Fundamentals of Fluid Mechanics and Transport Phenomena

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Preface

The study of fluid mechanics and transfer phenomena in flows involves the association of difficulties which are encountered in different disciplines: thermodynamics, mechanics, thermal conduction, diffusion, chemical reactions, etc. This book is not intended to be an encyclopaedia, and we will thus not endeavour to cover all of the aforementioned disciplines in a detailed fashion. The main objective of the text is to present the study of the movement of fluids and the main consequences in terms of the transfer of mass and heat. The book is the result of many years of teaching and research, both theoretical and applied, in scientific domains which are often considered separately. In effect, the development of new disciplines which are at the same time specialized and universal was very much a characteristic of science in the 20th century. Thus, signal processing, system analysis, numerical analysis, etc. are all autonomous disciplines and indispensable means for students, engineers or researchers working in the domain of fluid mechanics and energetics. In the same way, various domains such as the design of chemical reactors, the study of the stars and meteorology require a solid knowledge of fluid mechanics in addition to that of their specific topics.

This book is primarily aimed at students, engineers and researchers in fluid mechanics and energetics. However, we feel that it can be useful for people working in other disciplines, even if the reading of some of the more theoretical and specialized chapters may be dispensable in this case. The science and technology of the first half of the 20th century was heavily rooted in classical mechanics, with concepts and methods which relied on algebra and differential and integral calculus, these terms being taken into account in the sense they were used at that time. Furthermore, scientific thought was fundamentally deterministic during this period, even if the existence of games of chance using mechanical devices (dice, roulette, etc.) seemed far from the philosophy of science or Cauchy’s theorem. Each time has its concepts, which are based on the current state of knowledge, and the science of fluid mechanics was reduced for the most part to semi-empirical engineering formulae and to particular analytical solutions. Between the 1920s and the 1950s, our ideas on boundary layers and hydrodynamic stability were progressively elucidated. Studies of turbulence, which began in the 1920s from a conceptual statistical point of view, have really only made further progress in the 1970s, with the writing of the balance equations using turbulence models with a physical basis. This progress remains quite modest, however, considering the immensity of the task which remains.

It should be noted that certain disciplines have seen a spectacular renewal since the 1970s for two main reasons: on the one hand, the development of information technology has provided formidable computation and experimental methods, and on the other hand, multidisciplinary problems have arisen from industrial necessities. Acoustics is a typical example: many problems of propagation had been solved in the 1950s-1960s and those which were not made only very slow progress. Physics focused on other fundamental, more promising sectors (semiconductors, properties of matter, etc.). However, in the face of a need to provide practical solutions to industrial problems (sound generated by fluid flow, the development of ultra-sound equipment, etc.), acoustics became an engineering science in the 1970s. Acoustics is indeed a domain of compressible fluid mechanics and it will constitute an integral part of our treatment of the subject.

Parallel to this, systems became an object of study in themselves (automatic control) and the possibilities of study and understanding of the complexity progressed (signal processing, modeling of systems with large numbers of variables, etc.). Determinism itself is now seen in a more modest light: it suffices to remember the variable level of our ambitions with regard to meteorological prediction in the last 30 years to see that we have not yet arrived at a point where we have a definite set of concepts. Meteorological phenomena are largely governed by fluid mechanics.

The conception of this book results from the preceding observations. The author refuses to get into the argument which consists of saying that the time of analytical solutions has passed and that numerical simulation will solve all our problems. The reality is clearly more subtle than this: analytical solution in the broad sense, that is, the obtaining of results derived from reasoning and mathematical concepts, is the basis of physical concepts. Computations performed by computers by themselves cannot provide any more insight than an experiment, although both must be performed with great care. The state of knowledge and of understanding of mechanisms varies depending on the domain studied. In particular, the science of turbulence is still at a somewhat embryonic stage, and the mystery of turbulent solutions of the Navier-Stokes equations is far from being thoroughly cleared up.

We are still at the stage of Galileo who attempted to understand mechanics without the ideas of differential calculus. Nobody can today say precisely what are the difficulties to be solved, and the time which will be required for their resolution (10 years, a century or 10 centuries). We will therefore present the state of our knowledge in the current scientific context by also considering some of the accompanying disciplines (thermodynamics, ideas related to partial differential equations, signal processing, system analysis) which are directly useful to the concepts, modeling, experiments and applications in fluid mechanics and energetics of flows. We will not cover specific combustion phenomena, limiting ourselves to a few simplified cases of physico-chemical reactions.

This book covers the necessary fundamentals for the study and understanding of the specific concepts and general properties of flows: the establishment and discussion of the balance equations of extensive quantities in fluid motions, the transport of these quantities by convection, wave-propagation or diffusion. These physical concepts are issued from the comprehension of theoretical notions associated with equations, such as characteristic curves or surfaces, perturbation methods, modal developments (Fourier series, etc.) and integral transforms, model reduction, etc. These mathematical aspects are either consequences of properties of partial differential equations or derived from other disciplines such as signal processing and system analysis, whose impact is important in every scientific or technological domain. They are discussed and illustrated by some elementary problems of fluid mechanics and thermal conduction, including measurement methods and experimental data processing This book is an introduction to the study of more specialized topics of fluid flow and transfer phenomena encountered in different domains of application: incompressible or compressible flow, dynamic and thermal boundary layers, natural or mixed convection, 3D boundary layers, physico-chemical reactions in flows, acoustics in flows, aerodynamic sound, thermoacoustics, etc.

Chapter 1 is devoted to a synthetic presentation of thermodynamics. After recalling the basics of the representation of material systems, thermostatics is covered in an axiomatic fashion which avoids the use of differential formulations and which allows for a simplified presentation of classical results. Taking entropy dynamics as a starting point, the thermodynamics of non-equilibrium states is then discussed using simple examples with phenomenological laws of linear thermodynamics.

The continuous medium at rest is obtained by taking the limit of discrete systems in Chapter 2. The exchange of extensive quantities is modeled by means of flux densities, and irreversible thermodynamics leads to the diffusion equations. Some reminders of fluid statics are given. We then discuss the difficulties specific to the diffusion of matter.

The association of mechanical phenomena with thermodynamics is briefly developed in Chapter 3 along with the formalism used for the description of the motion of continuous media. The elementary properties of viscosity are then discussed.

Chapter 4 is dedicated to the writing of the general equations of the dynamics of fluid and transfer. The integration of local equations in a domain enables the separation of sources and fluxes of extensive quantities, these fluxes being transfer phenomena involving definition of input-output mechanisms for that domain, considered as a system. The energy equation explicitly expresses the interactions between thermodynamics and the movement of matter. The main usual boundary conditions and similarity and its consequences are then discussed.

Chapter 5 discusses the classification of partial differential equations in fluid mechanics. The mathematical aspects at the basis of physical concepts are well understood, but unfortunately rarely taught. These are very important, both for the numerical solution of equations and for the understanding of physical phenomena. We will present them here without providing any thorough demonstrations. The reader who struggles with this chapter should nonetheless try to assimilate its content while leaving aside the details of certain calculations.

Chapter 6 is dedicated in the main to the influence of diffusion in the convection of linear or angular momentum. It firstly covers vortex dynamics, the transposition to continuous media of concepts used in solid body rotation. Vorticity often results from transitional processes which may be more or less viscous, but its transport is very often governed by the equations for an inviscid fluid. Lagrange's theorem introduces the idea of conservation of circulation of velocity which allows the rotation to be treated as a frozen material field. Elementary solutions of the 2D incompressible potential flows are quickly discussed. We then look at the quasi-1D approximation, which is particularly important in fluid mechanics, either for pipes or for flows in the vicinity of walls when a non-dimensional quantity becomes large. This last circumstance corresponds to a singular perturbation problem in the form of a boundary layer, which corresponds to the effects of viscous diffusion from the walls. The discussion of the boundary-layer equations reveals the separation mechanisms which are associated with the non-linear terms in steady flow equations.

The measurement of flow and transfer phenomena presents difficulties which are outlined in Chapter 7. The recent evolution of techniques based on the digitization of measurements, signal processing, analysis and reduction of models are naturally suited to applications in fluid mechanics and energetics. These methods have led to a renewal of progress in disciplines where unsteady phenomena are encountered, and in particular in the study of acoustic phenomena and turbulent flows.

Improvements in computing have of course also led to considerable progress in the modeling of phenomena. The use of these methods requires specialized techniques whose treatment is beyond the scope of this book. The elements of signal processing and system analysis which we provide are only intended to alert the reader to the possibilities and utility of these methods, but also to show their limits. The idea that computers will allow the resolution of all our problems remains too ubiquitous. *Computers only provide a tool to help us find the solutions we seek.* These recent methods, signal processing or system analysis, are also useful for the identification of physical concepts associated with phenomena and the representation of solutions.

In Chapter 7, we also indicate in a synthetic manner the essential ideas necessary for measurement and signal processing procedures which are most useful in the domains studied. The possibility of large computations in modeling and experimental data processing leads us to evoke the idea of conditioning of linear systems, which is a generalization of elementary calculations of errors and uncertainties.

Chapter 8 is dedicated to modeling which provides a general context for the study of the evolution of physical systems. However, automatic control is reasoning in a general way on models without taking account of the laws of thermodynamics. These are essential for the disciplines studied in this book. We will present a few points of view and methods developed in automatic control, directly applied to the balance equations of basic problems of thermal conduction. The approximation procedures for the balance equations are far from being equivalent depending on the way in which we proceed. In order to simplify the presentation and to clearly separate the difficulties, we will mainly limit ourselves here to the state representation which is derived from thermodynamic modeling, leaving aside models derived from the approximation of solutions which do not exactly satisfy the balance equations.

NOTE.

We have chosen to respect the usual notation of physical quantities in each discussed scientific domain, while trying to have consistent notations whenever possible.

At the same time, the notations for derivatives are different, depending on the domain covered (thermodynamics, mechanics or more mathematical developments) and the size of equations. They all are usual and well known:

– For functions $y(x)$ of one variable, they are marked $y'(x)$, $y''(x)$, $y'''(x)$, $y''''(x)$,..., $y^{(n)}(x)$.

– When discussing mechanical questions, the two first temporal derivatives of $x(t)$ are written with dots: $\dot{x}(t)$ and $\ddot{x}(t)$.

– The symbol $\dfrac{d}{dt}$ is used only for material (Lagrangian) derivatives, which are indeed derivatives with respect to time of compound functions in Euler variables; this is equivalent to the other usual notation $\dfrac{D}{Dt}$.

– For functions $f(x, y)$ of several variables, the two following notations are used according circumstances: either with symbol ∂ $(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^3 f}{\partial x^2 \partial y}$,...) or with indices marking the variables with respect of which derivations are performed: f_x , f_y, f_{xy} , f_{xxy}.

Integrals are always indicated by a simple integration sign, as the nature of this (single, double, triple, etc.) should be clear from the integration domain indicated and the differential element.

When tensor notation is used, vectors or matrices are denoted using upper case letters, their components being written in lower case letters. The convention of summation over repeated indices (Einstein's convention) will systematically be used.

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Chapter 1

Thermodynamics of Discrete Systems

The general objective of thermodynamics is to describe the properties of matter. After recalling the representational bases of material systems, thermostatics is dealt with by postulating the existence of a general equation of state which relates the extensive quantities. In this way we can forgo the need to delve into principles related to differential forms, and thereby simplify the presentation of traditional results. Then the thermodynamics of out of equilibrium systems are considered in terms of entropy dynamics, and discussed using simple examples. Finally, the phenomenological laws of linear thermodynamics are then considered.

1.1. The representational bases of a material system

1.1.1. *Introduction*

1.1.1.1. *Geometric Euclidean space and physical quantities*

The object of the physical sciences is the study of matter, for which the formulation of physical laws is necessary. However prior to the formulation of any such laws it is clearly necessary to characterize matter in terms of the various physical quantities which we can directly or indirectly measure. Matter is present all around us, and in a first instance we will limit ourselves to considering it in a static way, at a given instant which we can identify (this supposes a minimal definition of time); we perform geometric measurements in a 3D Cartesian coordinates system in order to identify the position and/or dimension of material elements. Measuring length presents no particular difficulty, excepting the choice of units. We will observe material elements in a geometric Euclidean space.

The geometric description of space is independent of the presence of matter; in other words the metric tensor does not depend on any physical quantity. This is not true for certain astrophysical phenomena which require us to place ourselves in the context of general relativity where geometric properties of space are no longer independent of the presence of matter. Simplistically put, the length of a meter depends on the mass found in its vicinity, which considerably complicates matters. In the following we exclude such phenomena, as they only become important at scales which greatly exceed those of our terrestrial physics.

We thus postulate *(Axiom 1) the existence of a geometric space whose structure is independent of the properties of matter* and the associated physical phenomena (gravitation, force fields, etc.).

We also admit *(Axiom 2) that this space is homogenous and isotropic*, which leads us to a traditional geometric Euclidean description of space $\mathbf{R}^3$ with its associated notions of length, surface and volume, whose scalar values are independent of the particular geometric frame of reference we choose to consider. This property of homogenity and isotropy will have important consequences for the expression of physical laws, which must not favor any given point or physical spatial direction. In particular, physical laws should neither favor any particular point in the universe, nor change as a result of a change in reference frame.

Finally, we suppose *(Axiom 3)* that matter can be characterized by *physical quantities which are measurable at each instant in time*, and not by mathematical entities (wavefunctions etc.) which allow, via mathematical operations, access to information of a probabilistic kind with regard to a physical quantity. This hypothesis of the possibility of directly measuring physical quantities supposes that the measure does not change the physical quantities of the material element considered. We therefore exclude microscopic phenomena relevant to quantum mechanics from our field of study, and we suppose the smallest material elements studied to contain a number of atoms or molecules sufficient for the neglect of statistical microscopic fluctuations to be justified.

1.1.1.2. *The existence of isolated systems and the definition of time*

The study of physical phenomena presupposes their reproducibility; the same effects should be observed under identical conditions. The establishment of physical laws thus supposes the definition of a time with the property of homogenity: in particular, quantifiable and reproducible observations of the evolution of a given material system must be possible.

The definition of time should thus be appropriately chosen. Previously associated with the length of the day, the definition of time has varied considerably between different individuals and epochs. For example, during the Roman period the lengths of the day and the night were respectively divided into seven and four parts, the Babylonians 2,000 years beforehand divided the day and the night each into 12 hours, which were clearly of unequal duration and varied according to the seasons. The Chinese and the Japanese divided each of the two cycles, from dawn to dusk and from dusk to dawn, into six periods. Japan only adopted the occidental system in 1873, but this did not prevent Japanese clockmakers from making mechanical clocks as early as the 17th century, these having quite complex mechanisms in order to accommodate the variable length of their hour.

The definition and measurement of time are thus not automatic operations for human beings. The relatively old notion of regular time (homogenous in the physical sense) is related to the use of indefinitely reproducible phenomena; this notion dates from the end of antiquity, the early Middle Ages and the invention of the clock (clepsydras, mechanical clocks, hourglass).

We will thus postulate *(Axiom 4)* that *physical phenomena are reproducible,* regardless of when an experiment is performed. Any evolutionary phenomenon which is considered reproducible will allow a time unit to be defined. A temporal dimension can be constructed simply by virtue of the reproducibility of a phenomenon, which amounts to admitting that time is homogenous, i.e. no instant in the universe is given any special privilege. This homogenity of time does not really exist in cosmological problems, and in particular during the time of the initial big bang. We exclude these kinds of problem.

Having long been attached to the average duration of a solar day, the definition of time is now effected using the vibration frequency of an atom of caesium 123 under the most stable conditions possible (at very low temperature).

1.1.1.3. *Causality and irreversibility*

We now dispose of a space-time coordinates system comprising three space dimensions and one time dimension. However, in contrast with geometric space, time is not isotropic. In effect, the definition of entropy (section 1.2.2.4) shows that an irreversible evolution exists in the universe with which we can associate a time variable (or one related to the age of the universe) in an attempt to characterize it. This irreversibility is explained by statistical mechanics whereby matter always tends to states in which it is maximally mixed: gas molecules in a volume will always be evenly dispersed over the volume. This is the most probable state in which the molecules will be found; while the probability of finding all of the molecules confined to the left half of the volume is not strictly zero, this situation is never observed.

The age of the universe is thus associated with a measure of its entropy on a very large scale (the universe or at least the earth). However, a time characterized by this scale has no guarantee of being homogenous. This “age” of the universe does not give us a useful indication of what time to use, and we will content ourselves with the time previously defined from the notion of reproducibility. The notion of entropy (or of the ageing of the universe) shows that time has a considerable anisotropy, manifest in *the distinction between the past, the present and the future.* The equations translating the physical laws and their consequences should not violate this anisotropy, the effect of which can be immediately seen if we change the direction of time by letting $t' = -t$.

Let us consider an isolated mechanical oscillator with friction, which can be described by the equation:

$$m\ddot{x}(t) + f\dot{x}(t) + kx(t) = 0 \qquad [1.1]$$

whose oscillatory solution takes the form $A\exp\left(-f\ t/2m\right)\cos\left(\varpi t + \varphi\right)$.

By multiplying equation [1.1] by $\dot{x}(t)$ and integrating with respect to time between 0 and T, the total variation of mechanical energy ΔE_m between these instants is:

$$\Delta E_m = \left[\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right]_0^T = -\int_0^T f\ \dot{x}^2 dt$$

The absolute value of this variation ΔE_m is always negative and increasing for a positive value of the friction coefficient. The quantity $\int_0^T f\ \dot{x}^2 dt$ is known as the *dissipation function* of the system.

Changing the direction of time would be equivalent to changing the term $f\dot{x}(t)$ to $-f\dot{x}(t)$, which implies a negative friction coefficient f leading to the solution $A\exp\left(+f\ t'/2m\right)\cos\left(\varpi t + \varphi\right)$ and to an increase in mechanical energy as a time function. This is impossible with an isolated oscillator and could only be made possible by the intervention of an exterior energy source. The preceding equation is clearly unstable in the sense that its solutions diverge analogously to the instabilities encountered in the local study of equilibrium.

Let us take as an example three equations representative of constant coefficient, second order partial differential equations (see Chapter 5):

- Laplace equation: $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial t^2}=0;$
- wave equation: $\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial t^2}=0;$
- heat equation: $\frac{\partial^2 f}{\partial x^2}-\frac{\partial f}{\partial t}=0.$

The general solution of Laplace's equation (which is elliptic) at a point requires that conditions be known at all points lying on a curve surrounding this point (Dirichlet condition). All points at the frontier of the domain exert an influence on the solution at a point (*x*,*t*). The result is that no physical phenomenon can be represented by Laplace's equation if time is chosen as a variable, since the solution in *t* would depend on smaller (earlier) and larger (later) values of the time variable.

The wave equation (which is hyperbolic) on the contrary is compatible with the definition of time. Its general solution:

$$f(x,t)=\phi\ (x+t)+\psi\ (x-t)$$

represents two waves which propagate along the x-axis with velocities +1 and –1. The value at a point *x* and instant *t* depends on what happens to each of the said waves to the left and the right of *x*, and before their arrival at time *t*. The wave equation is thus compatible with the non-influence of the future on the present.

The heat equation (which is parabolic) is also compatible with the non-influence of the future on the present, as we will see for heat conduction problems, since the initial conditions (or values from the past) suffice for a determination of the solution at any later time.

Another remark can be made here regarding the inversion of the direction of time. By replacing *t* with – *t*', we see that the wave equation remains unchanged, while the heat equation becomes:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial t'}=0$$

We will see similar behavior for the complete solution of the heat equation in a wall (Chapter 8) in which the inversion of the direction of time results in a change of a sum of temporally decaying exponential terms to a sum of temporally increasing exponential terms. Changing the direction of time in the heat equation leads to a physically inadmissible equation.

The preceding phenomena can be interpreted in a number of ways:

1) In terms of energy dissipation and of the creation of entropy

The wave equation represents a frictionless mechanical phenomenon, there is no creation of entropy over time; we have a reversible phenomenon and so an inversion of the direction of time is not incompatible with the laws of the universe. We should note however that the wave equation is only valid for relatively short times, for which the inevitable friction is not to have an influence. Acoustic waves are finally damped by diverse frictional forces after they have covered a very large distance; light waves are finally absorbed by matter in an irreversible process (the Joule effect) etc. Energy transfer creates entropy and is therefore compatible with the evolution of the universe.

2) In terms of information loss

The wave equation was earlier interpreted as a transmission of a signal by pure propagation. There is no loss of information during the transmission. The introduction of dissipation (creation of entropy) leads to the telegrapher's equation, which is no longer invariant under a change in the direction of time, and thus involves an attenuation of the signals during transmission, and then a subsequent loss of information.

The heat equation translates a smoothing of temperature distributions, which may initially be complex, to a more uniform field. The final state is often a constant temperature which has no memory of its initial distribution. We note again that an inversion of the time direction in heat diffusion problems does not allow for a retrieval of the information which has been lost. The same goes for an oscillator with friction, whose final state of rest precludes any knowledge of the initial conditions.

The notions of past and future, with respect to an event, introduce a fundamental asymmetry; the present does not depend on the future. This has certain consequences, both in the application of certain mathematical transformations (Fourier for example) on temporal signals, and in flow problems where the distinction between upstream and downstream is of the same nature as that between the past and the future.

1.1.1.4. *Causality and determinism*

The question of cause and effect is a very old philosophical problem (Aristotle, the scholastic philosophers of the middle-ages, Descartes, Leibniz, Spinoza, Hume, Kant, Schopenhauer, Bernard, etc.). We will not go into the complex philosophical distinctions related to causes (adequate, inadequate, efficient, final, formal, material, primary, secondary etc.). An effect is the result of and is produced by an efficient cause.

Kant upholds that the causality relation is “absolutely general and even necessary”. The general principle of causality is even more clear in determinism, which holds that all events can be rationally predicted, with a desired degree of precision, provided that past events and all of the laws of nature are known with sufficient precision. Such absolute and universal determinism is associated with a conception of a universe dominated by laws of celestial mechanics (Laplace). In other words, the same causes produce the same effects, and so our capacity to predict depends only on our scientific knowledge. Of course, quantum mechanics has brought this vision of things into question, but not on the scale of the phenomena studied here.

However, the question of determinism is not as simple as it might seem, in particular in situations where unstable phenomena intervene, or where chance plays a central role (chaos). Examples of such situations are usual in mechanical devices used for games of chance (dice, roulette, etc.) or in fluid mechanics whose equations have unstable solutions going through unpredictable evolutions in which flows are fluctuating in a chaotic way. This is the phenomenon of turbulence encountered in most practical flows; for example, atmospheric flows are results of such instabilities and then weather prediction is fundamentally impossible beyond a few days. Nevertheless, a statistical treatment of these turbulent flows leads to a more global kind of determinism ([LES 98]).

It is useful to note at this point that the conditions for prediction can be defined mathematically via theorems which treat of the existence of unique solutions for differential equations given a suitable set of initial conditions. The Cauchy-Lipschitz theorem is the best known, and deals with differential equations with real variables (x,y) of the form:

$$dy/dx = f(x,y)$$

The function $f(x,y)$ is only required to verify a Lipschitz condition[1]. This theorem establishes the existence of a unique solution $y = \varphi(x)$ which verifies the initial condition $y_0 = \varphi(x_0)$. This solution is continuous over the interval $(x_0, x_0 + h)$, where h is characterized by the interval of definition for x and an upper bound of $|f|$ in the rectangle considered. This theorem can be extended to systems of differential equations with the same kinds of conditions.

A similar theorem (Cauchy-Kovalevskaïa), but with stricter analycity conditions of the function $f(x,y)$ in the neighborhood of the point (x_0, y_0) (functions which can

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[1] I.e.: $|f(x,y) - f(x,y')| < A|y - y'|$, condition in which (x,y) and (x,y') are arbitrarily chosen in a rectangle where $f(x,y)$ is supposed to be continuous, A being a positive constant.

be developed in power series), leads to a unique analytic solution $y = \varphi(x)$ in the neighborhood of the point (x_0, y_0) with the initial condition $y_0 = \varphi(x_0)$. These results can be extended to systems of differential equations, linear partial differential equations, etc.

Cauchy's theorem thus translates a form of determinism, since given a cause (the initial condition $y_0 = \varphi\,(x_0)$), a unique solution $y = \varphi(x)$ exits. However, we see that there are certain limitations, in particular with the Cauchy-Kovalevskaïa theorem which imposes analyticity conditions, the physical realization of which has no reason to be assured for the function *f(x,y)* or any other perturbation which we may add in order to test the stability of the system

In all causal situations, the preceding Cauchy theorems lead to results of a local nature, that is to say over a short period of time, considering the variable x to represent time. In the middle to long term, numerous "mathematical accidents" may occur. The uniqueness of a local solution is not in contradiction with the impossibility of prediction of the evolution of this solution on a long enough period of time due to a chaotic behavior ([BER 84], [ORS 77]).

In conclusion, *the notions of determinism and causality are far from being universally applicable* in the domains which we will cover.

1.1.2. *Systems analysis and thermodynamics*

1.1.2.1. *Introduction*

The analysis of systems is a discipline which consists of constructing a model or a representation of a system characterized by observations and measurements, with a view to predicting the behavior of this system at a later stage, under conditions which may be different from those first encountered. We also attempt to contrive means of manipulating the system in order to cause it to evolve in a manner which we specify *a priori*. We thus enter into the domain of command and control, since it is now necessary to verify that the results are those sought, and if not, to perform the necessary corrections in order to obtain the desired results.

The characterization of a material device can thus vary depending on the objective which we seek to achieve. The device may be static and we may only be interested in its "state"; it may be dynamic, in which case it evolves as a time function.

In general, the objective of a system is to transform some input quantities *u(t)*, known thanks to some measurement (which provides the input variables), into some output quantities $y(t)$ which are also obtained via a measurement (output variables). For example, the input variables of a heating system are the available heating power, the desired temperature, and the output (controlled) variables are the power consumed and the temperature observed in the space to be heated. We also dispose of a command variable for the heating system. The input variables are thus the given conditions, while the output variables are the quantities obtained. Observations can be made for the time evolution of the various quantities in a continuous or sampled manner.

1.1.2.2. *External description (black box)*

The description of a system may be external, that is we satisfy ourselves to simply measuring the inputs and outputs of the system, the system itself remaining a "black box". We thereby ignore what goes on inside the system. As the system operates we measure $y(t)$ which depends on the input $u(t)$ and time t. Often, the system "has a history", and the output $y(t)$ cannot be represented as a function of the only two variables $u(t)$ and t.

The external description of a state is thus generally not sufficient. The difference between a raw egg and a hard-boiled egg is not visible to external measurements (size, mass, color, etc.); it is a result of internal variables (chemical composition) which cannot be measured directly, but which can be known indirectly (the rotational movement of a mass of solid and a mass of liquid are not the same), or by virtue of some previous known history (the egg was boiled).

From a mathematical point of view, the black box description corresponds to a direct relationship between the inputs and the outputs, in other words to calculations defined *a priori* on the input quantities. As long as the dynamic system is invariant in time, the formalism of transfer functions (or of impulse responses) is largely used. It is nonetheless necessary to pay close attention to questions of causality when using such approaches (see Chapter 7; for more detailed information, the reader is referred to works which deal with signal processing and automatic control theory).

1.1.2.3. *Internal description (state variable approach)*

In place of a "black box" description, we substitute a description of the internal state of the system using a number of state variables $X(t)$ (state vector). These characterize the "state" of the system, and when combined with a knowledge of the system inputs, knowledge of the system outputs can be obtained at every instant by means of evolution equations (ordinary or partial differential equations) which describe the conditions on geometric boundary of the system and the initial state.

The simplest dynamic systems are represented by constant-coefficient linear differential equations; these are known as *invariant linear systems* as their response does not depend on the initial instant chosen for the study of their evolution:

$$\frac{dX}{dt} = AX + BU(t); \qquad t = t_0 : X(t_0) = X_0$$

In order to identify a state representation, we can use purely mathematical considerations which are essentially based on the nature of the response of the system (system outputs) to a specific excitation (Dirac impulse or step function). If it is possible to identify the existence of different time constants, for example τ_1 and τ_2, then the behavior of the system can be considered to be second order, which implies the need for a description based on two state variables. State variables identified via an empirical modeling approach will not necessarily lend themselves to a clear physical interpretation. They are merely indicators which are linked in some way to the dominant physical quantities of the system. We will come back to this point when we discuss model reduction methods (Chapter 8).

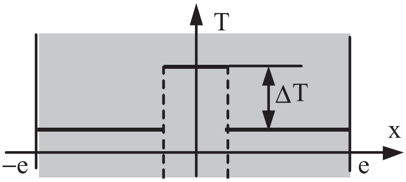

Figure 1.1. *Temperature pulse function unrealizable from imposed conditions on the walls (identical temperature on the two walls)*

Finally, we may wish to manipulate certain system variables in order to achieve a given desired state. From a mathematical point of view, boundary conditions must of course exist which allow a solution of the local equations (partial differential equations) corresponding to the evolution of the physical system towards such a final state. This condition is not always satisfied, as shown in Figure 1.1. In this example, the physical system considered is not controllable.

1.1.2.4. *Thermodynamics and mechanics*

This chapter and Chapter 2 are dedicated to a presentation of those basic physical laws which are valid regardless of the particular properties of the material elements considered. These basic laws constitute *thermodynamics* and *mechanics*; they need to be completed by means of other particular laws which may play a role in the behavior of the material elements, associated for example with physics (state equations of compressible fluids), chemistry, electricity, magnetism, electromagnetism, or any combination of these disciplines (laser-matter interactions, plasmas, chemical reactions or electrolysis in flows, etc.).

The laws of thermodynamics derive from the laws of mechanics applied to ensembles comprising a very large number n of molecules (*statistical mechanics*). The properties resulting from interactions between these n molecules cannot be exactly established for a variety of reasons (residual quantum effects, computations rendered impossible for very large numbers of particles, etc.). We therefore need to complete our microscopic mechanical models (kinetic theory of gas, molecular theory of liquids) by means of additional statistical axioms.

Thermostatics provides interpretations of physical quantities using the notion of balance via the intermediary of extensive quantities. This is the equivalent of imposing conservation principles for certain quantities, whose creation, disappearance or variation is not spontaneous, but which is associated with a clear cause that results in the transformation or displacement of the quantity considered. This static study of the properties of material systems is firstly made in a reference frame in which the material does not move, or at least under conditions such that the effects of movement have no effect on this material.

When considering balances, a knowledge of time only serves to localize various instants, while its definition is not important due to the infinitely slow nature of thermostatic transformations. On the contrary, the definition of time in thermodynamics is of great importance for the study and the prediction of the velocity of a system's temporal evolution. On the other hand, the equations of thermodynamics and its related disciplines must be associated with boundary and initial conditions which allow solutions that are actually observed in reality.

1.1.3. *The notion of state*

In thermodynamics, a state is a set of material elements which have well-defined properties. In order to characterize the state (a) of this ensemble, physical quantities G_i must be defined which can be measured (measurements g_i) and which allow us to distinguish between these and other material elements, or the same elements at another instant, after a transformation. From a mathematical point of view, a state is thus constituted by an ensemble of variables g_i which characterize the material contained in some entity or geometric domain. States thus defined obey the usual rules of the set theory ([GIL 64], [BOC 92]). We often refer to this material as being in state (a). It is clear that once defined as being in a given single state, the notion of a system does not supply any additional information with respect to the notion of state. The state of a system may be more or less complex and its description may require a more or less large number of variables, depending on the case considered.

As an example, let us consider *2n* contiguously arranged plates of a homogenous material (Figure 1.2a) distributed in three separate blocks by two thin thermally insulated layers P_1 and P_2. Suppose that the notion of temperature is known (for this example); half of these plates are at a temperature T_i which is greater than the temperature of the other half (Figure 1.2a). The description of this initial state thus requires that *2n* temperatures be given. Let us now cause this state to evolve, under the constraints imposed by the thermally insulated lateral faces. These *2n* variables are not necessarily required; the walls P_1 and P_2 play the role of a strong thermal resistance, the blocks of plates have an approximately uniform temperature at each instant (Figure 1.2b); these three temperatures suffice for a description of the state of the system and its subsequent temporal evolution. After a sufficiently long time, the state is at a uniform temperature $T_f = \frac{1}{2n}\sum_{i=1}^{2n} T_i$. This final state, which is described by a single variable is clearly in a state of equilibrium.

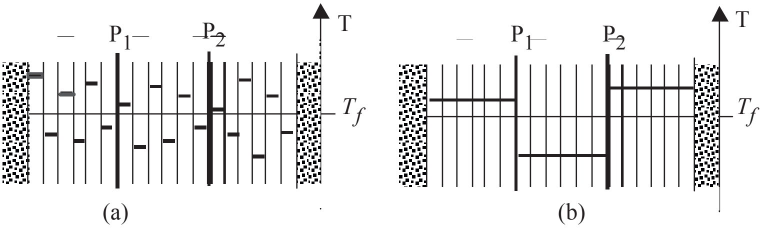

Figure 1.2. *(a) System with 2n variables; (b) system with 3 variables*

The general problem of describing a state comes down to finding the necessary variables. From the preceding example we see that the number of necessary variables depends largely on the physical situation we wish to describe. *The more complex the system considered, the greater the number of variables required.* We will frequently come back to this point, emphasizing it with respect to the specific objectives.

A state of equilibrium is in fact a succession of states for which all of the variables that constitute it conserve a constant value, physical exchanges with the exterior having ceased.

1.1.4. *Processes and systems*

1.1.4.1. *Definition of a process*

Certain authors define a process (a,b) as a pair of states: initial (a) and final (b). They are thereby led to distinguish between states which are possible and those which are not. Insofar as we limit ourselves to only consider processes which are truly observed (physical processes), the discussion of an axiomatization concerning impossible processes, being ill-defined, is beyond the scope and objectives of this book.

From a physical perspective, this means that a process can only be defined if the initial and boundary conditions are entirely determined during the process. It makes no sense to speak of a process which allows us to pass from a state (a) to state (b) unless the external conditions which constrain that process are specified. This is no longer a mathematical question, but rather a problem related to a determinism which amounts to admitting that an initial state (a), well-defined and always subjected to the same constraints, will always lead to the same final state (b). It will always be possible to relate two given states, under the condition that, on the one hand, exchanges with the exterior furnish the necessary physical quantities, and on the other, that the internal system processes which redistribute these quantities allow the desired distribution of these system quantities to be achieved. For example, it is not possible to realize a state consisting of a given mass whose temperature distribution comprises a central peak (Figure 1.1) by means of an action at the exterior walls. The necessary energy must be directly supplied to the central zone, which must be insulated from the adjacent regions.

By definition, a *process is a series of states.* This mathematical definition only has physical relevance for processes representative of real evolutions. While not precluding a choice of states with no link (a rabbit, a carrot, etc.), the obtention of physical evolution laws for matter implies a “certain continuity of content” for this ensemble of states. The same goes for all practical problems. A process is therefore a succession of states which must be uniquely defined. Apart from some exceptions (shocks), we will only consider processes comprising a continuous series of states, described by variables which must be continuous functions of time. We will however allow situations with discontinuities (shocks, shockwaves, deflagration) which momentarily violate this continuity condition.

1.1.4.2. *The notion of a system*

The notion of a system is a relatively vague one; it is in fact included in that of a process: a system is *an entity which we consider during a process.* As our considerations often take a differential form, the system is the principal part (zero order) on which we perform differential balances. The notion of a system is not clearly identified if we do not entirely state the conditions of the process under study.

Take the following examples:

1) The matter studied remains enclosed in a fixed volume (a cylindrical calorimetric bomb, for example). The observation domains *D* and *D′* at times *t* and *t′* are contained in the interior of the cavity *C* (Figure 1.3a). The presence of the barrier constituted by the rigid wall leads to the matter inside C being constrained to remain within the system (Figure 1.3a). This is a *closed system.*

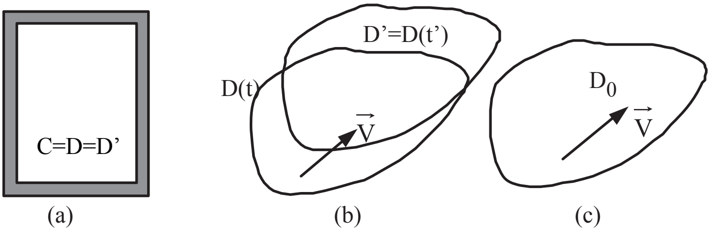

Figure 1.3. *(a) and (b): closed systems; (c) open system*

2) Now consider the case where matter is caused to move at a velocity $\vec{V}$ with respect to the used reference. The observation and description of this matter in movement can be performed:

– either, by following the matter in its movement, in which case the observation domain *D(t)* is displaced in time (Figure 1.3b); the matter contained within the domain *D(t)* constitutes a *closed system* in the sense just defined;

– or, by considering a fixed domain D_0 in which the matter is continually renewed; the ensemble of states contained within the fixed geometric domain is qualified as an *open system,* in other words a system which exchanges matter with the exterior (Figure 1.3c).

In Chapter 3 we will encounter these different ways of describing the movement of matter in the form of substantial (Lagrangian variables) and spatial (Eulerian variables) descriptions.

By definition, we will say that a process describing the evolution of material elements which are identified, and remain unchanged, operates on a *closed system,* in other words a system which does not exchange (does not provide or receive) matter with the exterior of the system. We can also use the denomination *material system.*

On the other hand, a process (a series of states) during which some matter passes from the inside of the domain to its outside corresponds to an *open system.*

1.1.4.3. *Types of processes and states*

We normally define the terminology of processes and states in the following way:

– a *natural process* is undergone by an isolated system which is not subjected to any external action;

– a *reversible process* is a process for which the direction of time can be changed. It is of course a process in which no entropy is created. Such processes occur over infinitely slow transformations;

– a *quasi-static* process is a succession of close infinitely equilibrium states;

– a *possible process* is a process where the constraints placed on entropy are obeyed. In the opposite case we speak of an *impossible process*;

– a *state of equilibrium* is in fact the result of a succession of states whose variables remain constant, any exchanges of its physical properties with the exterior having ceased.

1.1.4.4. *Enclosures and walls*

A *diathermic wall* is a wall which is permeable to heat and to external sources of entropy.

An *adiabatic wall* is impermeable to heat and does not allow the passage of entropy. It thermally insulates the system from the exterior.

1.2. Axioms of thermostatics

1.2.1. *Introduction*

The traditional presentation of thermodynamics usually begins with a direct definition of the various quantities (force, pressure, etc.) which are then used in the subsequent definition of elementary work and heat. The first and second principles (conservation of energy and entropy respectively) are then stated, to which further laws are then added (conservation of mass, chemical species, etc.). This all leads to the differential form of energy being written as the differential of a function E, energy. The result is a lack of coherence well known by students of thermodynamics. This situation can be avoided by means of a more structured presentation concerning the extensive quantities, among which energy plays a particular role in physics, whereas entropy is the basis of all considerations pertaining to irreversible thermodynamics. We will also more clearly outline the fact that the more complex and irreversible the evolution considered, the greater the number of variables required.

1.2.2. *Extensive quantities*

1.2.2.1. *Definition of extensive quantities*

Among the quantities used for the description of a state, we postulate (basic principle) that for every system S there exists an ensemble of *n extensive quantities* whose measure is proportional to the extension of the system, and which are always defined *regardless of the state considered.*

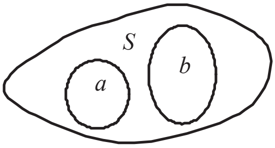

Figure 1.4. *Disjoint states*

Consider two disjoint states, i.e. states which have no matter in common; the extensive quantity X_i associated with the ensemble of the two states (a) and (b) is equal to the sum of the quantities corresponding to each state:

$$X_i(a \cup b) = X_i(a) + X_i(b)$$

If the sub-ensembles corresponding to (a) and (b) are not disjoint, we clearly have:

$$X_i(a \cup b) = X_i(a) + X_i(b) - X_i(a \cap b)$$

This definition only concerns the description of a state (a collection of matter) at a given instant. Under no circumstances does it imply the same property for two separated sub-systems (a) and (b) which we bring together in an externally applied field (force field, electromagnetic field, etc.), as this would constitute a process (series of states).

The index i takes on small integer values as these quantities are generally small in number for a state. The following quantities are extensive: mass, volume, number of moles, energy, entropy, force, etc.

1.2.2.2. *Natural processes*

The usual (historic) presentation of thermodynamics begins with a statement of the first law of thermodynamics, which sees energy play a central role compared with the other extensive quantities. This is in fact only one of the conservation principles among the axioms of thermostatics, which affirm the existence of conservation laws all possessing the same structure. Consider a material system (i.e. a collection of defined material elements) which evolves according to a *natural process* (without exchange with the exterior). The system is described in a Galilean (or inertial) reference frame in relation to which it is fixed (no dynamic effects in the sense of mechanics are authorized), and only geometric changes of reference frame are permitted.

The following principles can be established:

– The entropy S of the material system is non-decaying during the natural evolution of a process.

– The other extensive quantities X_i of a material system remain constant during the natural evolution of a process.

The preceding axioms actually express the existence of physical conservation laws, except for the entropy which can only increase during the spontaneous evolution of an isolated system. The axiom of extensive quantities also implies that the concept of variation in the extensive quantities of a system (a) is meaningful: it is possible to conceive of the quantity ΔX_i of which (a) has gained an amount X.

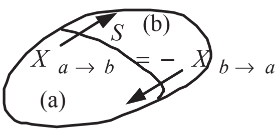

Figure 1.5. *Action and reaction in a system*

1.2.2.3. *Action and reaction*

The preceding definition implies a *principle of action and reaction*: let us divide the considered system into two sub-systems (a) and (b), separated by a surface S (Figure 1.5). The extensive quantity $X_{b \to a}$ *gained by* (a) *to the detriment of* (b) is opposed to the extensive quantity $X_{a \to b}$ *gained by* (b) *to the detriment of* (a). This amounts to saying that the action exercised by (a) on (b), the source of the transfer between (a) and (b), is opposed to the action of (b) on (a):

$$X_{a \to b} + X_{b \to a} = 0 \qquad [1.2]$$

To be more general, we can consider a process during which the sub-systems (a) and (b) receive in addition the quantities $X_{a\ ext}$ and $X_{b\ ext}$ from the exterior of the system S:

$$X_a = X_{a\,ext} + X_{b \to a}; \qquad X_b = X_{b\,ext} + X_{a \to b}$$

The conservation of the quantity X in the system implies that the total amount of the quantity X is equal to the amount of this quantity which is received from the exterior:

$$X_a + X_b = X_{a\,ext} + X_{b\,ext}$$

Equality [1.2] is once again the result.

The preceding reasoning is not applicable to entropy, for which we do not have conservation, but only non-decay. This results in the principle of action and reaction, which, as applied to the entropy, is written:

$$S_{a \to b} + S_{b \to a} \geq 0 \qquad [1.3]$$

This inequality implies that *one of the sub-systems has gained more entropy than the other has lost.*

The actions leading to a gain in the quantity X in system (a) can be classed into two categories:

– *volume actions* related to fields which can exert their influence from a distance: an electric field produces a displacement of electric charges, electromagnetic radiation can lead to a the creation of heat by absorption in the material medium, etc.;

– *contact actions* (on a surface) between (a) and (b) (Figure 1.5) essentially results from the action between the molecules in the immediate vicinity of the surface S (at distances of the order of the intermolecular distances or of the mean free path in a gas). As this volume is based on the surface S and is of small thickness, it can be modeled by the surface S, on which we can concentrate the interaction between (a) and (b). The usual contact actions are pressure, frictional force, thermal conduction, molecular diffusion, etc.

1.2.2.4. *States of equilibrium and extensive variables*

Let us consider a material system in a given initial state. Suppose that excepting its entropy, the values of its extensive quantities are known. We thus have a partial characterization which does not allow a full description of the state of the system. It is clear that a knowledge of the global values of mass, energy, etc., does not provide any understanding of how these quantities are distributed within the system.

Let us now isolate the system and leave it to evolve according to a natural process; the extensive quantities which it contains will naturally distribute themselves in a balanced way throughout the system. It is easy to understand that the final equilibrium state of the system is unique, as confirmed by experience. In other words, *the extensive quantities of a material system allow a description of the equilibrium state* achieved by a system at the end of all natural processes (provided the system remains isolated). Following the axiom pertaining to the entropy (section 1.2.2.2), the entropy associated with such a material system increases in order to attain a maximum value corresponding to this final state of equilibrium. We thereby deduce that this *entropy of the equilibrium state is a well-determined function of the other extensive quantities* for the system considered:

$$S = S(X_i) \tag{1.4}$$

Relation [1.4] constitutes the *entropic representation* of the system in equilibrium. It is in fact a state equation (in the thermodynamic sense) for equilibrium states which are only comprised of the extensive variables. In order not to confuse it with the usual state equations, we will refer to it as *the general equation of state of a system*. This relation is unique for a given material system, although numerous state equations, more or less dependent on it, can be derived from it. The *variance of a system* is the number of independent extensive quantities necessary for its representation.

The existence of this general equation of state leads to the properties of thermostatics.

1.2.2.5. *Homogenity*

We have said that the extensive quantities of a material system are proportional to its extension without stating exactly what this extension is. Let us suppose that these extensive quantities X_i and the entropy S are all associated with a given volume of ordinary 3D space. Taking two systems S_λ and S *w*ith material contents which are homothetic (or which are geometrically similar) with a ratio of λ, and which are constituted from the same matter, under the same physical conditions, at homologous points, we can say that the extensive quantities X_i are proportional to a cube with a reference dimension:

$$X_{\lambda i} = \lambda^3 X_i \qquad\qquad S_\lambda = \lambda^3 S$$

Denoting our two states by (a) and p (a), with p = λ^3, state relation [1.4] can be written:

$$S_p = S(pX_i) = pS(X_i) \tag{1.5}$$

The function $S(X_i)$ is a first degree homogenous function.

1.2.2.6. *Note*

This property no longer holds if the extensive quantities are associated with spaces of different dimensions. As an example consider a drop of water whose total energy is the sum of two terms, on the one hand an energy associated with its weight, E_g, proportional to its volume, and on the other hand an energy associated with its surface tension, E_T, proportional to its surface:

$$E = E_g + E_T$$

The total energy of the system S_λ becomes, considering a homothetic ratio λ:

$$E_\lambda = \lambda^3 E_g + \lambda^2 E_T$$

Other extensive quantities X_i (for example, mass or volume) of this drop remain proportional to the homothetic ratio λ^3 and respect relation [1.4]. It is therefore clear that relation [1.5] is no longer respected. We will not consider such cases in what follows.

1.2.3. *Energy, work and heat*

1.2.3.1. *Energetic representation of a system*

Among the extensive quantities, the energy, which we denote E, is often assigned a particular role. Depending on whether it is explicitly solved in S, or in E (energetic representation), the general equation of state [1.4] of a system can be written:

$$S = S(E, X_i) \qquad \text{or} \qquad E = E(S, X_i) \tag{1.6}$$

In practical applications at usual temperatures, energy and entropy only appear in terms of their variations. In these conditions, it is not necessary to define them in an absolute sense and it is sufficient to evaluate them to the nearest constant.

For a system at rest in a Galilean reference frame, the energy of a system is essentially comprised of its internal energy and the different forms of potential energy of the system elements. Changing the Galilean reference frame would amount to adding a constant amount of kinetic energy, which clearly does not change our description. This will no longer be the case if elements of the system have different movements, in which case their respective kinetic energies would need to be accounted for.

1.2.3.2. *Work and heat*

By calculating the differential of relation [1.6]:

$$dE = \frac{\partial E}{\partial S} dS + \sum_i \frac{\partial E}{\partial X_i} dX_i = \frac{1}{\dfrac{\partial S}{\partial E}} \left(dS - \sum_i \frac{\partial S}{\partial X_i} dX_i \right)$$

we can define the *elementary work dW* and the *elementary heat dQ* by the relations:

$$dQ = \frac{\partial E}{\partial S} dS = \frac{1}{\dfrac{\partial S}{\partial E}} dS \qquad dW = \sum_i \frac{\partial E}{\partial X_i} dX_i = -\frac{1}{\dfrac{\partial S}{\partial E}} \sum_i \frac{\partial S}{\partial X_i} dX_i \qquad [1.7]$$

such that we obtain the classic definition $dE = dW + dQ$, in which the differential forms dW and dQ are not differentials of a function, i.e. they are not exact differentials.

1.3. Consequences of the axioms of thermostatics

1.3.1. *Intensive variables*

1.3.1.1. *Definition and properties*

We will only study equilibrium states entirely characterized by the values of the extensive variables and the relation of state [1.4] or [1.6] between them.

We define *entropic intensive variables* using the relations:

$$Z_i = \frac{\partial S}{\partial X_i} \qquad \frac{1}{T} = \frac{\partial S}{\partial E} \qquad [1.8]$$

Z_i (resp. $1/T$) being called the conjugated variable of X_i (resp. E) with respect to S.

The *energetic intensive variables* are similarly defined:

$$Y_i = \frac{\partial E}{\partial X_i} \qquad T = \frac{\partial E}{\partial S} \tag{1.9}$$

Y_i (resp. T) is the conjugated variable of X_i (resp. S) with respect to E.

The differentials dE of the energy and dS of the entropy can be written:

$$dE = TdS + \sum_i Y_i dX_i \tag{1.10}$$

$$dS = \frac{1}{T}dE + \sum_i Z_i dX_i \tag{1.11}$$

From this, the relation between the energetic and entropic intensive variables, respectively Y_i and Z_i can be obtained:

$$Y_i = -TZ_i \tag{1.12}$$

Consider a fluid defined by the extensive variables (volume V, number of moles N, energy E and entropy S), the energetic intensive variables are the pressure p (to the nearest sign), the temperature T and the chemical potential μ:

$$dE = -pdV + TdS + \mu dN$$
$$\text{with} : p = -\left(\frac{\partial E}{\partial V}\right)_{N,S} \; ; \; T = \left(\frac{\partial E}{\partial S}\right)_{V,N} \; ; \; \mu = \left(\frac{\partial E}{\partial N}\right)_{V,S}$$

1.3.1.2. *Consequences of homogenity*

Differentiating relation [1.5] with respect to p, and letting $p = 1$, we obtain:

$$S = \frac{\partial S}{\partial E}E + \sum_i \frac{\partial S}{\partial X_i}X_i = \frac{E}{T} + \sum_i Z_i X_i \tag{1.13}$$

and:

$$E = TS + \sum_i Y_i X_i \tag{1.14}$$

We note that the general relation [1.13] or [1.14] does not allow the characterization of a system. It constitutes a differential equation which is satisfied by any homogenous function S of degree 1. For example, for a fluid, the following relation can be written:

$$E = TS - pV + \mu N$$

Differentiating relation [1.13] or [1.14], and comparing with expression [1.10] or [1.11], we obtain the relations:

$$SdT + \sum_i X_i dY_i = 0 \quad \text{(Gibbs-Duhem relation)} \qquad [1.15]$$

$$Ed\left(\frac{1}{T}\right) + \sum_i X_i dZ_i = 0 \qquad [1.16]$$

1.3.1.3. *The equations of state*

An expression relating the extensive and the intensive quantities which characterize a system is habitually called an *equation of state*.

The intensive variables, being by definition partial derivatives of a function $E(S, X,...)$, are not independent. In practice we rarely use the general equation of state [1.6], which only relates the extensive quantities, using instead the more usual equations of state which relate extensive and intensive variables, which can be more or less dependent.

For example, for a perfect gas, we know that there are two equations of state:

$$pV = NRT \qquad\qquad dE = mC_V dT$$

These are not independent as the first equation of state implies that the specific heat C_V can only be a function of the temperature (Joule's law; see elementary works on thermostatics).

1.3.2. *Thermodynamic potentials*

1.3.2.1. *Introduction*

While the preceding presentation assigns a particular role to the extensive variables, using them to represent a given system does not lead to the most useful means of studying that system. The temperature or pressure of a fluid are of a more direct interest than its volume or its energy when it comes to isobaric or isothermal processes. We often prefer a combination of intensive and extensive variables for studying a given system. From a mathematical point of view, such changing of variables can be relatively complex (e.g. contact or Legendre transformations [IGO 89], [COU 89], [BYU 02]).

Let us consider a simplified situation comprising a function $f(x,y)$ in two variables. Instead of taking the couple (x,y), we take the new variables $(x, \partial f/\partial y)$: we thus replace the variable y with the derivative $\partial f/\partial y$, the function f not being known *a priori*. We cannot therefore hope to obtain the explicit properties, only those of the differentials being accessible. We thus perform a change of function, replacing the differential of the extensive variable with the differential of the conjugate intensive variable. The simplest example is the introduction of the enthalpy H:

$$H(p,S,N) = E + pV$$

For a fluid, the differential dH can be written:

$$dH = d(E + pV) = Vdp + TdS + \mu dN$$

whence we see the intensive variable p appears in differential form. This is particularly convenient for the study of isobaric transformations or shaft work.

1.3.2.2. *Definition of thermodynamic potentials*

The preceding method is a general one. Suppose for instance that we want to choose the collection of independent variables (X_i, Y_j)[2]: we define the function $\overline{E}(X_i, Y_j)$:

$$\overline{E}(X_i, Y_j) = E - \sum_j X_j Y_j$$

Using relation [1.10][3] the differential of the function $\overline{E}(X_i, Y_j)$ can be written as:

$$d\overline{E} = dE - \sum_j (X_j dY_j + Y_j dX_j) = \sum_i Y_i dX_i - \sum_j X_j dY_j \qquad [1.17]$$

The function $\overline{E}$ is a *thermodynamic potential* whose partial derivatives:

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[2] The subscripts i and j correspond respectively to the different couples of extensive and intensive variables.

[3] Here, S is included in extensive variables X and is not explicitly written.

$$X_k = -\left(\frac{\partial \overline{E}}{\partial Y_k}\right)_{X_i, Y_{j \neq k}}$$

with respect to the intensive variables Y_k, are the corresponding extensive variables[4].

Using this expression in the definition of $\overline{E}$, we obtain the Gibbs-Helmholtz relation (a partial differential equation for $\overline{E}$ as a function of E):

$$\overline{E} - \sum_j \frac{\partial \overline{E}}{\partial Y_j} Y_j = E \qquad [1.18]$$

Remember that we frequently use:

– the *Helmholtz function* $F(V,T,N)$ (free energy);

$$F(V,T,N) = E - TS$$

hence: $dF = -pdV - SdT + \mu dN$

with: $p = -\frac{\partial F}{\partial V} \quad S = -\frac{\partial F}{\partial T} \quad \mu = \frac{\partial F}{\partial N}$

– the *Gibbs function* $G(p,T,N)$ (free enthalpy):

$$G(p,T,N) = E + pV - TS$$

hence: $dG = Vdp - SdT + \mu dN$

with: $V = \frac{\partial G}{\partial p} \quad S = -\frac{\partial G}{\partial T} \quad \mu = \frac{\partial G}{\partial N}$

The thermodynamic potentials are first order homogenous functions which satisfy Gibbs-Duhem relations (section 1.3.1.2). They are useful for the study of systems where certain intensive variables remain constant.

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[4] Notation $(\partial/\partial Y_k)_{X_i, Y_{j \neq k}}$ specifies that the chosen independent variables are X_i and Y_j.

For example, during a constant-pressure (p_a) evolution of a system whose volume increases by an amount ΔV, the system receives an amount of work $-p_a.\Delta V$ from the exterior. If $\overline{W}$ denotes the work (aside from that due to the pressure p_a), the variation of enthalpy ΔH of the system is (section 1.3.2.1):

$$\Delta H = \Delta\left(E + p_a.V\right) = \Delta E + p_a.\Delta V = \overline{W} + Q \qquad [1.19]$$

Generally speaking, the source of an extensive quantity X_j, for which the intensive variable Y_j is constant, providing the system with ΔX_j by means of a quasi-static process, leads to a variation of its thermodynamic potential equal to:

$$\Delta\overline{E} = \Delta E - Y_j \Delta X_j = \overline{W} + Q$$

($\overline{W}$ and Q being respectively the work and heat received from means other than the source, with Y_j being constant).

1.3.2.3. *Thermostatics and variables change*

The traditional practical problem of thermostatics consists of passing from a representation of a system with n variables to another representation with n variables related to the first by means of conjugation properties in the entropic and energetic representations. The number of possible combinations of n independent variables among the $2n$ extensive or intensive variables is clearly large. It is in fact the principal practical difficulty of thermodynamics. This multiplicity of possible independent variables leads to the numerous Maxwell relations between the coefficients of the differential forms of energy, entropy, enthalpy and the different thermodynamic potentials; the general form can be written by means of the differential $d\overline{E}$ [1.17]:

$$\frac{\partial X_i}{\partial X_j} = -\frac{\partial Y_j}{\partial Y_i} \qquad \frac{\partial Y_i}{\partial X_{i'}} = \frac{\partial Y_{i'}}{\partial X_i} \qquad \frac{\partial X_j}{\partial Y_{j'}} = \frac{\partial X_{j'}}{\partial Y_i}$$

We will leave it to the reader to work out examples involving the explicit functions given below.

Remember the rules for changing variables when using partial derivatives: if the three variables (x,y,z) are related, it is straightforward to show that:

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1 \qquad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1 \qquad [1.20]$$

1.3.2.4. *"Thermodynamic" coefficients*

For quasi-static closed process (*N=const*) we define the coefficients using units of mass or of volume. The *specific heats* at constant volume C_v and at constant pressure C_p, and the *calorific coefficients h* and *l* are:

$$dQ = TdS = NMC_v dT + ldV = NMC_p dT + hdp$$

from which we can derive the relations:

$$\begin{aligned} C_v &= \frac{T}{NM}\left(\frac{\partial S}{\partial T}\right)_{V,N} = -\frac{T}{NM}\frac{\partial^2 F}{\partial T^2} \\ C_p &= \frac{T}{NM}\left(\frac{\partial S}{\partial T}\right)_{p,N} = -\frac{T}{NM}\frac{\partial^2 G}{\partial T^2} \end{aligned} \qquad [1.21]$$

$$l = T\left(\frac{\partial S}{\partial V}\right)_{T,N} = -T\frac{\partial^2 F}{\partial V \partial T} = T\left(\frac{\partial p}{\partial T}\right)_{V,N} \qquad [1.22]$$

$$h = T\left(\frac{\partial S}{\partial P}\right)_{T,N} = -T\frac{\partial^2 G}{\partial P \partial T} = -T\left(\frac{\partial V}{\partial T}\right)_{P,N} \qquad [1.23]$$

In general, depending on the thermodynamic transformations studied, we may choose diverse independent variables. For example, we define the calorific coefficients λ and μ:

$$dQ = TdS = \lambda\, dp + \mu\, dv$$

and the isothermal and adiabatic compressibility coefficients, respectively χ_T and χ_S:

$$\chi_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \qquad \chi_S = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_S \qquad [1.24]$$

Recall the Reech relation (not derived here):

$$\frac{\chi_T}{\chi_S} = \frac{C_p}{C_V} = \gamma$$

The speed of sound is defined by the relation (ρ being specific mass):

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{SN} = \frac{1}{\rho \chi_S} \tag{1.25}$$

The coefficients of dilatation are defined by the relations:

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P = \frac{1}{V}\frac{\partial^2 G}{\partial T \partial p}; \quad \beta = \frac{1}{p}\left(\frac{\partial p}{\partial T}\right)_V = \frac{1}{p}\frac{\partial^2 F}{\partial T \partial V} \tag{1.26}$$

We recall the usual relations, which the reader can verify using identities [1.20]:

$$C_p - C_v = \frac{T}{NM}\left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial p}{\partial T}\right)_V; \quad \left(\frac{\partial C_v}{\partial V}\right)_T = \frac{T}{NM}\left(\frac{\partial^2 p}{\partial T^2}\right)_V;$$

$$\left(\frac{\partial C_p}{\partial p}\right)_T = -\frac{T}{NM}\left(\frac{\partial^2 V}{\partial T^2}\right)_p; \quad \lambda = NMC_v\left(\frac{\partial T}{\partial p}\right)_V; \quad \mu = NMC_p\left(\frac{\partial T}{\partial V}\right)_p;$$

$$c^2 = \frac{\beta}{\alpha}\frac{\gamma\, p}{\rho}; \quad \left(\frac{\partial s}{\partial \rho}\right)_p = -\left(\frac{\partial s}{\partial p}\right)_\rho c^2; \quad \left(\frac{\partial s}{\partial \rho}\right)_p = -\frac{C_p}{T}\left(\frac{\partial T}{\partial \rho}\right)_p = -\frac{C_p}{\alpha \rho T}$$

We deduce from this that the differential of the specific entropy (per unit mass) s as a function of the dp and $d\rho$ variables can be written in one of the following forms:

$$\begin{aligned} ds &= \left(\frac{\partial s}{\partial p}\right)_\rho dp + \left(\frac{\partial s}{\partial \rho}\right)_p d\rho = \left(\frac{\partial s}{\partial p}\right)_\rho \left(dp - c^2 d\rho\right) \\ &= \frac{C_v}{\beta\, pT} dp - \frac{C_p}{\alpha \rho T} d\rho = \frac{C_v}{\beta\, pT}\left(dp - c^2 d\rho\right) \end{aligned} \tag{1.27}$$

1.3.2.5. *Perfect gas*

Consider a perfect gas with constant specific heats C_v and C_p and of molar mass M. By reasoning using the unit of mass, state relation [1.6] between the extensive quantities can be written:

$$s = C_v Ln\left(\frac{e}{\rho^{\gamma-1}}\right) + \text{const} \qquad e = \rho^{\gamma-1}\exp\left(\frac{s}{C_v}\right) + \text{const}$$

$$\text{with:}\rho = m/V = NM/V; \quad s = S/m; \quad e = E/m \qquad h = H/m$$

From the intensive variable definitions, we can easily derive the usual state equations:

$$p = r\rho T \qquad e = C_v T + \text{const} \qquad \text{with: } R = r\text{M} = 8.32 \text{ joule/mole}$$

We can thereby obtain expressions for the extensive variables as functions of other variables, (e_0, h_0, s_0) being some constants:

$$e = C_v T + e_0 = \frac{p}{(\gamma-1)\rho} + e_0; \qquad h = C_p T + h_0 = \frac{\gamma p}{(\gamma-1)\rho} + h_0;$$

$$s = C_v Ln\left(\frac{T}{\rho^{\gamma-1}}\right) + s_0 = C_p Ln\left(\frac{T}{p^{(\gamma-1)/\gamma}}\right) + s_0 = C_v Ln\left(\frac{p}{\rho^{\gamma}}\right) + s_0$$

in addition to expressions for the usual coefficients given below:

$$l = p; \quad h = -\frac{1}{\rho}; \quad \alpha = \beta = \frac{1}{T}; \quad c^2 = \frac{\gamma p}{\rho} = \gamma r T = \rho \chi_S; \qquad [1.28]$$
$$C_p - C_v = r; \quad C_p / C_v = \gamma.$$

1.4. Out-of-equilibrium states

1.4.1. *Introduction*

The reasoning of section 1.2.2.4 shows how a system which is not in equilibrium does not have a general state relation outside of equilibrium conditions. The extensive quantities remain defined, but they are no longer sufficient to characterize the state of the system, as the entropy can no longer be defined as a function of these parameters alone. As seen in section 1.2.2.4, the distribution of extensive variables is uniquely defined for a system in equilibrium, whereas this is no longer the case for a system which is not in equilibrium: a more detailed description of the structure of the system is thus necessary. This implies that *more parameters will be necessary a priori for a description of an out-of-equilibrium system than for one which is in a state of equilibrium.*

The general method for describing an out-of-equilibrium system involves considering the system *as a collection of sub-systems, each of which is in a state of equilibrium* and which can thus be described by means of their extensive variables. We suppose (postulate) that such a procedure is always possible.

As the system is no longer in equilibrium, exchanges occur between the extensive variables of the various sub-systems, and the intensity of these exchanges must be characterized. It is now essential that time be homogenous, in other words, for an out-of-equilibrium system whose characteristics do not change, the amounts of quantities exchanged must be proportional to the duration of the transfers. At this point, it is sufficient to consider continuous matter at rest, i.e. in a fixed reference frame.

1.4.2. *Discontinuous systems*

1.4.2.1. *General principles*

A real system nearly always comprises a continuous variation of its physical properties. We therefore represent the latter using piecewise constant functions defined on a partition of the system in *P sub-systems, each of which is approximately in a state of equilibrium*, and to which we can therefore apply the properties of systems in equilibrium. Let *k* be the number of *independent* extensive variables required for a description of each sub-system (number of moles, volume, energy, entropy, etc.) in terms of an energetic or entropic representation.

A *knowledge of the state of the system S* requires a complete description of the *P* sub-systems, i.e. a total of *kP variables*. As seen earlier, certain extensive variables can be replaced by their corresponding intensive variables, which are defined for each sub-system as a result of the hypothesis that these sub-systems are in a state of equilibrium. This of course does not change the total number of independent variables, *kP*. For each sub-system *p*, we have the entropic form of the general equation of state (the energy of each sub-system, which is not individualized here, is included in the variables X_i):

$$S_p = S_p\left(X_{ip}\right) \qquad \left(p = 1,\dots,P;\; i = 1,\dots,k\right)$$

where X_{ip} designates the extensive quantity X_i contained in the sub-system *p*.

The *extensive quantities of the complete system* can be obtained by adding the corresponding extensive quantities of the sub-systems:

$$X_i = \sum_{p=1}^{P} X_{ip}; \quad S = \sum_{p=1}^{P} S_p\left(X_{ip}\right) \quad (p = 1,\dots,P)$$

Note that in general, it is obviously not possible to obtain a relation between the extensive quantities X_i of a system and its entropy S, since the kP variables X_{ip} must be eliminated from the $k + 1$ preceding equations. This fact bears witness to the absence of a general equation of state for an out-of-equilibrium system.

The extensive quantities X_i are generally functions of time. The flux φ_{ip} of the quantity X_i *received* by the component p is defined by:

$$\varphi_{ip} = \frac{dX_{ip}}{dt}$$

The origin of the quantities dX_{ip} and the flux φ_{ip} can be considered individually for each of the sub-systems. Let $dX_{ip,q}$ (resp. $dX_{iq,p}$) be the amount of extensive quantity X_i received by the sub-system p (resp. q) from the sub-system q (resp. p) in time dt. The amount of entropy $dS_{p,q}$ (resp. $dS_{q,p}$) received by each of the systems and associated with the aforementioned transfer is evaluated by means of the entropic representation differential of each of the sub-systems:

$$dS_{p,q} = \sum_{i=1}^{k} Z_{ip} dX_{ip,q} \qquad dS_{q,p} = \sum_{i=1}^{k} Z_{iq} dX_{iq,p}$$

The intensive variables Z_{ip} and Z_{iq} of two neighboring sub-systems will have different values if the sub-systems are not in equilibrium, and so the entropy variations $dS_{p,q}$ and $dS_{q,p}$ will have different absolute values. Indeed, there can only be an exact balance between the two sub-systems if the intensive variables Z_{ip} and Z_{iq} are equal.

With the exception of entropy, "exchanged" quantities obey the principle of action and reaction which results from the conservation principles (section 1.2.2.3):

$$dX_{ip,q} = -dX_{iq,p} \qquad dS_{p,q} + dS_{q,p} \geq 0 \qquad [1.29]$$

During an irreversible transfer of an extensive quantity, the entropy gained by a body is greater than the entropy lost by the body from which the extensive quantity is transferred (this statement is algebraically true).

The distribution of intensive variables is thus a constant function of space within the spatial bounds of a given sub-system, discontinuities at the frontier of each sub-system existing in proportion to the degree of thermodynamic imbalance within the system. Such imbalance leads to an exchange of the quantities $dX_{ip,q}$ between the sub-systems. It is thus necessary to introduce *relations between these causes and* *their effects*. Before dealing with the general problem of these relations, we will illustrate the methodology using the following elementary example, comprising two sub-systems which only exchange heat between themselves, or with the exterior.

1.4.2.2. *An insulated thermal system*

1.4.2.2.1. Entropy variation

The simplest example is an ensemble which is insulated from the exterior (Figure 1.6a), comprising two conducting blocks separated by a diathermic wall which creates a resistance to heat transfer (thermal resistance). Each of the subs-systems is, by assumption, characterized by its calorific energy content Q_1 (resp. Q_2) or its temperature T_1 (resp. T_2), assumed to be uniform at every instant. These quantities vary slowly with time on account of the thermal resistance of the system.

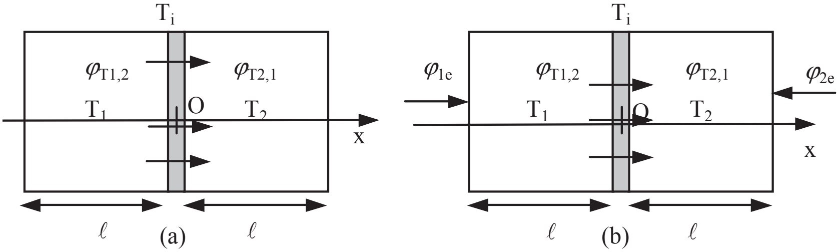


Figure 1.6. *Heat transfer in an (a) insulated or (b) uninsulated system*

The ensemble is not in equilibrium and is thus subject to a natural evolution during which the entropy increases, while the internal energy remains constant. We will calculate the variation in entropy between the state thus defined and the equilibrium state.

The entropy of a solid mass m of constant specific heat capacity C is $S = mC\,LnT + \text{const}$; the entropy of the ensemble of the two blocks, which are assumed to be identical, is thus:

$$S = S_1 + S_2 = mC\,LnT_1T_2 + \text{const}$$

As the ensemble is insulated from the exterior, the total amount of heat remains constant:

$$mCT_1 + mCT_2 = \text{const}$$

The natural evolution of the system S from initial temperatures T_{10} and T_{20} leads to the final temperature $T_f = \dfrac{T_{10} + T_{20}}{2}$, which is identical for the two blocks, and to the final entropy S of the system:

$$S = 2mC\, LnT_f + \text{const}$$

The variation in entropy ΔS between the final and initial instants is thus:

$$\Delta S = mcLn \frac{(T_{10} + T_{20})^2}{4T_{10}\, T_{20}} = mcLn\left(1 + \frac{(T_{10} - T_{20})^2}{4T_{10}\, T_{20}}\right)$$

It is always positive and independent of the intermediate evolution between the initial and final instants.

1.4.2.2.2. Entropy sources

Let $dQ_{p,q}$ be the quantity of heat received in time dt by the component p ($p = 1,2$) from the other component ($q = 1,2 \neq p$):

$$dQ_{1,2} + dQ_{2,1} = 0$$

The heat flow $\varphi_{Tp,q}$ received by the sub-system p is the quantity of heat $dQ_{p,q}/dt$ received per unit time by the component p ($p = 1,2$) from the other component. Thus, we have:

$$\frac{dS}{dt} = \frac{1}{T_1}\frac{dQ_{1,2}}{dt} + \frac{1}{T_2}\frac{dQ_{2,1}}{dt} = \frac{dQ_{1,2}}{dt}\left(\frac{1}{T_1} - \frac{1}{T_2}\right) \tag{1.30}$$

The *rate of entropy creation* dS/dt is always positive on account of the fact that the heat transfer naturally occurs from the hot body to the cold body:

$$T_1 > T_2 \Rightarrow dQ_{1,2} < 0 \quad \text{and} \quad T_1 < T_2 \Rightarrow dQ_{1,2} > 0$$

Let us consider the first situation ($T_1 > T_2$). The quantity $dQ_{2,1}$ is positive; the entropy (positive) dS_2 gained by sub-system 2 is greater than the entropy $-dS_1$ lost by sub-system 1. This is in accordance with the fact that for irreversible heat transfer, *the entropy gained by a system is greater than the entropy lost by the body which has provided the heat.*

The higher the degree of the imbalance, the greater the thermal flux, and the greater the rate of entropy creation.

1.4.2.3. *The insulated system and entropy creation*

Let us consider two bodies, each in equilibrium, and which exchange a quantity X across a wall permeable for this quantity. We define the intensive entropic parameters:

$$Z_1 = \frac{\partial S_1}{\partial X_1} \quad \text{or} \quad Z_2 = \frac{\partial S_2}{\partial X_2}$$

During a quasi-static transformation where the quantity $dX_{1,2} = -dX_{21}$ is exchanged in time dt, we obtain:

$$\frac{dS}{dt} = Z_1 \frac{dX_{1,2}}{dt} + Z_2 \frac{dX_{2,1}}{dt} = \frac{dX_{1,2}}{dt}\left(Z_1 - Z_2\right) \qquad [1.31]$$

As the entropy can only increase, the value of X increases (resp. decreases) in the sub-system for which the value of Z is greatest (smallest). T*he transfer of the quantity X occurs spontaneously from the sub-system with the smallest value of Z to the sub-system with the greatest value of Z.*

As in the preceding section, we see that *during an irreversible transfer of an extensive quantity, the entropy gained by a system is greater than the entropy lost by the body which has lost this extensive quantity.*

Expression [1.31] can be immediately generalized to a system constituted of P sub-systems characterized by I independent extensive quantities:

$$\frac{dS}{dt} = \sum_{i=1}^{I} \sum_{p=1}^{P} \sum_{q=1}^{P} \left[Z_{i,p} \frac{dX_{ip,q}}{dt} \right], \qquad q \neq p.$$

or, after grouping the opposing fluxes ($p < q$, so as to only count this component once):

$$\frac{dS}{dt} = \sum_{i=1}^{I} \sum_{p=1}^{P} \sum_{q=1}^{P} \left[\left(Z_{i,p} - Z_{i,q} \right) \frac{dX_{ip,q}}{dt} \right] \qquad p < q$$

1.4.2.4. *Systems with external exchanges and entropy source*

Let us first consider the same thermal systems as above (section 1.4.2.2), but where each of these receives a quantity of heat dQ_{pe} ($p = 1,2$) in time dt from the exterior. For the heat exchange we have (Figure 1.6b):

$$dQ_1 = dQ_{1,2} + dQ_{1e}; \quad dQ_2 = dQ_{2,1} + dQ_{2e}; \quad \text{with : } dQ_{1,2} + dQ_{2,1} = 0$$

The increase in system entropy is:

$$dS = \frac{dQ_{1e}}{T_1} + \frac{dQ_{2e}}{T_2} + \frac{dQ_{1,2}}{T_2} \left(\frac{1}{T_1} - \frac{1}{T_2} \right) \tag{1.32}$$

The external entropy supply terms $\frac{1}{T_1}\frac{dQ_{1e}}{dt}$ and $\frac{1}{T_2}\frac{dQ_{2e}}{dt}$ can now take any sign. The term in [1.32] which corresponds to irreversible internal heat exchange, which is always positive, constitutes an entropy source. We thus have:

$$\sum_{p=1,2} \frac{dQ_{pe}}{T_p} \le dS \tag{1.33}$$

Suppose, in addition, that the external *sources of heat are in thermostatic equilibrium*, such that their temperatures T_{1S} and T_{2S} are defined. These transfers are natural internal evolutions for any ensemble constituted of a sub-system and a corresponding external source. They are, as before, entropy generators; we can therefore write:

$$\sum_{p=1,2} \frac{dQ_{pe}}{T_p} - \sum_{p=1,2} \frac{dQ_{pe}}{T_{pS}} = \sum_{p=1,2} \frac{dQ_{pe}}{T_p} \left(\frac{1}{T_p} - \frac{1}{T_{pS}} \right) \ge 0 \tag{1.34}$$

By then evaluating the entropy lost by the external sources we obtain the Clausius inequality:

$$\sum_{p=1,2} \frac{dQ_{pe}}{T_{pS}} \leq \sum_{p=1,2} \frac{dQ_{pe}}{T_p} \leq dS \qquad [1.35]$$

Similarly, for internal and external exchanges of an extensive quantity *X*, we obtain, using the same notations as above:

$$\frac{dS}{dt} = Z_1 \frac{dX_{1e}}{dt} + Z_2 \frac{dX_{2e}}{dt} + \underset{\text{positive term}}{\frac{dX_{1,2}}{dt}\left(Z_1 - Z_2\right)}$$

The term $\frac{dX_{1,2}}{dt}\left(Z_1 - Z_2\right)$, which is always positive, is *the entropy source associated with internal exchanges.* The rate of entropy creation is proportional to the intensity $\frac{dX_{1,2}}{dt}$ of the internal exchanges and to the imbalance $\left(Z_1 - Z_2\right)$ between the two sub-systems.

In general, the internal production of entropy dS_{int}/dt (entropy source) is associated with the evolution of all of the extensive quantities in this system, which is made up of two sub-systems:

$$\frac{dS_{\text{int}}}{dt} = \sum_{i=1}^{I} \frac{dX_{i1,2}}{dt}\left(Z_{i1} - Z_{i2}\right) \qquad i = 1,2,....I \qquad [1.36]$$

Finally, let us recall that the entropic intensive quantities Zi are related to the energetic intensive quantities ([1.12]: $Y_i = -TZ_i$) and that it is possible to express the entropy source as a function of any other quantities.

The reasoning used above leads to an expression for entropy sources $\frac{dS_{\text{int}}}{dt}$ in a system made up of *P* sub-systems characterized by I independent extensive quantities:

$$\frac{ds_{\text{int}}}{dt} = \sum_{i=1}^{I} \sum_{p=1}^{P} \sum_{q=1}^{P} \left[(Z_{i,p} - Z_{i,p}) \frac{dx_{ip,q}}{dt} \right] \quad p < q \qquad [1.37]$$

Expression [1.37] only applies to internal entropy creation in the system considered. Entropy production is also associated with the transfer of extensive quantities from sources external to the system; this entropy production can be evaluated in the same way. For thermal transfers between external temperature sources T_{pS} ($p = 1,\ldots,P$) and the system, we have, as above, the Clausius inequality:

$$\sum_{p=1}^{P} \frac{dQ_{pe}}{T_{pS}} \leq \sum_{p=1}^{P} \frac{dQ_{pe}}{T_p} \leq dS$$

1.4.2.5. *The average intensive quantity*

1.4.2.5.1. Definition

An out-of-equilibrium system is characterized by a collection of intensive quantities whose values differ according to the sub-systems considered. It may be useful to characterize the system by a global intensive variable, which is an "average value" of the intensive variables of the sub-systems. In order to define this average value, we will refer to an "equivalent" equilibrium state of the system.

Consider an out-of-equilibrium system S made up of P sub-systems S_p each of which is in instantaneous equilibrium (quasi-static transformations). For each of these, we can define the intensive entropic quantities Z_{ip} associated with their N extensive quantities X_{ip}. The total amount of extensive quantity X_i contained in the system S is the sum $\sum_{p=1}^{P} X_{ip}$ of the extensive quantities of each sub-system.

It is clear that the system S cannot be described by any intensive quantity associated with X_i. We can however associate system S with an *average intensive quantity* Y_m or Z_m at any given instant t, defined as the *intensive quantity which the system* S *would attain following a natural evolution during which values* X_i should be constant (without any external contribution). Let us consider as an example the variables Z_m.

Suppose that during the transformations undergone by the system, certain extensive quantities X_{ip} ($i = 1,\ldots, I$; $\forall p$) of the sub-systems are exchanged, while the other $N - I$ extensive quantities remain constant in each of the sub-systems (for example mass, number of moles, volume, etc.). All intensive quantities of all sub-systems vary during the exchange of extensive quantities. In the final state of the previously defined system, *intensive quantities* Z_{ip} *corresponding to exchanged extensive quantities* have the uniform value Z_{im} for all sub-systems:

In each sub-system p, every intensive quantity Z_{ip} *corresponding to the extensive exchanged quantities* X_{ip} can be expressed as a function of the N extensive quantities (I extensive quantities X_{ip} exchanged with other sub-systems and $N - I$ other extensive quantities X_{jp} which are constant for the sub-system).

$$Z_{ip} = Z_{ip}\left(X_{ip}, X_{jp}\right) \quad (i = 1,\ldots, I;\ j = 1,\ldots, N - I;\ p = 1,\ldots, P)$$

By solving the preceding system of I equations for each sub-system p, with respect to the extensive quantities X_{ip}, we obtain:

$$X_{ip} = X_{ip}\left(Z_{ip}, X_{jp}\right) \quad (i = 1,\ldots, I;\ j = 1,\ldots, N - I;\ p = 1,\ldots, P)$$

The quantities X_i for the system S can be immediately obtained:

$$X_i = \sum_{p=1}^{P} X_{ip}\left(Z_{ip}, X_{jp}\right) \quad (i = 1,\ldots, I;\ j = 1,\ldots, N - I;\ )$$

The equilibrium conditions for the system can be written $Z_{ip} = Z_{im} \quad (i = 1,\ldots I;\ \forall p)$.

Considering the quantities X_i to be constant, we obtain a system of I equations from which we can evaluate the I average intensive quantities Z_{im} which characterize the out-of-equilibrium system:

$$X_i = \sum_{p=1}^{P} X_{ip}\left(Z_{im}, X_{jp}\right) \quad (i = 1,\ldots I;\ j = 1,\ldots, N - 1) \tag{1.38}$$

We laid down that certain extensive variables were not exchanged between the sub-systems. The problem can be easily discussed in the same manner with different conditions: for example, by fixing the uniform value to certain intensive quantities in the whole system (section 1.4.2.5.2, example 2), or by choosing different conditions according to certain ensembles of sub-systems.

1.4.2.5.2. Examples

We will consider two examples in order to illustrate the preceding procedure, which we will encounter again in the study of fluid mechanics:

1) *A thermal system* – consider a system S_1 (Figure 1.7a) whose P sub-systems exchange heat via a constant-volume process, such that the temperature T_p is the only variable intensive quantity of any of the sub-systems during the process. The energy E_p of each sub-system can be expressed as a function of its temperature and

its specific heat capacity Γ_p, which is assumed to be constant for the sake of simplicity:

$$E_p = \Gamma_p T_p$$

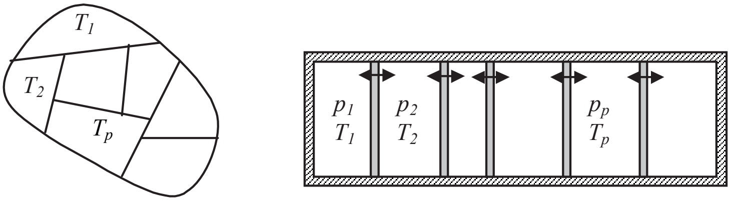
The energy E of the system and its average temperature T_m can be obtained:

$$E = \sum_{p=1}^{P} \Gamma_p T_p = \left(\sum_{p=1}^{P} \Gamma_p \right) T_m = \Gamma T_m; \qquad \text{with} : \Gamma = \sum_{p=1}^{P} \Gamma_p$$

The average temperature of the out-of-equilibrium system can thus be written:

$$T_m = \frac{1}{\Gamma} \sum_p \Gamma_p T_p$$

In this particular case, the average temperature is the average of the temperatures weighted by the specific heat capacities.



(a) System S_1 (b) System S_2

Figure 1.7. *Examples of out-of-equilibrium system: (a) incompressible thermal system S_1; (b) compressible thermal system S_2*

2) *A thermo-compressible system* – consider now a system S_2 (Figure 1.7b), which is comprised of sub-systems of variable volumes V_i, containing a perfect gas and susceptible to exchange heat between each other. The volumes V_i are separated by pistons which may be subject to fluid friction. The equations of state for a perfect gas (section 1.3.2.5) give the expression sought for the extensive quantities:

$$V_p = \frac{n_p R T_p}{p_p} \qquad E_p = \Gamma_{vp} T_p \quad (\Gamma_{vp}\text{: heat capacity at constant volume})$$

The global volume and internal energy balances for a natural transformation allow the average pressure and temperature to be defined:

$$V = \sum_{p=1}^{P} V_p = \sum_{p=1}^{P} \frac{n_p R T_p}{p_p} = \frac{n R T_m}{p_m}; \quad E = \sum_{p=1}^{P} \Gamma_{vp} T_p = \Gamma_v T_m$$

with: $n = \sum_{p=1}^{P} n_p$ and: $\Gamma_v = \sum_{p=1}^{P} \Gamma_{vp}$

We obtain the following expressions for the average temperature and pressure:

$$T_m = \frac{1}{\Gamma_v} \sum_{p=1}^{P} \Gamma_{vp} T_p \qquad \frac{1}{p_m} = \frac{1}{n T_m} \sum_{p=1}^{P} \frac{n_p T_p}{p_p}$$

The average values of the intensive quantities are no longer simply weighted arithmetic averages.

1.4.2.5.3. Some comments

1) The "equivalent" system used to define the average intensive quantities is in thermostatic equilibrium. It thus behaves according to a general state equation and may therefore constitute a reduced representation of the out-of-equilibrium system, which is clearly incomplete for a detailed description of the sub-systems. With the exception of entropy, this is coherent with the laws of thermostatics (we can also say that this is a "consistent" representation).

2) The average intensive quantities are only weighted arithmetic averages if the expressions for the extensive quantities X_{ip} are linear functions of X_{jp} and Y_{ip}. The reader can verify that the state equation for the equivalent complete system comprises the same linear properties.

3) In the examples of the last section we have considered two systems constituted of sub-systems with identical properties. The study of more complex systems, comprising combinations of sub-systems with different structures, can be effected in the same way.

4) The procedure for definition of average intensive quantities can also be applied to systems whose intensive quantities may be subjected to certain constraints. Let us reconsider the example of Figure 1.7a, in which the sub-systems are all at constant pressure equal to that of the atmosphere, rather than being at constant volume. We suppose nonetheless that the external walls of the system, which are in contact with the atmosphere, are adiabatic. The conservation of internal energy can be written by taking into account the work done by the atmospheric

pressure on the system. The reader can verify that this amounts to writing the conservation of enthalpy of the sub-systems.

1.4.2.6. *Phenomenological laws*

1.4.2.6.1. Introduction

In general, all causes of the same tensor nature act on all of the corresponding effects: we thus have a *coupling of irreversible phenomena*; the example of a thermocouple is well known. Phenomena of different tensorial order do not mutually interact (Curie's principle). The interested reader should refer to the specialized literature ([DEG 62], [BYU 02], [PRI 68]).

Consider two sub-systems p and q between which scalar quantities X can be exchanged. This exchange between the two sub-systems is assumed to be independent of other sub-systems, which is the case when exchanges only occur via direct contact, action at a distance not being possible. The irreversible evolution which occurs takes the form of a flux, $dX_{ip,q}/dt$ between the two sub-systems, and is caused by the existence of an imbalance characterized by the different values Z_{jp} and Z_{jq} *of all the intensive entropic quantities of the two sub-systems* ($j = 1,\ldots, I$).

We have a *relation between the causes* Z_{jp} *and* Z_{iq} *and the effects (the fluxes)* whose general form can be written as:

$$\frac{dX_{ip,q}}{dt} = F_i\left(Z_{jp}, Z_{jq}\right) \qquad \left(i, j = 1,\ldots, I\right)$$

The principle of action and reaction and the condition of zero flux at thermostatic equilibrium ($Z_{jp} = Z_{jq} = Z_{je}$) can be represented by the relations:

$$F_i\left(Z_{jp}, Z_{jq}\right) = -F_i\left(Z_{jq}, Z_{jp}\right); \qquad F_i\left(Z_{je}, Z_{je}\right) = 0$$

From this we can deduce a property of the derivatives of F_i which will be used in the following section:

$$\frac{\partial F_i}{\partial Z_{jp}}\left(Z_{jp}, Z_{jq}\right) = -\frac{\partial F_i}{\partial Z_{jq}}\left(Z_{jq}, Z_{jp}\right)$$

1.4.2.6.2. Linear thermodynamics

Provided the degree of thermodynamic imbalance is reasonably small, it is possible to perform a limited Taylor expansion of the functions $F_i\left(Z_{jp}, Z_{jq}\right)$ in the vicinity of the point $Z_{ie}, (ie = 1,...,I)$, of thermodynamic equilibrium. Taking into account the above equality we have:

$$F_i\left(Z_{jp}, Z_{jq}\right) = \left(Z_{jp} - Z_{jq}\right)\frac{\partial F_i}{\partial Z_{jp}}\left(Z_{ie}, Z_{ie}\right) + o\left(Z_{jp} - Z_{jq}\right), \quad \left(i, j = 1,..., I\right)$$

In the preceding relations, $L_{ij}\left(Z_{i0}\right) = \frac{\partial F_i}{\partial Z_{jp}}\left(Z_{ie}, Z_{ie}\right)$ is a square matrix of order *I*, the properties of which cannot be obtained via thermodynamic considerations. Only an experiment or a suitably adapted theory can provide these laws, which we here qualify as *phenomenological laws.*

Statistical reasoning shows that this matrix is often symmetric (Onsager relations). *This symmetry is no longer maintained if other intensive variables are used in place of intensive entropic variables.*

As long as the preceding approximation is valid, we say that we are dealing with linear thermodynamics of irreversible phenomena. We thus have, for each thermodynamic flux of an extensive quantity:

$$\frac{dX_{ip,q}}{dt} = L_{ij}\left(Z_{ie}\right)\left(Z_{jp} - Z_{jq}\right) \tag{1.39}$$

Internal entropy source [1.36], which exists at the interface of the two sub-systems considered, becomes:

$$\frac{dS_{\text{int}}}{dt} = \sum_i \sum_j L_{ij}\left(Z_e\right)\left(Z_{i1} - Z_{i2}\right)\left(Z_{j1} - Z_{j2}\right) \qquad i, j = 1,2,....n \tag{1.40}$$

This internal entropy source is thus a *positive quadratic form* whose principal diagonal elements are all positive ($L_{ii}\left(Z_e\right) > 0$).

We can also perform a change of variables which involves expressing the intensive entropic variables as a function of other thermodynamic variables, for example the *energetic intensive variables*. Consider two sub-systems 1 and 2. The differences $\left(Z_{i1} - Z_{i2}\right)$ are generally linear functions (to second order excepted) of the differences between the new variables which are chosen.

For example, with the intensive entropic variable $1/T$ which is associated with energy, we obtain:

$$\frac{1}{T_1}-\frac{1}{T_2}=\frac{T_2-T_1}{T_1T_2} \qquad [1.41]$$

and, generally, by using intensive energy variables $Y_i = -TZ_i$ in the place of intensive entropic variables Z_i:

$$Z_{i1}-Z_{i2}=-\frac{Y_{i1}}{T_1}+\frac{Y_{i2}}{T_2}=-\frac{T_1}{T_1T_2}\left(Y_{i1}-Y_{i2}\right)+\frac{Y_{i1}}{T_1T_2}\left(T_1-T_2\right) \qquad [1.42]$$

that is, to second order excepted, by linearizing in the vicinity of (T_0, Y_{i0}):

$$\frac{1}{T_1}-\frac{1}{T_2}=-\frac{T_1-T_2}{T_0^2}; \quad Z_{i1}-Z_{i2}=-\frac{1}{T_0}\left(Y_{i1}-Y_{i2}\right)+\frac{Y_{i0}}{T_0^2}\left(T_1-T_2\right) \qquad [1.43]$$

By substituting expression [1.43] into expression [1.39] for the thermodynamic flux in which the temperature appears, we obtain a linearized expression for the flux as a function of the intensive energy variables (including the temperature):

$$\frac{dX_{i1\,\mathrm{int}}}{dt}=L'_{ij}\left(T_0,Y_{i0}\right)\left(Y_{j1}-Y_{j2}\right)$$

in which the matrix L'_{ij} can be easily deduced from the matrix L_{ij}. It is easy to see that the matrix L'_{ij} is not symmetric, as noted earlier.

In addition, the *terms of the matrix diagonal* L'_{ij} *are negative*: the flux of the extensive quantities is in the same direction as the decaying intensive energy quantities.

Letting $i = j = 1$, the heat flux $\frac{dQ_{1,2}}{dt}$ and the temperature T, we define the thermal resistance R_T which is the inverse of the term L'_{11} of the first diagonal term of the matrix L'_{ij} :

$$\frac{dQ_{1,2}}{dt}=\frac{T_2-T_1}{R_T}$$

The entropy source at the interface of the two sub-systems can be written using the energy variables (accurate to second order excepted):

$$\frac{dS_{\text{int}}}{dt} = \sum_i \sum_j L'_{ij} \left(Y_j\right)\left(Y_{i1} - Y_{i2}\right)\left(Y_{j1} - Y_{j2}\right) \qquad i, j = 1,2,....I$$

1.4.2.6.3. Conditions for the application of irreversible linear thermodynamics

We have already noted that it is possible to linearize the phenomenological law which gives the flux of extensive variables if the imbalances are "small". This idea can be stated in the following usual way: if the imbalance corresponds to a weak variation of a "regular" process, then the linearization may be satisfactory. This leads us to invoke irreversible extensive quantity transfer mechanisms. Let us examine the case of thermal transfers, which are primarily due to two mechanisms.

1) *Intermolecular action within a material*

Molecular agitation, the intensity of which increases with temperature, results in the transmission of extensive properties via collisions between molecules, ions, etc. This is a statistical mechanism, which tends to cause a uniform distribution of the properties of a body. For example, the mechanical energy of molecules in hot regions is transmitted to molecules in cold regions via collisions between the molecules (gases), and/or by the action of intermolecular forces (liquids, solids). Within the context of kinetic theory in traditional mechanics, molecules are animated with a velocity in the order of the speed of sound; they cover a distance called the mean free path between successive collisions. Under ordinary conditions of pressure and temperature, this distance is in the order of 10^{-7} meters. Thermal energy is due to kinetic and potential energy of molecules. Let us take an example to evaluate the imbalance due to a temperature gradient. If we admit that statistically molecules lose one-thousandth of their energy with each collision, we can conclude that about 100 collisions are necessary in order for gas molecules to lose one-tenth of their energy. This loss corresponds to a temperature drop of about 30 Kelvin which will be produced over a distance of the order of 10^{-5} meters (10 μm). This corresponds to a considerable thermal imbalance. However, these collisions, which correspond to a tiny mean energy loss of 1/1,000, are clearly very small processes in comparison to two microscopic fluxes of mechanical energy due to the molecules going through any plane in one direction and also in the opposite direction. These opposite fluxes have nearly the same absolute value. The macroscopic mechanism for irreversible transfer of extensive quantities by molecular collision is thus statistically a tiny perturbation amongst the mechanisms of thermal agitation, and it is thus not surprising that the macroscopic processes are linear. Our experience verifies the arguments proposed by this rather simplistic reasoning.

However, we must realize that the collision properties can vary with temperature, even if the mean free path is not very temperature sensitive. This implies that the properties of thermal resistance can depend on the temperature chosen T_0 in relation [1.43], in order to evaluate the thermal resistance (the inverse of the first diagonal term of the matrix L'_{ij}). In fact, for large temperature differences, it is necessary to consider thermal resistance as a succession of isothermic thermal resistances in series and to perform an integration which takes the temperature variation into account:

$$\frac{dQ_{1,2}}{dt} = \int_{T_1}^{T_2} \frac{dT}{R_T(T)}$$

As a first approximation, we can often take the temperature T_0 as being equal to the mean temperature $\frac{T_1 + T_2}{2}$.

2) *Thermal radiation*

Thermal radiation is in fact an electromagnetic radiation, whose emitted power q_T per unit surface (thermal flux density) of a blackbody can be represented by Stefan's law:

$$q_T = \sigma\, T^4 \qquad (\sigma\text{: Stefan constant;T Kelvin})$$

Consider two parallel planes, face to face and respectively heated to temperatures of T_1 and T_2 ($T_1 < T_2$). The net heat flux density q_T received by plane 1 is equal to:

$$q_T = \sigma\left(T_2^4 - T_1^4\right)$$

The preceding relation will quickly deviate from a linear heat transfer law valid for small temperature differences. The reader can verify that in this case the preceding law can be represented by a thermal resistance R_T equal to $1/3\sigma\, T^3$, which varies strongly with temperature.

1.4.3. *Application to heat engines*

A heat engine is a device in which a fluid is made to *evolve according to a cycle C*: the material follows an evolution parameterized by the time *t*, after which the final state is identical to the initial state. Diverse extensive quantities are exchanged by this material with external sources of work and of heat. The objective of this machine is to produce certain desired quantities (work for a heat engine, heat for a heat pump, etc.) from external sources of other quantities.

The fluid which circulates in the engine passes through successive devices which either furnish or extract work and heat. Consider a mass m of this fluid whose quantities and in particular the temperature $T(t)$ evolve in a quasi-static manner as a time function t. Let δQ be an amount of heat which it receives between the instants t and $t+\delta t$. During the cycle C, the variations of the fluid's entropy and of its internal energy are zero:

$$\int_C dE = \int_C \left(\delta W + \delta Q\right) = W + Q = 0; \qquad \int_C dS = 0 \qquad [1.44]$$

Thus, the evolution process of the fluid over a cycle, as for all processes, creates entropy (see section 1.4.2.4). As the entropy variation of the fluid is zero over the cycle, a generalized version of relation [1.33] can be written, T being the *temperature of the fluid:*

$$\int_{\mathcal{C}} \frac{dQ}{T} \leq 0$$

On the other hand, the heat transfer from the sources at temperature $T_S(t)$ is also accompanied by a creation of entropy, the reasoning of relation shows that we have the Clausius inequalities:

$$\int_{\mathcal{C}} \frac{dQ}{T_S} \leq \int_{\mathcal{C}} \frac{dQ}{T} \leq \int_{\mathcal{C}} dS = 0$$

The preceding reasoning has the advantage of highlighting the entropy sources associated with the Clausius inequality; it also shows that the difference between the inequality terms is greater in proportion to the level of irreversibility.

We will not get into a detailed discussion of such cycles and the efficiency of heat engines, all of which can be deduced from the aforementioned inequalities. The reader will find such discussions in texts on applied thermodynamics.

Fundamentals of Fluid Mechanics and Transport Phenomena
Jean-Laurent Peube

Chapter 2

Thermodynamics of Continuous Media

The properties of continuous media can be obtained by a limiting process on variables of discrete systems. The exchange of extensive quantities is modeled by means of flux densities. Irreversible thermodynamics can be transposed in the same way, being represented by diffusion equations expressed in terms of intensive quantities (heat and diffusion equations of chemical species). The principal results of fluid statics are presented. The diffusion of material leads to the existence of several macroscopic reference frames, which brings with it specific difficulties which are discussed.

2.1. Thermostatics of continuous media

2.1.1. *Reduced extensive quantities*

The continuous medium is defined according to the usual method of passing from the discrete to the continuous by letting the elementary sub-systems tend towards zero, their number thereby increasing indefinitely. If the geometric dimensions of a system tend toward zero, the extensive quantities of that system also tend toward zero, whereas the intensive quantities do not change. A limiting process is therefore necessary for the study of continuous media.

We define *reduced extensive quantities,* i.e. extensive quantities per unit of mass, volume, or number of moles:

– a quantity per unit mass $g = \dfrac{G}{m}$,

– a volume quantity $\bar{g} = \dfrac{G}{V}$,

– a molar quantity $\tilde{g} = \dfrac{G}{N}$.

The amount of G which is contained in the domain D is:

$$G = \int_D \rho\, g dv = \int_D \bar{g} dv = \int_D n\tilde{g} dv$$

The concept of a local quantity corresponds to an average over a volume which is large on the microscopic scale, but small on the macroscopic scale. If we imagine that it is possible to measure an average quantity for a set of particles (molecules, for example) of a continuous medium, contained within a sphere of radius r and centered on a point M, the values obtained will only tend towards the value of the reduced extensive quantity if r is sufficiently large. For small values of r, noticeable fluctuations would be observed. Figure 2.1 shows the result of a measurement of the average density which would be obtained for a sphere whose radius r is of the order of the inter-molecular distances. On a larger scale we would of course observe the gradient (macroscopic) of the reduced extensive quantity.

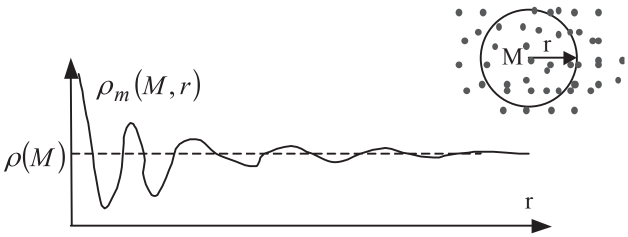


Figure 2.1. *Local value of the average specific mass in a sphere of radius r and fluctuations at a molecular scale*

2.1.2. *Local thermodynamic equilibrium*

The description of a given continuous medium can be performed by means of a field of reduced extensive quantities. We use the same methodology used for finite discrete systems, as the reduced extensive quantities chosen only allow for a description of the continuous medium if a *hypothesis of local thermodynamic equilibrium* is made: any very small volume obeys the general equation of state for a system in equilibrium, with all its consequences. We obtain the thermostatic relations for continuous media by a limiting process when the characteristic dimension of the discrete sub-systems tends to zero.

General equilibrium equation [1.6] for a discrete system is first degree homogenous; it can be applied immediately for reduced extensive quantities. By performing a trivial transposition of the reduced notation ($X \to x$, $E \to e$, $S \to s$, etc.), the notation S which here designates the mathematical function of formula [1.6], we have:

$$s = S(x, e) \quad \text{or: } \overline{s} = S(\overline{x}, \overline{e}) \quad \text{or: } \tilde{s} = S(\tilde{x}, \tilde{e}) \qquad [2.1]$$

The consequences of the hypothesis of local thermostatic equilibrium have already been described in Chapter 1; the diverse relations obtained being first degree homogenous relations, they are entirely transposable via a replacement of the extensive quantities by the corresponding reduced extensive quantities. We note however that the variance has decreased by one unit. We will often consider a *divariant fluid* (two independent state variables) which will often be a perfect gas with constant C_v, for which the equations of state become:

$$\frac{p}{\rho} = rT \qquad\qquad e = C_v T + \text{const}$$

The local thermodynamic equilibrium hypothesis translates the existence of two rapid dynamic processes which are opposed, such that locally there is a balanced exchange between them: in effect, the energy mechanism and momentum transmission on the molecular scale involves molecular collisions during which the exchange of extensive quantities occurs in both directions. The macroscopic transfer is no more than the residuum of these exchanges in opposing directions. Molecular displacements occur with a velocity in the order of the molecular agitation velocity (in the order of the speed of sound). The corresponding fluxes are individually very high, but they are in opposing directions and it is their very weak net outcome that we observe at the macroscopic scale for irreversible phenomena. These amount to a very weak perturbation of the local equilibrium of the continuous medium, which obeys thermostatic relations.

The local thermodynamic equilibrium hypothesis is based on the fact that the time required for the local gas equilibrium to be achieved (relaxation time) is small compared with the times associated with the macroscopic gas evolution. For the molecules of mean free path ℓ and molecular velocity c, the relaxation time is in the order of ℓ/c, if we consider that each molecular collision is efficient in the transfer of extensive molecular quantities (quantum conditions).

The molecular interactions which are at the heart of the aforementioned processes have a very short radius of action, whether we consider the intermolecular forces or the collisions characterized by their mean free paths. These actions are in fact volumetric at the scale of the given distances, and at the macroscopic scale they appear as contacts, i.e. flux densities of extensive quantities, which are determined by local conditions, i.e. surface forces.

The condition of local thermodynamic equilibrium is not always satisfied in the case of certain gases undergoing rapid changes (for example, tri-atomic gases during the passage of a series of shock waves, or in supersonic nozzles) and plasmas. We are therefore led to separate different populations (ions and electrons) or different forms of molecular energy (translation and vibration), for which we must introduce a further extensive quantity and supplementary hypotheses (relaxation law, etc.). Such separations into sub-systems are not only spatial. The interested reader can refer to [BAS 98] or texts dealing with ultrasound.

This condition of local thermodynamic equilibrium does not allow the entire medium considered to be in global thermodynamic equilibrium. It can be shown using statistical methods taken from kinetic gas theory, for example, that the net balance of the extensive molecular quantities gives, at first order, a distribution of Maxwell-Boltzmann velocities which correspond to a local statistic mechanical equilibrium ([CHA 91], [HIR 64]). At second order we have phenomena associated with the irreversible transfer of extensive quantities ([BIR 02]); in section 3.4.1.3 we will discuss the mechanism of this irreversible transfer in the case of momentum transfer. The statistical equations describing turbulence are identical to the preceding molecular statistical transfer equations, in which turbulent fluctuations play the role of molecular fluctuations for the transfer of these extensive quantities. Unfortunately, turbulent fluctuations do not verify general statistical laws and the preceding analogy is only formal ([COU 89], [MAT 00], [TEN 72]).

2.1.3. *Flux of extensive quantities*

2.3.1.1. *Flux density*

An extensive quantity G can be transferred in a continuous medium by different processes, the nature of which does not interest us here. The definition of a *flux* φ_{GS} *of the quantity G across the surface S has already been given for the surface of finite systems (the amount of the quantity G which crosses the surface S per unit time).*

We will show that *in a continuous medium, the flux of an extensive scalar quantity G is characterized by a flux density vector.*

Consider the tetrahedron $OA_1A_2A_3$ whose corners are the origin O and three points lying on the axes (Figure 2.2). Suppose that the surface dimensions of this tetrahedron are small, of order ε. Let $q_{G1}\ ds_1$, $q_{G2}\ ds_2$, $q_{G3}\ ds_3$ be, respectively, the fluxes of quantity G across the surfaces OA_2A_3, OA_1A_3 and OA_1A_2 (of respective surface areas ds_1, ds_2 and ds_3, of order ε^2) in the positive direction along the coordinate axes.

Let $d\varphi_G$ be the flux leaving the face $A_1A_2A_3$ of area *ds* of the tetrahedron. Suppose that the (algebraic) sources of the quantity *G* are volumetric. The amount of the quantity created in the tetrahedron is $O(\varepsilon^3)$. The balance of the extensive quantity G *leaving the tetrahedron* can be written:

$$d\varphi_G - q_{G1}ds_1 - q_{G2}ds_2 - q_{G3}ds_3 = O(\varepsilon^3)$$

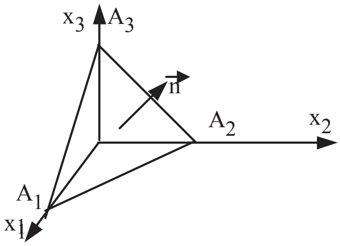


Figure 2.2. *Quantities balance on an elementary tetrahedron*

We obtain:

$$ds_i = \alpha_i ds \qquad (i = 1,2,3)$$

where α_i designates the directional cosines of the normal $\vec{n}$ oriented towards the exterior of the tetrahedron.

By replacing the surfaces ds_i with their previous expressions and by letting ε towards 0, the terms of order ε^2 should cancel out. By defining *vector* $\vec{q}_G$ or the *flux density of the quantity G,* with components (q_{G1}, q_{G2}, q_{G3}), we obtain the elementary flux of the quantity *G* across *ds* (the quantity of *G* crossing *ds* per unit time in the direction normal $\vec{n}$):

$$d\varphi_G = q_{Gi}\alpha_i ds = \vec{q}_G.\vec{n}\ ds \qquad [2.2]$$

The flux $\varphi_{G\Sigma}$ of the quantity G across the surface S can be written:

$$\varphi_{G\Sigma} = \int_S \vec{q}_G.\vec{n}\ ds \qquad [2.3]$$

By a similar argument, it can be shown that the transfer of a *vector* quantity G_i would be characterized by a *tensor* q_{Gij} whose flux across an elementary surface ds would give the vector quantity G_i which crossed ds per unit time:

$$d\varphi_{Gi} = q_{Gij} n_j ds = \overset{=}{q}_G.\vec{n}\ ds \qquad [2.4]$$

where the tensor $\overset{=}{q}_G$ is defined, as before, using the vector fluxes across the co-ordinate planes.

2.1.3.2. *Examples of flux of extensive quantities*

The preceding property did not require any hypothesis regarding the transfer mechanisms of the quantity G (action by contact or action at a distance): it results solely from the notion of balance for an extensive quantity. As an example, we can consider the flux of extensive quantities due either to propagative phenomena (example 1) or phenomena involving action by direct contact (examples 2 and 4):

1) Energy transfer by electromagnetic radiation (energy-flux density vector in lighting, infra-red heating, etc.) or by acoustic propagation (acoustic intensity vector).

2) Heat transfer in a material (molecular agitation energy at the molecular scale) is represented by the *thermal flux density* vector (section 2.1.5.3.1).

3) The diffusion of a chemical species in a material medium is represented by the *molar flux density* vector of the species considered (n: the number of moles/volume) (section 2.4.2).

4) For example, if the extensive quantity is a *force density* (or momentum density at the microscopic scale), the corresponding flux density is the *stress tensor* σ_{ij}. However, the convention for the orientation of the normal is here reversed. The elementary force is the *force exerted* on the surface ds, *by the material situated on the side of the normal* (side 1 in Figure 2.3a), *towards the other side*:

$$df_i = \sigma_{ij} n_j ds \qquad [2.5]$$

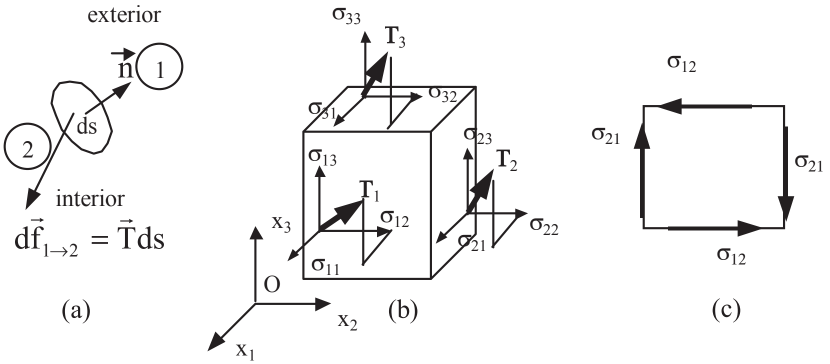


Figure 2.3. *Definition and symmetry of the stress tensor*

The force density at the point M on the surface *ds* is the *stress* $\vec{T}$:

$$\vec{T}(M,t) = \frac{d\vec{f}}{ds} = \overline{\overline{\sigma}}.\vec{n}$$

Figure 2.3b shows the stresses on the faces of a parallelepiped whose corners are parallel to the axes. The reader can easily verify that in the absence of volume sources of torques, the equality of the moments about the axis *Ox₃* of the stresses exerted on the four faces parallel to the axis *Ox₃* (see Figure 2.3c) leads to $\sigma_{12} = \sigma_{21}$. The same goes for the other components of the stress tensor ($\sigma_{ij} = \sigma_{ji}$).

2.1.3.3. *Volume source equivalent to the fluxes*

The balances in a material domain are clearly always effected *on closed surfaces* which comprise the boundary. By applying the Ostrogradski theorem the flux $\varphi_{G\Sigma}$ of the quantity *G* leaving a closed surface Σ can be written:

$$\varphi_{G\Sigma} = \int_{\Sigma} \vec{q}_G.\vec{n}\ ds = \int_D div\vec{q}_G dv \qquad \left(= \int_D \frac{\partial q_{Gi}}{\partial x_i} dv \right) \qquad [2.6]$$

This flux can thus be written in the form of a volume integral, which implies that the total flux of the quantity *G* on a closed surface is equivalent to the action of the *volume source div* $\vec{q}_G$ *of the quantity G.*

In the case of stresses in a continuous medium, a force volume source generated by a stress is equal to the vector $\overrightarrow{div}\,\overline{\overline{\sigma}}$:

$$\int_\Sigma \vec{T}(M,t)\,ds = \int_\Sigma \overline{\overline{\sigma}}.\vec{n}\,ds = \int_D \overrightarrow{div}\overline{\overline{\sigma}}\,dv$$

Reciprocally, all *volume sources* which are mathematically expressed by means of a *divergence operator* ($div\,\vec{q}$ for instance) can be interpreted as a transfer by the flux across a closed surface of vector flux density $\vec{q}$.

2.1.4. *Balance equations in continuous media*

2.1.4.1. *Balance equation of an extensive quantity*

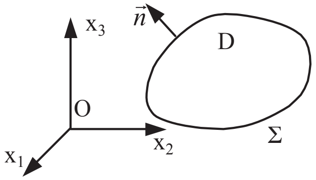


Figure 2.4. *Balance of an extensive quantity*

The balance equation of a volumetric extensive quantity $\overline{g}$ consists of writing that the variation of this quantity in a material domain D is due to contributions from outside, and which have thus crossed the closed surface Σ, which constitutes the boundary of D (the normal is directed outwards), and to volume sources of density σ_G in D:

$$\int_D \frac{\partial \overline{g}}{\partial t}\,dv = \int_D \sigma_G\,dv - \int_\Sigma q_{Gj} n_j\,ds \qquad [2.7]$$

Supposing this relation to be true regardless of the domain D, we can deduce the local equation:

$$\frac{\partial \overline{g}}{\partial t} = \sigma_G - \frac{\partial q_{Gj}}{\partial x_j} \qquad [2.8]$$

For a fixed medium, which can thus not be deformed, mass[1] must be strictly conserved, and we have:

$$\frac{\partial \rho}{\partial t} = 0$$

Balance equation [2.8] for the quantity G can also be written, using the mass quantity g:

$$\rho \frac{\partial g}{\partial t} = \sigma_G - \frac{\partial q_{Gj}}{\partial x_j} \qquad [2.9]$$

For a fixed continuous medium, the principal extensive quantities are internal energy (in its thermal form, and in the absence of possible physicochemical reactions), volume and the number of moles of a chemical species. Electric quantities may also be manifested; however, we will not deal with such questions in this work.

For a material of constant *specific heat* C, in the absence of chemical reactions, the variation of volumetric internal energy de is equal to ρCdT, and the balance equation can be written:

$$\rho C \frac{\partial T}{\partial t} = \sigma_T - \frac{\partial q_{Tj}}{\partial x_j} \qquad [2.10]$$

Heat-source volume σ_T is in fact a residuum of another form of energy which is present in the mass and which is degraded in the form of heat (Joule effect caused by the passage of an electric current (Ohm’s law), absorption of electromagnetic radiation, etc.). This result could be obtained in a general way, but such a discussion is beyond the scope of this work. We will come back to this question in Chapters 3 and 4 when it comes to dealing with mechanical energy.

For a chemical species comprising a number of moles n_i per unit volume (molar concentration), the balance equation can be written:

$$\rho \frac{\partial n_i}{\partial t} = \sigma_{ni} - \frac{\partial q_{nij}}{\partial x_j} \qquad [2.11]$$

---

[1] Excepting diffusion processes, for which we only give volumic balances in this chapter (section 2.4.4.2.2).

The molar flux densities q_{ni} and the sources σ_{ni} of chemical species are due, respectively, to diffusion phenomena and chemical reactions.

2.1.4.2. *Entropy source*

During the natural evolution of a system, entropy is not conserved and it can only increase. We have already described the entropy-creation mechanism in the case of two discrete sub-systems that are in contact and have different temperatures (section 1.4.2.3) and (section 1.4.2.4). The zone where entropy is created was localized in the contact zone between the two sub-systems. Here the distribution of entropic variables is continuous, and entropy creation will be diffused and associated with the existence of gradients with entropic variables.

On account of the local equilibrium hypothesis, relation [1.13] can be written, for the per mass unit quantities:

$$\rho\frac{\partial s}{\partial t} = \frac{\rho}{T}\frac{\partial e}{\partial t} + \rho Z_i\frac{\partial g_i}{\partial t} \tag{2.12}$$

By replacing the derivatives $\rho\dfrac{\partial g_i}{\partial t}$ in equation [2.12] by expressions obtained using balance equations [2.9], the following relation can be obtained:

$$\begin{aligned}\rho\frac{\partial s}{\partial t} &= \frac{\rho}{T}\frac{\partial e}{\partial t} + Z_k\left(\sigma_{Gk} - \frac{\partial q_{Gki}}{\partial x_i}\right)\\ &= \frac{\rho}{T}\frac{\partial e}{\partial t} + Z_k\sigma_{Gk} - \frac{\partial\left[Z_k q_{Gki}\right]}{\partial x_i} + q_{Gki}\frac{\partial Z_k}{\partial x_i}\end{aligned} \tag{2.13}$$

in which the surface flux terms associated with the divergence operator have been separated from those associated with the volume source as defined in section 2.1.3.3. The entropy is associated with the extensive quantities and the corresponding entropy addition are represented by two terms:

- $Z_k\sigma_{Gk} = \sigma_S$ *external entropy source* associated with the source σ_{Gk} of the extensive quantity *Gk*;
- $Z_k q_{Gki} = Z_S$ entropy flux associated with the flux of extensive quantities.

