(x_0, t), \quad \boldsymbol{\ell} = \ell_1(1, c). \tag{10.53}$$

Similarly, on the forward characteristic

$$\lambda = c, \quad \frac{dx_0}{dt} = c(x_0, t), \quad \boldsymbol{\ell} = \ell_1(1, -c). \tag{10.54}$$

Taking the scalar product of $\boldsymbol{\ell}$ with (10.37) we obtain the compatibility condition:

$$u_t - c u_{x_0} + c(v_t - c v_{x_0}) + \frac{c^2}{\gamma p_0}\left(\frac{\partial p_0}{\partial x_0}\right) v = 0, \tag{10.55}$$

for the backward characteristic. Similarly for the forward characteristic, the compatibility condition is:

$$u_t + c u_{x_0} - c(v_t + c v_{x_0}) + \frac{c^2}{\gamma p_0}\left(\frac{\partial p_0}{\partial x_0}\right) v = 0. \tag{10.56}$$

By using characteristic coordinates ξ and η, where

$$\xi_t + c\xi_{x_0} = 0, \quad \eta_t - c\eta_{x_0} = 0, \tag{10.57}$$

(10.55) and (10.56) become:

$$2\xi_t(u_\xi + c v_\xi) + \frac{c^2}{\gamma p_0}\frac{\partial p_0}{\partial x_0} v = 0, \tag{10.58}$$

$$2\eta_t(u_\eta - c v_\eta) + \frac{c^2}{\gamma p_0}\frac{\partial p_0}{\partial x_0} v = 0, \tag{10.59}$$

The compatibility condition (10.58) shows that ξ (the forward characteristic function) varies on the backward characteristic. Similarly η changes on the forward characteristic in (10.59). The compatibility conditions (10.58)–(10.59) are important in the numerical solution of the 1D Lagrangian wave equation by characteristics methods. This completes our discussion of 1D gas dynamics.

10.4   Vector Lagrangian Wave Equations

In the previous section we showed that the Lagrangian momentum equation could be written as a coupled wave equation system for $\mathbf{x} = \mathbf{x}(t, \mathbf{x}_0)$ and used it to discuss the MHD characteristic manifolds. It turns out that there is a more elegant formulation of the Lagrangian coupled wave equation system (10.23) derived by Golovin (2011), which we study below. This approach shows the connection between the Lagrangian momentum equation (10.15), Faraday's equation for the frozen in magnetic field, Gauss's law, and the mass continuity equation. Exact MHD solution methods

developed by Schief (2003), Rogers and Schief (2002), Bogoyavlenskij (2002) and
Golovin (2010, 2011) are related to this approach. Here we describe the approach of
Golovin (2011) in deriving a vector Lagrangian wave equation equivalent to (10.23)
by using a judicious choice of Lagrange labels (ξ^1, ξ^2, ξ^3) in the Lagrangian map
$\mathbf{x} = \boldsymbol{\gamma}(t, \boldsymbol{\xi})$ where the $\{\xi^i : 1 \le i \le 3\}$ are Lagrange labels advected with the fluid.

The Lagrangian MHD momentum equation (10.15) can be expressed in a more
elegant vector wave equation form (Golovin 2011). The analysis starts from the
Eulerian MHD momentum equation in the form:

$$\rho \left(\frac{d\mathbf{u}}{dt} + \nabla \Phi \right) + \nabla P - \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu_0} = 0, \tag{10.60}$$

where

$$P = p + \frac{B^2}{2\mu_0}, \tag{10.61}$$

is the total pressure (gas plus magnetic pressure). The mass continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{10.62}$$

Gauss's equation, and Faraday's equation:

$$\nabla \cdot \mathbf{B} = 0, \tag{10.63}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) = 0, \tag{10.64}$$

are two other basic MHD equations used in the analysis. It is convenient to set:

$$\mathbf{b} = \frac{\mathbf{B}}{\rho}. \tag{10.65}$$

Using the mass continuity equation (10.62), Faraday's equation (10.64) can be
written in the form:

$$\frac{\partial \mathbf{b}}{\partial t} + [\mathbf{u}, \mathbf{b}] = 0 \quad \text{where} \quad [\mathbf{u}, \mathbf{b}] = \mathbf{u} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u}, \tag{10.66}$$

is the Lie bracket of the vector fields $\mathbf{u}$ and $\mathbf{b}$. In terms of $\mathbf{b}$ and ρ Gauss's
equation (10.63) becomes:

$$\nabla \cdot (\rho \mathbf{b}) = 0, \tag{10.67}$$

which resembles the steady-state mass continuity equation.

The Lagrangian time derivative d/dt and the directional derivative $\mathbf{b} \cdot \nabla$ are denoted by the Lie derivative operators X_1 and X_2 where

$$X_1 = D_t = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad X_2 = \mathbf{b} \cdot \nabla = D_s. \tag{10.68}$$

Using Faraday's equation in the form (10.66) we find:

$$[X_1, X_2] = X_1 X_2 - X_2 X_1 \equiv \left(\frac{\partial \mathbf{b}}{\partial t} + [\mathbf{u}, \mathbf{b}] \right) \cdot \nabla = 0, \tag{10.69}$$

for $[X_1, X_2]$. Because X_1 and X_2 form an Abelian two-dimensional Lie algebra, then there exists, by Frobenius theorem, coordinates ξ^1 and t such that

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \xi^1}. \tag{10.70}$$

From (10.70) the Lie bracket condition $[X_1, X_2] = 0$ is equivalent to the equality of mixed partial derivatives, i.e.

$$[X_1, X_2] = \frac{\partial}{\partial t} \frac{\partial}{\partial \xi^1} - \frac{\partial}{\partial \xi^1} \frac{\partial}{\partial t} = 0, \tag{10.71}$$

where ξ^1 is a Lagrange label that corresponds to the affine parameter or distance along the field line. Thus, we can introduce the Lagrangian map:

$$x^\alpha = \gamma^\alpha(t, \xi^1, \xi^2, \xi^3) = (t, x, y, z), \tag{10.72}$$

where ξ^2 and ξ^3 are Lagrange labels independent of ξ^1. There is no strong constraint on ξ^2 and ξ^3, except that they are independent of each other and also independent of ξ^1. A natural choice for ξ^2 and ξ^3 are Euler potentials for the magnetic field, with $\mathbf{B} = \nabla \xi^2 \times \nabla \xi^3$ (in general the magnetic vector potential $\mathbf{A} = \xi^2 \nabla \xi^3 + \nabla \psi$ will not be globally defined for some magnetic field configurations with non-trivial topology, in which case ψ may contain jumps).

For the Lagrangian map (10.72) one can introduce a holonomic coordinate base:

$$\mathbf{e}_0 = \left(1, \frac{\partial \boldsymbol{\gamma}}{\partial t} \right), \quad \mathbf{e}_i = \left(0, \frac{\partial \boldsymbol{\gamma}}{\partial \xi^i} \right), \quad (1 \leq i \leq 3). \tag{10.73}$$

Because $X_1 = D_t = \partial / \partial t + \mathbf{u} \cdot \nabla$ and $X_2 = \mathbf{b} \cdot \nabla$, it follows that:

$$\mathbf{e}_0 = (1, \mathbf{u}) \quad \mathbf{e}_1 = (0, \mathbf{b}), \tag{10.74}$$

where we use the Cartesian (t, x, y, z) coordinate representation. The contravariant components of the vectors $\mathbf{u} = u^\alpha \mathbf{e}_\alpha$ and $\mathbf{b} = b^\alpha \mathbf{e}_\alpha$ in the curvilinear coordinate

system have the form:

$$u^0 = 1, \quad u^i = 0, \quad i = 1, 2, 3,$$
$$b^0 = 0, \quad b^1 = 1, \quad b^2 = b^3 = 0. \tag{10.75}$$

10.4.1   Equations in Coordinates (t, ξ^1, ξ^2, ξ^3)

The vector wave equation derived by Golovin (2011) is equivalent to the Lagrangian MHD momentum equation (10.15) derived by Newcomb (1962) except that special Lagrange label coordinates ξ^1, ξ^2 and ξ^3 are chosen which exploit the symmetries X_1 and X_2 in (10.68)–(10.69) of the MHD equations (see also Schief 2003).

The metric tensor $g_{\alpha\beta}$ for the base vectors (10.75) are:

$$g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta \quad \text{where} \quad \mathbf{e}_\alpha = \frac{\partial \boldsymbol{\gamma}}{\partial \xi^\alpha}, \tag{10.76}$$

The dual base vectors:

$$\mathbf{e}^\alpha = \frac{\partial \xi^\alpha}{\partial \boldsymbol{\gamma}}, \tag{10.77}$$

satisfies the orthogonality conditions:

$$\langle \mathbf{e}_\alpha, \mathbf{e}^\beta \rangle = \mathbf{e}_\alpha \cdot \mathbf{e}^\beta = \delta_\alpha^\beta. \tag{10.78}$$

The vector fields $\mathbf{e}_0$ and $\mathbf{e}_1$ in curvilinear (t, ξ^1, ξ^2, ξ^3)-space are given by:

$$\mathbf{e}_0 = (1, 0, 0, 0), \quad \mathbf{e}_1 = \mathbf{b} = (0, 1, 0, 0). \tag{10.79}$$

The space-like base vectors $\{\mathbf{e}^i\}$ and $\{\mathbf{e}_i\}$ are related by the equations:

$$\mathbf{e}^i = \frac{\epsilon_{ijk}}{\sqrt{g}} \mathbf{e}_j \times \mathbf{e}_k, \quad \mathbf{e}_i = \sqrt{g}\epsilon_{ijk} \mathbf{e}^j \times \mathbf{e}^k, \quad i, j, k = 1, 2, 3. \tag{10.80}$$

Noting that

$$g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu = \left(\frac{\partial \boldsymbol{\gamma}}{\partial \xi^\mu}\right)^T \left(\frac{\partial \boldsymbol{\gamma}}{\partial \xi^\nu}\right), \tag{10.81}$$

we obtain:

$$g = \det(g_{\mu\nu}) = \det\left(\frac{\partial \boldsymbol{\gamma}}{\partial \xi^\mu}\right)^T \det\left(\frac{\partial \boldsymbol{\gamma}}{\partial \xi^\nu}\right) = J^2. \tag{10.82}$$

Thus $J = \sqrt{g}$ if we take the positive square root in (10.82).

The Lagrangian mass continuity equation may be written in the form:

$$\frac{d}{dt}(\rho J) \equiv \frac{d}{dt}(\rho \sqrt{g}) = 0. \tag{10.83}$$

Similarly, the condition $\nabla \cdot \mathbf{B} = 0$ may be written as:

$$\nabla \cdot (\rho \mathbf{b}) = \frac{\partial}{\partial \xi^1}(\rho b^1) + \Gamma_{1s}^s \rho b^1 = \frac{\partial \rho}{\partial \xi^1} + \frac{1}{2g}\frac{\partial g}{\partial \xi^1}\rho \equiv \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^1}(\rho \sqrt{g}) = 0. \tag{10.84}$$

From (10.83)–(10.84) we obtain the integral:

$$\rho \sqrt{g} = \rho_0 = f(\xi^2, \xi^3), \tag{10.85}$$

where f is an arbitrary function of ξ^2 and ξ^3. The integral (10.85) encapsulates both the mass continuity law and Gauss's law $\nabla \cdot \mathbf{B} = 0$. In the derivation of (10.83)–(10.85) we have used the representation $\mathbf{b} = (0, 1, 0, 0)$ in the curvilinear coordinate system, and the affine connection results:

$$\frac{\partial \mathbf{e}_\alpha}{\partial \xi^\beta} = \Gamma_{\alpha\beta}^s \mathbf{e}_s,$$

$$\Gamma_{\alpha\beta}^s = \frac{1}{2}g^{s\mu}\left(g_{\mu\beta,\alpha} + g_{\mu\alpha,\beta} - g_{\alpha\beta,\mu}\right),$$

$$\Gamma_{\beta\alpha}^\alpha = \frac{1}{2g}\frac{\partial g}{\partial \xi^\beta}, \tag{10.86}$$

in which we use the notation $\psi_{,\alpha} = \partial\psi/\partial\xi^\alpha$ and $\xi^0 = t$. The condition (10.85) can also be written as:

$$\rho \det\left(\frac{\partial \gamma^i}{\partial \xi^j}\right) = \rho \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = \rho \sqrt{g} = \rho_0. \tag{10.87}$$

For incompressible flows ($\rho = const.$), (10.87) is a determinantal partial differential equation for $\boldsymbol{\gamma}(t, \boldsymbol{\xi})$, which in general is difficult to solve, but special solutions of this equation were obtained by Golovin (2011) which are of physical interest.

By noting:

$$\rho\frac{d\mathbf{u}}{dt} = \rho\frac{\partial^2 \boldsymbol{\gamma}}{\partial t^2} = \rho\Gamma_{00}^i \mathbf{e}_i,$$

$$\nabla\Phi = \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{\gamma}}\right)^T \cdot \frac{\partial \Phi}{\partial \boldsymbol{\xi}} = \mathbf{e}^i \cdot \frac{\partial \Phi}{\partial \boldsymbol{\xi}},$$

$$\rho D_s(\rho \mathbf{b}) = \rho\frac{\partial}{\partial \xi^1}\left(\rho\frac{\partial \boldsymbol{\gamma}}{\partial \xi^1}\right), \tag{10.88}$$

the momentum equation (10.60), can be written in the form:

$$\frac{\partial^2 \boldsymbol{\gamma}}{\partial t^2} - \frac{\partial}{\partial \xi^1}\left(\frac{\rho}{\mu_0}\frac{\partial \boldsymbol{\gamma}}{\partial \xi^1}\right) + \mathbf{e}^i\left(\frac{\partial \Phi}{\partial \xi^i} + \frac{1}{\rho}\frac{\partial P}{\partial \xi^i}\right) = 0, \tag{10.89}$$

or as:

$$\left(\frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\xi}}\right)^T \cdot \left[\frac{\partial^2 \boldsymbol{\gamma}}{\partial t^2} - \frac{\partial}{\partial \xi^1}\left(\frac{\rho}{\mu_0}\frac{\partial \boldsymbol{\gamma}}{\partial \xi^1}\right)\right] + \nabla_\xi \Phi + \frac{1}{\rho}\nabla_\xi P = 0. \tag{10.90}$$

which is the same as Eq. (9) of Golovin (2011) for the case $\Phi = 0$.

For the case of incompressible flow, ($\rho = const.$), P is regarded as an unknown function. The condition of incompressibility $\rho = const.$ implies that $J = \det(\partial \gamma^i / \partial \xi^j) = 1$, i.e.

$$\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = f(\xi^2, \xi^3). \tag{10.91}$$

For a compressible fluid with equation of state $p = F(\rho, S)$, the total pressure $P = p + B^2/(2\mu)$ has the form:

$$P = p + \frac{B^2}{2\mu_0} = p + \frac{\rho^2}{2\mu_0}\left(\frac{\partial \boldsymbol{\gamma}}{\partial \xi^1}\right)^2. \tag{10.92}$$

Note that (10.89) or (10.90) is a vector wave equation for $\boldsymbol{\gamma}(t, \boldsymbol{\xi})$. Equation (10.90) is the same as equation (9) of Golovin (2011).

By using (10.80) for the transformations between the bases $\mathbf{e}_\alpha$ and $\mathbf{e}^\alpha$, (10.89) may be written in the form:

$$\frac{\partial^2 \boldsymbol{\gamma}}{\partial t^2} - \frac{\partial}{\partial \xi^1}\left(\frac{\rho}{\mu_0}\frac{\partial \boldsymbol{\gamma}}{\partial \xi^1}\right) + \frac{1}{\rho_0}\left(\mathbf{e}_2 \times \mathbf{e}_3 \frac{\partial P}{\partial \xi^1} + \mathbf{e}_3 \times \mathbf{e}_1 \frac{\partial P}{\partial \xi^2} + \mathbf{e}_1 \times \mathbf{e}_2 \frac{\partial P}{\partial \xi^3}\right)$$
$$+ \frac{1}{\sqrt{g}}\left(\mathbf{e}_2 \times \mathbf{e}_3 \frac{\partial \Phi}{\partial \xi^1} + \mathbf{e}_3 \times \mathbf{e}_1 \frac{\partial \Phi}{\partial \xi^2} + \mathbf{e}_1 \times \mathbf{e}_2 \frac{\partial \Phi}{\partial \xi^3}\right) = 0, \tag{10.93}$$

which is equivalent to equation (14) of Golovin (2011) for the case $\Phi = 0$ and $\rho_0 = 1$ for the case of incompressible flow.

If $P = const.$, $\Phi = const.$ and $\rho = const.$ the vector wave equation (10.93) reduces to the wave equation:

$$\frac{\partial^2 \boldsymbol{\gamma}}{\partial t^2} - \frac{\partial \boldsymbol{\gamma}}{\partial (\tilde{\xi}^1)^2} = 0 \quad \text{where} \quad \tilde{\xi}^1 = \xi^1 \sqrt{\frac{\mu_0}{\rho}}. \tag{10.94}$$

It is instructive to note that the wave operator in (10.94) can be written as:

$$
\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial (\tilde{\xi}^1)^2} = \left(\frac{d}{dt} - \mathbf{V}_A \cdot \nabla \right) \left(\frac{d}{dt} + \mathbf{V}_A \cdot \nabla \right)
$$

$$
= \left(\frac{\partial}{\partial t} + (\mathbf{u} - \mathbf{V}_A) \cdot \nabla \right) \left(\frac{\partial}{\partial t} + (\mathbf{u} + \mathbf{V}_A) \cdot \nabla \right), \tag{10.95}
$$

which implies that the solutions of (10.94) correspond to backward and forward moving structures travelling at the Alfvén speed V_A relative to the fluid.

The general solution of (10.94) for $\boldsymbol{\gamma}$ has the form:

$$
\boldsymbol{\gamma} = \boldsymbol{\sigma} (t - \tilde{\xi}^1, \xi^2, \xi^3) + \boldsymbol{\tau} (t + \tilde{\xi}^1, \xi^2, \xi^3). \tag{10.96}
$$

The incompressibility constraint (10.85) with $\rho = 1$ reduces to:

$$
(-\boldsymbol{\sigma}_1 + \boldsymbol{\tau}_1) \cdot [(\boldsymbol{\sigma}_2 + \boldsymbol{\tau}_2) \times (\boldsymbol{\sigma}_3 + \boldsymbol{\tau}_3)] = f(\xi^2, \xi^3), \tag{10.97}
$$

where the subscript i denotes differentiation with respect to the ith argument. A simple choice in (10.97) is to set $f = 1$. The choice of f can be changed using equivalence transformations (Golovin 2011).

Stationary field aligned flows (Golovin 2011) are obtained by setting $\boldsymbol{\sigma} = 0$ in (10.96). In that case, one obtains

$$
\boldsymbol{\gamma} = \boldsymbol{\tau} (t + \tilde{\xi}^1, \xi^2, \xi^3), \quad \det \left(\partial \gamma^i / \partial \xi^j \right) = f. \tag{10.98}
$$

In this case $\mathbf{u} = \boldsymbol{\tau}_t = \boldsymbol{\tau}_1$ and $\boldsymbol{b} = \boldsymbol{\tau}_{\xi^1} \equiv \sqrt{\mu_0/\rho} \boldsymbol{\tau}_1$, i.e. $\mathbf{u} = \mathbf{V}_A$ (see also Chandrasekhar 1956, 1957)

The solutions of the determinantal equation (10.97) depend on the dimension of the vector $\boldsymbol{\sigma}$ and its functional form. For $\dim(\boldsymbol{\sigma}) = 2$ in which $\boldsymbol{\sigma}$ has the simple form:

$$
\boldsymbol{\sigma} = u(t - \tilde{\xi}^1)\boldsymbol{\eta}^1 + (t - \tilde{\xi}^1)\boldsymbol{\eta}^2, \tag{10.99}
$$

where $\boldsymbol{\eta}^1$, $\boldsymbol{\eta}^2$, and $\boldsymbol{\eta}^3$ are orthonormal Cartesian base vectors. The determinantal equation (10.97) implies:

$$
\boldsymbol{\eta}^1 \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) = 0 \quad \text{and} \quad \left(\boldsymbol{\tau}_1 - \boldsymbol{\eta}^2 \right) \cdot (\boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3) = f. \tag{10.100}
$$

The first condition (10.100) reduces to $\partial(\tau^2, \tau^3)/\partial(\xi^2, \xi^3) = 0$ which implies $\tau^3 = \tau^3(\tau^2, \tilde{\xi}^1, t)$, which is equivalent to the solution ansatz:

$$
\tau^2 = \tau^2(t + \tilde{\xi}^1, \lambda), \quad \tau^3 = \tau^3(t + \tilde{\xi}^1, \lambda) \quad \text{where} \quad \lambda = \lambda(t + \tilde{\xi}^1, \xi^2, \xi^3).
$$
$$
\tag{10.101}
$$

The solution for $\gamma = \sigma + \tau$ reduces to:

$$\gamma = \left[u(t - \tilde{\xi}^1) + \tau^1(t + \tilde{\xi}^1, \lambda) \right] \eta^1 + \tau^2(t + \tilde{\xi}^1, \lambda)\eta^2 + \tau^3(t + \tilde{\xi}^1, \lambda)\eta^3 + 2t\eta^2.$$

(10.102)

The $2t\eta^2$ term in (10.101) can be eliminated by using a Galilean transformation (Golovin 2011), i.e. $\tilde{\gamma} = \gamma - 2t\eta^2$ is also a valid solution of the Lagrangian fluid equations for γ. The Jacobian condition (10.97) or (10.100) reduces to:

$$\frac{\partial(\tau^2, \tau^3)}{\partial(\tilde{\xi}^1, \lambda)} \frac{\partial(\lambda, \tau^1)}{\partial(\xi^2, \xi^3)} = f(\xi^2, \xi^3).$$

(10.103)

The simplest solution in (10.102) is obtained by setting $u = 0$. Golovin (2011) chooses $\tau^1, \tau^2, \tau^3, \lambda$ and μ as:

$$\tau^1 = a(\mu) + C(\xi^2, \xi^3) \sin[\varphi(\mu) + A(\xi^2, \xi^3)],$$
$$\lambda = \{b(\mu) + C(\xi^2, \xi^3) \cos[\varphi(\mu) + A(\xi^2, \xi^3)]\}^{1/2},$$
$$\tau^2 = \lambda \cos(k\mu), \quad \tau^3 = \lambda \sin(k\mu), \quad \mu = t + \tilde{\xi}^1.$$

(10.104)

Using the solution ansatz (10.104), the Jacobian condition (10.103) becomes:

$$f(\xi^2, \xi^3) = \frac{kC}{2} \frac{\partial(A, C)}{\partial(\xi^2, \xi^3)},$$

(10.105)

which defines the function $f(\xi^2, \xi^3)$.

Golovin (2011) obtained solutions of (10.93) and (10.104) for incompressible MHD ($\rho = const.$) and for constant total pressure P for the case of no gravity $\Phi = 0$, which are illustrated in Fig. 10.1. In the left panel, the arbitrary functions in (10.104) are:

$$A = \xi^2, \quad C = \xi^3, \quad \varphi = 3\mu,$$
$$a = 0, \quad b = 2, \quad k = 2.$$

(10.106)

In the right panel, the arbitrary parameters are chosen as:

$$A = \xi^2, \quad C = \xi^3, \quad \varphi = 3\mu, \quad a = \sin 3\mu,$$
$$b = 3 + \cos 3\mu, \quad k = 2.$$

(10.107)

The magnetic surfaces in both the left and the right panel correspond to setting $\xi^3 = 1$. In the left panel of Fig. 10.1, the magnetic surface $\xi^3 = 1$ is a torus and a magnetic field line in the form of a trefoil knot lying on the magnetic surface is shown. In the right panel, the magnetic field surface $\xi^3 = 1$ consists of a trefoil knot, and a magnetic field line in the form of a trefoil knot lying on the surface is shown.
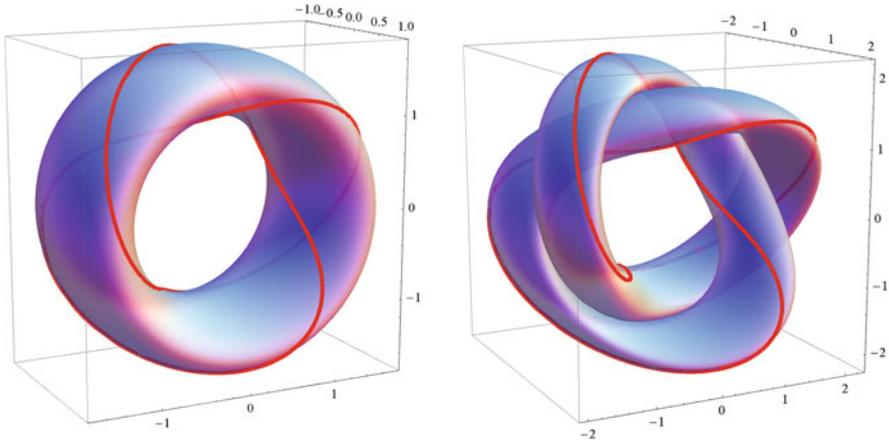
Fig. 10.1 Incompressible MHD solutions with $\mathbf{u} = \mathbf{V}_A$ described by (10.102)–(10.107). *Left Panel*: The magnetic surface $\xi^3 = 1$ in the form of a torus, with magnetic field line in the shape of a trefoil knot. *Right Panel*: Magnetic surface $\xi^3 = 1$ in the shape of a trefoil knot. A magnetic field line (in red) lies on the magnetic surface also has the form of a trefoil knot (from Golovin 2011)

These examples are steady-state field aligned flows with constant total pressure P. These solutions can be modified to cases of non-field aligned, time dependent flows (for more details, see Golovin 2011). The solutions illustrated in Fig. 10.1 have a non-trivial topology for the magnetic field. They are similar to the topological soliton solutions of Kamchatnov (1982) and Semenov et al. (2002) discussed in Chap. 6. Both the Golovin solutions and the topological soliton solutions of Chap. 6 correspond to a steady incompressible MHD flow, moving with velocity $\mathbf{u} = \pm\mathbf{V}_A$. The exact connection between these two different solution formulations is a question that could be interesting for further research.

Chapter 11
Symmetries and Noether's Theorem in MHD

In this chapter we discuss Noether's first theorem in MHD. The analysis is similar to that in Padhye (1998) and Webb et al. (2005b) We consider the Lagrangian form of the action (10.11), namely

$$A = \int \int \mathcal{L}^0 \, d^3 x_0 dt, \tag{11.1}$$

where the Lagrangian density $\mathcal{L}^0$ is given by (10.13). The general theory for Noether's theorems are discussed in Chap. 4. The Lagrangian action principle and the Lagrangian map are discussed in Chap. 10.

11.1 Noether's Theorem

Proposition 11.1.1 *Noether's theorem If the action (11.1) is invariant to $O(\epsilon)$ under the infinitesimal Lie transformations:*

$$x'^i = x^i + \epsilon V^{x^i}, \quad x_0'^j = x_0^j + \epsilon V^{x_0^j}, \quad t' = t + \epsilon V^t, \tag{11.2}$$

and the divergence transformation:

$$\mathcal{L}^{0'} = \mathcal{L}^0 + \epsilon D_\alpha \Lambda_0^\alpha + O(\epsilon^2), \tag{11.3}$$

(here $D_0 \equiv \partial/\partial t$ and $D_i \equiv \partial/\partial x_0^i$ are the total derivative operators in the jet-space consisting of the derivatives of $x^k(\mathbf{x}_0, t)$ and physical quantities that depend on $\mathbf{x}_0$ and t) then the MHD system admits the Lagrangian conservation law:

$$\frac{\partial I^0}{\partial t} + \frac{\partial I^j}{\partial x_0^j} = 0, \tag{11.4}$$

where

$$I^0 = \rho_0 u^k \hat{V}^{x^k} + V^t \mathcal{L}^0 + \Lambda_0^0, \tag{11.5}$$

$$I^j = \hat{V}^{x^k} \left[\left(p + \frac{B^2}{2\mu} \right) \delta^{ks} - \frac{B^k B^s}{\mu} \right] A_{sj} + V^{x_0^j} \mathcal{L}^0 + \Lambda_0^j, \tag{11.6}$$

In (11.5)–(11.6)

$$\hat{V}^{x^k(\mathbf{x}_0, t)} = V^{x^k(\mathbf{x}_0, t)} - \left(V^t \frac{\partial}{\partial t} + V^{x_0^s} \frac{\partial}{\partial x_0^s} \right) x^k(\mathbf{x}_0, t), \tag{11.7}$$

is the canonical Lie symmetry transformation generator corresponding to the Lie transformation (11.2) (i.e. $x'^k = x^k + \epsilon \hat{V}^{x^k}$, $t' = t$, $x_0'^j = x_0^j$).

Proof Using Noether's theorem (e.g. Bluman and Kumei 1989) we obtain:

$$I^0 = W^0 + V^t \mathcal{L}^0 + \Lambda_0^0 \equiv \frac{\partial \mathcal{L}^0}{\partial x_t^k} \hat{V}^{x^k} + V^t \mathcal{L}^0 + \Lambda_0^0,$$

$$I^j = W^j + \mathcal{L}^0 V^{x_0^j} + \Lambda_0^j \equiv \frac{\partial \mathcal{L}^0}{\partial x_{kj}} \hat{V}^{x^k} + \mathcal{L}^0 V^{x_0^j} + \Lambda_0^j, \tag{11.8}$$

for the conserved density I^0 and flux components I^j. Using (10.13) for $\mathcal{L}^0$ in (11.8) to evaluate the derivatives of $\mathcal{L}^0$ with respect to x_t^k and x_{ij} gives the expressions (11.5)–(11.6) for I^0 and I^j. Proofs of Noether's first theorem are given in Bluman and Kumei (1989) and Olver (1993) (see Webb et al. (2005b) for Noether's theorem for the MHD system, including the effects of fully nonlinear waves).   □

Remark 1 The condition for the action (11.1) to be invariant to $O(\epsilon)$ under the divergence transformation of the form (11.2)–(11.3) is:

$$\tilde{X}\mathcal{L}^0 + \mathcal{L}^0 \left(D_t V^t + D_{x_0^j} V^{x_0^j} \right) + D_t \Lambda_0^0 + D_{x_0^j} \Lambda_0^j = 0, \tag{11.9}$$

where

$$\tilde{X} = V^t \frac{\partial}{\partial t} + V^{x_0^s} \frac{\partial}{\partial x_0^s} + V^{x^k} \frac{\partial}{\partial x^k} + V^{x_t^k} \frac{\partial}{\partial x_t^k} + V^{x_{kj}} \frac{\partial}{\partial x_{kj}} + \cdots, \tag{11.10}$$

is the extended Lie transformation operator acting on the jet space of the Lie transformation (11.2). Note that $\tilde{X}$ gives the rules for transforming derivatives of $x^k(\mathbf{x}_0, t)$ under Lie transformation (11.2). From Ibragimov (1985):

$$\tilde{X} = \hat{X} + V^\alpha D_\alpha, \tag{11.11}$$

$$\hat{X} = \hat{V}^{x^k} \frac{\partial}{\partial x^k} + D_\alpha \left(\hat{V}^{x^k} \right) \frac{\partial}{\partial x_\alpha^k} + D_\alpha D_\beta \left(\hat{V}^{x^k} \right) \frac{\partial}{\partial x_{\alpha\beta}^k} + \cdots, \tag{11.12}$$

where $D_0 = \partial/\partial t$ $D_i = \partial/\partial x_0^i$ denote total partial derivatives with respect to t and x_0^i $(1 \leq i \leq 3)$, $V^0 \equiv V^t$ and $V^i \equiv V^{x_0^i}$ respectively. $\hat{X}$ is the extended Lie symmetry operator corresponding to the canonical Lie transformation $x'^k = x^k + \epsilon \hat{V}^{x^k}$, $t' = t$ and $x_0''^j = x_0^j$.

Remark 2 The basic conservation law (11.4) and the condition (11.9) for the action to be invariant under a divergence symmetry are a consequence of the identity:

$$\tilde{X}\mathcal{L}^0 + \mathcal{L}^0 D_\alpha V^\alpha + D_\alpha \Lambda_0^\alpha = \hat{V}^{x^i} E_{x^i}\left(\mathcal{L}^0\right) + D_\alpha\left(W^\alpha + \mathcal{L}^0 V^\alpha\right)$$
$$+ D_\alpha \Lambda_0^\alpha, \tag{11.13}$$

where $E_{x^i}(\mathcal{L}^0) = \delta A/\delta x^i$ is the variational derivative of A with respect to x^i in (10.14) and $W^\alpha = \hat{V}^{x^k} \partial \mathcal{L}^0/\partial x_\alpha^k$ is a surface vector term that arises in the proof of Noether's theorem.

To convert the Lagrangian conservation law (11.4) to its equivalent Eulerian form we use a result of Padhye (1998) given below.

Proposition 11.1.2 *The Lagrangian conservation law (11.4) can be written as an Eulerian conservation law of the form:*

$$\frac{\partial F^0}{\partial t} + \frac{\partial F^j}{\partial x^j} = 0, \tag{11.14}$$

where

$$F^0 = \frac{I^0}{J}, \quad F^j = \frac{u^j I^0 + x_{jk} I^k}{J}, \quad (j = 1, 2, 3), \tag{11.15}$$

are the conserved density F^0 and flux components F^j.

Proposition 11.1.3 *The Lagrangian conservation law (11.4) with conserved density I^0 of (11.5), and flux I^j of (11.6), is equivalent to the Eulerian conservation law:*

$$\frac{\partial F^0}{\partial t} + \frac{\partial F^j}{\partial x^j} = 0, \tag{11.16}$$

where

$$F^0 = \rho u^k \hat{V}^{x^k(\mathbf{x}_0,t)} + V^t \mathcal{L} + \Lambda^0, \tag{11.17}$$

$$F^j = \hat{V}^{x^k(\mathbf{x}_0,t)}\left(T^{jk} - \mathcal{L}\delta^{jk}\right) + V^{x^j}\mathcal{L} + \Lambda^j, \tag{11.18}$$

$$T^{jk} = \rho u^j u^k + \left(p + \frac{B^2}{2\mu}\right)\delta^{jk} - \frac{B^j B^k}{\mu}, \tag{11.19}$$

$$\Lambda^0 = \frac{\Lambda_0^0}{J}, \quad \Lambda^j = \frac{u^j \Lambda_0^0 + x_{js}\Lambda_0^s}{J}. \tag{11.20}$$

In (11.16)–(11.20) T^{jk} is the Eulerian momentum flux tensor (i.e. the spatial components of the stress energy tensor) and $\hat{V}^{x^k}(\mathbf{x}_0, t)$ is the canonical symmetry generator (11.6).

Remark Padhye and Morrison (1996a,b) and Padhye (1998) used Proposition 11.1.2 to convert Lagrangian conservation laws to Eulerian conservation laws. Linear waves in a non-uniform background flow were studied in Webb et al. (2005a), thus extending similar work by Dewar (1970) for WKB waves.

11.2   Lie Point Symmetries

The Lie point symmetry algebra of the ideal, compressible gas dynamic and MHD equations have been obtained by Fuchs (1991). The classification of the Lie algebra and sub-algebras of these equations have been carried out by Grundland and Lalague (1995). The Lie point symmetries of the equations obtained by Fuchs (1991) pertain to the Eulerian form of the equations. Golovin (2011) investigated the Lie point symmetries of the Lagrangian MHD equations and the equivalence transformations of the Lagrangian equations, which can be related to the Eulerian form of the symmetries (e.g. Webb and Zank 2007).

The MHD equations and gas dynamic systems admit the 10 parameter Galilei Lie group. This includes the space and time translation symmetries, the space rotations and the Galilean boosts. This group has the Lie algebra basis of vector fields:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z}, \quad X_4 = \frac{\partial}{\partial t}, \tag{11.21}$$

$$X_5 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u^x}, \quad X_6 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial u^y}, \quad X_7 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u^z}, \tag{11.22}$$

$$X_8 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + u^z\frac{\partial}{\partial u^y} - u^y\frac{\partial}{\partial u^z} + B^z\frac{\partial}{\partial B^y} - B^y\frac{\partial}{\partial B^z}, \tag{11.23}$$

$$X_9 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} + u^x\frac{\partial}{\partial u^z} - u^z\frac{\partial}{\partial u^x} + B^x\frac{\partial}{\partial B^z} - B^z\frac{\partial}{\partial B^x}, \tag{11.24}$$

$$X_{10} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + u^y\frac{\partial}{\partial u^x} - u^x\frac{\partial}{\partial u^y} + B^y\frac{\partial}{\partial B^x} - B^x\frac{\partial}{\partial B^y}, \tag{11.25}$$

In the above equations (t, x, y, z) refer to the time and rectangular, Cartesian space coordinates, $\mathbf{u}$ is the fluid velocity; $\mathbf{B}$ is the magnetic field induction; ρ is the gas density; and p is the gas pressure. The Lie symmetry operators $\{X_1, X_2, X_3\}$ represent the space translation symmetries, and correspond via Noether's theorem to the momentum conservation equations along the x, y and z axes respectively; X_4 is

the time translation symmetry and corresponds to the energy conservation equation; $\{X_5, X_6, X_7\}$ correspond to the Galilean boosts and give rise to the uniform center of mass conservation laws; $\{X_8, X_9, X_{10}\}$ correspond to rotational invariance about the x, y and z axes respectively and give rise to the angular momentum laws. These conservation laws are derived by Morrison (1982) using a non-canonical Poisson bracket formalism. They are also derived by Padhye (1998) and Webb et al. (2005b) using a Lagrangian form of the MHD action principle and Noether's first theorem.

