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Fridtjov Irgens
Department of Structural Engineering
Norwegian University of Science and Technology
Trondheim
Norway

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Preface

This book has originated from a compendium of lecture notes prepared by the author to a graduate course in Rheology and Non-Newtonian Fluids at the Norwegian University of Science and Technology. The compendium was presented in Norwegian from 1993 and in English from 2003. The aim of the course and of this book has been to give an introduction to the subject.

Fluid is the common name for liquids and gases. Typical non-Newtonian fluids are polymer solutions, thermoplastics, drilling fluids, granular materials, paints, fresh concrete and biological fluids, e.g., blood.

Matter in the solid state may often be modeled as a fluid. For example, creep and stress relaxation of steel at temperature above ca. 400 °C, well below the melting temperature, are fluid-like behaviors, and fluid models are used to describe steel in creep and relaxation.

The author has had great pleasure demonstrating non-Newtonian behavior using toy materials that can be obtained from science museum stores under different brand names like Silly Putty, Wonderplast, Science Putty, and Thinking Putty. These materials exhibit many interesting features that are characteristic of non-Newtonian fluids. The materials flow, but very slowly, are highly viscous, may be formed to a ball that bounces elastically, tear if subjected to rapidly applied tensile stress, and break like glass if hit by a hammer.

The author has been involved in a variety of projects in which fluids and fluid-like materials have been modeled as non-Newtonian fluids: avalanching snow, granular materials in landslides, extrusion of aluminium, modeling of biomaterials as blood and bone, modeling of viscoelastic plastic materials, and drilling mud used when drilling for oil.

Rheology consists of Rheometry, i.e., the study of materials in simple flows, Kinetic Theory of Macromaterials, and Continuum Mechanics.

After a brief introduction of what characterizes non-Newtonian fluids in Chap. 1 some phenomenal characteristic of non-Newtonian fluids are presented in Chap. 2. The basic equations in fluid mechanics are discussed in Chap. 3. *Deformation Kinematics*, the kinematics of shear flows, viscometric flows, and extensional flows are the topics in Chap. 4. *Material Functions* characterizing the behavior of fluids in special flows are defined in Chap. 5. *Generalized Newtonian Fluids* are the most common types of non-Newtonian fluids and are the subject in Chap. 6. Some linearly viscoelastic fluid models are presented in Chap. 7. In Chap. 8 the concept of tensors is utilized and advanced fluid models are introduced. The book is concluded with a variety of 26 problems.

Trondheim, July 2013 Fridtjov Irgens

Contents

Chapter 1
Classification of Fluids

1.1 The Continuum Hypothesis

Matter may take three aggregate forms or phases: solid, liquid, and gaseous. A body of solid matter has a definite volume and a definite form, both dependent on the temperature and the forces that the body is subjected to. A body of liquid matter, called a *liquid*, has a definite volume, but not a definite form. A liquid in a container is formed by the container but does not necessarily fill it. A body of gaseous matter, called a *gas*, fills any container it is poured into.

Matter is made of atoms and molecules. A molecule usually contains many atoms, bound together by *interatomic forces*. The molecules interact through *intermolecular forces*, which in the liquid and gaseous phases are considerably weaker than the interatomic forces.

In the liquid phase the molecular forces are too weak to bind the molecules to definite equilibrium positions in space, but the forces will keep the molecules from departing too far from each other. This explains why volume changes are relatively small for a liquid.

In the gaseous phase the distances between the molecules have become so large that the intermolecular forces play a minor role. The molecules move about each other with high velocities and interact through elastic impacts. The molecules will disperse throughout the vessel containing the gas. The pressure against the vessel walls is a consequence of molecular impacts.

In the solid phase there is no longer a clear distinction between molecules and atoms. In the equilibrium state the atoms vibrate about fixed positions in space. The solid phase is realized in either of two ways: In the *amorphous state* the molecules are not arranged in any definite pattern. In the *crystalline state* the molecules are arranged in rows and planes within certain subspaces called crystals. A crystal may have different physical properties in different directions, and we say that the crystal has *macroscopic structure* and that it has *anisotropic mechanical properties.* Solid matter in crystalline state usually consists of a disordered collection of crystals, denoted grains. The solid matter is then polycrystalline. From a macroscopic point of view polycrystalline materials may have *isotropic*

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mechanical properties, which mean that the mechanical properties are the same in all directions, or may have structure and anisotropic mechanical properties.

Continuum mechanics is a special branch of Physics in which matter, regardless of phase or structure, is treated by the same theory. Special macroscopic properties for solids, liquids and gases are described through *material* or *constitutive equations*. The constitutive equations represent macromechanical models for the real materials. The simplest constitutive equation for a solid material is given by Hooke's law: $\sigma = E\varepsilon$, used to describe the relationship between the axial force N in a cylindrical test specimen in tension or compression and the elongation ΔL of the specimen of length L and cross-sectional area A. The force per unit area of the cross-section is given by the *normal stress* $\sigma = N/A$. The change of length per unit length is represented by the *longitudinal strain* $\varepsilon = \Delta L/L$. The material parameter E is the *modulus of elasticity* of the material.

Continuum Mechanics is based on the *continuum hypothesis*:

> *Matter is continuously distributed throughout the space occupied by the matter. Regardless of how small volume elements the matter is subdivided into, every element will contain matter. The matter may have a finite number of discontinuity surfaces, for instance fracture surfaces or yield surfaces in solids, but material curves that do not intersect such surfaces, retain their continuity during the motion and deformation of the matter.*

The basis for the hypothesis is how we macroscopically experience matter and its macroscopic properties, and furthermore how the physical quantities we use, as for example pressure, temperature, and velocity, are measured macroscopically. Such measurements are performed with instruments that give average values on small volume elements of the material. The probe of the instrument may be small enough to give a local value, i.e., an intensive value, of the property, but always so extensive that it registers the action of a very large number of atoms or molecules.

1.2 Definition of a Fluid

A common property of liquids and gases is that they at rest only can transmit a pressure normal to solid or liquid surfaces bounding the liquid or gas. Tangential forces on such surfaces will first occur when there is relative motion between the liquid or gas and the solid or liquid surface. Such forces are experienced as frictional forces on the surface of bodies moving through air or water. When we study the flow in a river we see that the flow velocity is greatest in the middle of the river and is reduced to zero at the riverbank. The phenomenon is explained by the notion of tangential forces, *shear stresses*, between the water layers that try to slow down the flow.

The volume of an element of flowing liquid is nearly constant. This means that the *density*: mass per unit volume, of a liquid is almost constant. Liquids are therefore usually considered to be incompressible. The compressibility of a liquid, i.e., change in volume and density, comes into play when convection and acoustic phenomena are considered.

Gases are easily compressible, but in many practical applications the compressibility of a gas may be neglected, and we may treat the gas as an incompressible medium. In elementary aerodynamics, for instance, it is customary to treat air as an incompressible matter. The condition for doing that is that the characteristic speed in the flow is less than 1/3 of the speed of sound in air.

Due to the fact that liquids and gases macroscopically behave similarly, the equations of motion and the energy equation for these materials have the same form, and the simplest constitutive models applied are in principle the same for liquids and gases. A common name for these models is therefore of practical interest, and the models are called fluids. A *fluid* is thus a model for a liquid or a gas. *Fluid Mechanics* is the macromechanical theory for the mechanical behavior of liquids and gases. Solid materials may also show fluid-like behavior. Plastic deformation and creep, which is characterized by increasing deformation at constant stress, are both fluid-like behavior. Creep is experienced in steel at high temperatures (>400 °C), but far below the melting temperature. Stones, e.g., granite, may obtain large deformations due to gravity during a long geological time interval. All thermoplastics are, even in solid state, behaving like liquids, and therefore modeled as fluids. In continuum mechanics it is natural to define a fluid on the basis of what is the most characteristic feature for a liquid or a gas. We choose the following definition:

A fluid is a material that deforms continuously when it is subjected to anisotropic states of stress.

Figure 1.1 shows the difference between an *isotropic state of stress* and *anisotropic states of stress*. At an isotropic state of stress in a material point all material surfaces through the point are subjected to the same normal stress, tension or compression, while the shear stresses on the surfaces are zero. At an *anisotropic state of stress* in a material point most material surfaces will experience shear stresses.

As mentioned above, solid material behaves as fluids in certain situations. Constitutive models that do not imply fluid-like behavior will in this book be called *solids*. Continuum mechanics also introduces a third category of constitutive models called *liquid crystals*. However these materials will not be discussed in this book.

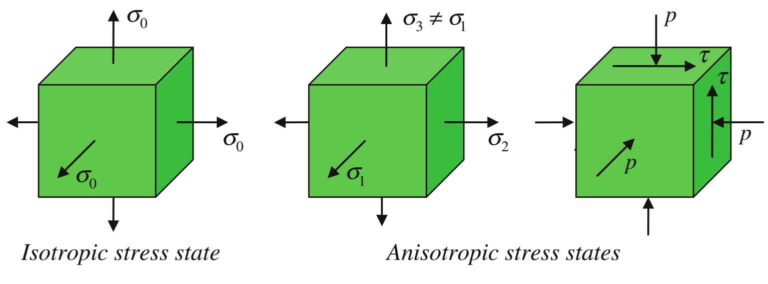


Fig. 1.1 Isotropic state of stress and anisotropic states of stress

1.3 What is Rheology?

The term *rheology* was invented in 1920 by Professor Eugene Bingham at Lafayette College in Indiana USA. He was inspired by a colleague, Martin Reiner, a professor in Classical Languages and History. Bingham, a professor of Chemistry, studied new materials with strange flow behavior, in particular paints. The syllable Rheo is from the Greek word "rhein", meaning flow, so the name rheology was taken to mean *the theory of deformation and flow of matter*. Rheology has also come to include the constitutive theory of highly viscous fluids and solids exhibiting viscoelastic and viscoplastic properties. The term Rheo was inspired by the quotation "ta panta rhei", everything flows, mistakenly attributed to Heraclitus [ca. 500–475 BCE], but actually coming from the writings of Simplicius [490–560 CE].

Newtonian fluids are fluids that obey Newton's linear law of friction, Eq. (1.4.5) below. Fluids that do not follow the linear law are called non-Newtonian. These fluids are usually highly viscous fluids and their elastic properties are also of importance. The theory of *non-Newtonian fluids* is a part of rheology. Typical non-Newtonian fluids are polymer solutions, thermo plastics, drilling fluids, paints, fresh concrete and biological fluids.

1.4 Non-Newtonian Fluids

We shall classify different real fluids in categories according to their most important material properties. In later chapters we shall present fluid models within the different categories. In order to define some simple mechanical properties to be used in the classification, we shall consider the following experiment with different real liquids.

Figure 1.2 shows a *cylinder viscometer*. A cylinder can rotate in a cylindrical container about a vertical axis. The annular space between the two concentric cylindrical surfaces is filled with a liquid. The cylinder is subjected to a torque M and comes in rotation with a constant angular velocity ω. The distance h between the two cylindrical surfaces is so small compared to the radius r of the cylinder that the motion of the liquid may be considered to be like the flow between two parallel plane surfaces, see Fig. 1.3. It may be shown that for moderate ω-values the velocity field is given by:

$$v_x = \frac{v}{h} y, \quad v_y = v_z = 0, \quad v = \omega r \tag{1.4.1}$$

$v_x, v_y,$ and v_z are velocity components in the directions of the axes in a local Cartesian coordinate system $Oxyz$. The term $v = \omega r$ is the velocity of the fluid particle at the wall of the rotating cylinder.

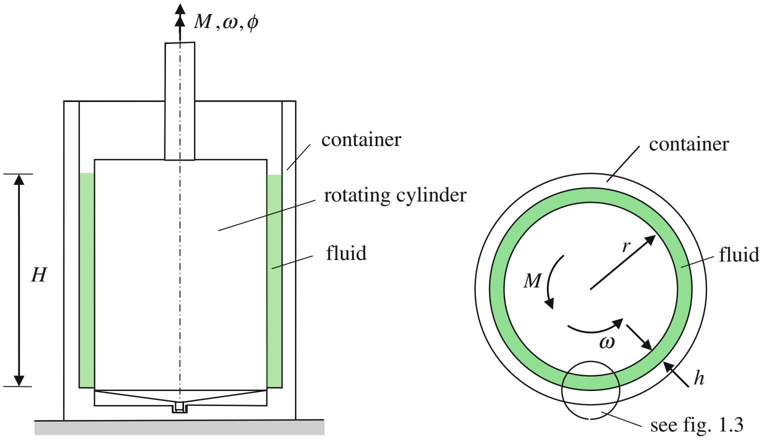


Fig. 1.2 Cylinder viscometer

Fig. 1.3 Simple shear flow

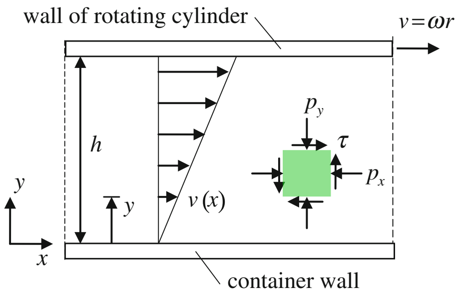
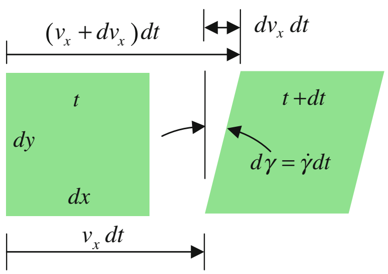


Fig. 1.4 Fluid element from Fig. 1.3



A volume element having edges $dx, dy,$ and dz, see Fig. 1.4, will during a short time interval dt change its form. The change in form is given by the *shear strain* $d\gamma$:

$$d\gamma = \dot{\gamma} dt = \frac{dv_x\, dt}{dy} = \frac{dv_x}{dy} dt = \frac{v}{h} dt = \frac{\omega r}{h} dt$$

The quantity:

$$\dot{\gamma} = \frac{dv_x}{dy} = \frac{v}{h} = \frac{r}{h}\omega \tag{1.4.2}$$

is called the *rate of shear strain*, or for short the *shear rate*. The flow described by the velocity field (1.4.1) and illustrated in Fig. 1.3, is called *simple shear flow*.

The *fluid element* in Fig. 1.4 is subjected to *normal stresses* on all sides and a *shear stress* τ on four sides, see Fig. 1.3. The shear stress may be determined from the balance law of angular momentum applied to the rotating cylinder. For the case of steady flow at constant angular velocity ω the torque M is balanced by the shear stresses τ on the cylindrical wall with area $2\pi rH$. Thus:

$$(\tau r)(2\pi rH) = M \quad \Rightarrow \quad \tau = \frac{M}{2\pi r^2 H} \tag{1.4.3}$$

The viscometer records the relationship between the torque M and the angular velocity ω. Using formulas (1.4.2) for the shear rate $\dot{\gamma}$ and (1.4.3) for the shear stress τ, we obtain a relationship between the shear stress τ and the shear rate $\dot{\gamma}$. We shall now discuss such relationships.

A fluid is said to be *purely viscous* if the shear stress τ is a function only of the shear rate:

$$\tau = \tau(\dot{\gamma}) \tag{1.4.4}$$

An incompressible *Newtonian fluid* is a purely viscous fluid with a linear constitutive equation:

$$\tau = \mu\dot{\gamma} \tag{1.4.5}$$

The coefficient μ is called the *viscosity* of the fluid and has the unit $\text{Ns/m}^2 = \text{Pa} \cdot \text{s}$, *pascal-second*. Alternative units for viscosity are *poise* (P) and *centipoise* (cP):

$$10\ \text{P} = 1000\ \text{cP} = 1\ \text{Pa} \cdot \text{s}. \tag{1.4.6}$$

The unit poise is named after Jean Lois Marie Poiseuille [1797–1869].

The viscosity varies strongly with the temperature and to a certain extent also with the pressure in the fluid. For water $\mu = 1.8 \times 10^{-3}\text{Ns/m}^2$ at 0 °C and $\mu = 1.0 \times 10^{-3}\text{Ns/m}^2$ at 20 °C Usually a highly viscous fluid does not obey the linear law (1.4.5) and belongs to the *non-Newtonian fluids*. However, some highly viscous fluids are Newtonian. Mixing glycerin and water gives a Newtonian fluid with viscosity varying from 1.0×10^{-3} to 1.5Ns/m^2 at 20 °C, depending upon the concentration of glycerin. This fluid is often used in tests comparing the behavior of a Newtonian fluid with that of a non-Newtonian fluid.

For non-Newtonian fluids in simple shear flow a *viscosity function* $\eta(\dot{\gamma})$ is introduced:

$$\eta(\dot{\gamma}) = \frac{\tau}{\dot{\gamma}} \tag{1.4.7}$$

The viscosity function is also called the *apparent viscosity*. The constitutive equation (1.4.4) may now be rewritten to:

$$\tau = \eta(\dot{\gamma})\dot{\gamma} \tag{1.4.8}$$

The most commonly used model for the viscosity function is given by the *power law*:

$$\eta(\dot{\gamma}) = K|\dot{\gamma}|^{n-1} \tag{1.4.9}$$

The *consistency parameter K* and the *power law index n* are both functions of the temperature. Note that the power law function (1.4.9) gives:

$$\begin{aligned} \eta_0 &\equiv \eta(0) = \infty \text{ for } n<1, \quad \text{and} \quad = 0 \text{ for } n>1 \\ \eta_\infty &\equiv \eta(\infty) = 0 \text{ for } n<1, \quad \text{and} \quad = \infty \text{ for } n>1 \end{aligned} \tag{1.4.10}$$

This is contrary to what is found in experiments with non-Newtonian fluids, which always give:

$$\eta_0 \equiv \eta(0) = \text{finite value } > 0, \quad \eta_\infty \equiv \eta(\infty) = \text{ finite value} > 0 \tag{1.4.11}$$

The parameters η_0 and η_∞ are called *zero-shear-rate-viscosity* and *infinite shear-rate- viscosity* respectively. The power law is the basic constitutive equation for the *power law fluid model* presented in Sect. 6.1. Table 1.1 presents some examples of K- and n-values.

In order to include elastic properties in the description of mechanical behavior of real fluids we may first imagine that the test fluid in the container solidifies. The torque M will not manage to maintain a constant angular velocity ω, but the cylinder will rotate an angle ϕ. Material particles at the rotating cylindrical wall will approximately obtain a rectilinear displacement $u = \phi r$. The volume element in Fig. 1.4 will be sheared and get a shear strain:

$$\gamma = \frac{u}{h} = \frac{r}{h}\phi \tag{1.4.12}$$

Table 1.1 Consistency parameter K and power law index n for some fluids

Fluid	Region for $\dot{\gamma}$ [s^{-1}]	K [Ns^n/m^2]	n
54.3 % cement rock in water, 300 °K	10–200	2.51	0.153
23.3 % Illinois clay in water, 300 °K	1800–6000	5.55	0.229
Polystyrene, 422 °K	0.03–3	1.6×10^5	0.4
Tomato Concentrate, 90 °F 30 % solids		18.7	0.4
Applesauce, 80 °F 11,6 % solids		12.7	0.4
Banana puree, 68 °F		6.89	0.28

A material is said to be *purely elastic* if the shear stress is only a function of the shear strain and independent of the shear strain rate, i.e.:

$$\tau = \tau(\gamma) \tag{1.4.13}$$

For a *linearly elastic material*:

$$\tau = G\gamma \tag{1.4.14}$$

where G is the *shear modulus*.

For many real materials, both liquids and solids, the shear stress may be dependent both upon the shear strain and the shear strain rate. These materials are called *viscoelastic*. The relevant constitutive equation may take the simple form:

$$\tau = \tau(\gamma, \dot{\gamma}) \tag{1.4.15}$$

But usually we have to apply more complex functional relationships, which take into consideration the *deformation:history* of the material. We shall see examples of such relationships below.

Fluid models may be classified into three main groups:

A. Time independent fluids
B. Time dependent fluids
C. Viscoelastic fluids

We shall briefly discuss some important features of the different groups. In the Chaps. 6–8 general constitutive equations for some of these materials will be presented.

1.4.1 Time Independent Fluids

This group may further be divided into two subgroups

A1. Viscoplastic fluids
A2. Purely viscous fluids

Figure 1.5 shows characteristic graphs of the function $\tau(\dot{\gamma})$ for viscoplastic materials. The material models are solids when the shear stress is less than the *yield shear stress* τ_y, and the behavior is elastic. For $\tau > \tau_y$ the material models are fluids. When the material is treated as a fluid, it is generally assumed that the fluid is incompressible and that the material is rigid, without any deformations, when $\tau < \tau_y$. The simplest viscoplastic fluid model is the *Bingham fluid*, named after Professor Bingham, the inventor of the name Rheology. The model behaves like a Newtonian fluid when it flows, and the constitutive equation in simple shear is:

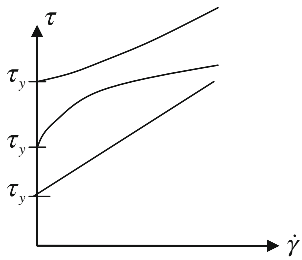


Fig. 1.5 Viscoplastic fluids

$$\tau(\dot{\gamma}) = \left[\mu + \frac{\tau_y}{|\dot{\gamma}|}\right]\dot{\gamma} \text{ when } \dot{\gamma} \neq 0, \quad \tau(\dot{\gamma}) \leq \tau_y \text{ when } \dot{\gamma} = 0 \tag{1.4.16}$$

Examples of fluids exhibiting a yield shear stress are: drilling fluids, sand in water, granular materials, margarine, toothpaste, some paints, some polymer melts, and fresh concrete. General constitutive equations for the Bingham fluid model are presented in Sect. 6.1.

The velocity profile of the flow of a viscoplastic fluid in a tube is shown in Fig. 1.6. The flow is driven by a pressure gradient. The central part of the flowing fluid has a uniform velocity and flows like a plug. When toothpaste is squeezed from a toothpaste tube, a *plug-flow* is clearly observed.

Purely viscous fluids have the constitutive equation (1.4.4) or (1.4.8) in simple shear flow. A purely viscous fluid is said to be shear-thinning or *pseudoplastic* if the viscosity expressed by the viscosity function (1.4.7) decreases with increasing shear rate, see Figs. 1.7 and 1.8. Most real non-Newtonian fluids are *shear-thinning fluids*. Examples: nearly all polymer melts, polymer solutions, biological fluids, and mayonnaise. The word "pseudoplastic" relates to the fact the viscosity function of a shear-thinning fluid has somewhat the same character as for the viscoplastic fluid models, compare Figs. 1.5 and 1.7. The power-law (1.4.9) describes the shear-thinning fluid when $n < 1$.

For a relatively small group of real liquids "the apparent viscosity" $\tau/\dot{\gamma}$ increases with increasing shear rate. These fluids are called *shear-thickening fluids* or *dilatant fluids* (expanding fluids). The last name reflects that these fluids often increase their volume when they are subjected to shear stresses. While the two effects are phenomenological quite different, a fluid with one of the effects also usually has the other. The power law (1.4.9) represents a shear-thickening fluid for $n > 1$.

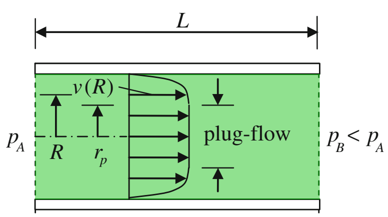


Fig. 1.6 Plug-flow in a tube

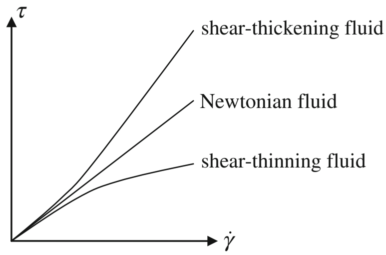


Fig. 1.7 Purely viscous fluids

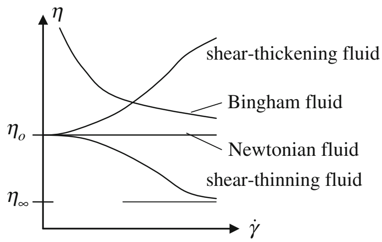


Fig. 1.8 Constant shear rate test

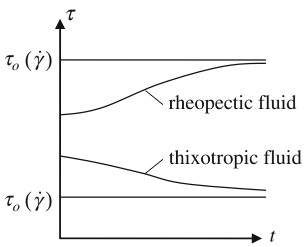


Fig. 1.9 The viscosity function $\eta(\dot{\gamma})$

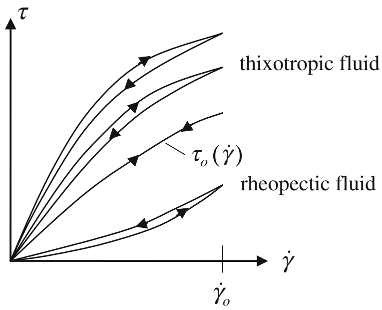
1.4.2 Time Dependent Fluids

These fluids are very difficult to model. Their behavior is such that for a constant shear rate $\dot{\gamma}$ and at constant temperature the shear stress τ either increases or decreases monotonically with respect to time, towards an asymptotic value $\tau(\dot{\gamma})$, see Fig. 1.9. The fluids regain their initial properties some time after the shear rate has returned to zero. The time dependent fluids are divided into two subgroups:

B1. *Thixotropic fluids*: At a constant shear rate the shear stress decreases monotonically.
B2. *Rheopectic fluids*: At a constant shear rate the shear stress increases monotonically. These fluids are also called *antithixotropic fluids*.

There is another fascinating feature with these fluids. When a thixotropic fluid is subjected to a shear rate history from $\dot{\gamma} = 0$ to a value $\dot{\gamma}_0$ and back to $\dot{\gamma} = 0$, the graph for the shear stress τ as a function of $\dot{\gamma}$ shows a *hysteresis loop*, see Fig. 1.10.

Fig. 1.10 Shear rate histories



For repeated shear rate histories the hysteresis loops get less steep and slimmer, and they eventually approach the graph for $\tau_o(\dot{\gamma})$. Examples of thixotropic fluids are: drilling fluids, grease, printing ink, margarine, and some polymer melts. Some paints exhibit both viscoplastic and thixotropic response. They have gel consistency and become liquefied by stirring, but they regain their gel consistency after some time at rest. Also for rheopectic fluids we will see hysteresis loops when the fluids are exposed to shear rate histories, see Fig. 1.10. Relatively few real fluids are rheopectic. Gypsum paste gives an example.

1.4.3 Viscoelastic Fluids

When an undeformed material, solid or fluid, is suddenly subjected to a state of stress history, it deforms. An instantaneous deformation is either elastic, or elastic and plastic. The initial elastic deformation disappears when the stress is removed, while the plastic deformation remains as a permanent deformation. If the material is kept in a state of constant stress, it may continue to deform, indefinitely if it is a fluid, or asymptotically towards a finite configuration if it is a solid. This phenomenon is called *creep*. When a material is suddenly deformed and kept in a fixed deformed state, the stresses may be constant if the material behaves elastically, but the stress may also decrease with respect to time either toward an isotropic state of stress if the material is fluid-like or toward an asymptotic limit anisotropic state of stress if the material is solid-like, This phenomenon is called *stress relaxation*. Creep and stress relaxation are due to a combination of an elastic response and internal friction or viscous response in the material, and are therefore called viscoelastic phenomena. If the material exhibits creep and stress relaxation, it is said to behave viscoelastically. When the material is subjected to dynamic loading, viscoelastic properties are responsible for damping and energy dissipation.

Propagation of sound in liquids and gases is an elastic response. Fluids are therefore in general both viscous and elastic, and the response is *viscoelastic*. However, the elastic deformations are very small compared to the viscous deformations.

Many solid materials that under "normal" temperatures may be considered purely elastic, will at higher temperatures respond viscoelastically. It is customary to introduce a *critical temperature* Θ_c for these materials, such that the material is considered to be viscoelastic at temperatures $\Theta > \Theta_c$. For example, for common structural steel the critical temperature $\Theta_c$is approximately 400 °C. For plastics a *glass transition temperature* Θ_c is introduced. At temperatures below the glass transition temperature the materials behave elastically, more or less like brittle glass. Established plastic materials have Θ_c-values from −120 to +120 °C. Some plastics behave viscoelastically within a certain temperature interval: $\Theta_g < \Theta < \Theta'$. For temperatures $\Theta < \Theta_g$ and $\Theta > \Theta'$ these materials are purely elastic. Vulcanized rubber is an example of such a material.

In order to expose the most characteristic properties of real viscoelastic materials, we shall now discuss typical results from tests in which the material, liquid or solid, is subjected to simple shear. The test may be performed with the cylinder viscometer presented in Fig. 1.2.

In a *creep test* a constant torque M_0 is introduced, and the angle of rotation as a function of the torque and of the time t, i.e., $\phi = \phi(M_0, t)$, is recorded. The resulting shear stressτ_0 is found from equation (1.4.3) as:

$$\tau_0 = \frac{M_0}{2\pi r^2 H} \tag{1.4.17}$$

The shear strain$\gamma(\tau_0, t)$is found from equation (1.4.12) as:

$$\gamma(\tau_0, t) = \frac{r}{h}\phi(M_0, t) \tag{1.4.18}$$

Figure 1.11 shows the result of a creep test. The diagram may be divided into the following regions (Fig. 1.12):

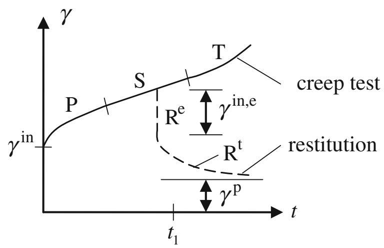
I: *Initial shear strain* $\gamma^{in} = \gamma^{in,e} + \gamma^{in,p}$. Almost instantaneously the material gets an initial shear strain which may be purely elastic or contain an elastic part $\gamma^{in,e}$and a plastic part $\gamma^{in,p}$.

P: *Primary creep*. The time rate of shear strain $\dot{\gamma} = d\gamma/dt$ is at first relatively high, but decreases towards a stationary value.

S: *Secondary creep*. The rate of strain $\dot{\gamma} = d\gamma/dt$ is constant.

T: *Tertiary creep*. If the material is under constant shear stress for a long period of time, the rate of shear strain $\dot{\gamma} = d\gamma/dt$ may start to increase.

Fig. 1.11 Creep test in shear $\gamma(\tau_0, t)$ and restitution after unloading



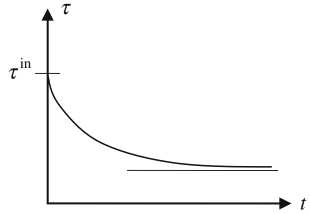


Fig. 1.12 Relaxation test in shear $\tau(\gamma_0, t)$

The diagram in Fig. 1.11 also shows a *restitution* of the material after the torque M_0 has been removed at the time t_1 :

R^e: *Elastic restitution*. At sudden unloading by removing the torque M_0, the initial elastic shear strain $\gamma^{in,e}$ disappears momentarily

R^t: *Time dependent restitution* also called *elastic after-effect*

After the restitution is completed, in principle it may take infinitely long time, the material has got a permanent or plastic shear strain γ^p. The different regions described above are more or less prominent for different materials. Tests show that an increase in the stress level or of the temperature will lead to increasing shear strain rates in all the creep regions.

In a stress relaxation experiment with the cylinder viscometer a constant angle of rotation ϕ_0 is introduced, and the resulting torque M as a function of the constant angle ϕ_0 and of the time t is recorded, i.e., $M = M(\phi_0, t)$. The angle of rotation results in a constant shear strain γ_0, and the torque gives a shear stress as a function of the shear strain γ_0 and of time:$\tau = \tau(\gamma_0, t)$, with an initial value τ^{in}. The shear stress $\tau = \tau(\gamma_0, t)$ decreases with time asymptotically towards a value, which for a fluid is zero.

A viscoelastic material may be classified as a solid or a fluid, see Fig. 1.13. The creep diagram for a *viscoelastic solid* will exhibit elastic initial strain, primary creep, and complete restitution without plastic strain. The primary creep will after sufficiently long time reach an "elastic ceiling", which is given by the *equilibrium shear strain* $\gamma_e(\sigma_0)$. In a relaxation test of a viscoelastic solid the stress decreases towards an *equilibrium shear stress* $\tau_e(\gamma_0)$. The creep diagram for a *viscoelastic liquid* may exhibit all the regions mentioned in connection with Fig. 1.11. The relaxation graph of a viscoelastic fluid approaches the zero stress level asymptotically. For comparison Fig. 1.13 also presents the response curves for an elastic material and a purely viscous material, for example a Newtonian fluid.

In a creep test the constant shear stress τ_0 may be described by the function:

$$\tau(t) = \tau_0 H(t) \tag{1.4.19}$$

where *H(t)* is the *Heaviside unit step function*, Oliver Heaviside [1850–1925]:

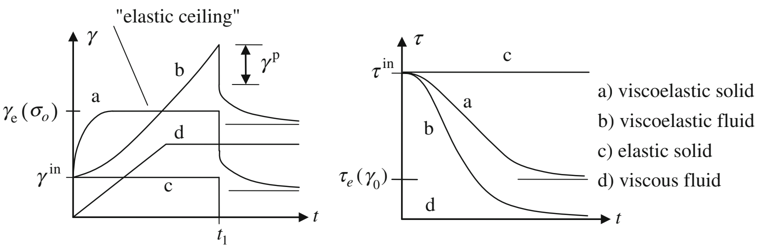


Fig. 1.13 Solid and fluid response in creep and relaxation

$$H(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases} \tag{1.4.20}$$

The result of the creep test may be described by a *creep function in shear* $\alpha(\tau_0, t)$, such that the shear strain becomes:

$$\gamma(\tau_0, t) = \alpha(\tau_0, t)\, \tau_0\, H(t) \tag{1.4.21}$$

In a relaxation test the material is subjected to a sudden shear strain γ_0 such that:

$$\gamma(t) = \gamma_0\, H(t) \tag{1.4.22}$$

The shear strain results in a shear stress:

$$\tau(\gamma_0, t) = \beta(\gamma_0, t)\, \gamma_0\, H(t) \tag{1.4.23}$$

$\beta(\gamma_0, t)$ is called the *relaxation function in shear*. The functions $\alpha(\tau_0, t)$ and $\beta(\gamma_0, t)$are temperature dependent, but for convenience the temperature dependence is not indicated here.

If the creep test and the relaxation test of a material indicate that it is reasonably to present the creep function and the relaxation function as independent of the shear strain:

$$\alpha = \alpha(t), \quad \beta = \beta(t) \tag{1.4.24}$$

we say that the material shows *linearly viscoelastic response*. A linearly viscoelastic material model may be used as a first approximation in many cases.

The instantaneous response of a linearly viscoelastic material is given by the *glass compliance* $\alpha_g = \alpha(0)$and the *glass modulus*, also called the *short time modulus*, $\beta_g = \beta(0)$.

The parameters $\alpha_e \equiv \alpha(\infty)$ and $\beta_e \equiv \beta(\infty)$ are called respectively the *equilibrium compliance* and the *equilibrium modulus* or the *long time modulus*. For a viscoelastic fluid $\alpha_e \equiv \infty$ and $\beta_e \equiv 0$.

The parameters α_g, α_e, β_g, and β_e are all temperature dependent. Because it is the same whether we set $\sigma(0^+) = \beta_g\,\varepsilon_0$ or $\varepsilon(0^+) = \alpha_g\,\sigma_0$ and $\sigma(\infty)=\beta_e\,\varepsilon_0$ or $\varepsilon(\infty) = \alpha_e\,\sigma_0$,we have the result:

$$\alpha_g\beta_g = 1, \quad \alpha_e\beta_e = 1 \tag{1.4.25}$$

Tests with multiaxial states of stress show that viscoelastic response is primary a shear stress-shear strain effect. Very often materials subjected to isotropic stress will deform elastically. This fact agrees well with common conception that there is a close micro-mechanical correspondence between viscous and plastic deformation, and that plastic deformation is approximately volume preserving. Thus, general stress–strain relationships may be obtained by combining shear stress tests and tests with isotropic states of stress.

It will be demonstrated in Chap. 7 that the response of a linearly viscoelastic fluid in simple shear flow may be represented by the constitutive equation:

$$\tau(t) = \int_{-\infty}^{t} \beta(t-\bar{t})\,\dot{\gamma}(\bar{t})d\bar{t} \tag{1.4.26}$$

The function $\dot{\gamma}(\bar{t})$ for $-\infty < \bar{t} \leq t$ is the *rate of shear strain history* that the fluid has experienced up to the present time t.

The *Maxwell fluid*, James Clerk Maxwell [1813–1879], is a constitutive model of a linearly viscoelastic fluid. The *response equation* for simple shear flow is:

$$\frac{\tau}{\mu} + \frac{\dot{\tau}}{G} = \dot{\gamma} \tag{1.4.27}$$

μ is a viscosity and G is a shear modulus. The response equation is obtained by assuming that the total rate of shear strain rate $\dot{\gamma}$ is a sum of a viscous contribution $\dot{\gamma}_v = \tau/\mu$ and an elastic part $\dot{\gamma}_e = \dot{\tau}/G$. The Eq. (1.4.27) may be rewritten to:

$$\tau + \lambda\dot{\tau} = \mu\dot{\gamma} \tag{1.4.28}$$

The parameter $\lambda = \mu/G$ is called the *relaxation time*. It will be shown in Chap. 7 that the creep function and the relaxation function for the Maxwell fluid are:

$$\alpha(t) = \frac{1}{G}\left[1 + \frac{t}{\lambda}\right], \quad \beta(t) = G\exp(-t/\lambda) \tag{1.4.29}$$

The functions $\alpha(t)$ and β(t) are derived from the response equation (1.4.28). Figure 1.14 shows the results of a creep test and a relaxation test on a Maxwell fluid. The relaxation time λ is illustrated in the figure.

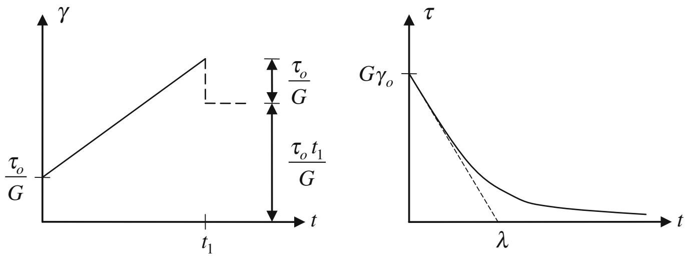


Fig. 1.14 Creep and relaxation in simple shear of a Maxwell fluid

1.4.4 The Deborah Number

In order to characterize the intrinsic fluidity of a material or how "fluid-like" the material is, a number *De* called the *Deborah number* has been introduced. The number is defined as:

$$De = \frac{t_c}{t_p}$$

$$t_c = \text{stress relaxation time, e.g. } \lambda \text{ in Fig. 1.14}$$

$$t_p = \text{characteristic time scale in a flow, an experiment, or a computer simulation} \qquad (1.4.30)$$

A small Deborah number characterizes a material with fluid-like behavior, while a large Deborah number indicates a material with solid-like behavior.

Professor Markus Reiner coined the name for Deborah number. Deborah was a judge and prophetess mentioned in the Old Testament of the Bible (Judges 5:5). The following line appears in a song attributed to Deborah: "The mountain flowed before the Lord".

1.4.5 Closure

In general any equation relating stresses to different measures of deformation is called a *constitutive equation*. Both Eq. (1.4.26) and Eq. (1.4.28) are constitutive equations. However, it is convenient to call the special differential form that relates stresses and stress rates to strains, strain rates, and other deformation measures a *response equation*. Equation (1.4.28) is an example of a response equation.

In this chapter we have classified real liquids in fluid categories according to their response in simple shear flow. Furthermore, we have for simplicity only discussed the relationship between shear stress, shear strain, and shear strain rate. In the Chaps. 5–8 we shall also discuss normal stress response and the effect of other measures of deformation.

Chapter 2
Flow Phenomena

The purpose of this chapter is to present some examples of flows in which there are significant differences between the behavior of Newtonian fluids and non-Newtonian fluids. In the figures to follow the Newtonian fluid is indicated with an "N" and the non-Newtonian fluid is marked with "nN". These examples and some others are discussed in greater details in the book "Dynamics of Polymeric Liquids", vol 1. Fluid Mechanics, by Bird, Armstrong and Hassager [3].

2.1 The Effect of Shear Thinning in Tube Flow

Figure 2.1 shows two vertical tubes, one filled with a Newtonian fluid (N) of *viscosity* μ, and the other filled with a *shear-thinning fluid* (nN) with a *viscosity function* $\eta(\dot{\gamma})$. The tubes are open at the top but closed with a plate at the bottom. The two fluids are chosen to have the same density and such that they have approximately the same viscosity at low shear rates: $\eta(\dot{\gamma}) \approx \mu$ for small $\dot{\gamma}$. For example, the situation may be realized by using a glycerin-water solution as the Newtonian fluid and then adjust the viscosity by changing the glycerin content until two small identical spherical balls fall with the same velocity through the tubes, Fig. 2.1a.

Figure 2.1b indicates what happens after the plate has been removed. The tubes are emptied, but the shear-thinning fluid accelerates to higher velocities than the Newtonian fluid. At the relatively high shear rates $\dot{\gamma}$ that develop near the tube wall, the *apparent viscosity* $\eta(\dot{\gamma})$ is smaller than the constant viscosity μ of the Newtonian fluid, i.e., $\eta(\dot{\gamma}) < \mu$. The shear stress from the wall that counteracts the driving force of gravity is therefore smaller in the shear thinning fluid, leading to higher accelerations. The shear-thinning fluid leaves the tube faster than the Newtonian fluid.

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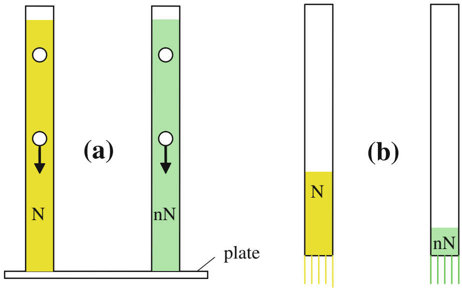

Fig. 2.1 **a** Falling spheres in a Newtonian fluid (N) and a shear-thinning fluid (nN). **b** Tube flow of the two fluids

2.2 Rod Climbing

Figure 2.2 illustrates two containers with fluids and with a vertical rod rotating at a constant angular velocity. The container in Fig. 2.2a is filled with a Newtonian fluid (N). The fluid sticks to the container wall and to the surface of the rod, and the fluid particles obtain a circular motion about the rod. Due to centrifugal effects the free surface of the fluid shows a depression near the rod. The container in Fig. 2.2b is filled with a non-Newtonian viscoelastic fluid (nN). This fluid will start to climb the rod until an equilibrium condition has been established. The phenomenon is explained as a consequence of tensile stresses in the circumferential direction that develop due to the shear strains in the fluid. The tensile stresses counteract the centrifugal forces and squeeze the fluid towards the rod and up the rod. Long, thread-like molecular structures are stretched in the directions of the circular stream lines and thus create the tensile stresses. The phenomenon may be observed in a food processor when mixing waffle dough.

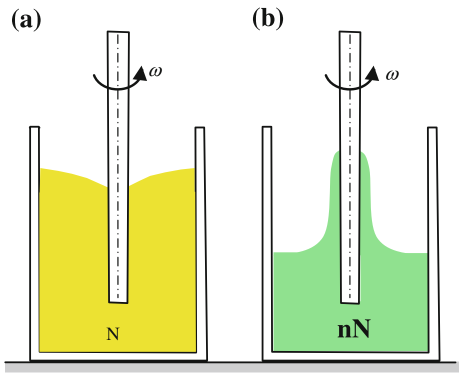

Fig. 2.2 Rod climbing

2.3 Axial Annular Flow

We investigate the axial laminar flow of fluid in the *annular space* between two concentric circular cylindrical surfaces, Fig. 2.3. The pressure is measured at a point A at the inner surface and at a point B at the outer surface in the same cross section of the container. Measurements then show that the two pressures are the same when the fluid in Newtonian, while for a non-Newtonian fluid a small pressure difference is observed. The general result of this experiment is:

$$p_A = p_B \text{ for Newtonian fluids,} \quad p_A > p_B \text{ for non - Newtonian fluids} \tag{2.1}$$

The measured pressure in this experiment is the difference between the *thermodynamic pressure* p in a compressible fluid, or any undetermined isotropic pressure p in an incompressible fluid, and the *viscous normal stress* τ_{RR} in the radial direction. In Chap. 3 the difference between the pressure p and the pressure $(p - \tau_{RR})$ will be discussed in detail.

2.4 Extrudate Swell

A highly viscous fluid flows under pressure from a large reservoir and is extruded through a tube of diameter d and length L (Fig. 2.4). The extruded fluid exiting the tube swells and obtains a diameter d_e that is larger than the inner diameter d of the tube. A 200 % increase in diameter is reported in tests. The ratio d_e/d is decreasing with increasing length L of the tube. A comparable Newtonian fluid, with viscosity μ and density ρ, will under similar conditions not exhibit any immediate change in diameter, i.e.: $d_e/d = 1$. For high *Reynolds numbers*, $\text{Re} = \rho v d/\mu$, where v is the mean velocity in the tube, it may be shown that d_e is somewhat less then d. This latter effect is of course is due to gravity.

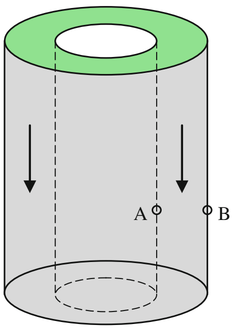

Fig. 2.3 Axial annular flow

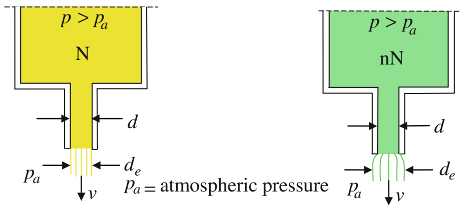

Fig. 2.4 Swelling at extrusion. *Newtonian fluid* N. *Non-Newtonian fluid* nN

The swelling phenomenon may be explained based on two effects:

(1) The non-Newtonian fluid is compressed elastically in the radial direction upon entering the relative narrow tube. In the tube the fluid is responding by expanding in the axial direction, while after leaving the tube the fluid is restituting by expanding radially. The fluid has a kind of memory of the deformation history it has experienced in passing into the tube, but this memory is fading with time. The longer back in time a deformation was introduced, the less of it is remembered. The fluid is said to possess *a fading memory*. The longer the tube is, the lesser will the restitution effect have for the swelling phenomenon.

(2) The shear strains introduced during the tube flow introduce elastic tensile stresses in the axial direction. We may imagine that these tensile stresses are due to long molecular structures in the fluid that are stretched elastically in the direction of the flow. Upon leaving the tube the fluid seeks to restitute itself in the axial direction. Due to the near incompressibility of the fluid, it will then swell in the radial direction.

2.5 Secondary Flow in a Plate/Cylinder System

Figure 2.5 illustrates a circular plate rotating on the surface of a fluid in a cylindrical container. The motion of the plate introduces a primary flow in the fluid in which the fluid particles move in circular paths. The particles closer to the plate move faster than the particles nearer the bottom of the cylinder. The effect of centrifugal forces therefore increases with the distance from the bottom. In a Newtonian fluid this effect introduces a *secondary flow* normal to the primary flow, as shown in Fig. 2.5a.

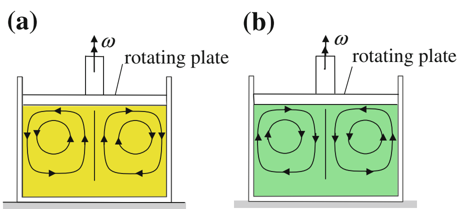

Fig. 2.5 Secondary flow in a plate/cylinder system. **a** Newtonain fluid. **b** non-Newtonain fluid

In a non-Newtonian fluid the secondary flow may be opposite of that in the Newtonian fluid, as shown in Fig. 2.5b. This phenomenon is a consequence of the tangential tensile stresses introduced by the primary flow, and related to the rod climbing phenomenon. The tensile stresses increase with the distance from the bottom of the container and counteract the centrifugal forces.

2.6 Restitution

Figure 2.6 shows a tube with a *visco- elastic fluid*. In Fig. 2.6a the fluid is at rest, the pressures at the ends of the tube are the same: $p_B = p_A$. Using a colored fluid (black) a material diametrical line is marked in the fluid. The pressure p_A is increased and flow starts. The black material line deforms as shown in Fig. 2.6b. The pressure p_A is then reduced to p_B. The flow is retarded, the fluid comes to rest, and then starts to move for a short while in the opposite direction. The black material line is seen to retract as the fluid is somewhat restituted, Fig. 2.6c.

The same phenomenon may be observed when a fluid is set in rotation in a container at rest. The fluid sticks to the container wall and bottom, and the flow of the fluid is slowed down. Eventually the fluid comes to rest and then starts to rotate slightly in the opposite direction. In this case the fluid motion and the restitution may be observed by introducing air bubbles into the fluid and study their motion. The bubbles will move in circles, stop, and then start to move in the reverse direction.

2.7 Tubeless Siphon

Figure 2.7a illustrates a vessel with fluid and a tube bent into a siphon. If the fluid is Newtonian the flow through the tube will stop as soon has the siphon has been lifted up such that the end of the tube stuck into fluid in the container has left the

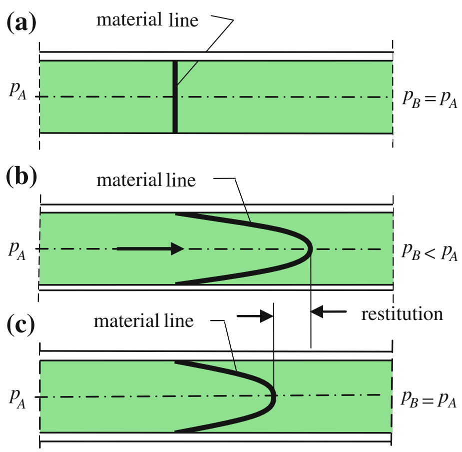

Fig. 2.6 Restitution in a viscoelastic fluid

surface of the fluid. A highly viscoelastic fluid, however, will continue to flow even after the tube end has left the fluid surface. It is also possible to empty the container without the siphon if the container is tilted to let the fluid start to flow over the edge. The elasticity of the fluid will then continue to lift the fluid up to the edge and over it. This is illustrated in Fig. 2.7b. Another way of starting the flow is to use a finger to draw the fluid up and over the edge.

2.8 Flow Through a Contraction

A low Reynolds number flow of a Newtonian fluid through a tube contraction, as illustrated in Fig. 2.8a, will have stream lines that all go from the region with the larger diameter to the region with smaller diameter. A non-Newtonian fluid may have stream lines as shown in Fig. 2.8b. Large eddies are formed and instabilities may occur, with the result that the main flow starts to oscillate back and forth across the axis of the tube.

2.9 Reduction of Drag in Turbulent Flow

Small amounts of polymer resolved in a Newtonian fluid in turbulent flow may reduce the shear stress at solid boundary surfaces dramatically. Figure 2.9 shows results from tests with pipe flow of water. The parameter f is called the *Fanning friction number* and is defined by:

$$f = \frac{1}{4}\frac{D}{L}\frac{\Delta p}{\rho v^2/2} \tag{2.2}$$

D = pipe diameter, Δp = the pressure difference over a pipe length L, and v = the mean velocity in the pipe. The amounts of polymer, given in parts per million [ppm] by weight, are added to the water. The curves show that the drag reduction occurs in the turbulent regime. For the *Reynolds number* $\mathrm{Re} = \rho v D/\mu = 10^5$, where ρ is the density and μ is the viscosity of water, and a polymer concentration of 5 ppm, the Fanning number f is reduced by 40 %. The viscosity in

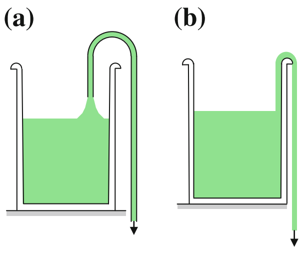

Fig. 2.7 Tubeless siphon. Non-Newtonian fluid

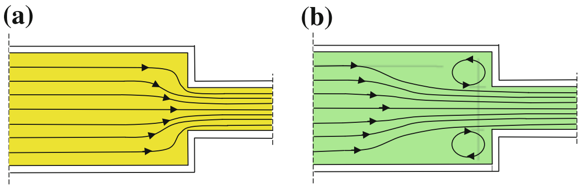

Fig. 2.8 Flos through a contraction. **a** Newtonain fluid. **b** non-Newtonain fluid

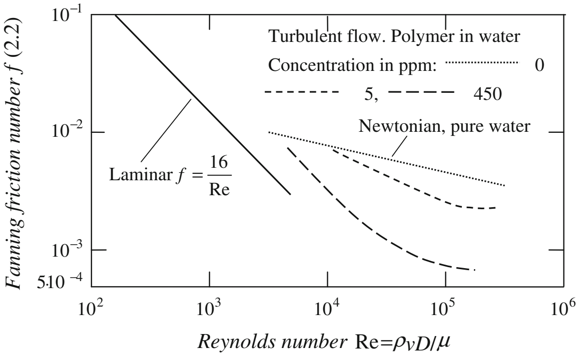

Fig. 2.9 The Fanning friction number for pipe flow

the fluid mixture is changed only slightly. For the present example the viscosity μ is only increased by 1 % relative to that of water. The reason why very small amounts of polymer additives to a Newtonian fluid like water have such a large effect on drag, is not completely understood. What is known is that the effects of different types of polymers are very different. Polymers having long unbranched molecules and low molecular weight give the greatest drag reduction.

The applications of drag reduction using polymer additives are many. One example is in long distance transport of oil in pipes.

Figure 2.9 is adapted from Fig. 3.11-1 in Bird et al. [3]. The curves are based on original data from P.S.Virk, Sc.D. Thesis. Massachusetts Institute of Technology, 1961.

Chapter 3
Basic Equations in Fluid Mechanics

3.1 Kinematics

A portion of fluid is called a *body*. The body has at any time t a volume V and a surface A. A material point in the body is called a *particle*. In order to localize particles and be able to describe their motions, we introduce a *reference frame*, for short called a *reference Rf*, and a *Cartesian coordinate system Ox*, fixed in the reference. See Fig. 3.1. The point O is the origin of the coordinate system and x indicates the axes $x_1, x_2,$ and x_3, which alternatively will be denoted x, y, and z. While the notation x_i for $i = 1, 2,$ or 3 will be used in the general presentation and development of the theory, the notation x, y, and z will be more convenient in applications to special examples.

A *place* in the three-dimension physical space is localized by three coordinate values x_i for i = 1, 2, and 3. We introduce the conventions:

$$place = (x_1, x_2, x_3) \equiv x_i \equiv x \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \equiv \{x_1\, x_2\, x_3\} \tag{3.1.1}$$

In the last two representations x is a *vector matrix*. By the representations (3.1.1) we mean that x_i and x may represent all three coordinates collectively. However, the symbol x_i may also indicate anyone of the three coordinates x_1, x_2, or x_3. We shall use the expression "*the place x*".

At an arbitrarily chosen *reference time* t_0 the place of a particle in the fluid body is given by set of coordinates X_i. We choose to attach the coordinate set X to the particle and use it as an identification of the particle. Thus:

$$particle = (X_1,\, X_2,\, X_3) \equiv X_i \equiv X \equiv \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \equiv \{X_1\, X_2\, X_3\} \tag{3.1.2}$$

We shall use the expression "*the particle X*". The set of places X that represents the body at the reference time t_o, is called the *reference configuration* K_0 of the body.

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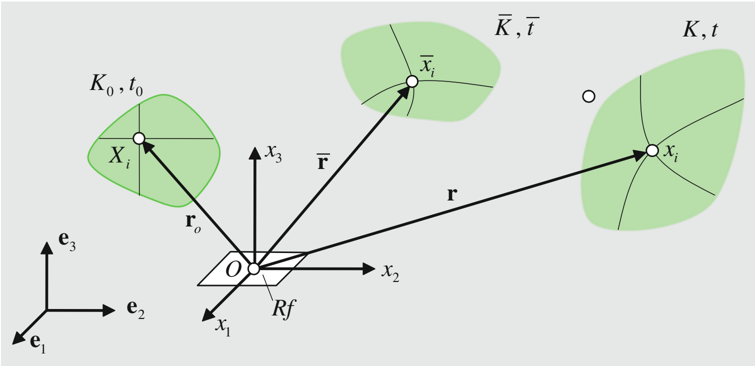


Fig. 3.1 Reference Rf, coordinate system Ox, configurations $K_o, \bar{K}$, and K, and base vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$

The fluid is observed or investigated at the *present time t*. At that time the particle X has moved to the place x. The set of places x that represents the body at the present time t, is called the *present configuration K* of the body. Every particle X, i.e. a place in K_0, has a place x in K, and to every place in x at the time t corresponds one and only one particle X in K_0. Thus the following relationships exist:

$$x_i = x_i(X_1, X_2, X_3, t) \equiv x_i(X, t) \quad \Leftrightarrow \quad X_i = X_i(x_1, x_2, x_3, t) \equiv X_i(x, t) \qquad (3.1.3)$$

These relationships represent a one-to-one mapping between the particles X_i in K_0 and the places x in K. The functions $x_i(X, t)$ represent the *motion* of the fluid.