The thermodynamic imbalance is characterized by the gradients $\overrightarrow{grad Z_i}$, which we designate under the label *thermodynamic forces*. The additional term $q_{Gki} \dfrac{\partial Z_k}{\partial x_i}$ of equation [2.13] is the *entropy source associated with the local macroscopic imbalance*. It is always positive. Its expression is analogous to the expression for the creation of entropy $\dfrac{dX_{1,2}}{dt}\left(Z_1 - Z_2\right)$ from formula [1.31].

2.1.5. *Phenomenological laws*

2.1.5.1. *Introduction*

An irreversible evolution is characterized by flux density fields of the quantity $\vec{q}_{Gk}$ in a continuous medium. These thermodynamic fluxes are associated with gradients of intensive entropic quantities Z_i. The transfer of a quantity G_i can occur via relatively different kinds of processes:

1) *Molecular* agitation leads to an exchange of extensive quantities between the particles involved. These intermolecular actions occur from place to place, because it is these microscopic particles themselves which transport the extensive quantity considered (mass, momentum, energy, chemical species, etc.). The interaction zone between two material domains D_1 and D_2 separated by the surface S (Figure 2.5) is limited to a thickness in the order of 2d (d: intermolecular distance in liquids or mean free path in gases). This extremely thin zone is modeled on the macroscopic scale by the surface S, on which we can consider *contact actions*.

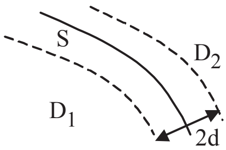


Figure 2.5. *Interaction zone between material domains* D$_1$ *and* D$_2$

2) On the macroscopic scale, and for turbulent flows, we observe chaotic velocity fluctuations which lead to fluxes of extensive quantities convected by the fluid. The effective interaction zone between two material domains is no longer limited to a surface as before. The flux depends on the structure of the whole turbulent zone. Considerable difficulties result from this situation in which the cause of the flux of an extensive quantity can no longer be modeled with a general local phenomenological law ([COU 89], [MAT 00], [TEN 72]).

3) In other situations, the flux of an extensive quantity (essentially energy) is due to the presence of an electromagnetic field. This is the case for energy transfer by thermal radiation in semi-transparent media, which both emit and absorb at all points and whose local state results from the emission balance in a macroscopic volume surrounding the point considered. Here again, we can no longer localize the cause of the extensive quantity flux on a single surface.

2.1.5.2. *Contact actions and thermodynamic forces*

The interaction zone between two material domains D_1 and D_2 is modeled by the surface *S*. The thermodynamic forces, represented by $\overrightarrow{grad Z_i}$, are the cause of thermodynamic fluxes.

As with discrete systems (section 1.4.2.6) *all* causes of the same tensorial nature act on *all* the corresponding effects and we have a *coupling of irreversible phenomena*: for example, a temperature gradient leads to a material flux (thermal diffusion). Phenomena of different tensorial orders do not interact.

A rudimentary explanation of these facts can be provided from context of kinetic gas theory. A gas is a set of molecules which are subjected to a thermal agitation. Irreversible phenomena are the macroscopic result of this *spontaneous action*. Molecules with different properties (mass, type, kinetic energy, etc.) do not respond in the same way to non-symmetries in the mean properties of the medium. A molecular concentration of a given species will be progressively diluted in the rest of the gas; a temperature gradient (gradient of the molecular kinetic energy) will not act in the same way on different species of molecules and so may create a concentration gradient. For example, at equal energy, we notice that smaller, and therefore faster, molecules can slip in a gas comprising larger molecules, hence the phenomenon of *thermal diffusion*. On the other hand, it is difficult to see how the static scalar properties of a gas which is macroscopically at rest can spontaneously generate a vector macroscopic momentum (i.e. a bulk movement) in the absence of an external influence.

There thus exists a relation between thermodynamic forces and thermodynamic fluxes of the same tensorial rank. Since in the absence of thermodynamic forces, the thermodynamic fluxes are zero, the general form of this relation can be written as:

$$\vec{q}_{Gk} = \vec{F}_k\left(\overrightarrow{gradZ_l}\right) \qquad (k,l = 1,...,K) \qquad \text{with: } \vec{F}_i\left(\vec{0},\vec{0},...\right) = \vec{0} \qquad [2.14]$$

On account of the principle of action and reaction, the function $\vec{F}_k$ is odd ($\vec{F}_k\left(\overrightarrow{gradZ_l}\right) = -\vec{F}_k\left(-\overrightarrow{gradZ_l}\right)$). Relation [2.14] must verify properties of homogeneity and of spatial and material isotropy. In what follows, we will limit our discussion to cases where the difference from thermodynamic equilibrium is relatively small, so that we can justify a first order Taylor expansion of the functions $\vec{F}_k\left(\overrightarrow{grad Z_l}\right)$:

$$\vec{F}_k\left(\overrightarrow{grad Z_l}\right) = L_{kl}\,\overrightarrow{grad Z_l}$$

For a set of K extensive quantities, L_{kl} is a square matrix of order K whose terms are functions of the values of the K intensive quantities Z_k at the point Z_{ke}, about which linearization is effected. When the preceding approximation is valid, we consider that we are dealing with *linear thermodynamics* of irreversible phenomena.

If the medium is considered to be isotropic, the matrix is reduced to a matrix of dimension K (see section 2.1.5.3.1).

Thermodynamics does not provide access to any properties associated with the matrix L_{kl}. However, by means of statistical reasoning the matrix can be shown to be symmetric (Onsager's relations). This *symmetry is only verified if the entropic variables Z_k are used* to define this matrix.

In practice, the local imbalance is characterized using simpler variables than the intensive entropic variables Z_k. For example, we use temperature in place of the entropic intensive variable $1/T$ (section 1.3.1.1). Expressions for linearized thermodynamic force are equivalent to first order, but the matrix coefficients L_{ij} are modified, such that the symmetric properties generally disappear. An analogous situation has been observed in the case of discontinuous media (section 1.4.2.6.2).

2.1.5.3. *Some simplifying laws for irreversible transfer*

2.1.5.3.1. Fourier's law and thermal conduction

Consider firstly a case where the medium comprises a pure body, such that thermal transfer occurs alone, without any coupling with diffusion or electrical conduction. The relation between the thermal flux density and the thermal gradient can be written in the context of linear thermodynamics (Fourier's law) as:

$$\vec{q}_T = -\overline{\overline{\lambda}}.\overrightarrow{gradT} \quad \text{or} \quad q_{Ti} = \lambda_{ij}\frac{\partial T}{\partial x_j}$$

The principal axes of the 3*3 tensor λ_{ij} are the symmetric axes of the medium. If the medium is homogenous (fluids, non-crystalline solids, etc.), the three eigenvalues of the tensor λ_{ij} are equal and this one is spherical ($\lambda_{ij} = \lambda\delta_{ij}$); it can thus be expressed solely as a function of the *thermal conductivity* λ of the medium. Fourier's law can be written as:

$$q_{Ti} = \lambda \frac{\partial T}{\partial x_i} \qquad \vec{q}_T = -\lambda\, \overrightarrow{gradT} \qquad [2.15]$$

The thermal conductivity of the medium λ, is expressed in Watts/meter.K.

Heat is the macroscopic form of mechanical energy related to thermal agitation of a medium. Its transfer in the medium is due to interactions between the microscopic entities (molecules, atoms, ions, electrons). The values of the coefficient λ depends on the nature of the medium.

Metallic media are excellent thermal conductors, thermal conduction being principally assured by electrons which have a high mobility. The values of the coefficient λ lie in a range spanning from a few tens to a few hundred W/m.K.

Solid crystalline media are generally good conductors: as the crystalline structure presents a reasonably strong coherence on account of its organization, energy can be transmitted in a vibrational form (phonons). The coefficient values λ are in the order of a few W/m.K.

Solid amorphous media or composites have weaker conductivity: well under 1 W/m.K for fibrous materials.

Liquids, comprising a looser structure, are poorer conductors of heat than solids (of the order of 0.1 to 0.2 W/m.K): intermolecular forces here assure conduction.

Energy exchange in *gases* only occurs by means of molecular collisions in the gaseous medium; the corresponding values of the coefficient λ are of the order of 0.01 to 0.02 W/m.K. Insulating materials are constituted of a matrix (fibers, wools, foams, etc.) which is as light as possible, thermal insulation being assured by the gaseous interstices.

The thermal flux across a surface *S* can thus be written as:

$$\varphi_T = \int_S \vec{q}_T.\vec{n}ds = -\int_S \lambda\, \overrightarrow{gradT}.\vec{n}ds = -\int_S \lambda \frac{\partial T}{\partial n}.ds$$

We can also define the *thermal diffusivity a*:

$$a = \frac{\lambda}{\rho\, C}$$

The reader can easily verify that this quantity a can be expressed in m^2/sec; we will see the important role that this quantity plays in the heat equation (section 2.3.1).

The thermal diffusivity takes on the following values at 20ºC:

– air: 0.19×10^{-4} m^2. s^{-1};

– water: 1.4×10^{-7} m^2. s^{-1};

– metals: $\sim 10^{-4}$ m^2. s^{-1}.

2.1.5.3.2. Fick's law and the diffusion of chemical species

As the extensive variable is here the number of molecules of a chemical species, the corresponding intensive entropic variable is $-\frac{\mu}{T}$ (where μ is the chemical potential (section 1.3.1.1)). In ideal solutions or perfect gases, we replace the intensive entropic variables with a molar or mass concentration variable. This simplified approach will suffice here for an exposition and discussion of the phenomena which we will examine. It will allow us to outline the principal difficulties associated with fluid movement. The additional complications which arise when we take account of more complex thermodynamic and chemical properties are beyond the scope of this work.

The presence of a concentration gradient leads to the existence of a molecular flux towards regions of weaker concentration. Taking the molecular concentration n_1 (the number of moles per unit volume) of a component, Fick's law, which characterizes the diffusion of the species considered, can be written:

$$\vec{q}_{n_1} = -D\,\overrightarrow{grad}\; n_1 \qquad [2.16]$$

where D is the *diffusion coefficient*.

Diffusion is a complex phenomenon, as it implies that the material is not immobile, and in such cases it is not possible to effect a generalized reasoning using a single constituent. We will come back to this point at a later stage (section 2.4), and we will specify the conditions under which law [2.16] is valid.

2.1.5.3.3. Electrical conduction and Ohm's law

Electric conduction presents certain analogies with thermal conduction and diffusion; however, the underlying physics is a little different. Here we are dealing with an electric field derived from an electric potential V_{el} which exerts a force on mobile electric charge carriers. Ohm's law has the same form as the preceding laws:

$$\vec{j} = -\sigma_{el}\,\overrightarrow{grad}\ V_{el}$$

where $\vec{j}$ is the electric current density and σ_{el} is the electrical conductivity.

Thermal molecular agitation, which grows with temperature, slows down the movement of electric charge-carriers because of collisions; thus, electrical conductivity is a decreasing function of temperature, whereas for other irreversible phenomena, thermal agitation is the driving factor of thermal and diffusional fluxes.

2.1.5.3.4. General case: coupled transfer between diffusion and thermal conduction

In general, the flux of a scalar quantity (energy, chemical species) depends on all the local thermodynamic forces $\overrightarrow{grad}\,T, \overrightarrow{grad}\,n_i, \overrightarrow{grad}\,p,$ etc., associated with these scalar quantities.

We can schematically write under certain conditions the following relations for the diffusion of a constituent of binary mixing in the presence of heat transfer:

$$\vec{q}_T = -\lambda\ \overrightarrow{gradT} - K\,\overrightarrow{grad}\ n_1$$

$$\vec{q}_{n1} = -D_1^T\,\frac{\overrightarrow{gradT}}{T} - D\,\overrightarrow{grad}\ n_1$$

The coefficient D_1^T characterizes the thermal diffusion (Soret effect), i.e. the existence of a flux of chemical species which is associated with a temperature gradient. The symmetric effect of a heat flux caused by a concentration gradient (Dufour effect) and characterized by the coefficient *K* is generally much weaker.

From a physical point of view, the problem is more complex than this, as each chemical species introduces its own thermal energy (enthalpy at constant pressure, etc.). Definitions of the preceding coefficients may vary, and we will not provide a complete discussion of these phenomena, as this would require lengthy discussions concerning chemical thermodynamics ([BIR 01], [DEG 62], [PRI 68]).

2.2. Fluid statics

2.2.1. *General equations of fluid statics*

2.2.1.1. *Stresses in a fluid at rest*

We designate as a fluid any body which is in either a liquid or a gaseous state. In contrast to solids, fluids do not have any intrinsic shape. They adapt to the shape of the container in which they are found. On the microscopic scale, there is no longer a crystalline structure, and the molecules, ions, etc., interact by means of intermolecular forces and collisions. On the macroscopic scale, fluids generally have isotropic properties; the forces in a fluid at rest are thus represented at each point by a spherical stress tensor $p\delta ij$ which depends on a scalar quantity p, designating pressure. The force dfi exerted on an elementary surface ds by the material situated on the same side as the positive normal is (Figure 2.6a):

$$df_i = -p\delta_{ij} n_j ds = -pn_i ds \qquad \vec{df} = -p\,\vec{n}\,ds$$

2.2.1.2. *Conditions for the existence of equilibrium in a fluid*

A fluid in equilibrium is only subjected to pressure forces and to external forces of mass density g_i. By considering the net force balance on a domain D, we obtain:

$$-\int_{\Sigma} p.\vec{n}ds + \int_{D} \rho\ \vec{g}dv = 0 \qquad [2.17]$$

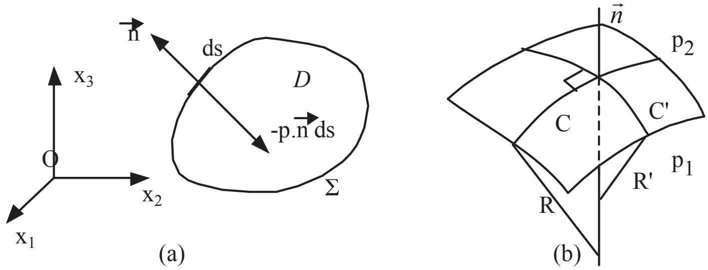


Figure 2.6. *(a) Pressure force on a surface; (b) Laplace's law*

Using the identity $\int_{\Sigma} f.\vec{n}ds = \int_{D} \overrightarrow{grad} f.dv$, we obtain:

$$-\frac{\partial p}{\partial x_i} + \rho\, g_i = 0 \qquad [2.18]$$

Equation [2.18] can only be solved if the vector ρg_i can be derived from a potential, in other words if the following condition is satisfied:

$$\overrightarrow{rot}(\rho \vec{g}) = \rho \, \overrightarrow{rot} \, \vec{g} + \overrightarrow{grad} \rho \wedge \vec{g} = 0$$

We can see that for a fluid of constant density, *an equilibrium situation can only exist if the force field* $\vec{g} = -\overrightarrow{grad}\, U$ *derives from a potential.*

Supposing that *the force field* g_i *can be derived from a potential* U *(*$\vec{g} = -\overrightarrow{grad}\, U$ *)* the preceding condition can be written:

$$\overrightarrow{grad} \; \rho \wedge \overrightarrow{grad}\, U = 0$$

This leads to the fact that surfaces of ρ *=const. and U=const.* are identical.

Equation [2.18] then becomes:

$$\overrightarrow{grad}\, p + \rho \; \overrightarrow{grad U} = 0 \qquad [2.19]$$

This also leads to the identity of isobaric and equipotential surfaces that are identical. If the fluid is *divariant*, with an equation of state in the form $p = p(\rho, T)$, the said surfaces are also *isothermal*.

For example, in the case of gravitational forces, equilibrium is only possible if the horizontal surfaces are isothermal. If this condition is not satisfied, *natural convection* will occur.

The resultant pressure force on the exterior of the closed surface Σ can be immediately deduced from relation [2.17]:

$$-\int_\Sigma p.\vec{n} ds = -\int_D \rho \, \vec{g} dv$$

This force is the opposite of the sum of the forces exerted by the force field $\vec{g}$ on the fluid contained in *D*.

Let us now consider the case of gravity. As with equation [2.17], the balance moment of forces on domain *D* – where Π is the center of gravity of this domain *D* filled with the fluid – can be written:

$$-\int_{\Sigma}\overrightarrow{\Pi M}\wedge p.\vec{n}ds=-\int_{D}\overrightarrow{\Pi M}\wedge\rho\,\vec{g}dv=\vec{g}\wedge\int_{D}\rho\,\overrightarrow{\Pi M}\,dv=0$$

which is zero from the definition of the center of gravity.

So, the collection of the pressure forces is equivalent to a single force, equal to the weight of fluid displaced and applied at the center of gravity Π of the domain supposed to be filled with the fluid. The point Π is called the *center of buoyancy*. This result constitutes *Archimedes' theorem*.

Now, considering a vertical cylinder with identical horizontal bases, we see that *the difference in pressure forces between the lower and the upper bases is equal to the weight of fluid contained in the cylinder*.

2.2.1.3. *Solution to the general hydrostatic equation*

If the preceding conditions are satisfied, the pressure satisfies equation [2.19]. The conditions for the existence of the solution (stratification of the space into identical surfaces U=const, p=const, ρ=const, etc.) for a fluid whose equation of state is $p=p(\rho,T)$ leads, under equilibrium conditions, to the relation $\rho=f(p)$ between p and ρ. We often say that such a fluid is *barotropic* (which is not strictly true, as this is not a property of the fluid, but of the configuration studied).

We can thus define the function $h(p)=\int\frac{dp}{\rho}$ such that:

$$dh=\frac{dp}{\rho}\qquad\left(\overrightarrow{grad}h=\frac{\overrightarrow{grad}p}{\rho}\right)$$

If the specific fluid entropy is constant over the entire domain studied, it is said to be homoentropic, and the function h is therefore the specific fluid enthalpy. General equation [2.19] can be written as:

$$\overrightarrow{grad}(h+U)=0$$

and its solution is: $h+U=\text{const}$. The pressure p can thus be obtained by inverting the function *h(p).*

In the case of gravity, the function U is equal to gz, where z designates an increasing vertical coordinate.

2.2.1.4. *Equilibrium of an incompressible fluid*

2.2.1.4.1. Hydrostatic equations

Such a fluid is generally either a liquid or a gaseous domain of limited vertical extent, such that its density can be considered constant. We thus consider density to be independent of pressure, which may eventually be a function of the coordinate z. If the fluid is considered to be of uniform density ρ_0, we have:

$$p + \rho_0 U = p_g = \text{const} \qquad \text{(U = gz for gravity potential)}$$

So, it is this quantity $p + \rho_0 U$ (in the case of gravity $p + \rho_0 gz$) and not the pressure alone which is constant in a motionless fluid. Hence, *variations of this quantity must be the causes of movement*. For this reason, we will call it "driving pressure" and we will note it as $p_g = p + \rho_0 gz$ in the case of gravity.

In fact, for such an incompressible homogenous fluid, we will find (see Chapter 4) that, under certain conditions, movement is associated with variations of this driving pressure alone, without explicit intervention from pressure.

EXERCISES –

1) Equilibrium of a homogenous liquid in a rotating reference frame of axis Oz. Give the expression for pressure as a function of the coordinates x and y and z. Show that in a reference frame which rotates with angular velocity ω about the vertical axis Oz, the isobar surfaces are paraboloids of revolution about the axis Oz.

(Answer: $p = \frac{1}{2}\rho\omega^2(x^2 + y^2) - \rho gz + \text{const}$)

2) Equilibrium of a homogenous liquid in a reference frame which is undergoing a constant horizontal acceleration γ. Derive an expression for pressure as a function of the abscissa x parallel to the acceleration and the altitude z. Show that in this reference frame isobar surfaces are inclined planes.

(Answer: $p = -\rho\left(gz + \gamma x\right) + \text{const}$)

2.2.1.4.2. Conditions at fluid-fluid interfaces

Continuity of the pressure between two immiscible fluids is only guaranteed if the interfacial surface is a plane. If this is not the case, the interface behaves in many respects like a membrane; its constant surface tension is characterized by the set of the two fluids, if they are pure, and generally decreases with temperature. The pressure discontinuity $p_1 - p_2$ across the interface is given by Laplace's law:

$$p_1 - p_2 = \sigma\left(\frac{1}{R} + \frac{1}{R'}\right)$$

R and R' are the radii of curvature of two interfacial surface sections by perpendicular planes which contain the normal to this interface at the considered point of this one. The pressure p_1 is greater on the concave side of the surface (Figure 2.6b).

On the other hand, the contact angle between an interface and wall is a physical characteristic between these three domains [BEN 06]. A liquid wetting the wall perfectly is joining tangentially to another liquid (water on a clean wall made of glass, for instance). Surface active substances generally ensure a perfect wetting of the surface, but they modify the value of the surface tension.

2.2.1.5. *Equilibrium of the atmosphere*

By limiting ourselves to the troposphere (lower limit of the stratosphere), the atmosphere can be considered to comprise a homogenous mixture of nitrogen and oxygen, and it is characterized by a rapidly decaying temperature with increasing altitude. It is not in thermodynamic equilibrium, because as oxygen is more dense than nitrogen, it should be more concentrated in the lower layers. In fact, the atmosphere is constantly undergoing movements whose characteristic time is small compared with the time required for the temperature to become uniform. The atmosphere is in fact *perfectly mixed*: in the movement of large air masses, the atmosphere undergoes rapid compression and rarefaction, which occur isentropically, and so during their movement air masses do not have the time to exchange heat by thermal conduction. This mixing leads to an atmosphere at constant entropy. Experiments confirm that the Earth’s atmosphere is in a state which is very nearly homoentropic. This condition defines the standard atmosphere, whose differences from reality, in the order of a few percent, are due to meteorological phenomena.

EXERCISES –

Calculate the pressure p as a function of the altitude z:

1) in an isothermal atmosphere of temperature T_0: ($p = p_0 \exp\left(-\frac{g(z - z_0)}{rT_0}\right)$);

2) in a homoentropic atmosphere, where the temperature at the ground is equal to T_0.

2.2.1.6. *Stability of equilibrium solutions in stratified fluids*

A stratified fluid with horizontal surfaces of constant density is in an equilibrium situation in the gravitational field. The stability of this fluid can be tested in an elementary way by displacing a fluid particle and observing whether or not it undergoes a restoring force. Let us first of all consider an *incompressible* fluid whose density, which depends only on temperature, increases with increasing altitude. A fluid particle which is displaced in the upward direction finds itself in a region of heavier fluid; it thus undergoes a greater thrust, and its movement upward is amplified.

The same reasoning applied to a *compressible* fluid (the atmosphere for example) leads to the following situation: as the displacement occurs isentropically, it is the specific entropy of the particle which must be compared with that of the local environment into which the fluid particle is moved. If the local entropy of ambient fluid falls as the fluid particle rises, the temperature of the particle is greater than that of this ambient fluid: the Archimedes force thus increases and the fluid particle comes up.

In conclusion, we have *equilibrium stability of a fluid if its temperature (or its entropy) increases with increasing altitude*. In the opposite case, the fluid system is unstable.

2.2.2. ***Pressure forces on solid boundaries***

2.2.2.1. *Constant pressure*

A constant pressure situation occurs in numerous applications: pipe-systems and compressed-air reservoirs, water-distribution networks, etc., where the effects of an external force field is often negligible as a result of the high pressure levels which occur within the system.

Consider a surface S (Figure 2.7a) with normal $\vec{n}$ on which uniform pressure p is exerted, the pressure force on this surface S is:

$$\vec{F} = -\int_S p\,\vec{n}\,ds$$

Letting ds' be the projection of the elementary surface ds on the plane perpendicular to the direction Ox, the component Fx on the axis Ox can be obtained from:

$$F_x = -\int_S p\,\vec{x}.\vec{n}\,ds = -\int_{S'} p ds' = -p.S'$$

The component of the uniform pressure force in the direction Ox is equal to the pressure force on the surface S', which is a projection of S on the plane perpendicular to Ox.

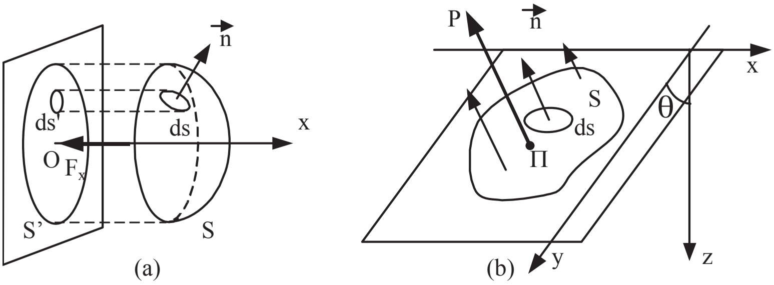


Figure 2.7. *Pressure forces: (a) arbitrary surface with constant pressure; (b) forces due to a liquid on a plane surface*

In practice, this surface is an element of a pipe, a reservoir, etc. Let us take the example of a cylindrical pipe with radius *R*. The stress *T* on the wall, due to the overpressure Δp in the pipe, must balance the pressure forces on a half-cylinder. It is thus equal to ΔpR per unit length: *the stress on the wall increases with the radius R.* For a tube of radius 1 cm transporting compressed air pressurized to 100 atmospheres, this is equal to 10^5 N/meter.

Pipeline and reservoir dimensions (*diameters*) ought to be as small as possible, in order to avoid technical difficulties related to the wall resistances.

2.2.2.2. *Hydrostatics*

The equilibrium of a homogenous liquid (often water) in a gravitational field is of considerable practical importance, particularly in cases such as water reservoir walls, sluice gates and boat hulls.

The *pressure forces on a plane surface*, one side of which is covered by a liquid, are parallel to the wall normal: they are thus equivalent to a single vector applied at a given point on the wall, the *center of pressure* Π. Consider a plane surface *S* identified by the coordinates *Oxy* in its plane (Figure 2.7b). Suppose that the free surface of the water is at $z = 0$. The hydrostatic pressure is equal to $\Delta p = \rho g z$. Denoting the center of inertia of the surface *S* as *G*, the *thrust P* (the resultant pressure force) is equal to:

$$P = \int_S \rho gzds = \rho g \cos\theta \int_S yds = \rho g \cos\theta . y_G S = \rho\, gz_G S = \Delta p_G S$$

The *thrust P* is equal to the *product of the pressure at the center of inertia G of the homogenous surface and its surface S.* The center of pressure, which can be easily calculated by taking the moment of the pressure forces with respect to *Oz*, is beneath the inertia centers.

The forces exerted by water on the walls of large reservoirs and dams are often considerable; the construction of dams thus involves massive elevations of earth whose weight blocks the mass of water (weight dams), or large concrete constructions which resist by returning the thrust on the rocky walls (mountain arch dams).

The reader can easily verify that the *horizontal component* (along *Ox*) of the hydrostatic pressure force on a curved surface is equal to the hydrostatic pressure force on the surface *S'*, which is the projection of *S* on a plane perpendicular to *Ox*. This result clearly makes no sense for a vertical component.

EXERCISES –

1) Calculate the coordinates of the pressure center Π on the surface *S* of Figure 2.7b. (Answer: $x_\Pi = x_G$, $y_\Pi = y_{Gy} + I_{G_y} / S$, I_{Gy} giving the inertia moment of the surface taking with respect to a straight line parallel to *Oy* and passing through *G*.)

2) Calculate the load-force generated on a vertical rectangular wall, 4 m in width and containing a mass of water 3 m in height. Determine the position of the center of pressure. (Answer: 18.0×10^4 newtons, 1 m above the bottom.)

3) Answer the same questions for a dam in the form of a 50 m high equilateral triangle.

2.2.2.3. *Floaters*

A floater is a body placed on the surface of a liquid, and whose weight is less than that of an equivalent volume of the same liquid. In this kind of situation there is always an equilibrium position, such that the force exerted by the fluid on the body balances its weight. However, the stability of this position is not guaranteed. We will not study hull-stability problems ([BAR 01]) which requires geometrical knowledge related to surface curvature and normal fields ([KRE 91]).

We will only discuss a simple example in order to illustrate the origin of such problems. Consider a homogenous, elliptical cylinder whose density is half that of water, such that at equilibrium, the center of the ellipse is always in the plane free surface of the water. While in equilibrium, the upward thrust which the cylinder experiences, *P*, applied at the center of buoyancy Π, opposes that of its weight, *P*, applied at the center of gravity of the ellipse, *G*, which lies in a vertical section of the meridian plane of the cylinder. There are two equilibrium positions ((a) and (c) in Figure 2.8) corresponding to the vertical positions for each axes of the ellipse. The equilibrium stability is studied in the usual way: a small clockwise rotation of the cylinder is performed (positions (b) and (d) in Figure 2.8).

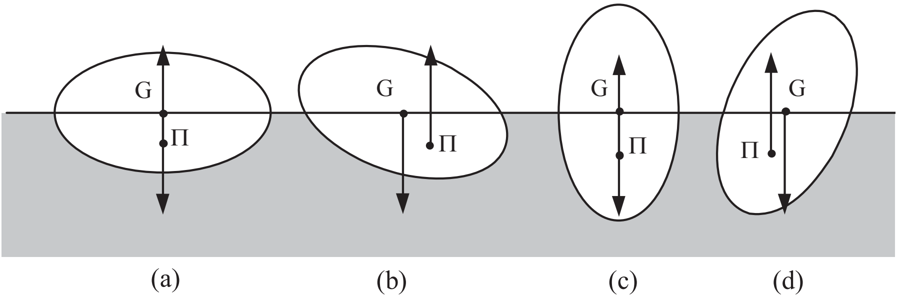


Figure 2.8. *Stability study of a floater whose cross-section is elliptic*

We assume that the submerged volume (beneath the flotation plane) and the thrust *P* remain constant during the displacement; however as the distribution of the volume submerged has changed, the center of buoyancy Π is moved to the side where the submerged volume has increased (with respect to the vertical passing through *G*), thus creating either a restoring moment (situation (b)) or an amplifying moment (situation (d)), depending on the situation: position (a) is stable, whereas position (c) is unstable.

In fact for a real hull we must also take account of the way it is loaded, and possible movements of this load. For the preceding elliptical floater, we see that we can add a weight to the upper surface of the cylinder: the system will be stable as long as the restoring moment is less than the moment generated by the extra weight. On the other hand, a circular cylinder is in a state of neutral equilibrium if it is not loaded; it obviously becomes unstable if any weight is added to the upper side. Escorting floating logs during their floatation down rivers is not a straightforward use of fluid statics!

2.3. Heat conduction

2.3.1. *The heat equation*

By inserting expression [2.15] into relation [2.10] we obtain a thermal balance equation:

$$\rho\, C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial T}{\partial x_i} \right) + \sigma_T$$

which, if the thermal conductivity of the medium λ is constant, becomes:

$$\rho\, C \frac{\partial T}{\partial t} = \lambda\, \Delta T + \sigma_T \qquad \text{with } \Delta = \frac{\partial^2 T}{\partial x_i \partial x_i}$$

In the absence of thermal power sources ($\sigma_T = 0$), we obtain the heat equation:

$$\frac{\partial T}{\partial t} = a \Delta T$$

The thermal diffusivity $a = \lambda / \rho C$ can be expressed in meter^2/second.

2.3.2. *Thermal boundary conditions*

Here we will deal with the principal problems of heat conduction, which we will later encounter in a similar form when we deal with viscous fluid mechanics. Thermal boundary conditions can generally be classed in one of the following categories:

1) *Dirichlet condition: the temperature distribution T(M) is known* at all points on the boundary. This is often the case for recipient walls which contain an agitated fluid, walls which are in contact with highly conductive solids (metals, etc.); the corresponding practical problems relate to heating, thermal insulation, etc.

2) *Neumann condition: the heat-flux density on the boundary* $q_{Tp}(M)$ *is known* at all points M on the boundary. These conditions are found in the presence of heat sources resulting from the dissipation of other forms of energy, such as heating by Joule effect, heat produced by nuclear reactors, radiation absorption, etc.

3) *Mixed condition:* where a linear relationship exists between the wall flux-density and the temperature difference between the wall temperature *Tp(M) and a given temperature* T_0:

$$q_{Tp}(M) = -\lambda\left(\frac{\partial T}{\partial n}\right)_p = h\left(T_p - T_0\right)$$

The coefficient *h* of the above relation is called the *heat-transfer coefficient* of the wall. It can represent diverse phenomena, such as an external wall, heat transfer across the boundary layer of a flowing fluid, etc.

Furthermore, as we have already said, it is necessary to fix the initial temperature distribution in the domain being studied.

In addition to a detailed knowledge of temperature fields, other important unknown quantities are:

– either the wall heat-flux-density and the thermal power issuing from the domain studied when the wall temperature is given;

– or the wall temperature in the case where the heat-flux wall is given.

Finally, in the case where chemical reactions occur at the boundary (evaporation, fusion, catalytic reactions, oxidation, electrolytic reactions, etc.), we have more complex conditions. So, it is often at the boundary that heat is released (positively or negatively) with a power proportional to the speed of a reaction. As the reaction is often limited with respect to the transfer towards the wall of certain reactants present in the fluid, we have a coupling between a thermal problem and a problem related to the diffusion of the fluid constituents in which the presence of flows plays an essential role. For example, in evaporation of a wet wall in an air flow (temperature measured by a wet-bulb thermometer), the quantity of liquid evaporated and heat absorbed in the wall surface are increasing functions of the flow velocity, but the wall temperature is quite independent of the flow velocity ([BIR 01]).

2.4. Diffusion

2.4.1. *Introduction*

2.4.1.1. *Definition*

When a fluid F_1 is carefully introduced to a recipient which contains a fluid F_2 which is miscible with F_1, after a certain duration we notice that movements related to mixing have ceased, and that hydrostatic equilibrium is attained; however, an inhomogeneity can persist in the concentration (this can be observed by differences in color for example). This indicates that the phenomenon is an extremely slow function of time. This *migration of the components with respect to one another* constitutes a phenomenon known as *material diffusion* which is characterized by the fact all of the *components* of a mixture do not have exactly the same speed, and these differences lead to variations in the composition of the mixture.

The molecular velocity of a body of given chemical species which makes up a mixture is thus engendered by two causes: the diffusion and the bulk movement of the mixture (convection). The separation of these two causes must be very carefully considered, as the bulk movement results from the behavior of all of the constituents. Interactions between the two kinds of phenomena (convection and diffusion) are encountered in all mixing processes with or without chemical reactions (for example, sugar, which dissolves in coffee).

2.4.1.2. *Microscopic interpretation of diffusion*

This difference in the behavior of the two components is obviously related to molecular agitation, which leads to a macroscopic displacement and is different for the two component*s*. The molecular agitation can be directly visible under certain conditions in the presence of very small particles (in the order of a few micrometers) in suspension in a fluid. Examined under a microscope these particles present erratic behavior: this is known as *Brownian movement*, which is a macroscopic manifestation of molecular agitation. These movements obey diffusion equations.

Let us consider two molecular species (black and white molecules in Figure 2.9a). By supposing that the two categories of molecules have the same energy (in other words the same temperature), each molecular species moves as a result of collisions. These collisions have a random character; for a resting gas made up of one or many species of molecule, uniformly distributed, we see that the molecules are statistically stationary, because each species only permutes between themselves. During collisions and molecular interactions in the volume shown in Figure 2.9a, there are more black (or respectively white) molecules at the bottom (or at the top), but they will end up uniformly distributed in the container considered. The statistical aspects of the second principle of thermodynamics leads to a complete mixing of the two species, whose concentrations become uniform.

However, in this container, where the pressure is supposed constant, the total molar concentration remains constant at all points, with the species whose concentration decreases being replaced by the other species whose concentration increases. This pressure constraint leads *to a relation between migrations of the two species.*

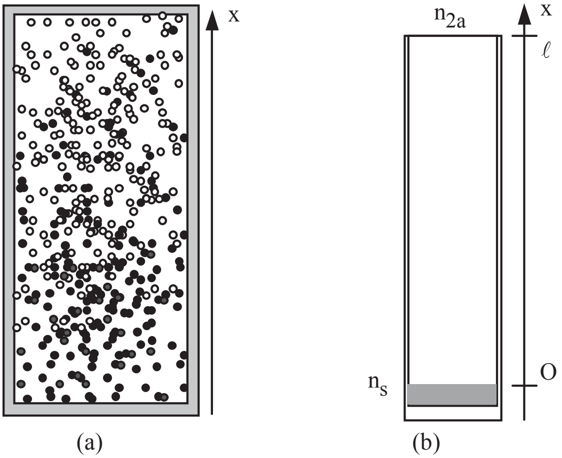


Figure 2.9. *(a) Gaseous diffusion in a volume; (b) evaporation of a liquid*

Evaporation towards the atmosphere of a liquid contained in a tube (Figure 2.9b) is another example of gaseous diffusion where we have a vapor flux from the liquid towards the exterior across air, which is at rest in the tube. Note that the vapor crosses the free surface of the liquid (at the lower wall where evaporation takes place) and diffuses towards the exterior air, whereas the air, which does not dissolve in the liquid, does not cross the free surface.

For the sake of simplicity, we will consider the case of mixing between two perfect gases: gravity here often plays a negligible role at the laboratory scale. Kinetic gas theory allows the modeling of phenomena at the molecular scale. We will limit the discussion to cases where the temperature of the systems studied is uniform; if this was not the case, it would be necessary to introduce a further phenomenon: thermal diffusion. The interested reader should refer to classic texts on the subject (see section 2.4.5).

It should nevertheless be noted that, depending on the problem studied, the influence of external forces or accelerations on diffusive phenomena should be taken into account (centrifugation for example).

2.4.1.3. *Extensive variables of a mixture*

The composition of a continuous medium where k chemical species are present can be characterized by the local reduced extensive concentration variables which can be either:

– the *number of moles* n_i of each species *per unit volume* (molar concentration);

– the *partial density* ρ_i of the species i, which is the mass per unit volume ($\rho_i = n_i M_i$) of the molecules of species i, of molar mass M_i (mass concentration);

– the *partial pressure* p_i associated with the molecules of species i supposed only to occupy an elementary local volume of the mixture.

As the preceding quantities are additive, we can define the total number of moles n (or molecules) per unit volume, and the density ρ of the mixture:

$$n = \sum_{i=1}^{k} n_i; \qquad \rho = \sum_{i=1}^{k} \rho_i = \sum_{i=1}^{k} n_i M_i \qquad [2.20]$$

If the system comprises a perfect gas, which we assume in what follows, the *total pressure is equal to the sum of the partial pressures* of the different constituents.

We also consider the concentrations of the species i by means of relative values:

– molar fraction γ_i;

– mass fraction ω_i:

$$\gamma_i = n_i / n; \ \omega_i = \rho_i / \rho \qquad [2.21]$$

According to [2.20], these variables respect the relations:

$$\sum_{i=1}^{k} \gamma_i = 1; \qquad \sum_{i=1}^{k} \omega_i = 1$$

While the density ρ or the number of moles n per unit volume of a mixture appear in the global mixing equations, the composition of the mixture is in fact characterized by the given concentrations of only $k - 1$ components.

In a liquid medium, the *molar concentration* C_i (also called molarity) is expressed as the number of moles of a species contained in a volume equal to 1 liter.

It often happens that the mixture contains a constituent 1, which is dominant, and a weak proportion of a species i:

$$n_i << n_1 \quad \text{or:} \quad p_i << p_1 \quad \text{or:} \quad \rho_i << \rho_1 \quad ....$$

All relative concentrations of species 1 then have the same first order value. For example, it is thus possible to confuse the molar fraction $\gamma_i = n_i / n$ with the quantity $\gamma_i = n_i / n_1$, the errors thus committed being in the order of $(n_i / n_1)^2$. The same goes for the mass fraction $\omega_i = \rho_i / \rho$ and the quantity ρ_i / ρ_1, the partial pressure p_i / p and p_i / p_1. All these variables of relative concentration are proportional. We will note as c_i the species concentration *i* present in small quantities in a mixture.

2.4.2. *Molar and mass fluxes*

2.4.2.1. *Flux of a component*

We have seen (see section 2.1.3.1) that for any scalar extensive quantity *G*, we can identify a corresponding vector *flux density* $\vec{q}_G$, whose flux across the surface (*S*) (equation [2.3]) represents the quantity of *G* which crosses *S* per unit time. The extensive quantities of a pure body (volume, energy, etc.) are associated with the matter which we suppose fixed in the reference frame. In the presence of diffusion, not only does this reference not exist, but the flux densities of the components, as a result of their movement, depend on the particular reference which is chosen. We will return to this choice at a later stage...

It can easily be seen that the matter which crosses a surface *ds* at a velocity $\vec{V}$ over a period of length δt, comprises a cylinder of length $\vec{V}\delta t$ whose volume is equal to $\vec{V}.\vec{n}\,\delta t$.

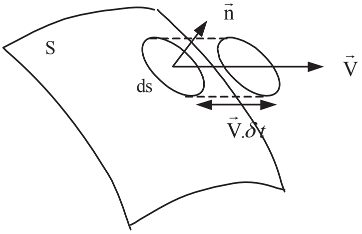


Figure 2.10. *Balance in an elementary displacement across ds*

Letting $\bar{g}$ be the volume density of the quantity G (see section 2.1.1), we can derive expressions for the vector flux density $\vec{q}_g$ of G[2] and the flux φ_{GS} of the quantity G across the surface S:

$$\vec{q}_g = \bar{g}\vec{V} \qquad \varphi_{GS} = \int_S \vec{q}_G.\vec{n}ds = \int_S \bar{g}\,\vec{V}.\vec{n}ds$$

By considering $\bar{g}$ to be the number of moles per unit volume, we obtain respectively for the *molar flux density* $\vec{q}_{ni}$ and the *molar flux* φ_{ni} across (S) of the constituent i with velocity $\vec{V}_i$:

$$\vec{q}_{ni} = n_i\vec{V}_i; \qquad \varphi_{ni} = \int_S \vec{q}_{n_i}.\vec{n}ds = \int_S n_i\,\vec{V}_i.\vec{n}ds \qquad [2.22]$$

If we now choose mass to be the extensive quantity, we obtain, respectively for the *mass flux density* $\vec{q}_{mi}$ and the *mass flux* φ_{mi} of the component i across (S):

$$\vec{q}_{mi} = \rho_i\vec{V}_i \qquad \varphi_{mi} = \int_S \vec{q}_{mi}.\vec{n}\,ds = \int_S \rho_i\vec{V}_i.\vec{n}\,ds. \qquad [2.23]$$

The molar flux φ_{ni} and the mass flux φ_{mi} represent, respectively, the number of moles and the component mass i crossing S per unit time.

2.4.2.2. *Balance equations*

The balance equation of each constituent i (i =1, 2,..., k) can be obtained by applying equation [2.8] to the volumic number of moles n_i:

$$\frac{\partial n_i}{\partial t} = \sigma_{ni} - \frac{\partial q_{nij}}{\partial x_j} \qquad \left(\frac{\partial n_i}{\partial t} = \sigma_{ni} - div\vec{q}_{ni}\right)$$

The volume source σ_{ni} is the (volumetric) volumic number of moles in component i created by homogenous chemical reactions. By replacing the molar flux density with expression [2.22]:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x_j}\left(n_i u_{ij}\right) = \sigma_{ni} \qquad i = 1,2,..k. \qquad [2.24]$$

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[2] We shall study convective fluxes in more detail later (section 3.3.3.2).

Balance equations can be written for the density ρ_i, which are equivalent to [2.24], by multiplying each of these by the molar mass M_i of the corresponding component:

$$\frac{\partial \rho_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho_i u_{ij}\right) = M_i \sigma_{ni} \qquad i = 1,2,..k. \qquad [2.25]$$

2.4.2.3. *Global fluxes and mean velocities*

For the set of components of a mixture, we define:

– the *total density* $\vec{q}_n$ *of the molar flux* and the total molar flux φ_n:

$$\vec{q}_n = \sum_{i=1}^{k} \vec{q}_{ni} = \sum_{i=1}^{k} n_i \vec{V}_i \qquad \varphi_n = \int_S \vec{q}_n . \vec{n}\, ds \qquad [2.26]$$

– the *total density* $\vec{q}_m$ *of the mass flux* and the *total mass flux* φ_m:

$$\vec{q}_m = \sum_{i=1}^{k} \vec{q}_{mi} = \sum_{i=1}^{k} \rho_i \vec{V}_i \qquad \varphi_m = \int_S \vec{q}_m . \vec{n}\, ds \qquad [2.27]$$

The global molar and mass balances for the mixture allow average mixing velocities to be defined according to the methodology described in section 1.4.2.5 for the intensive quantities[3]. We thus define:

– the *molar average velocity* $\vec{V}^*$ (which is independent of the molar mass) using the molar balance [2.26]:

$$\vec{V}^* = \frac{1}{n} \sum_{i=1}^{k} n_i \vec{V}_i \qquad [2.28]$$

– the *mass average velocity* $\vec{V}$ (velocity of the inertia center of a fluid particle[4]), using mass balance [2.27]:

$$\vec{V} = \frac{1}{\rho} \sum_{i=1}^{k} \rho_i \vec{V}_i = \frac{\vec{q}_m}{\rho} \qquad [2.29]$$

where $\rho_i = n_i M_i$ is the partial density of the species *i* of molar mass M_i.

---

[3] The velocity is an intensive quantity (see section 1.2.1).

[4] See section 3.2.1 and following sections.

The velocities $\vec{V}$ and $\vec{V}^*$ are the same only if the continuous medium comprises a homogenous mixture of constant composition or of identical molecules (a pure substance).

The molar fluxes φ_n and φ_m represent, respectively, the number of moles and the total mass of a mixture which are crossing a surface *S* per unit time:

$$\varphi_n = \int_S \vec{q}_n.\overrightarrow{ds} = \int_S n\vec{V}^*.\overrightarrow{ds} \qquad \varphi_m = \int_S \vec{q}_m.\overrightarrow{ds} = \int_S \rho\vec{V}.\overrightarrow{ds}$$

By summing equations [2.24], term by term, and taking account of definition [2.28], we obtain the balance equation for the total volumic number of moles *n*:

$$\frac{\partial n}{\partial t} + div\left(n\vec{V}^*\right) = \sum_{i=1}^{k} \sigma_{ni} \qquad [2.30]$$

The quantity $\sum_{i=1}^{k} \sigma_{ni}$ is the number of moles created by chemical reactions.

We do the same for equations [2.25] and [2.29]; however, the sum $\sum_{i=1}^{k} M_i \sigma_{ni}$ is zero as there is no mass source. We obtain the conservation equation for the total mass:

$$\frac{\partial \rho}{\partial t} + div\left(\rho\vec{V}\right) = 0 \qquad [2.31]$$

We will encounter the preceding equation again in the next chapter, during our study of fluid flows, where it plays a fundamental role.

The reader will note that the *k* equation [2.24] (or [2.25]) is equivalent to the system of *k-1* equation [2.24] (or [2.25]) and equation [2.30] (or [2.31]). In other words, it suffices to describe *k-1* component balance equations in addition to the global balance equation for the mixture.

2.4.3. *Choice of reference frame*

2.4.3.1. *Introduction*

In this chapter, we have already studied the transfer of diverse quantities *G* (energy, entropy, etc.) with respect to matter which was supposed stationary; in

Chapter 3 we will examine the transport of a quantity G associated with convection of matter, in other words associated with a bulk movement of the continuous medium.

In an inhomogenous mixture, each component i has its own velocity $\vec{V}_i$. There is therefore no preferred reference frame in which all of the matter is stationary. It is for this reason that we did not specify the reference frame used in the preceding definitions of flux.

As diffusion is a *differential phenomenon between the components*, it is necessary to identify a global condition which defines the bulk movement of the matter, in other words it is necessary to fix the local mean velocities field for the matter, as diffusion is a phenomenon which must be considered relative to a mean velocity. The reference frame chosen for a study of diffusion is thus determined by the choice of reference frame for the mean velocity. It depends on the kind of problem studied, as the following examples demonstrate.

The quantities which characterize diffusion (velocities, flux densities), which depend on the reference frame $\mathcal{R}$ chosen for their study, will be represented between brackets $[\ ]^*$ with a superscript (*,G or 1) indicating the reference frame.

2.4.3.2. *Diffusion in a fluid at mechanical rest*

Let us now consider an isothermal fluid mixture containing k components. Diffusion phenomena are here characterized by very slow velocities associated with negligible accelerations. The local composition is characterized by the number of moles n_i per unit volume for each species. Suppose that the total number of moles n per unit volume is constant at all points in the mixture (for example, in an ideal isothermal gas). In the absence of chemical reactions, equation [2.30] can thus be written:

$$div\vec{V}^* = 0 \qquad [2.32]$$

We will limit our discussion to a simple case involving a configuration in stable equilibrium (section 2.2.1.6), in which the concentration is constant in horizontal planes, the diffusion velocities being vertical and parallel to the axis Ox. The field $\vec{V}^*(x,t)$, which thus respects the condition $dV^*/dx = 0$, derived from [2.32], is uniform at each instant and the molar average velocity $\vec{V}^*(t)$ is only a function of time.

The preceding situation is observed for isothermal gaseous mixtures whose *pressure is uniform and constant.* In a gravitational field, diffusion only occurs without natural convection for fluids in situations of 1D equilibrium. In other cases, the inevitable density gradients created by variations in the component concentrations must be taken into account: these induce complex movements due to natural convection in the fluid.

2.4.3.3. *Diffusion in a closed container*

Let us now consider a perfect, isothermal, gaseous mixture, contained in a fixed container (Figure 2.9a), whose total volumic number of moles *n* is constant. We will suppose that the walls are impermeable; this results in a *molar average velocity which is zero in the reference frame of the container*, which can thus be considered a preferred reference frame:

$$n\vec{V}^* = \sum_{i=1}^{k} n_i \left[\vec{V}_i\right]^* = 0, \qquad \text{with: } \left[\vec{V}_i\right]^* = \vec{V}_i - \vec{V}^*$$

This results in the molar flux densities playing an important role in this reference frame:

$$\left[\vec{q}_{ni}\right]^* = n_i \left[\vec{V}_i\right]^*, \qquad \text{with: } \sum_i \left[\vec{q}_{ni}\right]^* = 0$$

2.4.3.4. *Diffusion in steady evaporation*

Let us now examine the 1D problem of steady evaporation of a liquid contained in a vertical tube as shown in Figure 2.9b. As before, the pressure is constant in the tube. We suppose that the temperature is constant and the composition is independent of time, but not uniform in space. The total volumic number of moles *n* and the molar velocity $\vec{V}^*$ (section 2.4.3.2) are steady and uniform in space. The molar flux densities of air (species 1) and vapor (species 2) are steady (section 2.4.3.2).

As the free surface of the liquid at the bottom of the tube is impermeable to air, the flux density and velocity of the air are both zero at all points within the tube: *the air is at rest in the reference frame of the tube* ($\left[\vec{V}_1\right]^1 = 0$) and so diffusion velocity of vapor in air can be written:

$$\left[\vec{V}_2\right]^1 = \vec{V}_2 - \vec{V}_1$$

Since we have a steady evaporation regime, we can conclude that the *flux of liquid evaporated* $\left[n_2 V_2\right]^1$ across a cross-section of the tube is independent of the vertical position of this section (otherwise, we should have some accumulation of vapor between two sections with different fluxes, which would increase with time).

2.4.3.5. *Diffusion in a moving medium*

By virtue of the laws of dynamics, the characteristics of the inertia center of a material system can be obtained through knowledge of the external forces acting on the system. The same goes for fluid particles of a continuous medium if the external forces acting on the fluid particle are known.

Diffusive phenomena amount to differences in velocity between different components of a fluid. The separation of bulk movement and diffusive phenomena can be achieved by identifying the *diffusive velocities in a reference frame associated with the inertia center G of fluid particles* (mass or barycentric reference frame) which move with velocity $\vec{V}_G$ (mass velocity) and we will denote simply as $\vec{V}$, as obtained from the equations of motion which will be covered in Chapter 3. From definition [2.29] we have, for the local inertia center:

$$\rho\vec{V} = \sum_{i=1}^{k} \rho_i \vec{V}_i \qquad [2.33]$$

Quantities measured in a reference frame $\mathcal{R}^{\mathrm{G}}$ associated with this inertia center are denoted $[\ ]^G$ (between brackets and with the superscript *G*). Relation [2.33] can be written:

$$\sum_{i=1}^{k} \rho_i \left[\vec{V}_i\right]^G = 0, \qquad \text{with: } \left[\vec{V}_i\right]^G = \vec{V}_i - \vec{V} \qquad [2.34]$$

This relation expresses a particular importance of the mass flux densities:

$$\left[\vec{q}_{mi}\right]^G = \rho_i \left[\vec{V}_i\right]^G, \qquad \text{with: } \sum_{i=1}^{k} \left[\vec{q}_{mi}\right]^G = 0 \qquad [2.35]$$

which describe the diffusive fluxes in this reference frame.

2.4.3.6. *Diffusion of components in weak concentration*

The preceding difficulties disappear when a mixture is almost entirely comprised of one of the components (for example species 1). Supposing that the number of moles of the other components takes on a small value, of order ε, we have seen in section 2.4.1.3 that all concentration definitions were equivalent for the first order. It can be shown in the same way from expressions [2.28] and [2.29] that the velocities $\vec{V}$, $\vec{V}^*$ and $\vec{V}_1$ are equal to order ε, so that *the predominant component constitutes the natural reference frame.*

In this case, the diffusion velocity of a component is its velocity with respect to the predominant species, and any extensive quantities can be used to define a flux density, as the reference frame is no longer described by a balance equation.

2.4.3.7. *General methodology: diffusion in an arbitrary reference frame*

The preceding examples show that the velocity of a component is not the only characteristic of its diffusion and we always have a global condition between the ensemble of diffusion velocities which characterize the particular problem considered. Diffusion itself is characterized by *diffusion velocities and flux densities in the reference frame best adapted to the problem considered*. The flux density of the extensive quantity associated with this velocity also depends on the choice of reference frame.

The diffusion velocities and corresponding flux densities (molar, barycentric or otherwise) of each component are summarized in Table 2.1 for different reference frames, which are denoted by indices corresponding to the preceding quantities:

– the superscript * in the reference frame $\mathcal{R}^*$ corresponding to the molar average velocity $\vec{V}^*$;

– the superscript ^{G} in the reference frame $\mathcal{R}^G$ corresponding to the mass average velocity[5] $\vec{V}$ (G being the inertia center);

– the superscript ^{1} in the reference frame $\mathcal{R}^1$ of fluid 1.

---

[5] The mass speed $\vec{V}$ is the local speed of fluid medium intervening in the mechanical equations ($\vec{V}^G = \vec{V}$).

Reference frame	Extensive quantity	Diffusion velocity	Diffusion flux densities
$\mathcal{R}^*$ molar average velocity $\vec{V}^*$	n_i	$\left[\vec{V}_i\right]^* = \vec{V}_i - \vec{V}^*$	$[\vec{q}_{ni}]^* = n_i\left[\vec{V}_i\right]^* = n_i\left(\vec{V}_i - \vec{V}^*\right)$ $\sum_{i=1}^{k}[\vec{q}_{ni}]^* = 0$
$\mathcal{R}^G$ mass average velocity $\vec{V}$	ρ_i	$\left[\vec{V}_i\right]^G = \vec{V}_i - \vec{V}$	$[\vec{q}_{mi}]^G = \rho_i\left[\vec{V}_i\right]^G = \rho_i\left(\vec{V}_i - \vec{V}\right)$ $\sum_{i=1}^{k}[\vec{q}_{mi}]^G = 0$
$\mathcal{R}^1$ component 1 velocity $\vec{V}_1$	n_i ρ_i	$\left[\vec{V}_i\right]^1 = \vec{V}_i - \vec{V}_1$ $\left[\vec{V}_1\right]^1 = 0$	$[\vec{q}_{mi}]^1 = \rho_i\left[\vec{V}_i\right]^1, i \neq 1$ $[\vec{q}_{ni}]^1 = n_i\left[\vec{V}_i\right]^1, i \neq 1$ $[\vec{q}_{m1}]^1 = [\vec{q}_{n1}]^1 = 0$
weak concentrations of components other than 1 $\mathcal{R}^* \cong \mathcal{R}^G \cong \mathcal{R}^1$	$c_i \quad i \neq 1$	$\left[\vec{V}_i\right]^1 = \left[\vec{V}_i\right]^G$ $= \left[\vec{V}_i\right]^* \quad i \neq 1$ $\left[\vec{V}_1\right]^1 = 0$	$[\vec{q}_{ci}]^1 = c_i\left[\vec{V}_i\right]^1 = c_i\left(\vec{V}_i - \vec{V}^1\right)$ $i \neq 1$

Table 2.1. *Variables characterizing diffusion for different reference frames*

The situations described in Table 2.1 correspond to those most frequently encountered. Other more complex situations may arise (unsteady evaporation, macroscopic mixing of numerous components with walls which are permeable to only some of these, etc.). In all cases, a global analysis is necessary for an appropriate choice of representation for the diffusion equations.

2.4.4. *Binary isothermal mixture*

2.4.4.1. *Expressions for diffusion velocities and flux densities*

We have reviewed the different definitions which are possible for extensive quantities in different reference frames. For *n* flux-density vectors, diffusion phenomena are characterized by *n-1* independent equations for the flux-densities or the diffusion velocities, and by an equation which describes a global condition. The *n*-1 differences in diffusion velocities, two by two are obviously independent of the reference frame used, and they can be linked to causes of imbalance, such as concentration gradients, by phenomenological relations. We here limit our treatment of the problem to a binary mixture; the reader should consult texts treating the problem of irreversible thermodynamics for developments associated with diffusion in mixtures with more than two components.

For a binary mixture, the only quantity which is independent of the reference frame and diffusion characteristic is the velocity difference $\vec{V}_1 - \vec{V}_2$, of species 1 and 2 with respect to an arbitrary reference frame. Diffusion velocities in the reference frames $\mathcal{R}^*$ or $\mathcal{R}^G$ can then be easily expressed as a function of $\vec{V}_1 - \vec{V}_2$.

In effect, in the reference frame corresponding to the mean molar velocity $\vec{V}^*$, we obtain relation [2.28]:

$$n_1\left[\vec{V}_1\right]^* + n_2\left[\vec{V}_2\right]^* = 0 \quad \text{with:} \quad \left[\vec{V}_i\right]^* = \vec{V}_i - \vec{V}^*$$

which can be written as:

$$\frac{\left[\vec{V}_1\right]^*}{\dfrac{1}{n_1}} = -\frac{\left[\vec{V}_2\right]^*}{\dfrac{1}{n_2}} = \frac{\left(\vec{V}_1 - \vec{V}_2\right)}{\dfrac{1}{n_1} + \dfrac{1}{n_2}} = \frac{n_1 n_2}{n}\left(\vec{V}_1 - \vec{V}_2\right) \qquad [2.36]$$

or:

$$\left[\vec{q}_{n1}\right]^* = n_1\left[\vec{V}_1\right]^* = -\left[\vec{q}_{n2}\right]^* = -n_2\left[\vec{V}_2\right]^* = \frac{n_1 n_2}{n}\left(\vec{V}_1 - \vec{V}_2\right) \qquad [2.37]$$

In the reference frame $\mathcal{R}^*$ associated with the molar velocity, the diffusion velocities and molar flux densities of each component can be written as a function of the velocity difference $\left(\vec{V}_1 - \vec{V}_2\right)$.

Similarly, using relation [2.34] between the mass flux densities in reference frame $\mathcal{R}^G$ we obtain:

$$\left[\vec{q}_{m1}\right]^G = \rho_1\left[\vec{V}_1\right]^G = -\left[\vec{q}_{m2}\right]^G = -\rho_2\left[\vec{V}_2\right]^G = \frac{\rho_1 \rho_2}{\rho}\left(\vec{V}_1 - \vec{V}_2\right) \qquad [2.38]$$

Comparing [2.36] and [2.38] gives:

$$\left[\vec{V}_1\right]^G = \frac{\rho_2}{\rho}\frac{n}{n_2}\left[\vec{V}_1\right]^* = \frac{(n_1+n_2)M_2}{n_1M_1+n_2M_2}\left[\vec{V}_1\right]^* = \frac{M_2}{M}\left[\vec{V}_1\right]^* \qquad [2.39]$$

$$\left[\vec{V}_2\right]^G = \frac{M_1}{M}\left[\vec{V}_2\right]^* \qquad [2.40]$$

where $M = \dfrac{n_1M_1+n_2M_2}{(n_1+n_2)}$ is the local average molar mass of the mixture.

In summary, the diffusion velocity in a given reference frame can be expressed as a function of the velocity difference between the two components by means of the law which defines the reference frame. It is thus *velocity differences which characterize binary diffusion phenomena independently of the reference frame which is chosen.*

2.4.4.2. *Isothermal diffusion*

2.4.4.2.1. Fick's law

Isothermal diffusion is caused by the existence of concentration gradients which we will characterize by a volumic number of moles.

In order to define a diffusion coefficient we will use the relation:

$$\vec{V}_1 - \vec{V}_2 = -D\frac{\overrightarrow{grad}\,\dfrac{n_1}{n_2}}{\dfrac{n_1}{n_2}} \qquad [2.41]$$

the ratio $\dfrac{n_1}{n_2}$ being dimensionless. It is equal to the concentrations ratio of two species, expressed using a proportional definition for the two species[6]:

$$\begin{aligned}\vec{V}_1 - \vec{V}_2 &= -D\frac{\overrightarrow{grad}(\gamma_1/\gamma_2)}{\gamma_1/\gamma_2} = -D\frac{\overrightarrow{grad}(\rho_1/\rho_2)}{\rho_1/\rho_2} \\ &= -D\frac{\overrightarrow{grad}(\omega_1/\omega_2)}{\omega_1/\omega_2} = -D\frac{\overrightarrow{grad}(p_1/p_2)}{p_1/p_2}\end{aligned} \qquad [2.42]$$

---

[6] Using partial pressures is only correct for perfect gases and ideal solutions.

Relation [2.41] shows that the diffusion coefficient can be expressed in $m^2.\text{s}^{-1}$. For a gas at atmospheric pressure, it is more or less independent of concentration. For mixtures with air, it usually lies between 1.0×10^{-5} and 2.5×10^{-5} $m^2.\text{s}^{-1}$, the lowest values corresponding to heavy molecules (carbon dioxide, ethanol, benzene, etc.). Very light molecules (hydrogen, helium) give larger values (up to 13.2×10^{-5} $m^2.\text{s}^{-1}$ for hydrogen-helium mixtures).

In liquid mixtures, the diffusion coefficient is in the order of 10^{-9} $m^2.\text{s}^{-1}$. Diffusion in solids results from different mechanisms, depending on whether we are dealing with diffusion of impurities which move from a free position in one crystalline structure to another, or with particles (atoms, etc.) capable of moving around the structural grid. The diffusion coefficient in solids varies from 10^{-12} to 10^{-14} $m^2.\text{s}^{-1}$.

Expression [2.41] shows that the quantity $D\overrightarrow{grad}$ has the dimension of velocity and that this gives an order of magnitude of D/ℓ for the diffusion velocities, where ℓ is the distance over which the concentration gradient is extended. Taking for example $\ell = 0.1$ meter, we can see that the diffusion velocity is in the order of 10^{-4} $m.\text{s}^{-1}$ in gases and 10^{-8} $m.\text{s}^{-1}$ in liquids. These velocities increase considerably if the distance ℓ is significantly diminished; as for momentum transfer (section 6.5.3), convection effect can reduce this quantity to values comparable with the thickness of a boundary layer, leading to a significant increase in diffusion velocity ([BIR 01]).

We define the *Lewis number Le* as the dimensionless ratio between the diffusion coefficient and thermal diffusivity a:

$$\text{Le} = \frac{D}{a}$$

Excluding instances of extremely strong force fields or accelerations, the total number of moles n per unit volume is often nearly constant ($n_1 + n_2 \cong$ constant) under standard conditions (in particular for ideal gases). Thus, we have:

$$\overrightarrow{grad}\, n_1 + \overrightarrow{grad}\, n_2 = 0 \qquad [2.43]$$

Substituting [2.43] into Fick's law [2.41], the velocity difference can be written as a function of the concentration n_1 alone:

$$\vec{V}_1 - \vec{V}_2 = -D\frac{n}{n_1 n_2}\overrightarrow{grad}\, n_1 = -D\frac{n}{n_1\left(n - n_1\right)}\overrightarrow{grad}\, n_1 \qquad [2.44]$$

2.4.4.2.2. Diffusion in the local reference frame of molar velocity

We use the local reference frame $\mathcal{R}^*$ of molar velocity for diffusion in a closed fixed volume. From [2.41] and [2.44] we obtain an expression for the molar flux in the reference frame $\mathcal{R}^*$:

$$\left[\vec{V}_1\right]^* = -D\frac{\overrightarrow{grad\, n_1}}{n_1} \qquad \left[\vec{V}_2\right]^* = -D\frac{\overrightarrow{grad\, n_2}}{n_2} \qquad [2.45]$$

From this we can deduce the molar flux densities of the two species:

$$\left[\vec{q}_{n1}\right]^* = n_1\left[\vec{V}_1\right]^* = -\left[\vec{q}_{n2}\right]^* = -n_2\left[\vec{V}_2\right]^* = -D\overrightarrow{grad}\, n_1 = D\overrightarrow{grad}\, n_2 \qquad [2.46]$$

Instead of representing the concentrations by n_1 and n_2, diffusion velocities in the fixed reference frame can be expressed as a function of quantities of the two gases which are proportional to these concentrations, such as the partial specific masses ρ_1 and ρ_2, the partial pressures p_1 and p_2, and the molar fractions γ_1 and γ_2:

$$\begin{aligned}\left[\vec{V}_1\right]^* &= -D\frac{\overrightarrow{grad\, \rho_1}}{\rho_1} = -D\frac{\overrightarrow{grad\, p_1}}{p_1} = -D\frac{\overrightarrow{grad\, \gamma_1}}{\gamma_1}\\ \left[\vec{V}_2\right]^* &= -D\frac{\overrightarrow{grad\, \rho_2}}{\rho_2} = -D\frac{\overrightarrow{grad\, p_2}}{p_2} = -D\frac{\overrightarrow{grad\, \gamma_2}}{\gamma_2}\end{aligned} \qquad [2.47]$$

The mass fractions cannot be directly used as concentration variables, because the specific mass of the mixture varies within the volume.

NOTE – *Expression* [2.47], *using the molar fractions as variables*, *is exact*, even if the volumic number of moles n has a non-zero gradient. This results from the definition:

$$\gamma_1 + \gamma_2 = 1 \quad \text{and:} \quad \overrightarrow{grad}\gamma_1 + \overrightarrow{grad}\gamma_2 = 0$$

and from expression [2.42] of Fick's law using molar fractions which can be written:

$$\vec{V}_1 - \vec{V}_2 = -\frac{D}{\gamma_1\gamma_2}\overrightarrow{grad}\gamma_1 \qquad [2.48]$$

From [2.47], we obtain the densities of molar flux:

$$\left[\vec{q}_{n1}\right]^* = -n\,D\overrightarrow{grad}\gamma_1 \qquad \left[\vec{q}_{n2}\right]^* = -n\,D\overrightarrow{grad}\gamma_2 \tag{2.49}$$

2.4.4.2.3. Application to diffusion in a medium at macroscopic rest

Here we consider a medium at macroscopic rest, in other words a medium whose very slight movements are related to diffusion phenomena. We assume that the medium satisfies the equations of fluid statics and that differences of specific mass do not lead to movements associated with natural convection.

In the case of a closed impermeable container in which a fluid is at rest, the molar average velocity is assumed to be zero at all points of the medium in *a cartesian reference frame associated with the container.*

The diffusion equation of the components can be obtained by applying [2.8] for the number of moles n_1 and n_2,

$$\frac{\partial n_1}{\partial t} = \sigma_G - \frac{\partial q_{n1j}}{\partial x_j}$$

which, on accounting for expression [2.46], gives:

$$\frac{\partial n_1}{\partial t} = \frac{\partial}{\partial x_j}\left(D\frac{\partial n_1}{\partial x_j}\right) + \sigma_G \qquad \left(\frac{\partial n_1}{\partial t} = div\left(D\overrightarrow{grad}\,n_1\right) + \sigma_G\right) \tag{2.50}$$

The volume source σ_G of species 1 expresses the number of moles per unit volume of this species created by a homogenous chemical reaction inside the container. It is fixed by chemical kinetics. It is zero in the absence of any chemical reaction.

In the common case where the diffusion coefficient is constant and where there is no chemical reaction, equation [2.50] reduces to the heat equation:

$$\frac{\partial n_1}{\partial t} = D\,\Delta n_1 \tag{2.51}$$

The boundary conditions for impermeable walls can be represented by a zero normal component $\overrightarrow{grad}\,n_1$ at the walls ($\vec{n}_w$: normal to the wall):

$$\vec{n}_w \cdot \overrightarrow{grad}\, n_1 = \left(\frac{\partial n_1}{\partial n}\right)_w = 0$$

2.4.4.2.4. Diffusion with respect to a fixed component

The discussion of the steady evaporation problem given in section 2.4.3.4 (Figure 2.9b) showed us that gas 1 is fixed. In this case it suffices to directly apply Fick's law in the form [2.44]:

$$\vec{V}_2 = -D \frac{n}{n_1 n_2} \overrightarrow{grad}\, n_2$$

The flux density of species 2 can thus be written as:

$$\vec{q}_{n2} = n_2 \vec{V}_2 = -D\, n \frac{\overrightarrow{grad}\, n_2}{n - n_2}$$

The conservation equation of species 2 can be written:

$$div\left(D\, n \frac{\overrightarrow{grad}\, n_2}{n - n_2}\right) = 0 \qquad [2.52]$$

For the 1D evaporation problem of Figure 2.9b, we obtain, by integrating [2.52]:

$$q_{n2} = -\frac{D\, n}{n - n_2} \frac{dn_2}{dx} = \text{constant} \qquad [2.53]$$

We consider the following boundary conditions:

– at the surface of the liquid ($x = 0$), the number of moles per unit volume n_S, is that of the saturated vapor at the experimental temperature;

– at the tube extremities ($x = \ell$), we assume that the very slight movements of atmospheric air suffice to eliminate all vapor, such that we can write that the vapor concentration n_2 is equal to the water vapor concentration n_{2a} of the atmospheric air.

The distribution of the water vapor concentration can be obtained by integrating equation [2.53]:

$$n - n_2 = (n - n_S)\exp\left(\frac{x\, q_{n2}}{D\, n}\right)$$

Expressing, at the abscissa ℓ, the concentration condition for the ambient air we obtain a relation giving the flux density of the vapor:

$$\frac{q_{n2}\ell}{Dn} = \ln\left(\frac{n - n_{2a}}{n - n_S}\right)$$

For relatively weak concentrations we have:

$$q_{n2} = \frac{D\left(n_S - n_{2a}\right)}{\ell} \tag{2.54}$$

Numerical application

Take a tube of height $\ell = 0.10$ m. We base our reasoning on a density which is proportional to the volumic number of moles for water vapor at 20ºC, assuming the external air to be perfectly dry ($n_{2a} = 0$) and using the following data: $D = 2.6 \times 10^{-5}$ $m^2.s^{-1}$: $\rho_s = 0.018$ $kg.m^{-3}$. We find, for the flux density evaporated:

$$q_{m2} = \frac{2.6 \times 10^{-5} \times 0.018}{0.1} = 4.68 \times 10^{-6}\, kg.m^{-2}.s^{-1}$$

This corresponds to a reduction in the level of the free surface of about 0.4 mm per day. For acetone, whose saturated vapor pressure is 10 times greater, the evaporation is 10 times more rapid.

Evaporation rate in static conditions is very weak. Diffusion in a stationary medium is a very slow process, which only becomes efficient in the presence of convection (section 2.4.4.2.1).

2.4.4.2.5. Diffusion in a reference frame related to the local inertia center

Equation [2.38] gives an expression for the mass flux in a reference frame associated with the local inertia center:

$$\left[\vec{q}_{m1}\right]^G = \rho_1\left[\vec{V}_1\right]^G = -\left[\vec{q}_{m2}\right]^G = -\rho_2\left[\vec{V}_2\right]^G = \frac{\rho_1\rho_2}{\rho}\left(\vec{V}_1 - \vec{V}_2\right) \tag{2.55}$$

The expression of Fick's law [2.42] which is best adapted to this reference frame uses mass fractions for the concentration variables:

$$\vec{V}_1 - \vec{V}_2 = -D\,\frac{\overrightarrow{grad}\,\dfrac{\omega_1}{\omega_2}}{\dfrac{\omega_1}{\omega_2}} = -D\left[\frac{\overrightarrow{grad}\omega_1}{\omega_1} - \frac{\overrightarrow{grad}\omega_2}{\omega_2}\right]$$

From the definition of mass fractions, we obtain:

$$\omega_1 + \omega_2 = 1 \qquad \text{and} \qquad \overrightarrow{grad}\omega_1 + \overrightarrow{grad}\omega_2 = 0$$

and thus:

$$\vec{V}_1 - \vec{V}_2 = -\frac{D}{\omega_1\omega_2}\,\overrightarrow{grad}\omega_1 \qquad [2.56]$$

Substituting [2.56] into [2.55] gives:

$$[\vec{q}_{m1}]^G = -\rho\, D\overrightarrow{grad}\omega_1 \qquad [\vec{q}_{m2}]^G = -\rho\, D\overrightarrow{grad}\omega_2 \qquad [2.57]$$

There are almost no static diffusion problems which lead to a motionless local inertia center. Expression [2.57] will be used in the diffusion equation in the presence of convection in Chapter 4.

NOTE – Using relations [2.39] and [2.40], diffusion velocities in the reference frame $\mathcal{R}^{\mathrm{G}}$ can be expressed, as before, as a function of the variables n_1 *and* n_2, as well as ρ_1 with ρ_2, p_1 and p_2 or γ_1 and γ_2:

$$[\vec{V}_1]^G = -\frac{DM_2}{M}\frac{\overrightarrow{grad}\,\rho_1}{\rho_1} = -\frac{DM_2}{M}\frac{\overrightarrow{grad}\,p_1}{p_1} = -\frac{DM_2}{M}\frac{\overrightarrow{grad}\,\gamma_1}{\gamma_1} \qquad [2.58]$$

$$[\vec{V}_2]^G = -\frac{DM_1}{M}\frac{\overrightarrow{grad}\,\rho_2}{\rho_2} = -\frac{DM_1}{M}\frac{\overrightarrow{grad}\,p_2}{p_2} = -\frac{DM_1}{M}\frac{\overrightarrow{grad}\,\gamma_2}{\gamma_2} \qquad [2.59]$$

With these concentration variables, diffusion velocities can no longer be expressed by means of the same diffusion coefficient and so we have two different diffusion coefficients, in which the molar average mass M is variable:

$$D_{G1} = \frac{DM_2}{M} \qquad D_{G2} = \frac{DM_1}{M}$$

2.4.4.2.6. Unsteady evaporation

From a practical point of view, the consequences of the developments outlined in the preceding sections are that the exact formulation of equations for diffusion problems should only be performed after an explicit choice of reference frame when possible; this allows us to obtain precise expressions for $[\vec{V}_1]$ and $[\vec{V}_2]^1$. For example, in certain problems (steady evaporation, see sections 2.4.1.2 and 2.4.4.2.4) one of the gases, say G_1, is stationary, and only gas G_2 moves during diffusion. Taking a reference frame $\mathcal{R}_1$ linked with G_1, the velocity $[\vec{V}_2]^1$ of gas G_2 follows immediately from [2.41].

On the other hand, if the *evaporation is no longer steady*, then the air (species 1) is no longer stationary. Consider the case where the tube in Figure 2.9b is filled only with air at the initial instant; the total pressure remains constant; the rise of vapor (species 2) in the tube moves the molar quantity corresponding to the air which is now in movement and is thus no longer the favored reference frame.

Let us reconsider the evaporation problem for the unsteady 1D case, in an atmosphere at constant temperature and pressure (Figure 2.9b). The total volumetric number of moles n is constant. This leads to a situation where the molar average velocity V^* and the total molar flux $q_n = nV^*$ are independent of the abscissa x. However, *a priori* they do depend on time. All that remains is to write the balance equation for the vapor in the reference frame of the tube:

$$\frac{\partial n_2}{\partial t} + \frac{\partial}{\partial x}\left(n_2 V_2\right) = 0 \qquad [2.60]$$

The flux density of the vapor $n_2 V_2$ is now a function of the two variables x and t. It can be obtained by decomposing the velocity, $\vec{V}_2 = \left[\vec{V}_2\right]^* + V^*$, and by replacing $[\vec{V}_2]^*$ by its expression in [2.45]:

$$q_{n2} = n_2 V_2 = -D\frac{\partial n_2}{\partial x} + q_n \frac{n_2}{n} \qquad [2.61]$$

By substituting into equation [2.60], we obtain the diffusion equation:

$$\frac{\partial n_2}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_n}{n} n_2 - D\frac{\partial n_2}{\partial x}\right) = 0 \qquad [2.62]$$

As the velocity V_1 of the air is zero at the abscissa $x = 0$ (impermeability of the liquid surface), the total molar flux $q_n = nV^*$ is equal to the molar flux $\left[n_2 V_2\right]_{x=0}$ of the vapor at the origin. This can be obtained by means of expression [2.44] of Fick's law and the condition $n_2(0, t) = n_s$:

$$q_n = nV^* = \left[n_2 V_2\right]_{x=0} = -\frac{Dn}{n - n_S} \left.\frac{\partial n_2}{\partial x}\right|_0 \qquad [2.63]$$

The molar concentration n_2 satisfies equation [2.62] and condition [2.63] with which we can associate the conditions used in section 2.4.4.2.4 for positions 0 and ℓ:

$$n_2(0,t) = n_s \qquad n_2(\ell,t) = n_{2a}$$

in addition to the concentration distribution $n_2(x,0)$ at the initial time.

We will go no further with these calculations. Let us just *recover the steady regime* studied earlier (section 2.4.4.2.4) by observing that in this case, gas 1 is stationary; we therefore have, for all x:

$$q_n = nV^* = n_2 V_2$$

Substituting this expression into [2.61] gives us:

$$n_2 V_2 = -D \frac{n}{n - n_2} \frac{\partial n_2}{\partial x}$$

Equation [2.62] is thus identical to [2.53]:

$$\frac{\partial}{\partial x}\left(\frac{n}{n - n_2}\left[-D\frac{\partial n_2}{\partial x}\right]\right) = 0$$

If evaporation occurs *at constant volume* in a container in which initially no vapor is present, the pressure will increase during evaporation, which also produces a displacement of the mixture. We will leave aside such static problems where the equations of motion are to be solved.

2.4.4.2.7. Case of weak concentrations

When the mixture contains a weak proportion of species 2:

$$n_2 << n_1 \quad \text{or;} \quad p_2 << p_1 \quad \text{or;} \quad \rho_2 << \rho_1 \quad ....$$

it is then possible to confuse the different units of concentration (section 2.4.1.3).

By comparing relations [2.28] and [2.29], we can see that the three reference frames $\mathcal{R}^*$, $\mathcal{R}^G$ and $\mathcal{R}^1$ become identical to the first order. We can note that expressions [2.47], or [2.58] and [2.59], show that the diffusion velocities $\left[\vec{V}_1\right]^*$ or $\left[\vec{V}_1\right]^G$ of component 1, which is excessively large, are negligible compared with the diffusion velocity $\left[\vec{V}_2\right]^*$ or $\left[\vec{V}_2\right]^G$ of component 2.

It results in the preceding approximations that:

– the molecular diffusion velocity $\vec{V}_2$ of species 2 is thus the same in any one of the three reference frames $\mathcal{R}^*$, $\mathcal{R}^G$ and $\mathcal{R}^1$;

– any variable amongst those defined in section 2.4.1.3 (number of moles per unit volume n_2, molar or mass concentration c_2 or ρ_2, partial pressure p_2, molar or mass fraction γ_2 or ω_2, etc.) can be used to represent the concentration c_2 of component G_2 to the second order excepted.

Then we have, in the three reference frames previously chosen to characterize the diffusion the relation:

$$\vec{V}_2 = -D\frac{\overrightarrow{grad}\, c_2}{c_2} \tag{2.64}$$

In these conditions, we can define the flux density of species 2 by means of the variable c_2:

$$\vec{q}_{c2} = c_2\vec{V}_2 = -D\overrightarrow{grad}\, c_2 \tag{2.65}$$

The flux φ_{c2} of the variable c_2 across the surface (S) related to any one of the three mentioned reference frames is:

$$\varphi_{c2} = \int_S \vec{q}_{c2}.\overrightarrow{ds} = -\int_S D\, \overrightarrow{grad}\, c_2.\overrightarrow{ds} \tag{2.66}$$

This approximation of law [2.41] in balance equation [2.8] allows the diffusion equation to be obtained:

$$\frac{\partial c_2}{\partial t} = \frac{\partial}{\partial x_j}\left(D \frac{\partial c_2}{\partial x_j} \right) + \sigma_2 \qquad \left(\frac{\partial c_2}{\partial t} = div\left(D\overrightarrow{grad}\, c_2 \right) + \sigma_2 \right) \qquad [2.67]$$

The volume source σ_2 of species 2 expresses, with the units of concentration c_2, the quantity per unit of time of this species created by a homogenous chemical reaction.

For weak concentrations, we can usually assume that the coefficient of diffusion D is constant; the diffusion equation is then reduced to the heat equation:

$$\frac{\partial c_2}{\partial t} = D \frac{\partial^2 c_2}{\partial x_j \partial x_j} + \sigma_2 \qquad \left(\frac{\partial c_2}{\partial t} = D\Delta c_2 + \sigma_2 \right) \qquad [2.68]$$

This approximation for weak concentrations is often used, either on account of experimental conditions or because it allows for a simplified approximate treatment of the problem, particularly in situations where it is associated and coupled with a heat release and with chemical reactions. On the other hand, diffusion often occurs in flows which present difficulties, and thus the preceding approximation is necessary.

2.4.5. *Coupled phenomena with diffusion*

2.4.5.1. *Binary non-isothermal mixtures*

The elementary discussion of the last section is not entirely rigorous, even in the absence of movement in the gas, because all phenomena characterized by scalar quantities are coupled (sections 2.1.5.2 and 2.1.5.3.4): the diffusion of a species in another species is accompanied by a thermal flux and a temperature gradient (Dufour effect). However, this is very often negligible in the absence of thermal conditions leading to a temperature gradient. Similarly, a temperature gradient (differences in the energy of molecular agitation of diluted gases) will induce a concentration gradient (i.e. a difference in the ways two different species of molecule will move in a gaseous mixture, some moving to the cold regions, others moving in the opposite direction).

The action of a mass force field (gravity, inertial forces, etc.) on the two species is manifest in a static equilibrium comprising a pressure gradient (or of the volumic number of moles), as we saw in section 2.2.1.5 (atmospheric equilibrium). In such a situation, the lighter molecules tend to rise. The thermodynamic force responsible for this effect is the pressure gradient. In practice, the diffusion of pressure is entirely negligible in the atmosphere, where the quantity $\overrightarrow{grad}\, n$ is nearly always negligible compared with $\overrightarrow{grad}\, n_1$ or $\overrightarrow{grad}\, n_2$; diffusion therefore occurs as if there was no force field. The diffusion of pressure must be accounted for in mixtures subjected to high accelerations, as in centrifuges.

Another cause for the molecular migration of a chemical species with respect to molecules of another species is a difference in the forces exerted between the two, for example between ions with different charges placed in an electric field. This thermodynamic force is a vector field, in other words of the same rank as the gradient of a scalar quantity. These phenomena are encountered, for example, during the electrolysis of solutions or in an ionized gas in the presence of an electric field.

Kinetic gas theory allows us to construct the theory of diffusive phenomena. The principle involves seeking distribution functions for neighboring velocities of the Maxwell-Boltzman distribution (Chapman-Enskog method). We find that first order deviations from this equilibrium distribution lead to the irreversible processes previously discussed. This theory also shows a decoupling between imbalances of different tensor orders such as thermal, mass or viscous fluxes. We will not develop the corresponding equations. For velocity differences in a gas ([PEN 55]) we obtain:

$$\begin{aligned}\left[\vec{V}_1\right]-\left[\vec{V}_2\right] &= -D\frac{n_2}{n_1}\,\overrightarrow{grad}\,\frac{n_1}{n_2}+\frac{Dn\left(M_1-M_2\right)}{p\rho}\,\overrightarrow{grad}\;p\\ &\quad+\frac{n^2D}{n_1n_2}\frac{\rho_1\rho_2}{p\rho}\left(\vec{F}_1-\vec{F}_2\right)-\frac{n^2}{n_1n_2}D_T\,\frac{\overrightarrow{grad}\,T}{T}\end{aligned} \qquad [2.69]$$

D_T is the thermal diffusion coefficient; we usually define the thermal diffusion ratio k_T by the relation:

$$D_T = k_T D$$

The thermal diffusion ratio k_T is more or less independent of temperature, but varies strongly with concentration: in particular, it tends to zero with each of the concentrations. Its maximal value, attained when the concentrations of components are in the same order of magnitude, is of the order of 0.1-0.2.

The preceding expression shows that a temperature gradient leads to additional diffusion phenomena (thermal diffusion) which can be used to separate two components of a gaseous mixture when chemical or other physical methods are not practicable (for example, in gaseous mixtures comprising two isotopes).

2.4.5.2. *Mixtures with several components*

Once a mixture is comprised of many components, the number of reduced extensive variables increases, each component requiring a concentration variable. There thus exist as many mass or molar flux densities as there are species present. For each species, we have a balance equation [2.24].

Following the principle outlined in sections 1.4.2.6 and 2.1.5.2, all diffusive fluxes depend on all of the thermodynamic forces of the same tensor rank. We thus have a matrix of diffusion coefficients. The general discussion concerning the choice of reference frames which characterize the diffusion processes is identical to that outlined earlier.

As we have already said (section 2.4.2.3), the *k* balance equations for each component leads to a global mixing equation that describes the conditions under which the mixture will evolve (in movement, at rest in a fixed container, during evaporation). The *k* equations are generally replaced by this global equation and *k* – 1 equations characterizing the components of the mixture.

The interested reader should consult textbooks covering problems of irreversible thermodynamics ([BIR 02], [BOC 92], [CHA 91], [DEG 62], [DOU 01], [EU 92], [GER 94], [LEV 62], [PRI 68]).

2.4.6. *Boundary conditions*

In the absence of chemical reactions, the boundary conditions can be identical to those of the thermal problem (section 2.3.2). The existence of a heterogenous reaction on the wall P leads to the production or absorption of the components. Chemical kinetics provides the law for the reaction speed for the components concerned. The flux density of a component at the wall must be equal to that produced or absorbed by the chemical reaction, for example for a reaction of order m:

$$-D\left(\frac{\partial c_1}{\partial n}\right)_P = kc_{1P}^m \qquad [2.70]$$

The coefficient *k* is often an expression of the form $k = k_0 \exp(-U/RT)$, where U is the activation energy of the reaction, *T* designating the absolute temperature. The form of this relation shows a strong coupling between the temperature and the reaction speed. The heat release due to heterogenous chemical reactions must be taken into account in the boundary conditions of the energy equation ([BOR 00], [PEN 55], [WIL 65]).

Fundamentals of Fluid Mechanics and Transport Phenomena
Jean-Laurent Peube

Chapter 3

Physics of Energetic Systems in Flow

In the preceding chapters, we examined the physical and mechanical properties of matter independent of the dynamic effects induced by motion. Before dealing in the next chapter with the general equations of fluid dynamics and the transfer of quantities in flows, we will first recall the basic laws of mechanics and their role in thermodynamics; we will then outline the formalism used to describe the motion of continuous media and finally we will examine the mechanical properties of moving fluids.

3.1. Dynamics of a material point

3.1.1. *Galilean reference frames in traditional mechanics*

As geometric space is homogenous and isotropic, the translational motion of an isolated material point is necessarily rectilinear and uniform. In effect, for any other kind of trajectory, a favored direction could be defined and any non-uniform movement of an isolated material point would imply an inhomogenous time. We thus postulate the existence of Galilean reference frames in which the distance traveled by an isolated material particle is a linear function of time. The laws of physics should be the same in all Galilean the reference frames. We must now change the reference frame to where the transformation matrix is a function of time and where all uniform translational movement are required to have the same properties in the new reference frame. For Cartesian reference frames this results in new coordinate systems which are in uniform rectilinear translation with respect to one another, and which form a *group*.

In *traditional mechanics*, changes in the reference frame which conserve distances and time belong to the Galilean group. They have the form:

$$x'_i = x_i + V_i t \qquad (i = 1,2,3), \qquad t' = t$$

The presence of the time variable in these reference frame changes leads to specific properties of temporal derivatives. As time is the same in all Galilean reference frames, the values of extensive scalar quantities are independent of the Galilean reference frame used. On the other hand, components of vector or tensor quantities vary in reference frame changes according to the usual formulae (covariant or contravariant according to the case considered). We say that these quantities are invariant for (geometric) changes of the Galilean reference frame (see texts on linear or tensor algebra). However, certain vectors (position or velocity of a particle) are defined with respect to a given Galilean reference frame; the evaluation of their temporal derivatives depends on the reference frame chosen for this definition. For example, the components in reference frame R' of the velocity of a point defined in reference frame R are not equal to the components in reference frame R' of the velocity of the point defined in reference frame R'. In a general manner, so-called cinematic operations (such as calculations of temporal derivatives velocity, acceleration) lead to formulae of changing reference frames dependent on their relative motion. We will assume that these ideas are known to the reader.

Let us recall the following elementary formulae, which will subsequently prove useful:

$$\vec{V}_a = \vec{V}_e + \vec{V}_r, \qquad \vec{\gamma}_a = \vec{\gamma}_e + \vec{\gamma}_r + 2\vec{\omega} \wedge \vec{V}_r$$

in which the indices *a* and *r* indicate that derivatives with respect to time are calculated in a Galilean reference frame *(a)* or in some other reference frame *(r)* which is in motion relative to the former. In the expression of acceleration the first term *e* represents the drag term (centripetal acceleration in the case of a rotating frame) and the third term is the *Coriolis acceleration*.

In the following, where not indicated otherwise, momentum, velocity and acceleration will be calculated in a Galilean reference frame.

3.1.2. *Isolated mechanical system and momentum*

We have seen in Chapter 2 that we characterize motionless matter by convenient extensive properties and that the assumption of thermodynamic equilibrium leads to the existence of relations between thermodynamic properties.

As in Chapter 1, we will begin by considering the simplest discrete system having mechanical properties: the material particle or the material point. This choice is made on account of the fact that a geometric point suffices to define its position. A material system, like any thermodynamic system, is made up of many simple systems, each comprised of an ensemble of material particles. These are described by systems whose dimensions are sufficiently small.

The extensive quantity associated with the motion of a material particle is momentum. The momentum of a material system is the sum of the momentum of its components. This vector quantity is proportional to the extension of the particle.

The momentum of an isolated mechanical system remains constant.

3.1.3. *Momentum and velocity*

As space is homogenous, the momentum of an isolated system is a function neither of its coordinates nor of time. Its expression must be identical in all Galilean reference frames; this property will impose an expression for the momentum.

We must note firstly that the idea of entropy does not exist in the mechanics of a particle. It only appears for systems in which the thermodynamic properties are defined and in which mechanical energy is transformed into heat. We will come back to this point a little later (section 3.2.5).

The *intensive quantity associated with the momentum*, for an elementary system, is the *velocity* $\vec{V}$, with components u_i, which represents the intensity of motion in a Galilean reference frame. Velocity depends on the reference frame used.

In a Galilean reference frame, momentum can only be a function of velocity in that reference frame, and not of position or time. The relationship between momentum and velocity can be derived from the preceding principles.

In effect, consider an isolated system composed of *two identical particles*. These particles, of constant initial velocities $\vec{V}_1$ and $\vec{V}_2$ in a Galilean reference frame R, interact through a collision (whose details are not important) such that after the collision they have the same constant velocity $\vec{V}'$ (a soft impact) in this same reference frame. If $\vec{p}(\vec{V})$ is the momentum of each particle, the momentum conservation of the two particle system can be written in R as:

$$\vec{p}(\vec{V}_1) + \vec{p}(\vec{V}_2) = 2\vec{p}(\vec{V}')$$

This equality must be true in all Galilean reference frames moving with constant velocity $\vec{V}$ ($\vec{V}$ having any value) with respect to the reference frame R. The following identity is derived, true for all $\vec{V}$:

$$\vec{p}\left(\vec{V}_1+\vec{V}\right)+\vec{p}\left(\vec{V}_2+\vec{V}\right)=2\vec{p}\left(\vec{V}'+\vec{V}\right) \qquad \forall\vec{V}$$

This equality implies that the function $\vec{p}\left(\vec{V}\right)$ is a linear function (take $\vec{V}$ successively equal to $-\vec{V}_1$ and $-\vec{V}'$, and then derive the identity $\vec{p}\left(\vec{X}\right)+\vec{p}\left(\vec{Y}\right)=\vec{p}\left(\vec{X}+\vec{Y}\right)$, characteristic of a linear relationship).

The momentum of a particle can thus be written in the form:

$$p_i = m_{ij}u_j \qquad \vec{p} = \overset{=}{m}.\vec{V}$$

where $\overset{=}{m}$ is a constant tensor which characterizes the material particle. As space is isotropic, this tensor is necessarily spherical ($m_{ij} = m\delta_{ij}$).

A material particle is thus characterized by a scalar m (mass) which leads to the relation:

$$p_i = m\delta_{ij}u_j = m_i u_i \qquad (\vec{p} = m\vec{V}) \qquad [3.1]$$

Similar reasoning using a Lorentz group for changing reference frames leads to a different expression of mass in special relativity.

3.1.4. *Definition of force*

As momentum is an extensive quantity it must satisfy a balance equation. When a material particle is subjected to an external action, its momentum changes. We call this external action *force* $\vec{f}$; it constitutes a source of momentum for this particle, whose rate of change $\dfrac{d\vec{p}}{dt}$ is:

$$\vec{f} = \frac{d\vec{p}}{dt} \qquad [3.2]$$

Forces are independent of the Galilean reference frame chosen for their description (this is no longer true in relativistic mechanics).

The effect of a force during a time interval (t', t) is characterized by a vector equal to the momentum variation $\vec{p}\big|_{t'}^{t}$ between these two instants. This is, by definition, the *impulse* $\vec{I}$ received by the particle:

$$\vec{I} = \int_{t'}^{t} \vec{f}dt = \vec{p}\big|_{t'}^{t} \qquad [3.3]$$

The idea of an impulse is useful in the study of collisions, which are often very rapid events and the detailed modeling of which is replaced by more or less global assumptions regarding their nature (soft impact, elastic, etc.). The *external force* $\vec{f}$ acting on a material particle is the *impulse quantity* $\frac{d\vec{I}}{dt}$ received by the particle per unit time.

As for the other thermodynamic quantities, this definition is only of interest in so far as we are able to explain the expression of this force as a function of external parameters. As with the other thermodynamic quantities, we can distinguish:

– *volume forces* representing actions at a distance, which are due to force fields: gravity, electromagnetic forces, etc.;

– *contact forces*, due to interactions on the microscopic scale between molecules, ions, atoms, etc. As for other contact actions, these are modeled by *surface forces*: stresses, pressure, viscous or dry friction, etc. We will come back to this point in the sections covering continuous media.

NOTES –

1) A force is associated with the material element on which it acts: forces are localized vectors (bi-points in mathematics) which can only be composed for each application point considered separately, except in the case of a rigid solid.

2) Forces are actions which obviously exist independent of motion.

3.1.5. *The fundamental law of dynamics (closed systems*)

Consider a closed material particle, i.e. one that does not exchange matter with the exterior; its mass is thus constant. Substituting [3.1] into [3.2] we obtain:

$$\vec{f} = \frac{d\left(m\overline{V}\right)}{dt} = m\frac{d\overline{V}}{dt} \qquad [3.4]$$

This equation, called the *fundamental law of dynamics* (or *Newton's second law of motion*), involves the acceleration $\vec{\gamma} = d\vec{V} / dt$; the quantity $m\vec{\gamma}$ is called the quantity of acceleration.

The quantity of acceleration is only the momentum's time derivative for systems of constant mass. A rocket which ejects a certain momentum in order to propel itself is not a closed system. In fluid mechanics, we will often reason in terms of open systems which exchange momentum with the exterior.

3.1.6. *Kinetic energy*

Every extensive quantity has a corresponding form of energy. The term for the energy differential (kinetic energy E_c) associated with the extensive quantity $\vec{p}$ for the preceding system's motion is:

$$dE_c = u_i dp_i \qquad [3.5]$$

which, taking account of expression [3.1] for a *particle of constant mass m*, is:

$$dE_c = mu_i du_i = d\left(\frac{mV^2}{2}\right) \qquad [3.6]$$

Kinetic energy can thus be expressed by the relation:

$$E_c = \frac{mV^2}{2}$$

The *kinetic energy theorem* can be obtained by taking the scalar product of the two sides of equation [3.2] with the velocity $\vec{V}$:

$$\frac{dE_c}{dt} = \vec{V}.\frac{d\vec{p}}{dt} = \vec{f}.\vec{V} \qquad [3.7]$$

The quantity $\vec{f}.\vec{V}$ is the power of the force $\vec{f}$ in the reference frame considered. As with the kinetic energy, it depends on the reference frame used for its evaluation.

3.2. Mechanical material system

3.2.1. *Dynamic properties of a material system*

A mechanical material system will be constructed, as for any thermodynamic system, by decomposition into *n* sub-systems which are points (or nearly points) in separate equilibriums (here at uniform velocity). The sub-systems interact amongst themselves according to the principle of action and reaction which results from the extensive nature of momentum. This quantity and the kinetic energy associated with the movement for the whole system of points are additive:

$$\vec{p} = \sum_{i=1}^{n} \vec{p}_i \qquad E_c = \sum_{i=1}^{n} \frac{m_i V_i^2}{2} \qquad [3.8]$$

The conservative nature of extensive quantities entails that total momentum remains constant in an isolated system. The velocity, which is the corresponding intensive variable, is of course not defined for the complete system if this one is not in a uniform state.

Quantity $\vec{p}$ is called the *total linear momentum* or *momentum of the system.* We also define the *total angular momentum* or *angular momentum* $\vec{\sigma}_O$ *at a point O* by the relation:

$$\vec{\sigma}_O = \sum_{i=1}^{n} \overrightarrow{OM}_i \wedge \vec{p}_i$$

Introducing the inertia center of the material system, and using formula [3.1], relation [3.8] can be written:

$$\vec{P} = m\vec{V}_G$$

Internal actions between components of the material system balance one another two by two like all internal actions. Let us separate the forces acting on the particle M_i into external forces $\vec{F}_{ext\,i}$ and internal forces $\vec{F}_{\mathrm{int}\,i}$ representing interactions between M_i and the other material particles of the system. Applying equation [3.2] gives:

$$\frac{d\vec{p}_i}{dt} = \vec{F}_{ext\,i} + \vec{F}_{\mathrm{int}\,i} \qquad i = 1,2,....n \tag{3.9}$$

The n vector equations [3.9] enable a detailed study of the mechanical system. By taking their sum, we obtain a vector equation which eliminates the internal forces and describes the effect of external action on the sum of the quantities of particle acceleration $m_i\vec{\gamma}_i = d\vec{p}_i/dt$ in the system, which is called *dynamic resultant* $\sum\limits_i m_i\vec{\gamma}_i = d\vec{P}/dt$

$$\frac{d\vec{P}}{dt} = m\vec{\gamma}_G = \sum_{i=1}^{n} \vec{F}_{ext\,i} \tag{3.10}$$

Equations [3.10] state that *the* dynamic resultant $d\vec{P}/dt$ *is equal to the sum of the external forces*. Internal forces do not participate and the momentum of an isolated system remains constant.

We proceed in similar fashion for the moments of the internal forces, which cancel one another out, two by two; the moment $\vec{C}$ at a given point in the ensemble of forces, naturally reduces to the moment of the external forces. The sum of the moments of the quantities of acceleration $d\vec{p}_i/dt$ at O, also called *dynamic moment* $\vec{\delta}_O$ *at a point* O, *is equal to the sum* $\vec{M}_{ext}$ *of the moments of the external forces acting on the points* M_i *and considered at the same point.*

We know that in taking moments, either at a fixed point O, or at the inertia center of the system, the dynamic moment $\vec{\delta}_O$ is equal to the time derivative of the total angular moment $\vec{\sigma}_O$, this last moment being the sum of the momentum moments:

$$\vec{\delta}_O = \frac{d\vec{\sigma}_O}{dt} = \sum_{i=1}^{n} \vec{M}_{exti} \tag{3.11}$$

The conservation of linear momentum for an isolated system is thus a consequence of the properties of internal forces.

Two particular vectors $\vec{P}$ and $\vec{\sigma}_O$ define an entity which replaces the detailed distribution of localized momentums only in the case of rigid solids. The reader should refer to texts on mechanics for a further study [AME 58].

3.2.2. *Kinetic energy of a material system*

3.2.2.1. *Kinetic energy theorem*

Kinetic energy theorem [3.7] is applied for each particle of the system. For the material system it can be obtained by taking the sum of all the kinetic energy equations corresponding to the particles M_i (obtained by taking the scalar product of each equation [3.9] with $\vec{V}_i$):

$$\frac{dE_c}{dt} = \frac{d}{dt}\left[\sum_{i=1}^{n} \frac{m_i V_i^2}{2}\right] = \mathscr{P}_{f\ \text{int}} + \mathscr{P}_{mec} \qquad [3.12]$$

with: $\mathscr{P}_{f\ \text{int}} = \sum_{i=1}^{n}\left(\vec{V}_i.\vec{F}_{\text{int}\ i}\right)$, the *power of the internal forces*;

$\mathscr{P}_{mec} = \sum_{i=1}^{n}\left(\vec{V}_i.\vec{F}_{ext\ i}\right)$, *the power of the external forces* (mechanical power provided to the mechanical system from the exterior).

The power of the internal forces is not generally zero.

3.2.2.2. *The power of internal forces*

The power of internal forces is independent of the reference frame used for its evaluation.

In order to show this, it suffices to regroup the internal forces by couples of opposed forces $\vec{F}_{ij}$ and $\vec{F}_{ji}$ interacting between the points M_i and M_j. The work W_{ij} of these forces is:

$$W_{ij} = \vec{F}_{ij}\vec{M}_j + \vec{F}_{ji}\vec{M}_i = \vec{F}_{ij}\left(\vec{M}_j - \vec{M}_i\right) = \vec{F}_{ij}.\overrightarrow{M_i M_j}$$

The two vectors of the scalar product W_{ij} are independent of the reference frame chosen (Galilean or otherwise) and the scalar W_{ij} takes on the same value in all reference frames; the same is true for the total work done by the internal forces. As time is identical in all reference frames, the power of the internal forces is thus independent of the reference frame used.

Certain internal forces derive from a potential, such as Newtonian forces or elastic forces; the same is not true for frictional forces.

3.2.2.3. *The power of external forces*

Let us consider a case where the *external forces and certain internal forces derive from a potential U*, the other internal forces being designated by $\vec{F}_{f\,\text{int}\,i}$. Equation [3.12] can be written:

$$\frac{d}{dt}\left[\sum_{i=1}^{n}\left(\frac{m_i V_i^2}{2}+U(Mi)\right)\right]=\sum_{i=1}^{n}\vec{V}_i.\vec{F}_{f\,\text{int}\,i} \qquad [3.13]$$

We define the *mechanical energy* E_m of the system by the relation:

$$E_m=\sum_{i=1}^{n}\left(\frac{m_i V_i^2}{2}+U(M_i)\right)$$

Equation [3.13] can be written:

$$\frac{dE_m}{dt}=\mathscr{P}_{f\,\text{int}} \qquad [3.14]$$

The total mechanical energy of the isolated material system is not conserved because the internal forces provide a non-zero power, except in certain particular cases (rigid, solid, etc.).

Consider the following example: two particles of the same mass *m* are elastically linked with friction; they constitute an oscillator where each of their opposed elongations are centered around the inertia center *G*.

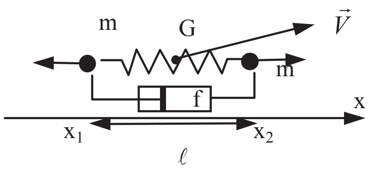


Figure 3.1. *Isolated oscillator*

We assume that there are no external forces. The kinetic energy of the whole is the kinetic energy of the inertia center animated by the constant velocity V_G, which is increased by the kinetic energy of the oscillator in a reference frame related to the inertia center G. The total mechanical energy can be obtained by adding the potential energy of the oscillator. This decreases on account of the friction and is transformed into heat, thus increasing the internal energy by heat addition. On the other hand, the momentum of the ensemble has not changed.

The *power of the internal forces* depends only on the relative particle velocity and *not on the chosen reference frame.* On account of the internal friction of the system, it decays as well as the total mechanical energy of the system.

3.2.3. *Mechanical system in thermodynamic equilibrium: the rigid solid*

We will now examine how motion intervenes in thermostatics and thermodynamics. Consider a material system, assumed to be in movement and isolated, in other words not surrounded by any other material system which exerts an action on it; it will finish by being "rigidified": the relative movements of its different constituents disappear under the effect of internal dissipative forces (viscous damping or internal friction, exchange of momentum and collisions between the particles comprising the system, etc.). In other terms, the relative distances between the different material particles making up the system remain constant. The mechanical system in thermodynamic equilibrium has become a *rigid system* in which there are no longer any entropy sources associated with friction. The system is in a state of thermostatic equilibrium if its other extensive quantities, defining its internal state, are no longer susceptible to change.

We obtain the same final state for a material system placed in a force field derived from a potential. The fluids contained in such a material system are thus in hydrostatic equilibrium. We have already discussed, in section 2.2.1.2, the restrictive conditions necessary for the existence of such a situation. However, if the external force field does not derive from a potential, the exterior will always be able to communicate an energy to compensate the friction to the fluid (by an electromagnetic action in a conducting fluid, for example).

The rigid solid thus plays an important role from the point of view of thermostatics, since it is the ultimate state (thermostatic equilibrium) towards which all isolated mechanical systems will tend. It is furthermore what is observed for a gaseous or liquid mass enclosed in a container which is not subject to any external action; the same is true for all collections of solids (a pile of sand, etc.), including solar bodies: the moon no longer rotates, always showing the same side to the Earth; the daily rotation of the Earth slows continually due to diverse dissipative effects (mostly tidal movements of the oceans, which lead to a dissipation of energy on account of a gravitational Earth-sun and Earth-moon interactions).

We know that the momenta of a rigid solid are characterized by linear momentum and angular momentum, which are constant for an isolated rigid solid. Recall that its motion is not characterized by an instantaneous constant velocity vector of rotation; (this motion is a Poinsot motion around its inertia center (see texts on mechanics [AME 58])).

3.2.4. *The open mechanical system*

3.2.4.1. *Introduction*

We can also perform a balance for an open material system whose momentum varies because of an input of matter (for example a water bomber collecting water) or an output of matter (the propulsion of a rocket). In fluid mechanics, the balances of extensive quantities, including momentum, are performed for open systems. However, the physical conservation laws for extensive quantities are applied to a given material quantity, i.e. to closed systems. The balance equations for open systems must obviously take account of the change of definition in the system where the balance is performed (section 1.1.4.2), i.e. take into account the input or output of extensive quantities associated with the input or loss of matter.

3.2.4.2. *Momentum of a rocket*

As an example, consider the vertical motion of a rocket in a gravitational field. Its mass *m(t)* decreases with the ejection of a constant mass flow of gas $q_m = -\dfrac{dm}{dt}$, at a velocity U with respect to the rocket.

Let *m(t)* be the rocket mass at instant *t*. We consider a fixed reference frame Oz and perform an elementary momentum balance between time *t* and *t* + *dt* on the mass *m* comprised by the rocket at time *t*.

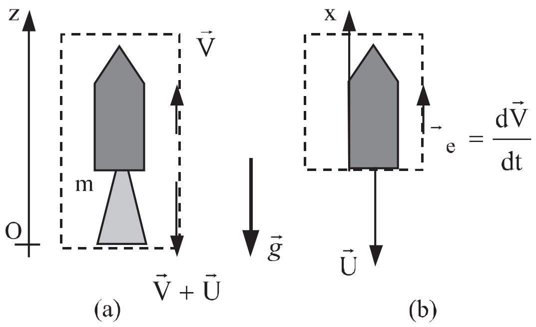


Figure 3.2. *Momentum balance: (a) closed system; rocket + gas, (b) open system; rocket alone*

At the instant $t + dt$ the rocket mass has changed from m to $m + dm$, whereas the velocity in the fixed reference frame has increased from $\overline{V}$ to $\overline{V} + d\vec{V}$, and the mass $-dm$ of gas has been ejected. The mass ejected, initially moving at a velocity $\overline{V}$ has now changed to $\vec{V} + \vec{U}$ at time $t + dt$ in the fixed reference frame. Between these two instants, the momentum variation of the mass m can be calculated as:

$$d\vec{p} = (m + dm)(\vec{V} + d\vec{V}) - dm(\vec{V} + \vec{U}) - m\vec{V} = m\,d\vec{V} - \vec{U}\,dm \qquad [3.15]$$

Equality [3.15] highlights:

– the change in momentum $d\vec{p}_f = d(m\vec{V})$ of the rocket (of variable mass m):

$$d\vec{p}_f = (m + dm)(\vec{V} + d\vec{V}) - m\vec{V} = d(m\vec{V})$$

– the momentum $d\vec{p}_g$ of the mass $-dm$ ejected from the rocket:

$$dp_g = -dm\,(\vec{V} + \vec{U})$$

We have, per unit time:

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{V})}{dt} - \frac{dm}{dt}(\vec{V} + \vec{U}) = \frac{d\vec{p}_f}{dt} + \frac{d\vec{p}_g}{dt} = m\frac{d\vec{V}}{dt} - \vec{U}\frac{dm}{dt} \qquad [3.16]$$

The time derivative of the momentum $\vec{p}_f$ of the rocket is not equal to the quantity of acceleration $m\dfrac{d\vec{V}}{dt}$.

The quantity $d\vec{p}_g / dt$ is associated with a matter which is constantly being generated at the exit of the rocket exhaust: it is a *momentum flux*.

In order to simplify matters, we will consider that the only external force is constituted by gravity; taking account of [3.16], equation [3.10] for the momentum at constant mass m can be written:

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{V}}{dt} - \vec{U}\frac{dm}{dt} = m\vec{g}$$

or:

$$m\frac{d\vec{V}}{dt} = -\vec{U}\, q_m + m\vec{g} \tag{3.17}$$

Equation [3.17] shows that we can apply the fundamental dynamic relation to a system of variable mass by considering the *quantity of acceleration* for that system, the ejection of burned gases generating a force on the rocket opposed to the momentum flux $\vec{U}q_m$ (relative to the rocket) generated by the propulsion system.

We can verify that the opposite of this force, acting for a time dt, corresponds to the impulsion $\vec{U}q_m dt$ necessary to give to the mass $q_m dt$ ejected in time dt a variation in velocity equal to $\vec{U}$. This force is the *rocket thrust*.

The preceding balance can also be written in the non-Galilean reference frame of the rocket (whose acceleration is $\dfrac{dV}{dt}$). In this reference frame, the rocket ejects momentum $\vec{U}q_m$ per unit time (momentum flux) which balances the inertial force $-m\dfrac{d\vec{V}}{dt}$ and the gravitational force $m\vec{g}$.

A momentum flux is associated with the existence of a force. This observation is the basis of many concepts in fluid mechanics. We will see it again when we study the momentum flux theorem.

The preceding results can be extended to moments (of momentum, or of quantities of acceleration) taken about a fixed point O or about the inertia center of the system.

3.2.4.3. *Kinetic energy balance*

The variation of kinetic energy dE_c of a system comprising mass m can be calculated in a similar manner to that used to calculate the momentum. We have:

$$\frac{dE_c}{dt} = m\frac{d}{dt}\left(\frac{V^2}{2}\right) + q_m\left[\frac{1}{2}\left(\vec{V}+\vec{U}\right)^2 - \frac{\vec{V}^2}{2}\right] = m\frac{d}{dt}\left(\frac{V^2}{2}\right) + q_m\left[\vec{V}.\vec{U} + \frac{U^2}{2}\right]$$

By taking the scalar product of equation [3.17] with the velocity $\vec{V}$ we find the kinetic energy equation of the rocket:

$$m\frac{d}{dt}\left(\frac{V^2}{2}\right) = -\vec{U}.\vec{V}\,q_m + m\vec{g}.\vec{V}$$

This gives, for the kinetic energy variation dE_c:

$$\frac{dE_c}{dt} = q_m\frac{U^2}{2} + m\vec{g}.\vec{V} \qquad [3.18]$$

The term $m\vec{g}.\vec{V}$ is the power of the gravitational force. The power $\mathscr{P}_{\text{fint}}$ of the internal forces is here equal to $q_m\dfrac{U^2}{2}$ which represents the quantity of kinetic energy per unit time created by combustion in the fixed reference frame of the rocket where chemical energy (reaction enthalpy) is transformed into mechanical energy.

This preceding reasoning will be used in a more general manner in Chapter 4, where the balance equations in a moving fluid will be established. We will use Eulerian variables in order to clearly explain the phenomena associated with the concept of convection.

3.2.5. *Thermodynamics of a system in motion*

3.2.5.1. *Energy equation*

Consider a material system, described by means of the variables of a certain number *p* of sub-systems. The internal energy of each component is the energy of its matter in a reference frame in which it is at rest. The differential [3.5] *dE* of the internal energy can be expressed as a function of the extensive quantities X_i and the intensive quantities Y_i previously defined:

$$dE = \sum_{i=1}^{p} \left(T_i dS_i + Y_i dX_i \right)$$

Now suppose that the system components are in motion. The total energy contained in the system must take account of all energy forms, including its kinetic energy E_c. In other words, the mechanical properties constitute just one particular aspect of the physical phenomena to be considered. We denote as E_t this "total" energy in order to distinguish it from the internal energy, denoted *E*:

$$E_t = E + E_c$$

The conservation principle is thus applied, not to the internal energy alone, but to the total energy, whose variation ΔE_t is equal to the *quantities of heat Q and of work W received from the exterior*, including the work W_{mec} received by the system and coming from the external forces:

$$\Delta E_t = \Delta E + \Delta E_c = W + Q$$

The extensive and intensive scalar variables, and in particular the internal energy, do not depend on the reference frame chosen: for example, the tension of an elastic string and its stretching, the pressure of a gas and its volume, the number of moles of a chemical species and its chemical potential, etc. Interpretations of internal energy in diverse physical theories are often made using forces whose details are of little importance, but as these are internal forces, their resultant moments are zero; we have seen (section 3.2.2.2) that the power $\mathscr{P}_{\text{fint}}$ of the internal forces of a system (or the corresponding work) is independent of the reference frame chosen for its evaluation. This power of the internal forces is in fact already described in the differential *dE* of the internal energy (work done by pressure forces, etc.).

The kinetic energy and the total energy of a system depends on the reference frame chosen for their evaluation.

For a system which receives from exterior sources of a purely mechanical kind, we thus have:

$$\frac{d}{dt}(E+E_c)=\mathcal{P}_{mec}+\mathcal{P}_{th} \tag{3.19}$$

$\mathcal{P}_{\text{mec}}$ and $\mathcal{P}_{\text{th}}$ being the mechanical and thermal powers provided to the system from the exterior.

Consider first a mechanical system comprised of a material "particle" which is animated, for the sake of simplicity, by a macroscopic translational motion of velocity V. Now suppose that this particle receives, from the exterior, the mechanical power $\mathcal{P}_{mec}=\vec{V}.\vec{F}$ and the thermal power $\mathcal{P}_{\text{th}}$. The balance equation for the (total) energy of the material system can be written:

$$\frac{d}{dt}\left(E+\frac{mV^2}{2}\right)=\mathcal{P}_{th}+\mathcal{P}_{mec}$$

Now consider a system of n material particles of velocities $\vec{V}_k$. Letting E denote the internal energy of the system (which is not in equilibrium), the balance equation can be written, taking account of motion, as:

$$\frac{d}{dt}\left(E+\sum_{k=1}^{n}\frac{m_kV_k^2}{2}\right)=\mathcal{P}_{th}+\mathcal{P}_{mec} \tag{3.20}$$

Subtracting the kinetic energy equation [3.12] from [3.20], we obtain the energy equation (E, the internal energy) of the material system:

$$\frac{dE}{dt}=\mathcal{P}_{th}-\mathcal{P}_{f\,\text{int}} \tag{3.21}$$

The rate of internal energy variation of a system is equal to the difference between the thermal power which it has received and the power of its internal forces.