The Galilei symmetries do not depend on the equation of state of the gas. However, there is a class of Lie point symmetries of the Eulerian gas dynamic and MHD equations that apply if the gas has a polytropic equation of state of the form:

$$\varepsilon = \frac{p}{\gamma - 1}, \quad S = C_v \ln\left[\frac{p}{p_1}\left(\frac{\rho_1}{\rho}\right)^\gamma\right], \quad p = p_1\left(\frac{\rho}{\rho_1}\right)^\gamma \exp(\bar{S}), \qquad (11.26)$$

where ρ_1 and p_1 are constant, characteristic values of the density and gas pressure, ε is the internal energy density of the gas, S is the entropy ($\bar{S} = S/C_v$), C_v is the specific heat of the gas at constant volume, and γ is the adiabatic index of the gas. Using $(t, x, y, z, u^x, u^y, u^z, B^x, B^y, B^z, p, \rho)^t$ as variables in the Eulerian MHD equations, Fuchs (1991) obtained the symmetries:

$$X_{11} = t\frac{\partial}{\partial t} - u^x\frac{\partial}{\partial u^x} - u^y\frac{\partial}{\partial u^y} - u^z\frac{\partial}{\partial u^z} + 2\rho\frac{\partial}{\partial \rho}, \qquad (11.27)$$

$$X_{12} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u^x\frac{\partial}{\partial u^x} + u^y\frac{\partial}{\partial u^y} + u^z\frac{\partial}{\partial u^z} - 2\rho\frac{\partial}{\partial \rho}, \quad (11.28)$$

$$X_{13} = B^x\frac{\partial}{\partial B^x} + B^y\frac{\partial}{\partial B^y} + B^z\frac{\partial}{\partial B^z} + 2\rho\frac{\partial}{\partial \rho} + 2p\frac{\partial}{\partial p}, \qquad (11.29)$$

We also note that for $\gamma = 5/3$, the ideal fluid dynamics equations admit the symmetry:

$$X_{14} = tx^\alpha\frac{\partial}{\partial x^\alpha} + \left(x^i - tu^i\right)\frac{\partial}{\partial u^i} - 3t\rho\frac{\partial}{\partial \rho} - 5tp\frac{\partial}{\partial p}. \qquad (11.30)$$

In (11.30) $x^\alpha = (t, x, y, z)^t$, and we use the Einstein summation convention for repeated indices. The Greek indices $\alpha = 0, 1, 2, 3$ correspond to the space time coordinates (t, x, y, z), and the Latin indices $i = 1, 2, 3$ pertain to the space coordinates (x, y, z).

The Lie algebra of the symmetries $\{X_j : 1 \leq j \leq 14\}$ and the classification of the subgroups and conjugacy classes of the Lie algebra are given in Grundland and Lalague (1995). Grundland and Lalague use the notation:

$$P_\mu = \frac{\partial}{\partial x^\mu}, \quad K_i = x^0\frac{\partial}{\partial x^i} + \frac{\partial}{\partial u^i}, \quad J_k = \epsilon_{kij}\left(x^i\frac{\partial}{\partial x^i} + u^i\frac{\partial}{\partial u^i} + B^j\frac{\partial}{\partial B^i}\right),$$
$$(11.31)$$

to describe the Galilei group (11.21)–(11.25). The symmetries P_μ ($\mu = 0, 1, 2, 3$) correspond to the time and space translation symmetries (11.21) where $x^\mu = (t, x, y, z)$. The K_i ($i = 1, 2, 3$) are the Galilean boosts (11.22) and the J_k are the rotational symmetries (11.23)–(11.25). ϵ_{ijk} is the Levi-Civita symbol or antisymmetric tensor density. Grundland and Lalague use the symmetries: $F = X_{11} + X_{12}$, $G = X_{13} - X_{11}$ and $H = X_{13}$ as alternative basis vector fields instead of X_{11}, X_{12} and X_{13}, and use the symbol $C \equiv X_{14}$ for the projective symmetry X_{14}. An inspection of the Lie point symmetries using the Lagrangian action principle shows that Golovin (2011) also obtains the stretch symmetry $F = X_{11} + X_{12}$.

Webb and Zank (2007) and Webb et al. (2009) obtained conservation laws related to the scaling symmetries (11.27)–(11.29). These conservation laws were obtained by a judicious linear combination of the Lie symmetries (11.27)–(11.29) that leave the Lagrangian form of the action invariant. They converted the Eulerian symmetries to Lagrange label space, in which the Eulerian position coordinate $\mathbf{x}$ is a function of the Lagrange fluid labels $\mathbf{x}_0$ and time t (i.e. $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$). Each Eulerian Lie point symmetry of the Galilean group was mapped onto an infinite class of symmetries in Lagrange label space, associated with the fluid relabelling symmetries The infinitesimal symmetry generators V^t, V^x, V^y, V^z are the same in both the Eulerian and Lagrangian symmetry operators, where the symmetry generator V^{x_0} for the fluid relabeling symmetry satisfies an auxiliary set of equations in Lagrange label space. Conditions for the scaling symmetries to be a divergence or variational symmetry of the action were used to obtain conservation laws using Noether's theorem. The latter conservation laws only apply for special initial data for the gas entropy and magnetic field distribution, and have a complicated form.

The above synopsis does not include higher order Lie symmetries, e.g. the potential symmetries of the equations. Sjöberg and Mahomed (2004) obtained nonlocal symmetries and conservation laws for the planar, one dimensional gas dynamic equations from the cover system, consisting of the original equations, supplemented by known conservation laws and their associated pseudo-potentials (see also Akhatov et al. (1991), Ibragimov et al. (1998), Anco and Bluman (2002a,b), Bluman (2008) and Cheviakov and Anco (2008) for related approaches). Noether's theorem can be used to derive conservation laws, if the differential equation system admits a variational formulation or action principle. Anco and Bluman (2002a,b) used a direct method of finding conservation laws of a system of partial differential equations (see Chap. 4) that applies for equations with no variational principle. Olver and Nutku (1988) obtained higher order conservation laws and multi-Hamiltonian structures for the planar, one dimensional gas dynamic equations for the case of an isentropic polytropic equation of state. Webb and Zank (2009) used the work of Sjöberg and Mahomed (2004) to derive nonlocal conservation laws associated with the scaling symmetries.

11.2.1   Galilei Group Conservation Laws

Below, we illustrate the use of Noether's first theorem in deriving the conservation laws associated with the Galilei Lie point symmetries (11.21)–(11.25). We use a result relating the Eulerian Lie point symmetries (11.31) to their Lagrangian counterparts given in the proposition:

Proposition 11.2.1 *The Galilei group Eulerian symmetries P_μ, K_i and J_k in (11.31) correspond to the Lagrangian symmetry operators:*

$$P_\mu^L = \frac{\partial}{\partial x^\mu} + V^{\mathbf{x}_0} \cdot \nabla_0, \quad K_i^L = t\frac{\partial}{\partial x^i} + V^{\mathbf{x}_0} \cdot \nabla_0, \quad J_k^L = \epsilon_{kij}x^j\frac{\partial}{\partial x^i} + V^{\mathbf{x}_0} \cdot \nabla_0, \quad (11.32)$$

where the $V^{\mathbf{x}_0}$ satisfy the fluid relabelling symmetry equations

$$\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) = 0, \quad V^{\mathbf{x}_0} \cdot \nabla_0 S = 0, \quad D_t(\rho_0 V^{\mathbf{x}_0}) = 0,$$
$$\nabla_0 \times (V^{\mathbf{x}_0} \times \mathbf{B}_0) = 0, \quad \nabla_0 \cdot \mathbf{B}_0 = 0, \quad \Lambda_0^\alpha = 0, \quad (11.33)$$

where $\alpha = 0, 1, 2, 3$. Equations (11.60) are Lie determining equations for the fluid relabelling symmetries obtained by Padhye (1998) and Webb et al. (2005b). These equations are discussed in more detail in (11.60) et seq.

Remark The above proposition is proved in Webb and Zank (2007). For our present purposes, we set $V^{\mathbf{x}_0} = 0$. This is also the approach to the Galilean group conservation laws adopted by Padhye (1998).

Proposition 11.2.2 *The time translation symmetry of the Lagrangian action (11.1) with:*

$$V^t = 1, \quad V^{x^i} = 0, \quad V^{x_0^s} = 0, \quad \Lambda_0^\alpha = 0, \quad (11.34)$$

where $i, s = 1, 2, 3$, and $\alpha = 0, 1, 2, 3$ is a variational symmetry of the action (11.1). The corresponding conservation law using Noether's first theorem (Proposition 11.1.3) is the energy conservation law:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho|\mathbf{u}|^2 + \varepsilon(\rho, S) + \frac{B^2}{2\mu_0} + \rho\Phi(\mathbf{x})\right] + \nabla \cdot \left[\rho\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2 + h + \Phi(\mathbf{x})\right) + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}\right] = 0,$$
$$(11.35)$$

where $h = (\varepsilon + p)/\rho)$ is the gas enthalpy, $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ is the electric field strength and $\mathbf{E} \times \mathbf{B}/\mu_0$ is the Poynting flux.

Proof The canonical or evolutionary Lie symmetry generator $\hat{V}^{\mathbf{x}}$ from (11.7) is:

$$\hat{V}^{x^i} = V^{x^i} - V^{x_0^s}x_{is} - V^t x_{,t}^i \equiv -u^i. \quad (11.36)$$

Noting that

$$\hat{X} = -\left(u^i \frac{\partial}{\partial x^i} + \frac{du^i}{dt} \frac{\partial}{\partial u^i} + \frac{\partial u^i}{\partial x_0^j} \frac{\partial}{\partial x_{ij}} + \dots \right),$$

$$D_t = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + \frac{du^i}{dt} \frac{\partial}{\partial u^i} + \frac{\partial u^i}{\partial x_0^j} \frac{\partial}{\partial x_{ij}} + \dots, \tag{11.37}$$

we obtain:

$$\tilde{X}\mathcal{L}^0 = \left(\hat{X} + V^t D_t \right) \mathcal{L}^0 \equiv \frac{\partial \mathcal{L}^0}{\partial t} = 0, \tag{11.38}$$

since $\mathcal{L}^0$ does not depend explicitly on t. Thus, the condition (11.9) for the Lie invariance of the action (11.1) is satisfied. Using (11.34)–(11.36) in (11.17)–(11.20) we obtain:

$$F^0 = -\left[\frac{1}{2} \rho |\mathbf{u}|^2 + \varepsilon + \frac{B^2}{2\mu_0} + \rho \Phi(\mathbf{x}) \right],$$

$$F^j = -\left[u^j \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \varepsilon + p + \rho \Phi \right) + \frac{B^2}{\mu_0} u^j - \frac{B^j B^k}{\mu_0} u^k \right], \tag{11.39}$$

for the conserved density F^0 and flux F^j. The conservation law (11.16) reduces to the energy conservation law (11.32). This completes the proof.   □

Proposition 11.2.3 *In the absence of a gravitational field (i.e. $\Phi = 0$), the action (11.1) is invariant under the space translation symmetry with*

$$V^t = 0, \quad V^{x^j} = \delta_j^i, \quad V^{x_0^s} = 0, \quad \Lambda_0^\alpha = 0, \tag{11.40}$$

($j, s = 1, 2, 3$, $\alpha = 0, 1, 2, 3$). The conservation law associated with this symmetry (11.16) reduces to the momentum conservation equation:

$$\left\{ \frac{\partial}{\partial t} (\rho \mathbf{u}) + \nabla \cdot \left[\rho \mathbf{u} \otimes \mathbf{u} + \left(p + \frac{B^2}{2\mu_0} \right) \mathsf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right] \right\}^i = 0. \tag{11.41}$$

Proof Using (11.7) we obtain:

$$\hat{V}^{x^j} = V^{x^j} = \delta_j^i, \quad \hat{V}^{x_{jk}} = D_{x_0^j} \left(\hat{V}^{x^j} \right) = 0,$$

$$\hat{V}^{u^j} = D_t \left(\hat{V}^{x^j} \right) = 0. \tag{11.42}$$

The Lie invariance condition (11.9) reduces to:

$$\hat{X}\mathcal{L}^0 \equiv \tilde{X}\mathcal{L}^0 = -\rho_0 \frac{\partial \Phi}{\partial x^i} = 0, \tag{11.43}$$

since we assume that $\nabla\Phi = 0$. Thus, (11.40) is a variational symmetry of the action in this case. Computing the conserved density F^0 and flux F^j from (11.17) and (11.18) gives the momentum conservation law (11.41). This completes the proof.

$\square$

Remark The case of an external gravitational field, is different than the problem of self-consistent Newtonian gravity. In the latter case, the gravitational potential Φ satisfies Poisson's equation:

$$\nabla^2\Phi = -4\pi G\rho(\mathbf{x}), \tag{11.44}$$

where G is the universal gravitational constant. The solution of the Poisson equation (11.44) is given by:

$$\Phi = G\int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3x'. \tag{11.45}$$

In this book, we do not address the problem of self consistent gravity.

Proposition 11.2.4 *The rotational symmetry with generators:*

$$V^{x^i} = \epsilon_{ijk}\Omega^j x^k \equiv (\mathbf{\Omega}\times\mathbf{x})^i, \quad V^{x_0^s} = 0, \quad \Lambda_0^\alpha = 0, \tag{11.46}$$

($i, s = 1, 2, 3$, $\alpha = 0, 1, 2, 3$) corresponds via Noether's theorem (Proposition 11.1.3), to the angular momentum conservation law:

$$\frac{\partial}{\partial t}\left[\mathbf{\Omega}\cdot(\mathbf{x}\times\mathbf{M})\right] + \nabla\cdot\left[\mathbf{\Omega}\cdot(\mathbf{x}\times\mathsf{T})\right] = 0, \tag{11.47}$$

where

$$\mathbf{M} = \rho\mathbf{u}, \quad (\mathbf{x}\times\mathsf{T})^{pj} = \epsilon_{pqk}x^q T^{kj}. \tag{11.48}$$

The condition on the gravitational potential needed for the conservation law (11.47) to apply is that

$$\tilde{X}\mathcal{L}^0 = -\rho_0(\mathbf{\Omega}\times\mathbf{x})\cdot\nabla\Phi = 0. \tag{11.49}$$

The condition (11.49) is tantamount to the existence of an axis of symmetry for the gravitational potential $\Phi(\mathbf{x})$. For example if $\mathbf{\Omega} = \Omega\mathbf{e}_z$ is directed along the z-axis, then (11.49) is satisfied if $\partial\Phi/\partial\phi = 0$ where ϕ is the spherical polar azimuthal angle about the z-axis. Thus, in this case the gravitational potential is independent of the azimuthal angle ϕ. The condition is also satisfied for a spherically symmetric potential Φ. The condition is also satisfied if $\Phi = 0$ (i.e. gravity is negligible).

Proof For the Lie symmetry generators (11.46),

$$\hat{V}^{x^i} = V^{x^i} = \epsilon_{ijk}\Omega^j x^k, \quad \hat{V}^{u^i} = \epsilon_{ijk}\Omega^j u^k, \quad \hat{V}^{x_{ij}} = \epsilon_{isk}\Omega^s x_{kj}. \tag{11.50}$$

We find:

$$\hat{V}^{x_{ij}}\frac{\partial\mathcal{L}^0}{\partial x_{ij}} = \left(p + \frac{B^2}{2\mu_0}\right)\delta_k^i\epsilon_{isk} - J\mathbf{B}\cdot(\mathbf{\Omega}\times\mathbf{B}) = 0, \quad \hat{V}^{\mathbf{u}}\cdot\frac{\partial\mathcal{L}^0}{\partial\mathbf{u}} = (\mathbf{\Omega}\times\mathbf{u})\cdot\rho_0\mathbf{u} = 0. \tag{11.51}$$

The condition (11.9) for a variational symmetry of the action reduces to (11.49). Using (11.17) and (11.18) to compute the conserved density F^0 and flux F^j gives the angular momentum conservation equation (11.47).   □

Proposition 11.2.5 *The Galilean boost symmetry, with generators:*

$$V^{x^i} = \Omega^i t, \quad V^{x_0^s} = 0, \quad V^t = 0, \quad \Lambda_0^0 = -\rho_0(\mathbf{x}_0)\mathbf{\Omega}\cdot\mathbf{x}, \quad \Lambda_0^i = 0, \tag{11.52}$$

(i, s = 1, 2, 3) admits the center of mass conservation law:

$$\frac{\partial}{\partial t}\left[\mathbf{\Omega}\cdot\rho(\mathbf{u}t - \mathbf{x})\right] + \nabla\cdot(\mathbf{\Omega}\cdot[t\mathsf{T} - \rho\mathbf{x}\otimes\mathbf{u}]) = 0, \tag{11.53}$$

provided the gravitational potential Φ satisfies the condition (11.9), which reduces to:

$$\tilde{X}\mathcal{L}^0 - \frac{d}{dt}\left(\rho_0(\mathbf{x}_0)\mathbf{\Omega}\cdot\mathbf{x}\right) = -\rho_0(\mathbf{x}_0)t\mathbf{\Omega}\cdot\nabla\Phi = 0. \tag{11.54}$$

where d/dt is the Lagrangian time derivative with $\mathbf{x}_0$ held constant. Thus, the conservation law (11.53) applies if $\mathbf{\Omega}\cdot\nabla\Phi = 0$.

Proof Using (11.7) we obtain:

$$\hat{V}^{x^i} = V^{x^i} = \Omega^i t, \quad \hat{V}^{u^i} = D_t\left(\hat{V}^{x^i}\right) = \Omega^i, \quad \hat{V}^{x_{ij}} = D_{x_0^j}\left(\Omega^i t\right) = 0. \tag{11.55}$$

Using (11.55), the condition (11.9) for a divergence symmetry of the action reduces to (11.54). Using the results (11.55) in (11.17) and (11.18) to determine the conserved density F^0 and flux F^j gives the center of mass conservation law (11.53), where $\mathsf{T} = T^{jk}\mathbf{e}_j\otimes\mathbf{e}_k$ are the spatial components of the MHD stress energy tensor (11.19).   □

Remark 1 The conservation law (11.53) can also be written as:

$$\frac{\partial}{\partial t}[\mathbf{\Omega}\cdot\rho(\mathbf{u}t - \mathbf{x})] + \nabla\cdot\left[\mathbf{\Omega}\cdot\left\{\rho(\mathbf{u}t - \mathbf{x})\otimes\mathbf{u} + t\left[\left(p + \frac{B^2}{2\mu_0}\right)\mathsf{I} - \frac{\mathbf{B}\otimes\mathbf{B}}{\mu_0}\right]\right\}\right] = 0. \tag{11.56}$$

Remark 2 From (11.54) the Galilean boost $V^{\mathbf{x}} = \boldsymbol{\Omega} t$ must be perpendicular to $\nabla \Phi$ in order to obtain a conservation law. If $\boldsymbol{\Omega} t \cdot \nabla \Phi \neq 0$ there is no conservation law, because of symmetry breaking.

11.3   Fluid Relabelling Symmetries

Consider infinitesimal Lie transformations of the form (11.2)–(11.3), with

$$V^t = 0, \quad V^{\mathbf{x}} = 0, \quad V^{\mathbf{x}_0} \neq 0, \tag{11.57}$$

which leave the action (11.1) invariant. The extended Lie transformation operator $\tilde{X}$ for the case (11.57) has generators:

$$\hat{V}^{\mathbf{x}} = -V^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{x}, \quad V^{\mathbf{x}_t} = -D_t \left(V^{\mathbf{x}_0} \right) \cdot \nabla_0 \mathbf{x},$$
$$V^{\nabla_0 \mathbf{x}} = -\nabla_0 \left(V^{\mathbf{x}_0} \right) \cdot \nabla_0 \mathbf{x}. \tag{11.58}$$

The condition (11.9) for a divergence symmetry of the action reduces to:

$$\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) \left(\frac{1}{2} |\mathbf{u}|^2 - \Phi(\mathbf{x}) - \frac{\varepsilon + p}{\rho} \right) - J \frac{\partial \varepsilon(\rho, S)}{\partial S} V^{\mathbf{x}_0} \cdot \nabla_0 S$$

$$- D_t \left(\rho_0 V^{\mathbf{x}_0} \right) \cdot \nabla_0 \mathbf{x} \cdot \mathbf{u} - \frac{1}{\mu J} (\nabla_0 \mathbf{x}) \cdot (\nabla_0 \mathbf{x})^T : \left[\left(V^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{B}_0 \right) \mathbf{B}_0 \right.$$

$$\left. + \mathbf{B}_0 \mathbf{B}_0 \nabla_0 \cdot V^{\mathbf{x}_0} - \left(\mathbf{B}_0 \cdot \nabla_0 V^{\mathbf{x}_0} \right) \mathbf{B}_0 \right] = -\partial \Lambda_0^\alpha / \partial x_0^\alpha. \tag{11.59}$$

where $x_0^\alpha = (t, x_0, y_0, z_0)$ is the spatial four-vector in Lagrange label space, and $\alpha = 0, 1, 2, 3$ and $x_0^0 \equiv t$.

A simple class of solutions of (11.59) with $\Lambda_0^\alpha = 0$ ($\alpha = 0, 1, 2, 3$) are obtained by setting:

$$\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) = 0, \quad V^{\mathbf{x}_0} \cdot \nabla_0 S = 0, \quad D_t (\rho_0 V^{\mathbf{x}_0}) = 0,$$
$$\nabla_0 \times (V^{\mathbf{x}_0} \times \mathbf{B}_0) = 0, \quad \nabla_0 \cdot \mathbf{B}_0 = 0, \quad \Lambda_0^\alpha = 0, \tag{11.60}$$

where $\alpha = 0, 1, 2, 3$. Equations (11.60) are Lie determining equations for the fluid relabelling symmetries obtained by Padhye (1998) and Webb et al. (2005b). These equations apply for a general equation of state for the gas with $\varepsilon = \varepsilon(\rho, S)$ and also apply in an external gravitational field described by the gravitational potential Φ. The fluid relabelling symmetries do not change the Eulerian physical variables ρ, $\mathbf{u}$, S, p, $\mathbf{B}$ under the Lie transformation (11.57).

However, the solutions of (11.60) do not give the most general solutions describing the fluid relabelling symmetries. To investigate other possible solutions

of (11.59) it is useful to convert the fluid relabelling divergence symmetry condition to its corresponding Eulerian form given below.

Proposition 11.3.1 *The condition (11.59) for a divergence symmetry of the action converted to Eulerian form is:*

$$\nabla \cdot \left(\rho \hat{V}^{\mathbf{x}} \right) \left(h + \Phi(\mathbf{x}) - \frac{1}{2} |\mathbf{u}|^2 \right) + \rho T \hat{V}^{\mathbf{x}} \cdot \nabla S + \rho \mathbf{u} \cdot \left(\frac{d\hat{V}^{\mathbf{x}}}{dt} - \hat{V}^{\mathbf{x}} \cdot \nabla \mathbf{u} \right)$$

$$+ \frac{\mathbf{B}}{\mu_0} \cdot \left[-\nabla \times \left(\hat{V}^{\mathbf{x}} \times \mathbf{B} \right) + \hat{V}^{\mathbf{x}} \nabla \cdot \mathbf{B} \right] = -\nabla_\alpha \Lambda^\alpha, \tag{11.61}$$

where

$$\nabla_\alpha \Lambda^\alpha = \frac{\partial \Lambda^0}{\partial t} + \frac{\partial \Lambda^i}{\partial x^i}, \tag{11.62}$$

is the four divergence of the four dimensional vector $\mathbf{\Lambda} = \left(\Lambda^0, \Lambda^1, \Lambda^2, \Lambda^3 \right)$. *The four vector* $\mathbf{\Lambda}$ *is related the Lagrange label space vector* Λ_0^α *by the transformations:*

$$\Lambda^\alpha = \Lambda_0^\beta B_{\beta\alpha} \equiv \Lambda_0^\beta \frac{x_{\alpha\beta}}{J}, \tag{11.63}$$

where $x_{\alpha\beta} = \partial x^\alpha / \partial x_0^\beta$, $J = \det(x_{ij})$ *and* $B_{\alpha\beta} = cofac(\partial x_0^\alpha / \partial x^\beta)$ *(the transformations (11.63) are the same as those in (11.20); note that* α, β *have values* 0, 1, 2, 3).

Proof Using (11.1)–(11.8) and using the transformations (11.58) relating $\hat{V}^{\mathbf{x}}$, $\hat{V}^{\mathbf{x}_t}$ and $\hat{V}^{\mathbf{x}_{ij}}$ to $V^{\mathbf{x}_0}$, we obtain the transformations:

$$\nabla_0 \left(\rho_0 V^{\mathbf{x}_0} \right) = -J\nabla \cdot \left(\rho \hat{V}^{\mathbf{x}} \right), \quad -V^{\mathbf{x}_0} \cdot \nabla_0 S = \hat{V}^{\mathbf{x}} \cdot \nabla S,$$

$$-\frac{d}{dt} \left(\rho_0 V^{\mathbf{x}_0^k} \right) x_{pk} u^p = \rho J \mathbf{u} \cdot \left(\frac{d\hat{V}^{\mathbf{x}}}{dt} - \hat{V}^{\mathbf{x}} \cdot \nabla \mathbf{u} \right), \tag{11.64}$$

for the gas dynamical terms in (11.59). Similarly, the magnetic field terms in (11.59) transform as:

$$-\frac{1}{\mu_0 J} (\nabla_0 \mathbf{x}) \cdot (\nabla_0 \mathbf{x})^T : \left[(V_0^{\mathbf{x}} \cdot \nabla_0) \mathbf{B}_0 \mathbf{B}_0 + \mathbf{B}_0 \mathbf{B}_0 (\nabla_0 \cdot V^{\mathbf{x}_0}) - (\mathbf{B}_0 \cdot \nabla_0 V^{\mathbf{x}_0}) \mathbf{B}_0 \right]$$

$$= \frac{J\mathbf{B}}{\mu_0} \cdot \left[-\nabla \times (\hat{V}^{\mathbf{x}} \times \mathbf{B}) + \hat{V}^{\mathbf{x}} (\nabla \cdot \mathbf{B}) \right]. \tag{11.65}$$

Substitution of (11.64)–(11.65) in (11.59) results in the Eulerian divergence symmetry condition (11.61). This completes the proof of (11.61).   □

Proposition 11.3.2 *The divergence symmetry condition (11.61) has solutions:*

$$\hat{V}^{\mathbf{x}} = \mathbf{u},$$

$$\Lambda^0 = -\left(\frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \rho\Phi(\mathbf{x}) - \frac{B^2}{2\mu_0}\right) - \rho f(\mathbf{x}_0),$$

$$\Lambda^i = -\rho u^i f(\mathbf{x}_0), \tag{11.66}$$

where $f(\mathbf{x}_0)$ is an arbitrary function of $\mathbf{x}_0$. The gauge potential $\Lambda^0 = -L - \rho f(\mathbf{x}_0)$ where L is the Eulerian MHD Lagrangian density, including an external gravitational potential $\Phi(\mathbf{x})$. The corresponding conservation laws associated with the solutions (11.66) are the MHD energy conservation equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \varepsilon(\rho, S) + \rho\Phi(\mathbf{x}) + \frac{B^2}{2\mu_0}\right) + \nabla\cdot\left[\rho\mathbf{u}\left(\frac{1}{2}|\mathbf{u}|^2 + h + \Phi\right) + \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right] = 0, \tag{11.67}$$

and the conservation law:

$$\frac{\partial}{\partial t}[\rho f(\mathbf{x}_0)] + \nabla\cdot[\rho\mathbf{u}f(\mathbf{x}_0)] = 0 \quad or \quad \left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)f(\mathbf{x}_0) = 0. \tag{11.68}$$

Remark 1 The MHD energy conservation equation (11.67) is usually associated with the time translation symmetry of the action (a Lie point symmetry), for which $V^t = 1$, $V^{\mathbf{x}} = 0$, $V^\psi = 0$ (ψ is any of the MHD physical variables ρ, $\mathbf{u}$, $\mathbf{B}$ and S), and $\Lambda^\alpha = 0$ ($\alpha = 0, 1, 2, 3$). The result (11.67) shows that the energy conservation law (11.67) also arises as a gauge or divergence symmetry of the action associated with the fluid relabelling symmetry.

Remark 2 The conservation law (11.68) states that an arbitrary function $f(\mathbf{x}_0)$ of the Lagrange labels $\mathbf{x}_0$ is advected with the flow. This is a fairly obvious conservation law, since $d\mathbf{x}_0/dt = 0$ for the Lagrange labels $\mathbf{x}_0$. Non-trivial examples of this conservation law are obtained for:

$$f_1(\mathbf{x}_0) = \frac{\mathbf{B}\cdot\nabla S}{\rho} \equiv \frac{\mathbf{B}_0(\mathbf{x}_0)\cdot\nabla_0 S(\mathbf{x}_0)}{\rho_0(\mathbf{x}_0)},$$

$$f_2(\mathbf{x}_0) = \frac{\mathbf{A}\cdot\mathbf{B}}{\rho} \equiv \frac{\mathbf{A}_0(\mathbf{x}_0)\cdot\mathbf{B}_0(\mathbf{x}_0)}{\rho_0(\mathbf{x}_0)}, \tag{11.69}$$

where $\mathbf{A}$ is chosen so that $\mathbf{A}\cdot d\mathbf{x} = \mathbf{A}_0(\mathbf{x}_0)\cdot d\mathbf{x}_0$ is advected with the flow.

Proof To obtain the solutions (11.66) of the Lie determining equations (11.61) for a divergence symmetry of the action, we note that with $\hat{V}^{\mathbf{x}} = \mathbf{u}$, (11.61) reduces to:

$$\nabla\cdot(\rho\mathbf{u})\left(h + \Phi(\mathbf{x}) - \frac{1}{2}|\mathbf{u}|^2\right) + \rho T\mathbf{u}\cdot\nabla S + \rho\mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial t}$$

$$+ \frac{\mathbf{B}}{\mu_0}\cdot[-\nabla\times(\mathbf{u}\times\mathbf{B}) + \mathbf{u}(\nabla\cdot\mathbf{B})] = -\nabla_\alpha\Lambda^\alpha. \tag{11.70}$$

Next we use the identities:

$$T_1 = \rho\mathbf{u} \cdot \frac{\partial\mathbf{u}}{\partial t} - \frac{1}{2}|\mathbf{u}|^2\nabla\cdot(\rho\mathbf{u}) \equiv \frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\mathbf{u}|^2\right) - \frac{1}{2}|\mathbf{u}|^2\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right],$$

$$T_2 = \nabla\cdot(\rho\mathbf{u})h + \rho T\mathbf{u}\cdot\nabla S = \nabla\cdot(\rho\mathbf{u}h) - \mathbf{u}\cdot\nabla p \equiv -\frac{\partial\varepsilon}{\partial t},$$

$$T_3 = \nabla\cdot(\rho\mathbf{u})\Phi(\mathbf{x}) \equiv -\frac{\partial\rho}{\partial t}\Phi = -\frac{\partial}{\partial t}[\rho\Phi(\mathbf{x})],$$

$$T_4 = \frac{\mathbf{B}}{\mu_0}\cdot[-\nabla\times(\mathbf{u}\times\mathbf{B}) + \mathbf{u}(\nabla\cdot\mathbf{B})] \equiv -\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right), \tag{11.71}$$

In (11.71) use of the mass continuity equation (2.1) gives $T_1 = \partial((1/2)\rho|\mathbf{u}|^2)/\partial t$. The term T_2 in (11.71) reduces to $-\partial\varepsilon/\partial t$, where we have used the internal energy evolution equation for the gas:

$$\frac{\partial\varepsilon}{\partial t} + \nabla\cdot(\rho\mathbf{u}h) = \mathbf{u}\cdot\nabla p, \tag{11.72}$$

where $\varepsilon = \varepsilon(\rho, S)$. The expression T_4 in (11.71), using Faraday's equation reduces to $-\partial(B^2/2\mu_0)/\partial t$. The latter result may also be obtained by using Poynting's theorem.

Using the results (11.71), (11.70) reduces to:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \rho\Phi(\mathbf{x}) - \frac{B^2}{2\mu_0}\right) = -\left(\frac{\partial\Lambda^0}{\partial t} + \frac{\partial\Lambda^i}{\partial x^i}\right). \tag{11.73}$$

Equation (11.73) has solutions of the form (11.66).

The total energy conservation law (11.67) and the Lagrangian advection conservation law (11.68) now follow by using the symmetry results (11.66) in Noether's theorem (Propositions 9.1.1–9.2.1). From (11.17)–(11.18) we find:

$$F^0 = \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \varepsilon + \rho\Phi + \frac{B^2}{2\mu_0}\right) + \rho f(\mathbf{x}_0), \tag{11.74}$$

$$\mathbf{F} = \left[\rho\mathbf{u}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + h + \Phi\right) + \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right] + \rho\mathbf{u}f(\mathbf{x}_0), \tag{11.75}$$

where $\mathbf{F} = (F^1, F^2, F^3)$ is the spatial flux and $\mathbf{E} = -(\mathbf{u}\times\mathbf{B})$ is the electric field. The MHD energy conservation law (11.67) is obtained by setting $f(\mathbf{x}_0) = 0$ in (11.74)–(11.75) and using (11.74)–(11.75) for F^0 and $\mathbf{F}$ in (11.16). Similarly, the conservation law (11.68) for $f(\mathbf{x}_0)$ is obtained by using (11.16). This completes the proof.   □

Proposition 11.3.3 *The fluid relabelling symmetry and divergence symmetry of the action defined by:*

$$\hat{V}^{\mathbf{x}} = \mathbf{b}, \quad \Lambda^0 = (\mathbf{B} \cdot \nabla S)r, \quad \Lambda^j = \Lambda^0 u^j, \tag{11.76}$$

satisfies the condition (11.61) for a divergence symmetry. It gives rise to the nonlocal cross helicity conservation law (3.67) discussed in Sect. 3.3 in Chap. 3.

11.4   Casimirs and Fluid Relabelling Symmetries

In this section, we discuss the analysis of Padhye and Morrison (1996a,b) that the fluid relabelling symmetry determining equations (11.60) are related to the Casimir equations (8.125)–(8.126). Henyey (1982) used a Clebsch variable formulation of MHD, and investigated the connection between gauge symmetries and Casimirs in MHD.

The Poisson bracket for MHD using the canonical Poisson bracket from (10.17)–(10.18) is given by:

$$\{F, G\} = \int \sum_{k=1}^{3} \left(\frac{\delta F}{\delta q^k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q^k} \frac{\delta F}{\delta p_k} \right) d^3 x_0. \tag{11.77}$$

where $q^k = x^k(\mathbf{x}_0, t)$ and $p_k \equiv \Pi_k = \partial \mathcal{L}^0 / \partial \dot{x}^k$ are canonically conjugate variables. Using the Poisson bracket (11.77) we obtain

$$\{F, q^k\} = -\frac{\delta F}{\delta p_k} \quad \text{and} \quad \{F, p_k\} = \frac{\delta F}{\delta q^k}. \tag{11.78}$$

If the Hamiltonian dynamics of interest takes place on a Casimir surface $C[q, p] = const.$ then

$$\delta C = \int \left(\frac{\delta C}{\delta q^k} \delta q^k + \frac{\delta C}{\delta p_k} \delta p_k \right) d^3 x_0 = 0. \tag{11.79}$$

Writing

$$\delta q^k = \epsilon \hat{V}^{q^k} \quad \text{and} \quad \delta p_k = \epsilon \hat{V}^{p_k}, \tag{11.80}$$

(11.79) becomes:

$$\delta C = \int \left(\hat{V}^{q^k} \frac{\delta C}{\delta q^k} + \hat{V}^{p_k} \frac{\delta C}{\delta p_k} \right) d^3 x_0 = 0. \tag{11.81}$$

Equation (11.81) is satisfied by the choice:

$$\hat{V}^{q^k} = \{C, q^k\} = -\frac{\delta C}{\delta p_k} \quad \text{and} \quad \hat{V}^{p_k} = \{C, p_k\} = \frac{\delta C}{\delta q^k}. \tag{11.82}$$

Thus, the vectors $\left\{\hat{V}^{q^k}, \hat{V}^{p_k}\right\}$ and $\{\delta C/\delta q^k, \delta C/\delta p_k\}$ are orthogonal. We can write:

$$\delta C = \int \hat{X}(C)\, d^3x_0 = \int \{C, q^k\} \frac{\delta C}{\delta q^k} + \{C, p_k\} \frac{\delta C}{\delta p_k}\, d^3x_0 = 0. \qquad (11.83)$$

Using formula (11.12) for the canonical (or evolutionary) Lie symmetry $\hat{X}$ (which describes Lagrangian variations with $\mathbf{x}_0$ fixed), we obtain:

$$\{C, \Pi_k\} = \frac{\delta C}{\delta x^k} = \hat{X}\Pi_k = \rho_0 \hat{V}^{\mathbf{x}} \cdot \nabla u^k \equiv \Delta\Pi_k,$$

$$\{C, x^k\} = -\frac{\delta C}{\delta \Pi_k} = \hat{X}x^k = \hat{V}^{x^k} \equiv \Delta x^k. \qquad (11.84)$$

In (11.84) and below, we use the notation $\Delta\psi = \partial\psi[\mathbf{x}(\mathbf{x}_0, t); \epsilon]/\partial\epsilon$ at $\epsilon = 0$ to denote the Lagrangian variation of the physical quantity ψ with $\mathbf{x}_0$ held fixed. In the derivation of the formula for $\hat{X}\Pi_k$ in (11.84) we used the result:

$$\hat{X}\Pi_k = \rho_0 \hat{V}^{x^k_{,t}} \equiv \rho_0 D_t\left(-V^{x^s_0} x_{ks}\right) = -\rho_0 V^{\mathbf{x}_0} \cdot \nabla_0 u^k = \rho_0 \hat{V}^{\mathbf{x}} \cdot \nabla u^k, \qquad (11.85)$$

(note that $D_t(V^{\mathbf{x}_0}) = 0$ from the fluid relabelling symmetry equations (11.60)). Equations (11.84) relate the variational derivatives $\delta C/\delta x^k$ and $-\delta C/\delta\Pi_k$ to the fluid relabelling symmetry infinitesimal transformations $\hat{X}\Pi_k$ and $\hat{X}x^k$.

Similarly, using the evolutionary symmetry operator $\hat{X}$ we obtain the results:

$$\Delta\mathbf{B} = \hat{X}\mathbf{B} = \mathbf{B} \cdot \nabla\hat{V}^{\mathbf{x}} - \mathbf{B}\left(\nabla \cdot \hat{V}^{\mathbf{x}}\right), \quad \Delta\rho = \hat{X}\rho = -\rho\nabla \cdot \hat{V}^{\mathbf{x}},$$

$$\Delta\mathbf{M} = \hat{X}(\rho\mathbf{u}) = \Delta(\rho\mathbf{u}) = -\mathbf{M}\left(\nabla \cdot \hat{V}^{\mathbf{x}}\right) + \rho\hat{V}^{\mathbf{x}} \cdot \nabla\mathbf{u},$$

$$\Delta\sigma = -\sigma\nabla \cdot \hat{V}^{\mathbf{x}}, \quad \hat{X}J = J(\nabla \cdot \hat{V}^{\mathbf{x}}). \qquad (11.86)$$

To derive the Casimir equations (8.125)–(8.126) from the fluid relabelling symmetry equations (11.60) we consider Lagrangian variations of C using both the canonical variables x^k and Π_k and also the physical variables ρ, $\mathbf{M}$, σ and $\mathbf{B}$, using the variational equation:

$$\int \left(\frac{\delta C}{\delta x^k}\Delta x^k + \frac{\delta C}{\delta\Pi_k}\Delta\Pi_k\right) d^3x_0$$

$$= \int J\, d^3x_0 \left(\frac{\delta C}{\delta\rho}\Delta\rho + \frac{\delta C}{\delta\mathbf{M}} \cdot \Delta\mathbf{M} + \frac{\delta C}{\delta\sigma}\Delta\sigma + \frac{\delta C}{\delta\mathbf{B}} \cdot \Delta\mathbf{B}\right), \qquad (11.87)$$

where $J = \det(\partial x^i/\partial x_0^j)$ is the Jacobian of the Lagrangian map.