The motion of the fluid body from K_0 to K will in general lead to a deformation of the body. Material lines, surfaces and volume elements may change form and size during the motion. The deformation is illustrated in Fig. 3.1 by material lines, which in K_0 are parallel to the coordinate axes.

In the motion of fluids the deformations are usually very large and it is only possible to compare the present configuration K with neighbor configurations a short time before or after the present timer t. It is therefore convenient to choose K as reference configuration. Since the reference configuration K changes with time, it is now called a *relative reference configuration*. A *current configuration* $\bar{K}$ at time $\bar{t}$, where $-\infty < \bar{t} \leq t$, is then used to describe the deformation process of the fluid before the present time t. See Fig. 3.1. The place of the particle X at the current time $\bar{t}$ is given by the coordinates $\bar{x}_i$.

The particle X and the places x and $\bar{x}$ are also represented by the position or place vectors $\mathbf{r}_0$, $\mathbf{r}$, and $\bar{\mathbf{r}}$. The unit vectors $\mathbf{e}_i$ in the direction of the x_i − axes are called the *base vectors of the coordinate system Ox*, Fig. 3.1. Then:

$$\mathbf{r}_0 = X_1\,\mathbf{e}_1 + X_2\,\mathbf{e}_2 + X_3\,\mathbf{e}_3 = \sum_{i=1}^{3} X_i\,\mathbf{e}_i, \quad \mathbf{r} = \sum_{i=1}^{3} x_i\,\mathbf{e}_i, \quad \bar{\mathbf{r}} = \sum_{i=1}^{3} \bar{x}_i\,\mathbf{e}_i \qquad (3.1.4)$$

We introduce the *Einstein summation convention*, Albert Einstein [1879–1955]:

A letter index repeated once and only once in a term, implies summation over the number range of the index.
Thus we may write:

$$\mathbf{r}_0 = X_i \mathbf{e}_i, \quad \mathbf{r} = x_i \mathbf{e}_i, \quad \bar{\mathbf{r}} = \bar{x}_i \mathbf{e}_i \tag{3.1.5}$$

In continuum mechanics we work with *fields*, which are functions of place and time:

$$f = f(x_1, x_2, x_3, t) \equiv f(x, t) \tag{3.1.6}$$

The fields are *intensive quantities*, and examples are pressure p, temperature Θ, density $\rho =$ mass per unit volume, and velocity $\mathbf{v}$. An intensive quantity defined per unit mass is called a *specific quantity*. The velocity vector $\mathbf{v}$ may be considered to be a *specific linear momentum*. An intensive quantity defined per unit volume is called a *density*. The quantity ρ, which for short is called the density, is then really the *mass density*.

Intensive quantities are either expressed as functions of the *particle coordinates* X_i and the present time t or by the *place coordinates* x_i and the present time t. The four coordinate (X,t) are called *Lagrangian coordinates*, named after Joseph Louis Lagrange [1736–1813], while the four coordinates (x,t) are called *Eulerian coordinates*, named after Leonhard Euler [1707–1783]. A function of Lagrangian coordinates $f(X, t)$ is called a *particle function*. Confer the notation in Eq. (3.1.3). A function of Eulerian coordinates $f(x, t)$ is called a *place function*. Confer the notation in Eq. (3.1.6).

For a particular choice of coordinate set X, an intensive quantity $f(X, t)$ is related to the particle X. The time rate of change of f when related to X, is called the *material derivative* of f and is denoted by f supplied by a “superdot”:

$$\dot{f} = \frac{\partial f(X,t)}{\partial t} \equiv \partial_t f(X, t) \tag{3.1.7}$$

Other names for $\dot{f}$, used in the literature, is the substantial derivative, the particle derivative, and the individual derivative.

The *velocity* of a fluid particle is defined by:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{r}(X,t)}{\partial t} \equiv \partial_t \mathbf{r}(X,t) = v_i \mathbf{e}_i \quad \Leftrightarrow \quad v_i = \dot{x}_i = \frac{\partial x_i(X,t)}{\partial t} \equiv \partial_t x_i(X,t) \tag{3.1.8}$$

v_i are the velocity components in the directions of the coordinate axes.

In fluid mechanics it is usually most convenient to work with Eulerian coordinates (x,t). For a particular choice of place x a place function $f(x, t)$ is related to the place x. The particle velocity $\mathbf{v}(x,t)$ then represents the velocity of the particle X passing through the place x at time t.

To find the material derivative of an intensive quantity represented by a place function $f(x,\, t)$, we replace the place coordinates x by the functions $x(X,\, t)$ to obtain a particle function:

$$f = f(x(X,\, t),\, t) \tag{3.1.9}$$

Using the chain rule of differentiation, we may write:

$$\dot{f} = \frac{\partial f(x,\, t)}{\partial t} + \frac{\partial f(x,t)}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial f(x,t)}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial f(x,t)}{\partial x_3}\frac{\partial x_3}{\partial t} \tag{3.1.10}$$

We introduce the notations:

$$\frac{\partial f(x,t)}{\partial t} \equiv \partial_t f, \quad \frac{\partial f(x,t)}{\partial x_i} \equiv f_{,i}\,, \quad \frac{\partial^2 f(x,t)}{\partial x_j \partial x_i} \equiv f_{,ij} \equiv f_{,ji} \tag{3.1.11}$$

Furthermore we recognize the terms $\partial x_i/\partial t$ in Eq. (3.1.10) as the velocity components v_i in the Eq. (3.1.8). Hence we may express the material derivative of an intensive quantity $f(x,t)$ as:

$$\dot{f} = \partial_t f + f_{,i}\, v_i \tag{3.1.12}$$

The *del-operator* ∇ is a vector operator defined by:

$$\nabla \equiv \mathbf{e}_i \frac{\partial}{\partial x_i} \equiv \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \tag{3.1.13}$$

The scalar product of the velocity vector $\mathbf{v}$ and the del-operator is the scalar operator:

$$\mathbf{v} \cdot \nabla \equiv v_i \frac{\partial}{\partial x_i} \tag{3.1.14}$$

The expression for the material derivative of the place function $f(x,\, t)$ may now be presented by the formula:

$$\dot{f} = \partial_t f + \mathbf{v} \cdot \nabla f \equiv \partial_t f + (\mathbf{v} \cdot \nabla) f \tag{3.1.15}$$

Acceleration is defined as the time rate of change of velocity. The *acceleration* **a** of a fluid particle X passing through the place x at the present time t is then:

$$\mathbf{a} = \dot{\mathbf{v}} = \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} \quad \Leftrightarrow \quad a_i = \dot{v} = \partial_t v_i + v_k v_{i,k} \tag{3.1.16}$$

The quantities $v_{i,k}$ are called *velocity gradients*. The first term on the right-hand side of the Eq. (3.1.16), $\partial_t \mathbf{v} \quad \Leftrightarrow \quad \partial_t v_i$, is called the *local acceleration*, while the last term, $(\mathbf{v} \cdot \nabla)\mathbf{v} \quad \Leftrightarrow \quad v_k v_{i,k}$, is called the *convective acceleration*.

The concept of *streamlines* is introduced to illustrate fluid flow. The streamlines are vector lines to the velocity field $\mathbf{v}(x,\, t)$, i.e. lines that have the velocity vector as a tangent in every point in the space of the fluid. The stream line pattern of a *non-steady flow* $\mathbf{v} = \mathbf{v}(x,t)$ will in general change with time, see Problem 2. In a *steady flow* $\mathbf{v} = \mathbf{v}(x)$ the streamlines coincide with the particle trajectories, called the *pathlines*. For a given velocity field the streamlines are determined from the differential equations:

$$d\mathbf{r} \times \mathbf{v}(x,t) = \mathbf{0} \quad \Leftrightarrow \quad \frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} \quad \text{at constant time } t \tag{3.1.17}$$

The *vorticity vector* or for short the *vorticity* $\mathbf{c}(x, t)$ of the velocity field is defined as:

$$\mathbf{c} = \nabla \times \mathbf{v} \equiv \text{ rot } \mathbf{v} \equiv \text{ curl } \mathbf{v} \tag{3.1.18}$$

The significance of this concept will be discussed in Sect. 4.1. Some important flows are vorticity free, i.e. $\mathbf{c}(x,t) = \mathbf{0}$, in a region of the flow. A *vorticity free flow* is also called an *irrotational flow*, and it may be shown that the velocity field in such a case may be developed from a *velocity potential* $\phi(x,t)$:

$$\mathbf{v} = \nabla\phi, \quad \phi = \phi(x,t) \tag{3.1.19}$$

This fact provides a third name *potential flow* for this type of flow. See Problem 6.

3.2 Continuity Equation: Incompressibility

In classical mechanics *mass* of a body is conserved. This conservation principle is applied in continuum mechanics by the statement that the mass of any body of a continuous medium is constant. According to the continuum hypothesis the mass of a fluid body of volume V is continuously distributed in the volume such that it is possible to express the mass as a volume integral:

$$m = \int_V \rho \, dV \tag{3.2.1}$$

$\rho = \rho(x,t)$ is the *density*, i.e. mass per unit volume, and dV is a volume element, i.e. a small part of the body with the volume dV. At the present time t the volume element is chosen as the element marked with t and shown in two-dimensions in Fig. 3.2, and with the volume $dV = dx_1 \, dx_2 \, dx_3$. By increasing the time by a short time increment dt the volume element is deformed, and Fig. 3.2 shows the element at the time $(t + dt)$. The angles of rotation $v_{1,2} \, dt$ etc. of the edges are very small, and we may to the first order state that the lengths of the edges of the element are changed from dx_1 to $dx_1 \cdot (1 + v_{1,1} \, dt)$ etc. The volume of the element at the time $(t + dt)$ becomes:

$$dV + \Delta dV = [dx_1 \cdot (1 + v_{1,1} \, dt)][dx_2 \cdot (1 + v_{2,2} \, dt)][dx_3 \cdot (1 + v_{3,3} \, dt)] \quad \Rightarrow$$
$$dV + \Delta dV = dV[1 + v_{1,1} \, dt + v_{2,2} \, dt + v_{3,3} \, dt + \text{higher order terms}] = dV[1 + v_{i,i} \, dt] \quad \Rightarrow$$
$$\Delta dV = v_{i,i} \, dV \, dt$$

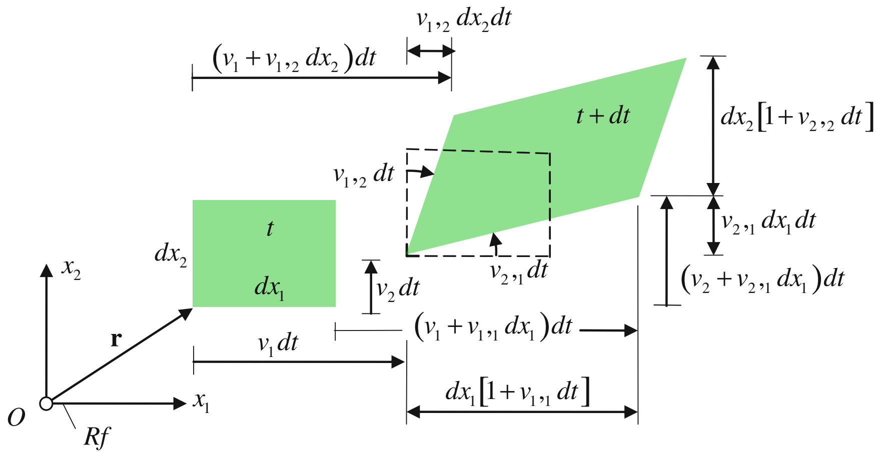


Fig. 3.2 Deformation of a fluid element

The result implies that the time rate of change of the element volume dV is:

$$d\dot{V} = \frac{\Delta dV}{dt} = v_{i,i}\, dV = (\mathrm{div}\, \mathbf{v})dV \equiv (\nabla \cdot \mathbf{v})dV \tag{3.2.2}$$

The divergence, $\mathrm{div}\, \mathbf{v}$, of the velocity field $\mathbf{v}$ is seen to represent *the change of volume per unit volume and per unit time*. Since the mass $\rho\, dV$ of the fluid element is constant, we may write:

$$\frac{d}{dt}[\rho\, dV] = \dot{\rho}\, dV + \rho\, d\dot{V} = [\dot{\rho} + \rho \nabla \cdot \mathbf{v}]dV = 0 \quad \Rightarrow$$

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad \dot{\rho} + \rho\, v_{i,i} = 0 \tag{3.2.3}$$

This is the *continuity equation*. The expression (3.1.15) for the material derivative applied to the density ρ, provides an alternative expression of the continuity equation:

$$\partial_t \rho + \nabla \cdot (\rho\, \mathbf{v}) = 0 \quad \Leftrightarrow \quad \partial_t \rho + (\rho\, v_i)_{,i} = 0 \tag{3.2.4}$$

For an *incompressible fluid* the equation of continuity is replaced by the *incompressibility condition*:

$$\mathrm{div}\, \mathbf{v} \equiv \nabla \cdot \mathbf{v} = 0 \quad \Leftrightarrow \quad v_{i,i} = 0 \tag{3.2.5}$$

This is a statement of no time rate of change of volume of a volume element. Eq. (3.2.5) is also called the *continuity equation for an incompressible fluid*.

3.3 Equations of Motion

A fluid body is subjected to two kinds of forces, see Fig. 3.3:

(1) *Body forces*, given as force **b** per unit mass,
(2) *Contact forces* on the surface of the body, given as force **t** per unit area. The vector **t** is called the *stress vector* or the *traction*.

The most common body force is the constant *gravitational force* $g = 9.81$ N/kg, representing a homogeneous gravity field. Other examples of body forces are electrostatic forces, magnetic forces, and centrifugal forces.

The *resultant force* **F** and the *resultant moment* $\mathbf{M}_O$ about a point O of the forces on a fluid body of volume V and surface A are expressed by:

$$\mathbf{F} = \int_V \mathbf{b}\rho\, dV + \int_A \mathbf{t}\, dA \tag{3.3.1}$$

$$\mathbf{M}_O = \int_V \mathbf{r} \times \mathbf{b}\rho\, dV + \int_A \mathbf{r} \times \mathbf{t}\, dA \tag{3.3.2}$$

The fluid body has a *linear momentum* **p** and an *angular momentum* $\mathbf{L}_O$ about the point O:

$$\mathbf{p} = \int_V \mathbf{v}\rho\, dV, \quad \mathbf{L}_O = \int_V \mathbf{r} \times \mathbf{v}\rho\, dV \tag{3.3.3}$$

The fundamental laws of motion for a body of continuous matter or a system of particles are the following *Euler's axioms*:

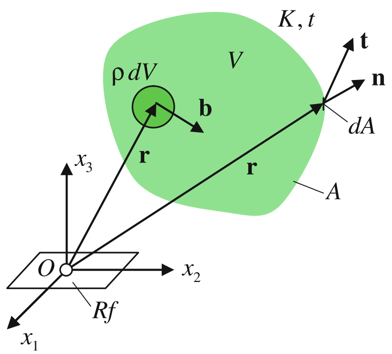


Fig. 3.3 Fluid body subjected to body force **b** and contact force **t**

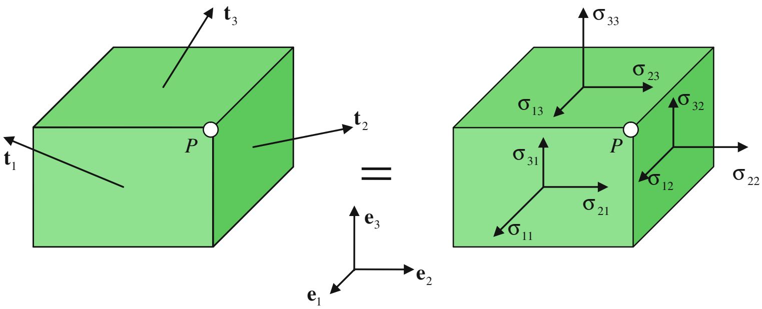


Fig. 3.4 Stress vectors $\mathbf{t}_i$ and coordinate stresses σ_{ik} on material coordinate planes and coordinate stresses

$$\mathbf{F} = \dot{\mathbf{p}} \quad 1.\text{ axiom}, \quad \mathbf{M}_O = \dot{\mathbf{L}}_O \quad 2.\text{ axiom} \tag{3.3.4}$$

The first axiom, which states that *the resultant force on a body is equal to the time rate of change of the linear momentum of the body*, includes Newton's 2. law of motion for a *mass particle*, i.e. a body with finite mass but with negligible extent. The second axiom, which states that *the resultant moment of forces about O is equal to the time rate of change of the angular momentum about O of the body*, may, for a system of mass particles, be derived from Newton's 2. law of motion.

The stress vectors $\mathbf{t}_i$ on three orthogonal material coordinate planes through a particle have components given by the *coordinate stresses* σ_{ik}, see Fig. 3.4:

$$\mathbf{t}_k = \sigma_{ik}\mathbf{e}_i \tag{3.3.5}$$

$\sigma_{11}, \sigma_{22},$ and σ_{33} are *normal stresses*, and σ_{ik} for $i \neq k$ are *shear stresses*. It will be shown in Sect. 3.3.6 that Euler's 2. axiom implies that:

$$\sigma_{ik} = \sigma_{ki} \tag{3.3.6}$$

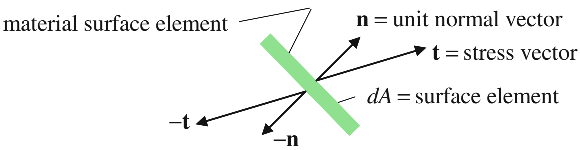
Thus only three of the six coordinate shear stresses may be different. The symbol T is used to denote the *stress matrix* whose elements are the coordinate stresses. Thus we write:

$$T \equiv (\sigma_{ik}) \equiv \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{3.3.7}$$

Equation (3.3.6) shows that the *stress matrix* $T = (\sigma_{ik})$ *is symmetric*.

It follows from the axioms of Euler that the stress vectors on the two sides of a material surface are of equal magnitude but of opposite direction, see Fig. 3.5.

Fig. 3.5 Stress vectors on both sides of a material surface



3.3.1 Cauchy's Stress Theorem

The stress vector **t** on a surface with unit normal **n** through a particle may be determined from the coordinate stresses σ_{ik} for the particle. The components t_i of the stress vector $\mathbf{t} = t_i \mathbf{e}_i$ are given by:

$$t_i = \sigma_{ik} n_k \text{ Cauchy's stress theorem} \tag{3.3.8}$$

n_k are the components of the unit normal vector $\mathbf{n}: \ \mathbf{n} = n_k \mathbf{e}_k$. This result is called *Cauchy's stress theorem*, Augustin Louis Cauchy [1789–1857], and will be derived below. Equation (3.3.8) is a relationship between the components of two coordinate invariant quantities: the stress vector **t** and the unit normal **n**. This fact implies that the coordinate stresses σ_{ik} also represent a coordinate invariant quantity. We say that σ_{ik} are the components of the *stress tensor* **T,** also called *Cauchy's stress tensor*. The stress tensor **T** is in the coordinate system Ox represented by the stress matrix $T = (\sigma_{ik})$. In general a tensor is a coordinate invariant quantity represented in any Cartesian coordinate system by a matrix.

Fig. 3.6 shows a small body in the shape of a tetrahedron. Of the four triangular planes of the surface three are planes through the particle P and parallel to coordinate planes, while the fourth plane has the unit normal **n** and has a distance h from the particle P. The tetrahedron is called the *Cauchy tetrahedron*. The body is subjected to the body force **b**, stress vectors $(-\mathbf{t}_i)$ on three material planes, and the stress vector **t** on the fourth material plane. The velocity of the particle is **v** and the density at the particle is ρ. Referring to Fig. 3.6 we may express the volume of the body alternatively by:

$$dV = \frac{1}{3} dA \cdot h = \frac{1}{3} dA_1 \cdot h_1 = \frac{1}{3} dA_2 \cdot h_2 = \frac{1}{3} dA_3 \cdot h_3 \tag{3.3.9}$$

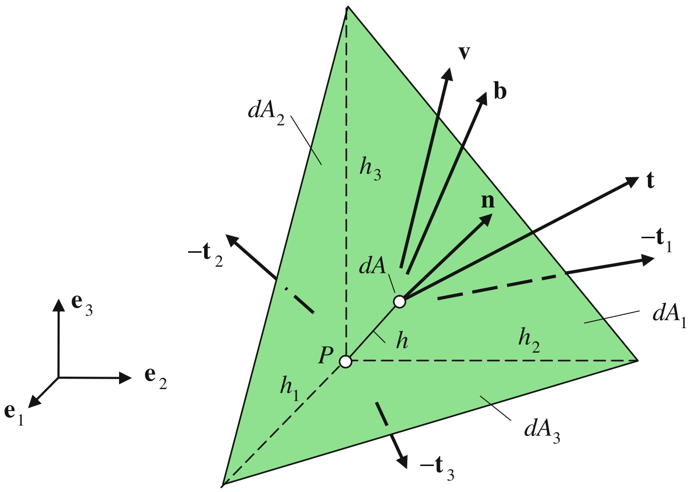


Fig. 3.6 The Cauchy tetrahedron

Since the body is small we may take $\mathbf{v}$ as the velocity of the body, $\rho\, dV$ as the mass of the body, and $\mathbf{v}\rho\, dV$ as the linear momentum of the body.

Since $\mathbf{n}$ is a unit vector, we may write:

$$n_k = \mathbf{n} \cdot \mathbf{e}_k = \cos\theta_k = \frac{h}{h_k} \tag{3.3.10}$$

θ_k is the angle between $\mathbf{n}$ and the $x_k -$ direction. From the Eqs. (3.3.9) and (3.3.10) we obtain the relation:

$$dA_k = dA\, n_k \tag{3.3.11}$$

Euler's 1. axiom for the tetrahedron implies that:

$$\mathbf{b}\,\rho\, dV + (-\mathbf{t}_k \cdot dA_k) + \mathbf{t}\, dA = \frac{d(\mathbf{v}\,\rho\, dV)}{dt} = \frac{d(\mathbf{v})}{dt}\rho\, dV + \mathbf{v}\frac{d(\rho\, dV)}{dt} = \dot{\mathbf{v}}\,\rho\, dV \tag{3.3.12}$$

Here we have used result that since the mass ρV of the body is constant, then $d(\rho dV)/dt = 0$. Substitution into equation (3.3.12) dV from equation (3.3.9) and dA_k from equation (3.3.11), followed by a division by dA, leads to the result:

$$\mathbf{b}\,\rho\,\frac{h}{3} - \mathbf{t}_k n_k + \mathbf{t} = \dot{\mathbf{v}}\,\rho\,\frac{h}{3} \tag{3.3.13}$$

If we let $h \to 0$ such that $\mathbf{t}$ becomes the stress vector on a plane dA through the particle P and with unit normal vector $\mathbf{n}$, the result (3.3.13) is reduced to:

$$\mathbf{t} = \mathbf{t}_k\, n_k \tag{3.3.14}$$

Now the relations (3.3.5) are applied and Eq. (3.3.14) yields:

$$\mathbf{t} = \sigma_{ik}\mathbf{e}_i n_k = t_i \mathbf{e}_i \quad \Rightarrow \quad t_i = \sigma_{ik}\, n_k \tag{3.3.15}$$

Cauchy's stress theorem (3.3.8) is thus proved.

3.3.2 Cauchy's Equations of Motion

We are now ready to derive the equations of motion for a particle X at the place x. Euler's 1. axiom (3.3.4) implies for the fluid element of volume dV shown in Fig. 3.7 that:

$$\begin{aligned} &\mathbf{b}\rho\, dV + (\mathbf{t}_{1,1}\, dx_1)dx_2 dx_3 + (\mathbf{t}_{2,2}\, dx_2)dx_3 dx_1 + (\mathbf{t}_{3,3}\, dx_3)dx_1 dx_2 \\ &= \frac{d(\mathbf{v}\rho dV)}{dt} = \dot{\mathbf{v}}\rho dV \Rightarrow \rho\,\mathbf{b} + \mathbf{t}_{k,k} = \rho\dot{\mathbf{v}} \end{aligned} \tag{3.3.16}$$

The stress vectors $\mathbf{t}_k$ are introduced from Eq. (3.3.5), and the component form of Eq. (3.3.16) becomes (after a trivial change of indices):

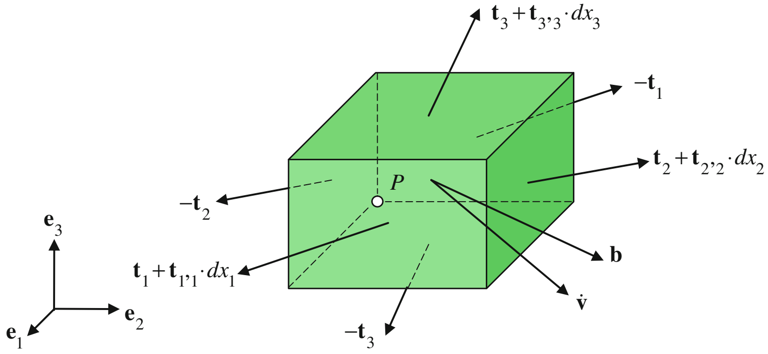


Fig. 3.7 Fluid body with body forces and contact forces

$$\rho \dot{v}_i = \sigma_{ik},_k + \rho b_i \tag{3.3.17}$$

These three equations are *Cauchy's equations of motion* for a continuum.

For fluids it is convenient to express the coordinate stresses as a sum of an *isotropic pressure p* and *extra stresses* τ_{ik}:

$$\sigma_{ik} = -p\delta_{ik} + \tau_{ik} \tag{3.3.18}$$

The symbol δ_{ik} is called a *Kronecker delta*, named after Leopold Kronecker [1823–1891], and is defined by:

$$\delta_{ik} = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \quad \Leftrightarrow \quad (\delta_{ik}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv 1 \tag{3.3.19}$$

Thus δ_{ik} represent the elements of a 3×3 *unit matrix* 1, which in the coordinate system Ox represents the *unit tensor* **1**. The extra stresses τ_{ik} are elements in the *extra stress matrix* T':

$$T' = (\tau_{ik}) = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{3.3.20}$$

The extra stress matrix T' and the extra stresses τ_{ik} represent in the coordinate system Ox a coordinate invariant quantity called the *extra stress tensor* $\mathbf{T}'$. The relationship (3.3.18) may now be written alternatively as:

$$\sigma_{ik} = -p\delta_{ik} + \tau_{ik} \quad \Leftrightarrow \quad T = -p1 + T' \tag{3.3.21}$$

$$\mathbf{T} = -p\mathbf{1} + \mathbf{T}' \tag{3.3.22}$$

The *matrix* equation (3.3.21) is the representation in the Cartesian coordinate system Ox of the *tensor equation* (3.3.22). In another Cartesian coordinate system $\bar{O}\bar{x}$

the tensors $\mathbf{T}, \mathbf{1}$, and $\mathbf{T}'$ are represented by matrices $\bar{T}, \bar{1} = 1$, and $\bar{T}'$. The tensor concept and tensor algebra will only sparsely be used in this book before Chap. 8

For a compressible fluid the pressure p is the *thermodynamic pressure* and is in general a function of the *density* ρ and the *temperature* Θ of the fluid particle:

$$p = p(\rho, \Theta) \tag{3.3.23}$$

For an incompressible fluid the isotropic pressure is constitutively undetermined and must be found from the equations of motion and the boundary conditions of the flow problem. In a fluid that has been at rest for some period of time, the state of stress is solely given by the pressure. The extra stresses τ_{ik} are due to the deformation process of the fluid.

Using the expressions (3.3.18) for the coordinate stresses σ_{ik} and Eq. (3.1.16) for the accelerations $\dot{v}_i$, we obtain from the Cauchy's equations of motion (3.3.17) the *general equations of motion for a fluid*:

$$\rho(\partial_t v_i + v_k v_{i,k}) = -p_{,i} + \tau_{ik,k} + \rho b_i \tag{3.3.24}$$

In applications it is often convenient to use the equations of motion in other coordinate systems than the Cartesian system. First we recognize the Eq. (3.3.24) as the three component equations of a vector equation. The coordinate invariant, vector form is written as:

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}] = -\nabla p + \nabla \cdot \mathbf{T}' + \rho \mathbf{b} \tag{3.3.25}$$

The symbol ∇p is the *gradient* of the pressure p. The vector $\nabla \cdot \mathbf{T}'$ is called the *divergence* of the *stress tensor* $\mathbf{T}$ and is a vector with components $\tau_{ik,k}$ in the Cartesian coordinate system Ox. The component form of the vector equation (3.3.25) in Cartesian coordinate (x,y,z), cylindrical coordinates (R, θ, z), and spherical coordinates (r, θ, ϕ) are now listed for future reference.

3.3.3 Cauchy's Equations in Cartesian Coordinates (X,Y,Z)

$$\rho\left[\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right] = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho b_x \quad \text{etc.} \tag{3.3.26}$$

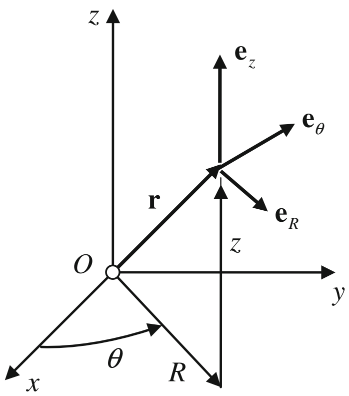
3.3.4 Extra Stress Matrix, Extra Coordinate Stresses, and Cauchy's Equations in Cylindrical Coordinates (R, θ, Z)

$$T' = \begin{pmatrix} \tau_{RR} & \tau_{R\theta} & \tau_{Rz} \\ \tau_{\theta R} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zR} & \tau_{z\theta} & \tau_{zz} \end{pmatrix} \tag{3.3.27}$$

$$\begin{aligned} \rho\left[\frac{\partial v_R}{\partial t} + v_R\frac{\partial v_R}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_R}{\partial \theta} + v_z\frac{\partial v_R}{\partial z} - \frac{v_\theta^2}{R}\right] &= -\frac{\partial p}{\partial R} \\ + \frac{1}{R}\frac{\partial}{\partial R}(R\tau_{RR}) + \frac{1}{R}\frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{\partial \tau_{Rz}}{\partial z} - \frac{\tau_{\theta\theta}}{R} + \rho b_R \end{aligned} \tag{3.3.28}$$

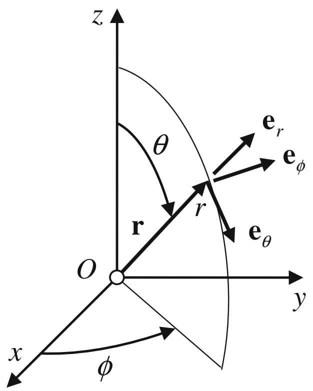
$$\begin{aligned} \rho\left[\frac{\partial v_\theta}{\partial t} + v_R\frac{\partial v_\theta}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_R v_\theta}{R}\right] &= -\frac{1}{R}\frac{\partial p}{\partial \theta} \\ + \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\tau_{R\theta}\right) + \frac{1}{R}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \rho b_R \end{aligned} \tag{3.3.29}$$

$$\begin{aligned} \rho\left[\frac{\partial v_z}{\partial t} + v_R\frac{\partial v_z}{\partial R} + \frac{v_\theta}{R}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_R}{\partial z}\right] &= -\frac{\partial p}{\partial z} \\ + \frac{1}{R}\frac{\partial}{\partial R}(R\tau_{zR}) + \frac{1}{R}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \rho b_z \end{aligned} \tag{3.3.30}$$



3.3.5 Extra Stress Matrix, Extra Coordinate Stresses, and Cauchy's Equations in Spherical Coordinates (r, θ, φ)

$$T' = \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \tau_{\phi\phi} \end{pmatrix} \tag{3.3.31}$$



$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right] = -\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tau_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\tau_{r\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi r}}{\partial\phi} - \frac{\tau_{\theta\theta}+\tau_{\phi\phi}}{r} + \rho b_r \tag{3.3.32}$$

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2}{r^2}\cot\theta\right] = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\tau_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\tau_{\theta\theta})}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\theta\phi}}{\partial\phi} + \frac{\tau_{\phi\phi}}{r}\cot\theta + \rho b_\theta \tag{3.3.33}$$

$$\rho\left[\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi}{r}\cot\theta\right] = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\tau_{\phi r}\right) + \frac{1}{r\sin^2\theta}\frac{\partial\left(\sin^2\theta\tau_{\theta\phi}\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + \rho b_\phi \tag{3.3.34}$$

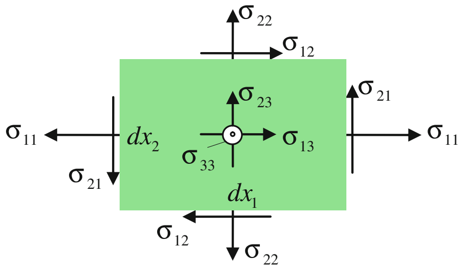


Fig. 3.8 Fluid element $dV = dx_1\,dx_2\,dx_3$ subjected to a homogeneous state of stress σ

3.3.6 Proof of the Statement

Let $T = (\sigma_{ik})$ represent the state of stress at a particle X. In the neighborhood of X the state of stress may then be denoted $T + \Delta T = (\sigma_{ik} + \Delta\sigma_{ik})$. Since $T = (\sigma_{ik})$ now represents a homogeneous state of stress in the neighborhood of the particle X, only the stress increment $\Delta T = (\Delta\sigma_{ik})$ must satisfy the Cauchy equation (3.3.17):

$$\rho\,\dot{v}_i = \Delta\sigma_{ik},_k + \rho b_i$$

A fluid element subjected to the homogeneous stress field T is in equilibrium, but we must insure that this stress field satisfies Euler's 2. axiom. For the fluid element, $dV = dx_1\,dx_2\,dx_3$, shown in Fig. 3.8 the axiom requires moment equilibrium about any axis. For any axis parallel to the x_3 − axis this equilibrium equation is:

$$[\sigma_{12}\,(dx_1\;dx_3)\cdot dx_2] - [\sigma_{21}\,(dx_2\;dx_3)\cdot dx_1] = 0 \quad\Rightarrow\quad \sigma_{12} = \sigma_{21}$$

In general $\sigma_{ik} = \sigma_{ki} \Leftrightarrow \textit{Equation}(3.3.6)$

3.4 Navier–Stokes Equations

In order to show how the equations of motion appear for a particular fluid model, we need constitutive equations defining the fluid model. The *constitutive equations for a linearly viscous fluid,* also called *a Newtonian fluid* are given by the Eqs. (3.3.18), (3.3.23), and:

$$\tau_{ik} = \mu(v_i,_k + v_k,_i) + \left(\kappa - \frac{2\mu}{3}\right)(\nabla\cdot\mathbf{v})\delta_{ik} \tag{3.4.1}$$

μ is the (*shear* or *dynamic*) *viscosity* and κ is the *bulk viscosity*.

The bulk viscosity is difficult to measure and is hard to find values for in the literature. For monatomic gases it is reasonable to set $\kappa = 0$, while for other gases and for all liquids κ is larger than and often much larger than μ. However, the divergence of the velocity vector, $\nabla \cdot \mathbf{v}$, is for most flows so small that the term $(\kappa - 2\mu/3)\nabla \cdot \mathbf{v}$ in the Eq. (3.4.1) may be neglected. The bulk viscosity has dominating importance for dissipation and absorption of sound energy. For an *incompressible fluid* or an *isochoric flow*, i.e. is a volume preserving flow, the divergence of the velocity vector vanishes, $\nabla \cdot \mathbf{v} = 0$. The Newtonian fluid model will be further discussed in the Sects. 4.3 and 6.1.

For simple steady shear flow between two parallel plates the velocity field may be expressed by: $v_1 = \dot{\gamma} x_2$, $v_2 = v_3 = 0$, where $\dot{\gamma}$ is a constant shear strain rate. From the constitutive Eqs. (3.3.18) and (3.4.1) the state of stress for simply steady shear flow becomes:

$$\sigma_{11} = \sigma_{22} = \sigma_{22} = -p, \quad \tau_{12} = \mu\dot{\gamma}, \quad \tau_{23} = \tau_{31} = 0 \tag{3.4.2}$$

For an isotropic state of stress, see Fig. 1.1:

$$\sigma_{ik} = \sigma_o\,\delta_{ik} = -p\,\delta_{ik} + \tau_{ik}, \quad \tau_{ik} = (\sigma_o + p)\,\delta_{ik} \tag{3.4.3}$$

the Eq. (3.4.1) give:

$$\tau_{kk} = (\sigma_o + p)\delta_{kk} = \mu(v_{k,k} + v_{k,k}) + \left(\kappa - \frac{2\mu}{3}\right)(\nabla \cdot \mathbf{v})\delta_{kk} \tag{3.4.4}$$

Because $\delta_{kk} = \delta_{11} + \delta_{11} + \delta_{11} = 3$ and $v_{k,k} = \nabla \cdot \mathbf{v}$, we obtain:

$$\sigma_o + p = \kappa\,\nabla \cdot \mathbf{v} \tag{3.4.5}$$

Thus we see that the bulk viscosity κ expresses the resistance of the fluid toward rapid changes of volume. It is fairly difficult to measure κ for a real fluid, and little information about bulk viscosities is found in the literature. For incompressible fluids $\nabla \cdot \mathbf{v} = 0$, and the bulk viscosity κ has no meaning.

The equations of motion for a Newtonian fluid are called the *Navier–Stokes equations*, Claude L. M. H. Navier [1785–1836], George Gabriel Stokes [1819–1903]. These equations are obtained by substituting the constitutive equation (3.4.1) into the general equations of motion (3.3.24). The resulting equations are:

$$\rho(\partial_t v_i + v_k v_{i,k}) = -p_{,i} + \mu\,v_{i,kk} + \left(\kappa + \frac{\mu}{3}\right)v_{k,ki} + \rho\,b_i \tag{3.4.6}$$

The coordinate invariant form of the Navier–Stokes equations is:

$$\rho(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = -\nabla p + \mu\,\nabla^2\mathbf{v} + \left(\kappa + \frac{\mu}{3}\right)\nabla(\nabla \cdot \mathbf{v}) + \rho\,\mathbf{b} \tag{3.4.7}$$

3.5 Modified Pressure

It is often convenient to combine the pressure term ∇p and the body force $\rho\mathbf{b}$ in the general equations of motion (3.3.25) by introducing the *modified pressure* P. This is done by solving the equations of motion for the fluid at rest, i.e. to solve for the *static pressure* p_s determined from the equilibrium equation (no extra stresses due to deformation and no acceleration):

$$\mathbf{0} = -\nabla p_s + \rho\mathbf{b} \tag{3.5.1}$$

The modified pressure P is defined by the expression:

$$P = p - p_s \tag{3.5.2}$$

For example, let the body force $\mathbf{b}$ be the gravitational force $-g\mathbf{e}_z$, where z is the vertical height above a reference level at which the pressure is p_o. The Eq. (3.5.1) then yields:

$$p_s = p_o - \rho g z \tag{3.5.3}$$

When the pressure p in Eq. (3.3.25) is replaced by $P + p_s$ and Eq. (3.5.1) is applied, we get a new and simplified set of equations of motion:

$$\rho(\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}) = -\nabla P + \nabla\cdot\mathbf{T}' \quad \Leftrightarrow \rho(\partial_t v_i + v_k v_{i,k}) = P_{,i} + \tau_{ik,k} \tag{3.5.4}$$

As will be shown in Sect. 3.9, it is not always practical to introduce the modified pressure into a problem.

3.6 Flows with Straight, Parallel Streamlines

This type of flow is very simple to analyze and occurs, sometimes only approximately, in many practical applications. In the Sects.3.7, 3.8, and 3.9 special cases of such flows will be discussed.

It will now be assumed that the fluid is incompressible and that the velocity field takes the form:

$$v_x = v_x(x, y, z, t), \quad v_y = v_z = 0 \tag{3.6.1}$$

The streamlines are then straight lines parallel to the x-axis. The incompressibility condition (3.2.5) applied to the velocity field (3.6.1) provides the result:

$$\frac{\partial v_x(x, y, z, t)}{\partial x} = 0 \quad \Rightarrow \quad v_x = v_x(y, z, t) \tag{3.6.2}$$

Thus the velocity field is independent of the x – coordinate.

It is reasonable to assume that the extra stresses τ_{ik}, which are due to the deformations of the fluid resulting from the velocity field in Eqs. (3.6.1) and (3.6.2), are independent of the $x-$coordinate. The equations of motion (3.5.4) are then reduced to:

$$\rho\frac{\partial v_x}{\partial t} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \tag{3.6.3}$$

$$0 = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}, \quad 0 = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{3.6.4}$$

Because the stresses τ_{ik} are independent of x, the equations (3.6.4) imply that:

$$\frac{\partial^2 P}{\partial x \partial y} = \frac{\partial^2 P}{\partial x \partial z} = 0 \tag{3.6.5}$$

which by application of the Eqs. (3.6.3) and (3.6.2), shows that:

$$\frac{\partial P}{\partial x} = c \text{ (a constant) or a function of time } c(t) \tag{3.6.6}$$

For steady flows the pressure gradient (3.6.6) is a constant. This result may be stated as follows:

In flows with straight, parallel streamlines the gradient in the streamwise direction of the modified pressure P is constant for steady flow and a function of time for unsteady flows.

3.7 Flows Between Parallel Planes

Figure 3.9 illustrates a flow of a fluid between two parallel plates a distance h apart. One of the plates is at rest while the other moves with a constant velocity v_1. The flow is driven by the motion of the plate, by a pressure gradient in the $x-$direction, and by the gravitational force g. The constitutive equations of the fluid will be specified below.

A steady *laminar flow* is assumed with the velocity field:

$$v_x = v(y), \quad v_y = v_z = 0 \tag{3.7.1}$$

This is a special case of the flow presented in the previous section. It is further assumed that the fluid sticks to both plates, which provides the boundary conditions:

$$v(0) = 0, \quad v(h) = v_1 \tag{3.7.2}$$

The acceleration is zero and it follows from Eq. (3.6.6) that the modified pressure gradient $\partial P/\partial x$ is constant:

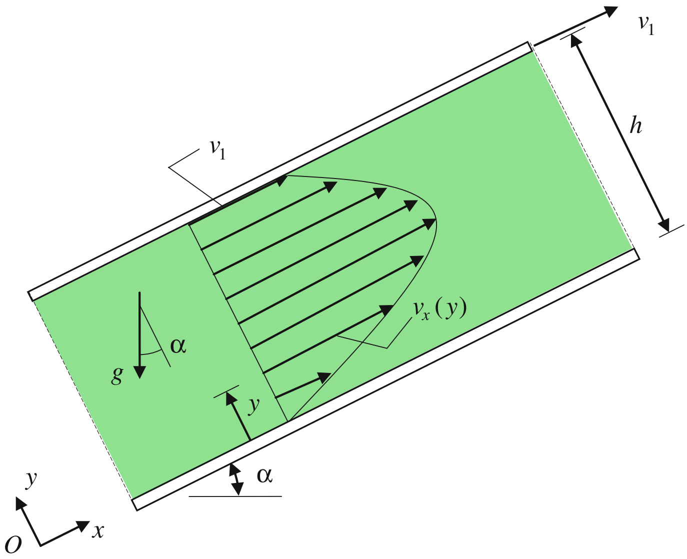


Fig. 3.9 Flow between parallel planes

$$\frac{\partial P}{\partial x} = c \text{ (a constant)} \tag{3.7.3}$$

Due to the assumed velocity field (3.7.1) the extra stresses τ_{ik} are only functions of y, and due to symmetry τ_{xz} and τ_{yz} are zero:

$$\tau_{ik} = \tau_{ik}(y), \quad \tau_{xz} = \tau_{yz} = 0 \tag{3.7.4}$$

The equations of motion (3.6.3, 3.6.4) are now reduced to three equations of equilibrium:

$$0 = -c + \frac{d\tau_{xy}}{dy}, \quad 0 = -\frac{\partial P}{\partial y} + \frac{d\tau_{yy}}{dy}, \quad 0 = -\frac{\partial P}{\partial z} \tag{3.7.5}$$

The first equation is integrated to:

$$\tau_{xy}(y) = cy + \tau_0 \tag{3.7.6}$$

τ_0 is the unknown shear stress at the fixed boundary surface $y = 0$. For the modified pressure P the last two of the equations (3.7.5) and equation (3.7.3) yield:

$$P = P(x, y) = cx + \tau_{yy}(y) + P_0 \tag{3.7.7}$$

P_0 is an unknown constant pressure.

From this point in the analysis we need to specify a fluid model by introducing constitutive equations. Three fluid models will be considered: the Newtonian fluid, a power law fluid, and the Bingham fluid.

Newtonian Fluid. The constitutive equation. (3.4.1) give:

$$\tau_{xy} = \mu \frac{dv}{dy}, \quad \tau_{yy} = 0 \tag{3.7.8}$$

From the last of these equations and Eq. (3.7.7) it follows that:

$$P = P(x) = cx + P_0 \tag{3.7.9}$$

The result implies that the total pressure $p = P + p_s$ varies linearly over any cross section $x =$ constant of the flow, as in a fluid at rest. The first of the equations (3.7.8) and equation (3.7.6) are combined, and the result is integrated:

$$\frac{dv}{dy} = \frac{c}{\mu} y + \frac{\tau_0}{\mu} \quad \Rightarrow \quad v(y) = \frac{c}{2\mu} y^2 + \frac{\tau_0}{\mu} y + C_1 \tag{3.7.10}$$

The constant shear stress τ_0 at the fixed boundary surface and the constant of integration C_1 are determined from the boundary conditions (3.7.2):

$$v(0) = 0 \quad \Rightarrow \quad C_1 = 0, \quad v(h) = v_1 \quad \Rightarrow \quad \tau_0 = -\frac{ch}{2} + \frac{\mu v_1}{h} \tag{3.7.11}$$

The velocity field (3.7.10) is now found as:

$$v(y) = -\frac{ch^2}{2\mu}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right] + v_1 \frac{y}{h} \tag{3.7.12}$$

The volumetric flow through a cross section $x =$ constant is per unit width of the flow:

$$Q = \int_0^h v(y)\, dy = -\frac{ch^3}{12\mu} + v_1 \frac{h}{2} \tag{3.7.13}$$

Figure 3.10 shows the velocity profiles for some characteristic special cases:

(a) Zero modified pressure gradient: $c = 0 \Rightarrow$

$$v(y) = v_1 \frac{y}{h}, \quad Q = \frac{v_1 h}{2} \tag{3.7.14}$$

(b) Positive modified pressure gradient: $c > 0 \Rightarrow$

$$Q = 0 \text{ for } v_1 = \frac{c h^2}{6\mu} \tag{3.7.15}$$

(c) Negative modified pressure gradient: $c < 0 \Rightarrow$

$$Q = \frac{|c| h^3}{12\mu} + \frac{v_1 h}{2} \tag{3.7.16}$$

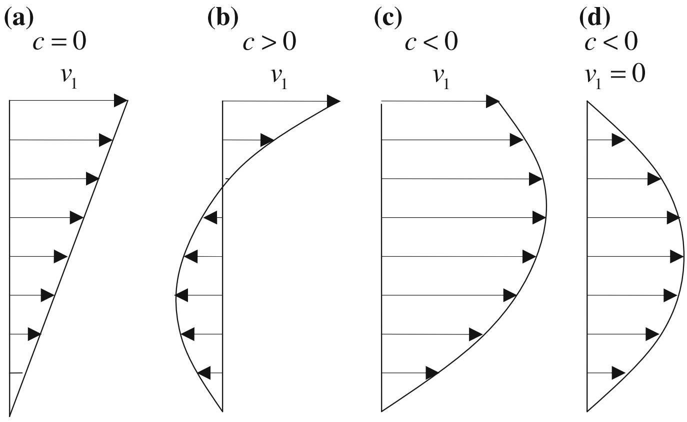


Fig. 3.10 Velocity profiles for a Newtonian fluid

(d) Negative modified pressure gradient and two fixed plates: $c < 0$ and $v_1 = 0 \Rightarrow$

$$v(y) = \frac{|c|h^2}{2\mu}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right], \quad Q = \frac{|c|h^3}{12\mu} \tag{3.7.17}$$

Power Law Fluid. This fluid model is a special *generalized Newtonian fluid* and the general presentation of the constitutive equations for the fluid model will be given in Sect. 6.1. For the special flow defined in Eq. (3.7.1), the constitutive equations are reduced to:

$$\tau_{xy} = \eta\frac{dv}{dy}, \quad \tau_{yy} = 0, \quad \eta = \eta(\dot{\gamma}) = K|\dot{\gamma}|^{n-1}, \quad \dot{\gamma} = \frac{dv}{dy} \tag{3.7.18}$$

The *consistency parameter* K and the *power law index* n are two material parameters.

The general result (3.7.7) and the Eq. (3.7.18) yield the modified pressure function (3.7.9). The result implies that the total pressure $p = P + p_s$ varies linearly over any the cross section $x =$ constant of the flow, as in a fluid at rest.

The Eqs. (3.7.6) and (3.7.18) may combine to give:

$$K\left|\frac{dv}{dy}\right|^{n-1}\frac{dv}{dy} = cy + \tau_0 \tag{3.7.19}$$

When this equation is integrated, it is convenient to distinguish between the following two situations that may occur:

(a) The velocity gradient dv/dy has the same sign in the interval $0 \le y \le h$.
(b) The velocity gradient dv/dy changes sign in the interval $0 \le y \le h$.

However, we shall for simplicity choose the conditions: $v_1 = 0$ and $c < 0$. Then the velocity gradient dv/dy is ≥ 0 in the region $0 \leq y \leq h/2$ and $dv/dy \leq 0$ in the region $h/2 \le y \le h$. The flow is now symmetric about the plane $y = h/2$. The boundary conditions (3.7.2) for $v(y)$ are replaced by:

$$v(0) = 0, \quad \left.\frac{dv}{dy}\right|_{y=h/2} = 0 \tag{3.7.20}$$

The latter of these conditions implies together with Eq. (3.7.19) the condition:

$$\tau_0 = \frac{|c|h}{2} \tag{3.7.21}$$

This condition also follows from (3.7.6) and the symmetry condition: $\tau_{xy} = 0$ at $h/2$. In the interval $0 \le y \le h/2$ the Eq. (3.7.19) may be written as:

$$\frac{dv}{dy} = \left[\frac{|c|}{K}\left(\frac{h}{2} - y\right)\right]^{1/n} \tag{3.7.22}$$

Integration of Eq. (3.7.22) followed by application of the condition $v(0) = 0$ and symmetry in the interval $0 \le y \le h$ yields:

$$v(y) = \left[\frac{|c|h}{2K}\right]^{1/n} \frac{nh}{2(1+n)} \left\{1 - \left|1 - \frac{2y}{h}\right|^{1+1/n}\right\} \tag{3.7.23}$$

The volumetric flow is:

$$Q = 2\int_0^{h/2} v(y)\,dy = \left[\frac{|c|h}{2K}\right]^{1/n} \frac{nh^2}{2(1+2n)} \tag{3.7.24}$$

Figure 3.11 shows the velocity profile for a power law fluid with the power law index $n = 0.2$. If we choose $K = \mu$ and $n = 1$, the solutions (3.7.23) and (3.7.24) are reduced to the solution (3.7.17) for a Newtonian fluid.

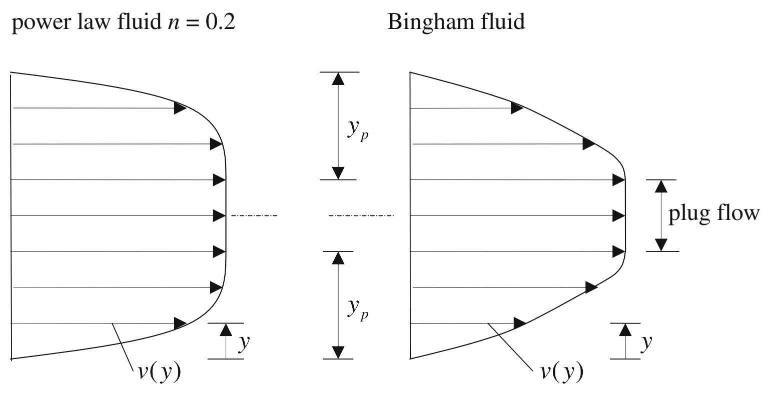


Fig. 3.11 Velocity profile for a power law fluid and for a Bingham fluid

Bingham Fluid. As we shall see in Sect. 6.1, this fluid model may be considered to be a special kind of a *generalized Newtonian fluid*. The relevant constitutive equations for the flow assumed by the velocity field (3.7.1) are:

$$\tau_{xy} = \left[\mu + \frac{\tau_y}{|dy/dy|}\right]\frac{dv}{dy} \quad \text{when } \frac{dv}{dy} \neq 0, \quad |\tau_{xy}| \leq \tau_y \quad \Leftrightarrow \frac{dv}{dy} = 0$$
$$\tau_{yy} = 0 \tag{3.7.25}$$

The *yield shear stress* τ_y and the viscosity μ are material parameters.

As in the previous example we choose for simplicity the special case $v_1 = 0$ and $c < 0$. The flow is symmetric about the plane $y = h/2$, and we obtain from Eq. (3.7.6):

$$\tau_{xy}(h) = -\tau_{xy}(0) \quad \Rightarrow \quad ch + \tau_0 = -\tau_0 \quad \Rightarrow$$
$$\tau_0 = \frac{|c|h}{2} \tag{3.7.26}$$

The shear stress formula (3.7.6) is rewritten to:

$$\tau_{xy}(y) = |c|\left[\frac{h}{2} - y\right] \tag{3.7.27}$$

The shear stress is equal to the yield shear stress τ_y for two values of y: y_p and $h - y_p$, where y_p is determined from:

$$\tau_{xy}(y_p) = \tau_y = |c|\left[\frac{h}{2} - y_p\right] \quad \Rightarrow$$
$$y_p = \frac{h}{2} - \frac{\tau_y}{|c|} \tag{3.7.28}$$

The fluid layer in the region $y_p \leq y \leq h - y_p$ flows as solid, undeformed material. The flow in the region is called a *plug flow*, see Fig. 3.11.

In the region $0 \leq y \leq y_p$ the velocity gradient dv/dy is positive, and the Eqs. (3.7.25) and (3.7.27) give:

$$\frac{dv}{dy} = \frac{|c|}{\mu}\left[\frac{h}{2} - y\right] - \frac{\tau_y}{\mu}, \quad y \leq y_p \tag{3.7.29}$$

Integration of Eq. (3.7.29), followed by the boundary condition $v(0) = 0$, gives:

$$v(y) = \frac{|c|h^2}{2\mu}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right] - \frac{\tau_y}{\mu}y, \quad y \leq y_p \tag{3.7.30}$$

The velocity of the plug $v_p = v(y_p)$ becomes:

$$v_p = \frac{|c|h^2}{8\mu} - \frac{h}{2\mu}\tau_y + \frac{1}{2\mu|c|}\tau_y^2 \tag{3.7.31}$$

The volumetric flow becomes:

$$Q = v_p\left(h - 2y_p\right) + 2\int_0^{y_p} v(y)\,dy \quad \Rightarrow \quad Q = \frac{|c|h^3}{12\mu} - \frac{h^2}{4\mu}\tau_y + \frac{1}{2\mu\,c^2}\tau_y^3 \qquad (3.7.32)$$

3.8 Pipe Flow

Laminar flow of fluid in a pipe may be treated as a flow with straight, parallel stream lines. A condition for this is that the diameter of the pipe does not vary too much and that the radius of curvature of the pipe is very large compared to the diameter of the pipe.

In this section we shall determine the velocity profile over a cross section of the pipe and derive a formula for the volumetric flow Q through the pipe, when it is assumed that the fluid may be modeled as a Newtonian fluid, a power law fluid, and a Bingham fluid. The flow is driven by a pressure gradient.