3.2.5.2. *The entropy form of the energy equation*

It is useful to explain the preceding phenomena using an example. Consider the system already discussed in section 3.2.2.3 (Figure 3.1), comprised of two masses linked by elastic under tension $\tau(\ell)$ and equipped with a viscous friction system. The tension force τ derives from a potential.

The masses are subjected to external forces $\vec{F}_j$ (*j=1,2*); we assume that gravity does not directly intervene. The viscous friction system is assumed to have no mass and the heat released by the friction heats the two masses which are assumed to be at the same temperature. The dynamic equation of motion for the masses in the reference frame of the inertia center can be written:

$$m\vec{\gamma}_1 = \vec{F}_{ext1} + \vec{F}_f + \vec{\tau} \qquad m\vec{\gamma}_2 = \vec{F}_{ext2} - \vec{F}_f - \vec{\tau}$$

in which the tension $\tau(\ell)$ of the elastic link and the viscous friction force F_f, aligned with the axis Ox, are, by definition, the forces applied on each mass:

$$F_f = f\frac{d(x_2 - x_1)}{dt} = f\dot{\ell}$$

By adding the terms of the two equations after multiplication of each by its corresponding velocity $\vec{V}_k\,(k = 1,2)$, we obtain the kinetic energy equation. The power of the internal forces can be immediately calculated:

$$\mathcal{P}_{f\,int} = (\tau + F_f)(\vec{V}_1 - \vec{V}_2) = -\tau\dot{\ell} - f\dot{\ell}^2 \qquad [3.22]$$

Energy equation [3.21] can be written:

$$\frac{dE}{dt} = \left(\tau\dot{\ell} + f\dot{\ell}^2\right) + \mathcal{P}_{th} \qquad [3.23]$$

By specifying the differential of the internal energy $dE = TdS + \tau\, d\ell$ for the system, we finally obtain the energy balance equation in its entropic form:

$$T\frac{dS}{dt} = f\dot{\ell}^2 + \mathcal{P}_{th} \qquad [3.24]$$

We see that in this entropic *form of the energy equation* the mechanical terms have been eliminated and *all that remains is the irreversible part of the motion which is transformed into heat*. The dissipation function $f\ddot{x}^2$ is a heat source which is always positive. It appears in equation [3.24] as an *internal entropy source*.

The preceding developments can be repeated for a system of *n* particles at the same temperature, between which there exist elastic forces and friction forces. We thus obtain an entropic form of the energy equation, analogous to [3.24], with a dissipation function for all the internal frictions of the system. The reader will note that this reasoning is only valid if the entropy of the system studied is a function of the other extensive quantities, i.e. if the system is in a state of thermostatic equilibrium at every instant.

We will see these developments (sections 3.1 and 3.2) once again when we study flowing continuous media, albeit with a formalism which is naturally more complex.

3.3. Kinematics of continuous media

3.3.1. *Lagrangian and Eulerian variables*

A problem of flowing matter involves the study of the influences of external conditions on the fluid medium, specifically the many different ways forces exerted on the medium (presence of fixed boundaries with respect to the observer, moving boundaries such as propellers blades, turbine blades, etc., pressure differences between two reservoirs, external forces such as gravity and electrical volume forces due to some external device, etc.). The properties of a flow thus result from the action of external causes which modify the mechanical and physical properties of the matter (velocity and acceleration, internal stresses, pressure, temperature, chemical composition, etc.) and inversely the flowing medium exerts stresses on walls or modifies the boundary properties (stresses induced, temperature, chemical properties, etc.).

The questions posed and the results expected from a study, an experiment or the operation of a device or system can vary considerably. Here are some examples:

– in a water treatment station or in a chemical reactor, we are naturally concerned with the product being treated; the practical problem is thus to design walls, materials and diverse processes so as to obtain the desired result concerning the product which is treated;

– in meteorological applications, the objective is to predict the weather at a given place and time, i.e. to predict the motion, and the physical and chemical properties of large air masses (velocity, temperature, humidity, presence of pollutants, etc.);

– for a boat, a plane or a vehicle in motion, the essential properties are the external forces exerted on the solid boundaries in contact with the flow when the vehicle is displaced at a given velocity. The same is true for the flow of a river or fluids in an industrial pipe network for which suitable constructions are necessary (dams, turning vanes, pipes, open channels, pumps, turbines, etc.).

The study of a physical problem begins with a definition of its variables. The description of moving matter involves the localization of a material particle which is identifiable in a given reference frame by means of coordinates. The physical properties of this particle are associated with it. In particle mechanics, we give numbers to the different particles studied and to the physical quantities which characterize them. We thus define the trajectory of a particle as an ensemble of the successive positions which it occupies over the course of time.

When studying the physics and mechanics of continuous media, this means of description is no longer suitably adapted, since the set of the material particles of a continuous medium is not denumerable. We thus need to identify them with respect to a continuous ensemble which is simply the ensemble of positions x_{i0} which this material occupies at a given reference time. The variables g associated with the matter are thus functions of this position x_{i0} occupied at an initial instant and of time:

$$g = g(x_{i0}, t)$$

This is called *material representation* by means of *Lagrangian variables*. Problems for moving continuous media where the matter must be clearly individualized are generally treated using Lagrangian variables; this description is often used in the mechanics and physics of solids.

The disadvantage of Lagrangian variables is that such a description does not have any particular interest for the observer in many situations: the fisherman at the water's edge sees the passage of water particles which he will never again see, and the same is true for the plumber examining a radiator, etc. As the fluid matter passes by continuously, the identification of fluid particles is not particularly useful. This is not the case however when we are interested in identifying the local physico-chemical properties of the matter. Examples include the progress of a chemical reaction in a closed reactor, the presence of clouds and the knowledge of humid air masses in weather applications or the presence and movement of pollutants downstream of a pollution source, etc.

Frequently, in fluid mechanics we consider a preferred observer reference frame, which is most often associated with solid boundaries between which or in the vicinity of which the fluid matter passes (pipe networks, solid obstacles, the wings of an aircraft, etc.). We thus represent the matter *spatially* by means of a field where quantities associated with a material particle are functions of the coordinates x_i which the particles occupy at the observation time t. This kind of representation involves *Eulerian variables*.

In physics, the study of particles contained in and subject to a force field generally involves the combination of these two representations, the particle being described by its coordinates as a function of time (Lagrangian variables) whereas the external field (gravity, etc.) is defined using its spatial properties.

Fluids in motion are generally described using Eulerian variables, the observer reference frame being associated with the solid boundaries. A considerable difficulty thus appears, as the formulation of the laws of thermodynamics and mechanics requires the *balance of extensive quantities associated with matter.* In the following sections we will write these balances by "following matter" over material domains represented using Eulerian variables.

3.3.2. *Trajectories, streamlines, streaklines*

In Eulerian variables the kinematics of a flow is characterized by the velocity field, which is a function of time:

$$u_i = u_i\left(x_j, t\right) \quad \left(i, j = 1,2,3\right) \quad \text{or} \quad \vec{V} = \vec{V}\left(\vec{M}, t\right).$$

In a velocity field, we can define the following curve families:

– the *streamlines at time t* are lines whose tangents are the velocity vectors at this instant.

These give a *vision of the flow at instant t* and their differential equations are:

$$\frac{dx_1}{u_1(x_i,t)} = \frac{dx_2}{u_2(x_i,t)} = \frac{dx_3}{u_3(x_i,t)} \qquad [3.25]$$

– *a trajectory* is the locus of a fluid particle; trajectories are characterized by their spatial coordinates which are a function of time (Lagrangian description); their differential equations (here *t* is a variable) are:

$$\frac{dx_1}{u_1(x_i,t)} = \frac{dx_2}{u_2(x_i,t)} = \frac{dx_3}{u_3(x_i,t)} = dt \qquad [3.26]$$

– a *streakline* at time *t* is the locus of particles which have passed by a given point earlier than *t*, such as the cloud of smoke issuing from a chimney or the colored line obtained in a moving fluid by injection of a colored liquid or a trickle of smoke. The process of flow visualization involves the generation of visible emission lines (Figure 3.3) by injection of markers (dyes, smoke, small particles).

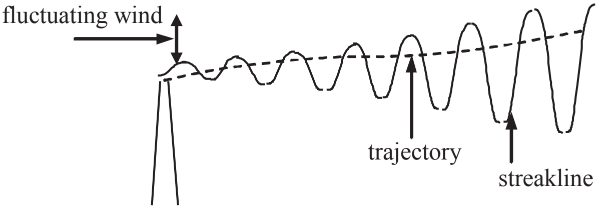


Figure 3.3. *Trajectory of a smoke particle and streakline from a chimney when wind is blowing*

When a velocity field does not explicitly depend on the variable t, $u_i(x_j,t) = u_i(x_j)$ and we have a *steady flow*. The streamlines, which no longer depend on time, are fixed in space: the appearance of the flow does not change. The preceding definitions and equations show that *streamlines, trajectories and streaklines are all the same in steady flows*.

EXERCISE – Find the equations for the streamlines, the trajectories and the streaklines of the following 2D velocity field: u = constant, $v = a \cos \omega t$ t. Interpret the results.

3.3.3. *Material (or Lagrangian) derivative*

3.3.3.1. *Material (or Lagrangian) derivative of a quantity g*

Let $g(x_i,t)$ be the field of a continuous scalar quantity associated with a medium of fluid particles moving with a velocity field $u_i(x_j,t)$ defined in a given reference frame. The *material derivative* (also called the *total* or *Lagrangian* or *substantial* derivative) is defined as the time derivative of *the quantity g associated with the material particle* of coordinates $x_j(t)$ and velocity $u_i(x_j,t)$. It can be obtained by calculating the time derivative of g in which the coordinates are functions of time and describe the trajectory:

$$\frac{dg}{dt} = \frac{\partial g}{\partial t} + u_j \frac{\partial g}{\partial x_j} \quad \text{or} \quad \frac{dg}{dt} = \frac{\partial g}{\partial t} + \vec{V}.\overrightarrow{grad}\ g$$

We will denote the Lagrangian derivative $\frac{d}{dt}$, which is by its very definition the derivative of a *function of time only for a fluid particle*. It is also often denoted $\frac{D}{Dt}$ in fluid mechanics.

In particular, for components of the acceleration, we have:

$$\gamma_i = \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \quad \text{or} \quad \vec{\gamma} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \left(\overline{\overline{grad}}\, \vec{V} \right) \vec{V} \qquad [3.27]$$

Acceleration [3.27] can also be written in the form:

$$\vec{\gamma} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \overrightarrow{grad} \frac{V^2}{2} + 2\vec{\omega} \wedge \vec{V} \qquad [3.28]$$

where $\vec{\omega} = \frac{1}{2}\, r\vec{o}t\, \vec{V}$ is the *rotation vector*. The *vorticity vector* is twice the rotation vector.

NOTE – The material derivative $\frac{dg}{dt}$, *defined in a Galilean reference frame*, is a physical quantity which does not depend on the reference frame chosen for its evaluation; on the contrary, the temporal derivative $\frac{\partial g}{\partial t}$ of the spatial representation corresponds to an observation of the quantity g at a fixed point and it does depend on the reference frame chosen to represent the field. Consider two Cartesian reference frames of coordinates $\left(x_j, t\right)$ and $\left(\tilde{x}_j, \tilde{t}\right)$:

$$\tilde{x}_j = x_j - U_j t; \quad \tilde{t} = t.$$

The reader can easily verify the relation:

$$\frac{\partial g}{\partial t} = \frac{\partial \tilde{g}}{\partial \tilde{t}} - U_j \frac{\partial \tilde{g}}{\partial \tilde{x}_j}$$

where we let $g\left(x_i, t\right) = \tilde{g}\left(\tilde{x}_i, \tilde{t}\right)$.

The time derivative calculated in a reference frame depends on the choice of reference frame. Its value is notably associated with spatial inhomogenities of the quantity considered and which is displaced with respect to this reference frame.

For example, the reader can verify that the field of the quantity g, steady in the reference frame (x_1, t) and with spatial period λ

$$g(x_1,t) = A\cos\left(\frac{2\pi\, x_1}{\lambda}\right)$$

is seen by a fixed observer in the reference frame $\left[\tilde{x}_1 = x_1 - Ut;\ \tilde{t} = t\right]$ as an oscillating field with temporal period $\tilde{T} = \lambda/U$. We will come back to this phenomenon, which is known as the *Doppler-Fizeau effect* in acoustics.

3.3.3.2. *Flux of the quantity G*

We saw (section 2.1.3.1) that the transfer of an extensive quantity G is characterized by a *flux density vector* $\vec{q}_G$ whose flux across a surface is equal to the amount of the quantity G which crosses this surface per unit time. The demonstration of the existence of this vector is independent of the transfer mechanisms considered. In particular, the movement of matter with respect to a surface leads to a flux of the quantities associated with the matter across this surface, which we often refer to as a convective flux of the quantity G.

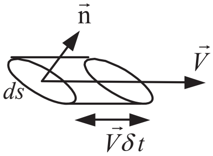


Figure 3.4. *Balance in a flow through ds for an elementary displacement*

When a quantity G is transported by moving matter, the quantity G which crosses a surface ds of unit normal $\vec{n}$ in time δt occupies an oblique cylinder of length $\vec{V}\delta t$ (Figure 3.4). We thus derive an expression for the flux density vector of G:

$$\vec{q}_g = \overline{g}\vec{V} \qquad (\overline{\text{g}}\text{: quantity of } G \text{ per volume unit}) \qquad [3.29]$$

The convective flux of the quantity G crossing a surface S per unit time is thus equal to:

$$\varphi_G = \int_S \vec{q}_g.\vec{n}\, ds = \int_S \overline{g}\vec{V}.\vec{n}\, ds \qquad \left(\varphi_G = \int_\Sigma \overline{g}u_j n_j ds\right) \qquad [3.30]$$

The volume and mass transported by the matter's motion are respectively characterized by the volume flux density vector $\vec{V}$ and the mass flux density vector $\rho\vec{V}$ which correspond, respectively, to $\overline{g} = 1$ and $\overline{g} = \rho$.

The elementary volume flux dq_v and the elementary mass flux dq_m are[1]:

$$dq_v = \vec{V}.\vec{n}\ ds \qquad \text{and} \qquad dq_m = \rho\vec{V}.\vec{n}\ ds$$

The *volume flux* q_v (or volume flow rate) and the *mass flux* q_m (or mass flow rate) across a surface S can be written:

$$q_v = \int_S \vec{V}.\vec{n}\ ds \quad \text{and} \quad q_m = \int_S \rho\vec{V}.\vec{n}\ ds$$

When the quantity $\vec{G}$ is a vector (for example the momentum $m\vec{V}$), the preceding results can simply be applied to each of the components $\overline{g}_i$. We thus define a *flux density tensor* $\overline{\overline{q}}_G$ of the quantity $\vec{G}$ by the relation:

$$q_{Gij} = \overline{g}_i u_j \quad \text{or} \quad \overline{\overline{q}}_G = \vec{g} \otimes \vec{V} \qquad [3.31]$$

The flux of the quantity $\vec{G}$ crossing the surface S is thus the vector $\vec{\varphi}_{GS}$:

$$\vec{\varphi}_{GS} = \int_S \overline{\overline{q}}_G \vec{n} ds$$

3.3.3.3. *Material derivative of a volume integral*

Let $\overline{g}(x_i, t) = \rho g(x_i, t)$ be the volume density of the quantity G (g designates the corresponding mass density). The amount of the quantity G contained in a domain $\mathfrak{D}$ of fluid inside the closed surface Σ is:

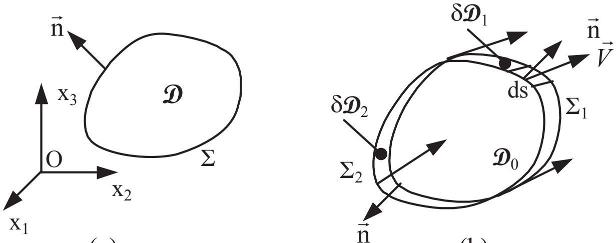
$$G = \int_{\mathfrak{D}} \overline{g}(x_i, t)\ dv$$

The laws of physics imply the balance of the extensive quantities (mass, momentum, etc.) associated with the matter which, in a flow, is in motion. It is therefore necessary to calculate the variation of the quantity G associated with a material domain $\mathfrak{D}$ in motion using the usual Eulerian representation. $\vec{n}$ designates the unit normal to the surface Σ directed towards the outside of the fluid domain $\mathfrak{D}$.

---

[1] We shall use the usual scalar notation q_v and q_m for volume or mass fluxes which evidently are fluxes and not flux densities; confusion is not possible, flux densities being at least vectors.

The extensive quantities are associated with the matter and the balance of the quantity G must be performed on a material domain. Consider (Figure 3.5a) the domain $\mathcal{D}(t)$ occupied by the matter at instant t, of external surface Σ, and then the domain $\mathcal{D}(t+\delta t)$ occupied by the matter at time $t+\delta t$. Between these two instants, the material domain is displaced such that it leaves Σ over the section Σ_1 whereas it is displaced towards the interior of $\mathcal{D}(t)$ over the surface Σ_2 (Figure 3.5b); let $\mathcal{D}_0$ be the volume common to $\mathcal{D}(t)$ and $\mathcal{D}(t+\delta t)$. Let $\delta\mathcal{D}_1$ and $\delta\mathcal{D}_2$ be the additional parts of $\mathcal{D}_0$ in $\mathcal{D}$(t) and $\mathcal{D}(t+\delta t)$ generated by the surfaces Σ_1 and Σ_2 and the vector displacement $\vec{V}\delta t$. The (positive) amounts of the quantity G contained in each of these domains are, respectively, $\int_{\Sigma_1}\overline{g}\vec{V}.\vec{n}\,\delta t ds$ and $-\int_{\Sigma_2}\overline{g}\vec{V}.\vec{n}\,\delta t ds$.



(a) (b)

Figure 3.5. *Balance of an extensive quantity in a flow: (a) domain at one instant; (b) displacement of the material domain*

The variation δG of the quantity G contained in $\mathcal{D}$ between the instants t and $t+\delta t$ is thus:

$$\begin{aligned}\delta G &= G(t+\delta t)-G(t)\\ &= \int_{\mathcal{D}_0}\left[\overline{g}(t+\delta t)-\overline{g}(t)\right]dv+\int_{\Sigma_1}\overline{g}\vec{V}.\vec{n}\,\delta t ds-\left(-\int_{\Sigma_2}\overline{g}\vec{V}.\vec{n}\,\delta t ds\right)\end{aligned}$$

By regrouping the surface integrals, taking the limit as $\delta t\to 0$, and applying Ostrogradski's theorem, we obtain:

$$\frac{dG}{dt}=\int_{\mathcal{D}_{\circ}}\frac{\partial\overline{g}}{\partial t}dv+\int_{\Sigma}\overline{g}\vec{V}.\vec{n}ds=\int_{\mathcal{D}_{\circ}}\left(\frac{\partial\overline{g}}{\partial t}+div\left(\overline{g}\vec{V}\right)\right)dv \quad [3.32]$$

or, using notation with indices:

$$\frac{dG}{dt} = \int_{\mathcal{D}} \frac{\partial \overline{g}}{\partial t} dv + \int_{\Sigma} \overline{g} u_i n_i ds = \int_{\mathcal{D}} \left(\frac{\partial \overline{g}}{\partial t} + \frac{\partial (\overline{g} u_i)}{\partial x_i} \right) dv \qquad [3.33]$$

The notation $\frac{dG}{dt}$ in equation [3.32] or [3.33] indicates a Lagrangian derivative.

Formula [3.33] shows that the flux of the quantity G across the surface Σ can be interpreted as due to the integration over the volume of the *local volume source* $div(\overline{g}\vec{V})$.

Material derivatives can also be expressed as a function of the mass density g:

$$\frac{dG}{dt} = \int_{\mathcal{D}} \frac{\partial \overline{g}}{\partial t} dv + \int_{\Sigma} \overline{g} dq_v = \int_{\mathcal{D}} \frac{\partial}{\partial t} (\rho\, g) dv + \int_{\Sigma} g dq_m$$

Let us take the elementary case where $\overline{g} = 1$. The quantity G is the volume of the domain $\mathcal{D}$. For a small domain of volume δv, we have:

$$\frac{d(\delta v)}{dt} = div\vec{V}.\delta v$$

The quantity $div\vec{V}$ is thus the local velocity of the volume expansion

$$\frac{1}{\delta v} \frac{d(\delta v)}{dt}.$$

If the quantity $\vec{G}$ is a vector (for example the volume momentum $\rho\vec{V}$), the preceding steps can simply be applied to each of the components g_i. We thereby define a *tensor flux of the quantity* $\vec{G}$ by the relation:

$$q_{Gij} = g_i u_j \quad \text{or} \quad \overline{\overline{q}}_G = \vec{g} \otimes \vec{V}, \quad (\text{with } \otimes : \text{Cartesian product}).$$

NOTE – A more mathematical demonstration of expression [3.33] could be performed by using the fact that the domain $\mathcal{D}(t + \delta t)$ can be derived from the domain $\mathcal{D}(t)$ by means of a one-to-one geometric transformation which is precisely defined by the displacement of fluid particles between these instants ([ZIL 06]).

3.3.3.4. *Material derivative of a flux integral*

Consider now the integral $\varphi_S = \int_S \vec{B}.\vec{n}ds$, in which the vector $\vec{B}$ and the surface S are *attached to the moving matter*. The surface *S(t)* moves to position $S(t+\delta t)$, which we will call S_1 at time $t+\delta t$, having thus swept the surface S_ℓ (Figure 3.6). The flux variation φ_S between these two instants is:

$$\delta\varphi_S = \varphi_S(t+\delta t) - \varphi_S(t) = \int_{S_1} \vec{B}(t+\delta t).\vec{n}ds - \int_S \vec{B}(t).\vec{n}ds \qquad [3.34]$$

Consider the domain $\mathcal{D}$ bounded by the surface $\Sigma = S_1 \cup S \cup S_\ell$; $d\vec{\tau}$ is the elementary arc of the curve *C* which encloses the surface *S;* we take the normal in the direction $-\vec{V} \wedge d\vec{\tau}$, on the lateral surface element $\left\|\vec{V}\delta t \wedge d\vec{\tau}\right\|$ swept out by $d\vec{\tau}$ during time δt.

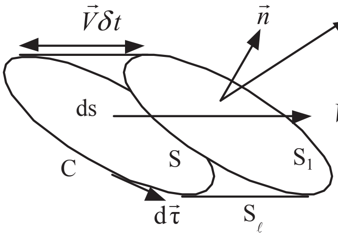


Figure 3.6. *Material derivative of a flux integral*

The flux of $\vec{B}$ leaving the closed surface Σ at time *t* can be written:

$$\int_\Sigma \vec{B}(t).\vec{n}ds = \int_{\mathcal{D}} div\vec{B}(t)dv = \int_{S_1} \vec{B}(t).\vec{n}ds - \int_S \vec{B}(t).\vec{n}ds - \int_C \delta t\vec{B}.\left(\vec{V} \wedge d\vec{\tau}\right)$$

Taking account of this expression, flux variation [3.34] can be written:

$$\begin{aligned}\delta\varphi_S = &\int_{S_1} \vec{B}(t+\delta t).\vec{n}ds - \int_{S_1} \vec{B}(t).\vec{n}ds \\ &+ \int_{\mathcal{D}} div\vec{B}(t)dv + \int_C \delta t\vec{B}.\left(\vec{V} \wedge d\vec{\tau}\right)\end{aligned}$$

Applying Ostrogradski's theorem, and letting δt tend to zero, we obtain the material derivative of the flux φ_S crossing *S*:

$$\frac{d\varphi_S}{dt} = \int_S \frac{\partial \vec{B}}{\partial t} . \vec{n} ds + \int_C \vec{B} . \left(\vec{V} \wedge d\vec{\tau} \right) + \int_S \left(\vec{V} . \vec{n} \right) div \vec{B} ds$$

which is, after the application of Stokes' theorem:

$$\frac{d\ \varphi_S}{d\ t} = \int_S \left(\frac{\partial\ \vec{B}}{\partial\ t} + r\vec{o}t \left(\vec{B} \wedge \vec{V} \right) + \vec{V} div \vec{B} \right) . \vec{n}\ ds \tag{3.35}$$

If the *field* $\vec{B}$ *is conservative* ($div\vec{B} = 0$), the total derivative of the flux φ_S can be written:

$$\frac{d\ \varphi_S}{d\ t} = \int_S \left(\frac{\partial\ \vec{B}}{\partial\ t} + r\vec{o}t \left(\vec{B} \wedge \vec{V} \right) \right) . \vec{n}\ ds \tag{3.36}$$

3.3.3.5. *Unsteady and quasi-steady flows*

In Eulerian variables, a flow (or the transfer of a quantity *G*) is considered unsteady if the time *t* appears as a variable in the description of the flow:

$\partial u_i / \partial t \neq 0$ or $\partial g / \partial t \neq 0$

In Lagrangian variables, the idea of a flow's unsteadiness does not have any direct meaning.

If the term $\partial u_i / \partial t$ (respectively $\partial g / \partial t$) is very small and negligible compared with the other terms involving the corresponding material derivative, the flow (respectively the transfer of quantity *G*) is quasi-steady and the differential character with respect to time no longer features in the balances. The time *t* thus becomes a parameter, and the *unsteady properties of the problem studied are those of a steady flow* (or of the transfer of the quantity *G*). The *physical causes of variations of the quantity G in the matter are thus solely balanced by convective fluxes.*

Following the note made in section 3.3.3.1, the flow can only be steady in a given, particular reference frame.

3.3.4. *Deformation rate tensors*

The determination of the stresses undergone by the matter in a flow is related to our understanding of the manner in which the matter is deformed. Any deformation of a solid requires the action of a stress. A fluid which does not have any particular shape or form can be deformed easily without stresses, on condition that the corresponding action occurs slowly.

In a fluid stresses are thus associated with the structure of the velocity field. However, velocity fields corresponding to changes of reference frame, or to displacements, will not be associated with stresses.

The study of the local structure of a velocity field at any time is made by considering two neighboring points $\vec{M}$ and $\vec{M} + d\vec{M}$ at the respective coordinates x_i and $(x_i + dx_i)$ (Figure 3.7) whose velocities are $u_i(x_j)$ and $u_i(x_j + dx_j)$.

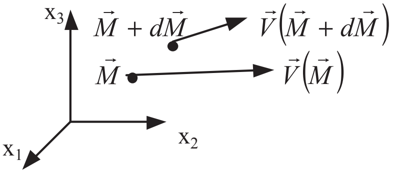


Figure 3.7. *Velocity field at two neighboring points*

The study of the velocities at these two points will allow the nature of the motion of the matter to be determined. We perform a Taylor expansion about the point $\vec{M}$, at coordinates x_j:

$$u_i(x_j + h_j) = u_i(x_j) + \frac{\partial u_i}{\partial x_j} h_j + o(h)$$

We can separate the symmetric and anti-symmetric components of the tensor $\partial u_i / \partial x_j$:

$$\frac{\partial u_i}{\partial x_j} = \Omega_{ij} + \varepsilon_{ij} \quad \text{with: } \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right); \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad [3.37]$$

The anti-symmetric part $\Omega_{ij} h_j$ of the velocity field du_i can be written:

$$\Omega_{ij} h_j = \begin{pmatrix} 0 & -\Omega_{21} & \Omega_{13} \\ \Omega_{21} & 0 & -\Omega_{32} \\ -\Omega_{13} & \Omega_{32} & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} \omega_2 h_3 - \omega_3 h_2 \\ \omega_3 h_1 - \omega_1 h_3 \\ \omega_1 h_2 - \omega_2 h_1 \end{pmatrix}$$

with:

$$\omega_1 = \Omega_{32} \qquad \omega_2 = \Omega_{13} \qquad \omega_3 = \Omega_{21}$$

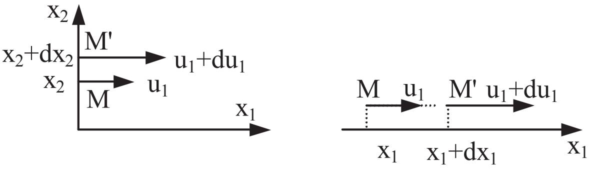
The preceding expression $\Omega_{ij} h_j$, written in vector form $\vec{\omega} \wedge \vec{h}$, expresses the velocity field due to the instantaneous rotational velocity vector $\vec{\omega}$:

$$\vec{\omega} = \frac{1}{2}\overrightarrow{rot}\vec{V} \tag{3.38}$$

The remaining symmetric tensor ε_{ij} characterizes the *deformation rates (strain rates)*, as the displacement velocities have already been accounted for.

We can easily verify that the diagonal terms of the tensor ε_{ij} are expansion (or compression) rates, the other terms corresponding to shear (or angular) expansion rates. By considering for example the velocity field $[u_1(x_2), u_2 = u_3 = 0]$ we see that the deformation generated comprises a transverse motion of the segment MM' (Figure 3.8a) with respect to the velocity.

If the segment MM' is parallel to the velocity $(u_1(x_1), u_2 = u_3 = 0)$, the segment MM' is expanded or compressed according to the sign of du_1 (Figure 3.8b).



(a): $\partial u_1/\partial x_2 \neq 0$ (b): $\partial u_1/\partial x_1 \neq 0$

Figure 3.8. *Deformation rates: (a) shear* $\partial u_1 / \partial x_2 \neq 0$; *(b) expansion* $\partial u_1/\partial x_1 \neq 0$

The volume expansion rate $div\vec{V}$ (section 3.3.3.3) is furthermore always the trace of the matrix ε_{ij}, i.e. the sum of the linear expansion rates following the three axes. We will leave it to the reader to verify this.

3.4. Phenomenological laws of viscosity

3.4.1. *Definition of a fluid*

3.4.1.1. *Introduction*

We saw in Chapter 2 that the stresses in a continuous medium can be represented by a tensor σ_{ij}. A fluid particle undergoing a bulk movement (translation and rotation) is at rest in a moving Cartesian reference frame. It is not subject to stresses other than those induced by pressure forces. On the other hand, this is no longer the case in the presence of strain. The internal stresses in this continuous medium do not depend on the relative position of the fluid particles, but on their relative velocities: an infinitely slow fluid movement will not generate any stresses, contrary to what occurs in solid bodies. This distinction between fluids and solids does not always exist: certain bodies can have the properties of elastic solids for motions which occur at the scale of seconds or fractions of seconds, and their shape may be changed by a flow at the scale of many hours or many days (e.g. viscoelasticity, creep in solids).

3.4.1.2. *Viscous fluids*

A viscous fluid is one in which the stresses at a given instant are a function only of the deformation rates at that instant. We will separate pressure stresses and the viscous stress tensor τ_{ij} using the relation:

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \qquad (\text{Kronecker symbol}: \delta_{ij} = 0 \ si \ i \neq j, \delta_{ii} = 1) \qquad [3.39]$$

This definition corresponds with the idea of a fluid as defined in fluid statics (section 2.2.1.1). In keeping with what has already been said, the viscous stress tensor depends only on the strain rates, and not the deformations. There exist “visco-elastic” bodies in which these two kinds of stress generation can co-exist. We will not cover the more complex cases ([FRE 64], [GER 94], [TAN 00], [COI 97]), limiting our attention to the study of viscous fluids.

As in other domains of physics, the “relationship” between viscous stresses and strain rates become complex when the structure is complex at the molecular level. In particular, the flow may provoke changes in the “molecular cohesion”, or preferred orientations in the presence of macromolecules. The establishment of a flow in a fluid (gas or liquid) containing solid particles which are more or less dispersed may lead to their becoming suspended. For example, the wind can carry sand in the desert or on beaches, in many industrial applications powders are often transported using fluidized beds (airflows which are sufficiently energetic to raise and transport solid particles, these particles being deposed when the fluid velocity drops sufficiently), etc. A snow avalanche is a heavy fluid which flows, whereas the snow comprised a solid structure beforehand. The polymerization of a liquid progressively increases its viscosity until it becomes solid.

The properties of fluids may thus depend on their history, or on the way we excite them (this is the case for quicksand, yoghurt stirred with a spoon, soils liquefaction, etc.), and the chemical transformations which they undergo (cooking of food, polymerization during the generation of plastics, for example). Such behaviors have significant practical importance, as they are often encountered in the chemical industry, in the food industry and in many other natural phenomena. The complete representation of these properties is a difficult problem which is beyond the scope of this book; we will here limit ourselves to the presentation of just some of the laws governing the complex viscous behavior of simple 1D flows.

The general form of the relation between the viscous stress tensor and the strain-rate tensor must be invariant in changing reference frames (in particular, geometric space is homogenous and isotropic). Moreover, a fluid is often a body with isotropic physical properties (with the exception of liquid crystals, nematic and smectic liquids, etc.), so the relationship between the viscous stresses characterized by the tensor τ_{ij} and the strain rates characterized by the tensor ε_{ij} should not have any favored direction. This means that the two tensors should have the same principal axes. The general expression of the laws governing non-linear fluid behavior, which obeys the necessary invariance in changing reference frames, is beyond the scope of this book and we will limit ourselves in a first instance to the study of 1D flows, and then to a more general study of Newtonian fluids which correspond to the linear approximation of irreversible thermodynamics. For other cases the discussion becomes quite complex and the reader should refer to texts concerning the rheology of non-Newtonian fluids ([FRE 64], [GER 94], [TAN 00], [VER 97]).

Finally, recall that the action of the stress tensor on matter is equivalent to the existence of the volume source $\overrightarrow{div}\overline{\overline{\sigma}}$ (see section 2.1.3.2) which can be written:

$$\overrightarrow{div}\overline{\overline{\sigma}} = -\overrightarrow{grad}\, p + \overrightarrow{div}\overline{\overline{\tau}} \qquad [3.40]$$

3.4.1.3. *Physical origin of viscosity*

A fluid is comprised of matter which does not have any particular structure: the relative positions of the particles which make up the matter are not fixed with respect to any reference structure. These particles can move around freely, rather like a person moving around in a crowd which is confined to a closed space: this person can have a relatively autonomous motion in the crowd, but he or she must follow the general motion.

Macroscopic forces in a fluid are either forces induced by a force field at the microscopic level (intermolecular forces), or the result (or manifestation) of momentum transfers at the molecular level (thermal agitation). For an ordinary fluid at macroscopic rest, the effects of molecular forces and thermal agitation reduce to pressure forces.

A simple physical rationale can be used to explain the origin of viscous phenomena. Consider a gas in which the molecules are sufficiently sparsely distributed for their interactions to be entirely localized in the vicinity of the intermolecular collisions. Consider schematically two layers P and P′ in the vicinity of molecules, of average velocities u and $u + du$ (Figure 3.9). The forces exerted directly (from a distance) by each of the layers on the other are zero, on account of the preceding assumption. In addition to their mean velocities, all of the molecules are subjected to a thermal agitation whose action does not have any privileged direction. In particular, the molecules of the lower layer move through the upper layer and vice versa. While a molecule moves from one layer to the other due to a transverse motion it conserves its longitudinal momentum. Thus, the molecules of the lower layer P′, of mean velocity u, are slower than those of the upper layer P into which they arrive. This layer, P, will exert a force on these slower molecules in order to increase their velocity to $u+du$. The molecules thus accelerated will obviously have exerted an opposing force on the layer P which is opposite to the velocity. An analogous effect is produced in the reverse direction for molecules which descend into the plane P′. This constant exchange means that the lower layer slows the upper layer, while the upper layer transports the lower layer.

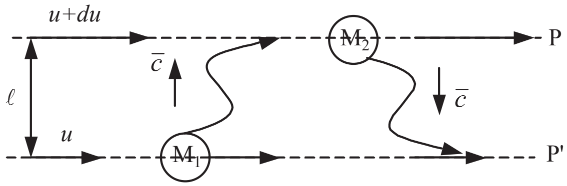


Figure 3.9. *Viscous friction and momentum transfer*

Let us propose a simple model, supposing that the molecules are exchanged between two layers separated by a distance equal to the mean free path λ. Let $\bar{c}$ be the mean transfer velocity of molecules towards the plane P (mean quadratic velocity). A molecule of mass m moving from the lower layer to the upper layer arrives with a momentum deficit equal to $-mdu$. Now, over the distance λ, the variation of the velocity is equal to $\lambda\, du/dy$. The mass flux of the molecules, per unit surface, arriving from the lower layer into P is equal to $nM\bar{c}/2 = \rho\bar{c}/2$ (where

n is the volume number of moles), where the factor 1/2 comes from the simplifying assumption that half of the molecules move from the lower to the upper layer. This results in a force per unit surface, exerted by these molecules on the plane P, equal to $-\dfrac{1}{2}\rho\bar{c}\lambda\dfrac{du}{dy}$. We note that the friction force increases with the temperature in the same way as the mean velocity does.

In other words, the viscous stress τ is proportional to the velocity gradient, and the order of magnitude of the proportionality coefficient (the dynamic viscosity μ defined in section 3.4.3) is:

$$\mu \approx \frac{1}{2}\rho\bar{c}\lambda \qquad [3.41]$$

The preceding rationale can be rigorously applied in the context of kinetic gas theory. It cannot be directly applied to liquids in which intermolecular forces act directly between the fluid layers. The effect of viscous friction is thus directly related to these forces and to the geometry of the molecules, particularly when these are complex (long macromolecules which may be oriented, stretched or broken under the shearing action of a velocity gradient). Furthermore, these mechanisms show that an increase in temperature, on account of the increased thermal agitation, will decrease the effect of these intermolecular forces and thence the level of viscous friction.

The origin of viscosity is thus seen to be a microscopic momentum-mixing phenomenon. At the macroscopic scale, the same phenomenon is encountered in turbulent flows where considerable fluctuations occur. However, the scale of these fluctuations is no longer that of the mean free path or the intermolecular distance, but that of the flow. This leads to considerable difficulties which completely change the nature of the problem ([MAT 00], [TEN 72]).

3.4.2. *Viscometric flows*

3.4.2.1. *Introduction*

We consider here the flow of fluids of constant specific mass in which the *acceleration is everywhere equal to zero*. The trajectories and streamlines are thus *straight lines traveled over at constant speed* by fluid particles which possess a uniform motion. The conservation of the mass requires that these straight lines be parallel, all convergence or divergence of these lines obviously implying an acceleration or deceleration of particles in the medium which is supposed continuous. The corresponding dynamic equations are obtained by writing, as in fluid statics, that the sum of the external forces acting on the fluid is zero. In the presence of a mass force of density g_i and using expression [3.39], we obtain:

$$\rho\, g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} = 0 \quad \text{or} \quad \rho\, \vec{g} - \overrightarrow{grad}\, p + \overrightarrow{div}\, \overline{\overline{\tau}} = 0 \qquad [3.42]$$

If the force $\vec{g}$ derives from a potential U, we can, in such flows, replace the pressure with the driving pressure $p_g = p + \rho_0 g z$ (z, vertical upwards coordinate; see section 2.2.1.4.1):

$$-\frac{\partial p_g}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} = 0 \quad \text{or} \quad -\overrightarrow{grad}\, p_g + \overrightarrow{div}\, \overline{\overline{\tau}} = 0 \qquad [3.43]$$

As the axis Ox is taken parallel to and in the same direction as the velocity $u(y,z)$ of the flow, the strain-rate tensor in Cartesian coordinates is reduced to two components:

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\frac{\partial u}{\partial y} \qquad \varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\frac{\partial u}{\partial z}$$

Let us assume that for a fixed flow geometry, the shear velocities do not lead to normal and transverse stresses (no effect analogous to a Poisson coefficient in elasticity). All that remains therefore are the viscous shear stresses, τ_{xy} and τ_{xz}. Using these assumptions with the equations for viscous fluids, we see that there is no variation of the driving pressure in the directions *Oy* and *Oz*. There remains only a single equation in the direction *Ox* for the force balance:

$$0 = -\frac{\partial p_g}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \qquad [3.44]$$

We can immediately see that as the viscous stresses are only functions of the coordinates (y,z), the *driving pressure is necessarily a linear function of the direction x.* The gradient dp/dx is equal to the ratio $-\Delta p_g / L$, where Δp_g is the pressure drop observed over the distance L (for example, in a tube (Figure 3.10)). Supposing that we know the driving pressure, then equation [3.44] contains two unknown viscous stresses which can only be calculated if the physical law which relates them to the velocity gradient is given.

Flows of this type will often involve plane or cylindrical boundaries aligned with *Ox*, and whose viscous properties are independent of physical factors which are a function of the *x*-direction. These flows are fully immersed, i.e. without free surfaces.

3.4.2.2. *2D viscometric flows*

Axisymmetric or 2D plane problems can be easily solved, as there is only a single unknown viscous stress τ_{xy}, which we will denote simply as τ in order to simplify notations. Equation [3.44] can thus be integrated and the *distribution of viscous stresses is independent of the phenomenological laws governing* the viscous stresses.

For a 1D flow with plane symmetry we obtain:

$$0 = -\frac{dp_g}{dx} + \frac{\partial \tau}{\partial y}$$

By integrating we can show that the distribution of viscous stresses is a linear function of the *y*-direction:

$$\tau = y\frac{dp_g}{dx} + \text{const} \qquad [3.45]$$

We obtain an analogous result for an axisymmetric problem of axis O*x*, either writing equation [3.44] with cylindrical coordinates or making a balance of forces exerted from the exterior on the cylinder whose radius is *r* (Figure 3.10b):

$$0 = -\frac{dp}{dx}\pi r^2 + 2\pi r\tau(r)$$

By integrating with respect to the radial coordinate r, we obtain the distribution of viscous stress:

$$\tau(r) = \frac{dp}{dx}\frac{r}{2} + \text{const} \qquad [3.46]$$

Under the assumption that there is no variation of the driving pressure in the medium, the viscous stress τ is constant. The flow can thus only be produced by the viscous entrainment of the fluid due to friction on some cylindrical walls which are moving at a different velocity. In the 2D plane, a Couette flow is produced between the two plane surfaces (Figure 3.10a); such a flow is also generated by two circular cylinders undergoing axial displacement, but these configurations are of little practical interest.

The flow between two fixed planes or in a cylindrical tube requires a driving pressure gradient.

In the preceding cases, *the viscous stress distribution is independent of the laws of viscosity*. In particular, these results will be valid for *turbulent flows* for which there is no local physical law for the stresses ([COU 89], [MAT 00], [TEN 72]).

When a fluid is *purely viscous*, the phenomenological law takes the form:

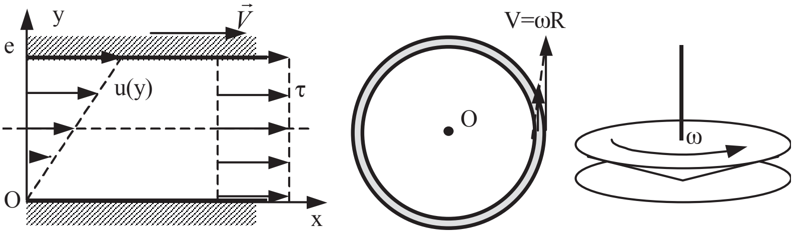
$$\tau = \tau(u'(y)) \quad \text{or} \quad \tau = \tau(u'(r)) \qquad [3.47]$$

Substituting this relation into the expression for the viscous stress distribution [3.45] or [3.46], we obtain a differential relation which permits the calculation of the longitudinal velocity distribution *u* and of the mass flux across a cross-section. The volume flow rate in a circular cylinder can therefore be obtained by integration over the cross-section; for example, for the problem of revolution in the circular cylinder of radius R:

$$q_v = \int_0^R 2\pi r u dr$$

3.4.2.3. *The Couette flow*

Assuming that there is no variation of the driving pressure in the medium, it is straightforward to verify that with the preceding assumptions the viscous stress is constant. For the plane geometry, we have a viscous flow generated by entrainment of the fluid by friction, which occurs between a fixed plane *y=0* and a mobile plane *y=e* which is moving at velocity *V* (Figure 3.10a). The velocity field *u(y)* is parallel to the axis *Ox* and the viscous stress tensor reduces to the components $\tau_{xy} = \tau_{yx} = \tau$ which cause the fluid situated above the plane to exert a shear stress on the fluid situated beneath the plane. This tension is constant in the thickness of fluid contained between the two planes.



(a) (b) (c)

Figure 3.10. *Couette flow: (a) principle, (b) Couette rheometer, (c) rheometer with rotating cone*

The practical realization of such a flow involves two co-axial cylinders, one of radius *R*, which is fixed, and a second, of radius *R*+*e*, which is animated by a rotational motion of angular velocity ω about its axis (Figure 3.10b). The moment of the torque measured on the fixed cylinder gives the value of the viscous stress.

Another realization is the plane-cone rheometer which is comprised of a cone turning over a plane (Figure 3.10c); a fluid placed between these is then studied. Locally this gives a Couette flow. As the thickness *e* between the walls and the velocity of the cone wall are proportional to the distance from the axis, the shear velocity is constant at all points.

These two devices are particularly well-adapted to studying the phenomenological law of a fluid whose *viscous stress is only a function of the shear velocity gradient* u′(y). The result of this is that the shear velocity is also constant and the velocity distribution is linear regardless of the phenomenological laws governing the fluid studied (see Figure 3.10), *y* being the distance of a point situated between the two cylinders from the inner cylinder:

$$u(y) = V\frac{y}{e}$$

The Couette flow allows us to obtain in *a direct and simple manner* the viscous law of a fluid for viscometric flows. As the general law for the viscous behavior of fluids is of a tensorial nature, the preceding relationship is a simplified law, which is uniquely valid for this kind of flow.

3.4.2.4. *Principal physical laws for viscous behavior of a fluid*

3.4.2.4.1. Introduction

We have just discussed a means of simultaneously measuring the velocity gradient $u'(y)$ and the viscous stress τ, in other words, means of establishing the phenomenological law for the viscous behavior of a fluid. Fluids comprised of molecules of a sufficiently small size generally have a linear behavior, viscous stress being proportional to shear velocity (Newton’s law). Fluids containing macromolecules, particles which are more or less solid in suspension, that is to say pasty substances, may have relatively varied viscous laws. In general, the non-linearity of the viscous law is closely related to the complexity of the molecular structure. The viscous behavior may be a function of time, in other words of the mechanical excitations and motions which the fluid has most recently undergone.

Due to the tensorial nature of the viscous stresses and strain-rates, there may exist transverse effects associated with a 1D effect (normal stresses for a shear velocity). This is comparable to the properties of solids (Poisson coefficient in elasticity). This type of effect is mainly manifest in liquids with complex structures. Furthermore, elastic effects may occur for relatively weak stresses in viscoelastic bodies; these may or may not be time dependent: the body may present the properties of a solid for a stress whose modulus is less than some level τ_0, beyond which it becomes liquid (Figure 3.11b and Figure 3.11c). In what follows we will only discuss some simple cases from the vast domain of rheology.

3.4.2.4.2. Time-dependent fluids

We will distinguish:

– the *thixotropic* (Figure 3.11a) situation in which the excitations and the imposed stresses reduce the viscosity (ketchup, yoghurt, gels, drilling mud, quicksand, etc.);

– *rheopexy*, which is less frequently encountered (Figure 3.11b), which corresponds to an opposite effect, where the viscosity increases with agitation (suspensions of Indian corn starch in water).

The time dependence effect of the viscous behavior leads to hysteresis phenomena (Figure 3.11c).

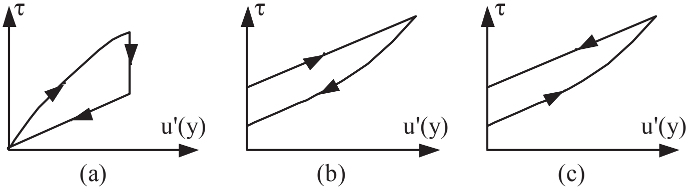


Figure 3.11. *Hysteresis cycles: thixotropic fluid (a) without or (b) with elastic properties; (c) rheopexic fluid with elastic properties*

These effects are obviously associated with local modification for the particles or macromolecules which are more or less in contact at the microscopic level. The explanation for thixotropy is schematically as follows. In suspensions of solid grains in liquid, the grains, after a sufficiently long duration of immobility, finish by coming into contact with one another; this leads to solid friction. After a certain level of agitation (vibration, flow) and the resultant stresses imposed on the ensemble, all solid-solid contact has been undone, and a lubrication occurs such that the grains now slide over one another when a velocity gradient is imposed. This corresponds to the phenomenon of "quicksand". Further to these purely mechanical effects, physico-chemical interactions may occur between the grains, leading to a structural macroscopic organization. This is progressively destroyed under the effects of a velocity gradient. The possible complexity of these phenomena is such that explanations of the same nature may, depending on the circumstances, often lead to inverse effects (rheopexy).

3.4.2.4.3. Purely viscous fluids

Their viscous behavior is characterized by a single physical law which is independent of time for a fluid of a given structure and composition. The most commonly encountered types of fluid are:

– The *Newtonian fluid* which obeys a linear law (curve 1 of Figure 3.12):

$$\tau = \mu u'(y) \qquad (\mu\text{: the coefficient of dynamic viscosity}) \tag{3.48}$$

– The *Ostwald-de Waele fluid*. Behavior is no longer linear for fluids containing molecules with complex structures or particles in suspension and of strong concentration:

$$\tau = m\left|u'(y)\right|^{n-1} u'(y) \tag{3.49}$$

When the viscous stress increases faster than the velocity gradient (n>1) the fluid is said to be dilatant or rheothickening (curve 3 of Figure 3.12); in the opposite case (n<1), it is known as pseudoplastic or rheofluidifying (curve 4 of Figure 3.12). The first case occurs in concentrated suspensions of solid particles which are in contact with one another, when the velocity gradients are steep. The second situation is observed in flows of polymers of high molecular mass, whose linear molecules are more or less intertwined and trap a certain quantity of water. Under agitation, the velocity shear aligns the molecules along their axes, thus freeing the trapped water.

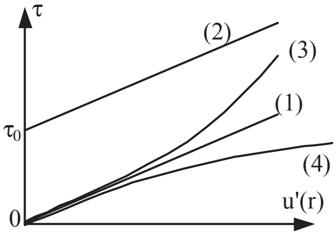


Figure 3.12. *(1) Newtonian fluid, (2) Bingham, (3) rheothickening, (4) rheofluidifying*

– The *Bingham fluid*. This kind of a fluid is characterized by a cohesion such that if the modulus of the stress τ is less than a given threshold τ_0, no strain rate exists in the fluid, which therefore remains rigid; the sliding of liquid layers over one another only occurs when the stress exceeds the threshold value, and the relationship between the stress and the velocity gradients then becomes linear (curve 2 of Figure 3.12):

$$\begin{array}{ll} |\tau| \leq \tau_0 & u'(y) = 0 \\ |\tau| > \tau_0 & \tau = \mu u'(y) \mp \tau_0 \end{array} \qquad [3.50]$$

This law is used in particular for the flow of pastes and certain types of mud.

As discussed above, fluids containing solid particles in suspension have a complex behavior. However, when the solid particles are in weak volumic concentration c, the fluid thus constituted is Newtonian: its viscosity μ is modified with respect to that of the pure fluid according to the (Einstein) relation, valid for any 3D flow:

$$\mu = \mu_0 \left(1 + \frac{5}{2} c \right)$$

3.4.2.5. *Poiseuille flow*

3.4.2.5.1. Flow in a circular cylinder

The flow is here due entirely to a pressure gradient and the conduit is assumed sufficiently long for the flow to become fully established, in other words such that phenomena associated with the tube entrance are not present. Furthermore, the fluid is assumed to be incompressible such that pressure variations do not lead to changes in the fluid density.

We will consider the flow of a fluid in a circular tube of radius R (using the notation of Figure 3.13). The flow, directed in the positive direction along the axis Ox, leads to a constant negative axial pressure gradient, corresponding to a drop in the driving pressure equal to Δp_g for a length L of the pipe.

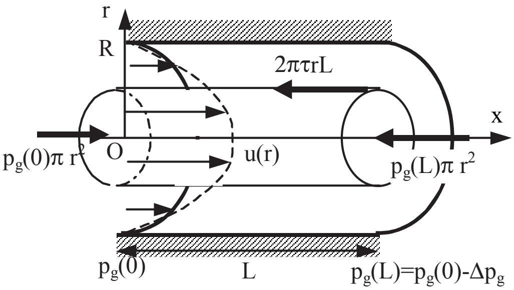


Figure 3.13. *Established flow in a circular tube*

The velocity distribution can be calculated using equation [3.46], along with the phenomenological law for the viscous fluid. Using physical law [3.47] in relation [3.46], we obtain an expression for the derivative *u'(y)* as a function of the radius r and the pressure gradient; integrating and taking the velocity equal to zero at the wall ($r = R$), we obtain a parabolic velocity distribution. The mass flow q_v can be obtained as a function of the pressure gradient by integrating the velocity over a cross-section.

We define the mean mass flux velocity u_q using the mass flux q_v:

$$u_q = \frac{q_v}{S}$$

This leads to the following results:

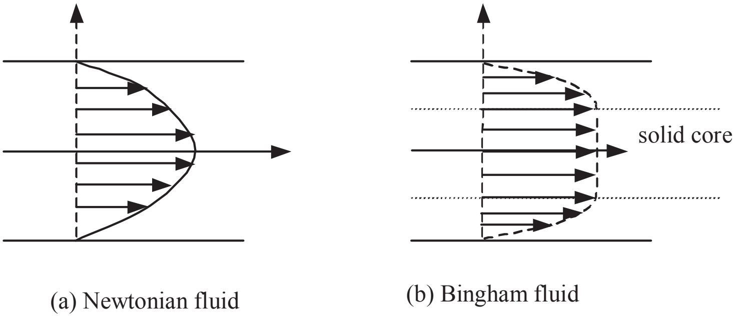
– *Newtonian fluid* with constant viscosity: μ=*cte*. The velocity distribution is parabolic (Figure 3.14a):

$$u = -\frac{dp}{dx}\frac{\left(r^2 - R^2\right)}{4\mu} \qquad [3.51]$$

We can deduce the volume flow rate q_v in the tube and the flow velocity u_q as a function of the pressure gradient (Poiseuille's law):

$$q_v = -\frac{dp}{dx}\frac{\pi R^4}{8\mu} = -\frac{\Delta p}{L}\frac{\pi R^4}{8\mu} \qquad u_q = \frac{q_v}{\pi R^2} = \frac{\Delta p}{L}\frac{R^2}{8\mu} \qquad [3.52]$$

– *Bingham fluid*. Using the relationship between the stresses and the velocity gradient for the fluid ([3.50]), the velocity profile can be calculated as for Newtonian fluids. We find a velocity distribution comprising a uniform central section (corresponding to a bulk displacement) and two half-parabolas (Figure 3.14b).



(a) Newtonian fluid

(b) Bingham fluid

Figure 3.14. *Velocity profile for (a) a Newtonian fluid and (b) a Bingham fluid*

– *Ostwald-de Waele model*. The physical law is no longer linear for fluids containing strong concentrations of particles in suspension:

$$\tau = m\left|\frac{du}{dr}\right|^{n-1}\frac{du}{dr}$$

The velocity distribution thus calculated looks like a parabolic distribution, but pointier if n is greater than 1, and flatter if n is less than 1.

3.4.2.5.2. The Rabinowitsch-Mooney relation

The law for the drop in driving pressure as a function of the mass flow in a Poiseuille flow depends in a complex way on the law governing the viscous behavior of the fluid, as this law must be integrated over the radius. The measurement of pressure-drop in capillary tubes does not directly give the physical law governing viscous behavior; such measurements must be specially treated in order to determine the physical laws. We will here outline the principle of the method (Rabinowitsch-Mooney) for obtaining the relationship between the viscous stress τ and the velocity gradient $u'(r)$.

We note first of all that the pressure gradient measured can be expressed using [3.46] as a function of the wall friction τ_p (which is defined as the stress exerted *by the fluid on the wall,* $-\tau(R)$), from which we obtain:

$$\frac{\tau}{\tau_p} = -\frac{r}{R} \quad \text{with:} \quad \tau_p = -\tau(R) = -\frac{dp}{dx}\frac{R}{2} \tag{3.53}$$

The measurement gives a relation between the flow rate q_v and the pressure gradient or, according to [3.53], between the viscous wall stress τ_p and the flow rate q_v, which can be written:

$$q_v = q_v\left(\tau_p\right) \tag{3.54}$$

We obtain a relation between the flow rate and the velocity gradient using integration by parts on the defining relation for the flow rate:

$$q_v = \int_0^R 2\pi r u dr = \pi \int_0^R u d\left(r^2\right) = -\pi \int_0^R r^2 \frac{du}{dr} dr$$

Formula [3.53] allows us to use the viscous stress τ as a variable in place of the radius r in the expression for the flow rate, which can be written (noting that $\tau_R = \tau$(R)):

$$\frac{q_v \tau_R^3}{\pi R^3} = -\int_0^{\tau_R} \tau^2 \frac{du}{dr} d\tau \tag{3.55}$$

Then, from [3.54], the relation $q_v\left(\tau_p\right)$ is known; we thus differentiate the two sides of [3.55] with respect to τ_R:

$$-\left.\frac{du}{dr}\right|_R = \frac{1}{\pi R^3 \tau_R^2} \frac{d\left(q_v \tau_R^3\right)}{d\tau_R}$$

which finally gives:

$$\left|u'(r)\right|_R = \frac{3q_v}{\pi R^3} + \frac{\tau_p}{\pi R^3}\frac{dq_v}{d\tau_p} \tag{3.56}$$

Relation [3.56] between the velocity gradient $\left|u'(R)\right|$ and the stress τ_p is the relation sought between the velocity gradient and the viscous stress.

3.4.2.5.3. Diverse remarks

1) *The flow between two fixed parallel planes* separated by a distance *2e* can be treated in a similar fashion. We find, for a Newtonian fluid of dynamic viscosity μ, the following velocity distribution $u(y)$, flow rate q_v and mass-flow velocity u_q:

$$u = -\frac{dp}{dx}\frac{y^2 - e^2}{2\mu}; \qquad q_v = -\frac{dp}{dx}\frac{2e^3}{3\mu}; \qquad u_q = \frac{q_v}{2e} = -\frac{dp}{dx}\frac{e^2}{3\mu}$$

Flows of Bingham or Ostwald-de Waele fluids produce results analogous to those obtained in the case of the circular cylinder.

The preceding results remain valid when the thickness *e* varies slowly, in other words when the planes are weakly non-parallel. These results are the basic laws used for the theory of dynamic lubrication flows in a thin fluid film (oil or gas) of slightly variable thickness between a moving wall and a fixed wall; such a flow generates pressures high enough to support heavy rotating devices without solid contacts in a bearing ([GUY 01]).

2) It is necessary to note that experimental results agree with the preceding results provided the mass-flow velocity is not too great. In fact, the condition which must be satisfied involves the Reynolds number *Re* which, for a circular tube of diameter *D* equal to *2R*, is written:

$$Re = \frac{\rho u_q D}{\mu}$$

The value of the Reynolds number *Re* must be less than about 1,850 for the preceding theoretical results to agree with experiments. Above this value the flow becomes unstable and turbulent, and this entirely changes the momentum transfers, as stated earlier in section 3.4.1.3 ([MAT 00], [SCH 99], [TEN 72])

3.4.3. *The Newtonian fluid*

3.4.3.1. *Definition of viscosity*

We will place ourselves in the context of linear thermodynamics where the relationship between the causes (the strain-rate tensor ε_{ij}) and effects (the viscous stress tensor τ_{ij}) is linear. Furthermore, as the fluid matter considered has isotropic properties, and as geometric space is homogenous and isotropic, this relationship cannot involve any preferred direction. As a consequence of this, the two preceding tensors must have the same principal axes. Hence, they must be proportional, excluding the addition of an isotropic tensor proportional to the trace $div\vec{V}$ of the strain tensor. The linear relationship sought between the two preceding tensors (Newton's law) can thus be written:

$$\tau_{ij} = 2\mu\varepsilon_{ij} + \left(\eta - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\delta_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\eta - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\delta_{ij} \quad [3.57]$$

Linear relation [3.57] introduces *dynamic viscosity* μ and *bulk viscosity* η. The latter corresponds to the friction introduced by a purely spherical expansion, for which the diagonal terms of the strain-rate tensor are equal, and the off-diagonal terms are zero:

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} = \frac{\partial u_3}{\partial x_3} = \frac{1}{3}div\vec{V}$$

Thus:

$$\tau_{11} = \tau_{22} = \tau_{33} = \eta div\vec{V}$$

This relation shows that the viscosity η is indeed associated with a expansion velocity, and that this coefficient must be positive in order to represent friction. In what follows we will neglect the effects of bulk viscosity, which is only important for phenomena comprising very strong expansion rates, such as shock waves, very high frequency ultrasound, etc.

3.4.3.2. *Properties of viscosity*

Dynamic viscosity μ is associated with friction in the sliding of fluid layers over one another. It is expressed in Pascal.second (Poiseuille or decapoise).

The values of dynamic viscosity under normal conditions are, for air and water:

– air: $\mu_0 = 17.08 \times 10^{-6}$ Pa.s (same order of magnitude for gases);

– water: $\mu_0 = 1.793 \times 10^{-3}$ Pa.s.

The variability in the order of magnitude of the viscosity of liquids compared to the viscosity of water is a little greater for usual liquids, with the exception of oils and glycerine, which have very high viscosity (from 10^{-2} to 1 Pa.s).

In section 3.4.1.3 we have given a rough estimation of the viscosity of a gas. As thermal agitation increases with temperature, changes occur in momentum exchange at the molecular level. The viscosity increases with increases in the absolute temperature of gases in accordance with the following (Sutherland) formula, in which C is a constant which depends on the gas (C=142K for air):

$$\frac{\mu}{\mu_0} = \sqrt{\frac{T}{T_0}}\,\frac{1+C/T_0}{1+C/T} \qquad (T_0 = 273.15\,K)$$

The viscosity decreases quite rapidly with temperature for liquids, because the intermolecular forces responsible for viscosity in condensed media have a less vigorous action on account of increased thermal agitation (section 3.4.1.3).

The quantity $\nu = \frac{\mu}{\rho}$, called the *kinematic viscosity*, is expressed in $m^2.s^{-1}$.

EXERCISE –

Calculate the components of the viscous stress tensor τ_{ij}:

– at all points of the flow defined by the velocity field $u_1=kx_2,\ u_2=u_3=0$;

– in the vicinity of a solid boundary.

3.4.3.3. *Expression of viscous volume forces*

We have seen (section 3.4.1.2) that the action of viscous stresses in matter is equivalent to a volume source $\overrightarrow{div\overline{\overline{\tau}}}$ (or $\partial\tau_{ij}/\partial x_j$ in tensor notation). Assuming a variable viscosity, we obtain, from [3.57]:

$$\frac{\partial\tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right] + \frac{\partial}{\partial x_i}\left[\left(\eta-\frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\right]$$

At constant viscosity this simplifies to:

$$\frac{\partial\tau_{ij}}{\partial x_j} = \mu\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \left(\eta+\frac{1}{3}\mu\right)\frac{\partial}{\partial x_i}\left[\frac{\partial u_k}{\partial x_k}\right]$$

The first term of $\partial\tau_{ij}/\partial x_j$ is equal to $\mu\Delta u_i$, where Δu_i are the components of the Laplacian vector $\overrightarrow{\Delta V}$:

$$\overrightarrow{\Delta V} = -\overrightarrow{rot}\left(\overrightarrow{rot\vec{V}}\right) + \overrightarrow{grad}\left(div\vec{V}\right)$$

Replacing $\overrightarrow{\Delta V}$ with the above expression gives, in vector notation:

$$\overrightarrow{div}\,\overline{\overline{\tau}} = \mu\overrightarrow{\Delta V} + \left(\eta + \frac{\mu}{3}\right)\overrightarrow{grad}\left(div\vec{V}\right) = -\mu\,\overrightarrow{rot}\left(\overrightarrow{rot\vec{V}}\right) + \left(\eta + \frac{4\mu}{3}\right)\overrightarrow{grad}\left(div\vec{V}\right) \quad [3.58]$$

Usually expansion velocities are weak compared with shear velocities and only the first term $\mu\,\overrightarrow{rot}\left(\overrightarrow{rot\vec{V}}\right)$ of $\overrightarrow{div\,\overline{\overline{\tau}}}$ remains:

$$\overrightarrow{div}\,\overline{\overline{\tau}} = -\mu\,\overrightarrow{rot}\left(\overrightarrow{rot\vec{V}}\right) \quad [3.59]$$

With the preceding approximation, formula [3.59] can be written in Cartesian coordinates:

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu\,\frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad [3.60]$$

Fundamentals of Fluid Mechanics and Transport Phenomena
Jean-Laurent Peube

Chapter 4

Fluid Dynamics Equations

This chapter is devoted to general equations describing the dynamics of fluid flows and of the associated transfers. Writing the balance equations for extensive quantities leads to the equations of fluid dynamics with heat or mass transfers which take on either local or global forms, and which allow the separation of input and output mechanisms and sources of extensive quantities. The energy equation then allows us to discern the interactions between thermodynamics and the movement of the fluid matter. The main boundary conditions which must be associated with the partial differential equations are then discussed. Because of the invariance of physical laws with respect to unit systems, similarity relations allow us to characterize dynamic and energy problems by means of non-dimensional parameters.

4.1. Local balance equations

4.1.1. *Balance of an extensive quantity G*

4.1.1.1. *The global balance equation*

In matter, the physical phenomena associated with the scalar quantity G are governed by the volume sources σ_G and fluxes characterized by the flux density vector $\vec{q}_G$. We perform a balance for G over the domain $\mathfrak{D}$ associated with matter in motion, in other words on a closed system in the thermodynamic sense (no exchange of matter with the exterior). This balance can be written in the same way as in a fixed medium, the material derivative (section 3.3.3.3) replacing the temporal derivative (section 2.1.4.1):

$$\frac{d}{dt}\left(\int_{\mathcal{D}} \bar{g}\,dv\right)=\int_{\mathcal{D}}\frac{\partial\bar{g}}{\partial t}dv+\int_{\Sigma}\bar{g}u_j n_j\,ds=\int_{\mathcal{D}}\sigma_G\,dv-\int_{\Sigma}q_{Gj}n_j\,ds \qquad [4.1]$$

The first term $\int_{\mathcal{D}}\frac{\partial\bar{g}}{\partial t}dv$ of the left-hand side of equation [4.1] represents the variation rate in the amount of the quantity G contained in $\mathcal{D}$, expressing an *accumulation* of G in $\mathcal{D}$.

The second term $\int_{\Sigma}\bar{g}u_j n_j\,ds$ is the flux (convective flux) of the quantity G leaving the surface Σ.

The first term of the right-hand side $\int_D \sigma_G\,dv$ is a *volume source*, in other words a creation of the quantity G, most often by means of the transformation of another extensive quantity.

The second term $\int_{\Sigma} q_{Gj} n_j\,ds$ is a *convective flux of the quantity G leaving the surface* Σ characterized by physical quantities localized on the surface Σ.

4.1.1.2. *Local balance equation*

We can now transform the surface integrals of equation [4.1] into volume integrals:

$$\int_{\mathcal{D}}\left(\frac{\partial\bar{g}}{\partial t}+\frac{\partial\left(\bar{g}u_j\right)}{\partial x_j}\right)dv=\int_{\mathcal{D}}\left(\sigma_G-\frac{\partial q_{Gj}}{\partial x_j}\right)dv \qquad [4.2]$$

As balance integral [4.2] is true in any domain $\mathcal{D}$, we can derive a local relation under the usual regularity conditions of the functions considered:

$$\frac{\partial\bar{g}}{\partial t}+\frac{\partial\left(\bar{g}u_j\right)}{\partial x_j}=\sigma_G-\frac{\partial q_{Gj}}{\partial x_j} \qquad [4.3]$$

When the volume quantity is a vector, $\vec{\bar{g}}$ (or $\bar{g}_i,\ i=1,2,3$), we have a balance equation for each of its components, which can be written in the vector form:

$$\frac{\partial\bar{g}_i}{\partial t}+\frac{\partial\left(\bar{g}_i u_j\right)}{\partial x_j}=\sigma_{Gi}-\frac{\partial q_{Gij}}{\partial x_j} \quad \text{or} \quad \frac{\partial\vec{\bar{g}}}{\partial t}+\overrightarrow{div}\left(\vec{\bar{g}}\otimes\vec{V}\right)=\vec{\sigma}_G-\overrightarrow{div}\,\bar{\bar{q}}_G$$

4.1.2. *Interpretation of an equation in terms of the balance equation*

The terms in equation [4.3] each have a specific form corresponding to physical mechanisms which we have discussed earlier (section 2.1.3.3). Consider a partial differential equation which can be written in the form [4.3]:

$$\frac{\partial f}{\partial t} = \sigma - \frac{\partial q_j}{\partial x_j} \qquad [4.4]$$

We can integrate equation [4.3] over a *geometric domain D* to give:

$$\int_{\mathcal{D}} \frac{\partial f}{\partial t} dv = \int_{\mathcal{D}} \sigma dv - \int_{\Sigma} q_j n_j ds \qquad [4.5]$$

We will interpret f as the volume density of a quantity F, the amount of which present in D is equal to $\int_{\mathcal{D}} f\, dv$. The first term $\int_{\mathcal{D}} \frac{\partial f}{\partial t} dv$ of equation [4.4] represents the variation rate in the amount of the quantity F contained in $\mathcal{D}$.

On the right-hand side of equation [4.4], we have separated the terms which can be written in the form of a divergent vector q_i expressed using data or functions of the problem, and those which cannot be written in this form. In integrating over the domain D, the terms written in the form of a divergence have been transformed into *flux integrals of the vector q_i on the external surface Σ of the domain D*: they can thus be interpreted as transfers of the quantity F by the vector q_i. These transport terms are thus the input-output of the quantity F in D, in other words quantities gained by D from (or lost to) the exterior through the surface Σ.

On the other hand, if the vector q_i can be written in the form $q_i = f\, v_i$, we interpret the term $\frac{\partial q_j}{\partial x_j}$ as representing the convective transport of the quantity F by the velocity field v_i, whose physical interpretation is not important here. In fact, the vector q_i will most often be the sum of a variety of terms with different physical interpretations. In general, a flux term will only transport a quantity without creating any; in balance equations which are integrated over a large domain bounded by a surface Σ, *fluxes which only have local effects will not appear*.

Terms which cannot be written in the form of the divergence of a vector having a physical meaning consistent with the problem considered cannot be interpreted as flux terms. Furthermore, take in this case a balance equation by integration on a domain D which encloses a zone containing the phenomena to be studied. As the volume integral of non-divergent terms cannot be written as a surface integral, it is not possible to characterize as an input or an output the detailed repartition of quantity of F injected in this domain by the source σ. This distinction between these two kinds of terms is very often used in turbulence theory ([COU 89], [MAT 00], [TEN 72]).

We will have occasion to use this interpretation.

4.2. Mass balance

4.2.1. *Conservation of mass and its consequences*

4.2.1.1. *The equation of mass conservation*

Mass is strictly conserved (no sources or fluxes of mass in a barycentric reference frame); we let $\overline{g} = \rho$. We obtain from [4.3] the equation (known as the continuity equation):

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0 \qquad [4.6]$$

or alternatively:

$$\frac{d\rho}{dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \qquad [4.7]$$

which shows that the variation rate of the density $\frac{1}{\rho}\frac{d\rho}{dt}$ of a fluid particle is compensated by the volume expansion rate $\frac{\partial u_i}{\partial x_i}$.

The flow of an incompressible fluid (liquid or gas moving at low velocity) of variable density satisfies $d\rho/dt = 0$: matter is displaced with its specific mass on a trajectory (obvious *a priori*).

4.2.1.2. *Consequences of the mass conservation equation*

4.2.1.2.1. Writing the balance of a mass quantity

Let g be a mass quantity ($\overline{g} = \rho g$); by means of a simple calculation the following two useful relations can be immediately verified:

$$\frac{d}{dt}\left(\int_{\mathcal{D}} \rho g \, dv\right) = \int_{\mathcal{D}} \rho \frac{dg}{dt} dv \qquad [4.8]$$

$$\frac{\partial \overline{g}}{\partial t} + \frac{\partial(\overline{g}u_i)}{\partial x_i} = \frac{\partial(\rho g)}{\partial t} + \frac{\partial(\rho g u_i)}{\partial x_i} = \rho \frac{dg}{dt} \qquad [4.9]$$

The balance equation of the quantity G can be written using the mass quantity g:

$$\rho \frac{dg}{dt} = \rho\left(\frac{\partial g}{\partial t} + u_i \frac{\partial g}{\partial x_i}\right) = \frac{\partial(\rho g)}{\partial t} + \frac{\partial(\rho g u_i)}{\partial x_i} = \sigma_G - \frac{\partial q_{Gj}}{\partial x_j} \qquad [4.10]$$

4.2.1.2.2. Stream functions

In the particular case of problems with two variables, the solution of equation [4.6] or [4.7] can immediately be written by means of a *stream function* ψ:

a) For a 1D compressible flow:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad \Leftrightarrow \quad \rho = -\frac{\partial \psi}{\partial x} \qquad \rho u = \frac{\partial \psi}{\partial t} \qquad [4.11]$$

b) For a steady plane 2D flow:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \quad \Leftrightarrow \quad \rho u = \rho_0 \frac{\partial \psi}{\partial y} \qquad \rho v = -\rho_0 \frac{\partial \psi}{\partial x} \qquad [4.12]$$

It is easy to verify that, in this case, the lines ψ = const *are* streamlines *of the flow* (we have $\rho \vec{\mathrm{V}}.\overrightarrow{\mathrm{grad}}\,\psi = 0$).

The mass flow between two streamlines passing through the points A and B is equal to the difference $\rho_0(\psi(B) - \psi(A))$. The arc element $d\vec{\ell}\,(dx, dy)$ of the curve joining A and B allows the elementary vector $\vec{n}d\ell\,(dy,-dx)$ of the normal of this curve to be expressed, from which we can derive the mass flow:

$$q_m = \int_{AB} \rho \vec{V}.\vec{n}d\ell = \int_{AB} \rho_0 \left(\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy\right) = \int_{AB} \rho_0 d\psi = \rho_0(\psi(B) - \psi(A))$$

c) For a *steady axisymmetric flow* (where O*z* is the axis of symmetry), the mass conservation equation in a steady regime is equivalent to the following definition for the stream function ψ:

$$\frac{\partial(\rho r u)}{\partial r}+\frac{\partial(\rho r w)}{\partial z}=0 \qquad \Leftrightarrow \qquad \rho u=\frac{\rho_0}{r}\frac{\partial\psi}{\partial z}, \quad \rho w=-\frac{\rho_0}{r}\frac{\partial\psi}{\partial r}$$

d) For flows with constant density ρ, the densities disappear from the definition of the stream function; in a 2D plane flow, we have:

$$u_1=\frac{\partial\psi}{\partial x_2} \quad u_2=-\frac{\partial\psi}{\partial x_1}$$

4.2.1.2.3. Velocity fields of a viscous fluid near a solid boundary

The property which we will discuss here only depends on the mass balance and is of course independent of the reference frame which is chosen (Galilean or otherwise). We will consider a solid boundary P and study the velocity field in the vicinity of some point O on this boundary. Taking rectangular axes such that the plane O*xz* is tangential to the surface (whose normal is O*y*) at the point O. Assuming the fluid to be viscous, the velocity $\vec{V}$ with components (*u, v, w*) is zero at the surface (y = 0). The *u* and *w* components of the velocity $\vec{V}$, parallel to the wall, can be expanded about *y* in the vicinity of this point:

$$u(x,y,z,t)=y\,u_1(x,z,t)+y^2u_2(x,z,t)\ldots;$$
$$w(x,y,z,t)=y\,w_1(x,z,t)+y^2w_2(x,z,t)\ldots$$

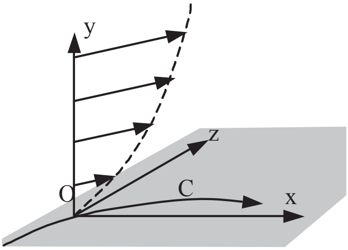


Figure 4.1. *Flow near a rigid wall*

Substituting into the mass conservation equation [4.7] we obtain:

$$\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{d\rho}{dt}-y\left(\frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z}\right)+O(y^2)$$

or, by integrating: $v = -\int_0^y \frac{1}{\rho}\frac{d\rho}{dt}dy - \frac{y^2}{2}\left(\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z}\right) + O(y^3)$

When the expansion rate is negligible or equal to zero (*incompressible fluid* (section 4.2.2)), the first non-zero term of the expansion of the v component is $O(y^2)$. The normal component v tends to zero more rapidly in y than the components tangent to the wall: *the velocity in the vicinity of the wall is parallel to the wall.* The wall streamline C is the streamline limit when y tends to zero (Figure 4.1).

4.2.1.2.4. The strain-rate tensor near a solid boundary

Replacing the preceding expressions for the velocity components into the expression for strain rate tensor [3.37], we obtain:

$$\bar{\bar{\varepsilon}} = \begin{pmatrix} 0 & u_1/2 & 0 \\ u_1/2 & -(d\rho/dt)/\rho & w_1/2 \\ 0 & w_1/2 & 0 \end{pmatrix} + O(y)$$

We note the existence of shear velocities associated with a velocity field which is *quasi*-parallel to the wall, and a expansion velocity which is essentially normal to the wall.

4.2.1.2.5. Acceleration near a solid boundary

From the preceding expressions for the velocity we can derive the expressions of the fluid acceleration components (in the wall reference frame). Let us first consider the component $\gamma_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$. In the general case where $d\rho/dt$ is not equal to zero, by substituting the velocity components with their preceding expressions, we obtain:

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = y^2\left[u_1\frac{\partial u_1}{\partial x} + w_1\frac{\partial u_1}{\partial z}\right] + O(y^3)$$

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} = y\frac{\partial u_1}{\partial t} - u_1\int_0^y \frac{1}{\rho}\frac{d\rho}{dt}dy + O(y^2)$$

Fluid expansion and unsteady flow variations involve the existence of terms of order y near the wall.

In the same way, for the second tangential component $\gamma_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$ we obtain:

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = y^2\left[u_1\frac{\partial w_1}{\partial x} + w_1\frac{\partial w_1}{\partial z}\right] + O(y^3)$$

$$\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial y} = y\frac{\partial w_1}{\partial t} - w_1\int_0^y \frac{1}{\rho}\frac{d\rho}{dt}dy + O(y^2)$$

We finally derive the following expressions for the acceleration component $\gamma_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$ normal to the wall:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \left(\frac{1}{\rho}\frac{d\rho}{dt} - \frac{\partial}{\partial t}\right)\left[\int_0^y \frac{1}{\rho}\frac{d\rho}{dt}dy\right] + O(y^2)$$

$$u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z} = -y\left(u_1\frac{\partial}{\partial x} + w_1\frac{\partial}{\partial z}\right)\int_0^y \frac{1}{\rho}\frac{d\rho}{dt}dy + O(y^3)$$

The acceleration is a linear function of the distance from the wall.

For an *unsteady flow of an incompressible fluid*, the expansion rate is equal to zero so that we have $d\rho/dt = 0$ (section 4.2.2). The preceding expressions for acceleration component γ_y normal to the wall is O(y^2), the tangential components remaining O(y).

For a *steady flow of an incompressible fluid*, we have:

$$\gamma_x = y^2\left(\frac{u_1}{2}\frac{\partial u_1}{\partial x} - \frac{u_1}{2}\frac{\partial w_1}{\partial z} + w_1\frac{\partial u_1}{\partial z}\right) + O\left(y^3\right)$$

$$\gamma_y = \frac{y^3}{2}\left[\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z} - \left(u_1\frac{\partial}{\partial x} + w_1\frac{\partial}{\partial z}\right)\right]\left(\frac{\partial u_1}{\partial x} + \frac{\partial w_1}{\partial z}\right) + O(y^4)$$

$$\gamma_z = y^2\left(u_1\frac{\partial w_1}{\partial x} - \frac{w_1}{2}\frac{\partial u_1}{\partial x} + \frac{w_1}{2}\frac{\partial w_1}{\partial z}\right) + O\left(y^3\right)$$

In the absence of a expansion rate (incompressible fluid), the acceleration normal to the wall is of higher order than the tangential acceleration components.

4.2.1.2.6. Velocity and acceleration near a fixed surface of path lines

Consider a fixed surface which is the locus of trajectories which may be variable. As we will later see, the practical realization of such surfaces frequently involves walls or flow separation surfaces on which the flow of a non-viscous fluid slides. We proceed as above, performing an expansion following the surface normal, for an *incompressible* fluid:

$$u(x,y,z,t) = u_0(x,y,z,t) + y\,u_1(x,y,z,t) + O(y^2)....$$

$$w(x,y,z,t) = w_0(x,y,z,t) + y\,w_1(x,y,z,t) + O(y^2)$$

$$\frac{\partial v}{\partial y} = -\left(\frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial z}\right) + O(y)$$

The normal velocity component and the acceleration components can be directly calculated:

$$v = -y\left(\frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial z}\right) + O(y^2)$$

$$\gamma_x = \frac{du}{dt} = \frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} + w_0\frac{\partial u_0}{\partial z} + O(y^2)$$

$$\gamma_y = \frac{dv}{dt} = y\left[\frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial z} - \left(\frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + w_0\frac{\partial}{\partial z}\right)\right]\left(\frac{\partial u_0}{\partial x} + \frac{\partial w_0}{\partial z}\right) + O(y^2)$$

$$\gamma_z = \frac{dw}{dt} = \frac{\partial w_0}{\partial t} + u_0\frac{\partial w_0}{\partial x} + w_0\frac{\partial w_0}{\partial z} + O(y)$$

The expansion is accounted for as before.

4.2.1.2.7. Application to the study of quasi-1D flows

We often encounter flows with one or more preferred directions because these are guided by walls presenting small angles of divergence or convergence (pipes) or by a wall with a weak curvature (boundary layer). The same can be true in certain zones of flows generated from almost parallel flows (jets, mixing-layers, etc.).

The preceding developments show that the velocity components and accelerations normal to these privileged directions are small compared with the other components. It is thus possible to neglect these components under certain conditions: for example, the momentum or acceleration is essentially longitudinal in a 1D flow and there can obviously be no transverse component except in the presence of an external action. Thus, a fluid with a trajectory associated with a curved wall will have a centripetal acceleration which is compensated by a pressure gradient.

On the other hand, in a steady flow, the transverse component v ensures the conservation of mass in any evolution which is not strictly parallel. Examination of mass conservation equation [4.6] shows that *the term* $\partial v / \partial y$ *is necessary for the existence of a longitudinal variation* $\partial u / \partial x$ *of the u component.*

The existence of a weak transverse velocity component leads to a transverse displacement of the longitudinal momentum associated with and transported by the fluid matter (sections 4.5.4 and 6.5).

4.2.2. *Volume conservation*

Volume conservation is not a physical law of matter; it results from particular physical conditions. The flow of a liquid in ordinary conditions is a transformation which occurs at practically constant volume, except in the case of natural convection. We will see that the same goes for gas flows at a small Mach number. The conservation of volume is expressed by a zero expansion velocity:

$$\frac{\partial u_k}{\partial x_k} = 0 \qquad (\text{or} \quad div\, \vec{V} = 0)$$

Taking the previous relation into account, mass balance equation [4.7] may be written for an incompressible fluid:

$$\frac{d\rho}{dt} = 0$$

4.3. Balance of mechanical and thermodynamic quantities

4.3.1. *Momentum balance*

4.3.1.1. *Dynamic equations*

The quantity considered ($\vec{p} = m\vec{V}$) is a vector quantity: ($\overline{g}_i = \rho u_i$ or $g_i = u_i$). Hence we obtain the dynamic equations (i, j = 1, 2, 3):

$$\frac{d}{dt}\left(\int_{\mathfrak{D}} \rho u_i \, dv\right) = \int_{\mathfrak{D}} \rho \frac{du_i}{dt} dv = \int_{\mathfrak{D}} \rho f_i dv + \int_{\Sigma} \sigma_{ij} n_j ds$$

with: f_i the force mass density (for example gravity g_i) and σ_{ij} the stress tensor (section 2.1.3.2) Transforming the surface integral of the stress tensor using Ostrogradsky's theorem (section 2.1.3.3), and assuming continuous and differentiable properties, we obtain the local dynamic equations:

$$\rho \frac{du_i}{dt} = \rho f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \quad (i, j = 1,2,3) \quad or \quad \rho \frac{d\vec{V}}{dt} = \rho \vec{f} + \overrightarrow{div}\, \overline{\overline{\sigma}} \qquad [4.13]$$

Replacing the stress tensor with expression [3.39], we obtain the *Navier-Stokes* equations:

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j} \quad (i, j = 1,2,3)$$

$$\text{or} \quad \rho \frac{d\vec{V}}{dt} = -\overrightarrow{grad}\, p + \rho \vec{f} + \overrightarrow{div}\, \overline{\overline{\tau}} \qquad [4.14]$$

In a gas flow of *sufficient velocity* (at least a few meters per second), the effects of gravity are negligible and the corresponding term can be ignored. On the contrary, pressure gradient is negligible for natural convection ([LAN 89], [SCH 99], [YIH 77]).

Case of an external force deriving from a potential U

The force f_i may derive from a potential U ($f_i = -\partial U_i / \partial x_i$), essentially for gravitation, electrostatic forces and a few simple cases of inertial entrainment forces (when the equations are written in a non-Galilean reference frame). Navier-Stokes equations [4.14] can then be written:

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho \frac{\partial U}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\text{or} \quad \rho \frac{d\vec{V}}{dt} = -\overrightarrow{grad}\, p - \rho \overrightarrow{grad} U + \overrightarrow{div}\, \overline{\overline{\tau}} \qquad [4.15]$$

In many cases of practical interest, the external force field is gravity which derives from the potential gz, where z is a vertically ascending direction; we have:

$$\rho \frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho g \frac{\partial z}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \quad \text{or} \quad \rho \frac{d\vec{V}}{dt} = -\overrightarrow{grad}\, p + \rho \overrightarrow{grad} gz + \overrightarrow{div}\, \overline{\overline{\tau}}$$

When the fluid has constant density ρ (often incorrectly referred to as an "incompressible fluid"), the dynamic equations can be written with the driving pressure $p_g = p + \rho gz$, as previously defined (section 2.2.1.4.1):

$$\rho \frac{du_i}{dt} = -\frac{\partial p_g}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad (i, j = 1,2,3)$$

The form of the equations shows that the *cause of movement* (the local resultant of external forces) is the *gradient of the driving pressure*. This elimination of gravity from the dynamic equations is only of interest if the boundary conditions can also be expressed as a function of the driving pressure alone (section 4.4.2.4).

NOTE – The force f_i does not derive from a potential in many very important configurations, such that:

– the electromagnetic Laplace force which results from the interaction of an electric current of density $\vec{j}$ with a magnetic field $\vec{B}$: this produces the *volume force* $\vec{j} \wedge \vec{B}$;

– the Coriolis force in a reference frame moving with an angular velocity $\vec{\Omega}$, which leads to a *mass force* $2\vec{\Omega} \wedge \vec{V}$.

We note that these two forces have a similar structure; thus, this produces some analogous properties between flows in a rotating frame and flows of a fluid conducting electricity in a magnetic fluid..

4.3.1.2. *Other expressions of the Navier-Stokes equations*

Using vector expression [3.28] for acceleration, we have:

$$\frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + \overrightarrow{grad}\frac{V^2}{2} + \overrightarrow{rot}\,\vec{V} \wedge \vec{V} = -\frac{\overrightarrow{grad\, p}}{\rho} - \overrightarrow{gradU} + \frac{1}{\rho}\overrightarrow{div}\,\overline{\overline{\tau}} \qquad [4.16]$$

For a divariant fluid, we can express the pressure as a function of the specific enthalpy h and the specific entropy s:

$$dh = \frac{dp}{\rho} + Tds \qquad \left(\text{or}: \overrightarrow{grad}\, h = \frac{\overrightarrow{grad\, p}}{\rho} + T\,\overrightarrow{grad}\, s\right) \qquad [4.17]$$

This gives:

$$\frac{\partial \vec{V}}{\partial t} + \overrightarrow{grad}\,\frac{V^2}{2} + \overrightarrow{rot}\,\vec{V} \wedge \vec{V} = -\overrightarrow{grad}\,h + T\,\overrightarrow{grad}\,s - \overrightarrow{grad U} + \frac{1}{\rho}\overrightarrow{div}\,\overline{\overline{\tau}} \qquad [4.18]$$

or:

$$\frac{\partial \vec{V}}{\partial t} + \overrightarrow{grad}\left(\frac{V^2}{2} + U + h\right) = -\overrightarrow{rot}\,\vec{V} \wedge \vec{V} + T\,\overrightarrow{grad}\,s + \frac{1}{\rho}\overrightarrow{div}\,\overline{\overline{\tau}} \qquad [4.19]$$

We note that despite the presence of enthalpy and entropy in equation [4.19], the origin of this equation is purely mechanical (we have so far not considered any energy balance).

4.3.1.3. *The case of a Newtonian fluid*

The term $\overrightarrow{div}\;\overline{\overline{\tau}}$, which represents the viscous stresses, can be expressed using one of the forms discussed in section 3.4.3.3:

$$\frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] + \frac{\partial}{\partial x_i}\left[\left(\eta - \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}\right]$$

In the case where the viscosity coefficients are constant, we have expression [3.58] and the Navier-Stokes equation is written as:

$$\rho\frac{d\vec{V}}{dt} = -\overrightarrow{grad}\,p - \rho\,\overrightarrow{grad U} - \mu\,\overrightarrow{rot}\left(\overrightarrow{rot}\vec{V}\right) + \left(\eta + \frac{2\mu}{3}\right)\overrightarrow{grad}\left(div\vec{V}\right) \qquad [4.20]$$

The expansion velocity is usually small and we have, from [3.60]:

$$\rho\frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial U}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad [4.21]$$

4.3.1.4. *Inviscid fluids – the Euler equations*

When viscous terms are negligible in [4.21], we obtain *Euler* equations:

$$\rho\frac{du_i}{dt} = -\frac{\partial p}{\partial x_i} + \rho\,f_i; \qquad \left(\rho\frac{d\vec{V}}{dt} = -\,\overrightarrow{grad}\,p + \rho\,\vec{f}\right) \qquad [4.22]$$

For a compressible fluid, we can express the pressure as a function of enthalpy and entropy ([4.17]):

$$\frac{\partial \vec{V}}{\partial t}+\overrightarrow{grad}\left(\frac{V^2}{2}+U+h\right)=-\overrightarrow{rot}\,\vec{V}\wedge\vec{V}+T\,\overrightarrow{grad}\,s \quad (\text{gravity:}\, U=g\,z) \qquad [4.23]$$

Assuming a compressible inviscid fluid we have a situation where the fluid particles undergo isentropic transformations.

If the *flow is homoentropic* (s is constant in all the fluid), we have:

$$\frac{\partial \vec{V}}{\partial t}+\overrightarrow{grad}\left(\frac{V^2}{2}+U+h\right)=-\overrightarrow{rot}\,\vec{V}\wedge\vec{V} \quad (\text{gravity:}\, U=gz) \qquad [4.24]$$

Note the presence of the rotation vector $\vec{\omega}$ (or vorticity $\overrightarrow{rot}\vec{V}=2\vec{\omega}$) on the right-hand side of the preceding equations; we will consider the consequences later.

If the fluid flow has constant and uniform density, the Euler equations can be written with the *driving pressure* $p_g = p+\rho\,gz$:

$$\rho\frac{du_i}{dt}=-\frac{\partial p_g}{\partial x_i} \qquad \left(\rho\frac{d\vec{V}}{dt}=-\overrightarrow{grad}\,p_g\right) \qquad [4.25]$$

This form of dynamic equation shows that the cause of movement (local result of external forces) is indeed the driving pressure gradient.

4.3.2. *Kinetic energy theorem*

4.3.2.1. *Local equation*

The kinetic energy theorem is a *consequence of the Navier-Stokes equations*; the local form of the kinetic energy equation is obtained as before (section 3.2.2.1) by taking the scalar product of dynamic equations [4.13] and the velocity vector:

$$\rho u_i\frac{du_i}{dt}=\rho\frac{d}{dt}\left(\frac{V^2}{2}\right)=\rho\, f_i u_i+u_i\frac{\partial \sigma_{ij}}{\partial x_j} \qquad [4.26]$$

Kinetic energy is not an extensive quantity for which a conservation principle applies; we can nevertheless consider [4.26] as a balance equation (section 3.2.4.3).

Replacing the stress tensor with expression [3.39] and assuming that the external force f_i derives from a potential U, we obtain, after grouping similar terms:

$$\rho \frac{d}{dt}\left(\frac{V^2}{2}\right) = -u_i \frac{\partial p}{\partial x_i} - \rho u_i \frac{\partial U}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j}$$

or:

$$\rho \frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) + \rho u_i \frac{\partial}{\partial x_i}\left(\frac{V^2}{2} + U\right) = -u_i \frac{\partial p}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j} \qquad [4.27]$$

4.3.2.2. *Enthalpic form of the kinetic energy equation*

For a divariant fluid we have the thermodynamic relation:

$$dh = \frac{dp}{\rho} + Tds \qquad \left(\text{or: } \frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial h}{\partial x_i} - T\frac{\partial s}{\partial x_i}\right)$$

Substituting the preceding expressions into equation [4.27], we obtain:

$$\rho \frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) + \rho u_i \frac{\partial}{\partial x_i}\left(\frac{V^2}{2} + U + h\right) = \rho T u_i \frac{\partial s}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j} \qquad [4.28]$$

The reader will note that equation [4.28] is an equation of essentially mechanical origin, even if it involves thermodynamic functions.

4.3.2.3. *Applications*

4.3.2.3.1. The constant density fluid

When the density of a fluid is uniform, we can introduce the driving pressure p_g previously defined in equation [4.27]. We then derive the kinetic energy equation:

$$\rho \frac{\partial}{\partial t}\left(\frac{V^2}{2}\right) + u_i \frac{\partial}{\partial x_i}\left(\rho\frac{V^2}{2} + p_g\right) = u_i \frac{\partial \tau_{ij}}{\partial x_j} \qquad (i, j = 1,2,3) \qquad [4.29]$$

Note the absence of any coupling between the mechanical and the thermodynamic phenomena for a constant density fluid.

Equation [4.29] leads to the definition of *total pressure* p_t and total *driving pressure* p_{gt}:

$$p_{gt} = p_g + \rho \frac{V^2}{2} \qquad p_t = p + \rho \frac{V^2}{2}$$

If we assume that the flow is steady and the viscous stresses are zero, the first term on the left-hand side and the right-hand side of equation [4.29] can be removed, giving:

$$u_j \frac{\partial}{\partial x_j}\left(p_g + \rho \frac{V^2}{2} \right) = 0 \quad \text{or} \quad \frac{d}{dt}\left(p_g + \rho \frac{V^2}{2} \right) = 0 \qquad [4.30]$$

Bernoulli's first theorem: in steady inviscid flow, the total driving pressure $p_{gt} = p_g + \rho \dfrac{V^2}{2}$ is constant on a trajectory or here on a streamline.

Bernoulli's first theorem can also be written:

$$V = \sqrt{2 \frac{p_{gt} - p_g}{\rho}} \qquad [4.31]$$

We thus have, in differential (Lagrangian) form: $dp_g + \rho V dV = 0$.

This expression shows that the driving pressure is a decaying function of the velocity modulus.

4.3.2.3.2. The flow of compressible fluids

With the exception of flows generated by natural convection, which are due to a density gradient, the *effects of gravity are negligible in gas flows*. Kinetic energy can be written:

$$\rho \frac{\partial}{\partial t}\left(\frac{V^2}{2} \right) + \rho u_i \frac{\partial}{\partial x_i}\left(\frac{V^2}{2} + h \right) = \rho T u_i \frac{\partial s}{\partial x_i} + u_i \frac{\partial \tau_{ij}}{\partial x_j} \qquad [4.32]$$

For steady flows of inviscid fluid undergoing isentropic transformations, this relation can be written as:

$$u_j \frac{\partial h_t}{\partial x_j} = 0 \quad \text{or :} \frac{\mathrm{d}}{\mathrm{dt}}\left(h + \frac{V^2}{2}\right) = 0$$

This is Bernoulli's first theorem: *total enthalpy* $h_t = h + V^2/2$ *is constant on streamlines in steady inviscid flow*. It is equal to the enthalpy generation h_0 which corresponds to zero velocity. We have:

$$h_t = h_0 = h + \frac{V^2}{2} \qquad or : V = \sqrt{2(h_t - h)} \tag{4.33}$$

In differential form, following a streamline, we obtain:

$$dh_t = \frac{dp}{\rho} + VdV = 0 \tag{4.34}$$

Here again, pressure is a decaying function of the velocity modulus along a streamline.

4.3.2.3.3. Case of a perfect gas: the Saint-Venant relation

Consider a divariant fluid evolving according to an *isentropic transformation* along its trajectory; the quantities ρ and c (sound velocity) are then functions of one variable, p for example. Relation [4.34] is a differential equation whose integration gives the pressure-velocity relationship. Consider the case of a perfect gas for which we have:

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma}$$

Substituting this expression into [4.34] gives:

$$\frac{dp}{\rho} + VdV = \frac{p_0^{1/\gamma}}{\rho_0} \frac{dp}{p^{1/\gamma}} + VdV = 0$$

Integrating and taking as reference (p_0, ρ_0, T_0) the initial or *generation conditions,* where the velocity is zero, we obtain the *Saint-Venant relation*:

$$V = \sqrt{\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}\left[1-\left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]} \qquad [4.35]$$

As the pressure and thermodynamic temperature cannot be negative, the Saint-Venant relation implies the existence of a *maximum velocity* V_{max}:

$$V_{\max} = \sqrt{\frac{2\gamma}{\gamma-1}\frac{p_0}{\rho_0}} = c_0\sqrt{\frac{2}{\gamma-1}} .$$

The local sound speed c is given by relation [1.25] $c^2 = (\partial p/\partial \rho)_S$. We obtain:

$$c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma r T} \qquad [4.36]$$

Let us denote by c_* the velocity V in conditions (called critical conditions) where the *fluid velocity is equal to the local sound speed*: $V = c = c_*$. Considering a perfect gas of specific enthalpy $h = C_p T$, and substituting the preceding definitions into [4.33] expressing the conservation of total enthalpy, we obtain:

$$C_p T_0 = C_p T + \frac{V^2}{2} = \frac{c^2}{\gamma-1} + \frac{V^2}{2} = \frac{V_{\max}^2}{2} = \frac{\gamma+1}{2(\gamma-1)} c_*^2$$

The velocities c_* and V_{max} are defined by generation conditions.

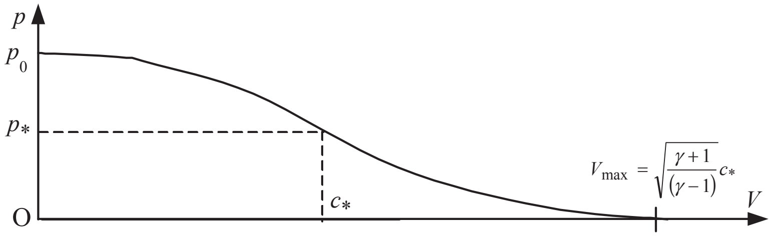


Figure 4.2. *Saint-Venant relation*

Solving the Saint-Venant relation with respect to pressure, $c_c = (\gamma p_0/\rho_0)^{1/2}$ being sound velocity in generation conditions, we obtain:

$$\frac{p}{p_0} = \left[1 - \frac{\gamma-1}{2\gamma}\frac{\rho_0}{p_0}V^2\right]^{\frac{\gamma}{\gamma-1}} = \left[1 - \frac{\gamma-1}{2\gamma}\frac{V^2}{c_0^2}\right]^{\frac{\gamma}{\gamma-1}}$$

The development of pressure to fourth order with respect to Mach number $\mathrm{M}_0 = V/c_0$ is:

$$p_0 - p = \rho_0\frac{V^2}{2} - \frac{1}{8}\rho_0\frac{V^4}{c_0^2} + ... = \rho_0\frac{V^2}{2}(1 - \frac{1}{4}\frac{V^2}{c_0^2} + ...)$$

For values of p close to p_0, i.e. for small enough Mach numbers, the Saint-Venant relation reduces to formula [4.31] for an incompressible fluid, whose application to compressible fluid induces a relative error amounting only to $\mathrm{M}_0^2/4$. Then by using Bernoulli's theorem to air flowing at 70 m.s^{-1} ($\mathrm{M}_0 \# 0.2$), error is only 1%. For pressure calculations, Bernoulli's incompressible relation can be used for many industrial or domestic problems of gas flowing such as ventilation, wind effects, etc.

We will look at some other consequences of the Saint-Venant relation in section 5.5.

4.3.2.3.4. Variation of the mass flux density and the Hugoniot relation

The mass flux density (mass flow per unit section of a stream tube) is equal to ρV. Replacing dp in equation [4.34] by its expression $dp = c^2 d\rho$, we obtain:

$$d\rho = -\rho\frac{V}{c^2}dV \qquad [4.37]$$

or, introducing the Mach number $M = V / c$:

$$d(\rho V) = \rho\left(1 - M^2\right)dV \qquad [4.38]$$

We see that the *mass flux density ρ V increases with the Mach number in subsonic flows while it decreases in supersonic flows*; it is at maximum for

$V = c = c_*$. Variations of $\dfrac{\rho V}{\rho_* \, c_*}$ as a function of the velocity V are shown in Figure 4.3 (ρ_* is the critical density). Note that two velocity values correspond to the preceding ratio, one subsonic, the other supersonic.

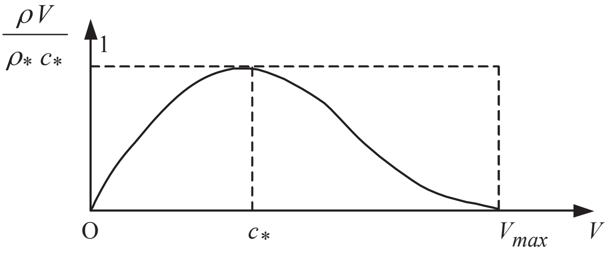


Figure 4.3. *Variation of the mass flux density (with reference to critical conditions) as a function of the flow velocity*

Consider a stream tube of variable cross-section S, small enough for the properties (p, ρ, V) of the fluid to be uniform in any cross-section (this is the approximation of the flow by slices).

For an *incompressible fluid* ($d\rho = 0$) the mass conservation SV = const is manifest in the fact that the cross-section and the velocity are inversely proportional. In the stream tube shown in Figure 4.4a, the velocity increases to a maximum which occurs at the throat of the minimal section S_c, then decays. According to [4.34], the driving pressure is minimal at the throat.

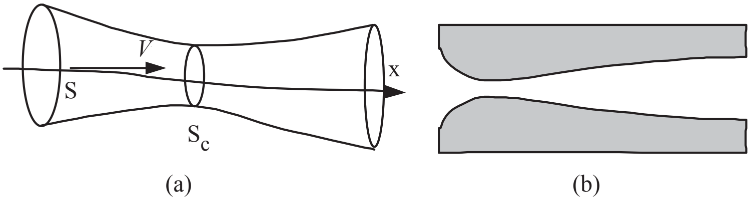


Figure 4.4. *(a) Flow in a stream tube; (b) nozzle*

Consider now the case of a *divariant compressible fluid* which undergoes isentropic transformations along a stream tube. The thermodynamic and mechanical quantities of the fluid now only depend on one variable (pressure, velocity, cross-section, etc.). We can write the conservation of mass flow ρSV in differential form in this stream tube ($d(\rho VS) = 0$); taking account of [4.38], we obtain Hugoniot's differential relation:

$$\frac{d(\rho VS)}{\rho VS} = \left(1 - \frac{V^2}{c^2}\right)\frac{dV}{V} + \frac{dS}{S} = 0 \qquad [4.39]$$

Equation [4.39] cannot be solved with respect to the derivative dV/dS in a velocity interval in which we have the velocity $V = c$ (critical conditions), this value corresponding to a singular point of this differential equation.

On the other hand, we can always solve equation [4.39] with respect to dS/dV, showing that the cross-section S of the tube is a decreasing function of the velocity in subsonic flow (V<c), but increasing for supersonic flows (V>c).

The minimum cross-section corresponds to:

– either a velocity *maximum in a subsonic flow*, as in the incompressible case;

– or a *velocity equal to the local sound speed.*

As the mass flow rate ρSV is constant, the area S of the cross-section varies inversely with the product ρV studied earlier (Figure 4.3). The maximum mass flux density ρV occurs in the throat of the nozzle, where the cross-section is smallest. However, this quantity ρV can only take values smaller than the critical value $\rho_* c_*$. So, the mass flow rate cannot exceed the critical value $\rho_* S_c c_*$ calculated in the throat cross-section. The mass flow rate may be obviously less than this maximum value, the velocity in cross-section S_c then being smaller than the sound velocity.

This configuration is characteristic of nozzles (Figure 4.4b). Note that for a given value of the cross-section S or the quantity ρSV, there are two corresponding velocities, which explains the difficulty encountered when we try to solve equation [4.39] for the velocity V. We will return to look at the consequences of this situation in section 5.5.4.

4.3.3. *The vorticity equation*

We have seen that for a mechanical system, we can consider a dynamic moment (section 3.2.1) taken about the inertia center of the system in a reference frame which is parallel to a Galilean reference frame. The instantaneous rotational movements in a fluid are characterized by the vorticity vector $2\vec{\omega} = \overrightarrow{rot}\,\vec{V}$ ([3.38]). The vorticity equation can be obtained by taking the curl of the Navier-Stokes equations [4.14]. Assuming body forces to derive from a potential, and using the identity:

$$\overrightarrow{rot}\left(a.\vec{A}\right) = a.\overrightarrow{rot}\vec{A} + \overrightarrow{grad}\,a \wedge \vec{A} \qquad [4.40]$$

we obtain, for any fluid:

$$\frac{\partial\left(\overrightarrow{rot}\,\vec{V}\right)}{\partial t} + \overrightarrow{rot}\left(\overrightarrow{rot}\,\vec{V} \wedge \vec{V}\right) = \frac{1}{\rho^2}\overrightarrow{grad}\,\rho \wedge \overrightarrow{grad}\,p + \overrightarrow{rot}\left(\frac{1}{\rho}\overrightarrow{div}\,\overline{\overline{\tau}}\right) \qquad [4.41]$$

For a divariant fluid, we have, applying formulae [4.19] and [4.40]:

$$\frac{\partial\left(\overrightarrow{rot}\,\vec{V}\right)}{\partial t} + \overrightarrow{rot}\left(\overrightarrow{rot}\,\vec{V} \wedge \vec{V}\right) = \overrightarrow{grad}\,T \wedge \overrightarrow{grad}\,s + \overrightarrow{rot}\left(\frac{1}{\rho}\overrightarrow{div}\,\overline{\overline{\tau}}\right) \qquad [4.42]$$

Assuming a Newtonian fluid with constant viscosity and density, the vorticity equation can be written, with [4.21]:

$$\frac{\partial\vec{\omega}}{\partial t} + \overrightarrow{rot}\left(\vec{\omega} \wedge \vec{V}\right) = \upsilon\overrightarrow{\Delta\omega} \qquad [4.43]$$

For a 2D velocity field, the vorticity vector $\vec{\omega}$ has only one component. This fact leads to very important specific properties, in particular in the study of turbulence ([COU 89], [MAT 00], [TEN 72])

The reader will note that the vorticity equations *in inviscid flows* are identical for *homoentropic* and *incompressible* fluids.

4.3.4. *The energy equation*

4.3.4.1. *The total energy equation*

4.3.4.1.1. General expression as a function of the stress tensor

We will here apply the same procedure which we applied for an elementary material system (section 3.2.5.1).

The quantity g is here the total specific energy $e + V^2/2$ whose balance is obtained by writing that its variation in a domain $\mathscr{D}$ is due to the external addition of energy:

$$\frac{d}{dt}\left(\int_{\mathscr{D}} \rho\left(e+\frac{V^2}{2}\right)dv\right)=\mathscr{P}_{th}+\mathscr{P}_{mec} \qquad [4.44]$$

The external additions of energy are:

– *mechanical power* $\mathscr{P}_{mec}$ provided from the exterior, which includes:

- the *specific mechanical power* $f_i u_i$ due to the specific force f_i in the domain $\mathscr{D}$;

- the *surface power* $\sigma_{ij}\, n_j\, u_j$ due to *external stresses* $\sigma_{ij}\, n_j$ on the outer surface Σ of the domain $\mathscr{D}$;

$$\mathscr{P}_{mec}=\int_{\mathscr{D}} \rho\, f_i u_i\, dv+\int_{\Sigma}\sigma_{ij} n_j u_i ds=\int_{\mathscr{D}} \rho\, f_i u_i\, dv+\int_{\mathscr{D}}\frac{\partial}{\partial x_j}\left(\sigma_{ij}u_i\right)dv$$

– *thermal power* $\mathscr{P}_{\text{th}}$ received from the exterior, which includes:

- volume power σ_T generated by volume source in the domain $\mathscr{D}$;

- *thermal fluxes across the outer surface* Σ due to the thermal flux vector density $\vec{q}_T$:

$$\mathscr{P}_{th}=\int_{\mathscr{D}}\sigma_T\, dv-\int_{\Sigma} q_{Tj} n_j\, ds=\int_{\mathscr{D}}\sigma_T\, dv-\int_{\mathscr{D}}\frac{\partial q_{Tj}}{\partial x_j}dv$$

Replacing the powers $\mathscr{P}_{\text{th}}$ and $\mathscr{P}_{\text{mec}}$ in [4.44] with their values, we obtain:

$$\begin{aligned}\frac{d}{dt}\left(\int_{\mathscr{D}} \rho\left(e+\frac{V^2}{2}\right)dv\right)&=\mathscr{P}_{th}+\mathscr{P}_{mec}=\int_{\mathscr{D}}\sigma_T\, dv-\int_{\mathscr{D}}\frac{\partial q_{Tj}}{\partial x_j}dv\\&+\int_{\mathscr{D}} \rho\, f_i u_i\, dv+\int_{\mathscr{D}}\frac{\partial\left(\sigma_{ij}u_i\right)}{\partial x_j}dv\end{aligned} \qquad [4.45]$$

Using Ostrogradsky's theorem to transform the surface integrals, and relation [4.45], we get the local equation:

$$\rho\frac{d}{dt}\left(e+\frac{V^2}{2}\right)=\rho\, f_i\, u_i+\frac{\partial\left(\sigma_{ij}u_i\right)}{\partial x_j}+\sigma_T-\frac{\partial q_{Tj}}{\partial x_j} \qquad [4.46]$$

or by introducing the viscous stress tensor [3.39]:

$$\rho \frac{d}{dt}\left(e+\frac{V^2}{2}\right)=\rho f_i u_i-\frac{\partial(pu_i)}{\partial x_i}+\frac{\partial(\tau_{ij}u_i)}{\partial x_j}+\sigma_T-\frac{\partial q_{Tj}}{\partial x_j} \qquad [4.47]$$

4.3.4.1.2. Expression using the total enthalpy

Taking the divergent form (see [4.10]) of the left-hand side of [4.47], we have:

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\frac{\partial}{\partial x_j}\left[\rho\left(e+\frac{V^2}{2}\right)u_j+pu_j\right]$$
$$=\rho f_i u_i+\frac{\partial(\tau_{ij}u_i)}{\partial x_j}+\sigma_T-\frac{\partial q_{Tj}}{\partial x_j}$$

Using the definition of enthalpy ($h=e+p/\rho$) then gives:

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\frac{\partial}{\partial x_j}\left[\rho\left(h+\frac{V^2}{2}\right)u_j\right]$$
$$=\rho f_i u_i+\frac{\partial(\tau_{ij}u_i)}{\partial x_j}+\sigma_T-\frac{\partial q_{Tj}}{\partial x_j} \qquad [4.48]$$

4.3.4.1.3. The internal energy equation

Subtracting equation [4.26] from equation [4.46] gives:

$$\rho\frac{de}{dt}=\frac{\partial(\rho e)}{\partial t}+\frac{\partial(\rho e u_j)}{\partial x_j}=\sigma_{ij}\frac{\partial u_i}{\partial x_j}+\sigma_T-\frac{\partial q_{Tj}}{\partial x_j} \qquad [4.49]$$

Expanding the stress tensor, we obtain:

$$\rho\frac{de}{dt}=-p\frac{\partial u_i}{\partial x_i}+\tau_{ij}\frac{\partial u_i}{\partial x_j}+\sigma_T-\frac{\partial q_{Tj}}{\partial x_j} \qquad [4.50]$$

4.3.4.1.4. Enthalpic and entropic forms of the energy equation

Equation [4.47] can also be written as a function the other extensive variables: the internal energy *e*, the enthalpy *h* or the entropy *s*, defined by the thermodynamic relations:

$$de=-pd\left(\frac{1}{\rho}\right)+Tds=dh-d\left(\frac{p}{\rho}\right)$$

or, taking account of relation [4.7]:

$$\frac{de}{dt}=-p\frac{d}{dt}\left(\frac{1}{\rho}\right)+T\frac{ds}{dt}=\frac{p}{\rho^2}\frac{d\rho}{dt}+T\frac{ds}{dt}=-\frac{p}{\rho}\frac{\partial u_i}{\partial x_i}+T\frac{ds}{dt}=\frac{dh}{dt}-\frac{d}{dt}\left(\frac{p}{\rho}\right)$$

The energy equation can thus be written in one of the following forms:

$$\left.\begin{array}{r}\rho\dfrac{de}{dt}+p\dfrac{\partial u_i}{\partial x_i}\\ \rho\dfrac{dh}{dt}-\dfrac{dp}{dt}\\ \rho T\dfrac{ds}{dt}\end{array}\right\}=\tau_{ij}\frac{\partial u_i}{\partial x_j}+\sigma_T-\frac{\partial q_{Tj}}{\partial x_j} \qquad [4.51]$$

The third equation of [4.51] allows us to define the *dissipation function* Φ:

$$\Phi=\tau_{ij}\frac{\partial u_i}{\partial x_j} \qquad [4.52]$$

which represents the thermal power released locally per unit volume by viscous friction.

4.3.4.1.5. Entropy balance

The entropic form of energy equation [4.51] is of some interest, as it in fact constitutes an entropy balance, which is not surprising since on account of the assumption of local equilibrium, the entropy is a state function related to the internal energy. This equation contains the flux terms $-\partial q_{Tj}/\partial x_j$ (divergence terms), and volume source terms associated with $\tau_{ij}\ \partial u_i/\partial x_j$ and σ_T . The entropy balance equation can thus be written:

$$\rho\frac{ds}{dt}=\frac{\tau_{ij}}{T}\frac{\partial u_i}{\partial x_j}+\frac{\sigma_T}{T}-\frac{1}{T}\frac{\partial q_{Tj}}{\partial x_j} \qquad [4.53]$$

4.3.4.1.6. Other forms of the energy equation

The left-hand side of the energy equation can also be written using other thermodynamic variables. For example, formula [1.27] allows us to write (β is expansion coefficient [1.26] at constant volume):

$$\frac{ds}{dt} = \frac{C_v}{\beta\, pT}\left(\frac{dp}{dt} - c^2\,\frac{d\rho}{dt}\right) \qquad [4.54]$$

Finally, equation [4.6] for the conservation of mass allows us to replace $\dfrac{\partial u_i}{\partial x_i}$ with $-\dfrac{1}{\rho}\dfrac{d\rho}{dt}$ and to obtain other forms for the left-hand side of equation [4.51]:

$$\left.\begin{array}{l} \rho\dfrac{de}{dt} + p\dfrac{\partial u_i}{\partial x_i} = \rho\dfrac{de}{dt} - \dfrac{p}{\rho}\dfrac{d\rho}{dt} \\ \qquad\qquad \rho\dfrac{dh}{dt} - \dfrac{dp}{dt} \\ \rho T\dfrac{ds}{dt} = \dfrac{\rho C_v}{\beta\, p}\left(\dfrac{dp}{dt} - c^2\,\dfrac{d\rho}{dt}\right) \\ \dfrac{\rho C_v}{\beta\, p}\left(\dfrac{dp}{dt} + c^2\rho\dfrac{\partial u_j}{\partial x_j}\right) \end{array}\right\} = \tau_{ij}\,\frac{\partial u_i}{\partial x_j} + \sigma_T - \frac{\partial q_{Tj}}{\partial x_j} \qquad [4.55]$$

The *heat convection equation* can be obtained from an expression of equation [4.51] for a transformation:

– *at constant pressure* (*dp/dt* = 0) using specific enthalpy *h*;

– *at constant volume* ($\partial u_j / \partial x_j = 0$) using specific internal energy *e*.