Using (11.84)–(11.86) in the variational equation (11.87) we obtain:

$$
\int \left(\frac{\delta C}{\delta x^k} \hat{V}^{x^k} + \frac{\delta C}{\delta \Pi_k} \rho_0 \hat{V}^{x^k} \cdot \nabla u^k \right) d^3 x_0
$$

$$
= \int d^3 x_0 J \left\{ \frac{\delta C}{\delta \rho} \left(-\rho \nabla \cdot \hat{V}^x \right) + \frac{\delta C}{\delta \mathbf{M}} \cdot \left(-\mathbf{M} \left(\nabla \cdot \hat{V}^x \right) + \rho \hat{V}^x \cdot \nabla \mathbf{u} \right) \right.
$$

$$
\left. + \frac{\delta C}{\delta \sigma} (-\sigma \nabla \cdot \hat{V}^x) + \frac{\delta C}{\delta \mathbf{B}} \cdot \left[\mathbf{B} \cdot \nabla \hat{V}^x - \mathbf{B} \left(\nabla \cdot \hat{V}^x \right) \right] \right\}. \tag{11.88}
$$

Equating the components of $\hat{V}^{x^k} \cdot \nabla u^k$ in (11.88) gives the balance equation:

$$
\frac{\delta C}{\delta \mathbf{M}} = \frac{\delta C}{\delta \Pi}. \tag{11.89}
$$

The balance equation (11.89) is consistent with (11.88) and leads to the Casimir invariant determining equation (8.126). However, it is not the only possible balance (see, the Eulerian variational approach to the relation between Casimirs and fluid relabelling symmetries in the next section).

Noting that $d^3 x = J d^3 x_0$, balance of the remaining terms in (11.88) gives:

$$
\int \frac{\delta C}{\delta \mathbf{x}} \cdot \hat{V}^x \, d^3 x_0
$$

$$
= -\int d^3 x \left(\rho \frac{\delta C}{\delta \rho} + \mathbf{M} \cdot \frac{\delta C}{\delta \mathbf{M}} + \sigma \frac{\delta C}{\delta \sigma} + \mathbf{B} \cdot \frac{\delta C}{\delta \mathbf{B}} \right) \nabla \cdot \hat{V}^x + \int d^3 x \, \mathbf{B} \cdot \nabla \hat{V}^x \cdot \frac{\delta C}{\delta \mathbf{B}}. \tag{11.90}
$$

Integrating the right hand-side of (11.90) by parts, dropping surface terms and noting that $\hat{V}^x \cdot \nabla C[\mathbf{M}, \mathbf{B}, \rho, \sigma] = 0$, (11.90) implies:

$$
\frac{\delta C}{\delta \mathbf{x}} = J \left\{ \rho \nabla \left(\frac{\delta C}{\delta \rho} \right) + M^j \nabla \left(\frac{\delta C}{\delta M^j} \right) + \sigma \nabla \left(\frac{\delta C}{\delta \sigma} \right) + \mathbf{B} \times \left[\nabla \times \left(\frac{\delta C}{\delta \mathbf{B}} \right) \right] - \nabla \cdot \mathbf{B} \frac{\delta C}{\delta \mathbf{B}} \right\}. \tag{11.91}
$$

From (11.84) and (11.89) we obtain a further equation for $\delta C/\delta \mathbf{x}$:

$$
\hat{V}^x = -\frac{\delta C}{\delta \Pi} = -\frac{\delta C}{\delta \mathbf{M}}
$$

$$
\frac{\delta C}{\delta \mathbf{x}} = \rho_0 \hat{V}^x \cdot \nabla \mathbf{u} = -\rho J \frac{\delta C}{\delta \mathbf{M}} \cdot \nabla \mathbf{u}. \tag{11.92}
$$

Substituting $\delta C/\delta \mathbf{x}$ from (11.92) in (11.91) gives the Casimir determining equation (8.126):

$$
M^j \nabla C_{M^j} + \rho C_{\mathbf{M}} \cdot \nabla (\mathbf{M}/\rho) + \rho \nabla C_\rho + \sigma \nabla C_\sigma + \mathbf{B} \times (\nabla \times C_{\mathbf{B}}) - (\nabla \cdot \mathbf{B}) C_{\mathbf{B}} = 0. \tag{11.93}
$$

(in (8.126), $\nabla \cdot \mathbf{B} = 0$). The remaining Casimir equations (8.125) are equivalent to the fluid relabelling equations (11.60). This completes our exposition of the work of Padhye and Morrison (1996a,b) on the link between the fluid relabelling symmetry equations (11.60) and the Casimir equations (8.125)–(8.126). As mentioned in (11.89) et seq., there is a more general formulation of the problem, which we address below.

11.4.1   Eulerian Variations

The above derivation of the Casimir determining equations (11.93) can also be carried out more directly by using Eulerian variations of C:

$$\delta C = \int \left(\frac{\delta C}{\delta \rho} \delta\rho + \frac{\delta C}{\delta \sigma} \delta\sigma + \frac{\delta C}{\delta \mathbf{M}} \cdot \delta\mathbf{M} + \frac{\delta C}{\delta \mathbf{B}} \cdot \delta\mathbf{B} \right) d^3x = 0, \qquad (11.94)$$

The Eulerian variation of a physical quantity ψ, $\delta\psi$, is related to its Lagrangian variation $\Delta\psi$ by the equation:

$$\delta\psi = \Delta\psi - \Delta\mathbf{x} \cdot \nabla\psi \equiv \Delta\psi - \hat{V}^{\mathbf{x}} \cdot \nabla\psi, \qquad (11.95)$$

where $\hat{V}^{\mathbf{x}} = \Delta\mathbf{x} = (\partial\mathbf{x}/\partial\epsilon)|_{\mathbf{x}_0}$ (in the Euler-Poincaré development in Sect. 7.2, we took $\epsilon = t$ and $\Delta\mathbf{x} = \mathbf{u}$). Using the Lagrangian variations (11.86) we obtain:

$$\delta\rho = -\nabla \cdot (\rho\hat{V}^{\mathbf{x}}), \quad \delta\sigma = -\nabla \cdot \left(\sigma\hat{V}^{\mathbf{x}} \right),$$

$$\delta\mathbf{B} = \nabla \times \left(\hat{V}^{\mathbf{x}} \times \mathbf{B} \right) - \hat{V}^{\mathbf{x}}\nabla \cdot \mathbf{B}, \quad \delta\mathbf{M} = -\mathbf{u}\nabla \cdot \left(\rho\hat{V}^{\mathbf{x}} \right). \qquad (11.96)$$

Using the variational formulae (11.96) in (11.94) we obtain:

$$\delta C = \int \left\{ \frac{\delta C}{\delta\rho} \left[-\nabla \cdot (\rho\hat{V}^{\mathbf{x}}) \right] + \frac{\delta C}{\delta\sigma} \left[-\nabla \cdot \left(\sigma\hat{V}^{\mathbf{x}} \right) \right] + \frac{\delta C}{\delta\mathbf{M}} \cdot \left[-\mathbf{u}\nabla \cdot \left(\rho\hat{V}^{\mathbf{x}} \right) \right] \right.$$

$$\left. + \frac{\delta C}{\delta\mathbf{B}} \cdot \left[\nabla \times \left(\hat{V}^{\mathbf{x}} \times \mathbf{B} \right) - \hat{V}^{\mathbf{x}}\nabla \cdot \mathbf{B} \right] \right\} d^3x. \qquad (11.97)$$

Integration of (11.97) by parts gives:

$$\delta C = \int \left\{ \hat{V}^{\mathbf{x}} \cdot \left[\rho\nabla C_\rho + \sigma\nabla C_\sigma + \rho\nabla(\mathbf{u} \cdot C_{\mathbf{M}}) + \mathbf{B} \times (\nabla \times C_{\mathbf{B}}) - C_{\mathbf{B}}\nabla \cdot \mathbf{B} \right] \right.$$

$$\left. + \nabla \cdot \left[-\hat{V}^{\mathbf{x}} \left(\rho C_\rho + \sigma C_\sigma + \mathbf{M} \cdot C_{\mathbf{M}} \right) + \left(\hat{V}^{\mathbf{x}} \times \mathbf{B} \right) \times C_{\mathbf{B}} \right] \right\} d^3x. \qquad (11.98)$$

Dropping the surface terms in (11.98) gives:

$$\delta C = \int \hat{V}^{\mathbf{x}} \cdot \left[\rho \nabla C_\rho + \sigma \nabla C_\sigma + \rho \nabla (\mathbf{u} \cdot C_{\mathbf{M}}) + \mathbf{B} \times (\nabla \times C_{\mathbf{B}}) - C_{\mathbf{B}} \nabla \cdot \mathbf{B}\right] d^3 x. \quad (11.99)$$

Equation (11.99) can be written in the form:

$$\delta C = \delta C_1 + \delta C_2, \quad (11.100)$$

where

$$\delta C_1 = \int d^3 x \, \hat{V}^{\mathbf{x}} \cdot \left[M^j \nabla C_{M^j} + \rho C_{\mathbf{M}} \cdot \nabla (\mathbf{M}/\rho) + \rho \nabla C_\rho + \sigma \nabla C_\sigma \right.$$

$$\left. + \mathbf{B} \times (\nabla \times C_{\mathbf{B}}) - C_{\mathbf{B}} \nabla \cdot \mathbf{B} \right], \quad (11.101)$$

$$\delta C_2 = -\int d^3 x \, \rho \hat{V}^{\mathbf{x}} \cdot \boldsymbol{\omega} \times C_{\mathbf{M}}. \quad (11.102)$$

Thus, $\delta C = 0$ if both $\delta C_1 = 0$ and $\delta C_2 = 0$. The condition $\delta C_1 = 0$ is equivalent to the Casimir determining equation (11.93). The condition $\delta C_2 = 0$ implies an extra constraint on the fluid relabelling symmetry equations, in order to obtain a Casimir invariant (i.e. the fluid relabelling symmetries are more general than the Casimir invariants).

One possible solution for $\rho \hat{V}^{\mathbf{x}}$ that satisfies $\delta C_2 = 0$ is:

$$\rho \hat{V}^{\mathbf{x}} = \nabla \times \boldsymbol{\psi} = \alpha \boldsymbol{\omega} + \beta C_{\mathbf{M}}, \quad (11.103)$$

where $\nabla \cdot (\rho \hat{V}^{\mathbf{x}}) = \nabla \cdot \nabla \times \boldsymbol{\psi} = 0$. The choice $\beta = -\rho$ and $\alpha = 0$ in (11.103) recovers the solution (11.92) obtained by Padhye and Morrison (1996a,b). The choice $\beta = 0$ and $\alpha = 1$ gives $\rho \hat{V}^{\mathbf{x}} = \nabla \times \mathbf{u} = \boldsymbol{\omega}$. The latter choice is clearly related to the potential vorticity conservation law. However, more generally the equation $\rho \hat{V}^{\mathbf{x}} = \nabla \times \boldsymbol{\psi}$ involves the arbitrary function $\boldsymbol{\psi}$. This is analogous to the situation obtained for Noether's second theorem, where an arbitrary function and its derivatives are involved in a variational equation. In this latter case, we obtain:

$$\delta C_2 = -\int_V (\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{\omega} \times C_{\mathbf{M}} \, d^3 x$$

$$= -\int_V \{\nabla \cdot [\boldsymbol{\psi} \times (\boldsymbol{\omega} \times C_{\mathbf{M}})] + \boldsymbol{\psi} \cdot \nabla \times (\boldsymbol{\omega} \times C_{\mathbf{M}})\} \, d^3 x$$

$$\equiv -\int_V \boldsymbol{\psi} \cdot \nabla \times (\boldsymbol{\omega} \times C_{\mathbf{M}}) \, d^3 x, \quad (11.104)$$

where in the last line we assume that the surface term obtained by using Gauss's theorem vanishes on the boundary ∂V. Because $\boldsymbol{\psi}$ is arbitrary, the du-Bois Reymond lemma implies:

$$\nabla \times (\boldsymbol{\omega} \times C_{\mathbf{M}}) = 0, \tag{11.105}$$

as an extra condition imposed on the fluid relabelling symmetries, in order that C is a Casimir invariant with $\delta C = 0$. Equation (11.105) is satisfied if

$$\boldsymbol{\omega} \times C_{\mathbf{M}} = \nabla \Psi, \tag{11.106}$$

where $\Psi(\mathbf{x})$ is a potential. If $\nabla \Psi \neq 0$ then $\boldsymbol{\omega}$ and $C_{\mathbf{M}}$ are non-parallel. Note that the fluid helicity is a Casimir for the non-canonical fluid bracket only for the barotropic gas case for which $p = p(\rho)$.

Chapter 12
MHD Stability

In this chapter our main concern is the analysis of stability for MHD flows and magnetostatic equilibria. The linear stability of magnetostatic equilibria was investigated in the seminal paper by Bernstein et al. (1958) who derived sufficient conditions for magneto-static equilibria, based on the so-called energy principle. A sufficient, but not necessary condition for magnetostatic equilibria is that the potential energy functional $W(\boldsymbol{\xi}, \boldsymbol{\xi})$ satisfies $\delta^2 W(\boldsymbol{\xi}, \boldsymbol{\xi}) > 0$, where $\boldsymbol{\xi}$ is the Lagrangian displacement of the fluid element. A generalization of the energy principle for steady MHD flows was obtained by Frieman and Rotenberg (1960). They noted that for steady flows, the perturbed MHD equations could be written in the form:

$$\rho \boldsymbol{\xi}_{tt} + 2\rho \mathbf{u} \cdot \nabla \boldsymbol{\xi}_t = \mathbf{F}(\boldsymbol{\xi}), \tag{12.1}$$

where $\mathbf{F}(\boldsymbol{\xi})$ is the generalized perturbation force acting on the plasma. The operator $\mathbf{F}(\boldsymbol{\xi})$ is self-adjoint (i.e. an Hermitean operator). However, the operator on the left hand side of (12.1) is in general non-self-adjoint due to the $\mathbf{u} \cdot \nabla \boldsymbol{\xi}_t$ term. The non-self-adjointness of this latter term makes the analysis of the stability of steady flows much more complicated than the case of magneto-static equilibria with $\mathbf{u} = 0$, since the eigen-functions (eigenmodes) are more complicated for the case $\mathbf{u} \neq 0$. Work on the stability of incompressible shear flows in ideal fluid mechanics (Balmforth and Morrison 1999, 2002 and Balmforth et al. 2013 shows the importance of singular eigen-functions with a continuous spectrum in the Hamiltonian description of the perturbed flow). The singular eigen-functions involved are analogous to the Van-Kampen modes in plasmas or the Case eigenfunctions in radiative transfer theory and in solutions of the BGK Boltzmann equation (Webb et al. 2000). Hirota and Fukumoto (2008a,b) write the Frieman and Rotenberg (1960) equations in Hamiltonian form, using 'accessible' variations associated with the non-canonical Poisson bracket of Morrison and Greene (1980, 1982) (see also Hameiri 2003, 2004). The Hirota and Fukumoto (2008a,b) analyses describe the effects of the singular eigen-

functions due to the continuous spectrum (e.g. there are singular eigen-functions associated with the Alfvén wave continuum). Ilgisnos and Pastukhov (2000) develop variational approaches to plasma stability which uses the concept of negative energy waves and perturbations.

Arnold (1966) developed a version of the Euler-Poincaré equations for an ideal incompressible fluid, and showed that resultant Euler Lagrange equations could be thought of as geodesic spray equations for the group $Sdiff(\mathbf{R}^3)$. Ono (1995a,b) obtained the corresponding geodesic spray equation formulation of the incompressible and compressible MHD equations. Araki (2015, 2017) obtained similar equations for the incompressible Hall MHD (i.e. XMHD) equations. Araki (2016) develops a normal mode expansion based on the geodesic spray formulation. The curvature of the geodesic metric is negative for unstable flows. Thus, the geodesic spray equations formulation can be used in stability analyses (e.g. Araki 2015, 2017). We do not describe in detail this approach to MHD stability in this book.

In the next section (Sect. 12.1) we give an elementary derivation of the Frieman and Rotenberg equations. This is followed (in Sect. 12.2) by a variational principle derivation of the Frieman and Rotenberg (1960) equations using the Lagrangian map, and expanding the MHD Lagrangian as a power series in $\boldsymbol{\xi}$ and ΔS where ΔS is the Lagrangian entropy perturbation. This method is related to that used by Dewar (1970) for WKB waves in a non-uniform flow, and its generalization by Webb et al. (2005a) for non WKB, MHD waves. An alternative equivalent derivation of the equations (Sect. 12.3) is to take the first and second variations of the MHD action using Eulerian variations of the physical variables. This latter approach is essentially the same as the Euler-Poincaré variational approach adopted in Chap. 7, where the Lagrangian displacement of the fluid element is defined as:

$$\Delta \mathbf{x} \equiv \boldsymbol{\xi} = \frac{\partial \mathbf{x}(\mathbf{x}_0, \epsilon)}{\partial \epsilon}. \tag{12.2}$$

In the Euler-Poincaré approach of Chap. 7, the variable ϵ is replaced by the time variable t. In that case $\Delta \mathbf{x} \rightarrow \mathbf{u}$ where $\mathbf{u}$ is the fluid velocity. From the variational principle for the second variations of the action, one can identify the Hamiltonian for the system, and write down the Hamiltonian evolution equations that are equivalent to the Frieman and Rotenberg equations. In Sect. 12.4 we derive the Frieman and Rotenberg equations, using accessible variations which are based on the non-canonical Poisson bracket formulation of linearized MHD. This latter approach was developed by Hameiri (2003, 2004), and is related to the Casimir equations. The resultant Hamiltonian form of the Frieman and Rotenberg equations is that given by Hirota and Fukumoto (2008a,b) who discuss the role of the singular MHD eigenmodes, which are omitted in most discussions of linearized MHD, but which are essential for a proper description of the MHD eigenmodes. Holm et al. (1985) describes the nonlinear stability and Lyapunov stability calculations of MHD stability that uses the Casimirs as part of the stability analysis (see also Arnold and Khesin 1998).

12.1   The Frieman and Rotenberg Equations

Frieman and Rotenberg (1960) generalized the energy principle of Bernstein et al. (1958) to study the stability of steady MHD flows including the effects of gravity. The energy principle of Bernstein et al. (1958) applies only to magnetostatic equilibria. Similar equations were also used by Ferraro and Plumpton (1958) in a study of MHD wave propagation in the gravitationally stratified, solar atmosphere. Frieman and Rotenberg's perturbation equations for ξ_b, can be obtained by perturbing the MHD momentum equation:

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla\left(p + \frac{B^2}{2\mu}\right) + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu} + \rho\mathbf{g}, \tag{12.3}$$

where $\mathbf{g} = -\nabla\Phi$ is the acceleration due to gravity.

The Eulerian perturbations $\delta\psi$ and the Lagrangian perturbation $\Delta\psi$ of a physical quantity ψ are related by the equation:

$$\delta\psi = \Delta\psi - \boldsymbol{\xi}\cdot\nabla\psi, \tag{12.4}$$

where $\boldsymbol{\xi}$ is the Lagrangian displacement of the fluid element (e.g. Newcomb 1962; Lundgren 1963). The Lagrangian perturbations Δp, $\Delta\rho$, $\Delta\mathbf{u}$, and $\Delta\mathbf{B}$ in linear perturbation theory are given by:

$$\Delta p = p_S\Delta S - a^2\rho\nabla\cdot\boldsymbol{\xi}, \quad \Delta\rho = -\rho\nabla\cdot\boldsymbol{\xi},$$
$$\Delta\mathbf{u} = \dot{\boldsymbol{\xi}} = \boldsymbol{\xi}_t + \mathbf{u}\cdot\nabla\boldsymbol{\xi}, \quad \Delta\mathbf{B} = \mathbf{B}\cdot\nabla\boldsymbol{\xi} - \mathbf{B}\nabla\cdot\boldsymbol{\xi}. \tag{12.5}$$

The corresponding Eulerian perturbations using (12.4) are given by:

$$\delta p = p_S\Delta S - a^2\rho\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p, \quad \delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}).$$
$$\delta\mathbf{u} = \boldsymbol{\xi}_t + \mathbf{u}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{u}, \quad \delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}). \tag{12.6}$$

Linearizing the momentum equation (12.3) using Eulerian perturbations, gives the perturbed momentum equation:

$$\mathbf{P}^{(FR)} \equiv \rho\boldsymbol{\xi}_{tt} + 2\rho\mathbf{u}\cdot\nabla(\boldsymbol{\xi}_t) - \mathbf{F}(\boldsymbol{\xi}) = 0, \tag{12.7}$$

where the force-like term $\mathbf{F}(\boldsymbol{\xi})$ does not depend on $\boldsymbol{\xi}_t$, and has the form:

$$\mathbf{F}(\boldsymbol{\xi}) = -\nabla\cdot\boldsymbol{\Pi} + \frac{\mathbf{B}\cdot\nabla\delta\mathbf{B} + \delta\mathbf{B}\cdot\nabla\mathbf{B}}{\mu} - \mathbf{g}\nabla\cdot(\rho\boldsymbol{\xi})$$
$$+\nabla\cdot\left(\rho\boldsymbol{\xi}\frac{d\mathbf{u}}{dt} - \rho\mathbf{u}\mathbf{u}\cdot\nabla\boldsymbol{\xi}\right) - \frac{\partial}{\partial t}(\rho\mathbf{u})\cdot\nabla\boldsymbol{\xi}, \tag{12.8}$$

$$\boldsymbol{\Pi} = p_S\Delta S - a^2\rho\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p + \frac{\mathbf{B}\cdot\delta\mathbf{B}}{\mu}. \tag{12.9}$$

For the case of a steady background flow, $\partial \mathbf{u}/\partial t = 0$ and $(\rho \mathbf{u})_t = 0$, and for the case of zero entropy perturbations, $\Delta S = 0$. In this case the perturbed momentum equation (12.7) reduces to that obtained by Frieman and Rotenberg (1960).

12.2   Variational Method Using the Lagrangian Map

In this section, we derive the Frieman and Rotenberg (1960) equations using the approach of Webb et al. (2005a), which uses the Lagrangian map: $\mathbf{x}^* = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$ where $\boldsymbol{\xi}$ is the Lagrangian displacement of the fluid element representing waves. The analysis also uses the perturbation quantity ΔS, representing entropy waves, in which ΔS is advected with the background flow (the standard MHD case of Frieman and Rotenberg sets $\Delta S = 0$). A similar expansion of the action was used by Dewar (1970) to describe WKB, MHD waves in a non-uniform background flow.

The first step in the analysis is to write down the action for the combined system of waves and background plasma in the form:

$$A = \int d^3x^* \int dt \, \mathcal{L}^*, \tag{12.10}$$

where

$$\mathcal{L}^* = \frac{1}{2}\rho^* u^{*2} - \varepsilon(\rho^*, S^*) - \frac{B^{*2}}{2\mu} - \rho^* \phi(\mathbf{x}^*), \tag{12.11}$$

is the Lagrangian density for the system. In (12.11), the terms in the Lagrangian density $\mathcal{L}^*$ correspond respectively to the kinetic energy of the plasma flow ($u = |\mathbf{u}|$ is the magnitude on the fluid velocity $\mathbf{u}$); the internal energy density ε, the magnetic energy density ($\mathbf{B}$ is the magnetic field induction, and μ is the magnetic permeability); and the gravitational potential energy $\rho\phi$. The position coordinate $\mathbf{x}^* = \mathbf{x} + \boldsymbol{\xi}(\mathbf{x}, t)$ where $\mathbf{x}$ is the position of the background plasma element, and $\boldsymbol{\xi}$ is the Lagrangian displacement of the fluid element due to the waves. The entropy $S^* = S + \Delta S$ in (12.10), where ΔS is the Lagrangian entropy perturbation. The volume element

$$d^3x^* = J^* d^3x, \tag{12.12}$$

where

$$J^* = \det\left(\frac{\partial x^{*i}}{\partial x^j}\right) = \det\left(\delta_j^i + \partial \xi^i/\partial x^j\right)$$

$$= 1 + \nabla \cdot \boldsymbol{\xi} + \frac{1}{2}\left[(\nabla \cdot \boldsymbol{\xi})^2 - \nabla \boldsymbol{\xi} : \nabla \boldsymbol{\xi}\right]$$

$$+ \frac{1}{6}\left[(\nabla \cdot \boldsymbol{\xi})^3 + 2(\nabla \boldsymbol{\xi} \cdot \nabla \boldsymbol{\xi}) : \nabla \boldsymbol{\xi} - 3(\nabla \cdot \boldsymbol{\xi})\nabla \boldsymbol{\xi} : \nabla \boldsymbol{\xi}\right], \tag{12.13}$$

is the Jacobian of the transformation between $\mathbf{x}^*$ and $\mathbf{x}$ (see e.g. Kumar et al. 1994). The Lagrangian transformations:

$$\rho^* = \frac{\rho}{J^*}, \quad B^{*i} = \frac{\partial x^{*i}}{\partial x^j} \frac{B^j}{J^*}, \tag{12.14}$$

correspond to mass continuity, and Faraday's law (e.g. Newcomb 1962). Using (12.14) in (12.10) we obtain the action in the form:

$$A = \int d^3x \int dt \mathcal{L} \quad \text{where} \quad \mathcal{L} = J^* \mathcal{L}^*. \tag{12.15}$$

The exact Lagrangian density $\mathcal{L}$ can be written more explicitly in the form:

$$\mathcal{L} = \frac{1}{2}\rho \left(|\mathbf{u}|^2 + 2\mathbf{u} \cdot \dot{\boldsymbol{\xi}} + |\dot{\boldsymbol{\xi}}|^2 \right) - J^* \varepsilon \left(\frac{\rho}{J^*}, S + \Delta S \right) - \frac{1}{2\mu J^*} \left(x_j^{*i} B^j x_s^{*i} B^s \right) - \rho \phi (\mathbf{x} + \boldsymbol{\xi}). \tag{12.16}$$

The transformation for $\mathbf{B}$ in (12.14) is the frozen in field theorem in magnetohydrodynamics (see e.g. Parker 1979, Ch. 4, for a detailed exposition). Using the transformations (12.13) and (12.14), we obtain the expansion

$$A = \int d^3x \int dt \left[\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + O(\xi^3) \right], \tag{12.17}$$

for the action of the system, where

$$\mathcal{L}_0 = \frac{1}{2}\rho u^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu} - \rho \phi, \tag{12.18}$$

$$\mathcal{L}_1 = \rho \mathbf{u} \cdot \dot{\boldsymbol{\xi}} - (\rho T \Delta S - p\nabla \cdot \boldsymbol{\xi}) + \frac{B^2}{2\mu}\nabla \cdot \boldsymbol{\xi} - \frac{\mathbf{B} \cdot \nabla \boldsymbol{\xi} \cdot \mathbf{B}}{\mu} - \rho \boldsymbol{\xi} \cdot \nabla \phi, \tag{12.19}$$

$$\mathcal{L}_2 = \frac{1}{2}\rho |\dot{\boldsymbol{\xi}}|^2 - \frac{1}{2} \left[(\rho a^2 - p)(\nabla \cdot \boldsymbol{\xi})^2 + p\nabla \boldsymbol{\xi} : \nabla \boldsymbol{\xi} - 2p_S \Delta S (\nabla \cdot \boldsymbol{\xi}) + \varepsilon_{SS}(\Delta S)^2 \right]$$
$$+ \frac{\mathbf{B} \cdot \nabla \boldsymbol{\xi} \cdot \mathbf{B}}{\mu}\nabla \cdot \boldsymbol{\xi} - \frac{(\mathbf{B} \cdot \nabla \boldsymbol{\xi})^2}{2\mu} - \frac{B^2}{4\mu} \left((\nabla \cdot \boldsymbol{\xi})^2 + \nabla \boldsymbol{\xi} : \nabla \boldsymbol{\xi} \right)$$
$$- \frac{1}{2}\rho \boldsymbol{\xi} \boldsymbol{\xi} : \nabla \nabla \phi. \tag{12.20}$$

In (12.20)

$$\dot{\boldsymbol{\xi}} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\xi} \quad \text{and} \quad a = \left(\frac{\partial p}{\partial \rho} \right)^{1/2}, \tag{12.21}$$

denote the Lagrangian velocity perturbation, moving with the fluid (note $\mathbf{u}^* = \mathbf{u} + \dot{\boldsymbol{\xi}}$) and the adiabatic sound speed respectively. Dewar considered the case of an

adiabatic gas, with adiabatic index γ, in which case $\varepsilon = p/(\gamma - 1)$. In the derivation of (12.17), it is assumed that the entropy S and the Lagrangian entropy perturbation ΔS are advected with the flow, i.e.,

$$\frac{dS}{dt} = 0, \quad \frac{d\Delta S}{dt} = 0, \tag{12.22}$$

where $d/dt = \partial_t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative moving with the flow.

In the absence of waves, the total Lagrangian $\mathcal{L} \equiv \mathcal{L}_0$ in (12.17) and (12.18), and the variational principle (12.17) obtained by varying the background plasma, taking into account the Lagrangian constraints (i.e., the mass continuity equation, Faraday's equation, and the entropy advection equation in Lagrangian form) yields the MHD momentum equation for the background plasma (Newcomb 1962). Newcomb obtained: (1) both the Lagrangian and Eulerian form of the MHD momentum equation by using both Lagrangian and Eulerian forms of the variational principle; (2) the energy principle for static, MHD equilibria of Bernstein et al. (1958); and (3) an energy principle for some steady, azimuthal MHD flows. Dewar (1970) applied the variational principle (12.17) to derive equations for WKB, MHD waves in a non-uniform background flow. He used an averaged Lagrangian method, similar to that used by Whitham (1965), in which the Lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + O(\xi^3)$ is averaged over the periodic, fast variations of the wave phase ϕ. Variations of the wave amplitude, in the averaged action principle using the averaged Lagrangian density $\langle \mathcal{L}_2 \rangle$, results in the MHD wave eigenvector equations and dispersion relation, whereas slow variations of the wave phase (i.e. of $\mathbf{k} = \nabla\phi$ and $\omega = -\phi_t$) results in the wave action equation.

12.2.1   Linear Waves in Non-uniform Flows

We now consider equations for linear waves in non-uniform flows, in which the wave amplitudes are supposed to be sufficiently small, that the waves do not affect the background flow. The action principle (12.17) can be written as

$$A = \int d^3x \int dt \left[\mathcal{L}_b + \mathcal{L}_w + O(\xi^3) \right], \tag{12.23}$$

where

$$\mathcal{L}_b = \mathcal{L}_0, \quad \mathcal{L}_w = \mathcal{L}_1 + \mathcal{L}_2 + O(\xi^3), \tag{12.24}$$

represent the background Lagrangian density $\mathcal{L}_b$ and the wave Lagrangian density $\mathcal{L}_w$. We also use the notation:

$$A_j = \int d^3x \int dt \, \mathcal{L}_j, \quad (j = 0, 1, 2) \tag{12.25}$$

to denote the action components due to $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Using (12.19) we find:

$$\frac{\delta A_1}{\delta \boldsymbol{\xi}} = -\left[\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla \cdot \left(\rho \mathbf{u}\mathbf{u} + \left(p + \frac{B^2}{2\mu}\right)\mathbf{I} - \frac{\mathbf{BB}}{\mu}\right) + \rho \nabla \phi\right] = 0.$$
(12.26)

The equation $\delta A_1/\delta \boldsymbol{\xi} = 0$ is recognizable as the momentum equation for the undisturbed background flow. Equation (12.26) can also be obtained by varying the background variables in the action $A_0 = \int d^3x \int dt \mathcal{L}_0$ (see e.g. Newcomb 1962). Variations of the action A_2 with respect to $\boldsymbol{\xi}$, and setting $\mathbf{P}^{(D)} = -\delta A_2/\delta \boldsymbol{\xi} = 0$, gives the linearized momentum equation:

$$\mathbf{P}^{(D)} = \frac{\partial}{\partial t}\left(\rho \dot{\boldsymbol{\xi}}\right) + \nabla \cdot \left\{\rho \mathbf{u}\dot{\boldsymbol{\xi}} + ((p - \rho a^2)\nabla \cdot \boldsymbol{\xi} - p_S \Delta S)\mathbf{I} - p(\nabla \boldsymbol{\xi})^t\right.$$

$$+ \left(\frac{\mathbf{B} \cdot \nabla \boldsymbol{\xi} \cdot \mathbf{B}}{\mu} - \frac{B^2}{2\mu}\nabla \cdot \boldsymbol{\xi}\right)\mathbf{I} - \frac{B^2}{2\mu}(\nabla \boldsymbol{\xi})^t$$

$$\left.+ \frac{\mathbf{B}}{\mu}((\nabla \cdot \boldsymbol{\xi})\mathbf{B} - \mathbf{B} \cdot \nabla \boldsymbol{\xi})\right\} + \rho \boldsymbol{\xi} \cdot \nabla \nabla \phi = 0.$$
(12.27)

In (12.27) we use the notation $\mathbf{P}^{(D)}$ to denote the linearized momentum flux, where the superscript D, refers to Dewar's variational principle. Equation (12.27), coupled with the advection equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right)\Delta S = 0,$$
(12.28)

for the Lagrangian entropy perturbation, are the fundamental equations governing the interaction of linear MHD waves and the entropy wave in non-uniform background flows, in the presence of an external gravitational potential $\phi(\mathbf{x})$. For the case of an ideal gas, with adiabatic index γ, the thermodynamics of the gas are governed by the equations:

$$\varepsilon = \frac{p}{\gamma - 1}, \quad p = p_0 \left(\frac{\rho}{\rho_0}\right)^\gamma \exp\left(\frac{S - S_0}{C_v}\right), \quad S = C_v \ln\left(\frac{p}{\rho^\gamma}\right),$$
(12.29)

where C_v is the specific heat of the gas at constant volume, in which case $p_S = p/C_v$ in (12.27). It is interesting to compare the perturbed momentum equation $\mathbf{P}^{(FR)} = 0$ in (12.7) (the superscript FR refers to Frieman and Rotenberg), with the perturbed momentum equation $\mathbf{P}^{(D)} = 0$, obtained in (12.27) from Dewar's variational principle. From (12.27) and (12.7) we find:

$$\mathbf{P}^{(D)} - \mathbf{P}^{(FR)} = \nabla \cdot \left\{\boldsymbol{\xi}\left[\rho\frac{d\mathbf{u}}{dt} + \nabla\left(p + \frac{B^2}{2\mu}\right) - \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{\mu} + \rho \nabla \phi\right]\right\}$$
(12.30)

If the background momentum equation is unaffected by the waves, then the right-hand side of (12.30) vanishes by virtue of the background MHD momentum equation (12.24). Hence in this case, $\mathbf{P}^{(FR)} = 0$ is equivalent to $\mathbf{P}^{(D)} = 0$. In cases where $\Delta S \neq 0$, the perturbed momentum equation (12.7) is coupled with the advection equation (12.28), $d\Delta S/dt = 0$.

For the steady flows considered by Frieman and Rotenberg (1960) ($\partial/\partial t = 0$ and $\Delta S = 0$), (12.7) has solutions of the form: $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}(\mathbf{r})\exp(i\omega t)$ where $\tilde{\boldsymbol{\xi}}(\mathbf{r})$ satisfies the equation:

$$-\omega^2 \rho\tilde{\boldsymbol{\xi}} + 2i\omega\rho\mathbf{u}\cdot\nabla\tilde{\boldsymbol{\xi}} - \mathbf{F}(\tilde{\boldsymbol{\xi}}) = 0. \tag{12.31}$$

In (12.31), $i\rho\mathbf{u}\cdot\nabla$ is a Hermitean operator (i.e., it is a self-adjoint operator, with respect to the complex inner product $\langle f, g\rangle = \int fg^* d^3x$). The operator $\mathbf{F}$ is a self-adjoint operator (i.e. $\int_{-\infty}^{\infty}\tilde{\boldsymbol{\eta}}\mathbf{F}(\tilde{\boldsymbol{\xi}})d^3x = \int_{-\infty}^{\infty}\tilde{\boldsymbol{\xi}}\mathbf{F}(\tilde{\boldsymbol{\eta}})d^3x$).

The proof that $\mathbf{F}$ is self-adjoint is facilitated by noting $\mathbf{P}^{(D)} \equiv \mathbf{P}^{(FR)}$, using integration by parts, and dropping surface terms. Because $\mathbf{P}^{(D)} - \mathbf{P}^{(FR)} = 0$ for solutions of the MHD equations (see (12.30)) it follows that

$$\mathbf{F}(\boldsymbol{\xi}) \equiv \rho\boldsymbol{\xi}_{tt} + 2\rho\mathbf{u}\cdot\nabla\boldsymbol{\xi}_t - \mathbf{P}^{(D)}(\boldsymbol{\xi}). \tag{12.32}$$

For $\Delta S = 0$

$$\int \boldsymbol{\eta}\cdot[\rho\boldsymbol{\xi}_{tt} + 2\rho\mathbf{u}\cdot\nabla\boldsymbol{\xi}_t]\ d^3x\, dt = \int \boldsymbol{\xi}\cdot[\rho\boldsymbol{\eta}_{tt} + 2\rho\mathbf{u}\cdot\nabla\boldsymbol{\eta}_t]\ d^3x\, dt,$$

$$\int \boldsymbol{\eta}\cdot\mathbf{P}^{(D)}(\boldsymbol{\xi})\ d^3x\, dt = \int \boldsymbol{\xi}\cdot\mathbf{P}^{(D)}(\boldsymbol{\eta})\ d^3x\, dt,$$

$$\int \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})\ d^3x\, dt = \int \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})\ d^3x\, dt. \tag{12.33}$$

In the derivation of (12.33) it is useful to note in particular that:

$$\int \boldsymbol{\eta}\cdot\left\{\nabla\cdot\left[\frac{\mathbf{B}\cdot\nabla\boldsymbol{\xi}\cdot\mathbf{B}}{\mu_0}\mathsf{I} + \frac{\mathbf{B}\mathbf{B}}{\mu_0}\nabla\cdot\boldsymbol{\xi}\right]\right\}\ d^3x = \int \boldsymbol{\xi}\cdot\left\{\nabla\cdot\left[\frac{\mathbf{B}\cdot\nabla\boldsymbol{\eta}\cdot\mathbf{B}}{\mu_0}\mathsf{I} + \frac{\mathbf{B}\mathbf{B}}{\mu_0}\nabla\cdot\boldsymbol{\eta}\right]\right\}\ d^3x.$$
$$\tag{12.34}$$

and similar results for the other integrals involved.