We assume steady laminar flow with the velocity profile in cylindrical coordinates (R, θ, z):

$$v_z = v(R), \quad v(d/2) = 0, \quad v_R = v_\theta = 0 \qquad (3.8.1)$$

When the velocity profile $v(R)$ has been found, the volumetric flow is determined as follows. The volume of fluid flowing through a ring element dA, see Fig. 3.12, is per unit time: $dQ = v(R) \cdot dA = v(R) \cdot (2\pi R \cdot dR)$. Then:

$$Q = 2\pi \int_0^{d/2} R\,v(R)\,dR \qquad (3.8.2)$$

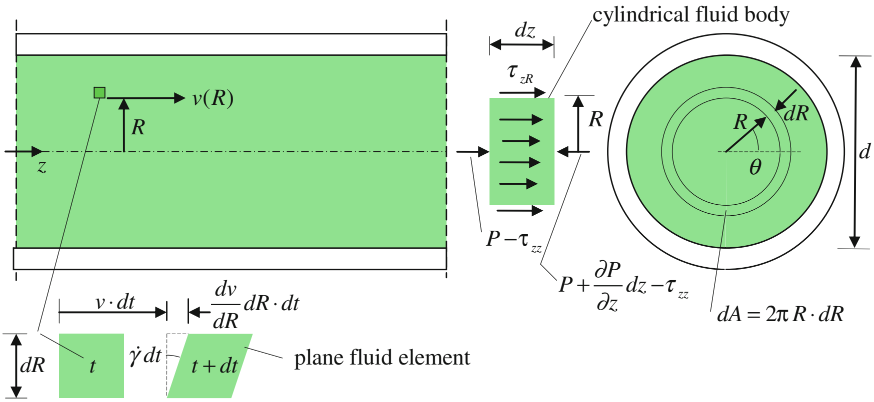


Fig. 3.12 Pipe flow

Due to the symmetry of the flow $\tau_{z\theta} = \tau_{R\theta} = 0$. Because the velocity field is a function of R only, the stresses are also functions of R alone. Thus:

$$\tau_{RR}(R), \quad \tau_{\theta\theta}(R), \quad \tau_{zz}(R), \quad \tau_{zR}(R), \quad \tau_{z\theta} = \tau_{R\theta} = 0 \tag{3.8.3}$$

We use modified pressure P and apply the general result from Sect. 3.6 that the gradient of the modified pressure is constant in the flow direction, see Eq. (3.6.6):

$$\frac{\partial P}{\partial z} = c \text{ (a constant)} \tag{3.8.4}$$

Figure 3.12 shows a cylindrical fluid body with radius R and length dz, and subjected to stresses. The velocity field (3.8.1) gives zero acceleration. Because the extra stress τ_{zz} is only dependent on R, Euler's 1 axiom provides the following equilibrium equation for the cylindrical fluid body:

$$\tau_{zR} \cdot (2\pi R \cdot dz) - \frac{\partial P}{\partial z} dz \cdot \left(\pi R^2\right) = 0 \quad \Rightarrow$$

$$\tau_{zR} = \tau_{zR}(R) = \frac{c}{2} R \tag{3.8.5}$$

We assume the flow to be in the positive z-direction. That implies that the pressure gradient c must be negative. The shear stress from the pipe wall is therefore:

$$\tau_0 = |\tau_{zR}(d/2)| = \frac{|c|d}{4} \tag{3.8.6}$$

The result (3.8.5) may also be obtained from the equations of motion (3.3.28–3.3.30) in cylindrical coordinates, which in the present case are reduced to:

$$0 = -\frac{\partial P}{\partial R} + \frac{1}{R}\frac{d}{dR}(R\tau_{RR}) - \frac{1}{R}\tau_{\theta\theta}, \quad 0 = -\frac{1}{R}\frac{\partial P}{\partial \theta}, \quad 0 = -\frac{\partial P}{\partial z} + \frac{1}{R}\frac{d}{dR}(R\tau_{zR}) \tag{3.8.7}$$

To obtain these equations we have taken the following into consideration:

(1) The velocity field is independent of the coordinates z and θ, i.e. the equation (3.8.1).
(2) The extra stresses, being the result of the velocity field, are also independent of z and θ, i.e. the equation (3.8.3).
(3) The shear stresses $\tau_{\theta z}$ and $\tau_{R\theta}$ on planes through the axis of the pipe are zero due to symmetry, i.e. the equation (3.8.3).

The equations of motion (3.8.7) imply the result (3.8.4) and that $P = P(R,z)$. The third of the equation (3.8.7) then gives:

$$\frac{d}{dR}(R\tau_{zR}) = cR \quad \Rightarrow \quad R\tau_{zR} = \frac{c}{2}R^2 + C_1 \tag{3.8.8}$$

The constant of integration C_1 is determined from the symmetry condition: $\tau_{zR}(0) = 0$. This gives $C_1 = 0$, and τ_{zR} is given by the expression in Eq. (3.8.5).

The equilibrium equation (3.8.5) will now be used to determine the velocity profile $v(R)$ and the volumetric flow Q through the pipe for three fluid models: the Newtonian fluid, the power law fluid, and the Bingham fluid.

Newtonian Fluid. Figure 3.12 shows the deformation during a time increment dt of a small, plane fluid element. From the figure we derive the shear rate $\dot{\gamma} \equiv \dot{\gamma}_{zR} = dv/dR$, which is the only non-zero deformation rate in the $R\theta z-$coordinate system. In order to use the constitutive equation (3.4.1) we may imagine a local Cartesian coordinate system with axes coinciding with the edges of the small plane element in Fig. 3.12. Then the following extra stresses are obtained:

$$\tau_{zR} = \tau_{Rz} = \mu \frac{dv}{dR}, \quad \tau_{RR} = \tau_{\theta\theta} = \tau_{zz} = \tau_{R\theta} = \tau_{\theta R} = \tau_{\theta z} = \tau_{z\theta} = 0 \tag{3.8.9}$$

Compare with equation (3.7.8). The general set of constitutive equations of a Newtonian fluid in a cylindrical coordinate system can be found in Sect. 4.3.

The Eqs. (3.8.5) and (3.8.9) are combined to give:

$$\frac{dv}{dR} = \frac{cR}{2\mu} \tag{3.8.10}$$

With the sticking condition $v(d/2) = 0$, the equation is integrated to give:

$$v(R) = v_0\left[1 - \left(\frac{2R}{d}\right)^2\right], \quad v_0 = \frac{|c|\,d^2}{16\mu} \tag{3.8.11}$$

Figure 3.13a shows the velocity profile for a Newtonian fluid. The volumetric flow Q is determined from Eq. (3.8.2):

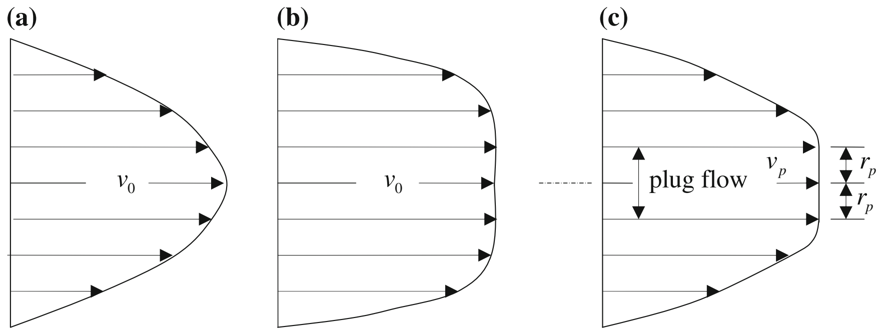


Fig. 3.13 Velocity profiles in pipe flow. **a** Newtonian fluid, **b** power law fluid $n = 0.2$, **c** Bingham fluid

$$Q = 2\pi v_0 \int_0^{d/2} \left[R - \frac{4R^3}{d^2} \right] dR \quad \Rightarrow$$

$$Q = \frac{v_0}{2} A, \quad A = \frac{\pi d^2}{4} \quad \Rightarrow \quad Q = \frac{\pi d^4}{128\mu} |c| \quad \text{the } \textit{Hagen} - \textit{Poiseuille formula} \tag{3.8.12}$$

The formula was developed independently by Gotthilf Heinrich Ludwig Hagen [1797–1884] and Jean Louis Marie Poiseuille [1797–1869].

Power Law Fluid. The general constitutive equations for this fluid are presented in Chap. 6. For the flow (3.8.1) the extra stresses are:

$$\tau_{Rz} = \tau_{zR} = \eta \frac{dv}{dR}, \quad \eta = K \left| \frac{dv}{dR} \right|^{n-1}, \quad \tau_{RR} = \tau_{\theta\theta} = \tau_{zz} = \tau_{R\theta} = \tau_{\theta R} = 0 \tag{3.8.13}$$

The Eqs. (3.8.5) and (3.8.13) are now combined to give:

$$\frac{dv}{dR} = - \left[\frac{|c| R}{2K} \right]^{1/n} \tag{3.8.14}$$

Here we have used the assumption that the pressure gradient c is negative. Integration and use of the boundary condition $v(d/2) = 0$ yield:

$$v(R) = v_0 \left[1 - \left(\frac{2R}{d} \right)^{1/n+1} \right], \quad v_0 = \left(\frac{|c| d}{4K} \right)^{1/n} \frac{nd}{2(1+n)} \tag{3.8.15}$$

Figure 3.13b shows the velocity profile for the power law index $n = 0.2$.

The volumetric flow Q is determined from the Eq. (3.8.2):

$$Q = 2\pi \int_0^{d/2} R\, v(R)\, dR = 2\pi v_o \int_0^{d/2} \left[R - \left(\frac{2}{d} \right)^{1+1/n} (R)^{2+1/n} \right] dR = v_o \frac{1+n}{1+3n} \frac{\pi d^2}{4} \quad \Rightarrow$$

$$Q = v_o \frac{1+n}{1+3n} \frac{\pi d^2}{4} = \left(\frac{|c| d}{4K} \right)^{1/n} \frac{n}{1+3n} \frac{\pi d^3}{8} \tag{3.8.16}$$

For a Newtonian fluid $n = 1$ and K is replaced by μ, and we see that Eq. (3.8.15) gives Eq. (3.8.11) and Eq. (3.8.16) gives Eq. (3.8.12).

Bingham Fluid. The general form of the constitutive equations for this fluid model is presented in Sect. 6.1. The relevant constitutive equations for the flow given by Eq. (3.8.1) are:

$$\begin{aligned}\tau_{Rz} &= \tau_{zR} = \left[\mu + \frac{\tau_y}{|dv/dR|}\right] \frac{dv}{dR} \text{ when } \frac{dv}{dR} \neq 0 \ , \ |\tau_{Rz}| = |\tau_{zR}| \leq \tau_y \text{ when } \frac{dv}{dR} = 0 \\ \tau_{RR} &= \tau_{\theta\theta} = \tau_{zz} = \tau_{R\theta} = \tau_{\theta R} = 0\end{aligned} \tag{3.8.17}$$

τ_y is a *yield shear stress*. From the equilibrium equation (3.8.5) it follows that:

$$|\tau_{zR}| \leq \tau_y \quad \text{when } R \leq r_p = \frac{2\tau_y}{|c|} \tag{3.8.18}$$

The Eqs. (3.8.18) and (3.8.17) show that inside a cylindrical surface of radius r_p the material flows like solid plug.

In order to determine the velocity profile $v(R)$ we combine the constitutive equations (3.8.17) and the equilibrium equation (3.8.5). Because $c < 0$ and $dv/dR \leq 0$, we obtain:

$$\frac{dv}{dR} = -\frac{|c|}{2\mu} R + \frac{\tau_y}{\mu} \tag{3.8.19}$$

Integration and the boundary condition $v(d/2) = 0$ yield:

$$v(R) = \frac{|c|d^2}{16\mu}\left[1 - \left(\frac{2R}{d}\right)^2\right] - \frac{\tau_y d}{2\mu}\left[1 - \frac{2R}{d}\right], \quad r_p \leq R \leq \frac{d}{2} \tag{3.8.20}$$

The velocity of the solid plug is:

$$v_p = v(r_p) = \frac{|c|d^2}{16\mu}\left[1 - \frac{4\tau_y}{|c|d}\right]^2 \tag{3.8.21}$$

It follows from this result that no flow results if $|c| < 4\tau_y/d$:

$$|c| \leq \frac{4\tau_y}{d} \quad \Rightarrow \quad v_p = 0 \text{ and } r_p = \frac{d}{2} \quad \Leftrightarrow \quad \text{no flow} \tag{3.8.22}$$

Figure 3.13c shows the velocity profile for a Bingham fluid.

The volumetric flow Q is determined from Eq. (3.8.2).

$$\begin{aligned} Q &= v_p \cdot \pi r_p^2 + 2\pi \int_{r_p}^{d/2} v(R)\, R\, dR \quad \Rightarrow \\ Q &= \frac{|c|\pi d^4}{128\mu}\left[1 - \frac{16}{3}\frac{\tau_y}{|c|d} + \frac{256}{3}\left(\frac{\tau_y}{|c|d}\right)^4\right]\end{aligned} \tag{3.8.23}$$

For the special $\tau_y = 0$ the Eqs. (3.8.20) and (3.8.23) correspond with the Eqs. (3.8.11) and (3.8.12) for a Newtonian fluid.

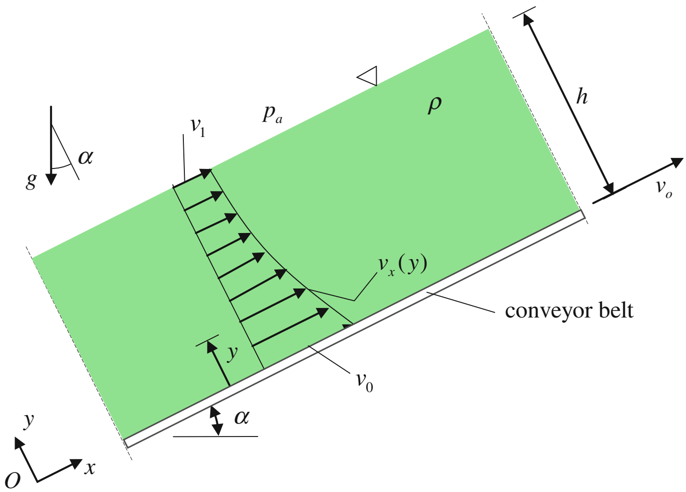


Fig. 3.14 Film flow on a conveyor belt

3.9 Film Flow

Figure 3.14 is illustrating the transport of fluid as a film with constant thickness h on a wide conveyor belt. The width of the belt is b. The belt is inclined an angle α with respect to the horizontal plane and moves with constant velocity v_0. The fluid density is ρ.

We assume the velocity field:

$$v_x = v(y), \quad v_y = v_z = 0 \tag{3.9.1}$$

and want to determine the velocity profile $v(y)$ and the volumetric flow Q. It is assumed that the fluid sticks to the belt. The atmospheric pressure on the free surface is p_a. The boundary conditions for the flow are then:

$$v_x(0) = v_0, \quad \sigma_{yy} = -p + \tau_{yy} = -p_a \quad \text{and} \quad \tau_{xy} = 0 \quad \text{at } y = h \tag{3.9.2}$$

Due to the presence of the free surface at $y = h$, which does not exist when the fluid is at rest, it is not practical in this case to introduce the modified pressure P in the equations of motion (3.3.24). The velocity field (3.9.2) implies no acceleration and the body force is:

$$\mathbf{b} = -g \sin\alpha\, \mathbf{e}_x - g\cos\alpha\, \mathbf{e}_y \tag{3.9.3}$$

Because of the assumption (3.9.2) we may assume that the extra stresses τ_{ik} are functions of y only, and that τ_{zy} and τ_{zx} vanish due to symmetry.

$$\tau_{ik} = \tau_{ik}(y), \quad \tau_{zy} = \tau_{zx} = 0 \tag{3.9.4}$$

The equations of motion (3.3.24) provide the following three equations of equilibrium:

$$0 = -\frac{\partial p}{\partial x} + \frac{d\tau_{xy}}{dy} - \rho g \sin\alpha, \quad 0 = -\frac{\partial p}{\partial y} + \frac{d\tau_{yy}}{dy} - \rho g \cos\alpha, \quad 0 = -\frac{\partial p}{\partial z} \tag{3.9.5}$$

From these equations we shall first derive a general expression for the pressure p. The Eqs. (3.9.5) imply that $p = p(x,y)$ and that:

$$\begin{gathered}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) = 0, \quad \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = 0 \quad \Rightarrow \\ \frac{\partial p}{\partial x} = \text{ constant} = c\end{gathered} \tag{3.9.6}$$

Integrations of Eq. (3.9.6) and the second of the Eqs. (3.9.5) yield:

$$p = p(x,y) = \tau_{yy}(y) - (\rho g \cos\alpha)y + cx + C_1 \tag{3.9.7}$$

From the boundary conditions (3.9.2) and the results (3.9.4) and (3.9.7) we obtain:

$$\begin{gathered}-p(x,h) + \tau_{yy}(h) = -p_a \quad \Rightarrow \\ c = \frac{\partial p}{\partial x} = 0, \quad C_1 = p_a + (\rho g \cos\alpha)h) - \tau_{yy}(h)\end{gathered} \tag{3.9.8}$$

The general expression for the pressure is therefore:

$$p = p(y) = p_a + \tau_{yy}(y) - \tau_{yy}(h) + (h-y)\rho g \cos\alpha \tag{3.9.9}$$

The pressure is thus only a function of y.

With this expression for the pressure and $\tau_{xy}(h) = 0$, from the boundary conditions (3.9.2), the first of the equilibrium equations (3.9.5) is integrated to give:

$$\tau_{xy}(y) = -(h-y)\rho g \sin\alpha \tag{3.9.10}$$

Next, we shall investigate the flow of three different fluid models: the Newtonian fluid, the power law fluid, and the Bingham fluid.

Newtonian Fluid and Power Law Fluid. For the flow (3.9.1) the constitutive equations for a power law fluid are given by the Eq. (3.7.18). The Newtonian fluid model is obtained by setting $K = \mu$ and $n = 1$.

The Eqs. (3.9.10) and (3.7.18) are combined to give:

$$K\left|\frac{dv}{dy}\right|^{n-1}\frac{dv}{dy} = -(h-y)\rho g \sin\alpha \tag{3.9.11}$$

Because the shear stress τ_{xy} according to Eq. (3.9.10) is negative, the velocity gradient is negative. Thus Eq. (3.9.11) may be rewritten to:

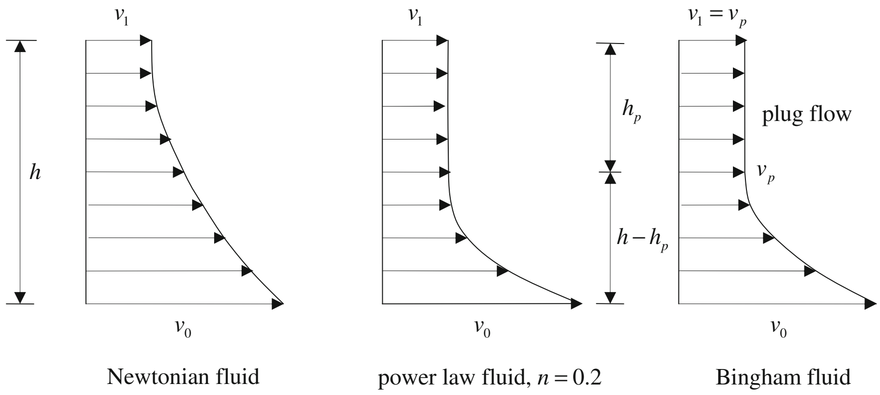


Fig. 3.15 Velocity profiles for film flow

$$\frac{dv}{dy} = -\left[\frac{\rho g \sin \alpha}{K}\right]^{1/n} (h - y)^{1/n} \tag{3.9.12}$$

The equation is integrated and the boundary condition $v_x = v(0) = v_o$ is used. The result is:

$$v(y) = v_0 - \left[\frac{\rho g \sin \alpha}{K}\right]^{1/n} \frac{n h^{1+1/n}}{1+n} \left[1 - \left(1 - \frac{y}{h}\right)^{1+1/n}\right] \tag{3.9.13}$$

Figure 3.15 shows the velocity profile for the Newtonian fluid and for the power law fluid. The velocity of the free surface of the fluid is:

$$v_1 = v(h) = v_o - \left[\frac{\rho g \sin \alpha}{K}\right]^{1/n} \frac{n h^{1+1/n}}{1+n} \tag{3.9.14}$$

The volumetric flow Q becomes:

$$Q = b \int_0^h v(y)\, dy = v_o bh - \left[\frac{\rho g \sin \alpha}{K}\right]^{1/n} \frac{n}{1+2n} bh^{2+1/n} \tag{3.9.15}$$

The flow of the Newtonian fluid , $K = \mu$ and $n = 1$, is represented by:

$$v(y) = v_o - \frac{\rho g h^2 \sin \alpha}{2\mu} \left[2\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$
$$v_1 = v_o - \frac{\rho g h^2 \sin \alpha}{2\mu}, \quad Q = v_o bh - \frac{\rho g \sin \alpha}{3\mu} bh^3 \tag{3.9.16}$$

Bingham Fluid. The relevant constitutive equations for the flow (3.9.1) are given by the Eq. (3.7.25).

The equilibrium equation (3.9.10) and the constitutive equation (3.7.25) imply that a top layer of the fluid film flows as a solid plug. The thickness h_p of the plug is determined from Eq. (3.9.10) for $y = h - h_p$:

$$\left|\tau_{xy}(h - h_p)\right| = \rho g \sin\alpha \, h_p = \tau_y \quad \Rightarrow \quad h_p = \frac{\tau_y}{\rho g \sin\alpha} \tag{3.9.17}$$

The equilibrium equation (3.9.10) and the constitutive equation (3.7.25) are now combined. Because dv/dy is negative, we write the result of the combination as:

$$\frac{dv}{dy} = -\frac{\rho g \sin\alpha}{\mu}(h - y) + \frac{\tau_y}{\mu}, \quad y \le h - h_p \tag{3.9.18}$$

Integration, followed by application of the boundary condition $v(0) = v_0$, gives:

$$\begin{aligned} v(y) &= v_0 + \frac{\tau_y}{\mu} y - \frac{\rho g \sin\alpha \, h^2}{2\mu}\left[\frac{2y}{h} - \left(\frac{y}{h}\right)^2\right], \quad y \le h - h_p \\ v_1 &= v_p = v(h_p) = v_0 + \frac{\tau_y h}{\mu} - \frac{\tau_y^2}{2\mu\rho g \sin\alpha} - \frac{\rho g \sin\alpha \, h^2}{2\mu} \end{aligned} \tag{3.9.19}$$

v_p is the plug velocity. Figure 3.15 shows the velocity profile, which consists of a rectangular part and a parabolic part. The volumetric flow Q may be therefore be computed as:

$$Q = v_p \, bh + \frac{1}{3}(v_0 - v_p) b (h - h_p) \quad \Rightarrow$$

$$Q = v_0 \, bh - \frac{\rho g \sin\alpha \, bh^3}{3\mu} + \frac{\tau_y b h^2}{2\mu} - \frac{\tau_y^3 \, b}{6\mu(\rho g \sin\alpha)^2} \tag{3.9.20}$$

3.10 Energy Equation

A fluid body of volume V, Fig. 3.16, has the *internal energy*:

$$E = \int_V e\rho \, dV \tag{3.10.1}$$

$e = e(x, t)$ is the *specific internal energy e(x,t)* and represents the internal energy per unit mass.

Heat is supplied to the fluid body by:

1. Heat conduction through the surface area A of the body, and

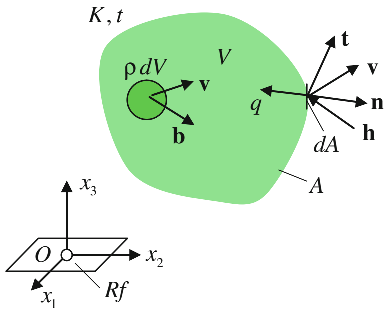


Fig. 3.16 Fluid body

2. Heat generated from heat sources in the body and/or by radiation into the body from external distant radiating bodies.

We shall include only the first kind of heat supply. The heat conducted through the surface per unit time and per unit area is given by the *heat flux q*. Then the total heat supplied per unit time to the body is:

$$\dot{Q} = \int_A q\, dA \tag{3.10.2}$$

The quantity $Q\cdot$ is called the *heat power* supplied to the body.

The work done on the body per unit time, is expressed by the *mechanical power*:

$$P = \int_V \mathbf{b} \cdot \mathbf{v}\, \rho\, dV + \int_A \mathbf{t} \cdot \mathbf{v}\, dA \tag{3.10.3}$$

The mechanical power may change the *kinetic energy K* of the body:

$$K = \int_V \frac{1}{2} v^2 \rho\, dV \tag{3.10.4}$$

The *first law of thermodynamics* may be expressed by the equation:

$$P + \dot{Q} = \dot{E} + \dot{K} \tag{3.10.5}$$

The sum of the mechanical power and the heat power supplied to the body per unit time is equal to the time rate of change of the internal energy and the kinetic energy of the body.

By writing the first law of thermodynamics for the *Cauchy tetrahedron* in Fig. 3.6, we can show that the heat flux per unit area q may be expressed by a *heat flux vector* $\mathbf{q}(x,t)$ such that, see Fig. 3.16:

$$q = -\mathbf{h} \cdot \mathbf{n} \tag{3.10.6}$$

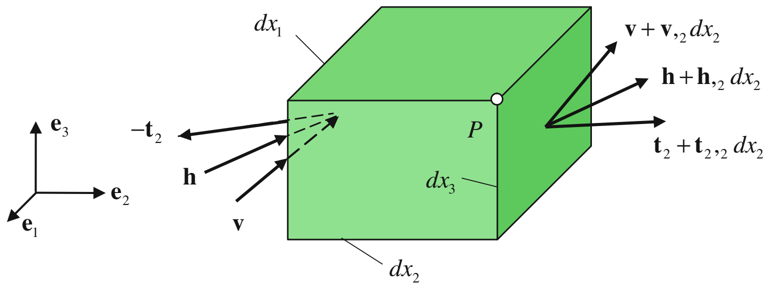


Fig. 3.17 Fluid element subjected to forces and heat flux

The negative sign on the left-hand side is introduced so that the vector gives the direction of the heat flow.

The first law (3.10.5) will now be applied to the fluid element illustrated in Fig. 3.17. The element has volume $dV = dx_1\, dx_2\, dx_3$ and mass $\rho\, dV$. For simplicity the figure is only supplied with the relevant quantities on two surfaces of the element in addition to the body force **b** and the velocity **v**. The net mechanical power supplied to the element is now:

$$\begin{aligned} dP =& \mathbf{b}\cdot\mathbf{v}\,\rho\, dV + [-\mathbf{t}_2\cdot\mathbf{v}\, dx_3\, dx_1 + (\mathbf{t}_2 + \mathbf{t}_{2,2}\, dx_2)(\mathbf{v} + \mathbf{v}_{,2}\, dx_2)(dx_3\, dx_1)] \\ &+ [-\mathbf{t}_1\cdot\mathbf{v}\, dx_2\, dx_3 + (\mathbf{t}_1 + \mathbf{t}_{1,1}\, dx_1)(\mathbf{v} + \mathbf{v}_{,1}\, dx_1)(dx_2\, dx_3)] \\ &+ [-\mathbf{t}_3\cdot\mathbf{v}\, dx_1\, dx_2 + (\mathbf{t}_3 + \mathbf{t}_{3,3}\, dx_3)(\mathbf{v} + \mathbf{v}_{,3}\, dx_3)(dx_1\, dx_2)] \end{aligned}$$

Removing higher order terms, we obtain:

$$dP = [(\mathbf{b}\rho + \mathbf{t}_{k,k})\cdot\mathbf{v} + \mathbf{t}_k\cdot\mathbf{v}_{,k}]\, dV \tag{3.10.7}$$

The heat power supplied to the element is:

$$\begin{aligned} &d\dot{Q} = -(\mathbf{q}_{,2}\, dx_2)\cdot\mathbf{e}_2(dx_3\, dx_1) - (\mathbf{q}_{,1}\, dx_1)\cdot\mathbf{e}_1(dx_2\, dx_3) - (\mathbf{q}_{,3}\, dx_3)\cdot\mathbf{e}_3(dx_1\, dx_2) \Rightarrow \\ &d\dot{Q} = -q_{k,k}\, dV = -\nabla\cdot\mathbf{q}\, dV \end{aligned} \tag{3.10.8}$$

The time rate of change of internal energy and the time rate of change of kinetic energy for the element are:

$$d\dot{E} = \dot{e}\rho dV, \quad d\dot{K} = \frac{d}{dt}\left[\frac{v^2}{2}\rho\, dV\right] = \frac{d}{dt}\left[\frac{v^2}{2}\right]\rho\, dV = \mathbf{v}\cdot\dot{\mathbf{v}}\rho\, dV \tag{3.10.9}$$

The first law of thermodynamics (3.10.5) yields: $dP + d\dot{Q} = d\dot{E} + d\dot{K} \quad \Rightarrow$

$$\begin{aligned} &[(\mathbf{b}\rho + \mathbf{t}_{k,k})\cdot\mathbf{v} + \mathbf{t}_k\cdot\mathbf{v}_{,k}]\, dV - \nabla\cdot\mathbf{q}\, dV = \dot{e}\,\rho\, dV + \mathbf{v}\cdot\dot{\mathbf{v}}\,\rho\, dV \quad \Rightarrow \\ &[\mathbf{b}\rho + \mathbf{t}_{k,k} - \dot{\mathbf{v}}\,\rho]\cdot\mathbf{v} + \mathbf{t}_k\cdot\mathbf{v}_{,k} - \nabla\cdot\mathbf{q} = \dot{e}\,\rho \end{aligned} \tag{3.10.10}$$

The term $\omega = \mathbf{t}_k \cdot \mathbf{v}_{,k}$ is called the *stress power per unit volume*, or the *deformation power per unit volume*, and represents the work done by the stresses on the fluid per unit time and per unit volume. Using Eq. (3.3.5), we may express the stress power by coordinate stresses, and using the expressions (3.3.18) for the coordinate stresses in a fluid, we obtain:

$$\omega = \mathbf{t}_k \cdot \mathbf{v}_{,k} = \sigma_{ik} v_{i,k} = (-p\,\delta_{ik} + \tau_{ik}) v_{i,k} = -p\,\delta_{ik} v_{i,k} + \tau_{ik} v_{i,k} = -p\,v_{k,k} + \tau_{ik} v_{i,k} \quad \Rightarrow$$
$$\omega = \mathbf{t}_k \cdot \mathbf{v}_{,k} = \sigma_{ik} v_{i,k} = -p\,v_{k,k} + \tau_{ik} v_{i,k} = -p\,\nabla \cdot \mathbf{v} + \tau_{ik} v_{i,k} \tag{3.10.11}$$

The physical interpretation of the stress power will be presented in Sect. 4.1.3.

The terms in the brackets in Eq. (3.10.10) vanish due to the equation of motion (3.3.16), and we are left with the *thermal energy balance equation for a fluid particle*:

$$\rho \dot{e} = -\nabla \cdot \mathbf{q} + \omega \tag{3.10.12}$$

The result may also be called the *thermal energy balance equation at a place*.

For thermal isotropic fluids *Fourier's heat conduction equation* applies, Jean Baptiste Joseph Fourier [1768–1830]:

$$\mathbf{q} = -k \nabla \Theta \tag{3.10.13}$$

$\Theta = \Theta(x, t)$ is the *temperature*. k is the *thermal conductivity* and is a function of the temperature, but is often taken to be a constant parameter. In the following k will be treated as a constant.

For an *incompressible fluid* we may assume that the specific internal energy is a function of the temperature alone:

$$e = e(\Theta) \tag{3.10.14}$$

Specific heat c is defined as the change in internal energy per unit mass and per unit temperature:

$$c = \frac{de}{d\Theta} \tag{3.10.15}$$

The specific heat c varies only slightly with temperature. By the definitions (3.10.14) and (3.10.15) the thermal energy balance equation (3.10.12) may be rewritten to:

$$\rho c \dot{\Theta} = k \nabla^2 \Theta + \omega \tag{3.10.16}$$

For an incompressible Newtonian fluid the extra stresses τ_{ik} are presented by the constitutive equation (3.4.1) with $\nabla \cdot \mathbf{v} = 0$. Hence the stress power per unit volume, equation (3.10.11) becomes:

$$\begin{aligned}\omega &= -p\nabla\cdot\mathbf{v} + \tau_{ik}v_{i,k} = \tau_{ik}v_{i,k} = \mu(v_{i,k}+v_{k,i})\\ v_{i,k} &= \frac{\mu}{2}[(v_{i,k}+v_{k,i})(v_{i,k}+v_{k,i})] \quad \Rightarrow\\ \omega &= \frac{\mu}{2}[(v_{i,k}+v_{k,i})(v_{i,k}+v_{k,i})]\end{aligned} \tag{3.10.17}$$

The expression in the brackets is a sum of squares. Thus the stress power per unit volume is always positive and represents a dissipation of mechanical energy into thermal energy.

3.10.1 Energy Equation in Cartesian Coordiantes (x, y, z)

$$\begin{aligned}&\rho c\left[\frac{\partial\Theta}{\partial t} + v_x\frac{\partial\Theta}{\partial x} + v_y\frac{\partial\Theta}{\partial y} + v_z\frac{\partial\Theta}{\partial z}\right]\\ &= k\left[\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2}\right] + \tau_{xx}\frac{\partial v_x}{\partial x} + \tau_{xy}\left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right] + \tau_{yy}\frac{\partial v_y}{\partial y}\\ &\quad + \tau_{yz}\left[\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right] + \tau_{zx}\left[\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right] + \tau_{zz}\frac{\partial v_z}{\partial z}\end{aligned} \tag{3.10.18}$$

3.10.2 Energy Equation in Cylindrical Coordinates (R, θ, z)

$$\begin{aligned}&\rho c\left[\frac{\partial\Theta}{\partial t} + v_R\frac{\partial\Theta}{\partial R} + \frac{v_\theta}{R}\frac{\partial\Theta}{\partial\theta} + v_z\frac{\partial\Theta}{\partial z}\right]\\ &= k\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Theta}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Theta}{\partial\theta^2} + \frac{\partial^2\Theta}{\partial z^2}\right] + \tau_{RR}\frac{\partial v_R}{\partial R} + \tau_{R\theta}\left[\frac{1}{R}\frac{\partial v_R}{\partial\theta} + R\frac{\partial}{\partial R}\left(\frac{v_\theta}{R}\right)\right]\\ &\quad + \tau_{zR}\left[\frac{\partial v_z}{\partial R} + \frac{\partial v_R}{\partial z}\right] + \tau_{\theta\theta}\left[\frac{1}{R}\frac{\partial v_\Theta}{\partial\theta} + \frac{v_R}{R}\right] + \tau_{\theta z}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{R}\frac{\partial v_z}{\partial\theta}\right] + \tau_{zz}\frac{\partial v_z}{\partial z}\end{aligned} \tag{3.10.19}$$

3.10.3 Temperature Field in Steady Simple Shear Flow

A Newtonian fluid flows between two parallel plates, Fig. 3.18. The bottom plate is at rest, while the top plate moves with a constant velocity v_1. Both plates are kept at a constant temperature Θ_0. The modified pressure gradient c is zero, and the viscosity μ and the thermal conductivity k are assumed to be constants. Steady

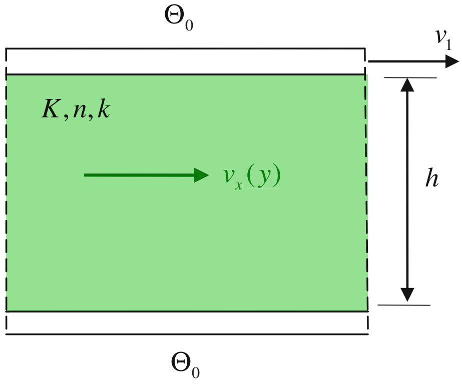


Fig. 3.18 Steady shear flow

laminar flow is assumed. We want to derive the expression for the *temperature field*, which we assume to be of the form $\Theta(y)$.

From Sect. 3.7 we obtain the velocity field, Fig. 3.10a:

$$v_x(y) = \frac{v_1}{h} y, \quad v_y = v_z = 0 \tag{3.10.20}$$

and the extra stresses τ_{ik} of which only one is different from zero, equation (3.7.8):

$$\tau_{xy} = \mu \frac{dv_x}{dy} = \frac{\mu v_1}{h} \tag{3.10.21}$$

The energy equation (3.10.18) is by the Eqs. (3.10.20) and (3.10.21), and the assumption $\Theta = \Theta(y)$, reduced to:

$$k \frac{d^2\Theta}{dy^2} = -\left(\frac{\mu v_1}{h}\right)\left(\frac{v_1}{h}\right) \tag{3.10.22}$$

Two integrations yield:

$$\Theta(y) = -\frac{\mu}{k}\left(\frac{v_1}{h}\right)^2 \frac{y^2}{2} + C_1 y + C_2$$

The constants of integration C_1 and C_2 are determined from the boundary conditions:

$$\Theta(0) = \Theta_0 \quad \Rightarrow \quad C_2 = \Theta_0, \quad \Theta(h) = \Theta_0 \quad \Rightarrow \quad C_1 = \frac{\mu v_1^2}{2hk}$$

Hence the temperature field is found to be:

$$\Theta(y) = \Theta_0 + \frac{\mu v_1^2}{2k}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right], \quad \Theta_{max} = \Theta_0 + \frac{\mu v_1^2}{8k} \quad \text{at } y = \frac{h}{2}$$

Chapter 4
Deformation Kinematics

The surface of a fluid body that has been at rest for a long time, is only subjected to the isotropic pressure p. Motion and deformation of the fluid result in extra stresses normal to and tangential to the surface of the fluid body, and are due to the deformation history of the fluid. The *constitutive equations* defining the fluid models express the relationship between the stresses in a fluid and the deformation history of the fluid, which in turn is given by the history of the velocity field. In the present chapter we shall first analyze the properties of a general velocity field. Then some standard types of laminar flows will be presented. These flows occur in many practical problems and are also characteristic of flows in experiments used to determine material parameters for a particular fluid.

4.1 Rates of Deformation and Rates of Rotation

Figure 4.1 illustrates a fluid particle P at the place $\mathbf{r} = x_i \mathbf{e}_i$ at the time t. The particle has the velocity $\mathbf{v}(x,t)$ and gets during a short time increment dt the displacement $\mathbf{v}dt$. A neighbor particle $\bar{P}$ at the place $\mathbf{r} + d\mathbf{r} = (x_i + dx_i)\mathbf{e}_i$ gets the displacement $(\mathbf{v} + d\mathbf{v})dt$. The differential $d\mathbf{v}$ has the components:

$$dv_i = \frac{\partial v_i}{\partial x_k} dx_k = v_{i,k}\, dx_k \tag{4.1.1}$$

The quantities $v_{i,k}$ are called the *velocity gradients* and the matrix:

$$L = (L_{ik}) \equiv (v_{i,k}) \tag{4.1.2}$$

is the *velocity gradient matrix.*

We shall now investigate the deformation of a small fluid element due to the displacement field:

$$(\mathbf{v} + d\mathbf{v})dt = (v_i + v_{i,k}\, dx_k)dt\, \mathbf{e}_i \tag{4.1.3}$$

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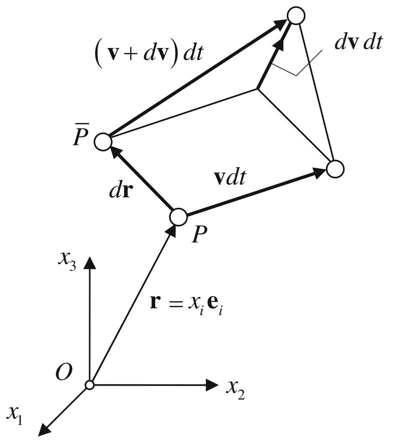


Fig. 4.1 Displacement of fluid particles

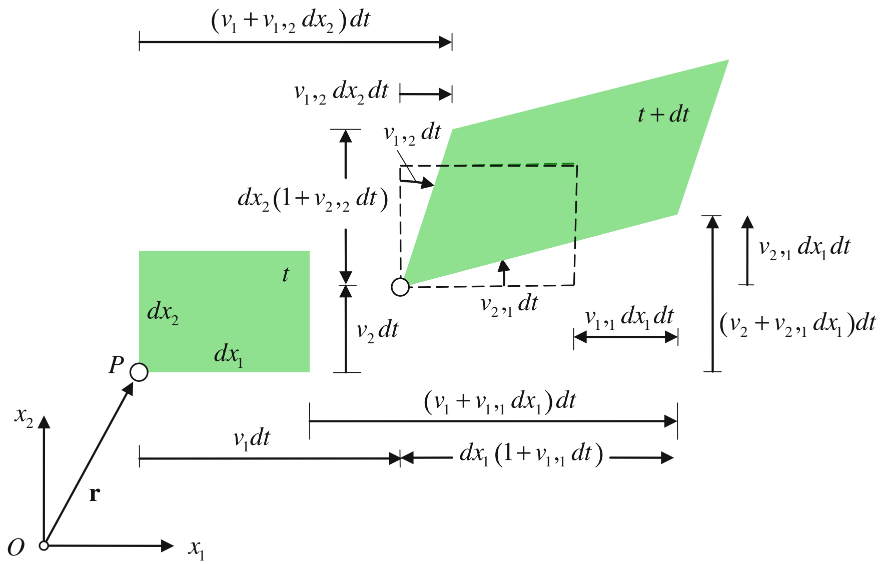


Fig. 4.2 Deformation of a fluid element

As shown in Fig. 4.2 the element is at the time t extended from three material line elements dx_1, dx_2, and dx_3 from the particle P. For simplicity, Fig. 4.2 illustrates the situation in two dimensions, and the element is shown at the time t and the time $t + dt$.

The change in length per unit length and per unit time of a material line element extended from particle P is called the *rate of longitudinal strain* at P in the direction of the line element. The line element dx_1 in Fig. 4.2 gets a change in length $v_{1,1}\, dx_1 dt$. The rate of longitudinal strain at P in the x_1–direction is therefore:

$$\dot{\varepsilon}_1 = \frac{v_{1,1}\, dx_1 dt}{dx_1 dt} = v_{1,1} \tag{4.1.4}$$

The negative change per unit time of a material right angle at particle P is called a *rate of shear strain*, or for short the *shear rate* at P. The right angle between the line element dx_1 and dx_2 is reduced by $v_{1,2}\, dt + v_{2,1}\, dt$. The shear rate at P for the two directions x_1 and x_2 is therefore:

$$\dot{\gamma}_{12} = v_{1,2} + v_{2,1} \tag{4.1.5}$$

With respect to the line elements dx_1, dx_2, and dx_3 there are 3 rates of longitudinal strain:

$$\dot{\varepsilon}_i = v_{i,i}\,, \quad i = 1, 2, \text{ or } 3 \tag{4.1.6}$$

and 3 rates of shear strain:

$$\dot{\gamma}_{ik} = v_{i,k} + v_{k,i}\,, \quad i \neq k \tag{4.1.7}$$

Note that $\dot{\gamma}_{ik}$ is symmetric: $\dot{\gamma}_{ik} = \dot{\gamma}_{ki}$.

The change in volume per unit volume and per unit time at a particle P is called the *rate of volumetric strain* at P. For the fluid element in Fig. 4.2 the volume at time t is:

$$dV(t) = dx_1 dx_2 dx_3 \tag{4.1.8}$$

At time $t + dt$ the volume of the same fluid element has become:

$$\begin{aligned} dV(t+dt) &= [dx_1(1 + v_{1,1}\, dt)][dx_2(1 + v_{2,2}\, dt)][dx_3(1 + v_{3,3}\, dt)] \\ &= [dx_1(1 + \dot{\varepsilon}_1 \cdot dt)][dx_2(1 + \dot{\varepsilon}_2 \cdot dt)][dx_3(1 + \dot{\varepsilon}_3 \cdot dt)] \\ &= dV(t) + (\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3) dt\, dV(t) + \text{higher order terms} \quad \Rightarrow \end{aligned}$$

$$dV(t+dt) = dV(t) + (\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3) dt\, dV(t) \tag{4.1.9}$$

The rate of volumetric strain thus becomes:

$$\dot{\varepsilon}_V = \frac{dV(t+dt) - dV(t)}{dV(t) \cdot dt} \quad \Rightarrow$$

$$\dot{\varepsilon}_V = \dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3 = v_{k,k} = div\, \mathbf{v} \equiv \nabla \cdot \mathbf{v} \tag{4.1.10}$$

The formulas (4.1.6), (4.1.7), and (4.1.10) show that the seven characteristic measures of rates of deformation: $\dot{\varepsilon}_i, \dot{\gamma}_{ik}$, and $\dot{\varepsilon}_V$, may be expressed by the velocity gradients $v_{i,k}$. The strain rates $\dot{\varepsilon}_i$ and $\dot{\gamma}_{ik}$ are presented in the *rate of deformation matrix*:

$$D = \left(\frac{1}{2}\dot{\gamma}_{ik}\right) = \begin{pmatrix} \dot{\varepsilon}_1 & \frac{1}{2}\dot{\gamma}_{12} & \frac{1}{2}\dot{\gamma}_{13} \\ \frac{1}{2}\dot{\gamma}_{21} & \dot{\varepsilon}_2 & \frac{1}{2}\dot{\gamma}_{23} \\ \frac{1}{2}\dot{\gamma}_{31} & \frac{1}{2}\dot{\gamma}_{32} & \dot{\varepsilon}_3 \end{pmatrix} \qquad (4.1.11)$$

Due to the symmetry in the shear rates $\dot{\gamma}_{ik}$ the rate of deformation matrix has only 6 distinct components, i.e., D is a symmetric matrix:

$$D = D^T \quad \Leftrightarrow \quad D_{ik} = D_{ki} \qquad (4.1.12)$$

D^T denotes the *transposed matrix* to D, i.e., D^T is obtained from D by interchanging rows and columns. The elements in the matrix D may according to the formulas (4.1.6), (4.1.7), and (4.1.11) be expressed as:

$$D_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i}) \quad \Leftrightarrow \quad D = \frac{1}{2}\left(L + L^T\right) \qquad (4.1.13)$$

In general the velocity gradient matrix L contains 9 distinct elements $v_{i,k}$. Because the rate of deformation matrix D only contains 6 distinct elements in the formula (4.1.13), only 6 of the "informations" in L have been used. The other 3 "informations" in L represent rotation of the fluid. This fact will now be demonstrated.

The matrix L is decomposed into a symmetric matrix D and an antisymmetric matrix W:

$$W = \frac{1}{2}\left(L - L^T\right) = -W^T \quad \Leftrightarrow \quad W_{ik} = \frac{1}{2}(v_{i,k} - v_{k,i}) = -W_{ki} \qquad (4.1.14)$$

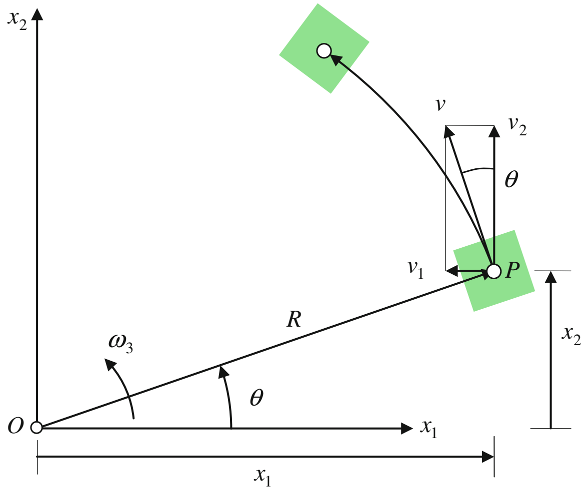
It follows that:

$$L = W + D \quad \Leftrightarrow \quad L_{ik} = W_{ik} + D_{ik} \qquad (4.1.15)$$

The matrix W is called the *rate of rotation matrix* and the elements W_{ik} are the rate of rotation components. Other names of W found in the literature are the *vorticity matrix* and the *spin matrix*, and the elements W_{ik} are called *vorticities* or *spins*. The antisymmetry of W implies that the matrix only contains 3 distinct elements. Thus W represents the three "informations" in the velocity gradient matrix L mentioned above. It will now be demonstrated that W represents a rate of rotation.

Before we demonstrate the meaning of the rate of rotation matrix W for general flow, we consider the special case in Fig. 4.3 of a velocity field that corresponds to a *rigid-body rotation* about the x_3-axis. The angular velocity is ω_3.The fluid particles move in circular, concentric paths with the velocity v proportional with the distance R from the axis of rotation: $v = \omega_3 R$. In vector notation the angular velocity $\boldsymbol{\omega}$, the velocity vector $\mathbf{v}$, and the velocity components v_i in the Ox-system are:

Fig. 4.3 Rigid-body rotation



$$\boldsymbol{\omega} = \omega_3 \mathbf{e}_3, \quad \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = v_i \mathbf{e}_i$$
$$v_1 = -v \sin\theta = -v\frac{x_2}{R} = -\omega_3 x_2, \quad v_2 = v\cos\theta = v\frac{x_1}{R} = \omega_3 x_1, \quad v_3 = 0 \tag{4.1.16}$$

The velocity gradient matrix L, the rate of rotation matrix W, and the rate of deformation matrix D become:

$$L = W = \begin{pmatrix} 0 & -\omega_3 & 0 \\ \omega_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D = 0 \tag{4.1.17}$$

A fluid element surrounding a particle P, see Fig. 4.3, rotates as a rigid body about the x_3-axis.

As a generalization we consider a *rigid-body rotation* about the origin with the angular velocity vector $\boldsymbol{\omega} = \omega_i \mathbf{e}_i$, and we find:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad \Rightarrow \quad v_1 = \omega_2 x_3 - \omega_3 x_2, \quad v_2 = \omega_3 x_1 - \omega_1 x_3, \quad v_3 = \omega_1 x_2 - \omega_2 x_1$$
$$L = (v_{i,k}) = \frac{1}{2}(v_{i,k} - v_{k,i}) = W = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad D = 0 \tag{4.1.18}$$

The three distinct elements of W are in this case given by the components of the angular velocities vector.

Now we shall discuss the implications of any general velocity field $\mathbf{v}(x,t)$. The velocity gradient matrix L:

$$L = \begin{pmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{pmatrix} \tag{4.1.19}$$

is decomposed into three matrices: W, D_1, and D_2, such that:

$$L = W + D_1 + D_2, \quad D = D_1 + D_2 \tag{4.1.20}$$

$$D_1 = \begin{pmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{pmatrix}, \quad D_2 = \frac{1}{2}\begin{pmatrix} 0 & \dot{\gamma}_{12} & \dot{\gamma}_{13} \\ \dot{\gamma}_{21} & 0 & \dot{\gamma}_{23} \\ \dot{\gamma}_{31} & \dot{\gamma}_{32} & 0 \end{pmatrix} \tag{4.1.21}$$

The rate of rotation matrix W is defined by Eq. (4.1.14) and presented by W in Eq. (4.1.18) with:

$$\begin{aligned} \omega_1 &= \frac{1}{2}(v_{3,2} - v_{2,3}) = W_{32}, \quad \omega_2 = \frac{1}{2}(v_{1,3} - v_{3,1}) = W_{13}, \\ \omega_3 &= \frac{1}{2}(v_{2,1} - v_{1,2}) = W_{21} \end{aligned} \tag{4.1.22}$$

The vector $\boldsymbol{\omega} = \omega_i \mathbf{e}_i$ is called the *angular velocity of a fluid particle* and will be interpreted below. It follows from the Eq. (4.1.22) that:

$$\boldsymbol{\omega} = \frac{1}{2}\text{rot } \mathbf{v} \equiv \frac{1}{2}\nabla \times \mathbf{v} \tag{4.1.23}$$

In fluid mechanics it is customary to use the *vorticity vector* **c** rather than the angular velocity vector **w** :

$$\mathbf{c} = \text{rot } \mathbf{v} \equiv \nabla \times \mathbf{v} = 2\boldsymbol{\omega} \tag{4.1.24}$$

In order to demonstrate the physical implication of W, the displacement vector $(\mathbf{v} + d\mathbf{v})dt$ is presented in its component form as follows.

$$(v_i + v_{i,k}\, dx_k)dt = v_i\, dt + L_{ik}\, dx_k\, dt = v_i dt + (W_{ik} + D_{1ik} + D_{2ik})dx_k\, dt \tag{4.1.25}$$

The right-hand side of Eq. (4.1.25) is a sum of four contributions, each representing a characteristic motion of the fluid in the neighborhood of a particle P. Figure 4.4 shall illustrate these motions.

In Fig. 4.4 the cubic fluid element $dV = dx_1\, dx_2\, dx_3$, where $dx_1 = dx_2 = dx_3$, surrounds the particle P. During a short time interval dt the velocity **v** of the particle gives the volume element a translation $\mathbf{v}dt$. The rate of rotation matrix W contributes with a rigid-body rotation about the particle P. The two deformation rate matrices D_1 and D_2 represent a motion of the fluid relative to the particle P: The matrix D_1 gives a motion that is symmetrical with respect to the planes through P parallel to the sides of the element. Each of the three distinct elements in the matrix D_2 represents a motion that is symmetrical with respect to diagonal planes, one of which is shown in Fig. 4.4. The "net rotation" of the fluid in the neighborhood of the P is thus expressed by the rate of rotation W.

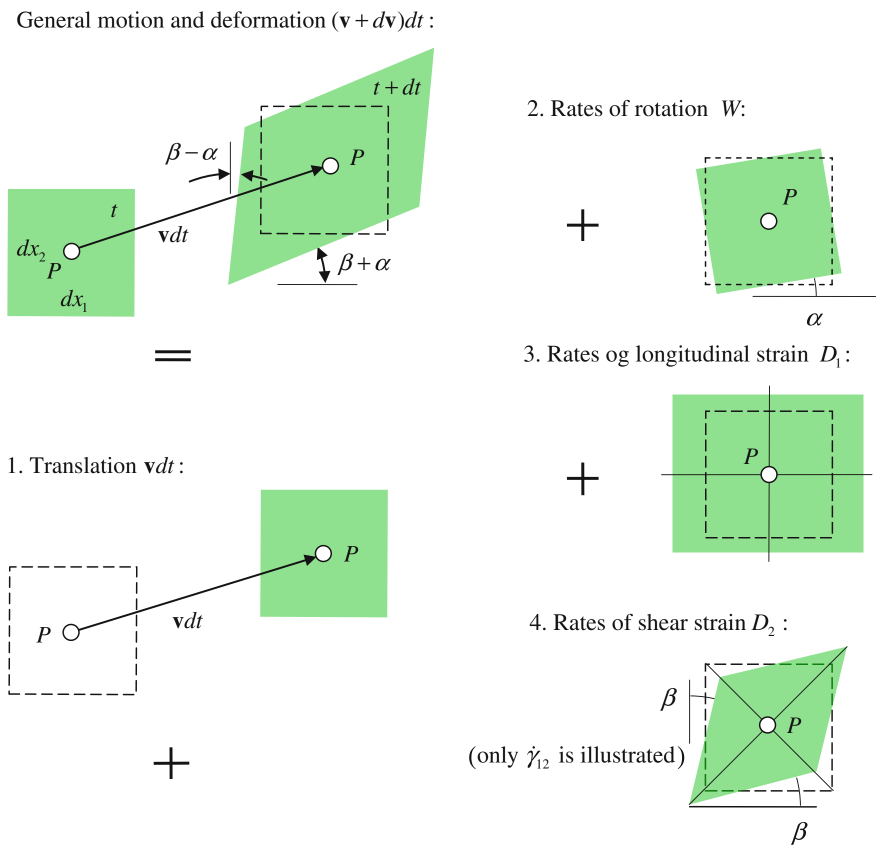


Fig. 4.4 Decomposition of the motion and deformation of a fluid element around a fluid particle P

A further investigation of the deformation kinematics in the neighborhood of any particle P will show that there exist three orthogonal material line elements through P that remain orthogonal after the displacement $(\mathbf{v} + d\mathbf{v})dt$, and that the three line elements rotate with the angular velocity $\boldsymbol{\omega}$, defined by Eq. (4.1.23). The shear strain rates with respect to each pair of these line elements are zero. A differential volume element with edges parallel to the three line elements retains its orthogonal form during the displacement $(\mathbf{v} + d\mathbf{v})dt$. The Sects. 4.1.1 and 4.1.2 will illustrate these properties. The three orthogonal line elements are said to represent the *principal direction of the rates of deformation*.

4.1.1 Rectilinear Flow with Vorticity: Simple Shear Flow

A fluid flows between two parallel plane surfaces. One surface is at rest, while the other surface moves with a constant velocity v, Fig. 4.5.