For a perfect gas or an incompressible liquid, these quantities can be expressed as a temperature function ($dh = C_p dT$ and $de = C_v dT$) and [4.51] can be written in the following form, where, depending on the case considered, we take either C_p or C_v as the specific heat C:

$$\rho C\frac{dT}{dt} = \tau_{ij}\,\frac{\partial u_i}{\partial x_j} + \sigma_T - \frac{\partial q_{Tj}}{\partial x_j} = \tau_{ij}\,\frac{\partial u_i}{\partial x_j} + \sigma_T + \frac{\partial}{\partial x_j}\left(\lambda\,\frac{\partial T}{\partial x_j}\right)$$

4.3.4.1.7. Expression of the dissipation function for a Newtonian fluid

Substituting [3.57] for the viscous tensor of a Newtonian fluid into [4.52], we obtain ($\theta = div\vec{V}$):

$$\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} = \mu\frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \left(\eta - \frac{2}{3}\mu\right)\left(\frac{\partial u_i}{\partial x_i}\right)^2$$

$$= 4\mu\left(\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2\right) + 2\mu\left(\varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{33}^2\right) + \left(\eta - \frac{2}{3}\mu\right)\theta^2$$

Very often, the shear velocities are much greater than the expansion velocities and the expression for the dissipation function is simplified:

$$\Phi = 4\mu\left(\varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2\right)$$
$$= \mu\left[\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)^2 + \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)^2\right] \qquad [4.56]$$

4.3.5. *Balance of chemical species*

Equation [2.25] for the mass balance of a chemical species can be written using the partial density ρ_i and the local velocity $\vec{V}_i$ of the component *i*:

$$\frac{\partial \rho_i}{\partial t} + div\left(\rho_i \vec{V}_i\right) = \sigma_{mi} \qquad [4.57]$$

The volume source of mass σ_{mi} of species *i* is a homogenous chemical reaction. Introducing the local *mass average velocity* $\vec{V}$ [2.29] gives:

$$\frac{\partial \rho_i}{\partial t} + div\left(\rho_i \vec{V}\right) = \sigma_{mi} - div\left[\rho_i\left(\vec{V}_i - \vec{V}\right)\right] \qquad [4.58]$$

The mass flux density $q_{mi} = \rho_i\left(\vec{V}_i - \vec{V}\right)$ is associated with diffusive phenomena considered here in the reference frame of the inertia center (section 2.4.3.5). For the case of a binary mixture, this can be expressed using formula [2.69] for non-isothermal mixtures.

In the case of isothermal diffusion, the mass flux density of the species *i* is given by relation [2.57]. We thus obtain the diffusion equation of the flow:

$$\frac{\partial \rho_i}{\partial t} + div\left(\rho_i \vec{V}\right) = \sigma_{mi} + div\left[\rho\, D\overrightarrow{grad}\left(\frac{\rho_i}{\rho}\right)\right] \qquad (i = 1,2) \qquad [4.59]$$

In the case where species 1 is in weak concentration, a discussion analogous to that of section 2.4.4.2.7 leads to the following equation:

$$\frac{\partial c_1}{\partial t} + div\left(c_1 \vec{V}\right) = \sigma_{c1} + D\Delta c_1 \qquad [4.60]$$

Recall that for a binary mixture, diffusion is represented by the balance equation of one species, associated with the mass conservation equation for the mixture. In simple cases, we can consider that the creation of species 1 is due to a homogenous reaction of order *m*:

$$\sigma_{c1} = kc_1^m \qquad [4.61]$$

The coefficient *k* for the reaction kinetics varies according to a law of the form:

$$k = k_0 \exp\left(-\frac{U}{RT}\right)$$

The quantity *U* is the activation energy; *T* designates the absolute temperature. The form of the preceding relation shows a strong coupling between the temperature and the reaction speed.

4.4. Boundary conditions

4.4.1. *General considerations*

The partial differential equations satisfied by the preceding quantities are not sufficient for the definition of a particular problem. We must also specify boundary conditions of different kinds. The definition of the domain studied $\mathscr{D}$ constitutes the first step. It is defined by the surface which bounds $\mathscr{D}$ and on which we will impose conditions for the unknowns of the problem (boundary conditions). Because of the particular nature of the time variable, we must also specify the *initial conditions*.

In fluid mechanics the boundary conditions must be carefully considered, as we rarely encounter physical problems in entirely closed domains, since the flow must be generated by some appropriate device (a pump, a fan, a moving vehicle, a meteorological situation, an acoustic cavity, etc.). The domain studied is generally limited by solid walls and zones which are connected to other zones being studied.

4.4.2. *Geometric boundary conditions*

4.4.2.1. *Solid walls*

The usual solid walls are relatively easy to treat, since the fluid touches the wall: *the velocities of the solid wall and the fluid are equal where they are in contact*. The reason for this fact is related to the roughness of the walls on the molecular level, and to the thermal excitation by which mean momentum is transferred from one medium to another (it is incidentally the same as the interpretation of viscosity and contact action). This interpretation is such that the physico-chemical nature of the wall does not influence the adherence condition.

This adherence condition of the fluid at the wall is always very well satisfied in ordinary conditions where the mean free path of the molecules is small compared with the roughness. The same is not true in the study of rarefied gases, where we must take account of the properties of the wall and introduce a slipping coefficient.

In certain cases, the walls are permeable, in other words they let some matter pass through them. This is the case when we suck or blow through a porous medium. In these conditions, the difference between the tangential fluid velocities and the wall are zero at the wall. The normal fluid velocity with respect to the wall depends on the fluid injection process. An analogous condition is encountered in the presence of *phase changes at the wall*: evaporation, fusion, and other heterogenous chemical reactions which consume or produce fluid.

The boundary conditions for thermal and diffusion problems were discussed in sections 2.3.2 and 2.4.6, which the reader can refer to.

4.4.2.2. *Flow entry and exit zones*

In addition to the walls which guide the flow, we generally need to specify the conditions at the entrance or exit of the domain studied. Real flows are always generated by machines (solid surface in movement) or by differences in conditions between upstream and downstream reservoirs. In practice, we know how to impose wall velocities, injection or extraction flow rates, unsteady forces on a wall (by means of electromagnetic devices). However, it should be noted that we do not know how to impose a given pressure or velocity distribution throughout a fluid.

In fact, the entrance of a flow into a domain is a rather particular zone, as we have an *initial condition* which is largely analogous to that of the temporal variable (it is in fact an initial condition in Lagrangian variables). In practice, such a zone is found at the exit of another fluid domain and we need to specify a velocity or pressure distribution which is compatible with the equations of motion. This is straightforward when we can take a uniform flow or zone at rest with a constant driving pressure in a section perpendicular to the velocity relatively far upstream of the walls which are to guide the flow. However, in general, this is not the case, and the choice of an incompatible velocity profile leads to difficulties ("numerical shocks") in the subsequent numerical solutions. The same difficulties are encountered in thermal and acoustic studies, where initial conditions for the entropy or other material quantities are required. We will come back to this point in section 5.6.2 when we consider the classification of partial differential equations.

Further to the preceding difficulties, we will see in Chapter 5 that certain material properties can also be transported by waves which propagate in different directions, including in the upstream direction. This means that the very idea of the flow entrance ("upstream" and "downstream") may depend on what it is we wish to study. Remember also that the idea of an extensive quantity leads to conservation properties which must be satisfied in the global balances, and it is not always obvious that this condition is respected.

Finally, for flows which possess unstable zones, it is not sufficient to specify velocity or pressure distributions: the perturbations which enter the domain must be defined so as to fully determine the problem to be solved.

In conclusion, it is clear that we are a long way from understanding how to proceed in all cases in order to obtain a well-posed problem.[1]

4.4.2.3. *Free surfaces*

The free surface of a flow[2] (or the interface between two immiscible fluids) is an unknown boundary, on which the two following properties need to be ensured:

1) The *free surface (or the interface) is a material boundary between two immiscible fluids*: it comprises the locus of fluid particle trajectories. Let z be the altitude of this free surface[3]:

$$z = \xi(x, y, t)$$

The w component of the velocity in the Oz direction satisfies the relation (known as the *kinematic relation*):

---

[1] We will here define a well-posed theoretical problem (to be solved analytically or numerically) or an experiment whose solution is reproducible, even if we change a calculation method or an experimental process.

[2] For instance, water flows in open channels, rivers, etc.

[3] The axis Oz is the opposite of the direction of external forces field, usually gravity field.

$$w = \frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y}$$

2) The physical nature of this surface must provide the conditions necessary for the stress tensor. The pressure discontinuity $p_1 - p_2$ across the surface is given by Laplace's law:

$$p_1 - p_2 = \sigma\left(\frac{1}{R} + \frac{1}{R'}\right)$$

where the quantities σ, R and R' are, respectively, the surface tension and the radii of curvature of the free surface at the point considered (section 2.2.1.4.2).

For nearly all industrial flows of water and aqueous solutions, surface tension does not play an important role: we can assume pressure continuity at the free surface.

The discontinuity of the tangential stresses is related to the physico-chemical and flow conditions of the free surface; in particular, if the free surface contains a surface active substance (a mono-molecular thickness layer can suffice) and is not regenerated, it behaves like a solid surface. In many cases of water flow, the free surface is regenerated and we have continuity of the tangential stresses between the two media. In hydraulics, we can consider that the viscous stress is nearly equal to zero at the free surface of a flow.

4.4.2.4. *Fully immersed flows*

Fully immersed flows are flows without free surface whose upstream and downstream conditions are hydrostatic. We have seen that in the Navier-Stokes equations, gravity can be eliminated by using a driving pressure variable $p_g = p + \rho gz$. This change of variables is only of interest so long as it does not result in gravity reappearing in the boundary conditions, which should be expressed in terms of the driving pressure and not pressure itself. This excludes flows with free surfaces which are not horizontal (sea swells, rivers, surface runoff, etc.) for which we have, for example, a constant pressure condition.

4.4.3. ***Initial conditions***

No physical problem is timeless. There is always a beginning to an experiment and therefore to its modeling. From this point of view, flow problems can be difficult, because even if the boundary conditions are steady (independent of time), the solution may present diverse and complex characteristics. Some examples will demonstrate the degree of these difficulties:

– a steady solution may be established after a transition period, but this solution may depend on the initial conditions (such properties are used in fluidics when we manipulate the eventual hysteresis of separated flow zones);

– no steady solution may exist, but a more or less complexly established unsteady flow regime may occur, which is determined and predictable;

– the flow may become more or less chaotic while maintaining a more or less organized unsteadiness;

– an established turbulence may be present in certain domains of the established flow: in other words the flow, while unsteady, may have stable statistical characteristics;

– weak perturbations, which are difficult to characterize, may be present in the initial or boundary conditions, and these may be of considerable importance for the evolution of the flow.

We will come back to some of these points in section 6.6.

4.5. Global form of the balance equations

4.5.1. *The interest of the global form of a balance*

Even the most rudimentary model must satisfy conservation laws for extensive quantities. The interest in a global balance is that it allows us to observe a system from the exterior. Balance equation [4.1] for a quantity G in the domain D can be written by replacing the material derivative by its expression (section 3.3.3.3); it can be written in one of the following two forms, as a function of the volume quantity $\overline{g}$ or of the mass quantity g:

$$\int_{\mathcal{D}_\circ} \frac{\partial \overline{g}}{\partial t} dv + \int_{\Sigma} \overline{g} u_i n_i ds = \int_{\mathcal{D}} \sigma_G \, dv - \int_{\Sigma} q_{Gj} n_j \, ds \qquad [4.62]$$

$$\int_{\mathcal{D}_\circ} \frac{\partial (\rho g)}{\partial t} dv + \int_{\Sigma} g dq_m = \int_{\mathcal{D}} \sigma_G \, dv - \int_{\Sigma} q_{Gj} n_j \, ds \qquad [4.63]$$

Note that it is equivalent to writing a Eulerian balance in a domain D *which is assumed fixed*. As we saw in section 1.1.4.2, this balance corresponds to an *open system*.

$$\int_D \frac{\partial \bar{g}}{\partial t}\, dv = \int_D \sigma_G \, dv - \underset{\text{“physical” flux}}{\int_\Sigma q_{Gj} n_j \, ds} - \underset{\text{convective flux}}{\int_\Sigma \bar{g} u_i n_i \, ds}$$

The left-hand side term corresponds to the possible accumulation of the quantity *G* in $\mathcal{D}$. This variation of the quantity *G* contained in this domain is the sum of:

– the production of the sources σ_G;

– the fluxes of the quantity *G* due to transfers through matter;

– the fluxes of quantity *G* associated in the matter which enters $\mathcal{D}$ by crossing the boundary Σ (convective fluxes).

In the preceding equations, we can interpret the surface and volume integrals in different ways:

– the surface integrals over Σ comprise *inputs* and *outputs* of the material system contained in $\mathcal{D}$, in other words we can evaluate them without knowing the internal structure of the flow within the system;

– the volume integrals in the domain $\mathcal{D}$ clearly imply the internal variables of the system for which a model is necessary.

IMPORTANT NOTES –

1) *The global balance equations do not require the assumption of a continuous medium*. They can be applied to any domain *D* constituted by matter, continuous or discontinuous: diverse media containing many different phases (liquid flows with bubbles, fluidized beds for transport of pulverized media, etc.), avalanches of solid substances (stones, snow, mud, etc.) and/or liquids (mud, etc.). The balance equation can be directly written without assumptions concerning the differentiable properties of the continuous medium. Only the assumption of integrability of the fluxes across the surface Σ is necessary. Even the time differentiation assumption is not necessary: it suffices to effect a balance of the amount of *G* contained in *D* between two instants t_1 and t_2:

$$G_D(t_2) - G_D(t_1) = -\overset{\text{quantity of } G}{\underset{\text{between } t_1 \text{ and } t_2}{\overline{\underline{\begin{matrix}\text{come out from } \Sigma \\ \text{by convection}\end{matrix} + \text{created in } D - \begin{matrix}\text{come out from } \Sigma \\ \text{by other process}\end{matrix}}}}}$$

When the quantity *G* is a vector (momentum), the same balance can be performed on its moments.

2) In all modeling we must have a global extensive variable *G(t)* internal to the domain *D*, defined from the amount of *G* contained in *D*, such that $\int_{\mathcal{D}_{\circ}} \frac{\partial \overline{g}}{\partial t} dv$ is equal to $\frac{dG}{dt}$; equation [4.63] can thus be written in the form:

$$\frac{dG}{dt} = \left\{ \begin{matrix} \text{source of} \\ G \text{ inside } \mathcal{D} \end{matrix} \right\} + \left\{ \begin{matrix} \text{inputs of} \\ G \text{ into } \mathcal{D} \end{matrix} \right\}$$

4.5.2. *Equation of mass conservation*

The expression of mass conservation for a fluid of density ρ inside the domain *D* during its movement, can be written from [4.62], with $\overline{g} = \rho$:

$$\int_{\mathcal{D}} \frac{\partial \rho}{\partial t} dv + q_{m\Sigma} = 0$$

where $q_{m\Sigma} = \int_{\Sigma} dq_m$ is the mass flux leaving the surface Σ.

4.5.3. *Volume balance*

The elementary volume flux dq_v crossing the surface *ds* is: $dq_v = \vec{V}.\vec{n}\ ds$. The volume flux leaving the domain *D* can be written:

$$\int_{\Sigma} dq_v = \int_{\mathcal{D}} div\,\vec{V}\, dv = \int_{\mathcal{D}} \frac{\partial u_i}{\partial x_i} dv$$

4.5.4. *The momentum flux theorem*

Applying formulae [4.8] and [3.33] to the quantity $\overline{g} = \rho \vec{V}$, we obtain the linear momentum theorem:

$$\int_{D} \frac{\partial (\rho u_i)}{\partial t} dv + \int_{\Sigma} \rho\, u_i u_j n_j\, ds = \sum F e_i$$

or, expressing the elementary mass flux: $dq_m = \rho\, u_j n_j ds$:

$$\int_D \frac{\partial(\rho u_i)}{\partial t} dv + \int_\Sigma u_i dq_m = \sum F_{ext\ i}$$

$$\text{or } \int_D \frac{\partial(\rho \vec{V})}{\partial t} dv + \int_\Sigma \vec{V} dq_m = \sum \vec{F}_{ext} \qquad [4.64]$$

Note that in *a steady flow the balance depends only on the input-output of the domain*. In unsteady flows, the first term corresponds to an eventual accumulation of momentum in *D*.

Thus, as discussed in section 3.2.1, we can also perform a balance using the moments (angular momentum). This gives:

$$\int_D \overrightarrow{OM} \wedge \frac{\partial(\rho \vec{V})}{\partial t} dv + \int_\Sigma \left(\overrightarrow{OM} \wedge \vec{V}\right) dq_m = \sum \vec{M}_{ext} \qquad [4.65]$$

The momentum flux theorem is particularly useful in applications. Let us consider as an example a propulsion device whose role is to increase the velocity of the fluid which crosses it (a propeller, a turbine, a pump which produces a jet, etc.). The reference frame is associated with the propulsion device, which is considered to operate in a steady regime. We will ignore the details of the system and we will consider the pressure to be constant on a surface Σ external to the device (Figure 4.5).

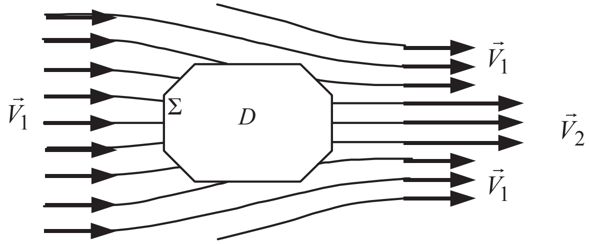


Figure 4.5. *Propeller thrust*

Let q_m be the mass flow rate passing through the propulsion device. The momentum flow rate *parallel to the upstream velocity* $\vec{V}_1$ and leaving the surface Σ is equal to:

$$\int_\Sigma \vec{V} dq_m = q_m\left(\vec{V}_2 - \vec{V}_1\right)$$

The momentum flux is balanced by the thrust it generates, equal to $P = -q_m\left(\vec{V}_2 - \vec{V}_1\right)$.

Let us note that conservation of mass flow rate demands the existence of an *incoming mass flow through the lateral sides* of surface Σ.

Very many practical problems can be treated in the same way, the changes in momentum corresponding to the forces which can be calculated from a knowledge of the mass flows and the velocities ([EVE 89], [GAR 06], [GUY 01], [SAG 66], [SPU 97]).

4.5.5. *Kinetic energy theorem*

The global form can be obtained by integrating the local form [4.27] over the domain $\mathcal{D}$. Assuming a potential *U which is independent of time*, we obtain:

$$\int_{\mathcal{D}} \rho \frac{d}{dt}\left(\frac{V^2}{2}+U\right)dv = \frac{d}{dt}\left[\int_{\mathcal{D}} \rho\left(\frac{V^2}{2}+U\right)dv\right] =$$

$$\int_{\mathcal{D}} \frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right)dv + \int_{\Sigma} \rho\left(\frac{V^2}{2}+U\right)u_j n_j ds = -\int_{\mathcal{D}} u_i \frac{\partial p}{\partial x_i}dv + \int_{\mathcal{D}} u_i \frac{\partial \tau_{ij}}{\partial x_j}dv$$

We transform the terms of the right-hand side by introducing the divergence of a vector, the corresponding integral being thus transformed into a surface integral Σ; we obtain, after moving a pressure term to the left-hand side:

$$\int_{\mathcal{D}} \frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right)dv + \int_{\Sigma}\left[p + \rho\left(\frac{V^2}{2}+U\right)\right]u_j n_j ds =$$

$$\int_{\mathcal{D}} p \frac{\partial u_i}{\partial x_i}dv + \int_{\Sigma} \tau_{ij} u_j n_j ds - \int_{\mathcal{D}} \tau_{ij} \frac{\partial u_i}{\partial x_j}dv$$

The first term describes the accumulation of kinetic energy in *D* in the transitional regime. The sum of this term and the following terms of mechanical energy flux (kinetic and pressure) across Σ is equal to the sum of the power $P_{v\Sigma}$ of the external viscous stresses exerted on *Σ,* of the power of pressure forces in *D*, and of the power $-P_{vD}$ dissipated in *D* by viscosity (dissipation function [4.52]):

$$\begin{aligned}\int_{\mathfrak{D}} \frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right) dv + \int_{\Sigma}\left[p + \rho\left(\frac{V^2}{2} + U\right)\right] u_j n_j ds \\ = \int_{\mathfrak{D}} p \frac{\partial u_i}{\partial x_i} dv + P_{v\Sigma} - P_{v\mathfrak{D}}\end{aligned} \qquad [4.66]$$

with: $P_{v\Sigma} = \int_{\Sigma} \tau_{ij} u_j n_j ds$ the external power provided by the viscosity on Σ;

$P_{v\mathfrak{D}} = -\int_{\mathfrak{D}} \tau_{ij} \frac{\partial u_i}{\partial x_j} dv$ the power dissipated in D by viscous friction.

In incompressible flows, we obtain:

$$\int_{\mathfrak{D}} \frac{\partial}{\partial t}\left(\frac{\rho V^2}{2}\right) dv + \int_{\Sigma}\left(p + \rho g h + \frac{\rho V^2}{2}\right) u_j n_j ds = P_{v\Sigma} - P_{v\mathfrak{D}} \qquad [4.67]$$

In incompressible inviscid flows, the sum of the temporal variation in the kinetic energy of D and the flux of mechanical energy (kinetic and pressure) leaving Σ is zero.

4.5.6. *The energy equation*

The integration of the energy equation over the domain D should be performed using a form of the equation which allows flux integrals to be written. For a compressible fluid, the form [4.48] with the total enthalpy is best suited. We obtain, neglecting gravity:

$$\begin{aligned}\int_{\mathfrak{D}} \frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] dv + \int_{\Sigma} \rho\left(h + \frac{V^2}{2}\right) u_j n_j ds = \\ \int_{\Sigma} \tau_{ij} u_i n_j ds + \int_{\mathfrak{D}} \sigma_T dv - \int_{\Sigma} q_{Tj} n_j ds\end{aligned} \qquad [4.68]$$

This equation has the inconvenience of not including the dissipation function Φ. Using one of the forms [4.51] is preferable for thermodynamic balances, but internal balances remain in the form of volume integrals:

$$\int_{\mathfrak{D}} \frac{\partial(\rho h)}{\partial t} dv + \int_{\Sigma} h dq_m - \int_{\mathfrak{D}} \left(\frac{\partial p}{\partial t} + u_j \frac{\partial p}{\partial x_j}\right) dv = \int_{\mathfrak{D}} \Phi \, dv + \int_{\mathfrak{D}} \sigma_G \, dv - \int_{\Sigma} q_{Tj} n_j ds$$

However, these forms become perfectly adapted for the study of thermal transfer at constant pressure or constant volume. *In such conditions*, we immediately obtain, from [4.51]:

$$\int_{\mathscr{D}} \frac{\partial(\rho g)}{\partial t} dv + \int_{\Sigma} g\, dq_m = \int_{\mathscr{D}} \Phi\, dv + \int_{\mathscr{D}} \sigma_T\, dv - \int_{\Sigma} q_{Tj} n_j\, ds \qquad [4.69]$$

where for *g* we must use either the specific internal energy *e* or the specific enthalpy *h*. For the usual case of a fluid of constant specific heat, the preceding equation becomes:

$$\int_{\mathscr{D}} \frac{\partial(\rho CT)}{\partial t} dv + \int_{\Sigma} \rho\, CT dq_m = \int_{\mathscr{D}} \Phi\, dv + \int_{\mathscr{D}} \sigma_T\, dv - \int_{\Sigma} q_{Tj} n_j\, ds \qquad [4.70]$$

where, depending on the case, we used either C_p or C_v for the specific heat C.

4.5.7. *The balance equation for chemical species*

Equation [4.58] can be immediately integrated to give:

$$\int_{\mathscr{D}} \frac{\partial \rho_i}{\partial t} dv + \int_{\Sigma} \rho_i\, \vec{V}.\vec{n}\, ds = \int_{\mathscr{D}} \sigma_{mi}\, dv - \int_{\Sigma} \vec{q}_{mi} \vec{n}\, ds \qquad [4.71]$$

For a binary isothermal mixture, using [4.59] we obtain:

$$\int_{\mathscr{D}} \frac{\partial \rho_i}{\partial t} dv + \int_{\Sigma} \rho_i\, \vec{V}.\vec{n}\, ds = \int_{\mathscr{D}} \sigma_{mi}\, dv + \int_{\Sigma} \rho\, D \overrightarrow{\vec{n}.grad}\left(\frac{\rho_i}{\rho}\right) ds \qquad [4.72]$$

In the case of weak concentrations, balance equation [4.60] gives:

$$\int_{\mathscr{D}} \frac{\partial c_i}{\partial t} dv + \int_{\Sigma} c_i\, \vec{V}.\vec{n}\, ds = \int_{\mathscr{D}} \sigma_{ci}\, dv + \int_{\Sigma} D\, \vec{n}.\overrightarrow{grad}\, c_i\, ds \qquad [4.73]$$

Homogenous or heterogeneous chemical reactions must be specified accordingly, in the volume source term $\int_{\mathscr{D}} \sigma_{ci}\, dv$ or in the flux term $\int_{\Sigma} D\, \vec{n}.\overrightarrow{grad}\, c_i\, ds$ at the walls. In general, chemical reactions also lead to heat release, which must be accounted for in a source term of the energy equation ([BIR 01]).

4.6. Similarity and non-dimensional parameters

4.6.1. *Principles*

4.6.1.1. *Invariance of physical laws*

The fundamental physical laws are described by relations between real numbers obtained as measures of physical quantities. A measurement is a comparison between the quantity studied and a similar quantity considered as a unit. Most units are "derived", as they depend on physical laws: for example, the surface unit is a square whose sides measure one unit of length. With such choices, physical laws are relations which comprise either dimensional coefficients which have a physical interpretation (the speed of sound for example) and which can be expressed using the units of the system, or non-dimensional coefficients which are independent of the system (for example, π for the surface of a circle). There are four fundamental units which can be arbitrarily taken for mass, length, time and temperature, and whose choice determines a coherent system of units. No specific physical phenomenon is used to govern the particular quantities chosen as units, and for practical reasons we can arbitrarily choose the four fundamental units of the international metric system (the meter, the kilogram, the second, the Kelvin).

With these choices, the mathematical relations representing physical phenomena are true regardless of the fundamental units which are used, which means that they possess an invariance with respect to changes in the units which are used, transformations which form a group. A given physical problem thus has an infinity of equivalent numerical representations. Similarly, a numerical problem can represent many different physical problems obtained using different systems of units.

4.6.1.2. *Similar problems*

A given problem must be repeatable, meaning that its definition must always lead to identical results, within a certain margin of error. The problem is only defined if analysis of the phenomena involved has been correct and complete, in other words if we know the partial differential equations, and the boundary and initial conditions which define the problem.

Two problems are *similar* if two systems of units exist, such that the measurements of all the quantities of one of the problems, using a given system of units, are equal to the measurements of the corresponding quantities of the other problem using another system of units.

The conditions for similarity may be obtained by searching for conditions in which the two problems will obey the same ensemble of equations and boundary conditions after an appropriate change of the system of units. Because we can choose the system of units, it is easier to write the equations and conditions of a problem with a *system of units which corresponds to the problem*; the equations obtained are non-dimensional, in other words they are independent of the units chosen for the physical measurements: calculating the area of a circle using its radius as a unit gives a result equal to π.

4.6.1.3. *Non-dimensional study of a problem*

4.6.1.3.1. Dynamic and thermal problem

We will now consider a flow problem involving a perfect gas with heat transfer around a circular obstacle of diameter L. We will assume that the specific heat C_p, the viscosity μ, and the thermal conductivity λ are constant. The problem is posed in the following manner:

– mass conservation:

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho u_i)}{\partial x_i}=0 \qquad [4.74]$$

– Navier-Stokes equations:

$$\rho\frac{du_i}{dt}=-\frac{\partial p}{\partial x_i}-\rho g\frac{\partial z}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \qquad (i, j = 1,2,3) \qquad [4.75]$$

– energy equation (in enthalpic form) and the equation of state:

$$\rho C_p\frac{dT}{dt}-\frac{dp}{dt}=\Phi(\varepsilon_{ij})+\lambda\frac{\partial^2 T}{\partial x_j \partial x_j}$$

$$\frac{p}{\rho}=rT \qquad [4.76]$$

– dynamic boundary conditions:

- on the body surface, velocity equal to zero: $u_i = 0$;
- at infinity, velocity $\vec{V}$ equal to $\vec{U}$;
- eventually a free surface condition on a horizontal surface;

– thermal boundary conditions:

- constant temperature T equal to T_w on the surface;
- constant temperature equal to T_0 at infinity;

– initial conditions:

- the fluid is assumed at rest at time *t=0*.

The results of the problem are the velocity field, the pressure field and the temperature field, in addition to the surface densities (friction stresses, pressure, thermal flux density, etc.) or the global quantities (drag, lift, heat flux, etc.). These are functions of the coordinates and the n data of the problem (U, L, ρ_0, T_0, T_w, μ, λ, g, C_p).

4.6.1.3.2. Non-dimensional equations

For the fundamental units we will use the data of the problem such as the velocity U, the length L and the reference density ρ_0, and we will define the following non-dimensional variables:

$$\tilde{x}_i = \frac{x_i}{L} \qquad \tilde{u}_i = \frac{u_i}{U} \qquad \tilde{p} = \frac{p}{\rho_0 U^2} \qquad \tilde{t} = \frac{Ut}{L} \qquad \tilde{\rho} = \frac{\rho}{\rho_0} \qquad \tilde{T} = \frac{T}{T_0} \tag{4.77}$$

We have:

$$\frac{\partial}{\partial x_i} = \frac{1}{L}\frac{\partial}{\partial \tilde{x}_i}; \qquad \frac{d}{dt} = \frac{U}{L}\frac{d}{d\tilde{t}}; \qquad \Delta = \frac{\partial^2}{\partial x_j \partial x_j} = \frac{1}{L^2}\frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_i} = \frac{1}{L^2}\tilde{\Delta}$$

Performing variable change [4.77] in equations [4.74] to [4.76] and dividing by the dimensional coefficients of the left-hand side in each equation leads to the following reduced equations:

$$\begin{aligned}
&\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial(\tilde{\rho}\,\tilde{u}_i)}{\partial \tilde{x}_i} = 0 \\
&\tilde{\rho}\frac{d\tilde{u}_i}{d\tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} - \tilde{\rho}\frac{gL}{U^2}\frac{\partial \tilde{z}}{\partial \tilde{x}_i} + \frac{\mu}{\rho_0 UL}\frac{\partial^2 \tilde{u}_{ij}}{\partial \tilde{x}_j \partial \tilde{x}_j} \qquad (i, j = 1,2,3) \\
&\tilde{\rho}\frac{d\tilde{T}}{d\tilde{t}} - \frac{U^2}{C_p T_0}\frac{d\tilde{p}}{d\tilde{t}} = \frac{\mu\, U}{\rho\, C_p T_0 L}\Phi\left(\tilde{\varepsilon}_{ij}\right) + \frac{\lambda}{\rho_0 C_p UL}\frac{\partial^2 \tilde{T}}{\partial \tilde{x}_j \partial \tilde{x}_j} \\
&\frac{\tilde{p}}{\tilde{\rho}} = \frac{rT_0}{U^2}\tilde{T}
\end{aligned} \tag{4.78}$$

where the dissipation function $\Phi\left(\varepsilon_{ij}\right)$ (which characterizes kinetic heating) is given by [4.56]: $\Phi\left(\varepsilon_{ij}\right)=\dfrac{\mu U^2}{L^2}\tilde{\Phi}\left(\tilde{\varepsilon}_{ij}\right)$.

The preceding equations contain the following (non-dimensional) similarity parameters:

- Reynolds number: $\mathrm{Re}=\rho_0 UL/\mu$
- Froude number: $\mathrm{Fr}=\ \mathrm{U}/\sqrt{g\ell}$
- Mach number: $\mathrm{M}_0=\mathrm{U}/c_0$
- Prandtl number: $\mathrm{Pr}=\mu\,\mathrm{C_p}/\lambda$
- Eckert number: $\mathrm{Ec}=\mathrm{U}^2/\mathrm{C_p}T_0=(\gamma-1)\mathrm{M}_0^2$

which allows us to write [4.78] in the form:

$$\begin{aligned}
&\frac{\partial\tilde{\rho}}{\partial\tilde{t}}+\frac{\partial(\tilde{\rho}\,\tilde{u}_i)}{\partial\tilde{x}_i}=0\\
&\tilde{\rho}\frac{d\tilde{u}_i}{d\tilde{t}}=-\frac{\partial\tilde{p}}{\partial\tilde{x}_i}-\tilde{\rho}\frac{1}{\mathrm{Fr}^2}\frac{\partial\tilde{z}}{\partial\tilde{x}_i}+\frac{1}{Re}\frac{\partial^2\tilde{u}_{ij}}{\partial\tilde{x}_j\partial\tilde{x}_j} \qquad (i,j=1,2,3)\\
&\tilde{\rho}\frac{d\tilde{T}}{d\tilde{t}}-\mathrm{Ec}\frac{d\tilde{p}}{d\tilde{t}}=\frac{\mathrm{Ec}}{\mathrm{Re}}\tilde{\Phi}\left(\tilde{\varepsilon}_{ij}\right)+\frac{1}{\mathrm{Re.Pr}}\frac{\partial^2\tilde{u}_{ij}}{\partial\tilde{x}_j\partial\tilde{x}_j}\\
&\frac{\tilde{p}}{\tilde{\rho}}=\frac{1}{\gamma\mathrm{M}_0^2}\tilde{T}
\end{aligned}\qquad [4.79]$$

The Eckert number and the Mach number are related by the perfect gas relation between C_p and r ($r\gamma=(\gamma-1)C_p$). We also define the Péclet number $Pe = Re.\ Pr$.

These similarity parameters characterize the importance of the non-dimensional terms by reference to the terms of the left-hand side of the non-dimensional equations. For example, we can write:

$$\mathrm{Re} = \frac{\rho_0 UL}{\mu} = \frac{\rho_0 U^2 / L}{\mu U / L} \approx \frac{\rho_0 du_i / dt}{\mu . \left(\partial^2 u_i / \partial x_j \partial x_j \right)} \approx \frac{\text{acceleration term}}{\text{viscosity term}}$$

$$\mathrm{Fr} = \frac{\mathrm{U}}{\sqrt{gL}} = \sqrt{\frac{\rho_0 U^2}{\rho_0 \, gL}} \approx \sqrt{\frac{\text{dynamic pressure}}{\text{hydrostatic pressure difference}}}$$

$$\mathrm{Ec} = \frac{\mathrm{U}^2}{\mathrm{C_p} T_0} = \frac{\rho_0 \, \mathrm{U}^2}{\rho_0 \, \mathrm{C_p} T_0} \approx \frac{\text{kinetic energy}}{\text{enthalpy}}$$

The Eckert number (or the Mach number) represents the influence of compressibility on the properties of the flow whose kinetic energy is due to enthalpy variations in the steady flow (conservation of the total enthalpy [4.33]).

The Reynolds number usually takes on very large values (from 10^3 to 10^8), which signifies the globally weak effects of viscosity and thermal conduction in most flows. We will see in Chapter 6 that this property is not true in zones close to solid boundaries (boundary layers).

Note that if the viscosity or other physical properties are not constant we must introduce further non-dimensional functions.

4.6.1.3.3. Non-dimensional boundary conditions

The treatment of boundary conditions is immediate: the conditions *U* and *L* are transformed into a single condition equal to 1. The thermal problem requires, for example, the temperature T_p, given at the wall to be transformed into a non-dimensional parameter T_p / T_0. If other velocity conditions or geometric dimensions are given, they provide supplementary similarity parameters (velocity ratio, shape factor, etc.).

The boundary conditions determine the order of magnitude of the similarity parameters and the associated phenomena. Simplifications can result from this. For example:

– if the Mach number is small (less than 0.3 in practice), it has to be shown (see section 4.3.2.3.3) that the density variations can be neglected for the study of fluid motion and that incompressible fluid relations can be used. Thermal transfer thus occurs at constant pressure and the dissipation function is generally negligible. The only unknown of the problem is therefore the temperature $T - T_0$, and we take as a reduced temperature $\widetilde{T} = \dfrac{T - T_0}{T_p - T_0}$;

– for fully immersed flows of a liquid (section 4.4.2.4), we have seen that gravity no longer has a direct effect on the flow; we therefore take the driving pressure $p_g = p + \rho g z$ to be the only variable, gravity disappearing from the equations and from the boundary conditions: the Froude number is no longer a parameter.

4.6.1.3.4. Non-dimensional expression of results

The non-dimensional quantities of a problem are functions of non-dimensional coordinates and similarity parameters. They allow the calculation of dimensional results which take on a specific form. For example, the pressure p can be written:

$$p = \rho_0 U^2 \tilde{p}(\tilde{x}_i, \tilde{t}, \text{ Re, M, ...})$$

Note that in place of n dimensional data, we now have only n – 4 similarity parameters (Re, Pr, Ec, Fr, T_p/T_0). For a purely dynamic problem (no thermodynamic equation), we would have only n – 3 similarity parameters. These results, which can be obtained from a dimensional analysis (without writing the specific equations of the problem), constitute the Vaschy-Buckingham theorem. Note that the dimensional analysis does not allow us to see if certain similarity parameters (Froude number) can be eliminated.

In practice, in place of the preceding non-dimensional parameters, we use other quantities defined by custom. For example, we define the *local pressure coefficient* $\mathrm{C_p} = \dfrac{p}{\rho_0 U^2/2}$ which is equal to $\tilde{p}/2$. The *local viscous friction stress* at a solid boundary, equal to $\tau_p = \mu \left.\dfrac{\partial u}{\partial n}\right|_{n=0} = \dfrac{\mu U}{L} \left.\dfrac{\partial \tilde{u}}{\partial \tilde{n}}\right|_{\tilde{n}=0}$ can be expressed by means of the *friction coefficient* $\mathrm{C_f} = \dfrac{\tau_p}{\rho_0 U^2/2}$, which can also be written:

$$C_f = \frac{2}{\mathrm{Re}} \left.\frac{\partial \tilde{u}}{\partial \tilde{n}}\right|_{\tilde{n}=0}$$

The *local thermal flux density* at the wall $q_{Tp} = -\lambda \left.\dfrac{\partial T}{\partial n}\right|_{n=0}$ can be written in non-dimensional form of the local *Nusselt number Nu*:

$$\mathrm{Nu} = -\left.\frac{\partial \tilde{T}}{\partial \tilde{n}}\right|_{\tilde{n}=0} = \frac{q_{Tp} L}{\lambda \left(T_p - T_0\right)}$$

The *global quantities* (components of the force exerted on an obstacle, thermal flux, etc.) can be obtained by integration of the local quantities over the wall surface. We here define the non-dimensional coefficients, which can be immediately deduced from the local coefficients. For example, the force X exerted on an obstacle in the x-direction can be calculated by integration over the surface S of the obstacle:

$$X = \int_S \left(\mu \left. \frac{\partial u}{\partial n} \right|_{n=0} \cos \vartheta - p \sin \theta \right) ds = \rho U^2 L^2 \int_{\widetilde{S}} \left(\frac{2}{\mathrm{Re}} \left. \frac{\partial \widetilde{u}}{\partial \widetilde{n}} \right|_{\widetilde{n}=0} \cos \vartheta - \widetilde{p} \sin \theta \right) ds$$

Instead of the non-dimensional integral in the above formula, we can use the *drag coefficient* $\mathrm{C_x}$ in the x-direction, often defined by means of the surface $\mathrm{S_c}$, projected frontal area (projection of S on to the plane perpendicular to the x-axis):

$$X = \mathrm{C_x} \frac{\rho_0 U^2}{2} S_c$$

4.6.1.3.5. Case of unsteady flows

The procedure is identical to that outlined above; the temporal conditions introduce further similarity parameters, for example the Strouhal number $\mathrm{St} = NL/U$, where N is a characteristic frequency which appears among the data of the problem.

However, we will see in section 6.6 that a problem posed with steady boundary conditions will often have an unsteady or turbulent solution: the non-dimensional solution is therefore a function of the variable $\widetilde{t}$. For example, if the wake of the cylinder discussed above comprises a preferred frequency N, the non-dimensional frequency $\mathrm{St} = NL/U$ will be a function of the similarity parameters St(Re,M,...).

4.6.1.3.6. Validity of the similarity approach and comparison with experiments

The posing of a problem in non-dimensional form constitutes a *mathematical model of the phenomena* studied, which is based on a choice of equations supposed to be representative. The experimental verification of the model involves ensuring that the non-dimensional results are correct, in other words that the non-dimensional coefficients are only functions of similarity parameters defined with the model. For example, in the fully immersed flow of an incompressible Newtonian fluid, the results should depend on the Reynolds number alone. The problem has been badly posed if this is not the case.

We can take an historic example: at the beginning of the 20th century, L. Prandtl and G. Eiffel independently measured the drag of a sphere in a wind tunnel. They found different drag coefficients for the same Reynolds number. This raised questions over the validity of the similarity approach. An understanding of this phenomenon was obtained 20 years later: one of the experiments comprised a laminar flow in the zone of the boundary layer where the wake was generated, whereas the turbulence had already been triggered in the other experiment. The solution of the Navier-Stokes equations was not the same in the two experiments because of instabilities generated by the residual present turbulence that is different in the two wind tunnels.

Questioning the similarity methodology amounts to denying the validity of the basic laws of classical physics, which seems, at the very least, both erroneous and presumptuous. It is clearly the specific model used which must be questioned when agreement is not obtained by means of the non-dimensional representation of the results (for example, the friction coefficient as a function of the Reynolds number).

4.6.1.3.7. Similarity in diffusion problems

We will limit ourselves to the case of weak concentrations [4.60] by considering the steady flow of a fluid of initial concentration c_0 which reacts with a reactive wall according to a law of the form [2.70]. The concentration c satisfies equation:

$$\frac{\partial}{\partial x_i}\left(c\,u_i\right) = D\Delta c$$

with the condition at the wall P: $-D\left.\frac{\partial c}{\partial n}\right|_{n=0} = k{c_P}^m$.

This can be written in non-dimensional form by letting $\tilde{c} = c/c_0$. The non-dimensional problem can be written:

$$\frac{\partial}{\partial \tilde{x}_i}\left(\tilde{c}\,\tilde{u}_i\right) = \frac{D}{UL}\tilde{\Delta}\tilde{c}$$

with the absorption condition for a constituent, due to heterogeneous reaction at the wall:

$$-\left.\frac{\partial \tilde{c}}{\partial \tilde{n}}\right|_{\tilde{n}=0} = -\frac{k c_0^{m-1} L}{D}{\tilde{c}_P}^{m-1} \qquad [4.80]$$

The similarity parameters of the diffusion problem are:

– the Péclet *diffusion number*: $Pe_D = \dfrac{UL}{D} = \text{Re.Sc}$;

– the *Schmidt number:* $\text{Sc} = \dfrac{\nu}{D}$ (ν kinematic viscosity);

– the *Damköhler number*: $\dfrac{kc_0^{m-1}L}{D}$ which characterizes the speed of the chemical reaction.

Condition [4.80] shows that if the number $\dfrac{kc_0^{m-1}L}{D}$ is large compared to 1, the wall concentration is zero, and the flux of the corresponding constituent is limited by diffusion. If, on the other hand, we have $\dfrac{kc_0^{m-1}L}{D} << 1$, we have $c_P = c_0$ and the reaction speed is equal to $kc_0{}^m$: diffusion is no longer important.

Fundamentals of Fluid Mechanics and Transport Phenomena
Jean-Laurent Peube

Chapter 5

Transport and Propagation

The objective of this chapter is to present the general properties of the equations which describe the flow of matter. The substantial derivative describes the physical idea of the properties associated with matter which is in motion. It results in a particular structure of the equations of fluid mechanics and the properties associated with the displacement of material quantities. This leads to a specific means of posing and numerically solving fluid mechanics problems. The corresponding mathematical techniques which underlie the physical concepts are well known, but unfortunately they are not widely taught; we will here recall them without providing detailed demonstrations.

5.1. General considerations

5.1.1. *Differential equations*

We will first quickly recall the general properties of a differential equation which is satisfied by *a scalar function*:

– *the order n of the equation*: the greater the maximum order n of the derivatives, the greater the complexity of the solution (n time constants correspond to n independent solutions for a linear differential equation with constant coefficients). We know that, in general, the solutions of a differential equation form a family of functions which depend on the n parameters. This means that the number of boundary conditions which must be specified in order to determine a unique solution is generally equal to the order n;

– *the general structure of the differential equation*: the solutions of a *linear differential equation* belong to a vector space, which means that we can express a solution by means of suitable linear combinations of other solutions. The dimension of the vector space of the solutions is equal to the order of the differential equation;

– *the properties of the coefficients* of the differential equation, and in particular their eventual singular properties: a coefficient which cancels itself out at some point very often leads to particular properties of the solutions at that point. Such is the case for solutions to problems of revolution about an axis for the point which corresponds to the axis (generally denoted r = 0). Other particular points can play an important role, as we will see with regard to stability problems;

– *the nature of the boundary conditions imposed* determines the kind of differential problem which is posed:

- a *Cauchy problem* is determined by *n boundary conditions given at a point* for the function and its *n* – 1 successive derivatives; such problems generally have a unique solution[1] in the vicinity of this point. This kind of problem is often encountered in mechanics for initial values of a motion. Regular behavior is only ensured in the vicinity of the point, and the regular behavior can eventually extend to the entire domain considered; however, in numerous cases we encounter accidents in the behavior of the system far from the said point (divergence of the solution, instabilities and random behavior, etc.),

- the *n boundary conditions* required may be specified at the two points of the extremities of an interval. This case is common in physics, for field problems of physical quantities (electromagnetic fields, velocity and displacement fields, etc.) for which the solid boundaries impose particular conditions. The existence and uniqueness of the mathematical solution can be obtained if the problems have been well posed in physical terms for the entire domain.[2]

In the preceding particular case where we impose zero conditions at two points, the differential equation generally has a zero solution. However, there may exist particular coefficient values for which a non-zero solution exists, and which therefore depends on some parameter. These particular values correspond to the eigenvalues of the differential operator. We will treat these problems in more detail in Appendix 4.

The case of a system of coupled differential equations can amount to the study of a higher order differential equation for one of the unknown functions after elimination of the other unknown functions. For example, the system of two second

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[1] I.e. when we have mathematical properties of regularity, for the rest more or less satisfied in application conditions of physics and mechanics.

[2] We wish to say that any approximation with notable physical consequences has been made when writing the equations for the problem.

order equations with two unknown functions $y(x)$ and $z(x)$ can immediately be transformed:

$$y''= f(z',z,y',y,x) \qquad z''= g(z',z,y',y,x)$$

Deriving z'' leads to the appearance in the expression for z''' the derivatives y'' and z'' which we replace with their earlier expression f and g, giving for z''' an expression of the form $h(z',z,y',y,x)$. Doing the same for z'''', we get a similar expression $m(z',z,y',y,x)$. The elimination of y, y' from the three equations:

$$z''= g(z',z,y',y,x), \qquad z'''= h(z',z,y',y,x), \qquad z''''= m(z',z,y',y,x)$$

gives a fourth order differential equation for the function *z(x)*.

Conversely the differential equation $z'''' = f(z''', z'', z, x)$ for the unknown function $z(x)$ is immediately transformed into a system of four first-order differential equations by letting $z'''= u, z''= v,\ z'= w$:

$$u'= f(u,v,w,z,x); \qquad v'= u; \qquad w'= v; \qquad z'.= w$$

More generally, the system of first-order differential equations:

$$x_i'(t) = f_i(x_j,t) \qquad (i,j = 1,\dots,n)$$

can be transformed into a differential equation of order n for one variable, for example x_1; in effect, by differentiating $x_1'(t)$, the two other derivatives $x_i'(t)$ appear, which we replace as a function of the quantities $x_\text{i}(t)$; we thus obtain:

$$x_1''(t) = x_j'(t)\frac{\partial f_i}{\partial x_j}(x_j,t) = g_2(x_j,t)$$

Doing the same for the subsequent derivatives up to order n, we obtain n equations which express the first n derivatives of the function x_1 as a function of all the functions x_i:

$$x_1^{(i)}(t) = g_i(x_j,t) \qquad (i,j = 1,2,\dots,n)$$

Eliminating the $n - 1$ functions $x_i(t)$ (i = 2, 3,…, n) between the n preceding equations leads to the n^{th} order differential equation satisfied by the function $x_1(t)$.

$$x_1^{(n)}(t) = F\left(x_1, x_1^{'}, x_1^{''}, .., x_1^{(n-1)}, t\right):$$

The initial conditions $x_1^{(k)}(0)\ (k = 0,...,n-1)$ can be obtained in the same way as a function of the initial values $x_i(0)$.

The preceding procedure is only appropriate if the equations of the system are suitably coupled. Otherwise, new variables X_i must be chosen, such that the equations are coupled. We will leave it to the reader to verify this for the elementary linear system (let: $X_1 = x_1. + x_2$ and $X_2 .= x_1 - x_2$):

$$\begin{pmatrix} x_1^{'} \\ x_2^{'} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

5.1.2. *The Cauchy problem for differential equations*

Consider the differential equation:

$$f'(x) = g(x, f) \qquad [5.1]$$

Solving the Cauchy problem involves calculating the value of the solution in the vicinity of the value x_0 of the variable x, at which the value $f_0 = f(x_0)$ is given. It is equivalent to saying that given the point (x_0, f_0) of the plane (x,f), we must find the variations dx and df which it is possible to calculate from equation [5.1]:

$$f'(x)dx = df \qquad [5.2]$$

It is clear that the system of equations [5.1] and [5.2] with the unknown $f'(x)$ should be of rank 1, hence the condition:

$$\begin{vmatrix} 1 & g \\ dx & df \end{vmatrix} = 0$$

The curve of the plane (x, f) on which the value of f extends (or is transmitted) is therefore given by the relation:

$$\frac{dx}{1} = \frac{df}{g(x, f)}$$

The preceding presentation may seem unnecessarily formal, but its interest is that it can be extended to the study of partial differential equations.

5.2. First order quasi-linear partial differential equations

5.2.1. *Introduction*

In expressing the laws of physics in a local form, we obtain *partial differential equations*, the functions representing the physical quantities depending on space and time variables. These equations have properties which are quite different from differential equations, because their general solution no longer comprises a family of functions with a finite number of parameters, but a family of functions which we can choose arbitrarily. Let us take the simple example of the equation:

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

whose solution depends on the two arbitrary functions φ and ψ:

$$f(x, y) = \varphi(x) + \psi(y)$$

This solution is only a general form of the function dependence with respect to the variables. The choice of these functions will depend on the boundary conditions which are specified.

As with the differential equation, we can define a Cauchy problem: for example, for a partial differential equation with two variables (x, y), the unknown function f is given on the curve C_0 of the plane (x,y) and we seek to evaluate this function in the vicinity of C_0.

In what follows, we will limit ourselves to a relatively elementary approach which only consists in verifying the existence of a series expansion in the place of a solution. We thus identify the essential properties of the solutions and the basic concepts which govern the general physics of the phenomena studied.

We will not demonstrate the uniqueness of the solution, as this requires advanced mathematical knowledge and precise assumptions which do not necessarily have physical reality, for example:

– do analytical functions (indefinitely differentiable) exist?;

– what is the nature of the "relation" between distributions and physical problems?;

– how can we verify the physical reality of a Lipschitz application (section 1.1.1.4)?

DEFINITION – A *quasi-linear partial differential equation* is an equation which is linear with respect to the partial derivatives of highest order, whose coefficients are functions of variables and derivatives of lower order.

The equations which we will encounter in mechanics and energy of continuous media will be quasi-linear.

5.2.2. *Geometric interpretation of the solutions*

We have previously seen that the substantial derivative in the equations of fluid mechanics describes transport of a quantity associated with matter. We will recover this interpretation of the substantial derivative by purely mathematical considerations.

Consider the partial differential equation which represents transport of the quantity F, in which the right-hand side is equal to *g(f,x,y,t)*:

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}=g(f,x,y,t) \qquad [5.3]$$

The coefficients u and v are given functions of (f, x, y, t). Equation [5.3] describes the balance of the scalar quantity F (mass, entropy, number of moles of a chemical species, etc.) associated with the corresponding specific quantity f (section 4.2.1.2.1). The function g can be considered a source of quantity F, as it only depends on the coordinates of the material particle and the associated value of the function f. There is however *no interaction with the neighboring particles*, and equation [5.3] does not contain any diffusive flux term for the quantity F. Balance equation [4.3] of the volume quantity $\bar{f}$ can thus be expressed in the form [5.3].

Let us ignore for the moment the physical interpretation of equation [5.3], of which we will here give a geometric interpretation. In the 4D space (t, x, y, z), consider the surface S described by the equation:

$$\Phi(t,x,y,z)=z-f\left(x,y,t\right)=0$$

We know that the vector $\overrightarrow{grad}\Phi$ of components $\left(-f_t,-f_x,-f_y,1\right)$ is normal to S. The equation of the plane tangent to the surface S at the point (t, x, y, z) can be written (designating the usual coordinates (T, X, Y, Z)):

$$-\frac{\partial f}{\partial t}(T-t)-\frac{\partial f}{\partial x}(X-x)-\frac{\partial f}{\partial y}(Y-y)+(Z-z)=0 \qquad [5.4]$$

Comparing equations [5.3] and [5.4] shows that *the vector* (1*, u, v, g*) *is located in the plane tangent to the surface S*. The solutions to the partial differential equation are thus represented by *surfaces tangential to the vector field* $(1, u, v, g)$ *in the said 4D space*. Now the curves tangent to the vector field (1, u, v, g) can be obtained by integration of the system of three differential equations:

$$dt=\frac{dx}{u}=\frac{dy}{v}=\frac{df}{g} \qquad [5.5]$$

The curves defined by system [5.5] are called *characteristic curves*. We note immediately that the first two differential equations define the trajectories of fluid particles (section 3.3.2). The third differential equation allows the unknown function f to be calculated on these curves. We thus obtain a Lagrangian (substantial) representation of the balance of the quantity f associated with the fluid particles.

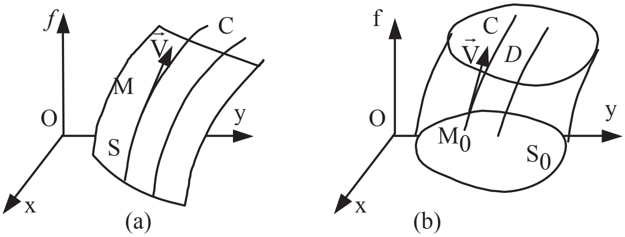


Figure 5.1. *C: characteristic curve; S: solution surface; S_0: surface with initial conditions (Cauchy problem)*

Note that the solution of equation [5.3] amounted to the solution of a system of *differential equations with initial conditions.* In the preceding 4D space, any surface S constituted of characteristic curves (characteristic surface) is tangent to the vector field $\vec{V}$(1, u, v, g); it is thus a solution of the partial differential equation (Figure 5.1a). If a trajectory point *C* belongs to a surface *S*, then the curve *C* lies entirely on *S*.

In summary, *any quasi-linear first order partial differential equation can be interpreted as a transport equation for a quantity on the trajectories associated with that equation.*

This property allows the definition of the ideas of *input* (the surface S_0 on which the initial conditions are given) and *output* (any surface derived from S_0 by "translation" following the characteristic curves). This introduces a *dissymmetry between the input and output*. Depending on the physical context of a problem, the ideas of input and output may correspond to upstream and downstream, or to initial and final conditions.

5.2.3. *Comments*

1) The preceding reasoning can be applied to any number of variables.

2) The preceding mathematical interpretation actually amounts to writing the balance of the quantity *F* in Lagrangian variables. For any *material particle M* which is displaced on this solution surface S of the 4D space, the associated volume quantity satisfies:

$$\frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy - df = 0 \quad \text{with} : dx = udt, \quad dy = vdt. \qquad [5.6]$$

The balance equation $\dfrac{df}{dt} = g$ for each fluid particle describes the compatibility between relation [5.6] and partial differential equation [5.3]. The preceding considerations show the *equivalence between the Lagrangian balance formulation in the form of differential equations* [5.5], and *the balance equation in Eulerian variables, expressed in the form of partial differential equation* [5.3].

3) In the *presence of diffusion of the quantity F*, the right-hand side of the balance equations is not of the form $g(f, x, y, t)$, rather it contains the second order transverse derivatives with respect to the characteristic curves: these derivatives express an interaction between neighboring characteristic curves due to diffusion of the quantity *F*. The introduction of higher order partial derivatives modifies the properties found earlier. However, these partial derivatives are associated with a coefficient which is often very small, and which leads to a reduction in the order of the equation, except in singular zones (sections 6.4.3 and 6.5.3): nearly everywhere, F is transported on the trajectories, with the source *g* taken into account.

5.2.4. *The Cauchy problem for partial differential equations*

We will reconsider the preceding problem in the mathematical form, which consists of solving the Cauchy problem where the value of the unknown function is given on the surface S_0, and where we seek to calculate its value in the neighborhood of S_0 by means of the partial differential equation. This problem can be solved by means of geometric interpretation (Figure 5.1b), provided that the surface S_0 does not comprise characteristic curves (characteristic surface).

In effect, we have seen that the calculation of the function *f* can be performed by integration over the characteristic curves (trajectories): in order for this calculation to be possible, these characteristic curves must obviously cross the surface S_0 on which the initial values of the function *f* are given.

Note that the solution of equation [5.3] is only defined in the domain *D* of the space containing the characteristic curves which cross the surface S_0: *D* is the *influence domain of the initial conditions given* on S_0. It is rigorously delimited. This property is specific to all quasi-linear first order partial differential equations.

In summary, the Cauchy problem for a first order partial differential equation was reduced to an ensemble of independent Cauchy problems for a system of ordinary differential equations on each of the characteristic curves (trajectories). In a transverse direction with respect to these curves, the partial differential equation gives no information regarding the function *f*: *the solution on characteristic curve C has no influence on its neighboring points*, except on *C*. The properties of the solution in the neighborhood of a characteristic curve are uniquely fixed by the initial conditions corresponding to this neighborhood. The solution space is thus found to be as a bundle of fibers.

We will later return to the Cauchy problem (section 5.3.5.1) in a more local manner, and one which is closer to the practical methods used for numerical calculations.

5.3. Systems of first order partial differential equations

5.3.1. *The Cauchy problem for n unknowns and two variables*

Consider now a quasi-linear system of first order partial differential equations with *n* unknown functions and which we will here limit to two independent variables (*x,t*). The simplest equations of fluid mechanics involve at least three thermodynamic quantities (for example, the density ρ, a velocity component *u* and the pressure *p*) which we will represent by the vector *F* with three components (f_1, f_2, f_3). Consider the system of three equations described in vector form (the case of *n* unknown functions can be treated in an identical manner):

$$A\frac{\partial F}{\partial t}+B\frac{\partial F}{\partial x}=G \qquad \text{with: } F=\begin{pmatrix} f_1\\ f_2\\ f_3 \end{pmatrix}, \quad G=\begin{pmatrix} g_1\\ g_2\\ g_3 \end{pmatrix} \qquad [5.7]$$

We will adopt the position of the Cauchy problem: we assume that the value of F is known on a curve C_0 of equation $\varphi(x,t)=0$ in the plane (x,t), and that from a point (x_0, t_0) of this curve, we seek to calculate the value of the function F in its neighborhood: this is possible if we know the value of the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial t}$ at the point (x_0, t_0). The calculation of these is possible using data on the curve C_0 and equation [5.7]. Designating the elemental arc $(\delta t, \delta x)$ of the curve C_0 and the growth δF of F on this arc, we have:

$$\frac{\partial F}{\partial t}\delta t+\frac{\partial F}{\partial x}\delta x=\delta F \qquad [5.8]$$

Eliminating $\frac{\partial F}{\partial t}$ between [5.7] and [5.8] we have the system:

$$\left(B-\frac{\delta x}{\delta t}A\right)\frac{\partial F}{\partial x}=G-A\frac{\delta F}{\delta t} \qquad [5.9]$$

If the determinant $Q=\left|B-\frac{\delta x}{\delta t}A\right|$ of the system [5.9] is non-zero, the unknown $\frac{\partial F}{\partial x}$ has a unique value. We can therefore obtain the value of F in the neighborhood of the considered point and the Cauchy problem has a unique solution. The preceding determinant is called the *characteristic determinant* of system [5.7]. Its value depends on the ratio $\delta x/\delta t=\lambda$, in other words on the choice of the curve C_0.

Suppose now that the determinant Q of the system [5.9] is zero, $\lambda=\delta x/\delta t$ is a root of characteristic equation [5.10]:

$$Q=\left|B-\frac{\delta x}{\delta t}A\right|=0 \qquad [5.10]$$

The corresponding curve C_0 is the *characteristic curve* associated with the root λ. When all the roots of the equation $Q = 0$ are real, system [5.7] is called *hyperbolic* (or totally hyperbolic).

When the characteristic determinant is zero, the system rank [5.9] has diminished by one unit and a non-zero solution exists for system [5.9], if we have a compatibility relation between the components of the vector $G - A\dfrac{\delta F}{\delta t}$ of the right-hand side.

This relationship can be obtained, for example, from the non-zero left solutions L (eigenfunctions) of the characteristic equation:

$$L(B - \lambda A) = 0 \qquad [5.11]$$

Multiplying on the left side [5.9] by L and taking account of [5.11], we then obtain the relationship sought between the components of δF on the characteristic curve concerned:

$$L\left(G - A\frac{\delta F}{\delta t}\right) = 0 \qquad [5.12]$$

In summary, characteristic curves of the plane (x,t), on which the characteristic determinant cancels out, are such that:

– the given values of F do not allow the Cauchy problem to be solved (calculation of F in their neighborhood);
– the unknown vector function F satisfies *particular differential relations*.

As in the case of solutions to quasi-linear partial differential equations, the last property can be used to study the solutions. We will come back to this point a little later.

NOTE – The relationship between the components of δF is of “Lagrangian” type in the sense that the value of this variation is calculated for corresponding values of δt and of $\delta x = \lambda\, d\, t$: the evaluation point of δF is displaced at “velocity” λ.

We will examine two particular cases which show the physical interest of these results.

5.3.2. *Applications in fluid mechanics*

5.3.2.1. *Unsteady 1D flow of a compressible inviscid fluid*

The equations of a compressible perfect fluid in unsteady inviscid flow can be written ([4.6], [4.22] and [4.55]):

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0\\
&\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial x}+\frac{\partial p}{\partial x}=0\\
&\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+\rho c^2\frac{\partial u}{\partial x}=0
\end{aligned} \qquad [5.13]$$

Letting: $F=\begin{pmatrix}\rho\\ u\\ p\end{pmatrix}$; $A=\overset{=}{I}$; $B=\begin{pmatrix}u & \rho & 0\\ 0 & u & 1/\rho\\ 0 & \rho c^2 & u\end{pmatrix}$; $G=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$, they can be written in vector form [5.7]:

$$A\frac{\partial F}{\partial t}+B\frac{\partial F}{\partial x}=G$$

Equation [5.9] can thus be written:

$$(B-\lambda\,\overset{=}{I})\frac{\partial F}{\partial x}=-\frac{\delta F}{\delta t} \qquad [5.14]$$

We can thus derive characteristic equation [5.10]:

$$Q=\begin{vmatrix}u-\lambda & \rho & 0\\ 0 & u-\lambda & 1/\rho\\ 0 & \rho c^2 & u-\lambda\end{vmatrix}=(u-\lambda)(u-\lambda+c)(u-\lambda-c)=0 \qquad [5.15]$$

The three roots $(u, \pm c+u)$ of equation [5.15] correspond, respectively, to a *displacement of the fluid matter on the trajectory* $(\delta x=u\delta t)$, and to *propagation at the speed of sound c with respect to the matter* which moves with velocity u $(\delta x=(\pm c+u)\delta t)$.

The system for the left eigenfunctions $L = \begin{pmatrix} L_1 & L_2 & L_3 \end{pmatrix}$ of [5.11] can here be written:

$$L\left(B - \lambda \bar{\bar{I}}\right) = \begin{pmatrix} L_1(u-\lambda) \\ L_1\rho + L_2(u-\lambda) + L_3\rho c^2 \\ L_2/\rho + L_3(u-\lambda) \end{pmatrix}^t = 0 \qquad [5.16]$$

The vector L corresponding to each of the eigenvalues can be easily calculated. Relation [5.12], which provides the relation between the variations δF of the components of the vector F, can thus be written:

$$L.\delta F = 0 \qquad [5.17]$$

1) For *the root* $\lambda = u$, we obtain: $L = \begin{pmatrix} -c^2 & 0 & 1 \end{pmatrix}$ and:

$$L\delta F = \begin{pmatrix} -c^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta u \\ \delta p \end{pmatrix} = \delta p - c^2 \delta\rho = 0 \qquad [5.18]$$

Here we recover the relationship which characterizes isentropic transformations (formula [1.27]): *the entropy remains constant during matter displacement*. The reader should note that the identification of this simple property required some mathematical developments.

2) For the eigenfunctions $\lambda = \pm c + u$ solution of system [5.16] gives:

$$L\left(B - \lambda \bar{\bar{I}}\right) = \begin{pmatrix} \mp L_1 c \\ L_1\rho \mp L_2 c + L_3\rho c^2 \\ L_2/\rho \mp L_3 c \end{pmatrix}^t = 0 \quad \text{i.e. } L = \begin{pmatrix} 0 & \pm\rho c & 1 \end{pmatrix}$$

We thus obtain:

$$L \begin{pmatrix} \delta\rho \\ \delta u \\ \delta p \end{pmatrix} = \begin{pmatrix} 0 & \pm\rho c & 1 \end{pmatrix} \begin{pmatrix} \delta\rho \\ \delta u \\ \delta p \end{pmatrix} = \delta p \pm \rho c \delta u = 0 \qquad [5.19]$$

These relationships correspond to acoustic waves which propagate in two directions at the speed of sound c with respect to the fluid matter which moves with velocity u.

NOTES –

1) Relation [5.16] for each of the roots λ can be written *in Lagrangian variables*. By means of linear combinations using initial system [5.13] of the equations of motion describing the flow of a compressible perfect fluid, we can easily obtain the system of equations:

$$\begin{aligned}
&\frac{dp}{dt} - c^2\frac{d\rho}{dt} = 0 && \frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} \\
&\frac{Dp}{Dt} + \rho c\frac{Du}{Dt} = 0 && \text{with : } \frac{D}{Dt} = \frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x} = \frac{d}{dt} + c\frac{\partial}{\partial x} \\
&\frac{\widetilde{D}p}{\widetilde{D}t} - \rho c\frac{\widetilde{D}u}{\widetilde{D}t} = 0 && \text{with : } \frac{\widetilde{D}}{\widetilde{D}t} = \frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x} = \frac{d}{dt} - c\frac{\partial}{\partial x}
\end{aligned} \qquad [5.20]$$

This system is the expression in Eulerian variables of relations [5.18] and [5.19]. We furthermore obtain the directions of characteristic curves from the preceding expression of derivatives in relations [5.20] [YIH 77, p.211].

2) The case of an incompressible fluid $\partial u/\partial x = 0$ can be obtained by letting the speed of sound tend to infinity. The eigenvalues of the acoustic propagation disappear, and only the eigenvalue $\delta x/\delta t = u$ exists with the characteristic variable $\delta\rho/\delta t = 0$. The reader can verify that this result can be obtained from [5.13].

5.3.2.2. *Steady 2D flow of an incompressible perfect fluid*

The equations for this kind of flow can be written ([4.6] and [4.22]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0;\quad \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = 0;\quad \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = 0 \qquad [5.21]$$

and can be put in the form [5.7] ($A\frac{\partial F}{\partial x} + B\frac{\partial F}{\partial y} = G$) with:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \rho u & 0 & 1 \\ 0 & \rho u & 0 \end{pmatrix};\quad F = \begin{pmatrix} u \\ v \\ p \end{pmatrix};\quad B = \begin{pmatrix} 0 & 1 & 0 \\ \rho v & 0 & 0 \\ 0 & \rho v & 1 \end{pmatrix};\quad G = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad [5.22]$$

Following the steps outlined in section 5.3.1 leads to the vector equation:

$$\left(B - \frac{\delta y}{\delta x} A\right)\frac{\partial F}{\partial x} = -A\frac{\delta F}{\delta x} \qquad [5.23]$$

From this we can calculate the characteristic determinant $Q = \left|B\delta x - A\delta y\right|$ and the roots $\lambda = \delta y/\delta x$. Equation [5.10] can here be written as:

$$Q = \begin{vmatrix} -\lambda & 1 & 0 \\ \rho(v-\lambda u) & 0 & -\lambda \\ 0 & \rho(v-\lambda u) & 1 \end{vmatrix} = \rho(u\lambda - v)(\lambda^2+1) = 0 \qquad [5.24]$$

Equation [5.24] only has one real root $\lambda = \delta y/\delta x = v/u$ which corresponds to the differential equation of the trajectories. For this root, system [5.11] can be written:

$$L\left(B - A\frac{v}{u}\right) = \begin{pmatrix} L_1 & L_2 & L_3 \end{pmatrix}\begin{pmatrix} -v/u & 1 & 0 \\ 0 & 0 & -v/u \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -L_1 v/u \\ L_1 \\ -L_2 v/u + L_3 \end{pmatrix}^t = 0$$

hence: $L = \begin{pmatrix} L_1 & L_2 & L_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & v/u \end{pmatrix}$.

Substituting into relation [5.12] gives:

$$LA\delta F = \begin{pmatrix} 0 & 1 & v/u \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \rho u & 0 & 1 \\ 0 & \rho u & 0 \end{pmatrix}\begin{pmatrix} \delta u \\ \delta v \\ \delta p \end{pmatrix} = \rho u\delta u + \rho v\delta v + \delta p = 0 \qquad [5.25]$$

The characteristic curves $\delta y/\delta x = v/u$ are the trajectories, and the variation of the quantity $\rho u\delta u + \rho v\delta v + \delta p$ is zero on these. We can recognize Bernoulli's first theorem for a perfect incompressible fluid (see section 4.3.2.3.1, relation [4.30]) which was obtained previously.

5.3.2.3. *Steady 2D flow of a inviscid compressible fluid*

The equations for a steady 2D plane of perfect compressible fluid can be written ([4.6], [4.22] and [4.55]):

$$u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+\rho\frac{\partial u}{\partial x}+\rho\frac{\partial v}{\partial y}=0; \qquad \rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}=0;$$
$$\rho u\frac{\partial v}{\partial x}+\rho v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}=0. \qquad u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+\rho c^2\frac{\partial u}{\partial x}+\rho c^2\frac{\partial v}{\partial y}=0 \tag{5.26}$$

or, in matrix form [5.7]:

$$\left(A\frac{\partial}{\partial x}+B\frac{\partial}{\partial y}\right)F=0$$

$$\text{with: } F=\begin{pmatrix}\rho\\ u\\ v\\ p\end{pmatrix}; \quad A=\begin{pmatrix}u & \rho & 0 & 0\\ 0 & \rho u & 0 & 1\\ 0 & 0 & \rho u & 0\\ 0 & \rho c^2 & 0 & u\end{pmatrix}; \quad B=\begin{pmatrix}v & 0 & \rho & 0\\ 0 & \rho v & 0 & 0\\ 0 & 0 & \rho v & 1\\ 0 & 0 & \rho c^2 & v\end{pmatrix}.$$

Characteristic equation [5.10] can be written:

$$Q=\left|B-\frac{\delta y}{\delta x}A\right|=\begin{vmatrix}v-\lambda u & -\rho\lambda & \rho & 0\\ 0 & \rho(v-\lambda u) & 0 & -\lambda\\ 0 & 0 & \rho(v-\lambda u) & 1\\ 0 & -\lambda\rho c^2 & \rho c^2 & v-\lambda u\end{vmatrix}=0$$

or:

$$\rho^2\,(v-\lambda\,u)^2\left[\lambda^2\,\left(u^2-c^2\right)-2\lambda\,uv+v^2-c^2\right]=0 \tag{5.27}$$

It has the same real root $\lambda = v/u$ as for the incompressible fluid, but here it is a double root.

The two other roots satisfy a second degree equation whose discriminant can be written:

$$\Delta' = c^2\left[\left(u^2+v^2\right)-c^2\right]\geq 0 \quad \text{if} \quad V^2=u^2+v^2\geq c^2 \tag{5.28}$$

These are *real only if the flow is supersonic* ($V > c$). If the *flow is subsonic, the roots and the characteristic curves are imaginary*, as in the case of the incompressible fluid. The values of the slopes of the characteristic curves are:

$$\lambda = \frac{\delta y}{\delta x} = \frac{uv \pm \sqrt{c^2[(u^2 + v^2) - c^2]}}{u^2 - c^2} \qquad [5.29]$$

The *velocity* $\vec{V}$ *is the bisector of the characteristic curves;* in effect, taking the axis Ox parallel to the velocity $\vec{V}$ (i.e. $v = 0$), slopes [5.29] of the characteristic curves in the plane (*x,y*) are opposed:

$$\lambda = \frac{\delta y}{\delta x} = \pm\sqrt{\frac{c^2}{u^2 - c^2}} = \pm\frac{1}{\sqrt{M^2 - 1}}$$

where $M = V/c$ designates the *local Mach number* defined using the local sound speed c.

The reader can easily verify that the characteristic curves lie at an angle θ with respect to the velocity vector (or the trajectory) defined by: $\sin\theta = 1/M$ (we have $\lambda = \tan\theta$); these curves are thus called *Mach lines*.

Identification of the relationship between the components of the differential δF on each of the characteristic curves can be effected as previously by finding a vector L which is a left solution of equation [5.11] and by substituting it into equation [5.12]. We will here only show the calculation for the value $\lambda = v/u$:

$$L\left(B - A\frac{v}{u}\right) = \begin{pmatrix} L_1 & L_2 & L_3 & L_4 \end{pmatrix}\begin{pmatrix} 0 & -\rho\frac{v}{u} & \rho & 0 \\ 0 & 0 & 0 & -\frac{v}{u} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{v}{u}\rho c^2 & \rho c^2 & 0 \end{pmatrix} = 0$$

or:

$$-\rho\frac{v}{u}L_1 - \frac{v}{u}L_4\,\rho c^2 = 0; \quad \rho L_1 + L_4\,\rho c^2 = 0; \quad -L_2\frac{v}{u} + L_3 = 0.$$

From this we can calculate L_3 and L_4 as a function of the two arbitrary values of L_1 and L_2:

$$L_1 + L_4\ c^2 = 0 \quad -L_2\frac{v}{u} + L_3 = 0$$

Substituting into condition [5.12] gives:

$$L_1 u\left(\delta\rho - \frac{\delta p}{\rho c^2}\right) + L_2\left(\rho u\,\delta u + \rho v\delta v + \delta p\right) = 0$$

As this equation is satisfied regardless of the values of L_1 and L_2, we have the two relations:

$$\left(\delta\rho - \frac{\delta p}{\rho c^2}\right) = 0; \qquad \rho u\,\delta u + \rho v\delta v + \delta p = 0$$

which describe the Lagrangian conservation of entropy and mechanical energy on the trajectories (Bernoulli's first theorem).

5.3.3. *Cauchy problem with n unknowns and p variables*

Now consider an n dimensional vector function F with components f_i $\left(i,= 1,..., n\right)$ functions of p variables x_j $\left(j = 1,..., p\right)$. The function F satisfies the n first order quasi-linear partial differential equations:

$$A_{ij\ell}\frac{\partial f_\ell}{\partial x_j} - g_i = 0 \qquad \begin{matrix}(i, \ell = 1,..., n;\\ j = 1,...; p)\end{matrix} \qquad \left(A\frac{\partial F}{\partial x} - G = 0\right) \qquad [5.30]$$

We will now solve the Cauchy problem, in other words we will calculate the variation δF of the function F in the neighborhood of a point M of a hypersurface S_0 of dimension $p - 1$ on which the values of F are given. This results in our knowing $p - 1$ independent derivatives of the function F calculated following the surface directions. The derivative of the vector function F following a transverse direction S_0 is not given, but it can be calculated from system [5.30].

In order to individualize this direction, we will perform a change of coordinates $x_j = x_j\left(\xi_k\right) \quad \left(j, k = 1,..., p\right)$.

The Wronskian (functional determinant) $\left|\dfrac{\partial x_j}{\partial \xi_k}\right|$ is assumed non-zero such that we can also write:

$$\xi_k = \xi_k\left(x_j\right) \quad \left(j, k = 1,\ldots, p\right)$$

This gives the following relationships between the partial derivatives of F with respect to the two sets of variables:

$$\frac{\partial f_\ell}{\partial x_j} = \frac{\partial f_\ell}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j} \qquad \begin{matrix}(\ell = 1,\ldots, n;\\ j, k = 1,\ldots p)\end{matrix} \qquad \left(\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \xi}.\frac{\partial \xi}{\partial x}\right) \qquad [5.31]$$

System [5.30] can thus be written:

$$A_{ij\ell}\frac{\partial f_\ell}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j} - g_i = 0 \qquad \begin{matrix}i, \ell = 1,\ldots, n;\\ j, k = 1,\ldots p\end{matrix} \qquad \left(A\frac{\partial F}{\partial \xi}.\frac{\partial \xi}{\partial x} - G = 0\right) \qquad [5.32]$$

We will choose the coordinates ξ_k $(k = 2,3,\ldots, p)$ such that they describe the surface (curvilinear coordinates on S_0 (Figure 5.2)). The curve of coordinate ξ_1 crosses the surface S_0 whose equation is $\xi_1 = \xi_1\left(\xi_2,\ldots\xi_p\right) = 0$. The derivatives $\partial f_\ell / \partial \xi_k$ are known from the functions f_ℓ on the surface S_0.

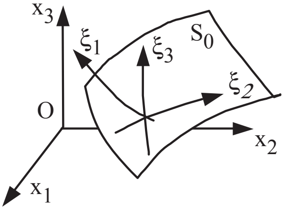


Figure 5.2. *Initial surface of the Cauchy problem*

All that remains is to determine the n unknown derivatives $\partial f_\ell / \partial \xi_1$ in the direction ξ_1 by means of system [5.30] in which all the other terms in $\partial f_\ell / \partial \xi_k$ $\left(k \neq 1\right)$ are now known; system [5.30] can be written:

$$A_{ij\ell}\frac{\partial \xi_1}{\partial x_j}\frac{\partial f_\ell}{\partial \xi_1} + \psi_i\left(\frac{\partial f_\ell}{\partial \xi_k},\ldots\right) - g_i = 0 \qquad \begin{matrix}i, \ell = 1,\ldots, n;\\ j = 1,\ldots p; k = 2,\ldots p\end{matrix}$$

$$\left(\text{or: } \tilde{A}\frac{\partial F}{\partial \xi_1} + \Psi - G = 0 \quad \text{with: } \tilde{A} = A_{ij\ell}\frac{\partial \xi_1}{\partial x_j}\right) \qquad [5.33]$$

where the function ψ_i is known from the derivatives which are already given $\partial f_\ell / \partial \xi_k \ (k \neq 1)$.

If the determinant of the matrix $\widetilde{A}$ is non-zero, system [5.33] with unknowns $\partial f_\ell / \partial \xi_1$ is of rank n and the n derivatives $\partial f_\ell / \partial \xi_1$ can be uniquely determined. The Cauchy problem thus possesses a unique solution.

If, on the other hand, we have $\left|\widetilde{A}\right| = 0$, system [5.33] is of rank $n-1$ and the determination of the n unknown derivatives $\partial f_\ell / \partial \xi_1$ is no longer possible.

Equation $\left|\widetilde{A}\right| = 0$ is a partial differential equation for the scalar function $\xi_1(x_j)$ of p variables:

$$\left|\widetilde{A}\right| = \left| A_{ij\ell} \frac{\partial \xi_1}{\partial x_j} \right| = 0 \qquad [5.34]$$

We thus have a problem such as that described in section 5.2.4. The vector $\partial \xi_1 / \partial x_j$ associated with the point M and satisfying equation [5.34] belongs to a hypersurface (the cone C with summit M) of dimension $p-1$ of the geometric space $(x_1, \ldots, x_p)$. At the point M, this vector is *the direction of the surface normal* S_0 *of the equation* $\xi_1 = 0$; the tangent plane is normal to it, and *the surface* S_0 *is surrounded by tangent planes whose normals belong to the cone C*. This surface S_0 is itself a cone of summit M to which the characteristic curves passing through M are tangent.

As before, in the case $\left|\widetilde{A}\right| = 0$, there exists a non-zero vector L such that $L_i A_{ij\ell} \dfrac{\partial \xi_1}{\partial x_j} = 0$. Equation [5.33] thus results in a linear relation between the partial derivatives $\partial f_i / \partial \xi_k \ (k \neq 1)$:

$$L_i \left[\Psi_i \left(\frac{\partial f_\ell}{\partial \xi_k}, \ldots \right) - g_i \right] = 0 \qquad (i, \ell = 1, \ldots, n;\ k = 2, \ldots p)$$

5.3.4. *Partial differential equations of order n*

The same reasoning can be applied for an n^{th} order quasi-linear partial differential equation with p variables, or for a system of such equations. For example, consider the quasi-linear second order equation whose coefficients and the right-hand side term g depend on the function f, on its first derivatives and on the variables x_i:

$$A_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} = g \quad (i,j = 1,....,p) \tag{5.35}$$

Suppose now that the *values of the function f and of its first derivatives* are given at all points M of a hypersurface S_0 of dimension $p - 1$. The values of the second derivatives *taken on this surface* are known. We can perform the change of coordinates described in section 5.3.3, such that the second derivatives of the function are known on the surface S_0 of equation $\xi_1 = \xi_1\left(\xi_2,...\xi_p\right) = 0$ in the reference frame $\left(\xi_i\right)$. The change of variables [5.31] allows the terms of equation [5.35] to be written in the form:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial \xi_k \partial \xi_m}\frac{\partial \xi_k}{\partial x_j}\frac{\partial \xi_m}{\partial x_i} + ...(i, j = 1,..., p)$$

The calculation is performed as previously, and the equation thus transformed should allow the calculation of the unknown value of $\dfrac{\partial^2 f}{\partial \xi_1^{\,2}}$ at the point where we seek a solution of the Cauchy problem. We obtain:

$$A_{ij}\frac{\partial \xi_1}{\partial x_i}\frac{\partial \xi_1}{\partial x_j}\frac{\partial^2 f}{\partial \xi_1^{\,2}} + \psi\left(f, \frac{\partial f}{\partial \xi_i}, \frac{\partial^2 f}{\partial \xi_i \partial \xi_k},..\right) - g = 0 \quad \begin{matrix}(i, j = 1,..., p) \\ (k = 2,..., p)\end{matrix} \tag{5.36}$$

where the function ψ is a function of f, of its first derivatives and of its second derivatives $\dfrac{\partial^2 f}{\partial \xi_i \partial \xi_k}$ $\left(k \neq 1\right)$, which are known from the initial data given at all points M of the hypersurface S_0. The coefficient of $\partial^2 f / \partial \xi_1^2$ must obviously be non-zero for the calculation to be possible. If it is zero, we find a direction ξ_1 corresponding to the characteristic surface $\xi_1\left(\xi_2,...\xi_p\right) = 0$. These characteristic directions $\alpha_i = \dfrac{\partial \xi_1}{\partial x_i}$ satisfy the equation:

$$A_{ij}\frac{\partial \xi_1}{\partial x_i}\frac{\partial \xi_1}{\partial x_j} = A_{ij}\alpha_i\alpha_j = 0; \quad (i, j = 1,..., p) \qquad [5.37]$$

The left-hand side of equation [5.37] is a quadratic form which represents the equation of a second degree cone comprising the normals to the characteristic surfaces. If it is imaginary, we have an *elliptic equation*. If on the other hand it is real, the equation is *hyperbolic*. The preceding equation, which is easy to write, allows us to reduce our study of the type of a differential equation to the identification of the nature of a quadratic form. Instead of searching for the principal directions and performing an orthogonal change of reference frame, we can follow the simpler procedure of directly identifying a reduced form for this quadratic form. We will look at some applications in the next section.

The compatibility relation of equation [5.36] can be written for the solutions of [5.37]:

$$\psi\left(f, \frac{\partial f}{\partial \xi_i}, \frac{\partial^2 f}{\partial \xi_i \partial \xi_k}, ..\right) - g = 0 \quad \begin{matrix}(i, j = 1,..., p)\\(k = 2,..., p)\end{matrix}$$

5.3.5. *Applications*

5.3.5.1. *Quasi-linear partial differential equation*

We will apply the preceding results to the flow of an inviscid fluid, limiting ourselves to the identification of characteristic surfaces for the two-variable examples already studied.

We immediately recover the results of section 5.2, by applying the preceding results to equation [5.3]. Characteristic equation [5.37] can be written:

$$\frac{\partial \xi_1}{\partial t} + u\frac{\partial \xi_1}{\partial x} + v\frac{\partial \xi_1}{\partial y} = 0$$

Its solutions $\xi_1 = \xi_1(t, x, y) = 0$ are equations of surfaces comprising the trajectories (the equation shows in effect that the surface normals are normal to the vector (1, u, v) which is thus situated in the tangent plane of the surface).

5.3.5.2. *System of partial differential equations with two independent variables*

The characteristic curves obtained in section 5.3.1 can also be found using characteristic equation [5.34]. This can be written as:

$$\left| A_{i\ell} \frac{\partial \xi_1}{\partial t} + B_{i\ell} \frac{\partial \xi_1}{\partial x} \right| = 0 \quad \left(i, \ell = 1, \ldots, n\right) \qquad [5.38]$$

In two dimensions, the characteristic "surfaces" are the curves $\xi_1\left(t, x\right) = 0$ on which we have $\delta \xi_1 = \frac{\partial \xi_1}{\partial t} \delta t + \frac{\partial \xi_1}{\partial x} \delta x = 0$. Eliminating $\frac{\partial \xi_1}{\partial t}$ and $\frac{\partial \xi_1}{\partial x}$ between this relation and [5.38], we recover equation [5.10] for the characteristic curves:

$$\left| B_{i\ell} - \frac{\delta x}{\delta t} A_{i\ell} \right| = 0$$

5.3.5.3. *Unsteady 2D flow of a perfect compressible fluid*

Let us reconsider the study of section 5.4.2.3. with the two spatial coordinates (*x,y*). The equations for an inviscid compressible fluid can be written ([4.6], [4.22] and [4.54]):

$$\begin{aligned} &\frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} = 0; \quad \rho \frac{du}{dt} + \frac{\partial p}{\partial x} = 0 \\ &\rho \frac{dv}{dt} + \frac{\partial p}{\partial y} = 0; \qquad -c^2 \frac{d\rho}{dt} + \frac{dp}{dt} = 0 \\ &\text{with:} \frac{d}{dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \end{aligned} \qquad [5.39]$$

or in matrix form:

$$\begin{pmatrix} \frac{d}{dt} & \rho \frac{\partial}{\partial x} & \rho \frac{\partial}{\partial y} & 0 \\ 0 & \rho \frac{d}{dt} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \rho \frac{d}{dt} & \frac{\partial}{\partial y} \\ 0 & \rho c^2 \frac{\partial}{\partial x} & \rho c^2 \frac{\partial}{\partial y} & \frac{d}{dt} \end{pmatrix} \begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix} = 0$$

The unknown function f_i which has four components (ρ, u, v, p) is assumed to be given on the surface S_0 defined in the preceding section and characterized by the coordinates $\xi_k \ \left(k \neq 1\right)$. Setting $\left(x_1 = ct, x_2 = x, x_3 = y\right)$, we can perform the change of variables [5.31]:

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial x_j} + \frac{\partial f_i}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j} \qquad (i = 1,..,4;\ k = 2,3)$$

Separating the unknown derivatives $\frac{\partial f_i}{\partial \xi_1}$ we have:

$$\begin{pmatrix} \frac{d\xi_1}{dt} & \rho\frac{\partial \xi_1}{\partial x} & \rho\frac{\partial \xi_1}{\partial y} & 0 \\ 0 & \rho\frac{d\xi_1}{dt} & 0 & \frac{\partial \xi_1}{\partial x} \\ 0 & 0 & \rho\frac{d\xi_1}{dt} & \frac{\partial}{\partial y} \\ 0 & \rho c^2\frac{\partial \xi_1}{\partial x} & \rho c^2\frac{\partial \xi_1}{\partial y} & \frac{d\xi_1}{dt} \end{pmatrix} \frac{\partial}{\partial \xi_1}\begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix} + \begin{matrix} \text{terms} \\ \text{similar to} \\ \text{(known)} \end{matrix} \frac{\partial}{\partial \xi_k}\begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix} = 0 \quad (k = 2,3)$$

This system of equations does not have a unique solution $\frac{\partial f_i}{\partial \xi_1}$ if its determinant is zero:

$$\begin{vmatrix} \frac{d\xi_1}{dt} & \rho\frac{\partial \xi_1}{\partial x} & \rho\frac{\partial \xi_1}{\partial y} & 0 \\ 0 & \rho\frac{d\xi_1}{dt} & 0 & \frac{\partial \xi_1}{\partial x} \\ 0 & 0 & \rho\frac{d\xi_1}{dt} & \frac{\partial}{\partial y} \\ 0 & \rho c^2\frac{\partial \xi_1}{\partial x} & \rho c^2\frac{\partial \xi_1}{\partial y} & \frac{d\xi_1}{dt} \end{vmatrix} = \rho^2\left(\frac{d\xi_1}{dt}\right)^2\left[\left(\frac{d\xi_1}{dt}\right)^2 - c^2\left(\frac{\partial \xi_1}{\partial y}\right)^2 - c^2\left(\frac{\partial \xi_1}{\partial x}\right)^2\right] = 0$$

The characteristic equation can be decomposed into two first order partial differential equations:

1) A quasi-linear equation $\frac{d\xi_1}{dt} = 0$ whose solution we have already studied in section 5.2.2. We have seen that the characteristic curves are the trajectories, the corresponding characteristic surfaces being defined from these.

2) The partial differential equation:

$$\left(\frac{1}{c}\frac{d\xi_1}{dt}\right)^2 - \left(\frac{\partial \xi_1}{\partial y}\right)^2 - \left(\frac{\partial \xi_1}{\partial x}\right)^2 = 0$$

which can be easily interpreted by taking a reference frame for which velocity (u, v) of a point M is zero at time t (*this reference frame has the speed of the matter of point M at the considered instant*). In this last frame the previous substantial derivative $\left.\frac{d\xi_1}{dt}\right|_{M,t}$ is equal to $\left.\frac{\partial\xi_1}{\partial t}\right|_{M,t}$, so that the previous equation relation between derivatives of function ξ_1 is written:

$$\frac{1}{c^2(M,t)}\left(\left.\frac{\partial\xi_1}{\partial t}\right|_{M,t}\right)^2-\left(\left.\frac{\partial\xi_1}{\partial x}\right|_{M,t}\right)^2-\left(\left.\frac{\partial\xi_1}{\partial y}\right|_{M,t}\right)^2=0$$

showing that the vector $\left[\left.\frac{\partial\xi_1}{\partial t}\right|_{M,t},\left.\frac{\partial\xi_1}{\partial x}\right|_{M,t},\left.\frac{\partial\xi_1}{\partial y}\right|_{M,t}\right]$ is located on the revolving cone of summit M, of axis Ot and whose equation is:

$$\frac{1}{c^2(M,t)}(t-t_M)^2-(x-x_M)^2-(y-y_M)^2=0$$

The tangent planes at point M to the characteristic surfaces are normal to the vector $\left[\left.\frac{\partial\xi_1}{\partial t}\right|_{M,t},\left.\frac{\partial\xi_1}{\partial x}\right|_{M,t},\left.\frac{\partial\xi_1}{\partial y}\right|_{M,t}\right]$, which makes an angle $Arc\tan[1/c(M,t)]$ with the axis Ot. They envelope the complementary cone of revolution whose angle with axis Ot is equal to $Arc\tan[c(M,t)]$ and whose equation is:

$$c^2(M,t)(t-t_M)^2-(x-x_M)^2-(y-y_M)^2=0$$

The characteristic curves comprised by the cone satisfy the relations $dr^2-c^2dt^2=0$ which describe radial propagation at the speed of sound c with respect to the matter.

5.3.6. *Physical interpretation of propagation*

The partial differential equations of fluid dynamics and transfer are balance equations; we have now outlined the essential ideas which govern the manner in which material quantities are displaced on the characteristic curves either by convection (transport of material quantities by fluid particles) or by propagation.

The latter mode results from the exchange of extensive quantities between fluid particles, from one to the next and so on. It can be simply interpreted.

Consider a string of coupled oscillators comprising the masses m and springs of stiffness k and length Δx. Let x_n be the coordinate where the nth mass is at rest, and let ξ_n be its displacement with respect to this rest position (Figure 5.3).

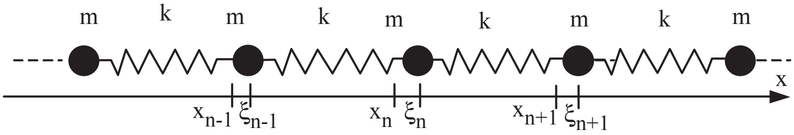


Figure 5.3. *Propagation on a line of mass-spring oscillators*

The equation of motion for the nth mass can be written:

$$m\ddot{\xi}_n + k\left(2\xi_n - \xi_{n-1} - \xi_{n+1}\right) = 0 \qquad [5.40]$$

Consider this string to be a model for a continuous medium with spatial discretization $x_{n+1} - x_n = x_n - x_{n-1} = ... = \Delta x$.

The second spatial derivative can be approximated by:

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\xi_{n-1} + \xi_{n+1} - 2\xi_n}{\Delta x^2}$$

such that by letting $c = \Delta x\sqrt{k/m}$[3], equation [5.40] becomes a wave equation:

$$\frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} - \frac{\partial^2 \xi}{\partial x^2} = 0$$

The *propagation of waves results from interactions between the mass of the medium and its compressibility* (or what is equivalent, its elasticity). In the continuous compressible medium, the mass and the elasticity are uniformly distributed.

---

[3] The reader can verify that stiffness per unit length is $k\Delta x$ and mass per unit length is $m/\Delta x$, so that by identifying the mass and the stiffness with corresponding properties of gas pressure and specific mass), we obtain the value of the sound velocity $c = \left(\partial p/\partial \rho\right)_S$.

5.4. Second order partial differential equations

5.4.1. *Introduction*

We have just shown that a system which flows generally presents at least one family of characteristic curves (its trajectories) which correspond to the transport of matter and on which there exists a balance relation for the extensive quantities (entropy, mechanical energy, etc.). In many practical cases, the flowing fluid may possess properties of homogenity, either dynamic (absence of vorticity) or physical (constant entropy). If we have strict conservation of this quantity everywhere in the flow, then the corresponding partial differential equation can be immediately integrated. For example the steady 1D flow studied in section 5.3.2.1 is homoentropic, which leads to the existence of the relation $\rho = \rho(p)$.

We can thus often obtain a quasi-linear second order partial differential equation (i.e. linear with respect to the second derivatives) for one of the quantities of the problem by using a system of first order partial differential equations.

For example, let us assume for the sake of simplicity that the density and velocity variations are small enough for the linearization of equations [5.13] to be possible in a constant entropy medium:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} = 0; \qquad \rho \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0$$

Using the definition of the speed of sound $c^2 = (\partial p / \partial \rho)_S$, here assumed to be constant on account of the linearization, we obtain the wave equation:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0$$

We will now reconsider quasi-linear second order partial differential equations with two variables, in a form largely used in practice when a velocity potential exists (the homentropic flow of a compressible fluid, waves on the surface of liquids, etc.; see Chapter 6). Furthermore, as their characteristic equation is of second degree, it is locally of a well-defined type, elliptic or hyperbolic depending on whether the roots are imaginary or real. This facilitates a discussion of a problem's boundary conditions.

Consider the quasi-linear second order partial differential equation with two variables:

$$A\,r + 2\,B\,s + C\,t = D \tag{5.41}$$

with the usual notation:[4]

$$f_x = p \qquad f_y = q \qquad f_{xx} = r \qquad f_{xy} = s \qquad f_{yy} = t$$

The coefficients *A, B, C* and *D* are functions of the unknown function *f* and of its first derivatives *p* and *q*:

$$A = A(f, p, q, x, y); \quad B = B(f, p, q, x, y);$$
$$C = C(f, p, q, x, y); \quad D = D(f, p, q, x, y).$$

5.4.2. *Characteristic curves of hyperbolic equations*

Because of the practical applications of this formulation, we will reconsider the Cauchy problem, which consists here of determining the solution using data for *f* and its derivatives (*p,q*) on a curve C_0 in the plane (*x,y*). In order to know the function *f* at all points in the neighborhood of the point (*x,y*) in C_0, it suffices to know the second derivatives (*r*, *s*, *t*) at that point (we can then calculate the higher order derivatives in a similar fashion by successive differentiations of equation [5.41]). The functions (*r*, *s*, *t*) satisfy the relations:

$$Ar + 2Bs + Ct = D; \quad r\,\delta x + s\,\delta y = \delta p; \quad s\,\delta x + t\,\delta y = \delta q \qquad [5.42]$$

or, in matrix form:

$$\begin{pmatrix} A & 2B & C \\ \delta x & \delta y & 0 \\ 0 & \delta x & \delta y \end{pmatrix} \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} D \\ \delta p \\ \delta q \end{pmatrix} \qquad [5.43]$$

As a function of the known variations $(\delta p, \delta q)$ of (p, q) on the curve C_0 *between the points* $(x + \delta x, y + \delta y)$ *and (x, y) on this one*, we can generally calculate the quantities (r, s, t) at (*x*, *y*), except if the characteristic determinant of system [5.43] is zero:

$$\begin{vmatrix} A & 2B & C \\ \delta x & \delta y & 0 \\ 0 & \delta x & \delta y \end{vmatrix} = 0 \quad \text{or:} \; C\,\delta x^2 - 2B\,\delta x\,\delta y + A\,\delta y^2 = 0 \qquad [5.44]$$

---

[4] In order to simplify the writing in section 4.6 and in any other case when it will be useful, partial derivatives ∂/∂*x*, ∂/∂*z* will be written f_x, f_z, a notation that does not allow any mistake in mathematical calculations.

The directions $\dfrac{\delta y}{\delta x} = \dfrac{B \pm \sqrt{B^2 - AC}}{A}$, solutions of the preceding equation, are the *characteristic directions* tangent at each point to the characteristic curves:

– if $B^2 - AC > 0$ the roots of the characteristic equation are real and the equation is of a *hyperbolic* kind;

– if $B^2 - AC < 0$ the roots of the characteristic equation are imaginary and the equation is of an *elliptic* kind; the Cauchy problem always possesses a unique solution in the neighborhood of any curve on which the values of the functions (f, p, q) are given;

– if $B^2 - AC = 0$ the characteristic equation possesses a double root, and the equation is of a *parabolic* kind.

In the hyperbolic case, the determination of (r, s, t), which is non-unique on the characteristic curves, is only possible if the system of equations [5.42] is of rank 2. There then exists a relation between D, δp and δq which can be obtained, as outlined in the preceding sections, by searching for a vector L which is a left solution of the system $LM = 0$, where M is the system matrix [5.43] without the right-hand side. We find:

$$L = (\delta x \delta y, -A\delta y, -C\delta x)$$

By multiplying the left-hand side of system [5.43] by L we obtain the following relation for each of the solutions $(\delta y/\delta x)_i$ of characteristic equation [5.44]:

$$D\left(\frac{\delta y}{\delta x}\right)_i - C\left(\frac{\delta q}{\delta x}\right)_i - A\left(\frac{\delta y}{\delta x}\right)_i\left(\frac{\delta p}{\delta x}\right)_i = 0 \quad i = 1,2. \qquad [5.45]$$

Relations [5.45] allow the function f and its derivatives (p, q) to be calculated from place to place in the following manner. Consider the subdivision ABDEF of an arc of the curve C_0 and trace at each of the points the two families of characteristics Γ_1 and Γ_2 of slope $(\delta y/\delta x)_i$, (i=1, 2). The different families of characteristics obtained from A and B intersect at G (Figure 5.4). We obtain relation [5.45] on each of the arcs AG and BG, which allows the values of p_G and q_G to be calculated as a function of their values on the curve C_0 at A and B. Assuming the arcs to be sufficiently small, these relationships can be written:

$$D\left(\frac{\delta y}{\delta x}\right)_1 (x_G - x_A) - C(q_G - q_A) - A\left(\frac{\delta y}{\delta x}\right)_1 (p_G - p_A) = 0$$

$$D\left(\frac{\delta y}{\delta x}\right)_2 (x_G - x_B) - C(q_G - q_B) - A\left(\frac{\delta y}{\delta x}\right)_2 (p_G - p_B) = 0$$

the elementary arcs AG and BG having respective slopes $\left(\frac{\delta y}{\delta x}\right)_1$ and $\left(\frac{\delta y}{\delta x}\right)_2$.

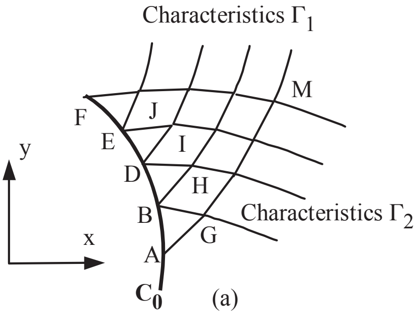


Figure 5.4. *Domain of influence of the arc AF of the initial curve C_0*

The preceding relations uniquely determine p_G and q_G as a function of the values of *f*, *p* and *q* at A and B. We thus see that the initial values on the curve C_0 propagate partially on each of the characteristics by virtue of relation [5.45]. The value of *f* at G is determined by the mean of the finite variations formula between the points A (or B) and G:

$$\delta f = f_G - f_A = p_A(x_G - x_A) + q_A(y_G - y_A).$$

The values f, *p* and *q* of the solution of the partial differential equation can thus be determined from place to place by the preceding procedure at any point within the curvilinear triangle AFM, which is delimited by the arc AF and the arcs AM and FM of the characteristic curves: in effect, it suffices for this point to be attained by progressing along the two families of characteristic curves starting from the initial arc AF. The inner domain of the curvilinear triangle AFM is called the *influence domain* of the arc AF of the initial curve C_0.

This method is not applicable to *parabolic* equations: by considering the case $B = C = 0$ in equation [5.41], relation [5.44] gives $\delta y = 0$, and system [5.45] gives no new relation on the characteristic curves comprising the axis *Ox*. We will return to discuss the parabolic equation in section 5.4.5.4.

In summary, the solution of the Cauchy problem is always possible for an elliptic equation regardless of the choice of initial data. On the other hand, the existence of real characteristic curves or surfaces implies a propagation of function values along these curves.

5.4.3. *Reduced form of the second order quasi-linear partial differential equation*

We will demonstrate that a second order quasi-linear partial differential equation can be locally changed, at all points, to a standard reduced form.

Recall firstly that a quadratic form $Am^2 + 2Bmn + Cn^2$ can be written in a reduced form by means of an appropriate change of basis. In effect, by considering the new variables (μ, ν):

$$m = c\mu + d\nu \qquad n = e\mu + f\nu$$

the preceding quadratic form can be written as a function of these:

$$Am^2 + 2Bmn + Cn^2 = \alpha\mu^2 + 2\beta\mu\nu + \gamma\nu^2 \qquad [5.46]$$

By appropriately choosing the coefficients (c, d, e, f) of basis change, we can eliminate β and make equal the absolute values of α and γ. Let us apply this procedure to equation [5.41], which can be associated with the quadratic form $Am^2 + 2Bmn + Cn^2$ whose coefficients are functions of the quantities (f, p, q, x, y).

The change of coordinates

$$\xi = \varphi(x, y) \qquad \eta = \psi(x, y) \qquad F(\xi, \eta) = f(x, y)$$

transforms equation [5.41] into another equation of the same kind. Showing *explicitly only those terms containing second order partial derivatives*, we have:

$$f_x = F_\xi \varphi_x + F_\eta \psi_x; \qquad f_y = F_\xi \varphi_x + F_\eta \psi_y$$

$$f_{xx} = F_{\xi\xi}\varphi_x^2 + 2F_{\xi\eta}\varphi_x\psi_x + F_{\eta\eta}\psi_x^2 + \ldots$$

$$f_{xy} = F_{\xi\xi}\varphi_x\varphi_y + F_{\xi\eta}\left(\varphi_x\psi_y + \varphi_y\psi_x\right) + F_{\eta\eta}\psi_x\psi_y + \ldots..$$

$$f_{yy} = F_{\xi\xi}\varphi_y^2 + 2F_{\xi\eta}\varphi_y\psi_y + F_{\eta\eta}\psi_y^2 + \ldots..$$

The left-hand side of equation [5.41] can be written:

$$\begin{aligned} &A f_{xx} + 2B f_{xy} + C f_{yy} = \alpha\ F_{\xi\xi} + 2\beta F_{\xi\eta} + \gamma F_{\eta\eta} + ... \\ &\text{with: } \beta = \phi_x \left(A\psi_x + B\psi_y\right) + \phi_y \left(B\psi_x + C\psi_y\right) \\ &\quad \alpha = A{\phi_x}^2 + 2B\phi_x\phi_y + C{\phi_y}^2 \\ &\quad \gamma = A{\psi_x}^2 + 2B\psi_x\psi_y + C{\psi_y}^2 \end{aligned} \qquad [5.47]$$

We can immediately verify the relation:

$$\alpha\gamma - \beta^2 = \left(AC - B^2\right)\left(\varphi_x\psi_y - \varphi_y\psi_x\right)^2$$

which shows that the discriminant of the quadratic form retains the same sign after the coordinates change.

The reduction to the normal form can be obtained by letting $\alpha = \gamma$ and $\beta = 0$. This last condition is satisfied by letting:

$$\varphi_x = M.\left(B\psi_x + C\psi_y\right) \qquad \varphi_y = -M.\left(A\psi_x + B\psi_y\right) \qquad [5.48]$$

By replacing in [5.47] the derivatives of φ with the preceding expressions, we can show that the coefficient α can be written:

$$\alpha = M^2\left(AC - B^2\right)\left(A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2\right) = M^2\left(AC - B^2\right)\gamma \qquad [5.49]$$

Depending on the value of $AC - B^2$, we can distinguish the following cases:

– *Elliptic case*: $AC - B^2 > 0$.

Letting $M^2\left(AC - B^2\right) = 1$, we have $\alpha = \gamma$. The characteristic equation [5.44] does not have a real solution and equation [5.41] is elliptic; as the coefficient α is non-zero, the second derivatives can be regrouped in the form of a Laplacian:

$$A\,r + 2\,B\,s + C\,t = Af_{xx} + 2Bf_{xy} + Cf_{yy} = \alpha\left(F_{\xi\xi} - F_{\eta\eta}\right) + ... = D$$

– *Hyperbolic case*: $AC - B^2 < 0$.

Letting $M^2\left(AC - B^2\right) = -1$, we have $\alpha = -\gamma$. The reduced form of the equation [5.41] can be written in the form of a wave equation:

$$A\,r + 2\,B\,s + C\,t = Af_{xx} + 2Bf_{xy} + Cf_{yy} = \alpha\left(-F_{\xi\xi} + F_{\eta\eta}\right) + ... = D\,.$$

Another reduced form of the hyperbolic equation can be obtained by alternatively choosing the functions φ and ψ to represent the two families of characteristic curves $\xi = \phi(x, y) = \text{const}$ and $\eta = \psi(x, y) = \text{const}$; in the place of relations [5.48], we let the tangent slopes of these curves equal the roots of characteristic equation [5.44] (the characteristic curves are taken as local coordinate curves):

$$\left(\frac{\delta y}{\delta x}\right)_1 = -\frac{\varphi_x}{\varphi_y}\ ; \qquad \left(\frac{\delta y}{\delta x}\right)_2 = -\frac{\psi_x}{\psi_y}\ .$$

It now follows from equation [5.44]:

$$\alpha = \gamma = C\,\varphi_y^2 + 2B\varphi_x\varphi_y + A\,\varphi_x^2 = A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2 = 0$$

By substituting the product and the sum of the ratios φ_x / φ_y and ψ_x / ψ_y by their expressions obtained from the characteristic equation [5.44], we obtain the non-zero coefficient β defined by [5.47]:

$$\beta = A\varphi_x\psi_x + B\left(\varphi_x\psi_y + \varphi_y\psi_x\right) + C\varphi_y\psi_y = 2\varphi_y\psi_y\left(C - \frac{B^2}{A}\right)$$

Equation [5.41], now considering the characteristic curves, can be written:

$$F_{\xi\eta} = D'$$

where D' is a function of coordinates, and of the values of F, and of its first derivatives.

–*Parabolic case*: $AC - B^2 = 0$.

We take φ such that $\varphi_x - \lambda_1\varphi_y = 0$ to obtain $\alpha = 0$; it follows that $\beta = 0$; and the normal form of the parabolic equation is then:

$$F_{\eta\eta} + \ldots = D$$

We will return to the properties of the parabolic equation when we consider it in terms of constant coefficients (section 5.4.5.4).

NOTE – As pointed out (section 5.4.1), we have verified that second order partial differential equations no longer have characteristic curves which represent trajectories, even though they do represent flows.

5.4.4. *Second order partial differential equations in a finite domain*

5.4.4.1. *The significance of the Cauchy problem*

In systems of quasi-linear partial differential equations, we can have mixed situations: for example, the flow of the incompressible fluid discussed in section 5.3.2.2 only represents a single family of characteristic curves, on which only mechanical energy is transported, but no other quantity propagates. On the other hand, the quasi-linear second order partial differential equations lead to two principal kinds of local situation:

– second order *elliptic equations* always possess a solution to the Cauchy problem, which implies that the initial data have a significant influence on the solution in their neighborhood;

– *hyperbolic equations* lead to a double structure associated with two families of characteristic curves on which the initial information is transmitted.

While elliptic equations distribute information in all directions, hyperbolic equations transmit it along the “fibers” of two bundles of curves. However, as *the elliptic or hyperbolic character is a local property,* an equation can be hyperbolic in one region of space and elliptic in another.

Our discussion of the Cauchy problem shows us that the simultaneous presence in a flow of subsonic and supersonic zones leads to *very different modes of transmitting information and to certain contradictions;* this results in important difficulties regarding the boundary conditions which must be imposed, which are different in the two cases (section 5.4.5). This situation often leads to the presence of shock waves. We will consider a simple example by studying the flow of a compressible fluid in a nozzle (section 5.5.4).

The understanding of these situations and of these properties is particularly important, not only for the discussion of physical phenomena, but also for numerical calculations whose algorithms must be chosen such that *numerical information is transmitted in a manner which is compatible with the general properties* which we have just outlined.

5.4.4.2. *Constant coefficient second order partial differential equations*

When the coefficients A, B and C are constants, the nature of the partial differential equation is identical in all parts of the domain studied.

Constant coefficients *elliptic equations* with no right-hand side can be expressed as a Laplace equation:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

This equation is encountered in problems of potential flow of incompressible fluids, and in problems of thermal conduction or mass diffusion in steady flows. The presence of a right-hand side (Poisson equation) implies the existence of sources for the quantities studied.

Constant coefficients *hyperbolic equations* with no right-hand side take the form of the wave equation:

$$\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}=0$$

in which the y variable is often the time. The characteristic curves are the straight lines $\xi = y - x = \text{constant}$ and $\eta = y + x = \text{constant}$ By performing the change of variables $(x, y) \to (\xi, \eta)$, the equation can be written:

$$\frac{\partial^2 f}{\partial \xi \, \partial \eta}=0$$

Its general solution can be written using two arbitrary functions Φ and Ψ:

$$f(\xi,\eta)=\Phi(\xi)+\Psi(\eta)=\Phi(x-y)+\Psi(x+y)$$

The reduced form of the *parabolic equation* is the heat equation:

$$\frac{\partial^2 f}{\partial x^2}-\frac{\partial f}{\partial y}=0$$

in which the y variable is very often the time.

5.4.5. *Second order partial differential equations and their boundary conditions*

5.4.5.1. *Introduction*

As physical problems do not usually occur in the form of a Cauchy problem, on account of the presence of boundaries (walls or other surfaces) surrounding more or less completely the fluid domain under study, and on which we must impose specific conditions. The properties discussed in the preceding sections regarding the characteristic curves or surfaces show that these depend on the nature of the system of partial differential equations. We will first examine the case of a quasi-linear second order equation with constant coefficients.

5.4.5.2. *Elliptic equations*

The Cauchy problem, which involves specifying the function values and its first derivatives on a curve C_0, generally allows the unique determination of the solution in the neighborhood of this curve C_0. However, physical problems are not given in this way for elliptic equations, and we generally have a problem posed in a domain on the boundary of which are given either:

– the *unknown function* (Dirichlet problem);

– *the function's normal derivative at the boundary* (Neumann problem);

– or a *mixed condition* in the form of a linear relation between the function and its normal derivative (mixed problem).

Problems involving *heat conduction in a steady flow regime* have already been discussed in section 2.3.2. Steady subsonic flow of an inviscid fluid about an obstacle is a Neumann problem (Chapter 6).

Elliptic problems cannot deal with propagative phenomena; they can generally only represent steady phenomena or unsteady situations where propagation plays a negligible role (domains which are small compared with a wavelength for example). We will come back to discuss this point on numerous occasions.

In the case of the Neumann problem, a compatibility condition exists as a result of Otrogradski's theorem. For example, the Poisson equation of a steady 2D heat conduction problem in a domain $\mathcal{D}$ of the plane (x,y):

$$div\left(\lambda \overrightarrow{gradT}\right) = \sigma_T$$

imposes on the curve C, which bounds D, the integral condition:

$$\int_C \lambda \frac{\partial T}{\partial n} d\ell = \int_C \lambda \, \overrightarrow{gradT}.\vec{n} \, d\ell = \int_D div\left(\lambda \, \overrightarrow{gradT}\right) ds = \int_D \sigma_T \, ds$$

This expresses the global conservation of energy, where the thermal flux crossing the curve C must be equal to the thermal power generated in D under steady conditions.

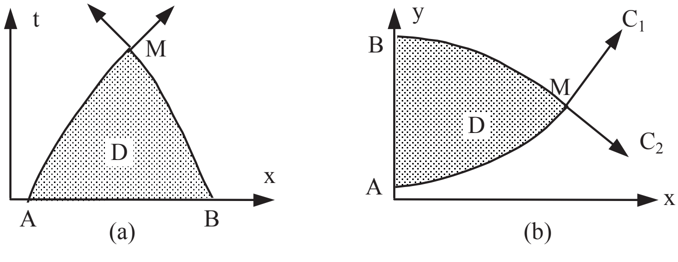
Under usual conditions, the solution of an appropriately posed problem, of one of the preceding kinds (Dirichlet, Neumann or mixed), is unique and elliptic equations do not lead to difficulties, provided of course that they correctly represent the physical problem studied.

5.4.5.3. *Hyperbolic equations*

For equations of this kind, the discussion of section 5.4.2 outlined the ideas of propagation and influence domain. The concept of the Cauchy problem is here degenerated to a certain extent, due to the structure of the solutions. As already discussed in section 5.2.4, the initial data have no influence in transverse directions with respect to the characteristic curves.

In addition to the initial conditions, we often impose other boundary conditions. For example, the presence of an obstacle in a flow leads to the conditions $\partial f/\partial n = 0$ on its wall, associated with the equation for the velocity potential f. This condition leads in particular to a reflection of waves on the considered wall, an idea which we will encounter for diverse phenomena (acoustic, supersonic flows, etc.).