Frieman and Rotenberg discuss sufficient conditions for stability and variational principles to determine the eigenvalues ω. Van der Holst et al. (1999) consider the problem of the stability of shear flows in gravitating plane plasmas, and investigate both the continuous spectra and the discrete spectra for ω as well as cluster spectra. A non-standard approach, for studying waves in shear flows, may be traced back to the work of Lord Kelvin (1887). The Kelvin modes are either periodic in, or independent of each space coordinate, but the wavenumber and amplitude associated with each mode are functions of time which depend on the shearing rate of the fluid. Examples of exact solutions for wave interactions in shear flows governed by the

incompressible Navier Stokes equations have been obtained, for example, by Craik and Criminale (1986). Related work on the interaction and transformation of MHD waves in shear flows, using this approach have been investigated by Chagelishvili et al. (1997), Poedts et al. (1998), Kaghashvili (1999), Kaghashvili (2002), Bodo et al. (2001), Gogberidze el. (2004) and Webb et al. (2007).

12.2.2 Characteristics for Linear Waves

Equations (12.27) and (12.28) describing linear MHD waves in a non-uniform flow, may be written in the form:

$$\mathbf{L}(\boldsymbol{\xi}) + \mathbf{R}(\boldsymbol{\xi}, \Delta S) = 0, \tag{12.35}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \Delta S = 0, \tag{12.36}$$

where

$$\mathbf{L}(\boldsymbol{\xi}) = \boldsymbol{\xi}_{tt} + 2\mathbf{u} \cdot \nabla(\boldsymbol{\xi}_t) + \mathbf{u}\mathbf{u} : \nabla\nabla\boldsymbol{\xi} - (a^2 + b^2)\nabla(\nabla \cdot \boldsymbol{\xi})$$
$$+ \mathbf{b} \cdot \nabla(\nabla\boldsymbol{\xi}) \cdot \mathbf{b} - \mathbf{b}\mathbf{b} : \nabla\nabla\boldsymbol{\xi} + [\mathbf{b} \cdot \nabla(\nabla \cdot \boldsymbol{\xi})]\mathbf{b}, \tag{12.37}$$

corresponds to the second derivatives of $\boldsymbol{\xi}$ in (12.27), and

$$\mathbf{R}(\boldsymbol{\xi}, \Delta S) = \frac{1}{\rho} \left\{ \nabla \cdot \boldsymbol{\xi} \left[\nabla(p - a^2\rho) + \frac{(\mathbf{B} \cdot \nabla)\mathbf{B}}{\mu} - \nabla\left(\frac{B^2}{2\mu} \right) \right] \right.$$
$$- \left[\nabla\boldsymbol{\xi} + (\nabla\boldsymbol{\xi})^t \right] \cdot \nabla\left(p + \frac{B^2}{2\mu} \right) + \nabla(p_S \Delta S)$$
$$\left. + \nabla\left(\frac{\mathbf{B}\mathbf{B}}{\mu} \right) : \nabla\boldsymbol{\xi} + \rho\boldsymbol{\xi} \cdot \nabla\nabla\phi \right\}, \tag{12.38}$$

corresponds to lower order derivatives of $\boldsymbol{\xi}$, terms linear in $\boldsymbol{\xi}$ (the gravitational term) and terms independent of $\boldsymbol{\xi}$ (the entropy wave contribution). In (12.38),

$$\mathbf{b} = \frac{\mathbf{B}}{\sqrt{\mu\rho}}, \tag{12.39}$$

is the Alfvén velocity and a is the adiabatic sound speed (12.21).

The concept of a characteristic manifold for a partial differential equation system can be defined as a manifold $\phi(\mathbf{x}) = const.$ (here $\mathbf{x}$ denote the independent variables), on which the Cauchy problem does not have a unique solution. In less technical jargon, this means that if the initial data is specified on a characteristic

manifold $\phi(\mathbf{x}) = const.$, then the problem does not have a unique solution. The characteristic manifolds of the wave equations (12.35) and (12.36) describing linear wave propagation and interaction in a non-uniform background flow, turn out to be equivalent to the characteristic manifolds for the fully nonlinear MHD equations. The characteristic manifolds for (12.35) and (12.36), thus correspond to the Alfvén waves, the fast and slow magnetoacoustic waves, and the entropy wave. Alternatively, one can think of the characteristic manifolds as corresponding to the wave fronts of short wavelength (WKB) disturbances in the medium (e.g. Whitham 1974).

Consider the Cauchy problem for (12.35) and (12.36) we introduce new independent variables $(\phi^0, \phi^1, \phi^2, \phi^3)$ where $\phi^j = \phi^j(\mathbf{x})$, and $\mathbf{x} = (t, x, y, z) = (x^0, x^1, x^2, x^3)$ are the independent variables. We have in mind, the problem of specifying initial data on the manifold $\phi^0(\mathbf{x}) = const.$, and determining when it is possible (or not possible) to obtain a unique solution for $\boldsymbol{\xi}$ and ΔS. At least locally, what is required to obtain a unique solution is that the Taylor series for the solution can be determined, using the initial data, and by calculating the higher order derivatives required from the differential equation system and its differential consequences. In the analysis below, we use the notation $\phi^0(\mathbf{x}) \equiv \phi(\mathbf{x})$, in order to emphasize that the initial data is specified on the manifold $\phi(\mathbf{x}) = const.$

In the new variables $\{\phi^j\}$, the entropy advection equation (12.36) for ΔS becomes:

$$\frac{\partial \Delta S}{\partial t} + \mathbf{u} \cdot \nabla \Delta S = \frac{\partial \Delta S}{\partial \phi} \left[\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi \right]$$

$$+ \sum_{j=1}^{3} \frac{\partial \Delta S}{\partial \phi^j} \left[\frac{\partial \phi^j}{\partial t} + \mathbf{u} \cdot \nabla \phi^j \right] = 0. \qquad (12.40)$$

If we choose ϕ such that

$$\frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \phi = 0, \qquad (12.41)$$

and specify initial data for ΔS and $\boldsymbol{\xi}$ on the surface $\phi(\mathbf{x}) = const.$ then it will not be possible to solve (12.40) for $\partial \Delta S/\partial \phi$, and higher order derivatives of ΔS with respect to ϕ, since the coefficient of $\partial \Delta S/\partial \phi$ is zero by virtue of the choice (12.41) for the evolution of ϕ. Thus, solutions of (12.41), correspond to the characteristic manifold for the entropy wave, and it is not possible to obtain a solution for ΔS off the characteristic surface $\phi = const.$, if the initial data was specified on $\phi = const.$ If in fact, we had specified initial data on a surface $\phi = const.$ To determine the characteristic manifolds of (12.35) we first re-write (12.35) in the form:

$$A_j^{i\alpha\beta} \xi_{\alpha\beta}^j + R^i(\boldsymbol{\xi}, \Delta S) = 0, \qquad (12.42)$$

where

$$A_j^{i\alpha\beta} = \delta_j^i \left[\delta_0^\alpha \delta_0^\beta + 2u^\beta \delta_0^\alpha + u^\alpha u^\beta - b^\alpha b^\beta \right]$$
$$- \left(a^2 + b^2 \right) \delta_i^\alpha \delta_j^\beta + b^\beta b^i \delta_i^\alpha + b^\alpha b^i \delta_j^\beta. \qquad (12.43)$$

In (12.43) we have defined $b^0 = 0$ and $u^0 = 0$ (i.e. **b** and **u** are vectors in 3D position space). In (12.42) and (12.43) the roman indices i,j take the values 1,2,3, but the greek indices α, β refer to the independent variables $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z)$ and take the values 0,1,2,3. The term $R^i(\boldsymbol{\xi}, \Delta S)$ in (12.42) is the ith component of the vector **R** in (12.38), which can be written in the form:

$$R^i = B_j^{i\alpha} \xi_\alpha^j + C_j^i \xi^j + D^i, \qquad (12.44)$$

and consists of first order derivatives of $\boldsymbol{\xi}$, linear terms in $\boldsymbol{\xi}$ and terms independent of $\boldsymbol{\xi}$. The detailed form of R^i does not play a role in the nature of the characteristic manifolds. Using new independent variables $\{\phi^\alpha(\mathbf{x})\}$, the wave equation (12.42) for $\boldsymbol{\xi}$ takes the form:

$$A_j^{i\alpha\beta} \frac{\partial \phi^\mu}{\partial x^\alpha} \frac{\partial \phi^\nu}{\partial x^\beta} \frac{\partial^2 \xi^j}{\partial \phi^\mu \partial \phi^\nu} + \left(A_j^{i\alpha\beta} \frac{\partial^2 \phi^\mu}{\partial x^\alpha \partial x^\beta} + B_j^{i\alpha} \frac{\partial \phi^\mu}{\partial x^\alpha} \right) \frac{\partial \xi^j}{\partial \phi^\mu} + C_j^i \xi^j + D^i = 0. \qquad (12.45)$$

For the purposes of characteristic analysis, we write (12.45) as:

$$A_j^{i\alpha\beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} \frac{\partial^2 \xi^j}{\partial \phi^2} + S^i = 0, \qquad (12.46)$$

where we have isolated off the second derivatives of ξ^j with respect to $\phi \equiv \phi^0$ and S^i represents the remaining terms in (12.45). As in our discussion of the characteristic manifold of the entropy advection equation in (12.40) et seq., we consider the initial value problem in which the data is specified on the manifold $\phi = const$. The initial data on the manifold $\phi = const.$ can be written in the form

$$\xi^j = \tilde{\xi}^j(\phi^1, \phi^2, \phi^3), \quad \xi_{,\phi}^j = \tilde{n}^j(\phi^1, \phi^2, \phi^3), \quad \Delta S = \tilde{s}(\phi^1, \phi^2, \phi^3), \qquad (12.47)$$

where $\tilde{\xi}^j$, $\tilde{n}^j$ and $\tilde{s}$ specify the initial data in terms of ϕ^1, ϕ^2, and ϕ^3. The initial data (12.47) is sufficient to determine the source term S^i in (12.46). To obtain a unique solution for $\partial^2 \xi^j / \partial \phi^2$ on the manifold $\phi = const.$, requires that the matrix:

$$\tilde{A}_j^i = A_j^{i\alpha\beta} \phi_\alpha \phi_\beta, \qquad (12.48)$$

to be non-singular, i.e. $\det(\tilde{\mathbf{A}}) \neq 0$. If $\det(\tilde{\mathbf{A}}) = 0$, then (12.46) does not possess a unique solution for $\xi_{\phi\phi}^j$. Thus,

$$\det\left(\tilde{\mathbf{A}}\right) \equiv \det\left(A_j^{i\alpha\beta} \phi_\alpha \phi_\beta\right) = 0, \qquad (12.49)$$

defines the characteristic manifolds $\phi = const.$ for the wave equation (12.35). The matrix $\tilde{\mathbf{A}}$ in (12.48) can be expressed in the form

$$\tilde{A}_j^i = \left[\omega'^2 - (\mathbf{b} \cdot \mathbf{k})^2\right]\delta_j^i - \left(a^2 + b^2\right)k^i k^j + (\mathbf{b} \cdot \mathbf{k})\left(b^i k^j + b^j k^i\right), \qquad (12.50)$$

where

$$\mathbf{k} = \nabla\phi, \quad \omega = -\phi_t, \quad \omega' = \omega - \mathbf{k} \cdot \mathbf{u}, \qquad (12.51)$$

are identified with the wave number $\mathbf{k}$ and frequency ω associated with the wave surface $\phi = const.$, and $\omega' = \omega - \mathbf{k} \cdot \mathbf{u}$ is the Doppler shifted frequency in the fluid frame. Taking the determinant of (12.50) we obtain

$$\det(\tilde{\mathbf{A}}) = \left[\omega'^2 - (\mathbf{b} \cdot \mathbf{k})^2\right]\left\{\omega'^4 - (a^2 + b^2)\omega'^2 k^2 + a^2 k^2 (\mathbf{b} \cdot \mathbf{k})^2\right\}. \qquad (12.52)$$

Thus, $\det(\tilde{\mathbf{A}}) = 0$, if

$$F_A \equiv \omega'^2 - (\mathbf{b} \cdot \mathbf{k})^2 = (\phi_t + \mathbf{u} \cdot \nabla\phi)^2 - (\mathbf{b} \cdot \nabla\phi)^2 = 0, \qquad (12.53)$$

corresponding to the Alfvén wave characteristic manifolds, or alternatively, $\det(\tilde{\mathbf{A}}) = 0$ if

$$\begin{aligned} F_{MS} &\equiv \omega'^4 - \left(a^2 + b^2\right)\omega'^2 k^2 + a^2 k^2 (\mathbf{b} \cdot \mathbf{k})^2 \\ &= (\phi_t + \mathbf{u} \cdot \nabla\phi)^4 - \left(a^2 + b^2\right)(\phi_t + \mathbf{u} \cdot \nabla\phi)^2 |\nabla\phi|^2 + a^2 (\mathbf{b} \cdot \nabla\phi)^2 |\nabla\phi|^2 \\ &= 0, \end{aligned} \qquad (12.54)$$

which defines the characteristic surfaces for the magnetosonic modes.

Equation (12.54) gives:

$$\omega' = \pm bk \cos\vartheta \quad \text{or} \quad V_p' = \frac{\omega'}{k} = \pm b \cos\vartheta, \qquad (12.55)$$

for the dispersion equations for the backward and forward Alfvén waves in the fluid frame, in which ϑ is the wave normal angle, i.e.

$$\cos\vartheta = b_n/b \quad \text{where} \quad b_n = \mathbf{b} \cdot \mathbf{n} = b \cos\vartheta. \qquad (12.56)$$

The dispersion equation for magneto-acoustic waves $F_{MS} = 0$ in (12.54) may be written in the form:

$$V_p'^4 - (a^2 + b^2)V_p'^2 + a^2 b_n^2 = 0, \qquad (12.57)$$

where $V_p' = \omega'/k$ is the wave phase speed in the fluid frame. Equation (12.57) has solutions:

$$V_p' = \pm c_s \quad \text{and} \quad V_p' = \pm c_f, \tag{12.58}$$

where

$$c_{f,s}^2 = \frac{1}{2}\left((a^2 + b^2) \pm \sqrt{(a^2 + b^2)^2 - 4a^2 b_n^2} \right), \tag{12.59}$$

define the fast and slow magnetosonic speeds. One can also express c_f and c_s in the form:

$$c_f = \frac{1}{2}(c_+ + c_-), \quad c_s = \frac{1}{2}(c_+ - c_-), \tag{12.60}$$

where

$$c_+ = |a\mathbf{n} + \mathbf{b}| \quad \text{and} \quad c_- = |a\mathbf{n} - \mathbf{b}|. \tag{12.61}$$

Note that:

$$c_+ = c_f + c_s \quad \text{and} \quad c_- = c_f - c_s, \tag{12.62}$$

gives $c_\pm$ in terms of c_f and c_s.

The phase and group velocities for the Alfvén and magnetoacoustic waves are described in Appendix D.

To sum up, the characteristic manifolds for linear MHD waves consist of the entropy wave manifold (12.41), the Alfvén wave manifold (12.53) and the magnetosonic waves manifolds (12.54). Initial value problems for the linear wave interaction equations (12.35) and (12.36) with initial data specified on a characteristic manifold does not have a unique solution. These manifolds correspond to the well known short wavelength WKB entropy, Alfvén and magnetoacoustic MHD waves.

12.3   Euler-Poincaré or Eulerian Variational Approach

The analysis of the stability of MHD flows used in this section gives results equivalent to the previous section. However, we do not expand the action explicitly as a power series in $\boldsymbol{\xi}$, but simply determine the first and second variations of the action:

$$J = \int\int d^3x\, dt \left[\frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu} - \rho\Phi(\mathbf{x}) \right] \equiv \int\int d^3x\, dt\, \ell. \tag{12.63}$$

by using Eulerian variations of the physical variables:

$$\delta\mathbf{u} = \frac{\partial\Delta\mathbf{x}}{\partial t} + \mathbf{u}\cdot\nabla\Delta\mathbf{x} - \Delta\mathbf{x}\cdot\nabla\mathbf{u} \equiv \frac{\partial\Delta\mathbf{x}}{\partial t} + [\mathbf{u}, \Delta\mathbf{x}],$$

$$\delta\rho = -\nabla\cdot(\rho\Delta\mathbf{x}), \quad \Delta S = -\Delta\mathbf{x}\cdot\nabla S,$$

$$\delta p = -a^2\rho\nabla\cdot\Delta\mathbf{x} - \Delta\mathbf{x}\cdot\nabla p,$$

$$\delta\mathbf{B} = \nabla\times(\Delta\mathbf{x}\times\mathbf{B}). \tag{12.64}$$

These formulas are analogous to the formulas (12.6) for the Eulerian variations, where $\Delta\mathbf{x}\to\boldsymbol{\xi}$. Here we omit the effect of Lagrangian variations of the entropy $S = S(\mathbf{x}_0)$. Note that the Eulerian variations (12.64) are the same as those used in the Euler-Poincaré analysis of Chap. 7 where $\Delta\mathbf{x}\to\mathbf{u}$. Note that $\Delta\mathbf{x} = (\partial\mathbf{x}/\partial\epsilon)_{\mathbf{x}_0}$ with $\epsilon = t$ gives $\Delta\mathbf{x} = \partial\mathbf{x}(\mathbf{x}_0, t)/\partial t = \mathbf{u}$.

Using the same notation as in Chap. 7, we obtain:

$$\frac{\delta\ell}{\delta a}\delta a = \frac{\delta\ell}{\delta\rho}\delta\rho + \frac{\delta\ell}{\delta S}\delta S + \frac{\delta\ell}{\delta\mathbf{B}}\cdot\delta\mathbf{B}$$

$$\equiv -\nabla\cdot\left(\rho\Delta\mathbf{x}\frac{\delta\ell}{\delta\rho}\right) + \nabla\cdot\left((\Delta\mathbf{x}\times\mathbf{B})\times\frac{\delta\ell}{\delta\mathbf{B}}\right)$$

$$+ \Delta\mathbf{x}\cdot\left\{\rho\frac{\delta\ell}{\delta\rho} - \nabla S\cdot\frac{\delta\ell}{\delta S} + \mathbf{B}\times\left[\nabla\times\left(\frac{\delta\ell}{\delta\mathbf{B}}\right)\right]\right\}. \tag{12.65}$$

The term in curly brackets, in the notation of Chap. 7, is identified as:

$$\frac{\delta\ell}{\delta a}\diamond a = \rho\frac{\delta\ell}{\delta\rho} - \nabla S\cdot\frac{\delta\ell}{\delta S} + \mathbf{B}\times\left[\nabla\times\left(\frac{\delta\ell}{\delta\mathbf{B}}\right)\right]. \tag{12.66}$$

Evaluation of $\delta\ell/\delta\rho$, $\delta\ell/\delta S$ and $\delta\ell/\delta\mathbf{B}$ gives:

$$\frac{\delta\ell}{\delta\rho} = \frac{1}{2}|\mathbf{u}|^2 - h - \Phi(\mathbf{x}), \quad \frac{\delta\ell}{\delta S} = -\rho T, \quad \frac{\delta\ell}{\delta\mathbf{B}} = -\frac{\mathbf{B}}{\mu_0}. \tag{12.67}$$

Thus,

$$\frac{\delta\ell}{\delta a}\diamond a \equiv \rho\left(T\nabla S - \nabla h\right) - \rho\nabla\Phi + \mathbf{J}\times\mathbf{B} + \rho\nabla\left(\frac{1}{2}|\mathbf{u}|^2\right). \tag{12.68}$$

A similar calculation for $(\delta\ell/\delta\mathbf{u})\cdot\delta\mathbf{u}$ gives:

$$\frac{\delta\ell}{\delta\mathbf{u}}\cdot\delta\mathbf{u} = \mathbf{m}\cdot\left(\frac{\partial\Delta\mathbf{x}}{\partial t} + \mathbf{u}\cdot\nabla\Delta\mathbf{x} - \Delta\mathbf{x}\cdot\nabla\mathbf{u}\right)$$

$$= \frac{\partial}{\partial t}(\mathbf{m}\cdot\Delta\mathbf{x}) + \nabla\cdot[\mathbf{u}(\mathbf{m}\cdot\Delta\mathbf{x})]$$

$$- \Delta\mathbf{x}\cdot\left(\frac{\partial\mathbf{m}}{\partial t} + \nabla\cdot(\mathbf{u}\otimes\mathbf{m}) + \mathbf{m}\cdot(\nabla\mathbf{u})^T\right), \tag{12.69}$$

where $\mathbf{m} = \delta\ell/\delta\mathbf{u} = \rho\mathbf{u}$ is the MHD momentum density or mass flux.

Using (12.65) and (12.69) in the evaluation of the first variation δJ of the action J is (12.63) gives:

$$\delta J = -\int\int d^3x\, dt\, \Delta\mathbf{x} \cdot \left[\frac{\partial\mathbf{m}}{\partial t} + \nabla\cdot(\mathbf{u}\otimes\mathbf{m}) - (-\nabla p - \rho\nabla\Phi + \mathbf{J}\times\mathbf{B}) \right]. \tag{12.70}$$

In (12.70) the surface terms have been dropped (i.e. the divergence terms from (12.65) and (12.69)) as they give rise to boundary surface integrals, which are assumed to vanish on the boundary ∂R of the integration region R. Equation (12.70) can also be written in the form:

$$\delta J = -\int\int d^3x\, dt\, \Delta\mathbf{x} \cdot \left[\rho\frac{d\mathbf{u}}{dt} - (-\nabla p - \rho\nabla\Phi + \mathbf{J}\times\mathbf{B}) \right]. \tag{12.71}$$

Thus, the vanishing of the first variation δJ of J gives the MHD momentum equation. This result was also established in Chap. 7 using the Euler-Poincaré equation formalism.

Proposition 12.3.1 *The second variation of the action $\delta^2 J$ obtained by taking the variational derivative of δJ in (12.71) can be written in the form:*

$$\delta^2 J = -\int\int d^3x\, dt\, \boldsymbol{\xi}\cdot[\rho(\boldsymbol{\xi}_{tt} + 2\mathbf{u}\cdot\nabla\boldsymbol{\xi}_t) - \mathbf{F}(\boldsymbol{\xi})], \tag{12.72}$$

where

$$\begin{aligned} \mathbf{F}(\boldsymbol{\xi}) = &-\nabla\delta p + \delta\mathbf{J}\times\mathbf{B} + \mathbf{J}\times\delta\mathbf{B} \\ &-\delta\rho\nabla\Phi + \nabla\cdot(\rho\boldsymbol{\xi}\mathbf{u}\cdot\nabla\mathbf{u} - \rho\mathbf{u}\mathbf{u}\cdot\nabla\boldsymbol{\xi}) \\ \equiv &\nabla\left(a^2\nabla\cdot(\rho\boldsymbol{\xi}) + \xi\cdot\nabla p - \frac{\mathbf{B}\cdot\delta\mathbf{B}}{\mu_0} \right) + \left(\frac{\mathbf{B}\cdot\nabla\delta\mathbf{B} + \delta\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0} \right) \\ &+\nabla\cdot(\rho\boldsymbol{\xi})(\nabla\Phi + \mathbf{u}\cdot\nabla\mathbf{u}) + \rho\boldsymbol{\xi}\cdot\nabla(\mathbf{u}\cdot\nabla\mathbf{u}) - \rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\nabla\boldsymbol{\xi}), \tag{12.73} \end{aligned}$$

is the generalized force in the Frieman and Rotenberg equations (12.7) and $\boldsymbol{\xi}\equiv\Delta\mathbf{x}$ is the Lagrangian displacement of the fluid element. The Eulerian variations $\delta\mathbf{u}$, $\delta\rho$, δp, $\delta\mathbf{B}$ are given by (12.64) and $\delta\mathbf{J} = \nabla\times\delta\mathbf{B}/\mu_0$ is the perturbed current. Suppose the plasma domain R is surrounded by a perfect conducting wall on which $\mathbf{n}\cdot\mathbf{u} = \mathbf{n}\cdot\mathbf{B} = 0$ ($\mathbf{n}$ is the outward normal to the boundary ∂R). Then for vector $\boldsymbol{\xi}$ satisfying the boundary conditions $\mathbf{n}\cdot\boldsymbol{\xi} = 0$, the force operator $\mathbf{F}$ is self adjoint. The variational derivative:

$$\frac{\delta(\delta^2 J)}{\delta\boldsymbol{\xi}} = -2[\rho(\boldsymbol{\xi}_{tt} + 2\mathbf{u}\cdot\nabla\boldsymbol{\xi}_t) - \mathbf{F}(\boldsymbol{\xi})] = 0, \tag{12.74}$$

gives the Frieman and Rotenberg (1960) equations.

Since two Lagrangians that differ by a pure divergence have the same Euler-Lagrange equations, it follows we may use:

$$\delta^2 J = \int \int d^3x \, dt \mathcal{L}, \tag{12.75}$$

where

$$\mathcal{L} = \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}) + \rho|\boldsymbol{\xi}_t|^2 + 2\rho(\mathbf{u} \cdot \nabla \boldsymbol{\xi}) \cdot \boldsymbol{\xi}_t. \tag{12.76}$$

We identify the canonical momentum p_k conjugate to ξ^k as:

$$p_k = \frac{\partial \mathcal{L}}{\partial \xi_t^k} = 2\rho \frac{d\xi^k}{dt} \equiv 2\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \xi^k. \tag{12.77}$$

Using the Legendre transformation:

$$\mathcal{H} = \sum_{k=1}^{3} p_k \xi_t^k - \mathcal{L} \equiv \rho|\boldsymbol{\xi}_t|^2 - \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}). \tag{12.78}$$

The corresponding Hamiltonian functional is

$$H = \int d^3x \, \mathcal{H} \equiv \int d^3x \left[\rho|\boldsymbol{\xi}_t|^2 - \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}) \right]. \tag{12.79}$$

In terms of the canonical variables $\boldsymbol{\xi}$ and $\mathbf{p}$, the Frieman and Rotenberg (1960) equations (12.74) can be expressed in the Hamiltonian form:

$$\boldsymbol{\xi}_t = \frac{\delta H}{\delta \mathbf{p}} \quad and \quad \mathbf{p}_t = -\frac{\delta H}{\delta \boldsymbol{\xi}}. \tag{12.80}$$

Proof Taking the variation of δJ in (12.71) we obtain the second variation δJ, which can be split into the sum of two terms:

$$\delta^2 J = I_1 + I_2, \tag{12.81}$$

where

$$I_1 = -\int d^3x \, \boldsymbol{\xi} \cdot \left\{ \delta\rho \frac{d\mathbf{u}}{dt} - \left(-\nabla\delta p + \delta\mathbf{J} \times \mathbf{B} + \mathbf{J} \times \delta\mathbf{B} - \delta\rho\nabla\Phi \right) \right\}, \tag{12.82}$$

$$I_2 = -\int d^3x \, \boldsymbol{\xi} \cdot \left\{ \rho \left(\frac{\partial}{\partial t} \delta\mathbf{u} + \delta\mathbf{u} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\delta\mathbf{u} \right) \right\}, \tag{12.83}$$

where $\boldsymbol{\xi} \equiv \Delta\mathbf{x}$. In (12.82), $d\mathbf{u}/dt = \mathbf{u} \cdot \nabla\mathbf{u}$, since the background flow is assume to be steady.

Following Hameiri (2003) it is advantageous to add the term:

$$\rho(\mathbf{u} \cdot \nabla\delta\tilde{\mathbf{u}} + \delta\tilde{\mathbf{u}} \cdot \nabla\mathbf{u}), \tag{12.84}$$

where

$$\delta\tilde{\mathbf{u}} = \mathbf{u} \cdot \nabla\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla\mathbf{u}, \tag{12.85}$$

to the integrand inside the curly brackets to (12.82) and to subtract the same quantity from (12.83). Thus,

$$\delta^2 J = I_1' + I_2', \tag{12.86}$$

where

$$I_1' = -\int d^3x \, \boldsymbol{\xi} \cdot \left\{ \delta\rho\frac{d\mathbf{u}}{dt} - \left(-\nabla\delta p + \delta\mathbf{J} \times \mathbf{B} + \mathbf{J} \times \delta\mathbf{B} - \delta\rho\nabla\Phi \right) \right.$$
$$\left. + \rho(\mathbf{u} \cdot \nabla\delta\tilde{\mathbf{u}} + \delta\tilde{\mathbf{u}} \cdot \nabla\mathbf{u}) \right\}, \tag{12.87}$$

$$I_2' = -\int d^3x \, \boldsymbol{\xi} \cdot \left\{ \rho\left(\frac{\partial}{\partial t}\delta\mathbf{u} + \delta\mathbf{u} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\delta\mathbf{u} \right) - \rho(\mathbf{u} \cdot \nabla\delta\tilde{\mathbf{u}} + \delta\tilde{\mathbf{u}} \cdot \nabla\mathbf{u}) \right\}, \tag{12.88}$$

Using (12.73) for $\mathbf{F}(\boldsymbol{\xi})$ and the variational formulae (12.64) we obtain:

$$I_1' = \int d^3x \, \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}),$$

$$I_2' = -\int d^3x \, \rho\boldsymbol{\xi} \cdot (\boldsymbol{\xi}_{tt} + 2\mathbf{u} \cdot \nabla\boldsymbol{\xi}_t). \tag{12.89}$$

Thus, the result (12.86) for $\delta^2 J$ reduces to the expression (12.72) for $\delta^2 J$ in the proposition. Taking the variational derivative of $\delta^2 J$ with respect to $\boldsymbol{\xi}$ gives the Frieman and Rotenberg equations (12.74).

The derivation of the Hamiltonian functional H in (12.79) is straightforward. Hamilton's equations (12.75) follow by noting:

$$H = \int d^3x \left[\rho\left| \frac{\mathbf{p}}{2\rho} - \mathbf{u} \cdot \nabla\boldsymbol{\xi} \right|^2 - \boldsymbol{\xi} \cdot \mathbf{F}(\boldsymbol{\xi}) \right],$$

$$\frac{\delta H}{\delta\mathbf{p}} = \frac{\mathbf{p}}{2\rho} - \mathbf{u} \cdot \nabla\boldsymbol{\xi} \equiv \boldsymbol{\xi}_t,$$

$$\frac{\delta H}{\delta\boldsymbol{\xi}} = -2\mathbf{F}(\boldsymbol{\xi}) + 2\rho\mathbf{u} \cdot \nabla\boldsymbol{\xi}_t, \quad \mathbf{p}_t = 2\rho(\boldsymbol{\xi}_{tt} + \mathbf{u} \cdot \nabla\boldsymbol{\xi}_t). \tag{12.90}$$

Use of the Frieman and Rotenberg equations (12.74) then gives $\mathbf{p}_t = -H_\xi$. This completes the proof.   □

Remark The Hamiltonian functional H is a constant of the motion, i.e.,

$$H_t = \frac{\delta H}{\delta \boldsymbol{\xi}} \cdot \boldsymbol{\xi}_t + \frac{\delta H}{\delta \mathbf{p}} \cdot \mathbf{p}_t = \frac{\delta H}{\delta \boldsymbol{\xi}} \cdot \frac{\delta H}{\delta \mathbf{p}} - \frac{\delta H}{\delta \mathbf{p}} \frac{\delta H}{\delta \boldsymbol{\xi}} = 0. \tag{12.91}$$

Note $F_t = \{F, H\}$ where the canonical Poisson bracket $\{F, G\}$ has the form:

$$\{F, G\} = \int d^3x \left(\frac{\delta F}{\delta \boldsymbol{\xi}} \cdot \frac{\delta G}{\delta \mathbf{p}} - \frac{\delta F}{\delta \mathbf{p}} \cdot \frac{\delta G}{\delta \boldsymbol{\xi}} \right). \tag{12.92}$$

Hirota and Fukumoto (2008a,b) have analyzed the Hamiltonian structure and eigen-modes of the Frieman and Rotenberg (1960) equations. They show that in the presence of a flow that is either non-parallel to the magnetic field, or supersonic at some places gives rise to singular eigenmodes with negative energy. The Alfvén and slow singular eigenmodes are neutrally stable, even in the presence of external potential fields (e.g. gravity), but may cause instability when coupled to another singular or non-singular eigenmode with opposite sign of the energy. A recent discussion of the MHD spectrum of stationary plasma flows for the Rayleigh Taylor and Kelvin Helmholtz instabilities is given by Goedbloed (2009). Singular eigenmodes for Vlasov, electrostatic oscillations were studied by Morrison and Pfirsch (1992), using a Hamiltonian formulation. Balmforth and Morrison (1999, 2002) study singular, continuum eigenmodes in shear flows which are analogous to the Van Kampen eigenmodes.

Below we present a form of Hamilton's equations (12.80) used by Hirota and Fukumoto (2008a,b) which they use to describe the singular and non-singular MHD eigenmodes of the Frieman and Rotenberg (1960) equations (12.74) and their role in MHD stability for steady flows.

Proposition 12.3.2 *The Hamiltonian equations (12.80) can be written in the form:*

$$\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{m} \end{pmatrix} = \mathcal{I} \Theta \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{m} \end{pmatrix}, \tag{12.93}$$

where

$$\mathbf{m} = \frac{1}{2} \mathbf{p} \equiv \rho \frac{d\boldsymbol{\xi}}{dt}, \tag{12.94}$$

and the matrix operators $\mathcal{I}$ and Θ are defined by the equations:

$$\mathcal{I} = \begin{pmatrix} \mathsf{O}_3 & \mathsf{I}_3 \\ -\mathsf{I}_3 & \mathsf{O}_3 \end{pmatrix},$$

$$\Theta = \begin{pmatrix} -\rho\mathbf{u}\cdot\nabla(\mathbf{u}\cdot\circ) - \mathbf{F} & \rho\mathbf{u}\cdot\nabla(\circ/\rho) \\ -\mathbf{u}\cdot\nabla & 1/\rho \end{pmatrix}, \tag{12.95}$$

where I_3 *is the unit* 3×3 *matrix and* O_3 *is the zero* 3×3 *matrix. The Hamiltonian functional in this formulation is:*

$$\begin{aligned} \frac{\delta^2 H}{2} &= \frac{1}{2} \int_R d^3 x \, (\boldsymbol{\xi}^T, \mathbf{m}^T)\Theta \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{m} \end{pmatrix} \\ &\equiv \frac{1}{2} \int_R d^3 x \left[\frac{1}{\rho} |\mathbf{m} - \rho\mathbf{u}\cdot\nabla\boldsymbol{\xi}|^2 - \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi}) \right], \end{aligned} \tag{12.96}$$

where $\mathbf{m} = \rho(\partial/\partial t + \mathbf{u}\cdot\nabla)\boldsymbol{\xi}$ *is the canonical momentum, and* Θ *is a self-adjoint operator* $(\Theta = \Theta^*)$ *with respect to the standard* L^2 *inner product.*

Proof Because $\mathbf{p} = 2\mathbf{m}$ Hamilton's equations (12.80) can be written as:

$$\frac{\partial}{\partial t} \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{m} \end{pmatrix} = \begin{pmatrix} \mathsf{O}_3 & \mathsf{I}_3 \\ -\mathsf{I}_3 & \mathsf{O}_3 \end{pmatrix} \begin{pmatrix} \frac{1}{2}\frac{\delta(\delta^2 H)}{\delta\boldsymbol{\xi}} \\ \frac{1}{2}\frac{\delta(\delta^2 H)}{\delta\mathbf{m}} \end{pmatrix}. \tag{12.97}$$

From (12.97) we obtain:

$$\boldsymbol{\xi}_t = \frac{\mathbf{m}}{\rho} - \mathbf{u}\cdot\nabla\boldsymbol{\xi}, \quad \mathbf{m}_t = \mathbf{F}(\boldsymbol{\xi}) - \rho\mathbf{u}\cdot\nabla\boldsymbol{\xi}_t. \tag{12.98}$$

Equations (12.98) can be combined to give (12.93). Using (12.95) we obtain:

$$(\boldsymbol{\xi}^T, \mathbf{m}^T)\Theta \begin{pmatrix} \boldsymbol{\xi} \\ \mathbf{m} \end{pmatrix} = \rho|\boldsymbol{\xi}_t|^2 - \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi}) + \nabla\cdot[\rho\mathbf{u}(\boldsymbol{\xi}\cdot\boldsymbol{\xi}_t)]. \tag{12.99}$$

Integration of (12.99) over the volume R and dropping the surface term and using the result in (12.96) establishes the result (12.96). This completes the proof.   □

12.4   Accessible Variations

In this section, we derive the Frieman and Rotenberg equations again using so-called accessible variations. The analysis is based on Hameiri (2003). Similar ideas were presented by Thiffeault and Morrison (2000). The main idea for accessible variations is that the variations should be compatible with the non-canonical Poisson bracket for the system (e.g. Morrison and Greene 1980, 1982; Holm and Kupershmidt 1983a,b). The non-canonical Poisson bracket for MHD is intrinsically related to the MHD Casimirs discussed in Chap. 8 (Sect. 8.5) and Chap. 11 (Sect. 11.4).