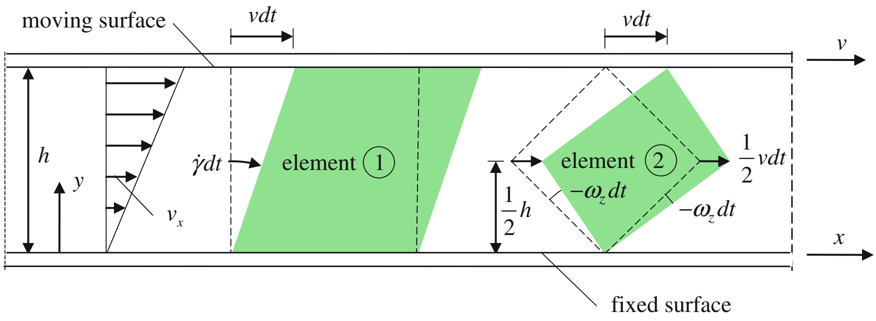


Fig. 4.5 Simple shear flow

We assume that the fluid particles move in straight parallel paths, that the fluid sticks to the solid surfaces, and such that the velocity field takes the form:

$$v_x = \frac{v}{h}y, \quad v_y = v_z = 0 \tag{4.1.26}$$

Referred to the *xyz*-coordinate system only one rate of deformation is different from zero, and only one component of the rate of rotation matrix is different from zero:

$$\dot{\gamma}_{xy} = \frac{v}{h}, \quad W_{xy} = \frac{v}{2h} \quad \Leftrightarrow \quad \omega_z = -\frac{v}{2h} \tag{4.1.27}$$

The flow is rectilinear but exhibits vorticity. Figure 4.5 illustrates that the fluid element 1 has a shear rate $\dot{\gamma}_{xy}$. The fluid element 2 has the angular velocity component $-\omega_z = -v/2h$ about the z-axis, and the element sides remain orthogonal during the short time interval dt. Note that the *rate of volumetric strain*: $\dot{\varepsilon}_V = \operatorname{div} \mathbf{v} = 0$ for this flow. The three matrices *L, D*, and *W* for this flow are:

$$L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{v}{h}, \quad D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{v}{2h}, \quad W = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{v}{2h} \tag{4.1.28}$$

4.1.2 Circular Flow Without Vorticity. The Potential Vortex

A solid circular cylinder rotating about its vertical axis with a constant angular velocity ω and surrounded by a *Newtonian fluid*, introduces a *vortex* in the fluid without vorticity. See Problem 12. The situation is illustrated in Fig. 4.6. The radius of the cylinder is a.The fluid particles move in concentric circular paths in the velocity field:

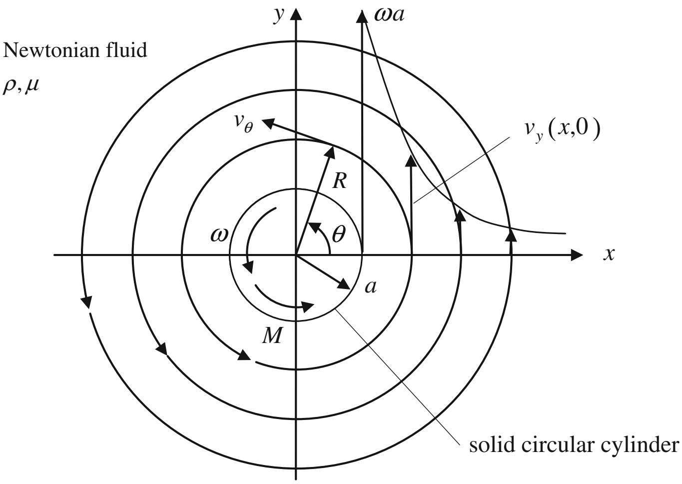


Fig. 4.6 Potential vortex due to a rotating solid cylinder in a Newtonian fluid

$$v_\theta = \frac{\alpha}{R}, \quad \alpha = \omega a^2, \quad v_R = v_z = 0 \tag{4.1.29}$$

In a Cartesian coordinate system $Oxyz$ the velocity components are:

$$v_x = -v_\theta \sin\theta = -v_\theta \frac{y}{R} = -\frac{\alpha y}{x^2 + y^2}, \quad v_y = v_\theta \cos\theta = v_\theta \frac{x}{R} = \frac{\alpha x}{x^2 + y^2} \tag{4.1.30}$$

The velocity gradients become:

$$\begin{aligned} \frac{\partial v_x}{\partial x} &= \frac{2\alpha yx}{(x^2+y^2)^2}, \quad \frac{\partial v_x}{\partial y} = -\frac{\alpha}{x^2+y^2} + \frac{2\alpha y^2}{(x^2+y^2)^2} \\ \frac{\partial v_y}{\partial x} &= \frac{\alpha}{x^2+y^2} - \frac{2\alpha x^2}{(x^2+y^2)^2}, \quad \frac{\partial v_y}{\partial y} = -\frac{2\alpha xy}{(x^2+y^2)^2} \end{aligned} \tag{4.1.31}$$

The non-trivial rate of rotationW_{xy}, the angular velocity component ω_z, and the rate of volumetric strain are zero:

$$W_{xy} = -\omega_z = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) = 0, \quad \dot{\varepsilon}_V = \operatorname{div} \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{4.1.32}$$

For the rates of strain we obtain:

$$\begin{aligned} &\dot{\varepsilon}_x = \frac{\partial v_x}{\partial x} = 0, \quad \dot{\varepsilon}_y = \frac{\partial v_y}{\partial y} = 0 \quad \text{both at } x = 0 \text{ and at } y = 0 \\ &\dot{\gamma}_{xy} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} = \frac{2\alpha(y^2 - x^2)}{(x^2+y^2)^2} = \begin{cases} \frac{2\alpha}{y^2} & \text{at } x = 0 \\ -\frac{2\alpha}{x^2} & \text{at } y = 0 \end{cases} \end{aligned} \tag{4.1.33}$$

In Sect. 4.1.2 the rate of strain components and rate of rotation components in cylindrical coordinates $(R, \theta, z)$are listed. From the formulas (4.2.1) and (4.2.2) we obtain for the velocity field (4.1.29):

$$\dot{\gamma}_{R\theta} = R\frac{d}{dR}\left(\frac{v_\theta}{R}\right) = -\frac{2\alpha}{R^2}, \quad W_{R\theta} = -\omega_z = -\frac{1}{2R}\frac{d}{dR}(Rv_\theta) = 0 \tag{4.1.34}$$

Figure 4.7 illustrates that a differential fluid element 1, subjected to the rate of shear strain $\dot{\gamma}_{xy} = -2\alpha/R^2$, will during a short time increment dt move symmetrically with respect to the indicated diagonal planes. The diagonals shown in Fig. 4.7 thus represent two of the principal directions of rates of deformation. Compare the deformation of this element with the result of the matrix D_2 in Fig. 4.4. The differential fluid element 2 in Fig. 4.7, which is oriented in the direction of the principal directions of rates of deformation, retains its right angles and does not rotate. To obtain the deformation figures of the two differential elements 1 and 2, the motion in the neighborhood of particle P in position: $x = R,\ y = 0,$ is decomposed into a rigid-body motion $v_{\theta 1}$ and a motion $v_{\theta 2}$:

$$v_{\theta 1} = \frac{\alpha}{R} + \frac{\alpha}{R^2}\Delta R, \quad v_{\theta 2} = -\frac{2\alpha}{R^2}\Delta R \tag{4.1.35}$$

$\Delta R$is a local radial coordinate measured from the particle P. To obtain this decomposition, the velocity v_θ in the neighborhood of P is expanded in a Taylor series, such that:

$$v_\theta = \frac{\alpha}{R + \Delta R} \approx \frac{\alpha}{R}\left(1 - \frac{\Delta R}{R}\right) = \frac{\alpha}{R} + \frac{\alpha}{R^2}\Delta R - \frac{2\alpha}{R^2}\Delta R = v_{\theta 1} + v_{\theta 2} \tag{4.1.36}$$

The flow (4.1.29) is a *potential flow*, see Problem 12. The *velocity potential* is $\phi = \alpha\theta,$ which results in the velocity field:

$$\mathbf{v} = \nabla\phi = \left(\mathbf{e}_R\frac{\partial}{\partial R} + \mathbf{e}_\theta\frac{1}{R}\frac{\partial}{\partial\theta} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\phi \quad \Rightarrow \quad v_R = 0\,, \quad v_\theta = \frac{\alpha}{R}, \quad v_z = 0 \tag{4.1.37}$$

in accordance with the Eq. (4.1.29). Hence the name potential vortex for this flow.

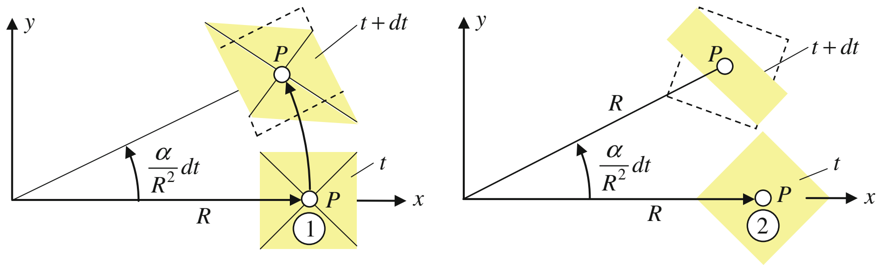


Fig. 4.7 Motion and deformation of a Newtonian fluid elements in a potential vortex

4.1.3 Stress Power: Physical Interpretation

The *stress power* per unit volume is defined by formula (3.10.11):

$$\omega = \mathbf{t}_k \cdot \mathbf{v}_{,k} = \sigma_{ik} v_{i,k} = -p v_{i,i} + \tau_{ik} v_{i,k} \tag{4.1.38}$$

A physical interpretation of the individual terms in the sum $\omega = \sigma_{ik} v_{i,k}$ will now be presented.
First we rewrite the sum. Using the symmetry property of the stress matrix, we obtain:

$$\omega = \sigma_{ik} v_{i,k} = \frac{1}{2}\sigma_{ik} v_{i,k} + \frac{1}{2}\sigma_{ki} v_{k,i} = \sigma_{ik}\frac{1}{2}(v_{i,k} + v_{k,i}) = \sigma_{ik} D_{ik} \quad \Rightarrow$$

$$\omega = \sigma_{ik} v_{i,k} = \sigma_{ik} D_{ik} \tag{4.1.39}$$

The sum contains two kinds of terms represented by:

$$\sigma_{11} D_{11} \quad \text{and} \quad \sigma_{12} D_{12} + \sigma_{21} D_{21}$$

Figure 4.8 shows a fluid element of volume $dV = dx_1\, dx_2\, dx_3$ subjected to the coordinate stresses σ_{ik} and the coordinate strains $D_{11}\, dt$ and $2D_{12}\, dt$. The work done on the element by the stresses due to the two strains is the sum of two terms:

$$\Delta W_{11} = (\sigma_{11}\, dx_2 dx_3)[(D_{11}\, dt) dx_1], \quad \Delta W_{12} = (\sigma_{12}\, dx_1 dx_3)[(2D_{12}\, dt) dx_2]$$

The total work by all the stresses σ_{ik} on the element when the element is subjected to the deformation $D_{ik}\, dt$ becomes:

$$\Delta W = (\sigma_{11} D_{11} + 2\sigma_{12} D_{12} dx_2 dx_3 + \cdots)(dx_1\, dx_2\, dx_3\, dt) = \sigma_{ik} D_{ik}\, dV\, dt \tag{4.1.40}$$

The stress power represents the work done by the stresses per unit volume and per unit time. For incompressible fluid, in which div $\mathbf{v} = v_{i,i} = 0$, the stress power is expressed by:

$$\omega = \tau_{ik} D_{ik} \tag{4.1.41}$$

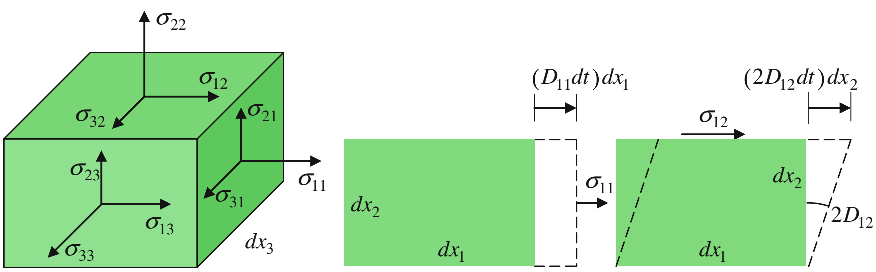


Fig. 4.8 Volume element dV with coordinate stresses σ_{ik} *and strains* $D_{11}dt$ *and* $2D_{12}dt$

4.2 Cylindrical and Spherical Coordinates

Two curvilinear coordinate systems are of particular importance in applications: cylindrical coordinates (R, θ, z) and spherical coordinates (r, θ, ϕ). Both systems represent orthogonal coordinates and at each place we may introduce a local Cartesian system with base vectors that are tangents to the coordinate lines. This makes it possible to form the *rate of deformation matrices* and the *rate of rotation matrices* for the two curvilinear coordinate systems, although the expressions of some of the elements of the matrices are not obtained directly from their Cartesian equivalents. The matrices and their elements are now presented without further explanations.

Cylindrical coordinates (R, θ, z), see the figure in Sect. 3.3.4.

$$D = \begin{pmatrix} \dot{\varepsilon}_R & \frac{1}{2}\dot{\gamma}_{R\theta} & \frac{1}{2}\dot{\gamma}_{Rz} \\ \frac{1}{2}\dot{\gamma}_{\theta R} & \dot{\varepsilon}_\theta & \frac{1}{2}\dot{\gamma}_{\theta z} \\ \frac{1}{2}\dot{\gamma}_{zR} & \frac{1}{2}\dot{\gamma}_{z\theta} & \dot{\varepsilon}_z \end{pmatrix}, \quad \dot{\varepsilon}_R = \frac{\partial v_R}{\partial R}, \quad \dot{\varepsilon}_\theta = \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R}{R}, \quad \dot{\varepsilon}_z = \frac{\partial v_z}{\partial z}$$

$$\dot{\gamma}_{R\theta} = \frac{1}{R}\frac{\partial v_R}{\partial \theta} + R\frac{\partial}{\partial R}\left(\frac{v_\theta}{R}\right), \quad \dot{\gamma}_{\theta z} = \frac{\partial v_\theta}{\partial z} + \frac{1}{R}\frac{\partial v_z}{\partial \theta}, \quad \dot{\gamma}_{zR} = \frac{\partial v_z}{\partial R} + \frac{\partial v_R}{\partial z} \tag{4.2.1}$$

$$W = \frac{1}{2}\begin{pmatrix} 0 & -\omega_z & \omega_\theta \\ \omega_z & 0 & -\omega_R \\ -\omega_\theta & \omega_R & 0 \end{pmatrix}, \quad W_{R\theta} = -\omega_z = \frac{1}{2R}\left[\frac{\partial v_R}{\partial \theta} - \frac{\partial}{\partial R}(R v_\theta)\right]$$

$$W_{\theta z} = -\omega_R = \frac{1}{2}\left[\frac{\partial v_\theta}{\partial x} - \frac{1}{R}\frac{\partial v_z}{\partial \theta}\right], \quad W_{R\theta} = -\omega_\theta = \frac{1}{2}\left[\frac{\partial v_z}{\partial R} - \frac{\partial v_R}{\partial z}\right] \tag{4.2.2}$$

Spherical coordinates (r, θ, ϕ), see the figure in Sect. 3.3.5.

$$D = \begin{pmatrix} \dot{\varepsilon}_r & \frac{1}{2}\dot{\gamma}_{r\theta} & \frac{1}{2}\dot{\gamma}_{r\phi} \\ \frac{1}{2}\dot{\gamma}_{\theta r} & \dot{\varepsilon}_\theta & \frac{1}{2}\dot{\gamma}_{\theta\phi} \\ \frac{1}{2}\dot{\gamma}_{\phi r} & \frac{1}{2}\dot{\gamma}_{\phi\theta} & \dot{\varepsilon}_\phi \end{pmatrix}. \tag{4.2.3}$$

$$\dot{\varepsilon}_r = \frac{\partial v_r}{\partial r}, \quad \dot{\varepsilon}_\theta = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}, \quad \dot{\varepsilon}_\phi = \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot\theta}{r}$$

$$\dot{\gamma}_{r\theta} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right), \quad \dot{\gamma}_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) \tag{4.2.4}$$

$$\dot{\gamma}_{\phi r} = r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}$$

$$W = \begin{pmatrix} 0 & -\omega_\phi & \omega_\theta \\ \omega_\phi & 0 & -\omega_r \\ -\omega_\theta & \omega_r & 0 \end{pmatrix}, \quad W_{r\theta} = -\omega_\phi = \frac{1}{2r}\left[\frac{\partial v_r}{\partial \theta} - \frac{\partial}{\partial r}(r v_\theta)\right]$$

$$W_{\theta\phi} = -\omega_r = \frac{1}{2r\sin\theta}\left[\frac{\partial v_\theta}{\partial \phi} - \frac{\partial}{\partial \theta}(\sin\theta\, v_\phi)\right] \tag{4.2.5}$$

$$W_{\phi r} = -\omega_\theta = \frac{1}{2}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi}\right]$$

4.3 Constitutive Equations for Newtonian Fluids

A Newtonian fluid is defined by the constitutive equation (3.4.1). These may now be rewritten to:

$$\tau_{ik} = 2\mu D_{ik} + \left(\kappa - \frac{2\mu}{3}\right) D_{jj}\,\delta_{ik} \quad \Leftrightarrow \quad T' = 2\mu D + \left(\kappa - \frac{2\mu}{3}\right)\mathrm{tr}D\;1 \tag{4.3.1}$$

tr$D \equiv$ trace of the matrix $D \equiv D_{jj} \equiv$ the sum of the diagonal elements in the matrix. The symbol 1 is the *unit matrix* with the *Kronecker delta* δ_{ik} as elements, see formula (3.3.19). The matrix form of the constitutive equations may be used in all orthogonal coordinate systems, when the proper rate of deformation matrix is used. In cylindrical coordinates and spherical coordinates we use the matrices in Eqs. (4.2.1) and (4.2.3). The trace of the matrix D becomes:

$$\mathrm{tr}D = D_{jj} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \text{ in Cartesian coordinates}$$

$$\mathrm{tr}D = D_{RR} + D_{\theta\theta} + D_{zz} = \frac{1}{R}\frac{\partial}{\partial R}(R v_R) + \frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \text{ in cylindrical coordinates} \tag{4.3.2}$$

For *incompressible Newtonian fluids* the equation (4.3.1) are reduced to:

$$\tau_{ik} = 2\mu D_{ik} \quad \Leftrightarrow \quad T' = 2\mu D \tag{4.3.3}$$

In Cartesian coordinates and in *cylindrical coordinates* these equations are:

$$\tau_{xx} = 2\mu D_{xx} = 2\mu\dot{\varepsilon}_x = 2\mu\frac{\partial v_x}{\partial x}, \quad \tau_{xy} = 2D_{xy} = \mu\dot{\gamma}_{xy} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \quad \text{etc.} \tag{4.3.4}$$

$$\tau_{RR} = 2\mu D_{RR} = 2\mu\dot{\varepsilon}_R = 2\mu\frac{\partial v_R}{\partial R}, \quad \tau_{\theta\theta} = 2\mu D_{\theta\theta} = 2\mu\dot{\varepsilon}_\theta = 2\mu\left(\frac{1}{R}\frac{\partial v_\theta}{\partial \theta} + \frac{v_R}{R}\right)$$
$$\tau_{zz} = 2\mu D_{zz} = 2\mu\dot{\varepsilon}_z = 2\mu\frac{\partial v_z}{\partial z}, \quad \tau_{R\theta} = \mu\dot{\gamma}_{R\theta} = \mu\left(\frac{1}{R}\frac{\partial v_R}{\partial \theta} + R\frac{\partial}{\partial R}\left(\frac{v_\theta}{R}\right)\right)$$
$$\tau_{\theta z} = \mu\dot{\gamma}_{\theta z} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{R}\frac{\partial v_z}{\partial \theta}\right), \quad \tau_{zR} = \mu\dot{\gamma}_{zR} = \mu\left(\frac{\partial v_z}{\partial R} + \frac{\partial v_R}{\partial z}\right) \tag{4.3.5}$$

The *stress power per unit volume* for an incompressible Newtonian fluid is previously presented in Eq. (3.10.17), which by the formulas (4.1.39) and (4.3.3) now may be rewritten to:

$$\delta = 2\mu D_{ik} D_{ik} \quad \Leftrightarrow \quad \delta = 2\mu\,\mathrm{tr}\left(D^2\right) \quad \{\text{in orthogonal coordinate systems}\} \tag{4.3.6}$$

4.4 Shear Flows

4.4.1 Simple Shear Flow

Steady flow between to parallel plates and without modified pressure gradient, ass in Fig. 4.9, is called *steady simple shear flow*. As this type of flow has some characteristic features common to many more complex flows important in applications, we shall take a closer look at the characteristic aspects of steady simple shear flow. The velocity field and the rate of deformation matrix are:

$$v_1 = \dot{\gamma} x_2, \quad v_2 = v_3 = 0, \quad D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{2}\dot{\gamma}, \quad \dot{\gamma} = v/h \tag{4.4.1}$$

The flow has the following characteristic features:

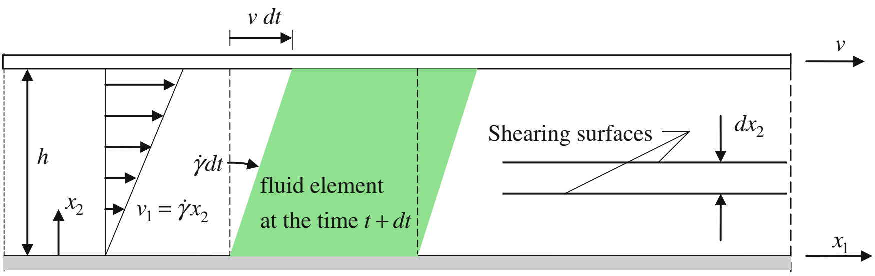


Fig. 4.9 Steady simple shear flow

(a) The flow is *isochoric*, i.e. volume preserving: $\nabla \cdot \mathbf{v} = \mathrm{tr}D = 0$.
(b) Material planes parallel to the x_1x_3-plane move in the x_1-direction without in-plane strains. We say that these planes represent a one-parameter family of *isometric planes*. The coordinate x_2 is the parameter defining each plane in the family. The word isometric is used to indicate that the distances between particles in the planes do not change during the flow. The planes are called *shearing surfaces*.
(c) The rate of deformation matrix D is given by Eq. (4.4.1). The characteristic parameter $\dot{\gamma}$ is called the *shear rate*.
(d) The shear rate $\dot{\gamma}$ in Eq. (4.4.1) is constant.

The traces of two shearing surfaces a distance dx_2 apart are shown in Fig. 4.9. The particles in the upper surface have the velocity $v_{\mathrm{rel}} = v_{1,2} \cdot dx_2$ relative to the lower surface. The streamlines related to the velocity field $v_{\mathrm{rel}} = v_{1,2} \cdot dx_2$ where dx_2 is small, are called *lines of shear*. The lines of shear are straight lines parallel to the x_1-axis. Because the fluid particles are fixed to the same line of shear at all times, *the lines of shear are material lines*.

4.4.2 General Shear Flow

The general shear flow has features parallel to those of the simple steady shear flow. A flow is a *shear flow* if the following conditions are fulfilled, Fig. 4.10:

(a) The flow is isochoric, i.e. volume preserving: $\nabla \cdot \mathbf{v} = \mathrm{tr}D = 0$.
(b) A one-parameter family of material surfaces exists, in which the surfaces move isometrically, i.e. without in-surface strains. These surfaces are called *shearing surfaces*.

The *streamlines related to the velocity field* $v_{\mathrm{rel}} = v_{1,2} \cdot dx_2$ of one shearing surface relative to a neighbor shearing surface are called *lines of shear*. The particles on one shear line at the time t will not in general stay on the same line of shear at a later time. In other words, the lines of shear are not necessarily material lines. The condition (a) implies that the distance between any two neighboring shearing surfaces is constant.

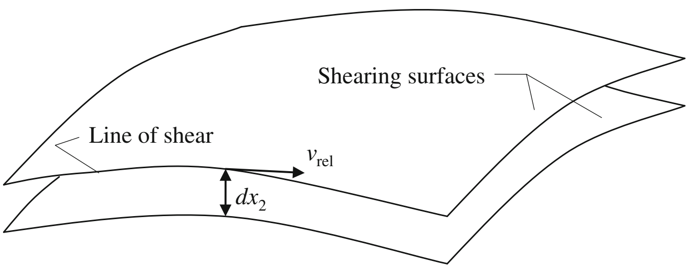


Fig. 4.10 Shear surfaces and line of shear

4.4.3 Unidirectional Shear Flow

A shear flow that in addition to the conditions (a) and (b) for a general shear flow, also satisfies the condition:

(c) The lines of shear are material lines,

is called a *unidirectional shear flow*. The material lines coinciding with the shear lines at a particular time will continue to be lines of shear as time passes. We may imagine that the lines of shear are "drawn" on the shearing surfaces, and these material lines would then represent the shear lines at later times. Unidirectional shear flow is the most common shear flow in applications and in particular in experiments designed to investigate the properties of non-Newtonian fluids.

The analysis of the deformation kinematics of shear flows in the neighborhood of a particle P is simplified by introducing a local Cartesian coordinate system Px at the particle, as shown in Fig. 4.11. The coordinate axes are chosen such that the base vector $\mathbf{e}_1$ and $\mathbf{e}_3$ are tangents to the shearing surface, with $\mathbf{e}_1$ in the direction of the relative velocity v_{rel} of the shearing surface relative to the neighbor shearing surface. The base vector $\mathbf{e}_1$ is thus tangent to the line of shear through the particle. The base vector $\mathbf{e}_2$ is normal to the shearing surface. The three vectors $\mathbf{e}_i$ are called the *shear axes*, and the vector $\mathbf{e}_1$ is the *shear direction*.

A fluid element $dV = dx_1 dx_2 dx_3$ is during a short time interval dt deformed as indicated in Fig. 4.12. The deformation is govern by one deformation rate: the *shear rate* $\dot{\gamma} = v_{1,2}$. The rate of deformation matrix D in the Px-system is therefore equal to the deformation rate matrix of a simple shear flow:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2} \dot{\gamma} \tag{4.4.2}$$

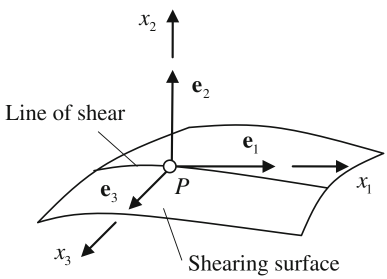


Fig. 4.11 Shear axes through particle P

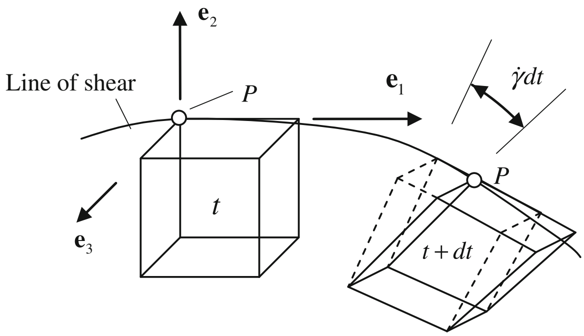


Fig. 4.12 Deformation of a fluid element

4.4.4 Viscometric Flow

A unidirectional flow that also satisfies the condition:

(d) For every particle the rate of shear $\dot{\gamma}$ is independent of time,

is called a *viscometric flow*. Another name of this kind of flow is *rheological steady flow*. The flow is not necessarily a steady flow as defined in fluid mechanics. The expression rheological steady means that the deformation rate of the fluid is not changing with time. Viscometric flows play an important role in investigating the properties of non-Newtonian fluids. We shall now present a series of important viscometric flows and identify shearing surfaces, lines of shear, and shear axes for each flow. Some of these flows will be further investigated later in relation to experimental situations.

4.4.4.1 Steady Axial Annular Flow: Steady Pipe Flow

The fluid flows in the *annular space* between two solid, concentric cylindrical surfaces, or as shown in Fig. 4.13, the fluid flows in a cylindrical pipe. The flow is

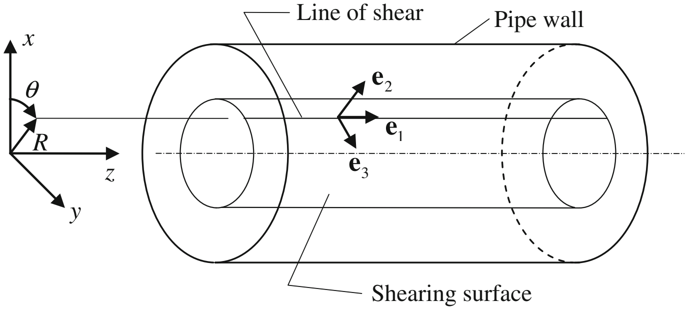


Fig. 4.13 Axial annular flow. Steady pipe flow

steady and the velocity is parallel to the axis of the cylindrical surfaces. The velocity field and the shear rate are:

$$v_z = v_z(R), \quad v_R = v_\theta = 0, \quad \dot{\gamma} = \dot{\gamma}_{zR} = \frac{dv_z}{dR} \tag{4.4.3}$$

The *shearing surfaces are concentric cylindrical surfaces*. The *lines of shear are straight lines* parallel to the axis of the cylindrical surfaces, and they coincide with the streamlines of the flow and with the pathlines of the fluid particles. The *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_z, \quad \mathbf{e}_2 = \mathbf{e}_R, \quad \mathbf{e}_3 = \mathbf{e}_\theta \tag{4.4.4}$$

The shear direction is parallel to the z-direction.

4.4.4.2 Steady Tangential Annular Flow

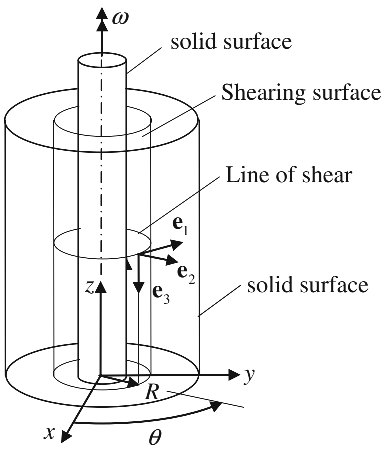
The fluid flows in the annular space between two concentric solid cylindrical surfaces. One of the solid surfaces rotates with a constant angular velocity ω. Figure 4.14 shows the case where the inner cylindrical surface rotates. The velocity field is:

$$v_\theta = v_\theta(R), \quad v_z = v_R = 0 \tag{4.4.5}$$

The *shearing surfaces are concentric cylindrical surfaces*. The *lines of shear are circles* with constant R and z, and they coincide with streamlines of the flow and the pathlines of the particles. The *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_\theta, \quad \mathbf{e}_2 = \mathbf{e}_R, \quad \mathbf{e}_3 = -\mathbf{e}_z \tag{4.4.6}$$

Fig. 4.14 Tangential annular flow



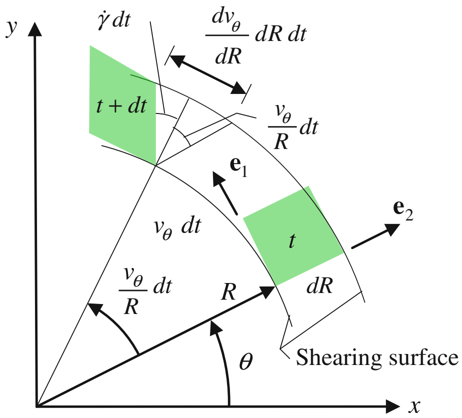


Fig. 4.15 *Derivation of the formula (4.4.7) for the rate of shear*

Figure 4.15 illustrates the derivation of the shear rate for this type of flow. We consider a fluid element between two shear surfaces and at times t and $t + dt$. From the figure we read:

$$\dot{\gamma}\,dt = \frac{1}{dR}\left(\frac{dv_\theta}{dR} \cdot dR \cdot dt\right) - \frac{v_\theta \cdot dt}{R} \quad \Rightarrow$$

$$\dot{\gamma} = \dot{\gamma}_{R\theta} = \frac{dv_\theta}{dR} - \frac{v_\theta}{R} \equiv R\frac{d}{dR}\left(\frac{v_\theta}{R}\right) \tag{4.4.7}$$

This result may also be obtained directly from the formulas (4.2.1).

It is found that for a Newtonian fluid the velocity field shown in Eq. (4.4.5) is unstable when the angular velocity ω is increased above a certain limit. The instability introduces a secondary flow with velocities both in the z- and R- directions, and is described as *Taylor vortexes*. Instability and Taylor vortexes occur when the parameter:

$$T_a \equiv \frac{\rho}{\mu}\omega r_1(r_2 - r_1)\sqrt{\frac{r_2}{r_1} - 1} \quad > 41.3 \tag{4.4.8}$$

ρ =density, μ =viscosity, r_1 and r_2 are the radii of the inner and outer solid boundary surfaces. At $T_a > 400$ the flow becomes turbulent. Similar instabilities can occur for non-Newtonian fluids.

4.4.4.3 Steady Torsion Flow

The fluid is set in motion between two plane parallel circular disks. One disk is at rest while the other disk rotates about its axis at a constant angular velocity ω.

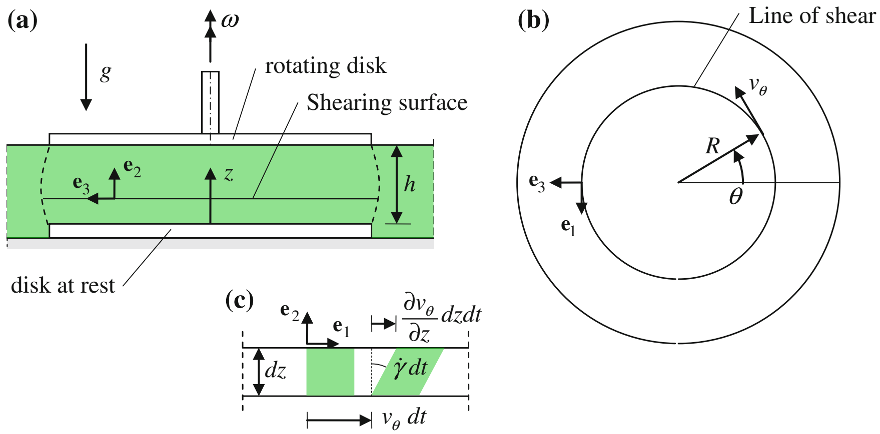


Fig. 4.16 Torsion flow between two parallel circular disks

Figure 4.16a, b illustrates the situation. The dashed curved line indicates a "free surface". In the case of a thick fluid this may really be a free surface, while in the case of a thin fluid, the disks are submerged in a fluid bath. The rotating disk is touching the free surface of the bath and the dashed line marks an artificial free surface. Only the fluid between the disks is considered in the analysis.

The velocity field is assumed to be:

$$v_\theta = v_\theta(R, z) = \frac{\omega R z}{h}, \quad v_R = v_z = 0 \tag{4.4.9}$$

The expression for the shear rate is found from Fig. 4.16c, which shows an unfolded part of the cylinder surface $R \cdot dz$ between two shearing surfaces a distance dz apart:

$$\dot{\gamma} = \dot{\gamma}_{\theta z} = \frac{\partial v_\theta}{\partial z} = \frac{\omega R}{h} \tag{4.4.10}$$

The result may also be obtained from the formulas (4.2.1). Based on the assumption that the fluid sticks to the solid disks, the velocity $v_\theta(R, z)$ satisfies the boundary conditions:

$$v_\theta(R, h) = \omega R, \quad v_\theta(R, 0) = 0 \tag{4.4.11}$$

The *shearing surfaces are planes normal to the axis of rotation*. The *lines of shear are concentric circles*, see Fig. 4.16b, and coincide with the streamlines of the flow and the pathlines of the particles. Figure 4.16c shows an unfolded part of the cylinder surface $R \cdot dz$ between two shear surfaces a distance dz apart. From the deformation of the fluid element shown in Fig. 4.16c, we conclude that the *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_\theta, \quad \mathbf{e}_2 = \mathbf{e}_z, \quad \mathbf{e}_3 = \mathbf{e}_R \tag{4.4.12}$$

4.4.4.4 Steady Helix Flow

The flow of the fluid in the annular space between to solid cylindrical surfaces is driven by the rotation and the axial translation of the inner cylindrical surface, see Fig. 4.17. The angular velocity ω and the axial velocity v are constants.

The velocity field is assumed as:

$$v_\theta = v_\theta(R), \quad v_z = v_z(R), \quad v_R = 0 \tag{4.4.13}$$

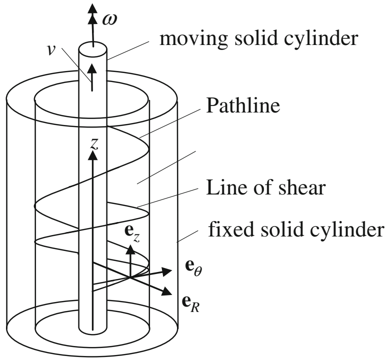
This kind of flow may also be obtained by a combination of a rotation of the inner cylinder and a constant modified pressure gradient dP/dz. The *shearing surfaces are concentric cylindrical surfaces*, which rotate and move in the axial direction. A fluid particle moves in a helix. Thus path lines and stream lines are helices. A fluid particle on a shearing surface moves relative to a neighbor shearing surface also in a helix. Hence the *lines of shear are helices*, but they do not coincide with the streamlines or the path lines. This is shown in Fig. 4.17. The rate of deformation matrix in cylindrical coordinates, the equation (4.2.1), contains only two independent elements for the assumed flow:

$$\dot{\gamma}_{R\theta} = R\frac{d}{dR}\left(\frac{v_\theta}{R}\right), \quad \dot{\gamma}_{zR} = \frac{dv_z}{dR} \tag{4.4.14}$$

The first of these formulas is identical to formula (4.4.7). The *shear axis* normal to the shearing surface is $\mathbf{e}_2 = \mathbf{e}_R$. To find the two shear axes $\mathbf{e}_1$ and $\mathbf{e}_3$ tangent to the shearing surface, we consider the relative motion of two fluid particles $P(R,\theta,z)$ and $Q(R+dR,\theta,z)$ in Fig. 4.18. The particles are on neighboring shearing surfaces. From the figure it follows that during the time increment dt particle P moves from position P to the position P', while the particle Q moves from its position Q the position Q'. Relative to the shearing surface through the particle P, the particle Q has a combination of two motions represented by the displacement vectors (Fig. 4.18a, b):

$$\begin{aligned} &(\dot{\gamma}_{R\theta}dt \cdot dR)\boldsymbol{e}_\theta \quad \text{parallel to the } xy-\text{plane}, \\ &(\dot{\gamma}_{zR}dt \cdot dR)\boldsymbol{e}_z \quad \text{parallel to the } Rz-\text{plane}, \end{aligned} \tag{4.4.15}$$

Fig. 4.17 *Helix flow*



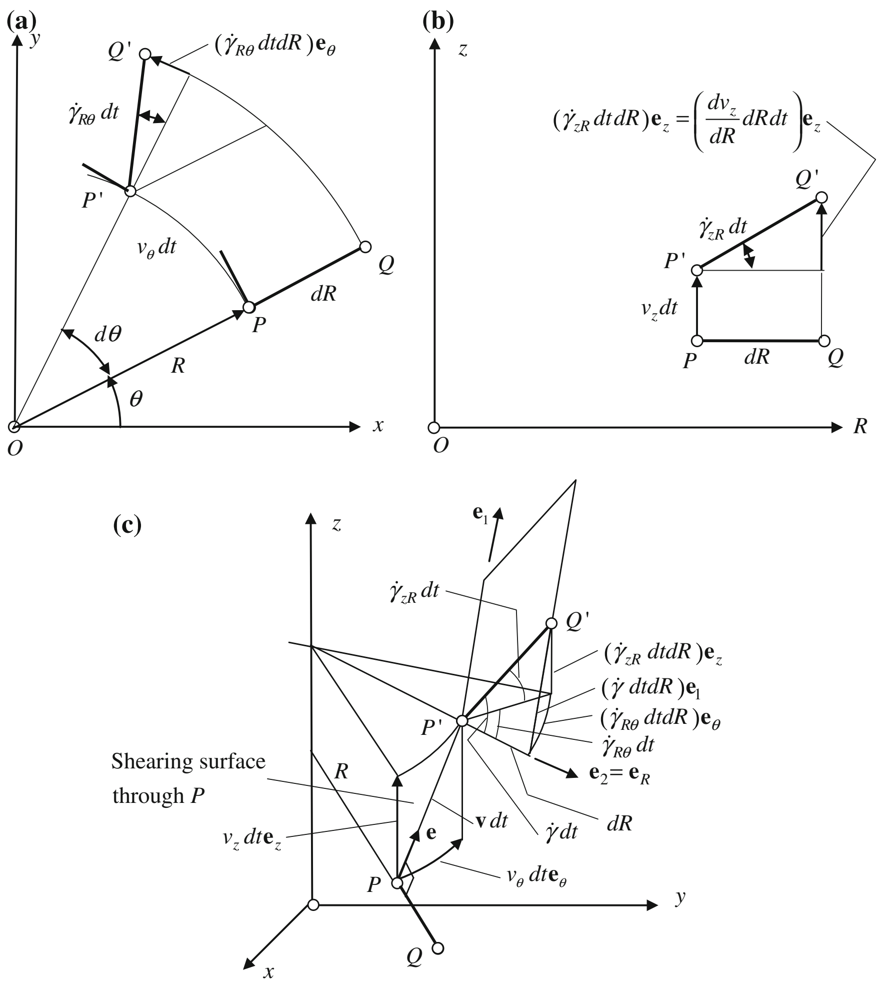


Fig. 4.18 Helix flow. Construction of shear rate and shear direction

The total displacement of Q relative to the shearing surface through P is given by the vector, see Fig. 4.18c:

$$(\dot{\gamma} dt \cdot dR)\mathbf{e}_1 = (\dot{\gamma}_{R\theta} dt \cdot dR)\mathbf{e}_\theta + \left(\dot{\gamma}_{zR} dt \cdot dR\right)\mathbf{e}_z \tag{4.4.16}$$

$\dot{\gamma}$ is the *shear rate* and $\mathbf{e}_1$ is the *shear direction* for this flow. From Eq. (4.4.16) we obtain the results:

$$\dot{\gamma} = \sqrt{(\dot{\gamma}_{R\theta})^2 + \left(\dot{\gamma}_{zR}\right)^2}, \quad \mathbf{e}_1 = \frac{\dot{\gamma}_{R\theta}}{\dot{\gamma}}\mathbf{e}_\theta + \frac{\dot{\gamma}_{zR}}{\dot{\gamma}}\mathbf{e}_z \tag{4.4.17}$$

The third axis of shear is then found to be:

$$\mathbf{e}_3 = \frac{\dot{\gamma}_{zR}}{\dot{\gamma}} \mathbf{e}_\theta - \frac{\dot{\gamma}_{R\theta}}{\dot{\gamma}} \mathbf{e}_z \tag{4.4.18}$$

That this result is true, follows from the fact that $\mathbf{e}_3$ satisfies that conditions:

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = 1, \quad \mathbf{e}_3 \cdot \mathbf{e}_1 = \mathbf{e}_3 \cdot \mathbf{e}_2 = 0$$
$$\mathbf{e}_1, \mathbf{e}_2, \text{ and } \mathbf{e}_3 \text{ form a right - handed system} \quad \Leftrightarrow \quad [\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3] \equiv (\mathbf{e}_1 \times \mathbf{e}_2) \cdot \mathbf{e}_3 = +1$$

4.5 Extensional Flows

4.5.1 Definition of Extensional Flows

As mentioned in Sect. 4.1, in any flow there exist through each particle three orthogonal material line elements that do not show rates of shear strain: The lines remain orthogonal after a short time increment dt. Confer the elements 2 in the Figs. 4.5 and 4.7. The three material line elements represent the principal directions of rates of deformation at the time t. In a local Cartesian coordinate system Px at the particle P, with base vectors $\mathbf{e}_i$ coinciding with the principal directions of rates of deformation at the time t, the rate of deformation matrix takes the form:

$$D = \begin{pmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{pmatrix} \tag{4.5.1}$$

For an incompressible fluid the strain rates must satisfy the incompressibility condition:

$$\dot{\varepsilon}_1 + \dot{\varepsilon}_2 + \dot{\varepsilon}_3 = 0 \tag{4.5.2}$$

A flow is called *an extensional flow if the same material line elements through each particle represent the principal directions of strain rates at all times.* The literature also *uses elongational flows* or *shear free flows* for this type of flow.

A simple extensional flow is given by the velocity field:

$$v_x = \dot{\varepsilon}_1(t)\, x, \quad v_y = \dot{\varepsilon}_2(t)\, y, \quad v_z = \dot{\varepsilon}_3(t)\, z \tag{4.5.3}$$

The deformation of a volume element in this flow is illustrated in Fig. 4.19. Material lines parallel to the coordinate axes represent the principal directions *PD* of the rates of deformation at all times. The principal directions are fixed in space for this flow.

Figure 4.20 shows that in a simple shear flow the material line elements *ML* representing the principal directions *PD* of the rates of deformation at a time t do

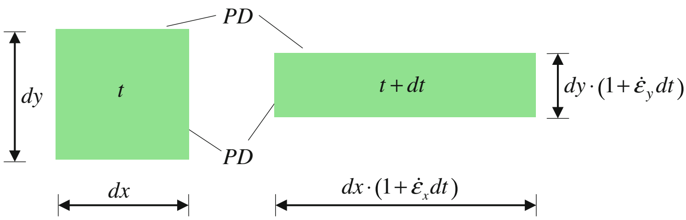


Fig. 4.19 Extensional flow PD = principal directions of strain rates = material lines

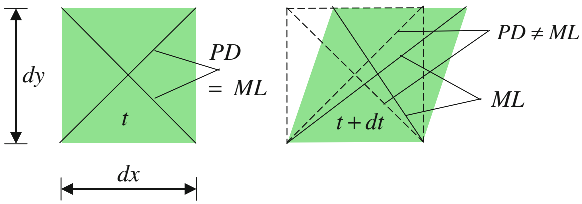


Fig. 4.20 Principal directions PD and material line elements ML in simple shear flow

not represent the principal directions at a later time $t + dt$. The principal directions are fixed in space but the material lines coinciding with the principal directions at one time, are not fixed in space. As we shall see in Chap. 5 on material functions this difference between shear flows and extensional flows is very important in modeling of non-Newtonian fluids.Extensional flows are important in experimental investigations of the properties of non-Newtonian fluids. These flows are also relevant in connection with forming processes for plastics, as for example in vacuum forming, blow molding, foaming operations, and spinning. In metal forming, extensional flows are important in milling and extrusion.

Figure 4.21 illustrates an extrusion apparatus. The fluid in the container is extruded through a die. A piston provides the pressure in the fluid. Along the container wall at some distance from the piston the flow is approximately shear flow. Near the symmetry axis of the container the flow is extensional, as indicated

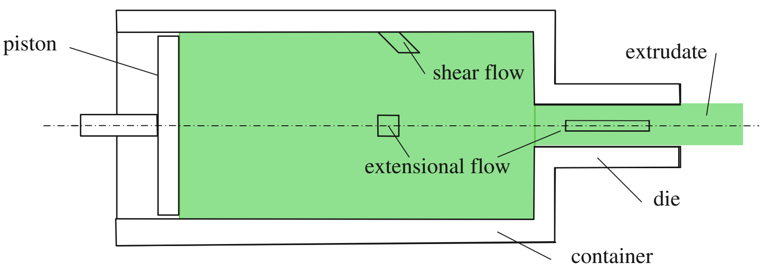


Fig. 4.21 *Extrusion*

by two material elements that each is represented in two configurations of the same material element at two different times. In the die, which in the present case is a short circular tube, most of the fluid is in a shear flow.

4.5.2 Uniaxial Extensional Flow

This type of extensional flow is characterized by the rate of deformation matrix:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\dot{\varepsilon}(t)}{2} \tag{4.5.4}$$

A *simple uniaxial flow* is given by the velocity field:

$$\begin{aligned} v_x &= \dot{\varepsilon}\, x, \quad v_y = -\frac{\dot{\varepsilon}}{2}\, y \\ v_z &= -\frac{\dot{\varepsilon}}{2}\, z, \quad \varepsilon = \varepsilon(t) \end{aligned} \tag{4.5.5}$$

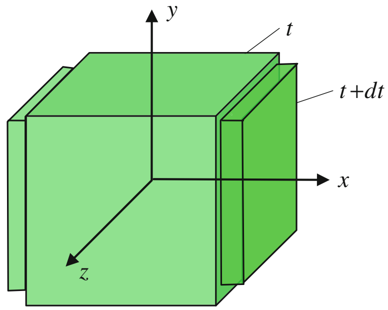
Figure 4.22 which shows the same fluid element at the times t and $t + dt$, illustrates this flow. Uniaxial extensional flow is relevant when the fluid is stretched axisymmetrically in one direction.

4.5.3 Biaxial Extensional Flow

When a fluid is stretched or compressed equally in two directions the flow may be characterized as biaxial extensional flow. The strain rate matrix for this type of flow is:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \dot{\varepsilon}(t) \tag{4.5.6}$$

Fig. 4.22 Uniaxial extensional flow



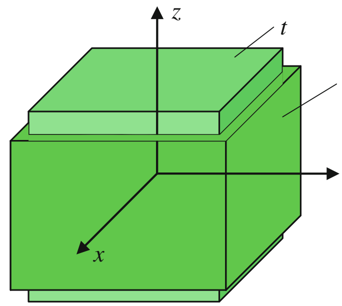


Fig. 4.23 Biaxial extensional flow

Figure 4.23 illustrates the deformation of a fluid element in biaxial flow. If the two types of flow in Figs. 4.22 and 4.23 are compared, and keeping in mind that the fluid is incompressible, it should be understood that the constitutive modeling will be identical. This fact will be addressed more closely in Sect. 5.6.

4.5.4 Planar Extensional Flow≡ Pure Shear Flow

This type of flow is characterized by the rate of deformation matrix:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\varepsilon}(t) \tag{4.5.7}$$

The flow is given the paradoxical name *pure shear flow* because we may find three orthogonal directions $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_z$, see Fig. 4.24, with respect to which the rate of deformation matrix is:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\varepsilon}(t) \tag{4.5.8}$$

This is the same rate of deformation matrix as for simple shear flow, see the Eq. (4.4.1).

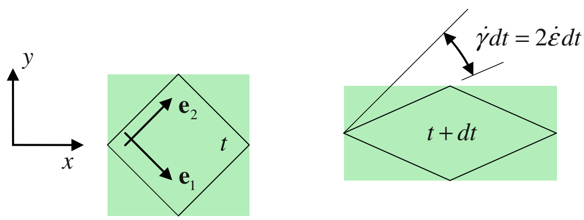


Fig. 4.24 Planar extensional flow. Pure shear flow

The major difference between a simple shear flow and pure shear flow is that the principal directions of rates of deformation in the case of pure shear flow are represented by the same material line elements at all times while this is not so in the case of a general shear flow. Compare the Figs. 4.20 and 4.24.

Extensional flows are important in experimental investigations of the properties of non-Newtonian fluids. These flows are also relevant in connection with forming processes for plastics, as for example in vacuum forming, blow molding, foaming operations, and spinning. In metal forming extensional flows are important in milling and extrusion.

Chapter 5
Material Functions

5.1 Definition of Material Functions

Relations between stress components and deformation components, like strains and strain rates, in characteristic and simple flows are expressed by *material functions*. The *viscosity function* $\eta(\dot{\gamma})$, defined by Eq. (1.4.7), the *creep function in shear* $\alpha(\tau_o, t)$, defined by Eq. (1.4.21) and the *relaxation function in shear*$\beta(\gamma_o, t)$, defined by Eq. (1.4.23), are examples of material functions for *simple shear flows*. The characteristic flows for which the material functions are defined occur in standard experiments designed to investigate the properties of non-Newtonian fluids. In general the material functions may be functions of stresses, rates of stress, rates of deformation, temperature, time, and other parameters.

The material functions are determined experimentally and are represented by data or mathematical functions representing these data. The material functions are also applied in classification of fluids, as discussed in Chap. 1.

In analyses of general flows fluid models are introduced. These models are defined by *constitutive equations*. A constitutive equation is a relationship between stresses and different measures of deformations, as strains, rates of deformation, and rates of rotation. A general constitutive equation is intended to represent a fluid in any flow, although it is experienced that most constitutive equations have limited applications and only to a few cases of flows. The material functions may enter the constitutive equations or are used to determine material parameters entering the constitutive equations.

It might be a goal when constructing a fluid model that the constitutive equations contain the material functions that are relevant for the special test flows that most resemble the actual flow the fluid model is intended for.

F. Irgens, *Rheology and Non-Newtonian Fluids*,
DOI: 10.1007/978-3-319-01053-3_5,

5.2 Material Functions for Viscometric Flows

We shall consider an *isotropic* and *incompressible fluid* in a general viscometric flow, as described in Sect. 4.4.4. Figure 5.1a shows a particle P and the shearing surface and the line of shear going through the particle at time t. A local Cartesian coordinate system Px is introduced such that the base vectors $\mathbf{e}_i$ are the shear axes, see Fig. 5.1b. The rate of deformation matrix becomes:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma} \tag{5.2.1}$$

The rate of deformation matrix D in Eq. (5.2.1) satisfies the condition of incompressibility. It is assumed that the extra stresses τ_{ik} are due to the shear flow and thus only a function of the shear rate $\dot{\gamma}$ and the temperature. The temperature dependence will not be reflected implicitly in the following. Thus we set:

$$\tau_{ik} = \tau_{ik}(\dot{\gamma}) \tag{5.2.2}$$

The condition of isotropy implies that the state of stress must have the same symmetry as the state of rate of deformation. The x_1x_2- plane is a symmetry plane. With reference to Fig. 5.1c, this means that the shear stresses $\tau_{13} = \tau_{31}$ and $\tau_{23} = \tau_{32}$ must be zero because these stresses act antisymmetrically with respect to the x_1x_2- plane.

The state of stress in the fluid is therefore given by the stress matrix:

$$T = (-p\,\delta_{ik} + \tau_{ik}) = \begin{pmatrix} \tau_{11} - p & \tau_{12} & 0 \\ \tau_{12} & \tau_{22} - p & 0 \\ 0 & 0 & \tau_{33} - p \end{pmatrix} \tag{5.2.3}$$

For an incompressible fluid the pressure level cannot influence the flow. Only pressure gradients are of importance. A thermodynamic equation of state for the pressure p is ignored when flows of incompressible fluids are discussed. Incompressibility implies therefore that the pressure p cannot be given by a constitutive

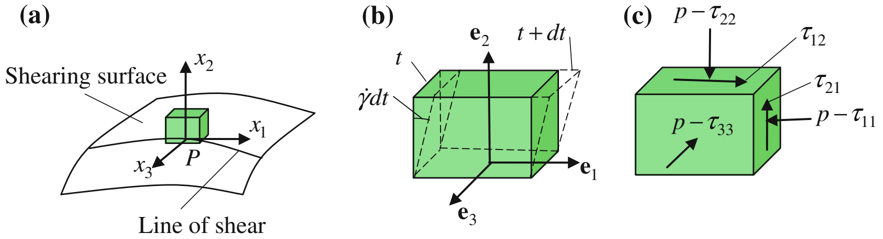


Fig. 5.1 Rate of deformation and stresses in the viscometric flow. **a** Shearing surface and line of shear, **b** Rate of deformation, **c** Stresses

equation but has to be determined from the equations of motion and the boundary conditions for the flow.