In practice, the Cauchy problem is rather academic, as physical problems are always posed in limited domains at the boundaries of which the boundary conditions are generally defined by external data. However, its study (section 5.4.2) has allowed us to define the characteristic curves and to show that these strictly limit the influence domain of conditions given on a bounding curve. *Conversely, the value of an unknown function at a point M cannot depend on data outside the influence domain formed by the characteristic curves which arrive at this point* (Figure 5.5). The notion of information transmission is closely associated with these considerations: the characteristic curves constitute the means of information transfer in the medium considered.



(a) (b)

Figure 5.5. *Domain D influencing the value of the unknown function f in M*

The boundary conditions associated with the wave equation can often be considered as initial conditions. However, the direction of travel of the characteristic curves is not important for a wave equation which is invariant if the time direction is changed (section 1.1.1.3). Figure 5.5a shows an example of initial data for an acoustic problem (given at time $t = 0$), while Figure 5.5b corresponds to velocity data given for a supersonic steady flow in the plane (x,y) (which can be reduced under certain particular conditions to a wave equation).

5.4.5.4. *Parabolic equations*

We have seen in section 5.4.2 that a double characteristic curve exists on which there is no longer a propagation relation. Taking it as a coordinate curve, we obtain a heat equation [5.50] (see section 5.4.3) whose behavior we will study (we let t be the parabolic variable y which we will interpret as time):

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t} \qquad [5.50]$$

We have already seen, in section 1.1.1.4, the consequences of asymmetry between the variables x and y; in particular, the variable y must evolve towards increasing values (from past to future, or from upstream to downstream, etc.).

We will now treat the problem of a *thermal shock*[5] *in a semi-infinite medium.* We will let f be the temperature in equation [5.50] and consider the thermal problem which corresponds to a unit temperature step function imposed at the initial instant $t = 0$, at the origin $x = 0$ of a semi-infinite medium $(x \geq 0)$, initially at temperature $f = 0$. We impose therefore the boundary conditions:

$$\begin{array}{lll} x \neq 0 & t = 0 & f(x,0) = 0 \\ x = 0 & t > 0 & f(0,t) = 1 \end{array} \qquad [5.51]$$

The scales X and Y corresponding to the variation F of f in [5.50] are:

$$\frac{F}{X^2} \approx \frac{F}{T} \quad \text{that is:} \quad X^2 \approx T.$$

The preceding estimate leads to the definition of the variable $\eta = x/2\sqrt{t}$ and we look for a solution of the form[6] $f(x,t) = \varphi(\eta)$. Substituting this function into equation [5.50], we have:

---

[5] The word "shock" is relative here to an initial condition and not to a discontinuity appearing in the solution.

[6] We have here a self-similar solution, i.e. which is invariant with respect to a group of affine transformations. Flow equations, which are exact or approximate for high Reynolds numbers, can have such solutions ([SCH 99], [YIH 77]).

$$f''+2\eta f'=0$$

from which we obtain:

$$f'(\eta)=A\exp\left(-\eta^2\right)$$

Taking account of boundary conditions [5.51] leads to:

$$f(x,t)=\frac{2}{\sqrt{\pi}}\int_{x/2\sqrt{t}}^{\infty}\exp\,(-u^2)\,du=1-erf(x/2\sqrt{t}) \qquad [5.52]$$

with: $erf\ x=\frac{2}{\sqrt{\pi}}\int_0^x\exp\,(-u^2)\,du\ (erf\infty=1)$.

Figure 5.6 shows the temperature diffusion imposed at the origin, whose influence can be felt in a zone, whose width, in the order of *X*, is proportional to $t^{1/2}$ (curves 1 to 4 for different increasing times). The dimensional presentation of the problem will be treated in section 8.3.2.2.2.

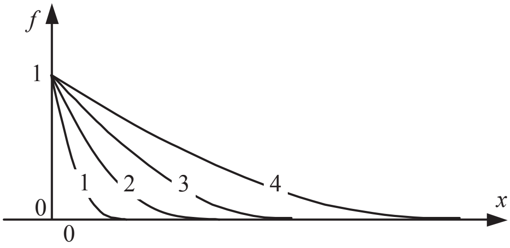


Figure 5.6. *Influence zone of a unit step function (of temperature) imposed at the origin*

We thus see that the behavior of the parabolic differential equation lies somewhere between that of the hyperbolic and elliptic equations:

– the double characteristic curve is a preferred axis for the transmission of information;

– the form of solution [5.52] shows an instantaneous action at all points of the axis *Ox:* the notion of propagation has completely disappeared;

– we have "diffusion" (transverse spreading) of the initial data about the characteristic curve.

The same initial boundary conditions applied to the wave equation would lead to an acceptable solution for negative values of *y,* which is not the case here: the heat equation describes an irreversible phenomenon, in other words the impossibility of reversing the time direction (section 1.1.1.3).

5.4.5.5. *Mixed equations*

In section 5.4.2 we saw that, depending on the quantity $AC - B^2$, quasi-linear partial differential equation [5.43]:

$$A\frac{\partial^2 f}{\partial x^2} + 2B\frac{\partial^2 f}{\partial x \partial y} + C\frac{\partial^2 f}{\partial y^2} = D$$

can be either elliptic, hyperbolic or parabolic. The nature of the solution depends therefore on the local coefficients value. The simplest example of a mixed equation is the Tricomi equation:

$$\frac{\partial^2 f}{\partial x^2} - x\frac{\partial^2 f}{\partial y^2} = 0$$

For negative values of x this equation is elliptic, while for positive values, it is hyperbolic. It is a summary model of the situation described in section 5.3.2.3, where, in a subsonic flow, the problem is elliptic, while a supersonic flow has a hyperbolic character. The passage from a subsonic flow to a supersonic flow happens in a transonic range, which may be locally represented by a Tricomi equation (the passage from the reduced form of the elliptic equation to the reduced form of the hyperbolic equation assumes that a coefficient changes sign when going to zero):

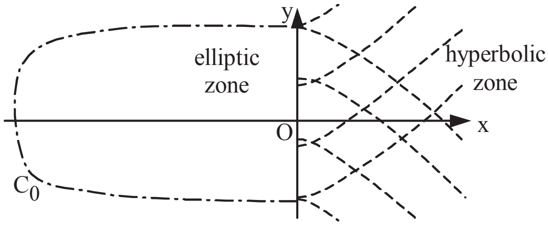


Figure 5.7 *Mixed problem (Tricomi equation)*

Consider a flow such that it is subsonic for negative values of x and supersonic for positive values. We assumed that the speed of sound is attained by the flow velocity at x = 0. Determination of the solution in the elliptic zone is possible using the boundary conditions on the curve C_0 (Figure 5.7) and the condition $AC - B^2 = 0$ here corresponds to sonic velocity on the boundary x = 0 between the subsonic zone and the hyperbolic zone. In the supersonic zone, the flow is determined by the upstream conditions. So, as any real subsonic flow is generated by an ensemble of conditions upstream and downstream, the presence of a supersonic zone leads to a contradiction corresponding to the presence of shock waves, which we will now examine.

Note that the coefficients of partial differential equation [5.42] are not only a function of the coordinates, but also of the unknown function and its first derivatives. Contrary to the Tricomi equation, the nature of the equation and the (associated) boundary conditions that must be associated are not easily predictable in general, as *they depend on the solution values*.

5.5. Discontinuities: shock waves

5.5.1. *General considerations*

The presence of shock waves may feature in the varied conditions of supersonic flows of compressible fluids. We will show, using examples, that the existence of characteristic curves or surfaces and the nature of the solutions to the hyperbolic equations lead to the possibility of discontinuities.

We will first of all study two examples (1D flow of an inviscid compressible fluid and steady supersonic flow) for which a discussion of the characteristic curves leads to the impossibility of a continuous flow. We will then consider a steady 1D flow of an inviscid fluid in a nozzle which is governed by a differential equation which again shows the necessity of shocks.

5.5.2. *Unsteady 1D flow of an inviscid compressible fluid*

Let us take the example of a one-dimensional flow of a compressible fluid verifying equations [5.13] (section 5.3.2.1). The characteristic transport curve $\delta x/\delta t = u$ ensures constant entropy at all points, which leaves only two independent variables p and u. This condition leads to the relations:

$$\frac{p(x,t)}{\rho^{\gamma}(x,t)} = \text{const}; \quad c = \sqrt{\gamma\, p/\rho} = \sqrt{\gamma \frac{p_0}{\rho_0}\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}} = c_0\left(\frac{p}{p_0}\right)^{(\gamma-1)/2\gamma}$$

in which (p_0, ρ_0) are the initial conditions of the fluid for a zero velocity.

Let us now apply the method of characteristics on a grid discretized as defined in section 5.4.2. Starting from two neighboring points R and S whose physical quantities (p_R, u_R) and (p_S, u_S) are known, and which are not located on the same characteristic curve (Figure 5.8a), we trace out two characteristic curve segments with respective slopes $\pm c + u$ which intersect in plane (x, t) at the point T whose the values of the unknowns (p_T, u_T) are calculated using the relations $\delta p \pm \rho c \delta u = 0$ (with notations of section 5.4.2) taken on the previous curves segments. These are written:

$$p_T - p_R - \rho_R c_R (u_T - u_R) = 0; \quad p_T - p_S + \rho_S c_S (u_T - u_S) = 0$$

These relations can be used for calculation of the values (p_T, u_T) for pressure and velocity at point T. At first order, we can take $\rho_R c_R \# \rho_S c_S \# \rho c$, so that we have:

$$p_T = \frac{1}{2}\left[p_R + p_S + \rho c(u_S - u_R)\right] \quad u_T = \frac{1}{2}\left[u_R + u_S - \frac{1}{\rho c}(p_S - p_R)\right]$$

We can then deduce from thermodynamic relations with constant entropy, the specific mass ρ_T and the sound velocity c_T at point T.

Let us consider the case where pressure and velocity take uniform values on the characteristic curve issued from point R, which having then a constant slope $c + u$, is a straight line. If R' is a neighboring point of R on this characteristic curve, at point T and at the intersection T' of characteristic curves issued from R' and S, we have:

$$p_T - p_R - \rho c (u_T - u_R) = 0; \qquad p_{T'} - p_{R'} - \rho c (u_{T'} - u_{R'}) = 0.$$

Taking out the first relation from the second relation, we obtain the relation on characteristic curve of slope $c + u$: between points T and T':

$$p_{T'} - p_T + \rho c (u_{T'} - u_T) = 0$$

So, between points T and T' we obtain the relation on characteristic curve of slope $c + u$:

$$p_{T'} - p_T + \rho c (u_{T'} - u_T) = 0$$

We then obtain the relation:

$$p_{T'} - p_T = \rho c \left(u_{T'} - u_T \right) = 0$$

which shows that pressure, velocity and sound velocity values are the same at points T and T'. Going on from place to place on characteristic curve of slope $c_S + u_S$ issued from S, we see that the flow properties are identical along this curve, which is also a straight line. The corresponding flow (*a simple progressive wave*) is then characterized by the propagation along axis Ox with velocity $c_S + u_S$ of pressure Δp and velocity Δu variations between the two characteristic curves going through R and S. These variations verify the relation:

$$\Delta p = \rho c \Delta u$$

The previous calculation can be continued on the next characteristic curves of the same family which will possess the same property of simple progressive waves. The reader can easily show that another family of simple progressive waves exists associated with the other family of characteristic curves of slope $- c + u$. For these waves, propagating towards the negative part of axis Ox, we have the relation $\Delta p = -\rho c \Delta u$.

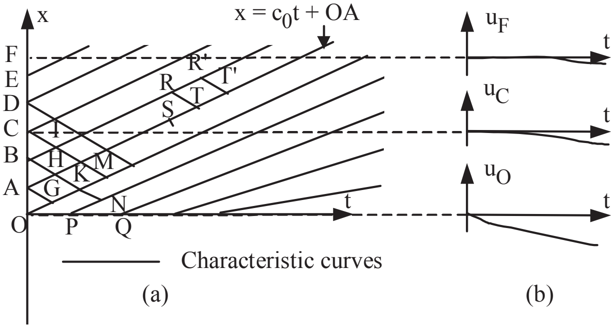


Figure 5.8. *(a) Application of the characteristics method and evolution of the characteristic straight lines for a decreasing pressure (or a negative velocity); (b) evolution of the velocity at the points O, C and F*

Now consider the domain formed by the first quadrant of the plane (x, t) with the following boundary conditions:

$$t = 0 : x > 0 \quad p(x,0) = p_0 \qquad u(x,0) = 0$$

$$t \geq 0 : x = 0 \quad p(0,t) = P(t) \quad u(0,t) = U(t)$$

The domain thus defined is the influence domain of the given conditions on the positive sides of the axes *Ox* and *Ot*. The positive side of the axis *Ox* is at rest at the initial instant and we apply pressure and velocity variations at the origin $x = 0$, as a function of time, which will then propagate on the *Ox* axis.

As previously, we can apply the method of characteristics by starting from the points O, A, B, C, D, E,…,P, Q, etc., of the *Ox* and O*t* axis (Figure 5.8a), and calculating the values of the unknowns *p* and *u* using the relations $\delta p \pm \rho c \delta u = 0$ taken on appropriate segments of the characteristic curves of slopes $\pm c + u$.

However, the initial state, being uniform and at rest, can be considered as a regime of simple progressive waves. Indeed, by applying previous relations on characteristic curves issued from points A, B, C, D, E, etc., we see that the initial rest data are transmitted to points H, I, etc., on the straight lines issued from previous points on all parts of the plane above the straight line $x = ct$ + OA.

Previous results show that propagation can only proceed with simple waves for increasing times. It can be verified that the values (p_O, u_O) at point O are also obtained from place to place for all points of the characteristic curve $x = c_O t$ issued from *O*, on which the fluid state is uniform. Calculation is then carried out at points *P*, *Q*, etc., of axis *Ot*, and it can be seen that we have a flow with simple progressive waves, propagating on the characteristic straight lines of a positive slope.

The difference of pressure and velocity values between two neighboring characteristic curves verify the relation $\Delta p = \rho c \Delta u$ of simple progressive waves. *This condition must be verified at points* P, Q, etc., of axis *Ot*, so as not to create an inverse propagation towards negative times on characteristic curves of negative slopes: this should be not acceptable according to the physical aspects of problem. *This compatibility condition is a consequence of the definition of the initial data curve itself, since there exist characteristic lines of a negative slope issued from axis O*x *intersecting axis Ot*, which involves the relation $\Delta p = \rho c \Delta u$.

The simplest practical realization of the preceding flow involves imposing a suitable velocity on the fluid matter by means of a moving piston. We will leave it to the reader to determine the domain thus defined in the plane (*x,t*) for this Lagrangian condition (we consider the trajectory of the piston).

When the *pressure and velocity variations are negative and decreasing* (*expansion*), the slope of the characteristic lines is a decreasing function of time (Figure 5.8a). These divergent straight lines thus form the shape of a fan, which involves a broadening of the wave front as it propagates; Figure 5.8b shows the form of the temporal velocity variation at the origin O, and then at points C and F.

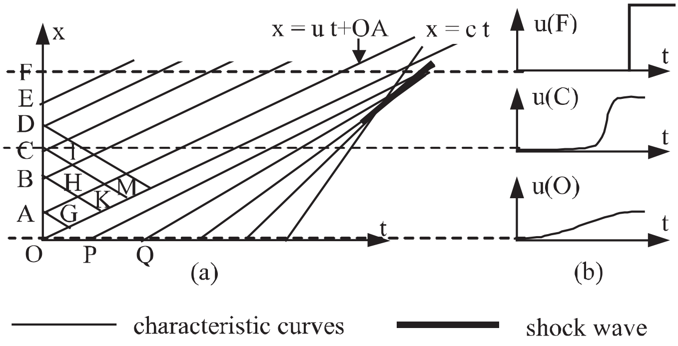


Figure 5.9. *(a) Application of the characteristics method and evolution of the characteristic straight lines for a positive and growing variation of pressure (or velocity); (b) evolution of the velocity t the points O, C and F*

On the other hand, when the *pressure and velocity variations are positive and increasing* (*compression*), the slope of the characteristic lines is an increasing function of time (Figure 5.9a). They thus form a beam of straight lines which tighten and finally intersect. This situation leads to a compression of the wave front as it propagates, which eventually leads to a discontinuity. Indeed, we note in this last case that it is impossible (equation [5.44]) to have more than one characteristic curve of each family at a given point. The result of this is that a *continuous solution cannot exist everywhere in the influence domain* of the boundary conditions which are specified. A discontinuity thus appears (a shock wave) downstream of which the calculation of the solution can only be achieved using the conditions which result from the shock wave. Figure 5.9b shows the form of temporal variation at the origin O, and then at points C and F.

The formation of a shock wave can be physically explained in the following elementary manner: the increasing pressure can be decomposed into a succession of elementary (acoustic) waves which propagate at the speed of sound. Each of these elementary waves corresponds to an isentropic compression which increases the temperature, and thence the speed of sound. Each elementary wave will thus travel slightly faster than its predecessor, which it will finally catch. These waves are thus concentrated at a point where they form a discontinuity. It is clear that when the pressure decreases, an inverse process occurs: the elementary waves are spread out.

5.5.3. *Plane steady supersonic flow*

We have already studied this flow in section 5.3.2.3, and so we will limit ourselves here to a qualitative discussion. Consider a uniform homentropic flow next to a wall in a semi-infinite medium (Figure 5.10).

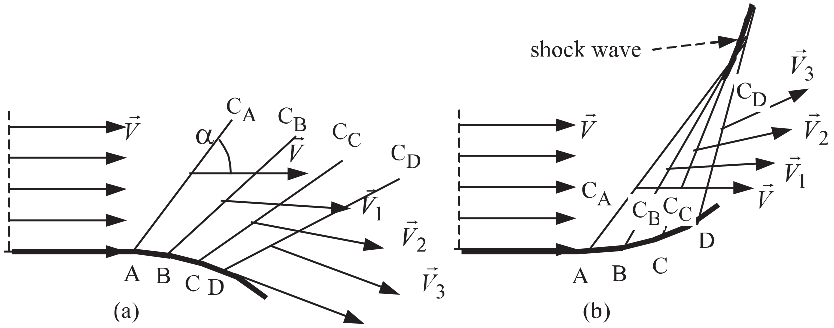


Figure 5.10. *Plane supersonic flow around (a) a convex wall and (b) a concave wall*

Bernoulli's first theorem is valid everywhere in the flow, as are the Saint-Venant and Hugoniot relations (section 4.3.2.3). The characteristic curves form an angle α with the streamlines, defined by $\sin\alpha = \pm 1/M$.

If the wall is convex (Figure 5.10a), the streamlines spread and the density decreases while the velocity $\vec{V}$ increases (supersonic expansion, section 4.3.2.3.4). The velocity (direction and modulus) propagates from the wall along the characteristic of a positive slope, and which is the only characteristic of consequence here (straight line C_A, C_B,... ... from A, B, etc.); the Mach number thus increases and the angle α decreases in the downstream direction. Similar to the previous example, we see divergent characteristics straight lines.

In the presence of a concave wall, an inverse situation occurs: the streamlines tighten up, leading to a reduction in the velocity associated with a compression; the Mach number decreases, and the characteristic lines intersect. Thus, a shock wave is formed (Figure 5.10b).

5.5.4. *Flow in a nozzle*

A nozzle is a truncated conduit comprised of a convergent section followed by a divergent section (Figure 5.11). When it separates two independent gaseous spaces, it supports a flow between an upstream region at pressure p_A and a downstream region at pressure p_E ($p_E < p_A$). We will assume that viscous friction effects at the nozzle walls are small enough to be negligible up to the exit. The flow exiting from the nozzle is thus in the form of a jet; the main viscous dissipation corresponds to the energy loss due to the pressure difference; this dissipation occurs in the jet downstream of the nozzle exit. Experience shows that we can consider that *the pressure in the exit plane is equal to p_E as long as the jet is subsonic.*

Assuming that the quantities associated with the gas are constant in a normal section, the Saint-Venant relation (section 4.3.2.3.3) provides an expression for the velocity as a pressure function using the generation conditions (initial conditions at zero velocity in the upstream domain) and in particular the velocity V_E in the exit section S_E. The isentropic transformation relation ($p/p^{y} = const$) determines the density ρ_E in the exit plane. From this we can deduce the mass flow $q_m = \rho_E V_E S_E$ in the nozzle.

However, we have shown (section 4.3.2.3.4) that a stream tube resulting from a given set of generation conditions has a maximum *flow rate* $q_{m\,\max} = \rho_c V_c S_c$ which occurs when the speed of sound c is attained in the smallest cross-section. We are thus faced with the following alternative:

– either the flow rate q_m evaluated at the exit plane is less than or equal to $q_{m\,\max}$ and we can calculate the continuous flow in the nozzle;

– or the flow rate q_m evaluated at the exit plane is greater than $q_{m\,\max}$ and *the problem thus posed does not have a solution*.

In the first case, the Hugoniot relation [4.39] in its differential form (section 4.3.2.3.4) shows that the velocity increases in the convergent part of the nozzle up to a value which is at most equal to the speed of sound c_* at the throat, and which then decreases such that its value at the exit plane V_E is that previously predicted. The flow is then everywhere subsonic ($V < c$). Figure 5.11 shows the pressure variations (contrary to the velocity variations) in the nozzle for regimes 1, 2 and 3 which are entirely subsonic. For pressures p_{E1} and p_{E2}, the velocity at the throat $V_{\max}$ is less than the speed of sound, while for the pressure at the exit p_{E3} the throat velocity is equal to the speed of sound c_*.

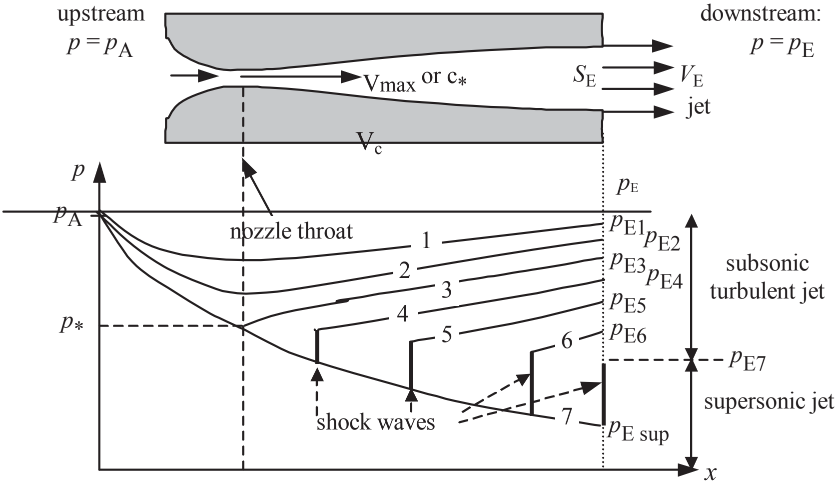


Figure 5.11. *Flow regimes in a nozzle*

In the second case, the flow in the convergent part of the nozzle is subsonic, then it becomes supersonic after passing through the throat where the velocity magnitude is equal to the speed of sound c_* and the critical conditions (p_* and ρ_*) are attained. However the continuous supersonic solution, calculated using the Saint-Venant relation in the exit plane using the generation conditions, is unique. It provides the value p_{Esup} for the exit pressure, which is not equal to the exit pressure imposed p_E.

However, the supersonic flow in the nozzle must match the exit conditions. This adaptation is achieved by means of a shock wave. So long as the shock wave remains within the nozzle, it is plane (from p_{E3} to p_{E7} in Figure 5.11). For lower pressures ($p_{E\ sup} < p_{E3} < p_{E7}$) we have a more or less complex system of shock waves in the jet (under-expanded jet). For p_E less that $p_{E\ sup}$, the adaptation is achieved by means of expansion waves (over-expanded jet).

The idea of characteristic curves and of propagation do not hold for the differential equation of the 1D model nozzle. We note only the non-existence of a continuous solution which verifies the boundary conditions at the exit. In fact, the flow in the nozzle is governed by the models outlined in section 5.3.2.3:

– for the *subsonic* part of the flow, the system of partial differential equations is elliptic and the flow is determined by the boundary conditions on all boundaries. Its solution assumes Neumann conditions which are here known in the upstream region, on the walls and in *the exit section of the subsonic flow*: depending on the regime, this is either at the nozzle exit or at the throat;

– from the throat of nozzle in the sonic regime, and up to the shock, the subsonic flow is governed by the 2D plane mode discussed in section 5.3.2.3, the fluid being assumed to be isentropic. The *supersonic zone of flow* belongs to the influence domain of the "upstream" conditions (section 5.4.5.3) which are here situated at the sonic throat. An exit condition cannot influence the supersonic flow, as the information cannot move upstream due to the characteristic curves which all have slopes $\pm (M^2 - 1)^{-1/2}$. In these conditions *no continuous supersonic solution can account for the conditions at the exit*.

The specific characteristics of the plane 2D supersonic model also generally correspond to the 1D model whose behavior is appropriate but without explaining the difficulties: we observe that data given at two conditions, one at the throat and the other at the exit, leads to an impossibility because of the existence of a singular point for M = 1 in [4.38] and [4.39]. Such difficulties are often encountered in fluid mechanics, where a global model can lead to contradictions (or to "paradoxes") that only a more refined model can explain.

The shock wave is a boundary between two spaces which cannot communicate completely, the upstream space not being able to receive information regarding the pressure at the exit. However, matter crosses the shock wave and the balance equations for the extensive quantities must be satisfied through the shock.

In conclusion, *a continuous solution of the 1D equations does not usually exist in isentropic compressible fluid*, for a nozzle whose throat velocity is sonic (critical velocity). From a physical point of view, we could also consider that the shock wave comprises an accumulation of pressure waves which travel from the downstream and which stop when they can no longer do so.

The shock wave is a dissipative structure which leads to an increase in entropy ([LAN 89], [YIH 77]), viscosity playing an important role at the scale of the mean free molecular path, for which a continuous viscous model is appropriate for the shock wave.

Let us finish by highlighting a particularly useful application for nozzles operating in the supersonic regime whose mass flow is fixed and depends only on the upstream conditions. Such a nozzle, placed upstream of an installation, perfectly regulates the flow if the upstream generation conditions are fixed, which is often the case in laboratory situations: downstream perturbations can have no influence on the mass flow of the device. The pressure loss of such a nozzle is relatively small (≈ 10^4 pascal).

5.5.5. *Separated shock wave*

A problem of the same kind is posed by a supersonic flow around an obstacle which imposes boundary conditions which cannot travel upstream in the supersonic flow. In particular, upstream of the obstacle there exists the stagnation point A, where the velocity is zero, and therefore a subsonic zone of flow in the region near the surface.

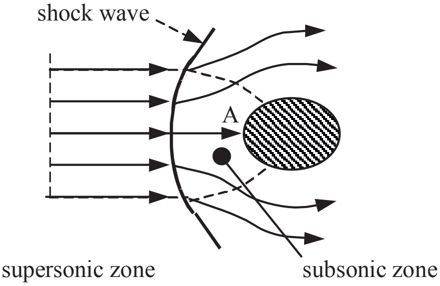


Figure 5.12. *Detached shock wave in front of a body in a supersonic flow*

Similar to the case of the nozzle, the adaptation of the supersonic flow to downstream conditions occurs by means of the shock wave.

5.5.6. *Other discontinuity categories*

Combustion phenomena in gaseous flows obey the fluid dynamics equations which we have already discussed in Chapter 4. We must however introduce the properties of chemical reactions using the methods of chemical thermodynamics which must be applied to the moving matter. Without containing new physical phenomena, the formalism obtained combines the difficulties of the two domains. We cannot address these questions in detail in this book (see [BOR 00], [KIR 67], [OPP 06], [WIL 65], [WIL 85]). Schematically, the possibility of a chemical reaction amounts to the introduction of a heat source associated with a local increase in temperature: we thus notice that a shock wave can trigger a chemical reaction which can augment its effects considerably, transforming the shock wave into a detonation wave.

Other domains of fluid mechanics also involve hyperbolic equations. Such is the case for flows including a free surface (see section 6.2.6) in which we observe the propagation of surface waves (swell in the sea) and flows of stratified fluids (vertical distribution of density) which present similar properties. We encounter analogous phenomena to those discussed for a gas, and in particular the possibility of discontinuities called hydraulic jumps [YIH 77] or tidal bores in estuaries which propagates against flowing water.

5.5.7. *Balance equations across a discontinuity*

A velocity discontinuity undergone by a material body in movement assumes infinite external forces and (finite) inertial effects associated with non-Galilean reference frames are therefore neglected. The equations for the shocks and collisions can be written in any reference frame, both in particle or solid body mechanics and in the dynamics of continuous media.

We will consider a discontinuous surface which is crossed by matter in movement and we will designate by the indices 1 and 2 the upstream and downstream quantities of the discontinuity. We apply the balance equations in global form in a thin volume *D* comprised of two parallel surfaces at the discontinuity *S*, of area *ds* and with normals oriented towards the exterior (Figure 5.13). The indices *n* and *t* designate the velocity components normal and tangent to the surface of the discontinuity.

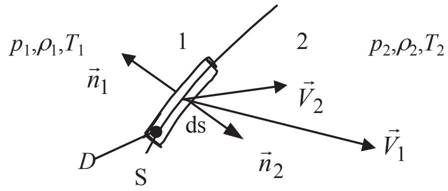


Figure 5.13. *Balance on a discontinuity surface (shock)*

We now write the balance equations for the extensive quantities in global form (section 4.5) in the domain *D* for the following quantities:

– mass (section 4.5.2):

$$\rho_1 V_{1n} = \rho_2 V_{2n} \tag{5.53}$$

– momentum (section 4.5.4):

$$\begin{aligned} p_1 + \rho_1 V_{1n}^2 &= p_2 + \rho_2 V_{2n}^2 \\ \rho_1 \vec{V}_{1t} V_{1n} &= \rho_2 \vec{V}_{2t} V_{2n} \end{aligned} \tag{5.54}$$

– total energy (enthalpic form) (section 4.5.6):

$$\rho_1 V_{1n}\left(h_1+\frac{V_1^2}{2}\right)=\rho_2 V_{2n}\left(h_2+\frac{V_2^2}{2}\right) \qquad [5.55]$$

– chemical species (i =1,2 …):

$$\rho_{i1} V_{i1n}=\rho_{i2} V_{i2n} \qquad [5.56]$$

We can immediately deduce the continuity of the tangential velocity components across the shock:

$$\vec{V}_{1t}=\vec{V}_{2t} \qquad [5.57]$$

NOTE – In the presence of a chemical reaction on the surface S (detonation or deflagration wave), balances [5.55] and [5.56] take the following form:

$$h_1+\frac{V_1^2}{2}+Q_r=h_2+\frac{V_2^2}{2} \qquad \rho_{i1} V_{i1n}+Q_{mi}=\rho_{i2} V_{i2n} \qquad [5.58]$$

where, Q_{mi} and Q_r denote the mass of species i and thermal surface power released by the chemical reaction for the mass flux $\rho_1 V_{1n}$ crossing the shock S, the enthalpies h_i being taken as equal to $C_{pi}\,T_i$ for perfect gases ("sensible enthalpy").

5.6. Some comments on methods of numerical solution

5.6.1. *Characteristic curves and numerical discretization schemes*

The numerical resolution of a system of first order differential equations *with initial conditions* (Cauchy) given at a point can be achieved from place to place: by discretizing the first derivatives, we calculate the value of the unknown vector function at a point using the values at the previous point. However, we often have conditions on either extremity of an interval instead of Cauchy conditions, in particular when the system is associated with a flow between solid boundaries. We therefore often use a shooting method: the missing initial values are determined by successive approximations so as to obtain suitable values at the other extremity of the interval.

Such an iterative procedure presents the advantage of not involving the inversion of a large matrix.

A numerical solution consists of replacing equations of a differential kind with finite difference algebraic equations obtained by means of a discretization of the derivatives which can be performed in many different ways which we will quickly evoke further. By way of a simple example, consider the transport equation in which the velocity u is constant:

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = 0 \qquad [5.59]$$

The axes Ox and Ot are discretized with step-sizes Δx and Δt such that $\Delta x = u.\Delta t$. We will here approximate the derivatives by formulae using the values at two points. We calculate the partial temporal derivative using the formula:

$$\frac{\partial f}{\partial t}(x_n, t_n) = \frac{1}{\Delta t}\left[f(x_n, t_n) - f(x_n, t_{n-1})\right] \qquad [5.60]$$

We approximate the spatial derivative $\frac{\partial f}{\partial x}$ at the point x_n by its value at the instant t_{n-1}, $\frac{\partial f}{\partial x}(x_n, t_{n-1})$ such that we have a simple, explicit scheme for the discussion. Take as an approximation of $\frac{\partial f}{\partial x}(x_n, t_{n-1})$ one of the following schemes, which are apparently locally equivalent:

– upwind scheme: $\frac{\partial f}{\partial x}(x_n, t_{n-1}) = \frac{1}{\Delta x}\left[f(x_n, t_{n-1}) - f(x_{n-1}, t_{n-1})\right]$;

– downwind scheme: $\frac{\partial f}{\partial x}(x_n, t_{n-1}) = \frac{1}{\Delta x}\left[f(x_{n+1}, t_{n-1}) - f(x_n, t_{n-1})\right]$;

– centered scheme: $\frac{\partial f}{\partial x}(x_n, t_{n-1}) = \frac{1}{2\Delta x}\left[f(x_{n+1}, t_{n-1}) - f(x_{n-1}, t_{n-1})\right]$.

Substituting these expressions into [5.59] we can calculate the value of the function $f(x_n, t_n)$ as a function of its values at the instant t_{n-1} at the points next to the axis Ox. We obtain for the different schemes:

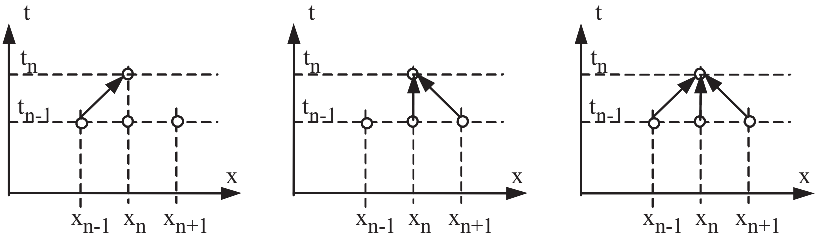
– upwind scheme: $f(x_n, t_n) = f(x_{n-1}, t_{n-1})$;

– downwind scheme: $f(x_n, t_n) = 2f(x_n, t_{n-1}) - f(x_{n+1}, t_{n-1})$;

– centered scheme:

$$f(x_n,t_n)=f(x_n,t_{n-1})+\frac{1}{2}f(x_{n-1},t_{n-1})-\frac{1}{2}f(x_{n+1},t_{n-1}).$$

Figure 5.14 schematically shows the transmission of numerical information between the points of the discretized network for the three numerical schemes.



(a): upwind scheme (b): downwind scheme (c): centered scheme

Figure 5.14. *Transmission of information in the numerical resolution following different discretization schemes*

Now, equation [5.59] is hyperbolic and its characteristics are the straight lines $x-ut=const$, which are trajectories of the uniform velocity field *u*. To simplify matters, the discretization ($\Delta x = u.\Delta t$) was chosen such that these straight characteristics pass through the points of the computation. Let us examine a particular case of the problem, defined by the boundary conditions which corresponds to transport at velocity *u* of a unit step function from the origin in a field with zero initial values:

$$\begin{aligned} &t=0, \quad x>0: \quad f(x,0)=0 \\ &t\geq 0, \quad x=0: \quad f(0,t)=1 \end{aligned} \qquad [5.61]$$

Table 5.1 indicates, on each line, the numerical values obtained by means of the three numerical schemes for the six points on the axis Ox at the six first instants (0, Δ*t*, 2Δ*t*, 3Δ*t*, etc. from the bottom of the table).

1	1	1	1	1	1	1	0	0	0	0	0	1.0	1.312	1.875	1.156	0.312	0.031
1	1	1	1	1	0	1	0	0	0	0	0	1.0	1.500	1.375	0.500	0.062	0
1	1	1	1	0	0	1	0	0	0	0	0	1.0	1.375	0.750	0.125	0	0
1	1	1	0	0	0	1	0	0	0	0	0	1.0	1.000	0.250	0	0	0
1	1	0	0	0	0	1	0	0	0	0	0	1.0	0.500	0	0	0	0
1	0	0	0	0	0	1	0	0	0	0	0	1.0	0	0	0	0	0
(a) upwind scheme						(b) downwind scheme						(c) centered scheme					

Table 5.1. *Calculated evolution of the function* $f(x,t)$ *for three numerical schemes*

The *downwind scheme cannot transmit* the numerical values in the direction of the flow. It is contrary to the physical nature of the problem studied. Regardless of the values specified on the straight line $x = 0$, we will have a discontinuity between these and the values of the function f on the neighboring points.

The *centered scheme* transmits boundary condition information at $x = 0$ in a partial and deformed manner: if the upstream and downstream values are very different, this scheme will favor the larger, and this may lead to numerical oscillations.

These examples, albeit rather rudimentary, show clearly that *it is not simply by increasing the order of the numerical schemes that we can hope to improve the results*. The material balance can only be assured by ensuring the transfer of information along the trajectories $u - ct = \text{const}$.

Regardless of the nature of a flow, *the characteristic curves constitute privileged lines of information transmission* and any numerical scheme which does not completely respect this constraint will lead to divergence and instability of the computation. As the trajectories and the characteristic curves of wave propagation are generally unknown, the respect of this condition introduces notable complications.

5.6.2. *A complex example*

We will now consider the more complex practical case of an unsteady 1D inviscid compressible flow governed by the equations of section 5.3.2.1. Let us indicate the experimental conditions of the problem modeled. The supply pipes of a thermal engine are dimensioned such that the quantity of air supplied is maximum at a suitable operating condition. The amplitudes of the velocity pulsations may be large on account of acoustic resonances which are contrived in order to ensure the maximum air supply. The *air entropy is not uniform* as it issues from zones which may be more or less heated, but *thermodynamic transformations are isentropic.*

We have instantaneous measurements of the pressure $p(x,t)$ and the velocities $u(x,t)$ at the extremities $x = 0$ and $x = \ell$ of the pipe system studied. We wish to calculate the distribution of physical quantities in the pipe (pressure, velocity, entropy, etc.) from the said measurements. The calculation domain was discretized and different high precision calculation methods were employed. As a general rule, with the exception of method of characteristics, these all led to the production of a *numerical shock* for the entropy values.

The three families of characteristics defined in section 5.3.2.1 for this system are:

– characteristics C_1: $\delta x - u\delta t = 0$ with: $\delta p - c^2\delta\rho = 0$;

– characteristics C_2: $\delta x - (c+u)\delta t = 0$ with: $\delta p + \rho c\delta u = 0$;

– characteristics C_3: $\delta x + (c-u)\delta t = 0$ with: $\delta p - \rho c\delta u = 0$.

These are shown in Figure 5.15 with their domain of influence: the families C_1 and C_2 transport their associated characteristic variables from the axis $x = 0$, whereas the family C_3 leads to the propagation of its characteristic variable $\delta p - \rho c\delta u = 0$ at the speed $-c + u$ from the straight line $x = \ell$. The characteristic variables are, by their nature, “input” variables in the region where their associated characteristic curves enter the domain. Their associated “exit” values at the other extremity of an interval cannot be given conditions without being in contradiction with the mathematical structure of the system of equations.

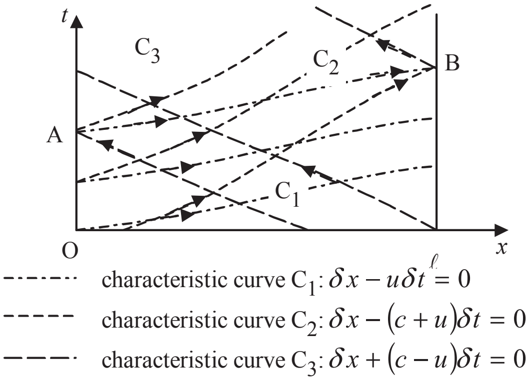


Figure 5.15. *Characteristic curves of an unsteady, subsonic, 1D flow*

The boundary conditions concerning p and u at the extremities of the interval $[0,\ell]$ are thus the sum of the "entry" data and the "exit" values: the values of p and u at the point A are initial conditions to be given on the characteristics C_1 and C_2, and the result of propagation on the characteristic C_3. The situation is similar at point B, but with a single datum on C_3. *Laying down three conditions on p and u at x = 0 and x =* ℓ *amounts to an implicit specification of the "entry" conditions on the three families of characteristics* C_1, C_2 *and* C_3. In order to *correctly lay down* the problem, it is thus necessary to specify the suitable information (as a function of time), i.e., preferentially, two boundary conditions on the left $(x = 0)$ and one boundary condition on the right $(x = \ell)$. *These boundary conditions are combinations of the "to be given" characteristic variables (entry variables) and unknowns (exit variables).*

Numerical information concerning any physical quantity at a point is issued from three different progresses, each bringing a partial contribution to the value of this quantity. Now, only an upwind discretization scheme is suitable for information to progress in one direction (section 5.6.1). It follows that any discretization of the physical quantity is necessarily inconsistent with at least one of the three progressions of the information. *Using a numerical scheme, be it of high precision or otherwise, which does not take into account the preceding physical (or mathematical) reality can only lead to difficulties in the calculation and to the appearance of numerical oscillations and discontinuities which are incompatible with the desired solution.* The solution can only be obtained by using a characteristics method with a suitable discretization scheme; we note that the solution of this discretized system cannot be obtained by a computation from place to place [SAN 97].

5.6.3. *Boundary conditions of flow problems*

We saw earlier how the values of the variables of a problem move along the characteristic curves. The preceding example shows the difficulties which can be encountered when we try to correctly write the boundary conditions of a flow problem which is often posed in an open domain. The physical quantities of the fluid entering the domain must be given. Even if we assume that there is no propagation in directions opposed to the trajectories (incompressible or supersonic flow), we still have to deal with three principle difficulties:

– in the region where the fluid enters the domain, the velocity and pressure fields must satisfy the dynamic equations which are used: this condition, which is satisfied by a uniform flow, is often difficult to meet for other kinds of flow, even if they are steady;

– furthermore, no characteristic curves must go out through the initial curves or surfaces unless suitable compatibility conditions are verified, both for compressible flows and incompressible flows; for instance, in certain near-wall regions (separation around the downstream part of an obstacle or in a divergent conduit (section 6.5.3.7)) the flow of a fluid can be in the opposite direction to the main flow: the corresponding entry zones, situated downstream, depend therefore on the structure of the solution of the problem;

– finally, in the case of real, measured, fluid values used as domain boundary conditions, measurement errors may lead to computational difficulties in so far as they may correspond to entry (or exit) conditions which are incompatible with the problem which is posed;

The considerations developed in this chapter concern flows of inviscid fluids which are represented by either elliptic or hyperbolic equations. The presence of viscous structures along certain trajectories in high Reynolds number flows (section 6.5.3) often leads to parabolic equations along these trajectories, so creating an enlargement of the influence domain of the initial conditions along these (see section 5.4.5.4).

In summary, with the exception of some simple situations, the specification of boundary conditions for a flow problem is often a delicate operation, and it is extremely difficult accomplish in a rigorous fashion.

Fundamentals of Fluid Mechanics and Transport Phenomena
Jean-Laurent Peube

Chapter 6

General Properties of Flows

In this chapter we will study some general physical properties of flows which result from the structure of the balance equations. Transport and propagation phenomena are always present, even in systems with uniform initial conditions. The dynamics of fluids and transfers imply coupled phenomena with multiple interactions. In the simplest cases, non-dimensional parameters can be identified which characterize the ratio of orders of magnitude between the terms corresponding to two phenomena, and this ratio is generally small or large with respect to 1. The dynamics of fluids and transfer is thus the domain of perturbation phenomena which lead to singular structures.

After examining the vortex properties, we will discuss flow properties associated with uniform initial conditions which lead to a relative simplification due to the existence of a potential. The third part of this chapter will deal with the study of orders of magnitude and perturbation problems. Quasi-1D approximations in pipes and boundary layers are then discussed. The last part will be dedicated to a short presentation of unsteady phenomena in flows.

6.1. Dynamics of vorticity

6.1.1. *Kinematic properties of the rotation vector*

6.1.1.1. *Definitions*

These are associated with definition [3.38] of the *rotation vector* $\vec{\omega} = \frac{1}{2}\, r\vec{o}t\ \vec{V}$ given in section 3.3.4. Let us recall that *vorticity vector* $2\vec{\omega}$ (section 4.3.3) has obviously the same properties as $\vec{\omega}$. At a given instant, we call:

– *surface of rotation or vortex surface*, a surface which at each of its points is tangent to the vector $\vec{\omega}$;

– *line of rotation* or *vortex line*, a line which is tangent to the vector $\vec{\omega}$ at each of its points;

– *tube of rotation* or *vortex tube*, a surface generated by the lines of rotation relying on a closed contour.

The divergence of the rotation vector is clearly zero, and the *flux of the vortex vector across any closed surface is also zero*.

6.1.1.2. *Circulation of the velocity vector* $\vec{V}$

Let Γ_{AB} be the circulation of the velocity vector along an arc *AB*:

$$\Gamma_{AB} = \int_{AB} \vec{V}.\vec{dl}$$

In the case where the curve *C* is closed, the circulation Γ_C is equal to the flux of the vector $\overrightarrow{rot}\,\vec{V}$ across a surface *S* relying on the contour *C* (Stokes' theorem):

$$\Gamma_C = \int_C \vec{V}.\vec{dl} = \int_S 2\,\vec{\omega}\,.\,\vec{n}\; ds \qquad [6.1]$$

The circulations Γ_{C1} and Γ_{C2} *along the two closed curves* C_1 *and* C_2 *situated on the same tube of rotation which they move around the same number of times are equal.*

In effect, let *D* be the inner domain of a tube of rotation which is bounded by two surfaces S_1 and S_2 whose contours on the rotating tube are, respectively, the closed curves C_1 and C_2. The flux of the vortex across the lateral surface of the tube and across the surface Σ which bounds the fluid domain *D* is zero; the result of this is that by orientating, continuously along the tube, the normals $\vec{n}$ across the surfaces S_1 and S_2, we have equality of the fluxes of the vector $\vec{\omega}$ across the two surfaces, and consequently across all sections of the vortex tube. This results in the equality of the circulations Γ_{C1} and Γ_{C2}.

The *intensity of a vortex tube* is defined by the circulation value of the velocity vector along a closed curve encircling the tube once counter-clockwise.

The *circulation* Γ_C can be calculated *by following the matter*. Using result [3.36] of section 3.3.3.4 concerning the material derivative of the integral of the flux of the conservative vector field $\vec{B}$ ($div\vec{B} = 0$) across the material surface S of fluid in movement, we can express the material derivative of the circulation Γ_C in the form:

$$\frac{d\Gamma_C}{d\,t} = 2\frac{d}{dt}\left(\int_S \vec{\omega}\,.\,\vec{n}\;ds\right) = 2\int_S \left(\frac{\partial\,\vec{\omega}}{\partial\,t} + r\vec{o}t\,\left(\,\vec{\omega}\wedge\vec{V}\right)\right).\,\vec{n}\;ds \qquad [6.2]$$

This expression will be useful for the demonstration of Lagrange's theorem.

6.1.1.3. *The Biot and Savart formula*

Any vector field can be decomposed into a field with zero divergence and an irrotational field. The latter is a gradient field derived from a potential. The zero-divergence field is a rotational field, defined to a near gradient. Knowledge of the rotation vector $\vec{\omega}$ allows the computation of the rotational part $\vec{V}_r$ of the velocity field $\vec{V}$. Letting $\overline{V_r} = \overrightarrow{rot}\vec{A}$ and taking into account $div\vec{A} = 0$ [1] gives:

$$2\vec{\omega} = \overrightarrow{rot}\,\overline{V_r} = \overrightarrow{rot}\left(\overrightarrow{rot}\,\vec{A}\right) = \overrightarrow{grad}\,div\vec{A} - \vec{\Delta}\vec{A} = -\vec{\Delta}\vec{A}$$

The components of the vector $\vec{A}$ in Cartesian coordinates satisfy the Poisson equation $\Delta A_i = -2\omega_i$ whose solution is:

$$A_i\left(x_j\right) = \frac{1}{2\pi}\int_D \frac{\omega_i\left(x_j'\right)}{R\left(x_j, x_j'\right)}dv' \quad \text{avec} : R^2 = \sum_{i=1}^{3}\left(x_j - x_j'\right)^2$$

Taking the curl of this expression gives:

$$\vec{V}_r = \frac{1}{2\pi}\int_D \vec{\omega}\wedge\frac{\vec{r}}{r^3}dv$$

which is the formula of Biot and Savart for a magnetic field, where the current density is equivalent to the vortex, and the velocity corresponds to the magnetic field. Note that this result, which is purely kinematic, is not related to any assumption regarding the nature of the fluid or the flow.

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[1] The vector $\vec{A}$ is not defined in a unique way; we can add any gradient at $\vec{A}$ without changing the values of $\vec{V}_r$. Thus, we choose this gradient vector so that $div\vec{A} = 0$.

6.1.1.4. *The velocity field induced by a vortex*

At a given instant, consider the tube of a cylindrical vortex, of radius r_0, and suppose that $\omega = \left\| \vec{\omega} \right\|$ is uniform over a cross-section S_0 of the tube and zero outside the tube.

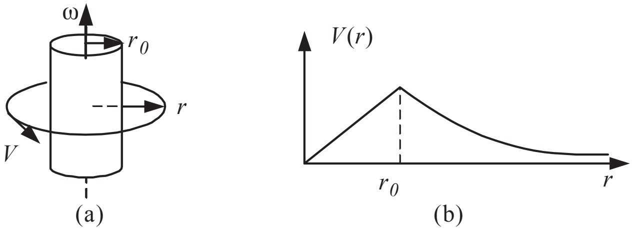

Figure 6.1. *Velocity around an uniform vortex tube*

Consider the circular surface S in the plane of a cross-section of the tube, centered on the axis of the tube, and of radius r (Figure 6.1a). The circulation $\Gamma_C = 2\pi Vr$ of the velocity along the circle C of radius r is equal to the vector flux $\overrightarrow{rot}\,\vec{V}$ across the surface of the circle of radius r. This flux can be expressed in two different ways, depending on the relative values of r and r_0. We can write:

$$r \le r_0 \quad V = \omega\ r; \qquad r \ge r_0 \quad V = \frac{\omega\ r_0^2}{r}$$

The velocity induced outside the vortex tube decays as 1/r, whereas it grows linearly with r on the inside of the vortex tube (Figure 6.1b).

6.1.1.5. *Material derivative of the rotation vector*

Let us introduce the material derivative of the rotation vector:

$$\frac{d\vec{\omega}}{dt} = \frac{\partial \vec{\omega}}{\partial t} + \left(\overline{\overline{grad}}\ \vec{\omega} \right).\vec{V} \qquad \left(\frac{d\omega_i}{dt} = \frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} \right)$$

We can write the following identities for the arbitrary vectors $\vec{A}$ and $\vec{B}$ [HAR 98]:

$$\overrightarrow{rot}\left(\vec{A} \wedge \vec{B} \right) = \left(\overline{\overline{grad}}\ \vec{A} \right).\vec{B} + \vec{A}.div\vec{B} - \left(\overline{\overline{grad}}\ \vec{B} \right).\vec{A} - \vec{B}.div\vec{A}$$

or, in this case ($div\vec{\omega} = 0$):

$$\overrightarrow{rot}\left(\vec{\omega}\wedge\vec{V}\right)=\left(\overline{\overline{grad}}\ \vec{\omega}\right).\vec{V}+\vec{\omega}.div\vec{V}-\left(\overline{\overline{grad}}\ \vec{V}\right).\vec{\omega}$$

$$=\overrightarrow{div}\left(\vec{\omega}\otimes\vec{V}\right)-\left(\overline{\overline{grad}}\ \vec{V}\right).\vec{\omega}$$

From expression [3.37] for $\overline{\overline{grad}}\ \vec{V}$ as a function of the tensor $\overline{\overline{\varepsilon}}$ of the strain-rates and of the anti-symmetric tensor $\overline{\overline{\Omega}}$, we can derive the relation: $\overline{\overline{grad}}\,\vec{V}.\vec{\omega}=\overline{\overline{\varepsilon}}.\vec{\omega}$; hence:

$$\overrightarrow{rot}\left(\vec{\omega}\wedge\vec{V}\right)=\left(\overline{\overline{grad}}\ \vec{\omega}\right).\vec{V}+\vec{\omega}.div\vec{V}-\overline{\overline{\varepsilon}}.\vec{\omega}=\overrightarrow{div}\left(\vec{\omega}\otimes\vec{V}\right)-\overline{\overline{\varepsilon}}.\vec{\omega}$$

Finally we obtain the expression:

$$\frac{\partial\vec{\omega}}{\partial t}+\overrightarrow{rot}\left(\vec{\omega}\wedge\vec{V}\right)=\frac{d\vec{\omega}}{dt}+\vec{\omega}.div\vec{V}-\overline{\overline{\varepsilon}}.\vec{\omega}=\frac{\partial\vec{\omega}}{\partial t}+\overrightarrow{div}\left(\vec{\omega}\otimes\vec{V}\right)-\overline{\overline{\varepsilon}}.\vec{\omega}\qquad[6.3]$$

We can associate the volume quantity $\vec{\omega}$ with the mass quantity $\dfrac{\vec{\omega}}{\rho}$; taking account of the mass conservation equation (see section 4.2.1.2.1 and formula [4.9]) leads to (Helmholtz):

$$\frac{\partial\vec{\omega}}{\partial t}+\overrightarrow{div}\left(\vec{\omega}\otimes\vec{V}\right)=\rho\frac{d}{dt}\left(\frac{\vec{\omega}}{\rho}\right)\qquad[6.4]$$

6.1.2. *Equation and properties of the rotation vector*

6.1.2.1. *The vorticity equation in the form of a balance equation*

Equation [4.41] (section 4.3.3), which is satisfied by the rotation vector $\vec{\omega}$ for *any given fluid*, can be written by taking account of expressions [6.3] and [6.4] in one of the two forms:

$$\left.\begin{array}{r}\dfrac{\partial\vec{\omega}}{\partial t}+\overrightarrow{div}\left(\vec{\omega}\otimes\vec{V}\right)\\[2ex] \rho\dfrac{d}{dt}\left(\dfrac{\vec{\omega}}{\rho}\right)\end{array}\right\}=\overline{\overline{\varepsilon}}.\vec{\omega}+\frac{1}{2\rho^{2}}\overrightarrow{grad}\,\rho\wedge\overrightarrow{grad}\,p+\overrightarrow{rot}\left(\frac{1}{2\rho}\overrightarrow{div}\,\overline{\overline{\tau}}\right)\qquad[6.5]$$

Equation [6.5] has the form of a balance equation of a volume vector quantity (equation [4.3], section 4.1.1.2). It can be interpreted as a balance of the rotation vector $\vec{\omega}$, considered as a volume density of an extensive quantity (with which we can associate the mass quantity $\vec{\omega}/\rho$).

For a divariant compressible, by taking rotational of $\overrightarrow{grad}\, h$ (relation [4.17]), we have:

$$\frac{1}{\rho^2}\overrightarrow{grad}\,\rho \wedge \overrightarrow{grad}\,p = \overrightarrow{grad}\,T \wedge \overrightarrow{grad}\,s$$

Vorticity equation [6.5] is thus a transport equation comprising:

– two volume source terms associated respectively:

  - with the usual properties of the *kinetic effects of the rotation* ($\overline{\overline{\varepsilon}}\,.\vec{\omega}$);
  - with the movement associated with mechanical or thermodynamic imbalance;

– a *viscous diffusion* term.

If the mechanical equilibrium condition ($\overrightarrow{grad}\,\rho \wedge \overrightarrow{grad}\,p = 0$) of a fluid in a force field is not satisfied, a rotational movement will result. For example, a horizontal pressure gradient in a fluid with a vertical density gradient, which is initially at rest, will create a horizontal acceleration inversely proportional to ρ and therefore a horizontal velocity gradient.

As for the viscous stresses, their role is essential in diffusing the rotation, as we will see in an example (section 6.1.2.4.1). We should furthermore note that the creation of a viscous flow (Poiseuille flow, boundary layer, etc.) from a non-viscous flow is accompanied by the creation of vorticity as a result of the adherence condition at the wall, which creates a shear flow in the vicinity of the wall (section 3.4.2.5), which is necessarily rotational. For the sake of simplicity, we will not give detailed expressions of the viscous stresses here.

For the case of an incompressible Newtonian fluid ($div\vec{V} = 0$) with constant viscosity and specific mass, we have $\overrightarrow{div}\,\overline{\overline{\tau}} = -\mu\overrightarrow{rot}\left(\overrightarrow{rot}\vec{V}\right)$ (section 3.4.3.3) and we obtain the following form of equation:

$$\frac{\partial\vec{\omega}}{\partial t}+\overrightarrow{rot}\,(\vec{\omega}\wedge\vec{V})=\nu\overrightarrow{\Delta\omega} \quad \text{or: } \frac{d\vec{\omega}}{dt}=\overline{\overline{\varepsilon}}.\vec{\omega}+\nu\overrightarrow{\Delta\omega} \qquad [6.6]$$

For *an inviscid fluid* which is homentropic or with constant specific mass, the preceding equations are simplified:

$$\begin{cases} \dfrac{\partial \vec{\omega}}{\partial t} + \overrightarrow{rot}\left(\vec{\omega} \wedge \vec{V}\right) = 0 \\ \dfrac{\partial \vec{\omega}}{\partial t} + \overrightarrow{div}\left(\vec{\omega} \otimes \vec{V}\right) = \rho \dfrac{d}{dt}\left(\dfrac{\vec{\omega}}{\rho}\right) = \overline{\overline{\varepsilon}}\,.\vec{\omega} \end{cases} \qquad [6.7]$$

6.1.2.2. *Interactions between vorticity and strain rates*

We will characterize the effects of the volume source term $\overline{\overline{\varepsilon}}\,.\vec{\omega}$ ($\omega_j.\varepsilon_{ij}$) which translates *an interaction between vorticity and the strain-rate tensor*. In order to simplify matters let us consider equation [6.7] for a constant specific mass:

$$\frac{d\vec{\omega}}{dt} = \overline{\overline{\varepsilon}}.\vec{\omega} + \ldots \quad \text{or:} \quad \frac{d\omega_i}{dt} = .\varepsilon_{ij}\omega_j + \ldots$$

Let us first of all examine the effect of *the first component* ω_1 *of the rotation vector on itself.* The corresponding equation can be written:

$$\frac{d\omega_1}{dt} = \omega_1.\varepsilon_{11} + \ldots$$

This demonstrates *the creation of* ω_1 *if the source term* $\omega_1.\varepsilon_{11}$ *is positive*, for example if ω_1 and ε_{11} are positive; thus, ε_{11} is the rate of increase of the component u_1 of the velocity in the x_1 direction (section 3.3.4), and it leads to a stretching of the matter along this axis: *a stretching of the matter along a given direction increases the corresponding component of the rotation vector* (Figure 6.2a). This result is in fact analogous to the intensity conservation of a vortex tube, the stretching rate being inversely proportional to the variation velocity of the tube cross-section.

Now consider the *source term* $\omega_1\varepsilon_{21}$ *in the equation for the component* ω_2:

$$\frac{d\omega_2}{dt} = \omega_1.\varepsilon_{21} + \ldots$$

The strain-rate ε_{21} amounts to a shearing of the type $\dfrac{\partial u_1}{\partial x_2}$ (section 3.3.4). We therefore see here the creation of the component ω_2, along the axis Ox_2, by the component ω_1 (Figure 6.2b), in other words a *tilting of the rotation vector* (a gyroscopic effect).

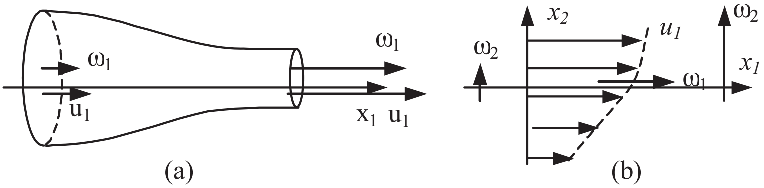

Figure 6.2. *Deformation of a vortex: (a) stretching in a lengthening velocity; (b) tilting in a shear velocity*

These properties are true for the three spatial directions; their effect is the *creation and maintenance of the 3D character of rotational flows*.

If the fluid is compressible, the second term of material derivative [6.3] contains the term $-\vec{\omega}.div\vec{V}$ which translates a reduction of $\vec{\omega}$ proportional to the expansion $div\vec{V}$, leaving the angular velocity constant in a *material volume*. This term is contained in the first term of vorticity equation [6.5] written with the mass quantity $\vec{\omega}/\rho$.

6.1.2.3. *The 2D plane flow*

The vorticity source term $\omega_j.\varepsilon_{ij}$ presents a marked 3D character but disappears in a 2D plane flow, because the rotation vector has only a component ω perpendicular to the velocity plane, and so we have:

$$\omega_j.\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} . = 0$$

In the 2D case, the rotation vector (or the vortex) satisfies the scalar equation:

$$\frac{d\ \omega}{d\ t} = \nu\ \Delta\omega \qquad [6.8]$$

In a 2D plane flow, *the vorticity equation is a convection-diffusion equation* which ensures the conservation of the rotation in all space (see definitions of diffusion terms in Chapter 2).

For an inviscid fluid, scalar equation [6.8] for the vorticity is an equation describing the transport of vorticity by the matter:

$$\frac{d\omega}{dt} = 0 \qquad [6.9]$$

6.1.2.4. *Diffusion of the vorticity in a viscous fluid*

By way of an example, consider equation [6.8] of an incompressible Newtonian fluid in plane two-dimensional flow. It has the form of a heat convection equation (section 4.3.4.1.6) or of the equation for the diffusion of chemical species (in weak concentration) with source terms and diffusion terms: *vorticity is diffused by viscous action*.

Let us examine the case of a vortex system of revolution about the axis Oz and whose velocity field, parallel to the plane Oxy and of zero radial component, has a tangential component equal to $V\theta(r,t)$. The vortex vector field $\omega(r,t)$, parallel to Oz, is a function of *r* and *t*. Equation [6.8] can be written:

$$\frac{\partial \omega}{\partial t} = \nu \, \Delta\omega \qquad [6.10]$$

or, in plane polar coordinates:

$$\frac{\partial \omega}{\partial t} = \frac{\nu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \omega}{\partial r} \right) \qquad [6.11]$$

The reader can easily verify that equation [6.11] has a class of solutions[2] in the form $t^n f(\eta)$ where we have introduced the new variable $\eta = r^2 / 4\, \nu\, t$. The function $f(\eta)$ satisfies a differential equation which depends on the parameter *n*.

The circulation $\Gamma(r,t)$ of the velocity vector along the circle centered on the origin and of radius r is equal to the flux of the vector 2ω across this circle:

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[2] These solutions are "self-similar" like solution [5.52] (see footnote 5 in section 5.4.5.4).

$$\Gamma(r,t) = 2t^n \int_0^r f\left(\frac{u^2}{4\nu\, t}\right) 2\pi\, r dr = 8\pi\,\nu\, t^{n+1} \int_0^{r^2/4\nu\, t} f(v) dv = 8\pi\,\nu\, t^{n+1} \int_0^{\eta} f(v) dv$$

Suppose that after the initial instant $t = 0$, there is *no source of vorticity anywhere in space*; the circulation on the circle whose radius tends to infinity remains constant, and this leads to the choice $n = -1$. Substituting this expression for ω into [6.11] we obtain the differential equation satisfied by the function $f(\eta)$:

$$(\eta\ f)' + (\eta\ f')' = 0$$

A first integration gives:

$$f + f' = \frac{\text{const}}{\eta}$$

Integrating a second time, taking the constant of integration to be zero such that the circulation remains constant for infinite η, the vorticity tending therefore to zero. This immediately gives the desired solution:

$$\omega(r,t) = \frac{A}{t}\exp(-\eta) = \frac{A}{t}\exp\left(\frac{-r^2}{4\,\nu\,t}\right) \qquad (\text{A} = \text{const})$$

The circulation $\Gamma(r,t)$ of the velocity vector along a circle of radius r can be expressed as:

$$\Gamma(r,t) = 8\,\pi\,A\,\nu\left[1 - \exp\left(\frac{-r^2}{4\,\nu\,t}\right)\right]$$

Its value is constant and equal to $8\,\pi\,A\,\nu$ for sufficiently large r. The velocity V_θ(r,t) can be calculated from the circulation:

$$V_\theta(r,t) = \frac{\Gamma(r,t)}{2\pi r} = \frac{4\,\nu\,A}{r}\left[1 - \exp\left(\frac{-r^2}{4\,\nu\,t}\right)\right]$$

This solution represents the diffusive spreading of a Dirac impulse of vorticity placed at the origin (Figure 6.3). The velocity distribution decays as 1/*r* outside the viscous zone (section 6.1.1.4) which contains the vorticity. The velocity gradient diminishes under the effect of viscosity which has no further real effect for a radius a little greater than that at which the maximum is located.

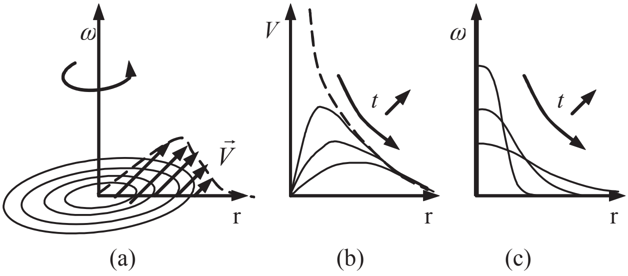


Figure 6.3. *(a) Vorticity zone, (b) diffusion of the velocity and (c) of the vorticity*

For a fixed radius *r*, when *t* increases, the circulation and the velocity decay so as to tend to zero as time tends to infinity: the rotation initially concentrated on the axis diffuses over time across the entire fluid under the action of viscosity. Defining the radius $R_v(t)$ of the viscous core using the condition that this contains 99% of the circulation of the velocity vector ($\exp(-\eta) = 0.01$ or $\eta =$ 4.605), we have: $R_v = 4.29\sqrt{\nu t}$.

6.1.2.5. *Lagrange's theorem*

Consider an inviscid fluid whose entropy is uniform if it is compressible. Taking account of vorticity equation [6.8], material derivative [6.2] of the circulation of the velocity on a closed curve *C* enclosing the surface *S* can be written:

$$\frac{d\,\Gamma_C}{d\,t} = 2\int_S \left(\frac{\partial\,\vec{\omega}}{\partial\,t} + \vec{rot}\left(\vec{\omega} \wedge \vec{V}\right)\right).\,\vec{n}\;ds = 0$$

Lagrange's theorem: the circulation of the velocity vector on a material curve, or the flux of a vorticity vector across a material surface, is conserved during movement. In particular, if the flux of the vorticity vector across a material surface S is zero at an instant t, it will remain zero thereafter.

As we are dealing with a material derivative, the surface *S* is constituted of fluid particles. It should not contain any singularity leading to the creation of vorticity (for example, a vortex whose intensity varies with time). We can deduce the following very important consequences:

– if, at a given instant, the flux φ_S is zero in a domain *D* of a flow regardless *of the surface S*, *the flow is irrotational in the domain D* (it suffices to take three elementary orthogonal surfaces to verify that the vector $\vec{\omega}$ is necessarily zero). From Lagrange's theorem we thus see that the flow remains irrotational afterwards in the material domain *D*. This situation is encountered when a flow issues from a fluid region at rest or of uniform velocity;

– a *vortex surface* (or *rotation* surface) is a surface to which the vorticity vector is tangent at an instant *t*; it *moves whilst remaining a vortex surface* (the flux of the vortex remains zero on all elementary surface of rotation). In particular, a vortex tube remains a vortex tube during any displacement of the matter of which it is constituted. Considering the circulation of the velocity along a curve situated on the tube and which encircles it once, we see that the intensity of a vortex tube remains constant during its displacement: *the vortex tube transports its circulation*; this can be easily seen in a rotational smoke ring (a closed rotation tube) in which smoke makes the motion of the matter and its rotation visible;

– a *vortex line* (*or rotation line*) at instant *t* can be considered as an intersection of two vortex surfaces: it therefore remains a vortex line. This results in *vortex lines being displaced with the matter.*

NOTES –

1) The notion of circulation on a closed material curve *C* is essential: in effect, it deforms during its displacement with the matter. It can eventually be divided into two curves C_1 and C_2 when passing an obstacle (Figure 6.4), but it cannot be transformed into a third curve C_3 (Figure 6.4). The sum of the circulations Γ_{C1} and Γ_{C2} over the curves C_1 and C_2 is equal to the circulation Γ_C over the curve *C*, whereas the circulation Γ_{C3} over the curve C_3 can take on any other value. We will use an elementary example (section 6.2.5.2.2) for which the circulations $\Gamma_{C,}$ Γ_{C1} and Γ_{C2} are zero, whereas Γ_{C3} is non-zero. This property is related to the structure of the surface *S* interior to C_3 which is not simply connected (to put it simply, it contains a "hole") and the vector field is not continuously differentiable on the interior of C_3.

2) Lagrange's theorem does not express the transport of the vorticity vector by the matter; it only expresses a more global property. For the reasons given above, we say that the vector field $\vec{\omega}$ which satisfies equation [6.7] is "frozen in the moving medium".

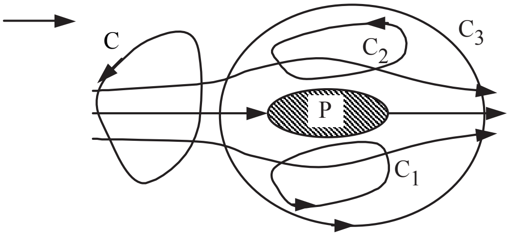


Figure 6.4. *Evolution of closed material curves in the flow around an obstacle*

3) The reader will note that once again we recover here a property of flow-information transfer over characteristic curves associated with convection.

6.2. Potential flows

6.2.1. *Introduction*

Lagrange's theorem expresses the property of transport of circulation of the velocity over any curve *C* in the flow of an inviscid, homentropic fluid if the fluid is compressible. When the circulation is zero, we can perform a partial integration of the equations of fluid mechanics over the family of characteristic curves constituted by the trajectories: the flow is therefore irrotational and the velocity field derives from a potential:

$$u_i = \frac{\partial \varphi}{\partial x_i} \quad \left(\text{or : } \vec{V} = \overrightarrow{grad}\varphi\right) \tag{6.12}$$

6.2.2. *Bernoulli's second theorem*

Dynamic equation [4.19] can be written in the form:

$$\frac{\partial \vec{V}}{\partial t} + \overrightarrow{grad}\left(\frac{V^2}{2} + U + h\right) = 0$$

with: *h* the specific enthalpy (for a perfect gas: $h = C_p T$) and *U* the potential of the gravitational forces ($U = gz$, height *z* being taken on a vertically ascending axis).

The existence of a velocity potential φ allows the immediate integration of the above equation with respect to the space variables, giving *Bernoulli's second theorem*:

$$\frac{\partial\phi}{\partial t}+\frac{V^2}{2}+gz+h=\text{const} \tag{6.13}$$

We note that equation [6.13] is valid everywhere in the domain of study and for an *unsteady flow,* contrary to Bernoulli's first theorem, which can only be applied in a steady flow over a streamline. For an incompressible fluid, it can be written:

$$\rho\frac{\partial\phi}{\partial t}+\rho\left(\frac{V^2}{2}+gz\right)+p=\text{const} \tag{6.14}$$

6.2.3. *Flow of compressible inviscid fluid*

The partial differential equation satisfied by the velocity potential φ is a quasi-linear second order equation and is derived from the Euler equations. As integration has already been performed on the trajectories, the equation has only two families of real or imaginary characteristic curves or surfaces which we have already seen (section 5.4.2).

The mass conservation equation in the form [4.7] can be written:

$$\frac{1}{\rho}\frac{d\rho}{dt}+\frac{\partial u_i}{\partial x_i}=0 \tag{6.15}$$

For a divariant fluid in homentropic flow, we have $dp = c^2 d\rho$ and $dp = \rho\, dh$. Replacing $d\rho$ as a function of dh in equation [6.15] gives:

$$\frac{1}{c^2}\frac{dh}{dt}+\frac{\partial u_i}{\partial x_i}=0 \qquad \left(\frac{1}{c^2}\frac{dh}{dt}+div\vec{V}=0\right) \tag{6.16}$$

The potential equation can therefore be obtained by replacing the enthalpy h in equation [6.16] by its expression as obtained from equation [6.13]:

$$\Delta\varphi-\frac{1}{c^2}\frac{d}{dt}\left(\frac{\partial\varphi}{\partial t}+\frac{\overrightarrow{grad}^2\varphi}{2}+gz\right)=0 \tag{6.17}$$

For the flow of a compressible fluid, we can in general neglect the gravitational term and potential equation [6.17] becomes:

$$\Delta\varphi - \frac{1}{c^2}\frac{d}{dt}\left(\frac{\partial\varphi}{\partial t} + \frac{\overrightarrow{grad}^2\,\varphi}{2}\right) = 0 \qquad [6.18]$$

or, by developing:

$$-\frac{1}{c^2}\left(\frac{\partial^2\varphi}{\partial t^2} + 2\frac{\partial^2\varphi}{\partial t\partial x_j}\frac{\partial\varphi}{\partial x_j}\right) - \frac{1}{c^2}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j} + \frac{\partial^2\varphi}{\partial x_i\partial x_i} = 0 \qquad [6.19]$$

where, by introducing the usual notation for the velocity components (u, v, w) along the axes (x, y, z):

$$\begin{aligned}\left(1-\frac{u^2}{c^2}\right)\varphi_{xx} + \left(1-\frac{v^2}{c^2}\right)\varphi_{yy} + \left(1-\frac{w^2}{c^2}\right)\varphi_{zz} - 2\frac{uv}{c^2}\varphi_{xy} - 2\frac{vw}{c^2}\varphi_{yz}\\ - 2\frac{wu}{c^2}\varphi_{zx} - \frac{1}{c^2}\varphi_{tt} - 2\frac{u}{c^2}\varphi_{tx} - 2\frac{v}{c^2}\varphi_{ty} - 2\frac{w}{c^2}\varphi_{tz} = 0\end{aligned} \qquad [6.20]$$

The potential equation for *the flow of an incompressible fluid* which can be obtained by letting the velocity sound tend to infinity in [6.17] reduces to Laplace's equation:

$$\frac{\partial^2\varphi}{\partial x_i\partial x_i} = \Delta\varphi = 0 \qquad [6.21]$$

6.2.4. *Nature of equations in inviscid flows*

Equation [6.19] or [6.20] is of the type studied in section 5.4. Writing explicitly the velocity components $u_i = \partial\varphi/\partial x_i$ and the Mach number $M = V/c$, equation [6.19] can be written:

$$-\frac{1}{c^2}\left(\frac{\partial^2\varphi}{\partial t^2} + 2u_j\frac{\partial^2\varphi}{\partial t\partial x_j}\right) - \frac{u_i u_j}{c^2}\frac{\partial^2\varphi}{\partial x_i\partial x_j} + \Delta\varphi = 0$$

The characteristic directions are given by equation [5.37] which corresponds to the preceding equation. Letting $\alpha_i = \partial\xi_1/\partial x_i$ and $\alpha_t = \partial\xi_1/\partial t$ (section 5.3.4), we obtain:

$$-\frac{1}{c^2}\left(\alpha_t^2 + 2\alpha_j\alpha_t u_j\right) - \frac{u_i u_j}{c^2}\alpha_j\alpha_i + \alpha_i\alpha_i = 0$$

This characteristic equation can be written in the reduced form:

$$-\frac{1}{c^2}\left(\alpha_t + \alpha_j u_j\right)^2 + \alpha_i\alpha_i = 0$$

This form is hyperbolic[3] and *the potential equation of a compressible fluid is in general of hyperbolic character*.

In the situation involving acoustic perturbations in a medium at rest, which is obtained when the Mach number approaches zero in equation [6.19] or [6.20], the wave equation is obtained:

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \qquad [6.22]$$

Considering now only *steady solutions*, the potential equation can be written:

$$\frac{\partial^2\varphi}{\partial x_i\partial x_i} - \frac{1}{c^2}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}\frac{\partial^2\varphi}{\partial x_i\partial x_j} = 0 \qquad [6.23]$$

Its characteristic equation can be obtained as before:

$$-\left(\alpha_j\frac{u_j}{c}\right)^2 + \alpha_i\alpha_i = 0$$

or, writing the indices explicitly:

$$\left(\frac{u_1}{c}\alpha_1 + \frac{u_2}{c}\alpha_2 + \frac{u_3}{c}\alpha_3\right)^2 - \alpha_1^2 - \alpha_2^2 - \alpha_3^2 = 0 \qquad [6.24]$$

---

[3] Letting $\alpha_T = \left(\alpha_t + \alpha_j u_j\right)/c$ and $\alpha_i\alpha_i = r^2$, the previous equation can be written $\alpha_T^2 - r^2 = 0$; it represents a cone of revolution around axis $O\alpha_T$ in a 4D space.

The nature of the quadratic form can be easily obtained by a geometric interpretation. Consider the vector $\overrightarrow{OA} = (\alpha_1, \alpha_2, \alpha_3)$. Equation [6.24] can be written in the form:

$$\left(\overrightarrow{OA}.\frac{\vec{V}}{c}\right)^2 - \overrightarrow{OA}^2 = 0 \qquad [6.25]$$

This equality shows that the projection of $\vec{V}/c$ on $\overrightarrow{OA}$ must be equal to OA. The existence of the non-zero vectors $\overrightarrow{OA}$ is possible only if $M = V/c$ is greater than 1, in other words if the flow is supersonic. We recover the result already obtained in section 5.3.2.3. We deduce from [6.25] the value of the angle β between the normal to the characteristic surface and the velocity direction:

$$\cos\beta = c/V = 1/M$$

The velocity thus makes the complementary angle θ with the characteristic surface; and so the result of section 5.3.2.3 is recovered ($\sin\vartheta = 1/M$).

6.2.5. *Elementary solutions in irrotational flows*

6.2.5.1. *Introduction*

We will now examine some elementary solutions in simple examples of potential equations. We will first consider the case of an incompressible fluid. The velocity potential satisfies Laplace's equation. Subsonic flows verifying an elliptic equation have similar properties, but are modified by the compressibility of the fluid ([YIH 77]).

The second case studied is the acoustic wave equation, which can be obtained via linearization and a suitable referential change in equation [6.18] and which represents the local properties of all second order hyperbolic equations.

6.2.5.2. *Irrotational 2D plane flow of an incompressible fluid*

6.2.5.2.1. Introduction

The problem comes down to the solution of Laplace's equation with free-slip conditions imposed at the solid boundaries. The best adapted means for the study of these flows involves the use of complex variables. In effect, relations [6.26] defining the velocity potential φ and the stream function ψ:

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \varphi}{\partial x} \qquad v = -\frac{\partial \psi}{\partial x} = \frac{\partial \varphi}{\partial y} \qquad [6.26]$$

are Cauchy relations between the derivatives of the real and imaginary parts, φ and ψ, of an analytic function $F(z)$ of the complex variable $z = x + jy$:

$$F(z) = \varphi(x, y) + j\psi(x, y) \qquad [6.27]$$

The function $F(z)$ is the *complex potential* of the flow considered. Its derivative $F'(z)$ with respect to z is the *complex velocity* of the expression:

$$F'(z) = u - jv \qquad [6.28]$$

If the function $F(z)$ is analytic, the function $-jF(z)$ is also. The velocity potential and the stream function of $-jF(z)$ are, respectively, ψ and $-\varphi$. The flows associated with the two potentials $F(z)$ and $-jF(z)$ are known as *conjugated flows*. Any analytical function of complex variables thus provides two solutions to the Laplace equation corresponding to two *conjugated flows* where the curves of potential lines of one flow are streamlines of the other.

Consider the integral $\int_C F'(z)dz$ taken *once counter-clockwise* on a closed path of the complex plane *(x, y)*. It can be written as a function of the velocity circulation Γ and of the volume flow rate $q_v = \int d\psi$ (section 4.2.1.2.2):

$$\int_C F'(z)dz = \int_C (udx + vdy + j(udy - vdx)) = \int_C (d\varphi + jd\psi) = \Gamma + jq_v \qquad [6.29]$$

If the closed path C does not surround any poles of the function $F'(z)$, then the function $F(z)$ is uniform: it takes on the same value after any excursion of the variable z on the contour C. The flow across C and the circulation of the velocity on C are zero. If C contains a pole of $F'(z)$, then the value of the function $F(z)$ increases by $\Gamma + jq_v$ with each excursion around C (see an example of the vortex in section 6.2.5.2.2).

As the velocity field is determined by the Laplace equation, the pressure is given by Bernoulli’s second theorem [6.14].

The simplest example of an irrotational flow is a *uniform velocity field* corresponding to the complex potential $F(z) = Uz + B$, where U and B are complex constants. The velocity Cartesian components $u = \mathrm{Re}(U)$ and $v = -\mathrm{Im}(U)$ can be immediately obtained from the complex velocity $F'(z) = U$.

NOTE – Time is not a variable in Laplace's equation, but it can be a parameter present in the boundary conditions and the coefficients of the solution. The result of this is that initial conditions do not have any meaning for this equation.

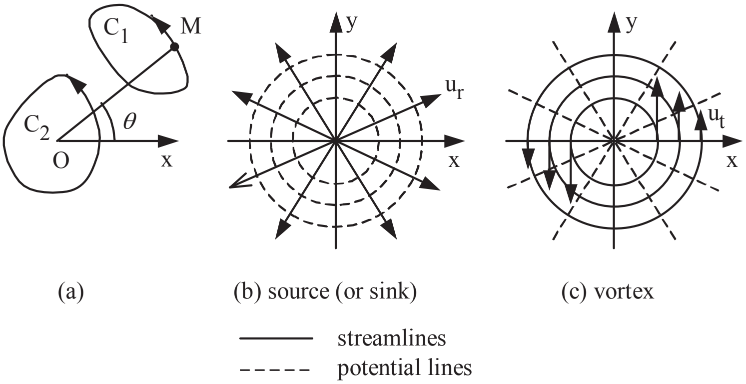
6.2.5.2.2. Source and vortex

A source and a vortex centered at the origin of the coordinate system are conjugated flows corresponding to the complex potential $F(z)$ and to the complex velocity $F'(z)$ expressed in plane polar coordinates (with $z = re^{j\theta}$):

$$F(z) = \frac{A}{2\pi}\ln z = \frac{A}{2\pi}(\ln r + j\theta) \qquad F'(z) = \frac{A}{2\pi z} \qquad [6.30]$$

The application of formula [6.29] to a closed contour C_1 (Figure 6.5a) which does not contain the origin leads to zero volume flow rate and circulation, since after one path counter clockwise, the variation of the polar angle θ is zero. On the other hand, if the closed contour C_2 (Figure 6.5a) contains the origin, the variation of the angle θ is equal to 2π and the integral $\int_C F'(z)dz$ is equal to jA:

$$\int_C F'(z)dz = \Gamma + jq_v = jA \qquad [6.31]$$



(a) (b) source (or sink) (c) vortex

Figure 6.5. *(a) Integral of complex velocity on a closed path, (b) flow of a source and (c) of a vortex*

When the quantity A *is real*, application of formula [6.31] over a closed contour C_2 containing the origin leads to a volume flow rate $q_v = A$, while the circulation Γ of the velocity is zero on all closed curves. The function $F(z)$ represents the *radial flow caused by a source* (positive A) or a *sink* (negative A) of volume flow rate q_v. The potential $F(z)$ and the complex velocity $F'(z)$ can be written:

$$F(z) = \frac{q_v}{2\pi}\ln z = \frac{q_v}{2\pi}\,(\ln r + j\theta) \qquad F'(z) = \frac{q_v}{2\pi z} \qquad [6.32]$$

We deduce from this the velocity potential φ, the stream function ψ and the radial and tangential components, u_r and u_θ, of the velocity vector:

$$\varphi = \frac{q_v}{2\pi}\ln r \quad \psi = \frac{q_v\theta}{2\pi} \quad u_r = \frac{\partial\varphi}{\partial r} = \frac{q_v}{2\pi r} \quad u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = 0 \qquad [6.33]$$

The *potential lines* are circles centered on the origin O and the *streamlines* (ψ = const) are straight lines lying on radii from the origin O (Figure 6.5b).

If the constant A is *imaginary*, application of formula [6.31] to a closed contour C_2 containing the origin leads to the circulation $\Gamma = jA$, or, $A = -j\Gamma$. The volume flow rate is zero across any closed surface C. The function $F(z)$ therefore represents the flow of an *irrotational point vortex centered on the origin* (Figure 6.5c). The potential $F(z)$ and the complex velocity $F'(z)$ can be written:

$$F(z) = -\frac{j\Gamma}{2\pi}\ln z = \frac{\Gamma}{2\pi}\,(\theta - j\ln r) \qquad F'(z) = -\frac{j\Gamma}{2\pi z} \qquad [6.34]$$

We can deduce from this the velocity potential φ, the stream function ψ and the radial and tangential components, u_r and u_θ, of the velocity vector:

$$\varphi = \frac{\Gamma}{2\pi}\theta; \quad \psi = -\frac{\Gamma}{2\pi}\ln r; \quad u_r = \frac{\partial\varphi}{\partial r} = 0; \quad u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \frac{\Gamma}{2\pi r}. \qquad [6.35]$$

The *potential lines* are straight radii coming from the origin and the *streamlines* are circles centered on the origin.

NOTES –

1) The values of the volume flow rate and circulation can be easily found from the components of the velocity by direct calculation; we will leave it to the reader to verify this.

2) The vorticity $\vec{\omega}$ of the irrotational point vortex is zero at all points, except at the origin where it takes on the value of a Dirac impulse, a multiplying factor excepted. The circulation Γ can be alternatively written as the flux of the rotation vector across the surface *S* enclosed by the curve *C*. The flux is only non-zero for surfaces containing the vorticity impulse.

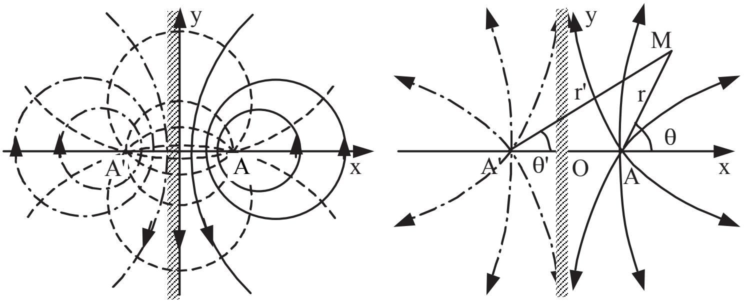
3) When the source, the sink or the vortex are placed at z_0 and not at the origin, the variable *z* in the functions $F(z)$ and $F'(z)$ is replaced by $z - z_0$.

6.2.5.2.3. Superposed flows

Any linear combination of harmonic functions or of analytic functions of complex variables is also a harmonic or an analytic function. We can therefore construct new solutions from known solutions. While the velocity fields can be superposed, the same is not true for the pressure fields, as Bernoulli's theorem is not linear. Let us consider some common simple examples.

The potential and the complex velocity of the *superposition of a source and a sink* with the same flow rate or of *two vortices of opposite circulation* positioned at the points A and A' of coordinates (0, ± *a*) (Figure 6.6a) can be obtained from equation [6.30]:

$$F(z) = \frac{K}{2\pi} \ln \frac{z-a}{z+a} = \frac{K}{2\pi}\left(\ln\left(r_A / r_{A'}\right) + j(\theta - \theta')\right); \quad F'(z) = \frac{K}{\pi} \cdot \frac{a}{z^2 - a^2} \qquad [6.36]$$



(a) (b)

Figure 6.6. *Flow of a vortex (a), of a source (b) placed near a solid wall*

Similarly, the potential and complex velocity of the *superposition of two identical sources* or of *two vortices of the same circulation* positioned at A and A' can be written:

$$F(z)=\frac{K}{2\pi}\ln\left(z^2-a^2\right)=\frac{K}{2\pi}\left(\ln r\,r'+j(\theta+\theta')\right);\qquad F'(z)=\frac{K}{\pi}\cdot\frac{z}{z^2-a^2}.$$

with (Figure 6.6b): $r=AM$; $r'=A'M$; $\theta=\left(\overrightarrow{Ax},\overrightarrow{AM}\right)$; $\theta'=\left(\overrightarrow{Ax},\overrightarrow{A'M}\right)$.

The constant A takes on the value q_v (respectively $-j\Gamma$) for the sources or sinks (respectively two vortices of identical or opposite circulation). The other quantities (φ, ψ and the velocity components) of these flows can also be obtained by taking the difference or sum of the corresponding values of the base flows.

When the axis Oy is a streamline which can be "solidified" (solid boundary with free-slip condition) we have a representation of the flow associated with a vortex or a source in the presence of a plane wall. These interesting specific cases (Figure 6.6) are obtained respectively with:

– *two vortices of opposite circulation* whose stream functions derived from [6.35] are equal to $\psi=-\dfrac{\Gamma}{2\pi}\ln\dfrac{r}{r'}$ and for which we obtain the axis Oy for $r=r'$;

– *two sources of equal flow rate* whose stream function (derived from [6.33]) is: $\psi=\dfrac{q_v}{2\pi}\left(\theta+\theta'\right)$ and for which we obtain the axis Oy for $\theta+\theta'=\pi$.

A *doublet* is a combination of a source and a sink of the same strength, in terms of their absolute value, of which the distance *2a* tends to zero such that the quantity $2aq_v$ is equal to C (moment of the doublet). A series development in a/z in formula [6.36] leads immediately to an expression for the complex potential of the doublet: $F(z)=-C/2\pi\,z$.

The velocity potential φ, the stream function ψ and the radial and tangential components, u_r and u_θ of the velocity vector can be derived:

$$\varphi=-\frac{C\cos\theta}{2\pi\,r}\qquad\psi=\frac{C\sin\theta}{2\pi\,r}\qquad u_r=\frac{C\cos\theta}{2\pi\,r^2};\qquad u_\theta=\frac{1}{r}\frac{\partial\varphi}{\partial\theta}=\frac{C\sin\theta}{2\pi\,r^2}$$

The *potential lines* (respectively the *streamlines*) are circles centered on the axis Ox (respectively Oy) and tangent to the axis Oy (respectively Ox).

6.2.5.2.4. Flow around a circular cylinder

Consider a straight circular cylinder of radius R, of unit extent, placed in a flow (Figure 6.7) whose velocity at infinity is equal to $U\vec{x}$ (U is a constant or a function of time). Let us use polar coordinates and consider the complex potential, a superposition of a uniform flow and a doublet at the origin:

$$F(z) = U_0\left(z + \frac{R^2}{z}\right) \qquad \text{with: } z = re^{i\theta} \tag{6.37}$$

The *circulation* of the velocity vector on a curve surrounding the cylinder is zero. The velocity potential φ, the stream function ψ and the radial u_r and tangential u_θ components of the velocity vector in polar coordinates can be derived from [6.37]:

$$\varphi = U\left(r + \frac{R^2}{r}\right)\cos\theta \qquad \psi = U\left(r - \frac{R^2}{r}\right)\sin\theta$$

$$u_r = \frac{\partial\varphi}{\partial r} = U(1 - \frac{R^2}{r^2})\cos\theta \qquad u_\theta = \frac{1}{r}\frac{\partial\varphi}{\partial\theta} = -U(1 + \frac{R^2}{r^2})\sin\theta$$

We can verify that for $r = R$ we have $\psi = 0 = cte$ and $u_r = 0$: the *circle of radius* R *is a streamline.*

Some particular values of the velocity components allow us to outline the form of the streamlines. In particular, on the circle $r = R$, we have: $u_\theta(R) = -2U\sin\theta$ and on the axis Ox ($\theta = 0$ or π), u_r is positive (negative) for r > R ($r < R$): there exist two points A and A' of zero velocity stagnation points, (section 6.2.5.2.5) of the flow on the cylinder (Figure 6.7a).

When z tends to infinity, $F(z)$ tends to Uz, the complex potential of a uniform flow. The velocity field presents a symmetry with respect to the axes Ox and Oy (between upstream and downstream).

We obtain an *irrotational flow with circulation* Γ around the cylinder of radius R by superposing the preceding flow and a point vortex, whose streamlines are circles centered on the origin. The complex potential and the complex velocity of this flow are:

$$F(z) = U(z + \frac{R^2}{z}) - \frac{j\Gamma}{2\pi}\ln z; \quad F'(z) = U(1 - \frac{R^2}{z^2}) - \frac{j\Gamma}{2\pi z} \tag{6.38}$$

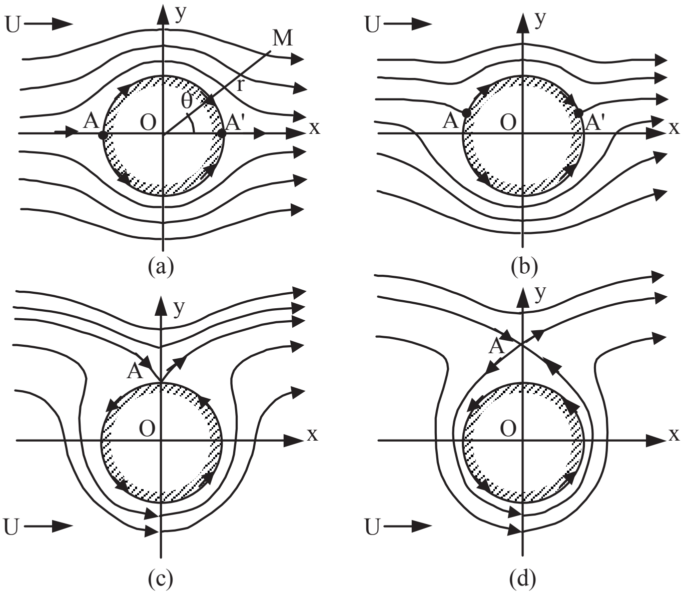


Figure 6.7. *Flow of an inviscid fluid around a circle: (a) circulation* $\Gamma = 0$; *(b)* $|\Gamma| < 4\pi RU$; *(c)* $|\Gamma| = 4\pi RU$; *(d)* $|\Gamma| > 4\pi RU$

The expression for the velocity $u_\theta(R) = -2U\sin\theta + \Gamma/2\pi R$ on the circle of radius R indicates that two stagnation points A and A' are found on the circle if $|\Gamma| < 4\pi RU$ (Figure 6.7b). These are joined for $|\Gamma| = 4\pi RU$ (Figure 6.7c).

For $|\Gamma| > 4\pi RU$, the points of zero velocity are the solutions of the equation $F'(z) = 0$; letting $z = j\, y_a$, we find for y_a two roots, only one of which is external to the circle of radius R (Figure 6.7d).

Bernoulli's second theorem allows us to calculate the pressure p_C on the cylinder from the velocity distribution on this one. We here limit ourselves to the case of a *steady flow*, as unsteadiness introduces secondary effects due to the added mass ([YIH 77]). We have:

$$p_c = -\rho\left(-2U_0\sin\theta + \Gamma/2\pi R\right)^2/2 + \text{constant} \qquad [6.39]$$

The force $\vec{F}$ exerted by the fluid on the cylinder (per unit extent) can be decomposed into the drag D and the lift L ($\vec{F} = T\,\vec{x} + P\,\vec{y}$). These can be calculated from expression [6.39] for the pressure. We find immediately:

$$D = -\int_0^{2\pi} p_c \cos\theta . R d\theta = 0; \qquad L = -\int_0^{2\pi} p_c \sin\theta . R d\theta = -\rho \Gamma U$$

Regardless of the form of the obstacle, these results are true for the drag (d'Alembert's paradox) and for the lift (Kutta-Joukowski theorem [YIH 77], [PAR 98]).

Comparisons with experiment

In the case where Γ = 0, this theoretical pressure distribution on the cylinder (Figure 6.8, curve a) can be compared with *experimental* results. In Figure 6.8, we have shown pressure variations $C_p = \frac{(p - p_\infty)}{\rho U^2 / 2}$ (difference between the wall pressure p and the pressure p_∞ in the uniform flow normalized by the dynamic pressure $\rho U^2 / 2$) as a function of the angle θ (defined in Figure 6.7a).

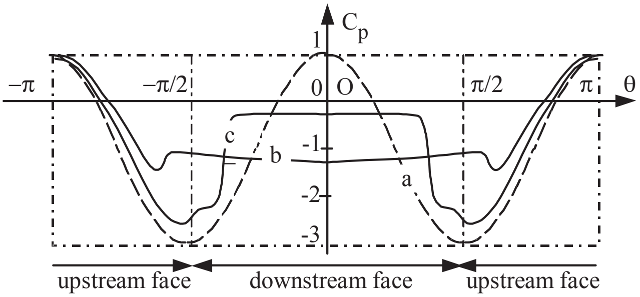


Figure 6.8. *Distribution of the pressure coefficient on a circular cylinder* (Γ = 0): *(a) irrotational flow*; *(b) laminar separation*; *(c) turbulent separation*

The values of the preceding calculation are relatively close to those measured on the upstream side of the cylinder, up to the angular position where a *separation of the flow* from the cylinder occurs and a wake is formed (section 6.5.3.7). The pressure measurements at locations situated on the wall where the flow has separated illustrate a strong flow dissymmetry between the upstream and downstream faces, which leads to a non-zero value for the drag (whence d'Alembert's paradox given to the "theoretical" result). The difference, which is quite small, between the calculation and the measurement on the upstream face of the cylinder comes from the fact that an inviscid fluid flow is not produced about the cylinder, rather it is produced about the ensemble constituted by both the cylinder and its wake.

The *non-zero drag force D* exerted by the fluid on the cylinder can be explained by the presence of the wake on the downstream face, on which a pressure force is exerted which is greater than that exerted on the upstream face. This force, known as *pressure drag,* is obviously proportional to the dynamic pressure $\rho U^2/2$. The value of the separation angle α (and therefore of the drag) is different ([SCH 99], [YIH 77]), depending on where the *boundary layer* (section 6.5.3) is *laminar* (*subcritical* flow, Figure 6.8, curve b) or has become *turbulent* (*supercritical* flow, Figure 6.8, curve c).

The *lift due to the circulation* (Kutta-Joukowski theorem) is indeed observed for wing profiles and for cylinders in rotation. However, the question as to the mechanism by which the circulation has been created has not been discussed. The latter is created by the beginning of the fluid movement about the airfoil as a result of viscous stresses on the wall (see section 6.6.4.1). However, the Kutta-Joukowski theorem is satisfied, and the effect of the lift is a curved trajectory for bodies being in rotation (the Magnus effect); this phenomenon is used in games with balloons and balls (the balls are "cut").

The lift of a stationary circular cylinder can also result from actions which generate dissymmetries of the wake by modification of viscous effects in the vicinity of the wall (dissymmetric sucking of the boundary layer).[4]

6.2.5.2.5. Kz^n potential flows

Consider the plane polar coordinate system (r,θ) and flows whose potentials and complex velocities are given by:

$$F(z) = Kz^n = K\,r^n \cos n\theta + jK\,r^n \sin n\theta$$
$$F'(z) = nKz^{n-1} = nKr^{n-1}e^{j(n-1)\theta}$$

The straight lines $\theta = \pi/n$ are streamlines terminating at, or issuing from, the zero velocity point $z = 0$ (for negative n). The case $n = 2$ corresponds to the usual stagnation point of a flow (points A and A' of Figures 6.7a, Figure 6.7b and point A in Figure 6.7d). The case $n = 3$ corresponds to a higher order stagnation point (point A in Figure 6.7c). Taking viscosity into account in these flows is possible with self-similar solutions of the boundary layer where n can take on any value ([SCH 99], [YIH 77]).

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[4] A ship with "sails", "l'Alcyon", has been built using this principle by Y. Cousteau and L. Malavard.

6.2.5.3. *The wave equation*

While the use of characteristics allows in principle the solution step by step of the acoustic wave equation, writing a complete solution in this way is generally difficult. We can immediately verify that a progressive plane wave is an elementary solution of [6.22]:

$$\varphi(x_i, t) = f\left(\overrightarrow{OM}.\vec{n} - ct\right) = f(x_i n_i - ct) \tag{6.40}$$

(n_i: direction cosines of the unit vector $\vec{n}$ normal to the plane wave ($\|\vec{n}\| = 1$)).

The velocity and the pressure fluctuation p' can be found from solution [6.40] and from a linearized version of Bernoulli's second theorem [6.13]:

$$u_i = n_i f'(x_i n_i - ct); \;\; V = \sqrt{u_i u_i} = f'; \;\; p' = -\rho \frac{\partial \varphi}{\partial t} = \rho c f'(x_i n_i - ct) = \rho c V$$

Taking the axis OX parallel to the normal $\vec{n}$, the quantity $x_i n_i$ of the problem of the function *f* is equal to *X*, and the function *f* can be written $\varphi(x_i, t) = f(X - ct)$. This form reveals the transmission without signal deformation (velocity potential, velocity or pressure). We note that if $f(X - ct)$ is a solution of [6.40], the same is true of $g(X + ct)$ which propagates in the opposite direction.

The superposition of certain waves can eventually lead to the disappearance of the propagative character, and we obtain stationary waves in which all points in space are in phase, as shown by the following simple example:

$$\cos(\omega t - kX) + \cos(\omega t + kX) = 2\cos \omega t \cos kX$$

Conversely, any suitable superposition of harmonic stationary waves can lead to one progressive harmonic wave:

$$\cos \omega t \cos kX + \sin \omega t \sin kX = \cos(\omega t - kX)$$

The ensemble of progressive plane waves is thus equivalent to the ensemble of stationary plane waves.

The equation for spherical waves $\varphi(r, t)$ can be immediately obtained (the expression for the divergence can be obtained by applying Ostrogradski's theorem between two spheres of radius *r* and *r* + *dr*). It is written:

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{1}{r}\left(\frac{\partial^2(r\varphi)}{\partial r^2} - \frac{1}{c^2}\frac{\partial^2(r\varphi)}{\partial t^2}\right) = 0 \qquad [6.41]$$

Its solution is analogous to that of the plane wave equation:

$$\varphi(r,t) = \frac{1}{r}f(r-ct) + \frac{1}{r}g(r+ct)$$

In the case where $g \equiv 0$, the expressions for the pressure and the velocity are:

$$p = -\rho\frac{\partial\varphi}{\partial t} = \frac{\rho c}{r}f'(r-ct); \qquad u = \frac{\partial\varphi}{\partial r} = \frac{1}{r}f'(r-ct) - \frac{1}{r^2}f(r-ct)$$

The expression for the velocity of spherical waves differs from the expression for plane waves by the term $-\frac{1}{r^2}f(r-ct)$, which dominates in the vicinity of the origin $r = 0$ (nearfield term). We have, for the volume flow rate q_v:

$$q_v(r,t) = 4\pi r^2\frac{\partial\varphi}{\partial r} = 4\pi\left[rf'(r-ct) - f(r-ct)\right]$$

We see that the flow rate at the origin $Q(t)$ is equal to
$Q(t) = q_v(0,t) = -4\pi f(-ct)$.

The velocity potential φ and the pressure p can be thus be written:

$$\varphi = -\frac{1}{4\pi r}Q\left(t - \frac{r}{c}\right); \qquad p = \frac{\rho c}{4\pi r}Q'\left(t - \frac{r}{c}\right)$$

6.2.6. *Surface waves in shallow water*

6.2.6.1. *2D equation for potential*

We will now consider a problem governed by an elliptic partial differential equation in space (x,y,z), but whose boundary conditions induce propagative phenomena in the plane (x,y).

Consider a horizontal plane Oxy, on which a layer of liquid is subjected to a vertical gravitational action. Let $\zeta = \zeta(x,y)$ be the height of the free surface of the liquid. If the movement of the liquid is considered irrotational, the velocity potential $\varphi(x, y, z, t)$ satisfies Bernoulli's equation:

$$\frac{\partial \varphi}{\partial t} + \frac{V^2}{2} + \frac{p}{\rho} + gz = const \tag{6.42}$$

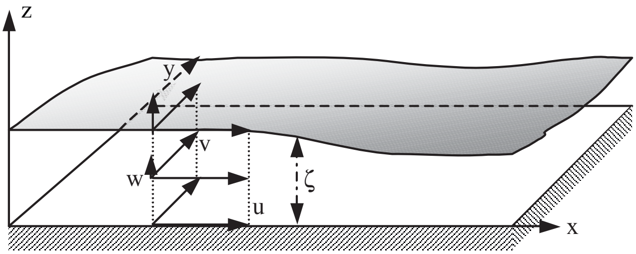


Figure 6.9. *Flow with free surface in shallow water*

As the liquid is inviscid, the velocity is horizontal at the bottom $z = 0$ where w is zero (slip condition). We will consider the case of *shallow water*: the flow is locally uniform (the horizontal components u and v are independent of z), and the thickness e is small compared with the horizontal distance L characteristic of variations of the velocity $\vec{V}$. The volume conservation equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial v} + \frac{\partial w}{\partial z} = 0 \tag{6.43}$$

allows the magnitude of the w component to be determined: the first two terms being of order V/L, we obtain $w \approx Ve/L$. The *vertical component of the velocity w is of second order* with respect to the velocity $\vec{V}$. The component w can be calculated by integration following z of equation [6.43]:

$$w = -z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial v}\right) \tag{6.44}$$

The velocity potential φ is thus the sum of two terms:

– a function $\phi(x,y,t)$ whose gradient comprises the u and v components of the velocity;

– a small term of order $-\frac{z^2}{2}\Delta\phi$ associated with the w component which can be written using [6.44] by expressing u and v as a function of ϕ:

$$w = -z\,\Delta\phi \tag{6.45}$$

The free surface is a material surface; its vertical velocity $w|_{z=\zeta}$ is the material derivative of the height ζ of the fluid particle being on this surface (kinematic condition):

$$w|_{z=\zeta} = \frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y}$$

Substituting into [6.45] gives:

$$\frac{d\zeta}{dt} = -\zeta\,\Delta\varphi \tag{6.46}$$

As this w component is small compared with u and v, Bernoulli's theorem leads to a z-distribution of the hydrostatic pressure, over a vertical section, which depends on the height $\zeta(x, y, t)$ of the free surface on which the atmospheric pressure p_a acts:

$$p + \rho\,gz = p_a + \rho\,g\zeta \tag{6.47}$$

Relation [6.42] can thus be written:

$$\frac{\partial\varphi}{\partial t} + \frac{u^2 + v^2}{2} + g\zeta = \text{const} \tag{6.48}$$

We replace ζ in [6.46] with its expression taken from [6.48] in order to obtain the potential ϕ:

$$\Delta\phi - \frac{1}{g\zeta}\frac{d}{dt}\left(\frac{\partial\phi}{\partial t} + \frac{V^2}{2}\right) = 0 \tag{6.49}$$

6.2.6.2. *Analogy with a compressible fluid*

Letting $c^2 = g\zeta$, we see that equation [6.49] can be written in a form identical to equation [6.18] for the potential of a compressible fluid with two spatial dimensions. This equation, which is hyperbolic, represents the propagation of surface waves of velocity amplitude V and height ζ.

Comparison between the corresponding equations [6.15] and [6.46] of these two problems shows that the height ζ is analogous to the density ρ. However, the expression for the velocity of sound $c^2 = \gamma p/\rho = k\rho^{\gamma-1}$ of the compressible fluid and that $c^2 = g\zeta$ of the free surface indicates an exponent γ equal to 2 for the equivalent compressible fluid.

In physical terms, compressing a gas or elevating the free surface of a fluid creates a reactive force in the form of a pressure increase (section 5.3.6) or of driving pressure corresponding to the "elastic" energy of an oscillator. The hyperbolic character of equation [6.49] leads to the existence of shock waves in the form of a hydraulic jump ([YIH 77]).

6.2.6.3. *Influence of surface tension*

We have previously assumed (section 6.2.6.1) the continuity of pressure across the free surface. However, waves of small wavelength require surface tension σ to be taken into account. The pressure difference δp_σ due to surface tension is given by Laplace's law (section 2.2.1.4.2) which can be written by expressing the average curvature, accurate to second order:

$$\delta p_\sigma = p - p_a = \sigma\left(\frac{1}{R} + \frac{1}{R'}\right) = -\sigma\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) = -\sigma \Delta\zeta$$

Condition [6.47] is replaced by:

$$p + \rho g z = p_a + \rho g \zeta + \delta p_\sigma = p_a + \rho g \zeta - \sigma \Delta\zeta$$

Substituting into Bernoulli's equation [6.42] gives the relation:

$$\frac{\partial \phi}{\partial t} + \frac{V^2}{2} + \frac{p}{\rho} + gz = \frac{\partial \phi}{\partial t} + \frac{V^2}{2} + g\zeta - \frac{\sigma}{\rho}\Delta\zeta = \text{const}$$

which, associated with [6.46], gives a complex system which we will not study here (see [YIH 77]).

6.3. Orders of magnitude

6.3.1. *Introduction and discussion of a simple example*

The mathematical variable properties of a continuous medium are relatively regular. The equations governing a continuous medium assume the continuity and the derivability to at least second order in the physical quantities, with the exception of regions where shocks or discontinuities occur. The validity of physical models (axioms of the continuous medium or models obtained from kinetic gas theory) implies that the physical quantities observed are solutions of ordinary or partial differential equations whose behavior is locally regular.

It is thus reasonable to admit that a quantity f undergoing *variations in the order of Δf on an interval of a time or space variable of length L*, possesses temporal or spatial derivatives of the order of $\Delta f / L$ and that their second derivatives under the same conditions are in order $\Delta f / L^2$. Such a hypothesis should be subsequently verified in discussing the results which can thence be obtained. The scale L corresponds to the interval over which the function f varies. For example, for an exponential function L is the characteristic dimension of the exponential variation (space or time constant).

The preceding considerations result from the fact that a derivative is the ratio limit of finite increases of the function and the variable, when the latter tends to zero. It is clear that to within a factor of at most a few units, this derivative is equal to the ratio of the finite increases in the region considered.

A partial differential equation (or an ordinary differential equation) is a numerical balance between a certain number of terms containing derivatives. If this equation only contains two terms, the absolute values of these are equal. On the other hand, if the equation contains a sufficiently large number of terms, certain of these are dominant in a given part of the domain, other terms being more important in other regions. Each zone of the domain can thus be characterized by the locally dominant physical phenomena.

Take the elementary example of the mass-spring oscillator with one degree of freedom:

$$m\ddot{x} + f\dot{x} + kx = 0 \qquad \text{with: } x(0) = x_0 ; \ \dot{x}(0) = 0 \qquad [6.50]$$

Suppose that we do not know the solution. We search first of all if a characteristic time τ exists for the phenomena described by this equation and corresponding to a movement of amplitude x_m.

The term kx of given order of magnitude kx_m is in the same order as at least one of the other two terms of the equation which are, respectively: $\mathcal{O}\left(mx_m/\tau^2\right)$ and $\mathcal{O}\left(f x_m/\tau\right)$.

Let us first suppose that $f\dot{x}$ is small compared to kx_m which is then of the same order as $m\ddot{x}$:

$$m\ddot{x} \approx k\,x \quad \Rightarrow \quad \frac{mx_m}{\tau^2} \approx k\,x_m \quad \Rightarrow \quad \tau \approx \sqrt{m/k}$$

The characteristic time τ necessary for the amplitude to vary from zero to x_m is of order $\sqrt{m/k}$ and we also have $f << \sqrt{mk}$.

To first approximation, equation [6.50] can be written:

$$mx + kx = 0 \quad \text{with: } x(0) = x_0\,; x(0) = 0 \qquad [6.51]$$

If, on the other hand, $m\ddot{x}$ is small, then it is the term $f\ddot{x}$ which balances kx; we thus have:

$$f\dot{x} \approx kx \quad \Rightarrow \quad \frac{f\,x_m}{\tau} \approx k\,x_m \quad \Rightarrow \quad \tau \approx f/k$$

The characteristic time τ necessary for the amplitude to vary from zero to x_m is of order $\tau \approx f\,/\,k$, with $f \gg \sqrt{mk}$; equation [6.50] thus reduces to:

$$f\dot{x} + kx = 0 \quad \text{with: } x(0) = x_0 \qquad [6.52]$$

The preceding order of magnitude analysis allows us to define the characteristic time τ for the phenomena described by the initial equation and to obtain an approximate equation for this particular time scale. Simple general considerations allow us to show that equation [6.51] represents an oscillatory movement, at least at the scale τ. For example, it is useful to discuss the equation in the plane (x, $\dot{x}$), often referred to as the phase-plane. The trajectory $[x(t), \dot{x}(t)]$ is there a curve parameterized by the time and a simple qualitative discussion using equation [6.51] allows us to easily see the form of this (Figure 6.10a), based on the fact that $\ddot{x}$ is the derivative of $\dot{x}$, and its sign indicates the direction of the variation $\dot{x}$.

Equation [6.52] corresponds to a damped aperiodic movement. The trajectory in the phase-plane is therefore a straight line with slope $-k/f$. We will later discuss the effect of the second initial condition $\dot{x}(0)=0$ which obviously had to be abandoned for the first order differential equation.

If we have equality in the orders of magnitude $f \approx \sqrt{mk}$, the three terms of the equation must be conserved, but we will pass gradually from the form of Figure 6.10a to that of Figure 6.10c.

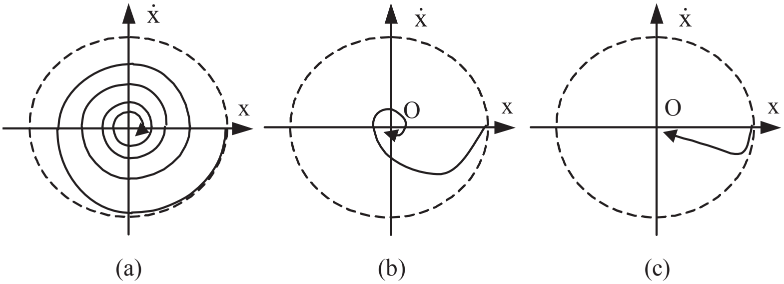


Figure 6.10. *Evolution of the oscillator in the plane (x, ẋ): (a) oscillator with a small damping; (b) damped oscillator; (c) aperiodic motion* (*strong damping)*

It then remains to study the influence of the small term neglected in each case: this is a perturbation problem which we will discuss a little later (section 6.4). In the first oscillatory case, we study the influence of friction with the balance equation for the mechanical energy derived from [6.50] by multiplying by $\dot{x}$ and integrating between 0 and t. We have, taking account of the initial conditions:

$$\frac{m\dot{x}^2}{2}+\int_0^t f\dot{x}^2\,dt+\frac{kx^2}{2}=\frac{kx_0^2}{2} \qquad [6.53]$$

Equation [6.53] immediately shows the following properties:

1) kinetic energy is localized in the vicinity of the origin x=0;

2) potential energy is localized in the vicinity of the extrema of the stretching motion;

3) kinetic energy can be transformed into potential energy such that the total mechanical energy decays with time;

4) dissipated energy $\int_0^t f\dot{x}^2 dt$ is an increasing function of time; the result of this is that the velocity must tend to zero if the integral is to remain finite; we can write equation [6.53] in the form:

$$\int_0^t f x^2 dt = -\left(\frac{mx^2}{2} + \frac{kx^2}{2}\right)_0^t \qquad [6.54]$$

Let τ_a be the damping time of the oscillations; let $\dot{x}_m$ denote the maximum of the velocity at the beginning of the movement which is considered only lightly damped; this leads to the following orders of magnitude:

$$\int_0^t f\dot{x}^2 dt \approx \tau_a f \dot{x}_m^2 \approx \frac{m\dot{x}_m^2}{2}$$

From this we can derive the order of magnitude of the damping time: $\tau_a \approx \dfrac{m}{f}$.

It should be noted that the preceding considerations are concerned with orders of magnitude which do not require an exact expression of equation [6.50]; they are in fact valid for any equation whose terms have the orders of magnitudes posed above. Therefore, *an approximate knowledge of results do not require complex mathematical procedures above and beyond the numerical discussion about monomes* ("rule of three"). By this procedure, which needs to be completed by a discussion of perturbation problems, we can identify the important terms in a system of equations, in other words, the dominant physical phenomena in each of the zones of the problem domain. The order of magnitude of the unknown quantities can also be deduced from this analysis.