Using the physical variables $\boldsymbol{\psi} = (\mathbf{M}^T, \mathbf{B}^T, \rho, S)^T$, where $\mathbf{M} = \rho\mathbf{u}$ is the MHD momentum density, the non-canonical Poisson bracket for MHD may be written in the form:

$$
\{F, G\} = \int_V d^3x \left[M^i \left(G_{\mathbf{M}} \cdot \nabla F_{M^i} - F_{\mathbf{M}} \cdot \nabla G_{M^i} \right) \right.
$$

$$
+ B^i \left(G_{\mathbf{M}} \cdot \nabla F_{B^i} - F_{\mathbf{M}} \cdot \nabla G_{B^i} \right) + F_{B^i} \left(\mathbf{B} \cdot \nabla G_{M^i} \right) - G_{B^i} \left(\mathbf{B} \cdot \nabla F_{M^i} \right)
$$

$$
\left. + \rho \left(G_{\mathbf{M}} \cdot \nabla F_\rho - F_{\mathbf{M}} \cdot \nabla G_\rho \right) + S \nabla \cdot \left(G_{\mathbf{M}} F_S - F_{\mathbf{M}} G_S \right) \right], \qquad (12.100)
$$

where we use the summation convention for repeated indices, and the integral is over a fixed volume V with perfectly conducting boundaries on which the normal components of $\mathbf{u}$ and $\mathbf{B}$ vanish. The Hamiltonian functional is given by:

$$
H = \int_V d^3x \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \varepsilon(\rho, S) + \frac{B^2}{2\mu_0} + \rho\Phi \right), \qquad (12.101)
$$

where $\varepsilon(\rho, S)$ is the internal thermodynamic energy density per unit volume, and $\Phi(\mathbf{x})$ is a gravitational potential due to an external gravitational field. In (12.100) $F_\psi \equiv \delta F/\delta \psi$ where $\delta F/\delta \psi$ is the variational derivative of F with respect to the physical variable ψ. The time evolution of ψ is given by the Hamiltonian Poisson bracket equation:

$$
\psi_t(\mathbf{x}, t) = \{\psi, H\}, \qquad (12.102)
$$

where we use the representation:

$$
\psi(\mathbf{x}, t) = \int_V \psi(\mathbf{x}', t)\delta(\mathbf{x}' - \mathbf{x}) \, d^3x', \qquad (12.103)
$$

for $\psi(\mathbf{x}, t)$ as a functional of ψ. Note that $\delta\psi(\mathbf{x}, t)/\delta\psi(\mathbf{x}', t) = \delta(\mathbf{x}' - \mathbf{x})$.

The energy Casimir method (Holm et al. 1985) makes substantial use of the Casimirs of the Hamiltonian system of interest to determine the linear and nonlinear stability of an equilibrium or equilibrium flow. The Casimirs are functionals $C[\psi]$ which are conserved under the action of any Hamiltonian K, i.e. $\{C, K\} = 0$ for all Hamiltonians K. The Casimirs correspond to null eigenvectors with $dC/d\tau = \{C, K\} = 0$. It expresses the idea that the phase space variables ψ are not all independent and that the system evolves on symplectic leaves of the manifold where $C = const$. The existence of Casimirs implies the existence of redundant variables. Because $\{F, C\} = 0$ for a Casimir C for a general functional F, then the evolution equation for F: $dF/dt = \{F, H + C\} \equiv \{F, H\}$ can be viewed as the evolution equation for F according to the Hamiltonian $H_C = H + C$. Both linear and nonlinear stability analyses, investigate the convexity of the Hamiltonian near an equilibrium point ψ_0. By using all possible Casimirs it is more likely to find conditions that

guarantee the convexity of H_C, rather than that of the original Hamiltonian H. Hameiri (2003) shows that the use of dynamically accessible variations enables a convexity investigation for linear stability, without the knowledge of the Casimirs themselves. The gist of his arguments are given below.

Using the 'artificial' evolution associated with the Hamiltonian K with evolution variable τ (i.e. $d\psi/d\tau$ is the variation of ψ associated with the Hamiltonian K), where $\tau = 0$ corresponds to the equilibrium state, we may write:

$$\frac{d\psi}{d\tau} = \{\psi, K\}. \tag{12.104}$$

Usually K is involved in (12.104) through its first variational derivative, i.e. through $\zeta(\tau) = \delta K/\delta\psi$. Hence we can replace K by $G(\psi, \tau) = \psi\zeta(\tau)$ since $\delta G/\delta\psi = \zeta(\tau) = \delta K/\delta\psi$. Thus G and K have the same first variational derivative near $\tau = 0$. However, the second variations of G and K will be different, but this piece of information will not be needed in the linear analysis. For example, use of the Poisson bracket (12.100) gives:

$$\left(\frac{d\rho}{d\tau}\right)_0 = \delta\rho = -\nabla \cdot \left(\rho\frac{\delta K}{\delta \mathbf{M}}\right) = -\nabla \cdot (\rho\zeta_{\mathbf{M}}), \tag{12.105}$$

and similar expressions involving $\zeta = \delta K/\delta\psi$ in other cases. Also note

$$\delta H = \left(\frac{dH}{d\tau}\right)_0 = \{H, K\} = -\{K, H\} = -\frac{dK}{dt} = 0, \tag{12.106}$$

because τ is the evolution variable for K and $dK/dt = 0$. Thus, H does not evolve away from $\tau = 0$ due to the variations associated with evolution in τ. Also using the Jacobi identity:

$$\delta^2 H = \frac{d^2 H}{d\tau^2} = \{\{H, K\}, K\} = \{\{H, K\}, G\}$$

$$= -\left[\{\{K, G\}, H\} + \{\{G, H\}, K\}\right] \tag{12.107}$$

The first term in (12.107) is zero as $\{K, G\} = 0$ since K can be replaced by G. Similarly, replacing K by G in (12.107) gives $\delta^2 H = -\{\{G, H\}, G\}$ which evaluated at $\tau = 0$ depends only on $\zeta(0)$. One can also show that all τ derivatives of C vanish at $\tau = 0$ because $dC/d\tau = \{C, K\} = 0$. Thus, one of the main virtues of dynamically accessible variations are that they preserve the Casimir invariants, and hence can be used to study linear and nonlinear stability using the energy-Casimir method (e.g. Holm et al. 1985; Morrison and Eliezer 1986).

12.4.1   Dynamically Accessible Variations

Consider dynamically accessible variations of the MHD Eulerian physical variables in MHD, based on the non-canonical Poisson bracket (12.100). For a given functional K we define the vector:

$$(\boldsymbol{\xi}, \boldsymbol{\eta}, \lambda, \sigma) = \left(K_{\mathbf{M}}, K_{\mathbf{B}}, K_\rho, K_S \right), \tag{12.108}$$

where we use the notation $K_\psi \equiv \delta K / \delta \psi$.

To determine the dynamically accessible variations (DAV) we use the Poisson bracket (12.100). Thus, for example:

$$\rho_\tau = \{\rho, K\} = \int d^3 x' \, \rho(\mathbf{x}') \left[K'_{\mathbf{M}} \cdot \nabla' \left(\frac{\delta \rho(\mathbf{x}')}{\delta \rho(\mathbf{x})} \right) - \frac{\delta \rho'}{\delta \mathbf{M}'} \cdot \nabla' \left(\frac{\delta K}{\delta \rho} \right) \right]$$

$$\equiv \int_R d^3 x' \, \rho(\mathbf{x}') [K_{\mathbf{M}} \cdot \nabla' \delta(\mathbf{x}' - \mathbf{x})] = \int_R d^3 x' \, \rho(\mathbf{x}') \boldsymbol{\xi}' \cdot \nabla \delta(\mathbf{x}' - \mathbf{x}). \tag{12.109}$$

Integrating (12.109) by parts and dropping the pure divergence surface term (note $\boldsymbol{\xi} \cdot \mathbf{n} = 0$ on the boundary ∂R) we obtain:

$$\rho_\tau = -\nabla \cdot (\rho \boldsymbol{\xi}). \tag{12.110}$$

Similarly, we obtain:

$$S_\tau = -\boldsymbol{\xi} \cdot \nabla S, \tag{12.111}$$

$$\mathbf{B}_\tau = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}), \tag{12.112}$$

$$\mathbf{u}_\tau = \boldsymbol{\xi} \times (\nabla \times \mathbf{u}) + \frac{(\nabla \times \boldsymbol{\eta}) \times \mathbf{B}}{\rho} + \nu \nabla S - \nabla \mu, \tag{12.113}$$

where

$$\nu = \frac{\sigma}{\rho}, \quad \mu = \lambda + \boldsymbol{\xi} \cdot \mathbf{u}. \tag{12.114}$$

Thiffeault and Morrison (2000) obtained equations for accessible variations which are equivalent to (12.110)–(12.114). The Thiffeault and Morrison (2000) equations for the accessible variations are written in the form:

$$\rho_\tau = \nabla \cdot \boldsymbol{\chi}_0, \quad S_\tau = \frac{\boldsymbol{\chi}_0}{\rho} \cdot \nabla S, \quad \mathbf{B}_\tau = \nabla \times \left(\mathbf{B} \times \frac{\boldsymbol{\chi}_0}{\rho} \right),$$

$$\rho \mathbf{u}_\tau = (\nabla \times \mathbf{u}) \times \boldsymbol{\chi}_0 + \mathbf{B} \times (\nabla \times \boldsymbol{\chi}_3) + \rho \nabla \chi_1 - \chi_2 \nabla S. \tag{12.115}$$

Here $\boldsymbol{\chi}_0$, χ_1, χ_2 and $\boldsymbol{\chi}_3$ are related to $\boldsymbol{\xi}, \mu$, ν and $\boldsymbol{\eta}$ by the equations:

$$\boldsymbol{\chi}_0 = -\rho\boldsymbol{\xi}, \quad \chi_1 = -\mu, \quad \chi_2 = -\nu\rho, \quad \boldsymbol{\chi}_3 = -\boldsymbol{\eta}. \tag{12.116}$$

It is straightforward to verify that:

$$\boldsymbol{\chi}_0 = -\tilde{K}_{\mathbf{u}}, \quad \chi_1 = -\tilde{K}_\rho, \quad \chi_2 = -\tilde{K}_S, \quad \boldsymbol{\chi}_3 = -\tilde{K}_{\mathbf{B}}. \tag{12.117}$$

where $\tilde{K}(\mathbf{u}, \mathbf{B}, \rho, S) = K(\mathbf{M}, \mathbf{B}, \rho, S)$ is the same functional as K, but written in terms of the variables $\mathbf{u}$, $\mathbf{B}$, ρ and S rather than $\mathbf{M}$, $\mathbf{B}$, ρ and S. Thiffeault and Morrison point out that $\boldsymbol{\xi} = -\boldsymbol{\chi}_0/\rho$ is not necessarily the Lagrangian displacement of the fluid element. The results (12.117) follow by noting:

$$\tilde{K}_{\mathbf{u}} = \rho K_{\mathbf{M}}, \quad \tilde{K}_\rho = K_\rho + \mathbf{u} \cdot K_{\mathbf{M}}, \quad \tilde{K}_S = K_S, \quad \tilde{K}_{\mathbf{B}} = K_{\mathbf{B}}. \tag{12.118}$$

It is instructive to investigate under what conditions the Eulerian velocity variation $\delta\mathbf{u} \equiv \mathbf{u}_\tau$ can be written in terms of the Lagrangian displacement $\boldsymbol{\xi}(\mathbf{x}_0, t) \equiv \tilde{\boldsymbol{\xi}}(\mathbf{x}, t)$, i.e. we require:

$$\delta\mathbf{u} = \mathbf{u}_\tau = \boldsymbol{\xi}_t + \mathbf{u} \cdot \nabla\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla\mathbf{u}, \quad \boldsymbol{\chi}_0 = -\rho\boldsymbol{\xi}. \tag{12.119}$$

Using (12.119) for $\mathbf{u}_\tau$, in (12.115) we obtain the equation:

$$\rho\left[\boldsymbol{\xi}_t + \mathbf{u} \cdot \nabla\boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla\mathbf{u} + (\nabla \times \mathbf{u}) \times \boldsymbol{\xi}\right] = \mathbf{B} \times (\nabla \times \boldsymbol{\chi}_3) + \rho\nabla\chi_1 - \chi_2\nabla S. \tag{12.120}$$

The latter equation can be reduced to the equation:

$$\rho\left[\boldsymbol{\xi}_t + \mathbf{u} \cdot \nabla\boldsymbol{\xi} - \boldsymbol{\xi} \cdot (\nabla\mathbf{u})^T\right] = \mathbf{B} \times (\nabla \times \boldsymbol{\chi}_3) + \rho\nabla\chi_1 - \chi_2\nabla S. \tag{12.121}$$

To show in a simple way, that equation (12.121) has consistent solutions for $\boldsymbol{\xi}$, χ_1, χ_2 and $\boldsymbol{\chi}_3$, consider the Euler-Poincaré variations used by Holm et al. (1998) in which:

$$\boldsymbol{\xi} = \Delta\mathbf{x} = \frac{\partial\mathbf{x}(\mathbf{x}_0, t)}{\partial t} = \mathbf{u}. \tag{12.122}$$

In this case (12.121) reduces to:

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla\mathbf{u} - \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right)\right] = \mathbf{B} \times (\nabla \times \boldsymbol{\chi}_3) + \rho\nabla\chi_1 - \chi_2\nabla S. \tag{12.123}$$

Thus,

$$\rho\frac{d\mathbf{u}}{dt} = [\nabla \times (-\boldsymbol{\chi}_3)] \times \mathbf{B} + \rho\nabla\left(\chi_1 + \frac{1}{2}|\mathbf{u}|^2\right) - \chi_2\nabla S. \tag{12.124}$$

However, the MHD momentum equation (including an external gravitational potential Φ), can also be written in the form:

$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{J} \times \mathbf{B} + \rho \left(T\nabla S - \nabla h \right) - \rho \nabla \Phi, \tag{12.125}$$

where h is the enthalpy [note $\rho(T\nabla S - \nabla h) = -\nabla p$]. Comparing (12.124) and (12.125) we obtain:

$$\chi_3 = -\frac{\mathbf{B}}{\mu_0}, \quad \chi_2 = -\rho T, \quad \chi_1 = -\left(h + \Phi + \frac{1}{2}|\mathbf{u}|^2 \right). \tag{12.126}$$

This example shows that the accessible variations contain the Euler-Poincaré Eulerian variations as a special case. Note that $\chi_0 = -\mathbf{M}$.

To proceed with the dynamical variations approach to stability (e.g. Hameiri 2003) we consider the first and second variations of the Hamiltonian H (12.101). The first variation $\delta H \equiv H_\tau$ from (12.101) is:

$$\frac{dH}{d\tau} = \int d^3x \left[\rho \mathbf{u} \cdot \mathbf{u}_\tau + \frac{1}{2}|\mathbf{u}|^2 \rho_\tau + \frac{\mathbf{B} \cdot \mathbf{B}_\tau}{\mu_0} + \varepsilon_\rho \rho_\tau + \varepsilon_S S_\tau + \rho_\tau \Phi \right]. \tag{12.127}$$

Using the DAV equations (12.111)–(12.114), (12.127) reduces to:

$$\frac{dH}{d\tau} = \int_R d^3x \left[\rho \mathbf{u} \cdot \left(\boldsymbol{\xi} \times (\nabla \times \mathbf{u}) + \frac{(\nabla \times \boldsymbol{\eta}) \times \mathbf{B}}{\rho} + \nu \nabla S - \nabla \mu \right) \right.$$
$$\left. - \left(h + \Phi + \frac{1}{2}|\mathbf{u}|^2 \right) \nabla \cdot (\rho \boldsymbol{\xi}) - \rho T \boldsymbol{\xi} \cdot \nabla S + \frac{\mathbf{B}}{\mu_0} \cdot [\nabla \times (\boldsymbol{\xi} \times \mathbf{B})] \right]. \tag{12.128}$$

Using integration by parts and dropping surface terms in (12.128) gives:

$$\frac{dH}{d\tau} = \int_R d^3x \left[\boldsymbol{\xi} \cdot (\rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathbf{J}_\tau \times \mathbf{B} + \rho \nabla \Phi) \right.$$
$$\left. + \boldsymbol{\eta} \cdot \nabla \times (\mathbf{B} \times \mathbf{u}) + \mu \nabla \cdot (\rho \mathbf{u}) + \nu \rho \mathbf{u} \cdot \nabla S \right]. \tag{12.129}$$

At an equilibrium point $\tau = 0$, each of the terms associated with the variations is zero, and hence $\delta H = H_\tau = 0$ at the equilibrium point $\tau = 0$. In the derivation of (12.128) the vanishing of the surface terms requires $\mathbf{B} \cdot \mathbf{n} = 0$, $\mathbf{u} \cdot \mathbf{n} = 0$ and $\boldsymbol{\xi} \cdot \mathbf{n} = 0$. It turns out to calculate $\delta^2 H = d^2H/d\tau^2 = 0$ requires $\mathbf{u}_\tau \cdot \mathbf{n} = 0$ on the boundary.

Taking the variation of (12.129) gives the second variation $d^2H/d\tau^2$ as:

$$\frac{d^2H}{d\tau^2} = \int d^3x\,\boldsymbol{\xi}\cdot\left\{\rho_\tau\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{J}_\tau\times\mathbf{B} - \mathbf{J}\times\mathbf{B}_\tau + \nabla p_\tau + \rho_\tau\nabla\Phi\right\}$$

$$+ \int d^3x\left\{\boldsymbol{\xi}\cdot\rho(\mathbf{u}_\tau\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_\tau) + \boldsymbol{\eta}\cdot\nabla\times[(\mathbf{B}\times\mathbf{u})_\tau]\right.$$

$$\left. + \mu\nabla\cdot[(\rho\mathbf{u})_\tau] + \nu(\rho\mathbf{u}\cdot\nabla S)_\tau\right\} \equiv I_1 + I_2, \qquad (12.130)$$

where I_1 and I_2 are the first and second integrals in (12.130). Adding the term

$$\rho\delta\tilde{\mathbf{u}}\cdot\nabla\mathbf{u} + \rho\mathbf{u}\cdot\nabla\delta\tilde{\mathbf{u}} \quad\text{where}\quad \delta\tilde{\mathbf{u}} = \mathbf{u}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\mathbf{u}, \qquad (12.131)$$

to the integrand of I_1 and subtracting off the same quantity from the integrand of I_2 gives $d^2H/d\tau^2 = I_1' + I_2'$ where

$$I_1' = \int d^3x\left\{\boldsymbol{\xi}\cdot\left[\rho_\tau\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{J}_\tau\times\mathbf{B} - \mathbf{J}\times\mathbf{B}_\tau + \nabla p_\tau + \rho_\tau\nabla\Phi\right] + \rho\delta\tilde{\mathbf{u}}\cdot\nabla\mathbf{u} + \rho\mathbf{u}\cdot\nabla\delta\tilde{\mathbf{u}}\right\}$$

$$I_2' = \int d^3x\left\{\boldsymbol{\xi}\cdot\rho(\mathbf{u}_\tau\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}_\tau) + \boldsymbol{\eta}\cdot\nabla\times[(\mathbf{B}\times\mathbf{u})_\tau]\right.$$

$$\left. + \mu\nabla\cdot[(\rho\mathbf{u})_\tau] + \nu(\rho\mathbf{u}\cdot\nabla S)_\tau - [\rho\delta\tilde{\mathbf{u}}\cdot\nabla\mathbf{u} + \rho\mathbf{u}\cdot\nabla\delta\tilde{\mathbf{u}}]\right\}. \qquad (12.132)$$

The integrals (12.132) reduce to:

$$I_1' = -\int_V d^3x\,\boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi}) \equiv \delta W(\boldsymbol{\xi}), \quad I_2' = \int d^3x\,\rho|\mathbf{u}_\tau + \boldsymbol{\xi}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\boldsymbol{\xi}|^2.$$

$$(12.133)$$

where $\mathbf{u}_\tau$ is given by (12.113). If $\mathbf{u}_\tau$ is also given by the Eulerian variational formula (12.119), then (12.133) gives:

$$\frac{d^2H}{d\tau^2} = \int_V d^3x\left[\rho|\boldsymbol{\xi}_t|^2 - \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\xi})\right], \qquad (12.134)$$

which is the Hamiltonian variational functional (12.79) obtained from the Euler-Poincaré (EPV) or Eulerian variational method. Since the accessible class of variations are a wider class of variations, the conditions for stability obtained from the accessible variations (DAV) (taking into account all possible variations) will be more stringent (i.e. closer to the actual stability threshold) than the EPV variations (e.g. Hameiri 2003).

Chapter 13
Concluding Remarks

The main aim of this book was to provide an overview of the different techniques used to determine conservation laws in ideal MHD and fluid dynamics. These methods consist of (a) the use of Noether's two theorems to derive conservation laws; (b) the Lie dragging techniques developed by Tur and Yanovsky (1993) and used by Webb et al. (2014a) and others (e.g. Besse and Frisch 2017) and (c) the direct method of Anco and Bluman (1997, 2002a,b), Cheviakov (2007, 2014) and Bluman et al. (2010), which consists in determining integrating factors for the equations. This process is in general a computer algebra intensive process. We did not, in fact use the direct method to derive fluid and MHD conservation laws. We illustrated this powerful method of deriving conservation laws, by deriving conservation laws for the KdV equation in Chap. 4. Pshenitsin (2016) derives infinite classes of conservation laws for the incompressible viscous MHD equations using this method.

Examples of the use of the magnetic helicity conservation (Woltjer 1958; Berger and Field 1984; Finn and Antonsen 1985; Moffatt and Ricca 1992) were discussed throughout the book. In Proposition 9.2.1, we pointed out the analysis of the evolution of the kink instability for solar magnetic flux ropes by using the result *Link = Twist + Writhe* which is equivalent to the conservation of magnetic helicity (Torok et al. 2014). Torok et al. (2010, 2014) investigated the evolution of a kinked magnetic flux rope emanating from the solar corona, using both numerical simulations and magnetic helicity theory to describe the evolution of the flux rope. The writhe component of magnetic helicity is the so-called self helicity of a twisted magnetic flux tube (it is sometimes described as the out of plane buckling of a knotted telephone cord), plus the twist helicity (which is responsible for the linking of separate flux tubes). The twist for a single flux rope measures how much the field lines wind around the magnetic axis of the flux rope, whereas the writhe measures the helical deformation of the magnetic axis of the flux rope. This process is thought to be important in the launching of coronal mass ejections (CME's) from the Sun, in which the uncoiling of the field provides extra energy needed to drive the CME (e.g.

Gibson and Low 1998). In Sect. 6.6, we provide other examples of the importance of magnetic helicity in topological fluid dynamics. We describe the relative magnetic helicity of the Parker (1958) Archimedean spiral magnetic field, as developed by Bieber et al. (1987) and Webb et al. (2010a). The magnetic helicity of the field, can be described in terms of the linkage of the toroidal and poloidal magnetic field fluxes (e.g. Kruskal and Kulsrud 1958). The magnetic helicity north of the heliospheric current sheet is negative ($H_r^N < 0$) and the relative magnetic helicity south of the current sheet is positive ($H_r^S > 0$). Berger and Ruzmaikin (2000) computed the helicity injection rate from the photosphere into the corona over the solar cycle. These results for the helicity injection rate were consistent with the relative helicity analysis of the Parker field by Bieber et al. (1987) and Webb et al. (2010a).

Large amplitude Alfvén waves have been observed in the solar wind, in which the magnitude of the field $B = const.$ throughout the wave. Observations of Alfvén wave hodographs in the solar wind (i.e. plots of (B_x, B_y, B_z)) show that the **B** vector at lowest order moves on the surface of the $B = const.$ sphere (e.g. Bruno et al. 2001; Roberts and Goldstein 2006; Matteini et al. 2015; Gosling et al. 2009). Webb et al. (2010b) derived the magnetic helicity of both shear and toroidal Alfvén waves in the solar wind, by using simple Alfvén wave solutions of the equations. Webb et al. (2012a) describe double Alfvén waves, which have two independent phases (double or multiple phase waves are possible in MHD). Webb et al. (2012b) obtain Hamiltonians and variational principles for Alfvén simple waves.

The magnetic helicity of MHD topological solitons, which are total pressure balance structures in which the fluid velocity $\mathbf{u} = \pm\mathbf{V}_A$ where V_A is the Alfvén velocity were derived by Kamchatnov (1982), and later investigated by Sagdeev et al. (1986), Semenov et al. (2002), and Thompson et al. (2014). These MHD solutions are sometimes referred to as Hopfions, since the magnetic helicity of the topological soliton is calculated by using the Hopf fibration, involving a map from the 3-sphere to the 2-sphere. An account of topological solitons based on the work of Kamchatnov (1982) and Semenov et al. (2002) is given in Sect. 6.6. Chanteur (1999) investigated the possibility of compressible Alfvénic topological solitons and found that the energy equation constrains the magnetic field of Alfvénic solutions to have a constant strength along the field lines. Some topological solitons in incompressible MHD do not have this property. Schief (2003) and Golovin (2010, 2011) obtain examples of pressure balance solutions with non-trivial topology. The above examples show the importance of magnetic helicity in solar wind and solar physics. However, only a minor part of the book concerns spacecraft plasma and magnetic field observations.

Webb et al. (2014a,b) derived cross helicity conservation laws for both barotropic and non-barotropic MHD. Yahalom (2013) interpreted the cross helicity conservation law for barotropic MHD and the magnetic helicity conservation law in MHD in terms of generalized Aharonov-Bohm effects. The Aharonov Bohm effect in quantum mechanics, describes the change in the phase of the wave function in the vicinity of an isolated magnetic flux island, depending on the path integral $\int \mathbf{A} \cdot d\mathbf{x}$ around the isolated magnetic flux (Aharonov and Bohm 1959). Yahalom expresses his results for the magnetic helicity in terms of the magnetic helicity

per unit magnetic flux and in terms of the magnetic metage (here metage is a measure of distance along a magnetic field line which is the intersection of two Euler surfaces). It turns out that for a non-zero magnetic helicity integral requires that there is a non-zero jump in the metage potential ψ around a closed path, where $\mathbf{A} = \lambda \nabla \mu + \nabla \psi$ and $\mathbf{B} = \nabla \lambda \times \nabla \mu$ are the magnetic vector potential and magnetic field induction. Similar results were also obtained for the cross helicity integral and the non-barotropic cross helicity integral per unit magnetic flux in Yahalom (2016a, 2017a,b) (see also Appendix E). Webb and Anco (2017) derived the generalized cross helicity conservation law and the magnetic helicity conservation law by using Noether's theorem and a version of gauge field theory for MHD.

The Euler-Poincaré variational approach; Hamiltonian Poisson bracket formulations of MHD (e.g. Morrison and Greene 1980, 1982; Morrison 1982; Holm and Kupershmidt 1983a,b); the multi-symplectic formulation of MHD using Clebsch variables (e.g. Webb et al. 2014c, 2015); the use of the Lagrangian map in MHD (Newcomb 1962; Webb et al. 2005a,b; Golovin 2010, 2011); Casimirs (e.g. Padhye and Morrison 1996a,b; Hameiri 2004; Holm et al. 1985); fluid relabelling symmetries (Padhye and Morrison 1996a,b; Webb and Zank 2007); and MHD stability analyses via Hamiltonian methods (e.g. Holm et al. 1985; Morrison 1998; Hameiri 2004); the Frieman and Rotenberg (1960) equations for stability of steady MHD flows; linear wave propagation equations in non-uniform background flows (Webb et al. 2005a) were treated as part of the analysis.

Conservation laws for turbulent fluids and MHD and the use of conservation laws in numerical MHD were not dealt with. The study of space craft observations of solar wind plasmas, although referred to, was not a major part of the book. So-called mathematically trivial conservation laws in fluids and MHD have been associated with sub-symmetries of the equations by Rosenhaus and Shankar (2016). These conservation laws with specific physical quantities for the conserved density give rise to physically significant conservation laws (e.g. potential vorticity in MHD Webb and Mace 2015).

There are in fact, certain advantages to the different approaches to conservation laws. For example, Tur and Yanovsky (1993) obtained a large number of conserved geometrical quantities that are Lie dragged with the fluid. This approach, in general requires less effort than the other approaches. Volkov et al. (1995), show that some of these conservation laws are due to a hidden super-symmetry of the hydrodynamic systems investigated. Some of the invariants in Tur and Yanovsky (1993) implicitly use the Clebsch variable formulation of MHD originally developed by Zakharov and Kuznetsov (1971). The Ertel invariant and the related Holmann invariant can easily be derived by the Lie dragging approach, in which vector fields, one-forms, 2-forms and 3-forms may be Lie dragged by the flow. Combinations of known forms and vector fields can then give rise to new invariants, by contracting forms with vector fields or taking the wedge product of invariant forms. Kats (2001) describes how to include jumps in Clebsch potentials in ideal fluid variational principles. These results are useful in dealing with multi-valued Clebsch potentials, that arise in solutions of the fluid and/or MHD equations, with non-trivial topology (see also Yahalom 2013, 2017a,b; Webb and Anco 2017). Kats (2003, 2004) derived

the analogue of the Ertel invariant for MHD by taking into account the magnetic part of the velocity $\mathbf{u}_M$ in the Clebsch variable expansion for $\mathbf{u}$. One aspect of the derivation of conservation laws using Noether's second theorem, that deserves further analysis is the assumption of vanishing fluxes on the spatial boundaries. Rosenhaus (2002) has investigated the role of non-vanishing fluxes and other spatial boundary conditions in Noether's second theorem.

Tur and Yanovsky (1993) discuss the Godbillon Vey topological invariant (see also Sect. 6.5). This invariant only arises for example, in MHD if the Lie dragged 1-form $\alpha = \tilde{\mathbf{A}} \cdot d\mathbf{x}$ is an integrable Pfaffian differential form. The condition for integrability of α is that $\alpha \wedge d\alpha = 0$ or in terms of vector Calculus $\tilde{\mathbf{A}} \cdot \nabla \times \tilde{\mathbf{A}} = 0$ (e.g. Sneddon 1957). In this case $p\,\tilde{\mathbf{A}} \cdot d\mathbf{x} = d\Phi$ where p is an integratng factor and $\Phi(x, y, z) = const.$ defines a foliation of integral surfaces with normal $\mathbf{n} = \tilde{\mathbf{A}}/|\tilde{\mathbf{A}}|$. In this case the 3-form $\omega_\eta^3 = \boldsymbol{\eta} \cdot \nabla \times \boldsymbol{\eta} \; d^3x$ where $\boldsymbol{\eta} = \tilde{\mathbf{A}} \times \mathbf{B}/|\tilde{\mathbf{A}}|^2$ is a topological charge for the volume element d^3x that is advected with the flow. This example shows, that the magnetic helicity does not always reveal the existence of topological structure that may be present (see also discussions by Bott and Tu (1982), Berger (1990), and Tur and Yanovsky (1993)). Explicit solutions of the MHD equations or knot configurations of flux tubes and vortex tubes exhibiting higher order topological invariants are clearly of interest in illustrating the possible complications (e.g. Akhmetiev and Ruzmaikin 1995; Berger 1990, 1991). Tur and Yanovsky (2017) describe coherent vortex structures in fluids and plasmas. They give three dimensional configurations of the velocity field and magnetic field in MHD that have nontrivial topology (e.g. topological solitons).

Noether's theorems and conservation laws using the method of moving frames has been developed by Goncalves and Mansfield (2012, 2016), in which the independent variables in the Lagrangian are themselves invariant under a symmetry group. This approach investigates the mathematical structure behind the Euler Lagrange equations. They give examples of variational problems that are invariant under semi-simple Lie algebras. The method of moving frames and its relation to Lie pseudo algebras was developed by Fels and Olver (1998).

The above synopsis of conservation laws for MHD and fluid dynamics concludes our analysis.

Appendix A
Lie Derivatives

In this appendix we obtain the Lie derivatives of: (a) a 0-form f (function), (b) a one form $\alpha = \mathbf{A} \cdot d\mathbf{x}$ and (c) a vector field $\mathbf{w}$ with respect to a vector field $\mathbf{u}$ from first principles.

The Lie derivative is directional derivative along a curve $\mathbf{x} = \mathbf{x}(\epsilon)$. The Lie derivative of a function f (0-form f) is:

$$\mathcal{L}_{\mathbf{u}} f = \frac{df}{d\epsilon} = \frac{d\mathbf{x}}{d\epsilon} \cdot \nabla f = \mathbf{u} \cdot \nabla f \quad \text{where} \quad \mathbf{u} = \frac{d\mathbf{x}}{d\epsilon}. \tag{A.1}$$

The Lie derivative of a 1-form $\alpha = A_i dx^i \equiv \mathbf{A} \cdot d\mathbf{x}$ is given by:

$$\begin{aligned}
\mathcal{L}_{\mathbf{u}} \alpha = \frac{d\alpha}{d\epsilon} &= \lim_{\epsilon \to 0} \left(\mathbf{A}' \cdot d\mathbf{x}' - \mathbf{A} \cdot d\mathbf{x} \right) / \epsilon \\
&= \frac{dA_i}{d\epsilon} dx^i + A_i \frac{d}{d\epsilon} \left(dx^i \right) \\
&= \frac{dx^j}{d\epsilon} \frac{\partial A_i}{\partial x^j} dx^i + A_i d \left(\frac{dx^i}{d\epsilon} \right) = \mathbf{u} \cdot \nabla \mathbf{A} \cdot d\mathbf{x} + A_i du^i \\
&= \mathbf{u} \cdot \nabla \mathbf{A} \cdot d\mathbf{x} + A_i \frac{\partial u^i}{\partial x^j} dx^j = [\mathbf{u} \cdot \nabla \mathbf{A} + \nabla \mathbf{u} \cdot \mathbf{A}] \cdot d\mathbf{x} \\
&\equiv [-\mathbf{u} \times (\nabla \times \mathbf{A}) + \nabla (\mathbf{u} \cdot \mathbf{A})] \cdot d\mathbf{x}. \tag{A.2}
\end{aligned}$$

To obtain the Lie derivative of a vector field $\mathbf{w}$ with respect to a vector field $\mathbf{u}$, we use the infinitesimal transformations:

$$\mathbf{x}' = \mathbf{x} + \epsilon \mathbf{u}, \quad \text{i.e.} \quad \frac{d\mathbf{x}}{d\epsilon} = \mathbf{u}, \quad \mathbf{x} = \mathbf{x}' - \epsilon \mathbf{u}. \tag{A.3}$$

Thus, to $O(\epsilon)$:

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x^j} = \frac{\partial(x'^j - \epsilon u'^j)}{\partial x'^i}\frac{\partial}{\partial x^j} \equiv \left(\delta_{ij} - \epsilon\frac{\partial u^j}{\partial x^i}\right)\frac{\partial}{\partial x^j},$$

$$w'^i = w^i + \epsilon u^j\frac{\partial w^i}{\partial x^j} \equiv w^i + \epsilon\mathbf{u}\cdot\nabla w^i,$$

$$\mathbf{w}'\cdot\nabla' = (\mathbf{w} + \epsilon\mathbf{u}\cdot\nabla\mathbf{w})\cdot(\mathsf{I} - \epsilon\nabla\mathbf{u})\cdot\nabla$$

$$\equiv \mathbf{w}\cdot\nabla + \epsilon\left[\mathbf{u}\cdot\nabla\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{u}\right]\cdot\nabla \equiv \mathbf{w}\cdot\nabla + \epsilon[\mathbf{u}, \mathbf{w}], \qquad (A.4)$$

Thus we obtain:

$$\mathcal{L}_{\mathbf{u}}(\mathbf{w}) = \lim_{\epsilon\to 0}\left(\mathbf{w}'\cdot\nabla' - \mathbf{w}\cdot\nabla\right)/\epsilon = [\mathbf{u}, \mathbf{w}]^i\nabla_i \equiv [\mathbf{u}, \mathbf{w}], \qquad (A.5)$$

where

$$[\mathbf{u}, \mathbf{w}] = [\mathbf{u}\cdot\nabla\mathbf{w} - \mathbf{w}\cdot\nabla\mathbf{u}]\cdot\nabla \qquad (A.6)$$

is the Lie bracket.

Appendix B
Weber Transformations

The classical Weber transformation uses the Lagrangian map: $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ to integrate the Eulerian momentum equation to get the Clebsch representation for $\mathbf{u}$. The Eulerian momentum conservation equation can be written as:

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla \cdot \left[\rho\mathbf{u} \otimes \mathbf{u} + p\mathsf{I} + \left(\frac{B^2}{2\mu_0}\mathsf{I} - \frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0}\right)\right] = 0, \tag{B.1}$$

or as:

$$\frac{d\mathbf{u}}{dt} = T\nabla S - \nabla h + \frac{\mathbf{J} \times \mathbf{B}}{\rho} + \mathbf{B}\frac{\nabla \cdot \mathbf{B}}{\mu_0\rho}. \tag{B.2}$$

Use:

$$\frac{d\mathbf{u}}{dt} = \frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} + \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) \quad \text{where} \quad \boldsymbol{\omega} = \nabla \times \mathbf{u},$$

$$\frac{d}{dt}(\mathbf{u} \cdot d\mathbf{x}) = \left[\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} + \nabla\left(|\mathbf{u}|^2\right)\right] \cdot d\mathbf{x}, \tag{B.3}$$

to get:

$$\frac{d}{dt}(\mathbf{u} \cdot d\mathbf{x}) = \left[T\nabla S + \nabla\left(\frac{1}{2}|\mathbf{u}|^2 - h\right) + \frac{\mathbf{J} \times \mathbf{B}}{\rho} + \mathbf{B}\frac{\nabla \cdot \mathbf{B}}{\mu_0\rho}\right] \cdot d\mathbf{x}. \tag{B.4}$$

On the right-hand side (RHS) of (B.4) for the magnetic terms we use:

$$\frac{d}{dt}\left[\left(\frac{(\nabla \times \boldsymbol{\Gamma}) \times \mathbf{B}}{\rho}\right) \cdot d\mathbf{x}\right] = -\left(\frac{\mathbf{J} \times \mathbf{B}}{\rho}\right) \cdot d\mathbf{x} \tag{B.5}$$

$$\frac{d}{dt}\left[\left(\frac{\nabla \cdot \mathbf{B}}{\rho}\right)\boldsymbol{\Gamma} \cdot d\mathbf{x}\right] = -\left(\frac{\nabla \cdot \mathbf{B}}{\rho}\right)\frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{x}. \tag{B.6}$$

On the right hand side of (B.4) for the gas bits we use:

$$\frac{d}{dt}(\nabla\phi \cdot d\mathbf{x}) = \nabla\left(\frac{1}{2}|\mathbf{u}|^2 - h\right) \cdot d\mathbf{x},$$

$$\frac{d}{dt}(r\nabla S \cdot d\mathbf{x}) = -T\nabla S \cdot d\mathbf{x}, \quad \frac{d}{dt}\left(\tilde{\lambda}\nabla\mu \cdot d\mathbf{x}\right) = 0,$$

$$\tilde{\lambda} = \frac{\lambda}{\rho}, \quad r = \frac{\beta}{\rho}. \tag{B.7}$$

to obtain the Clebsch representation $\mathbf{u} = \mathbf{u}_h + \mathbf{u}_M$ in (8.3)–(8.4).