In measuring directly or indirectly the normal stresses, it is not possible to distinguish between the pressure p and the contribution from the extra stresses due to the deformation of the fluid. The implication of this is that only normal stress differences may be expressed by constitutive equations. In a viscometric flow we seek constitutive equations for the following stress and stress differences:

$$\begin{aligned} &\textit{Shear stress}: && \tau_{12} \\ &\textit{Primary normal stress difference}: && N_1 = \sigma_{11} - \sigma_{22} \equiv \tau_{11} - \tau_{22} \\ &\textit{Secondary normal stress difference}: && N_2 = \sigma_{22} - \sigma_{33} \equiv \tau_{22} - \tau_{33} \end{aligned} \tag{5.2.4}$$

The third normal stress difference, $\sigma_{11} - \sigma_{33}$, is determined by the two others:

$$\sigma_{11} - \sigma_{33} = (\sigma_{11} - \sigma_{22}) + (\sigma_{22} - \sigma_{33}) = N_1 + N_2 \tag{5.2.5}$$

Three material functions, called *viscometric functions*, are introduced in a viscometric flow:

$$\begin{aligned} \eta(\dot{\gamma}) &= \textit{the viscosity function} \\ \psi_1(\dot{\gamma}) &= \textit{the primary normal stress coefficient} \\ \psi_2(\dot{\gamma}) &= \textit{the secondary normal stress coefficient} \end{aligned} \tag{5.2.6}$$

The viscosity function is also called the *apparent viscosity*. The viscometric functions are defined by the relations:

$$\begin{aligned} \tau_{12}(\dot{\gamma}) &= \eta(\dot{\gamma})\dot{\gamma} \quad &\Leftrightarrow \quad \eta(\dot{\gamma}) &= \tau_{12}(\dot{\gamma})/\dot{\gamma} \\ N_1(\dot{\gamma}) &= \psi_1(\dot{\gamma})\dot{\gamma}^2 \quad &\Leftrightarrow \quad \psi_1(\dot{\gamma}) &= N_1(\dot{\gamma})/\dot{\gamma}^2 \\ N_2(\dot{\gamma}) &= \psi_2(\dot{\gamma})\dot{\gamma}^2 \quad &\Leftrightarrow \quad \psi_2(\dot{\gamma}) &= N_2(\dot{\gamma})/\dot{\gamma}^2 \end{aligned} \tag{5.2.7}$$

The viscometric functions are all even functions:

$$\eta(-\dot{\gamma}) = \eta(\dot{\gamma}), \quad \psi_1(-\dot{\gamma}) = \psi_1(\dot{\gamma}), \quad \psi_2(-\dot{\gamma}) = \psi_2(\dot{\gamma}) \tag{5.2.8}$$

This property is a consequence of the assumption that the fluid is isotropic. This may be seen from the following reasoning, with reference to Fig. 5.2. Figure 5.2a shows the stresses resulting from a positive shear rate $\dot{\gamma}$, and Fig. 5.2b shows the stresses due to a negative shear rate $-\dot{\gamma}$. Isotropy implies that the stresses are the same as in Fig. 5.2b if the fluid, before it is subjected to shear rate $-\dot{\gamma}$, has been rotated 180° about the $\mathbf{e}_2$-direction. In the latter case Fig. 5.2.b may be interpreted as a mirror image of Fig. 5.2a. Thus the normal stresses in the two figures must be the same, while the shear stress must be equal in magnitude but opposite in direction. Therefore:

$$\begin{aligned} &\tau_{11}(-\dot{\gamma}) = \tau_{11}(\dot{\gamma}), \quad \tau_{22}(-\dot{\gamma}) = \tau_{22}(\dot{\gamma}), \quad \tau_{33}(-\dot{\gamma}) = \tau_{33}(\dot{\gamma}) \\ &\tau_{12}(-\dot{\gamma}) = -\tau_{12}(\dot{\gamma}) \end{aligned} \tag{5.2.9}$$

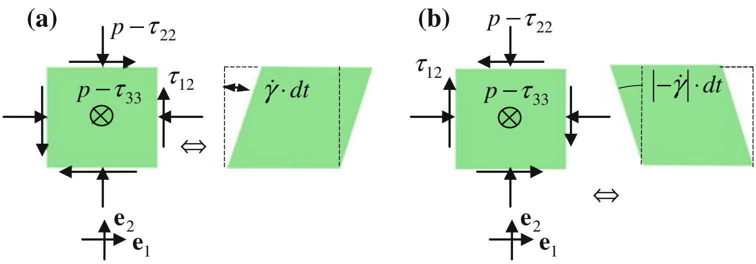


Fig. 5.2 Fluid element with shear rate and stresses. **a** positive shear rate, **b** negative shear rate

The results (5.2.8) now follow from the Eqs. (5.2.4) (5.2.7), and (5.2.9).

The viscosity function $\eta(\dot{\gamma})$ for a typical *shear-thinning fluid* is shown in Fig. 5.3. For low shear rates the viscosity $\eta(\dot{\gamma})$ is nearly constant and equal to $\eta_o = \eta(0)$, the *zero-shear-rate-viscosity*. For high shear rates the viscosity $\eta(\dot{\gamma})$ may approach asymptotically an *infinite-shear-rate viscosity* η_∞. For some fluids, for example highly concentrated polymer solutions and polymer melts, it may be impossible to measure η_∞. For these fluids the reason is that the polymer chains are destroyed at very high shear rates.

The primary normal stress $\psi_1(\dot{\gamma})$is positive, and is almost constant equal to $\psi_{1,0} = \psi_1(0)$ for low shear rates, and then decreases more rapidly with increasing shear rate than the viscosity function $\eta(\dot{\gamma})$. A lower bound for $\psi_1(\dot{\gamma})$ when $\dot{\gamma} \to \infty$ is not registered. Figure 5.3 shows a characteristic behavior of the primary normal stress coefficient. Figure 5.9 presents some experimental curves for $\psi_1(\dot{\gamma})$.

The secondary normal stress coefficient $\psi_2(\dot{\gamma})$ is usually negative and is found for polymeric fluids to have an absolute value of approximately 10 % of the primary normal stress coefficient $\psi_1(\dot{\gamma})$ for the same fluid. Figure 5.3 shows a characteristic behavior of the secondary normal stress coefficient $\psi_2(\dot{\gamma})$. Figure 5.9 presents some experimental curves for $\psi_2(\dot{\gamma})$.

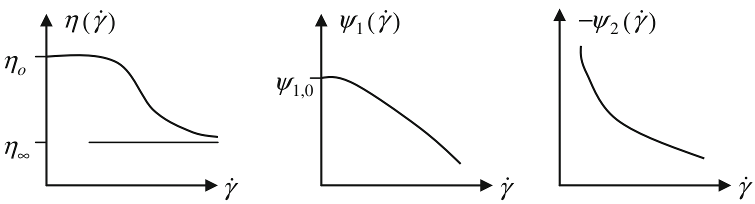


Fig. 5.3 Characteristic behavior of viscometric functions: viscosity function $\eta(\dot{\gamma})$, primary normal stress coefficient $\psi_1(\dot{\gamma})$, and secondary normal stress coefficient $\psi_2(\dot{\gamma})$

5.3 Cone-and-Plate Viscometer

The most commonly used viscometer for measuring all three viscometric functions for a fluid is of the cone-and-plate type. Figure 5.4 illustrates this viscometer which consists of a stationary circular horizontal plate and a rotating cone. The angle α_o between the conic surface and the plate is very small, usually less than 4°. The fluid to be investigated is placed in the space between the cone and the plate. The fluid surface at the edge of the instrument, which is marked by a dashed line in Fig. 5.4, may be a free surface if the fluid is sufficiently thick. If the consistency of the fluid is such that it has a tendency to flow out of the viscometer if the dashed marked surface was free, the viscometer is placed in a fluid bath. The influence of the fluid outside of the dashed marked surface is then neglected.

The cone is subjected to the torque M and the force F and is set to rotate about its vertical axis at constant angular velocity ω. The torque $M = M(\omega)$ is balanced by the shear stresses from the fluid, while the force $F = F(\omega)$ is transferred to the plate by the normal stresses from the fluid. The plate has narrow channels for pressure measurements by pressure transducers.

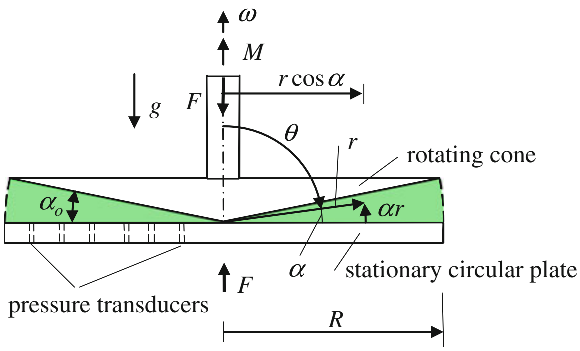
Kinematics of the Viscometer. We shall use *spherical coordinates* (r, θ, ϕ), Fig. 5.5, in the analysis of the viscometer. The only velocity component in the fluid is then v_ϕ. Because the angle α_o is very small, we may assume that v_ϕ varies linearly along a circular arc $\alpha_o r$, from zero at the plate to $\omega r \cos\alpha_o \approx \omega r$ at the cone. Using the angle $\alpha = \pi/2 - \theta$ as a variable, see Fig. 5.4, we may set:

$$v_\phi(r, \alpha) = \frac{\omega r}{\alpha_o}\alpha, \quad v_\theta = v_r = 0 \tag{5.3.1}$$

The flow is viscometric. The *shearing surfaces* are cones defined by the parameter α. Figure 5.6 shows a small fluid element on a spherical surface of radius r and between two neighboring shearing surfaces. The lines of shear are circles with radius $r\cos\alpha$. The *shear axes* are:

$$\mathbf{e}_1 = \mathbf{e}_\phi, \quad \mathbf{e}_2 = -\mathbf{e}_\theta, \quad \mathbf{e}_3 = \mathbf{e}_r \tag{5.3.2}$$

Fig. 5.4 The cone-and-plate viscometer



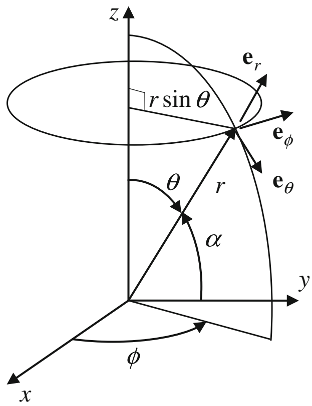


Fig. 5.5 Spherical coordinates

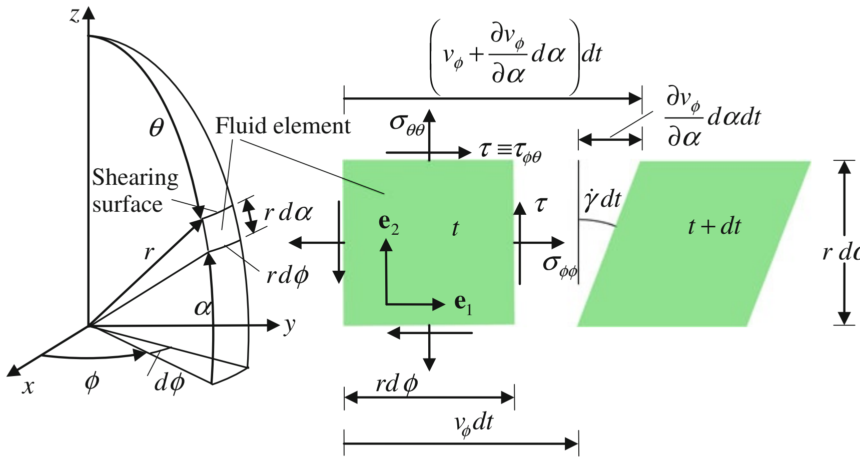


Fig. 5.6 Fluid element with stresses and shear rate

The fluid element is subjected to the normal stresses $\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}$, and the shear stress $\tau = \tau_{\theta\phi}$, of which the last three stresses are shown in Fig. 5.6. The *shear rate* $\dot{\gamma}$ will be found from Fig. 5.6. Approximately the shear rate becomes:

$$\dot{\gamma} = \left(\frac{\partial v_\phi}{\partial \alpha} d\alpha\, dt\right) \frac{1}{r\, d\alpha} \frac{1}{dt} = \frac{\omega}{\alpha_o} = \text{constant} \tag{5.3.3}$$

The exact expression for the shear rate $\dot{\gamma} = -\dot{\gamma}_{\theta\phi}$ may be found from equation (4.2.4):

$$\dot{\gamma}_{\theta\phi} = \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) = \frac{\sin\theta}{r}\frac{1}{\sin\theta}\frac{\omega r}{\alpha_o}(-1) + \frac{\sin\theta}{r}\frac{\omega r}{\alpha_o}\alpha\frac{-\cos\theta}{\sin^2\theta}$$
$$\approx -\frac{\omega}{\alpha_o} \quad \text{for} \quad \alpha_o << 1$$

The result (5.3.3) shows that in this viscometric flow the shear rate is the same everywhere in the fluid. Because the shear rate is constant throughout the flow, the extra stresses related to the shear axes, i.e. to the spherical coordinate system, are also constant in the fluid, since they by the assumption in Eq. (5.2.2), are functions of the shear rate.

Viscosity Function $\eta(\dot{\gamma})$. The shear stress is expressed by the viscosity function:

$$\tau = \tau_{\theta\phi} = \eta(\dot{\gamma})\dot{\gamma} = \text{constant through the fluid} \tag{5.3.4}$$

The torque M is equal to the resultant of the shear stress τ on the plate. Per unit area the contribution to the torque is $(\tau \cdot r)$. The ring element of the plate, shown in Fig. 5.7, has the radius r and the area $dA = 2\pi r \cdot dr$. The torque from this element is then $(\tau \cdot r) \cdot (2\pi r \cdot dr)$. Hence:

$$\int_0^R (\tau \cdot r) \cdot (2\pi r \cdot dr) = M \quad \Rightarrow$$

$$\tau = \frac{3M}{2\pi R^3} \tag{5.3.5}$$

From the Eqs. (5.3.3–5.3.5) we obtain the following formula for the viscosity function:

$$\eta(\dot{\gamma}) = \frac{3M}{2\pi R^3}\frac{1}{\dot{\gamma}}, \quad \dot{\gamma} = \frac{\omega}{\alpha_o} \tag{5.3.6}$$

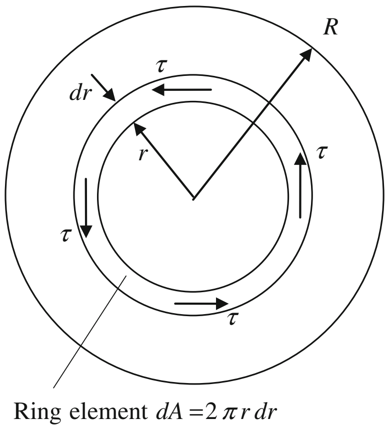


Fig. 5.7 Shear stress from the fluid on a ring element on the plate

Normal stress coefficients $\dot{\psi}_1(\dot{\gamma})$ *and* $\dot{\psi}_2(\dot{\gamma})$. We need the equation of motion in the r-direction. We assume that the velocities are very small and may therefore neglect centripetal accelerations. Furthermore, because $\alpha << 1$, a pressure field due to the gravitational force g is neglected. As stated above the extra stresses are all constant. Thus we obtain from the equation of motion (3.3.32) the following reduced equation:

$$0 = -\frac{\partial p}{\partial r} + \frac{1}{r}\left(2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi}\right) \tag{5.3.7}$$

Because the extra stress τ_{rr} is constant and $\sigma_{rr} = -(p - \tau_{rr})$, we may write:

$$\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}(p - \tau_{rr}) = -\frac{\partial \sigma_{rr}}{\partial r} \tag{5.3.8}$$

By definition of the normal stress coefficients in the Eqs. (5.2.4) and (5.2.7) we have:

$$\sigma_{\phi\phi} - \sigma_{\theta\theta} = \tau_{\phi\phi} - \tau_{\theta\theta} = \psi_1(\dot{\gamma})\dot{\gamma}^2 \tag{5.3.9}$$

$$\sigma_{\theta\theta} - \sigma_{rr} = \tau_{\theta\theta} - \tau_{rr} = \psi_2(\dot{\gamma})\dot{\gamma}^2 \tag{5.3.10}$$

from which we obtain:

$$2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi} = -(\psi_1 + 2\psi_2)\dot{\gamma}^2 \tag{5.3.11}$$

The equation of motion (5.3.7) may now be rewritten to:

$$\frac{\partial \sigma_{rr}}{\partial r} = \frac{1}{r}(\psi_1 + 2\psi_2)\dot{\gamma}^2 \tag{5.3.12}$$

The stress boundary condition at $r = R$ is:

$$\sigma_{rr}(R) = -p_a \tag{5.3.13}$$

p_a is the atmospheric pressure. Equation (5.3.12) is now integrated in the r-direction along the plate, $\alpha = 0$, and the boundary condition (5.3.13) is used. We then obtain:

$$\int_r^R \frac{\partial \sigma_{rr}}{\partial r} dr = \sigma_{rr}(R) - \sigma_{rr}(r) = (\psi_1 + 2\psi_2)\dot{\gamma}^2[\ln R - \ln r] \quad \Rightarrow$$

$$\sigma_{rr}(r) = (\psi_1 + 2\psi_2)\dot{\gamma}^2 \ln\frac{r}{R} - p_a \quad \text{at } \alpha = 0 \tag{5.3.14}$$

The pressure on the plate is $|\sigma_{\theta\theta}(r)| = -\sigma_{\theta\theta}(r)$ on the fluid side and p_a on the atmospheric side. The force F on the plate must balance the resultant of these pressures. The contribution to the force from the pressures on the ring element of area $dA = 2\pi r \cdot dr$, see Fig. 5.7, is: $dF = [-\sigma_{\theta\theta}(r) - p_a]dA$. The Eqs. (5.3.10) and (5.3.14) give:

$$-\sigma_{\theta\theta}(r) - p_a = -\psi_2\,\dot{\gamma}^2 - \sigma_{rr}(r) - p_a \quad \text{at } \alpha = 0 \quad \Rightarrow$$

$$-\sigma_{\theta\theta}(r) - p_a = -\psi_2\,\dot{\gamma}^2 - (\psi_1 + 2\psi_2)\dot{\gamma}^2 \ln\frac{r}{R} \quad \text{at } \alpha = 0 \tag{5.3.15}$$

Hence:

$$F = \int_0^R [-\sigma_{\theta\theta}(r) - p_a]2\pi r\,dr$$

$$= -2\pi\int_0^R \left[\psi_2\,\dot{\gamma}^2 + (\psi_1 + 2\psi_2)\dot{\gamma}^2 \ln\frac{r}{R}\right] r\,dr = -\psi_2\,\dot{\gamma}^2\pi R^2 + (\psi_1 + 2\psi_2)\dot{\gamma}^2\,\frac{\pi R^2}{2} \quad \Rightarrow$$

$$F = \psi_1\dot{\gamma}^2\,\frac{\pi R^2}{2} \tag{5.3.16}$$

The following formulas have been utilized in the derivation:

$$\int 2r\ln\frac{r}{R}dr = r^2\ln\frac{r}{R} - \frac{r^2}{2}, \quad 0 \cdot \ln 0 = 0$$

From the result (5.3.16) we get the expression for the primary normal stress coefficient.

$$\psi_1(\dot{\gamma}) = \frac{2F}{\pi R^2}\frac{1}{\dot{\gamma}^2}, \quad \dot{\gamma} = \frac{\omega}{\omega_o} \tag{5.3.17}$$

Next we shall develop a method to determine the secondary normal stress coefficient $\dot{\psi}_2(\dot{\gamma})$. Since the secondary normal stress difference by Eq. (5.3.10) is independent of r, we may use the result (5.3.12) to obtain the result:

$$\frac{d\sigma_{\theta\theta}}{dr} = \frac{1}{r}(\psi_1 + 2\psi_2)\dot{\gamma}^2 \quad \text{at} \quad \alpha = 0$$

which by application of the chain rule:

$$\frac{d\sigma_{\theta\theta}}{dr} = \frac{d\sigma_{\theta\theta}}{d(\ln r)}\frac{d(\ln r)}{dr} = \frac{d\sigma_{\theta\theta}}{d(\ln r)}\frac{1}{r}$$

may be rewritten to:

$$\frac{d\sigma_{\theta\theta}}{d(\ln r)} = (\psi_1 + 2\psi_2)\dot{\gamma}^2 \quad \text{at} \quad \alpha = 0 \tag{5.3.18}$$

From this equation we obtain the following formula for the secondary normal stress coefficient:

$$\psi_2(\dot{\gamma}) = \frac{1}{2\dot{\gamma}^2}\frac{d\sigma_{\theta\theta}}{d(\ln r)} - \frac{1}{2}\psi_1, \quad \dot{\gamma} = \frac{\omega}{\omega_o} \tag{5.3.19}$$

To find the term $d\sigma_{\theta\theta}/d(\ln r)$ we use recordings of the pressure $-\sigma_{\theta\theta}$ on the plate as a function of the radius r, from which the term can be calculated.

If the force F is not measured in the experiment using the cone and plate viscometer, the force may be calculated from the pressure measurements.

An alternative method to determine the two normal stress coefficients is as follows. From equation (5.3.10) and the boundary condition (5.3.13) we get:

$$\psi_2(\dot{\gamma}) = \frac{1}{\dot{\gamma}^2}(\sigma_{\theta\theta} + p_a) \quad \text{at } r = R \tag{5.3.20}$$

The expression in the parenthesis is determined from the pressure measurements on the plate. When $\dot{\psi}_2(\dot{\gamma})$ has been found from equation (5.3.20), formula (5.3.19) is used to obtain ψ_1.

Figure 5.8 shows a semi-log plot of results from pressure measurements on the plate of a cone and plate viscometer with a plate radius $R = 50$ mm. The fluid is a 2.5 % polyacrylamide solution and the figure is adapted from Bird et al. [3]. The slopes of the lines give the term $d\sigma_{\theta\theta}/d(\ln r)$ in the formula (5.3.19). Some experimental results for the two normal stress coefficients are shown in Fig. 5.9. In Problem 19 we are asked to determine points on the diagrams for the normal stress coefficients using the formulas (5.3.19 and (5.3.20) and experimental data in Fig. 5.8.

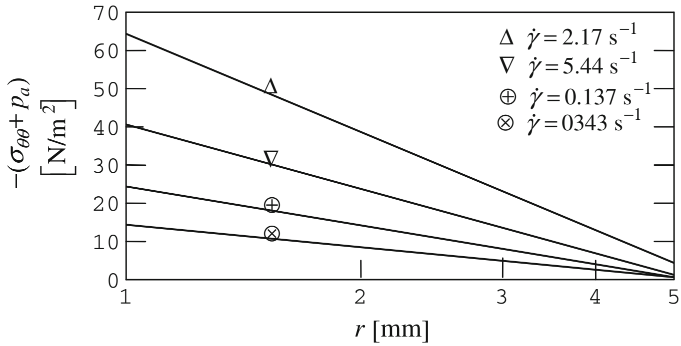


Fig. 5.8 Pressure measurements from a cone-and-plate viscometer. The fluid is 2.5 % polyacrylamide solution. The data are adapted from Bird et al. [3]

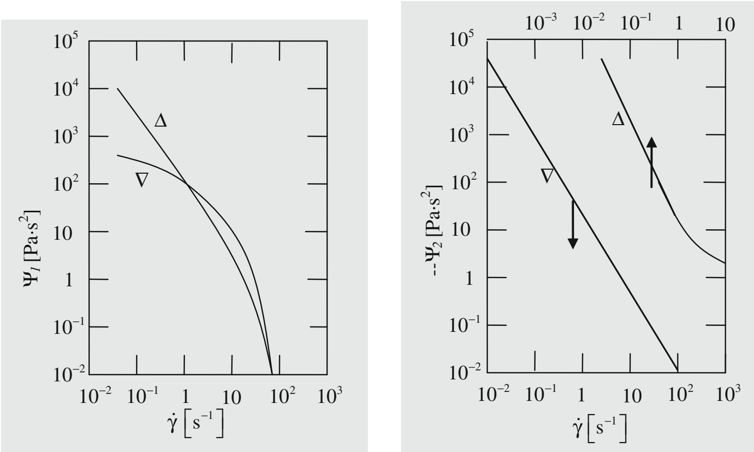


Fig. 5.9 Primary and secondary normal stress coeffisients ψ_1 *and* ψ_2 for two fluids

ψ_1 : ∇ *7 % aluminium laurate in decalin and m-cresol.*Δ *1.5 % polyacrylamide in water-glycerin solution.*

Based on a figure in Bird et al . [3] with data adopted from Huppler et al., *Trans. Soc. Rheol., 11, 159–179, 1967.*

ψ_2 :

∇ *2.5 % polyacrylamide in water-glycerin solution.*
Δ *3 %polyethyleneoxide in water-glycerin-isopropyl alcohol solution.*

Based on a figure in Bird et al. *[3] with data adopted from E. B. Christensen and W. R. Leppard, Trans- Soc. Rheol, 18. 65–86, 1974.*

5.4 Cylinder Viscometer

Figure 5.10 illustrates a cylinder viscometer. A circular cylinder rotates about a vertical axis in a cylindrical container. The cap h between the concentric cylindrical surfaces is very small as compared to the radii r_1 and r_2 :

$$h = r_2 - r_1 << r_1 \tag{5.4.1}$$

The lower part of the rotating cylinder is a cone. The angle α_0 between the conic surface of the rotating cylinder and the plane surface of the bottom of the cylindrical container is very small and will be specified further below. The fluid to be tested is poured into the space between the rotating cylinder and the container and fills the annular space between the cylindrical surfaces to the height H. The

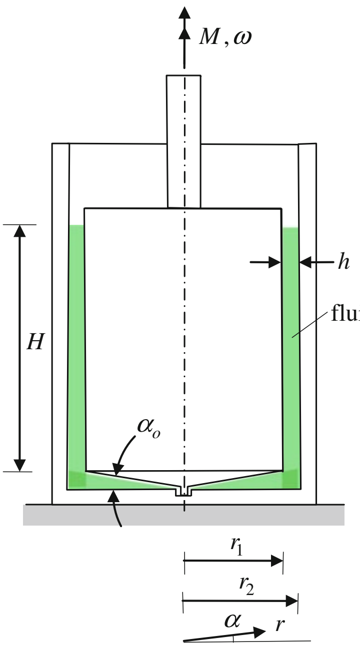


Fig. 5.10 The cylinder viscometer

viscometer may be used to find the *viscosity function* $\eta(\dot{\gamma})$ of the fluid. The rotating cylinder is subjected to a constant torque M and rotates with a constant angular velocity ω.

In the annular space between the two cylindrical surfaces the flow may be consider to be as a viscometric flow between two parallel plane surfaces, as shown in Fig. 1.3. The shear rate is according to formula (1.4.2):

$$\dot{\gamma} = \frac{r_1}{h}\omega \tag{5.4.2}$$

In the space between the conic surface of the rotating cylinder and the plane container bottom the flow is also viscometric and the shear rate is given by formula (5.3.3) as:

$$\dot{\gamma} = \frac{\omega}{\alpha_o} \tag{5.4.3}$$

To make the two shear rates (5.4.2) and (5.4.3) equal, the angle α_o is chosen to be:

$$\alpha_o = \frac{h}{r_1} \tag{5.4.4}$$

The shear stress τ on the cylindrical container wall and on the plane bottom of the cylinder will now be the same and given by:

$$\tau(\dot{\gamma}) = \eta(\dot{\gamma})\,\dot{\gamma} \tag{5.4.5}$$

The torque M is balanced by two contributions: M_c from the cylindrical surface and M_b from the bottom surface. Using the formulas (1.4.3) and (5.3.5), we get:

$$M = M_c + M_b, \quad M_c = 2\pi r_1^2 H\tau, \quad M_b = \frac{2\pi r_1^3}{3}\tau \tag{5.4.6}$$

By combining the equations (5.4.2), (5.4.5), and (5.4.6), we obtain the expression for the viscosity function:

$$\eta(\dot{\gamma}) \equiv \frac{\tau(\dot{\gamma})}{\dot{\gamma}} = \frac{h}{2\pi r_1^3 H(1 + r_1/3H)}\frac{M}{\omega} \tag{5.4.7}$$

5.5 Steady Pipe Flow

The *viscosity function* $\eta(\dot{\gamma})$ may be determined from tests with steady pipe flow. In this section we shall develop a set of formulas that first give the volumetric flow Q for any proposed viscosity function. Then it will be shown how these formulas may be used to determine the viscosity function from tests with steady pipe flow (Fig. 5.11).

We consider a steady pipe flow in a circular pipe of internal diameter d. The velocity field is assumed to be:

$$v_z = v(R), \quad v_R = v_\theta = 0 \tag{5.5.1}$$

This is a viscometric flow with the shear axes: $\mathbf{e}_1 = \mathbf{e}_z, \quad \mathbf{e}_2 = \mathbf{e}_R, \quad \mathbf{e}_3 = \mathbf{e}_\theta$ and rate of deformation matrix:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{\dot{\gamma}}{2}, \quad \dot{\gamma} = \frac{dv_z}{dR} = \frac{dv}{dR} \tag{5.5.2}$$

The stress matrix is, confer equation (5.2.3):

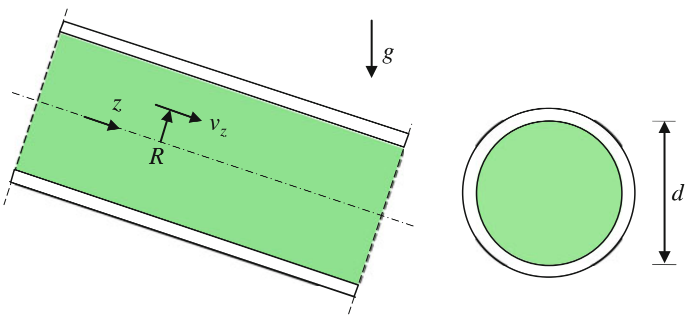


Fig. 5.11 Steady pipe flow

$$T = \begin{pmatrix} \tau_{zz} - P & \tau & 1 \\ \tau & \tau_{RR} - P & 0 \\ 0 & 0 & \tau_{\theta\theta} - P \end{pmatrix} \tag{5.5.3}$$

P is the modified pressure, and $\tau = \tau_{zR}$. The shear rate is according to equation (5.5.2) a function of the radius R only, and because the extra stresses are functions of the shear rate only, the extra stresses in the stress matrix (5.5.3) are also functions of the radius R only.

The analysis of steady pipe flow is presented in Sect. 3.8 from where we collect the formula (3.8.2) for the volumetric flow Q in a pipe of diameter d, and the formula (3.8.5) for the shear stress τ :

$$Q = 2\pi \int_0^{d/2} R\, v(R)\, dR, \quad \tau \equiv \tau_{zR} = \frac{c}{2} R, \quad c = \frac{\partial P}{\partial z} < 0 \tag{5.5.4}$$

The shear stress τ may be expressed by the shear stress τ_0 on the inner wall of the pipe:

$$\tau = -\frac{2\tau_0}{d} R, \quad \tau_0 = \frac{|c| d}{4} \tag{5.5.5}$$

Volumetric Flow Q . Partial integration of the integral in equation (5.5.4) yields:

$$Q = 2\pi \left[\frac{R^2}{2} v(R) \right]_0^{d/2} - 2\pi \int_0^{d/2} \frac{R^2}{2} \frac{dv}{dR} dR \tag{5.5.6}$$

It is now assumed that the fluid sticks to the pipe wall: $v(d/2) = 0$. This boundary condition reduces the formula for the volumetric flow to:

$$Q = -2\pi \int_0^{d/2} \frac{R^2}{2} \dot{\gamma}\, dR \tag{5.5.7}$$

At the end of the present section we shall see how we may extend the analysis in case slipping occurs at the pipe wall.

Now the variable in the integral in equation (5.5.7) is changed from the radius R to the shear stress τ from the equation (5.5.5). We get:

$$R^2 = \frac{d^2}{4\tau_0^2} \tau^2, \quad dR = -\frac{d}{2\tau_0} d\tau \tag{5.5.8}$$

This provides us with the following expression for the volumetric flow:

$$Q = \frac{\pi d^3}{8\tau_0^3} \int_0^{\tau_o} \tau^2 \dot{\gamma}\, d\tau, \quad \dot{\gamma} = \left| \frac{dv}{dR} \right| \tag{5.5.9}$$

Note that for convenience in the following development the negative rate of shear $\dot{\gamma}$ in equation (5.5.2) has been changed to the positive shear rate $\dot{\gamma} = |dv/dR|$. The formula is called the *Rabinowitsch equation*, after B. Rabinowitsch (1939). The formula (5.5.9) for Q may be used to compute the volumetric flow in a pipe when the relationship between the shear rate $\dot{\gamma}$ and the shear stress τ is known for the fluid. If the viscosity function $\eta(\dot{\gamma}) = \tau/\dot{\gamma}$ is given, it may be inverted to give $\dot{\gamma}(\tau)$. In most cases the integration in formula (5.5.9) must be executed numerically.

For a Newtonian fluid, with $\tau = \mu\dot{\gamma}$, equation (5.5.9) may be integrated analytically and gives the *Hagen-Poiseuille formula* in the equation (3.8.12):

$$Q = \frac{\pi d^3}{32\mu} \tau_0 = \frac{\pi d^4 |c|}{128\mu}, \quad \text{the } \textit{Hagen} - \textit{Poiseuille formula} \tag{5.5.10}$$

If the viscosity function is given by the power law: $\eta = K\dot{\gamma}^{n-1}$, the integration of formula (5.5.9) yields:

$$Q = \left(\frac{\tau_0}{K}\right)^{1/n} \frac{n}{1+3n} \frac{\pi d^3}{8} = \left(\frac{|c| d}{4K}\right)^{1/n} \frac{n}{1+3n} \frac{\pi d^3}{8} \tag{5.5.11}$$

This result is also found as equation (3.8.16) in Sect. 3.8.

Viscosity Function. We shall present a method by which the viscosity function $\eta(\dot{\gamma})$ may be determined from pipe flow tests. The starting point is the Rabinowitsch Equation (5.5.9), from which we obtain:

$$\int_0^{\tau_o} \tau^2 \dot{\gamma}\, d\tau = \frac{8Q}{\pi d^3} \tau_0^3 \tag{5.5.12}$$

This equation is differentiated with respect to τ_0, and the result is:

$$\tau_0^2 \dot{\gamma}_0 = \frac{d}{d\tau_0} \left[\frac{8Q}{\pi d^3} \tau_0^3 \right] \tag{5.5.13}$$

$\dot{\gamma}_o$ is the shear rate at the pipe wall. The viscosity function is then:

$$\eta(\dot{\gamma}_0) \equiv \frac{\tau_0}{\dot{\gamma}_0} = \tau_0^3 \left\{ \frac{d}{d\tau_0} \left[\frac{8Q}{\pi d^3} \tau_0^3 \right] \right\}^{-1} \tag{5.5.14}$$

The result will be transformed into a more practical formula. First we introduce two parameters:

$$\Gamma = \frac{32Q}{\pi d^3}, \quad \bar{n} = \frac{d(\log \tau_0)}{d(\log \Gamma)} \tag{5.5.15}$$

We may note that for a *Newtonian fluid* the expression (5.5.10) for the volumetric flow inserted into the first of the equation (5.5.15) yields:

$$\Gamma = \frac{\tau_0}{\mu} = \dot{\gamma}_0, \quad \bar{n} = 1 \tag{5.5.16}$$

For a *power law fluid* the expression (5.5.11) for the volumetric flow inserted into the equation (5.5.15) yields:

$$\Gamma = \frac{4n}{1+3n}\left(\frac{\tau_0}{K}\right)^{1/n} = \frac{4n}{1+3n}\dot{\gamma}_0, \quad \bar{n} = n \tag{5.5.17}$$

In a general case, when the viscosity function is unknown, we expand the term in the curly brackets in equation (5.5.14), using the parameter Γ in the equation (5.5.15).

$$\frac{d}{d\tau_0}\left[\frac{8Q}{\pi d^3}\tau_0^3\right] = \frac{d}{d\tau_0}\left[\frac{1}{4}\tau_0^3\Gamma\right] = \frac{3}{4}\tau_0^2\Gamma + \frac{1}{4}\tau_0^3\frac{d\Gamma}{d\tau_0} \tag{5.5.18}$$

The parameter $\bar{n}$ in equation (5.5.15), which represents the slope of the graph of log τ_0 versus log Γ, is also expanded.

$$\bar{n} = \frac{d(\log \tau_0)}{d(\log \Gamma)} = \frac{d(\log \tau_0)}{d\tau_0}\frac{d\tau_0}{d\Gamma}\frac{d\Gamma}{d(\log \Gamma)} = \frac{\Gamma}{\tau_0}\frac{d\tau_0}{d\Gamma} \tag{5.5.19}$$

We then obtain:

$$\tau_0^3\left\{\frac{d}{d\tau_0}\left[\frac{8Q}{\pi d^3}\tau_0^3\right]\right\}^{-1} = \tau_0^3\frac{1}{\frac{3}{4}\tau_0^2\Gamma + \frac{1}{4}\tau_0^3\frac{d\Gamma}{d\tau_0}} = \frac{4\frac{\Gamma}{\tau_0}\frac{d\tau_0}{d\Gamma}}{1+3\frac{\Gamma}{\tau_0}\frac{d\tau_0}{d\Gamma}}\frac{\tau_0}{\Gamma} \quad \Rightarrow$$

$$\tau_0^3\left\{\frac{d}{d\tau_0}\left[\frac{8Q}{\pi d^3}\tau_0^3\right]\right\}^{-1} = \frac{4\bar{n}}{1+3\bar{n}}\frac{\tau_0}{\Gamma} \tag{5.5.20}$$

By comparing the expressions (5.5.14) and (5.5.20), we see that the viscosity function is obtained as:

$$\eta(\dot{\gamma}_0) \equiv \frac{\tau_0}{\dot{\gamma}_0} = \frac{4\bar{n}}{1+3\bar{n}}\frac{\tau_0}{\Gamma} \quad \Leftrightarrow \quad \dot{\gamma}_0 = \frac{\tau_0}{\eta(\dot{\gamma}_0)} = \frac{1+3\bar{n}}{4\bar{n}}\Gamma \tag{5.5.21}$$

Based on the results from this analysis, we may design the following procedure to find the viscosity function $\eta(\dot{\gamma})$ for a fluid based on pipe flow tests.

1. Data sets for the modified pressure gradient c and the volumetric flow Q are recorded.
2. For each data set the shear stress τ_o at the pipe wall is calculated from the second of the formulas in (5.5.5) and the parameter Γ is found from the first of the formulas (5.5.15). The graph of $\log \tau_o$ versus $\log \Gamma$ is drawn.

3. The parameter $\bar{n}$ in the second of the formulas (5.5.15), which is the slope of the graph of $\log \tau_o$ versus $\log \Gamma$, is found at each data point on the graph, for instance by using a central difference formula.
4. Corresponding values for the viscosity function $\eta(\dot{\gamma})$ and the shear rate $\dot{\gamma}$ are calculated from the formulas (5.5.21). The graph of $\log \eta$ versus $\log \dot{\gamma}$ is drawn.
5. A suitable mathematical function for the viscosity function, for instance one of the analytical models presented in Sect. 6.1, is fitted to the graph of $\log \eta$ versus $\log \dot{\gamma}$.

The Case of Slip at the Pipe Wall. It may happen that the fluid does not stick to the wall of the pipe. A *slip velocity* v_o is then introduced. In fluids containing solid particles it may be that the particles have a tendency to move away from the wall and create a higher concentration near the center of the pipe. In the vicinity of the wall the fluid is then less viscous than in the central flow. This effect may be taken care of by introducing an *apparent slip velocity* at the wall and otherwise treat the fluid as homogeneous.

The slip velocity v_o may be found using the following procedure. First the formula (5.5.9) for volumetric flow is modified to:

$$Q = \frac{\pi d^2}{4} v_o + \frac{\pi d^3}{8\tau_o^3} \int_0^{\tau_o} \tau^2 \dot{\gamma}\, d\tau, \quad \dot{\gamma} = \left| \frac{dv}{dR} \right| \tag{5.5.22}$$

Then:

1. Pipes of different diameters d are used. The modified pressure gradient c is adjusted such that the shear stress at the pipe wall: $\tau_o = |c|d/4$ is the same for all the different diameters. The volumetric flow Q for the different diameters d is measured.
2. From equation (5.5.22) we find the following linear function of the variable $1/d$.

$$f(1/d) \equiv \frac{4Q}{\pi d^3} = \frac{v_o}{d} + \frac{1}{2\tau_o^3} \int_0^{\tau_o} \tau^2 \dot{\gamma}\, d\tau \tag{5.5.23}$$

   The graph of $f(1/d)$is drawn. The slip velocity v_o is then determined as the slope of the graph.

3. Points 1 and 2 are repeated using different values of the shear stress τ_o at the pipe wall.
4. The graph of the slip velocity v_o as a function of the wall shear stress τ_o is drawn, from which a suitable function: $v_o = v_o(\tau_o)$ is determined.

When the slip velocity $v_o = v_o(\tau_o)$ has been found, the parameter Γ in the first of the formulas (5.5.15) is replaced by:

$$\Gamma = \frac{32Q}{\pi d^3} - \frac{8v_o}{d} \tag{5.5.24}$$

Then the viscosity function $\eta(\dot{\gamma})$ may then be determined as describe above.

5.6 Material Functions for Steady Extensional Flows

Extensional flows are defined in Sect. 4.5. In a local coordinate system Px at a particle P with the x_i- axes parallel to the *principal directions of rates of deformation* the rate of deformation matrix is:

$$D = \begin{pmatrix} \dot{\varepsilon}_1 & 0 & 0 \\ 0 & \dot{\varepsilon}_2 & 0 \\ 0 & 0 & \dot{\varepsilon}_3 \end{pmatrix} \tag{5.6.1}$$

The principal directions are represented by the same material lines at all times. This implies that these directions also represent the *principal directions of strain*. For isotropic fluids these directions coincide with the *principal directions of stress*, i.e. no coordinate shear stresses in the Px – system, which again implies that the stress matrix in the Px – system is the diagonal matrix:

$$T = \begin{pmatrix} \tau_{11} - P & 0 & 0 \\ 0 & \tau_{22} - P & 0 \\ 0 & 0 & \tau_{33} - P \end{pmatrix} \tag{5.6.2}$$

For incompressible fluids, since the pressure is constitutively indeterminable, only normal stress differences may be modeled. It is customary two concentrate on:

$$\sigma_{11} - \sigma_{22} = \tau_{11} - \tau_{22}, \quad \sigma_{22} - \sigma_{33} = \tau_{22} - \tau_{33} \tag{5.6.3}$$

The third normal stress difference may be expressed by those two.

Three special cases of extensional flows are presented in Sect. 4.5, each defined by its characteristic rate of deformation matrix.

$$\textit{Uniaxial extensional flow}: \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{\dot{\varepsilon}}{2}, \quad \text{Figure 4.22} \tag{5.6.4}$$

$$\textit{Biaxial extensional flow}: \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \dot{\varepsilon}, \quad \text{Figure 4.23} \tag{5.6.5}$$

$$\textit{Planar extensional flow}: \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\varepsilon}, \quad \text{Figure 4.24} \tag{5.6.6}$$

The rate of deformation matrix for all these special extensional flows have only one characteristic strain rate. Two material functions may be introduced for these flows:

$$\bar{\eta}_1(\dot{\varepsilon}) = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}}, \quad \bar{\eta}_2(\dot{\varepsilon}) = \frac{\tau_{22} - \tau_{33}}{\dot{\varepsilon}} \tag{5.6.7}$$

Uniaxial Extensional Flow. The rate of deformation matrix (5.6.4) and isotropy implies that the normal stress in the $x_2 -$ and $x_3 -$ directions must be equal. Thus only one normal stress difference need be modeled and the material function is called the *extensional viscosity* or the *Trouton viscosity*, named after F. T. Trouton (1906):

$$\eta_{\mathrm{E}}(\dot{\varepsilon}) \equiv \bar{\eta}(\dot{\varepsilon}) = \bar{\eta}_1(\dot{\varepsilon}) = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}} \tag{5.6.8}$$

The alternative symbols for the extensional viscosity are used respectively by Barnes et al. [2] and by Bird et al. [3]. For some fluids the extensional viscosity is decreasing with increasing strain rate. This is called *tension-thinning*. When the extensional viscosity is increasing with increasing strain rate it is called *tension-thickening*.

For *incompressible Newtonian fluids*, the constitutive equations in Sect. 4.3, yield:

$$\tau_{ik} = 2\mu D_{ik} \quad \Rightarrow \quad \tau_{11} = 2\mu\,\dot{\varepsilon}, \quad \tau_{22} = -\mu\,\dot{\varepsilon}, \quad \tau_{33} = -\mu\,\dot{\varepsilon} \quad \Rightarrow$$

$$\tau_{11} - \tau_{22} = 3\mu\,\dot{\varepsilon}, \quad \eta_{\mathrm{E}} = 3\mu \tag{5.6.9}$$

This relationship between the extensional viscosity and the shear viscosity μ is in classical Newtonian fluid mechanics associated with the name Trouton.

The behavior of the extensional viscosity for a non-Newtonian fluid is frequently qualitatively different from that of the viscosity in shear flow. It is found that highly elastic polymer solutions that show shear-thinning often exhibit a dramatic tension-thickening. Experiments and further analysis in continuum mechanics show that as the strain rate $\dot{\varepsilon}$ approaches zero, the extensional viscosity approaches a value three times the value of the zero-shear-rate-viscosity as the shear strain rate approaches zero:

$$\eta_{\mathrm{E}}|_{\dot{\varepsilon}\to 0} = 3\eta|_{\dot{\gamma}\to 0} \quad \Leftrightarrow \quad \eta_{\mathrm{E}o} = 3\eta_o \tag{5.6.10}$$

Biaxial Extensional Flow. The rate of deformation matrix (5.6.5) for this flow and the rate of deformation matrix (5.6.4) for uniaxial extensional flow are equivalent. The extra stresses τ_{11} and τ_{22} are equal, and only one normal stress difference need be modeled. The material function is called the *biaxial extensional viscosity* η_{EB} :

$$\eta_{\mathrm{EB}}(\dot{\varepsilon}) \equiv \bar{\eta}_2(\dot{\varepsilon}) = \frac{\tau_{11} - \tau_{33}}{\dot{\varepsilon}}, \quad \bar{\eta}_1(\dot{\varepsilon}) = 0 \tag{5.6.11}$$

From the formula (5.6.8) for $\eta_{\mathrm{E}}(\dot{\varepsilon})$and the formula (5.6.11) for $\eta_{\mathrm{EB}}(\dot{\varepsilon})$ it follows that:

$$\tau_{11} - \tau_{33} = \dot{\varepsilon}\eta_{\mathrm{EB}}(\dot{\varepsilon}) = -(\tau_{33} - \tau_{11}) = -(-2\dot{\varepsilon})\eta_{\mathrm{E}}(-2\dot{\varepsilon}) \quad \Rightarrow$$

$$\eta_{\mathrm{EB}}(\dot{\varepsilon}) = 2\eta_{\mathrm{E}}(-2\dot{\varepsilon}) \tag{5.6.12}$$

Planar Extensional Flow. In this type of flow only one material function, called the *planar extensional viscosity*, is defined:

$$\eta_{\mathrm{EP}}(\dot{\varepsilon}) \equiv \bar{\eta}_1(\dot{\varepsilon}) \tag{5.6.13}$$

5.6.1 Measuring the Extensional Viscosity

It is much more difficult to measure the extensional viscosity $\eta_{\mathrm{E}}(\dot{\varepsilon})$ than the shear viscosity $\eta(\dot{\gamma})$. Figure 5.12 indicates an arrangement that is used. A cylindrical specimen of the fluid is stretched by a force $F = F(t)$ in the x_1 – direction.

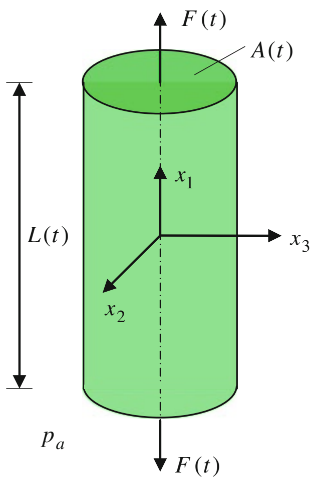
The length L of the specimen becomes an increasing function of time, while the cross-sectional area A becomes a decreasing function of time. The axial stress is:

$$\sigma_{11} = \tau_{11} - p = \frac{F(t)}{A(t)} - p_a \tag{5.6.14}$$

p_a is the atmospheric pressure. If body forces and accelerations are neglected, the general equations of motion (3.3.24) are reduced to:

$$0 = -p_{,i} + \tau_{ii,i} \quad i = 1, 2, 3 \tag{5.6.15}$$

Fig. 5.12 Fluid element



These equations imply that the stresses $\sigma_{ii} = \tau_{ii} - p$ are independent of the coordinate x_i. On the cylindrical surface of the specimen the atmospheric pressure p_a represents the stress vector, which implies that:

$$\sigma_{22} = \sigma_{33} = -p_a \quad \text{everywhere in the specimen} \tag{5.6.16}$$

Thus:

$$\tau_{22} - p = \tau_{33} - p = -p_a \quad \text{everywhere in the specimen} \tag{5.6.17}$$

From equations (5.6.14) and (5.6.17) we then get:

$$\tau_{11} - \tau_{22} = \frac{F(t)}{A(t)} \tag{5.6.18}$$

It will now be shown that to obtain a constant strain rate $\dot{\varepsilon}$ in the axial direction, the length $L(t)$ of the rod must be increased exponentially with time. Let the length and cross-sectional area of the rod at time t be L_o and A_o respectively. Then:

$$\dot{\varepsilon} = \frac{\dot{L}}{L} = \text{ constant} \quad \Rightarrow \quad \int_{L_o}^{L} \frac{dL}{L} = \dot{\varepsilon} \int_0^t dt \quad \Rightarrow \quad \ln\left(\frac{L}{L_o}\right) = \dot{\varepsilon}\, t \quad \Rightarrow$$

$$L(t) = L_o \exp(\dot{\varepsilon}\, t) \tag{5.6.19}$$

The volume of an incompressible fluid is constant: $L(t)\,A(t) = L_o\,A_o$, which implies that:

$$A(t) = A_o \exp(-\dot{\varepsilon}\, t) \tag{5.6.20}$$

To obtain a constant axial stress *F/A*, independent of time, the force must be adjusted such that:

$$F(t) = F_o(\dot{\varepsilon}) \;\exp(-\dot{\varepsilon}\, t) \tag{5.6.21}$$

The extensional viscosity may then be determined from the equations (5.6.8, 5.6.18, 5.6.20, 5.6.21). Hence:

$$\eta_{\mathrm{E}}(\dot{\varepsilon}) = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}} = \frac{F_o(\dot{\varepsilon})}{\dot{\varepsilon}\, A_o} \tag{5.6.22}$$

Chapter 6
Generalized Newtonian Fluids

6.1 General Constitutive Equations

The constitutive equations for an *incompressible Newtonian fluid* are defined by Eq. (4.3.3) in a Cartesian coordinate system *Ox*:

$$\tau_{ik} = 2\mu D_{ik} \tag{6.1.1}$$

$D = (D_{ik})$ is the rate of deformation matrix defined by:

$$D_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i}) \tag{6.1.2}$$

The viscosity μ is a function of the temperature and varies slightly with the pressure. However, the viscosity may in many cases be treated as a constant.

A *generalized Newtonian fluid* is an incompressible purely viscous fluid defined by the constitutive equations:

$$\tau_{ik} = 2\eta(\dot{\gamma}) D_{ik} \tag{6.1.3}$$

The *viscosity function* $\eta(\dot{\gamma})$ is a function of the *magnitude of shear rate* $\dot{\gamma}$, which is defined as:

$$\dot{\gamma} = \sqrt{2\,\mathrm{tr}\, D^2} \equiv \sqrt{2D_{ik} D_{ik}} \tag{6.1.4}$$

It may be shown that the magnitude of shear rate $\dot{\gamma}$ is an invariant, i.e. that the magnitude of shear rate is independent of the Cartesian coordinate system chosen to evaluate the rates of deformation D_{ik}. The definition (6.1.4) is easily extended to include also non-Cartesian coordinate system. In cylindrical coordinates (R, θ, z) and in spherical coordinates (r, θ, ϕ) we use the rate of deformation matrices in the Eqs. (4.2.1 and 4.2.3) respectively.

In a viscometric flow the rate of deformation matrix when related to the shear axes, takes the form:

F. Irgens, *Rheology and Non-Newtonian Fluids*,
DOI: 10.1007/978-3-319-01053-3_6,

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{\dot{\gamma}}{2} \tag{6.1.5}$$

The term $\dot{\gamma}$ is the shear rate of the flow. It follows from Eq. (6.1.4) that the magnitude of shear rate is equal to $|\dot{\gamma}|$ for viscometric flows. The constitutive equation (6.1.3) of a generalized Newtonian fluid give in viscometric flows the extra stresses:

$$\tau_{12} \equiv \tau = \eta(\dot{\gamma})\,\dot{\gamma}, \quad \text{other } \tau_{ik} = 0 \tag{6.1.6}$$

The viscosity function $\eta(\dot{\gamma})$ may thus be determined as the *viscosity function for viscometric flows*, see Eq. (5.2.7). On the other hand the primary and secondary normal stress coefficients for a generalized Newtonian fluid are both zero. That is contrary to experiments with most non-Newtonian fluids in steady shear flow. This discrepancy is a deficiency of the generalized Newtonian fluid model. For instance, the fluid model cannot describe the rod climbing phenomenon discussed in Chap. 2. Experience shows that the generalized Newtonian fluid is best suited for steady viscometric flows. However, it is also widely applied in more general types of flows, even for unsteady flows.

Figure 6.1 shows a characteristic graph for the viscosity function $\eta(\dot{\gamma})$ meant to fit data from experiments with a particular shear-thinning fluid in a viscometric flow. The value η_0 is the *zero-shear- rate-viscosity*, and the value η_∞ is the *infinite-shear-rate-viscosity*.

(a) *Power Law Fluid* (Ostwald-de Waele-fluid). The graph fitted to the experimental viscosity function of a shear-thinning fluid in a log–log diagram is often linear over the most interesting region of shear rates, as indicated in Fig. 6.1. The power law:

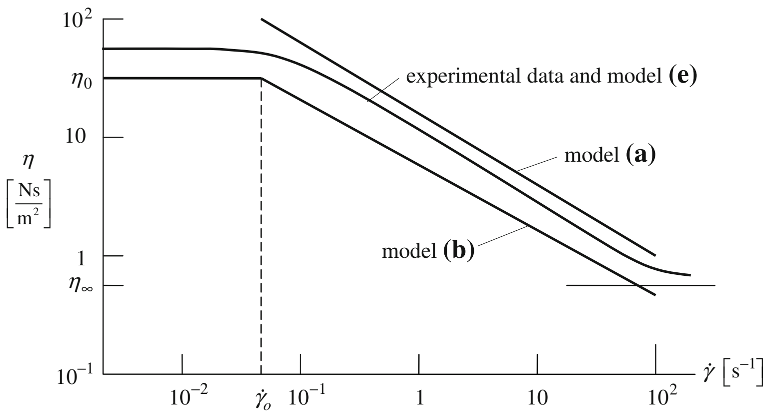


Fig. 6.1 Log-log plot of experimental viscosity function compared to proposed viscosity functions: (a) The power law fluid, (b) The Spriggs fluid, (e) The Carreau fluid, and an experimental graph

$$\eta(\dot{\gamma}) = K\dot{\gamma}^{n-1} \tag{6.1.7}$$

fits this linear portion of the viscosity function in the log–log plot. From Eq. (6.1.7) it follows that:

$$\log \eta = \log K + (n-1)\log \dot{\gamma} \tag{6.1.8}$$

The dimensionless *power law index n* is often found in the interval from 0.15 to 0.6. The *consistency parameter K* has the unity [Ns^n/m^2]. Examples of data pairs for the material parameters K and n are shown in Table 1.1. The dependence of K and n on the temperature Θ is often presented as:

$$K = K_0 \exp\left(-A\frac{\Theta - \Theta_0}{\Theta_0}\right), \quad n = n_0 + B\frac{\Theta - \Theta_0}{\Theta_0} \tag{6.1.9}$$

$K_0, A_0, n_0,$ and B are parameter values at the reference temperature Θ_0. The parameter B is often so small that the power law index may be taken to be constant. The power law was first suggested by W. Ostwald in Kolloid-Z., 36, 99−117 (1925) and A. de Waele in Oil and Color Chem. Assoc. Journal, 6, 33–88, (1923). An important objection to this model is the fact that it does not reflect the zero-shear-rate-viscosity. However, in many flow problems the regions of low shear rates are of lesser importance. A great advantage of the model is that it lends itself nicely to analytical solutions, and therefore the power law fluid model is widely in use in applications.

(b) *Spriggs Fluid* (The truncated power law-fluid) was suggested by T.W. Spriggs, Chem. Engr. Sci.,20, 931 (1965) and is defined by the viscosity function:

$$\eta(\dot{\gamma}) = \eta_0 \quad \text{when } \dot{\gamma} \le \dot{\gamma}_0, \quad \eta(\dot{\gamma}) = \eta_0\left(\frac{\dot{\gamma}}{\dot{\gamma}_0}\right)^{n-1} \quad \text{when } \dot{\gamma} \ge \dot{\gamma}_0 \tag{6.1.10}$$

$\dot{\gamma}_0$ is a reference shear rate.

(c) *Eyring Fluid* was proposed by H. Eyring and presented in F. H. Ree, T. Ree, and H. Eyring, Ind. Eng. Chem., 50, 1036–1040 (1959). The viscosity function is:

$$\eta(\dot{\gamma}) = \frac{\tau_0}{\dot{\gamma}}\arcsin\,(t_0\dot{\gamma}) \tag{6.1.11}$$

τ_0 is a characteristic shear stress and t_0 is a time constant.

(d) *Zener-Hollomon fluid*. This fluid model has been used in simulation of extrusion of aluminium by Sintef, Norway. The viscosity function is:

$$\eta(\dot{\gamma}) = \frac{1}{\sqrt{3}\,\alpha\,\dot{\gamma}}\arcsin\left[\left(\frac{Z}{A}\right)^{\frac{1}{n}}\right], \quad Z = \dot{\gamma}\,\exp\left(\frac{Q}{R\Theta}\right) \tag{6.1.12}$$

α, A, and n are material parameters, and Θ is the temperature. The material parameter Q is called the activation energy, and R is the universal gas constant.