6.3.2. *Obtaining approximate values of a solution*

6.3.2.1. *Principles*

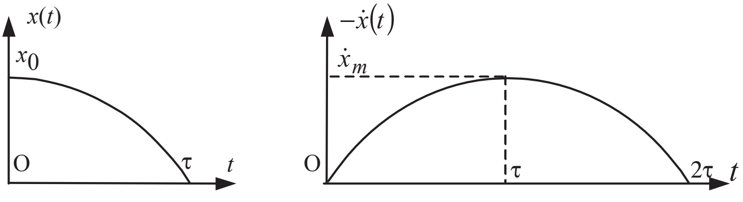
Most particular functions (called "elementary" or "special") are solutions of linear equations with simple algebraic coefficients. For linear partial differential equations (Laplace equation, wave equation, heat equation, Maxwell's equations, etc.), these functions are often useful in the search for solutions with particular boundary conditions. As the equations of fluid mechanics are not linear, the elementary or special functions are rarely directly useful.

Having simplified the equations and qualitatively discussed the various phenomena, we can obtain approximate values in a simple manner by searching for a *global solution in each zone where the equations can be simplified*. The method which we will use here can be generally applied, insofar as we suppose that we ignore the exact solution of the equations, and that the particularities of the equation linearity and the coefficient constancy are not used. The method consists in searching for a global condition which can be obtained by integrating the equations over each interval of the study. We then represent the form of the solution by a plausible function in which the main unknown value is a parameter which we can deduce from the preceding global condition. We will first apply this technique to the preceding example of the oscillator.

6.3.2.2. *Global solution for the linear oscillator*

Let us take for τ the first instant where the abscissa x is zero (Figure 6.11a), the velocity being then approximately maximum; this value τ is more or less equal to a quarter of the period T of the oscillatory movement. Let us consider equation [6.51] with the conditions:

$$x(0) = x_0; \quad \dot{x}(0) = 0; \quad x(\tau) = 0 \tag{6.55}$$



(a) (b)

Figure 6.11. *(a) Law of motion during the first quarter of a period; (b) variation of the velocity during half a period*

Taking these conditions into account, the integration of equation [6.51] between 0 and τ gives the global condition:

$$m\dot{x}(\tau) + k\int_0^\tau x(t)dt = 0 \tag{6.56}$$

First write the function $x(t)$ which in the form $x(t) = x_0\,\varphi(\eta)$ of a non-dimensional function of the variable $\eta = t/\tau$ satisfying [6.51] and conditions [6.55]:

$$\ddot{\phi} + \frac{\tau^2 k}{m}\phi = 0 \quad \text{with } \phi(0) = 1, \phi(1) = 0 \text{ and } \dot{\phi}(0) = 0 \tag{6.57}$$

Integrating [6.57] between 0 and 1, we obtain:

$$\dot{\varphi}(1)+\frac{\tau^2 k}{m}\int_0^1 \varphi(\eta)d\eta = 0 \qquad [6.58]$$

Equation [6.58] allows us to determine the unknown τ if the function φ is known. It remains to choose the function $\varphi(\eta)$, whose general shape we know in the phase plane from the discussion of section 6.3.1.1. The simplest algebraic function satisfying conditions [6.57] is:

$$\varphi = 1-\eta^2 \qquad [6.59]$$

Substituting into [6.58] we obtain:

$$-1+\frac{\tau^2 k}{3m} = 0$$

whose positive root $\sqrt{3m/k}$ only is acceptable. Thus, for the period $T = 4\sqrt{3m/k}$ we obtain the value $4\sqrt{3} = 6.928$ which is slightly larger than the exact value $2\pi = 6.28$ (error of 10%). The reader will note that the error level is quite small, given the crudeness of the computation.

However, we can hope to improve the result relatively easily by imposing *supplementary conditions resulting from equation* [6.57] on the second derivative $\ddot{\varphi}(\eta)$ at instants 0 and 1:

$$\ddot{\varphi}(0) = -\frac{\tau^2 k}{m}, \quad \ddot{\varphi}(1) = 0 \qquad [6.60]$$

Consider firstly second conditions [6.60]: $\ddot{\varphi}(1) = 0$. The simplest polynomial satisfying this and conditions [6.55] is of third order:

$$\varphi(\eta) = 1-\frac{3\eta^2}{2}+\frac{\eta^3}{2} \qquad [6.61]$$

Substituting into equation [6.58] gives:

$$-3+\frac{5}{4}\frac{\tau^2 k}{m} = 0$$

We therefore obtain $T = 4\tau = 4\sqrt{12m/5k}$; the value $4\sqrt{12/5} = 6.197$ only differs from the exact value by 1.4%.

The simplest polynomial satisfying the conditions [6.55] and [6.60] is of fourth order:

$$\varphi(\eta) = \frac{\tau^2 k}{m}\left(-\frac{\eta^2}{2} + \frac{5\eta^3}{6} - \frac{\eta^4}{3}\right) + 1 - 2\eta^3 + \eta^4$$

hence:

$$\dot{\varphi}(1) = \frac{\tau^2 k}{6m} - 2 \text{ and } \int_0^1 \varphi(\eta) d\eta = -\frac{\tau^2 k}{40m} + \frac{7}{10}$$

As the form of the curve representing the function φ depends on the unknown parameter $\tau^2 k/m$, we obtain, by substituting into [6.58] a second order equation:

$$-\frac{1}{40}\left(\frac{\tau^2 k}{m}\right)^2 + \frac{13}{15}\frac{\tau^2 k}{m} - 2 = 0$$

which has roots 2.497 and 32.16. It is easy to see that the second value is not suitable, φ having to be positive on the interval [0,1]. With the first value, the calculation of the period gives:

$$\tau = 1.58\sqrt{\frac{m}{k}} \qquad T = 4\tau = 6.32\sqrt{\frac{m}{k}}$$

The use of boundary conditions taken from the equation for the second derivative thus leads to an improvement (error less than 1%). It should be noted however that it is not possible to further improve the results of such a method, which only uses local data at the extremities of the interval considered.

6.3.2.3. *Damping of the oscillations*

In an oscillatory regime, the essential movement corresponds to an exchange between the kinetic and potential energy terms; only a small amount of the mechanical energy is dissipated in each period. The interest of the kinetic energy theorem [6.54] is that it gives an expression for the dissipated energy as a function of the variation of total mechanical energy: this is equal to the variation of maximum kinetic energy $m\dot{x}_m^2/2$, $\dot{x}_m(t)$ being the maximum velocity value at each stretching value of the mass m equal to zero. During a half-period 2τ (notation as in the preceding section) contained between two instants t and $t + 2\tau$ where the elongation is zero ($|\dot{x}| = \dot{x}_m$), equation [6.54] translates this mechanical energy variation:

$$-\delta\left(\frac{m\dot{x}^2}{2}\right) = -\frac{1}{2}m\dot{x}^2\Bigg|_{(2p-1)\tau}^{(2p+1)\tau} = \int_t^{t+2\tau} f\dot{x}^2 dt \qquad [6.62]$$

Suppose as before that the amplitude of the velocity $\dot{x}_m$ varies only slightly and that it can be considered as a constant during the half-period, while the velocity variation is symmetric with respect to the instant τ (Figure 6.11b). The law for the velocity $x(t) = x_0\,\varphi(\eta)$ results from the choice of the preceding function $\varphi(\eta)$:

$$\dot{x}(t) = x_0\dot{\phi}(t) = \dot{x}_m\frac{\dot{\phi}(t)}{\dot{\phi}(1)} \quad \text{with: } \dot{x}_m = x_0\dot{\phi}(1)$$

Taking the time origin at the beginning of the half-period, at the instant where $\dot{x} = 0$, and designating the velocity amplitude by $\dot{x}_m$, gives, after substitution into [6.62]:

$$-\frac{m}{2}\delta\left(\dot{x}_m^2\right) = \int_t^{t+2\tau} f\dot{x}^2 dt = \frac{2f\dot{x}_m^2\tau}{\dot{\varphi}^2(1)}\int_0^1 \dot{\varphi}^2(\eta)d\eta \qquad [6.63]$$

Relation [6.63] is a finite difference equation with a step 2τ for the amplitude $\dot{x}_m$, which we can replace with the differential equation:

$$-\frac{m}{2}\frac{d\left(\dot{x}_m^2\right)}{dt} = \alpha f\dot{x}_m^2 \quad \text{with: } \alpha = \frac{\int_0^1 \dot{\phi}^2(\eta)d\eta}{\dot{\phi}^2(1)}$$

or:

$$\frac{m}{\alpha f}\frac{d\dot{x}_m}{dt} + \dot{x}_m = 0 \qquad [6.64]$$

The differential equation [6.64] can be solved as before using a global method. We will here simply note that it represents a first order damped system with time constant $\tau_a = m/\alpha f$.

It remains to calculate the constant α. From the parabolic law of motion [6.59] we have:

$$\dot{\varphi} = -2\eta \quad \Rightarrow \quad \alpha = \int_0^1 \eta^2 d\eta = \frac{1}{3} \qquad \text{if: } \tau_a = \frac{m}{\alpha f} = \frac{3m}{f}$$

The third order law of motion [6.61] gives:

$$\dot{\varphi} = \frac{3}{2}\eta^2 - 3\eta \quad \Rightarrow \quad \alpha = \int_0^1 \dot{\varphi}^2(\eta) d\eta = \frac{8}{15} \qquad \text{if: } \tau_a = \frac{m}{\alpha f} = \frac{15m}{8f}$$

The exact solution for the damping is $\tau_a = \dfrac{2m}{f}$. The error is a little greater than for the calculation of the period: this is because a given approximation is always better for a function φ than for its derivative $\dot{\varphi}$.

6.4. Small parameters and perturbation phenomena

6.4.1. *Introduction*

The equations governing a physical phenomenon involve various non-dimensional parameters. Very often, some of these are small. The associated terms appear as a perturbation of the equations which are obtained when these terms are zero. Mathematical phenomena associated with these perturbation terms can be complex and their study should be carefully effected in order to understand their precise role. We will limit ourselves in this section to the usual elementary cases in fluid mechanics.

6.4.2. *Regular perturbation*

6.4.2.1. *Elementary example*

A *perturbation* is called *regular* if the effects resulting from the perturbations terms are everywhere in the same order of magnitude as the parameter which characterizes them. A good example involves a first order damped system governed by the following equation with the small parameter ε:

$$\dot{x} + x = (1 + \varepsilon) \quad x(0) = 0$$

whose solution is (Figure 6.12a): $x = (1+\varepsilon)(1-e^{-t})$.

This idea of a regular perturbation is associated with the mathematical idea of uniform convergence which we will recall here briefly: a family of functions $f(t,\varepsilon)$ converges uniformly towards the function $f(t,0)$ if the difference between the two functions is independent of the value of t on a closed interval, and tends to zero with ε. The limit $f(t,0)$ of the family of functions is thus continuous. This condition is visibly satisfied for the preceding example.

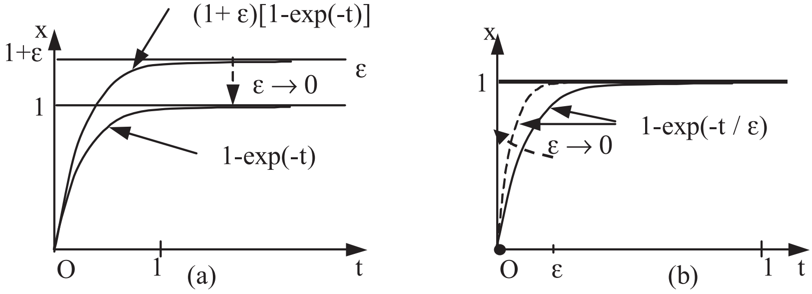


Figure 6.12. *(a) Regular perturbation; (b) singular perturbation*

6.4.2.2. *Regular perturbation of linear differential equations*

Consider the linear differential problem which depends on the parameter ε, whose solution $x(t,\varepsilon)$ converges uniformly towards $x(t,0)$ when the small parameter ε tends to zero; we can often search for $x(t,\varepsilon)$ in the form of a power series expansion of the parameter ε, if this parameter has been suitably chosen:

$$x(t,\varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots + \varepsilon^i x_i(t) + \ldots \qquad [6.65]$$

Substituting the preceding expression into the differential equation, we obtain a series in increasing powers of ε, the identification of whose coefficients in the two sides provides *successive differential equations*. Let us first consider a simple example of a linear differential equation in order to illustrate the computation[5] mechanism. Consider the differential equation representing the movement of a point subjected to a weak repulsive force, *t* varying on [0,1]:

---

[5] We assume that the domain of study corresponds to an time interval bounded *a priori* in order to ensure uniform convergence.

$$\ddot{x}(t) - \varepsilon\, x(t) = a \qquad x(0) = 0 \qquad \dot{x}(0) = 1 \tag{6.66}$$

Substituting development [6.65] into differential equation [6.66], we obtain the successive differential equations:

$$\begin{array}{lll}
\ddot{x}_0(t) = a & x_0(0) = 0 & \dot{x}_0(0) = 1 \\
\ddot{x}_1(t) = x_0(t) & x_1(0) = 0 & \dot{x}_1(0) = 0 \\
\ddot{x}_2(t) = x_1(t) & x_2(0) = 0 & \dot{x}_2(0) = 0 \\
\ldots\ldots\ldots\ldots & & \\
\ddot{x}_p(t) = x_{p-1}(t) & x_p(0) = 0 & \dot{x}_p(0) = 0 \\
\ldots\ldots\ldots\ldots & &
\end{array}$$

The first differential equation is the non-perturbed equation which corresponds to $\varepsilon = 0$; the successive differential equations only depend on the functions already calculated, of lower rank in the development. The *boundary conditions can be carried back into the first equation, provided they do not contain the parameter* ε.

The solution of the preceding system can be calculated immediately from place to place and we thus obtain a series development of the solution:

$$\begin{array}{l}
x_0 = a\dfrac{t^2}{2} + t, \quad x_1 = a\dfrac{t^4}{4!} + \dfrac{t^3}{3!}, \quad x_2 = a\dfrac{t^6}{6!} + \dfrac{t^5}{5!}, \\
\ldots., \quad x_p = a\dfrac{t^{2(p+1)}}{[2(p+1)]!} + \dfrac{t^{2p+1}}{(2p+1)!}, \quad \ldots\ldots..
\end{array} \tag{6.67}$$

However, a development including many terms is not of much interest, particularly if we consider the computational methods used by computers. Furthermore, if the parameter ε takes on values which require many terms of the development, the principal properties of the unperturbed equation are significantly modified: in other words *the unperturbed equation is no longer a sufficiently representative model* for the mathematical and physical properties *of the solution for these values of* ε.

For example, we easily recognize that [6.67] is the series development of the solution which is here easy to calculate directly:

$$x = \frac{a}{\varepsilon}\left[ch\left(t\sqrt{\varepsilon}\right) - 1\right] + \frac{sh\left(t\sqrt{\varepsilon}\right)}{\sqrt{\varepsilon}}$$

When ε is no longer small, the preceding series development is of limited practical interest, despite an infinite radius of convergence, as the behavior of the perturbed solution is too far from that of the unperturbed solution.

6.4.2.3. *Regular perturbation of non-linear differential equations*

Consider a given first order differential equation:

$$\dot{x} = f(t, x, \varepsilon) \quad x(0) = a$$

Calculating the development of the function $f(t, x, \varepsilon)$ in increasing powers of the small parameter ε becomes quickly complicated, and in general we are satisfied by a limited development. We will here only outline the principle of the method and the beginning of the calculation.

As before, we seek a solution of the form [6.65]:

$$x(t, \varepsilon) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \ldots + \varepsilon^i x_i(t) + \ldots$$

We have:

$$\begin{aligned}
f(t, x, \varepsilon) &= f(t, x, 0) + \varepsilon f_\varepsilon(t, x, 0) + \frac{\varepsilon^2}{2} f_{\varepsilon\varepsilon}(t, x, 0) \\
&= f\left(t, x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + .., 0\right) + \varepsilon f_\varepsilon\left(t, x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + .., 0\right) + \frac{\varepsilon^2}{2} f_{\varepsilon\varepsilon}(t, x_0 + \varepsilon x_1 + .., 0) \\
&= f(t, x_0, 0) + \varepsilon\left[f_\varepsilon(t, x_0, 0) + x_1 f_x(t, x_0, 0)\right] \\
&\qquad + \varepsilon^2\left[x_2 f_x(t, x_0, 0) + \frac{1}{2} f_{\varepsilon\varepsilon}(t, x_0, 0) + x_1 f_{\varepsilon x}(t, x_0, 0) + \frac{x_1^2}{2} f_{xx}(t, x_0, 0)\right] + \varepsilon^3 \ldots.
\end{aligned}$$

Substituting the preceding development into the differential equation and identifying the following increasing powers of the small parameter ε, we obtain:

$$\begin{aligned}
&\dot{x}_0 - f(t, x_0, 0) = 0 && x_0(0) = a \\
&\dot{x}_1 - x_1 f_x(t, x_0, 0) = f_\varepsilon(t, x_0, 0) && x_1(0) = 0 \\
&\dot{x}_2 - x_2 f_x(t, x_0, 0) = \frac{1}{2} f_{\varepsilon\varepsilon}(t, x_0, 0) + x_1 f_{\varepsilon x}(t, x_0, 0) + \frac{x_1^2}{2} f_{xx}(t, x_0, 0) && x_2(0) = 0
\end{aligned}$$

The first differential equation corresponds to the zero perturbation for ε. We note that *the successive differential equations are linear* for the corresponding unknown function, with a right hand side which only depends on the previous solutions. The equation linearity facilitates their numerical resolution. However, the expressions for the differential equations can quickly become complex.

The interest of regular perturbation methods is particularly evident for problems governed by partial differential equations, if particular solutions can be found which have a simple mathematical structure (for example, one which approaches differential equations), in which case the equations resulting from the application of the regular perturbation method have a structure analogous to the initial solution. We will see different examples of these methods applied to slightly unsteady flows, to thermal systems which do not vary too quickly, and to the problems where inertia terms due to geometric variations must be accounted for.

6.4.2.4. *Choice of a perturbation parameter*

The perturbation parameter chosen for a study is of the utmost importance. Suppose that a model leads to the following equation:

$$\dot{x} + \left(\sum a_n \varepsilon^n \right) x(t,\varepsilon) = b(t) \quad x(0) = A \qquad [6.68]$$

It is clearly possible to seek a solution in the form of a development of the solution in powers of ε:

$$x(t,\varepsilon) = x_0 \ (t) + \sum_{i=1}^{\infty} \varepsilon^i x_i(t)$$

However, it is more interesting to take $\alpha(\varepsilon) = \sum_{i=1}^{\infty} a_i \varepsilon^i$ as the perturbation parameter and to seek a solution of the differential equation:

$$\dot{x} + (a_0 + \alpha(\varepsilon)) x(t,\varepsilon) = b(t) \quad x(0) = A$$

in the form $\tilde{x}(t,\alpha) = x_0 \ (t) + \sum_{i=1}^{\infty} \alpha^i \tilde{x}_i(t).$

We find for the successive differential equations:

$$\tilde{x}_i + a_o x_i(t) = -\sum_{j=1}^{i} a_j x_{i-j}(t) \qquad x_j(0) = 0$$

$$\tilde{x}_i + a_o \tilde{x}_i(t) = -\tilde{x}_{i-1} \qquad \tilde{\mathrm{x}}_i(0) = 0$$

The system obtained is far simpler for $\widetilde{x}_i$ than for x_i. The preceding example is apparently rudimentary. However, it translates the fact that the model has attributed an important role to the parameter α, rather than to ε which was chosen in order to establish the model. It is clear that we can have a better development with the parameter α than with ε. This problem of parameter choice is often encountered in order to best represent the range of solutions, for example, for solutions of boundary layer equations ([SCH 99], [YIH 77]). The term $b(t)$ of [6.68] can depend on ε which must therefore express as a function of α.

The method can be applied to partial differential equations. We will later see some examples of this (section 6.4.2.6). Suitable variable changes also allows the modification or simplification of the differential problem (section 8.5.3.2).

The practical limits of the preceding method are determined by the convergence of the entire series, but *even more by the speed of convergence of the series obtained.* Methods for accelerating the convergence can be used here ([ABR 65] p. 16, [BRE 91]).

6.4.2.5. *Regular perturbations and orders of magnitude*

In the domain of studies where *the orders of magnitude of the terms are fixed*, knowledge of a solution (exact or approximate) of an unperturbed problem allows the calculation of correctional terms for the solutions in the neighborhood of the base solution.

The successive differential equations obtained are linear equations which all have the same linear operator, the right hand sides being known at each stage from the preceding solutions. Numerical solution is thus simplified. The computation of higher order terms of the solution by means of analytical developments, is generally difficult in practice, on account of its complexity.

6.4.2.6. *Applications in fluid mechanics*

With the exception of viscous stresses, taking account of other phenomena in fluid mechanics, when these are relatively weak, very often leads to regular perturbations: unsteady effects in established flow (in other words a flow which is independent of its initial conditions), effects of compressibility in steady flow, weak geometric changes, etc.

Consider the established flow of an incompressible fluid with constant viscosity in a rectilinear pipe of arbitrary cross-section (Figure 6.13), and let us suppose that we have a Poiseuille flow, with a driving pressure gradient $-\partial p/\partial x$, parallel to the velocity in the direction *Ox*, which is a given function of time; the velocity satisfies the following equation (from [4.21]) and boundary conditions:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \,\Delta u; \quad \text{with: } (y,z) \in C : \; u(y,z,t) = 0 \tag{6.69}$$

We will consider the following non-dimensional parameters and variables:

$$\tilde{t} = \frac{t}{T} \quad \tilde{u} = \frac{u}{U} \quad (\tilde{y},\tilde{z}) = \left(\frac{y}{D},\frac{z}{D}\right) \quad \varepsilon = \frac{\rho D^2}{\mu T} \quad -\frac{D^2}{\mu U}\frac{\partial p}{\partial x} = f(\tilde{t})$$

where D and T are, respectively, a reference length for the cross-section and a unit of time to be defined.

Equation [6.69] can be written:

$$\varepsilon \frac{\partial \tilde{u}}{\partial \tilde{t}} = f(\tilde{t}) + \tilde{\Delta}\tilde{u}; \qquad \left(\tilde{\Delta} = \frac{\partial^2}{\partial \tilde{y}^2} + \frac{\partial^2}{\partial \tilde{z}^2}\right) \tag{6.70}$$

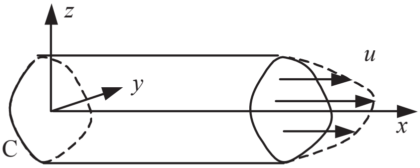


Figure 6.13. *Established flow in a cylindrical tube*

Assume ε to be small and consider that equation [6.70] is the result of a regular perturbation of this equation for $\varepsilon = 0$. We seek the solution of [6.70] in the form $\tilde{u} = \sum_{i=0}^{\infty} \varepsilon^i \tilde{u}_i$. Substituting into equation [6.70] and identifying terms following the increasing powers of ε, we obtain the system:

$$0 = f(\bar{t}) + \tilde{\Delta}\tilde{u}_0; \quad \frac{\partial \tilde{u}_0}{\partial \tilde{t}} = \tilde{\Delta}\tilde{u}_1; \quad \frac{\partial \tilde{u}_1}{\partial \tilde{t}} = \tilde{\Delta}\tilde{u}_2; \ldots; \quad \frac{\partial \tilde{u}_i}{\partial \tilde{t}} = \tilde{\Delta}\tilde{u}_{i+1}; \ldots$$

with the boundary conditions: $\tilde{u}_i(y,z) = 0 \quad (\tilde{y},\tilde{z}) \in \tilde{C} \;\; i = 0,1,2,\ldots,n,\ldots$

It is immediately verifiable from place to place that the functions $\widetilde{u}_i$ are variables of separated functions and that they are proportional to the successive derivatives of *f*(*t*):

$$\widetilde{u}(\widetilde{y},\widetilde{z},\widetilde{t}) = \sum_{i=0}^{\infty} g_i(\widetilde{y},\widetilde{z}) f^{(i)}\left(\widetilde{t}\right) \qquad [6.71]$$

The functions $g_i(\widetilde{y},\widetilde{z})$ are solution of the system:

$$\tilde{\Delta} g_0 = -1; \quad \tilde{\Delta} g_1 = g_0; \quad \tilde{\Delta} g_2 = g_1; \quad \ldots \tilde{\Delta} g_n = g_{n-1}, \ldots$$

with the boundary conditions: $g_i(\tilde{y},\tilde{z}) = 0 \quad (\tilde{y},\tilde{z}) \in C; \quad i = 0,1,\ldots,n\ldots$

The preceding equations can be solved successively with increasing *i*, and the first equation corresponds to the Poiseuille flow (section 3.4.2.5). Let $\varepsilon = 1$, which amounts to taking $T = D^2/\nu$ as a unit of time. The expression for the velocity becomes:

$$u(y,z,t) = U\widetilde{u}(\widetilde{y},\widetilde{z},\widetilde{t}) = U \sum_{i=0}^{\infty} f^{(i)}\left(\widetilde{t}\right) g_i(\widetilde{y},\widetilde{z})$$

The volume flow rate can thus be easily expressed as a function of the successive derivatives of the pressure gradient:

$$q_v = UD^2 \int_S \widetilde{u}(\widetilde{y},\widetilde{z},\widetilde{t}) d\widetilde{s} = UD^2 \sum_{i=0}^{\infty} f^{(i)}\left(\widetilde{t}\right) \int_S g_i(\widetilde{y},\widetilde{z}) d\widetilde{s}$$

Defining the dimensionless coefficients $G_i = \int_S g_i(\widetilde{y},\widetilde{z}) d\widetilde{s}$, it can be written in the form:

$$q_v = UD^2 \sum_{i=0}^{\infty} G_i f^{(i)}\left(\widetilde{t}\right)$$

Let $\Delta p(t)$ be the driving pressure loss for the length *L*; the pressure gradient can be written in the form: $\dfrac{\partial p}{\partial x} = -\dfrac{\Delta p(t)}{L}$ and we have:

$$f^{(i)}\left(\widetilde{t}\right) = \frac{D^2}{\mu U}\left(\frac{D^2}{\nu}\right)^i \frac{\partial^i}{\partial t^i}\left(\frac{\Delta p}{L}\right)$$

From this we can derive an expression for the volume flow rate as a function of the pressure loss and its temporal derivatives:

$$q_v = \frac{D^4}{\mu} \sum_{i=0}^{\infty} G_i \left(\frac{D^2}{\nu} \right)^i \frac{\partial^i}{\partial t^i} \left(\frac{\Delta p}{L} \right)$$

For example, in the case of a circular cross-section C ($\tilde{\Delta} = \frac{1}{\tilde{r}} \frac{\partial}{\partial \tilde{r}} \left(\tilde{r} \frac{\partial}{\partial \tilde{r}} \right)$) of radius $\tilde{r} = 1$, we easily find:

$$g_0(\tilde{r}) = \frac{1}{4}\left(1 - \tilde{r}^2\right); \qquad g_1(\tilde{r}) = \frac{1}{16}\left(-\frac{3}{4} + \tilde{r}^2 - \frac{\tilde{r}^4}{4} \right);$$

$$g_2(\tilde{r}) = \frac{1}{256}\left(\frac{19}{9} - 3\tilde{r}^2 + \tilde{r}^4 - \frac{\tilde{r}^6}{9} \right); \quad \ldots$$

The physical interpretation of the series is not so simple, as the physical significance of the higher order derivatives implies a long-term property. It suffices to consider a function $f(t)$ and a polynomial approximation of this over an interval of time. The higher order derivatives of these functions are very different, and so, consequently, are the preceding corresponding series. In fact, we can easily verify the convergence of the series for exponential or sinusoidal functions whose time or frequency constants are relatively small, but the preceding representation is of limited use for these particular cases.

For the example considered, we can show that we have convergence of the series if the series of the general term $\left| f^{(n)}(\tilde{t}) \right| / 4^n$ converges. Recall, however, that, from a mathematical point of view, the solutions established for the partial differential equation does not require the existence of derivatives for the given function $f(t)$. The preceding results can equally be obtained by means of a Laplace transform (Appendix 1) in the asymptotic approximation for large periods of time.

The regular perturbation of the solution to a non-linear partial differential equation is more difficult (section 6.4.2.3), and in general we can only obtain the first terms of the development. Examples of this kind of development (Blasius series, Görtler series, etc.) are found for boundary layer equations ([SCH 99], [YIH 77]).

The method of regular perturbations can also be applied in acoustics and energetics. Consider the geometry of Figure 6.13 in which the cylinder of base C is a solid, with constant physical properties, subjected to a heat release of volume power *P(t)* and whose external wall is maintained at constant temperature $T = 0$.

The thermal conduction equation for this solid and the associated boundary conditions:

$$\rho\, C \frac{\partial T}{\partial t} = P(t) + \lambda\, \Delta T; \quad \text{with: } (y,z) \in C\text{: } T(y,z,t) = 0$$

can be immediately reduced to equation and boundary conditions given in [6.69].

6.4.3. *Singular perturbations*

6.4.3.1. *Introduction*

In section 6.4.2 we assumed that the differential equation could be solved with respect to the first derivative such that the unperturbed equation (ε=0) possesses a differentiable solution. This is not the case if the coefficient of the derivative tends to zero with ε faster than the other coefficients. Let us examine the elementary example of the following differential equation in which the parameter ε is now the coefficient of the derivative $\dot{x}(t)$:

$$\varepsilon\, x'(t) + x(t) = 1 \qquad x(0) = 0 \tag{6.72}$$

The solution is: $x(t) = 1 - \exp(-t/\varepsilon)$.

For ε to tend to zero, it is easy to follow the evolution of the solution towards its discontinuous limit (Figure 6.12b) made up of the half-line ($x = 1$, $t > 0$) and the origin O. Letting ε equal to zero in equation [6.72], we obtain the solution $x = 1$ which is *valid nearly everywhere*, and we can no longer satisfy the initial condition posed at the origin for this part of the solution (called the *external solution*). We note that the family of functions $x(t, \varepsilon)$ does not converge uniformly towards its limit for $\varepsilon = 0$.

It is therefore necessary to perform a particular study over a small interval in the neighborhood of the origin where the solution undergoes significant variations over a small distance (*internal solution*) in order to recover the boundary condition which is not satisfied.

The external solution can also undergo a regular perturbation due to the small parameter ε. For example, the solution of the following differential equation:

$$\dot{x} + x = (1 + \varepsilon) \qquad x(0) = 0$$

does only differs from the preceding equation by the coordinate 1 + ε of the horizontal asymptote.

6.4.3.2. *Methods for studying singular perturbations*

The existence of a singular perturbation is related to the presence of a singularity for the limit solution of a system of differential equations. This phenomenon is encountered in diverse circumstances and notably in the case of the lowering of the order of a differential system in which we impose ε as zero, or when the limit solution or one of the derivatives involves a discontinuity on the interval of the study. We will limit ourselves here to the study of a very simple case in order to illustrate the general method by which we discuss such problems.

The direct numerical study of a problem involving singular perturbations often presents difficulties on account of the very large disparity in the characteristic scales found in different regions of the study. This point can obviously involve difficulties for numerical methods in fluid mechanics. It is therefore particularly interesting to directly obtain information independent of the exact solution (often very difficult) of the differential problem which is posed. We will illustrate this using a very simple case; the general analysis method involves a number of stages:

1) *searching for the external solution*: a study of the order of magnitude of the solution which is assumed to vary regularly over the interval under study allows us to obtain equations which are satisfied by the external solution;

2) we then try to identify a contradiction with respect to the data of the problem in order to *identify the singularity to study*;

3) a new examination of the order of magnitudes of the terms of the equation provides information pertaining to the *scales of the singularity to be studied*;

4) it is then possible to identify the *equations of the differential problem in the singular zone* (*internal solution*), these resulting from simplifications of the complete problem;

5) we then discuss the boundary conditions to be associated with the two preceding problems for matching their solutions.

6.4.3.3. *An elementary example*

The example of section 6.4.3.1 is too simple because the internal solution and the exact solution are identical. Consider the following differential problem:

$$\varepsilon\,\dot{x}(t) + x(t) = 1 + \alpha\, t \qquad x(0) = 0 \tag{6.73}$$

Ignoring the results of the last example, let us apply the methodology described above:

1) We first look for the *external solution* by supposing that the unknown function has variations of the order of one on the intervals of the same order of magnitude. To $\mathcal{O}(\varepsilon)$, differential equation [6.73] reduces to:

$$x(t) = 1 + \alpha\, t$$

The equation obtained is here an algebraic relation (a zero order differential equation) which explicitly gives the external solution which is valid nearly everywhere.

2) The condition that $x(t)$ be zero at the origin is not fulfilled. We must therefore complete the preceding solution by an internal solution on an interval with a different scale to the interval where the external solution is valid, otherwise we will only recover the latter. Let δ be the order of magnitude of the length scale of the interval over which the neglected term $\varepsilon\,\dot{x}$ must be taken into account.

The interval sought of scale δ must be *situated at the origin*. In effect, if this was not the case, we would have between this interval $o(1)$ and the origin a finite interval on which the results of the external zone would be applicable; the function x would be equal to $1+\alpha t$ and would not therefore be zero at the origin, which would imply that nothing had been solved. The only reasonable possibility is therefore to place our small interval δ in the neighborhood of the origin.

3) The order of magnitude of δ can be obtained by supposing that the term $\varepsilon\, x'$ is of the same order as that of the variations of x, i.e. of the order of 1:

$$\begin{array}{llll} \varepsilon\,\dot{x}(t) & +\; x(t) & = & 1 + \alpha\, t \\ \approx \dfrac{\varepsilon}{\delta} & \approx 1 & & \approx 1 + \alpha\delta \end{array}$$

We can deduce from this that $\delta = \varepsilon$, and we need to study the behavior of the solution in an interval of amplitude ε close to the origin.

4) We perform the change of variable:

$$\eta = \frac{t}{\varepsilon} \qquad x(t) = X(\eta)$$

Neglecting the terms of order ε, equation [6.73] can be written:

$$\dot{X}(\eta) + X(\eta) = 1 \qquad X(0) = 0$$

Its solution satisfying the condition at the origin can be written:

$$X(\eta) = A\left(1 - e^{-\eta}\right)$$

It tends to A when η tends to infinity.

5) We now have two pieces of the solution (Figure 6.14b) which we need to match. This procedure appears relatively empirical and we will consider only small values of ε. We see that as ε tends to zero, the value $1 + \alpha\varepsilon$ of the external solution for $x = \varepsilon$ tends to 1. We therefore adopt the following simple matching rule.

The limit of the internal solution $F(\eta)$ *for* η *becoming infinite is equal to the limit of the external solution* $x(t)$ *when t tends to 0*:

$$\lim_{\eta \to \infty} X(\eta) = \lim_{t \to 0} x(t)$$

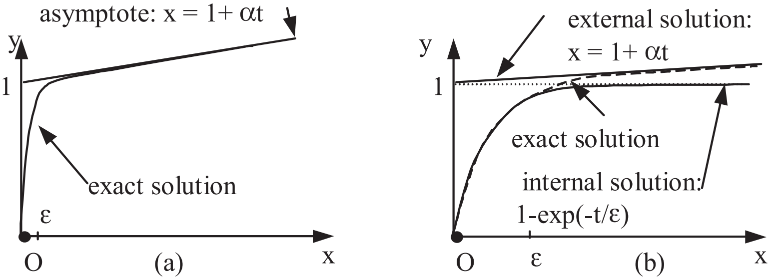


Figure 6.14. *(a) Exact solution of differential problem [6.73]; (b) matching of external and internal solutions*

This rule provides the boundary conditions which were missing for the internal solution and leads to a value of 1 being assigned to the constant *A*. We thus obtain the internal solution:

$$X(\eta) = 1 - e^{-\eta} \quad \text{with: } \eta = t / \varepsilon$$

In fluid mechanics, the internal solution is called a *boundary layer*. Such singularities are encountered in all domains of physics (skin effect in electromagnetism, penetration depth of an evanescent electromagnetic or acoustic wave, etc.). We will see other examples in problems of heat transfer or chemical reactions with or without flow.

The singularity can also consist of a discontinuity of the first or second derivatives.

Perturbation problems can also present other characteristics. Consider for example, the following lightly damped oscillator (section 6.3.1.1):

$$\ddot{x} + \varepsilon \dot{x} + x = 0$$

The friction term $\varepsilon \dot{x}$ is a perturbation of the undamped equation. The modification of the oscillatory solution due to a damping term $e^{-\varepsilon t/2}$ has an effect of order ε over a duration $\mathcal{O}(1)$, but the final result is the suppression of the movement. The perturbation effects due to friction are cumulative.

More complete developments on singular perturbations can be found in specialized textbooks ([COL 68], [HIN 91], [NAY 81], [VAN 75]).

6.5. Quasi-1D flows

6.5.1. *General properties*

6.5.1.1. *Assumptions*

Many flows can present a quasi-1D character, in other words the evolution scales of phenomena differ depending on whether we follow the principal direction of the flow or the transverse directions.

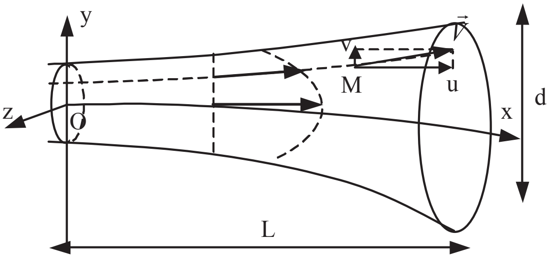


Figure 6.15. *Quasi-1D flows*

The basic assumption of a viscous quasi-1D flow is that the cross-section and direction of a stream tube vary quite slowly (Figure 6.15). The thickness *d* of the domain of study is small compared to its length *L* (*d* << *L*). We consider the coordinate on a curve oriented along O*x*, more or less parallel to the stream tubes (the axis of a pipe, jet or wall, etc.) and the orthogonal coordinates in cross-sections orthogonal to Ox. The local coordinates defined by the curvilinear abscissa x and the orthogonal coordinates (y,z) in cross-sections constitute a *coordinate system which is locally Cartesian*. These assumptions allow us to write the *balance equations in Cartesian form* in this zone.

We furthermore assume that the variations δf of the quantities f studied are more or less of the same order of magnitude over the distances d and L.

6.5.1.2. *Approximations*

The following properties result from the preceding assumptions:

1) Partial derivatives of a quantity in the directions y or z are an order of magnitude greater than the corresponding axial derivative in the direction Ox. In effect $\partial f/\partial x \approx \delta f/L$ and $\partial f/\partial y \approx \delta f/d$ lead to:

$$\frac{\partial f}{\partial x} \approx \frac{d}{L}\frac{\partial f}{\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} \approx \frac{d^2}{L^2}\frac{\partial^2 f}{\partial y^2}$$

This results in vectors of the form $k\,\overrightarrow{grad}\,f$ (for example, the thermal or mass flux densities $\vec{q}_G$, etc.) are perpendicular to the axis Ox and that we only take into account the two components $k\,\partial f/\partial y$ and $k\,\partial f/\partial z$ of the flux density $\vec{q}_G$.

2) The diffusion term $-div\,\vec{q}_G$ of the balance equation of a scalar quantity G is therefore reduced to the sum of the transverse derivatives $-\partial q_{Gy}/\partial y$ and $-\partial q_{Gz}/\partial z$.

3) The *transverse components v and w of the velocity are small compared with the axial component u (*$v \ll u$ and $w \ll u$*)*. In effect, assuming the flow to be incompressible, the volume balance can be written:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If variations of the cross-section exist, the transverse velocity is not zero, but it is very small, in order to ensure the inclination of the velocity with respect to the axis Ox, of order *d/L*. Assuming the 2D problem ($w = 0$) in order to simplify matters, the two derivatives $\partial u/\partial x$ and $\partial v/\partial y$ have the same value (sign excepted). An order of magnitude analysis shows that:

$$\frac{\partial u}{\partial x} \approx \frac{U}{L} \approx \frac{\partial v}{\partial y} \approx \frac{v}{d}$$

i.e., sign excepted, $v \approx Ud/L$.

No simplification is possible in the mass conservation equation.

The result of this evaluation of the velocity components is that the *dominant terms of the viscous stress tensor* are the tangential components $\mu\,\partial u/\partial y$ and $\mu\,\partial u/\partial z$ (friction forces in the Ox direction).

4) *The material derivative of a scalar quantity conserves its general expression in the quasi-1D approximation.*

Consider the material derivative $\dfrac{dg}{dt} = \dfrac{\partial g}{\partial t} + u\dfrac{\partial g}{\partial x} + v\dfrac{\partial g}{\partial y}$ of the quantity *g*. The two convection terms are of the same order:

$$u\frac{\partial g}{\partial x} \approx U\frac{\delta g}{L} \qquad v\frac{\partial g}{\partial y} \approx U\frac{d}{L}\frac{\delta g}{d} = U\frac{\delta g}{L}$$

5) On the other hand, an analogous order of magnitude calculation shows that the small value of the transverse components of the velocity leads to the *transverse accelerations dv/dt and dw/dt being negligible with respect to the axial acceleration du/dt.*

6.5.1.3. *Local balance equations*

Taking account of the preceding approximations, balance equation [4.3] for the volume quantity $\overline{g}$ can be written:

$$\frac{\partial \overline{g}}{\partial t} + div\left(\vec{V}\overline{g}\right) = \rho\frac{dg}{dt} = \sigma_G - \frac{\partial q_{Gy}}{\partial y} - \frac{\partial q_{Gz}}{\partial z} \qquad [6.74]$$

By way of example, the dynamic equation following O*x* [4.21], for a Newtonian fluid, can be written:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \rho f_x + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) \qquad [6.75]$$

In a plane 2D flow, we obtain for a fluid with *constant physical properties* (where p_g is the driving pressure):

$$\rho \frac{du}{dt} = \rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = - \frac{\partial p_g}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad [6.76]$$

Finally, as the transverse accelerations dv/dt and dw/dt are negligible with respect to the axial acceleration du/dt, the viscous stress terms of the transverse equations are equally negligible with respect to the corresponding term in the axial dynamic equation. The transverse equations reduce to:

$$\frac{\partial p_g}{\partial y} = \frac{\partial p_g}{\partial z} = 0$$

The pressure (or driving pressure) distribution in a cross-section is hydrostatic, but it depends on x and t.

Energy equation [4.51] can thus be written in the quasi-1D approximation:

$$\left. \begin{array}{r} \rho \dfrac{de}{dt} + p \dfrac{\partial u_i}{\partial x_i} \\ \rho \dfrac{dh}{dt} - \dfrac{dp}{dt} \\ \rho T \dfrac{ds}{dt} \end{array} \right\} = \mu \left(\frac{\partial u}{\partial y} \right)^2 + \mu \left(\frac{\partial u}{\partial z} \right)^2 + \sigma_T + \frac{\partial}{\partial y} \left(\lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left(\lambda \frac{\partial T}{\partial z} \right) \qquad [6.77]$$

Dissipation function [4.56] can be simplified by only considering the dominant strain rates $\partial u / \partial y$ and $\partial u / \partial z$. We find immediately:

$$\Phi = \mu \left(\frac{\partial u}{\partial y} \right)^2 + \mu \left(\frac{\partial u}{\partial z} \right)^2 \qquad [6.78]$$

6.5.1.4. *Parabolic character of the quasi-1D equations*

Consider first convection equation [6.74] in which the velocity field is assumed to be given. This presents a structure analogous to the heat equation, i.e. it has a dissymmetry in the order of the derivations with respect to the variables: second derivatives with respect to y and z are encountered, the derivatives with respect to x and t being only first order. This results in a parabolic behavior with respect to the variables x and t: *the temperature distribution in a cross-section of coordinate x at* *the instant t depends only on data from the past or the upstream region.* This explains the fact that the state of the matter in the cross-section depends on its anterior state on the trajectory; the diffusion term of quantity G only provides a limited action in this section, the suppression of the conduction term $-\partial q_{Gx}/\partial x$ amounting to the suppression of all flux in the upstream direction.

The velocity components u and v are the unknowns of dynamic equation [6.76]. The demonstration of the parabolic character can be effected by introducing the stream function ψ such that the mass conservation is satisfied:

$$u = \partial\psi/\partial y \qquad v = -\partial\psi/\partial x$$

Equation [6.76] can thus be written:

$$\rho\frac{du}{dt} = \rho\left[\frac{\partial^2\psi}{\partial t\partial y} + \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right] = -\frac{\partial p_g}{\partial x} + \mu\,\frac{\partial^3\psi}{\partial y^3} \qquad [6.79]$$

In equation [6.79] the derivations with respect to x and t are of order 1, whereas the derivation with respect to y is of order 3, which indicates the parabolic character with respect to the variables x and t.

In applications, the hypotheses of section 6.5.1.1 are very often encountered. Quasi-1D flows can be produced:

– when the geometric boundary conditions impose such an evolution: in flows in pipes this kind of approximation exists for most macroscopic physical phenomena (electric, electromagnetic, thermal, etc.);

– when diffusion phenomena in flows lead to weak fluxes of extensive quantities in the axial direction Ox compared with the convection fluxes of these. In inviscid fluids transport or propagation phenomena governed by the characteristics are, in fact, perturbed by contact actions (viscosity, thermal conduction, diffusion, etc.), and this leads to a *transverse migration* of the extensive quantities. The balance equations contain high-order derivatives which “perturb” the convective transport terms. In these situations, there exist non-dimensional parameters (Reynolds, Peclet numbers, etc.) which take on high values.

We will successively examine two categories of problem in which the data are different.

6.5.2. *Flows in pipes*

6.5.2.1. *Nature of the problem*

A pipe is a stream tube Σ materialized by a wall. The dominant velocity component *u* in a cross-section is directed along its axis. The definition of the problem to solve can be obtained as usual by combining the local balance equations (mass, axial momentum, energy, etc.) with the initial and usual boundary conditions on the wall Σ_L.

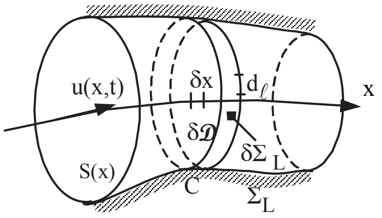


Figure 6.16. *Balance in a pipe*

The geometric elements of the pipe are given, such that integration of the dynamic equation leads to a relation between the volume (or mass) flow and a pressure difference between two cross-sections. Calculation of the velocity distribution (or of a quantity *G*) in a cross-section is an *internal problem* posed in the interior domain of the stream tube Σ (Figure 6.16).

6.5.2.2. *Global balance equation for an extensive quantity in a pipe*

The normal of a cross-section S(*x*) is here oriented parallel to Ox (orientation by continuity). This convention requires a change in sign at the time of application of Osstrogradski's theorem to a closed surface Σ containing cross-sections. By assumption, the *lateral wall* Σ_L *is supposed impermeable to the flow*, i.e. $\int_{\Sigma_L} g\overline{V}.\vec{n}ds = 0$. Consider the small domain $\delta\mathcal{D}$ of the pipe comprised between the sections of abscissas x and $x + \delta x$ of the lateral surface $\delta\Sigma_L$ (Figure 6.16).

Formula [4.62] for the global balance of a volume quantity $\overline{g}$ can be written:

$$\int_{\mathcal{D}} \frac{\partial \overline{g}}{\partial t} dv + \int_{\Sigma} \overline{g} u_i n_i ds = \int_{\mathcal{D}} \sigma_G \; dv - \int_{\Sigma} q_{Gj} n_j \, ds \qquad [6.80]$$

Taking account of the preceding assumptions, we have:

$$\int_{\Sigma} \overline{g} u_i n_i ds = \int_S guds\Big|_x^{x+\delta x} ; \qquad \int_{\mathfrak{D}} \sigma_G \; dv = \delta x . \int_S \sigma_G \, ds$$

The density flux $\vec{q}_G$ is essentially normal to the lateral wall (section 6.5.1.2) for irreversible changes (viscosity, conduction, diffusion) and the integral $\int_{\Sigma} q_{Gj} n_j ds$ is equal to $\delta x . \int_C q_{Gw} d\ell$ taken over the contour C of the cross-section (q_{Gw} is the flux density of G, normal to the wall).

Substituting these expressions into [6.80] and dividing by δx gives:

$$\int_S \frac{\partial \overline{g}}{\partial t} ds + \frac{\partial}{\partial x}\left(\int_S u\overline{g} ds\right) = \int_S \sigma_G \; ds - \int_C q_{Gw} d\ell \tag{6.81}$$

In the place of the volume quantity $\overline{g} = \rho g$, let us take the massive quantity g; we obtain:

$$\int_S \frac{\partial (\rho g)}{\partial t} ds + \frac{\partial}{\partial x}\left(\int_S \rho ugds\right) = \int_S \sigma_G \; ds + \int_C q_{Gw} d\ell \tag{6.82}$$

The quantity $\varphi_{GS} = \int_S \rho ugds = \int_S u\overline{g}ds$ is the *flow of quantity G* across the cross-section *S*.

6.5.2.3. *Applications*

Taking $\overline{g} = 1$, we have the *volume flow rate* $q_{vS} = \int_S uds$ across *S*.

Taking $g = 1$, we have the equation for the *mass balance*:

$$\int_S \frac{\partial \rho}{\partial t} ds + \frac{\partial q_m}{\partial x} = 0 \tag{6.83}$$

with: $q_m = \int_S \rho uds$, the mass flow in the rate cross-section *S*.

The *momentum balance* along Ox can be obtained by applying formula [6.82] of the quantity $g = u$. Supposing (section 6.5.1.3) that the driving pressure gradient $\partial p_g / \partial x$ (here, the term σ_G) is constant over the cross-section *S*, and designating by τ_w the *viscous stress exerted by the fluid on the lateral wall*, we have:

$$\int_S \frac{\partial(\rho u)}{\partial t} ds + \frac{\partial}{\partial x}\left(\int_S \rho u^2 ds\right) = -S\frac{\partial p_g}{\partial x} - \int_C \tau_w d\ell \tag{6.84}$$

Proceeding as before, kinetic energy equation [4.66] can immediately be written, by noting that the power of the viscous forces on the external surface of *δ𝔇* is zero because of the assumptions which have been made (zero velocity on the wall, and quasi-1D approximation on *S*), we obtain:

$$\int_S \frac{\partial}{\partial t}\left(\frac{\rho u^2}{2}\right) ds + \frac{\partial}{\partial x}\left[\int_S \left(p_g + \frac{\rho u^2}{2}\right) u ds\right] = - P_{vS} \tag{6.85}$$

with: $P_{vS} = \int_S \Phi ds$, the power dissipated by viscosity per unit length of the pipe.

The first term of [6.85] gives the accumulation of kinetic energy in *D* in transitional regime and the second expresses the flow rate of mechanical energy across S.

The different forms of the *energy equation* seen in section 4.3.4 can be integrated over the surface *S* (or in the domain *δ𝒟*), but it is not possible to write down an energy flow in the cross-section, except if we use initial formula [4.68] of the balance equation which contains the flow of total enthalpy $\varphi_{ht} = \int_s [h + (V^2/2)] dq_m$ across the section *S*.

Furthermore, the integral over *δD* of the term $\dfrac{\partial(\tau_{ij} u_i)}{\partial x_j}$ can be transformed into a surface integral, which is zero, as we have already said.

Proceeding as before for the thermal flux density, we obtain:

$$\int_S \frac{\partial}{\partial t}\left[\rho\left(e + \frac{u^2}{2}\right)\right] ds + \frac{\partial \varphi_{ht}}{\partial x} = \int_S \rho f_x u ds + \int_S \sigma_T ds - \int_C q_{Tw} d_\ell \tag{6.86}$$

Recall that the power of an external force field is often negligible for a perfect gas. The expressions for the internal specific energy *e* and the specific enthalpy *h* for a perfect gas can be written:

$$\rho e = \rho C_v T = \frac{C_v}{r} p = \frac{p}{\gamma - 1}; \quad \rho h = \rho C_p T = \frac{C_p}{r} p = \frac{\gamma p}{\gamma - 1}$$

The energy equation for an incompressible fluid of constant specific heat C (section 4.3.4.1.6 and equation [4.70]) can be written:

$$\int_S \frac{\partial(\rho CT)}{\partial t} ds + \frac{\partial}{\partial x}\left(\int_S \rho uCTds\right) = P_{vS} + \int_S \sigma_T \; ds - \int_C q_{Tw} d\ell \tag{6.87}$$

where, depending on the case, we used either C_p or C_v for the specific heat C.

6.5.2.4. *Average values of intensive quantities*

As we have already said in section 1.4.2.5, to be consistent, the definition of mean intensive values is effected such that the balance of the corresponding extensive quantities is verified for the system studied. The application of this general principle is expressed here by writing that values of the fluxes of extensive quantities (mass, momentum, energy, etc.) in the cross-section S are identical either by integration of local values or by using these mean values for balances. Then, we take the following definitions:

– the mean density ρ_m: $\rho_m = \frac{1}{S}\int_S \rho \, ds$

– the mean velocity u_q: $u_q = \frac{1}{\rho_m S}\int_S \rho u \; ds$

– the mean temperature T_m (C = const): $T_m = \frac{1}{\rho_m u_q S}\int_S \rho uT \; ds$

– and in general the mean quantity g_m: $g_m = \frac{1}{\rho_m u_q S}\int_S \rho ug \; ds$

These quantities are often called the *average mixing values* (or *mean mixing values*), as they correspond to the intensive value represented by the variable g under the assumption that the flow of G across the section S would directly fill a volume where it should be mixed without any external input. The preceding definition of T_m supposes that the specific heat is independent of the temperature.

In cases where the same quantity contributes differently to several mean values, we introduce a suitable coefficient, for example:

– the momentum coefficient β: $\beta u_q^2 = \frac{1}{\rho_m S}\int_S \rho u^2 \; ds$

– the kinetic energy coefficient α: $\alpha u_m^3 = \frac{1}{\rho_m S} \int_S \rho u^3 \; ds$

These mean values and the preceding coefficients allow the equations for the quasi-1D model of flow in a pipe to be written very simply in a form analogous to the 1D slice approximation with uniform properties in the cross-section (see section 4.3.2.3.4). However, the system of differential equations obtained only determines a solution if the preceding coefficients constitute data, which must be chosen more or less empirically from assumptions derived from the velocity, temperature or concentration profiles.

We will leave it to the reader to verify that in a laminar flow we have the following values:

– uniform flow: $\alpha = \beta = 1$;

– Poiseuille flow in a circular tube: $\alpha = 2$; $\beta = 4/3$.

In industrial pipe systems, the values of α and β are often of the order of 1.1 to 1.3 ([ASH 89], [IDE 99]). If the differences between the local velocity u and the mean velocity u_q are small, the reader can verify that we have approximately[6] $\beta - 1 = 3(\alpha - 1)$.

The mechanical energy balance [6.85] (generalized Bernoulli's theorem) in a pipe for an incompressible fluid can be written with the definition of α:

$$\frac{\partial}{\partial t}\left(\beta \frac{\rho u_q^2}{2}\right) + \frac{\partial}{\partial x}\left[\left(p_g + \alpha \frac{\rho u_q^2}{2}\right) q_v\right] = - P_{vS} \qquad [6.88]$$

We can thus define the *total mean driving pressure*:

$$p_{tm} = p + \rho g h + \alpha \frac{\rho u_q^2}{2} .$$

The quantity $q_v p_{tm}$ represents the *flow rate of mechanical energy across a cross-section S*. Equation [6.88] becomes, on neglecting the power of the viscous forces on Σ (approximation 2 of section 6.5.1.2):

---

[6] Let $u = u_q + u'$ and neglect the term in u'^3 in the calculation of β.

$$\frac{\partial}{\partial t}\left(\beta\frac{\rho u_q^2}{2}\right)+q_v\frac{\partial p_{tm}}{\partial x}=-P_{vS}$$

Integrating along the axis in the domain D included between two cross-sections S_1 and S_2 gives:

$$\int_{\mathfrak{D}}\frac{\partial}{\partial t}\left(\beta\frac{\rho u_q^2}{2}\right)dv+q_v\left(p_{tm2}-p_{tm1}\right)=-P_{v\mathfrak{D}} \qquad [6.89]$$

This equation is a model which reveals the inflows and outflows for the studied domain of the pipe; it is particularly useful in steady flows for evaluating the mechanical energy dissipated, using measurements of velocity distributions in the sections S_1 and S_2 [IDE 99].

6.5.2.5. *Local equations*

The local equations in the cross-section S of the stream tube are identical to the local equations of the boundary layer which are developed in the following section.

6.5.3. *The boundary layer in steady flow*

6.5.3.1. *Introduction*

We will limit our discussion in this section to the case of a steady flow of an incompressible fluid of constant viscosity. The Navier-Stokes equations ([4.74] and [4.75]) can be written with non-dimensional variables (section 4.6.1.3), $\widetilde{p}$ being here the non-dimensional driving pressure:

$$\widetilde{u}_j\frac{\partial\widetilde{u}_i}{\partial\widetilde{x}_j}=-\frac{\partial\widetilde{p}}{\partial\widetilde{x}_i}+\frac{1}{Re}\widetilde{\Delta}\widetilde{u}_i;\qquad\frac{\partial\widetilde{u}_j}{\partial\widetilde{x}_j}=0;\qquad(i,j=1,2,3) \qquad [6.90]$$

Under the usual conditions, the Reynolds number is very large compared to 1, and the term $\widetilde{\Delta}\widetilde{u}_i$ is weighted by a coefficient which is very small compared to 1. This therefore appears to be a perturbation quantity whose nature we will study using the procedure outlined in section 6.4.

6.5.3.2. *External solutions and the Euler equations*

Assuming that the solution of the preceding equations and its derivatives vary at the scale of 1, all the non-dimensional derivatives are in the order of 1, and the term $\frac{1}{Re}\tilde{\Delta}\tilde{u}_i$ is therefore very small compared to 1.

The dynamic equations can be reduced to the Euler equations:

$$\tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} \qquad (i, j = 1,2,3)$$

These are one order less, and require weaker boundary conditions than the Navier-Stokes equations. It is clear from a physical point of view that we must abandon the adherence condition, since the viscosity no longer exists, and the fluid can therefore slide over the walls. We thus find ourselves in the singular perturbation situation described in section 6.4.3.

6.5.3.3. *Finding a singular perturbation zone*

Following the preceding reasoning, this zone cannot concern a zone of scale 1 in all three dimensions. At least one of the dimensions of this zone must be small in order for the value of a derivative to be sufficiently large to compensate the coefficient 1/*Re*. Where can such a zone be found? We note firstly that on account of the transport of fluid and its properties, it is difficult for such a zone to spontaneously appear in the heart of the flow. An exterior intervention is then necessary in order to create a viscous phenomenon sufficiently large which then develops. This can only happen when the flow of an inviscid fluid encounters an obstacle on the singular streamline which contains the stagnation point A of zero velocity (Figure 6.17).

The subsequent velocity evolution on the wall streamlines of the inviscid fluid lead to a non-zero sliding velocity which increases downstream of the point A. It is then in the neighborhood of the wall that the viscosity must necessarily act.

The length of this zone is in the order of obstacle dimension L and its thickness δ is necessarily $o(L)$, otherwise we are back in the preceding situation. We must therefore study a thin zone in the vicinity of the walls where we can make the approximation of a quasi-1D flow. We will here consider a plane 2D flow over an obstacle placed in a flow of uniform velocity U (Figure 6.17), and we will allow the radii of curvature of the walls to be large compared with the thickness δ.

6.5.3.4. *Boundary layer equations*

Let us consider an obstacle inside a uniform flow of an inviscid fluid at speed *U*. Let us take a locally Cartesian coordinate system (x, y) defined in the following way (Figure 6.17): x is the curvilinear abscissa evaluated algebraically on the wall downstream of the stagnation point A on the obstacle, the coordinate y being evaluated along the wall normal $\vec{n}$. The velocity components are designated by (u,v) in the coordinate system fixed to the wall and in its immediate vicinity.

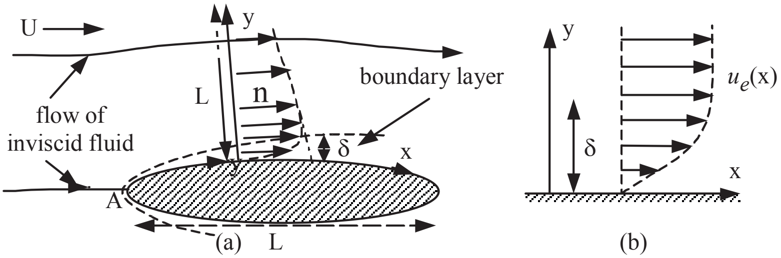


Figure 6.17. *Boundary layer on the wall of an obstacle in uniform flow: (a) figure on the scale of* L; *(b) figure on the scale of* δ

The discussion is as per section 6.5.1.3. The longitudinal velocity and acceleration components are nearly parallel to the wall; the normal acceleration component is negligible compared to the longitudinal component: the pressure, constant across the thickness of the boundary layer is here only a function of the abscissa x (section 6.5.1.3). However, in the dynamic equation, it is not possible to neglect the v component in the material derivative which ensures a part of the momentum transport (section 6.5.1.2). The dimensional equations of the 2D boundary layer can thus be written:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\mu\frac{\partial^2 u}{\partial y^2};\qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad [6.91]$$

The order of magnitude δ of the boundary layer thickness can be obtained by writing that the material derivative and the viscous stress term are of the same order of magnitude (section 6.5.1.3):

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\approx\frac{\rho U^2}{L}\approx\mu\frac{\partial^2 u}{\partial y^2}\approx\frac{\mu U}{\delta^2}$$

where $\delta \approx \sqrt{\mu L / \rho U}$, or by defining the Reynolds number $Re_L = \rho UL/\mu$ with the length L:

$$\delta/L \approx \sqrt{\mu/\rho UL} = \mathrm{Re}_L^{-1/2} \qquad [6.92]$$

Introducing the stream function ψ ($u = \partial\psi/\partial y$; $v = -\partial\psi/\partial x$), equations [6.91] can be reduced to the equation:

$$\rho\left(\frac{\partial\psi}{\partial y}.\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}.\frac{\partial^2\psi}{\partial y^2}\right) = -\frac{dp}{dx} + \mu\frac{\partial^3\psi}{\partial y^3} \qquad [6.93]$$

The order of the derivatives with respect to x is less than the order of the derivatives with respect to y in equation [6.93] which is parabolic: the *distribution of the velocity at a given abscissa x_0 only depends on the upstream conditions of the external velocity $u_e(x)$,* corresponding to values of x less than x_0.

6.5.3.5. *Boundary conditions*

The boundary layer equations are clearly simpler than the Navier-Stokes equations. We have already seen that the suppression of the transverse dynamic equation leads to pressure being a function of the x direction only. We must now express the adherence condition of the fluid at the wall, as this was our objective in the introduction to the boundary layer.

$$y = 0 \quad u(x,0) = v(x,0) = 0$$

We must now match the boundary layer and the external inviscid fluid flow. We proceed in a first approximation as per section 6.4.3.3 by writing that the velocity at the outer edge of the boundary layer is equal to the velocity $u_e(x)$ of the *inviscid fluid on the wall in the absence of a boundary layer*:

$$\frac{y}{\delta} \to \infty \quad u(x,y) \to u_e(x) \qquad [6.94]$$

The velocity $u_e(x)$ and p_e (x) satisfy Bernoulli's theorem on the wall for an inviscid fluid:

$$p_e(x) + \rho\frac{u_e^2(x)}{2} = p_\infty + \rho\frac{U^2}{2} \qquad [6.95]$$

where p_∞ designates the static pressure in the external uniform flow of velocity U.

Finally, we must fix the initial upstream condition for the velocity distribution at a given abscissa often taken as the origin $x = 0$. As we have explained in section 6.5.3.3, the stagnation point A of the flow of a inviscid fluid over the obstacle (Figure 6.17) is the departure point of the boundary layer that can be easily calculated in this zone where its thickness is quite small ([SCH 99], [YIH 77]).

If *the leading edge of the obstacle is of negligible thickness* (plane or wedge-shaped wall), the thickness of the boundary layer is here taken to be zero.

6.5.3.6. *General properties of boundary layers*

6.5.3.6.1. Physical interpretation of the boundary layer

We can immediately note that *the condition of zero velocity on the wall also leads to a condition of zero acceleration*. Conversely, *the viscous stress becomes zero at the outer edge of the boundary layer*. This is therefore a zone where the pressure gradient, which is constant, sees its action balanced by the acceleration in the outer region and by the viscous friction at the wall. This situation is shown schematically in Table 6.1 which indicates the dominant terms following the height in the boundary layer.

The solution previously obtained in section 5.4.5.4 for the heat equation is of the same kind as that of the boundary layer, with time replaced by the x coordinate. The zero-velocity condition imposed on the wall leads to *viscous diffusion of the viscous stress and of the vorticity*. This results in an augmentation of the boundary layer thickness as an abscissa function, as indicated by the order of magnitude $\delta/L \approx \sqrt{\mu/\rho UL} = \mathrm{Re}_L^{-1/2}$ found earlier [6.92].

height in the boundary layer	**acceleration quantity** $\rho\dfrac{du}{dt}$	**pressure gradient** $-\dfrac{dp}{dx}$	**viscous stress** $\mu\dfrac{\partial^2 u}{\partial y^2}$
outside (inviscid fluid)	X ◄——	——► X	0
inside boundary layer	x ◄- - -	- - -► X ◄- -	- - - -► x
wall	0	X ◄——	——► X

Table 6.1. *Balance of "forces" in the boundary layer*

This diffusion can be counterbalanced by an aspiration of fluid across the wall. Consider the simple case of a plane wall placed in a uniform flow of velocity u_e and realize a suction of fluid across the wall at constant velocity $-\ v_0$ (Figure 6.18). We see immediately that the solution:

$$u = u_e \exp\left(-\frac{v_0}{\nu}\, y \right) \qquad [6.96]$$

satisfies equations [6.91] of the boundary layer and corresponds to an established solution where the thickness $\delta = \nu/v_0$ is constant. Figure 6.18 represents this flow.

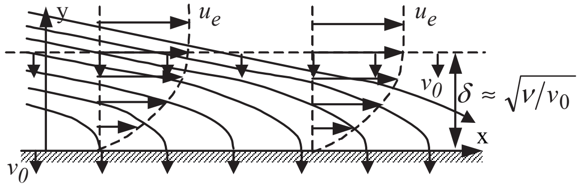


Figure 6.18. *Established boundary layer with suction through a wall in an uniform flow*

We note that in the case where we have blowing rather than suction ($-\ \mathrm{v}_0$ positive), the solution is not acceptable: the boundary layer no longer exists if the blowing is sufficiently strong ([SCH 99], [YIH 77]).

6.5.3.6.2. Non-dimensional equations

Let us write equations [6.91] in non-dimensional form with the following change of variables:

$$\widetilde{u} = u/U \quad \widetilde{v} = vRe^{1/2}/U \quad \widetilde{x} = x/L \quad \widetilde{y} = yRe^{1/2}/L \quad \widetilde{u}_e = u_e(x/L)/U$$

This gives:

$$\widetilde{u}\frac{\partial \widetilde{u}}{\partial \widetilde{x}} + \widetilde{v}\frac{\partial \widetilde{u}}{\partial \widetilde{y}} = -\widetilde{u}_e \frac{d\widetilde{u}_e}{d\widetilde{x}} + \frac{\partial^2 \widetilde{u}}{\partial \widetilde{y}^2}; \qquad \frac{\partial \widetilde{u}}{\partial \widetilde{x}} + \frac{\partial \widetilde{v}}{\partial \widetilde{y}} = 0 \qquad [6.97]$$

Equations [6.97] do not contain the parameter $\mathrm{Re} = \rho UL/\mu$. This results in the non-dimensional solution *being independent of the Reynolds number*. So, all bodies which have the same form have identical velocity distributions $\widetilde{u}_e(\widetilde{x})$ at the wall; they therefore have the same boundary layer development. We note however that the Reynolds number comes into play as soon as we recast the problem using dimensional data for the coordinate y and notably for the thickness of the boundary layer (formula [6.92]).

6.5.3.7. *Separation of steady flows*

The preceding discussion shows that, in the external part of the boundary layer, the flow is close to the flow of an inviscid fluid where the pressure gradient is determined by Bernoulli's theorem: the modulus of the velocity decreases if the pressure increases.

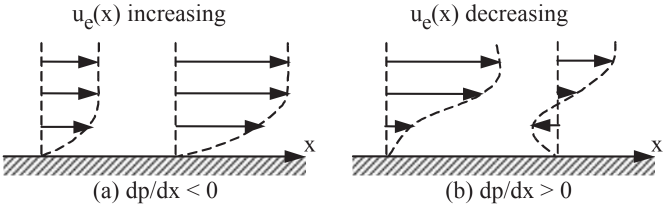
Close to the wall, the weak kinetic energy no longer plays an appreciable role and the variations of the viscous stresses are opposed to the pressure gradient in the boundary layer equations. The flow which is nearly purely viscous close to the wall results from two causes:

1) The *pressure gradient* leads to a tendency of the flow towards decaying pressures as in a Poiseuille flow (section 3.4.2.5). The velocity curvature profile at the wall is equal, according to [6.91] and [6.95] to:

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = \frac{1}{\mu}\frac{dp_e}{dx} = -\frac{\rho}{\nu}u_e\frac{du_e}{dx}$$

2) The *external velocity* $u_e(x)$ imposed creates a viscous entrainment in the boundary layer in a manner analogous to the corresponding phenomena in a Couette flow (section 3.4.2.3).

When the flow external to the boundary layer $u_e(x)$ increases with the abscissa x, the longitudinal pressure gradient is negative and the velocity curvature profile has a constant sign (Figure 6.19a). In the opposite case (Figure 6.19b), the sign of the curvature of the profile changes, and this can lead, close to the wall, to a flow which is reversed with respect to the external flow $u_e(x)$.



(a) dp/dx < 0 (b) dp/dx > 0

Figure 6.19. *Velocity profile near the wall with a (a) negative or (b) positive pressure gradient*

The appearance of a flow from downstream to upstream is in *contradiction with the parabolic properties* of the boundary layer equations: the velocity distribution in the zone of reversed flow no longer depends only on the upstream conditions the external velocity $u_e(x)$ considered, but also on the downstream conditions. The boundary layer equations are thus no longer applicable in such a zone.

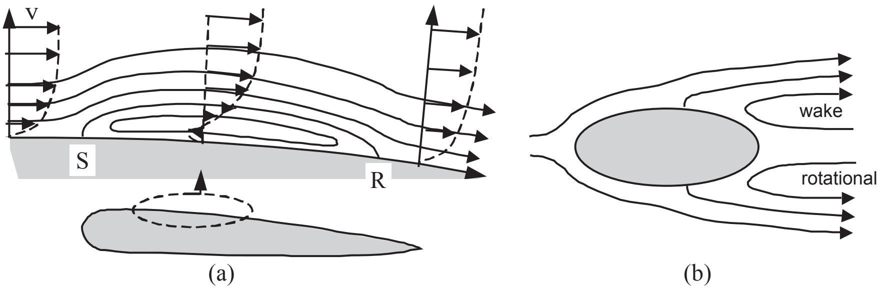


Figure 6.20. *(a) Separation with reattachment in the boundary layer (profiled obstacle); (b) separation and creation of an open wake (unprofiled obstacle)*

There therefore exists a back stagnation point S in a viscous flow where the friction stresses on the wall are zero (Figure 6.20a). In fact, the flow, which is reversed with respect to the main flow, comes from a rotational zone which can be manifested in two forms:

– either a relatively thin rotational zone is generated within the boundary layer, but reattaches at point R (Figure 6.20a); following what was said above, this can only happen if the effect of the pressure gradient is sufficiently weak for the viscous entrainment to constitute the dominant effect. This is the case for *profiled obstacles* whose reducing section in the downstream direction is very gradual;

– or a completely open wake can be generated downstream of an obstacle which is not profiled (Figure 6.20b).

Note that the *position of the separation point is independent of the Reynolds number*, provided the velocity distribution of the inviscid fluid at the wall does not depend on the Reynolds number either. This is the case for the irrotational inviscid fluid around a profiled obstacle which is not modified (to second order) and obviously does not depend on the viscosity. For the unprofiled obstacle, the real velocity field is different from the field calculated in an inviscid fluid on account of the presence of a rotational wake which is fed from downstream; however, if we assume that the structure of this wake is independent of the Reynolds number, we see that it is consistent to assume that the position of the separation point is also independent of this parameter: this is confirmed by experiments, *provided the boundary layer is not turbulent* ([SCH 99], [YIH 77]).

6.6. Unsteady flows and steady flows

6.6.1. *Introduction*

The temporal evolution of the properties of matter is fundamentally based on the balance laws of the associated extensive quantities. We have already discussed in Chapter 2 the difficulties of representing the continuous medium which we encounter depending on whether we choose to use a Lagrangian (substantial) description of the fluid particles or a Eulerian (spatial) representation of the flow. We must now return to the fundamental difficulties which arise when we use Eulerian variables.

The fields to which matter is subjected are furthermore always due to actions at a distance performed by other material elements: a gravitational field is caused by the presence of mass, an electric field results from the presence of charges, an electromagnetic field is due to electric charges in movement at either the macroscopic or the microscopic scale. A field is described by functions of space-time variables in a reference frame (known as the laboratory reference frame) associated with a flow device or an object moving with respect to a fluid (vehicle, plane, etc.). There are numerous situations for this observer in which the velocity fields and the material quantities are not functions of time, but only of space. The corresponding phenomena are therefore steady. This *terminology only has meaning in reference to this privileged reference frame, the quantities attached to the material particles being always functions of time* (Lagrangian representation).

However, these steady phenomena, when they exist, always arise as a result of the evolution of a transitional regime. Thus, in many situations, the transitional regimes do not lead to steady flows and we observe complex phenomena which we will describe very briefly here.

In order to simplify the discussion, we will consider in what follows *an inviscid or Newtonian fluid of constant density*, unless otherwise stated. The variations of the physical properties, if they are not too great, do not significantly modify the structure of the phenomena which we will discuss.

We will leave aside questions related to the existence and to the uniqueness of solutions of the Navier-Stokes equations, the understanding of which requires a more advanced course in mathematical analysis. In this domain many questions remain open, and the physical aspects of the phenomena which we describe in this textbook demonstrate the formidable complexity of such an eventual theory.

6.6.2. *The existence of steady flows*

For a flow and the associated transfers to be steady, it is necessary for the boundary conditions describing the corresponding problem to be steady; in particular, *actions on the flow by fixed elements in this reference frame* should be independent of time. For example, the flow between upstream and downstream infinite reservoirs at constant pressure and connected by a nozzle (section 5.5.4) can be independent of time if the system is described in the reference frame of the nozzle. However, before being observed at a given flow rate, the flow was created from a zero pressure difference and it followed an evolution through the following states: a subsonic regime with increasing velocity, then a sonic regime and finally a supersonic regime with the progressive appearance of a shock wave which descends in the divergent part of the pipe until the pressure gradient is stabilized.

The solution of the problem defined by the steady boundary conditions is not always unique. The nature of the boundary conditions to be used is often a source of considerable difficulty (section 5.6.3). Consider for example the flow between two cross-sections S_1 and S_2 ($S_1 > S_2$) in a divergent pipe (Figure 6.21). Under the assumption of an inviscid incompressible fluid and with the approximation of the flow by slices, Bernoulli's theorem can be written:

$$p_1 + \rho \frac{U_1^2}{2} = p_2 + \rho \frac{U_2^2}{2}$$

The flow rate $q_v = \rho V S$ in this section of the pipe can be immediately obtained:

$$q_v = \pm \sqrt{\frac{p_1 - p_2}{\mathrm{B}\rho}} \quad \text{with: } B = \frac{1}{2S_2^2} - \frac{1}{2S_1^2}.$$

There exist two opposite flow rate values for this steady flow. We will see later that only one of these is really acceptable in the context of the preceding assumptions.

Let us now perform an experiment with a plane pipe whose divergent has a sufficiently large angle (of the order to 10 degrees or so). Here we observe that the steady flow follows one of the walls while it separates from the other. The flow chooses the wall to follow as a result of particular circumstances of the transitional regime. For example, in the configuration in Figure 6.21a, we have included in the walls two suction orifices A_1 and A_2 which are not activated in the steady regime and which we can use in order to create a small additional transitional depression in order to "suck" the flow towards a chosen wall. This suction does not need to be constant: once the flow has attached to one of the walls, it will remain so after the suction has been stopped. Such flows with two stable positions can be used to control flows in fluid circuits (fluidics command and control).

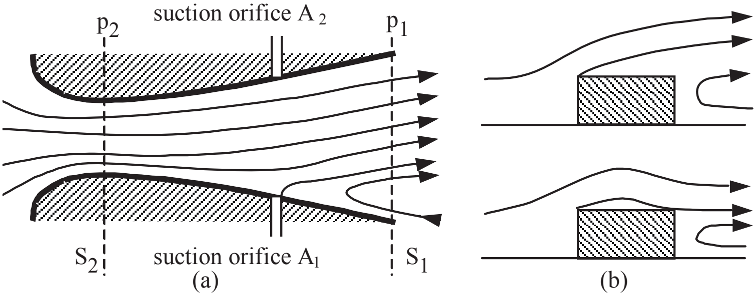


Figure 6.21. *Steady flows with two configurations*

For the two preceding flows, the pressure is nearly constant in each of the sections S_1 and S_2. Writing the boundary conditions for the steady flow of a viscous fluid leads to considerable difficulties, because there no longer exists a velocity potential and we know that we cannot independently specify the entry and exit conditions of a flow since the transport properties on the trajectories are dominant (section 5.6).

Figure 6.21b shows another configuration where two flow structures in a pipe are possible depending on whether the flow reattaches or not on an obstacle placed on its wall. The existence of reattachment is also related to the transitional regime which leads to the fully established flow. Examples of the same kind exist for sheets of water over spillways, which may flow above cavities either ventilated (i.e. filled with air) or not ([CHA 04] p. 399, [JAI 01] p. 264).

The preceding problems are examples of systems presenting hysteresis (the state of a system depends on its history). From a physical point of view, we can note that we have here a "retroaction" in the upstream direction which leads to the existence of a memory for the flow.

In summary, a flow problem with *steady boundary conditions does not necessarily have a unique steady solution*. We could also say that a steady problem is not necessarily a well-posed problem.

6.6.3. *Transitional regime and permanent solution*

6.6.3.1. *Relation between pressure and flow rate in a fixed stream tube*

Depending on the imposed conditions, pressure can be the cause or the consequence of movement of a fluid. Aside from cases where the viscosity plays a dominant role (low Reynolds number flows), the pressure (or the driving pressure depending on the case) balances the acceleration, and for steady flow of an inviscid fluid, Bernoulli's first theorem treats the pressure (or the total enthalpy) as a component of the total mechanical energy which is conserved for a fluid particle. This local property no longer exists in unsteady flows.

Consider the flow of an *inviscid fluid of constant density* whose fluid trajectories are fixed; they are thus coincident with the streamlines (and the emission lines). A stream tube is therefore a surface on which the inflow and outflow of fluid occurs over the cross-sections S_1 and S_2 with velocities V_1 and V_2 (Figure 6.22). The assumption of incompressibility leads to a volume flow $q_V(t) = SV$ circulating in the stream tube which is independent of the cross-section used to evaluate it; we will treat it thus as a variable. Let us apply kinetic energy theorem [6.85] to the interior domain D of the stream-tube element limited by the surfaces S_L, S_1 and S_2 (Figure 6.22); we have:

$$\int_D \frac{\partial}{\partial t}\left(\rho \frac{V^2}{2}\right) dv + \int_{S_2}\left(p + \rho\frac{V^2}{2}\right)\vec{V}.\vec{n}ds - \int_{S_1}\left(p + \rho\frac{V^2}{2}\right)\vec{V}.\vec{n}ds = 0$$

where p designates the driving pressure by way of simplification.

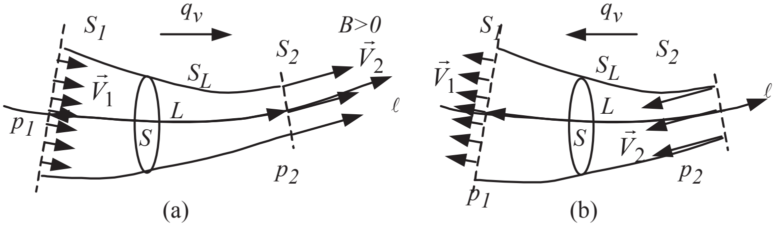


Figure 6.22. *Flow by slices inside a stream tube: (a) convergent tube; (b) divergent tube*

In the slice approximation of the flow, the quantities are uniform in all cross-sections, and after simplification by q_v, we immediately obtain:

$$\rho \frac{dq_v}{dt} \int_0^L \frac{d\ell}{S} + \rho \left(\frac{U_2^2}{2} - \frac{U_1^2}{2} \right) + p_2 - p_1 = 0$$

Let: $A = \int_0^L \frac{d\ell}{\rho S}$ and $B = \frac{1}{2\rho} \left(\frac{1}{S_2^2} - \frac{1}{S_1^2} \right)$

The relation between the flow rate q_v and the pressure difference $p_1 - p_2$ can be written:

$$A \frac{dq_v}{dt} + B q_v^2 = p_1 - p_2 \qquad [6.98]$$

This equation shows that the pressure is balanced by two *acceleration* terms, one unsteady, $A \frac{dq_v}{dt}$, and the other, $B q_v^2$, corresponding to the steady flow, *proportional to the square of the flow rate and independent of the direction of the flow*. This separation is related to the *Eulerian description* of the phenomena, and from a physical point of view there are not two kinds of acceleration for the fluid particles. As we have already said (section 3.3.3.5), this description is always associated with a favored reference frame resulting from the existence of boundary conditions on the particular surfaces which are here the sides of the stream tube.

We should note that the coefficient A of equation [6.98] has an order of magnitude proportional to the length, contrary to the coefficient B which only depends on the values of the inflow and outflow sections. Furthermore, the 1/S dependence of the integral A shows that a severe intermediate narrowing will not modify the values S_1 and S_2 but will lead to a considerable increase of the coefficient A, in other words to the pressure difference necessary for the transitional acceleration in the Eulerian representation. The reason for this is the existence of a strong acceleration in sections of small dimension where the velocity takes on high values in order to conserve the flow rate.

6.6.3.2. *Properties of the solutions*

Let us take B to be positive ($S_1 > S_2$), which does not restrict the generality of the reasoning. Equation [6.98] can only therefore possess *steady solutions if the difference* $p_1 - p_2$ *is positi*ve. In this case, we have the two steady solutions

$q_v = \pm\sqrt{(p_1 - p_2)/B}$, corresponding to an established flow, either in the positive direction, or in the negative direction of the axis ℓ (section 6.6.2). However, *the possibility of realizing such a solution depends on the existence of a transitional regime which can lead to this kind of situation.*

In the unsteady regime, the pressure difference $p_1 - p_2$ can be negative. However, an examination of equation [6.98] shows that if it is always negative, the same goes for the derivative $\frac{dq_v}{dt}$; we can thus see that the flow rate q_v decays indefinitely, which is physically unacceptable.

Suppose now that the difference $p_1 - p_2$ is positive and let $Q(t) = \sqrt{(p_1 - p_2)/B}$; $Q(t)$ is the positive value of the flow rate under the assumption that the unsteady term is negligible. Equation [6.98] can be written:

$$A\frac{dq_v}{dt} = B\left(Q^2(t) - q_v^2\right) \qquad [6.99]$$

We see that:

– if the instantaneous flow rate q_v is greater than $Q(t)$, the derivative $\frac{dq_v}{dt}$ is negative: the flow rate q_v decays and approaches the value $Q(t)$;

– if q_v lies between $-Q$ and $+Q$, it therefore increases, and approaches the value $Q(t)$ once again;

– if the instantaneous flow rate q_v is less than $-Q(t)$, the derivative dq_v/dt is negative: the flow rate q_v decays and moves away from the value $-Q(t)$.

This property leads to the flow rate being bounded if the positive quantity Q is bounded. If $Q(t)$ tends to a limit for infinite t, the same goes for the instantaneous *flow rate q_v if the latter remains always greater than $-Q$*. Under the assumption of constant pressure difference $p_1 - p_2$, we see that the transitional regime does not allow the solution $q_v = -Q(t) = -\sqrt{(p_1 - p_2)/B}$ to be attained.

The inviscid fluid model does not apply for negative flow rates, and the viscosity must be introduced. The study of viscous fluid flow between two infinite divergent planes can be performed (the Jeffery-Hamel exact solution of the Navier-Stokes equations). We find that a separation occurs for a Reynolds number which tends to infinity, even if the angle of the planes tends to zero ([SCH 99], p. 104). In these conditions, the uniqueness of the solution is not assured.

6.6.3.3. *Comparison with experiments and consequences*

In practice, flow is possible in a divergent channel, but it is not possible to model this using an inviscid fluid, even for very large Reynolds number values. A *section of convergent pipe transforms the mechanical power* $q_v(p_1 - p_2)$ *provided by the pressure difference into an increase in the kinetic energy flow* $\alpha \rho S u_m^3 / 2$ between the inflow and outflow sections (section 6.5.2.4). A *divergent section leads to the opposite transformation: kinetic energy is partially recovered in the pressure.*

We define the efficiency of these transformations using the kinetic energy flow $\alpha \rho S u_m^3 / 2$ between the inflow and outflow sections (section 6.5.2.4) and the mechanical power $q_v(p_1 - p_2)$ provided or recovered by the pressure difference. Whereas a pipe with a convergent section in the flow direction transforms pressure into kinetic energy with good efficiency, the inverse effect in a divergent pipe occurs with a non-negligible dissipation. For a given ratio S_1 / S_2 of the sections S_1 and S_2, the efficiency is close to 1 in a converging flow for a short length ℓ whereas in a divergent pipe, the efficiency goes through a maximum of the order of 0.7-0.8 for an angle of about 7º. A compromise must be found between a small angle and a large length over which there exists a notable viscous dissipation, and a larger angle leading to the formation of a separated flow at the wall, whose kinetic energy is nearly entirely lost.

The approximation of the flow by slices (uniform velocity distribution in the stream tube constituted by the pipe) is always a good approximation for steady flow in a convergent pipe. On the other hand, this approximation is a poor model in a divergent element, because numerous phenomena can occur which contradict the assumption of uniform velocity and lead to the loss of mechanical energy (tendency for the flow to separate close to the wall in the boundary layer (section 6.6.3.7), instabilities close to the wall leading to the generation of turbulence, etc.).

The experiment always amounts to an observable process, and thus to a well-posed problem provided a suitable analysis is performed. As we have just seen, the same is not true for theoretical models for which the initial conditions must be ascertained by experiment. In particular, methods of numerical solution of the Navier-Stokes equations often relies on the computation of a velocity field from a pressure field, which allows, at the next iteration, the computation of a new pressure field, hence a velocity field and so on. These intermediate fields are not solutions of the equations of motion, and the intermediate pressure fields can thus correspond to initial conditions which lead to a rapid divergence of the velocity field at the following approximation. This is an important and unfortunately common difficulty.

6.6.3.4. *Separation in unsteady flow*

We have seen (section 6.5.3.7) that separation is due to the quadratic term in the Bernoulli's theorem for an inviscid fluid. The argument thus invoked is no longer valid in a strongly unsteady flow. Consider an irrotational unsteady flow for which we have Bernoulli's second theorem ([6.13]).

When the dynamic pressure $V^2/2$ is smaller than the term $\partial\varphi/\partial t$, the phenomenon of flow separation at the wall no longer occurs. If the flow is oscillatory, the velocity profile may present changes of sign, but there no longer generally exists a *rear stagnation point S at the origin of a streamline towards the heart of the fluid (Figure 6.20a), which is characteristic of a 2D separation.*

6.6.4. *Non-existence of a steady solution*

Even if we impose steady conditions at the fluid domain boundaries, we do not encounter steady flows in very many situations. The wakes observed behind obstacles belong to this category. Let us reconsider the example of a circular cylinder of diameter D (without circulation) in a velocity field which is assumed to be uniform and steady at infinity. The structure of the flow depends on the Reynolds number $\mathrm{Re} = UD/\nu$.

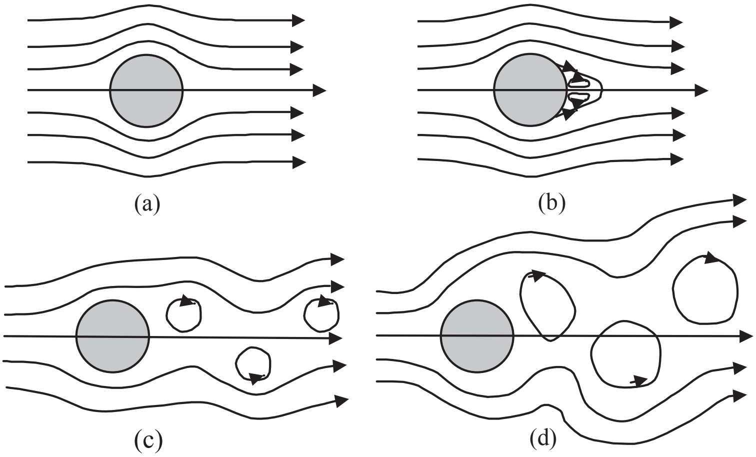


Figure 6.23. *Real flow around a circular cylinder placed in a uniform steady velocity field*

Figure 6.23 shows the evolution of this flow as the Reynolds number increases. For values of the Reynolds number small compared to 1, the flow is symmetric (Figure 6.23a); the dissymmetry between upstream and downstream increases with the Reynolds number; for larger values (of the order of 0.2-20) we observe the appearance of a fixed zone of reversed flow just downstream of the cylinder (attached wake (Figure 6.23b)) comprising two symmetric vortices). For values greater than 20 (Figure 6.23c), we observe the emission of alternate vortices at a well-defined frequency. Beyond values of Re of the order of 1,000, the periodicity of the vortices is attenuated (Figure 6.23c) and random fluctuations appear.

6.6.4.1. *The creation of the circulation*

We have studied the flow around a circular cylinder with circulation (section 6.2.5.2.4). The origin of this circulation cannot be explained by a rotation of the cylinder since the fluid is inviscid. Experiment furthermore shows that an important difference exists between the circulation calculated in the fluid and the circulation calculated on the cylinder using the velocity at the wall. This difference shows that the circulation is not created directly by viscous diffusion from the wall, which is not significant in an inviscid fluid.

The *circulation is in fact created during the transitional regime* of the flow. It is due to the viscosity of the fluid. Consider a fluid initially at rest about a profiled body at incidence. In the initial acceleration regime of the flow around the body, a dissymmetry appears between the intrados (the under-side) and the extrados (the upper-side) of the profile which generates at the trailing edge (downstream of the boundary layers) a sheared viscous flow at the origin of a vortex, which grows (Figure 6.24a and Figure 6.24b) until it separates from the obstacle and is carried away by the flow (Figure 6.24c). The viscosity is only important for a small zone and Lagrange's theorem can be applied in the inviscid fluid external to the ensemble of the obstacle and the vortex. As the circulation is initially zero, it remains so: the vortex carries away the circulation Γ, and the opposite circulation – Γ is established about the obstacle.

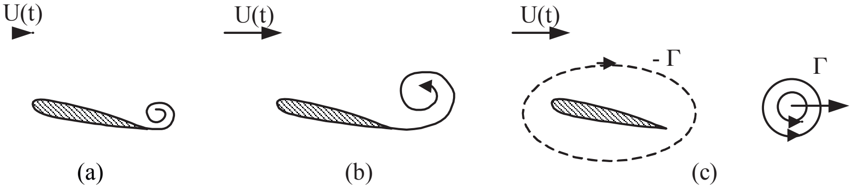


Figure 6.24. *Creation of circulation by initial emission of a vortex around a profiled body*

The circulation about a non-profiled obstacle is created by the same transitional mechanism, to which is added a turbulent wake analogous to that described in section 6.6.3.4, and which does not create and additional mean circulation.

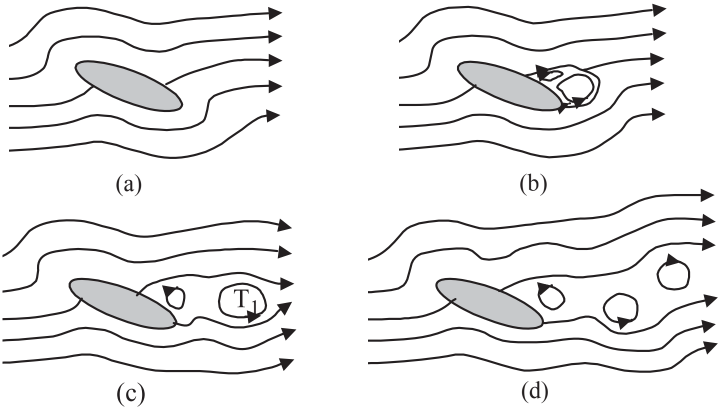


Figure 6.25. *Creation of circulation by initial emission of a vortex around a non-profiled body: (a) and (b) development of the initial vortex; (c) and (d) the circulation is installed after the emission of the initial vortex T_1*

6.6.4.2. *Instabilities and turbulence*

The phenomena described in the preceding sections result from flow instabilities, in other words to phenomena associated with the amplification of perturbations of varying amplitudes. Situations of instability are numerous in fluid mechanics and the viscosity is not necessarily a damping factor for these phenomena. On the other hand, in so far as the most unstable structures correspond to high Reynolds numbers, the characteristic time constant of the amplification is much lower than the damping time.

Instabilities evolve in general towards a variety of more of less complex situations; for high Reynolds numbers corresponding to the flows encountered in practice, we see the appearance of random fluctuations the details of which are unpredictable (chaos). The velocity fluctuations of fluid particles also lead to the fluctuations of other extensive quantities, and the transfer mechanisms of irreversible phenomena are significantly changed ([SCH 99], [YIH 77]).

Fundamentals of Fluid Mechanics and Transport Phenomena
Jean-Laurent Peube

Chapter 7

Measurement, Representation and Analysis of Temporal Signals

The measurement of flows and transfer mechanisms presents specific difficulties related to fluid movement. We will review the principles of the main measurement methods. A synthesis of the different signal processing procedures commonly used in the domains studied, and the various numerical techniques used in order to implement these, will be presented.

7.1. Introduction and position of the problem

The modeling of macroscopic phenomena can be more or less detailed. The representation of the continuous medium by means of fields of continuous quantities was obtained by extending systems of finite dimensions in thermodynamic equilibrium to their limits. Writing an infinite number of values of a continuous field is obviously impossible and constitutes the infinite limit of a practical realization; this can only be achieved if we dispose of a procedure for computing the value at every point, which can be trivial and consist of the simple condition of a constant value. Furthermore, discrete systems in thermodynamic equilibrium constitute a more or less good approximation to a real system. The information necessary for the description of a continuous system always therefore comprises an ensemble, necessarily finite, of numerical values and a procedure for the computation of quantities at all points, which amounts to an interpolation.

A material ensemble is described by means of quantities characterized by measurements which permit the representation of its evolution. The equations are exact equalities between real (or complex) numbers, which represent the physical quantities studied. Now, these are never known exactly and this exact knowledge is furthermore neither practically obtainable nor necessary. We work with approximations or mathematical representations (transformed signals) which are more convenient to use. The values themselves can be synthesized by a transformation procedure allowing their number to be reduced. The representation of physical quantities is thus a basic problem associated with our knowledge of the phenomena of interest. These considerations can apply both to space and time variant quantities. In what follows we will reason in terms of a temporal representation of a signal, but the problems discussed are analogous regardless of the nature of the representation and the number of variables.

Time is a particular variable, since the cause of a phenomenon observed at time t_0 can only be a function of the state variables from earlier times: only the past influences the future. In reality, this property is also encountered for spatial variables in flows, since the extensive properties remain attached to the matter in movement. The notion of *causality*, largely developed in signal processing can in fact by applied in space-time over characteristic curves or the succession of characteristic surfaces. While causality appears obvious for the time-direction considered in isolation, the problem is far less simple when we perform a Fourier transform. We will not cover problems of causality which appear in integral transformations: the reader should refer to the specialized literature ([AND 99], [BEE 03], [DEB 06], [PRE 91]).

The idea of continuous phenomena has only really been used by some modern calculation methods. The first analog calculators produced a physical representation of the equations, the adjustment of the variables and parameters being achieved by modification of physical quantities (for example, voltage for electronic calculators or pressure for hydraulic and pneumatic systems). With numerical calculation, we operate using numerical values, which requires equations to be discretized.

7.2. Measurement and experimental data in flows

7.2.1. *Introduction*

Knowledge of a physical system requires the measurement of its quantities. In general, the local intensive quantities are easier to measure than the extensive quantities which are most often measured indirectly using equations of state, phenomenological laws or relationships obtained theoretically (for example, Bernoulli's theorem). As we cannot here provide detailed description of measurement methods, we will simply indicate the principles on which these are based. The quantities measured are primarily the pressure, the velocity, the temperature and the concentration of a constituent.

7.2.2. *Measurement of pressure*

As pressure is a force normal to a surface, its direct measurement can be related to the measurement of a force; as the force is the energetic intensive quantity associated with a displacement in a fixed discrete system, we are dealing with a measurement of displacement which is in fact achieved by the intermediary of associate electrical or electromagnetic phenomena. Let us consider a few examples of such measurement devices: *piezoresistive* probes where deformation of the silicon membrane produces variations in electrical resistance, *capacitive probes* (Figure 7.1b) which comprise a fixed electrode opposite the membrane, the movement of the latter is accompanied by a variation of the capacitance, or *piezoelectric probes* (Figure 7.1a) that comprise a piezoelectric material on which the action of a force creates a difference of potential.

For steady flow, a measurement which is less and less frequently employed consists of using liquid manometers, based on hydrostatics laws, to find the difference in height Δh between the free surfaces in two tubes implying a pressure difference equal to $\rho g \Delta h$, or between two free surfaces if the two tubes communicate directly, or, between the pressure taps in the liquid. We measure here a difference of level (a length), but the real difficulty comes from the accurate identification of the position of the free surface and the effects of surface tension (section 2.2.1.4.2).

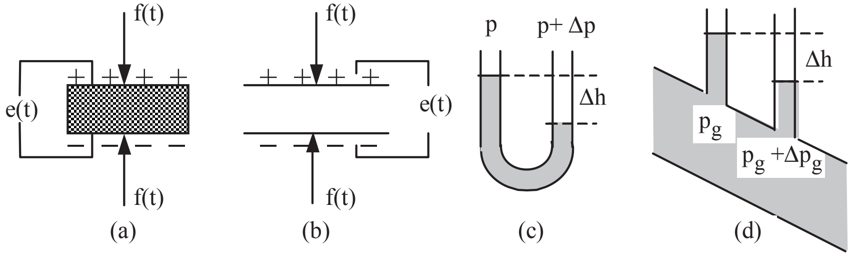


Figure 7.1. *Probes for pressure (a), (b) and (c) and driving pressure (d) measurements*

The pressure measured by a probe is the surface force on the sensitive member of the probe. Pressure measurement at a point in a fluid requires in principle that the probe is positioned such that it does not itself create a perturbation: it must therefore be carried with the matter in movement. This is clearly unrealistic, unless some optical means of pressure measurement could be made – and this is generally not possible. In practice, it is only possible to use some instrument, which is integrated into a solid surface. It is thus necessary to *extract* the pressure and *transmit* it to the sensitive surface of the probe by means of a pneumatic or hydraulic link.

The extraction of pressure can be performed by means of an orifice at the tube extremity used for its transmission. The tube extremity comprises an obstacle in a flow, and this must be considered. We can assume that the pressure in an orifice pierced in a wall is the pressure in the fluid at that point. We can in fact consider that the flow is here parallel to the wall, without any velocity component normal to the wall; the balance of transverse forces is zero on any fluid element, hence the local uniformity of pressure along the wall normal. However, the pressure tube must be contained in a suitable obstacle, such that it does not modify the flow and therefore the pressure value. We will come back to this point a little later when we deal with velocity measurements.

Furthermore, the transmission of pressure fluctuations by a tube is a delicate problem when the length of the tube is more than a few centimeters, as it is necessary to avoid problems of acoustic resonance [GAB 98]. However, this problem is beyond the scope of this work.

7.2.3. ***Anemometric measurements***

7.2.3.1. *Introduction*

On a wall, the velocity of a fluid in motion is equal to that of the wall. It is obviously only of interest to measure the velocity at points at the heart of the flow. When making direct measurements of velocity, it is necessary to consider individual fluid particles, of a given size, and to obtain an electrical or optical signal which results from a phenomenon associated with the velocity of this fluid particle. Velocity measurement devices (anemometers) which allow direct measurement are mainly laser Doppler anemometers, cup anemometers, etc.). Indirect measurement of velocity can be obtained by means of pressure measurements by application of the laws of fluid mechanics (Pitot tubes, total pressure probes, etc.) or from calibrated thermal measurements (hotwire anemometers). The same is true for measurements of the flow rate, which are not generally made by directly measuring the volume or mass which flowed during some given time.

As for the pressure, it is necessary to measure the local property of a material element (Lagrangian measurement) or to introduce a probe which is sensitive to the local velocity but which will necessarily disturb the flow (Eulerian measurement).

7.2.3.2. *Lagrangian anemometry*

In general we want to know the velocity at an observation point which we define with respect to the laboratory reference frame (Euler variables). Fluid particles are not individually visible and it is necessary in all cases to introduce solid elements, which are entrained by the fluid. The problem of solid particle entrainment by a fluid is therefore essential.

In the case of *cup anemometers* (Figure 7.2a), we use the dissymmetry of a rotating device, which presents a resistance to the fluid movement in a given direction (cup C_1) and very low resistance in the other direction (cup C_2). The result is that the cup situated at position C_1 tends to follow the flow, whereas the other, C_2 induces a very low torque on the moveable system. Alternatively a propeller can be used (Figure 7.2b). As the elements of these devices are not entrained by the fluid with the fluid velocity, it is necessary to calibrate them.

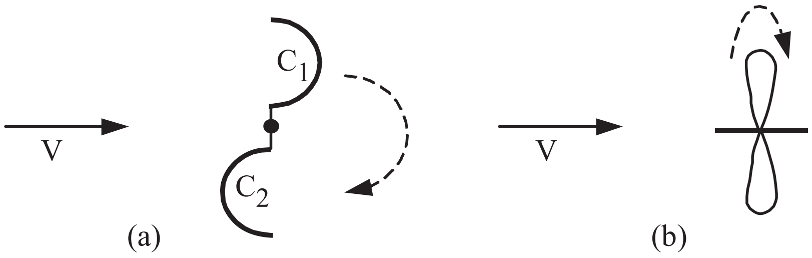


Figure 7.2. *(a) Cup anemometer or (b) propeller anemometer*

Laser Doppler Anemometry (LDA) involves the creation of a system of interference fringes by means of a laser whose beam is split into two beams focused by a lens (Figure 7.3a). Very tiny particles crossing the fringes are alternatively illuminated as they cross the fringes. It is possible to measure the scattered light by means of a photomultiplier. The frequency value *n* of the light signal measured allows a calculation of the particle velocity, equal to *i.n*, where *i* is the inter-fringe spacing, easily obtained from the optical characteristics of the device. The denomination Doppler comes from the fact that the procedure can also be explained by considering the variation of the optical frequency which occurs when the particle scatters the light provided by each of the coherent light beams.

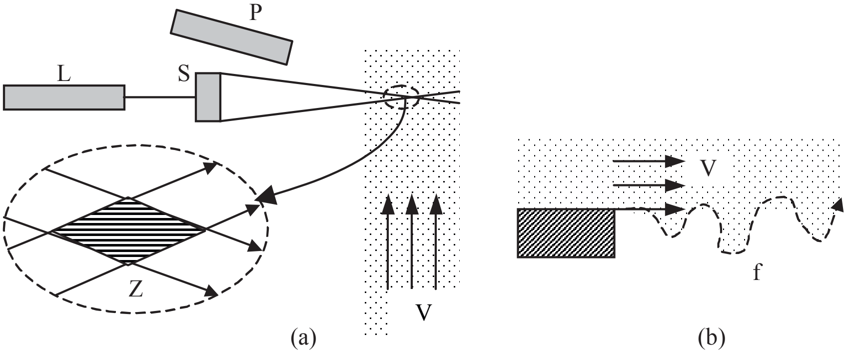


Figure 7.3. *(a) Principle of anemometric measurement by ALD (L, laser; S, beam-splitter and focusing lens; Z, fringes in the measurement zone; P, photodetector centered on the zone Z); (b) bias in ALD measurement in a non-homogenous flow seeding*

The principle of anemometric measurement assumes the presence of small particles, which are in suspension in the fluid [DUR 81]. These nearly always exist in water, but it is necessary to "seed" gaseous flows by means of droplets obtained by the pulverization of some suitable oil, or by very light smoke obtained by combustion of some suitable substance (incense for example) so as to obtain particles whose dimensions are in the order of 1 μm The introduction of particles needs to be effected so as to produce as uniform a distribution as possible in the flow. Figure 7.3b shows a region of flow between a stream of velocity V and a region of flow at rest; the frontier f between the two oscillates unpredictably; only fluid particles coming from the flow containing seeding will provide a measurement signal: we thus see a systematic bias in the measurement of the velocity which will "favor" the same kinds of flow structures.

The LDA procedure provides a local and instantaneous measurement in most conditions. It is well suited to the laboratory study of small flow structures and it even allows us to obtain acoustic velocities (from 1 mm/s to 1 cm/s) [PEU 92]. Cup anemometers only allow measurement in a volume in the order of some considerable fraction of a dm^3, and therefore these are only suitable for quite global measurements (wind velocity in meteorology or air-conditioning units).

Let us finally recall anemometric measurements at the atmospheric scale, which are obtained by releasing balloons that move at constant altitude and whose position can be tracked by satellite. This method provides trajectories (Lagrangian viewpoint) and not streamlines (Eulerian representation of meteorological charts).

7.2.3.3. *Eulerian anemometry*

We here use a device, which is fixed in space, on which the flow acts by creating a phenomenon whose intensity (overpressure, temperature, etc.) depends on the velocity of the flow.

In the steady flow of an ideal incompressible fluid for which Bernoulli's theorem is valid, the difference in driving pressure between two points whose velocities are respectively equal to V and 0 is equal to:

$$\Delta p_g = \rho \frac{V^2}{2}$$

The measurement of velocity is thus realized by means of a measurement of a pressure difference. A zero velocity is obtained when a flow is stopped by an obstacle, on the upstream edge of which a stagnation point A occurs (Figure 7.4a); it suffices to measure the total driving pressure (or the total pressure for a gas) at this point A by connecting the orifice to a pressure probe. This orifice needs to be of very small size in order to obtain a well-defined pressure value. However, if the flow is not correctly aligned with respect to the obstacle, the orifice will not be at the stagnation point and will not therefore measure the total pressure (Figure 7.4b). The effect of the angle of incidence can be reduced by using a tube with a divergent opening (Prandtl tube, Figure 7.4c).

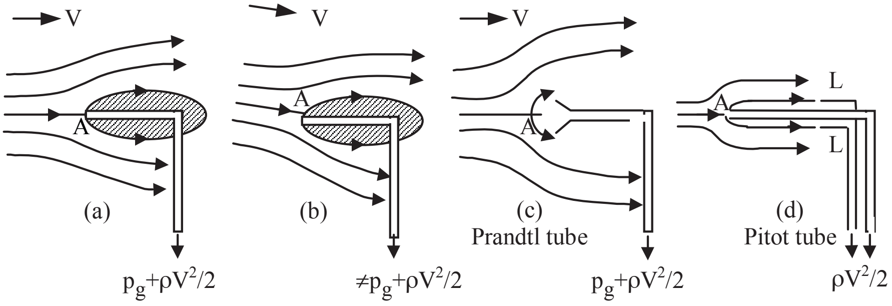


Figure 7.4. *Measurement of stagnation pressure in a flow*

By placing the total pressure tube inside a streamlined cylindrical tube (Figure 2.4d), the flow quickly returns to a uniform state (after 5 or 6 external diameters); lateral orifices L placed on the external wall will measure the static driving pressure p_g (or static pressure in a gas) and the difference in pressure between the two tubes provides a direct estimate of the dynamic pressure $\rho V^2/2$. This device, known as a

Pitot tube, constitutes a somewhat cumbersome obstacle (it is at least a few centimeters in length); it assumes that the flow velocity is defined and uniform at this scale.

Bernoulli's theorem is also applicable for a compressible fluid in subsonic flow, provided of course that the compressible form of Bernoulli's equation is used. The problem is more difficult in supersonic flows because of the existence of a shock wave upstream of the tube, across which specific laws need to be applied.

Hot wire anemometry is based on a very different principle: the thermal power evacuated by a heated body is an increasing function of velocity. If the body is electrically heated, this power can be known by means of a measurement of intensity. It can be used in a flow in which velocity fluctuations exist with very small hotwire diameters (1 to 5 μm) whose thermal inertia is compensated by a suitable supply (constant temperature anemometer). It is thus possible to measure velocity fluctuations whose frequencies may be as large as 1,000 to 2,000 Hz. With two or three differently inclined hotwires it is possible to measure the instantaneous velocity vector. We therefore have in the hotwire anemometer a powerful means of knowing the velocity fluctuations provided these remain smaller than the mean velocity, as it is clearly not possible to distinguish the velocity direction from a thermal power measurement alone.

7.2.4. *Temperature measurements*

The local measurement of temperature at the heart of a continuous medium can be achieved by optical methods (measurements at a distance) or by means of probes introduced into the flow. Optical procedures are based on emission phenomena, which are associated with absorption phenomena, the medium considered being thus semi-transparent for the wavelengths used. This leads to calculation methods which are often complex and we direct the reader to specialized texts for more detail (see for example ([BRU 95], [JOH 98], [MAR 99], [MCG 88]).

In a manner analogous to the introduction of pressure or velocity measurement probes, the introduction of temperature measurement probes provokes a modification of the flow and associated thermal phenomena only of importance at high velocities. As with the pressure, the temperature of a fluid will depend on the position of the measurement element on the wall of the probe; these measurements are generally performed either at a stagnation point, or on a wall which is parallel to the flow. For an ideal gas in adiabatic flow, we have conservation of total enthalpy (section 4.3.2.3.2), or, with T_a the stagnation temperature:

$$C_p T + \frac{V^2}{2} = C_p T_a$$

However, the measurement of temperature presents additional problems, as the temperature of the sensible part of the probe is the result of a thermal balance between the heat flux transmitted by the flow, the heat flux in the metallic electric wires and in the support structure, and the thermal radiation caused by the surrounding walls. As the last two quantities are independent of the flow, their influence is less important on the temperature measurements at higher velocities. We will note that for high enough velocities, thermal dissipation in the vicinity of the obstacle, which is constituted by the temperature probe, may be a non-negligible factor ([SCH 99]). *Temperature measurements in flows are always difficult* and should not be attempted without a careful discussion of the different thermal phenomena which may be present.

7.2.5. *Measurements of concentration*

The measurement of the concentration of a mixture is quite difficult. Very often a fluid sample is taken via local suction by a tubular probe. The probe geometry is not important here, but the suction velocity needs to be suitably chosen so as not to alter the fluid trajectories in the extraction zone (isokinetic sampling) ([BRU 95], [JOH 98], [CHE 88], [LIP 05]). An analysis of the fluid sample is then performed using physico-chemical methods adapted to the mixture under study.

In certain cases, it is possible to perform measurements from a distance by means of optical procedures associated with radiation emission. It is also possible to use physico-chemical reactions on a surface placed in the flow, but the specific procedure employed is always dependent on the particular mixture studied, and often requires modeling of the heat and mass transfer phenomena involved. For example, we have already said that evaporation phenomena induce a variation of temperature: a wet thermometer does not measure temperature of air (section 2.3.2). In a conducting liquid medium, the use of electrodes allows the measurement of concentration in certain circumstances (diffusion phenomena have to be negligible).

7.2.6. *Fields of quantities and global measurements*

7.2.6.1. *Introduction to field quantities*

In practice, local quantities are of limited interest, unless they are representative of the device being studied. The dimensioning of a device or the control of its operation requires knowledge of global quantities (mass flux, thermal power, etc.)

or of certain local quantities, which may be associated with possible damage (temperature at a hot spot in thermal systems). These two kinds of data require knowledge of the associated field quantities, either for integration (computation of a flow rate for example) or for identifying the point where critical values may be attained. In many situations, knowledge of field quantities can only be obtained by exploring the domain by means of local measurement probes. This is clearly only of interest if the concerned experiment is reproducible, which can only be the case if the flow is perfectly defined. It is then possible to obtain velocity, pressure, temperature fields, etc. This situation does not include poorly defined flows regardless of their origin (instabilities, turbulence, etc.).

Steady flows do not present particular difficulties, since the instant of measurement is unimportant. For unsteady flows, this is not the case however, as the fields of a quantity $g(x_{i,}t)$ now depend on time. The installation of an ensemble of probes, which instantaneously and simultaneously measure the quantity g over the entire domain considered, is not generally realistic. Only methods which allow the obtaining of full instantaneous fields by optical means are possible; such techniques have seen significant progress in recent years.

7.2.6.2. *Direct obtaining of field quantities of a flow*

7.2.6.2.1. Visualizations

The visualization of a flow can consist of “photographing” visible material elements which have been placed into the flow without disturbing it at fixed locations with respect to the observer. For instance, it is possible to place *pieces of light wires* on a grid inside a flow; these will then be oriented depending on the direction of the local velocity. The same procedure can be used on a wall in order to obtain the direction of streamlines and to visualize separated zones. The visible effect is here a *Eulerian representation* (observation of streamlines).

It is also possible to introduce *streams of smoke into the air* or to inject *colored dyes in a liquid medium.* This kind of injection requires certain precautions (suitable injection velocity, density of colored dye equal to that of the liquid medium, etc.). Diverse particles can also be introduced and entrained by the flow (which has not at some point stopped to observe the flow structures visible in a river, a gutter, etc.). However, it is important to remember that this kind of *visualization is Lagrangian*, and it shows streaklines which may be very different from the streamlines and trajectories (section 3.3.2).

Finally, *Eulerian visualizations* can be performed on walls by means of physical processes (entrainment of a coating comprising particles for example) or physico-chemical processes producing a differential deposit on wall streamlines (evaporation of the coating solvent, or electrochemical deposit under particular conditions). For a more complete and practical information, see ([HAN 05], [MER 87], [YAN 01]).

7.2.6.2.2. Obtaining instantaneous fields

The preceding visualization methods are rather qualitative. We can make these quantitative by introducing tracer particles and photographing these with suitable lighting, either over a period short enough for visualization of the distance covered during the lighting-time or by means of two successive light pulses. It then remains to measure the segment covered by each particle in order to ascertain the velocity. It is obviously necessary to know the position of the particles in space, and this is done by generating a light sheet of small thickness, particles external to this light sheet not being visible. This method requires a seeding of particles with very small diameter (1 to 3 µm) which is not biased (section 7.2.3.2). Such approaches are known as particle image velocimetry (PIV). Modern techniques for the generation of two close light pulses of short duration (5 to 10 ns) by a suitable laser and the processing of images have made this technique less fastidious than in the past ([RAF 98], [STA 00], [STA 00]). However, in turbulent flows, the obtaining of a mean velocity field requires statistics to be gathered from a large number of identical experiments (sections 7.2.6.3.2 and 7.2.6.3.3).

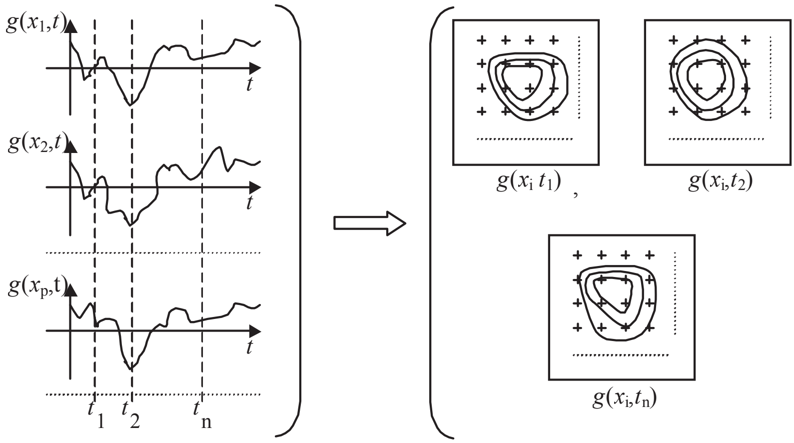
Optical methods of temperature and concentration measurement provide field quantities; as mentioned earlier, the semi-transparent aspect of the medium considered (which is both emissive and absorbent) leads to rather global measurements which depend on thickness, and a deconvolution of signals acquired in a band of wavelengths is necessary in order to reconstruct the value of the quantity in each slice. These optical procedures are used in observations by meteorological satellites.