Using (B.5)–(B.7) in (B.4) gives:

$$\frac{d}{dt}(\mathbf{w} \cdot d\mathbf{x}) = 0, \tag{B.8}$$

where

$$\mathbf{w} = \mathbf{u} - \left(\nabla\phi - r\nabla S - \frac{\nabla \times \mathbf{\Gamma}}{\rho} \times \mathbf{B} - \left(\frac{\nabla \cdot \mathbf{B}}{\rho}\right)\mathbf{\Gamma}\right). \tag{B.9}$$

Integration of (B.8) gives

$$\mathbf{w} \cdot d\mathbf{x} = f_0(\mathbf{x}_0)^k dx_0^k \quad \text{or} \quad w^j = f_0(\mathbf{x}_0)^k \partial x_0^k / \partial x^j. \tag{B.10}$$

Using the initial data: $w^j = f_0(\mathbf{x}_0)^j = f_{00}(\mathbf{x}_0)\partial g_{00}/\partial x_0^j$ at $t = 0$ gives

$$\mathbf{w} = -\mu\nabla\nu, \tag{B.11}$$

where $\mu = -f_{00}$ and $\nu = g_{00}$. Equations (B.9)–(B.10) then give:

$$\mathbf{u} = \nabla\phi - \mu\nabla\nu - r\nabla S - \frac{\nabla \times \mathbf{\Gamma}}{\rho} \times \mathbf{B} - \left(\frac{\nabla \cdot \mathbf{B}}{\rho}\right)\mathbf{\Gamma}, \tag{B.12}$$

which is the Clebsch representation (8.3)–(8.4) for $\mathbf{u}$.

The proof of (B.5) is sketched below. Note that $\mathbf{b} = \mathbf{B}/\rho$ is an advected vector field, satisfying (5.23) with $\mathbf{J} \to \mathbf{b}$. The one form on the LHS of (B.5) can be written as

$$\alpha = \mathbf{b}_\lrcorner\left(\tilde{\mathbf{\Gamma}} \cdot d\mathbf{S}\right) = (\tilde{\mathbf{\Gamma}} \times \mathbf{b}) \cdot d\mathbf{x} \equiv [(\nabla \times \mathbf{\Gamma}) \times \mathbf{B}/\rho] \cdot d\mathbf{x}. \tag{B.13}$$

where $\tilde{\boldsymbol{\Gamma}} = \nabla \times \boldsymbol{\Gamma}$. The RHS of (B.5) is:

$$\frac{d\alpha}{dt} = \frac{d\mathbf{b}}{dt} \lrcorner \tilde{\boldsymbol{\Gamma}} \cdot d\mathbf{S} + \mathbf{b} \lrcorner \frac{d}{dt}(\tilde{\boldsymbol{\Gamma}} \cdot d\mathbf{S})$$

$$= 0 - \mathbf{b} \lrcorner (\mathbf{J} \cdot d\mathbf{S}) \equiv -\frac{\mathbf{J} \times \mathbf{B}}{\rho} \cdot d\mathbf{x}. \tag{B.14}$$

This establishes (B.5). There are similar proofs for (B.6) and (B.7).

Appendix C
Cauchy Invariant $\mathbf{b} = \mathbf{B}/\rho$

In this appendix, we discuss the results (10.3)–(10.5) for the density ρ and magnetic field induction $\mathbf{B}$ in Lagrangian MHD derived in Newcomb (1962) (see also Parker 1979). The solutions for ρ and $\mathbf{B}$ are expressed in terms of the Lagrangian map $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$. One method, to derive (10.5) is to write Faraday's equation (2.4) in terms of the quantity:

$$\mathbf{b} = \frac{\mathbf{B}}{\rho}, \tag{C.1}$$

giving the equation:

$$\left(\frac{\partial}{\partial t} + \mathcal{L}_{\mathbf{u}} \right) \mathbf{b} = \left(\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{b} - \mathbf{b} \cdot \nabla \mathbf{u} \right) \equiv \left(\frac{\partial \mathbf{b}}{\partial t} + [\mathbf{u}, \mathbf{b}] \right) = 0, \tag{C.2}$$

where $[\mathbf{u}, \mathbf{b}]$ is the Lie bracket of $\mathbf{u}$ and $\mathbf{b}$. The condition (C.2) states that the vector field $\mathbf{b}$ is Lie dragged by the vector field $\mathbf{u}$. Thus, $\mathbf{b}$ is a Lie dragged invariant, which implies:

$$\mathbf{b} \cdot \nabla = \mathbf{b}_0 \cdot \nabla_0, \tag{C.3}$$

where $\mathbf{b}_0 = \mathbf{b}_0(\mathbf{x}_0)$ depends only on the Lagrange labels $\mathbf{x}_0$.

From (C.3), we obtain:

$$b^i \frac{\partial}{\partial x^i} = b_0^j \frac{\partial}{\partial x_0^j} \quad \text{so that} \quad b_0^j = b^i \frac{\partial x_0^j}{\partial x^i}. \tag{C.4}$$

The quantity b_0^j is a Cauchy invariant, i.e. $db_0^j/dt = 0$ where $b_0^j = b_0^j(\mathbf{x}_0)$ depends only on the Lagrangian labels $\mathbf{x}_0$. To verify this result, we note:

$$\frac{db_0^j}{dt} = \frac{d}{dt}\left(b^i y_{ji}\right) = \frac{db^i}{dt}y_{ji} + b^i \frac{dy_{ji}}{dt}, \tag{C.5}$$

where $y_{ji} = \partial x_0^j/\partial x^i$. By noting that:

$$\frac{d}{dt}y_{ji} = -\frac{\partial u^s}{\partial x^i}\frac{\partial x_0^j}{\partial x^s} = -\frac{\partial u^s}{\partial x^i}y_{js}, \tag{C.6}$$

we obtain:

$$\frac{db_0^j}{dt} = \frac{\partial x_0^j}{\partial x^i}\left[\frac{\partial \mathbf{b}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\mathbf{u}\right]^i = 0, \tag{C.7}$$

because $\mathbf{b}$ satisfies (C.2). This proves that b_0^j is a Lie dragged invariant (i.e. a Cauchy invariant). From (C.2) we obtain:

$$\frac{B_0^j}{\rho_0} = \frac{B^i}{\rho}y_{ji} \quad \text{and hence} \quad \frac{B^i}{\rho} = x_{ij}\frac{B_0^j}{\rho_0}. \tag{C.8}$$

Using the Lagrangian mass continuity equation: $\rho d^3x = \rho_0 d^3x_0$ We obtain:

$$\rho = \frac{\rho_0}{J} \quad \text{and} \quad B^i = \frac{x_{ij}B_0^j}{J}, \tag{C.9}$$

where $J = \det(x_{ij})$ is the Jacobian of the Lagrangian map. This establishes (10.3)–(10.5).

Appendix D
Magnetosonic N-Waves

In this appendix, we discuss the phase and group velocity for the magneto-acoustic and Alfvén waves described by (12.54) and (12.53), namely:

$$F_{MS} = \omega^4 - \left(a^2 + b^2\right)\omega^2 k^2 + a^2 k^2 \left(\mathbf{b} \cdot \mathbf{k}\right)^2 = 0, \tag{D.1}$$

$$F_A = \omega^2 - \left(\mathbf{b} \cdot \mathbf{k}\right)^2 = 0. \tag{D.2}$$

We use cylindrical coordinates $\mathbf{r} = (x_1, x_2 \cos\theta, x_2 \sin\theta)^T$ where x_1 is distance along the field $\mathbf{B}$ and x_2 is cylindrical radius about $\mathbf{B}$. The corresponding coordinates in $\mathbf{k}$-space are $\mathbf{k} = (k_1, k_2 \cos\Theta, k_2 \sin\Theta)^T$. The wave frequency ω and wave number $\mathbf{k}$ are defined by the equations:

$$\mathbf{k} = \nabla S \quad \text{and} \quad \omega = -S_t, \tag{D.3}$$

where S is the wave phase or wave eikonal function (In this appendix S is the wave eikonal function, and not the entropy of the gas). From (D.1)–(D.3) the fast magneto-acoustic wave dispersion equation and the Alfvén dispersion equations may be written as:

$$
\begin{aligned}
F_{MS} &= \omega^4 - (a^2 + b^2)\omega^2 \left(k_1^2 + k_2^2\right) + a^2 b^2 k_1^2 \left(k_1^2 + k_2^2\right) = 0, \\
F_A &= \omega^2 - b^2 k_1^2 = 0.
\end{aligned} \tag{D.4}
$$

In terms of S and its derivatives, the dispersion equations are:

$$
\begin{aligned}
F_{MS} &= S_t^4 - (a^2 + b^2)\left(S_{x_1}^2 + S_{x_2}^2\right) S_t^2 + a^2 b^2 S_{x_1}^2 \left(S_{x_1}^2 + S_{x_2}^2\right) = 0, \\
F_A &= S_t^2 - b^2 S_{x_1}^2 = (S_t - b S_{x_1})(S_t + b S_{x_1}) = 0.
\end{aligned} \tag{D.5}
$$

Equations $F_{MS} = 0$ and $F_A = 0$ in (D.5) are first order, nonlinear partial differential equations for S, which may be integrated by using the method of characteristics (Sneddon 1957). In general, there are both multi-parameter, complete integral solutions, and envelope type solutions. In the MHD case, the envelope solutions corresponds to the group velocity surface for the waves. Equations (D.5) are the magnetosonic and Alfvén wave eikonal equations. One can also write the magneto-acoustic and Alfvén dispersion equations in the form:

$$\omega_m - \Omega_m(\mathbf{k}, \mathbf{x}) = 0, \tag{D.6}$$

where the subscript m identifies the wave mode of interest.

Writing $k^0 = S_t$ and $(t, x, y, z) = x^\alpha$ ($\alpha = 0, 1, 2, 3$), the characteristics of the wave eikonal equation $F = 0$ (here $F = F_{MS}$ or $F = F_A$) are given by:

$$\frac{dx^\alpha}{d\tau} = \frac{\partial F}{\partial k^\alpha}, \quad \frac{dS}{d\tau} = k^\alpha \frac{\partial F}{\partial x^\alpha}, \quad \frac{dk^\alpha}{d\tau} = -\left(\frac{\partial F}{\partial x^\alpha} + k^\alpha \frac{\partial F}{\partial S}\right), \quad \alpha = 0, 1, 2, \tag{D.7}$$

(Sneddon 1957), where τ is the affine parameter along the characteristics. Because $\partial F/\partial S = 0$, the characteristics (D.6) for the magnetoacoustic modes satisfy Hamilton's equations:

$$\frac{dx^\alpha}{d\tau} = \frac{\partial F}{\partial k^\alpha}, \quad \frac{dk^\alpha}{d\tau} = -\frac{\partial F}{\partial x^\alpha}, \quad \frac{dS}{d\tau} = 4F = 0, \tag{D.8}$$

where $F \equiv F_{MS}$. The equation $dS/d\tau = 4F$ follows by noting that $F(\lambda\mathbf{k}, \mathbf{x}) = \lambda^4 F(\mathbf{k}, \mathbf{x})$ for $F = F_{MS}$ where $k^\alpha = \partial S/\partial x^\alpha$ for $\alpha = 0, 1, 2$. Thus, F_{MS} is a homogeneous function of degree 4 in k^α. Thus the wave eikonal function S does not change moving along the characteristics.

Alternatively, if we separate off the individual wave modes in the form $F = \omega - \Omega(\mathbf{k}, \mathbf{x}) = 0$ where $\mathbf{k}$ is the spatial $\mathbf{k}$-vector then, the characteristics or ray equations take the form:

$$\frac{dx^i}{d\tau} = \frac{\partial \Omega}{\partial k^i}, \quad \frac{dk^i}{d\tau} = -\frac{\partial \Omega}{\partial x^i},$$
$$\frac{dt}{d\tau} = 1, \quad \frac{dS}{d\tau} = \omega \frac{\partial F}{\partial \omega} - \mathbf{k} \cdot \frac{\partial \Omega}{\partial \mathbf{k}} = 0, \quad \frac{d\omega}{d\tau} = \frac{\partial \Omega}{\partial t}. \tag{D.9}$$

The evolution equations for x^i, k^i and ω are Hamilton's equations of classical mechanics, with Hamiltonian Ω. To prove $dS/d\tau = 0$ in (D.8) we note:

$$\omega = \Omega = kV_p(\mathbf{n}, \mathbf{x}, t), \quad \mathbf{n} = \frac{\mathbf{k}}{k},$$

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = V_p\mathbf{n} + (\mathbf{I} - \mathbf{nn}) \cdot \nabla_\mathbf{n} V_p, \tag{D.10}$$

which implies:

$$\frac{dS}{d\tau} = \omega - \mathbf{k} \cdot \mathbf{V}_g = \omega - kV_p = \omega - \Omega = 0. \tag{D.11}$$

For the magneto-acoustic modes the wave dispersion equation (D.1) can be expressed in the form:

$$c^4 - \left(a^2 + b^2\right) c^2 + a^2 \left(\mathbf{b} \cdot \mathbf{n}\right)^2 = 0 \quad \text{where} \quad \mathbf{b} \cdot \mathbf{n} = b \cos \vartheta. \tag{D.12}$$

The group velocity

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = c\mathbf{n} - \frac{a^2 (\mathbf{b} \cdot \mathbf{n}) [\mathbf{b} - \mathbf{b} \cdot \mathbf{n}\mathbf{n}]}{c \left[2c^2 - (a^2 + b^2)\right]}. \tag{D.13}$$

Alternatively using $\omega = kc(\vartheta)$ where $\cos \vartheta = \mathbf{n} \cdot \mathbf{e}_1$ and $\mathbf{e}_1 = \mathbf{B}/B$ is the unit vector along $\mathbf{B}$ we obtain:

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = c(\vartheta)\mathbf{n} + c'(\vartheta)\mathbf{e}_\vartheta, \tag{D.14}$$

where $c'(\vartheta) = \partial c/\partial \vartheta$ and

$$\mathbf{n} = \cos \vartheta \mathbf{e}_1 + \sin \vartheta \mathbf{e}_2, \quad \mathbf{e}_\vartheta = -\sin \vartheta \mathbf{e}_1 + \cos \vartheta \mathbf{e}_2. \tag{D.15}$$

From (D.13)–(D.15) the group velocity surface $\mathbf{r} = \mathbf{V}_g t$ can be written in the form:

$$\begin{aligned} x_1 &= V_{g1}t = \left[c(\vartheta) \cos \vartheta - c'(\vartheta) \sin \vartheta\right] t, \\ x_2 &= V_{g2}t = \left[c'(\vartheta) \cos \vartheta + c(\vartheta) \sin \vartheta\right] t, \end{aligned} \tag{D.16}$$

where

$$c'(\vartheta) = \frac{a^2 b^2 \cos \vartheta \sin \vartheta}{c \left[2c^2 - (a^2 + b^2)\right]}. \tag{D.17}$$

the derivation of (D.17) follows by implicit differentiation of the dispersion equation (D.12).

The group velocity surface (D.16) also follows by determining the envelope of the family of plane waves described by the eikonal solution:

$$S = k (x_1 \cos \vartheta + x_2 \sin \vartheta) - kc(\vartheta)t, \tag{D.18}$$

obtained by requiring $S = 0$ and $S_\vartheta = 0$ simultaneously. These two conditions give the solutions (D.16) for $x_1(\vartheta, t)$ and $x_2(\vartheta, t)$ for the group velocity surface.
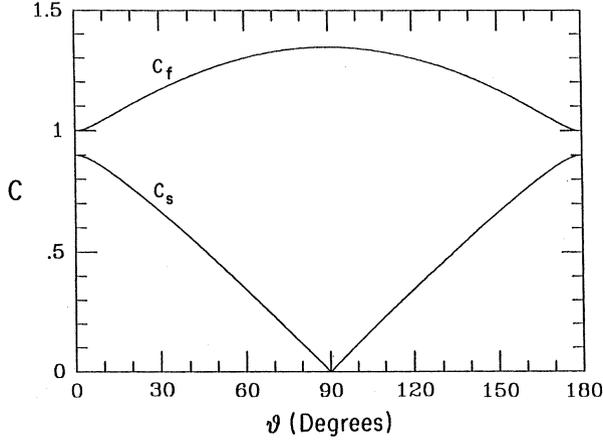
Fig. D.1 The phase velocities c_f and c_s from (D.19) for the fast and slow magnetosonic waves versus the angle ϑ between $\mathbf{k}$ and $\mathbf{B}$ ($\mathbf{k} \cdot \mathbf{B} = kB \cos \vartheta$) for the case $a = 1$ and $b = 0.9$

Figure D.1 shows plots of the fast and slow magnetosonic speeds, $c_f(\vartheta)$ and $c_s(\vartheta)$ versus ϑ for the case $a = 1$ and $b = 0.9$. The fast and slow magnetosonic modes are given by:

$$c_{f,s}^2 = \frac{1}{2} \left[(a^2 + b^2) \pm \left[(a^2 + b^2)^2 - 4a^2 b^2 \cos^2 \vartheta \right]^{1/2} \right]. \tag{D.19}$$

The main points to note are that $c_f > c_s$ for all ϑ ($0 < \vartheta < \pi$). The slow speed $c_s = 0$ at $\vartheta = \pi/2$ and the fast mode speed is maximal at $\vartheta = \pi/2$ where $c_f = (a^2 + b^2)^{1/2}$.

Figure D.2 shows the group velocity surface (D.16) for the fast and slow magnetosonic waves for the case $a = 1$ and $b = 0.9$. The group velocity $\mathbf{r} = \mathbf{V}_g t$ ($t = 1$) is the vector $\mathbf{OP}$ from the origin to a point $P(x_1, x_2)$ on the surface. The wave vector $\mathbf{k} = \nabla S$ is normal to the surface. The outer ellipsoidal-like surface is the fast magnetosonic group velocity surface, and the two cusped triangular-shaped surfaces are the slow mode surfaces (see e.g. Webb et al. 1993, 1994, 2001 for more detail). The formulation (D.16) of the MHD group velocity surfaces is similar to that of Whitham (1974) (equations (7.92)–(7.96)).

The dispersion equation and phase speed for the forward Alfvén wave from (D.4) are:

$$\omega = bk \cos \vartheta \quad \text{and} \quad V_A = \frac{\omega}{k} = b \cos \vartheta. \tag{D.20}$$

The group velocity for the forward Alfvén wave from (D.14) is:

$$\mathbf{V}_{gA} = \frac{\partial \omega}{\partial \mathbf{k}} = bk \cos \vartheta \, \mathbf{n} - bk \sin \vartheta \, \mathbf{e}_\vartheta = b\mathbf{e}_1 \equiv \frac{\mathbf{B}}{\sqrt{\mu \rho}}. \tag{D.21}$$
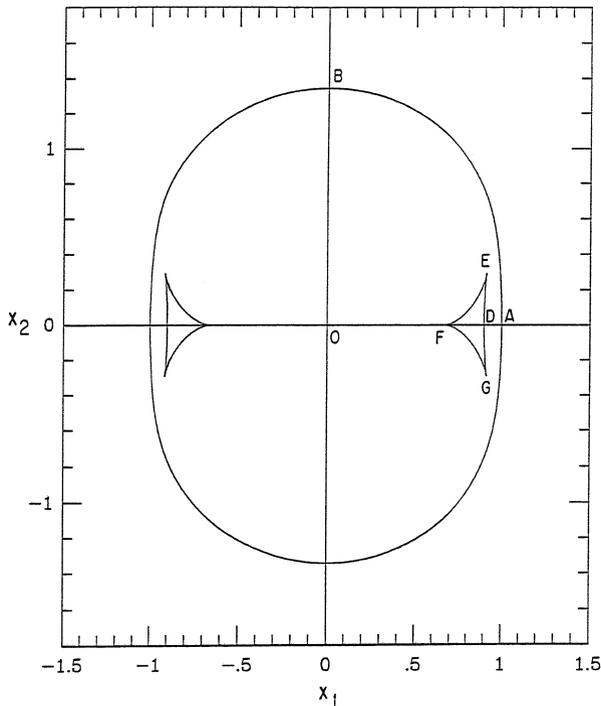
Fig. D.2 The group velocity surfaces for (a) the fast magnetosonic wave (outer ellipsoidal shaped curve) and (b) the slow magnetosonic group velocity surfaces (triangular shaped curves) for the case $a = 1$ and $b = 0.9$. x_1 is distance along the magnetic field and x_2 is distance perpendicular to **B**. The group velocity surfaces are calculated using the formulae (D.16) with $t = 1$

Thus, the Alfvén wave group velocity is directed along the magnetic field **B**, but the phase velocity $V_{pA} = V_A \cos \vartheta$ is parallel to the wave normal **n**.

An alternative approach to obtaining the group velocity surface developed by Lighthill (1960) is to plot the dispersion equation $F_{MS} = 0$ in the form:

$$\left(\frac{k_\perp}{\omega}\right)^2 = -\frac{\left[(k_\parallel/\omega)^2 - 1/a^2\right]\left[(k_\parallel/\omega)^2 - 1/b^2\right]}{\left[(k_\parallel/\omega)^2 - (1/a^2 + 1/b^2)\right]}. \tag{D.22}$$

Thus the wave number surface, or the slowness surface is the plot of $k_\perp/\omega$ versus $k_\parallel/\omega$ for a fixed ω (the slowness is defined as $\mathbf{k}/\omega$). From Whitham (1974), Section 11.4, one can identify the condition for stationary phase:

$$\delta S = \delta(\mathbf{k} \cdot \mathbf{r} - \omega t) = \delta \mathbf{k} \cdot (\mathbf{r} - \omega_{\mathbf{k}} t) = 0 \tag{D.23}$$

with the group velocity surface:

$$\mathbf{r} - \mathbf{V}_g t = 0 \quad \text{where} \quad \mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}}. \tag{D.24}$$

Differentiation of the dispersion equation $D(\omega, \mathbf{k}) = 0$ gives:

$$\mathbf{V}_g = \frac{\partial \omega}{\partial \mathbf{k}} = -\frac{D_{\mathbf{k}}(\mathbf{k}, \omega)}{D_\omega(\mathbf{k}, \omega)}. \tag{D.25}$$

In Lighthill's method of stationary phase (Lighthill 1960) the wave number surface $D(\mathbf{k}, \omega) = 0$ with ω fixed, has normal: $\mathbf{n} = -D_{\mathbf{k}}(\mathbf{k}, \omega)/|D_{\mathbf{k}}(\mathbf{k}, \omega)|$. Because $\mathbf{V}_g = V_g \mathbf{n}$, the group velocity surface can be written in the form:

$$\mathbf{r} = \frac{(\mathbf{k} \cdot \mathbf{r})\mathbf{n}}{\mathbf{k} \cdot \mathbf{n}} = \frac{\phi \mathbf{n}}{\mathbf{k} \cdot \mathbf{n}}, \tag{D.26}$$

where

$$\phi = \mathbf{r} \cdot \mathbf{k} \equiv S + \omega t. \tag{D.27}$$

Because S is stationary and ω is fixed, the phase $\phi = S + \omega t$ is a constant at a fixed t. This allows one to geometrically construct the group velocity surface from the wave-number surface (in our case the wave number surface is the slowness surface (D.22)). At a given point P on the wavenumber surface ($\mathbf{k} = \mathbf{OP}$ where $\mathbf{O}$ is the origin in $\mathbf{k}$-space), determine the wave normal $\mathbf{n} = -D_{\mathbf{k}}/|D_{\mathbf{k}}|$. Draw the tangent to the wave number surface through P, and find the perpendicular distance $OT = \mathbf{k} \cdot \mathbf{n}$ from the tangent plane to the origin. The group velocity surface from (D.26) is then given by

$$\mathbf{r} = \frac{\phi \mathbf{n}}{OT}. \tag{D.28}$$

The shape of the group velocity surface is given by $\mathbf{r} = \mathbf{n}/OT$ which is the reciprocal polar, or pedal curve of the wave number surface (because ϕ is constant we can set $\phi = 1$ in (D.28)).

To obtain the group velocity surface (D.28) from the wave number surface (D.22) note that the slowness:

$$\bar{k} = \frac{k}{\omega} = \frac{1}{c(\vartheta)}, \tag{D.29}$$

where $c(\vartheta)$ is the wave phase speed. The wave phase is:

$$\phi = \bar{\mathbf{k}} \cdot \mathbf{x} = \bar{k} r \cos(\vartheta - \chi), \tag{D.30}$$

where

$$\mathbf{x} = (x_1, x_2) = r(\cos\chi, \sin\chi),$$
$$\bar{\mathbf{k}} = (\bar{k}_1, \bar{k}_2) = \bar{k}\,(\cos\vartheta, \sin\vartheta)\,, \tag{D.31}$$

For stationary phase variations:

$$\delta\phi = \frac{\partial\phi}{\partial\vartheta}\delta\vartheta = \delta\vartheta\left[-\bar{k}r\sin(\vartheta - \chi) + \frac{\partial\bar{k}}{\partial\vartheta}r\cos(\vartheta - \chi)\right] = 0. \tag{D.32}$$

From (D.32)

$$\frac{\bar{k}'(\vartheta)}{\bar{k}(\vartheta)} = \tan(\vartheta - \chi) = \tan(\zeta), \tag{D.33}$$

which implies $\zeta = \vartheta - \chi$ and

$$\tan\chi = \tan(\vartheta - \zeta) = \frac{\tan\vartheta - k'/k}{1 + \tan\vartheta\,(k'/k)}$$
$$= \frac{\tan\vartheta + c'/c}{1 - \tan\vartheta\,(c'/c)} = \frac{c\sin\vartheta + c'\cos\vartheta}{c\cos\vartheta - c'\sin\vartheta} = \frac{V_{g2}}{V_{g1}}. \tag{D.34}$$

From (D.34):

$$\cos\chi = \frac{[c\cos\vartheta - c'(\vartheta)\sin\vartheta]}{[c^2 + c'^2]^{1/2}} \quad \sin\chi = \frac{[c\sin\vartheta + c'(\vartheta)\cos\vartheta]}{[c^2 + c'^2]^{1/2}} \tag{D.35}$$

and (D.26) gives:

$$\mathbf{r} = \phi\left[c\cos\vartheta - c'(\vartheta)\sin\vartheta, c\sin\vartheta + c'(\vartheta)\cos\vartheta\right] = \mathbf{V}_g\phi, \tag{D.36}$$

which is the group velocity surface (D.16) for the case $\phi = t$.

Figures D.3 and D.4 show the magnetic field lines for the linear magnetosonic N wave arising from an initial delta function pressure distribution $\delta p = A\delta(\mathbf{x})$ at time $t = 0$, which is the analog of the acoustic N-wave described by Whitham (1974). In the acoustic N-wave, the solution consists of an N-wave in which there is a compression followed by a rarefaction located in the vicinity the sonic group velocity surface at the leading edge of the wave. The detailed form of the wave depends on whether the geometry is planar, cylindrical or spherical. The MHD analog of the acoustic $N-$ wave was obtained by Webb et al. (1993) in which the initial uniform magnetic field $\mathbf{B} = B_0\mathbf{e}_{x1}$ lies along the x_1-axis. The main point of interest in Figs. D.3 and D.4 is that there are singular N-wave type disturbances located on both the slow mode and fast mode group velocity surfaces. The magnetic
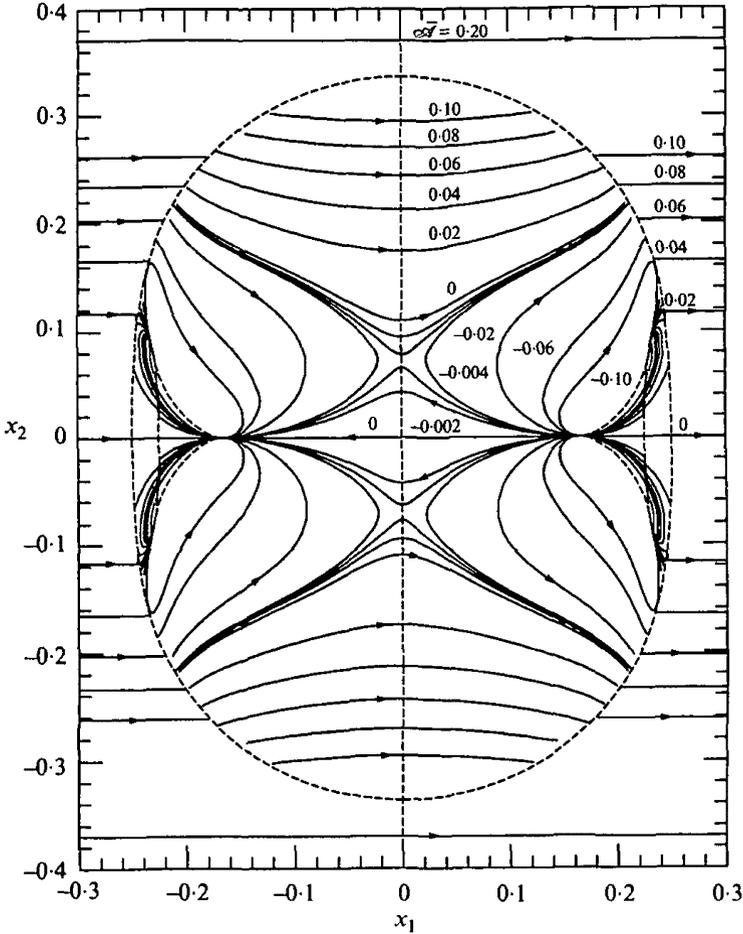
Fig. D.3 Magnetic field lines (contours of wave potential $\mathcal{A}$) for the magnetosonic N-wave for $a = 1$ and $b = 0.9$ at time $t = 0.25$. The perturbation parameter $\epsilon = A/p_0 = 0.2$. The fast and slow mode magnetosonic eikonals are given by the dashed curves (from Webb et al. 1993, Fig. 11)

field structure evolves with time. The fields (according to linear theory) reconnect at time $t = 0.31376$ after which times the slow mode cusps pull apart (right panel of Fig. D.4). The deltoid shaped slow magnetoacoustic surfaces act as sources and sinks for the magnetic field. The forward slow magnetoacoustic cusp point acts as a source and the backward slow magnetoacoustic cusp point acts as a sink. These points together, at early times, behave like a dipole, in which the forward point is a north pole and the backward slow cusp is the south pole. The polarity of the dipole is opposite to the uniform background magnetic field. In the left panel of Fig. D.4, the dipole collapses to a single field line connecting the north and south poles, and the
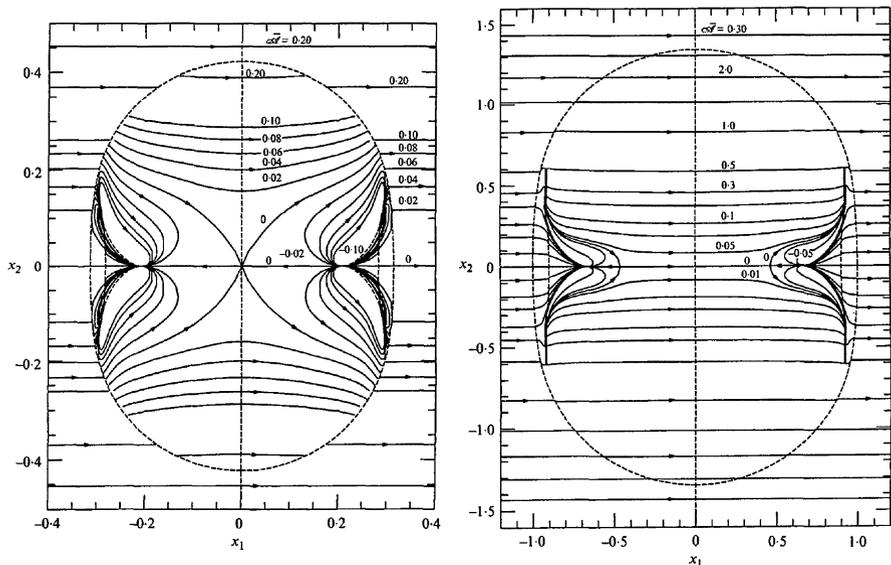
Fig. D.4 Magnetic field lines for the magnetosonic *N*-wave for the same parameters as in Fig. D.3 but at later times $t = 0.31376$ (left) and $t = 1$ (right) (from Webb et al. 1993, Figures 12 and 13)

external uniform magnetic field begins to dominate the solution. In the right panel of Fig. D.4, a straight line uniform magnetic field segment connecting the backward and forward slow mode disturbances develops, involving two neutral points at the ends of the segment.

We are not aware of numerical MHD simulations of the magnetoacoustic *N*-wave, but we expect that for small initial disturbances of δp confined near the origin should give rise to a magnetic field structure similar to that in Figs. D.3 and D.4. there is no disturbance outside the fast mode group velocity surface.

In Fig. D.2 the group velocity for the slow mode at the cusp F:

$$\mathbf{V}_c = \frac{ab}{\sqrt{a^2 + b^2}} \mathbf{e}_B, \tag{D.37}$$

is along the field. The cusp speed V_c describes surface waves on magnetic flux tubes (e.g. Roberts and Mangeney 1982; Roberts 1985). For slab magnetic field geometry the weakly nonlinear tube wave or surface wave is described by the Benjamin-Ono equation (Roberts and Mangeney 1982). For cylindrical flux tubes, weakly nonlinear long wavelength tube waves are governed by the Leibovich-Roberts equation (e.g. Roberts 1985; Weishaar 1989; Bogdan and Lerche 1988; Ruderman 2006).

Appendix E
Aharonov Bohm Effects in MHD

This appendix discusses the formulation of Yahalom (2013, 2016a, 2017a,b) of magnetic helicity H_M and non-barotropic cross helicity H_{CNB}. Yahalom developed a Clebsch variable variational principle for MHD that uses the action:

$$
\mathcal{A} = \int \left\{ \left(\frac{1}{2}\rho u^2 - \rho e(\rho, S) + \frac{B^2}{2\mu_0} \right) \right.
$$
$$
+ \phi \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] - \rho\alpha \frac{d\chi}{dt} - \rho\beta \frac{d\eta}{dt} - \rho\sigma \frac{dS}{dt}
$$
$$
\left. - \frac{\mathbf{B}}{\mu_0} \cdot \nabla\chi \times \nabla\eta \right\} d^3x\, dt. \tag{E.1}
$$

The stationary point requirements $\delta\mathcal{A}/\delta\mathbf{B} = 0$ and $\delta\mathcal{A}/\delta\mathbf{u} = 0$ gives the Clebsch expansions:

$$
\mathbf{B} = \nabla\chi \times \nabla\eta,
$$
$$
\mathbf{u} = \nabla\phi + \alpha\nabla\chi + \beta\nabla\eta + \sigma\nabla S, \tag{E.2}
$$

for $\mathbf{B}$ and $\mathbf{u}$ ($r \equiv -\sigma$ in our formulation). One can write down the other variational equations by varying ρ, S and the Clebsch variables in the variational principle (see e.g. Yahalom 2017a,b).

The $\mathbf{B}$ field Clebsch variable expansion (E.2) is also used by Sakurai (1979). $\mathbf{A}$ and $\mathbf{B}$ have the forms:

$$
\mathbf{A} = \chi\nabla\eta + \nabla\zeta, \quad \mathbf{B} = \nabla\chi \times \nabla\eta. \tag{E.3}
$$

For a non-trivial $\mathbf{B}$ field topology there does not exist a global $\mathbf{A}$ (i.e. χ, η, ζ are not global single valued functions of $\mathbf{x}$). From (E.2) the magnetic helicity density

$h_m = \mathbf{A} \cdot \mathbf{B}$ is given by:

$$h_m = \mathbf{A} \cdot \mathbf{B} = \nabla\zeta \cdot \nabla\chi \times \nabla\eta = \frac{\partial(\zeta, \chi, \eta)}{\partial(x, y, z)}. \tag{E.4}$$

Thus $h_m \neq 0$ only if χ, η and ζ are independent functions of $\mathbf{x}$. Semenov et al. (2002) show that the field topology changes due to jumps in ζ in magnetic fields with non-trivial topology for generalized versions of the MHD topological soliton (c.f. Kamchatnov 1982). A jump in ζ also occurs in the non-global $\mathbf{A}$ for the magnetic monopole (Urbantke 2003).

Yahalom (2013, 2017a,b) introduces an independent magnetic field potential, $\zeta(\mu)$, (μ is called the metage). The metage μ represents distance or affine parameter along the magnetic field line formed by the intersection of the $\eta = const.$ and $\chi = const.$ Euler potential surfaces. We obtain:

$$\nabla\zeta = \frac{\partial\zeta}{\partial\chi}\nabla\chi + \frac{\partial\zeta}{\partial\eta}\nabla\eta + \frac{\partial\zeta}{\partial\mu}\nabla\mu. \tag{E.5}$$

Using (E.5) in (E.4) gives:

$$h_m = \mathbf{A} \cdot \mathbf{B} = \frac{\partial\zeta}{\partial\mu}\nabla\mu \cdot \nabla\chi \times \nabla\eta = \frac{\partial\zeta}{\partial\mu}\left|\frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}\right|. \tag{E.6}$$

The magnetic helicity is given by:

$$H_M = \int_V \mathbf{A} \cdot \mathbf{B}\, d^3x = \int_V \frac{\partial\zeta}{\partial\mu}\, d\chi \wedge d\eta \wedge d\mu. \tag{E.7}$$

However, the differential of the magnetic flux:

$$d\Phi_B = \mathbf{B} \cdot d\mathbf{S} = (\nabla\chi \times \nabla\eta) \cdot d\mathbf{S} = d\chi d\eta. \tag{E.8}$$

Equation (E.6) follows by noting that:

$$d\mathbf{S} = \mathbf{r}_\chi \times \mathbf{r}_\eta\, d\chi d\eta \quad \text{and} \quad \mathbf{B} = \nabla\chi \times \nabla\eta,$$
$$\mathbf{B} \cdot d\mathbf{S} = (\nabla\chi \times \nabla\eta) \cdot (\mathbf{r}_\chi \times \mathbf{r}_\eta)\, d\chi d\eta = d\chi d\eta. \tag{E.9}$$

Equation (E.9) follows by setting $(q^1, q^2, q^3) = (\chi, \eta, \mu)$ and noting:

$$\frac{\partial q^a}{\partial x^k}\frac{\partial x^k}{\partial q^b} = \delta_b^a \quad \text{which implies} \quad \mathbf{e}^a \cdot \mathbf{e}_b = \delta_b^a, \tag{E.10}$$

where $\mathbf{e}^a = \nabla q^a$ and $\mathbf{e}_b = \partial\mathbf{r}/\partial x^b$.