Z is called the *Zener-Hollomon parameter*, and is a temperature compensated magnitude of shear rate.

The viscosity function $\eta(\dot{\gamma})$ in Eq. (6.1.12) is based on a suggestion in a paper by C. Zener, and J. H. Hollomon, J. H., J. Appl. Phys., Vol. 15, p. 22, 1944. Clearly, this fluid model is a variant of the Eyring fluid.

(e) *Carreau Fluid* was proposed by P.J. Carreau in his Ph.D. Thesis from University of Wisconsin 1968. The viscosity function is:

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)\left[1 + (\lambda\dot{\gamma})^2\right]^{\frac{n-1}{2}} \tag{6.1.13}$$

and contains both the *zero-shear-rate-viscosity* η_0 and the *infinite-shear-rate-viscosity* η_∞. In addition the model contains the time parameter λ and the power law index n. As seen from Fig. 6.1 the Carreau function (6.1.13) can be adjusted to give a very good fit to the experimental graph for the viscosity function over the whole range of $\dot{\gamma} -$ values.

(f) *Bingham Fluid* is attributed to E. C. Bingham and was suggested in his book Fluidity and Plasticity, McGraw-Hill, New York, 1922. This is a *viscoplastic fluid model*, behaving like a solid at low level of maximum shear stress and as a purely viscous fluid when the maximum shear stress $\tau_{\max}$ in the fluid exceeds the *yield shear stress* τ_y. The viscosity function may be presented as:

$$\eta(\dot{\gamma}) = \mu + \frac{\tau_y}{\dot{\gamma}} \text{ when } \tau_{\max} \geq \tau_y, \quad \eta(\dot{\gamma}) = \infty \text{ when } \tau_{\max} \leq \tau_y \tag{6.1.14}$$

This model is often used to model drilling mud, applied as a lubricant and a medium for conveying drill chips. It follows that when $\tau_{\max} \leq \tau_y$ the magnitude of shear rate $\dot{\gamma} = 0$.

Drilling fluid may consist of a mixture of small solid particles suspended in a liquid. The viscosity μ and the yield shear stress τ_y are functions of the volume fraction ϕ of solid particles, the diameter of the particles D_p, and the viscosity μ_l of the liquid. D. G. Thomas (A.I. Ch. E. Journal, 7, 431–437 (1961) and 9, 310-316 (1963)) has proposed these empirical formulas:

$$\tau_y = 312.5\frac{\phi^3}{D_p}, \quad \mu = \mu_l \exp\left[\frac{5}{2} + \frac{14}{\sqrt{D_p}}\right] \tag{6.1.15}$$

In these formulas the particle diameter D_p has the unit micrometer, and the yield stress τ_y is given in Pa $(= \text{N/m}^2)$.

(g) *Casson Fluid* was proposed by N. Casson in Rheology of Disperse Systems (C.C. Mills, Ed.), Pergamon Press, New York, 1959 to describe the flow of mixtures of pigments and oil. The viscosity function is:

$$\eta(\dot{\gamma}) = \left(\sqrt{\mu} + \sqrt{\frac{\tau_y}{\dot{\gamma}}}\right)^2 = \mu + \frac{\tau_y}{\dot{\gamma}} + 2\sqrt{\mu\frac{\tau_y}{\dot{\gamma}}} \text{ when } \tau_{\max} \geq \tau_y$$
$$\eta(\dot{\gamma}) = \infty \text{ when } \tau_{\max} \leq \tau_y \qquad (6.1.16)$$

The Casson fluid model is often used to describe blood flow. Blood consists of plasma and blood cells. The volume fraction of cells is called *hematocrit* and denoted H. Blood plasma is a Newtonian fluid. At high values of the magnitude of shear rate $\dot{\gamma}$, blood behaves as Newtonian fluid. At low values of the magnitude of shear rate $\dot{\gamma}$, blood is a non-Newtonian fluid. The Casson fluid model seems to be appropriate when $\dot{\gamma} < 10 \text{ s}^{-1}$ and $H < 40\,\%$. A detailed discussion of the mechanical properties of blood may be found in the book Biomechanics by Fung [6].

A few simple, but practically important, flow problems may be solved analytical using generalized Newtonian fluid models following the same procedures as for a Newtonian fluid. Applications with the power law fluid and the Bingham fluid have been presented in Sects. 3.7–3.9. However, most flow problems with non-Newtonian fluids must be solved numerically. In the next section we shall see how easily we run into practical problems when we try to solve a fairly simple problem analytically.

6.2 Helix Flow in Annular Space

For all the flow examples discussed in Sects. 3.7–3.9 the magnitude of shear rate $\dot{\gamma}$ is equal to the absolute value of the only rate of deformation used in the analysis of the flow. In the present section we shall discuss an example of a flow where this is not the case. The example will clearly demonstrate how soon computational problems arise in analysis of non-Newtonian fluid flows.

Figure 6.2a, b shall illustrate the flow of non-Newtonian fluid of density ρ in the annular space between two concentric cylindrical surfaces with a vertical axis. The distance h between the surfaces is small compared to radii r_1 and r_2 of the cylindrical surfaces. An example of this situation is given by the flow of drilling fluid around a drilling pipe during drilling for oil. The drilling mud flows downwards inside of the drilling pipe and returns upwards in the annular space between the pipe and the drilling hole carrying the drilling chips made by the drill crown.

The flow is driven by a constant negative modified pressure gradient $\partial P/\partial z$ and a rotation of the inner cylindrical surface, which rotates with a constant angular velocity ω. As a model for the fluid we select the *power law fluid*, and our task is to find an expression for the volumetric flow Q in the axial direction. We shall discover that the development of an analytical solution is possible up to a point from which we need in the most general case to apply a numerical solution. If the distance h is not small compared to the cylinder radii, an analytical solution to

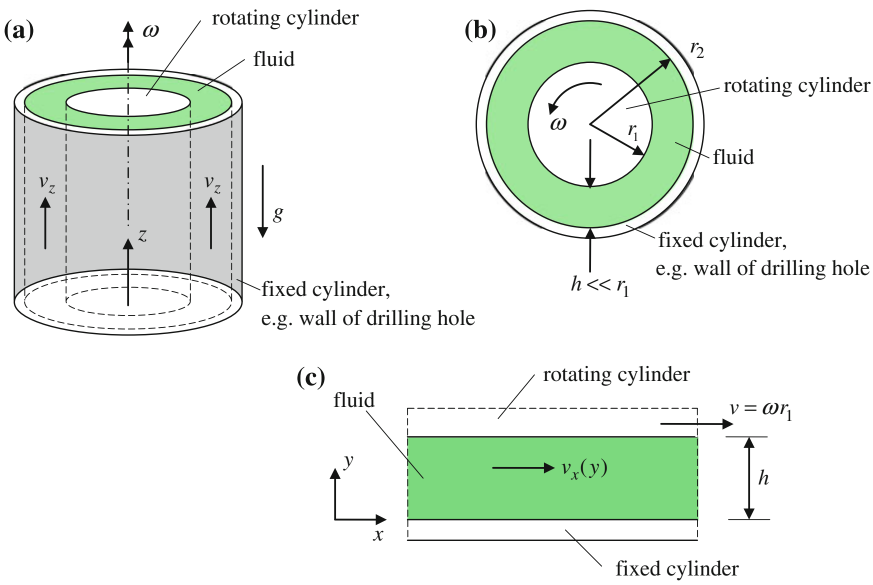


Fig. 6.2 Helix flow in annular space

the problem becomes impossible, and a numerical solution to the relevant differential equations must be performed at the very start of the solution.

Because $h \ll r_1$, we may treat the flow as flow between two parallel plates, Fig. 6.2c. The velocity field is assumed as:

$$v_z = v_z(y), \quad v_x = v_x(y), \quad v_y = 0 \tag{6.2.1}$$

Assuming that the fluid sticks to the solid surfaces, and referring to Fig. 6.2b, c, we may state the boundary conditions as:

$$(1)\ v_x(0) = 0, \quad (2)\ v_x(h) = \omega\, r_1, \quad (3)\ v_z(0) = 0, \quad (4)\ v_z(h) = 0 \tag{6.2.2}$$

The velocity field (6.2.1) gives the rate of deformation matrix:

$$D = \begin{pmatrix} 0 & \frac{1}{2}\frac{dv_x}{dy} & 0 \\ \frac{1}{2}\frac{dv_x}{dy} & 0 & \frac{1}{2}\frac{dv_z}{dy} \\ 0 & \frac{1}{2}\frac{dv_z}{dy} & 0 \end{pmatrix} \tag{6.2.3}$$

from which we compute the *magnitude of shear rate* $\dot{\gamma}$:

$$\dot{\gamma} = \sqrt{2D_{ik}D_{ik}} = \sqrt{\left(\frac{dv_x}{dy}\right)^2 + \left(\frac{dv_z}{dy}\right)^2} \tag{6.2.4}$$

The extra stresses are calculated from the constitutive equation (6.1.3) with the formula (6.1.7) for the viscosity function, and the stresses become functions of the coordinate y:

$$\tau_{ik}(y) = 2K\,\dot{\gamma}^{n-1} D_{ik} \quad \Rightarrow$$

$$\tau_{xy}(y) = K\,\dot{\gamma}^{n-1}\frac{dv_x}{dy}, \quad \tau_{zy}(y) = K\,\dot{\gamma}^{n-1}\frac{dv_z}{dy}, \quad \tau_{zx} = \tau_{xx} = \tau_{yy} = \tau_{zz} = 0 \tag{6.2.5}$$

The velocity field (6.2.1) results in no accelerations and the equations of motion (3.5.4) are reduced to the following equilibrium equations:

$$0 = -\frac{\partial P}{\partial x} + \frac{d\tau_{xy}}{dy}, \quad 0 = -\frac{\partial P}{\partial y}, \quad 0 = -\frac{\partial P}{\partial z} + \frac{d\tau_{zy}}{dy} \tag{6.2.6}$$

Due to symmetry with respect to the z-axis the modified pressure P must be independent of the tangential direction, which means that in the local coordinate system in Fig. 6.2c the modified pressure P must be independent of the x-coordinate. Then by the second of the equilibrium equations (6.2.6) the modified pressure is only a function of the z-coordinate. The third of the equilibrium equations (6.2.6) then implies that the pressure gradient in the z–direction must be a constant.

$$\frac{dP}{dz} = -c \text{ (} c \text{ is a positive constant)} \tag{6.2.7}$$

Substitution of the stresses (6.2.5) into the equilibrium Eq. (6.2.6) yields:

$$\begin{aligned} \frac{d}{dy}\left\{K\left[\left(\frac{dv_x}{dy}\right)^2 + \left(\frac{dv_z}{dy}\right)^2\right]^{\frac{n-1}{2}}\frac{dv_x}{dy}\right\} &= 0, \\ \frac{d}{dy}\left\{K\left[\left(\frac{dv_x}{dy}\right)^2 + \left(\frac{dv_z}{dy}\right)^2\right]^{\frac{n-1}{2}}\frac{dv_z}{dy}\right\} &= -c \end{aligned} \tag{6.2.8}$$

Integrations of the two equations give, with A and B as constants of integration:

$$\left[\left(\frac{dv_x}{dy}\right)^2 + \left(\frac{dv_z}{dy}\right)^2\right]^{\frac{n-1}{2}}\frac{dv_x}{dy} = A, \quad \left[\left(\frac{dv_x}{dy}\right)^2 + \left(\frac{dv_z}{dy}\right)^2\right]^{\frac{n-1}{2}}\frac{dv_z}{dy} = -\frac{c}{K}y + B \tag{6.2.9}$$

These equations are solved with respect to the velocity gradients and the result is:

$$\frac{dv_x}{dy} = \left[A^2 + \left(B - \frac{c}{K}y\right)^2\right]^{\frac{1-n}{2n}} A, \quad \frac{dv_z}{dy} = \left[A^2 + \left(B - \frac{c}{K}y\right)^2\right]^{\frac{1-n}{2n}}\left(B - \frac{c}{K}y\right) \tag{6.2.10}$$

The next integration must in the general case be performed numerically.

However, for $n = 1/3$ an analytical solution is found as:

$$v_x = \int \left[A^2 + \left(B - \frac{c}{K}y\right)^2\right] A\,dy = A^3 y + A\left(B - \frac{c}{K}y\right)^3 \left(\frac{K}{-3c}\right) + C$$

$$v_z = \int \left[A^2 + \left(B - \frac{c}{K}y\right)^2\right] \left(B - \frac{c}{K}y\right) dy \tag{6.2.11}$$

$$= A^2\left(B - \frac{c}{K}y\right)^2 \left(\frac{K}{-2c}\right) + \left(B - \frac{c}{K}y\right)^4 \left(\frac{K}{-4c}\right) + D$$

C and D are constants of integrations. To determine the four constants of integration A, B, C, and D we use the boundary conditions (6.2.2). Boundary conditions number (3) and number (4) yield:

$$D = \frac{A^2 B^2 K}{2c} + \frac{B^4 K}{4c}, \quad 2A^2\left(B - \frac{ch}{K}\right)^2 + \left(B - \frac{ch}{K}\right)^4 - 2A^2B^2 - B^4 = 0 \tag{6.2.12}$$

The last equation is of third degree for the unknown constant B, and the only real root is:

$$B = \frac{ch}{2K} \tag{6.2.13}$$

The boundary conditions (6.2.2) number (1) and number (2) now yield:

$$C = \frac{AB^3K}{3c}, \quad A^3 + \frac{1}{12}\left(\frac{ch}{K}\right)^2 A - \frac{\omega r_1}{h} = 0 \tag{6.2.14}$$

The only real root of the third degree equation for A is:

$$A = \left(\frac{\omega r_1}{h}\right)^{\frac{1}{3}} \left\{ \left[\frac{1}{2} + \sqrt{\frac{1}{4} + E}\right]^{\frac{1}{3}} + \left[\frac{1}{2} - \sqrt{\frac{1}{4} + E}\right]^{\frac{1}{3}} \right\}, \quad E = \left(\frac{c^2h^2}{36K^2}\right)^3 \left(\frac{h}{\omega r_1}\right)^2 \tag{6.2.15}$$

The four constants of integration A, B, C, and D are now given by the Eqs. (6.2.12–6.2.15), and the velocity field is determined. The result is:

$$v_x(y) = A^3 y + \frac{Ac^2h^3}{12K^2}\left[4\left(\frac{y}{h}\right)^3 - 6\left(\frac{y}{h}\right)^2 + 3\frac{y}{h}\right]$$

$$v_z(y) = \left(A^2 + \frac{c^2h^2}{4K^2}\right)\frac{ch^2}{2K}\frac{y}{h} - \left(A^2 + \frac{3c^2h^2}{4K^2}\right)\frac{ch^2}{2K}\left(\frac{y}{h}\right)^2 + \frac{c^3h^4}{2K^3}\left(\frac{y}{h}\right)^3 - \frac{c^3h^4}{4K^3}\left(\frac{y}{h}\right)^4 \tag{6.2.16}$$

The volumetric flow in the axial direction is:

$$Q = \left[\int_0^h v_z \, dy\right] \cdot 2\pi r_1 = \frac{\pi r_1 h^2}{40}\left(\frac{ch}{K}\right)^3\left[1 + \frac{20}{3}\left(\frac{AK}{ch}\right)^2\right] \tag{6.2.17}$$

In order to get a better understanding of the expression for the volumetric flow Q for large values of the pressure gradient c, we develop a power series expansion using the formulas:

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \ldots, \quad (1 \pm x)^{1/3} = 1 \pm \frac{1}{3}x + \ldots, \quad (|x| < 1) \tag{6.2.18}$$

First we write:

$$\frac{AK}{ch} = \frac{K}{h}\left(\frac{\omega r_1}{h}\right)^{\frac{1}{3}}\left\{\left[\frac{1}{2c^3} + \sqrt{\frac{1}{4c^6} + E}\right]^{\frac{1}{3}} + \left[\frac{1}{2c^3} - \sqrt{\frac{1}{4} + E}\right]^{\frac{1}{3}}\right\}$$

Using the expansions (6.2.18), we obtain the approximate formula:

$$\frac{AK}{ch} \approx 12\left(\frac{\omega r_1}{h}\right)\left(\frac{K}{ch}\right)^3 \tag{6.2.19}$$

From the Eqs. (6.2.17) and (6.2.19) we finally get:

$$Q = \frac{\pi r_1 h^2}{40}\left(\frac{ch}{K}\right)^3\left[1 + 960\left(\frac{\omega r_1}{h}\right)^2\left(\frac{K}{ch}\right)^6\right] \tag{6.2.20}$$

This formula shows to what extent the rotation of the inner cylindrical surface enlarges the volumetric flow in the axial direction. The reason is that the *apparent viscosity* of the fluid:

$$\eta(\dot{\gamma}) = K\dot{\gamma}^{n-1} = K/\dot{\gamma}^{2/3} \tag{6.2.21}$$

is decreased when the magnitude of shear rate (6.2.4) is increased by the contribution from the tangential shear rate $\dot{\gamma}_{R\theta} \approx \dot{\gamma}_{xy} = dv_x/dy$. Note that for a Newtonian fluid the axial flow and the tangential flow will be uncoupled, confer Problem 17.

6.3 Non-Isothermal Flow

With the exception of the example in Sect. 3.10.3, in all examples and flow cases we have discussed up to now, it has been tacitly assumed that the temperature in the fluid is constant. In other words, we have only treated *isothermal flows*. However, due to dissipation heat is created in the fluid and a temperature field will be developed. In addition the temperature field will influence the material parameters of the constitutive equations. We shall now briefly discuss non-isothermal flows in a *power law fluid*.

For simplicity we assume that it is acceptable as a first approximation to take the fluid density ρ and the coefficient of heat conduction k to be constants. To assume constant density implies that free convection is not considered.

First we shall write the thermal energy balance equation (3.10.16):

$$\rho c\dot{\Theta} = k\nabla^2\Theta + \omega \tag{6.3.1}$$

for incompressible fluids using the expression (3.10.11): for the stress power:

$$\omega = \tau_{ik} v_{i,k} \tag{6.3.2}$$

Due to the symmetry in the extra stresses, we may write:

$$\omega = \tau_{ik} v_{i,k} = \frac{1}{2}\tau_{ik} v_{i,k} + \frac{1}{2}\tau_{ki} v_{k,i} = \tau_{ik}\frac{1}{2}\left(v_{i,k} + v_{k,i}\right) \quad \Rightarrow$$

$$\omega = \tau_{ik} D_{ik} \tag{6.3.3}$$

For a generalized Newtonian fluid represented by constitutive equation (6.1.3) the stress power becomes:

$$\omega = \tau_{ik} D_{ik} = 2\eta(\dot{\gamma}) D_{ik} D_{ik} = 2\eta(\dot{\gamma}) D^2 \tag{6.3.4}$$

The thermal energy balance equation (6.3.1) is then for a generalized Newtonian fluid and for a power law fluid, respectively:

$$\rho c\dot{\Theta} = k\nabla^2\Theta + \omega = k\nabla^2\Theta + 2\eta D_{ik} D_{ik} \tag{6.3.5}$$

$$\rho c\dot{\Theta} = k\nabla^2\Theta + \omega = k\nabla^2\Theta + 2K\dot{\gamma}^{n-1} D_{ik} D_{ik} \tag{6.3.6}$$

The other basic fluid mechanics equations are: The *incompressible condition*:

$$\operatorname{div}\mathbf{v} = 0 \tag{6.3.7}$$

and the *Cauchy equations of motion*:

$$\rho\,\dot{\mathbf{v}} = -\nabla p + \nabla\cdot\mathbf{T}' + \rho\,\mathbf{b}, \quad \rho\,\dot{v}_i = -p_{,i} + \tau_{ik,k} + \rho b_i \tag{6.3.8}$$

6.3.1 Temperature Field in a Steady Simple Shear Flow

A fluid between two parallel horizontal plates flows steadily due to a constant velocity v of one plate relative to the other plate. Both plates are kept at constant temperature Θ_0, Fig. 6.3.

We want to determine the temperature field $\Theta(y)$ in the fluid due to the flow. The fluid is modeled as a *power law fluid*, and we assume for simplicity that the two material parameters K and n of the model are assumed to be constants.

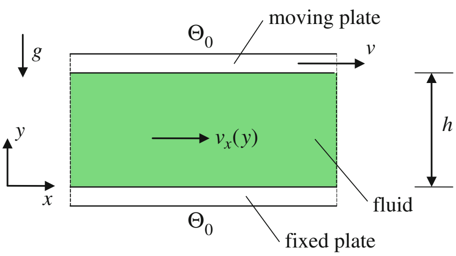


Fig. 6.3 Steady shear flow between two parallel plates

We assume the velocity field:

$$v_x = v_x(y), \quad v_y = v_z = 0 \tag{6.3.9}$$

The accelerations are zero, and the rate of deformation matrix and the magnitude of shear rate become:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\frac{dv_x}{dy}, \quad \dot{\gamma} = \sqrt{2D_{ik}D_{ik}} = \left|\frac{dv_x}{dy}\right| \tag{6.3.10}$$

The pressure p and the extra stresses are functions of the coordinate y:

$$p = p(y), \quad \tau_{ik} = 2K\dot{\gamma}^{n-1}D_{ik} \quad \Rightarrow$$
$$\tau_{xy} = K\left|\frac{dv_x}{dy}\right|^{n-1}\frac{dv_x}{dy} = K\left(\frac{dv_x}{dy}\right)^n, \quad \text{other } \tau_{ik} = 0 \tag{6.3.11}$$

The equations of motion (6.3.8) are reduced to:

$$0 = -\frac{dp}{dy} - \rho g \quad \Rightarrow \quad p(y) = -\rho g y + p(h), \quad 0 = \frac{d\tau_{xy}}{dy} \quad \Rightarrow \quad K\frac{d}{dy}\left(\frac{dv_x}{dy}\right)^n = 0 \tag{6.3.12}$$

The unknown pressure $p(h)$ at the moving plate is introduced as a boundary condition. Integration of the second of the equation (6.3.12) yields the general result:

$$\left(\frac{dv_x}{dy}\right)^n = C_1^n \quad \Rightarrow \quad \frac{dv_x}{dy} = C_1 \quad \Rightarrow \quad v_x(y) = C_1 y + C_2 \tag{6.3.13}$$

The constants of integration C_1 and C_2 are determined by the boundary conditions:

$$v_x(0) = 0 \quad \Rightarrow \quad C_2 = 0, \quad v_x(h) = v \quad \Rightarrow \quad C_1 = \frac{v}{h} \tag{6.3.14}$$

The velocity field is then:

$$v_x(y) = \frac{v}{h}y \tag{6.3.15}$$

The energy equation (6.3.6) is first reduced to:

$$0 = k\frac{d^2\Theta}{dy^2} + 2K\dot{\gamma}^{n-1}\left[\left(\frac{1}{2}\frac{dv_x}{dy}\right)^2 + \left(\frac{1}{2}\frac{dv_x}{dy}\right)^2\right] \quad \Rightarrow$$

$$k\frac{d^2\Theta}{dy^2} + K\left(\frac{dv_x}{dy}\right)^{n+1} = 0 \tag{6.3.16}$$

Then the velocity field (6.3.15) is introduced, which gives:

$$\frac{d^2\Theta}{dy^2} = -\frac{K}{k}\left(\frac{v}{h}\right)^{n+1} \quad \Rightarrow \quad \Theta(y) = -\frac{K}{k}\left(\frac{v}{h}\right)^{n+1}\frac{y^2}{2} + C_3 y + C_4 \tag{6.3.17}$$

The constants of integration C_3 and C_4 are determined by the boundary conditions:

$$\Theta(0) = T_o \quad \Rightarrow \quad C_4 = \Theta_0, \quad \Theta(h) = \Theta_0 \quad \Rightarrow \quad C_3 = \frac{K}{k}\left(\frac{v}{h}\right)^{n+1}\frac{h}{2} \tag{6.3.18}$$

The temperature field is thus:

$$\Theta(y) = \Theta_0 + \frac{Kh^2}{2k}\left(\frac{v}{h}\right)^{n+1}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right], \quad \Theta_{\max} = \Theta\left(\frac{h}{2}\right) = \Theta_0 + \frac{Kh^2}{8k}\left(\frac{v}{h}\right)^{n+1} \tag{6.3.19}$$

A more realistic solution to this problem in which the material parameter K is temperature dependent as in the formulas (6.1.9), may follow the same solution procedure as above, but since the equations of motion and the energy equation now will be coupled, the solution which has to be numerical, is much more complex.

Chapter 7
Linearly Viscoelastic Fluids

7.1 Introduction

If the stresses in a fluid depend both on strains and on strain rates, the fluid is characterized as *viscoelastic*. In general the stresses in a viscoelastic fluid depend on the *deformation history* the fluid has been subjected to. All real fluids are really viscoelastic because the pressure p is always a function of the volumetric strain. But if we limit the discussion to incompressible fluid models, we may distinguish between viscoelastic models and purely viscous models.

When we wish to analyze a non-steady flow problem for a viscoelastic fluid we must often use very complex constitutive equations to obtain acceptable results. In this chapter we discuss linearly viscoelastic fluid models that are relatively simple to apply, but which have a relatively limited range of applications. We have to assume that the strains are small in the time span of investigation and that the velocity gradients are small. The latter assumption implies that both the rates of deformation and the rates of rotation are small. The models may be used in analyses of small deformations of plastics, but also in some real fluid flow problems. The linearly viscoelastic fluid models are also of interest because the models may be "extended", as we shall see in the Sects. 8.6 and 8.7, to be relevant to real fluids in general flows.

7.2 Relaxation Function and Creep Function in Shear

The *relaxation function* $\beta(\gamma_0, t)$ *in shear* and the *creep function* $\alpha(\tau_0, t)$ *in shear* are material functions entering constitutive equations of linearly viscoelastic fluid models. These functions, resulting from two special so-called static tests with viscometric flows, were defined in Sect. 1.4.3 and the definition will be repeated below.

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It is convenient in the mathematical presentation to introduce two special functions that are related to one another. The *Heaviside unit step function* is defined in Eq. (1.4.20):

$$H(t) = \begin{cases} 0 & \\ & \text{for} \\ 1 & \end{cases} \begin{matrix} t \leq 0 \\ \\ t > 0 \end{matrix} \tag{7.2.1}$$

The *Dirac delta function* $\delta(t)$, named after Paul A. M. Dirac [1902–1984], is defined by the following properties:

$$\delta(t) \equiv \dot{H}(t) = \begin{cases} 0 & \\ & \text{for} \\ \infty & \end{cases} \begin{matrix} t \neq 0 \\ \\ t = 0 \end{matrix}$$
$$\int_{t_1}^{t_2} f(t)\delta(t)dt = f(0)[H(t_2) - H(t_1)] \tag{7.2.2}$$

$f(t)$ is any function of time t.

In a relaxation test in shear the fluid is subjected to a constant shear strain γ_o from the time $t = 0$. Using the Heaviside unit step function, we may express the strain history by:

$$\gamma(t) = \gamma_0 H(t) \tag{7.2.3}$$

The shear stress τ then becomes a function of the shear strain level γ_0 and of time t, and the general *relaxation function in shear* $\beta(\gamma_o, t)$ is now defined by:

$$\tau(\gamma_0, t) = \beta(\gamma_0, t)\gamma_0 H(t) \tag{7.2.4}$$

In a creep test the shear strain γ is registered as a result of a constant shear stress τ_0 from the time $t = 0^+$. The shear stress history is presented as:

$$\tau(t) = \tau_0 H(t) \tag{7.2.5}$$

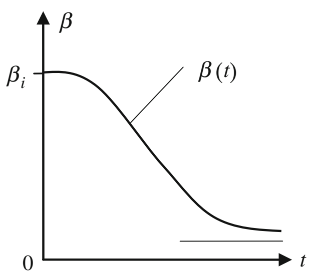


Fig. 7.1 The relaxation function in shear $\beta(t)$

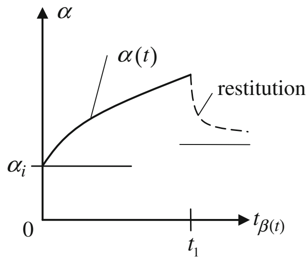


Fig. 7.2 The creep function in shear $\alpha(t)$

The resulting shear strain history defines the general *creep function in shear* $\alpha(\tau_0, t)$:

$$\gamma(\tau_0, t) = \alpha(\tau_0, t)\tau_0 H(t) \tag{7.2.6}$$

For *linearly viscoelastic fluids* the relaxation function in shear and the creep function in shear are functions of time only, Figs. 7.1 and 7.2:

$$\beta = \beta(t), \quad \alpha = \alpha(t) \tag{7.2.7}$$

The shear stress that results from a shear strain increment γ_0 at the time $\bar{t}$, i.e. from the strain history $\gamma_0 H(t - \bar{t})$, is independent of the level of strain prior to the time $\bar{t}$. Likewise, the shear strain that results from a shear stress increment τ_0 at the time $\bar{t}$, i.e. from the stress history $\tau_0 H(t - \bar{t})$, is independent of the level of stress prior to the time $\bar{t}$. These aspects may be used to develop constitutive equations for linearly viscoelastic fluids. We shall also see how the two material functions $\beta(t)$ and $\alpha(t)$ are related. In the Eq. (1.4.25) the following relationships were presented:

$$\alpha_g \beta_g = 1, \quad \alpha_e \beta_e = 1 \tag{7.2.8}$$

$\alpha_g \equiv \alpha(0)$ is the *glass compliance*, $\beta_g \equiv \beta(0)$ is the *glass modulus*, $\alpha_e \equiv \alpha(\infty)$ is the *equilibrium compliance*, and $\beta_e \equiv \beta(\infty)$ is the *equilibrium modulus*.

Now we want to determine an expression for the shear stress $\tau(t)$ due to a *shear strain history*:

$$\gamma = \gamma(\bar{t}), \quad \infty < \bar{t} \leq t \tag{7.2.9}$$

We assume that $\gamma(\bar{t}) = 0$ for $\bar{t} \leq t_0$. The time interval $[t_0, t]$ is divided into n equal time increments Δt, Fig. 7.3:

$$\Delta t = t_i - t_{i-1}, \quad i = 1, 2, 3, \ldots, n \tag{7.2.10}$$

In the interval $[t_{i-1}, t_i]$ we can find a time $\bar{t}_i$ such that:

$$\dot{\gamma}(\bar{t}_i) = \frac{\gamma_i - \gamma_{i-1}}{t_i - t_{i-1}} = \frac{\Delta\gamma_i}{\Delta t} \Rightarrow \Delta\gamma_i = \dot{\gamma}(\bar{t}_i)\Delta t \tag{7.2.11}$$

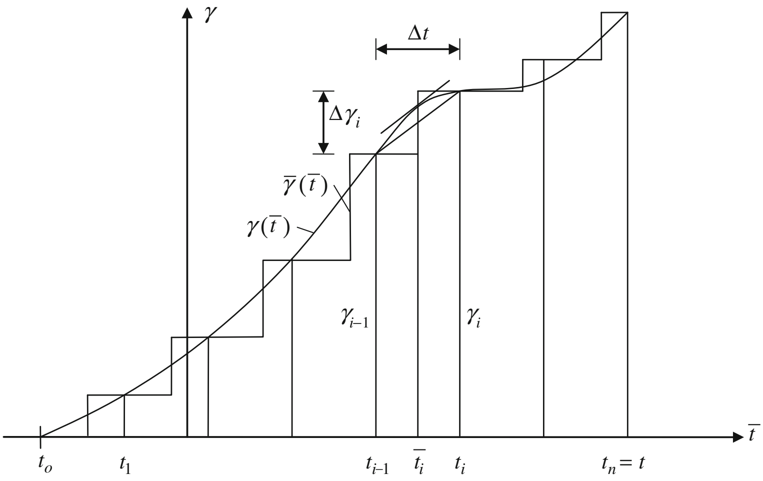


Fig. 7.3 Shear strain history

The shear strain history $\gamma(\bar{t})$ may then approximately be replaced by the step function:

$$\bar{\gamma}(\bar{t}) = \sum_{i=1}^{n} \Delta\gamma_i \, H(\bar{t} - \bar{t}_i) = \sum_{i=1}^{n} \dot{\gamma}(\bar{t}_i) \, H(\bar{t} - \bar{t}_i) \, \Delta t \tag{7.2.12}$$

See Fig. 7.3. Every element in the sums corresponds to a constant strain increment, and the response of the fluid is a shear stress expressed by the relaxation function $\beta(t)$ and the shear strain increment. The shear strain increment $\Delta\gamma_i$ results in the shear stress:

$$\tau_i(t) = \beta(t - \bar{t}_i)\Delta\gamma_i = \beta(t - \bar{t}_i)\dot{\gamma}(\bar{t}_i)\Delta t \tag{7.2.13}$$

Because $t > \bar{t}_i$ for all $i = 1, 2, 3, \ldots, n,$ the Heaviside function is left out in Eq. (7.2.13). The shear strain history gives the shear stress history:

$$\sum_{i=1}^{n} \tau_i(t) = \sum_{i=1}^{n} \beta(t - \bar{t}_i)\dot{\gamma}(\bar{t}_i) \, \Delta t \tag{7.2.14}$$

When we let $n \to \infty,$ the approximate shear strain history approaches the given shear strain history $\gamma(\bar{t})$, and the shear stress approaches:

$$\tau(t) = \int_{t_o}^{t} \beta(t - \bar{t})\dot{\gamma}(\bar{t}) \, d\bar{t} \Rightarrow \tau(t) = \int_{-\infty}^{t} \beta(t - \bar{t})\dot{\gamma}(\bar{t}) \, d\bar{t} \tag{7.2.15}$$

The implication in Eq. (7.2.15) follows from the assumption that $\dot{\gamma}(\bar{t}) = 0$ when $\bar{t} < t_o$. This method to arrive at the result (7.2.15) is called the *Boltzmann* *superposition principle* and was introduced by Ludwig Boltzmann [1844–1906] in (1874).

A similar procedure may be used to derive the following expression for the shear strain history $\gamma(t)$ if the *shear stress history* $\tau(t)$ is given:

$$\gamma(t) = \int_{-\infty}^{t} \alpha(t - \bar{t})\dot{\tau}(\bar{t})\, d\bar{t} \tag{7.2.16}$$

The two material functions, the relaxation function $\beta(t)$ and the creep function $\alpha(t)$, are related to one another. To obtain this relationship the strain history $\gamma(t) = \alpha(t)\tau_0 H(t)$ in a creep test and the corresponding stress history $\tau(t) = \tau_0 H(t)$ are introduced into the constitutive equation (7.2.15). The result is:

$$\begin{aligned} \tau_0 H(t) &= \int_{-\infty}^{t} \beta(t - \bar{t})[\alpha(\bar{t})\tau_0\, \delta(\bar{t}) + \dot{\alpha}(\bar{t})\tau_0\, H(\bar{t})]\, d\bar{t} \\ &= \beta(t)\alpha(0)\tau_0 + \int_{-\infty}^{t} \beta(t - \bar{t})[\dot{\alpha}(\bar{t})\tau_0\, H(\bar{t})]\, d\bar{t} \Rightarrow \\ 1 &= \beta(t)\alpha_g + \int_{0}^{t} \beta(t - \bar{t})\dot{\alpha}(\bar{t})\, d\bar{t} \end{aligned} \tag{7.2.17}$$

A similar procedure applied to the constitutive equation (7.2.16) yields:

$$1 = \alpha(t)\beta_g + \int_{0}^{t} \alpha(t - \bar{t})\dot{\beta}(\bar{t})\, d\bar{t} \tag{7.2.18}$$

From either of the Eqs. (7.2.17) or (7.2.18) we obtain the result:

$$1 = \alpha_g \beta_g \tag{7.2.19}$$

For $t = \infty$ the Eq. (7.2.18) yields:

$$\begin{gathered} 1 = \alpha(\infty)\beta_g + \alpha(\infty)[\beta(\infty) - \beta(0)] \Rightarrow \\ 1 = \alpha_e\, \beta_e \end{gathered} \tag{7.2.20}$$

7.3 Mechanical Models

In order to get a physical understanding of viscoelastic response it is customary to compare the behaviour of the viscoelastic material in uniaxial stress to that of mechanical models. Figure 7.4a shows a test specimen of viscoelastic material subjected to an axial tensile force N. The cross-sectional area of the specimen is A, and the length is L_0 when $N = 0$. The specimen has uniaxial stress $\sigma = N/A$, and the strain in the axial direction becomes ε. For a linearly elastic material Hooke's law applies:

$$\sigma = E\varepsilon \tag{7.3.1}$$

E is the *modulus of elasticity*. The mechanical model shown in Fig. 7.4b is a linear spring with spring constant k. The force $N = \sigma A$ results in an elongation $\Delta L = \varepsilon L_o$ such that:

$$N = k\Delta L \quad \Rightarrow \quad \sigma A = k\varepsilon L_0 \quad \Rightarrow, \quad \sigma = \frac{kL_0}{A}\varepsilon = E\varepsilon \quad \Rightarrow \quad k = \frac{EA}{L_0} \tag{7.3.2}$$

Thus Eq. (7.3.1) represents the response to the mechanical model in Fig. 7.4b and a linearly elastic material in uniaxial stress. Similarly the *linear dashpot* (damper) model in Fig. 7.4c has the same response as a linearly viscous material in uniaxial stress and the response equation is:

$$\sigma = c\dot{\varepsilon} \tag{7.3.3}$$

The parameter c is a viscosity.

Figure 7.4d shows the simplest viscoelastic mechanical model relevant for fluid modelling: the *Maxwell model*, James Clerk Maxwell [1831–1879]. The spring provides the strain contribution: $\varepsilon_1 = \sigma/E$, and the dashpot results in a strain rate

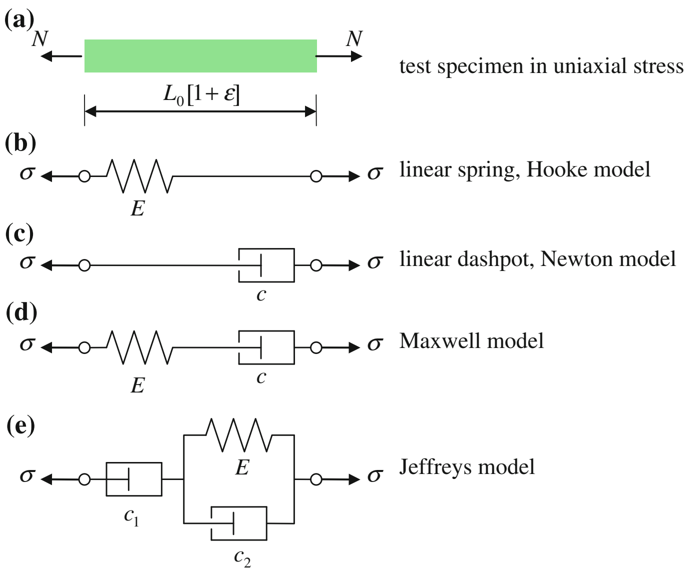


Fig. 7.4 Mechanical models

contribution: $\dot{\varepsilon}_2 = \sigma/c$. The total strain rate for the Maxwell mechanical model is then:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{c} \tag{7.3.4}$$

This equation is rearranged to the *response equation for the Maxwell model*:

$$\sigma + \frac{c}{E}\dot{\sigma} = c\dot{\varepsilon} \tag{7.3.5}$$

Figure 7.4e shows the *Jeffreys model*, Harold Jeffreys [1891–1989]. The response equation for this model is developed as follows. The dashpot on the left-hand side in Fig. 7.4c contributes the strain rate:

$$\dot{\varepsilon}_1 = \frac{\sigma}{c_1} \tag{7.3.6}$$

The parallel element of the spring and the dashpot contributes the strain ε_2 resulting in the stress $E\varepsilon_2$ in the spring and the stress $c_2\dot{\varepsilon}_2$ in the dashpot. The total stress in the parallel element is therefore:

$$\sigma = E\,\varepsilon_2 + c_2\,\dot{\varepsilon}_2 \tag{7.3.7}$$

The total strain in the Jeffreys model is:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{7.3.8}$$

From the Eqs. (7.3.6–8) we obtain:

$$\begin{gathered}\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}_1) + c_2(\ddot{\varepsilon} - \ddot{\varepsilon}_1) = E\left(\dot{\varepsilon} - \frac{\sigma}{c_1}\right) + c_2\left(\ddot{\varepsilon} - \frac{\dot{\sigma}}{c_1}\right) \quad \Rightarrow \\ \sigma + \frac{c_1 + c_2}{E}\dot{\sigma} = c_1\dot{\varepsilon} + \frac{c_1 c_2}{E}\ddot{\varepsilon}\end{gathered} \tag{7.3.9}$$

This is the *response equation for the Jeffreys model*.

The mechanical models presented in Fig. 7.4 are most appropriate for visco-elastic solids. However, the models may give a physical understanding for vis-coelastic behaviour of both solid and liquids. Viscoelastic response of liquids is usually investigated by shear tests, for instance with a cylinder viscometer. This fact will be reflected in the presentation of the general constitutive equations in the following section.

7.4 Constitutive Equations

In this section we shall present the most commonly used models for isotropic, linearly viscoelastic, and incompressible fluids. Each model is related to a corre-sponding mechanical model.

The *Maxwell fluid* is defined by the following response equations, which is analogous to Eq. (7.3.5) for a mechanical *Maxwell model*.

$$\tau_{ik} + \lambda\dot{\tau}_{ik} = 2\mu D_{ik} \tag{7.4.1}$$

λ is a time parameter called the *relaxation time*, and μ is a viscosity. For simple shear flow the Eq. (7.4.1) are reduced to the response equation for the shear stress τ and shear strain rate $\dot{\gamma}$:

$$\tau + \lambda\,\dot{\tau} = \mu\dot{\gamma} \tag{7.4.2}$$

The relaxation function in shear $\beta(t)$ is determined by integration of the response equation (7.4.2) for the shear strain history:

$$\gamma(t) = \gamma_0 H(t) \tag{7.4.3}$$

with the shear stress condition: $\tau(t) = 0$ when $t \leq 0$. Substitution of the shear strain history (7.4.3) into the response equation (7.4.2), results in the differential equation:

$$\tau + \lambda\,\dot{\tau} = \mu\gamma_0\,\delta(t) \tag{7.4.4}$$

A particular solution of the homogeneous equation $\tau + \lambda\,\dot{\tau} = 0$, is $\exp\,(-t/\lambda)$. Using the method of variation of parameters, we assume as the general solution of Eq. (7.4.4):

$$\tau(t) = C(t)\exp\left(-\frac{t}{\lambda}\right) \tag{7.4.5}$$

The function $C(t)$ is determined from the inhomogeneous equation (7.4.4). When the solution (7.4.5) is substituted into Eq. (7.4.4), we obtain:

$$\dot{C}(t) = \frac{\mu}{\lambda}\gamma_o\exp\left(\frac{t}{\lambda}\right)\,\delta(t) \tag{7.4.6}$$

Because $\tau(t) = 0$ for $t < 0$, $C(0) = 0$. Thus for any $t_0 < 0$:

$$C(t) = \int_{t_0}^{t} \dot{C}(\bar{t})\,d\bar{t} = \frac{\mu}{\lambda}\gamma_0 \int_{t_0}^{t} \exp\left(\frac{\bar{t}}{\lambda}\right)\,\delta(\bar{t})\,d\bar{t} = \frac{\mu}{\lambda}\gamma_0\,H(t) \tag{7.4.7}$$

The response to the shear strain history: $\gamma(t) = \gamma_0 H(t)$ is therefore, according to Eq. (7.4.5) and the result (7.4.7):

$$\tau(t) = \frac{\mu}{\lambda}\gamma_0\exp\left(-\frac{t}{\lambda}\right)H(t) \tag{7.4.8}$$

This is the solution of the differential equation (7.4.4) under the stress condition $\tau(t) = 0$ when $t < 0$. The *relaxation function in shear for the Maxwell fluid* is thus:

$$\beta(t) = \frac{\mu}{\lambda}\exp\left(-\frac{t}{\lambda}\right) \tag{7.4.9}$$

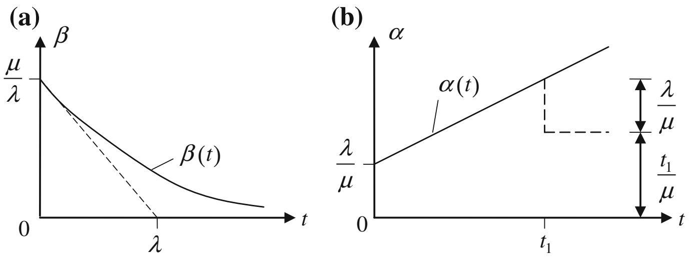


Fig. 7.5 The relaxation function $\beta(t)$ and the creep function $\alpha(t)$ in shear for the Maxwell fluid

The graph $\beta(t)$ of is shown in Fig. 7.5a. The relaxation time λ is illustrated in Fig. 7.5a.

The creep function in shear $\alpha(t)$ is determined by integration of the response Eq. (7.4.2) for the shear stress history:

$$\tau(t) = \tau_0 H(t) \tag{7.4.10}$$

with the shear strain condition: $\gamma(t) = 0$ when $t \leq 0$. Substitution of the shear stress history (7.4.10) into the response equation (7.4.2), results in the differential equation:

$$\dot{\gamma} = \left[\frac{1}{\mu} H(t) + \frac{\lambda}{\mu} \delta(t)\right] \tau_0 \tag{7.4.11}$$

Integration of this equation from any time $t_0 \leq 0$ to the present time t yields:

$$\gamma = \int_{t_0}^{t} \dot{\gamma}\, d\bar{t} = \frac{\tau_0}{\mu} \int_{t_0}^{t} H(\bar{t})\, d\bar{t} + \frac{\lambda \tau_0}{\mu} \int_{t_0}^{t} \delta(\bar{t})\, d\bar{t} = \frac{1}{\mu}(t + \lambda)\, \tau_0 H(t)$$

From which we extract the *creep function in shear for a Maxwell fluid*:

$$\alpha(t) = \frac{\lambda}{\mu}\left(1 + \frac{t}{\lambda}\right) \tag{7.4.12}$$

The graph of $\alpha(t)$ is shown in Fig. 7.5b, which also indicates the restitution when the stress τ_0 is removed at time t_1.

The *Jeffreys fluid* is defined by the following response equations, which is analogous to Eq. (7.3.9) for a mechanical *Jeffreys model*.

$$\tau_{ik} + \lambda_1 \dot{\tau}_{ik} = 2\mu D_{ik} + 2\mu \lambda_2 \dot{D}_{ik} \tag{7.4.13}$$

λ_1 and λ_2 are time parameters, and μ is a viscosity. For simple shear flow the Eq. (7.4.13) are reduced to the response equation:

$$\tau + \lambda_1 \dot{\tau} = \mu \dot{\gamma} + \mu \lambda_2 \ddot{\gamma} \tag{7.4.14}$$

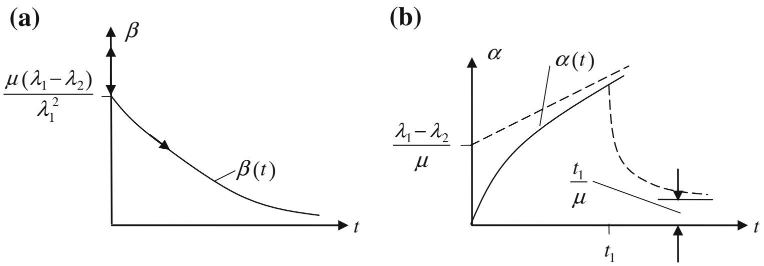


Fig. 7.6 The relaxation function $\beta(t)$ and the creep function $\alpha(t)$ in shear for the Jeffreys fluid

The *relaxation function in shear* $\beta(t)$ and the *creep function in shear* $\alpha(t)$ *for the Jeffreys fluid* are determined as for the Maxwell fluid. The result is:

$$\begin{aligned}\beta(t) &= \frac{\mu}{\lambda_1}\left[\left(1-\frac{\lambda_2}{\lambda_1}\right)\exp\left(-\frac{t}{\lambda_1}\right)+\lambda_2\,\delta(t)\right],\\ \alpha(t) &= \frac{t}{\mu}+\frac{\lambda_1-\lambda_2}{\mu}\left[1-\exp\left(-\frac{t}{\lambda_2}\right)\right]\end{aligned} \tag{7.4.15}$$

The graph of $\beta(t)$ and $\alpha(t)$ are shown in Fig. 7.6.

For a simple shear flow a general constitutive equation of a linearly viscoelastic fluid with a *relaxation function in shear* $\beta(t)$ is given by Eq. (7.2.15). A generalization of this equation yields the following general constitutive equation of an incompressible and isotropic linearly viscoelastic fluid when subjected to the rate of deformation history $D_{ik}(t)$.

$$\tau_{ik}(t) = 2\int_{-\infty}^{t} \beta(t-\bar{t})D_{ik}(\bar{t})\,d\bar{t} \tag{7.4.16}$$

It is convenient to rewrite the integral in Eq. (7.4.16) by introducing the concept of a "past time" $s = t-\bar{t}$, which measures time backwards from the present time t to the current time $\bar{t}$:

$$s = t-\bar{t} \quad \Rightarrow \quad s = t \text{ at } \bar{t} = 0, \quad s = 0 \text{ at } \bar{t} = t \text{ and } d\bar{t} = -ds \tag{7.4.17}$$

Then we obtain:

$$\begin{aligned}\tau_{ik}(t) &= 2\int_{\infty}^{0} \beta(s)D_{ik}(t-s)\,(-ds) \quad \Rightarrow\\ \tau_{ik}(t) &= 2\int_{0}^{\infty} \beta(s)D_{ik}(t-s)\,ds\end{aligned} \tag{7.4.18}$$

These equations are the general *constitutive equations for linearly viscoelastic fluids*.

Alternative constitutive equations may be found as a generalization of the shear flow equation (7.2.16), which is based on the creep function in shear $\alpha(t)$. However, these alternative equations are not particularly useful for viscoelastic fluids.

7.5 Stress Growth After a Constant Shear Strain Rate

We consider a viscoelastic fluid between two parallel horizontal plates, Fig. 7.7. The fluid is at rest at times $t \le 0$. The lower plate is at rest at all times. At the time $t = 0$ the upper plate starts suddenly to move with a constant velocity v_o, such that the horizontal plate velocity is given by:

$$v(t) = v_0 \, H(t) \tag{7.5.1}$$

We assume simple shear flow in the fluid:

$$v_x(x,t) = v(t)\frac{y}{h} = v_0 \, H(t)\frac{y}{h}, \qquad v_y = v_z = 0 \tag{7.5.2}$$

The only non-zero rate of deformation is:

$$D_{xy}(t) = \frac{1}{2}\frac{dv_x}{dy} = \frac{1}{2}\dot{\gamma}_0 \, H(t), \quad \dot{\gamma}_0 = \frac{v_0}{h} \tag{7.5.3}$$

The general constitutive equations (7.4.18) for a linearly viscoelastic fluid yield only one non-zero extra stress:

$$\tau(t) \equiv \tau_{xy}(t) = 2\int_0^\infty \beta(s)\, D_{xy}(t-s)\, ds = \int_0^\infty \beta(s)\dot{\gamma}_0 \, H(t-s)\, ds \quad \Rightarrow$$

$$\tau(t) \equiv \tau_{xy}(t) = \dot{\gamma}_0 \left[\int_0^t \beta(s)\, ds\right] H(t), \quad \dot{\gamma}_0 = \frac{v_o}{h} \tag{7.5.4}$$

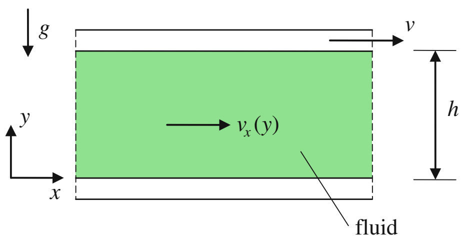


Fig. 7.7 Simple shear flow

After a while, theoretically after infinitely long time, the flow becomes steady, and the shear stress is:

$$\tau_0 = \dot{\gamma}_0 \int_0^\infty \beta(s) ds \tag{7.5.5}$$

The integral represents the viscosity μ_0 in the case of a steady shear flow.

$$\mu_0 = \int_0^\infty \beta(s) ds \tag{7.5.6}$$

The value of μ_0 is equal to the area under graph of the relaxation function in shear, as represented in Fig. 7.1. The shear stress in a steady flow is: $\tau_0 = \mu_0 \, \dot{\gamma}_0$.

For a *Maxwell fluid* with the relaxation function in shear (7.4.9) we get:

$$\int_0^t \beta(s)\, ds = \int_0^t \frac{\mu}{\lambda} \exp\left(-\frac{s}{\lambda}\right) ds = \left[\frac{\mu}{\lambda}(-\lambda) \exp\left(-\frac{s}{\lambda}\right)\right]_0^t = \mu\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right] \quad \Rightarrow$$

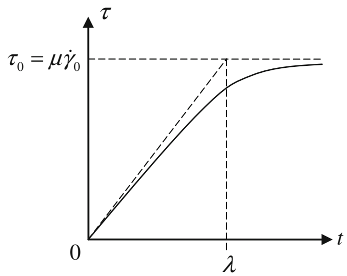
$$\int_0^t \beta(s)\, ds = \mu\left[1 - \exp\left(-\frac{t}{\lambda}\right)\right], \quad \mu_0 = \int_0^\infty \beta(s)\, ds = \mu \tag{7.5.7}$$

The viscosity in steady flow μ_o is thus equal to the viscosity parameter μ. From the equation (7.5.4) and (7.5.7) we obtain:

$$\tau(t) = \tau_0 \left[1 - \exp\left(-\frac{t}{\lambda}\right)\right] H(t), \quad \tau_0 = \tau(\infty) = \mu \dot{\gamma}_0 \tag{7.5.8}$$

The graph of the shear stress $\tau(t)$ is shown in Fig. 7.8.

Fig. 7.8 Shear stress $\tau(t)$



7.6 Oscillations with Small Amplitude

We consider a viscoelastic fluid between two parallel plates as in Fig. 7.7. The lower plate is at rest at all times, while the upper plate oscillates horizontally with the velocity:

$$v(t) = v_0 \cos \omega t, \quad v_0 = \text{constant} \tag{7.6.1}$$

The parameter ω is the *angular: frequency* of the oscillations. We assume that the velocity field in the fluid is:

$$v_x(y,t) = \frac{v(t)}{h} y = \frac{v_0}{h} y \cos \omega t, \quad v_y = v_z = 0 \tag{7.6.2}$$

This assumption implies that we neglect propagation of velocity waves between the oscillating plate and the stationary plate. The only non-zero rate of deformation in this flow is:

$$D_{xy}(t) = \frac{1}{2}\frac{dv_x}{dy} = \frac{1}{2}\dot{\gamma}_0 \cos \omega t, \quad \dot{\gamma}_0 = \frac{v_0}{h} \tag{7.6.3}$$

The only extra stress in this flow is the shear stress $\tau(t) \equiv \tau_{xy}(t)$, which is to be determined from the constitutive equations of the fluid. This procedure will be demonstrated below.

When the shear stress $\tau(t) \equiv \tau_{xy}(t)$, has been found, we can use the result to calculate the torque $M(t)$ in the cylinder viscometer in Fig. 5.10 if the rotating cylinder is subjected to a harmonically angular velocity:

$$\Omega(t) = \Omega_0 \cos \omega t, \quad \Omega_0 = \text{constant} \tag{7.6.4}$$

The non-zero rate of deformation in the viscometer becomes, see Eq. (5.4.2):

$$D_{xy}(t) = \frac{1}{2}\dot{\gamma}(t) = \frac{1}{2}\frac{r_1}{h}\,\Omega(t) = \frac{1}{2}\dot{\gamma}_0 \cos \omega t, \quad \dot{\gamma}_0 = \frac{r_1 \Omega_0}{h} \tag{7.6.5}$$

The torque $M(t)$ is obtained from the Eq. (5.4.6):

$$M(t) = \frac{2\pi r_1^3}{3}\left(1 + \frac{3H}{r_1}\right)\tau(t) \tag{7.6.6}$$

The constitutive equations (7.4.18) for linearly viscoelastic fluids give:

$$\tau(t) = \int_0^\infty \beta(s)\,\dot{\gamma}(t-s)\,ds = \dot{\gamma}_0 \int_0^\infty \beta(s)\,\cos[\omega(t-s)]\,ds \tag{7.6.7}$$

The trigonometric formula: $\cos[\omega(t-s)] = \cos \omega t\, \cos \omega s + \sin \omega t\, \sin \omega s$, is substituted into the integral and the result is:

$$\tau(t) = [\eta'(\omega)\cos\omega t + \eta''(\omega)\sin\omega t]\dot{\gamma}_0 \tag{7.6.8}$$

$$\eta'(\omega) = \int_0^\infty \beta(s)\ \cos\omega s\, ds, \quad \eta''(\omega) = \int_0^\infty \beta(s)\ \sin\omega s\, ds \tag{7.6.9}$$

The two parameters η', called the *dynamic viscosity*, and η'', are *material functions for shear flow oscillations*.