7.2.6.3. *Application in unsteady flows*

7.2.6.3.1. Unsteady repeatable flows

We here assume that the flow can be reproduced at will, in other words there exists an initial instant with identical initial and boundary conditions. This is true for transitional regimes, which precede the established regimes, and also that of periodic flows.

We repeat the flow regime which is to be studied as many times as necessary, and measure the value of the quantity $g(x, t)$ each time (Figure 7.5a) at different points (x_1, x_2,…, x_p,…). We then represent the fields (Figure 7.5b) at successive instants (tx_1, t_2,…, t_n,…) which we have chosen. These operations, which are in principle quite simple, clearly require the use of numerical calculations ([PEU 79], [PEU 89], [PEU 91]).



(a) time representation of *g(x,t)*

(b) space representation of *g(x,t)*

Figure 7.5. *Obtaining instantaneous fields of quantity g from point measurements as a function of time*

7.2.6.3.2. “Steady” turbulence flows

A turbulent flow presents random fluctuations. We will here call a “steady” turbulent flow any flow whose mean value of the measured quantity, over a large enough time, does not depend on the initial instant chosen to begin the measurement. We see immediately that an analogous procedure to that described in the last section allows the obtention of quantities derived from the curves $g(x, t)$ by means of appropriate operations (mean values, fluctuations, variance of the quantity g, etc.) used in the study of turbulent flows ([SCH 99], [YIH 77]).

7.2.6.3.3. Unsteady “reproducible” turbulent flows

This category of flow is obtained by imposing fixed initial and boundary conditions that are defined functions of time. As before, we repeat successive measurements at different points by reproducing the experiment exactly. As the flow is maintained in a turbulent state, in other words it fluctuates from one experiment to the next, it is necessary to perform the measurement a sufficient number of times at each point, for the associated statistical quantities to be calculated; these statistical quantities are the only ones which are meaningful for this ensemble of experiments.

7.2.6.4. *Measurement of global quantities*

Global quantities such as mass or volume flux, internal energy or enthalpy flux are very important in industry. Precise knowledge of these can be obtained by integrating local measurements performed on a grid that covers the region of interest. This method of integration is used for establishing norms for precise measurement. However, these methods are time-consuming, expensive and often cumbersome. It is thus necessary to implement more global procedures based on laws of fluid mechanics and transfer. The corresponding devices require calibration in laboratory conditions. Each of these measurement procedures is based on a particular physical phenomenon, for example, the flow rate in a duct can be obtained from the mean spatial velocity deduced from the frequency of vortices emission behind a cylinder, after a calibration of the used device.

7.2.7. *Errors and uncertainties of measurements*

7.2.7.1. *Introduction*

Errors, which occur in the measurement of physical quantities, arise as a result of diverse factors. It is first of all necessary to appreciate the experimental conditions, which lead to an experiment always being of an approximate nature. The control of experimental conditions (temperature and velocity, etc.) are factors which can significantly diminish these errors, although without eliminating them. In other words, the data of a problem are only known within the bounds of some uncertainty. Measurements are always accompanied by errors of various kinds, depending on the kind of methods that are used:

- with analog devices errors come from the sensitivity of the components which are used, parasite phenomena (influence of temperature, etc.), dry friction in devices with moving parts, aging of components, etc.;

- the numerical treatment of an electric signal introduces no error (with the exception of rounding errors); it does however introduce errors in the signal acquisition to be treated (sampling and digitization mainly: for example, the digitization on 8 bits leads to an additional uncertainty equal to $½^8$ in absolute value).

Because they retain many decimal places, numerical devices are too often considered to be capable of a precision that they do not necessarily provide. They constitute most often only the last visible phase of a complex process of measurement, the precision of which is not representative: the precision of the numerical presentation is not directly related to the uncertainty on the quantity measured and *the error in the measurement of a voltage by means of a measurement system having a numerical display is not simply the digit of the last decimal place.*

7.2.7.2. *General properties*

A measurement error is not always exactly known, otherwise it would be possible to eliminate it. It is *systematic* when due to a particular flaw in the measurement device; this flaw is generally not known, otherwise a correction would be possible. It is random, when it is associated with parasite phenomena (noise from electronic circuits, diverse fluctuations, etc.). Regardless of their origin, errors will always exist, and can never be exactly evaluated. The only quantity that is really available is the *uncertainty* of a measurement or numerical value, in other words an *upper bound on the absolute value of the error.* Uncertainties moreover verify rules that correspond to properties of distances as they are defined in mathematics (triangular inequality, etc.).

The evaluation of uncertainty of a measurement should always be made after *error calculation,* which is performed in accordance with the usual rules for calculating small variations in the variables of the problem. For example, we calculate a quantity $s(a,b)$ using measurements of signals *a* and *b*. The error δ can immediately be obtained as a function of the errors δa and δb:

$$\delta s = s_a' \delta a + s_b' \delta \mathrm{b}$$

The formulae obtained in the calculation of errors allow us to obtain the uncertainty of the quantity considered by bound estimation for the errors. For the quantity *s*, it is possible to obtain the uncertainty Δs as a function of the uncertainties Δa and Δb:

$$\Delta s = \left| s_a' \right| \Delta a + \left| s_b' \right| \Delta b \qquad [7.1]$$

We draw the reader's attention to this last point, which is very important in practice. It is indeed regrettable that these elementary ideas are so often completely forgotten in the beginning of higher level education.

Let us note that depending on the content of the mathematical expressions, the uncertainties may be partially compensated or notably worsened, as we will see in the following example, where we evaluate a quantity $s(a,b)$ from measurement of signals *a* and *b* each of which has the same uncertainty Δa . We will leave it to the reader to calculate the relative uncertainty which results in *s* in the two following cases:

$$1/s = \frac{a+b}{a-b} \qquad \text{(answer: } \Delta s = \frac{2b\Delta a + 2a\Delta b}{(a-b)^2}\text{)}$$

$$2/s = a - \frac{a^2}{b} \qquad \text{(answer: } \Delta s = \left|\left(1 - \frac{2a}{b}\right)\right| \Delta a + \frac{a^2}{b^2} \Delta b \text{)}$$

In the second case, we see that if b is equal to $2a$, the measurement of a does not cause any uncertainty in s (to second order). The function s thus presents an extremum with respect to the variable a, making it stationary. The situation is thus quite optimal for measuring the quantity s. In general, expression [7.1] shows that the *lower the sensitivity of the quantity measured to the variations of the measurement parameters, the better the precision of the measurement.* Note that systems that are close to instability are particularly sensitive to perturbations and that the measurement of associated quantities is thus particularly difficult.

These elementary ideas are sufficient in simple cases. However, they must be completed in the case of complex measurements requiring a large number of unknowns and of quantities to be measured.

7.2.7.3. *Errors and conditioning of a linear system*

These ideas of errors and uncertainty have been broadly studied for the numerical solution of linear systems of various sizes. The development of computer technology has led to the possibility of measuring large numbers of quantities simultaneously and of deducing interesting physical quantities by solving the equations that characterize the measurement process. The problems posed here are identical to those encountered in the numerical study of systems of equations obtained from modeling.

Let us assume that the evaluation of a vector quantity X is performed via the measurement of another vector quantity B of the same size. The vector of unknowns X and the vector of given data B are assumed to be related by a linear system in which the square matrix A models the measurement process:

$$AX = B \qquad [7.2]$$

As we are here interested in discussing errors that are assumed small, the linear system can always be obtained by linearizing the equations about the experimental conditions and the matrix A is constant. It is invertible (otherwise the measurement is not meaningful). The precision which can be obtained for the solution $X = A^{-1}B$ depends on the sensitivity of the system: if a small variation in the right-hand side δB leads to a large variation of the solution, we must expect poor precision in obtaining the solution. We quantify the idea of sensitivity by means defining the *condition number* $\kappa_p(A)$ of the matrix A by the relation:

$$\kappa_p(A) = \|A\|_p \left\|A^{-1}\right\|_p$$

The symbol $\|.\|_p$ designates a matrix norm defined most often by means of a vector norm. For example, the norm $\|A\|_1$ of the matrix A is the maximum value taken as the sum of absolute element values of each column of A. A norm $\|.\|_p$ satisfies the conditions:

$$\|A + B\|_p \le \|A\|_p + \|B\|_p \; ; \quad \|\lambda A\|_p = |\lambda|.\|A\|_p \; ; \quad \|A.\mathrm{B}\|_p \le \|A\|_p.\|B\|_p$$

Using this condition, the reader can easily verify the following inequalities:

$$1 \le \|A\|_p.\left\|A^{-1}\right\|_p ; \quad \frac{\|\delta X\|_p}{\|X\|_p} \le \|A\|_p.\left\|A^{-1}\right\|_p \frac{\|\delta B\|_p}{\|B\|_p}$$

This inequality shows that the condition number $\kappa_p(A)$ constitutes an upper bound of the relative amplification error between the data B and the unknown X in solution of system [7.2]. The greater this value, the greater the sensitivity of the solution to variations in B. The precise condition number value depends on the choice of the norm. *A well-conditioned system (*$\kappa_p(A)$ *of the order of 1) allows the obtention of a good accuracy in the solution.*

In order to illustrate the influence of the condition number we will consider a rudimentary example where X has two components. Consider the matrix A with which we associate the norm $\|A\|_1$ (defined above):

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \qquad [7.3]$$

The reader can certify that we obtain:

$$A^{-1} = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}; \quad \|A\|_1 = \left\|A^{-1}\right\|_1 = 1 + \alpha; \quad \kappa_1(A) = (1 + \alpha)^2$$

hence:

$$X = A^{-1}B = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} b_1 - \alpha b_2 \\ b_2 \end{pmatrix} \qquad [7.4]$$

When α is equal to 0, the system is perfectly conditioned ($\kappa_1(A) = 1$). In this case the solution is the identity ($X = B$), or: ($(x_1 = b_1, x_2 = b_2)$). We thus directly measure the unknowns, which is ideal.

On the other hand, for $\alpha = 100$, we have an ill-conditioned system and small variations of measurement B can lead to large variations in solution X. The reader can easily verify that, for the preceding example, changing the vector B from: $\begin{pmatrix} 100 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 100 \\ 0 \end{pmatrix}$ will cause solution X to change from $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 100 \\ 0 \end{pmatrix}$: a variation of 1% in the modulus of the inputs data B leads to a variation of a factor of 100 in the vector X, which has also rotated by 90°. They can also calculate the relative variation $\|\delta X\|_1 / \|X\|_1$ of the solution and that of the right-hand side, in addition to the amplification factor of the relative error between B and X.

These considerations show that the conditioning of a linear system of equations deteriorates as its matrix becomes filled and its coefficients are large. A full matrix of large size is thus very poorly conditioned.

Poor conditioning can be improved by a suitable change in the way the equation is written. The system of the last example [7.3] is optimal. The system of the last example [7.3] can be written in the solved form [7.4] whose conditioning is optimal. The accuracy problem has not been resolved however, as we see that it is the quantity $b_1 - \alpha b_2$ which must be measured directly with good precision. Changing the way the system of equations is written without changing the measurement method clearly improves nothing.

In conclusion, this idea of conditioning is essential in order to appreciate the quality of an experiment, a model or a numerical processing of information. Considerable theoretical and practical progress has been made in this domain. We refer the reader to [DEM 97], [PRE 07] and to manuals of computation tools (MATLAB for example; instructions “cond” and “condest”).

7.2.7.4. *Consequences for measurement techniques*

In general, the evaluation of a quantity is effected by measuring other quantities that are related to the first by relations that characterize the measurement procedure. The matrix A of the corresponding system of equations must be well conditioned in order that the uncertainty is as small as possible. In cases of poor conditioning of the matrix A, the very principle of the method of measuring the unknown quantities is questionable and needs to be modified. In simple terms, the preceding example amounts to saying that if two measurements are determined for the most part by means of a single quantity, it will not be possible to obtain an indication of the value of another quantity with good accuracy and it will be necessary to change the measurement method.

For example, uncertainties always accumulate for independent measurements; we should always avoid obtaining the value of a quantity g by means of the difference in measurements between two much larger quantities. Thus, the value for a small air velocity is obtained using Bernoulli's theorem from the pressure difference Δp between two cross-sections that are quite close together, but of different sections. It is clear that Δp should be directly measured by means of a differential manometer which is directly sensitive to this pressure difference and not from the independent measurement of two pressures.

In general, the quantities measured should be in the same order of magnitude as the unknown. Let us take another example, the propagation velocity c of sound signals with respect to matter is large (about 340 m/s in air or 1,500 m/s in water). If the matter is in movement at velocity U (a few m/s), a signal emitted at A (Figure 7.6) will arrive at B (AB parallel to the velocity; $\|AB\| = d$) with a delay equal to $d/(c+U)$, whereas a signal emitted at B will arrive at A with delay of $d/(c-U)$. The separation τ between the propagation times of the two signals is thus equal to $2Ud/\left(c^2 - U^2\right)$. In principle, the measurement of τ allows the measurement of the flow velocity U.

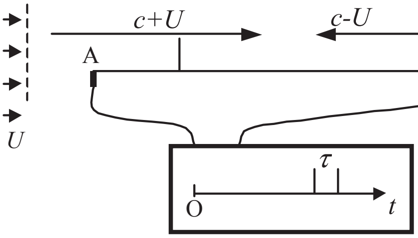


Figure 7.6. *Direct measurement of the delay between the arrival of two synchronous signals in A and B*

However, it is important not to measure the propagation times of the two signals, but rather the time separation τ; for this the two signals should be emitted simultaneously, and the time difference can be measured for example by starting a stopwatch when the first signal arrives. Proceeding in this way, errors of discretization and digitization which are only concerned with a reduced value τ rather than much larger values will be avoided (in accordance with the ratio c/U). If the measurement were performed by means of analogous measurement devices, the conclusion would be the same, as these have a relative accuracy, which is associated with the scale of the measurement.

In general, methods which involve ensuring that the quantity measured is zero in conditions very close to those in which we operate are known as zero methods. The electric-bridge methods used (Wheatstone bridge, etc.) to measure impedance are well-known zero methods.

7.3. Representation of signals

7.3.1. *Objectives of continuous signal representation*

7.3.1.1. *Introduction*

We have seen in the preceding chapters how a physical system can be represented by a continuous medium or modeled by associating components in a state of thermodynamic equilibrium. We have limited ourselves to thermodynamic aspects without really discussing the “quantity of information” necessary for knowledge of a system. This idea is quite difficult to define, as it depends in fact on the complexity of the system, the degree of approximation which can be tolerated in this knowledge, the sensitivity of the system towards perturbations and *the way the information is structured.* We have only indicated that the number of variables necessary is greater as the thermodynamic imbalances are more pronounced.

The continuous medium described in Chapter 2 is an indispensable mathematical limit wherein the equations describe the observable macroscopic properties, provided that the associated physical quantities have been included. In order to aid the discussion, we will assume that the physical system and its appropriate model (for example, the Navier-Stokes equations) are equivalent models: results obtained from each should be identical (Figure 7.7). It is clearly an optimistic departure point, but one which can afterwards be tempered by taking account of the errors associated with the choice of models, the measurement procedures, the numerical methods of calculation, etc.

The solutions of the theoretical model and the physical quantities of the corresponding experiments possess the same properties of continuity and differentiation. The representation of these solutions and quantities is the first *practical problem* to be solved. In what follows, we will limit ourselves to a function of one variable, which is for instance a temporal signal.

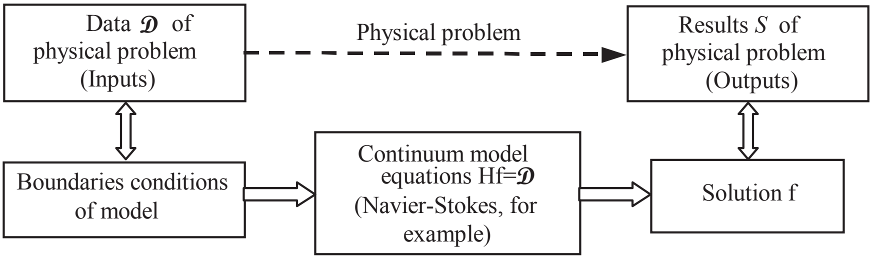


Figure 7.7. *Physical system and its mode*

In a procedure adopted over many years, any solution to the equations of continuous media, mechanical or electrical, are presented in the form of an analytical representation from which it is possible to calculate the value of a *measurable global quantity*, for example the flow rate in a pipe as a function of the pressure difference applied, the value of an electrical resistance of a conductor of a given form, the thermal flux resulting from a temperature difference imposed between two external surfaces of a medium, etc.

Thus, experiments have for a long time consisted of measuring certain local or global quantities which allow the validation of the model of the system studied. The appearance of computer techniques has not really changed this manner of proceeding, but it has allowed us to increase to an extraordinary extent the power of the means by which we compute quantities in the model. Progress in analog and numerical electronics has also allowed instantaneous measurements that were previously unimaginable. We thus find ourselves today confronted with an enormous quantity of information contained in the results of numerical computations or measurements, which need to be processed in as rational a manner as possible. These new technologies have radically changed both the way in which results are represented and used. Beyond these practical aspects, the comprehension of physical phenomena has been profoundly modified. An analogous mutation was produced at the beginning of the 19th century with the introduction of the Fourier series.

Finally, the problem representing a function arises both from the point of view of analysis and storage of measurement data, and that of the analytical or numerical resolution of modeling problems. The numerical calculation of solutions of differential equations was obtained either by performing finite difference calculations or by means of tables, these having been calculated for discrete values of the variables. In practice, these procedures required important and careful effort; only the use of numerical tables was familiar to the physicist or engineer who was mostly happy to calculate a numerical value by interpolation between two values of a table.

In this chapter, we will limit our discussion to general procedures for representing signals; the problems of representation associated with modeling will be discussed in Chapter 8.

7.3.1.2. *Objectives of signal representation*

In these conditions, the first question to be considered concerns the *practical utility of a field of a continuous quantity* and the reasons for using it. The model of Figure 7.7 is a *knowledge model*, which allows us to know everything about the problem that is posed. It contains variables which are continuous functions of time and which represent quantities of the associated physical system.

A variety of different reasons lead to the representation of a temporal signal:

– we can firstly *analyze it*, in other words obtain information which allows us to understand the physical process and eventually to modify it;

– the signal can also be *recorded in its totality with a view to later use;*

– we can also extract more or less condensed information from this signal that we will keep for a later, significant, signal restoration;

– in certain cases, we would like later to construct physical synthetic signals: music synthesizers, speech or image synthesizers, modeling of real devices by numerical simulation on computers, active control of phenomena, etc.;

– finally, we can perform a signal transformation by applying to it an algorithm with the objective of computing another signal.

Depending on the objective in mind, the nature and the quantity of the "information" which we intend to conserve and use, we proceed differently. In fact, we often find ourselves confronted by a double problem of storage and/or of interpreting information contained in a physical signal. Conserving billions of numerical values without understanding what they represent is not very helpful; this operation has a certain associated cost, even if this is becoming less and less. In certain difficult and expensive experimental cases, it is of interest to conserve large masses of data, which is possible using modern equipment, in the hope that we will know how to extract pertinent information which we do not know how to extract at the moment of the experiment; but the difficulty is storing the data in a suitable form.

With the exception of measurements performed with a specific objective, any representation of a function of time $s(t)$ should allow the signal to be recalculated. The practical needs associated with the storage and reproduction of information concerns all domains; beyond science and technology, the conservation of music, speech and image (for historians, etc.) has a social interest. Signal processing is a complex discipline that we will not consider in detail; the reader should refer to specialized texts

([BAH 01], [CAS 06], [JAC 91], [MEA 91], [PRI 91]). We will here only give certain general indications, which should allow the reader to appreciate the information processing problems that arise in the treatment of physical phenomena encountered in acoustics, fluid mechanics and thermodynamics.

7.3.2. *Analytical representation*

A signal $s(t)$ can be represented by a simple “analytical” function, in other words a compact expression which defines a process at each instant of an interval of study, either by means of predefined functions such as circular functions, polynomials, Bessel functions, etc. or by means of formulae which imply one or many known methods (integration, differentiation, convolution, etc.). The “analytical” term is not here to be taken in the strict mathematical sense, despite the fact that the function used can satisfy the mathematical definition of analyticity.

This analytical representation, exact or approximate, can be obtained in different ways:

– an exact explicit solution of a system of equations that constitute a model, although in practice this is rarely possible for continuous media in flow;

– an approximate global solution of the same system of equations by a procedure which consists of satisfying the averaged equations (weak solution). Different ways of proceeding exist; for example, we can replace the equations with integral conditions which constitute a simpler system of equations containing fewer variables and to which it is possible to find an analytical solution (see elementary examples discussed in sections 6.2.6, 6.3.1.2 and 6.5.2.2) or a solution of a form which is given *a priori* and for which certain coefficients can be obtained by least square methods (error minimization, etc.);

– interpolation functions (polynomial or other function) obtained from punctual measurement data, graphical recordings, etc.

The analytical representation of a signal $s(t)$ thus consists of defining the class of functions used and the parameters that characterize this particular function. In general, predefined elementary or special functions allow a particular synthetic knowledge, which a numerical representation does not provide. Knowledge of their properties often allows interpretations of the solution thus obtained and reasoning regarding the relations of cause and effect by means of known analytical properties. It is thus possible to derive particular properties or other analytical forms without any numerical computation in the context of the theories used. These analysis possibilities only exist if the analytical representation comprises only a handful of coefficients: the properties of a full series are too general to be useful, except if they represent known functions or if they are defined by laws of recurrence.

Finally, certain exact solutions of the equations of fluid mechanics, of thermodynamics and acoustics, can be brought to the solution of linear differential equations if the partial differential equation is linear, or even non-linear in the most cases. We have already seen examples (sections 5.4.5.4 and 6.1.1.2.4), which most of the time correspond to a well-defined physical evolution.

In summary, a curve that is measured or numerically calculated or *a table of n numerical values gives raw, unstructured information* that is apt to represent many kinds of signals. On the other hand, an analytical formula, which characterizes the curve, provides a structured information. *An analytical function with few numerical coefficients often allows clear physical concepts to be associated with the information*; obviously *this analytical formula can only represent very specific kinds of phenomena*. The development of computer technology has unfortunately led to analytical results being neglected, which frequently allow analysis and sometimes predictions based on physical arguments. Numerical calculations are of course not to be neglected; on the contrary, it must not be forgotten that they provide knowledge of the same kind as an experiment.

7.3.3. *Signal decomposition on the basis of functions; series and elementary solutions*

7.3.3.1. *Representation in the form of a series of functions*

Exact mathematical representations can also be obtained in the form of series or integrals, but in so far as general procedures are concerned, their physical interest is often limited. Thus, a function of a real variable can be decomposed in terms of a set of basis functions, of which there are many kinds. For example, a signal can be represented by a power series development (Taylor) around an instant t_0:

$$s(t) = \sum_{0}^{\infty} \frac{(t - t_0)^n}{n!} s^{(n)}(t_0)$$

This series is often unusable in physics, as the property of infinite differentiability does not exist; furthermore, the convergence of the series of an analytical function is often limited to a finite interval. We will nonetheless note the following particularity of power series developments: all information concerning the function of time to be represented can be found concentrated at a given instant t_0. This assumes that the future of the function is completely determined from that instant: the fact of indicating data in the long term creates problems of accuracy which are manifest in the numerical value of the higher order derivatives. Such a representation, which is theoretically possible for large values of time (the power series cos ωt converges regardless of t), is nonetheless unusable in practice.

A function $s(t)$ can also be developed as a *Fourier series* on a finite interval $[0,T]$ of length T:

$$s(t) = \sum_{0}^{\infty} \left(a_n \cos \frac{2\pi nt}{T} + b_n \sin \frac{2\pi nt}{T} \right)$$

The coefficients a_n and b_n being given by the classical formulae:

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} s(t) . dt \qquad a_n = \frac{2}{T} \int_{-T/2}^{T/2} s(t) . \cos(2\pi nt / T) dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} s(t) . \sin(2\pi nt / T) dt$$

The Fourier series represents a periodic function of period T; it is *badly adapted to the representation of very rapid variations and discontinuities.* As the harmonic functions of period T/n are orthogonal (where n is an integer), we have:

$$\frac{1}{T} \int_{0}^{T} s^2(t) dt = a_0^2 + \frac{1}{2} \sum_{1}^{\infty} \left(a_n^2 + b_n^2 \right)$$

This property can be interpreted as a conservation of energy: the energy of the signal during time T is equal to the sum of the energies of the harmonics (Parseval's theorem).

There exist many other basis functions sets on a finite interval: Bessel functions, Legendre and Tchehychev polynomials, etc. (see mathematical texts). The interest in these is often related to the nature of the considered problem. Certain functions such as real decaying exponentials are particularly useful for the study of damped systems, but they do not form an orthogonal basis set. Finally, basis functions are not necessarily continuous, as we will see in a later example (section 8.3.2.3).

7.3.3.2. *Representation by combinations of elementary solutions*

The series evoked above are often chosen as a function of simple and universal mathematical properties (Taylor series, etc.) without any prior consideration of the physical properties of the system studied. Another manner of constructing solution representations consists of combining elementary solutions of the equations studied. For example, for problems associated with Laplace's equation (electrostatics, steady conduction of heat or electricity, irrotational fluid flow, etc.), we can seek an *exact or approximate solution that satisfies the boundary conditions in the form of a linear combination of monopoles, dipoles and vortices*. In general, such methods are possible for linear problems in different forms (singularities and multipole distribution methods, Green’s functions, boundary element methods, etc.). These methods often use a mathematical formalism which can be quite involved and which we cannot cover in this text.

We will encounter this kind of methodology in the synthesis of musical sounds; we can however already note that a musical score constitutes a combination of elementary solutions (the notes of the instruments and indications regarding their interpretation) which allow the representation of a musical sequence in a manner equivalent to the numerical values recorded on a CD.

7.3.3.3. *Signal reconstruction*

The reconstruction of a signal $s(t)$ represented in the form of a series or as a combination of functions is an intricate exercise, which can sometimes be quite difficult. Finding a simple analytical representation of a series is an almost impossible task, except in very specific cases; only a numerical reconstruction of the signal is possible.

Let us recall that *a power series is associated with a function independent of the value of its radius of convergence and it is characterized by the infinite sequence of coefficients*. Because of this, calculating the value of a function thus represented is a normal mathematical operation, even if the series diverges. The simplest way to obtain the values of the developed function $s(t)$ is to numerically calculate the sum of the series when possible. In the case of series divergence (or poor convergence), there exist procedures which allow the divergence problem to be contoured or the speed of convergence to be improved in order that its sum be efficiently calculated (analytical prolongation, Euler algorithms, Cesaro or Féjer sums for Fourier series, etc. The interested reader should refer to mathematical texts ([ABR 65] p. 16, [BRE 91], [PRE 07])).

In summary, a series development of a function *s(t)* replaces the function by a denumerable sequence of numerical values (series coefficients) and knowledge of the basis functions. However, such developments are only of interest if the function can be calculated with a small number of terms thus obtained, which is often the case when the basis functions are solutions of the equations which correspond to the model of the physical problem (Appendix 4).

7.3.4. *Integral transforms*

7.3.4.1. *Introduction*

An integral transform is a generalization of the idea of a series, which amounts to developing a function on a continuous infinity of basis functions $g(\nu,t)$ that depend on the parameter ν. Instead of obtaining a denumerable sequence of coefficients, the result of the integral transform $\mathcal{J}$ of the function $s(t)$ is the function $\mathcal{J}_s(\nu)$ of the variable ν; as the information provided by the function is greater than that contained in a denumerable sequence of coefficients, we see that the conditions for applying integral transforms are far broader than those of series developments. We can often associate an integral transform with a series development. The integral transforms can comprise complex values.

An integral transform can often be inversed, in other words there exits another integral transform $\mathcal{J}^{-1}$ whose result $\mathcal{J}^{-1}(\mathcal{J}(s(t)))$ is equal to the original function $s(t)$. The exact or approximate reconstruction of the initial signal is the obvious condition for using an integral transform to store a signal. This condition is not necessary for its analysis.

The *correlation coefficient* C_{fg} of two temporal functions $f(t)$ and $g(t)$ is defined by the relation:

$$C_{fg} = \frac{< f, g >}{\sqrt{< f^2 >}\sqrt{< g^2 >}} \qquad \text{with} : < f, g > = \frac{1}{T}\int_0^T f(t)g(t)dt$$

In the usual definition, we consider the infinite interval T by taking the limit. In practice, this coefficient is always calculated on as large a finite interval as possible. *An integral transform is thus a factor excepted no more than the ensemble of correlations between a function s(t) and the family of basis functions* $g(\nu, t)$.

An integral transform only provides interesting information and simplifications if a large correlation is obtained for a small number of functions $g(\nu, t)$. In such cases significant interpretations may be possible. *The use of an integral transform is only of interest if a reduced set of basis functions represents a notable part of the properties of the signal studied.* There exist many integral transforms. We will give some examples of commonly used transforms with only some of their basic properties, chosen on account of their physical interest. The reader will find a more exhaustive discussion in specialized books ([AND 99], [BEE 03], [DEB 06], [GUP 83], [JER 92], [WOL 79]).

7.3.4.2. *Fourier transforms*

7.3.4.2.1. Definition and properties

We will leave aside the rigorous definition and conditions for existence of the Fourier transform. Consider a real-valued temporal signal $x(t)$; its Fourier transform $F_x(\nu)$ is a complex function of the frequency ν defined by the relation:

$$F_x(\nu) = \int_{-\infty}^{+\infty} x(t)\, e^{-2\pi j\nu t} dt$$

The real and imaginary parts of this transform $F_x(\nu)$ are:

$$\mathrm{Re}\big[F_x(\nu)\big] = \int_{-\infty}^{+\infty} x(t)\, \cos 2\pi\nu t \ dt; \quad \mathrm{Im}\big[F_x(\nu)\big] = -\int_{-\infty}^{+\infty} x(t)\, \sin 2\pi\nu t \ dt$$

The modulus $\left|F_x(\nu)\right|$ and the phase $\varphi\big[F_x(\nu)\big]$ of $F_x(\nu)$, respectively known as the amplitude spectrum and the phase spectrum, can be written:

$$\left|F_x(\nu)\right| = \sqrt{\mathrm{Re}\big[F_x(\nu)\big]^2 + \mathrm{Im}\big[F_x(\nu)\big]^2} \qquad \varphi\big[F_x(\nu)\big] = Arctg\left(-\frac{\mathrm{Im}\big[F_x(\nu)\big]}{\mathrm{Re}\big[F_x(\nu)\big]}\right)$$

The temporal signal is reconstructed by the inverse Fourier transform:

$$x(t) = \int_{-\infty}^{+\infty} F_x(\nu)\, e^{2\pi j\nu t} d\nu$$

$F_x(\nu)$ and $x(t)$ are two different and equivalent representations of the same quantity, respectively in frequency space and in the temporal domain.

The main properties of the Fourier transform are:

1) the property of evenness:

$x(t)$ real $\Leftrightarrow$ $\mathrm{Re}[F_x(\nu)]$ even, $\mathrm{Im}[F_x(\nu)]$ uneven

$x(t)$ real even $\Leftrightarrow$ $F_x(\nu)$ real even

$x(t)$ real odd $\Leftrightarrow$ $F_x(\nu)$ imaginary odd

$x(-t)$ $\Leftrightarrow$ $F_x(-\nu) = F_x(\nu)^*$

2) a stretching of the timescale of the function $x(t)$ which leads to a contraction of the frequency scale for the transform $F_x(\nu)$ and vice versa (similarity):

$$F_{x(at)}(\nu) = \frac{1}{|a|} F_{x(t)}\left(\frac{\nu}{a}\right)$$

3) a translation τ of the timescale of the function $x(t)$ which leads to a phase rotation equal to $-2\pi\tau\nu$ for the transform $F_x(\nu)$ and vice versa:

$$F_{x(t-\tau)}(\nu) = e^{-2\pi j\tau\nu} F_{x(t)}(\nu)$$

4) the Dirac distribution $\delta(\nu)$ is defined by the relation:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \text{ with: } t \neq 0,\ \delta(t) = 0$$

It satisfies:

$$\int_{-\infty}^{+\infty} \delta(t) f(t)dt = f(0).$$

We can easily show that its Fourier transform is equal to one:

$$F_{\delta(t)}(\nu) = \int_{-\infty}^{+\infty} \delta(t)e^{-2\pi j\nu t}dt = 1$$

The Fourier transform of a Dirac distribution centered at the origin is a constant function of the variable ν, equal to one on the interval [-∞, +∞];

5) the Dirac distribution $\delta(t-\tau)$ centered at the instant τ verifies:

$$\int_{-\infty}^{+\infty} \delta(t-\tau) f(t)dt = f(\tau)$$

From the relation of translation of the timescale, we obtain its Fourier transform:

$$F_{\delta(t-\tau)}(\nu) = \int_{-\infty}^{+\infty} \delta(t-\tau)e^{-2\pi j\nu t}dt = e^{-2\pi j\tau\nu}$$

6) the inverse Fourier transform of the Dirac distribution $\delta(\nu - N)$ in frequency space is:

$$F^{-1}_{\delta(\nu-N)}(t) = \int_{-\infty}^{+\infty} \delta(\nu - N)\ e^{2\pi j\nu t} d\nu = e^{2\pi jNt}$$

$$F^{-1}_{\delta(\nu)}(\int_{-\infty}^{+\infty} \delta(\nu)\ e^{2\pi j\nu t} d\nu = 1$$

7) the Fourier transform of the function $e^{2\pi jNt}$ is then the Dirac distribution centered on frequency $+N$:

$$F_{\exp(2\pi jNt)} = \delta(\nu - N)$$

8) the Fourier transforms of the functions $\cos 2\pi Nt$ and $\sin 2\pi Nt$ are the half-sum and the half-difference of two Dirac distributions centered on the frequencies $+\alpha$ and $-\alpha$.

The function $A\cos(2\pi Nt+\varphi)$, continuous in temporal space, is thus represented by only three numerical values A, N and φ in frequency-space;

9) the Fourier transform of the product of two functions $x_1(t)$ and $x_2(t)$ is equal to a convolution product $F_{x1}(\nu)*F_{x2}(\nu)$ of the Fourier transforms $F_{x1}(\nu)$ and $F_{x2}(\nu)$, and vice versa;

10) Parseval's theorem expresses that *the energy of a signal s(t) is conserved by the Fourier transform:*

$$E_S = \int_{-\infty}^{\infty} |s(t)|^2\, dt = \int_{-\infty}^{\infty} |F_S(\nu)|^2\, d\nu$$

We see *completely different distributions of information in temporal and frequency space:* the non-zero temporal signal and the infinite duration of the cosine function is found concentrated in four non-zero values (frequencies $\pm\nu$, amplitude and phase or complex amplitude), whereas the information of the Dirac concentrated at the time axis origin is found spread over the entire frequency domain.

7.3.4.2.2. Interpretation of the Fourier transform

Using the Fourier transform amounts to seeking the *correlation between a signal x(t) and the harmonic signals of frequency* ν, which can be expressed in the form $e^{j2\pi\nu t}$. This interpretation is valuable because it can be shown that the better this correlation, the closer the signal is to a harmonic signal. For a signal mainly comprising harmonic discrete signals, the correlation will be high for corresponding frequencies (spectrum of lines) and the signal will be characterized by the values of these lines. We thus obtain a small number of numerical values in the place of the values obtained by temporal discretization of the signal. The Fourier transform consists of representing the functions $x(t)$ using a basis-set of harmonic functions.

The application of the Fourier transform on a temporal signal of infinite length poses two problems that are difficult to reconcile *a priori* with the idea of irreversible time associated with the second law of thermodynamics:

– on the one hand, we can only know the transform after a very long period of time;

– on the other hand, it assumes that the beginning of the signal is situated a long time ago in the past.

The first inconvenience makes it difficult to follow the evolution of a phenomenon without a long enough delay; this can be remedied by means of observation windows of limited duration. The second difficulty is more serious, as it implies that the signals obtained by inverse Fourier transform are not possible to realize if they occupy the time interval [-∞, +∞]. Only a non-zero temporal signal which starts from a given instant is realizable: we refer to this as a *causal signal*, which defines a possible action in time.

7.3.4.3. *Laplace transform*

The Laplace transform possesses properties similar to those of the Fourier transform, and is defined by the relation:

$$L_f(p) = \int_0^\infty e^{-pt} f(t)dt$$

Laplace transformation is not fundamentally different from Fourier transformation, as it consists of taking in this Fourier transform imaginary values for the variable in complex plane; it has the same properties, except for inverse transformation.

This transform is of considerable interest, notably for the study of damped systems in automatic control, the family of functions for comparison with the signal $s(t)$ being precisely the ensemble of aperiodic damped modes of linear invariant systems of first order. The reader can refer to texts on the dynamics and control of systems ([BEE 03], [DEB 06], [GUP 83], [WOL 79]). We will see in Chapter 8 the application of the Laplace transform for the solution of linear systems with constant coefficients (Appendix 1).

7.3.4.4. *The Hilbert transform*

The objective of the Hilbert transform is to define the amplitude and the instantaneous frequency of a movement which is close to a harmonic movement, by with variable characteristics. We here have a generalization of the definition of the complex exponential $e^{j2\pi nt}$ (Appendix 2). We define the Hilbert transform $H(t)$ of the signal $H(t)$ by the relation[1]:

$$H(t) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi} VP\int_{-\infty}^{\infty} x(u).\frac{du}{t-u} \qquad [7.5]$$

---

[1] The notation *VP* means that the integral has to be understood in terms of Cauchy principal value; notation $*$ indicates the convolution product of two functions.

The Hilbert transform of $\cos 2\pi\nu t$ is equal to $\sin 2\pi\nu t$; in general, the effect of the Hilbert transform is to introduce a phase lag of $\pi/2$ in the initial harmonic function. For any real signal $x(t)$, we can associate the complex analytical signal $\alpha_x(t)$:

$$\alpha_x(t) = x(t) + jH(t) \tag{7.6}$$

The modulus $a(t)$ and the argument $\varphi(t)$ of the analytical signal $\alpha_x(t)$ can be defined as the *amplitude and the instantaneous phase* of the signal $x(t)$:

$$x(t) = \mathrm{Re}(\alpha_x(t)) = \mathrm{Re}\left[a(t)e^{j\varphi(t)}\right]$$

The *instantaneous frequency* ν_i is defined as the derivative of the instantaneous phase $\nu_i = \frac{1}{2\pi}.\frac{d\varphi}{dt}$. The idea of instantaneous frequency is only meaningful for functions that are close to harmonic functions, in other words for relatively narrowband signals. We will note that these signals possess two very different timescales, one rapid timescale corresponding to a "carrier" and another much slower scale that characterizes a modulation. Figure 7.8 shows an example of a signal modulated in amplitude (a) or in frequency (b).

Most musical signals are characterized by a fixed frequency, which is modulated in amplitude, and in phase. They constitute the basis for the synthesis of sounds in musical synthesizers. We will come back to this point a little later in section 7.4.3.4.

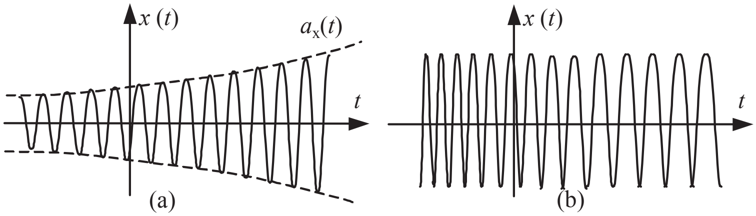


Figure 7.8. *Signal modulated in amplitude (a) or in phase (b)*

7.3.4.5. *Cepstrum*

The principle of cepstrum consists of taking the logarithm of a spectral density, and then performing an inverse transform. Ordinary products and convolution products are respectively transformed into sums and products; a harmonic modulation of the spectrum is transformed into a Dirac in the inverse transform. These properties are often used for signal processing applications such as:

– the suppression of echoes in acoustic signals;

– the characterization of vibrations in rotating machinery and more particularly in obscuring the operation of machines (for example, in looking for defects in rotating parts which result from the wear of bearings and which leads to the appearance of lines in the spectrum or by modulation frequencies which are difficult to see in a simple spectrum).

We will see examples of applications of the cepstrum in Appendix 3. Depending on the problem, different definitions of the cepstrum are used ([DES 00], [JUR 08], [NOR 03], [WAI 90]).

7.3.4.6. *Short time Fourier transforms*

The Fourier transform is defined on an infinite interval. For diverse reasons, we can only record a signal over a limited duration, and this modifies the Fourier transform. It is thus necessary to find a compromise between the volume of information required and the accuracy of the results obtained. Recording a signal for a finite duration *T*, consists of multiplying it by a non-zero function over this interval and by zero outside of this interval. Following the usual terminology, we will say that this function, called a gate function $\Pi_T(t)$, is a particular case (rectangular function) of a window function. The gate function $\Pi_T(t)$ centered at the origin can be written:

$$\Pi_T(t) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & t < -T/2 \quad T/2 < t \end{cases}$$

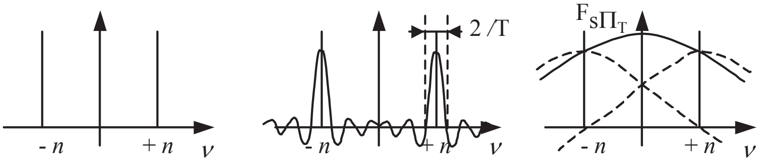
The signal transform thus truncated is written:

$$F_{x\Pi_T}(\nu) = \int_{-\infty}^{+\infty} \Pi_T(t) x(t)\, e^{-2\pi j \nu t} dt = \mathrm{F_x}(\nu) * F_{\Pi_T}(\nu)$$

The Fourier transform of the truncated signal $x(t).\Pi_T(t)$ is equal to the convolution product of the Fourier transforms of the signal $x(t)$ on the infinite interval and the gate $\Pi_T(t)$. The Fourier transform $F_{\Pi_T}(\nu)$ of this one is a cardinal sine function:

$$F_{\Pi_T}(\nu) = T \frac{\sin \pi \nu T}{\pi \nu T}$$

Now consider the effect of recording the signal $\cos 2\pi n t$ using a gate of duration *T*. The Fourier transform of a cosine of infinite duration is composed of two Diracs (Figure 7.9a): the spectrum of the cosine convolved with the transform of the gate of duration T is comprised of two cardinal[2] sinusoids centered on the frequencies $\pm n$ of the two preceding Dirac distributions (Figure 7.9b). Considering that the width of the central peak of the cardinal sinusoid is characterized by its first zero, the widening of the Dirac function peak is equal to $2/T$, which is, for a *window which records 10 periods*, a widening in frequency of $0.1n$ on either side of the frequency n.



(a) (b) (c)

Figure 7.9. *Fourier transform of: (a)* $\cos 2\pi\, nt$; *(b)* $\Pi_T(t).\cos 2\pi nt$ *with 1/T << 2n; (c)* $\Pi_T(t).\cos 2\pi nt$ *with 1/T = 2n*

We see that the shorter the window, the wider the frequency band obtained: *an insufficient observation will reduce the accuracy of the Fourier transform by "clouding" the signal.* We note that as a recording which does not disturb the spectrum should be applied with a window which is the inverse transform of the Dirac distribution, in other words the unrealizable infinite width time window, which we wanted to avoid.

According to the usual Rayleigh criterion, we consider that the frequency peaks become unidentifiable if the first zero of the cardinal sinusoid centered at the frequency $+n$ is found at the frequency $-n$ (Figure 7.9c), i.e.:

$$1/T \leq 2n$$

The perception of a frequency n requires thus an observation horizon of duration T greater than $1/2n$: the recording duration T should include at least a half period of the lowest visible frequency of the short-time Fourier transform. Another result of these considerations is that we can only distinguish two frequencies ν and $\nu+\Delta\nu$ if they are separated by at least the value $1/2T$. From a physical point of view, an insufficient recording of information can only give bad results (see formula [7.8] and Shannon's sampling theorem (section 7.3.6.4)). We will note that the spectrum obtained by such a transform has not only lost the details concerning the peak; but it also contains low frequencies that do not exist physically.

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[2] We obtain: $\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$.

The Fourier transform is thus only a correlation performed between a signal and a family of harmonic reference signals. This observation allows a simple physical interpretation of the widening of peaks, which results from the use of a finite window. In effect, the correlation between two harmonic functions is zero over an infinite time, unless their frequencies are equal. However, the same correlation will be increased, as the window size become progressively smaller and as the frequencies are nearer. We will leave it to the reader to verify these observations.

In reality, the widening of the Dirac spectrum by the cardinal sinusoid function is not limited to the central peak of this function and the smaller but non-negligible amplitudes of the lateral lobes can also lead to a net increase in the width of the spectrum obtained. This last inconvenience is a problem, in particular for analyses of acoustic signals on account of the sensitivity of the ear (the logarithmic decibel scale clearly leads to a smaller scale of the amplitude variations). As the energy of the secondary lobes is quite weak, we try to re-center it on the main lobe, even if this means widening it slightly. This can be achieved by replacing the gate function $\Pi_T(t)$ by a window function $\Phi_T(t)$, which leads to much smaller amplitudes of the lateral spectrum peaks than those obtained with a rectangular window:

$$F_{x\Phi_T}(\nu) = \int_{-\infty}^{+\infty} \Phi_T(t)x(t)\, e^{-2\pi j\nu t} dt = F_x(\nu) * F_{\Phi_T}(\nu) \qquad [7.7]$$

The recording of a raw signal over a limited duration therefore leads to deformations of the Fourier transform consisting of two kinds of distributive modification of spectral energy: the widening of the central peak and the appearance of secondary lobes. This widening of the signal spectrum can be studied and characterized for each window by taking the "moment of the signal energy" $\int t^2 \left|x(t)\right|^2 dt$ and of its transform $\int \nu^2 \left|F_x(\nu)\right|^2 d\nu$. General considerations ([BLA 98], [FLA 98], [HIG 93], [STR 96]) allow the demonstration of the Heisenberg-Gabor inequality:

$$\Delta t . \Delta \nu \geq \frac{1}{4\pi} \qquad [7.8]$$

In this inequality Δt and $\Delta\nu$ are respectively the duration of the energy content of the temporal signal and the width of the frequency band in which the energy is contained; Δt is of the order of $T/2$ and $\Delta\nu$ is analogous to the quantity $2/T$ defined above for the rectangular gate. The equality is obtained for a Gauss window, which corresponds thus to an optimum of the preceding minimization criterion. The limitation of the preceding principle is related to the basic uncertainty of quantum mechanics, but the physical analogy is far from complete, the interpretation of quantities being very different in the two domains.

In practice, the effective choice of a window results from the ensemble of various considerations. For a presentation and comparison of commonly used windows, the reader should refer to ([BAH 01], [MAD 98], [NOR 03], [STR 96] [MAX 96]).

7.3.4.7. *Continuous wavelet transforms*

7.3.4.7.1. Introduction

The Fourier transform with a window function can also be interpreted as an integral transform whose kernel is the product $\Phi_T(t)e^{j2\pi\nu t}$ of the complex exponential $e^{j2\pi\nu t}$ by the window $\Phi_T(t)$. In other words, we no longer perform a Fourier transform over a limited duration; rather, we perform *a transform with a function* $g(t,\nu) = \Phi_T(t)e^{j2\pi\nu t}$ that is different from the complex exponential. Such a function is often called a *wavelet*.

This point of view can be generalized by considering functions of time that contain additional parameters such as window width, which can depend on the frequency. An example is the Gabor transform in which we decompose the signal studied according to a basis of functions comprising the product of the complex exponential $e^{j2\pi\nu t}$ and Gauss windows of width α, centered here at the origin:

$$G_x(\alpha,\nu) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\,\alpha}}\, e^{-t^2/4\alpha}\, e^{j2\pi\nu t} x(t)dt$$

In the Gabor transform, we thus have as an additional parameter the width α of the window; we obtain, after transformation, a function with two variables which provide redundant information for representation of the signal $x(t)$. The frequency representation studied in section 7.3.4.6 can be obtained by fixing the value of window width; by varying this, we can obtain complementary information. Consider for example a signal comprising one or two periods of a harmonic function of a given frequency that is centered at the origin. Let us apply the Gabor transform, we obtain a new transform which is not so different from the previous one, so long as the analysis window has a width α of the order of that of the preceding signal, but if this window is widened, the value of the transform will be reduced. A Gabor transform with variable width allows the length of a signal to be identified.

7.3.4.7.2. Timescale transforms

We thus arrive at the idea that by using window width as a variable of the analysis wavelet, we can obtain information regarding the timescales of the signal. We thus use an analysis wavelet whose form is conserved when we dilate the window width. It is clear that by proceeding in this way, the physical idea of frequency is no longer so clear. If the window comprises little oscillation, then it is its size that will be the pertinent variable for the transform.

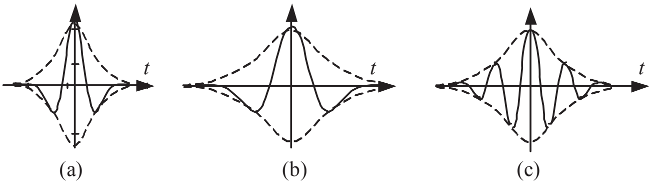

Figure 7.10. *Wavelets with variable width ((a) and (b)) or variable frequency ((b) and (c))*

We define the transform $T_s\,(a)$ of the signal $s(t)$ by the wavelet $h(t/a)$:

$$T_s(a) = |a|^{-1/2} \int_{-\infty}^{\infty} s(t) h(t/a) dt$$

in which the transform variable is the scale a.

A simple wavelet is the “Mexican hat” wavelet, which is the second derivative of the Gauss function:

$$h(t,a) = \frac{1}{\sqrt{2\pi} a} \left(1 - \frac{t^2}{a^2} \right) e^{-t^2 / 2a^2}$$

Figure 7.10 shows the window variations for a timescale analysis (windows a and b) and for a time-frequency analysis (windows b and c).

Diverse problems arise in the choice of wavelets and in particular for the reconstruction of a signal from its transform ([ALL 04], [FLA 77], [MIS 07], [STR 96]).

7.3.5. *Time-frequency (or timescale) representations*

7.3.5.1. *General principles*

The representation of a signal $x(t)$ by means of an integral transform is useful so long as this is not too different from the transform kernel: very short signals centered at the origin will be well represented by wavelets, whereas the Fourier representation is better suited to periodic (or nearly periodic) signals. For other signals the preceding representations are numerically ill-conditioned.

Let us take the example of a centered signal of length 100 and which comprises two sequences of oscillation (Figure 7.11) separated by a sequence in which the signal is zero. Its representation by means of a Fourier series on the interval [-50,50] (after change of time origin) is theoretically possible, but it will require a series with so many coefficients that if will not be very useful.

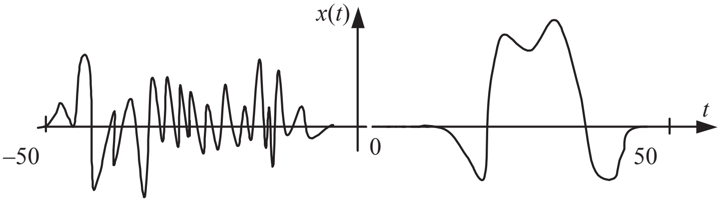

Figure 7.11. *Intermittent signal comprising two different sequences*

The use of a Fourier transform in the same window will hardly be more satisfactory. The use of a Fourier transform with window functions would be even worse on account of signal attenuation towards the window edges. Furthermore, the (complex) transforms will not indicate in a simple manner the fact that the signal comprises two sequences of different frequencies at distinct instants, rather than a uniformly distributed frequency content. We can also think of performing two Fourier transforms on the separate intervals, but such a procedure is directly related to the signal structure and so cannot be easily generalized.

The most direct representation thus consists of recording the signal by means of a window function which is centered on the instant τ, and then performing a short-time Fourier transform. We are thus led to perform sliding transforms, which allow the identification of the spectral content of signals contained in the window $\Phi\,(t - \tau)$. This sliding Fourier transform is expressed:

$$F_{x\Phi}(\nu,\tau) = \int_{-\infty}^{+\infty} \Phi(t-\tau)x(t)\, e^{-2\pi j\nu t}\, dt$$

The transform $F_{x\Phi}(\nu,\tau)$ is a function of two variables τ and ν which we will qualify as a *time-frequency representation.* The spectral distributions become functions of time. However, such a continuous representation contains much redundant information concerning the signal $x(t)$ since knowledge of a single transform on the time interval of the signal suffices for a reconstruction of the initial signal.

7.3.5.2. *Practical applications*

A natural solution for reducing the size of the time-frequency representation thus consists of performing a partition of the total study interval into *n* temporal segments each of a length Δt in the order of the window width, and only conserving the Fourier transforms performed at the center of the segments. We thus obtain a succession of spectra, each of which can be attributed to a given instant.

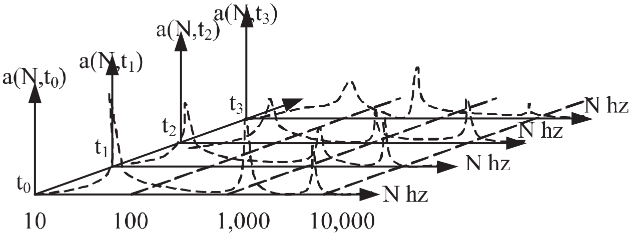

Figure 7.12. *Spectrogram*

The representation of these spectra in the form of a succession of spectra as a function of time is known as a spectrogram (Figure 7.12). However, a spectrogram can quickly become quite complex and difficult to read for long signals. We often prefer a representation, known as a *sonogram*, where the values are shown in the time-frequency plane where the use of color or grayscale allows the indication of the signal level. In this way the information contained in the signal can be visualized in terms of the evolution of the frequencies and amplitudes.

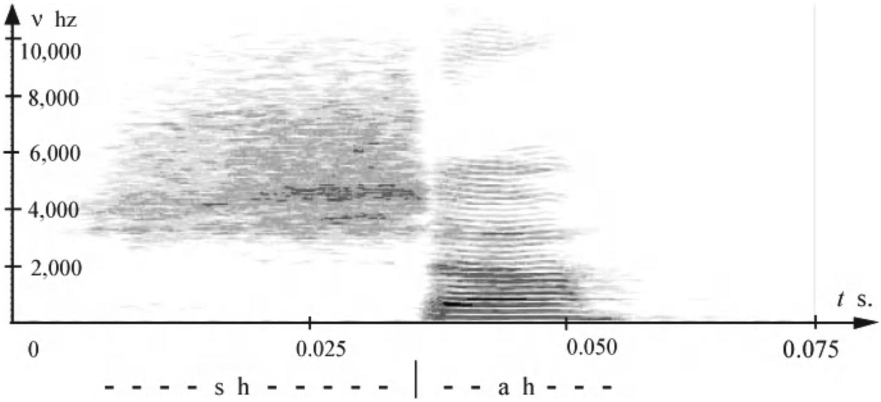

Figure 7.13. *Sonogram of the word "shah" pronounced by a person*

Sonograms (often known as spectrograms) are very often used for the analysis of complex signals; Figure 7.13 shows the sonogram of an acoustic signal corresponding to the word “shah”. We can note that the ensemble of letters “sh” corresponds to higher frequencies than the ensemble “ah”.

The considerations of section 7.3.4.6, relating to the time-frequency uncertainty, here applies and the Heisenberg-Gabor inequality indicated earlier remains valid: a precise temporal localization and an accurate analysis are incompatible. This phenomenon is illustrated by Figure 7.14. which presents sonograms of a signal composed of a silence of 0.05 second duration, followed by a harmonic oscillation of frequency 4 kHz during 0.05 seconds, and then, once again a silence. The analyses performed by means of the software COOL EDIT with windows of width ℓ_F respectively equal to 0.031 and 0.002 seconds show that the first value allows a suitable identification of the frequency, whereas the second allows a better temporal localization of the portions of the signal. The Blackmann-Harris window used gave better results than the other windows that were available; however, the reader should remember that the choice of window depends on empirical considerations ([ALL 04], [JUR 08], [MAD 98], [STR 96], [WAI 90]).

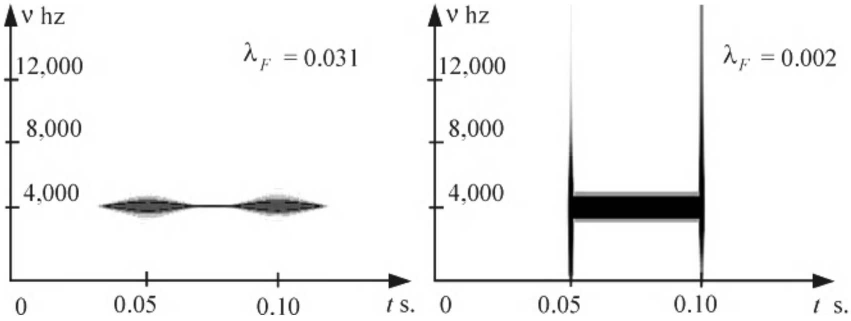

Figure 7.14. *Influence of the width ℓ_F of the Fourier window on the sonogram (4,000 Hz)*

The explanations which have just been provided for the time-frequency analysis can be transposed to timescale analysis. This domain is more common in image treatment; the interested reader should see specialized texts on this subject. The principle of time-frequency representation can be applied in two dimensions, the frequency or scale spectra being thus functions of two variables.

7.3.5.3. *Regarding multiple timescales*

The fact that spatial or temporal variations of a phenomenon take place at two completely different scales has important but variable consequences depending on the mathematical form of the variations. If these are relatively uniform and non-oscillatory, we have a singular perturbation problem (see section 6.4.3). For oscillations of a linear system, the preceding time-frequency analysis is only meaningful if the notion of frequency can be defined, in order to allow separate identification of the rapid component (short oscillation period) and the slow component (characteristic time of the amplitude and phase variation, etc. (section 7.3.4.4)).

These notions of fast and slow scales are not however so easy to define. Let us consider the example of a signal $s(t)$, which is the sum of two harmonic signals of close frequencies $N_0 - \delta N$ and $N_0 + \delta N$:

$$s(t) = \cos 2\pi\left(N_0 - \delta N\right)t + \cos 2\pi\left(N_0 + \delta N\right)t = \frac{1}{2}\cos 2\pi N_0 t . \cos 2\pi\,\delta N\,t$$

This expression shows that $s(t)$ can be considered as a harmonic signal $\cos 2\pi N_0 t$ with fast variations, whose *amplitude is slowly modulated* by the function $\cos 2\pi\,\delta N\,t$. The two timescales are here clearly $1/N$ and $1/\delta N$. The amplitude modulation can be any broadband slow signal $a(t)$ as in the transmission of radio waves.

Frequency modulation is also a system with two timescales, with a low-frequency carrier signal N_0 and a slow variation of frequency δN:

$$s(t) = \cos 2\pi\left(N_0 + \delta N(t)\right)t.$$

The spectrum of this signal modulated with frequency $\delta N((t) = \delta N_0 \cos \omega t$ is not very different from the amplitude modulated spectrum: in addition to the two close frequencies $N_0 - \delta\, N_0$ and $N_0 + \delta\, N_0$, it contains only a small number of weak peaks.

The amplitude spectra of amplitude and frequency modulated signals are nearly similar, but the temporal signals and the phase spectra have very different structures. This comparison illustrates the difficulty of reconstructing, with its detailed characteristics, the time signal which corresponds to an amplitude or energy spectrum: theoretically only the temporal and complex spectral representations are completely equivalent. More generally, a signal similar to a harmonic signal of frequency N_0 can be studied with Hilbert transform, amplitude and frequency modulations being defined from its analytical signal (section 7.3.4.4).

In practice, signals used in telecommunications are modulated in amplitude or in frequency by a *low frequency signal,* which contains information to be transmitted and decoded. In mechanical or energetic systems, these modulations are the result, either of the presence of two close frequencies, or of a slow oscillatory variation of the physical characteristics of an oscillator (stiffness or length of a vibrating system, for example, or interaction between structures of different frequencies in fluid mechanics or acoustics).

We will re-encounter the problem of multiple time (and space) scales in turbulent fluids, at the heart of which are fluctuations of central importance. An everyday example can be found for flow in the atmosphere: the wind presents instantaneous fluctuations with very short periods (a couple of seconds for gusts of wind), and longer periods (from many hours to a day for perturbations (in the meteorological sense)) which are relatively individual structures at the atmospheric scale. These two categories of scale are generally distinct and we can thus separate meteorological and turbulence phenomena by performing statistics for the mean values at the scale of the turbulent fluctuations. Meteorological predictions and data provide the mean velocity, the effect of fluctuations due to gusts being smoothed at timescales of a fraction of an hour.

On the other hand, turbulence frequencies have a continuous spectrum and the temporal separation of phenomena is hazardous in proportion to the degree of non-linearity of the turbulence mechanisms. Furthermore, the corresponding flows can nonetheless be subject to temporal variations at a slower scale due to influences other than that of the turbulence.

7.3.5.4. *Study of intermittency*

We call *intermittency* that property of a signal that comprises structures of the same nature, which occur at intervals that may be more or less regular or random. The signal is comprised of “packets” with a specific structure different from the characteristics of the rest of the signal (Figure 7.15).

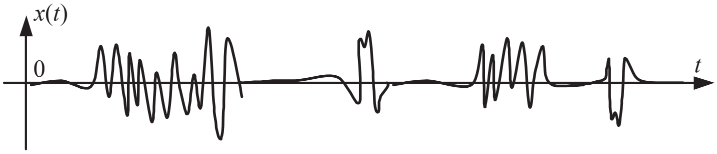

Figure 7.15. *Example of intermittent signal*

Intermittency is encountered in many flows; for example, Figure 7.16 shows that a fixed observer at the horizontal level N will perform measurements characterized by intermittency for liquid packets in a water jet (a), the external turbulent zones of a boundary layer (b), or ascending currents in clouds which are carried by the wind (c).

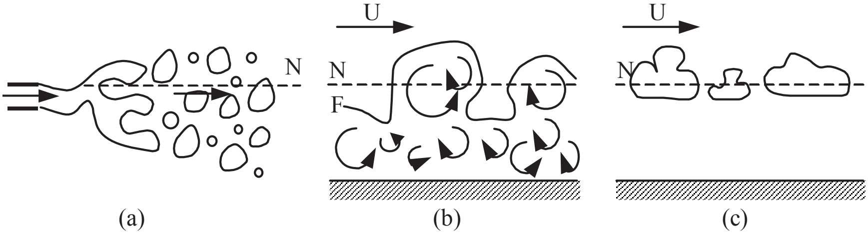

Figure 7.16. *Intermittency phenomena in: (a) a liquid jet; (b) in the frontier zone of a turbulent boundary layer; (c) in atmospheric flow with cumulus*

Figure 7.17 shows the sonogram of a sound recording of a water flow that issues from a floater tap in which a rapid liquid jet bursts in an air pocket which is trapped at the top of a vertical pipe before filling a reservoir. The impact of the liquid packets on the walls is particularly loud and is translated by dark regions on the sonogram. When the air pocket is purged, all that remains is the sound produced by the turbulence in the water which is much less noisy, and the sonogram is thus a relatively uniform shade of gray.

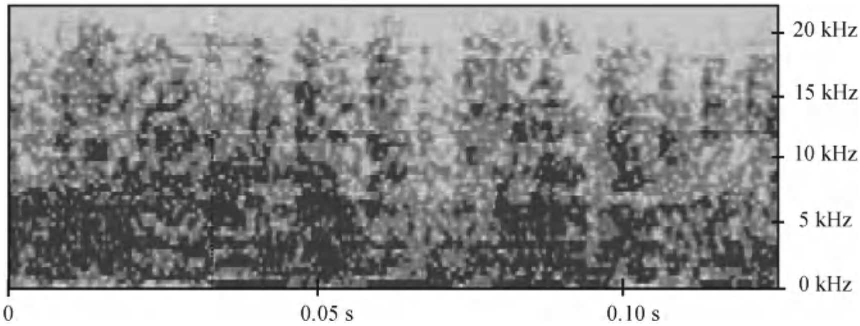

Figure 7.17. *Sonogram of the noise of a flow of water with air pockets downstream a tap*

In general, the emission of sound in a flow is always associated with unsteady structures. This is a frequent nuisance, *easily identified by an attentive ear*, that we seek to eliminate. The analysis of flow noise with the signal processing techniques discussed above, and in particular time-frequency analysis, often allows the identification and improvement of noisy regions of a flow. These procedures merit broader use, but they require relatively complete knowledge in the domains of fluid mechanics and acoustics.

7.3.6. *Discretized signals*

7.3.6.1. *Evolution of techniques*

While physical quantities are represented by real or complex numbers with continuous values, the results of experiments are truncated decimal approximations, which are subject to the uncertainties of the experiment. Since the earliest scientific developments, it is the experimenter himself who discretizes the values of the measurements he performs. In the same way numerical applications of the equations, which result from exact or empirical theories, can only be used with discretized, truncated numerical values. The introduction of computer technology has completely modified both experimental techniques and the practice of system modeling.

The evolution of experimental and measurement methods has been marked by two types of devices, depending on the nature of the electronic treatment:

– *analog devices* in which the input signal (voltage or electric current) is a continuous function of time which is then transformed by electronic circuits into an electrical output signal, which is itself a continuous function of time; in these devices, mathematical discontinuities don’t really exist. These measurement devices generally have a relative accuracy associated with the scale chosen for the measurement;

– *digital devices,* which use information technology, and operate on signals constituted of a finite sequence of truncated numerical values (which are encoded in a number of “bits” or “bytes”). Here, once again, digitization errors (quantification) are in direct relation with the maximum value chosen for the representation of numbers (12 or 16 bits for example).

The interest of digital techniques lies in the ease of adaptation and the near infinite possibilities: any modification in the treatment of a signal is performed through the modification of a computer program, on the contrary, the modification of an electric analog circuit can only be achieved through modifying the physical properties of the components, some of which may be variable by construction anyway.

The techniques used in practice are often mixed for different reasons:

– as the initial signals are often continuous, it is necessary to construct a table of numerical values which we wish to retain by sampling (this is the job of the analog-to-digital converter);

– the signals which result from a treatment are often destined to be used to act on analog materials which require a certain power (power amplifiers, actuators, loudspeakers, etc.); the discontinuities of the numerical values must therefore be interpolated by means of a digital-to-analog converter which delivers a continuous electrical signal.

7.3.6.2. *Sampling of continuous signals*

The representation of continuous signals by tables of numerical values is only useful if the initial signal can be reconstructed exactly without any loss of information. We replace the continuous signal by a sequence of values that are generally recorded at regular intervals. It is clear that the signal should not have varied too much between two successive values, such that its reconstruction in a continuous form can be performed with suitable precision.

The temporal representation of a signal $s(t)$ amounts to its decomposition into a basis of Dirac functions according to the following property:

$$f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau$$

Sampling a signal $s(t)$ comes down to multiplying it by the sampling function $\text{Ш}_T(t)$ (also known as Dirac comb or impulse train) defined by the sum of Dirac impulses positioned at the points kT (k integer):

$$\text{Ш}_T(kT) = \sum_{k=-\infty}^{k=+\infty} \delta(t\text{-}kT),\ k \text{ integer}$$

We thus have:

$$s(kT) = \sum_{k=-\infty}^{k=+\infty} \int_{-\infty}^{\infty} \delta\left(t\text{�}kT\right) s(t)dt,\ k \text{ integer}$$

The sampling of temporal signals is a complex operation ([BAH 01], [BEL 00], [CAS 06], [HIG 93], [MAD 98], [MAX 96]).

7.3.6.3. *Fourier transform of a discretized signal*

The sampling function is a periodic function of period *T*. Its development as a Fourier series can be immediately written by calculating the Fourier coefficients (section 7.3.3.1) of the Dirac function on interval [-T/2, T/2]:

$$\text{Ш}_T(t) = \sum_{k=-\infty}^{k=+\infty} \delta\left(t\text{-}kT\right) = \frac{1}{T} \sum_{k=-\infty}^{k=+\infty} e^{-2\pi j \frac{t}{T} k} \qquad k \text{ integer}$$

Using the previous second expression of function $\text{Ш}_T(t)$ and results from section 7.3.4.2.1, it is easy to obtain:

$$F_{\text{Ш}_T}(\nu) = \frac{1}{f_0} \sum_{k=-\infty}^{k=+\infty} \delta(\nu - kf_0), \quad k \text{ integer}, \quad f_0 = \frac{1}{T}$$

For a discretized signal of finite duration, which can be written as:

$$s(kT) = \sum_{k=-N}^{k=+N} \int_{-\infty}^{\infty} \delta\left(t\text{-}kT\right) s(t) dt \text{ , } k \text{ integer}$$

the Fourier transform is equal to:

$$\begin{aligned} F_{s.\text{Ш}_T} &= \sum_{k=-\infty}^{k=+\infty} F_s * F(e^{-2\pi j \frac{t}{T} k}) = \frac{1}{f_0} \sum_{k=-\infty}^{k=+\infty} F_s\left(\nu\right) * \delta\left(\nu \text{ - } kf_0\right) \\ &= \frac{1}{f_0} \sum_{k=-\infty}^{k=+\infty} F_s\left(\nu \text{ - } kf_0\right) \qquad k \text{ integer} \end{aligned}$$

The Fourier transform of the sampled signal is the transform of the signal *s(t)* which is completed by a *periodization on the frequency axis*, with a period equal to $f_0 = 1/T$.

The other integral transformations studied earlier can also be applied to discretized signals. The essential physical notions already discussed for these apply equally to the discretized signals; of course it is necessary to take account of the numerical aspects associated with the discretization ([BEE 03], [BEL 02], [FLA 98], [JER 92], [PRI 91]).

7.3.6.4. *Errors associated with digital techniques*

The digitization of analog signals can be performed by means of devices comprising a fixed number of digits (8, 12, 16, etc., binary digits). A *quantification* (or rounding) error results, which is all the greater as the number of significant digits is small. For example, the representation on 12 bits allows the representation of 2^{12} (=4,096) numerical values; thus a relative precision far greater than 1/1,000 cannot be hoped for, and this is on condition that the 12 bits are nearly all used. For example, if the 12 bits allow the representation of numerical values between 0 and 100, the absolute quantification error of a given number is in the order of 100.2^{-12}, i.e. about 0.03; only 5 and 6 bits will be used to represent the number 1.378, corresponding to a relative accuracy of the order of 2%. The preceding error is similar to the reading error of measurements on analog apparatus, for which it is necessary that the quantity to be measured is near of the full scale used.

The *errors associated with discretization* are greater in proportion to the evolutionary rapidity of the phenomena as in impulse signals. In practice, an impulse is applied over a certain duration; if this duration is in the order of a very small number of temporal points, the measurement error may be very great. This loss moreover corresponds to *irreversible information loss.*

In section 7.3.6.3, we saw that signal discretization leads to the periodization of the Fourier transform of continuous signal. In order to reconstruct this initial signal, it is thus necessary to conserve one period of the spectrum, and this assumes that the Fourier transforms of each period do not overlap: if N is the sampling frequency, the signal spectrum should not contain frequencies greater than N*/2*. This result constitutes *Shannon's theorem.*[3] In these conditions, by taking the inverse Fourier transform of the central period of the spectrum, we exactly reconstruct the initial discretized signal without any information loss.

The result of this, given that any signal will contain parasite noise of various origins and different characteristics, is that it is necessary to suppress the higher frequencies vis-à-vis Shannon's theorem. This is the role of low-pass filters positioned before the analog-to-digital converter; these filters are called anti-aliasing filters (as they prevent energy at frequencies higher than the Shannon limit from appearing in the lower frequencies, this being due to the partial overlapping of two successive periods of periodized spectrum). We cannot discuss these problems in greater detail; the interested reader should refer to texts on signal processing.

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[3] A usual and equivalent statement of Shannon's theorem is that a signal comprising only frequencies up to *p* Hz must be sampled at 2*p* Hz at least, so as not lose any information.

7.3.6.5. *Discrete transformations of discretized signals*

The transforms described above operate on signals with continuous or discrete values, but they provide continuous values. These need to be discretized in order that they can be stored or transformed numerically. In most practical cases, numerical calculations of integral transforms are performed by means of algorithms applied to discretized signals that directly provide discrete values.

As a result of binary representation of numerical values, transform algorithms can be greatly simplified for ensembles of values whose number is a power of two. The *fast Fourier transform* is performed by means of a fast algorithm proposed by Tuckey. Any signal *s(t)*, which is discretized into 2^n values, will have a discrete Fourier transform that contains as many values in the frequency domain as the discretized temporal signal contained. There has thus not been any data reduction compared with the initial signal. However, for a real signal, on account of the properties of evenness, the number of significant values is equal to 2^{n-1}.

Furthermore, the discretization of an unknown signal can create difficulties. Consider a temporal signal composed of *p* narrow impulses. Its analysis and representation are clearly quite simple in the continuous temporal domain. However, the discretization obtained by temporal sampling will constitute a rather poor representation of the signal *s(t) if the impulses do not occur at the measurement instants*. On the other hand, the Fourier transform of the discretized signal is a sum of complex exponentials the properties of which are not immediately obvious.

This difficulty is also encountered in the frequency domain for the identification of one or several isolated peaks (harmonic oscillations) if the central frequencies of the peaks are not equal to one of the frequencies of the sampled spectrum obtained. It is for this reason that if we want an accurate measure of a spectrum's peak amplitude (a sound spectrum for example), it is important to choose window functions which broaden the peak, while at the same time suppressing the secondary lobes of the Fourier transform of the window ([BEL 98], [HIG 93], [MAD 98]).

7.3.7. *Data compression*

7.3.7.1. *Introduction*

Files containing raw numerical values can be extremely large, even after transformation. Images for example, in comparison with sound and text, consume an enormous quantity of data when digitized. Experimental modern measurements also produce large quantities of numerical data. It is thus necessary to reduce the size of the files for storage by exploiting the power of processors, and it would not be appropriate to discuss signal representation without recalling certain specific procedures of *data compression.* This is achieved by exploiting *particular properties of data* that result from arithmetic or numerical properties of the signal studied or of one of its transforms.

Data compression procedures may or may not be accompanied by data loss. Their efficiency can be evaluated by means of a degree of compression (volume of the compressed file/volume of initial file). These are classed according to two main categories depending on the arithmetic or analytic nature of the characteristics that are exploited.

7.3.7.2. *Arithmetic methods of data compression*

A sequence of numerical values often presents particular combinational properties: for example, real numbers have a periodic decimal development. Any data file comprises a sequence of numerical values, which is not the result of pure chance, as they always represent some specific information. This implies that the file belongs to a certain class of files and it therefore possesses certain arithmetic particularities, which can be demonstrated. For example, an image is not made up of numerical values chosen at random – it includes certain structures (contours, color-ranges, etc.) which correspond to the images it represents. An image constituted of random pixels will most likely not represent anything at all. Similarly, a sequence of letters chosen randomly has a very small probability of representing a text. To clarify this, consider the following: the number of permutations, without repetition, of the 26 letters of the alphabet (equal to 26!) is of the order of 4×10^{26}. If we admit repetition, the number of combinations is much greater. It is clear that this number is far greater than the number of meaningful sentences comprising only 26 letters that it is possible to write in a given language. Text files thus form a particular class of files.

The combinatory particularities of a class of files can be used in order to define appropriately adapted representation conventions. It is therefore possible to estimate that such an encoding of a file will allow the reduction of its size. Of course, the reverse decoding operation must be possible without any ambiguity.

Arithmetic methods of compression involve searching for numerical structures in a sequence of values of a file and exploiting the multiplicity of their occurrence. Let us consider three examples:

– *the method of repetition* involves “factorizing” the sequences which are repeated one after another; for example, a sequence of 30 identical values (96 for example) for consecutive pixels of an image will be denoted 96*30 instead of (96, 96, 96, etc.); a suitable convention for writing the file is obviously necessary;

– *the dictionary method* involves recognizing the structure of values which are repeated and are thus indexed; the name “dictionary” derives from the fact that a text is not a random sequence of letters, but rather a sequence of words which we sort and store in a dictionary, indexing them by order of their appearance in the dictionary. The compressed file is thus the dictionary, and the text is encoded with the order numbers of the words used. While the words of a text are easy to read, numerical structures (sequences of similar bytes) of an image file must be sought with a suitable algorithm. It is also possible to establish a partial dictionary by not encoding isolated values, which are not recognized as part of a repeated structure. This method is applicable to all types of files.

– *the Huffman method*, which is entirely statistical, is based on the fact that in language, all the letters are not used with the same frequency. In French, for example, the probability of encountering the vowel "a" is 17.3%, whereas that of encountering the consonant "w" is 0.05%. Now, letters are encoded on 8 bits (ASCII characters). In general, a byte file contains variable occurrences for the different bytes which are possible, while the Huffman method consists of encoding the bytes encountered in a source file with variable binary lengths such that the most frequent data are encoded on a very short binary length, rare bytes being represented by a binary length which is greater than the average. The few bits lost on the rare bytes are quickly recovered for the more frequent bytes ("a" is 346 times more frequent than "w"). As the number of bits encoded is now variable, it is necessary to establish a criterion that allows us to distinguish between successive encoded elements. The encoded file will finally comprise the used source code file and the encoded message. Its establishment requires the implementation of a suitable algorithm; data reading, in other words the reconstruction of the initial file, is performed by means of a decoding algorithm (decompression).

The Huffman method is applicable to all kinds of file (text, image, music, etc.) since it can establish a table of byte frequencies when the file is read. Despite its age (it dates from 1952), this method remains competitive, as research has improved its capacity to compress data.

All of the above methods of data compression are *no-loss* methods, as it is possible to completely reconstruct the initial file. *They do not use any underlying "physical" property of the file structure*, the algorithms detecting the structure of repetitions in a purely logical manner. A suitable data compression code leads to a reduction of the file volume. Its efficiency is related to the degree of repetition of the file entities (bits, bytes, structures, etc.).

7.3.7.3. *Analytical methods*

Let us note first of all that the representation of a signal by an analytical formula can be considered as signal encoding, its decoding being performed by numerical calculation with formulae used for analytical representation. However, in most situations, the signal is constituted by a sequence of numerical values that cannot be represented by an analytical formula.

Any kind of understanding of the physical structure of the phenomena represented can allow the identification of numerical particularities. Transformations, which are *adapted to the physical processes encountered*, will reduce the amount of data significant for that phenomenon: for example, the Fourier transform of a harmonic function only gives non-zero values for the amplitude and phase for one frequency. The broadening due to the finite length of the observation window will increase the number of values in the vicinity of the signal frequency. In these conditions it suffices to retain the data with sufficiently high values, by adopting the convention that frequencies which are not retained have zero amplitude. This manner of writing the results of the Fourier transform constitutes *data compression*.

MP3 (Mpeg Audio Layer 3) encoding for sound files is based on a psychoacoustic model and uses 10 to 12 times less data than a standard sound file. Recall that one second of stereo sound on a CD comprises 2*44,200 values (the sensitivity of the ear is 0-20 kHz, the sampling frequency which must be respected (Shannon) is greater than 40 kHz). This encoding is a little destructive, but this loss is nearly imperceptible to the human ear:

– we eliminate the sounds of a sequence which will not be perceived by the ear, the frequencies being close to those of the dominant sound whose energy is much greater (“masking effect”);

– we also eliminate, by means of Fletscher and Munsen curves which determine the perceptual limits of the ear, all those sounds for which the level and frequencies are weaker than the values of these curves: thus for an average level, the sensitivity of the ear is maximum between 1,000 and 5,000 Hz; it decays strongly below 300-400 Hz, and it also decays from 8,000 Hz. Sounds outside a certain range are thus eliminated;

– in the case of stereo recording, we compress the data in mono in the low frequencies, the difference in phase of the low frequency sound between the two ears being so small as to be imperceptible to the listener.

– finally we use Huffman compression, without loss, which associates an encoding which is shorter in proportion to the frequency of data sequences. The single use of this method provides 20-25% of the compression.

– the source signal is also decomposed into sub-bands during the Fourier transform. The psychoacoustic model is used in these sub-bands which are quantified by thresholds. The assembly of the sub-bands is then realized.

Data compression is thus based on quite general properties [SAY 00]) either due to purely arithmetic statements, or to some general structural property, such as the oscillatory phenomena for music, speech data, or the specific occurrence of letters in a text. Obviously, a Fourier transformation will not have any value for compression of text file data.

7.4. Choice of representation and obtaining pertinent information

7.4.1. *Introduction*

Knowledge is structured by successive levels of systems which are more or less interlinked, and to each of which there corresponds a category of properties, and whose properties are attributed at each level in order that they can be synthesized at the next level. Epistemologically speaking, these are elaborated by means of a "system-analysis" methodology, which is discussed in Chapter 8. It begins with formal logic and extends as far as the science of living systems. The idea of pertinent information is thus difficult to define in an absolute sense, as it depends on the context and the objective of the analysis performed. We will position ourselves here at the level of the physics of systems of continuous media in flows, acoustic and transfer phenomena included.

The representation of physical phenomena, arising from simulation or experimentation, now consists of tables of numerical values that contain *all the corresponding information*. Let us assume for example that we have a recording of the instantaneous velocity modulus of a turbulent flow at 20 points and with 400 measurements per second for 1 minute. We obtain an ensemble of numerical values containing 40 x 20 x 60 = 48,000 values. The reading of these values is of little interest to the human mind. At most, we might note that the velocity varies over certain intervals, and that these values seem to be associated here or there as broad groups. The graphical representation of the velocity values as a function of time provides curves whose reading may provide some further indications to the trained eye, but the information content thus recognized will appear rather small. For at least a century, we have observed turbulent fluctuations, and despite this, the science of turbulence remains incomplete.

The choice of representation obviously depends on the objective that is in view. The *detailed reproduction of a signal* can be realized from a numerical recording, eventually using a lossless compression of the data. We will tolerate some small losses for the approximate reproduction (MP3 procedures for music, JPEG for images, etc.). We may eventually choose a variable step size in the discretization so as to suitably represent the rapid variations of phenomena in different zones (in certain compression procedures, or in finite element calculation).

A signal may similarly be studied with a view of its *analysis* in order to recognize certain global properties that are represented by synthetic information. An integral transform does not generally lead to an economy in the numerical representation of the initial signal; this is only possible by means of *specific transforms which allow an adapted representation* which simplifies the presentation of information (Fourier transforms, etc.). An *integral transform is only of interest in cases where the physical properties of the signal are analogous to its kernel* (section 7.3.4). An adapted representation of a signal thus results from a knowledge of its properties and, as this is often achieved thanks to a suitable form of representation, we find ourselves confronted by certain difficulties. Hence, we have to find the answer to a question which is always phrased in a manner corresponding to our known concepts. If the answer needs other concepts unknown to us, it will be very difficult to find in what direction the solution is: the human mind is progressing from a known place to a nearly known place. In other words, w*e should never forget that we will only find what we set out to find, and we only ever seek with ideas which we know!* Could Galileo imagine or ever understand the laws of mechanics in the 17th century without the knowledge of the derivative?

We will study signal-analysis problems and information processing in the case of audible sound signals, which are less complex than turbulence, at least in appearance.

7.4.2. ***An example: analysis of sound***

7.4.2.1. *Introduction*

Sound amounts to pressure fluctuations that propagate in a fluid medium. Their origin may be due to internal fluid mechanisms (aerodynamic sound for example), or to the vibration of solid surfaces which are in contact with the fluid, or to the interaction of fluid with solid surfaces. The structure of a sound field depends both on the production mechanism and the conditions of propagation. Sound signals are complex and very often contain important information whose scientific analysis can be difficult. We will here limit our discussion to audible sounds whose spectral content is contained between 20 Hz and 20 kHz.

The complexity of an audible sound signal results from Shannon’s theorem, which requires at least 40,000 values per second in order for the sound to be characterized. The reader can calculate the (very large) number of sound sequences that are possible in one second by assuming that each numerical value is encoded on 8 bits.

Audible sounds have the particularity of sending the human brain a signal with which a sensation is associated; this is then compared to memories and eventually interpreted. It is clear that these mechanisms of recognition are the consequence of training with respect to the usual sounds of our environment, which constitutes only a very small part of the ensemble of possible sound sequences. It is precisely because we already have an intuitive understanding of these sounds that we will take them as an example for discussing signal analysis and synthesis methodologies.

The interpretation of audible sounds by the brain concerns the domain of *psychoacoustics*. We will not discuss the structure of the human hearing system. For questions associated with acoustic signals, it is sufficient to note that *the response of the ear is non-linear,* sensations being moreover recorded on a logarithmic scale. Now any action of a linear filter on a linear combination of harmonic signals leads to another linear combination of harmonic signals of the same frequencies. On the other hand, *any non-linear operation on a linear combination of harmonic signals of discrete frequencies will create new frequency components* (harmonics and sub-harmonics); for example, consider the signal $s(t)$ which is the sum of harmonic signals of frequencies $2n$ and $3n$ ($\omega = 2\pi\ n$):

$$s(t) = a(\cos 2\omega\ t + \cos 3\omega\ t)$$

Its square s^2 can be written:

$$s^2(t) = a^2\left[1 + \cos\omega\ t + \frac{1}{2}(\cos 4\omega\ t + \cos 6\omega\ t) + \cos 5\omega\ t\right]$$

The signal thus obtained does not contain the initial frequencies $2n$ and $3n$ which are replaced by the harmonics $4n$, $5n$, $6n$, and the sub-harmonics of the frequency n. However, the spectral composition in relative value is here independent of the amplitude a. In general this is not the case, and the global amplitude variation of a sound signal modifies the spectral content as soon as the non-linear operation is not simple. The reader can easily verify that the spectral composition of the signal $s^2 + \alpha\ s$ depends on the constant value α.

This leads to the *human ear hearing frequencies that do not have any physical existence*: this is the question of the "missing fundamental", the ear hearing the previous sub-harmonics ([BER 90], [KIN 82]). Navier-Stokes equations being non-linear, turbulence evolution is a strongly non-linear dynamic process in which frequencies associated with turbulent fluctuations are increasing (turbulent energy cascade [MAT 00]).

7.4.2.2. *Hearing and time-frequency analysis*

As the ear has a spectral response that is rather logarithmic for sound levels, it is not a good instrument for evaluating the spectral content of a sound. The spectral content of sounds perceived by the human ear does not vary significantly if we modify the sound level; this can be easily verified by listening to music on a high-quality system (except at very low sound level). However, the elementary calculations of the last section show how the ear can hear sounds whose frequencies do not exist in the sound spectrum. This phenomenon has been known for a long time in music, where suitable harmonic combinations can lead to the belief in the presence of a fundamental frequency that does not exist. The interested reader can see texts treating psychoacoustics and musical acoustics. Fourier analysis of sound differs from the perception that the ear can have of sound; in what follows we will leave aside the problem of sound sensation, and we will limit our results to the physical analysis performed using time-frequency techniques.

7.4.2.3. *“Natural” sounds*

We will designate by the term “natural” sounds all those sounds that are generated mechanically in our environment. These are produced by vibration of solid bodies (plates, shells, membranes, etc.), by oscillations in fluid velocity (musical wind instruments, speech, the wind, etc.) or by interactions between solids and fluids (wavemakers, vibrating walls under the influence of a flow, etc.). These sounds result from the properties of the movement of fluids and solids. There is incidentally no physical difference between musical sounds (which are in principle agreeable to the ear) and industrial sounds that are often a nuisance. The two categories of sound are produced by means of impacts (percussion instruments, a hammer, etc.) friction (violin, squeaking of brakes, etc.), by airflows (flute, pipes which blow, speech, etc.). These properties of natural or forced vibration are quite well known in many relatively un-complicated instances.

“Natural” sounds are thus particular categories of sound signals whose function or utility is quite varied:

– familiar sounds are part of the environment and the context of normal life; any modification of these is immediately perceived as new information; the absence of any sound can quickly become oppressive, as we can experience by spending time in an anechoic chamber;

– suitable musical sounds have a relaxing and agreeable effect, which may vary depending on the individual;

– sounds emitted by a sound source allow the identification of the position and nature (at least partially) of its source; an anomaly in the content of a sound can serve to identify an anomaly in the functioning of the source (for example, listening to the sound of the engine of a car or of an industrial process): we have here a *diagnostic function* which is beginning to be used in certain software of preventive maintenance ([BOU 98], [WAN 06]);

– finally speech allows the transmission of information between individuals in the form of language.

The scientific problem which is posed is that of *defining a methodology* for obtaining the numerical characteristics corresponding to sound signals. The understanding of mechanisms leading to these characteristics (phase of analysis) will allow the subsequent implementation of devices that permit the synthesis and control of the natural signals (music, speech, etc.). We will examine here the case of musical sounds, which are more "standardized" and better understood than industrial sounds, before describing the kinds of field quantities encountered in flows.

7.4.3. *Analysis of musical signals*

7.4.3.1. *Introduction*

Consider the example of a piece of music one minute long; we know that the complete reproduction of the *sound signal perceived by the ear* of a listener needs to be sampled at least 40 kHz, in order that no information be lost (for high-definition listening for example). A minute of music is represented by at least 40,000 x 60 = 240,000 numerical values. It is this ensemble of values that we record on to an ordinary audio CD. In fact, we record two channels in stereo (one for each ear) and we sample at 44.2 kHz instead of the 40 kHz necessary according to Shannon's theorem. From the scientific point of view, such a numerical table can be considered as a complete musical score.

The listener or the musician does not perceive this table as such, but he feels the impressions, which do not really correspond to such a quantity of information. The reaction time of the brain is much less than the sampling time of a piece of music, as a consequence of which a certain number of modifications to the numerical table of values can be made without the listener noticing. In fact, the listening apparatus is a natural receiver which is adapted to the reception of ambient sounds: acoustic vibrations are perceived by the organs of the inner ear, which transforms these into electrical signals that are transmitted to the brain, via an analog measurement "device" (or rather "evaluation" device). The brain performs an analysis of the sound signal and deduces information (origin and causes of the sound, etc.); this is an expertise which is based on pre-training (memories of similar sounds already encountered) and the brain tries then to identify the *global sound structures* in a complex ensemble and to compare these to "known" sounds. We thus recognize the music of a piano, the firing of a cannon, the sound of a train, etc. We will now describe the physical mechanisms of musical sounds and the analysis methods which allow us to characterize these.

7.4.3.2. *Instruments and sound structures*

In fact, each sound structure comes from an individual mechanical system, which we call an instrument in the domain of music. The production of “natural” sounds is a complex phenomenon, which assumes the creation of a vibrational energy and its transformation into sounds that propagate through the atmosphere. It is interesting to study these in order to better understand the links that exist between the physical mechanisms to be analyzed and the analysis tools that need to be implemented.

Since the origin of humanity, sounds have been emitted by mechanical vibrations produced by bodies in motion. Musical instruments, the human voice, natural sounds due to the wind or the flow of water, etc., each constitute what can be termed a *mechanical musical instrument* in which a form of a more or less continuous mechanical energy is transformed into sound energy.

Sound is thus a “by-product” of a mechanical system in which occurs a transformation of mechanical energy into vibrations, often highly complex, and which are localized in a region of restricted dimensions that we might designate as a *primary acoustic source*. It is for example the contact zone between a solid and a body which strikes it, the flow region behind an obstacle where vortices are generated, the contact zone between a wheel and the road or a rail, the contact zone between a bow and the string of a violin (or between a brake pad and disc), etc. The musician acts essentially in this zone by producing an impulse (percussion instruments), a continuous movement or a continuous airflow which produces more or less periodic vibrations (emission of vortices, relaxation oscillations in bowed string instruments, etc.). This primary source often has a highly non-linear behavior which varies in time. It may also be periodic (imbalance in wheel rotation or purr of a transformer for industrial noise, etc.) The oscillatory mechanical energy created is essentially localized here and only a small part of this is transformed into acoustic energy.

Let us now take the example of traditional classical music. The primary acoustic source excites the rest of the musical instrument, which is generally larger, and whose role is to “filter” the excitation, in other words to transform it *without creating additional vibrational energy*. The resonant parts of the instrument are the *apparent acoustic source* for the listener, which can be referred to as *secondary source* (Figure 7.18a). These allow the localized oscillatory energy to be transformed into acoustic energy that propagates through the air (in fact we are dealing here with an impedance adaptation mechanism (see horns in [KIN 82]). Furthermore, we know the importance of certain construction details of a musical instrument for the quality of the sound that is obtained. The essential role of the instrument is to provide a very weakly damped filter which supports oscillations of very small amplitude and the equations for which are thus linear with coefficients which are independent of time, at least as long as the instrument geometry is not changed.

From a mathematical point of view (Chapter 5), the main mechanical energy of a wind instrument is contained in the part of the solution associated with the convective characteristic curve, the *sound corresponding to the propagative characteristic curves.*

Then, the listener listens to these sound vibrations in a given environment, which also possesses particular reverberation properties. Finally, the vibrational characteristics of the energy-creation zone are far from being the same as those of the sounds which we hear: the final sound signal which results from these successive operations, between which there may be retroaction, at least so far as the primary and secondary sources are concerned. More exceptionally, this retroaction may exist between the surrounding environment and the primary source, as seen, for example, in the Larsen effect between a loudspeaker and a microphone connected to the same sound system.

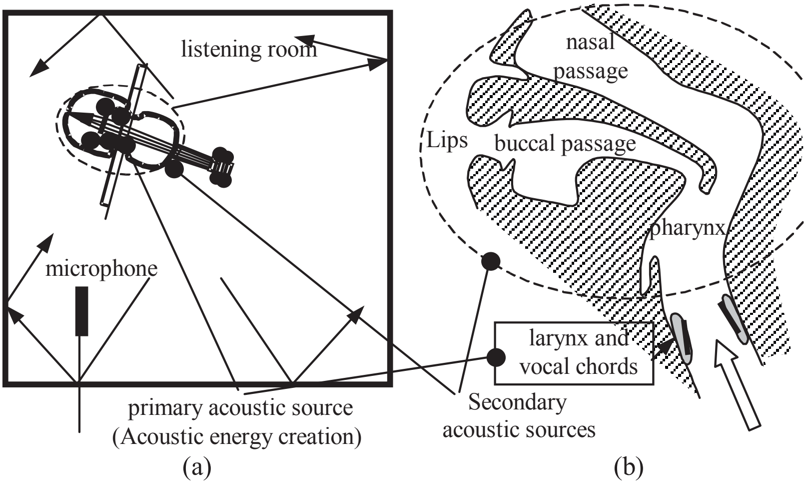

Figure 7.18. *Acoustic mechanisms: (a) emission of the sound of a violin in a room; (b) speech emission*

The human voice is a musical wind instrument (Figure 7.18b). The initial mechanical energy of the airflow coming from the lungs produces an oscillatory energy in the larynx in the vicinity of the vocal cords (the primary acoustic source) which comprise vibrating obstacles. The secondary source in contact with the outside environment is an “acoustic filter” comprising the nasal and buccal passages and the pharynx. Important geometric modulations with slow variations are produced by movement of the lips in particular, and of the tongue, and this constitutes a considerable complication compared with the instrument that produces fixed sounds. The acoustic filter thus realized is not representable by a linear constant-coefficient differential equation system.

To summarize, musical signals are comprised of combinations of *weakly damped sinusoidal functions and by modulations, which are more or less variable at a timescale which is greater than the period of the emitted harmonic signals.* They thus constitute a particular class of intermittent signals which correspond to the eigenfrequencies of the acoustic system.

7.4.3.3. *The importance of harmonic signals*

Musical instruments produce periodic oscillations which are more or less variable (amplitude, phase and frequency variations, etc.). As a periodic function is decomposable in a Fourier series, sinusoidal functions play an essential role in temporal representation. These are also found in different domains of physics (electricity, electromagnetism, optics, etc.). The analysis of periodic and harmonic signals has been used for over 150 years in the domain of physics. However, the study of speech has only become possible with the use of devices which allow a graphical representation of the amplitude of sound vibrations as a function of frequency and of time (the first sonograms appeared before the 1950s).

Let us return to our musical signal, represented by 44,200 numerical values per second. We need a means of easily recognizing functions that are *a priori* not so different from harmonic functions, which are in fact wave-packets. Time-frequency analysis (section 7.3.5) is thus well adapted to this interpretation. As each note of a musical instrument is quite well characterized by its pitch (fundamental frequency) and its harmonic content, it will appear on the sonogram as an ensemble of parallel bands with respect to the time axis. Figure 7.13 shows this, the human voice being a musical instrument, which is slightly complicated by its variable geometry. This representation shows the separation of two phonemes “sh” and “ah”, which is not truly visible in a temporal representation of the corresponding sound.

A musical signal is thus illustrated in the time-frequency domain by geometric structures whose forms are associated with physical characteristics and whose levels are given by a grayscale or a system of colors. In general, these forms vary very little as the note is played (fundamental frequency) or the sound level is changed. We can thus say that this global structure is characteristic of a musical instrument.

7.4.3.4. *Applications to the synthesis of musical signals*

7.4.3.4.1. General principles

The information which characterizes musical signals is encoded in the occidental world by means of musical scores: it is classically characterized by the tempo (duration of a black note), the notes to be played, the names of the instruments used by the musicians, the indicated sound level (forte, piano, etc.) and some indications of temporal variations (crescendo, etc.) and interpretation (glissando, rubato, etc.). The data of the score are used by composers in order to write the music and for the musicians to interpret it; the listeners hardly recognize any more than this. The number of numerical values that can be encoded in the musical score of an orchestra for a second is quite restricted (at most a few dozen).

On the other hand, the preceding time-frequency analysis allows the quantification in a musical sound of:

– the characteristics and the general structure of a musical instrument characterized by its harmonic content (its timbre);

– the note played, characterized by pitch and duration, this being noted explicitly on the musical partition;

– the interpretation of the musician (sound intensity, eventual variations of frequency) indicated more or less completely on the score.

We will now examine how these ideas can be used in order to reconstruct pieces of synthesized music, which amounts to the reconstruction of a musical signal that has been stored in a compressed form. The procedure is here very different from data compression techniques described earlier, as we directly use the structure of the musical sounds, by reproducing in a digitized form the interpretation of a musical score with synthetic instruments (a "virtual" realization). The score and its interpretation are realized in a MIDI file, the instruments being, as in reality, realized independently of the score in distinct analog or digital modules (synthesizers).

7.4.3.4.2. The MIDI system

The MIDI ("musical instrument digital interface") system is essentially a tool for the control and management of information allowing the control of musical instruments such as samplers or synthesizers, which are integrated into computers in the form of cards. This amounts to the transcription into computer language of an ensemble of data which is characteristic of a musical score with its interpretation and diverse additional parameters. The MIDI system also allows the control of mixing consoles, effect processors and recording systems. An international norm defines the MIDI system with certain extensions and variants ([ROT 95]). We will here limit ourselves to a description of the principles.

The MIDI system is a serial control interface that was originally dedicated to musical systems. However, it rapidly went beyond its original vocation and can today be found in the control of a broad range of material that is not only audio, but also dedicated to theatrical lighting. The MIDI protocol integrates numerous control parameters, most of which can be freely manipulated by the user depending on his needs. It is possible to control polyphonic musical instruments in pseudo-real time: transmission times are imperceptible in most cases. It is also possible to address numerous devices using the same MIDI data.

The basic data of the system are messages, which comprise:

– a status (engagement and release of a note, "sustain" pedal, modulation, polyphonic pressure, continuous control, change of program, weak variation of pitch ("pitchbend"), etc.);

– data characterizing the action indicated in the status (note played, amplitude of variation in level of pitch applied by the rowel of pitch variation, speed of key engagement, etc.).

MIDI data are most often elaborated using an electromechanical interface for the inputs, this being a keyboard which looks like that of a musical instrument (piano, accordion, saxophone, etc.), and on which the user plays as if it was a real instrument. The rapidity of the transmission of digitized data allows the control of 16 instruments quasi-simultaneously; we have in fact a single line crossing all of the instruments played, on which the MIDI messages serve as indicators of which device they are to be addressed to; each of these transmits all of the messages and only takes account of the messages with which it is concerned (MIDI channel). The temporal delay, which results from this in-series configuration, is small enough not to be perceived by the listener.

A schematic of the MIDI system is shown in Figure 7.19; the data of a MIDI partition can be written either by playing keyboards 1 or 2 or by writing the MIDI files directly from a computer. In the same manner sound restitution can be obtained directly from the keyboards, or off-line from a MIDI file (.mid file extension mid). The MIDI instructions, which define the sound signals, are then transformed into continuous electrical signals by sound generators (synthesizers).

The reader will very likely find a MIDI file reader in the accessories of his computer (Windows Media Player, etc.), associated with a numerical sound synthesizer. It is easy to verify that the volume of such files is quite small (e.g. 5 kbytes for one minute of music, whereas the same musical piece sampled at 44.2 kHz contains over 200 Mbytes).

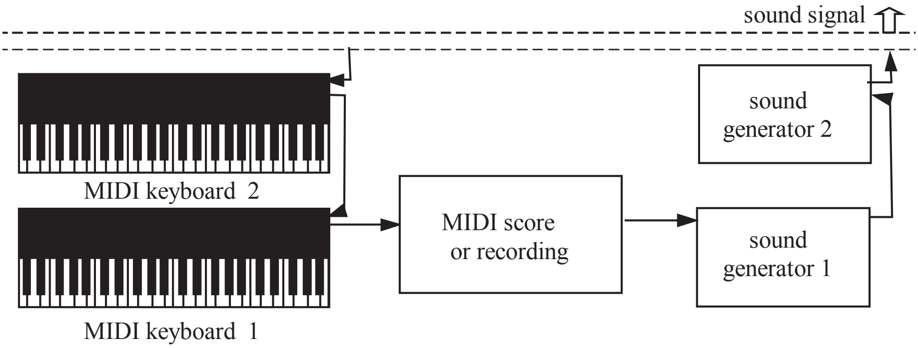

Figure 7.19. *Diagram of links of the MIDI system*

MIDI constitutes an economical means of controlling and storing sound information. We take advantage of this information link for the transmission of other kinds of data (message systems). However, the greatest advantage of the MIDI system remains the ease of correcting and editing MIDI scores, any parameter being individually and quickly modified.

7.4.3.4.3. Synthesizing musical instruments

It now remains to study the analysis and synthesis of musical instruments, which can be analogous or digital. We have seen previously that a musical instrument is characterized by its timbre (harmonic content). We must add to this a temporal variation of sound, which takes account of the evolution of sound amplitude emitted, which depends on the nature of the instrument and its mode of excitation (Figure 7.20): the sound of a wind instrument can last a very long time, whereas the sound emitted by a chord which is struck (piano) or plucked (guitar) is essentially transitional.

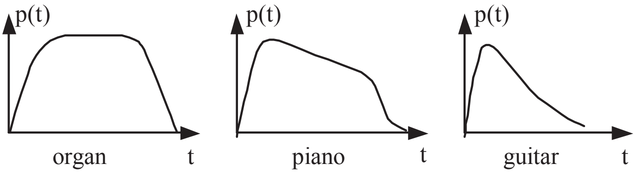

Figure 7.20. *Time evolution (amplitude envelope) of the amplitude of an organ, piano and guitar sound*

In fact, synthesis, which is based on the realization of a fundamental and its harmonics, is not sufficient for the ear to have the impression of a real instrument. Even if the sound obtained seems similar enough to that produced by an instrument, it is not identical to a note played from the real instrument. The reality is more complex than the approach outlined earlier from study of the sonogram. The sound emission is associated with the resonance of the instrument body, but also to certain more complex transitional properties of the sound which depend on certain details of the excitation: in the sound emission of a flute we first hear the sound of blowing. Finally, weak modulations of frequency or amplitude can be produced for diverse reasons related to the sound amplitude, the way of playing, etc. These can be deliberate on the part of the musician (player or composer).

Let us quickly describe the parameters of the usual synthesizer. We define the amplitude envelope by a small number of values indicating the durations T_1 and T_2 of the attack sound, the duration T_m of the sound established if it exists, and two parameters T_3 and T_4 for the duration of the extinction of the sound; we associate these values with an amplitude curve, which is piece-wise linear in practice.

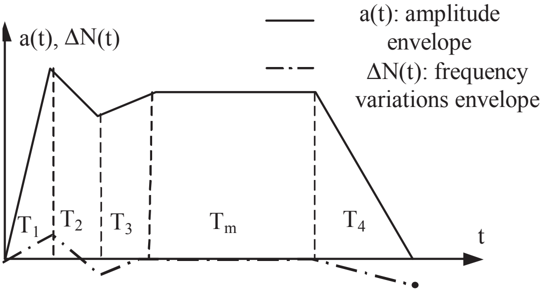

Figure 7.21. *Examples of amplitude and frequency variation envelopes*

These parameters allow the representation of characteristics of two principal kinds of sound which can be excited permanently (wind instruments, string instruments: flute, organ, trumpet, violin, etc.) or by impulse (percussion instruments or instruments with plucked strings: drum, piano, guitar, xylophone, etc.). These also allow the characterization of transitional regimes.

It then remains to realize the spectral sound content sought. For the established regime, the harmonic content can be obtained by different means: additive synthesis using harmonics: superposition of signals which are rich in harmonics (triangle, etc.) which we may eventually filter in order to only conserve the first harmonics, subtractive or multiplicative synthesis, frequency modulation, etc. We can introduce temporal variations of the harmonic or amplitude content (modulation) associated with the envelope parameters and diverse effects. We will note that these are distinct from variations related to the preceding envelope curves. We can also add recorded sounds that are difficult to synthesize.

Another technique consists of recording the sound of an instrument and eventually modifying it by filtering in order to improve it. However, the variation in pitch of the note by the MIDI command can give imperfect results, as the sound of the real instrument can vary during the duration of the note.

Finally, we obtain reasonable results for instruments of fixed geometry. The result is less satisfactory for instruments where it is up to the musician to generate the note (violin, for example), as the sound of the violin is harmonious only thanks to small variations in frequency due to the musician, and that depend on the piece played. This effect cannot be accounted for in the sound generator; it can theoretically be included in the MIDI file through the introduction of a suitable controller, but this risks notably increasing the volume of data.

7.4.3.4.4. Regarding the musical sound structures

Synthesizers and sound generators can also create new sounds by manipulating the system parameters. We will note that variety music contains many synthesized sounds. However, this synthesized music remains in the context of music, which is measured and based on the equal temperament.

The preceding technologies do not allow the exploration of the vast domain of electronic music that is different from the preceding context, being based on sound structures which are not related to the harmonics of a fundamental frequency or the modal frequencies. From this perspective, the first stage of research is to define agreeable sounds, which can allow us to make the distinction between music and noise. The reader can easily imagine the difficulties which may be encountered by the composers of *electronic music* whose objective is to generate new sound structures which do not result from the resonant properties of vibrating strings or resonant cavities. Regardless of whether their sound is agreeable or not, it should be possible to characterize them using a small number of parameters, in order to define a musical notation of a new kind which would allow their representation by means of a musical score. We are here dealing with a relatively unexplored domain, which is well beyond the scope of this book. It is worth noting, however, that a first difficulty is to succeed in specifying the domain to be explored and to find a concept that can replace the Pythagorean basis of tonal music ([LIC 02]).

In conclusion, the time-frequency analysis concept has allowed significant progress in the analysis of sound signals, in particular musical sounds, and also speech analysis, recognition and synthesis, topics we could not discuss here.

7.4.4. *Signal analysis in aero-energetics*

7.4.4.1. *Introduction*

We have examined (section 7.2) the different ways in which signals can be represented and analyzed, these two notions being related as we have seen. An exhaustive treatment of this subject is beyond the scope of this book. Many other analysis and extraction procedures also exist: in particular, filtering procedures consist of the extraction or suppression of certain components of a signal. For example, we can extract a coherent signal from a random background noise; these procedures are largely used, but it is always necessary to characterize what it is we are looking for. The preceding example of acoustic signals has shown the links that exist between physical analysis and signal processing techniques. However, the local measurement of quantities is the result of modifications of the fluid medium on the ensemble of its characteristic curves or surfaces (Chapter 5) which move with the matter, or propagate “acoustically”.

7.4.4.2. *Effects of turbulence*

The solutions of the equations of fluid mechanics are very often unstable and they present a chaotic aspect with random fluctuations (known as “turbulent”) whose properties determine the flow properties, heat and mass transfer ([SCH 99], [YIH 77]). A detailed understanding of turbulence is far from complete. On the other hand, all linear measurements allow the user to obtain mean values of flow quantities (velocity, pressure, etc.). The measurements using devices with a non-linear response need to be treated prior to the application of statistical analysis or integral transforms. The presence of turbulence may considerably hinder instantaneous measurements, as we will see in the following example.

7.4.4.3. *Separation of causes of phenomena in measurements*

Classifying the phenomena encountered in flows and discussing the possible interactions between the various physical mechanisms is not easy (sections 5.3 to 5.6). Obtaining relations or properties depends on the specific dominant phenomena. An important particular case concerns low Mach numbers (velocities less than 100 $m.s^{-1}$ in air) which correspond to air or water flow conditions in many industrial, domestic or environmental problems. All fluids are compressible and we have seen in Chapter 5 that modifications of properties are transferred either by convection or by means of acoustic waves. The orders of magnitude of these two kinds of phenomena are here generally very different: the acoustic component is very often weak compared to the dynamic or thermal effects which result from the boundary conditions. For example, acoustic variations of velocity are less than 1 mm/s, whereas the velocities of the matter are very often between 10 and 100 m/s[1]. The variations of temperature that result from dynamic effects are of the order of 1°C (at about 100 m/s^{-1}), whereas the smallest problem of heat transfer generally involves fluctuations of some about 10°C. For example, a local, instantaneous measurement sensor of velocity or of temperature characterizes phenomena associated with the flow and the heat transfer depending on the circumstances (section 7.2.4).

On the other hand, the *respective contributions* δp_a *of the acoustic modes and* δp_c *of the convective modes to pressure variations* may be of the same order. In effect, we have seen (section 5.3.2) that we have:

– on the acoustic characteristic curve:

$$\delta p_a \approx \rho c \delta u_a$$

– on the convective characteristic curve:

$$\delta p_c + \rho V \delta V = 0$$

The ratio of these pressure fluctuations is thus:

$$\frac{\delta p_a}{\delta p_c} \approx \frac{c}{V}\frac{\delta u_a}{\delta V} = \frac{1}{M}\frac{\delta u_a}{\delta V}$$

As the Mach number is small at low velocity (M ~ 0.05 to 0.15 in ventilation problems for example), the factor $1/M$ is quite large and it compensates the small acoustic velocity fluctuations compared to the turbulent velocity fluctuations. This results in *pressure fluctuations due to local turbulence effects, which are of the same order of magnitude as acoustic pressure fluctuations.*

A pressure probe (microphone) *measures the local pressure independently of its origin* and without knowing on which characteristic curve it is transmitted: it is no longer possible to distinguish acoustic fluctuations from those due to local velocity fluctuations. This problem is well known to those who measure sound: a microphone placed in a flow does not only measure noise. Listening to the pressure signal acquired by a microphone placed in the wind we are immediately aware of a parasite “sound” due to the wind. This “sound” is not of an acoustic nature, but rather a signal that corresponds to local pressure fluctuations associated with the flow created around the microphone. Measurement of sound in the presence of wind requires suitable precautions in order to reduce the impact of parasitic phenomena (for example, “anti-wind” foam covers on microphones).

The separation of turbulence and acoustic pressure components is an important and extremely difficult problem, since it corresponds to Navier-Stokes solutions for which the theory of characteristics curves is in practice not applicable. This separation between acoustic and turbulence signatures can only be made possible if additional information is available: for example the periodic sound due to a propeller can be extracted from a signal containing random turbulence fluctuations.

7.4.4.4. *The study of unsteady flows*

The velocity, pressure or temperature fields associated with unsteady flows are now accessible by means of experimental methods described in section 7.2.6.3; on the other hand, the different methods of time-frequency analysis allows the temporal and spectral characteristics of flows to be identified. These modern methods allow us to consider the vast domain of experimental unsteady fluid mechanics, provided that experiments can be performed in a reproducible manner (which means that different realizations obtained are identical (section 7.2.6.3)). The identification of unsteady flow structures and, in particular, the dynamics of vortex interactions constitutes a domain of study which is relatively unexplored.

We will see that turbulence, being by its nature unsteady and broadly 3D, is largely the result of interactions between inviscid fluid structures (turbulent energy cascade [MAT 00]). The origin of turbulence is furthermore associated with the development of flow instabilities issued from pre-existing perturbations. The elementary mechanisms of these highly unsteady interactions are poorly understood. For these reasons, we have focused on signal-processing and experimental methods that can be used in a broad range of applications where unsteady phenomena are encountered in possibly rotational flows These methods allowed for example the acquisition of experimental data for the problem discussed in section 5.6.2.

Fundamentals of Fluid Mechanics and Transport Phenomena
Jean-Laurent Peube

Chapter 8

Thermal Systems and Models

This chapter is dedicated to the modeling of systems. The perspective developed in automatic control and the corresponding methods are outlined. In addition to the mathematical properties of thermodynamic balance equations and the measurement and signal processing problems, these provide a general framework for the study of the representation and evolution of physical systems. Using some basic heat conduction problems in media at rest, the methods can be easily proved on account of the linear properties that these present.

8.1. Overview of models

8.1.1. *Introduction and definitions*

In Chapters 1 to 4 we studied how to describe a physical system in the context of thermodynamics. The equations for continuous media are hardly useable except for very simple problems and any more complex system can only be represented by making approximations, which it must be possible to evaluate. We will limit ourselves in this chapter to state representation in which a given thermodynamic system is represented as a group of sub-systems in instantaneous equilibrium. This makes it possible to define the *state variables of each sub-system*. The partial differential equations which represent the continuous medium are thus replaced by partial differential equations describing the evolution of a restricted number of variables. The general study of systems is derived from automatic control and signal processing which provide, as in thermodynamics, general principles which cannot be ignored. We will recall here the ideas that are necessary for the establishment of models in fluid dynamics and heat and mass transfer, by limiting ourselves to the study of thermal systems, whose damped character simplifies the discussion.

A system is a material ensemble that exchanges quantities with the exterior. We act on the system(s) by means of inputs, which are imposed variables (a temperature, a mass flow, pressures, etc.); these inputs lead to modifications of the system which are manifest in observed and measured effects, which constitute the outputs of the system. The inputs are therefore *causes* and the outputs are *consequences*.

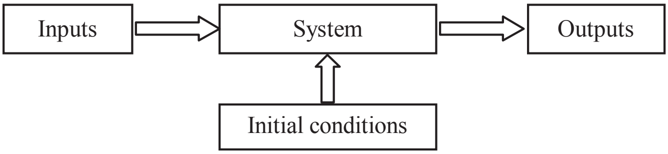

Figure 8.1. *Scheme of a system*

A system is said to be causal (with respect to a variable) if the conditions which determine its state and its outputs for a value of the variables x_0 result from data for values of the variable x which are less than or equal to x_0. This idea of *causality* simply translates the mathematical properties associated with the parabolic or hyperbolic character of the system towards certain variables or groups of variables of the model that is used. While it is obvious for the time dimension (section 1.1.1.3), this idea, which is used in automatic control and signal processing, can also be applied to transport and propagation phenomena (sections 5.3.2 and 5.4.5.4).

The description and knowledge of a system can be achieved by means of a model, which can only be established if the system is *observable*, in other words if we can measure the data necessary for its establishment. This condition obviously depends on the nature of the system and the objective of the modeling.

The control of a system is an action that is often necessary for diverse reasons: control of temperature in a zone, of a flow structure, a sound level, etc. A system is said to be *controllable* if, from any initial state of the system, it is possible to act on certain inputs in order to bring the system to a desired state.

A system is said to be *instantaneous* if its state and its outputs at a given instant depend only on the inputs at that instant. This definition can be applied to any system whose sub-systems are in mutual thermodynamic equilibrium. A system is said to be *dynamic* if it is not instantaneous.