Using (E.6) in (E.5) and integrating over μ, we obtain:

$$H_M = \int [\zeta] \, d\chi d\eta \equiv \int [\zeta] \, d\Phi_B, \qquad (E.11)$$

where $[\zeta]$ is the jump in ζ between the two ends of the field line (in the Aharonov Bohm problem, the path is a closed path). Equation (E.11) lead to the invariant:

$$[\zeta] = \frac{dH_M}{d\Phi_B}, \qquad (E.12)$$

which is the magnetic helicity per unit magnetic flux. Thus, for a closed field line, the jump in $[\zeta]$ is non-zero for a non-trivial magnetic helicity. Yahalom (2013, 2016a, 2017a,b) refers to (E.12) as the MHD 'magnetic Aharonov-Bohm effect', in analogy with the Aharonov-Bohm effect in quantum mechanics.

Yahalom (2013, 2016a, 2017a,b) and Webb et al. (2014a,b) developed conservation laws for generalized cross helicity for both barotropic and non-barotropic MHD. The cross helicity H_C defined as:

$$H_C = \int_V \mathbf{u} \cdot \mathbf{B} \, d^3x, \qquad (E.13)$$

is conserved for non-barotropic flows. The differential form of the cross helicity evolution equation from (3.62) is:

$$\frac{\partial}{\partial t}(\mathbf{u} \cdot \mathbf{B}) + \nabla \cdot \left[(\mathbf{u} \cdot \mathbf{B})\mathbf{u} + \mathbf{B}\left(h + \Phi - \frac{1}{2}u^2 \right) \right] = T(\mathbf{B} \cdot \nabla S). \qquad (E.14)$$

Integration of (E.14) over the volume V_m co-moving with the fluid, and assuming $\mathbf{B} \cdot \mathbf{n} = 0$ on ∂V, where $\mathbf{n}$ is the outward normal to ∂V, gives the helicity evolution equation:

$$\frac{dH_C}{dt} = \int_V T(\mathbf{B} \cdot \nabla S) \, d^3x. \qquad (E.15)$$

Thus, $dH_C/dt = 0$ for barotropic flows where $\nabla S = 0$.

For non-barotropic flows, we define the generalized cross helicity as:

$$H_{CNB} = \int_V (\mathbf{u} - \sigma \nabla S) \cdot \mathbf{B} \, d^3x, \qquad (E.16)$$

(note $\sigma = -r$ in Webb et al. (2014a,b) and Webb and Anco (2017); and in the definition of H_{CNB} in 3.68). Equation (E.14) gives:

$$\frac{dH_{CNB}}{dt} = 0, \quad \text{where} \quad \frac{d\sigma}{dt} = T(\mathbf{x}, t). \qquad (E.17)$$

Using the Clebsch expansions (E.2) for **u** and **B**, we obtain:

$$H_C = \int \mathbf{B} \cdot \nabla \phi \, d^3x + \int \sigma \mathbf{B} \cdot \nabla S \, d^3x,$$

$$\equiv \int [\phi] d\Phi_B + \int \sigma \frac{\partial S}{\partial \mu} d\mu d\Phi_B. \tag{E.18}$$

Also

$$H_{CNB} = H_C - \int \sigma \mathbf{B} \cdot \nabla S \, d^3x = H_C - \int \sigma \frac{\partial S}{\partial \mu} d\mu d\Phi_B. \tag{E.19}$$

Here $[\phi]$ is the jump in ϕ across the discontinuity surface Σ. For simplicity we assume that there is one such surface, Σ, but there could be many such surfaces. From (E.18) and (E.19)

$$\frac{dH_C}{d\Phi_B} = [\phi] + \int \sigma \frac{\partial S}{\partial \mu} d\mu \equiv [\phi] + \oint \sigma dS, \quad \frac{dH_{CNB}}{d\Phi_B} = [\phi]. \tag{E.20}$$

Thus, $dH_{CNB}/d\Phi_B$ is an advected topological invariant (note $d[\phi]/dt = 0$ follows from the variational equation $\delta \mathcal{A}/\delta \rho = 0$). A more detailed analysis is given by Yahalom (2017a,b).

Appendix F
Equivalence Transformations

In this appendix we give a definition of equivalence transformations for a system of partial differential equations:

$$R^\sigma(x, u, \partial u, \partial^2 u, \ldots, \partial^k u) = 0, \quad \sigma = 1, \ldots, N. \tag{F.1}$$

(see also Bluman et al. 2010, p. 21). The differential equation system (F.1) is assumed to involve L constitutive parameters for functions $(K_1, K_2, \ldots K_N)$, which may depend on both the dependent and independent variables and on the derivatives of the dependent variables. A one parameter Lie group of equivalence transformations of a family $\mathcal{S}_k$ of differential equations of the type (F.1) of the form:

$$\tilde{x}^i = f^i(x, u, \epsilon), \quad \tilde{u}^p = g^p(x, u, \epsilon), \quad \tilde{K}_s = G_s(x, u, k, \epsilon), \tag{F.2}$$

maps members of the differential equation system (F.1) onto another member of the system $\mathcal{S}_k$ in the same family.

Appendix G
Covariant, Non-relativistic MHD

In this appendix we discuss the use of generalized Eulerian coordinates q^k and Lagrangian labels a^k in the Lagrangian action principle described in Chap. 10. This approach has some similarities with the general relativistic MHD action principle developed by Achterberg (1983).

From Chap. 10, (Sect. 10.1), the Lagrangian map $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ is obtained by formally integrating the differential equation system:

$$\frac{\partial x^i(\mathbf{x}_0, t)}{\partial t} = u^i(\mathbf{x}, t) \quad \text{where} \quad x^i(\mathbf{x}_0, 0) = x_0^i, \tag{G.1}$$

and the Eulerian fluid velocity $u^i(\mathbf{x}, t)$ is assumed known. We also use generalized coordinates $q^i(\mathbf{a}, t)$ to describe the Lagrangian map, where

$$\frac{\partial q^i}{\partial t} = w^i(\mathbf{q}, t), \quad \text{where} \quad q^i(\mathbf{a}, 0) = a^i. \tag{G.2}$$

For example, we could use spherical polar coordinates $\mathbf{q} = (r, \theta, \phi)$ rather than Cartesian coordinates to describe the Eulerian position of the fluid element. w^i gives the generalized velocity corresponding to $\partial q^i / \partial t$.

We use holonomic coordinate bases vectors $\{\mathbf{e}_k\}$ and its dual basis $\{\mathbf{e}^k\}$, where

$$\mathbf{e}_k = \frac{\partial \mathbf{x}}{\partial q^k}, \quad \mathbf{e}^k = \frac{\partial q^k}{\partial \mathbf{x}}, \tag{G.3}$$

satisfy the orthogonality relations:

$$\langle \mathbf{e}_k, \mathbf{e}^m \rangle = \frac{\partial x^i}{\partial q^k} \frac{\partial q^m}{\partial x^i} = \delta^m{}_k, \tag{G.4}$$

The metric tensors g_{mk} and g^{mk} are defined by the equations:

$$g_{mk} = \mathbf{e}_m \cdot \mathbf{e}_k \quad \text{and} \quad g^{mk} = \mathbf{e}^m \cdot \mathbf{e}^k. \tag{G.5}$$

Using (G.3)–(G.5) we obtain:

$$\mathbf{e}^s = g^{sp}\mathbf{e}_p, \quad \mathbf{e}_s = g_{sp}\mathbf{e}^p, \tag{G.6}$$

relating the bases $\{\mathbf{e}_s\}$ and $\{\mathbf{e}^s\}$. From the definition of the determinant $J = \det(\partial x^i/\partial q^j)$ it follows that

$$\mathbf{e}_a \cdot \mathbf{e}_b \times \mathbf{e}_c = J\varepsilon_{abc} \quad \text{where} \quad J = \det\left(\frac{\partial x^i}{\partial q^j}\right). \tag{G.7}$$

Note that

$$g = \det(g_{ab}) = \det\left(\frac{\partial x^i}{\partial q^a}\delta_{ij}\frac{\partial x^j}{\partial q^b}\right) = J^2 \quad \text{and} \quad J = \sqrt{g}. \tag{G.8}$$

The formulae:

$$\mathbf{e}^b \times \mathbf{e}^c = \frac{\varepsilon_{bca}}{\sqrt{g}}\mathbf{e}_a \quad \text{or} \quad \mathbf{e}_a = \sqrt{g}\varepsilon_{abc}\mathbf{e}^b \times \mathbf{e}^c,$$

$$\mathbf{e}_b \times \mathbf{e}_c = \sqrt{g}\varepsilon_{bca}\mathbf{e}^a \quad \text{or} \quad \mathbf{e}^a = \frac{\varepsilon_{abc}}{\sqrt{g}}\mathbf{e}_b \times \mathbf{e}_c. \tag{G.9}$$

connect the two bases.

The mass continuity equation may be written in the form:

$$\rho d^3x = \rho_0 d^3x_0 \quad \text{or} \quad \rho J(\mathbf{x}, \mathbf{x}_0) = \rho_0, \tag{G.10}$$

where the determinant J is given by:

$$J(\mathbf{x}, \mathbf{x}_0) = \det\left(\frac{\partial x^i}{\partial x_0^j}\right). \tag{G.11}$$

By noting that

$$d^3x = J(\mathbf{x}, \mathbf{x}_0)d^3x_0, \quad d^3x_0 = \sqrt{g_0}d^3a, \quad d^3x = J(\mathbf{x}, \mathbf{x}_0)d^3x_0, \tag{G.12}$$

and the rule:

$$J(\mathbf{x}, \mathbf{x}_0) = J(\mathbf{x}, \mathbf{q})J(\mathbf{q}, \mathbf{a})J(\mathbf{a}, \mathbf{x}_0), \tag{G.13}$$

for the composition of determinants, we obtain:

$$J(\mathbf{x}, \mathbf{x}_0) = \sqrt{g}J(\mathbf{q}, \mathbf{a})/\sqrt{g_0}. \tag{G.14}$$

Thus the mass continuity equation may be written in the form:

$$\rho = \frac{\rho_0}{J(\mathbf{x}, \mathbf{x}_0)} \equiv \frac{\rho_0\sqrt{g_0}}{\sqrt{g}J(\mathbf{q}, \mathbf{a})}. \tag{G.15}$$

Following Newcomb (1962) we introduce the derivative transformation matrices:

$$q^i_{\ j} = \frac{\partial q^i}{\partial a^j}, \quad y^b_{\ j} = \frac{\partial a^b}{\partial q^j}, \tag{G.16}$$

associated with the transformations $\mathbf{q} \to \mathbf{a}$ and the inverse transformations $\mathbf{a} \to \mathbf{q}$. Note that

$$q^i_{\ k}y^k_{\ j} = \delta^i_{\ j}, \quad y^k_{\ j} = \frac{A_j^{\ k}}{J(\mathbf{q}, \mathbf{a})}, \tag{G.17}$$

where

$$A_j^{\ k} = \text{cofac}\left(\frac{\partial q^j}{\partial a^k}\right), \tag{G.18}$$

is the cofactor of the matrix $q^j_{\ k}$.

The conservation of magnetic flux moving with the flow, is equivalent to Faraday's equation (e.g. Parker 1979; Webb et al. 2014a), which can be written in the form:

$$B_0^\alpha dS_{0\alpha} = B^\beta dS_\beta. \tag{G.19}$$

The area elements $dS_{0\alpha}$ and dS_β are defined by the equations:

$$d^3x = \sqrt{g}d^3q = \sqrt{g}d\sigma_\alpha dq^\alpha = dS_\alpha dq^\alpha,$$
$$d^3x_0 = \sqrt{g_0}d^3a = \sqrt{g_0}d\sigma_{0\beta}dq^\beta = dS_{0\beta}da^\beta. \tag{G.20}$$

Using $d^3x = J(\mathbf{x}, \mathbf{x}_0)d^3x_0$, (G.20) gives:

$$dS_{0\beta} = \frac{q^\alpha_{\ \beta}}{J(\mathbf{x}, \mathbf{x}_0)}dS_\alpha \equiv \frac{\sqrt{g_0}}{\sqrt{g}}\frac{q^\alpha_{\ \beta}}{J(\mathbf{q}, \mathbf{a})}dS_\alpha. \tag{G.21}$$

Using (G.19) and (G.21) we obtain:

$$B^\alpha = \frac{\sqrt{g_0}}{\sqrt{g}} \frac{q^\alpha{}_\beta}{J(\mathbf{q}, \mathbf{a})} B_0^\beta \equiv \frac{q^\alpha{}_\beta B_0^\beta}{J(\mathbf{x}, \mathbf{x}_0)}, \tag{G.22}$$

as the relationship between the magnetic field components B^α and B_0^α. Equation (G.22) is equivalent to Faraday's equation.

A simpler derivation of (G.22) follows by noting that $\mathbf{b} = \mathbf{B}/\rho$ is a lie dragged vector field, i.e.

$$b^\alpha \frac{\partial}{\partial q^\alpha} = b_0^\beta \frac{\partial}{\partial a^\beta} \quad \text{or} \quad b^\alpha = b_0^\beta \frac{\partial q^\alpha}{\partial a^\beta}. \tag{G.23}$$

Then using $\mathbf{b} = \mathbf{B}/\rho$ and $\mathbf{b}_0 = \mathbf{B}_0/\rho_0$ the Cauchy invariant relation (G.23) implies the transformation (G.22).

Using (G.22) we obtain:

$$\nabla \cdot \mathbf{B} = \nabla_\alpha B^\alpha = \frac{1}{J(\mathbf{x}, \mathbf{x}_0)} \nabla_{0\alpha} B_0^\alpha, \tag{G.24}$$

where

$$\nabla_\alpha B^\alpha = \frac{\partial B^\alpha}{\partial q^\alpha} + \Gamma^\alpha_{s\alpha} B^s = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^\alpha} \left(\sqrt{g} B^\alpha \right), \tag{G.25}$$

$$\frac{\partial \mathbf{e}_\alpha}{\partial q^\beta} = \Gamma^s_{\alpha\beta} \mathbf{e}_s, \quad V^\alpha_{;\beta} = \frac{\partial V^\alpha}{\partial q^\beta} + \Gamma^\alpha_{s\beta} V^s, \tag{G.26}$$

In (G.25) $\nabla \cdot \mathbf{B} = 0$ requires that $\nabla_0 \cdot \mathbf{B}_0 = 0$ for Gauss's law to hold.

From Chap. 10, the MHD action, taking into account the Lagrangian map, has the form:

$$\mathcal{A} = \int d^3x \int dt\, L, \tag{G.27}$$

where

$$L = \frac{1}{2} \rho u^2 - \varepsilon(\rho, S) - \rho \Phi(\mathbf{x}) - \frac{B^2}{2\mu_0}, \tag{G.28}$$

is the Lagrangian. Using generalized coordinates $q^\alpha = q^\alpha(\mathbf{a}, t)$, the action (G.27) can be written in the form:

$$\mathcal{A} = \int \sqrt{g} d^3q \int dt\, L = \int d^3a \int dt\, L_0, \tag{G.29}$$

where

$$L_0 = J(\mathbf{q}, \mathbf{a})\sqrt{g}L \equiv \sqrt{g_0}J(\mathbf{x}, \mathbf{x}_0)\, L. \tag{G.30}$$

Using the mass continuity equation (G.15) and the frozen in field theorem (G.22) we find:

$$L_0 = \sqrt{g_0}\left\{ \rho_0\left[\frac{1}{2}g_{\alpha\beta}\frac{dq^\alpha}{dt}\frac{dq^\beta}{dt} - U\left(\frac{\rho_0}{J(\mathbf{x}, \mathbf{x}_0)}, S\right) - \Phi(\mathbf{x})\right] \right.$$

$$\left. - \frac{g_{\mu\nu}q^\mu{}_\alpha q^\nu{}_\beta B_0^\alpha B_0^\beta}{2\mu_0 J(\mathbf{x}, \mathbf{x}_0)} \right\}, \tag{G.31}$$

as the form of the Lagrangian in generalized coordinates, where $U(\rho, S) = \varepsilon(\rho, S)/\rho$ is the internal energy density per unit mass.

The Euler-Lagrange equations for the action (G.29) with Lagrangian (G.31) are:

$$\frac{\delta\mathcal{A}}{\delta q^i} = \frac{\partial L_0}{\partial q^i} - \frac{\partial}{\partial t}\left(\frac{\partial L_0}{\partial q_t^i}\right) - \frac{\partial}{\partial a^j}\left(\frac{\partial L_0}{\partial q_j^i}\right) = 0. \tag{G.32}$$

Evaluating the partial derivatives of L_0 in (G.32) we obtain:

$$\frac{\delta\mathcal{A}}{\delta q^i} = -\sqrt{g}J(\mathbf{q}, \mathbf{a})\left\{ \rho g_{i\mu}\left(\frac{dw^\mu}{dt} + \Gamma_{\alpha\beta}^\mu w^\alpha w^\beta\right) \right.$$

$$\left. + \frac{\partial}{\partial q^i}\left(p + \frac{B^2}{2\mu_0}\right) - \frac{\nabla_\alpha\left(B^\alpha B_i\right)}{\mu_0} + \rho\frac{\partial\Phi}{\partial q^i}\right\} = 0, \tag{G.33}$$

where

$$w^\mu = \frac{\partial q^\mu(\mathbf{a}, t)}{\partial t} \equiv \frac{dq^\mu}{dt}, \tag{G.34}$$

is the generalized velocity corresponding to q^μ. The affine connection coefficients $\Gamma_{\alpha\beta}^\mu$ in (G.33) are given by the standard formulae:

$$\Gamma_{\alpha\beta}^\mu = \frac{1}{2}g^{\mu\nu}\left[g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}\right], \tag{G.35}$$

where $g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$ is the metric of (G.5). In (G.33),

$$A^\mu = \frac{dw^\mu}{dt} + \Gamma_{\alpha\beta}^\mu w^\alpha w^\beta, \tag{G.36}$$

is the acceleration vector of the fluid (i.e. $d/dt(w^\mu \mathbf{e}_\mu) = A^\mu \mathbf{e}_\mu$). From (G.33) the covariant form of the momentum equation is:

$$\rho A_i = -\nabla_i \left(p + \frac{B^2}{2\mu_0} \right) + \frac{\nabla_\alpha (B^\alpha B_i)}{\mu_0} - \rho \nabla_i \Phi, \qquad (G.37)$$

where $A_i = g_{i\mu} A^\mu$ is the covariant form of the acceleration vector. It is straightforward to write down the contravariant form of the momentum equation (G.37). The frame independent form of (G.37) is given by:

$$\rho \mathbf{A} = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\frac{\mathbf{B} \otimes \mathbf{B}}{\mu_0} \right) - \rho \nabla \Phi, \qquad (G.38)$$

where $\mathbf{A} = A_i \mathbf{e}^i \equiv A^j \mathbf{e}_j$ is the acceleration vector of the fluid.

References

Achterberg, A.: Variational Principle for Relativistic Magnetohydrodynamics. Phys. Rev. A 28(4), 2499–2458 (1983)

Aharonov, Y., Bohm, D.: Significance of Electromagnetic Potentials in Quantum Theory. Phys. Rev. 115, 485–491 (1959)

Akhatov, I., Gazizov, R., Ibragimov, N.: Nonlocal Symmetries, Heuristic Approach (English Translation). J. Sov. Math. 55(1), 1401 (1991)

Akhmetiev, P., Ruzmaikin, A.: A Fourth Order Topological Invariant of Magnetic or Vortex Lines. J. Geom. Phys. 15, 95–101 (1995)

Aldinger, J., Klapper, I., Tabor, M.: Formulae for the Calculations and Estimation of Writhe. J. Knot Theory Ramif. 4, 343–372 (1995)

Anco, S.C., Bluman, G.: Derivation of Conservation Laws from Nonlocal Symmetries of Differential Equations. J. Math. Phys. 37(5), 2361–2375 (1996)

Anco, S.C., Bluman, G.: Direct Construction of Conservation Laws from Field Equations. Phys. Rev. Lett. 78(15), 2869–2873 (1997)

Anco, S.C., Bluman, G.W.: Direct Construction Method for Conservation Laws of Partial Differential Equations. Part I: Examples of Conservation Law Classification. Eur. J. Appl. Math. 13, 545–566 (2002a)

Anco, S.C., Bluman, G.W.: Direct Construction Method for Conservation Laws of Partial Differential Equations. Part II: General Treatment. Eur. J. Appl. Math. 13, 567–585 (2002b)

Anile, A.M.: Relativistic Fluids and Magneto-FLUIDS: With Applications in Astrophysics and Plasma Physics. Cambridge University Press, Cambridge (1989)

Araki, K.: Differential-Geometrical Approach to the Dynamics of Dissipationless, Incompressible Hall MHD: I. Lagrangian Mechanics on Semi-direct Products of Two Volume Preserving Diffeomorphisms and Conservation Laws. J. Phys. A: Math. Theor. 48, 175501 (2015)

Araki, K.: Particle Relabeling Symmetry, Generalized Vorticity, and Normal Mode Expansion of Ideal, Incompressible Fluids and Plasmas in Three-Dimensional Space (2016). arXiv:1601.05477v1

Araki, K.: Differential-Geometrical Approach to the Dynamics of Dissipationless Incompressible Hall Magnetohydrodynamics: II, Geodesic Formulation and Riemannian Curvature Analysis of Hydrodynamic and Magnetohydrodynamic Stabilities. J. Phys. A 50, 235501 (32 pp.) (2017)

Arnold, V.I.: Sur la geometrie differentielle des groups de Lie de dimension infinie et ses applications á l'hydrodynamique des fluides parfaits. Ann. Inst. Fourier Grenoble 16, 319–361 (1966)

Arnold, V.I.: The Asymptotic Hopf Invariant and Its Applications. Translated by R.P. Boas, English Translation in Sel. Math. Sov. 5(4), 326–345 (1974/1986)

Arnold, V.I., Khesin, B.A.: Topological Methods in Hydrodynamics. Springer, New York (1998)

Arrayaás, M., Bouwmeester, D., Trueba, J.L.: Knots in Electromagnetism. Phys. Rep. 667, 1–16 (2017)

Balmforth, N.J., Morrison, P.J.: A Necessary and Sufficient Instability Condition for Inviscid Shear Flow. Stud. Appl. Math. 102, 309–344 (1999)

Balmforth, N.J., Morrison, P.J.: Hamiltonian Description of Shear Flow. In: Large Scale Atmosphere-Ocean Dynamics II, pp. 117–142. Cambridge University Press, Cambridge (2002)

Balmforth, N.J., Morrison, P.J., Thiffeault, J.L.: Pattern Formation in Hamiltonian Systems with Continuous Spectra; a Normal form Single-Wave Model (2013). arXiv:1303.0065 (condensed matter)

Balogh, A., Lanzerotti, L.J., Suess, S.T. (eds.): The Heliosphere Through the Solar Activity Cycle. Springer-Praxis Publishing/Praxis Publishing Ltd., New York/Chichester (2008)

Balsara, D., Kim, J.S.: An Intercomparison Between Divergence Cleaning and Staggered Mesh Formulations for Numerical Magnetohydrodynamics. Astrophys. J. 602, 1079–1090 (2004)

Banerjee, R., Kumar, K.: New Approach to Nonrelativistic Ideal Magnetohydrodynamics (2016). arXiv:1601.03944v1 [physics.flu-dyn]

Barnes, A.: On the Non-existence of Plane Polarized, Large Amplitude Alfvén Waves. J. Geophys. Res. Space Phys. 81, 281 (1976)

Barnes, A.: Interplanetary Alfvénic Fluctuations: A Stochastic Model. J. Geophys. Res. 86(A9), 7498–7506 (1981)

Beckenstein, J.D.: Helicity Conservation Laws for Fluids and Plasmas. Astrophys. J. 319, 2017–214 (1987)

Beckenstein, J.D., Oron, E.: New Conservation Laws in General-Relativistic Magnetohydrodynamics. Phys. Rev. D 18(6), 1809–1819 (1978)

Berger, M.A.: Topological Invariants Rooted in Planes. Geophys. Astrophys. Fluid Dyn. 34, 265–281 (1986)

Berger, M.A.: An Energy Formula for Nonlinear Force Free Magnetic Fields. Astron. Astrophys. 201, 355–361 (1988)

Berger, M.A.: Third-Order Link Integrals. J. Phys. A. Math. Gen. 23, 2787–2793 (1990)

Berger, M.A.: Third Order Braid Invariants. J. Phys. A 24, 4027–4036 (1991)

Berger, M.A.: Magnetic Helicity in Space Physics. In: Brown, M., Canefield, R., Petsov, A. (eds.) Helicity in Space and Laboratory Plasmas. Monograph Series 111, pp. 1–9. AGU, Washington, DC (1999a)

Berger, M.A.: Introduction to Magnetic Helicity. Plasma Phys. Controlled Fusion 41, B167–B175 (1999b). https://doi.org/10.1088/0741-3335/41/12B/312

Berger, M.A., Field, G.B.: The Topological Properties of Magnetic Helicity. J. Fluid. Mech. 147, 133–148 (1984)

Berger, M.A., Prior, C.: The Writhe of Open and Closed Curves. J. Phys. A Math. Gen. 39, 8321–8348 (2006)

Berger, M.A., Ruzmaikin, A.: Rate of Helicity Production by Solar Rotation. J. Geophys. Res. 105(A5), 10481–10490 (2000)

Bernstein, I., Frieman, E.A., Kruskal, M.D., Kulsrud, R.M.: An Energy Principle for Hydromagnetic Stability Problems. Proc. R. Soc. A 244, 17–40 (1958)

Besse, N., Frisch, U.: Geometric Formulation of the Cauchy Invariants for Incompressible Euler Flow in Flat and Curved Spaces. J. Fluid Mech. 825, 412–478 (2017). arXiv:1701.01592v1

Bieber, J.W., Evenson, P.A., Matthaeus, W.H.: Magnetic Helicity of the Parker Field. Astrophys. J. 315, 700 (1987)

Bila, N.: Symmetries of PDEs Systems in Solar Physics and Contact Geometry (1999). arXiv:math-ph/9910042v1

Bluman, G.: Nonlocal Extensions of Similarity Methods. J. Nonlinear Math. Phys. 15, 1–24 (2008)

Bluman, G.W., Kumei, S.: Symmetries and Differential Equations. Applied Mathematical Sciences 81. Springer, New York (1989)

Bluman, G.W., Cheviakov, A.F., Anco, S.C.: Applications of Symmetry Methods to Partial Differential Equations. Applied Mathematical Sciences Series 168. Springer, New York (2010)

Bodo, G., Poedts, S., Rogava, A., Rossi, P.: Spatial Aspect of Wave Transformations in Astrophysical Flows. Astron. Astrophys. 374, 337–347 (2001)

Bogoyavlenskij, O.I.: Symmetry Transforms for Ideal Magnetohydrodynamics Equilibria. Phys. Rev. E 66(11), 056410 (2002)

Bogdan, T.J., Lerche, I.: A Note on Solitary Wave Solutions of the Leibovich-Roberts Equation. Q. Appl. Math. XLVI, 365 (1988)

Boillat, G.: Simple Waves in N-Dimensional Propagation. J. Math. Phys. 11, 1482–1483 (1970)

Boozer, A.H.: Physics of Magnetically Confined Plasmas. Rev. Mod. Phys. 76, 1071 (2004)

Bott, R., Tu, L.W.: Differential Forms in Algebraic Topology. Springer, New York (1982)

Boyd, T.J.M., Sanderson, J.J.: In: Jeffrey, A. (ed.) Plasma Dynamics. Applications of Mathematics Series. Barnes and Noble, New York (1969)

Brading, K.: Which Symmetry? Noether, Weyl and Conservation of Electric Charge. Stud. Hist. Philos. Mod. Phys. 33, 3–22 (2002)

Bridges, T.J.: Spatial Hamiltonian Structure, Energy Flux and the Water Wave Problem. Proc. R. Soc. Lond. A 439, 297–315 (1992)

Bridges, T.J.: Multi-Symplectic Structures and Wave Propagation. Math. Proc. Camb. Philos. Soc. 121, 147–190 (1997a)

Bridges, T.J.: A Geometric Formulation of the Conservation of Wave Action and Its Implications for Signature and Classification of Instabilities. Proc. R. Soc. A 453, 1365–1395 (1997b)

Bridges, T.J., Reich, S.: Numerical Methods for Hamiltonian PDEs. J. Phys. A Math. Gen. 39(22), 5287–5320 (2006)

Bridges, T.J., Hydon, P.E., Reich, S.: Vorticity and Symplecticity in Lagrangian Fluid Dynamics. J. Phys. A Math. Gen. 38, 1403–1418 (2005)

Bridges, T.J., Hydon, P.E., Lawson, J.K.: Multi-Symplectic Structures and the Variational Bi-Complex. Math. Proc. Camb. Philos. Soc. 148, 159–178 (2010)

Brio, M., Zakharian, A.R., Webb, G.M.: Numerical Time-Dependent Partial Differential Equations for Scientists and Engineers. In: Chui, C.K. (ed.) Mathematics in Science and Engineering 123, 1st edn. Elsevier, Burlington (2010)

Bruno, R., Carbone, V.: The Solar Wind as a Turbulence Laboratory. Living Reviews in Solar Physics (2005). Available online at http://www.livingreviews.org/lrsp-2005-4

Bruno, R., Carbone, V., Veltri, P., Pieropaolo, E., Bavasonno, B.: Identifying Intermittency Effects in the Solar Wind. Planet. Space Sci. 49, 1201 (2001)

Calkin, M.G.: An Action Principle for Magnetohydrodynamics. Can. J. Phys. 41, 2241–2251 (1963)

Calugareanu, G.: L'integral de Gauss et l'analyse des Noeds Tridimensionels. Rev. Math. Pure Appl. 4, 5–20 (1959)

Campbell, J., Berger, M.A.: Helicity, Linking, and Writhe in a Spherical Geometry. J. Phys. C (Conference Series) 544, 012001 (p. 9) (2014)

Carenina, J.F., Crampin, M., Ibort, L.A.: On the Multisymplectic Formalism for First Order Field Theories. In: Differential Geometry and Its Applications 1, pp. 345–374. North Holland, Amsterdam (1991)

Cary, J.R., Littlejohn, R.G.: Noncanonical Hamiltonian Mechanics and Its Application to Magnetic Field Line Flow. Ann. Phys. 151, 1–34 (1983)

Cendra, H., Marsden, J.E., Pekarsky, S., Ratiu, T.S.: Variational Principles for Li-Poisson and Hamilton-Poincaré Equations. Mosc. Math. J. 3(3), 837–867 (2003)

Chagelishvili, G.D., Chanishvili, R.G., Lominadze, J.G., Tevzadze, A.G.: Magnetohydrodynamic Waves Linear Evolution in Parallel Shear Flows: Amplification and Mutual Transformations. Phys. Plasmas 4(2), 259–269 (1997)

Chandrasekhar, S.: On the Stability of the Simplest Solution of the Equations of Magnetohydrodynamics. Proc. Natl. Acad. Sci. USA 42, 273–276 (1956)

Chandrasekhar, S.: On Cosmic Magnetic Fields. Proc. Natl. Acad. Sci. USA 43, 25–27 (1957)

Chandrasekhar, S.: Hydrodynamic and Hydromagnetic Stability. Oxford University Press/Clarendon Press, Oxford (1961)

Chandre, C.: Casimir Invariants and the Jacobi Identity in Dirac's Theory of Constraints of Constrained Hamiltonian Systems. J. Phys. A Math. Theor. 46(37), 375201 (2013)

Chandre, C., Morrison, P.J., Tassi, E.: On the Hamiltonian Formulation of Incompressible Ideal Fluids and Magnetohydrodynamics via Dirac's Theory of Constraints. Phys. Lett. A 376, 737–743 (2012)

Chandre, C., de Guillebon, L., Back, A., Tassi, E., Morrison, P.J.: On the Use of Projectors for Hamiltonian Systems and Their Relationship with Dirac Brackets. J. Phys. A. Math. Theor. 46(12), 125203 (2013)

Chanteur, G.: Localized Alfvénic Solutions of Nondissipative and Compressible MHD. Nonlinear Process. Geophys. 6, 145–148 (1999)

Cheviakov, A.F.: GeM Software Package for Computation of Symmetries and Conservation Laws of Differential Equations. Comput. Phys. Commun. 176, 48–61 (2007)

Cheviakov, A.F.: Conservation Properties and Potential Systems of Vorticity-Type Equations. J. Math. Phys. 55, 033508 (16 pp.) (2014) (0022-2488/2014/55(3)/033508/16)

Cheviakov, A.F., Anco, S.C.: Analytical Properties and Exact Solutions of Static Plasma Equilibrium Systems in Three Dimensions. Phys. Lett. A 372, 1363–1373 (2008)

Cheviakov, A.F., Oberlack, M.: Generalized Ertel's Theorem and Infinite Hierarchies of Conserved Quantities for Three-Dimensional Time Dependent Euler and Navier-Stokes Equations. J. Fluid. Mech. 760, 368–386 (2014)

Cotter, C.J., Holm, D.D.: On Noether's Theorem for Euler Poincaré Equation on the Diffeomorphism Group with Advected Quantities. Found. Comput. Math. (2012). https://doi.org/10.1007/S10208-012-9126-8

Cotter, C.J., Holm, D.D., Hydon, P.E.: Multi-Symplectic Formulation of Fluid Dynamics Using the Inverse Map. Proc. R. Soc. Lond. 463, 2617–2687 (2007)

Courant, R., Hilbert, D.: Methods of Mathematical Physics 2. Wiley, New York (1989)

Craik, A.D.D., Criminale, W.O.: Evolution of Wavelike Disturbances in Shear Flows: A Class of Exact Solutions of the Navier Stokes Equations. Proc. R. Soc. Lond. A 406, 13–26 (1986)

D'Avignon, E., Morrison, P.J., Pegararo, F.: Action Principle for Relativistic Magnetohydrodynamics. Phys. Rev. D 91, 084050 (2015)

de Donder, T.: Theorie Invariantive du Calcul des Variations. Gauthier Villars, Paris (1930)

Dedner, A., Kemm, F., Kröner, D., Munz, C., Schnitzer, T., Wesenberg, M.: Hyperbolic Divergence Cleaning for the MHD Equations. J. Comput. Phys. 175, 645–673 (2002)

Dewar, R.L.: Interaction Between Hydromagnetic Waves and a Time Dependent Inhomogeneous Medium. Phys. Fluids 13(11), 2710–2720 (1970)

Dewar, R.L., Yoshida, Z., Bhattacharjee, A., Hudson, S.R.: Variational Formulation of Relaxed and Multi-Region Relaxed Magnetohydrodynamics. J. Plasma Phys. 81, 515810604 (2015)

Dewar, R.L., Hudson, S.R., Bhattacharjee, A., Yoshida, Z.: Multi-Region Relaxed Magnetohydrodynamics in Plasmas with Slowly Changing Boundaries-Resonant Response of a Plasma Slab. Phys. Plasmas 24, 042507 (2017)

Elsässer, W.M.: Hydrodynamic Dynamo Theory. Rev. Mod. Phys. 28, 135 (1956)

Evans, C.R., Hawley, J.F.: Simulation of Magnetohydrodynamic Flows: A Constrained Transport Method. Astrophys. J. 332, 659–677 (1988)

Fecko, M.: Differential Geometry and Lie Groups for Physicists. Cambridge University Press, Cambridge (2006)

Fels, M., Olver, P.J.: Moving Frames I. Acta Appl. Math. 51, 161–312 (1998)

Ferraro, V.C.A., Plumpton, C.: Hydromagnetic Waves in a Horizontally Stratified Atmosphere. Astrophys. J. 127, 459–476 (1958)

Finn, J.H., Antonsen, T.M.: Magnetic Helicity: What Is it and What Is it Good for? Comments Plasma Phys. Contr. Fusion 9(3), 111 (1985)

Finn, J.M., Antonsen, T.M.: Magnetic Helicity Injection for Configurations with Field Errors. Phys. Fluids 31(10), 3012–3017 (1988)

Flanders, H.: Differential Forms with Applications to the Physical Sciences. Academic, New York (1963)

Forger, M., Paufler, C., Römer, H.: A General Construction of Poisson Brackets on Exact Multisymplectic Manifolds. Rep. Math. Phys. 51, 187–195 (2003)

Frankel, T.: The Geometry of Physics, An Introduction. Cambridge University Press, Cambridge (1997)

Frieman, E.A., Rotenberg, M.: On Hydromagnetic Stability of Stationary Equilibria. Rev. Mod. Phys. 32(4), 898–902 (1960)

Fuchs, J.C.: Symmetry Groups of Similarity Solutions of the MHD Equations. J. Math. Phys. 32, 1703–1708 (1991)

Gelfand, I.M., Fomin, S.V.: Calculus of Variations, Translated and Edited by Richard A Silverman, Dover edition 2000. Dover, New York (1963)

Gibson, S.E., Low, B.C.: A Time Dependent Three-Dimensional Magnetohydrodynamic Model of the Coronal Mass Ejection. Astrophys. J. 493(20), 460–473 (1998)

Gilbert, N.D., Porter, T.: Knots and Surfaces. Oxford University Press, Oxford/New York (1994)

Gleeson, L.J., Webb, G.M.: The Propagation of Cosmic Rays in the Interplanetary Region (the Theory). Fund. Cosmic Phys. 6, 187–312 (1980)

Goedbloed, J.P.: New Construction of the Magnetohydrodynamic Spectrum of Stationary Plasma Flows. II-Rayleigh-Taylor and Kelvin-Helmholtz Instability. Phys. Plasmas 16, 122111 (13 pp.) (2009)

Gogoberidze, G., Chagelishvili, G.D., Sagdeev, R.Z., Lominadze, D.G.: Linear Coupling and Over-Reflection Phenomena of Magnetohydrodynamic Waves in Smooth Shear Flows. Phys. Plasmas 11(10), 4672–4685 (2004)

Goldstein, H.: Classical Mechanics, 2nd edn. Addison Wesley, Reading (1980)

Golovin, S.V.: Analytical Description of Stationary Ideal MHD Fluid Flows with Constant Total Pressure. Phys. Lett. A 374, 901–905 (2010)

Golovin, S.V.: Natural Curvilinear Coordinates for Ideal MHD Equations. Non-stationary Flows with Constant Pressure. Phys. Lett. A c375, 283–290 (2011)

Goncalves, T.M.N., Mansfield, E.L.: On Moving Frames and Noether's Conservation Laws. SIAM 128(1), 1–29 (2012). https://doi.org/10.1111/j.1467-9590.2011.00522.x