It is customary to introduce the *complex viscosity function*:

$$\eta* = \eta' - i\eta'', \quad i = \sqrt{-1} \tag{7.6.10}$$

The result (7.6.9) can now be presented as the real part of the complex shear stress $\tau * (t)$.

$$\tau * (t) = \eta * (\omega)\ \dot{\gamma} * (t), \quad \dot{\gamma} * (t) = \dot{\gamma}_0 \exp(i\omega t), \quad \exp(i\omega t) = \cos\omega t + i\sin\omega t \tag{7.6.11}$$

We may express the complex viscosity function alternatively as:

$$\eta* = \int_0^\infty \beta(s)\ \exp(-i\omega s)\, ds \tag{7.6.12}$$

The shear strain rate is given by the real part of the complex shear strain rate, and the shear stress is given by the real part of the complex stress:

$$\mathrm{Re}\{\dot{\gamma} * (t)\} = \mathrm{Re}\{\dot{\gamma}_0 \exp(i\omega t)\} = \dot{\gamma}_0 \cos\omega t$$

$$\tau(t) = \mathrm{Re}\{\tau * (t)\} = \mathrm{Re}\{\eta * (\omega)\ \dot{\gamma} * (t)\} = [\eta'(\omega)\cos\omega t + \eta''(\omega)\sin\omega t]\dot{\gamma}_o \tag{7.6.13}$$

For a Maxwell fluid with the relaxation function in shear (7.4.9) we obtain, using an integral table, the results:

$$\eta\prime(\omega) = \frac{\mu}{\lambda}\int_0^\infty \exp\left(-\frac{s}{\lambda}\right)\ \cos\omega s\, ds = \frac{\mu}{\lambda}\left[\frac{\frac{1}{\lambda}}{\frac{1}{\lambda^2} + \omega^2}\right] \quad \Rightarrow$$

$$\eta\prime(\omega) = \frac{\mu}{1 + (\lambda\omega)^2} \tag{7.6.14}$$

$$\eta\prime\prime(\omega) = \frac{\mu}{\lambda}\int_0^\infty \exp\left(-\frac{s}{\lambda}\right)\ \sin\omega s\, ds = \frac{\mu}{\lambda}\left[\frac{\omega}{\frac{1}{\lambda^2} + \omega^2}\right] \quad \Rightarrow$$

$$\eta''(\omega) = \frac{\mu\lambda\omega}{1 + (\lambda\omega)^2} \tag{7.6.15}$$

7.7 Plane Shear Waves

A linearly viscoelastic fluid fills the semi-infinite space: $\infty < x < \infty$, $0 \leq y < \infty$, Fig. 7.9. At $y = 0$ the fluid sticks to a plane, horizontal plate. The plate oscillates in the $x-$direction with the velocity $v_0 \cos \omega t$, and a velocity wave is propagated from the plate and into the fluid. In order to simplify the solution procedure we shall use complex quantities. Assuming that the complex velocity field in the fluid is:

$$v_x(y,t) = v(y)\exp(i\omega t), \quad v_y = v_z = 0 \tag{7.7.1}$$

we shall determine the velocity in the $x-$direction as the real part of the complex velocity $v_x(y,t)$.

The boundary conditions are:

$$\text{Re}\{v_x(0,t\} = v_0 \cos \omega t, \quad \text{Re}\{v_x(\infty,t)\} = \text{finite} \tag{7.7.2}$$

The non-zero rate of deformation in the flow is:

$$D_{xy}(y,t) = \frac{1}{2}\frac{\partial v_x}{\partial y} = \frac{1}{2}\frac{dv}{dy}\exp(i\omega t) \tag{7.7.3}$$

The constitutive equations (7.4.18) yield for the non-zero complex extra stress:

$$\tau_{xy}(y,t) = 2\int_0^\infty \beta(s) D_{xy}(t-s)\,ds = \frac{dv}{dy}\exp(i\omega t)\int_0^\infty \beta(s)\exp(-i\omega s)\,ds \tag{7.7.4}$$

By the Eqs. (7.7.4) and (7.6.12) the stress is expressed as:

$$\tau_{xy}(y,t) = \eta * \ \frac{dv}{dy}\ \exp(i\omega t) \tag{7.7.5}$$

The Cauchy equations of motion (3.3.26) are in this case reduced to:

$$\rho\frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}, \quad 0 = -\frac{\partial p}{\partial y} - \rho g, \quad 0 = -\frac{\partial p}{\partial z} \tag{7.7.6}$$

These three equations, the conditions $v_x = v_x(y,t)$ and $\tau_{xy} = \tau_{xy}(y,t)$, and the condition that the pressure p must be finite when $x \to \infty$, imply the result:

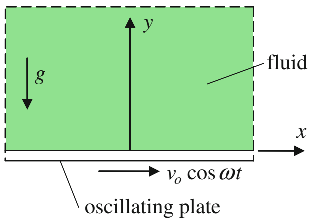


Fig. 7.9 Fluid flow in a semi-infinite space over an oscillating plate

$$p(y,t) = -\rho g y + C(t) \tag{7.7.7}$$

$C(t)$ is an unknown function of time. The equation of motion in the $x-$ direction, i.e. the first of the equation (7.7.6) is now reduced to:

$$\rho \frac{\partial v_x}{\partial t} = \frac{\partial \tau_{xy}}{\partial y} \tag{7.7.8}$$

By the Eqs. (7.7.1) and (7.7.5) the above equation is transformed to an ordinary differential equation:

$$\frac{d^2 v}{dy^2} - \frac{i\omega\rho}{\eta *} v = 0 \tag{7.7.9}$$

The general solution of this equation is:

$$v(y) = C_1 \exp\left(\sqrt{\frac{i\omega\rho}{\eta *}}\, y\right) + C_2 \exp\left(-\sqrt{\frac{i\omega\rho}{\eta *}}\, y\right) \tag{7.7.10}$$

C_1 and C_2 are constants of integration. We define a complex parameter with two positive real constants ϕ_1 and ϕ_2:

$$\sqrt{\frac{i\omega\rho}{\eta *}} = \phi_1 + i\,\phi_2 \tag{7.7.11}$$

Then the velocity $v_x(y,t)$ becomes:

$$\begin{aligned} v_x(y,t) &= v(y) \exp(i\omega t) \\ &= [C_1 \exp(\phi_1 y) \exp(i\,\phi_2 y) + C_2 \exp(-\phi_1 y) \exp(-i\phi_2 y)] \exp(i\omega t) \end{aligned} \tag{7.7.12}$$

With the boundary conditions (7.7.2) the following real solution is obtained for the fluid velocity in the $x-$ direction:

$$v_x(y,t) = v_o \exp(-\phi_1 y) \cos(\omega t - \phi_2 y) \tag{7.7.13}$$

This velocity field represents a wave with exponentially decreasing amplitude and, as will be demonstrated, with the *wave velocity*:

$$c = \frac{\omega}{\phi_2} \tag{7.7.14}$$

The wave propagates from the oscillating plate into the fluid in the $y-$ direction. Fig. 7.10 shows the graph of the wave at two different times: t and $t + \Delta t$. The zero point P moves in the $y-$ direction during the time interval Δt a distance Δy obtained from the expression:

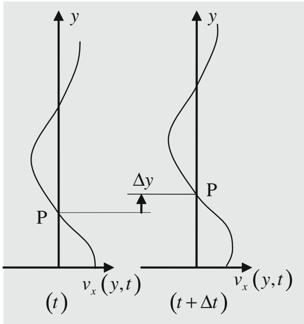


Fig. 7.10 Plane shear wave

$$\cos[\omega(t+\Delta t)-\phi_2(y+\Delta y)]=\cos(\omega t-\phi_2 y)\quad\Rightarrow\quad\omega\,\Delta t-\phi_2\,\Delta y=0\quad\Rightarrow$$
$$\Delta y=\frac{\omega}{\phi_2}\Delta t=c\,\Delta t \tag{7.7.15}$$

The result implies that the velocity propagates into the fluid with the wave velocity given by equation (7.7.14).

For a Newtonian fluid with viscosity μ we find, using the formula: $\sqrt{i}=(1+i)/\sqrt{2}$:

$$\eta*=\eta'=\mu,\quad\eta''=0,\quad\phi_1=\phi_2=\sqrt{\frac{\rho\,\omega}{2\mu}} \tag{7.7.16}$$

The velocity field for the shear wave in the Newtonian fluid becomes:

$$v_x(y,t)=v_o\exp\left(-\sqrt{\frac{\rho\,\omega}{2\mu}}y\right)\cos\left(\omega t-\sqrt{\frac{\rho\,\omega}{2\mu}}y\right) \tag{7.7.17}$$

The wave velocity is expressed by:

$$c=\sqrt{\frac{2\mu\omega}{\rho}} \tag{7.7.18}$$

Chapter 8
Advanced Fluid Models

8.1 Introduction

Chapter 6 has presented the simplest and most commonly used models of non-Newtonian fluids, the *generalized Newtonian fluids*. These models are well suited for steady shear flow and in particular steady viscometric flows, and are also used for unsteady flows of purely viscous fluids. However, a main objection to these models is that they do not reflect normal stress differences in shear flows.

The linearly viscoelastic fluid models in Chap. 7 are only applicable in flows with small deformations, i.e. small strains and small rotations. Also the strain rates have to be small for these models to apply. The fundamental reasons for these restrictions can be found in the books [2, 3, 10 and 15] or in books on continuum mechanics, but is beyond the scope of the present book to present these objections in any detail.

The aim of the present chapter is to present some of the mostly used advanced fluid models of non-Newtonian fluids. The models in Chaps. 6 and 7 will appear as special cases of the more advanced models. However, it is not possible to give a detailed presentation and motivation of the individual models. The references [1, 3, 5, 10, 11 and 15] give a more comprehensive discussion of the models presented in this chapter, and of other useful but even more complex models.

The book "Engineering Rheology" by R. I. Tanner [15] provides interesting evaluations of the models presented in this chapter and other models not included here. Tanner also discusses the models in relation to special flow situations.

Before the advanced models are introduced, we shall review some of the basic concepts used in continuum mechanics related to fluids, and even introduce a few new concepts needed for the presentation of the advanced models.

Figure 8.1 presents a fluid body at three different times: K_0 is a reference configuration of the body at the reference time t_0. K is the present configuration of the body at the present time t, and $\bar{K}$ is a configuration at the current time $\bar{t}$: $-\infty < \bar{t} \leq t$. The motion of the fluid is represented by the functions $x_i(X, t)$, and the particle velocity is:

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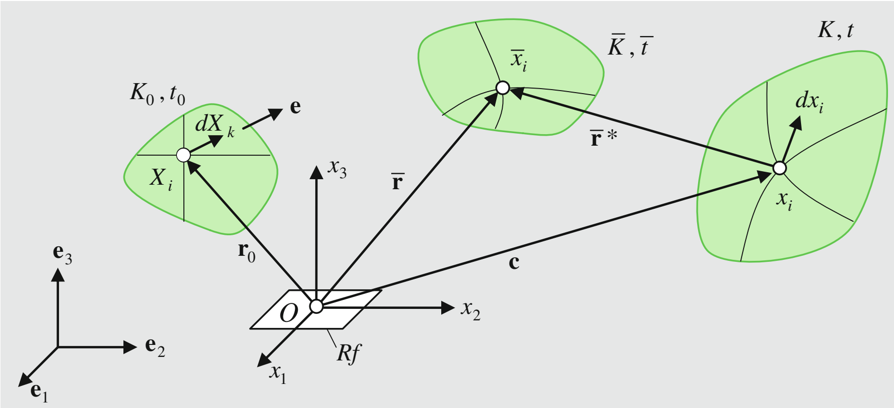

Fig. 8.1 Reference Rf, coordinate system Ox, and configurations $K_0, \bar{K},$ *and* K, and base vectors $\mathbf{e}_i$

$$v_i = \frac{\partial x_i}{\partial t} \tag{8.1.1}$$

We now introduce a new concept: the *deformation gradients*:

$$F_{ik} = \frac{\partial x_i(X,t)}{\partial X_k} \tag{8.1.2}$$

The inverse F^{-1} of the *deformation gradient matrix* F has the elements:

$$F_{kj}^{-1} = \frac{\partial X_k(x,t)}{\partial x_j} \tag{8.1.3}$$

such that:

$$FF^{-1} = 1 \quad \Leftrightarrow \quad F_{ik}F_{kj}^{-1} = \frac{\partial x_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \delta_{ij} \tag{8.1.4}$$

The matrix F^{-1} is called the *inverse matrix* of the matrix F.

The velocity gradients are defined in the Eq. (4.1.2) by:

$$L_{ik} = \frac{\partial v_i(x,t)}{\partial x_k} \equiv v_{i,k} \tag{8.1.5}$$

The relationship between the matrices F and L is found as follows:

$$L_{ik} = \frac{\partial v_i}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial x_i}{\partial t}\right) = \frac{\partial}{\partial X_j}\left(\frac{\partial x_i}{\partial t}\right)\frac{\partial X_j}{\partial x_k} = \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial X_j}\right)\frac{\partial X_j}{\partial x_k} = \frac{\partial}{\partial t}(F_{ij})F_{jk}^{-1} \quad \Leftrightarrow$$
$$L_{ik} = \dot{F}_{ij}F_{jk}^{-1} \quad \Leftrightarrow \quad L = \dot{F}F^{-1} \quad \Leftrightarrow \quad \dot{F} = LF \tag{8.1.6}$$

The rate of deformation matrix, D defined in Eq. (4.1.13) and the rate of rotation matrix W defined in Eq. (4.1.14) are given by:

$$D = \frac{1}{2}\left(L + L^T\right), \quad W = \frac{1}{2}\left(L - L^T\right) \tag{8.1.7}$$

Although the concept of strain is not of primary interest in fluid flow, we need a few concepts from the strain analysis to get a satisfactory understanding of the constitutive equations of some of the more advanced fluid models. By strain we mean local deformation of a material. There are three primary concepts of strain *longitudinal strain* expressing the change of the length of material line elements, *shear strain* (or angular strain) expressing the change of the angle between material line elements, and *volumetric strain* representing the change in the volume of the material. We shall include here a very brief introduction to longitudinal strain. A differential material line element in the direction of the unit vector $\mathbf{e} = e_k \mathbf{e}_k$ at the particle X, see Figure 8.1, is in the reference configuration K_0 expressed by the vector dX_k of length ds_0. The same material line element is in the present configuration K expressed by the vector dx_i of length ds. We find:

$$e_k = \frac{dX_k}{ds_0}, \quad dx_i = \frac{\partial x_i}{\partial X_k} dX_k = F_{ik} \frac{dX_k}{ds_0} ds_0 = F_{ik} e_k ds_0 \tag{8.1.8}$$

$$ds_0^2 = dX_k dX_k, \quad ds^2 = dx_i dx_i = (F_{ik} e_k ds_0)(F_{il} e_l ds_0) = e_k F_{ik} F_{il} e_l ds_0^2 \tag{8.1.9}$$

It follows that:

$$\left(\frac{ds}{ds_0}\right)^2 = e_k F_{ik} F_{il} e_l \tag{8.1.10}$$

We now introduce the *strain matrix*:

$$E_{kl} = \frac{1}{2}(F_{ik} F_{il} - \delta_{kl}) \quad \Leftrightarrow \quad E = \frac{1}{2}\left(F^T F - 1\right) \tag{8.1.11}$$

Then:

$$\frac{ds^2 - ds_o^2}{ds_o^2} = e_k F_{ik} F_{il} e_l - 1 = 2\, e_k E_{kl} e_l \tag{8.1.12}$$

The *longitudinal strain* in the direction $\mathbf{e}$ is defined as the change in length of a material line element per unit length, and is mathematically expressed by:

$$\varepsilon = \frac{ds - ds}{ds_0} = \frac{ds}{ds_0} - 1 = \sqrt{1 + 2 e_k E_{kl} e_l} - 1 \tag{8.1.13}$$

The two other primary strain measures, the shear strain and the volumetric strain, can also be expressed by the strain matrix. However, these two measures are not needed in the following exposition.

The time rate of change of the strain matrix E is the *rate of strain matrix*:

$$\dot{E} = \frac{1}{2}\left(\dot{F}^T F + F^T \dot{F}\right) = \frac{1}{2}\left(F^T L^T F + F^T L F\right) \quad \Rightarrow \quad \dot{E} = F^T D F \tag{8.1.14}$$

where a result from the Eq. (8.1.6) has been used, followed by use of the expression in Eq. (8.1.7) for the rate of deformation matrix D. The literature uses both the name rate of strain matrix and the name rate of deformation matrix for D, probably because for small strains the two quantities: $\dot{E}$ in Eq. (8.1.14) and D coincide.

It has now been demonstrated that the matrices for strains, rates of strain, rates of deformation, and the rates of rotation all may be derived from the deformation gradient matrix F. The stresses in a material may depend on the complete history of the deformation the material has been subjected to. We therefore introduce the concept of *deformation history* F^t of a material:

$$F^t \equiv F^t(X, s) = F(X, \bar{t}) = F(X, t - s), \quad 0 \le s < \infty, \quad -\infty \le \bar{t} < t \tag{8.1.15}$$

The parameter s, called the "past time", was introduced in Eq. (7.4.17). We also need to express the *temperature history*: $\Theta^t(X, s)$.

In continuum mechanics a material model is called a *simple thermomechanical material* if it can be defined by constitutive equations of the type:

$$T_{ik} \equiv \sigma_{ik} = \underset{s=0}{\overset{s=\infty}{P_{ik}}}\left[F^t(X, s); \Theta^t(X, s); X, t\right] \tag{8.1.16}$$

The matrix P is a *functional*, i.e. a general operator, of the deformation history and the temperature history, and a function of the particle identification coordinates X_i and of time t. Two examples of fluid models that classify as simple materials according to the constitutive equations (8.1.16), have already been presented:

Generalized Newtonian fluids in Sect. 6.1 and defined by:

$$\begin{aligned} \sigma_{ik} &= -p\delta_{ik} + 2\eta(\dot{\gamma}) D_{ik} \\ D &= \frac{1}{2}\left(\dot{F}F^{-1} + F^{-T}\dot{F}^T\right), \quad D_{ik} = \frac{1}{2}\left(\dot{F}_{ij}F_{jk}^{-1} + F_{ij}^{-T}\dot{F}_{kj}\right) \\ \dot{\gamma} &= \sqrt{2D_{ik}D_{ik}} \equiv \sqrt{2\mathrm{tr}D^2} \end{aligned} \tag{8.1.17}$$

Linearly viscoelastic fluids in Sect. 7.4 and defined by Eq. (7.4.18):

$$\sigma_{ik} = -p\delta_{ik} + \int_0^\infty \beta(s) D_{ik}(t - s)\, ds \tag{8.1.18}$$

We shall now present a fundamental principle from continuum mechanics that all constitutive thermomechanical models are supposed to obey. First we shall agree that it seems reasonable to assume that the material properties of a continuum are not influenced by a rigid body motion of the material. For instance, it is a priori understood that the stiffness of a spring is the same regardless of the position and orientation of the spring, or in what kind of motion the spring is. A translation and rotation of a material may always be eliminated by changing the reference *Rf* to which the motion of the material is related. Thus, the material properties are expected to be invariant with respect to two conditions: (a) any change of the position and orientation of the material in space, and (b) any change of reference *Rf*. The condition (a) implies that the space is homogeneous and isotropic with respect to material properties, while the condition (b) implies that the material properties are reference invariant. For all practical purposes the two conditions are equivalent. The conditions may be formulated in:

> The *principle of material objectivity* (PMO): The constitutive equations of a material model must be reference invariant in the following sense: The stress vector $\mathbf{t}$ on a surface at a particle is independent of the reference *Rf* to which the constitutive equations are related, i.e. the stress vector $\mathbf{t}$ in the reference *Rf* is equal to the stress vector $\mathbf{t}^*$ related to the reference Rf^*. In other words: Constitutive equations that in one reference *Rf* give the stresses σ_{ik} at the time t, shall in any other reference Rf^* give the stresses $\sigma_{ik}^* = \sigma_{ik}$ if the coordinate systems used in the two references coincide at the time t.

The principle implies that the response of a fluid does not change if a rigid-body motion is added to the deformation field the fluid is subjected to. The rigid-body motion may be eliminated by changing reference from *Rf* to a reference that moves relative to *Rf* in accordance with the rigid-body motion.

8.2 Tensors and Objective Tensors

To really appreciate the implications of PMO we need to briefly introduce the concept of tensors. Figure 8.2 shows a fluid body in a current reference configuration $\bar{K}$ at the time $\bar{t} \leq t$, a reference *Rf* with a coordinate system Ox, and a reference Rf^* with a coordinate system O^*x^*. The two references are moving with respect to each other. To describe this motion we need the relations between the *base vectors* in the two coordinate systems:

$$\mathbf{e}_i^* = Q_{ik}\mathbf{e}_k \quad , \quad \mathbf{e}_k = Q_{ik}\mathbf{e}_i^* \tag{8.2.1}$$

where Q_{ik} is the cosine of the angle between the base vector $\mathbf{e}_i^*$ and the base vector $\mathbf{e}_k$:

$$Q_{ik} = Q_{ik}(\bar{t}) = \cos\left(\mathbf{e}_i^*, \mathbf{e}_k\right) \tag{8.2.2}$$

The elements Q_{ik} are called *direction cosines* for the base vectors. The matrix $Q = (Q_{ik})$ is called the *transformation matrix* for the coordinate transformation from Ox to O^*x^*. The transformation matrix is an *orthogonal matrix* in the sense that:

$$Q^T Q = 1 \quad \Leftrightarrow \quad Q_{ik}Q_{il} = \delta_{kl} \quad \Leftrightarrow \quad Q^T = Q^{-1} \tag{8.2.3}$$

That is: the *transpose* Q^T of the matrix Q is equal to the *inverse* Q^{-1} of Q:

$$Q^T_{ik} = Q_{ki}, \quad Q^{-1}Q = 1 \quad \Leftrightarrow \quad Q^{-1}_{ij}\, Q_{jk} = \delta_{ik} \tag{8.2.4}$$

To see that in fact $Q^T = Q^{-1}$, we write:

$$\mathbf{e}_k = Q_{ik}\mathbf{e}^*_i = Q_{ik}Q_{ij}\mathbf{e}_j \quad \Rightarrow \quad Q_{ik}Q_{ij} = \delta_{kj} \quad \Leftrightarrow \quad Q^T Q = 1 \tag{8.2.5}$$

A vector $\mathbf{a}$ is defined by its component set a_k in Ox, or by the components a^*_i in O^*x^*.

$$\mathbf{a} = a_k\mathbf{e}_k = a^*_i\mathbf{e}^*_i \tag{8.2.6}$$

The relations between the two sets of components are found by writing:

$$\begin{aligned} &\mathbf{a} = a^*_i\mathbf{e}^*_i = a^*_i Q_{ik}\mathbf{e}_k = a_k\mathbf{e}_k = a_k Q_{ik}\mathbf{e}^*_i \Rightarrow \\ &a^*_i = Q_{ik}a_k \quad \Leftrightarrow \quad a^* = Qa, \quad a_k = Q_{ik}a^*_i \quad \Leftrightarrow \quad a = Q^T a^* \end{aligned} \tag{8.2.7}$$

The symbols a^* and a are vector matrices of the components of the vector $\mathbf{a}$.

The motion of a fluid particle with respect to the references Rf and $Rf^*$is given by the *place vectors*:

$$\bar{\mathbf{r}} = x_k(X,\bar{t})\mathbf{e}_k, \quad \bar{\mathbf{r}}^* = x^*_k(X^*,\bar{t})\mathbf{e}^*_k \tag{8.2.8}$$

The coordinate vector matrices X and X^* relate to the same particle. From Fig. 8.2 we get the vector relationship:

$$\bar{\mathbf{r}}^* = \mathbf{c} + \bar{\mathbf{r}} \quad \Leftrightarrow \quad x^*_i\mathbf{e}^*_i = c^*_i\mathbf{e}^*_i + x_k\mathbf{e}_k = \left(c^*_i + x_k Q_{ik}\right)\mathbf{e}^*_i \tag{8.2.9}$$

from which we obtain the coordinate relations:

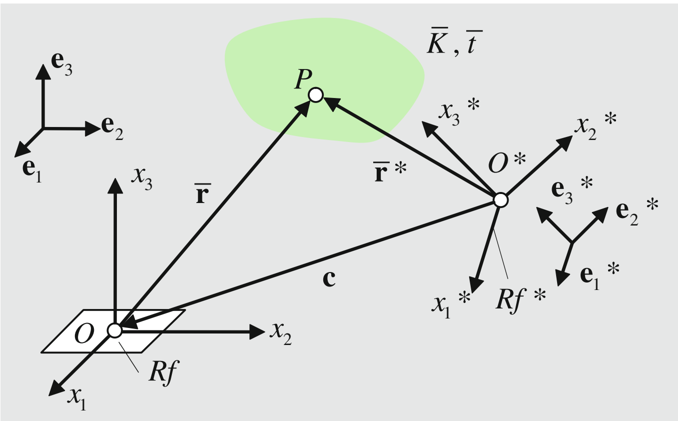

Fig. 8.2 The current configuration $\bar{K}$ of a fluid body relative to two references Rf and Rf^*

$$x_i^* = c_i^* + Q_{ik} x_k \quad \Leftrightarrow \quad x^* = c^* + Qx \quad \Leftrightarrow$$
$$x_k = -c_k + Q_{ik} x_i^* \quad \Leftrightarrow \quad x = -c + Q^T x^* \tag{8.2.10}$$

Let us first consider a situation in which the two references do not move relative to each other, i.e. the vector c and the transformation matrix Q are constants. The Cauchy stress theorem relates the components t_k in the Ox-system of the stress vector **t** on a surface at a particle to the coordinate stresses σ_{kl} and the components n_l of the unit normal vector **n** on the surface. In the $O^* x^* -$ system this is a relation between the components t_i^* of the stress vector, the coordinate stresses σ_{ij}^*, and the components n_j^* of the unit normal vector. Thus we have the relations:

$$t_k = \sigma_{kl} n_l \quad \Leftrightarrow \quad t = Tn, \quad t_i^* = \sigma_{ij}^* n_j^* \quad \Leftrightarrow \quad t^* = T^* n^* \tag{8.2.11}$$

The two sets of coordinate stresses are related, and the relationship is found as follows, by use of the components relations (8.2.7):

$$t_i^* = \sigma_{ij}^* n_j^* = Q_{ik} t_k = Q_{ik} \sigma_{kl} n_l = Q_{ik} \sigma_{kl} Q_{jl} n_j^* \quad \Rightarrow \quad \sigma_{ij}^* n_j^* = Q_{ik} \sigma_{kl} Q_{jl} n_j^*$$

Since the final result is valid for any choice of surface at the particle, i.e. any choice of the vector **n**, it follows that:

$$\sigma_{ij}^* = Q_{ik} \sigma_{kl} Q_{jl} \quad \Leftrightarrow \quad T^* = Q\,T\,Q^T \tag{8.2.12}$$

Equation (8.2.12), relating the stress matrices in two coordinate systems and which represent the same state of stress, is called the transformation rule for the components σ_{kl} and σ_{ij}^* of a second order tensor, namely the *stress tensor* **T**. We say that the stress tensor **T** is a *coordinate invariant quantity*, which in any coordinate system is represented by the stress matrix in that coordinate system, i.e. T in Ox and T^* in $O^* x^*$. The Cauchy stress theorem (3.3.8) may now be expressed in a coordinate invariant form as:

$$\mathbf{t} = \mathbf{T}\,\mathbf{n} \quad \Leftrightarrow \quad t = Tn, \quad t^* = T^* n^* \quad \text{Cauchy's stress theorem} \tag{8.2.13}$$

We will find that the velocity gradient matrix L, the rate of deformation matrix D, and the rate of rotation matrix W related to the coordinate system Ox represent tensors, i.e. the *velocity gradient tensor* **L**, the *rate of deformation tensor* **D** and the *rate of rotation tensor* **W** respectively, in the sense that in any other Cartesian coordinate system $O^* x^*$ the rates of deformation and the rates of rotation are given by the matrices L^*, D^*, and W^* such that:

$$L_{ij}^* = Q_{ik} L_{kl} Q_{jl} \quad \Leftrightarrow \quad L^* = QLQ^T$$
$$D_{ij}^* = Q_{ik} D_{kl} Q_{jl} \quad \Leftrightarrow \quad D^* = QDQ^T \tag{8.2.14}$$
$$W_{ij}^* = Q_{ik} W_{kl} Q_{jl} \quad \Leftrightarrow \quad W^* = QWQ^T$$

The Eqs. (8.2.12) and (8.2.14) show the general relationship between components and matrices of a second order tensor in two Cartesian coordinate system. Because the matrix D is symmetric, **D** is called a *symmetric tensor*, while because W is an antisymmetric matrix, W is an *antisymmetric tensor*.

For later reference we need to define the principal invariants of a symmetric second order tensor, e.g. **D**. First we show that the trace of the matrices D and D^* are equal. Then it follows that the trace of the matrices D^2 and D^{*^2} are equal and that the determinants of D and D^* are equal. From the Eq. (8.2.14) we obtain:

$$\begin{aligned} \operatorname{tr} D^* &\equiv D_{ii} = Q_{ik} D_{kl} Q_{il} = D_{kl}\delta_{kl} = D_{kk} \equiv \operatorname{tr} D \\ \operatorname{tr} D^{*^2} &\equiv D^*_{ik} D^*_{ki} = D^*_{ik} D^*_{ik} = (\operatorname{tr} D^*)^2 = (\operatorname{tr} D)^2 = \operatorname{tr} D^2 \\ \det D^* &= \det\left(QDQ^T\right) = \det Q \det D \det Q^T = \det D \end{aligned}$$

The three quantities $\operatorname{tr} D,\ \operatorname{tr} D^2,$ and $\det D$ are thus independent of the coordinate system Ox, i.e. they are coordinate invariants, and they are denoted:$\operatorname{tr}\mathbf{D}$, $\operatorname{tr}\mathbf{D}^2$, and $\det\mathbf{D}$. The three *principal invariants* of a symmetric second order tensor **D** are defined as:

$$I = \operatorname{tr}\mathbf{D}, \quad II = \frac{1}{2}\left((\operatorname{tr}\mathbf{D})^2 - \operatorname{tr}\mathbf{D}^2\right), \quad III = \det\mathbf{D} \tag{8.2.15}$$

Now we shall consider the situation when the reference Rf^* moves with respect to Rf. With reference to Fig. 8.2 and Eq. (8.2.2), the motion of Rf^* with respect to Rf is, given by the vector $\mathbf{c}(\bar{t})$ and the transformation matrix $Q(\bar{t})$. It is convenient to choose the coordinate systems Ox and O^*x^* such that they coincides at the present time t.

$$x_i^*(X^*, t) = x_i(X, t), \quad \mathbf{e}_i^*(t) = \mathbf{e}_i \quad \Leftrightarrow \quad Q_{ik}(t) = \delta_{ik}, \quad \mathbf{c}(t) = \mathbf{0} \tag{8.2.16}$$

According to the Eqs. (8.2.12) and (8.2.16) the two stress matrices representing the stress tensor **T** coincide at the present time t:

$$\sigma_{ij}^* = \sigma_{ij} \quad \Leftrightarrow \quad T^* = T \text{ at time } \bar{t} = t \tag{8.2.17}$$

According to the PMO the stress vector $\mathbf{t}^*$ observed in Rf^* is equal to the stress vector **t** observed in Rf, i.e. the stress vector is independent of the reference. It then follows from Cauchy stress theorem (8.2.13) that the stress tensor is also independent of the reference:

$$\mathbf{T}^* = \mathbf{T} \tag{8.2.18}$$

Because the stress vector and the stress tensor are not influenced by a change of reference they are called an *objective vector* and an *objective tensor* respectively. Any quantity that is not influenced by a change of reference is called an *objective quantity* or a *reference invariant quantity*. Thus the stress vector and the stress tensor are objective quantities.

The material derivative of the stress matrix is called the *stress rate matrix*. We shall now see that the stress rate matrices with respect to two references moving with respect to each other do not represent an objective quantity. First we shall show the time derivative of the transformation matrix Q is antisymmetric at the time $\bar{t} = t$:

$$\dot{Q}^T = -\dot{Q} \quad \Leftrightarrow \quad \dot{Q}_{ki} = -\dot{Q}_{ik} \quad \text{at} \quad \bar{t} = t \tag{8.2.19}$$

The matrix equation $Q^T Q = 1$ is differentiated with respect to the time $\bar{t}$ and evaluated at the present time t:

$$\frac{d}{d\bar{t}}\left(Q^T Q\right) = \dot{Q}^T Q + Q^T \dot{Q} = 0 \quad \Rightarrow \quad \dot{Q}^T + \dot{Q} = 0 \text{ at } \bar{t} = t \quad \Rightarrow$$
$$\dot{Q}^T = -\dot{Q} \quad \Rightarrow \quad \dot{Q}_{ki} = -\dot{Q}_{ik} \quad \text{at } \bar{t} = t \quad \text{QED}$$

Then we compute the material derivative of Eq. (8.2.12):

$$\dot{\sigma}^*_{ij} = \dot{Q}_{ik}\sigma_{kl}Q_{jl} + Q_{ik}\dot{\sigma}_{kl}Q_{jl} + Q_{ik}\sigma_{kl}\dot{Q}_{jl} \tag{8.2.20}$$

If the two coordinate systems were both fixed in Rf, the transformation matrix would have been time independent, and Eq. (8.2.20) gives the following relationship between the stress rate matrices:

$$\text{Constant } Q - \text{matrix} \ \Rightarrow \ \dot{\sigma}^*_{ij} = Q_{ik}\dot{\sigma}_{kl}Q_{jl} \quad \text{at} \quad \bar{t} < t, \quad \dot{\sigma}^*_{ij} = \dot{\sigma}_{kl} \quad \text{at} \quad \bar{t} = t \tag{8.2.21}$$

But that is not the case here since the $O^*x^* -$ system is moving with the reference Rf^*. At the present time t we obtain the following result, when the property (8.2.19) is applied:

$$\dot{\sigma}^*_{ij} = \dot{\sigma}_{ij} + \dot{Q}_{ik}\sigma_{kj} - \sigma_{ik}\dot{Q}_{kj} \quad \Leftrightarrow \quad \dot{T}^* = \dot{T} + \dot{Q}T - T\dot{Q} \text{ at } \bar{t} = t \tag{8.2.22}$$

From the relations (8.2.21) and (8.2.22) we may conclude: (1) The rate of stress matrices with respect to coordinate systems fixed in a reference Rf represent a coordinate invariant quantity, i.e. a second order tensor. (2) The rate of stress matrices with respect to coordinate systems fixed in two references moving with respect to each other do not represent a reference invariant or objective quantity, and do not therefore represent the same tensor in the two references. We say that the rate of stress matrices do not represent an objective tensor.

It is convenient to introduce the *rate of rotation tensor* **S** for the reference Rf relative to the reference Rf^* represented by the following matrices and components at the present time t:

$$\begin{aligned} &S_{kl} = \dot{Q}_{kl} \quad \Leftrightarrow \quad S = \dot{Q} \quad \text{in } Ox \text{ at } \bar{t} = t \\ &S^*_{ij} = Q_{ik}S_{kl}Q_{jl} = S_{ij} \quad \Leftrightarrow \quad S^* = S \quad \text{in } O^*x^* \text{ at } \bar{t} = t \end{aligned} \tag{8.2.23}$$

We shall now investigate the relations between the rate of deformation matrices and the rate of rotation matrices referred to the two references Rf and Rf^* moving relative to each other. First we calculate the velocities and velocity gradients in the two coordinate systems Ox and O^*x^*. We use a bar over the velocities and the velocity gradients to indicate the operations are performed at the current time $\bar{t}$.

$$\bar{v}_k = \dot{\bar{x}}_k, \bar{L}_{kl} = \frac{\partial \bar{v}_k}{\partial \bar{x}_l} \equiv \bar{v}_{kl}, \text{ at } \bar{\mathrm{t}} \leq \mathrm{t} \tag{8.2.24}$$

$$\begin{aligned} \bar{v}_i^* &= \dot{\bar{x}}_i^* = \dot{c}_i^* + \dot{Q}_{ik}\bar{x}_k + Q_{ik}\dot{\bar{x}}_k \text{ at } \bar{\mathrm{t}} \leq \mathrm{t} \\ \bar{L}_{ij}^* &= \frac{\partial \bar{v}_i^*}{\partial x_j^*} \equiv \dot{Q}_{ik}\frac{\partial \bar{\mathrm{x}}_k}{\partial x_j^*} + Q_{ik}\frac{\partial \bar{v}_i^*}{\partial x_j^*} \text{ at } \bar{\mathrm{t}} \leq \mathrm{t} \end{aligned} \tag{8.2.25}$$

At the present time t: $Q_{ik} = \delta_{ik}$ and $\dot{Q}_{ik} = S_{ik}$, and the results (8.2.25) give:

$$\begin{aligned} v_i^* &= \dot{x}_i^* = \dot{c}_i^* + S_{ik}x_k + \dot{x}_i \quad \Leftrightarrow \quad v^* = \dot{x}^* = \dot{c}^* + Sx + \dot{x} \\ L_{ij}^* &= S_{ij} + L_{ij} \quad \Leftrightarrow \quad L^* = S + L \end{aligned} \tag{8.2.26}$$

The rate of deformation matrices and the rate of rotation matrices at the present time t with respect to the two references become:

$$D_{ij} = \frac{1}{2}\left(L_{ij} + L_{ji}\right) \quad , \quad D_{ij}^* = \frac{1}{2}\left(L_{ij}^* + L_{ji}^*\right) = D_{ij} \tag{8.2.27}$$

$$W_{ij} = \frac{1}{2}\left(L_{ij} - L_{ji}\right), \quad W_{ij}^* = \frac{1}{2}\left(L_{ij}^* - L_{ji}^*\right) = W_{ij} + S_{ij} \tag{8.2.28}$$

Using tensor symbols we write:

$$\mathbf{L}^* = \mathbf{L} + \mathbf{S}, \quad \mathbf{D}^* = \mathbf{D}, \quad \mathbf{W}^* = \mathbf{W} + \mathbf{S} \tag{8.2.29}$$

From these results we conclude that while the rate of deformation matrices represent an objective: tensor, which is the *rate of deformation tensor* **D**, the velocity gradient matrices and the rate of rotation matrices represent *reference related tensors*, which are the *velocity gradient tensors* **L** and $\mathbf{L}^*$ and the *rate of rotation tensors* W and W^*. Because **D** is an objective: tensor the principal invariants *I*, *II*, and *III* of D, defined by the formulas (8.2.15), are also objective quantities, and they are called *objective scalars*.

To insure that constitutive equations of fluids do satisfy the objectivity principle, we may follow either of two general procedures:

1. The constitutive equations at a fluid particle are formulated in a reference Rf^{r} that rotates with the particle, such that the rate of rotation matrix W^{r} vanishes. Such a reference is called a *corotational reference* and is introduced in Sect. 8.4. The constitutive equations are then transformed to the reference *Rf* originally chosen to describe the flow. The implication of this procedure will be discussed in Sects. 8.4, 8.5, and 8.6.
2. The constitutive equations at a fluid particle are formulated in a coordinate system imbedded in the fluid. The system moves and deforms with the fluid and the coordinate system is curvilinear. Such coordinates are called *convected coordinates* or *codeforming coordinates*. The constitutive equations are then

transformed to the coordinate system fixed in the reference Rf and chosen to describe the flow. In the present context we are not in the position to use general curvilinear coordinates. However, implications of this procedure will be discussed in Sect. 8.7.

For curvilinear coordinate system and tensor analysis see Irgens: "Continuum Mechanics" [7].

8.3 Reiner-Rivlin Fluids

George Gabriel Stokes formulated four criteria for the relations between stresses and the velocity field in a viscous fluid:

1. The stresses are continuous functions of the rates of deformation D_{ik}.
2. The fluid is homogeneous such that the stresses are explicitly independent of the particle coordinates X_i.
3. When the rates of deformation are zero, i.e. $D_{ik} = 0$, the stresses are given by the isotropic thermodynamic pressure $p = p(\rho, \Theta)$.
4. Viscosity is an isotropic property, or in other words, the fluid is isotropic.

It may be shown that the first three criteria imply the following form of the constitutive equations of viscous fluids, called in general *Stokesian fluids*.

$$\begin{aligned} \sigma_{ik} &= -p(\rho,\Theta)\delta_{ik} + \tau_{ik}(D,\rho,\Theta), \quad \tau_{ik}(0,\rho,\Theta) = 0 \quad \Leftrightarrow \\ \mathbf{T} &= -p(\rho,\Theta)\mathbf{1} + \mathbf{T}'(\mathbf{D},\rho,\Theta), \quad \mathbf{T}'(\mathbf{0},\rho,\Theta) = \mathbf{0} \end{aligned} \tag{8.3.1}$$

Since the constitutive equations (8.3.1) involve only objective quantities, the principle of material objectivity is satisfied. Furthermore, it may be shown that the principle of material objectivity (PMO) implies that the fluid defined by Eq. (8.3.1) is isotropic. Thus the fourth criterion of Stokes is implied by the first three criteria. The fact that the constitutive equations (8.3.1) represent an isotropic material is formulated mathematically by stating that the stress tensor is an *isotropic function* of the rate of deformation tensor, see Irgens [7]. It then follows from a mathematical theorem that the extra stresses in general can be expressed as:

$$\tau_{ik} = \alpha\delta_{ik} + 2\eta D_{ik} + 4\psi_2 D_{ij}D_{jk} \quad \Leftrightarrow \quad \mathbf{T}' = \alpha\mathbf{1} + 2\eta\mathbf{D} + 4\psi_2\mathbf{D}^2 \tag{8.3.2}$$

α, η, and ψ_2 are scalar-valued functions of the three principal invariants I, II, and III of the tensor $\mathbf{D}$.

For incompressible fluids:

$$I = \text{tr}\mathbf{D} = \text{tr}D = D_{kk} = \dot{\varepsilon}_v = 0 \tag{8.3.3}$$

and the *magnitude of shear rate* $\dot{\gamma}$ may be expressed as:

$$\dot{\gamma} = \sqrt{2\,\mathrm{tr}\,D^2} = \sqrt{2\,\mathrm{tr}\,\mathbf{D}^2} = 2\sqrt{-II} \qquad (8.3.4)$$

For incompressible fluids the first term in the constitutive equations (8.3.2) cannot be distinguished from the isotropic pressure term in the total stress expressions (8.3.1). For incompressible fluids the extra stresses are therefore given as:

$$\tau_{ik} = 2\eta D_{ik} + 4\psi_2 D_{ij} D_{jk} \quad \Leftrightarrow \quad \mathbf{T}' = 2\eta \mathbf{D} + 4\psi_2 \mathbf{D}^2 \qquad (8.3.5)$$

These are the constitutive equations for a *Reiner-Rivlin fluid* (M. Reiner 1945, R. S. Rivlin 1948).

For simple shear flows, Sect. 4.1.1, we obtain:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right), \quad D^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right)^2, \quad \dot{\gamma} = \sqrt{2\mathrm{tr}D^2} = \frac{v}{h}$$

$$I = \mathrm{tr}\mathbf{D} = 0, \quad II = \frac{1}{2}\left((\mathrm{tr}\mathbf{D})^2 - \mathrm{tr}\mathbf{D}^2\right) = -\frac{1}{2}\dot{\gamma}, \quad III = \det \mathbf{D} = 0 \qquad (8.3.6)$$

The constitutive equations (8.3.5) give the extra stresses:

$$(\tau_{ik}) = 2\eta \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right) + 4\psi_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right)^2 \Rightarrow$$

$$\tau_{12} = \eta\dot{\gamma}, \quad \tau_{11} = \tau_{22} = \psi_2\dot{\gamma}^2, \quad \tau_{33} = 0$$

From the definitions of *viscometric functions* in Sect. 5.2 we see that η may be interpreted as the viscosity function, that the primary normal stress coefficient ψ_1 is zero, while the secondary normal stress coefficient may be interpreted as the scalar-valued function ψ_2 in Eq. (8.3.5).

Unfortunately no real fluids have been found that fit the Reiner-Rivlin fluid model. As mentioned in Chap. 5, it is often found for polymers that ψ_1 is about 10 times the absolute value of ψ_2. Due to this fact the literature has rejected the Reiner-Rivlin fluid in its general form. We see that the *generalized Newtonian fluid* model is a Reiner-Rivlin fluid with $\psi_2 = 0$ and the viscosity function η as a function of the second invariant II of $\mathbf{D}$, i.e. a function of the magnitude of shear rate $\dot{\gamma}$.

8.4 Corotational Derivative

We now introduce the concept of a *corotational reference* Rf^r, i.e. a reference rotating with the fluid particle that the constitutive equations are meant for. With respect to a corotational reference the rate of rotation matrix and the rate of rotation tensor are zero:

$$W^r = \left(W^r_{ik}\right) = 0 \quad \Leftrightarrow \quad \mathbf{W}^r = 0 \tag{8.4.1}$$

If Q is the transformation matrix representing the rotation of the corotational reference Rf^r with respect to the reference Rf chosen to finally present the constitutive equation, it follows from the Eqs. (8.2.23) and (8.2.28) that:

$$S_{ik} = \dot{Q}_{ik} = -W_{ik} \tag{8.4.2}$$

The material derivative of the stress matrix T^r with respect to the corotational reference is given by Eq. (8.2.22) as:

$$\dot{T}^r = \dot{T} - WT + TW \tag{8.4.3}$$

We take this matrix to define the matrix in Rf of an objective tensor $\partial_r\mathbf{T}$, called the *corotational derivative* of the stress tensor $\mathbf{T}$. In Rf the matrix of the tensor $\partial_r\mathbf{T}$ is thus:

$$\partial_r T = \dot{T} - W\,T + T\,W \tag{8.4.4}$$

In any other reference Rf^*, not necessarily the corotational reference, we obtain from the Eqs. (8.2.22, 8.2.29) and (8.4.2) that:

$$\begin{aligned}\partial_r T^* &= \dot{T}^* - W^* T^* + T^* W^* = \left(\dot{T} + \dot{Q}T - T\dot{Q}\right) - \left(W + \dot{Q}\right)T + T\left(W + \dot{Q}\right)\\ &= \dot{T} - W\,T + T\,W \Rightarrow\end{aligned}$$

$$\partial_r T^* = \dot{T} - W\,T + T\,W \tag{8.4.5}$$

The formulas (8.4.4) and (8.4.5) show that the two matrices $\partial_r T$ and $\partial_r T^*$ are equal, and thus they represent an objective tensor $\partial_r\mathbf{T}$, the *corotational derivative* of the stress tensor:

$$\partial_r\mathbf{T} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W} \tag{8.4.6}$$

This is a *tensor equation* and may be interpreted by its matrix representation (8.4.4) in the Ox – system.

We have seen in Eq. (8.2.29) that the rate of deformation matrix D represents an objective tensor, the *rate of deformation tensor* $\mathbf{D}$. The corotational derivative of the rate of deformation tensor is defined by the matrix:

$$\partial_r D = \dot{D} - WD + DW \tag{8.4.7}$$

In the next section we need expressions for $\partial_r D_{ik}$ in the case of steady simple shear flow. From Sect. 4.1.1 we have:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right), \quad W = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right), \quad \dot{\gamma} = \sqrt{2\text{tr}D^2} = \frac{v}{h} \tag{8.4.8}$$

We compute:

$$\begin{aligned} WD = \left(W_{ij}\, D_{jk}\right) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right)^2, \\ DW = \left(D_{ij}\, W_{jk}\right) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right)^2 \end{aligned} \tag{8.4.9}$$

When these results are introduced into Eq. (8.4.7), we obtain for the corotational derivatives of the rate of deformation matrix for simple shear flow:

$$\partial_r D = \dot{D} - WD + DW = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\ddot{\gamma} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma}^2 \tag{8.4.10}$$

8.5 Corotational Fluid Models

A *second-order fluid* is a model that has been used for steady flows. The constitutive equations are:

$$\tau_{ik} = 2\mu\left(D_{ik} + \lambda_5\, \partial_r D_{ik} - 2\lambda_6\, D_{ij} D_{jk}\right) \quad \Leftrightarrow \quad T' = 2\mu\left(D + \lambda_5\, \partial_r D - 2\lambda_6\, D^2\right) \tag{8.5.1}$$

μ, λ_5, and λ_6 are temperature dependent material parameters. μ is a viscosity with unit $\text{Ns/m}^2 = \text{Pa} \cdot \text{s}$, pascalsecond, while λ_5 and λ_6 are time parameters. The numbering of the λ's will be explained by Table 8.1 below.

In steady simple shear flow we use the results in the equations (8.3.6) and (8.4.10). Equation (8.5.1) now gives for the extra stresses:

$$T' = \begin{pmatrix} \mu(-\lambda_5 - \lambda_6)\dot{\gamma}^2 & \mu\dot{\gamma} & 0 \\ \mu\dot{\gamma} & \mu(\lambda_5 - \lambda_6)\dot{\gamma}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{aligned} \tau_{11} &= \mu(-\lambda_5 - \lambda_6)\dot{\gamma}^2 \\ \tau_{22} &= \mu(\lambda_5 - \lambda_6)\dot{\gamma}^2 \\ \tau_{12} &= \mu\dot{\gamma} \\ \tau_{33} &= \tau_{31} = \tau_{32} = 0 \end{aligned} \tag{8.5.2}$$

Table 8.1 Fluid models obtained from the Oldroyd 8-constant fluid model

Fluid models	Material parameters							
Oldroyd 8-constant fluid	μ	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
Newtonian fluid	μ							
Corotational Jeffreys fluid	μ	λ_1				λ_5		
Corotational Maxwell	μ	λ_1						
Upper-convected Maxwell fluid	μ	λ_1		λ_1				
Lower-convected Maxwell fluid	μ	λ_1		$-\lambda_1$				
Oldroyd A-fluid	μ	λ_1	λ_2	$-\lambda_1$		λ_5	$-\lambda_5$	
Oldroyd B-fluid	μ	λ_1	λ_2	λ_1		λ_5	λ_5	
Second-order fluid	μ					λ_5	λ_5	

The viscometric functions are:

$$\eta = \mu, \quad \psi_1 = (\tau_{11} - \tau_{22})/\dot{\gamma}^2 = -2\mu\lambda_5, \quad \psi_2 = (\tau_{22} - \tau_{33})/\dot{\gamma}^2 = \mu(\lambda_5 - \lambda_6) \tag{8.5.3}$$

We see that all three viscometric functions are represented by constants. The model may be used for low values of the shear rate.

A better model for steady flows is the *CEF fluid*, named after W.O. Criminale Jr., J.L. Ericksen, and G.L. Filbey (1958) [4]. The constitutive equations are:

$$\begin{aligned} \tau_{ik} &= 2\eta D_{ik} + (2\psi_1 + 4\psi_2) D_{ij} D_{jk} - \psi_1 \partial_r D_{ik} \Leftrightarrow \\ T' &= 2\eta D + (2\psi_1 + 4\psi_2) D^2 - \psi_1 \partial_r D \end{aligned} \tag{8.5.4}$$

η, ψ_1, and ψ_2 are the viscometric functions for viscometric flows as presented in Sect. 5.2, and thus functions of the magnitude of shear rate $\dot{\gamma}$ defined in equation (8.3.4).

A comparison of the equation (8.5.4) with the constitutive equations (6.1.3) of a generalized Newtonian fluid shows that the generalized Newtonian fluid is a special CEF fluid.

In a steady simple shear flow the magnitude of shear rate is equal to the shear deformation rate: $\dot{\gamma} = 2D_{12}$, the rate of deformation matrix D and the matrix D^2 are given by the equation (8.3.6), and the corotational derivative of the rate of deformation matrix $\partial_r D$ is given by equation (8.4.10). The extra stresses from equations (8.5.4) become:

$$T' = \eta \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{\gamma} + (2\psi_1 + 4\psi_2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left(\frac{1}{2}\dot{\gamma}\right)^2 - \psi_1 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma}^2 \Rightarrow$$

$$T' = \begin{pmatrix} (\psi_1 + \psi_2)\dot{\gamma}^2 & \eta\dot{\gamma} & 0 \\ \eta\dot{\gamma} & \psi_2\dot{\gamma}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Leftrightarrow \quad \begin{aligned} \tau_{12} &= \eta\dot{\gamma} \\ \tau_{11} - \tau_{22} &= \psi_1 \dot{\gamma}^2 \\ \tau_{22} - \tau_{33} &= \psi_2 \dot{\gamma}^2 \end{aligned} \tag{8.5.5}$$

The results (8.5.5) confirm that η, ψ_1, and ψ_2 are the three viscometric functions for viscometric flows.

It may be shown that the CEF fluid is a non-linear viscoelastic fluid. The viscoelastic properties are contained in the normal stress coefficients.

The *NIS fluid*, named after H. Norem, F. Irgens, and B. Schieldrop (1986) [12], is a *viscoelastic-plastic fluid* designed to be used for granular materials. A granular material consists of solid particles in a fluid suspension. If the volume fraction of solid particles is small, the material behaves approximately as the suspension fluid, very often as a Newtonian fluid. The NIS-model has been discussed and applied in F. Irgens, H. Norem (1996) [8] and F. Irgens, B. Schieldrop, C. Harbitz, U. Domaas, and R. Opsahl (1998) [9].

The flow of the granular material is characterized as a *macroviscous flow*. For high values of the volume fraction of solid particles, collisions between the particles as the granular material flow and deforms, result in non-Newtonian behavior. In steady simple shear tests the shear stress τ_{12} can be proportional to the square of the shear strain rate. Furthermore, it is found that for granular materials with predominantly dry, coarse particles, at low shear strain rates, the ratio of the shear stress τ_{12} and the normal stress σ_{22} is approximately constant and independent of the shear strain rate. This relationship may be expressed by:

$$\tau_{12} = |\sigma_{22}| \tan\theta \tag{8.5.6}$$

θ is an internal dry friction angle of the granular material.

In its most general form the NIS fluid model is defined by the constitutive equation:

$$\tau_{ik} = 2\frac{\alpha + \beta p_e}{\dot{\gamma}} D_{ik} + \tau_{ik}(\text{from equation 8.5.4}) \tag{8.5.7}$$

α represents cohesion, β is a dry friction coefficient, and p_e, called the *effective pressure*, is the part of the total pressure p that represents the direct contact between the solid particles in the suspension. The total pressure consists of the effective pressure and the *pore pressure* p_o.

A special version of the NIS fluid model has been used in simulations of snow avalanches, landslides, and in submarine slides. The viscometric functions are chosen as power laws:

$$\eta = \mu\dot{\gamma}^{n-1}, \quad \psi_1 = v_1\dot{\gamma}^{n-2}, \quad \psi_2 = v_2\dot{\gamma}^{n-2} \tag{8.5.8}$$

$\mu, n, v_1,$ and v_2 are constant material parameters. Based on experimental evidence, the power law index is chosen to be 2 for a granular material with a high volume fraction of solid particles. For cohesionless material, $\alpha = 0$, the ratio between the shear stress τ_{12} and the normal stress σ_{22} in a steady simple shear flow becomes:

$$\frac{\tau_{12}}{|\sigma_{22}|} = \frac{\beta p_e + \mu\dot{\gamma}^2}{p_e + \nu_2\dot{\gamma}^2} \tag{8.5.9}$$

This relationship agrees reasonably well with experiments.

8.6 Quasi-Linear Corotational Fluid Models

These models are developed from linear viscoelastic models. The response equations and constitutive equations at a fluid particle of the linear models are presented in a corotational reference for the particle, and then transformed to any convenient reference common to all fluid particles. These new models are therefore called quasi-linear and they satisfy the principle of material objectivity. The quasi-linear fluid models have a much wider application potential then the corresponding linearly viscoelastic counterparts. The reason for this is mainly the fact that the rotations of the principal axes of strains and the principal axes of strain rates are eliminated by introducing the corotational reference

From the Jeffreys fluid the *corotational Jeffreys fluid* is developed. The model is defined by the response equations:

$$\tau_{ik} + \lambda_1 \mathring{\partial}_r \tau_{ik} = 2\mu(D_{ik} + \lambda_5 \mathring{\partial}_r D_{ik}) \quad \Leftrightarrow \quad T' + \lambda_1 \mathring{\partial}_r T' = 2\mu(D + \lambda_5 \mathring{\partial}_r D) \tag{8.6.1}$$

$\mu, \lambda_1,$ and λ_5 are temperature dependent material parameters. In a corotational reference the response equations (8.6.1) reduce to the response equations (7.4.13) of the linear viscoelastic Jeffreys fluid. Constitutive equations for a corotational Jeffreys fluid on forms similar to the constitutive equations (7.4.16) for the linear Jeffreys fluid can be derived, but these will contain functions that are not easy to discuss in the present context.