Goncalves, T.M.N., Mansfield, E.L.: Moving Frames and Noether's Conservation Laws-the General Case. Forum Math. Sigma e1, 53 pp. (2016)

Gordin, V.A., Petviashvili, V.I.: Equation of Continuity for the Helicity in Media with an Infinite Conductivity. J. Exp. Theor. Phys. Lett. 45(5), 266–267 (1987)

Gordin, V.A., Petviashvili, V.I.: Lyapunov Instability of MHD Equilibrium of a Plasma with Nonvanishing Pressure. Sov. Phys. J. Exp. Theor. Phys. 68(5), 988–994 (1989)

Gosling, J.T., McComas, D.J., Roberts, D.A., Skoug, R.M.: A One Sided Aspect of Alfvénic Fluctuations in the Solar Wind. Astrophys. J. 695, L213–L216 (2009)

Gotay, M.J.: A Multisymplectic Framework for Classical Field Theory and the Calculus of Variations II: Space+Time Decomposition. Differ. Geom. Appl. 1, 375–390 (1991)

Gotay, M.J., Isenberg, J., Marsden, J.E., Montgomery, R. (with Sniatycki, J., Yasskin, P.B., collaborators): Momentum Maps and Classical Fields, Part I: Covariant Field Theory (2004a). arXiv:physics/9801019v2[math-ph]

Gotay, M.J., Isenberg, J., Marsden, J.E. (with Montgomery, R., Sniatycki, J., Yasskin, P.B. collaborators): Momentum Maps and Classical Fields, Part II: Canonical Analysis of Field Theories (2004b). arXiv:math-ph/0411032v1

Grundland, A.M., Lalague, L.: Lie Subgroups of Fluid Dynamics and Magnetohydrodynamics Equations. Can. J. Phys. 73, 463–477 (1995)

Grundland, A.M., Picard, P.: On Conditionally Invariant Solutions of Magnetohydrodynamic Equations Multiple Waves. J. Nonlinear Math. Phys. 11(1), 47–74 (2004)

Hameiri, E.: Variational Principles for Equilibrium States with Plasma Flow. Phys. Plasmas 5(9), 3270–3281 (1998). https://doi.org/1070-664X/98/5/9)/3270/12

Hameiri, E.: Dynamically Accessible Perturbations and Magnetohydromagnetic Stability. Phys. Plasmas 10(7), 2643–2648 (2003). https://doi.org/1070-664X/2003/10/7)/2643/6

Hameiri, E.: The Complete Set of Casimir Constants of the Motion in Magnetohydrodynamics. Phys. Plasmas 11(7), 3423–3431 (2004). https://doi.org/1070-664X/11/(7)/3423/9

Harrison, B.K., Estabrook, F.B.: Geometric Approach to Invariance Groups and Solution of Partial Differential Systems. J. Math. Phys. 12, 653–666 (1971)

Henyey, F.S.: Canonical Construction of a Hamiltonian for Dissipation-Free Magnetohydrodynamics. Phys. Rev. A 26(1), 480–483 (1982)

Hereman, W., Colagrosso, M., Sayers, R., Ringler, A., DeConninck, B., Nivala, M., Hickman, M.: Continuous and Discrete Homotopy Operators and the Computation of Conservation Laws. In: Wang, D., Zheng, Zh. (eds.) Differential Equations with Symbolic Computation. Trends in Mathematics, Chap. 15, pp. 249–285. Birkhäuser, Basel (2006)

Hirota, M., Fukumoto, Y.: Energy of Hydrodynamic and Magnetohydrodynamic Waves with Point and Continuous Spectra. J. Math. Phys. 49, 083101 (p. 28) (2008a). https://doi.org/0022-2488/2008/49(8)/083101/28

Hirota, M., Fukumoto, Y.: Action Angle Variables for the Continuous Spectrum of Ideal Magnetohydrodynamics. Phys. Plasmas 15, 122101 (p. 11) (2008b). https://doi.org/1070-664/2008/15(12)/122101/11

Hollmann, G.H.: Arch. Met. Geofys. Bioklimatol. A14, 1 (1964)

Holm, D.D.: Lagrangian Averages, Averaged Lagrangians, and the Mean Effects of Fluctuations in Fluid Dynamics. Chaos 12(2), 518–530 (2002)

Holm, D.D.: Geometric Mechanics, Part I: Dynamics and Symmetry. Imperial College Press, London (2008a). Distributed by World Scientific

Holm, D.D.: Geometric Mechanics, Part II: Rotating, Translating and Rolling. Imperial College Press, London (2008b). Distributed by World Scientific

Holm, D.D., Kupershmidt, B.A.: Poisson Brackets and Clebsch Representations for Magnetohydrodynamics, Multi-Fluid Plasmas and Elasticity. Phys. D 6D, 347–363 (1983a)

Holm, D.D., Kupershmidt, B.A.: Noncanonical Hamiltonian Formulation of Ideal Magnetohydrodynamics. Physica D 7D, 330–333 (1983b)

Holm, D.D., Marsden, J.E., Ratiu, T., Weinstein, A.: Nonlinear Stability of Fluid and Plasma Equilibria. Phys. Rep. (Review section of Phys. Rev. Lett.) 123(1 and 2), 1–116 (1985). https://doi.org/0370-1573/85

Holm, D.D., Marsden, J.E., Ratiu, T.S.: The Euler-Lagrange Equations and Semi-products with Application to Continuum Theories. Adv. Math. 137, 1–81 (1998)

Hydon, P.E.: Multi-Symplectic Conservation Laws for Differential and Differential-Difference Equations. Proc. R. Soc. A 461, 1627–1637 (2005)

Hydon, P.E., Mansfield, E.L.: Extensions of Noether's Second Theorem: From Continuous to Discrete Systems. Proc. R. Soc. A 467, 3206–3221 (2011). https://doi.org/10.1098/rspa.2011.0158

Ibragimov, N.H.: Transformation Groups Applied to Mathematical Physics. Reidel, Dordrecht (1985)

Ibragimov, N.H.: In: Ibragimov, N.H. (ed.) CRC Handbook of Lie Group Analysis of Differential Equations 1. CRC, Boca Raton (1994)

Ibragimov, N.H., Kara, A.H., Mahomed, F.M.: Lie-Bäcklund and Noether Symmetries with Applications. Nonlinear Dyn. 15, 115–136 (1998)

Ilgisnos, V.I., Pastukhov, V.P.: Variational Approaches to Problems of Plasma Stability and of Nonlinear Plasma Dynamics. J. Exp. Theor. Phys. Lett. 72(10), 530–540 (2000)

Jackson, J.D.: Classical Electrodynamics, 2nd edn. Wiley, New York (1975)

Janhunen, P.: A Positive Conservative Method for Magnetohydrodynamics Based on HLL and Roe Methods. J. Comput. Phys. 160, 649–661 (2000)

Jokipii, J.R., Thomas, B.T.: Effects of Drift on the Transport of Cosmic Rays IV: Modulation by a Wavy Interplanetary Current Sheet. Astrophys. J. 243, 1115–1122 (1981)

Kadomtsev, B.B., Pogutse, O.P.: Nonlinear Helical Perturbations of a Plasma in the Tokamak. J. Exp. Theor. Phys. 38, 283–290 (1974)

Kaghashvili, E.Kh.: On the Acceleration of the Solar Wind: Role of Inhomogeneous Flow. Astrophys. J. 512, 969–974 (1999)

Kaghashvili, E.Kh.: Mode conversion and wave particle interaction processes in the solar wind. Ph.D. Thesis, University of New Hampshire, Durham, NH (2002)

Kamchatnov, I.V.: Topological Soliton in Magnetohydrodynamics. Sov. Phys. 55(1), 69–73 (1982)

Kanatchikov, I.V.: On the Canonical Structure of the de-Donder-Weyl Covariant Hamiltonian Formulation of Field Theory I. Graded Poisson Brackets and Equations of Motion (1993). Preprint, arXiv:hep-th/9312162v1

Kanatchikov, I.V.: On Field Theoretic Generalizations of a Poisson Algebra. Rep. Math. Phys. 40, 225–234 (1997). hep-th/9710067

Kanatchikov, I.V.: Canonical Structure of Classical Field Theory in the Polymomentum Phase Space. Rep. Math. Phys. 41, 49–90 (1998). hep-th/9709229

Kara, A.H., Mahomed, F.M.: Relationship Between Symmetries and Conservation Laws. Int. J. Theor. Phys. 39(1), 23–40 (2000)

Kara, A.H., Mahomed, F.M.: A Basis of Conservation Laws for Partial Differential Equations. J. Nonlin. Math. Phys. 9, Suppl. 2, 60–72 (2002)

Kats, A.V.: Variational Principle and Canonical Variables in Hydrodynamics with Discontinuities. Physica D 152–153, 459–474 (2001)

Kats, A.V.: Variational Principle in Canonical Variables, Weber Transformation and Complete Set of Local Integrals of Motion for Dissipation-Free Magnetohydrodynamics. J. Exp. Theor. Phys. Lett. 77(12), 657–661 (2003)

Kats, A.V.: Canonical Description of Ideal Magnetohydrodynamic Flows and Integrals of Motion. Phys. Rev. E 69(4), 046303 (2004)

Kauffman, L.H.: On Knots. Annals of Mathematics Studies 115. Princeton University Press, Princeton (1987)

Kelbin, O., Cheviakov, A.F., Oberlack, M.: New Conservation Laws of Helically Symmetric, Plane and Rotationally Symmetric Viscous and Inviscid Flows. J. Fluid Mech. 721, 340–366 (2013)

Kelvin, L. (W. Thomson): Stability of Fluid Motion: Rectilinear Motion of Viscous Fluid Between Two Parallel Plates.Philos. Mag. 24(5), 188–196 (1887)

Kruskal, M.D., Kulsrud, R.M.: Equilibrium of a Magnetically Confined Plasma in a Toroid. Phys. Fluids 1, 265 (1958)

Kumar, P., Goldreich, P., Kerswell, R.: Effect of Nonlinear Interactions on P-Mode Frequencies and Line Widths. Astrophys. J. 427, 483–496 (1994)

Kuznetsov, E.A.: Vortex Line Representation for Hydrodynamic Type Equations. J. Nonlinear Math. Phys. 13(1), 64–80 (2006)

Kuznetsov, E.A., Ruban, V.P.: Hamiltonian Dynamics of Vortex Lines for Systems of Hydrodynamic Type. J. Exp. Theor. Phys. Lett. 67, 1076 (1998)

Kuznetsov, E.A., Ruban, V.P.: Hamiltonian Dynamics of Vortex and Magnetic Field Lines in the Hydrodynamic Type Models. Phys. Rev. E 61, 831–841 (2000)

Kuznetsov, E.A., Passot, T., Sulem, P.L.: Compressible Dynamics of Magnetic Field Lines for Incompressible MHD Flows. Phys. Plasmas 11, 1410 (2004)

Lichnerowicz, A.: Relativistic Hydrodynamics and Magnetohydrodynamics. Lectures on the Existence of Solutions. Benjamin, New York (1967)

Lighthill, M.J.: Studies on Magneto-Hydromagnetic Waves and Other Anisotropic Wave Motions. Philos. Trans. R. Soc. Lond. A 398–427 (1960)

Longcope, D.W., Malanushenko, A.: Defining and Calculating Self-helicity in Coronal Magnetic Fields. Astrophys. J. 674, 1130–1143 (2008)

Low, B.C.: Three-Dimensional Structures of Magnetostatic Atmospheres. Astrophys. J. 293, 31–43 (1985)

Low, B.C.: Magnetohydrodynamic Processes in the Solar Corona: Flares, Coronal Mass Ejections, and Magnetic Helicity. Phys. Plasmas 1, 1684 (1994)

Low, B.C.: Magnetic Helicity in a Two-Flux Partitioning of an Ideal Hydromagnetic Fluid. Astrophys. J. 646, 1288–1302 (2006)

Low, B.C.: Field Topologies in Ideal and Near-Ideal Magnetohydrodynamics and Vortex Dynamics. Sci. China Phys. Mech. Astron. 58(1), 015201 (2015)

Lundgren, T.S.: Hamilton's Variational Principle for a Perfectly Conducting Plasma Continuum. Phys. Fluids 6, 898–904 (1963)

Marsch, E., Mülhauser, K.H., Schwenn, R., Rosenbauer, H., Pilip, W., Neubauer, F.M.: Solar Wind Protons: Three Dimensional Velocity Distributions and Derived Plasma Parameters Measured Between 0.3 and 1 AU. J. Geophys. Res. 81(A1), 52–72 (1982)

Marsden, J.E., Ratiu, T.S.: Introduction to Mechanics and Symmetry. Springer, New York (1994)

Marsden, J.E., Shkoller, S.: Multi-Symplectic Geometry, Covariant Hamiltonians and Water Waves. Math. Proc. Camb. Philos. Soc. 125, 553–575 (1999)

Marsden, J.E., Ratiu, T., Weinstein, A.: Semidirect Products and Reduction in Mechanics. Trans. Am. Math. Soc. 281(1), 147–177 (1984)

Marsden, J.E., Pekarsky, S., Shkoller, S., West, M.: Variational Methods, Multisymplectic Geometry, and Continuum Mechanics. J. Geom. Phys. 38, 253–284 (2001)

Matteini, L., Horbury, T.S., Pantellini, F., Velli, M., Schwartz, S.J.: Ion Kinetic Energy Conservation and Magnetic Field Strength Constancy in Multi-Fluid Solar Wind and Alfvénic Turbulence. Astrophys. J. 802(11), 4 pp. (2015)

Matthaeus, W.H., Goldstein, M.L.: Measurement of the Rugged Invariants of Magnetohydrodynamic Turbulence in the Solar Wind. J. Geophys. Res. 87(A8), 6011–6028 (1982)

Misner, C.W., Thorne, K.S., Wheeler, J.A.: Gravitation. W.H. Freeman, San Francisco (1973)

Mobbs, S.D.: Some Vorticity Theorems and Conservation Laws for Non-barotropic Fluids. J. Fluid Mech. 81, 475–483 (1981)

Moffatt, H.K.: The Degree of Knottedness of Tangled Vortex Lines. J. Fluid. Mech. 35, 117 (1969)

Moffatt, H.K.: Magnetic Field Generation in Electrically Conducting Fluids. Cambridge University Press, Cambridge (1978)

Moffatt, H.K., Ricca, R.L.: Helicity and the Calugareanu Invariant. Proc. R. Soc. Lond. Ser. A 439, 411 (1992)

Moiseev, S.S., Sagdeev, R.Z., Tur, A.V., Yanovskii, V.V.: On the Freezing-in Integrals and Lagrange Invariants in Hydrodynamic Models. Sov. Phys. J. Exp. Theor. Phys. 56(1), 117–123 (1982)

Morrison, P.J.: Poisson Brackets for Fluids and Plasmas. In: Tabor, M., Treve, Y.M. (eds.) Mathematical Methods in Hydrodynamics and Integrability of Dynamical Systems. AIP Conference Proceedings 88, pp. 13–46. American Institute of Physics (1982)

Morrison, P.J.: Hamiltonian Description of the Ideal Fluid. Rev. Mod. Phys. 70(2), 467–521 (1998)

Morrison, P.J.: Magnetic Field Lines, Hamiltonian Dynamics and Nontwist Systems. Phys. Plasmas 7(6), 2279–2289 (2000)

Morrison, P.J., Eliezer, S.: Spontaneous Symmetry Breaking and Neutral Stability in the Noncanonical Hamiltian Formalism. Phys. Rev. A 33(6), 4205–4214 (1986)

Morrison, P.J., Greene, J.M.: Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal Magnetohydrodynamics. Phys. Rev. Lett. 45, 790–794 (1980)

Morrison, P.J., Greene, J.M.: Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal Magnetohydrodynamics (Errata). Phys. Rev. Lett. 48, 569 (1982)

Morrison, P.J., Hazeltine, R.D.: Hamiltonian Formulation of Reduced Magnetohydrodynamics. Phys. Fluids 27(4), 886–897 (1984)

Morrison, P.J., Pfirsch, D.: Dielectric Energy Versus Plasma Energy and Hamiltonian Action-Angle Variables for the Vlasov Equation. Phys. Fluids B4(10), 3038–3057 (1992)

Morse, P.M., Feshbach, H.: Methods of Theoretical Physics, Part I. McGraw Hill, New York (1953)

Newcomb, W.A.: Motion of Magnetic Lines of Force. Ann. Phys. N. Y. 3, 347 (1958)

Newcomb, W.A.: Lagrangian and Hamiltonian Methods in Magnetohydrodynamics. Nucl. Fusion Suppl. (Part 2), 451–463 (1962)

Nicole, D.A.: Solitons with Non-vanishing Hopf Index. J. Phys. G Nucl. Phys. 4(9), 1363–1369 (1978)

Noether, E.: Invariante Variations Probleme. Nachr. König. Gessell. Wissen. Göttingen Mathphys. Kl. 2, 235–257 (1918). See Transp. Theory Stat. Phys. 1, 186–207 (1971). For an English translation, posted at physics/0503066

Olver, P.J.: Applications of Lie Groups to Differential Equations, 2nd edn. Springer, New York (1993)

Olver, P.J., Nutku, Y.: Hamiltonian Structures for Systems of Hyperbolic Conservation Laws. J. Math. Phys. 29, 1610–1619 (1988)

Ono, T.: Riemannian Geometry of an Ideal Incompressible Magnetohydrodynamical Fluid. Physica D 81, 207–220 (1995a)

Ono, T.: A Riemannian Geometrical Description for Lie-Poisson Systems and Its Application to Idealized Magnetohydrodynamics. J. Phys. A 28, 1737–1651 (1995b)

Oughton, S., Matthaeus, W.H., Dmitruk, P.: Reduced MHD in Astrophysical Applications: Two Dimensional or Three Dimensional? Astrophys. J. 839(2), 13 pp. (2017)

Ovsjannikov, L.V.: Group Properties of Differential Equations. Siberian Section of the Academy of Science of U.S.S.R., Novisibirsk (1962). English translations by G.W. Bluman

Ovsjannikov, L.V.: Group Analysis of Differential Equations. Academic, New York (1982)

Padhye, N.S.: Topics in lagrangian and hamiltonian fluid dynamics: relabeling symmetry and ion acoustic wave stability. Ph.D. Dissertation, University of Texas at Austin (1998)

Padhye, N.S., Morrison, P.J.: Fluid Relabeling Symmetry. Phys. Lett. A 219, 287–292 (1996a)

Padhye, N.S., Morrison, P.J.: Relabeling Symmetries in Hydrodynamics and Magnetohydrodynamics. Plasma Phys. Rep. 22, 869–877 (1996b)

Panofsky, W.K.H., Phillips, M.: Classical Electricity and Electromagnetism, sect. 9.4, 2nd edn., p. 164. Wesley, Reading (1964)

Parker, E.N.: Dynamics of the Interplanetary Gas and Magnetic Field. Astrophys. J. 128, 664–676 (1958)

Parker, E.N.: Interplanetary Dynamical Processes. Interscience, New York (1963)

Parker, E.N.: Cosmic Magnetic Fields. Oxford University Press, New York (1979)

Parker, E.N.: Spontaneous Current Sheets in Magnetic Fields. Oxford University Press, New York (1994)

Pedlosky, J.: Geophysical Fluid Dynamics, 2nd edn., p. 710. Springer, New York (1987)

Poedts, S., Rogava, A.D., Mahajan, S.M.: Shear-Flow-Induced Coupling in the Solar Winds. Astrophys. J. 505, 369–375 (1998)

Poincaré, H.: Sue une forme nouvelle des equations de la mechanique. C.R. Acad. Sci. 132, 369–371 (1901)

Powell, K.G., Roe, P.L., Linde, T.J., Gombosi, T.I., De Zeeuw, D.: A Solution Adaptive Upwind Scheme for Ideal Magnetohydrodynamics. J. Comput. Phys. 154, 284–309 (1999)

Prim, R., Truesdell, C.: A Derivation of Zorawski's Criterion for Permanent Vector Lines. Proc. Am. Math. Soc. 1, 32–34 (1950)

Prior, C., Yeates, A.R.: On the Helicity of Open Magnetic Fields. Astrophys. J. 787(100), 13 pp. (2014)

Pshenitsin, D.: Conservation laws of magnetohydrodynamics and their symmetry transformation properties. Ph.D. Thesis, Department of Physics, Brock University, Saint Catharines (2016). Available at https://dr.library.ca.handle/10464/9801

Reich, S.: Multi-Symplectic Runge-Kutta Collocation Methods for Hamiltonian Wave Equations. J. Comput. Phys. 157, 473 (2000)

Roberts, B.: Solitary Waves in a Magnetic Flux Tube. Phys. Fluids 28, 3280 (1985)

Roberts, D.A., Goldstein, M.L.: The variance of solar wind magnetic fluctuations: solutions and further puzzles. AGU Fall Meeting Abstract SH14A-03 (2006)

Roberts, B., Mangeney, A.: Solitons in Solar Magnetic Flux Tubes. Mon. Not. R. Astron. Soc. 198, 7P (1982)

Rogers, C., Schief, W.K.: Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge (2002)

Rosenhaus, V.: Infinite Symmetries and Conservation Laws. J. Math. Phys. 43, 6129–6150 (2002). https://doi.org/10.1063/1.1517394

Rosenhaus, V., Shankar, R.: Second Noether Theorem for Quasi-Noether Systems. J. Phys. A Math. Theor. 49, 175205 (22 pp.) (2016). https://doi.org/1088/1751-8113/49/17/175205

Rosenhaus, V., Shankar, R.: Sub-symmetries I. Main Properties and Applications (2017a). Available on Researchgate publication/313892895

Rosenhaus, V., Shankar, R.: Sub-symmetries II. Sub-symmetries and Conservation Laws (2017b). Available on Researchgate

Rosner, R., Low, B.C., Tsinganos, K., Berger, M.A.: On the Relationship Between the Topology of Magnetic Field Lines and Flux Surfaces. Geophys. Astrophys. Fluid Dyn. 48, 251–271 (1989)

Ruderman, M.S.: Nonlinear Waves in the Solar Atmosphere. Philos. Trans. R. Soc. Lond. A Math. Eng. Sci. 364, 485 (2006)

Rust, D.: Spawning and Shedding of Magnetic Fields in the Solar Corona. Geophys. Res. Lett. 21, 241 (1994)

Ruzmaikin, A., Akhmetiev, P.: Topological Invariants of Magnetic Fields, and the Effect of Reconnections. Phys. Plasmas 1(2), 331–336 (1994)

Sagdeev, R.Z., Moiseev, S.S., Tur, A.V., Yanovsky, V.: Problems of the Theory of Strong Turbulence and Topological Solitons. In: Sagdeev, R.Z. (ed.) Nonlinear Phenomena in Plasma Physics and Hydrodynamics, pp. 137–182. Mir, Moscow (1986)

Sagdeev, R.Z., Tur, A.V., Yanovsky, V.V.: Construction of frozen in integrals, Lagrangian and topological invariants in hydrodynamical models. In: Moffatt, H.K., Tsinober, A. (eds.) Proceedings of the IUTAM Symposium. Topological Fluid dynamics, pp. 421–428. Cambridge University Press, Cambridge (1990)

Sakurai, T.: A New Approach to Force-Free Field and Its Application to the Magnetic Field of Solar Active Regions. Publ. Astron. Soc. Jpn. 31, 209 (1979)

Salmon, R.: Hamilton's Principle and Ertel's Theorem. AIP Conf. Proc. 88, 127–135 (1982)

Salmon, R.: Hamiltonian Fluid Mechanics. Ann. Rev. Fluid Mech. 20, 225–256 (1988)

Schief, W.K.: Hidden Integrability in Ideal Magnetohydrodynamics: The Pohlmeyer-Lund-Regge Model. Phys. Plasmas 10, 2677–2685 (2003)

Schutz, B.: Geometrical Methods in Mathematical Physics. Cambridge University Press, Cambridge (1980)

Semenov, V.S., Korvinski, D.B., Biernat, H.K.: Euler Potentials for the MHD Kamchatnov-Hopf Soliton Solution. Nonlinear Process. Geophys. 9, 347–354 (2002)

Sjöberg, A., Mahomed, F.M.: Nonlocal Symmetries and Conservation Laws for One Dimensional Gas Dynamics Equations. Appl. Math. Comput. 150, 379–397 (2004)

Sneddon, I.N.: Elements of Partial Differential Equations, International Student Edition. McGraw Hill, New York (1957)

Squire, J., Qin, H., Tang, W.M., Chandre, C.: The Hamiltonian Structure and Euler-Poincaré Formulation of the Vlasov-Maxwell and Gyrokinetic Systems. Phys. Plasmas 20, 122501 (14 pp.) (2013)

Stern, D.P.: The Motion of Magnetic Field Lines. Space Sci. Rev. 6, 143–173 (1966)

Strauss, H.R.: Nonlinear Three Dimensional Magnetohydrodynamics of Non-circular Tokamaks. Phys. fluids 19, 134–140 (1976)

Summners, D.W.: Knot theory and DNA. In: Summners, D.W. (ed.) New Scientific Applications of Geometry and Topology. Proceedings of Symposia in Applied Mathematics, vol. 45. American Mathematical Society, Providence (1992)

Tanehashi, K., Yoshida, Z.: Gauge Symmetries and Noether Charges in Clebsch-Parameterized Magnetohydrodynamics. J. Phys. A Math. Theor. 48, 495501 (20 pp.) (2015). https://doi.org/10.1088/1751-8113/48/49/495501

Taylor, J.B.: Relaxation of Toroidal Plasma and Generation of Reverse Magnetic Fields. Phys. Rev. Lett. 33, 1139–1141 (1974)

Taylor, J.B.: Relaxation and Magnetic Reconnection in Plasmas. Rev. Mod. Phys. 58, 741–763 (1986)

Thiffeault, J.-L., Morrison, P.J.: Classification and Casimir Invariants of Lie-Poisson Brackets. Physica D 136, 205–244 (2000)

Thompson, A., Sweargin, J., Wickes, A., Bouwmeester, D.: Constructing a Class of Topological Solitons in Magnetohydrodynamics. Phys. Rev. E 89, 043104 (2014)

Torok, T., Berger, M.A., Kliem, B.: The Writhe of Helical Structures in the Solar Corona. Astron. Astrophys. 516, A49 (p. 11) (2010)

Torok, T., Kliem, B., Berger, M.A., Linton, M.G., Demoulin, P., Van Driel-Gesztelyi, L.: The Evolution of Writhe in Kink-Unstable Flux Ropes and Erupting Filaments. Plasma Phys. Control. Fusion 56, 064012 (7 pp.) (2014)

Truesdell, C.: The Kinematics of Vorticity. Indiana University Press, Bloomington (1954)

Truesdell, C., Toupin, R.A.: In: Flugge, S. (ed.) The Classical Field Theories. Handbuch der Physik III/I, pp. 226–793. Springer, Heidelberg (1960)

Tsinganos, K.C.: Magnetohydrodynamic Equilibrium I. Exact Solutions of the Equations. Astrophys. J. 245, 764–782 (1981)

Tur, A.V., Yanovsky, V.V.: Invariants in Dissipationless Hydrodynamic Media. J. Fluid Mech. 248, 67–106 (1993)

Tur, A., Yanovsky, V.: Coherent Vortex Structures in Fluids and Plasmas. Springer Series in Synergetics. Springer, New York (2017)

Urbantke, H.K.: The Hopf Fibration-Seven Times in Physics. J. Geom. Phys. 46(2), 125–150 (2003)

Van der Holst, B., Nijober, R.J., Goedbloed, J.P.: Magnetohydrodynamic Spectrum of Gravitating Plane Plasmas with Flow. J. Plasma Phys. 61(2), 221–240 (1999)

Volkov, D.V., Tur, A.V., Janovsky, V.V.: Hidden Supersymmetry of Classical Systems (Hydrodynamics and Conservation Laws). Phys. Lett. A 203, 357–361 (1995)

Webb, G.M.: Multi-Symplectic, Lagrangian, One-Dimensional Gas Dynamics. J. Math. Phys. 56, 053101 (20 pp.) (2015). Also available at http://arxiv.org/abs/1408.4028v4

Webb, G.M., Anco, S.C.: Vorticity and Symplecticity in Multi-Symplectic, Lagrangian Gas Dynamics. J. Phys. A Math. Theor. 49, 075501 (44 pp.) (2016). https://doi.org/10.1008/1751-8113/49/075501

Webb, G.M., Anco, S.C.: On Magnetohydrodynamic Gauge Field Theory. J. Phys. A Math. Theor. 50, 255501, 34 pp. (2017)

Webb, G.M., Mace, R.L.: Potential Vorticity in Magnetohydrodynamics. J. Plasma Phys. 81, p. 18, 905810115 (2015). https://doi.org/10.1017/S0022377814000658. Preprint: http://arxiv/org/abs/1403.3133

Webb, G.M., Zank, G.P.: Fluid Relabelling Symmetries, Lie Point Symmetries and the Lagrangian Map in Magnetohydrodynamics and Gas Dynamics. J. Phys. A. Math. Theor. 40, 545–579 (2007). https://doi.org/10.1088/1751-8113/40/3/013

Webb, G.M., Zank, G.P.: Scaling Symmetries, Conservation Laws and Action Principles in One-Dimensional Gas Dynamics. J. Phys. A. Math. Theor. 42, 475205 (23 pp.) (2009)

Webb, G.M., Axford, W.I., Terasawa, T.: On the Drift Mechanism for Energetic Charged Particles at Shocks. Astrophys. J. 270, 537–553 (1983)

Webb, G.M., Woodward, T.I., Brio, M., Zank, G.P.: Linear Magnetosonic N-waves and Green's Functions. J. Plasma Phys. 49(Part 3), 465–513 (1993)

Webb, G.M., Brio, M., Zank, G.P.: Steady MHD Flows with an Ignorable Coordinate and the Potential Transonic Flow Equation. J. Plasma Phys. 52(Part 1), 141–188 (1994)

Webb, G.M., Brio, M., Zank, G.P.: Corrigendum: Steady MHD Flows with an Ignorable Coordinate and the Potential Transonic Flow Equation. J. Plasma Phys. 65(Part 3), 255–256 (1994)

Webb, G.M., Ratkiewicz, R., Brio, M., Zank, G.P.: Multi-Dimensional MHD Simple Waves. In: Winterhalter, D., Gosling, J.T., Habbal, S.R., Kurth, W.S., Neugebauer, M. (eds.) Proceedings of the 8th International Solar Wind Conference: Solar Wind Eight. AIP Conference Proceedings 382, pp. 335–338. AIP, New York (1996)

Webb, G.M., Pantazopoulou, M., Zank, G.P.: Multiple Scattering and the BGK Boltzmann Equation. J. Phys. A Math. Gen. 33, 3137–3160 (2000)

Webb, G.M., Zank, G.P., Kaghashvili, E.Kh., Ratkiewicz, R.E.: Magnetohydrodynamic Waves in Non-uniform Flows I: A Variational Approach. J. Plasma Phys. 71(6), 785–809 (2005a). https://doi.org/10.1017/S00223778050003739

Webb, G.M., Zank, G.P., Kaghashvili, E.Kh., Ratkiewicz, R.E.: Magnetohydrodynamic Waves in Non-uniform Flows II: Stress Energy Tensors, Conservation Laws and Lie Symmetries. J. Plasma Phys. 71, 811–857 (2005b). https://doi.org/10.1017/S00223778050003740

Webb, G.M., Kaghashvili, E.Kh., Zank, G.P.: Magnetohydrodynamic Wave Mixing in Shear Flows: Hamiltonian Equations and Wave Action. J. Plasma Phys. 73(Part 1), 15–68 (2007)

Webb, G.M., Pogorelov, N.V., Zank, G.P.: MHD Simple Waves and the Divergence Wave. In: Twelfth International Solar Wind Conference, St. Malo. AIP Conference Proceedings 1216, pp. 300–303 (2009). https://doi.org/10.1063/1.3396300

Webb, G.M., Hu, Q., Dasgupta, B., Zank, G.P.: Homotopy Formulas for the Magnetic Vector Potential and Magnetic Helicity: The Parker Spiral Interplanetary Magnetic Field and Magnetic Flux Ropes. J. Geophys. Res. (Space Phys.) 115, A10112 (2010a). https://doi.org/10.1029/2010JA015513. Corrections: J. Geophys. Res. 116, A11102 (2011). https://doi.org/10.1029/2011JA017286

Webb, G.M., Hu, Q., Dasgupta, B., Roberts, D.A., Zank, G.P.: Alfven Simple Waves: Euler Potentials and Magnetic Helicity. Astrophys. J. 725, 2128–2151 (2010b). https://doi.org/10.1088/0004-637X/725/2/2128

Webb, G.M., Zank, G.P., Burrows, R.H., Ratkiewicz, R.E.: Simple Alfven Waves. J. Plasma Phys. 77(Part 1), 51–93 (2011). https://doi.org/10.101/S00233377809990596

Webb, G.M., Hu, Q., Dasgupta, B., Zank, G.P.: Double Alfvén Waves. J. Plasma Phys. 78(Part 1), 71–85 (2012a). https://doi.org/10.1017/S0022377811000420

Webb, G.M., Hu, Q., le Roux, J.A., Dasgupta, B., Zank, G.P.: Hamiltonians and Variational Principles for Alfvén Simple Waves. J. Phys. A Math. Theor. 45, 025203 (33 pp.) (2012b)

Webb, G.M., Dasgupta, B., McKenzie, J.F., Hu, Q., Zank, G.P.: Local and Nonlocal Advected Invariants and Helicities in Magnetohydrodynamics and Gas Dynamics I: Lie Dragging Approach. J. Phys. A Math. Theor. 47, 095501 (33 pp.) (2014a). https://doi.org/10.1088/1751-8113/49/9/095501. Preprint available at http://arxiv.org/abs/1307.1105

Webb, G.M., Dasgupta, B., McKenzie, J.F., Hu, Q., Zank, G.P.: Local and Nonlocal Advected Invariants and Helicities in Magnetohydrodynamics and Gas Dynamics II: Noether's Theorems and Casimirs. J. Phys. A Math. Theor. 47, 095502 (31 pp.) (2014b). https://doi.org/10.1088/1751-8113/47/9/095502. Preprint available at http://arxiv.org/abs/1307.1038

Webb, G.M., McKenzie, J.F., Zank, G.P.: Multi-Symplectic Magnetohydrodynamics. J. Plasma Phys. 80(Part 5), 707–743 (2014c). https://doi.org/10.1017/S0022377814000257. Also at http://arxiv.org/abs/1312.4890v4

Webb, G.M., Burrows, R.H., Ao, X., Zank, G.P.: Ion Acoustic Travelling Waves. J. Plasma Phys. 80(Part 2), 147–171 (2014d). https://doi.org/10.1017/S0022377813001013. Preprint at http://arxiv.org/abs/1312.6406

Webb, G.M., McKenzie, J.F., Zank, G.P.: Multi-Symplectic Magnetohydrodynamics: II, Addendum and Erratum. J. Plasma Phys. 81, 905810610 (15 pp.) (2015). https://doi.org/10.1017/S0022377815001415

Weishaar, E.: Solitary Waves in Slender Magnetic Flux Tubes as Solutions of the Leibovich-Pritchard-Roberts Equation. Phys. Fluids A 1(8), 1406–1414 (1989)

Weyl, H.: Geodesic Fields in the Calculus of Variation for Multiple Integrals. Ann. Math. 36(3), 607–629 (1935). http://www.jstor.org/stable/1968645

Whitham, G.B.: A General Approach to Linear and Nonlinear Dispersive Waves Using a Lagrangian. J. Fluid Mech. 22, 273–283 (1965)

Whitham, G.B.: Linear and Nonlinear Waves. New York, Wiley (1974)

Woltjer, L.: A Theorem on Force-Free Magnetic Fields. Proc. Natl. Acad. Sci. 44, 489 (1958)

Yahalom, A.: Aharonov-Bohm Effects in Magnetohydrodynamics. Phys. Lett. A 377, 1898–1904 (2013)

Yahalom, A.: Simplified Variational Principles for Non-barotropic Magnetohydrodynamics. J. Plasma Phys. 82(2), 15 pp. (2016a). Article ID. 905820204

Yahalom, A.: Non-barotropic Magnetohydrodynamics as a Five Function Field Theory. Int. J. Geom. Methods Mod. Phys. 13(10) (2016b). ID. 1650130-139

Yahalom, A.: A Conserved Cross Helicity for Non-barotropic MHD. Geophys. Astrophys. Fluid Dyn. 111(2), 131–137 (2017a). Preprint. arXiv:1605.02537v1

Yahalom, A.: Non Barotropic Cross Helicity Conservation and the Aharonov-Bohm Effect in Magnetohydrodynamics. Fluid Dyn. Res. (2017b). https://doi.org/10.1088/1873-7005/aa6fc7

Yahalom, A., Lynden-Bell, D.: Simplified Variational Principle for Barotropic Magnetohydrody-
    namics. J. Fluid Mech. 607, 235–265 (2008)
Yeates, A.R., Hornig, G.: Unique Topological Characterization of Braided Magnetic Fields. Phys.
    Plasmas 20, 012102 (5 pp.) (2013)
Yoshida, Z.: Clebsch Parameterization: Basic Properties and Remarks on Its Applications. J. Math.
    Phys. 50, 113101 (2009)
Zakharov, V.E., Kuznetsov, E.A.: Variational Principle and Canonical Variables in Magnetohydro-
    dynamics. Sov. Phys. Dokl. 15(10), 913–914 (1971)
Zakharov, V.E., Kuznetsov, E.A.: Hamiltonian Formalism for Nonlinear Waves. Phys. Uspekhi
    40(11), 1087–1116 (1997)
Zank, G.P., Matthaeus, W.H.: The Equations of Reduced Magnetohydrodynamics. J. Plasma Phys.
    48(1), 85–100 (1992)
Zank, G.P., Dosch, A., Hunana, P., Florinski, V., Matthaeus, W.H., Webb, G.M.: The Transport
    of Low-Frequency Turbulence in Astrophysical Flows. I. Governing Equations. Astrophys. J.
    745(35) (20 pp.) (2012)
Zhou, Y., Matthaeus, W.H.: Transport and Turbulence Modeling of Solar Wind Fluctuations. J.
    Geophys. Res. 95(A7), 10291–10311 (1990a)
Zhou, Y., Matthaeus, W.H.: Models of Inertial Range Spectra of Interplanetary Magnetohydrody-
    namic Turbulence. J. Geophys. Res. 95(A9), 14881–14892 (1990b)
Zorawski, K.: Uber die Erhaltung der Wirbelbewegung. Bull. Acad. Sci. Cracov. C. R. 335 (1900)