It follows from the expressions (8.4.4) for the corotational derivatives that the response equations (8.6.1) are non-linear. The term quasi-linear refers to the fact that the equation (8.6.1) are locally linear at a particle if they are related to a corotational reference for that particle.

For the special case $\lambda_5 = 0$ in the equations (8.6.1), we obtain the response equations of the *corotational Maxwell fluid*.

$$\tau_{ik} + \lambda_1 \mathring{\partial}_r \tau_{ik} = 2\mu D_{ik} \quad \Leftrightarrow \quad T' + \lambda_1 \mathring{\partial}_r T' = 2\mu D \tag{8.6.2}$$

Compare these equations with the equation (7.4.1) for a linearly viscoelastic Maxwell fluid.

8.7 Oldroyd Fluids

A coordinate system that is embedded in the fluid and moves and deforms with the fluid is called a *convected coordinate system* and will naturally be a curvilinear coordinate system. Another name for convected coordinates is *codeforming coordinates*. In the present context we are in no position to discuss or use general curvilinear coordinates. A presentation of curvilinear coordinates and convected coordinates may be found in the book "Continuum Mechanics" by Irgens (2008) [7]. However, we need to consider some consequences of using convected coordinates in order to understand the definitions of the fluid models called *Oldroyd fluids*. J. G. Oldroyd (1950) [13] and (1958) [14] defined the fluid models that are named after him by constitutive equations defined in convected coordinates. Such constitutive equations automatically satisfy the principle of material objectivity: a rigid-body rotation superimposed on any deformation history can not be registered by equations written in convected coordinates.

In Sect. 8.2 it was shown that the material derivative of the stress matrix T did not represent an objective tensor. However, material differentiation of the stress matrix in a convected coordinate system does lead to tensors satisfying the objectivity principle. Transformation of the components to a reference fixed Cartesian coordinate system will result in either of two objective tensors: the *upper-convected derivative* $\mathbf{T}^{\nabla}$ of the tensor $\mathbf{T}$, and the *lower-convected derivative* $\mathbf{T}^{\Delta}$ of the tensor $\mathbf{T}$. The tensors $\mathbf{T}^{\nabla}$ and $\mathbf{T}^{\Delta}$ have the matrix representations:

$$T^{\nabla} = \dot{T} - LT - TL^{T} \quad \Leftrightarrow \quad \sigma_{ik}^{\nabla} = \dot{\sigma}_{ik} - L_{ij}\sigma_{jk} - \sigma_{ij}L_{kj} \tag{8.7.1}$$

$$T^{\Delta} = \dot{T} + L^{T}T + TL \quad \Leftrightarrow \quad \sigma_{ik}^{\Delta} = \dot{\sigma}_{ik} + L_{ji}\sigma_{jk} + \sigma_{ij}L_{jk} \tag{8.7.2}$$

$L_{ik} \equiv v_{i,k}$ are the velocity gradients. The two convected derivatives $\mathbf{T}^{\nabla}$ and $\mathbf{T}^{\Delta}$ are also called *Oldroyd derivatives*.

The convected derivatives $\mathbf{T}^{\nabla}$ and $\mathbf{T}^{\Delta}$, and the corotational derivative $\partial_r \mathbf{T}$ of a tensor $\mathbf{T}$ are related. From the definitions (8.4.4), (8.7.1), and (8.7.2) we obtain:

$$\partial_r T = \dot{T} - WT + TW = \dot{T} - \frac{1}{2}\left(L - L^{T}\right)T + T\frac{1}{2}\left(L - L^{T}\right) = \frac{1}{2}\left(\dot{T} - LT - TL^{T}\right) + \frac{1}{2}\left(\dot{T} + L^{T}T + TL\right) \quad \Rightarrow$$
$$\partial_r T = \frac{1}{2}\left(T^{\nabla} + T^{\Delta}\right) \tag{8.7.3}$$

In Problem 26 the following formulas are asked for:

$$\partial_r T = T^{\nabla} + TD + DT, \quad \partial_r T = T^{\Delta} - TD - DT \tag{8.7.4}$$

In order to demonstrate that the tensor defined by the matrix in equation (8.7.1) is objective we construct the expression for the matrix in a moving reference Rf^*. From the equations (8.2.17–8.2.20, 8.2.22, 8.2.23, 8.2.26) and we obtain:

$$T^* = T, \quad \dot{T}^* = \dot{T} + \dot{Q}T - T\dot{Q}, \quad L^* = L + \dot{Q}, \quad \dot{Q}^T = -\dot{Q} \tag{8.7.5}$$

Then:

$$\begin{aligned} &T^{*\nabla} = \dot{T}^* - L^*T^* - T^*L^{*T} = \left(\dot{T} + \dot{Q}T - T\dot{Q}\right) - \left(L + \dot{Q}\right)T - T\left(L^T + \dot{Q}^T\right) \quad \Rightarrow \\ &T^{*\nabla} = \dot{T} - LT - TL^T \end{aligned} \tag{8.7.6}$$

Thus, the upper convected derivatives T^∇ and $T^{*\nabla}$ of the stress matrix T with respect to any two references, Rf and Rf^*, and with coordinate systems Ox and O^*x^* coinciding at the present time, are represented by the same matrix and therefore represent an objective tensor. A similar analysis can be used to show that the lower convected derivative of the stress matrix T represents an objective quantity.

By transforming the response function (7.4.13) of the Jeffreys fluid from convected coordinates to Cartesian coordinates fixed in the reference Rf, Oldroyd obtained the response equations of two quasi-linear models: the *lower-convected Jeffreys fluid* or the *Oldroyd A-fluid* defined by:

$$\tau_{ik} + \lambda_1 \tau_{ik}^{\Delta} = 2\mu\left(D_{ik} + \lambda_5 D_{ik}^{\Delta}\right) \tag{8.7.7}$$

and the *upper-convected Jeffreys fluid* or the *Oldroyd B-fluid* defined by:

$$\tau_{ik} + \lambda_1 \tau_{ik}^{\nabla} = 2\mu\left(D_{ik} + \lambda_5 D_{ik}^{\nabla}\right) \tag{8.7.8}$$

In a steady simple shear flow the following *viscometric functions* η, ψ_1, and ψ_2 are found, and in a steady uniaxial extensional the following *extensional viscosity* η_E is found for the Oldroyd A-fluid and the Oldroyd B-fluid:
Oldroyd A-fluid:

$$\eta = \mu, \quad \psi_1 = 2\mu(\lambda_1 - \lambda_5), \quad \psi_2 = -\psi_1, \quad \eta_E(\dot{\varepsilon}) = 3\mu\frac{1 + \lambda_5\dot{\varepsilon} - 2\lambda_1\lambda_5\dot{\varepsilon}^2}{1 + \lambda_1\dot{\varepsilon} - 2\lambda_1^2\dot{\varepsilon}^2} \tag{8.7.9}$$

Oldroyd B-fluid:

$$\eta = \mu, \quad \psi_1 = 2\mu(\lambda_1 - \lambda_5), \quad \psi_2 = 0, \quad \eta_E(\dot{\varepsilon}) = 3\mu\frac{1 - \lambda_5\dot{\varepsilon} - 2\lambda_1\lambda_5\dot{\varepsilon}^2}{1 - \lambda_1\dot{\varepsilon} - 2\lambda_1^2\dot{\varepsilon}^2} \tag{8.7.10}$$

Because it is commonly found that $|\psi_2| << \psi_1$, the Oldroyd A-fluid overpredicts the second normal stress coefficient ψ_2 and has been judged in the literature not to be a useful model. The Oldroyd B-fluid has a zero second normal stress coefficient

ψ_2 and predicts a tension-thickening effect when $\lambda_1 > \lambda_5$, or a constant extensional viscosity $\eta_E = 3\mu$ when $\lambda_1 = \lambda_5$. For $\lambda_1 < \lambda_5$ the Oldroyd B-fluid predicts a negative primary normal stress coefficient ψ_1. Neither the Oldroyd A-fluid nor the Oldroyd B-fluid predict shear-thinning.

As shown in Sect. 7.4, by choosing $\lambda_5 = 0$ in the linearly viscoelastic Jeffreys fluid, we obtain the Maxwell fluid. If we set $\lambda_5 = 0$ in the Oldroyd A-fluid and Oldroyd B-fluid, two new fluid models are created:

$$\text{The } lower-convected\,Maxwell\,fluid: \quad \tau_{ik} + \lambda_1 \overset{\Delta}{\tau}_{ik} = 2\mu D_{ik} \tag{8.7.11}$$

$$\text{The } upper-convected\,Maxwell\,fluid: \quad \tau_{ik} + \lambda_1 \overset{\nabla}{\tau}_{ik} = 2\mu D_{ik} \tag{8.7.12}$$

As seen from the formulas (8.7.9) the lower-convected Maxwell fluid overpredicts the second normal stress coefficient ψ_2. As seen from the formulas (8.7.10) the upper-convected Maxwell fluid has a zero second normal stress coefficient ψ_2 and can predict tension thickening effect. None of the convected Maxwell fluids predict shear-thinning.

The most serious drawback with the four fluid models presented above is the fact that none of them are shear-thinning. To remedy this J. L. White and A. B. Metzner (1963) [16] proposed a fluid model, later named the *White–Metzner fluid*, and defined by the response equation:

$$\tau_{ik} + \frac{\eta(\dot{\gamma})}{\beta} \overset{\nabla}{\tau}_{ik} = 2\eta(\dot{\gamma}) D_{ik} \tag{8.7.13}$$

$\eta(\dot{\gamma})$ is the shear *viscosity function* and β is a *shear modulus*. The White–Metzner fluid contains both the shear-thinning features of a general Newtonian fluid and the viscoelastic, memory aspects of a Maxwell fluid. If the viscosity function is chosen to have a zero-shear-rate limit, the White-Metzner fluid behaves as an upper convected Maxwell fluid at low shear rates.

Oldroyd (1958) [14] proposed a fluid model that includes most of the fluid models presented in this chapter. Apart for the corotational derivatives of the extra stress matrix T' and the rate of deformation matrix D, the response equations include all possible terms that are linear in D and T', and quadratic in D. The model has been called *Oldroyd 8-constant fluid*. The response equations contain 8 constants, the viscosity μ and 7 time parameters $\lambda_1, \lambda_2, \ldots, \lambda_7$, or really 8 temperature dependent material parameters:

$$\begin{aligned} &T' + \lambda_1 \partial_r T' + \lambda_2 \mathrm{tr} T' D - \lambda_3 (T'D + DT') + \lambda_4 \mathrm{tr}(T'D) 1 \\ &\qquad = 2\mu \big(D + \lambda_5 \partial_r D - 2\lambda_6 D^2 + \lambda_7 \mathrm{tr} D^2 1\big) \quad \Leftrightarrow \\ &\tau_{ik} + \lambda_1 \partial_r \tau_{ik} + \lambda_2 \tau_{jj} D_{ik} - \lambda_3 \big(\tau_{ij} D_{jk} + D_{ij} \tau_{jk}\big) + \lambda_4 \tau_{jl} D_{jl} \delta_{ik} \\ &\qquad = 2\mu \big(D_{ik} + \lambda_5 \partial_r D_{ik} - 2\lambda_6 D_{ij} D_{jk} + \lambda_7 D_{jl} D_{jl} \delta_{ik}\big) \end{aligned} \tag{8.7.14}$$

Using the formulas (8.7.4), we can derive the alternative forms of the response equations (8.7.14):

$$\begin{aligned}\tau_{ik} + \lambda_1 \overset{\nabla}{\tau}_{ik} + \lambda_2 \tau_{jj} D_{ik} + (\lambda_1 - \lambda_3)\left(\tau_{ij} D_{jk} + D_{ij}\tau_{jk}\right) + \lambda_4 \tau_{jl} D_{jl}\delta_{ik} \\ = 2\mu\left(D_{ik} + \lambda_5 \overset{\nabla}{D}_{ik} + 2(\lambda_5 - \lambda_6) D_{ij} D_{jk} + \lambda_7 D_{jl} D_{jl}\delta_{ik}\right)\end{aligned} \tag{8.7.15}$$

$$\begin{aligned}\tau_{ik} + \lambda_1 \overset{\Delta}{\tau}_{ik} + \lambda_2 \tau_{jj} D_{ik} - (\lambda_1 + \lambda_3)\left(\tau_{ij} D_{jk} + D_{ij}\tau_{jk}\right) + \lambda_4 \tau_{jl} D_{jl}\delta_{ik} \\ = 2\mu\left(D_{ik} + \lambda_5 \overset{\Delta}{D}_{ik} - 2(\lambda_5 + \lambda_6) D_{ij} D_{jk} + \lambda_7 D_{jl} D_{jl}\delta_{ik}\right)\end{aligned} \tag{8.7.16}$$

Table 8.1 below shows that many of the fluid models presented in this chapter are represented by the Oldroyd 8-constant fluid model.

8.7.1 Viscometric Functions for the Oldroyd 8-Constant Fluid

The *viscometric functions* $\eta(\dot{\gamma})$, $\psi_1(\dot{\varepsilon})$, and $\psi_2(\dot{\varepsilon})$ for the *Oldroyd 8-constant fluid* will now be derived. For steady simple shear flow the matrices for the rate of deformation, the rate of rotation, and the matrix D^2 and the corotational derivative of the rate of deformation matrix are found in the equations (8.3.6, 8.4.8) and (8.4.10). In addition we compute the following matrices:

$$\partial_r T\prime = -WT\prime + T\prime W = \begin{pmatrix} -2\tau_{12} & \tau_{11} - \tau_{22} & -\tau_{23} \\ \tau_{11} - \tau_{22} & 2\tau_{12} & \tau_{13} \\ -\tau_{23} & \tau_{13} & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma} \tag{8.7.17}$$

$$(\mathrm{tr}T')D = \begin{pmatrix} 0 & \tau_{11} + \tau_{22} + \tau_{33} & 0 \\ \tau_{11} + \tau_{22} + \tau_{33} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma} \tag{8.7.18}$$

$$T'D + DT' = \begin{pmatrix} 2\tau_{12} & \tau_{11} + \tau_{22} & \tau_{23} \\ \tau_{22} + \tau_{11} & 2\tau_{12} & \tau_{13} \\ \tau_{23} & \tau_{13} & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma}, \quad \mathrm{tr}\left(D^2\right)1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{2}\dot{\gamma}^2 \tag{8.7.19}$$

The constitutive equations (8.7.14) for the Oldroyd 8-constant fluid now become:

$$\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} + \lambda_1 \begin{pmatrix} -2\tau_{12} & \tau_{11}-\tau_{22} & -\tau_{23} \\ \tau_{11}-\tau_{22} & 2\tau_{12} & \tau_{13} \\ -\tau_{23} & \tau_{13} & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma} + \lambda_2 \begin{pmatrix} 0 & \tau_{11}+\tau_{22}+\tau_{33} & 0 \\ \tau_{11}+\tau_{22}+\tau_{33} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma}$$
$$- \lambda_3 \begin{pmatrix} 2\tau_{12} & \tau_{11}+\tau_{22} & \tau_{23} \\ \tau_{22}+\tau_{11} & 2\tau_{12} & \tau_{13} \\ \tau_{23} & \tau_{13} & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma} + \lambda_4 \begin{pmatrix} \tau_{12} & 0 & 0 \\ 0 & \tau_{12} & 0 \\ 0 & 0 & \tau_{12} \end{pmatrix} \dot{\gamma}$$
$$= 2\mu \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma} + \lambda_5 \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\dot{\gamma}^2 - 2\lambda_6 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{4}\dot{\gamma}^2 + \lambda_7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{2}\dot{\gamma}^2 \right\}$$

From this matrix equation we obtain the following component equations:

$$\begin{aligned} &\tau_{11} + \tau_{12}(-\lambda_1 - \lambda_3 + \lambda_4)\dot{\gamma} = \mu(-\lambda_5 - \lambda_6 + \lambda_7)\dot{\gamma}^2 \\ &\tau_{22} + \tau_{12}(\lambda_1 - \lambda_3 + \lambda_4)\dot{\gamma} = \mu(\lambda_5 - \lambda_6 + \lambda_7)\dot{\gamma}^2 \\ &\tau_{33} + \tau_{12}\lambda_4\dot{\gamma} = \mu\lambda_7\dot{\gamma}^2 \end{aligned} \tag{8.7.20}$$

$$\tau_{12} + \frac{1}{2}\tau_{11}(\lambda_1 + \lambda_2 - \lambda_3)\dot{\gamma} + \frac{1}{2}\tau_{22}(-\lambda_1 + \lambda_2 - \lambda_3)\dot{\gamma} + \frac{1}{2}\tau_{33}\lambda_2\dot{\gamma} = \mu\dot{\gamma} \tag{8.7.21}$$

We solve the equation (8.7.20) for τ_{11}, τ_{22}, and τ_{33}:

$$\begin{aligned} \tau_{11} &= \mu(-\lambda_5 - \lambda_6 + \lambda_7)\dot{\gamma}^2 - \tau_{12}(-\lambda_1 - \lambda_3 + \lambda_4)\dot{\gamma} \\ \tau_{22} &= \mu(\lambda_5 - \lambda_6 + \lambda_7)\dot{\gamma}^2 - \tau_{12}(\lambda_1 - \lambda_3 + \lambda_4)\dot{\gamma} \\ \tau_{33} &= \mu\lambda_7\dot{\gamma}^2 - \tau_{12}\lambda_4\dot{\gamma} \end{aligned} \tag{8.7.22}$$

The expressions for τ_{11} and τ_{22} are substituted into equation (8.7.21), and we obtain the following equation for τ_{12}:

$$\begin{aligned} &\tau_{12}\left\{1 + \left[\lambda_1^2 + \lambda_2\left(\lambda_3 - \frac{3}{2}\lambda_4\right) - \lambda_3(\lambda_3 - \lambda_4)\right]\dot{\gamma}^2\right\} \\ &= \mu\dot{\gamma}\left\{1 + \left[\lambda_1\lambda_5 + \lambda_2\left(\lambda_6 - \frac{3}{2}\lambda_7\right) - \lambda_3(\lambda_6 - \lambda_7)\right]\dot{\gamma}^2\right\} \end{aligned} \tag{8.7.23}$$

The *viscosity function* is then:

$$\eta(\dot{\gamma}) = \frac{\tau_{12}}{\dot{\gamma}} = \mu\frac{1 + \left[\lambda_1\lambda_5 + \lambda_2\left(\lambda_6 - \frac{3}{2}\lambda_7\right) - \lambda_3(\lambda_6 - \lambda_7)\right]\dot{\gamma}^2}{1 + \left[\lambda_1^2 + \lambda_2\left(\lambda_3 - \frac{3}{2}\lambda_4\right) - \lambda_3(\lambda_3 - \lambda_4)\right]\dot{\gamma}^2} \tag{8.7.24}$$

From the equations (8.7.22) and (8.7.24) we obtain:

$$\begin{aligned} \tau_{11} - \tau_{22} = 2\tau_{12}\lambda_1\dot{\gamma} - 2\mu\lambda_5\dot{\gamma}^2 = (2\eta(\dot{\gamma})\lambda_1 - 2\mu\lambda_5)\dot{\gamma}^2 \quad \Rightarrow \\ \psi_1(\dot{\gamma}) = \frac{\tau_{11} - \tau_{22}}{\dot{\gamma}^2} = 2\eta(\dot{\gamma})\lambda_1 - 2\mu\lambda_5 \end{aligned} \tag{8.7.25}$$

From the equations (8.7.22, 8.7.24, 8.7.25) we get:

$$\tau_{22} - \tau_{33} = -(\tau_{11} - \tau_{22}) + \tau_{11} - \tau_{33}$$
$$= -\psi_1\dot{\gamma}^2 + \mu(-\lambda_5 - \lambda_6)\dot{\gamma}^2 - \eta(\dot{\gamma})(-\lambda_1 - \lambda_3)\dot{\gamma}^2 \quad \Rightarrow$$

$$\psi_2(\dot{\gamma}) = \frac{\tau_{22} - \tau_{33}}{\dot{\gamma}^2} = -\psi_1 + \eta(\dot{\gamma})(\lambda_1 + \lambda_3) - \mu(\lambda_5 + \lambda_6) \tag{8.7.26}$$

The expressions (8.7.8–8.7.9) for the *viscometric functions* for the Oldroyd A-fluid and Oldroyd B-fluid are contained in the formulas (8.7.24–8.7.26).

8.7.2 Extensional Viscosity for the Oldroyd 8-Constant Fluid

Uniaxial extensional flow is presented in Sect. 4.5.2 and is given by the velocity field:

$$v_1 = \dot{\varepsilon}x_1, \quad v_2 = -\frac{1}{2}\dot{\varepsilon}x_2, \quad v_1 = -\frac{1}{2}\dot{\varepsilon}x_3 \tag{8.7.27}$$

We want to find the extensional viscosity defined by:

$$\eta_E(\dot{\varepsilon}) = \frac{\tau_{11} - \tau_{22}}{\dot{\varepsilon}} \tag{8.7.28}$$

Due to symmetry: $\tau_{22} = \tau_{33}$, and:

$$\operatorname{tr} T' = \tau_{11} + \tau_{22} + \tau_{33} = \tau_{11} + 2\tau_{22} \tag{8.7.29}$$

We find the following matrices:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\frac{1}{2}\dot{\varepsilon}, \quad D^2 = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{4}\dot{\varepsilon}^2, \quad W = 0 \tag{8.7.30}$$

$$(\operatorname{tr} T')D = \begin{pmatrix} 2(\tau_{11} + 2\tau_{22}) & 0 & 0 \\ 0 & -(\tau_{11} + 2\tau_{22}) & 0 \\ 0 & 0 & -(\tau_{11} + 2\tau_{22}) \end{pmatrix}\frac{1}{2}\dot{\varepsilon} \tag{8.7.31}$$

$$T'D = DT' = \begin{pmatrix} 2\tau_{11} & 0 & 0 \\ 0 & -\tau_{22} & 0 \\ 0 & 0 & -\tau_{22} \end{pmatrix}\frac{1}{2}\dot{\varepsilon}, \quad (\operatorname{tr} D^2)1 = \frac{3}{2}\dot{\varepsilon}^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8.7.32}$$

$$\operatorname{tr}(T'D)1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}(\tau_{11} - \tau_{22})\dot{\varepsilon} \tag{8.7.33}$$

$$\dot{T}' = 0, \quad \dot{D} = 0, \quad \partial_r T' = 0, \quad \partial_r D' = 0 \tag{8.7.34}$$

From the constitutive equation (8.7.14) we now obtain the matrix equation:

$$\begin{aligned} &\begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{22} \end{pmatrix} + \lambda_2(\tau_{11} + 2\tau_{22}) \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{2}\dot{\varepsilon} \\ &- \lambda_3 \begin{pmatrix} 2\tau_{11} & 0 & 0 \\ 0 & -\tau_{22} & 0 \\ 0 & 0 & -\tau_{22} \end{pmatrix} \dot{\varepsilon} + \lambda_4(\tau_{11} - \tau_{22}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{\varepsilon} \\ &= 2\mu \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{1}{2}\dot{\varepsilon} - 2\lambda_6 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{4}\dot{\varepsilon}^2 + \frac{3}{2}\lambda_7 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \dot{\varepsilon}^2 \right\} \end{aligned} \tag{8.7.35}$$

From this matrix equation we obtain two component equations and the expression for $\operatorname{tr} T'$:

$$\begin{aligned} &\tau_{11} + \lambda_2 (\tau_{11} + 2\tau_{22})\dot{\varepsilon} - 2\lambda_3 \tau_{11}\dot{\varepsilon} + \lambda_4(\tau_{11} - \tau_{22})\dot{\varepsilon} = \mu\big(2\dot{\varepsilon} - 4\lambda_6 \dot{\varepsilon}^2 + 3\lambda_7 \dot{\varepsilon}^2\big) \\ &\tau_{22} - \frac{1}{2}\lambda_2(\tau_{11} + 2\tau_{22})\dot{\varepsilon} + \lambda_3 \tau_{22}\dot{\varepsilon} + \lambda_4(\tau_{11} - \tau_{22})\dot{\varepsilon} = \mu\left(-\dot{\varepsilon} - \lambda_6 \dot{\varepsilon}^2 + 3\lambda_7 \dot{\varepsilon}^2\right) \end{aligned} \tag{8.7.36}$$

Solving these two equations (8.7.37) for $(\tau_{11} - \tau_{22})$ we get the expression for the *extensional viscosity*:

$$\eta_E = \frac{(\tau_{11} - \tau_{22})}{\dot{\varepsilon}} = 3\mu \frac{1 - \lambda_6\dot{\varepsilon} + \big(\frac{3}{2}\lambda_2 - \lambda_3\big)(2\lambda_6 - 3\lambda_7)\dot{\varepsilon}^2}{1 - \lambda_3\,\dot{\varepsilon} + \big(\frac{3}{2}\lambda_2 - \lambda_3\big)(2\lambda_3 - 3\lambda_4)\dot{\varepsilon}^2} \tag{8.7.37}$$

The expressions for the *extensional viscosity* for the Oldroyd A-fluid and Oldroyd B-fluid are contained in the formula (8.7.37).

8.8 Non-Linear Viscoelasticity: The Norton Fluid

All metals show non-linearly viscoelastic response, except at very low levels of stress, whenever the temperature is higher than a *critical temperature* Θ_c. Also plastics must often be treated as non-linearly viscoelastic materials. For materials showing a dominant *secondary creep*, like metals, and that is subjected to weakly varying stress the following constitutive equations are often applied:

$$\frac{3}{2}\frac{\dot{\varepsilon}_c}{\sigma_c}\left(\frac{\sigma_e}{\sigma_c}\right)^{m-1} T'_{ik} + \frac{1}{2G}\left(\dot{T}_{ik} - \frac{v}{(1+v)}\dot{T}_{jj}\,\delta_{ik}\right) = D_{ik} \quad \Leftrightarrow$$
$$\frac{3}{2}\frac{\dot{\varepsilon}_c}{\sigma_c}\left(\frac{\sigma_e}{\sigma_c}\right)^{m-1} \mathbf{T}' + \frac{1}{2G}\left(\dot{\mathbf{T}} - \frac{v}{(1+v)}\left(\mathrm{tr}\dot{\mathbf{T}}\right)\mathbf{1}\right) = \mathbf{D} \tag{8.7.38}$$

$\dot{\varepsilon}_c$, σ_c, m, G and v are material parameters. G is the *shear modulus* of elasticity and v is called the *Poison's ratio*. For incompressible materials $v = 0.5$. The stress σ_e is an *effective stress* and is defined by:

$$\sigma_e = \sqrt{\frac{1}{2}T'_{ik}T'_{ik}} \tag{8.7.39}$$

The first term in the equation (8.7.38) reflects non-linear viscosity while the two last terms express linear elastic behavior.

For a uniaxial state of stress σ in the x_1 − direction the effective stress becomes equal to $|\sigma|$, and the constitutive equations (8.7.38) reduce to:

$$\frac{\dot{\varepsilon}_c}{\sigma_c}\left(\frac{\sigma}{\sigma_c}\right)^{m-1}\sigma + \frac{\dot{\sigma}}{E} = v_{1,1}\,, \quad -\frac{1}{2}\frac{\dot{\varepsilon}_c}{\sigma_c}\left(\frac{\sigma_e}{\sigma_c}\right)^{m-1}\sigma = v_{2,2} = v_{3,3} \tag{8.7.40}$$

$E = 2G(1+v)$ is the *modulus of elasticity*. The first of the equation (8.7.40) is called *Norton's law* for *secondary creep*, and the equations (8.7.38) define the *Norton Fluid*, named after F. H. Norton, Creep of Steel at high Temperature, McGraw-Hill (1929). Note that the linearly elastic contribution in the constitutive equations (8.7.38) presumes small strains. This implies that the Norton fluid model is mostly relevant for fluid-like behavior of solids, as in creep and stress relaxation. The first of the constitutive equations (8.7.40) resembles the response equation (7.3.5) for the mechanical *Maxwell model*, but with a non-linear viscous dashpot in series with a linear elastic spring.

Symbols

$A = (A_{ik}),\ A_{ik}$	3×3matrix, matrix elements
$A^T \Rightarrow A_{ik}^T = A_{ki}$	A^T transposed matrix
$A^{-1} \Rightarrow AA^{-1} = 1$	A^{-1} inverse matrix
$\mathbf{A}$, A_{ij}	tensor, tensor components
$\mathbf{a}$, a, a_i	vector, vector matrix, vector components
$\mathbf{b}$	body force
c	specific heat
$\mathbf{c}$	vorticity
$D = (D_{ik}),\ \mathbf{D}$	rate of deformation matrix, rate of deformation tensor
D_{ik}	rates of deformation
D_e	Deborah number
E	modulus of elasticity, internal energy
$\mathbf{E}$	Green strain tensor
e	specific internal energy
$\mathbf{e}_i$	base vectors
$\mathbf{F}$	deformation gradient
f	Fanning friction number
$f(x_1, x_2, x_3, t) \equiv f(x, t)$	place function
$f(X_1, X_2, X_3, t) \equiv f(X, t)$	particle function
$\dot{f}(X, t) = \frac{\partial f(X,t)}{\partial t} \equiv \partial_t f(X, t)$	material derivative of a particle function $f(X, t)$
$\dot{f}(x, t) = \partial_t f + f_{,i}\, v_i$	material derivative of a place function $f(x, t)$
$\dfrac{\partial f}{\partial x_i} \equiv f_{,i}\,,\ \dfrac{\partial^2 f}{\partial x_j \partial x_i} \equiv f_{,ij} \equiv f_{,ji}\,,\ \dfrac{\partial f}{\partial t} \equiv \partial_t f$	
$\sum_{i=1}^{3} \dfrac{\partial f(x, t)}{\partial x_i} v_i \equiv f_{,i}\, v_i$	Einstein summation convention

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G	shear modulus
g	gravitational force per unit mass
$\mathbf{h}(x,t)$	heat flux vector
$H(t)$	Heaviside unit step function
K	present configuration, consistency parameter
K_0	reference configuration
$\bar{K}$	current configuration
k	heat conduction coefficient
M	torque
m	mass
N_1, N_2	primary and secondary normal stress difference
n	power law index
P	mechanical power, modified pressure
p	pressure, thermodynamic pressure
p_e	effective pressure
p_0	pore pressure
p_s	static pressure
$Q = (Q_{ik})$, Q_{ik}	transformation matrix, direction cosines, volumetric flow
q	heat flux
R	radius, gas constant
$\mathbf{R}$	rotation tensor
$\mathbf{r}$	place vector
Rf	reference frame, reference
$\mathbf{S}$	rate of rotation tensor
$T' = \left(T'_{ik}\right) = (\tau_{ik})$, $\mathbf{T}'$	extra stress matrix, extra stress tensor
$T = (T_{ik}) = (\sigma_{ik})$, $\mathbf{T}$	stress matrix, stress tensor
T_{ik}	coordinate stresses
t	time
$\mathbf{t}$	stress vector
$\mathbf{v}$, v_i	velocity vector, velocity components
$\mathbf{W}$, W, W_{ik}	rate of rotation tensor, rate of rotation matrix, rates of rotation
$\boldsymbol{\omega}$	angular velocity vector
$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{r}(X,t)}{\partial t} \equiv \partial_t \mathbf{r}(X,t)$	velocity of particle X
$v_i = \dot{x}_i = \frac{\partial x_i(X,t)}{\partial t} \equiv \partial_t x_i(X,t)$	velocity components
x, y, and z	Cartesian coordinates
x_i	Cartesian particle coordinates in the present configuration, place

$x \equiv \begin{pmatrix} x \\ x_2 \\ x_3 \end{pmatrix} \equiv \{x_1\, x_2\, x_3\}$	place, vector matrix
X_i	Cartesian particle coordinates in the reference configuration, particle
$X \equiv \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \equiv \{X_1\, X_2\, X_3\}$	particle, vector matrix
$x_i(X_1, X_2, X_3, t) \equiv x_i(X, t)$	motion of particle X
$(x_1, x_2, x_3, t) \equiv (x, t)$	Eulerian coordinates
$(X_1, X_2, X_3, t) \equiv (X, t)$	Lagrangian coordinates
$\nabla \equiv \mathbf{e}_i \frac{\partial}{\partial x_i}$	del-operator in Cartesian coordinates
$\nabla\alpha \equiv \text{grad } \alpha = \mathbf{e}_i \frac{\partial \alpha}{\partial x_i}$	gradient of a scalar α
$\nabla \cdot \mathbf{a} \equiv \text{div } \mathbf{a} = \frac{\partial a_i}{\partial x_i} \equiv a_{i,i}$	divergence of a vector **a**
$\nabla \times \mathbf{a} \equiv \text{curl } \mathbf{a} \equiv \text{ rot } \mathbf{a}$	rotation of a vector **a**
1	unit matrix
$\mathbf{1}$	unit tensor
x, x_i, X, X_i	Cartesian coordinates
Z	Zener-Hollomon parameter
α, β, ...	scalars
α	cohesion
$\alpha(t)$	creep function in shear
α_g, α_e	glass compliance, equilibrium compliance in shear
β	friction coefficient
$\beta(t)$	relaxation function in shear
β_g, β_e	glass modulus, equilibrium modulus in shear
γ	shear strain
γ_e	equilibrium shear strain
$\dot{\gamma},\ \dot{\gamma} = \sqrt{2D_{ik}D_{ik}}$	shear rate, magnitude of shear rate
δ_{ij}	Kronecker delta
$\delta(t)$	Dirac delta function
$\varepsilon, \dot{\varepsilon}$	longitudinal strain, longitudinal strain rate
$\varepsilon_v, \dot{\varepsilon}_v$	volumetric strain, volumetric strain rate
$\eta(\dot{\gamma})$	viscosity function
$\eta_E(\dot{\varepsilon})$	extensional viscosity, Trouton viscosity
$\eta_{EB}(\dot{\varepsilon}),\ \eta_{EP}(\dot{\varepsilon})$	biaxial extensional viscosity, planar extensional viscosity
$\boldsymbol{\Theta}$	temperature
θ	internal friction angle

κ	bulk modulus, bulk viscosity
μ	viscosity
v	Poissons's ratio
ρ	density
σ	normal stress
σ_c	stress parameter
σ_e	effective stress
σ_{ik}	coordinate stresses
τ, τ_y	shear stress, yield shear stress
τ_e	equilibrium shear stress
τ_{ik}	extra stresses
ψ_1, ψ_2	primary and secondary normal stress coefficient
ω	angular frequency, stress power per unit volume
R, θ, z	cylindrical coordinates
r, θ, ϕ	spherical coordinates
I, II, III	principal invariants of the rate of deformation tensor **D**

Problems

Problem 1. A concentric cylinder viscometer is used to measure the viscosity μ of a Newtonian-fluid. The cylindrical container rotates with constant angular velocity ω. The inner cylinder is kept at rest by a constant torque M. The inner cylinder has a radius R, and the distance between the two cylindrical surfaces in contact with the fluid is $h\ (<<R)$. The fluid sticks to the walls. In order to eliminate the influence of shear stresses on the plane circular end surface of the inner cylinder, the torques is measured for two heights H_1 and H_2 of the thin fluid film between the cylindrical surfaces.

Determine the viscosity μ of the fluid when the following data are given:

$$R = 50 \text{ mm},\ h = 0.5 \text{ mm},\ \omega = 30\,\text{rad/s}$$
$$H = H_1 = 50 \text{ mm} \Rightarrow \quad M = M_1 = 0.45 \text{ Nm}$$
$$H = H_2 = 100 \text{ mm} \Rightarrow M = M_2 = 0.81 \text{ Nm}$$

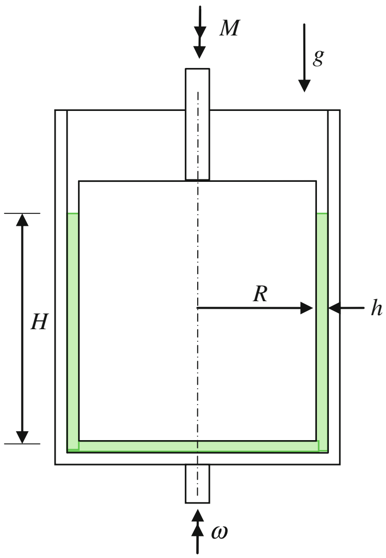

Fig. Problem 1

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Problem 2. A closed vessel filled with a fluid is given a translatoric motion defined by the velocity field:

$$v_1 = -v_o \sin \omega t, \quad v_2 = v_o \cos \omega t$$

v_o and ω are constants. The fluid moves with the vessel as a rigid body. Show the streamlines at time t are straight lines, and that the path lines are circles.

Problem 3. Show that the streamlines and the path lines coincide for the following type of non-steady two-dimensional flow:

$$v_1 = f(t)\, g(x,y), \quad v_2 = f(t)\, h(x,y), \quad v_3 = 0$$

$f(t)$, $g(x,y)$, and $h(x,y)$ are arbitrary functions of the variables: time t and Cartesian coordinates x and y.

Problem 4. Let $\alpha(\mathbf{r},t) = 0$ represent a fixed boundary surface A in a flow of a fluid. Show that the velocity field $\mathbf{v}(\mathbf{r},t)$ must satisfy the condition:

$$\mathbf{v} \cdot \nabla\alpha = 0 \quad \text{on } A$$

Problem 5. Let $\alpha(\mathbf{r},t) = 0$ represent a moving boundary surface A in a fluid flow. Show that the velocity field $\mathbf{v}(\mathbf{r},t)$ must satisfy the condition:

$$\partial_t\alpha + \mathbf{v} \cdot \nabla\alpha = 0 \text{ on } A$$

Problem 6. The following information is known about the velocity field of the flow of an incompressible fluid:

$$v_1 = v_1(x_1,t) = -\frac{\alpha x_1}{t_o - t}, \quad v_2 = v_2(x_2,t), \quad v_2(0,t) = 0, \quad v_3 = 0$$

α and t_o are constants.

(a) Determine the velocity component $v_2(x_2,t)$.
(b) Compute the local acceleration, the convective acceleration, and the particle acceleration.
(c) Show that the flow is irrotational, i.e. $\text{rot}\,\mathbf{v} = \mathbf{0}$, and determine the velocity potential ϕ from the formula $\mathbf{v} = \nabla\phi$.

Problem 7. Two vectors are defined by their Cartesian components:

$$\mathbf{a} = [1,2,-2], \quad \mathbf{b} = [2,1,-1]$$

Compute:

(a) the scalar product: $\mathbf{ab} = a_i\, b_i$
(b) the matrices: $A = \left(A_{ij}\right) = \left(a_i\, b_j\right), \quad B = \left(B_{ij}\right) = \left(b_i\, a_j\right)$
(c) the scalar: $\alpha = A_{ij}\, B_{ij}$
(d) the vector: $c_i = A_{ik} b_k$

Problem 8. The coordinate stresses in a particle X_i are given by:

$$\sigma_{ij} = -p\,\delta_{ij} + \tau_{ij}$$

Determine the formula for the normal stress on a surface with unit normal: $\mathbf{n} = [\cos\phi, \sin\phi, 0]$,where is the angle between $\mathbf{n}$ and the x_1- axis.

Problem 9. A capillary viscometer consists in principle of a container with a long straight circular thin tube (capillary tube). The container, which may be open or closed, as indicated in the figure, is filled with a fluid for which we will determine the viscous properties. For a given pressure p_o over the free fluid surface the volume flow Q through the tube is determined by measuring the amount of fluid flowing out of the tube in a certain time interval.

Assume static conditions in the container and that the fluid level h is approximately constant. Also neglect the special flow conditions at the inlet and the outlet of the tube. The atmospheric pressure at the outlet is p_a.

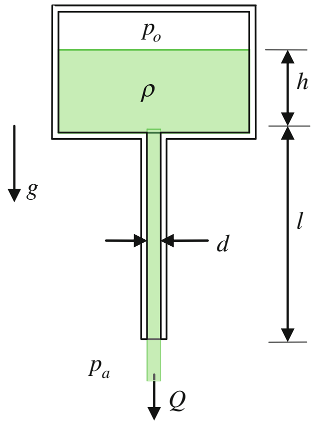

Fig. Problem 9

A fluid is modeled as a power-law fluid. The consistency K and power law index n shall be determined using the following procedure:

(a) Develop the formula:

$$\frac{p_o - p_a}{l} + \rho g\left(1 + \frac{h}{l}\right) = \left(\frac{8Q}{\pi d^3}\frac{3n+1}{n}\right)^n \frac{4K}{d}$$

(b) Set: h =20 cm, l = 50 cm, d = 5 mm, $\rho = 1{,}05 \cdot 10^3$ kg/m^3.

Determine K and n using the formula above and the following two sets of data:

(1) Q_1 =25 cm^3/s for $p_o - p_a$ = 8,92 kPa
(2) Q_2 =35 cm^3/s for $p_o - p_a$ = 11,73 kPa

Problem 10. The container in the capillary viscometer in problem 9 is now open and the pressure p_o is equal to the atmospheric pressure p_a. The container is filled with fluid to a height h = H. The container has the internal diameter D. Determine the time it takes to empty the container through the tube. Neglect inflow and outflow lengths in the tube, and assume static conditions in the container.

(a) The fluid is modeled as a Newtonian fluid with viscosity μ.
(b) The fluid is modeled as a power-law fluid with consistency K and index n.

Answer: a) $\frac{32\mu l D^2}{\rho g d^4}\ln\left(1+\frac{H}{l}\right)$, b) $\left(\frac{4K}{\rho g d}\right)^{\frac{1}{n}}\left[\left(1+\frac{H}{l}\right)^{\frac{n-1}{n}}-1\right]\frac{3n+1}{n-1}\frac{2lD^2}{d^3}$

Problem 11. A Newtonian fluid flows between two parallel planes a distance h apart. One of the planes is at rest and is kept at constant temperature T_o. The other plane moves with a constant velocity v_o and is insulated. The gradient of the modified pressure is constant equal to dP/dx in the flow direction. The viscosity μ and the thermal conductivity k are constants. Determine the temperature field $T(y)$ in the fluid.

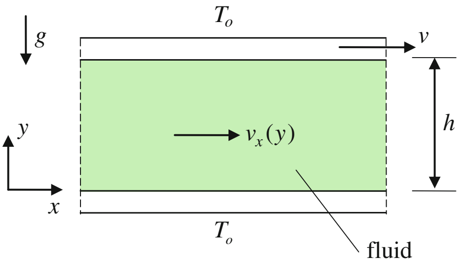

Fig. Problem 11

Problem 12. A Newtonian fluid moves in the annular space between two concentric cylindrical surfaces. The inner and outer radii of the annular space are r_1 and r_2, and the inner cylindrical surface is at rest. The outer cylindrical surface is subjected to a torque M and can rotate. Neglect end effects and assume steady laminar flow with the velocity field in cylindrical coordinates (R, θ, z):

$$v_\theta = v_\theta(R), \quad v_R = v_z = 0$$

(a) Show that: $v_\theta = \frac{\omega r_1}{\left[1-(r_1/r_2)^2\right]}\left[\frac{r_1}{r_2} - \frac{r_1 R}{r_2^2}\right], \quad M = \frac{4\pi\mu L\omega r_1^2}{\left[1-(r_1/r_2)^2\right]}$

(b) Show that the flow is irrotational when $b \to \infty$, and determine the velocity potensial ϕ such that $\mathbf{v} = \nabla\phi$.

Problem 13. The annular space between two concentric cylindrical surfaces is filled with a Bingham-fluid. The inner and outer radii of the annular space are r_1 and r_2, and the inner cylindrical surface is at rest. The outer cylindrical surface is subjected to a torque M and can rotate. Neglect end effects and assume steady laminar flow with the velocity field in cylindrical coordinates (R, θ, z):

$$v_\theta = v_\theta(R), \quad v_R = v_z = 0$$

(a) Find an expression for the shear stress $\tau_{R\theta}(R)$.

(b) Determine the minimum value of the torque M that make flow possible..
(c) Determine the velocity field and draw a graph of v_θ(R). Determine the angular velocity of the outer cylindrical surface.

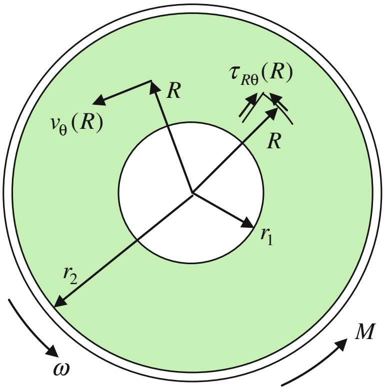

Fig. Problem 13

Problem 14. Determine the velocity field and draw the graph of $v_\theta(R)$ in problem 13 when the conditions are altered as follows: The inner cylindrical surface is subjected to the torque M, while the outer cylindrical surface is at rest.

Problem 15. A generalized Newtonian fluid with density ρ and viscosity function given by the power law has a steady, laminar flow in an annular space between two concentric cylindrical surfaces with vertical axis and radii r_1 and r_2. The flow is driven by a modified pressure gradient $\partial P/dz = -c$ in the axial $z-$ direction.

(a) Assume that the distance between the cylindrical surfaces: $h = r_2 - r_1 << r_1$. Determine the velocity field and the volume flow Q.
(b) Let $K = 18.7$ Ns^n/m^2, $n = 0.4$ for the power law parameters, and $h = 20$ mm. Determine Q as a function of the modified pressure gradient c.
(c) Determine the velocity field and the volume flow when h can not be assumed much less than r_1.

Problem 16. The figure illustrates a parallel-plate viscometer. A thick non-Newtonian fluid is placed between two parallel plates. The lower plate is at rest, while the upper plate is rotating with constant angular velocity ω. The torque M as a function of ω is recorded. A power-law fluid defined by the consistency parameter K and power law index n, is suggested as a fluid model. The velocity field in cylindrical coordinates (R, θ, z) is assumed to be:

$$v_\theta = Rf(z), \quad v_R = v_z = 0$$

(a) Show that if accelerations are neglected $f(z) = \omega z/h$.

(b) Derive the following formula relating M and ω : $M = 2\pi K\left(\frac{\omega}{h}\right)^n \frac{r^{n+3}}{n+3}$.

This formula may be used to evaluate the material parameters K and n.

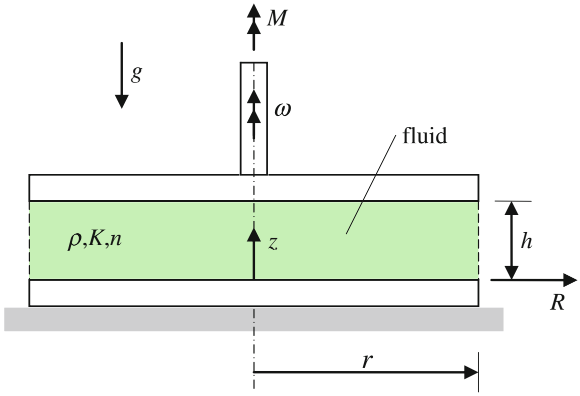

Fig. Problem 16

Problem 17. A Newtonian fluid with density ρ and viscosity μ has a steady, laminar flow in an annular space between two concentric cylindrical surfaces with a vertical axis. The flow is driven by a modified pressure gradient $\partial P/dz$ and a rotation of the inner cylindrical surface. The inner cylinder rotates with a constant angular velocity ω.

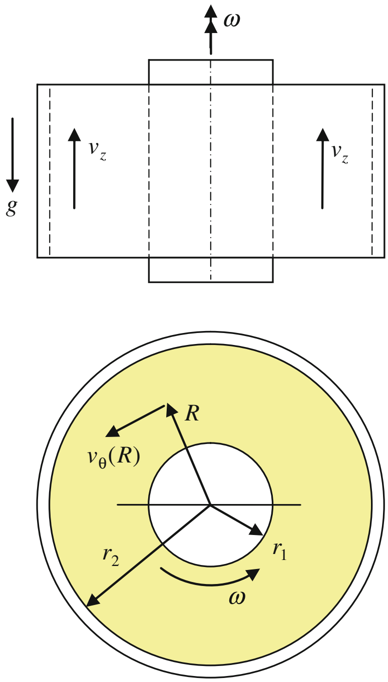

Fig. Problem 17

(a) Assume that $h = r_2 - r_1 << r_1$. Determine the velocity field and the volume flow in the axial direction.
(b) Determine the velocity field and the volume flow when h is not much less than r_1.

Problem 18. The viscosity function $\eta(\dot{\gamma})$ for steady unidirectional shear flow is to be determined experimentally for a polymer solution. The fluid flows through a circular capillary tube with internal diameter $d = 2$ mm. The reduction of the modified pressure ΔP along a length of $l = 100$ mm is measured. The Table. Problem 18 presents corresponding data for ΔP and the volume flow Q. The table is adapted from Bird et al. [3].

Use the method presented in Sect 5.5 and:

(a) Determine the shear stress at the tube wall: $\tau_o = -\Delta P\, d/4l$, and the parameter $\Gamma = 32Q/\pi d^3$. Draw the graph for $\log \tau_o$ versus $\log \Gamma$.
(b) Determine the parameter: $\bar{n} = \frac{d(\ln \tau_o)}{d(\ln \Gamma)} = \frac{d(\log \tau_o)}{d(\log \Gamma)}$. Compute:

$$\dot{\gamma}_o = \frac{3\bar{n} + 1}{4\bar{n}}, \quad \eta(\dot{\gamma}_o) = \frac{\tau_o}{\dot{\gamma}_o}$$

Draw the graph of $\log \eta$ versus $\log \dot{\gamma}$. Try to fit the experimental results to the viscosity function for the Carreau fluid model presented in Sect. 6.1

Table Problem 18

ΔP [mm H_2O]	Q [cm^3/s]
16.3	0.0157
40.8	0.0393
69.4	0.0785
108	0.157
173	0.393
240	0.785
306	1.57
398	3.93
490	7.85

Problem 19. Fig. 5.8 presents data from pressure measurements on the plate of a cone-and-plate viscometer. The plate has a radius of $R = 50$ mm. The fluid is a 2.5 % polyacrylamide solution. Let $\sigma_{rr}(R)$ be equal to the atmospheric pressure p_a. Determine the viscometric functions ψ_1, ψ_2, N_1, and N_2 for the fluid. See Fig. 5.9.

Problem 20. The pressure drop in a tube of length $L = 1$ m and diameter $d = 10$ mm is found to be: $\Delta P = -2.5$ kPa. for a test fluid. The fluid is modeled as Carreau fluid specified by the material parameters:

$$\eta_0 = 10.6 \text{ Ns/m}^2,\ \eta_\infty = 10^{-2} \text{ Ns/m}^2,\ \lambda = 8.04 \text{ s},\ n = 0.364.$$

Determine the volume flow Q from the Rabinowitsch-equation, Eq. (5.5.9):

$$Q = \frac{\pi d^3}{8\tau_o^3} \int_0^{\tau_o} \tau^2 \,\dot{\gamma}\, d\tau\,, \quad \tau_o = -\frac{\Delta P}{4L} d$$

Problem 21. A linearly viscoelastic fluid flows between two parallel plates. The distance between the plates is constant and equal to h. The fluid sticks to both plates. One of the plates can move with a velocity parallel the other plate, and this motion drives the flow, such that the flow is a simple shear flow with shear stress τ and shear rate $\dot{\gamma}$. The fluid has the relaxation function in shear $\beta(t)$.

(a) For $t < 0$ one of the plates moves with a constant velocity v_0. For $t > 0$ both plates are at rest. Derive the following expression for the shear stress.

$$\tau(t) = \frac{v_o}{h}\left(\int_0^\infty \beta(s)\, ds - H(t) \int_0^t \beta(s)\, ds\right)$$

(b) Determine the shear stress $\tau(t)$ for a Maxwell-fluid.

Problem 22. The annular space between two concentric cylindrical surfaces is filled with a generalized Newtonian fluid. The inner and outer radii of the annular space are r_1 and r_2. See Fig. Problem 13. The cylinder length is L. The inner cylindrical surface is fixed. The outer cylinder is subjected to an external constant torque M and can rotate. Neglect effects from the ends at $z = 0$ and $z = L$. The fluid sticks to both cylindrical walls. Assume steady, laminar flow with the velocity field given by:

$$v_\theta = v_\theta(R), \quad v_R = v_z = 0$$

The density of the fluid is ρ and the viscosity function is the power law:

$$\eta(\dot{\gamma}) = K\,\dot{\gamma}^{n-1}$$

(a) Determine the expression for the shear stress: $\tau_{R\theta}(R)$.
(b) Develop the following formula for the strain rate:

$$\dot{\gamma}_{R\theta} = \frac{dv_\theta}{dR} - \frac{v_\theta}{R} = R\frac{d}{dR}\left(\frac{v_\theta}{R}\right)$$

(c) Formulate the boundary conditions for the velocity field $v_\theta(R)$.
(d) Sketch the velocity field $v_\theta(R)$.

(e) Determine the velocity field $v_\theta(R)$.
(f) Find the expression for the angular velocity ω of the outer cylinder.

Problem 23. Determine the velocity field $v_\theta(R)$ in problem 22 when the situation is changed to: The inner cylinder is subjected to a constant torque and can rotate, while the outer cylinder is fixed.

Problem 24. A shear thinning fluid has steady flow through a circular pipe. The gradient $\partial P/dz$ in the axial direction of the modified pressure is constant. The fluid is modeled as a two-component-fluid: A central core of diameter d flows as a power law fluid with the viscosity function: $\eta(\dot{\gamma}) = K\dot{\gamma}^{n-1}$. In a thin layer of thickness $h << d$ between the core and the pipe wall the fluid is modeled as a Newtonian fluid with viscosity μ. The velocity $v_z(R)$ in the layer may be assumed to vary linearly with the radial distance R.

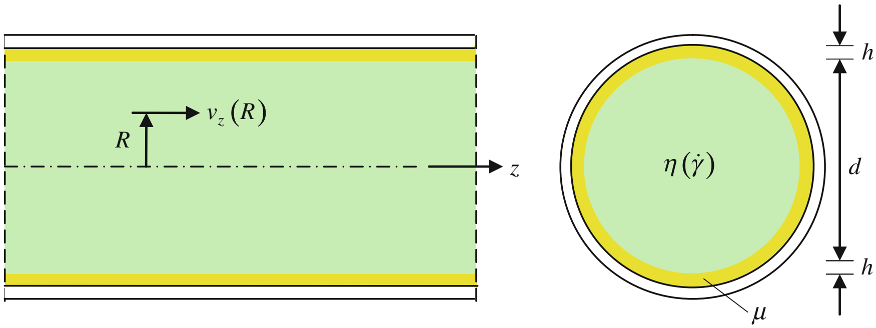

Fig. Problem 24

(a) Show that the shear stress: τ_{zR}in the fluid is everywhere given by: $\tau_{zR} = (R/2)dP/dz$.
(b) Formulate the boundary conditions for the velocity $v_z(R)$ at the wall and at the interface between the two fluid models.
(c) Determine the velocity $v_z(R)$.

Problem 25. The figure shows a rheometer for measuring the yield shear stress and the viscosity of a viscoplastic material. The cylinder is subjected to an external torque M and can rotate with a constant angular velocity ω. Assume that the fluid sticks to the rigid boundaries. A test fluid is to be modelled as an incompressible Bingham fluid with viscosity μ and yield shear stress τ_y. Just before the flow is initiated in the fluid the torque is M_y.

The velocity field in the test fluid is assumed as:

$$v_\theta = v(R), \quad v_R = v_z = 0 \quad \text{when } r \le R \le r + h$$
$$v_\theta = Rf(z), \quad v_R = v_z = 0$$
$$\text{when } R \le r + h, \quad 0 \le z \le a$$

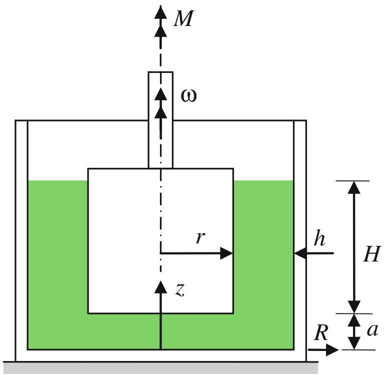

Fig. Problem 25

(a) Determine a formula for τ_y.
(b) Determine the strain rates $\dot{\gamma}_{R\theta}$ and $\dot{\gamma}_{\theta z}$, and the corresponding shear stresses for the assumed velocity field.
(c) Neglect all accelerations and body forces, and present the equations of motion in cylindrical coordinates and with the assumed velocity field. Formulate the boundary conditions for the velocity field.
(d) Use the equations of motion to show that $f(z) = \omega z/a$.
(e) Determine the region $r_o \leq R \leq r + h$where $v(R) = 0$, and sketch the velocity field.

Problem 26.

(a) Derive the formulas (8.7.4)
(b) Derive the formulas (8.7.15) and (8.7.16) from the formulas (8.7.14)

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