Systems exhibiting hysteresis present the particularity that the stationary solution corresponding to fixed inputs depends on their history (state variables at the initial state and the past evolution of the inputs). Let us take the example of an oscillator comprising a mass P placed between two springs and subjected to a friction force on a fixed wall. If the friction is caused by a fluid, the equilibrium position of the point P is unique, the tension in the springs being therefore equal. In return the existence of a solid source of friction in P means that the final position of the point P also depends on the movement history (Figure 8.2a).

Flow systems with hysteresis are encountered not only with fluids whose behavior laws involve a solid friction (Bingham fluid), but also for Newtonian fluids. This phenomenon is thus associated with the existence of *two possible steady solutions* for a flow with fixed conditions. This circumstance may be due to diverse phenomena such as the feeding of a siphon (Figures 8.2b and 8.2c), reattachment of a flow on a curved surface (Figure 8.2d and Figure 8.2e), the flow of water over a weir depending on whether the sheet of water is aerated (or ventilated) or not (Figure 8.2f and Figure 8.2g), etc. The observed solution can vary depending on the manner we seek to achieve it by.

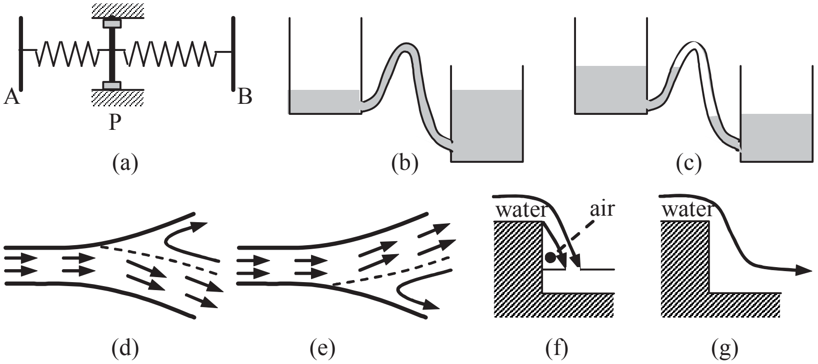

Figure 8.2. *Systems with hysteresis – (a) springs with dry friction; set of 2 tanks with feuded (b) or not (c) siphon; flows in a divergent pipe (d) and (e); weir with aerated water sheet (f) or non-aerated water sheet (g)*

In addition, unknown perturbations (noise) are always present to varying degrees in these inputs and can have diverse effects (“instabilities”). The idea of a *stable system* is difficult to define, because it depends on the nature of the applied perturbations and the properties of the response of the system. Stability is a complex problem and of extreme importance in fluid mechanics ([SCH 99], [YIH 77]).

A system is said to be *stationary* or time-invariant if the outputs corresponding to a given set of inputs do not depend on the instant at which the inputs are applied. The equations of a time-invariant system are not influenced by a change of the time origin. Systems whose structure and physical properties only depend on the state variables are time-invariant. The same is not true of systems whose parameters[1] depend on external conditions which vary with time (for example, the flow of a liquid in the process of polymerization, the action of an obstacle with an incidence angle $\alpha(t)$ which is imposed in accordance with some external law, etc.).

A linear time-invariant system (LTIS), which is also known as a *linear stationary system* (LSS), presents in addition the property of linearity: effects proportional to the causes which produce them. These systems are described by constant-coefficient linear models. Apart from some very specific cases, a system with a non-uniform flow is not linear. The heat transfer of mass by diffusion is often linear, even in flows, provided *the temperature or concentration does not cause the physical properties of the fluid to change*. Coupling between different phenomena and in particular chemical reactions in flows suppresses this linearity property. The reader can verify these properties for the general equations in Chapters 1 to 4.

8.1.2. *Modeling by state representation and choice of variables*

In automatic control, state representation of a discrete system is a model of the form [8.1] which represents its evolution:

$$\frac{dX}{dt} = A.X + B.U \qquad Y = D.X + E.U \tag{8.1}$$

The state vector X is a set of n variables which characterize the state of the system. The state representation is also called the internal representation. This definition is no different in principle to the definition which was given in Chapter 1 where extensive and intensive variables were used as state variables. However, in automatic control the components of the vector X can be *any variables* which characterize the state of the system. The matrix A is a square matrix of rank n. It characterizes the make-up of the system and of the sub-systems which correspond to the choice of the state vector X.

The vector U is the *input vector*; it is of dimension p and corresponds to different actions on the system. It does not in general have the same dimension as the vector X, and the matrix B is a rectangular matrix of dimension $n * p$.

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[1] I.e. the coefficients of operators in the equation; the inputs are not parameters of the system here.

The vector Y is the *output vector*; it is of dimension q and it comprises the different results desired for the problem posed. It does not in general have the same dimension as the vectors X and U. In what follows, we will take $E = 0$ (the system is thus called proper).

The representation of the state of a system is not unique as the variables can be chosen in different ways or can be modified by changes of variable. Furthermore, the components of the state vector can be related when we are limited to particular operation regimes as we will see on numerous occasions in the remainder of this chapter (section 8.3, section 8.5 and section 8.6). However, the systems studied in this work are constituted of a matter, which obeys the laws of thermodynamics. The form of differential system [8.1] is analogous to that of the balance equations, but the coherence with the laws of thermodynamics is only effective because of a suitable choice of state vector components (section 1.2.1).

We also saw in section 5.1.1 that a system of first order differential equations can be written in the form of a scalar differential equation of order n for one scalar variable. This equation can also be written in the form of differential system [8.1], the state vector $X(t)$ being replaced by a state vector having the same number of components, but constituted of a variable $x_1(t)$ and of its $n - 1$ first derivatives. By reconsidering the calculation of section 5.1.1 the reader will see that the inputs obtained for this differential system imply the initial inputs and their $n - 1$ first derivatives. This form of state-representation is known as the form of observability ([DOU 95], [GUP 83], [KUO 02]). We will use this for a particular case in section 8.4.2.2.

For a *non-linear system* the matrices A, B and D can be functions of the state vector X of the system.

For a *system involving an instantaneous response*, the derivative d/dt takes on small values compared with second order terms; the inertia of this system is weak and the extensive quantities which result from the inputs are instantaneously transferred to the sub-systems. The differential character and the idea of the initial state of system [8.1] have disappeared and the representation is reduced to an algebraic system:

$$A.X + B.U = 0, \qquad Y = D.X$$

We note that in the case of an impulse input, the response and the outputs of an instantaneous-response system involve a discontinuity. As nature does not contain discontinuities in such situations, we have in reality a continuous transitional regime involving a singular perturbation (section 6.4.3).

In the case of a *time-invariant system*, the coefficients matrices A, B and D are independent of time, but they may be functions of the state variables: the structure of the system remains unchanged over the course of time. If the system is furthermore linear, they are constant and we have a time-invariant linear system.

For a system made up of *continuous media*, the system of differential equations [8.1] is replaced by partial differential equations. The inputs are either boundary conditions or volume heat source terms. When the physical properties depend on the temperature, these equations are non-linear. On the other hand, if the properties are only a function of the coordinates, we have a *time-invariant linear system*. In the presence of a flow, the temporal derivative $\partial/\partial t$ becomes a material derivative d/dt in the case of a Eulerian representation.

8.1.3. *External representation*

The external representation of a system consists of considering it as a black box that links inputs and outputs. However, the conservation laws of extensive quantities must be considered between the inputs and outputs of a system. The permanent production of mechanical or thermal energy is only possible if energy is provided to the system; the same goes for mass, chemical species in the absence of chemical reactions, etc. The consequences of the second principle of thermodynamics are directly manifest in the constraints upon the entropy of the system studied. In other words, the system can only evolve within the bounds of possible processes. In fact, the coherence between automatic control and thermodynamics is only ensured if the global thermodynamic balances are satisfied for state representation [8.1] of the system.

In practice, a system is often characterized using incomplete data which are obtained by means of indicial experiments: we modify one variable by quickly increasing it by a small amount (Heaviside unit step) and we record the evolution of the outputs. If *the response of these is not instantaneous*, the system undergoes an internal evolution which must be characterized. For example, the progressive response of a system to an instantaneous increase of the mass flow rate or of the input temperature indicates the capacity of the system to accumulate matter or mass in its interior. We must therefore introduce an internal state variable to the system (volume or pressure for a compressible fluid, calorific capacity, etc.) or, equivalently, we must introduce terms involving temporal derivatives in the external representation (section 8.4.2.2).

The formulae used by engineers for over two centuries are *external representations* which correspond to operating conditions of the system, which are more or less well (or badly) defined. Models which use continuous-medium equations are state representations in which the inputs and outputs take the form of simple formulae corresponding to the system studied under particular conditions. These simplified external representations are then often used in more general conditions in which their degree of approximation is difficult to evaluate. For example, in a flow system comprising a pipe of length L and of diameter D, in which *the mass flow* q_m *is given* (as an input), an interesting output is the pressure difference Δp in the flow between the input and output sections. This will depend, for example, on the mass flux, on the viscosity, on the density of the fluid and on the dimension k of the roughness of the pipe. We will therefore have, in non-dimensional form for the pressure Δp :

$$\Delta p = \rho_e \frac{q_m^2}{2S^2} \frac{L}{D} \Lambda\left(\mathrm{Re}, \frac{k}{D}\right) \tag{8.2}$$

where S is the area of a reference cross-section. The non-dimensional coefficient Λ of the driving pressure loss is a function of the dimensionless data (Reynolds number Re and roughness parameter *k/D*) (section 4.6.1.3.4), this function having been determined by semi-empirical considerations and experiments.

Approximate external representations can be used in the study of more complex systems modeled by means of state representations. For example, in models of large dimension in continuous media studied by means of numerical solution of the Navier-Stokes equations by finite difference or finite volume methods, boundary layers in the vicinity of walls are too thin to be discretized (section 6.5.3.3). The friction on the walls is therefore taken as a boundary condition of certain elements and it is expressed by a global relation analogous to [8.2] as a function of the characteristics of the velocity field next to these elements.

8.1.4. *Command models*

The objective of a *knowledge model* is to describe quantitatively an evolving system and its internal and external transfers, with a precise enough description of the physics of processes both sufficient and pertinent.

A *command model* defines actions whose objective is to cause the system to evolve towards a given state. We seek only to represent the evolution of the system in the vicinity of a given state of operation; a linear relation between input and output variations is often sufficient. We must however note that this rather blind manner of proceeding assumes that the *internal structure of the system remains unchanged*.

The pragmatic approach, from the point of view of control and command of a system, often consists of considering only the observable outputs such as the temperature, the pressure, certain concentrations of chemical species, fluxes, etc., at points in an industrial system where measurement is possible. The objective of control is to fix the values of certain output variables (temperature, pressure, etc.) or certain state variables. The commands act on the input variables which may be of the same nature as the output variables. In practice, we content ourselves with the examination of the response of the system to perturbations such that we can estimate the corrections which need to be made. The study of a response curve allows us to define the global order of a system.

8.2. Thermodynamics and state representation

8.2.1. *General principles of modeling*

8.2.1.1. *Introduction*

System [8.1] is of a differential nature with respect to time. We must add an initial condition for the state variable $X(0)$. The procedure in automatic control consists of only considering the state-representation form and "forgetting" the origin or the physical interpretation of this. In practice, the real problem of establishing a model lies in the choice of variables for the state vector X, which must be performed in a manner which conforms to the laws of thermodynamics.

Our study of thermodynamics in Chapters 1 to 4 showed the following general mathematical structure:

– *extensive quantities* satisfying the conservation laws (balance equations) defined for each sub-system regardless of its state of imbalance;

– *intensive quantities* which are only defined for discrete systems in equilibrium, or for continuous media with an assumption of local equilibrium;

– *fluxes of extensive quantities* associated with thermodynamic imbalance characterized by differences or gradients of the intensive quantities;

– *relations between intensive and extensive variables* (equations of state) and phenomenological laws for the thermodynamic fluxes of the extensive quantities.

Modeling a sub-system leads us to define and choose the variables necessary for its dynamic description, and in particular *stocks and fluxes of extensive quantities* at each instant. The discrete equations result from the application of the balance equations to each of the sub-systems. The extensive quantities are additive and they are always defined for any given sub-system.

8.2.1.2. *Intensive variables and thermodynamic fluxes*

We have seen that if the extensive variables of a system or a sub-system are always defined, the same is not necessarily true of the intensive variables that only have meaning if the system is in equilibrium. The complete *thermodynamic representation* consists of representing the system with sufficient details, such that each of the elementary sub-systems is close to a state of instantaneous equilibrium. This condition is fulfilled if the size of the elements is not too great. For a finite sub-system which is never rigorously in equilibrium, we have discussed the way to define the mean intensive quantities (section 1.4.2.5 and section 6.5.2.4).

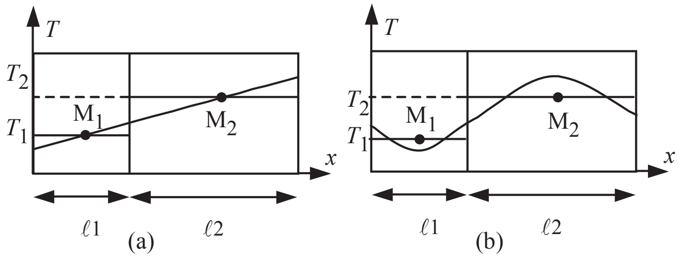


Figure 8.3. *Thermal flux between two neighboring elements*

The behavior laws necessary to close the system of equations of the system create a particular difficulty, as the thermodynamics fluxes which cross a surface are related to the local gradients of the intensive quantities of the local continuous medium. The problem is thus to express these by means of the intensive quantities of the neighboring sub-systems. There is no general solution to this problem. Let us take as an example the thermal resistance between two elements. Consider a fixed 1D continuous medium which is modeled as two blocks E_1 and E_2 which are homogenous and of width ℓ_1 and ℓ_2 (Figure 8.3). For a fixed homogenous medium, the mean temperatures T_1 and T_2 of each element are defined here by taking the mean of the temperatures. Attributing the mean temperature to the central points of the elements may appear natural. However, if the temperature gradient at the interface is equal to the slope of the segment M_1M_2 in the temperature distribution of Figure 8.3a, the same is not necessarily so in the case of Figure 8.3b.

The segmentation of the continuous medium is acceptable for the thermal imbalance of the first case, whereas it is too coarse for that of the second case. We encounter here a usual interpolation problem.

The modeling of thermal fluxes between the two sub-systems requires in addition a *suitable discretization of the intensive quantities between two neighboring sub-systems*. Let us consider two examples:

1) Consider a *1D solid* (Figure 8.4) subjected to a conductive thermal flux and discretized into n elements $\mathrm{E_i}$ of uniform thickness e and temperature T_i. (i=1,2,…,n). The element $\mathrm{E_i}$ is characterized either by its energy Q_i or by its mean temperature T_i:

$$Q_i = \int_{x_n}^{x_{n+1}} \rho C dT = \rho C e T_i$$

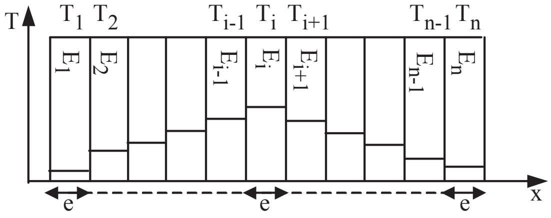


Figure 8.4. *Model of a 1D thermal transfer*

We must now express the thermal fluxes between two neighboring elements. Assuming that the middle of each element is at temperature T_i, we can write the thermal flux $q_{Ti,i+1}$ received by $\mathrm{E_i}$ from $\mathrm{E_{i+1}}$:

$$q_{Ti,i+1} = -\lambda \frac{\left(T_i - T_{i+1}\right)}{e}$$

We note that this expression satisfies the principal of action and reaction ($q_{Ti=1,i} = -q_{Ti+1,i}$) associated with the conservation of energy. If there is no lateral thermal flux, the balance equation can be written for each element $\mathrm{E_i}$:

$$\rho C e \frac{dT_i}{dt} = -\lambda \frac{\left(T_i - T_{i+1}\right)}{e} - \lambda \frac{\left(T_i - T_{i-1}\right)}{e} = -\lambda \frac{\left(2T_i - T_{i+1} - T_{i-1}\right)}{e} \qquad [8.3]$$

2) Consider *the established flow of a fluid in a cylindrical pipe*. We assume, in order to simplify matters, that the flux and the physical properties of the fluid are constant.

We have seen (section 6.5.2.2) that the balance equation of a quantity *g* in a pipe can be written by means of surface integrals on a cross-section and an integral over the contour of this section ([6.82]). We thus consider a cutting out of the pipe into *n* surfaces S_i with an inter-surface spacing of *e* (Figure 8.4). Let $u(t,M)$ and $T(t,M)$ be the velocity and temperature distributions in the cross-section; we define below the mean temperature T_i and the *mean mixing-temperature* T_{mi} for the section S_i:

$$T_i(t) = \int_{Si} T ds; \quad T_{mi}(t) = \frac{1}{q_v} \int_{Si} u\, T ds$$

The energy balance in the pipe can be written ([6.87]):

$$\int_S \frac{\partial(\rho CT)}{\partial t} ds + \frac{\partial}{\partial x}\left(\int_S \rho\, uCTds\right) = \int_S \sigma_T \; ds - \int_C q_{Tp} d\ell$$

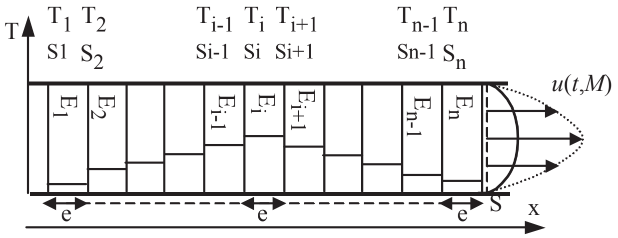


Figure 8.5. *Model of a pipe flow with thermal transfer*

The calculation of the convective thermal flux between two successive surfaces must be realized from upstream to downstream of the flow (section 5.6.1) and the derivative $\partial/\partial x$ should be discretized between the sections S_{i-1} and S_i. Let T_{pi} be the temperature of the wall associated with the surface S_i; the preceding equation [6.87] can thus be written:

$$\rho\, CSe \frac{dT_i}{dt} + \rho\, Cq_v \left(T_{m(i+1)} - T_{m(i-1)}\right) = e\chi\, h\left(T_{pi} - T_{mi}\right) \qquad [8.4]$$

where we denote by χ the perimeter of the cross-section. The exchange coefficient *h* between the fluid and the wall is defined as a function of the mean mixing-temperature T_{mi}; its value depending on the considered problem is obtained from experiments or theoretical evaluation in neighboring problems ([SCH 99], [YIH 77]).

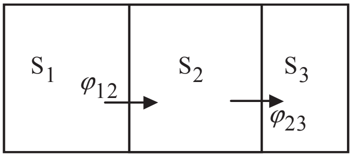
8.2.1.3. *Balance equations for the sub-systems*

The writing of balance equations for a system is not necessarily associated with the cutting out which is chosen in order to model the *n* disconnected sub-systems. Each finite sub-system is characterized by the extensive quantity *G* which it contains and the flux φ_{SG} which it receives from the neighboring sub-systems. The balance equations for the extensive mass quantity *g* in a domain *S* can be written ([2.7]):

$$\frac{\partial}{\partial t}\left(\int_S \rho\, g dv\right) = \varphi_{SG} + \int_S \sigma_G dv$$

The extensive quantity of a system is obviously the sum of the corresponding extensive quantities of its sub-systems only if these are disjoint. For two such sub-systems, such as S_1 and S_2, we have:

$$\int_{S1 \cup S2} \rho\, g dv = \int_{S1} \rho\, g dv + \int_{S2} \rho\, g dv$$



$$\begin{array}{l} \text{balance equation } S_1 \\ \text{balance equation } S_2 \\ \text{balance equation } S_3 \end{array} \Leftrightarrow \begin{array}{l} \text{balance equation } S_1 \\ \text{balance equation } (S_2 \cup S_3) \\ \text{balance equation } (S_3 \cup S_1) \end{array} \Rightarrow \begin{array}{l} \text{balance equation } S_1 \\ \text{balance equation } (S_2 \cup S_3) \end{array}$$

Figure 8.6. *Balance equations for three material adjacent sub-systems*

The balances can be obtained for sub-systems which are partially over-lapped. Figure 3.6 shows the case of three contiguous sub-systems S_1, S_2 and S_3: the sub-system $S_2 \cup S_3$ overlaps S_2, but it is possible to write the balance equations for the 3 disjoint sub-systems or for the three ensembles of sub-systems different two by two. This manner of proceeding is nonetheless limited by the fact that the system of equations obtained must be suitably enough conditioned (section 7.2.7.3), each equation providing specific and sufficient information. The sum of the internal fluxes exchanged between the sub-systems is zero (principle of action and reaction). In the construction of models and numerical schemes we must be careful that this property is ensured (conservative scheme).

8.2.1.4. *Coupling coefficients between two sub-systems*

Consider a system comprising *n* sub-systems E_i characterized by their extensive quantities g_i and their associated intensive energy quantities, or an equivalent (for example, quantity of heat and temperature). Let us assume that two elements *i* and *j* exchange a flux of the quantity *g* under the effect of a difference the intensive quantity $y_i - y_j$. The flux $\kappa_{ij}\left(y_j - y_i\right)$ of the quantity *x* received from the element *j* by the element *i* is opposed to the flux $\kappa_{ji}\left(y_i - y_j\right)$ of the quantity *x* received from the element *i* by the element *j* (action and reaction) and the positive coefficients κ_{ij} and κ_{ji} are equal. The balance equations of the quantity *g* for these two sub-systems can be written:

$$\frac{dg_i}{dt} = \ldots + \kappa_{ij}\left(y_j - y_i\right) + \ldots; \qquad \frac{dg_j}{dt} = \ldots + \kappa_{ji}\left(y_i - y_j\right) + \ldots$$

or, by classing the terms:

$$\frac{dg_i}{dt} = \ldots + \kappa_{ij} y_j + \ldots; \qquad \frac{dg_j}{dt} = \ldots + \kappa_{ji} y_i + \ldots$$

Denoting by *U* the group of sources of the quantity g_i and defining the state vectors *G* and *Y* of the extensive quantities g_i and the corresponding intensive quantities y_i, the balance equations can be written in the general form:

$$\frac{dG}{dt} = K\,Y + J\,U \qquad [8.5]$$

The matrix *J* translates the importance in each element of the external contributions. As the *coupling coefficients* κ_{ij} and κ_{ji} between the elements *i* and *j* are equal, *the matrix K is symmetric.* This feature is not a general constant, since the coefficients of κ_{ij} are functions of the state variables of the elements *i* and *j*.

So, the extensive variable g_i and the intensive variable y_i of a sub-system are related by a state equation. Passage to the form [8.1] of the state representation of the system can often be realized in thermal systems by expressing in each of the sub-systems the *extensive variable* g_i *as a function of the only corresponding intensive variable* y_i, $g_i = g_i\left(y_i\right)$. By defining the *diagonal matrix* Δ of positive elements $d_{ij} = \delta_{ij}\left(dg_i / dy_i\right)$, we have in this case:

$$\frac{dG}{dt} = \Delta \frac{dY}{dt} \tag{8.6}$$

Letting $Y = X$, system [8.5] can be written in the form [8.1]:

$$\frac{dX}{dt} = A.X + BU \quad \text{with : } A = \Delta^{-1}K \qquad B = \Delta^{-1}J. \tag{8.7}$$

It is easy to verify that the matrix $A = \Delta^{-1}.K$ of equation [8.1] is not symmetric, except if the value of each element d_{ij} is independent of the index i (identical subsystems). The reader can verify these considerations by means of the example in section 8.3.1.3 by comparing equations [8.23] and [8.24] (or [8.28] and [8.29]). The importance of this property for the linear case is discussed in section 8.2.2.2.

8.2.1.5. *Equivalence of inputs and initial conditions*

Inhomogenous system [8.8] *with impulse inputs, but with zero initial conditions*, can be reduced to a homogenous system with non-zero initial conditions. Let us verify this point in an elementary manner using second order differential equation [8.8] with zero initial conditions and having a Dirac distribution $c\delta(t)$ at instant 0 as the input:

$$\ddot{x}(t) + a_1\dot{x}(t) + x(t) = c\delta(t) \qquad x(0) = \dot{x}(0) = 0 \tag{8.8}$$

By integrating between 0 and a small time ε and assuming that the functions $x(t)$ *and* $\dot{x}(t)$ remain bounded, we obtain:

$$\dot{x}(\varepsilon) = c \qquad x(\varepsilon) = o(\varepsilon)$$

Problem [8.8] is thus reduced to a homogenous problem with an initial condition at the instant ε:

$$\ddot{x}(t) + a_1\dot{x}(t) + x(t) = 0; \qquad x(\varepsilon) = o(\varepsilon),\ \dot{x}(\varepsilon) = c \tag{8.9}$$

Letting ε tend to zero, we obtain the equivalent problem [8.10]:

$$\ddot{x}(t) + a_1\dot{x}(t) + x(t) = 0; \qquad x(0) = 0,\ \dot{x}(0) = c \tag{8.10}$$

The mechanical interpretation of this result (sometimes known as Schwartz's theorem) is clear, *given an initial velocity c is equivalent to the application of an impulse which produces this velocity at the initial instant*.

The reader can verify in the same way that non-linear problem [8.11] is equivalent to problem [8.12]:

$$\frac{dX}{dt} = A(X).X(t) + B.U(t) + X_0\delta(t) \qquad X(0) = 0 \qquad [8.11]$$

$$\frac{dX}{dt} = A(X).X(t) + B.U(t) \qquad X(0) = X_0 \qquad [8.12]$$

We can replace the initial conditions with these additional impulse inputs which act on the state variables. In certain cases, these can be realized by acting on the existing inputs *U*, except if the number of state variables *n* is greater than the number of independent inputs. In summary, it is often not possible to make clear conceptual distinctions between inputs and initial conditions (or boundary conditions) such that they appear in the mathematical representation of a model.

8.2.1.6. *Modeling and numerical solution of equations for continuous media*

We have reasoned so far in the context of "thermodynamic" modeling of systems by means of an ensemble of discrete sub-systems which are each in a state of thermodynamic quasi-equilibrium. Other approaches are possible for obtaining discretized equations from the balance equations for continuous media:

– *finite difference methods* consist of discretizing the domain under study and replacing the differential operators at the points of the grid thus obtained with finite difference operators;

– *finite element methods* represent the local solution by simple algebraic functions in very small sub-domains; the balance equations are integrated in the domain after multiplication by the weighting functions, *n* being the number of unknown parameters defining the ensemble of local solutions. Integrations by parts often allow the reduction of the order of the necessary derivatives, and the use of finite elements of standard form allows the realization of very general calculatory procedures. We thus obtain an integral formulation comprising *n* integral conditions for the determination of *n* unknowns;

– *finite volume methods* consist of the definition of finite elements in which the balance equations are integrated; the approximations of the solution in each element are simple algebraic functions.

In all cases, we obtain a system of equations which is analogous to the state representation, the preceding numerical schemes belong to two broad categories depending on whether they are *conservative* or not (a numerical scheme is said to be conservative if the discretized equations exactly satisfy the balance of an extensive quantity with the variables which are used to represent the system). A conservative scheme avoids the amplification of numerical errors which are integrated into the balance of physical quantities. It cannot of course correct the unstable character of a physical system. The finite element method is not conservative, contrary to the finite volume method. Finite difference schemes may or may not be conservative. Finally, it is important to remember that, regardless of the method used for discretizing the equations, numerical schemes and algorithms used should respect the rules of transmission of information from one point or element to another which results from the character elliptic, hyperbolic, parabolic or mixed of the equations (section 5.6.1).

8.2.2. *Linear time-invariant system (LTIS)*

8.2.2.1. *Introduction*

Differential system [8.7] for A constant and with initial conditions can be written:

$$\frac{dX}{dt} = A.X + B.U, \quad X(0) = X_0 \qquad [8.13]$$

The linearity of equation [8.13] makes it possible to use the usual methods of solving linear differential equations. The solution is the sum:

– of a *particular solution* $X_e(t)$ of the complete system [8.14]:

$$\frac{dX_e}{dt} = A.X_e + B.U \qquad [8.14]$$

– and a *solution* $\hat{X}(t)$ *of the homogenous system* [8.15] adjusted such that the solution satisfies the initial conditions:

$$\frac{d\hat{X}}{dt} = A.\hat{X}, \quad \hat{X}(0) + X_e(0) = X_0 \qquad [8.15]$$

The solutions of the homogenous system [8.15] are real decaying exponentials or damped sinusoidal functions if the system is stable. They represent *transitional regimes*.

A particular solution $X_e(t)$ is often chosen so as not to contain transitional terms and represents an *established regime* of the problem treated (assumed stable), *independent of the initial conditions*. *This established regime* plays a very important role in numerous cases, in particular when the inputs are either simple algebraic functions of time or harmonic excitations. This particular solution can be obtained in general by the variation of constants method (see mathematical texts).

The separation between transitional and established solutions is no longer theoretically so clear for *non-linear systems*; it can however be useful in numerous cases, as dissipation phenomena are often associated with a progressive elimination of the initial conditions, except of course in the case of a multiplicity of established solutions (section 6.6.2).

The above result can be immediately transposed to partial differential equations. In this case the particular solution $X_e(t)$ of the inhomogenous equation should satisfy all the zero and non-zero boundary conditions, the solutions of the homogenous equation satisfying zero boundary conditions. The solution obtained for the homogenous equation is chosen such that the complete solution satisfies the initial conditions.

8.2.2.2. *Recall on transitional regimes*

The general solution of homogenous equation [8.15] is a linear combination of terms of the form $\Theta e^{\Lambda t}$, Λ being a solution of the eigenvalue problem:

$$(A - \Lambda I)\Theta = 0 \qquad (I \text{ unit matrix }) \qquad [8.16]$$

Equation [8.16] only has non-zero solutions for particular values Λ_i, which are known as eigenvalues, of which there are *n*, including their eventual multiplicity. For each value Λ_i the solutions of equation [8.16] are of the form $c_i\Theta_i$, where Θ_i is an eigenvector associated with the eigenvalue Λ_i (c_i is some constant). An eigenvalue and its associated eigenvectors are known collectively as a mode. The eigenvalues are real and negative for aperiodic modes (in thermal systems in particular) or complex with a real part which may be negative for oscillatory damped modes or zero for unamped modes (in acoustics for example). By limiting ourselves here to thermal systems, the eigenvalues $\hat{X}_i(t)$ can be written:

$$\hat{X}_i(t) = \Theta_i \exp(\Lambda_i t)$$

The complete solution of a given problem is thus written:

$$X(t) = X_e(t) + \hat{X}(t) \quad \text{with : } \hat{X}(t) = \sum_{i=1}^{n} c_i\Theta_i \exp(\Lambda_i t) \qquad [8.17]$$

The scalar coefficients c_i are calculated from the initial condition $X(0)$ which is assumed to be known:

$$X(0) = X_e(0) + \sum_{i=1}^{n} c_i \Theta_i \qquad [8.18]$$

The transitional regimes are solutions comprising the return to rest of the homogenous system (or to the established regime $X_e(t)$ of the system with a right-hand side) from non-zero initial conditions (or eventually zero for an established regime $X_e(t)$). Having determined the eigenvectors Θ_i, the coefficients c_i are solutions of linear system [8.18], the matrix of which is full; for a large number n of equations this system is ill-conditioned (section 7.2.7.3). However, if a scalar product exists $< U|V >$ for which the matrix A is self-adjoint ($< AU|V >=< U|A\,V >$), the eigenvalues Θ_i are orthogonal (taking account of [8.16], if $\Lambda_i \neq \Lambda_j$, we have $< \Theta_i|\Theta_j >= \Lambda_i^{-1} < A\Theta_i|\Theta_j >= \Lambda_j^{-1} < \Theta_i|A\Theta_j >= 0$).

For the scalar product $\langle U|V \rangle = U^t \Delta V = V^t \Delta U$ of the vectors U and V, defined with the diagonal matrix Δ (relation [8.6]), K being symmetric, this leads to:

$$< AU|V >= U^t A^t \Delta V = U^t K^t \Delta^{-1} \Delta V = U^t KV = U^t \Delta\Delta^{-1} KV =< U|A\,V >$$

We can deduce from this, in a manner analogous to that outlined in section A.4.2.2 of Appendix 4:

$$c_i = \left(X(0) - X_e(0)\right)^t \Delta.X_i / X_i^t \Delta.X_i$$

NOTE – Expression [8.17] is the development at instant t of the function $\hat{X}(t)$ on the basis of eigenfunctions Θ_i with instantaneous coefficients $c_i \exp(-\Lambda_i t)$. After a time lapse t, which is large compared to $1/\Lambda_J$, the mode j is no longer present in the transitional solution; its coefficient there being equal to zero, this gives:

$$\hat{X}(t)^t \Delta.\Theta_j = \left(X(t) - X_e(t)\right)^t \Delta.\Theta_j = 0 \qquad [8.19]$$

8.3. Modeling linear invariant thermal systems

8.3.1. *Modeling discrete systems*

8.3.1.1. *Introduction*

The fundamental problem is the reasoned choice of the number of sub-systems or the variables necessary for representation of a system, the number of variables being greater as the thermodynamic imbalance is more pronounced. We will limit our discussion here to linear systems which only possess modal solutions allowing an analytical treatment. The methodologies for writing balance equations are independent of linearity properties. The variation of physical properties as a function of the state variables often leads to weak non-linearities, which do not change the general properties and the orders of magnitude obtained. We will first study three examples of simple discrete thermal systems, then two problems of continuous media (thermal walls).

8.3.1.2. *Models with two sub-systems at constant temperature*

Consider a time-invariant linear system which is composed of two blocks E_1 and E_2 of width ℓ, of specific heat C and separated by a thermal resistance R. Let us then consider a representation of this system by means of *two sub-systems of uniform temperatures* $T_1(t)$ and $T_2(t)$ (Figure 8.7). Each of these two sub-systems is separated from the exterior by another thermal resistance R'.

A thermal resistance is an element of small thickness which transmits heat in a quasi-instantaneous manner; the thermal flux φ which crosses this element is proportional to the temperature difference between its external faces. The fluxes $\varphi_{1,2}$, φ_{1w} and φ_{2w} received by each sub-system (Figure 8.7) are:

$$\varphi_{1,2} = \frac{T_1 - T_2}{R}; \quad \varphi_{1w} = \frac{T_{1ex} - T_1}{R'}; \quad \varphi_{2w} = \frac{T_{2ex} - T_2}{R'}. \qquad [8.20]$$

The initial temperatures $T_1(0)$ and $T_2(0)$ and the external temperatures $T_{1\text{ex}}(\text{t})$ and $T_{2\text{ex}}(\text{t})$ are given.

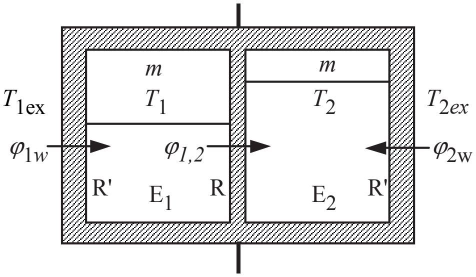


Figure 8.7. *Thermal conducting system with two equilibrium sub-systems*

Taking account of [8.20], the energy balance of each sub-system allows the obtaining of a differential system for the temperatures $T_1(t)$ and $T_2(t)$:

$$mC\frac{dT_1}{dt} = -\frac{T_1 - T_2}{R} - \frac{T_1 - T_{1ex}}{R'}; \quad mC\frac{dT_2}{dt} = -\frac{T_2 - T_1}{R} - \frac{T_2 - T_{2ex}}{R'}.$$

or in matrix form:

$$mC\frac{d}{dt}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{R} + \frac{1}{R'}\right) & \frac{1}{R} \\ \frac{1}{R} & -\left(\frac{1}{R} + \frac{1}{R'}\right) \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} + \frac{1}{R'}\begin{pmatrix} T_{1ex} \\ T_{2ex} \end{pmatrix} \qquad [8.21]$$

Searching for solutions of the form $\Theta\, e^{rt}$ for the homogenous system associated with [8.21] gives the eigenvalue equation (characteristic equation):

$$\begin{vmatrix} -\left(\frac{1}{R} + \frac{1}{R'}\right) - mCr & \frac{1}{R} \\ \frac{1}{R} & -\left(\frac{1}{R} + \frac{1}{R'}\right) - mCr \end{vmatrix} = \left(\frac{1}{R} + \frac{1}{R'} + mCr\right)^2 - \frac{1}{R^2} = 0$$

This has two roots (eigenvalues):

$$r_1 = -\frac{1}{mCR'}; \qquad r_2 = -\frac{1}{mC}\left(\frac{2}{R} + \frac{1}{R'}\right) \qquad [8.22]$$

From these we can find the components (Θ_{i1}, Θ_{i2}) of the eigenvectors Θ_i ($i = 1$, 2), which are solutions to the system of equations:

$$mCr\begin{pmatrix} \Theta_{i1} \\ \Theta_{i2} \end{pmatrix} = \begin{pmatrix} -\left(\frac{1}{R} + \frac{1}{R'}\right) & \frac{1}{R} \\ \frac{1}{R} & -\left(\frac{1}{R} + \frac{1}{R'}\right) \end{pmatrix}\begin{pmatrix} \Theta_{i1} \\ \Theta_{i2} \end{pmatrix}$$

We obtain:

– mode 1 (symmetric, $i = 1$, with $\Theta_{11} = \Theta_{12} = 1$):

$$r_1 = -\frac{1}{mCR'}; \qquad T_{11} = T_{12} = \exp(-t/mCR')$$

– mode 2 (antisymmetric, $i = 2$, with $\Theta_{21} = -\Theta_{22} = 1$):

$$r_2 = -\frac{1}{mC}\left(\frac{2}{R}+\frac{1}{R'}\right); \quad T_{21} = -T_{22} = \exp\left(-\frac{t}{mC}\left(\frac{2}{R}+\frac{1}{R'}\right)\right)$$

The general solution of the homogenous system can be written:

$$T_1 = c_1 \exp(r_1 t) + c_2 \exp(r_2 t), \quad T_2 = c_1 \exp(r_1 t) - c_2 \exp(r_2 t).$$

Let T_{1e}(t) and T_{2e}(t) be a particular solution of [8.21] ("established" solution); we obtain as a general solution to [8.21]:

$$\begin{aligned} T_1(t) &= c_1 \exp(r_1 t) + c_2 \exp(r_2 t) + T_{1e}(t) \\ T_2(t) &= c_1 \exp(r_1 t) - c_2 \exp(r_2 t) + T_{2e}(t) \end{aligned}$$

The integration constants c_1 and c_2 can be calculated with initial conditions;

$$\begin{aligned} c_1 &= [T_1(0) + T_2(0) - (T_{1e}(0) + T_{2e}(0))]/2 \\ c_2 &= [T_1(0) - T_2(0) - (T_{1e}(0) - T_{2e}(0))]/2 \end{aligned}$$

The decomposition of the solution into two modes can be written:

$$\begin{aligned} \theta_1(t) &= [T_1(t) + T_2(t)]/2 = c_1 \exp(r_1 t) + [T_{1e}(t) + T_{2e}(t)]/2 \\ \theta_2(t) &= [T_1(t) - T_2(t)]/2 = c_2 \exp(r_2 t) + [T_{1e}(t) - T_{2e}(t)]/2 \end{aligned}$$

The *fast mode* $\theta_2(t)$ corresponds here to the *establishment of internal equilibrium* of the system, while the *slow mode* $\theta_1(t)$ represents the *establishment of equilibrium between the system and the exterior*. For a system isolated from the exterior (infinite R'), we obtain:

$$T_1(t) + T_2(t) = T_1(0) + T_2(0), \quad T_1(t) - T_2(t) = (T_1(0) - T_2(0))\exp\left(-\frac{2t}{mCR}\right)$$

8.3.1.3. *System with three components in series*

Consider now the system shown in Figure 8.8 consisting of three sub-systems E_1, E_2 and E_3. The component E_3, isolated from the exterior, is in contact with

$E_1$and E_2 via the thermal resistances of the same value R/2. As before, the sub-systems E_1 and E_2 are each in contact via a thermal resistance of the same value R', with the external medium at temperature $T_{iex}(t)$ ($i = 1,2$). Sub-system E_3 is isolated from the exterior.

The energy balance equations can be written:

$$\begin{aligned}
mC\frac{dT_1}{dt} &= -T_1\left(\frac{2}{R}+\frac{1}{R'}\right)+\frac{2T_3}{R}+\frac{T_{1ex}}{R'};\\
mC\frac{dT_2}{dt} &= -T_2\left(\frac{2}{R}+\frac{1}{R'}\right)+\frac{2T_3}{R}+\frac{T_{2ex}}{R'};\\
m'C\frac{dT_3}{dt} &= \frac{2T_1}{R}+\frac{2T_2}{R}.-\frac{4T_3}{R}
\end{aligned}\qquad [8.23]$$

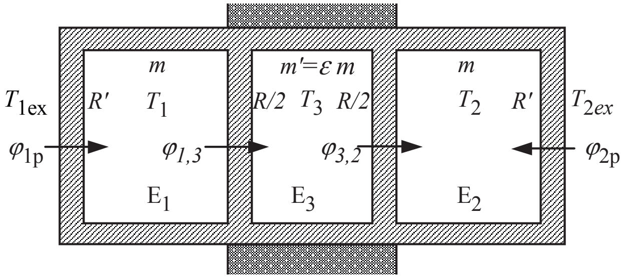


Figure 8.8. *System with three sub-systems (first example)*

The system of equations can be written in matrix form (with $\varepsilon = \frac{m'}{m}$):

$$\frac{d}{dt}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \frac{1}{mCR}\begin{pmatrix} -2-\frac{R}{R'} & 0 & 2 \\ 0 & -2-\frac{R}{R'} & 2 \\ 2/\varepsilon & 2/\varepsilon & -4/\varepsilon \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} + \frac{1}{mCR'}\begin{pmatrix} T_{1ex} \\ T_{2ex} \\ 0 \end{pmatrix} \qquad [8.24]$$

We will limit ourselves here to the discussion of solutions to the homogenous system associated with [8.24] in the form $\Theta_i e^{rt}$. By letting $a = 2 + \frac{R}{R'}$ and

$\Lambda = mCRr$, we obtain the non-dimensional eigenvalue equation (characteristic equation):

$$\begin{vmatrix} -a-\Lambda & 0 & 2 \\ 0 & -a-\Lambda & 2 \\ 2 & 2 & -4-\varepsilon\Lambda \end{vmatrix} = -(a+\Lambda)\left[\varepsilon\Lambda^2 + \Lambda(4+\varepsilon a) + 4a - 8\right] = 0 \quad [8.25]$$

The roots of equation [8.25] are:

$$\Lambda_1 = -a; \quad \Lambda_i = \frac{-(4+\varepsilon a) \pm \sqrt{16 + 8\varepsilon(4-a) + \varepsilon^2 a^2}}{2\varepsilon}, \quad (i = 2,3) \quad [8.26]$$

As the sum of the roots Λ_2 and Λ_3 of the equation are negative and their product is positive, the two roots are negative (a diffusive system is aperiodic).

The symmetry of the system allows us to immediately find the modes. A symmetric mode is characterized by $\Theta_{i1} = \Theta_{i2}$. Substituting this relation into the homogenous system derived from [8.24], we have:

$$(\Lambda + a)\Theta_{i1} = 2\Theta_{i3}; \quad (\varepsilon\Lambda + 4)\Theta_{i3} = 4\Theta_{i1}$$

Eliminating Θ_{i3} between these relations, we find that the eigenvalues Λ_i corresponding to these modes satisfy the second order trinomial of characteristic equation [8.25]. The values of Θ_{i1} and Θ_{i3} are of the same sign or of opposite sign depending on whether $\Lambda + a$ is positive or negative. Substituting $-a$ as the value of Λ in the preceding trinomial, we find that this takes on a negative value, which shows that the value $-a$ is situated between the roots of the trinomial. We can easily derive from this that the quantity $\Lambda + a$ is positive for the largest root Λ_3, whereas it is negative for the other root Λ_2. We can verify immediately that the root $\Lambda_1 = -a$ corresponds to the anti-symmetric mode $\Theta_{11} = -\Theta_{12}$ and $\Theta_{13} = 0$.

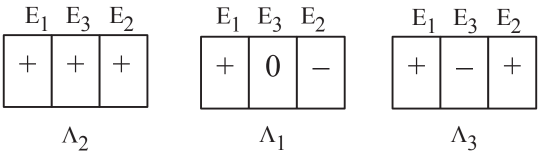


Figure 8.9. *Structure of normal modes classed with increasing modulus of the eigenvalues* $|r_2| < |r_1| < |r_3|$ *(decreasing time constants)*

The structure of the modes is shown in Figure 8.9. The time constant of a mode is equal to $1/|r|$. The preceding discussion shows that we have $|r_2| < |r_1| < |r_3|$. The slowest mode corresponds to the root r_2 for which the three sub-systems at the same temperature constitute a thermodynamic system in equilibrium, a structure for which the thermal inertia is greatest towards the thermal resistances *R'*. On the other hand, the most "agitated" mode corresponds to the smallest time constant $1/|r_3|$.

Suppose now that the mass *m'* of the sub-system E$_3$ is small compared to the mass *m* of the sub-systems E$_1$ and E$_2$ (i.e. $\varepsilon \ll 1$). Performing a series development of the roots [8.26], we obtain:

$$\Lambda_1 = -a; \quad \Lambda_3 = -\frac{4}{\varepsilon} + O(1) \quad \Lambda_2 = 2 - a + O(\varepsilon)$$

Returning to the dimensional values $r_i = \Lambda_i / mcR$, we obtain:

$$r_1 = -\frac{1}{mC}\left(\frac{2}{R} + \frac{1}{R'}\right) \quad r_3 = -\frac{4}{\varepsilon\, mCR} \quad r_2 = -\frac{1}{mCR'}$$

The values r_1 and r_2 are those already found ([8.22]) for the system studied in section 8.3.1. As the value of r_3 is large compared to r_1 and r_2, mode 3 is very quickly damped; after this damping, we have the relation [8.19] for $i = 3$, which can here be written $\hat{X}(t)^t \Delta.\Theta_3 = 0$. Replacing $\hat{X}(t)$, Δ (a matrix which allows us to pass from [8.23] to [8.24]) and Θ_3 by their values at small ε:

$$\hat{X}(t) = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}; \quad \Delta = \begin{pmatrix} mC & 0 & 0 \\ 0 & mC & 0 \\ 0 & 0 & \varepsilon\, mC \end{pmatrix}; \quad \Theta_3 = \begin{pmatrix} 1 \\ -1 \\ -2/\varepsilon \end{pmatrix}$$

we obtain:

$$\hat{X}(t)^t \Delta.\Theta_3 = mC(T_1 + T_2 - 2T_3) = 0 \qquad [8.27]$$

Relation [8.27] reduces to the balance equation of the sub-system E$_3$ (third equation [8.23]) in which the mass is negligible. We verify that in replacing T_3 in the first two equations of [8.23] with its value taken from [8.27] we recover equations [8.21]: *the three sub-systems model has been reduced to a model with two sub-systems by removal of the third sub-system of small mass.*

8.3.1.4. *Thermal systems with 3 components in the form of a star*

Now consider the system shown in Figure 8.10, which comprises three sub-systems E_1, E_2 and E_3 in contact two by two across the same thermal resistances R. Each of these is in contact across the same thermal resistance R' with the external temperature medium $T_{i\,ex}(t)$ (i = 1, 2, 3). We will only discuss the structure of the modes here. The energy balance equations can be written:

$$\begin{aligned} mC\frac{dT_1}{dt} &= \frac{-2T_1 + T_2 + T_3}{R} - \frac{T_1 - T_{1ex}}{R'}; \\ mC\frac{dT_2}{dt} &= \frac{T_1 - 2T_2 + T_3}{R} - \frac{T_2 - T_{2ex}}{R'} \\ m'C\frac{dT_3}{dt} &= \frac{T_1 + T_2 - 2T_3}{R} - \frac{T_3 - T_{3ex}}{R'} \end{aligned} \qquad [8.28]$$

or, in matrix form, letting: $a = \left(2 + \frac{R}{R'}\right)$ and $\varepsilon = \frac{m'}{m}$:

$$\frac{d}{dt}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} = \frac{1}{mCR}\begin{pmatrix} -a & 1 & 1 \\ 1 & -a & 1 \\ 1/\varepsilon & 1/\varepsilon & -a/\varepsilon \end{pmatrix}\begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix} + \frac{1}{mCR'}\begin{pmatrix} T_{1ex} \\ T_{2ex} \\ T_{3ex}/\varepsilon \end{pmatrix} \qquad [8.29]$$

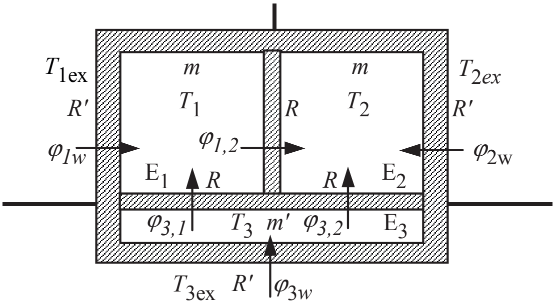


Figure 8.10. *System with three sub-systems (second example)*

Searching for solutions of the form $\Theta_i\,e^{rt}$ for the homogenous system associated with [8.29] gives the non-dimensional eigenvalue equation, where we have let $\Lambda = mCR\,r$:

$$\begin{vmatrix} -a-\Lambda & 1 & 1 \\ 1 & -a-\Lambda & 1 \\ 1 & 1 & -a-\varepsilon\Lambda \end{vmatrix} = 0$$

or:

$$-\varepsilon\Lambda^3 - \Lambda^2 a(1+2\varepsilon) + \Lambda(2+\varepsilon)\left(1-a^2\right) - (a-2)(a+1)^2 = 0 \qquad [8.30]$$

Taking account of the system symmetry, we see immediately that the anti-symmetric mode verifies the relations $\Theta_{11} = -\Theta_{12}$ and $\Theta_{13} = 0$. Substituting these relations into the equations of the homogenous system, we find that the corresponding eigenvalue is equal to $\Lambda_1 = -(a+1)$. The characteristic equation [8.30] can be written:

$$(\Lambda+1+a)[\varepsilon\Lambda^2+(\varepsilon(a-1)+a)\Lambda+(a+1)(a-2)]=0 \qquad [8.31]$$

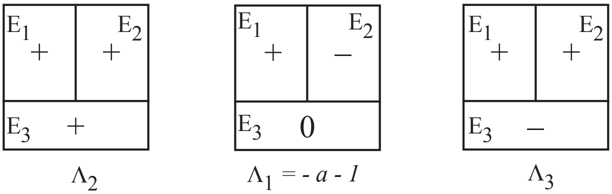
Its two other roots Λ_ι are ($i = 2.3$):

$$\Lambda_i = \frac{-a-\varepsilon(a-1) \pm \sqrt{a^2 - 2\varepsilon\left(a^2-a-4\right) + \varepsilon^2(1-a)^2}}{2\varepsilon}$$

The structure of the modes can be obtained by simple reasoning. Taking account of the system symmetry, the other eigenfunctions should present the symmetry $\Theta_{i1} = \Theta_{i2}$. Substituting this relation into the homogenous system derived from [8.29], we obtain:

$$(\Lambda_i + a - 1)\Theta_{i1} = \Theta_{i3}; \quad (\varepsilon\Lambda_i + a)\Theta_{i3} = 2\Theta_{i1}$$

We immediately find that the values Λ_ι which satisfy the preceding relations are the roots of the second order trinomial of characteristic equation [8.31]. The values of $\Theta_{\iota 1}$ and $\Theta_{\iota 3}$ have the same sign or opposite sign depending on whether $\varepsilon\Lambda_i + a$ is positive or negative. Substituting $-\varepsilon/a$ as the value of Λ_ι in the preceding trinomial, we find that this takes on a negative value, showing that this value is between the roots of the trinomial. The result is that the quantity $\varepsilon\Lambda_i + a$ is positive for the largest root Λ_3, whereas it is negative for the other root Λ_2. Figure 8.11 shows the structure of the three modes.



Λ_2 $\quad$ $\Lambda_1 = -a - 1$ $\quad$ Λ_3

Figure 8.11. *Structure of normal modes*

For small values of the calorific capacity $m'C$ of sub-system E_3, a development in ε gives the values:

$$\Lambda_3 = -\frac{a}{\varepsilon} + O(1) \qquad \Lambda_2 = -a + 1 + \frac{2}{a} + O(\varepsilon)$$

The corresponding values of $r_i = \Lambda_i / mCR$ are:

$$r_1 = \frac{1}{mC}\left(-\frac{3}{R} + \frac{1}{R'}\right) \qquad r_3 = -\frac{1}{mC\varepsilon}\left(\frac{2}{R} + \frac{1}{R'}\right) \qquad r_2 = -\frac{1}{mCR'}\left(\frac{R + 3R'}{R + 2R'}\right)$$

As in the case of three components in series, r_3 is much larger than r_1 and r_2, and mode 3 is thus rapidly damped. The third equation of system [8.28] can thus be written:

$$\frac{2T_3 - T_1 - T_2}{R} + \frac{T_3 - T_{3ex}}{R'} = 0$$

The reader can verify that the preceding relation is identical to condition [8.19]. Contrary to the case of three components in series, the limit of this system for $m' = 0$ is not the preceding system of two sub-systems (section 8.3.1.2).

8.3.2. *Thermal models in continuous media*

8.3.2.1. *Overview*

The idea of a continuous medium amounts to replacing an integer valued index (number of components of the state vector *X*) by a spatial variable of continuous values. The temporal and spatial behaviors of the system are thus continuous functions or piecewise continuous. However, we have seen in Chapter 5 that the theory of characteristics allows us to identify different behaviors for these variables. If the time is by nature irreversible, the spatial properties or spatio-temporal properties are different, depending on the elliptic, parabolic or hyperbolic nature of the system.

We will here consider problems of thermal conduction in homogenous media. The method can also be applied analogously in *heterogenous media with physical properties which depend on the spatial variables.*

8.3.2.2. *Wall problem*

8.3.2.2.1. Equations and solution of homogenous systems

Consider a wall of thickness 2ℓ comprising a homogenous material whose faces are supposed, for example, to be at a given, equal temperature $T_1(t)$. The *thermal diffusivity* a of the material is supposed constant. The distribution of temperature on the interval $[-\ell,\ell]$ satisfies the heat equation:

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} \quad \left(-\ell \le x \le \ell\right) \tag{8.32}$$

The given data are:

– the initial conditions: $T(0,x) = T_0(x)$;

– the boundary conditions (fixed temperature): $T(-\ell,t) = T(\ell,t) = T_1(t)$.

As equation [8.32] is linear, the methodology is the same as before: the complete solution can be obtained by superposition of the particular solution $T_e(x,t)$ of equation [8.32] satisfying the non-zero boundary conditions and a general solution of the heat equation with zero boundary conditions for the temperature. As in section 8.2.2.2, the coefficients of this solution are calculated such that the complete solution satisfies the initial conditions.

In order to simplify the notation of the general solution, let us take the non-dimensional variables:

$$\tilde{x} = x/\ell; \quad \tilde{t} = at/\ell^2 \tag{8.33}$$

Equation [8.32] becomes:

$$\frac{\partial T}{\partial \tilde{t}} = \frac{\partial^2 T}{\partial \tilde{x}^2} \quad \left(-1 \le \tilde{x} \le 1\right) \tag{8.34}$$

We seek a group of solutions with separated variables of the form: $T(\tilde{x},\tilde{t}) = f(\tilde{t})g(\tilde{x})$. Substituting this expression into [8.34], we obtain[2]:

$$\frac{f'(\tilde{t})}{f(\tilde{t})} = \frac{g''(\tilde{x})}{g(\tilde{x})} = \text{cte} = -\Lambda^2 \tag{8.35}$$

We derive from this the function *f*:

$$f(\tilde{t}) = \exp(-\Lambda^2 \tilde{t}) \tag{8.36}$$

and the eigenvalue problem for the function *g:*

$$g''(\tilde{x}) + \Lambda^2 g(\tilde{x}) = 0; \quad g(\pm 1) = 0 \tag{8.37}$$

The integration of equation [8.37] shows that the eigenfunctions are of the form $g(\tilde{x}) = A\cos(\Lambda\tilde{x} + B)$. The expression of boundary conditions [8.37] at $\tilde{x} = \pm 1$ gives the values of the constants *Λ* and *B*. We thus obtain:

$$\Lambda + B = \frac{\pi}{2} + p\pi; \quad -\Lambda + B = \frac{\pi}{2} + p'\pi \quad (p, p'\text{: integers})$$

or:

$$\Lambda = (p - p')\frac{\pi}{2}; \quad B = \frac{\pi}{2} + (p + p')\frac{\pi}{2}$$

In the preceding expressions $p - p'$ and $p + p'$ have the same parity:

– if $p - p'=2k$ is even, we have $\Lambda_{2k} = k\pi$ and the corresponding eigenfunctions are *odd:* $g_{2k} = \sin[k\pi\tilde{x}]$;

– if $p - p' = 2k + 1$ is odd, we have $\Lambda_{2k} = (2k + 1)\frac{\pi}{2}$ and the eigenvalues are *even:* $g_{2k+1} = \cos\left[(2k + 1)\frac{\pi}{2}\tilde{x}\right]$.

The first eigenfunctions of the wall problem at zero wall temperature are represented in Figure 8.12.

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[2] The system being damped, the constant of equation [8.36] is necessarily negative.

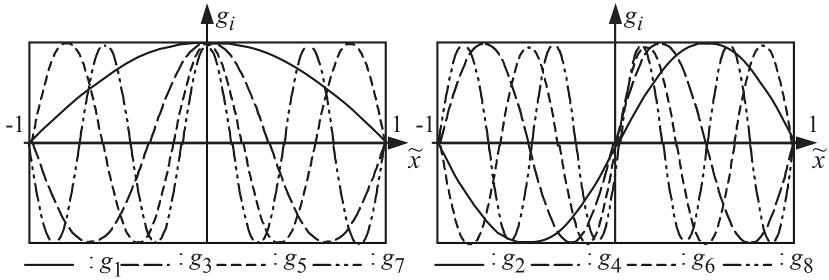

Figure 8.12. *First eigenfunctions of the homogenous problem inside a wall with constant temperature faces*

The preceding eigenfunctions form a representation basis of functions which are zero at the extremities of the interval [-1,1] and which possess sufficient regularity properties. Taking account of the condition $g(\pm 1) = 0$, the eigenfunctions of problem [8.37] are orthogonal. In effect, an integration by parts immediately gives:

$$\int_{-1}^{1} g_p g_q'' d\widetilde{x} = \int_{-1}^{1} g_p dg_q' = -\int_{-1}^{1} g_p' g_q' d\widetilde{x}$$

hence: $\int_{-1}^{1} \left[g_p g_q'' - g_q g_p'' \right] d\tilde{x} = 0.$

The operator of problem [8.37] is self-adjoint (Appendix 4), showing the orthogonality property. T_e being the established solution (section 8.2.2.2), we thus express the solution in the form:

$$T(\widetilde{x}, \widetilde{t}) = Te(\widetilde{x}, \widetilde{t}) + \sum_{0}^{\infty} c_i \exp(-\Lambda_i^2 \widetilde{t}) g_i(\widetilde{x}) \qquad [8.38]$$

The coefficients c_i are calculated at the instant $\widetilde{t} = 0$:

$$T(\widetilde{x}, 0) = T_0(\widetilde{x}) = T_e(\widetilde{x}, 0) + \sum_{0}^{\infty} c_i g_i(\widetilde{x})$$

hence (Appendix 4):

$$c_i = \frac{\int_{-1}^{+1}\left[T_0(\tilde{x}) - T_e(\tilde{x},0)\right]g_i(\tilde{x})d\tilde{x}}{\int_{-1}^{+1} g_i^2(\tilde{x})d\tilde{x}} \qquad [8.39]$$

8.3.2.2.2. Thermal shocks on the walls

Let us now consider the simple example of the formation of thermal shocks on the two faces of a wall whose initial temperature $T_0(x)$ is uniform and whose temperatures $T_1(\tilde{t})$ on the wall faces are constant and equal to T_1 for positive $\tilde{t}$. The established solution $T_e(\tilde{x},\tilde{t})$ is thus constant and equal to T_1. The coefficients c_{2k} corresponding to the odd eigenfunctions are zero. Formula [8.39] gives, for the coefficients of the even eigenfunctions:

$$c_{2k+1} = (T_0 - T_1)\frac{(-1)^k 4}{(2k+1)\pi}$$

From this we can derive the solution of the thermal shock problem $T_1 - T_0$ applied on the two faces of a wall of thickness 2ℓ:

$$\frac{T(\tilde{x},\tilde{t}) - T_0}{T_1 - T_0} = 1 - \sum_{k=0}^{\infty} \frac{(-1)^k 4}{(2k+1)\pi} e^{-(2k+1)^2 \frac{\pi^2}{4}\tilde{t}} .\cos\left[(2k+1)\frac{\pi}{2}\tilde{x}\right] \qquad [8.40]$$

The mean[3] temperature $T_m = \frac{1}{2}\int_{-1}^{1} T(\tilde{x},\tilde{t})d\tilde{x}$ can be expressed:

$$\frac{T_m(\tilde{t}) - T_0}{T_1 - T_0} = 1 - \sum_{k=0}^{\infty} \frac{8}{(2k+1)^2 \pi^2} e^{-(2k+1)^2 \frac{\pi^2}{4}\tilde{t}} \qquad [8.41]$$

This expression allows us to know the contribution of each mode (Table 8.1) to the thermal energy supplied by conduction, this being proportional to the relative amplitude of the mode (we have $\sum_{k=0}^{\infty} 8/(2k+1)^2 \pi^2 = 1$):

---

[3] Defined as the space mean temperature for homogenous media.

Mode number	0	1	2	3	4	5	6
Mode amplitude	0.8106	0.0901	0.0324	0.0165	0.0100	0.0067	0.0048

Table 8.1. *Distribution of thermal energy in the seven first modes*

The wall behaves in a manner similar to a first order system, since only 20% of the heat is exchanged more rapidly in modes higher than the first mode. The density of the thermal flux $q_{Tp}(-\ell,t) = -\lambda(\partial T/\partial x)_{x=-\ell}$ on the face of the wall at -ℓ can be derived from [8.40]. Infinite at time $t = 0$, it decays very rapidly in the first instants and it can immediately be written in non-dimensional form:

$$\frac{\ell q_{Tp}(-\ell,t)}{\lambda(T_1 - T_0)} = 2\sum_0^\infty e^{-(2k+1)^2\frac{\pi^2}{4}\frac{at}{\ell^2}}. \qquad [8.42]$$

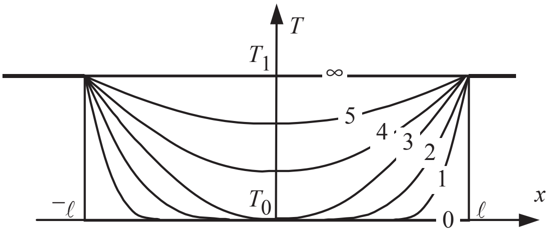

Figure 8.13. *Evolution of the temperature distribution during a thermal shock on the two faces of a wall*

Figure 8.13 shows the evolution of the temperature profiles with time (curves from 0 to 5). The curve 0 is the distribution at the initial instant where we impose the temperature T_1 on the walls ($x = \pm\ell$). The diffusion of this condition occurs progressively from the walls: the curves 1 and 2 represent the thermal shocks in a quasi-infinite medium from the walls. The thermal diffusion zones are then rejoined on the other curves (3 to 5). On curves 4 and 5, observed after a non-dimensional time in the order of $1/\Lambda_1$, only the first mode g_1 remains, and it damps until its amplitude falls to zero.

The temperature distribution of the thermal shock in a semi-infinite domain from a wall has already been obtained (section 5.4.5.4 and formula [5.52]). It can be written here for the thermal boundary layer on the wall $x = -\ell$, and by taking non-dimensional variables [8.33]:

$$\frac{T-T_0}{T_1-T_0}=1-erf\left(\frac{x+\ell}{2\sqrt{at}}\right)=1-erf\left(\frac{\tilde{x}+1}{2\sqrt{\tilde{t}}}\right)=1-\frac{2}{\sqrt{\pi}}\int_0^{(\tilde{x}+1)/2\sqrt{\tilde{t}}}e^{-u^2}\,du$$

The complete solution which represents the two boundary layers can be written:

$$\frac{T-T_0}{T_1-T_0}=2-erf\left(\frac{\tilde{x}+1}{2\sqrt{\tilde{t}}}\right)-erf\left(\frac{-\tilde{x}+1}{2\sqrt{\tilde{t}}}\right) \qquad [8.43]$$

The thermal flux density at the wall $\tilde{x}=-1$ can be derived from [8.43]:

$$q_{Tp}=-\lambda\left(\frac{\partial T}{\partial x}\right)_{x=-\ell}=\frac{\lambda}{\sqrt{\pi\,a\,t}}\left(T_1-T_0\right)$$

It can be written in non-dimensional form (section 4.6.1.3.4), with the reduced time $\tilde{t}$ [8.33]:

$$\frac{\ell\,q_{Tp}\left(-\ell,\tilde{t}\right)}{\lambda\left(T_1-T_0\right)}=\frac{1}{\sqrt{\pi\tilde{t}}} \qquad [8.44]$$

The mean temperature of the wall in the *thermal boundary layer regime* can be obtained by performing the energy balance in the wall between instants 0 and *t:*

$$\rho\left(T_m-T_0\right)C\ell=\int_0^t q_{Tp}(u)du=\left(T_1-T_0\right)\int_0^t\frac{\lambda\,du}{\sqrt{\pi\,au}}=\left(T_1-T_0\right)\frac{2\lambda\sqrt{t}}{\sqrt{\pi\,a}}$$

or:

$$\frac{T_m(t)-T_0}{T_1-T_0}=\frac{2}{\sqrt{\pi}}\sqrt{\tilde{t}} \qquad [8.45]$$

8.3.2.2.3. Composite representation by matched asymptotic expansions

The series expansion of eigenfunctions of the thermal shock problem on the faces of a wall is a poorly adapted representation in the first instants of the thermal shock, whereas the representation by thermal boundary layers captures the physics of the problem more satisfactorily. For small time values, the modal representation of the mean temperature is nearly acceptable with very few modes (taking account of a discontinuity at t = 0); on the other hand, modal expression [8.42] for the thermal flux density is unusable and we thus have recourse to expression [8.44] which is equivalent to the sum of the series [8.42] in these conditions.

We are thus led to search for a formula which contains the two different asymptotic expressions f_1(t) and f_2(t) of the function $f(t)$, which are valid for small and large values of t respectively following the time value. This can be obtained by means of a matching formula or a weight between the two temporal domains which gives *exact values for the function f and its temporal derivative* at the origin, and which respects the asymptotic behavior at infinity. A simple means consists of weighting the two formulae by a suitable auxiliary function $\varphi(t)$ close to 1 for small t and tending quite quickly to zero for t equal to infinity. The expression:

$$f(t) = \varphi(t) f_1(t) + (1 - \varphi(t)) f_2(t)$$

satisfies these conditions if the function $\varphi(t)$ at least satisfies the relations $\varphi(0) = 1,\ \varphi'(0) = \varphi(\infty) = 0$ and if $\varphi'(t)$ tends at infinity faster to zero than $f_2(t)$; we have:

$$f(0) = f_1(0) \quad f'(0) = f_1'(0) \quad f(\infty) = f_2(\infty) \quad f'(\infty) = f_2'(\infty)$$

The simplest weighting function is the Gaussian $\varphi(\tilde{t}) = e^{-\alpha\tilde{t}^2}$.

It remains to write a matching condition which can be defined at a point where the two approximations differ very little and where we require that the value of the function $\varphi(t)$ is equal to 0.5 (here, the function $f(t)$ is the mean of the values f_1(t) and f_2(t)).

Let us apply this procedure to obtain a quite simple expression of the solution valid all over the interval [-ℓ,+ℓ]. We take as our *asymptotic expression at infinity the modal solution limited to the first mode*, and for small t, the boundary layer solution. We will choose the mean temperature in order to determine the matching condition. A simple numerical calculation shows that the difference between the values of expression [8.41] limited to the first mode and formula [8.45] is minimal in the vicinity of $\tilde{t} = 0.20$. Taking as a weighting function the Gaussian $\varphi(\tilde{t}) = e^{-\alpha\tilde{t}^2}$ and taking $\varphi(0.2)$ equal to 0.5, we find $\alpha = 7.5$.

The mixed representation thus obtained for the first mode of the temperature T_m can be written:

$$\frac{T_m\left(\tilde{t}\right)-T_0}{T_1-T_0}=e^{-7.5.\tilde{t}^2}\,\frac{2}{\sqrt{\pi}}\sqrt{\tilde{t}}+\left(1-e^{-7.5.\tilde{t}^2}\right)\left[1-\frac{8}{\pi^2}.e^{-\frac{\pi^2}{4}\tilde{t}}\right] \qquad [8.46]$$

The corresponding expression for the wall's thermal flux density is thus:

$$\frac{\ell q_{Tp}\left(-\ell,t\right)}{\lambda\left(T_1-T_0\right)}=e^{-7.5.\tilde{t}^2}\,\frac{1}{\sqrt{\pi\,\tilde{t}}}+\left(1-e^{-7.5.\tilde{t}^2}\right)2.e^{-\frac{\pi^2}{4}\tilde{t}} \qquad [8.47]$$

The preceding formulae represent the exact solution to 1% accuracy.

The spatio-temporal temperature distribution can also be written in the same manner from expressions [8.40] and [8.43]:

$$\frac{T-T_0}{T_1-T_0}=e^{-7.5.\tilde{t}^2}\left[2-erf\left(\frac{\tilde{x}+1}{2\sqrt{\tilde{t}}}\right)-erf\left(\frac{-\tilde{x}+1}{2\sqrt{\tilde{t}}}\right)\right]$$
$$+\left(1-e^{-7.5.\tilde{t}^2}\right)\left[1-\frac{4}{\pi}e^{-\frac{\pi^2}{4}\tilde{t}}\cos\frac{\pi\,\tilde{x}}{2}\right]$$

The interest in simple analytic expressions is clear; furthermore, the precision of the approximations effected can be improved as much as we desire by conserving additional terms of the modal solution in the preceding composite solution.

8.3.2.3. *Thermal systems with continuous components*

8.3.2.3.1. Conduction in two walls separated by a thermal resistance

The separation of variables method can be applied to continuous media by individual segments which comprise discontinuities. We will reconsider the linear invariant system with two identical components discussed in section 8.3.1.2 with a continuous medium model of constant thermal diffusivity *a*; the sub-systems are separated by the thermal resistance *R per unit cross-section,* located at the origin (Figure 8.14). We will assume that the *thermal flux densities are given on the external faces,* the external thermal resistance *R'* being taken as zero.

With equation [8.32] on the intervals $[-\ell,0_-]$ and $[0_+,\ell]$, we must associate:

– the matching condition involving the thermal resistance between the two sub-systems:

$$-\lambda\left(\frac{\partial T}{\partial x}\right)_{0+}=-\lambda\left(\frac{\partial T}{\partial x}\right)_{0-}=-\frac{T(0_+)-T(0_-)}{R} \qquad [8.48]$$

– the boundary conditions of the system: thermal flux densities given (or eventually the temperature or thermal resistance) at the points $x = \pm\ell$;

– the initial temperature distribution $T(x,0)$.

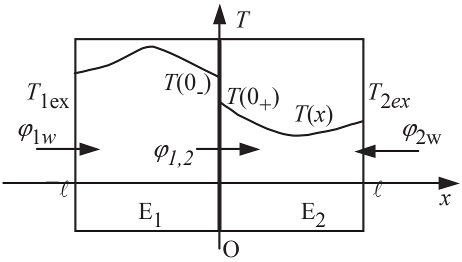

Figure 8.14. *Thermal conduction in a system of two continuous media separated by a thermal resistance*

As before, the complete solution can be obtained by superposition of a solution which satisfies the non-zero boundary conditions and a general solution of the heat equation with homogenous boundary conditions. The coefficients of this general solution are calculated such that the complete solution satisfies the initial conditions.

8.3.2.3.2. General solution of the homogenous problem

As an example, we will treat the *problem* where a *thermal flux is imposed at the interval extremities* $[-\ell,+\ell]$. The condition at $x = \pm\,\ell$ of the homogenous problem is thus a zero thermal flux. As above, we write the homogenous problem with non-dimensional variables [8.33]. We add to equation [8.34] the boundary conditions and thermal resistance matching condition [8.48], which can be written in non-dimensional form as:

$$\left(\frac{\partial T}{\partial \widetilde{x}}\right)_{\pm 1} = 0; \qquad -\left(\frac{\partial T}{\partial \widetilde{x}}\right)_{0+} = -\left(\frac{\partial T}{\partial \widetilde{x}}\right)_{0-} = -\frac{\ell}{\lambda R}\left[T(0_+) - T(0_-)\right] \qquad [8.49]$$

The problem depends on the parameter $P = \ell/\lambda R$, which represents the importance of the thermal resistance ℓ/λ of a component with respect to the thermal resistance *R*.

The solutions to equation [8.34] where the variables are separated as $T(\widetilde{x},\widetilde{t}) = f(\widetilde{t})g(\widetilde{x})$ again satisfy relations [8.35] and [8.36] for the function $f(\widetilde{t}) = \exp(-\Lambda^2\widetilde{t})$.

The eigenvalue problem for the eigenfunction g can now be written:

$$
\begin{aligned}
&g''(\tilde{x}) + \Lambda^2 g(\tilde{x}) = 0;\\
&g'(\pm 1) = 0; \quad g'(0_+) = g'(0_-) = \frac{\ell}{\lambda R}\left[g(0_+) - g(0_-)\right]
\end{aligned}
\qquad [8.50]
$$

The integration of equation [8.50] leads to eigenfunctions of the form $g(\tilde{x}) = A\cos\left(\Lambda\,\tilde{x} + B\right)$, the constants (A, B) taking on, respectively, the values (A_-, B_-) and (A_+, B_+) on the intervals (-1,0) and (0,1):

$$
\begin{aligned}
-1 \le \tilde{x} < 0: &\quad g(\tilde{x}) = A_- \cos\left(\Lambda\,\tilde{x} + B_-\right)\\
0 < \tilde{x} \le 1: &\quad g(\tilde{x}) = A_+ \cos\left(\Lambda\,\tilde{x} + B_+\right)
\end{aligned}
$$

The expression of the boundary conditions at $\tilde{x} = \pm 1$ gives:

$$A_-.\Lambda \sin\left(-\Lambda + B_-\right) = A_+.\Lambda \sin\left(\Lambda + B+\right) = 0$$

or:

$$\Lambda = -B_+ + p'\pi = B_- + p\pi \qquad \left(p, p'\text{: integers}\right) \qquad [8.51]$$

Matching condition [8.49] for the thermal flux densities at $x = 0$ can be written:

$$\Lambda A_+ \sin B_+ = \Lambda A_- \sin B_- = -P\left[A_+ \cos B_+ - A_- \cos B_-\right] \qquad [8.52]$$

Equations [8.51] and [8.52] possess two groups of solution[4]:

1) *Even eigenfunctions*. We find immediately the solutions to equations [8.51] and [8.52]:

$$B_- = B_+ = 0; \quad A_- = A_+; \quad \Lambda_{2k} = k\pi \quad (k \text{ integer}) \qquad [8.53]$$

The evenness of the eigenfunctions $g_{2k}(\tilde{x}) = \cos(k\pi\,\tilde{x})$ of the two-wall ensemble leads to the absence of thermal transfer between the two sub-systems which behave as a single symmetric block. The first four eigenfunctions (k = 0, 1, 2, 3) are represented in Figure 8.15a (we have taken $A_+ = A_- = 1$).

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[4] We will number the functions with even numbers (respectively odd) for even eigenvalues (respectively odd) with increasing order of the eigenvalues.

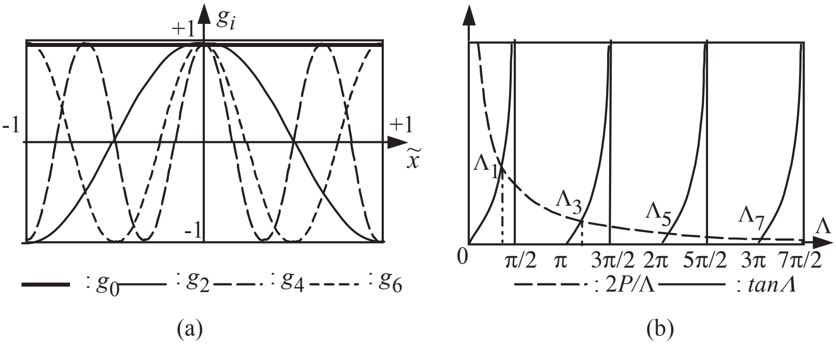


Figure 8.15. *(a) Even eigenfunctions of two walls separate by a thermal resistance; (b) graphical solution of the eigenvalue equation in the case of odd eigenfunctions*

2) *Odd eigenfunctions*. Substituting the values [8.51] of the pairs (B_-, B_+) into first relation [8.52], we obtain a second group of solutions ($B_- \neq 0$, $B_+ \neq 0$):

$$B_- = -B_+; \quad A_- = -A_+ \quad \text{or } B_- = -B_+ + \pi; \quad A_- = A_+$$

We derive the relation:

$$A_+ . \cos B_+ = -A_- . \cos B_-$$

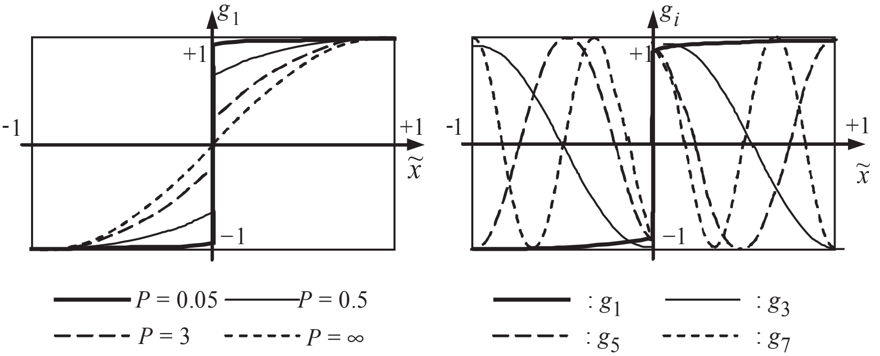
Substituting this expression into [8.52], and taking account of [8.51], we obtain the eigenvalue equation:

$$\Lambda \tan \Lambda = 2P \qquad [8.54]$$

whose graphical solution is shown in Figure 8.15b. The eigenfunctions $g_{2k+1}(\widetilde{x})$ which correspond to these eigenvalues are here odd and by taking $A_+ = -A_- = 1$ they present a discontinuity amplitude equal to $2\cos\Lambda_{2k+1}$ at the origin. They can be written:

$$0 < x \leq 1: \quad g_{2k+1}(\widetilde{x}) = A_{2k+1} \cos\Lambda_{2k+1}\left(\widetilde{x} - 1\right)$$

These discontinuous eigenfunctions depend on the dimensionless parameter P. We have represented, in Figure 8.16b, the first four odd eigenfunctions corresponding to $P = 0.05$ (eigenvalues Λ_1=0.3111, Λ_3=3.1731, Λ_5=6.2991, Λ_7=9.43538).



(a) Variations of the eigenfunction g_1 for different values of P.

(b) Discontinuous eigenvalues for $P = 0.05$.

Figure 8.16. *Odd eigenvalues for two walls separated by a thermal resistance*

8.3.2.3.3. Evolution of odd eigenfunctions

The value of thermal resistance placed between the two walls (in other words the value of P) has a strong influence on the discontinuity amplitude. Figure 8.16a shows the evolution of the first odd eigenfunction as a function of P. The small values of P (high thermal resistance R) lead to eigenvalues of the odd functions which are thus quite close to $k\pi$, which are the eigenvalues of the even functions (Figure 8.15b). The discontinuity at the origin $2\cos\Lambda_{2k+1}$ of the odd eigenfunctions is strong, and the thermal flux $\Lambda_{2k+1}\sin\Lambda_{2k+1}$ between the two blocks is all the smaller as k is large. The absolute values of the odd and even eigenfunctions are thus very close in each of the two continuous media. A homogenization of the temperature occurs in each block, the thermal flux $\Lambda_{2k+1}\sin\Lambda_{2k+1}$ between them being essentially limited to the first odd mode.

If the thermal resistance R tends to zero, P becomes large and the eigenvalues Λ_{2k+1} tend to the odd eigenvalues $\dfrac{\pi}{2}+k\pi$ of the total isolated system. We thus find the odd eigenfunctions $\sin\dfrac{(2k+1)\pi x}{2}$ of the conduction problem for an insulated wall (and of course, the temperature continuity at the origin).

In expression [8.17] of the modal development of the temperature, the dimensional time constant τ_k of each mode is directly related (formulae [8.33] and [8.36]) to the corresponding eigenvalue Λ_k:

$$\tau_k = \ell^2 / a\Lambda_k^2$$

For small values of P, the first term of the development of Λ_1 as a function of P is equal to $\sqrt{2P}$ (equation [8.54]). Substituting this value into the above expression and replacing P by $\ell / \lambda R$, we find for the time constant τ_1:

$$\tau_1 = \frac{\ell^2}{a\Lambda_1^2} = \frac{\ell^2}{2aP} = \frac{\rho C \ell R}{2}$$

We find the time constant of the thermodynamic model of section 8.3.1.2 in the particular case where we have $R' = 0$, with $m = \rho \ell$.

8.3.2.3.4. Expression of the solution

The complete solution of the problem can be obtained from formula [8.18] as the development of the difference $T(x,0) - T_e(x,0)$ as a series of eigenfunctions of the problem; these are orthogonal if the operator of the problem is self-adjoint (Appendix 4). The verification of this property can be checked easily by separately considering the intervals [-1,0_] and [0$_+$,1]; and taking account of the condition $g'(-1) = 0$, we have, after integration by parts, for p not equal to q:

$$\int_{-1}^{0} g_p g_q'' d\tilde{x} = \int_{-1}^{0} g_p dg_q' = g_p(0_-) g_q'(0_-) - \int_{-1}^{0} g_p' g_q' d\tilde{x}$$

Proceeding in a similar manner for the other interval, we obtain:

$$\int_{-1}^{1} \left[g_p g_q'' - g_q g_p'' \right] d\tilde{x} = \\ g_p(0_-) g_q'(0_-) - g_q(0_-) g_p'(0_-) - g_p(0_+) g_q'(0_+) + g_q(0_+) g_p'(0_+)$$

Taking account of conditions [8.48] for the thermal resistance at $x = 0$ and after replacing g_p'' and g_q'' by their expressions taken from equation [8.50], we see that the integral $\int_{-1}^{1} \left[g_p g_q'' - g_q g_p'' \right] d\tilde{x}$ is zero if p is not equal to q:

$$\int_{-1}^{1} \left[g_p g_q'' - g_q g_p'' \right] d\tilde{x} = \left(\Lambda_p^2 - \Lambda_q^2 \right) \int_{-1}^{1} g_p g_q dx = 0$$

The eigenfunctions g_p and g_q are orthogonal for the scalar product $< f, g >= \int_{-1}^{1} fg d\tilde{x}$.

The complete solution [8.18] can be written:

$$T(\tilde{x},\tilde{t}) = T_e(\tilde{x},\tilde{t}) + \sum_{k=0}^{\infty}\left(c_{2k} e^{-k^2\pi^2\tilde{t}} \cos(k\pi\tilde{x}) + c_{2k+1} e^{-\Lambda_{2k+1}^2\tilde{t}} g_{2k+1}(\tilde{x})\right) \quad [8.55]$$

$$\text{with } c_{2k+1} = \frac{\int_{-1}^{+1}\left[T(\tilde{x},0)-T_e(\tilde{x},0)\right]g_{2k+1}(\tilde{x})d\tilde{x}}{\int_{-1}^{+1} g_{2k+1}^2(\tilde{x})d\tilde{x}} \quad (k=0,1,2,\ldots,n,\ldots)$$

$$c_0 = \frac{1}{2}\int_{-1}^{+1}\left[T(\tilde{x},0)-T_e(\tilde{x},0)\right]d\tilde{x};\, c_{2k} = \int_{-1}^{+1}\left[T(\tilde{x},0)-T_e(\tilde{x},0)\right]cos(k\pi\tilde{x})d\tilde{x} \quad (k=1,2,\ldots,n,\ldots)$$

8.3.2.4. *Modal representation of systems and number of parameters*

The modal representation of a discrete system consists of replacing the n state variables $x_i(t)$, functions of time, with n coefficients c_i of the expression of the solution on the basis of the eigenfunctions whose time dependence is known. We thus see that, assuming the eigenfunctions to be known, *the number of numerical values (series development coefficients) which characterize the solution decreases with time*.

The same is true for the preceding models for continuous media, for which we replace the temperature distribution $T(x,t)$ in two continuous variables by a denumerable sequence of series development coefficients c_i [8.55] of eigenfunctions associated with the physical model used. In a manner analogous to the development of a periodic function in a Fourier series, the information necessary to characterize the solution has been considerably reduced through the use of basis functions adapted to the problem.

Furthermore, these developments offer the advantage that they provide an organization of information. We saw in Chapter 7 that the frequencies associated with the terms of a Fourier series are greater as the order of these terms is high. This thus results in criteria regarding: the nature of the approximation which has been made where only a limited number of terms are retained, and regarding the sampling of the corresponding temporal signals (Shannon's theorem). The modal representation of thermal systems indicates that the wealth of information decreases as time increases. For continuous media, this information diffuses from the wall and becomes progressively poorer. The results are that the representation using space-time numerical data (x,t) is very variable. During the first instants, we need to discretize the time and near-wall zones very densely, while the central zone of the wall is not subjected to any phenomenon. For larger times, the entire wall needs to be discretized, but less densely, and the same goes for the time variable. We will note that, as the modes vanish successively, the quantity of information contained in the solution decays with time, the effect of the second law of thermodynamics (increasing entropy) being to homogenize the temperature.

8.4. External representation of linear invariant systems

8.4.1. *Overview*

Transitional solutions of differential system [8.13] of n linear first order differential equations with n unknowns form a vector space of dimension n. They allow the characterization of all possible states described by [8.13]. In practice, it often happens that the initial data of a category of problems is of reduced number p, the system starting for example from a given standard state (rest for example). The transitional solutions which correspond to these initial conditions belong to a family which depends only on these p parameters. The state representation of a system of n state variables thus constitutes an overly detailed model. Rather than establishing a state representation, we can be satisfied with the study of system responses by means of an *external representation* by limiting ourselves to the effective inputs and initial conditions.

Consider for example a linear invariant system for which the inputs are zero. This is the modeling of an isolated homogenous system which evolves towards equilibrium and which is characterized by a zero value for the state vector and the outputs. Suppose that we impose as initial conditions given values (step functions for example) for p components ($p < n$) of the state vector, the other components being zero. This initial vector must be decomposed on eigenvectors of the homogenous system, and apart from some particular cases, none of these components is zero. The responses to these excitations comprise the n eigenmodes, but they belong to a family of only p parameters which form a vector sub-space of dimension p of the output space, which we can study directly.

8.4.2. *External description of linear invariant systems*

8.4.2.1. *Impulse response*

The solution corresponding to diverse inputs and initial conditions is the sum of solutions for each of the inputs and initial conditions taken separately, the others being considered zero. This superposition property for solutions of time-invariant linear systems allows us to separately study the response to different inputs and initial conditions.

Consider firstly a system with a single input u_1 to which we apply a Dirac distribution at the initial instant, the system being *initially at rest*; the output vector

$h_{Y1}(t)$ represents the evolution of the system outputs to this impulse on the first input. Now, any given input $u_1(t)$ can be considered as the superposition of Dirac distributions:

$$u_1(t) = \int_0^{\infty} u_1(\tau)\delta(t - \tau)d\tau$$

The response $Y_1(t)$ of the system outputs to the excitation $u_1(t)$ can be written:

$$Y_1(t) = \int_0^{\infty} h_{Y1}(t - \tau)u_1(\tau).d\tau \qquad [8.56]$$

The *impulse response* of the output vector therefore makes the characterization of the response of a stationary linear system to any input excitation possible.

Proceeding similarly for the group of *p* inputs, we obtain a rectangular matrix $(h_{Y1}(t), h_{Y2}(t), ..., h_{Yp}(t))$ of dimension $q * p$ formed with the *p* corresponding output vectors, the entire group of which constitutes the *matrix* $H_Y(t)$ *of the impulse response output vectors* of the system. This allows us to calculate the output corresponding to any given input vector $U(t)$:

$$Y(t) = \int_0^{\infty} H_Y(t - \tau).U(\tau).d\tau \qquad [8.57]$$

The *impulse response matrix* $H_Y(t)$ *constitutes an external description* of the system taken to be initially at rest. *Easily realizable experimentally,* it can be obtained using only impulse excitations, without any knowledge of the internal variables in the initial reference state. It also *allows us to describe the evolutions corresponding to the given non-zero initial states of the system* which are *obtainable by using suitable impulses on the inputs* (section 8.2.1.5).

As an input vector is usually of smaller dimension than the state vector, the *impulse response matrix* $H_Y(t)$ is *a reduced model of the system adapted to its operating conditions* and which is not equivalent to the state representation of section 8.1.2.

This matrix contains *p*q* time functions whose useful duration is equal to a number of times the largest damping time constant. The quantity of information is here much greater than in the case of the state representation: this is the result of the absence of a model. For the linear time-invariant[5] thermal system, each of the

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[5] Responses of stable systems usually studied in fluid mechanics and acoustics present an oscillatory damped character.

functions is a sum of decaying exponentials which we can seek to identify approximately, at least for the first of these. In the absence of a state representation, any interpretation of results is difficult and the obtaining of the structure of the eigenmodes of the system is problematic.

The result of formula [8.57] is that at a given instant, the state of a linear invariant system depends only on the inputs belonging to a past which is at most equal to the duration of the impulse response.

NOTE – The preceding procedure can also be applied to a state vector $X(t)$. The application of a Dirac impulse as an input of index i provides the state vector $h_{Xi}(t)$. Operating in a similar fashion on the ensemble of inputs, we obtain the *matrix* $H_X(t)$ *of impulse responses* of the state vector, of dimension $n * p$, which allows us to obtain the response of the state vector to any given input $U(t)$:

$$X(t) = \int_0^\infty H_X(t-\tau).U(\tau).d\tau$$

However, the direct measurement of state variables is not easy in general, and the impulse response matrix $H_X(t)$ of the state vector is only of limited interest.

8.4.2.2. *Inputs-outputs analytic representation*

The complete external representation of a system can be obtained from its state representation, cast in the form of a scalar differential equation of order n (section 5.1.1 and section 8.1.2) in a scalar state variable x (t). Let us take the simple example ($n = 3$) of the state representation with a single output $y(t)$:

$$a_0 x + a_1 x' + a_2 x'' + a_3 x''' = u; \quad y = d_0 x + d_1 x' + d_2 x'' \qquad [8.58]$$

Calculating the linear combination $d_0 u + d_1 u' + d_2 u''$ and eliminating $x(t)$ and its derivatives with [8.58], we obtain a differential equation for the output y (t):

$$a_0 y + a_1 y' + a_2 y'' + a_3 y''' = d_0 u + d_1 u' + d_2 u'' \qquad [8.59]$$

The method can be generalized to a differential equation of arbitrary order:

$$\sum_{i=0}^{n} a_i x^{(i)} = u; \quad y_j = \sum_{k=0}^{p} d_{jk} x^{(k)} \qquad [8.60]$$

We obtain:

$$\sum_{i=0}^{n} a_i y_j^{(i)} = \sum_{i=0}^{n} \sum_{k=0}^{p} a_i d_{jk} x^{(i+k)} = \sum_{k=0}^{p} d_{jk} u^{(k)} \qquad [8.61]$$

The inverse passage of the preceding external representation to a state representation can be achieved easily, the coefficients of relations [8.60] and [8.61] (or [8.58] and [8.59]) being identical.

The *representation of a dynamic system of n variables, effected by means of a differential equation of order n, can be replaced by a relation of a differential nature between the inputs and the outputs*. However, obtaining the preceding external representations by means of measurements, without any knowledge of the structure of the system, is impractical for complex systems: the evaluation of the temporal derivatives obtained from differences between measurements lead to errors which will be greater as the order of the differentiation is increased.

8.4.2.3. *Example of a thermal system with two variables*

As an example, consider the state representation of the two-component system of section 8.3.1.2:

$$T_1^{'} = a_{11}T_1 + a_{12}T_2 + u_1; \quad T_2^{'} = a_{21}T_1 + a_{22}T_2 + u_2 \qquad [8.62]$$

where the quantities u_{i} and a_{ij} are defined in equation [8.21]:

$$a_{11} = a_{22} = -\frac{1}{mC}\left(\frac{1}{R} + \frac{1}{R'}\right); \quad a_{12} = a_{21} = \frac{1}{mCR}; \quad u_i = \frac{T_{iex}}{mCR'}\,(i = 1{,}2).$$

Differentiating the first equation [8.62] and eliminating $T_2^{'}$ by means of second relation [8.62], we obtain a second order differential equation satisfied by T_1:

$$(a_{11}a_{22} - a_{12}a_{21})T_1 - (a_{11} + a_{22})T_1^{'} + T_1^{''} = -a_{22}u_1 + u_1^{'} + a_{12}u_2 \qquad [8.63]$$

Taking the output variable in the form [8.58]: $y = d_0 T_1 + d_1 T_1^{'}$, the external representation can be obtained from [8.59]:

$$(a_{11}a_{22} - a_{12}a_{21})y - (a_{11} + a_{22})y' + y'' = d_0 u + d_1 u' \qquad [8.64]$$

with: $u = -a_{22}u_1 + u_1^{'} + a_{12}u_2$.

8.4.2.4. *Laplace transforms and operational transfer matrices*

We have already seen in Chapter 7 the value of Fourier transforms for the study of oscillatory signals. The Laplace transform (Appendix 1) presents an analogous interest for the study of aperiodic signals and damped systems. The Laplace transform L_f of a function *f(t)* is defined by the relation:

$$L_f(p) = \int_0^\infty f(t) e^{-pt} dt \qquad [8.65]$$

It possesses diverse properties which are analogous to those of the Fourier transform (section 7.3.4.2); in particular, the Laplace transform of the derivative function *f* is written:

$$L_{f'}(p) = pL_f(p) - f(0)$$

This property allows us to transform the system of differential equations and the initial conditions into a system of algebraic equations between the transforms of the variables. For example, the differential equation transform:

$$m\ddot{x} + f\,\dot{x} + kx = u(t)$$

can be written:

$$\left(mp^2 + f\,p + k\right)L_x - px(0) - x'(0) = L_u$$

or:

$$L_x(p) = \frac{L_u(p) + px(0) + \dot{x}(0)}{mp^2 + f\,p + k} \qquad [8.66]$$

We thus obtain the transform of the complete solution written as a function of the initial conditions and the transform $L_u(p)$ of the right-hand side (the input). All that remains is to perform the inverse transform (Appendix 1).

The transform linearity allows it to be applied to a vector or to a matrix of functions. For the initial conditions $X(0)$, the Laplace transform of the state representation [8.1] of a linear invariant dynamic system can be written:

$$pL_X - X(0) = AL_X + B.L_U; \quad L_Y = D.L_X$$

We can thus deduce the transforms of the solution $X(t)$ and of the output vector $Y(t)$:

$$L_X(p)=(pI-A)^{-1}\left[BL_U(p)+X(0)\right]: \quad L_Y=D.L_X. \tag{3.67}$$

As the inverse of a matrix M is equal to the ratio between the transpose of its comatrix[6] and its determinant ([HAR 98]), the elements of the matrix $(pI-A)^{-1}$ of rank n are rational fractions whose denominator is equal to $|pI-A|$, the numerator being a polynomial in p whose order is at least equal to n. This rational fraction can be decomposed into simple elements associated with the *poles* (zeros of the denominator) which are the *eigenvalues* of the system studied. Each of these elements is the Laplace transform of the real or complex exponential functions which correspond to the eigenmodes of the system studied.

The matrix $H(p)=D.(pI-A)^{-1}B$ is the *operational transfer matrix* of the system:

$$L_Y(p)=D.(pI-A)^{-1}B.L_U(p)=H(p).L_U(p) \tag{3.68}$$

As the Laplace transform of a Dirac distribution at instant t = 0 is equal to 1, the transfer matrix $H(p)$ is the *Laplace transform of the impulse response matrix* $H_Y(t)$.

The Laplace transform is particularly useful in the domain of system controls which are essentially beyond the scope of this textbook, despite applications in the domain of flow and transfer phenomena.

8.5. Parametric models

8.5.1. *Definition of model parameters*

A parametric model can be defined as an exact or approximate solution whose state vector $X(t)$ can be written in the form of a combination (linear or otherwise) of some *state vectors $X_j(t)$ independent of the inputs*; the coefficients Pj of this combination are time functions which are deduced from the input vector $U(t)$ (the coefficients Pj are functionals of $U(t)$). For a linear combination, we have:

---

[6] I.e. the matrix of cofactors of each element.

$$X(t) = \sum_j X_j P_j(U(t)) \tag{8.69}$$

Expression [8.69] is quite analogous to a series development of eigenfunctions. Like these, the vectors X_j are *universal*, as they do not depend on any particular inputs of the problem. The coefficients P_j are parameters characterizing, in a simple manner, the solution represented at each instant as a basis composed of state vectors *Xj* which are often chosen in order to best represent the solution *for a class of given inputs*. Their non-dimensional expressions are often called *form parameters*, as they characterize the relative value distribution of the state vector components (see example in section 8.5.2). We have adopted the terminology *parametric model* in reference to the idea of form parameters often used in fluid mechanics.

The output vector $Y = D.X$ can be written in an analogous form [8.69] with the universal vectors $D\,X_j$ and the parameters P_j:

$$Y(t) = DX(t) = \sum_j DX_j P_j(U(t)) \tag{8.70}$$

We can note that the solution of the system obtained by solving equations [8.14] and [8.15] of the state representation using the variation of constants method is not in general of the form [8.69] or [8.70].

Solutions of parametric type can also be obtained in an approximate manner by means of global methods. Some examples of parametric solutions of linear systems will be covered in the following section. Such exact or approximate solutions are also used in fluid mechanics for the study of the boundary layer (Blasius and Howarth series, Görtler series, Karman-Polhausen global method, etc. ([SCH 99], [YIH 77])).

The value of the parametric methods is quite significant:

– *with the objective of understanding and interpreting physical phenomena:* the values of the parameters P_j allow in effect the classing of excitation situations by the function $U(t)$ by characterizing these situations using simple analytical forms of the solution. We will see examples of the application;

– *for writing models of complex systems:* expression [8.70] is an *external explicit representation* of a system in which the inputs and outputs are variables which characterize the interactions between neighboring systems; limited to a small number of terms, it constitutes an approximate representation which is particularly useful for modeling of interacting systems.

8.5.2. *Established regimes of linear invariant systems*

8.5.2.1. *Systems described by a state representation*

In section 8.2.2.1 we defined an *established regime* as a solution of a differential system whose expression does not contain transitional terms associated with the initial conditions. The solutions of discrete time-linear linear systems can be *cast* in a simple parametric form, at least when the inputs do not vary too rapidly. The guiding idea of the method has already been outlined in section 6.4.2.6; it consists of considering the unsteady term of state equation [8.1] as a perturbation [SAD 93]. By introducing the parameter ε which we will take to be equal to one in the following, the state equation [8.1] can be written:

$$\varepsilon \frac{dX}{dt} = A.X + B.U \qquad [8.71]$$

We seek the solution of equation [8.71] in the form:

$$X(t) = \sum_{0}^{\infty} \varepsilon^{i} X_i(t) \qquad [8.72]$$

By substituting expression [8.72] into equation [8.71], and by identification according to the increasing powers of the parameter ε, we obtain the system:

$$\begin{array}{ll} A.X_0(t) + B.U(t) = 0; & \ldots\ldots\ldots\ldots\ldots \\ A.X_1(t) = X_0'(t); & A.X_i(t) = X_{i-1}'(t); \\ A.X_2(t) = X_1'(t); & \ldots\ldots\ldots\ldots\ldots \end{array} \qquad [8.73]$$

Assuming the matrix A to be invertible, we deduce:

$$\begin{array}{ll} X_0(t) = -A^{-1}B.U(t); & \ldots\ldots\ldots\ldots\ldots \\ X_1(t) = -A^{-2}B.U'(t); & X_i(t) = -A^{-i-1}B.U^{(i)}(t); \\ X_2(t) = -A^{-3}B.U''(t); & \ldots\ldots\ldots\ldots\ldots \end{array}$$

Substituting and taking ε to be formally equal to 1, we finally obtain:

$$
\begin{aligned}
&X(t) = -A^{-1}B.U(t) - \sum_{i=1}^{\infty} A^{-i-1}B.U^{(i)}(t) \\
&Y(t) = C.X + D.U = \left[-CA^{-1}B. + D\right]U(t) + C.\left[-\sum_{1}^{\infty} A^{-i-1}B.U^{(i)}(t)\right]
\end{aligned}
\qquad [8.74]
$$

Series [8.74] is a general solution to the established regime of system [8.1]. Independent of the initial conditions, it converges if the successive derivatives of the input vector U do not vary too quickly, for example if the quantities $\left|u^{(i)}(t)\right|$ are bounded by $k\Lambda_1^i$ (Λ_1: smallest eigenvalue of the matrix A, k constant).

The idea of *form parameters* can be introduced by means of the simple example of a system with a scalar input $u(t)$. Solution [8.74] can thus be written:

$$
X(t) = -u(t)\left[A^{-1}B. + \sum_{i=1}^{\infty} A^{-i-1}B.\frac{u^{(i)}(t)}{u(t)}\right] \qquad [8.75]
$$

The parameters $u^{(i)}(t)/u(t)$ are form parameters of the problem and determine the structure of the solution.

The Laplace transform allows the preceding results to be retrieved by giving an explicit expression of the term $(pI - A)^{-1} BL_U(p)$ of the established regime of equation [8.67]. We obtain:

$$
(pI - A)^{-1} = -A^{-1}\left(I - pA^{-1}\right)^{-1} = -A^{-1}\left[I + pA^{-1} + p^2A^{-2} + .. + p^nA^{-n} + .\right]
$$

or:

$$
L_X = -(A - pI)^{-1}B.L_U = -A^{-1}\left[I + pA^{-1} + p^2A^{-2} + ... + p^nA^{-n} + ...\right]B.L_U
$$

This expression is indeed the Laplace transform of the first equation [8.74].

NOTES –

1) Temporal series of the form [8.74] (or [8.76]) which describes in parametric form how the established regimes can be derived from responses obtained by the convolution integral [8.57] in which we perform a Taylor series development. For example, for the state vector, taking account of the commutativity of the convolution product, we have:

$$X(t)=\int_0^{\infty} U(t-\tau).H(\tau)\,d\tau=\int_0^{\infty}\left[U(t)-\tau U'(t)+\frac{\tau^2}{2}U'(t)+\ldots\right]H(\tau)\,d\tau$$

$$=U(t)\int_0^{\infty} H(\tau)\,d\tau-U'(t).\int_0^{\infty}\tau H(\tau)\,d\tau+U''(t)\int_0^{\infty}\frac{\tau^2}{2}H(\tau)\,d\tau+\ldots$$

2) The perturbation method used can also be applied to non-linear equations.

8.5.2.2. *Case of representation by differential equations of order n*

The preceding method can be applied to write the established system and the solution to the differential equation [8.60] of order *n*, representing system [8.1], with a single state variable *x* (*t*):

$$\sum_{i=1}^{n} a_i x^{(i)} = u(t); \quad y_j = \sum_{k=0}^{n-1} d_{jk} x^{(k)}$$

We obtain, *for the established regime*:

$$\left(\sum_{i=1}^{n} a_i p^i\right) L_x(p) = L_u(p); \quad L_{yj} = \left[\sum_{k=0}^{n-1} d_{jk} p^k\right] L_x(p)$$

or:

$$L_x(p)=\frac{L_u(p)}{\sum_{i=1}^{n} a_i p^i}; \quad L_{yj}=\frac{\sum_{k=0}^{n-1} d_{jk} p^k}{\sum_{i=1}^{n} a_i p^i} L_u(p)$$

The established regime for small values of *p* can be obtained by calculating the above fraction series development following the increasing powers of *p*, for example, by means of a division following increasing powers.

By limiting ourselves to the third term, we obtain:

$$L_x(p)=\frac{L_u(p)}{a_0\left(1+\frac{a_i}{a_0}p^i\right)}=$$

$$\frac{1}{a_0}\left(1-\frac{a_1}{a_0}p+\left(\frac{a_1^2}{a_0^2}-\frac{a_2}{a_0}\right)p^2-\left(\frac{a_3}{a_0}-\frac{2a_2a_1}{a_0^2}+\frac{a_1^3}{a_0^3}\right)p^3+\ldots\right)L_u(p)$$

whence the parametric expression for the state variable $x(t)$:

$$x(t) = A_0 .u(t) + A_1 u'(t) + A_2 u''(t) + A_3 u'''(t) + \ldots \qquad [8.76]$$

with $A_0 = \dfrac{1}{a_0}$; $A_1 = -\dfrac{a_1}{a_0^2}$; $A_2 = \dfrac{a_1^2}{a_0^3} - \dfrac{a_2}{a_0^2}$; $A_3 = -\dfrac{a_3}{a_0^2} + \dfrac{2a_2 a_1}{a_0^3} - \dfrac{a_1^3}{a_0^4}$;....

8.5.2.3. *External representations*

The same procedure can be employed for the input-output representation [8.59]:

$$a_0 y + a_1 y' + a_2 y'' + a_3 y''' = d_0 u + d_1 u' + d_2 u''$$

for which the Laplace transform can be written:

$$\left(a_0 + a_1 p + a_2 p^2 + a_3 p^3\right) L_y(p) = \left(d_0 + d_1 p + d_2 p^2\right) L_u(p)$$

or:

$$L_y(p) = \frac{d_0 + d_1 p + d_2 p^2}{a_0 + a_1 p + a_2 p^2 + a_3 p^3} L_u(p)$$

By dividing the fraction in p following increasing powers, and then returning to the space-time domain, we obtain the differential representation:

$$y(t) = \frac{d_0}{a_0} u(t) + \left(\frac{d_1}{a_0} - \frac{d_0 a_1}{a_0^2}\right) u'(t) + \left(\frac{d_0 a_1^2}{a_0^3} - \frac{d_0 a_2}{a_0^2} - \frac{d_1 a_1}{a_0^2} + \frac{d_2}{a_0}\right) u''(t) + \ldots$$

8.5.2.4. *Harmonic established regimes*

A system can also be characterized in spectral space. We thus proceed in a manner analogous to the case of impulse excitation, but with harmonic excitations in complex form $e^{j\omega t}$. The established harmonic regimes of the linear invariant systems can be studied by means of the complex amplitudes method by searching for solutions in the form $X = \hat{X} e^{j\omega t}$, where we denote by $\hat{M}$ the constant complex matrix associated with the matrix function $M(t) = \hat{M} e^{j\omega t}$, the temporal derivation having been replaced by a multiplication by $j\omega$. System [8.1] can be written:

$$j\omega\hat{X} = A.\hat{X} + B\hat{U} \tag{8.77}$$

Solving linear algebraic system [8.77] makes it possible to calculate complex amplitudes of the state vector $\hat{X}(\omega)$ and the output vector $\hat{Y}(\omega)$ for all values of the pulsation ω. We obtain complex algebraic equations analogous to those obtained by Laplace transform for the established regimes, where the variable p is replaced by $j\omega$. The complex vector amplitudes $\hat{X}(\omega)$ and $\hat{Y}(\omega)$ can be written:

$$\hat{X}(\omega) = \left(j\omega I - A\right)^{-1} B.\hat{U}(\omega);$$
$$\hat{Y}(\omega) = \hat{H}(\omega).\hat{U}(\omega) \quad \text{with: } H\left(\omega\right) = D.\left(j\omega I - A\right)^{-1} B$$

The matrix $H(\omega)$ is the *transfer matrix* of the system between the inputs and the outputs. It is *the Fourier transform of the impulse response matrix*.

The harmonic regimes of *external representations* obtained above can be obtained in the same way. For example, from relation [8.59] we can derive the complex amplitude of the output $\hat{y}$ as a function of the input $\hat{u}(\omega)$:

$$\hat{y} = \frac{d_0 + jd_1\omega - d_2\omega^2}{a_0 - a_2\omega^2 + j\omega\left(a_1 - a_3\omega^2\right)}\hat{u}$$

The *parametric representations* obtained earlier can be written in the harmonic regime, by performing a series development in increasing powers of $j\omega$. For example, with the discrete system of equations [8.1], we have, from [8.74]:

$$\hat{X} = -\sum_{n=0}^{\infty}\left(j\omega\right)^n A^{-n-1}B.\hat{U}.; \quad \hat{Y}(\omega) = \left(D - C\sum_{n=0}^{\infty}\left(j\omega\right)^n A^{-n-1}B\right).\hat{U}$$

The preceding series developments in ω are a *valid approximation for the low frequencies*. We will obtain in section 8.5.3 an estimation of the radius of convergence of the preceding series in ω in an example.

In the study of certain vibration problems (mechanical, electrical and electromechanical, acoustics), we define impedances which are functions of the transfer between the intensive quantities (forces, electric potentials, acoustic pressure) and the flux of associated extensive[7] quantities (velocity, electrical intensity, acoustic flux) evaluated at the same point. These impedances are in fact a

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[7] These extensive quantities can be reduced, for instance in the case of mechanical impedances.

particular kind of transfer function. We will assume that these ideas are known to the reader at least in the domain of electricity.

NOTE – The different external representations discussed above are equivalent. In practice, we use the form which is best adapted to the problem under study.

8.5.3. *Established regimes in continuous media*

8.5.3.1. *Systems described by a continuous state representation*

Consider the cylinder of axis *Ox*, of cross-section limited by the arbitrary curve *C* (Figure 6.13 of section 6.4.2.6) and on the wall of which we impose the uniform temperature T_W(t). The 2D temperature distribution $T(y,z,t)$ satisfies the equations and conditions:

$$\frac{\partial T}{\partial t} = a\,\Delta T; \quad \text{with : } (y,z) \in C : \ T(y,z,t) = T_w(t) \qquad [8.78]$$

Let us take the dimensionless variables:

$$\tilde{y} = y/\ell; \quad \tilde{z} = z/\ell; \quad \tilde{t} = a\,t/\ell^2$$

Equation [8.78] becomes:

$$\frac{\partial T}{\partial \tilde{t}} = \tilde{\Delta} T \quad \text{with : } (\tilde{y},\tilde{z}) \in C : \ T(\tilde{y},\tilde{z},\tilde{t}) = T_w(\tilde{t}) \qquad [8.79]$$

Let us look for a solution in the form [PEU 84]:

$$T(\tilde{y},\tilde{z},\tilde{t}) = \sum_{i=0}^{\infty} g_i(\tilde{y},\tilde{z}) T_w^{(i)}(\tilde{t}) \qquad [8.80]$$

Substituting into [8.79] and identifying the terms corresponding to the derivatives $T_w^{(i)}(\tilde{t})$, we obtain:

$$\tilde{\Delta} g_0 = 0; \quad \tilde{\Delta} g_1 = g_0; \quad \tilde{\Delta} g_2 = g_1; \quad \dots \ \tilde{\Delta} g_n = g_{n-1}; \dots$$

with the boundary conditions: $(\tilde{y},\tilde{z}) \in C : g_0(\tilde{y},\tilde{z}) = 1; \ \ g_i(\tilde{y},\tilde{z}) = 0 \quad i = 1,\dots,n,\dots$

The functions $g_i(\tilde{y},\tilde{z})$ can thus be calculated from place to place. The reader can immediately verify that these are identical to the corresponding functions of the unsteady flow problem of a viscous fluid in a cylindrical tube of contour *C* (section 6.4.2.6), a difference of unity excepted for the value of index *i*.

The method used can be applied to *continuous systems containing discrete elements*. Let us take the system constituted of two walls separated by a thermal resistance (section 8.3.2.3.1). Consider here the case where one of the faces is maintained at constant temperature $T(-\ell) = 0$, the other face at abscissa $+\ell$ being subject to the variable temperature $T_w(t)$. Taking the dimensionless variables [8.33] ($\tilde{x} = x/\ell$ and $\tilde{t} = at/\ell^2$), we obtain the heat equation on the intervals [-1,0] and [0,1]:

$$\frac{\partial T}{\partial \tilde{t}} = \frac{\partial^2 T}{\partial \tilde{x}^2}; \quad \text{with:} -1 \leq \tilde{x} < 0,\ 0 < \tilde{x} \leq 1 \qquad [8.81]$$

As the thermal resistance *R* is characterized by relation [8.49] (with $P = \ell/\lambda R$), we have the boundary conditions (section 8.3.2.3.1):

$$T(-1) = 0;\ T(+1) = T_w(\tilde{t}); \quad \left(\frac{\partial T}{\partial \tilde{x}}\right)_{0+} = \left(\frac{\partial T}{\partial \tilde{x}}\right)_{0-} = P\,[T(0_+) - T(0_-)] \qquad [8.82]$$

We can easily verify that the equation and the preceding boundary conditions possess a solution analogous to [8.80]:

$$T(\tilde{x},\tilde{t}) = \sum_{i=0}^{\infty} g_i(\tilde{x}) T_w^{(i)}(\tilde{t}) \qquad [8.83]$$

The functions $g_i(\tilde{x})$ satisfy the following successive relations and conditions:

$$g_0''(\tilde{x}) = 0; \quad g_1''(\tilde{x}) = g_0(\tilde{x}); \quad g_2''(\tilde{x}) = g_1(\tilde{x}); \ \ldots\ldots \ g_n''(\tilde{x}) = g_{n-1}(\tilde{x}) \ldots$$

$$i = 1,\ldots,\infty : g_i(-1) = g_i(1) = 0; \quad g_i'(0_+) = g_i'(0_-) = \frac{\ell}{\lambda R}[g_i(0_+) - g_i(0_-)]$$

$$g_0(-1) = 0; \quad g_0(1) = 1;$$

We can easily calculate the first two functions:

$$\begin{aligned}
&-1 \leq \tilde{x} < 0 \quad g_0 = \frac{P(\tilde{x}+1)}{1+2P}; \quad g_1 = \frac{P(\tilde{x}+1)}{6(1+2P)}\left[(\tilde{x}+1)^2 - \frac{8P+6}{1+2P}\right] \\
&0 < x \leq 1 \quad g_0 = \frac{P(\tilde{x}-1)}{1+2P} + 1; \\
&\qquad g_1 = \frac{P(x-1)}{6(1+2P)}\left[(\tilde{x}-1)^2 - \frac{8P+6}{1+2P}\right] + \frac{\tilde{x}^2 - 1}{2}
\end{aligned} \tag{8.84}$$

The first function $g_0(\tilde{x})$ is the temperature distribution in the steady regime. The reader can easily certify that this corresponds to the thermal resistances in series associated with two continuous media and to the resistance R. The thermal flux density on the walls is derived from the following expressions:

$$\left.\begin{aligned}
q_{Tp}(-\ell) &= -\lambda\left(\frac{\partial T}{\partial x}\right)_{-\ell} = -\frac{\lambda}{\ell}\left(\frac{P\,T_p(t)}{1+2P} - \frac{P(4P+3)}{3(1+2P)^2}\frac{\ell^2}{a}T'_p(t) + \ldots\right) \\
q_{Tp}(\ell) &= -\lambda\left(\frac{\partial T}{\partial x}\right)_{\ell} = -\frac{\lambda}{\ell}\left(\frac{P\,T_p(t)}{1+2P} + \left[1 - \frac{P(4P+3)}{3(1+2P)^2}\right]\frac{\ell^2}{a}T'_p(t) + \ldots\right)
\end{aligned}\right| \tag{8.85}$$

More generally, M being an operator which only contains spatial derivatives, let us consider equation [8.86]:

$$M\,f = \frac{\partial f}{\partial t} \tag{8.86}$$

which possesses parametric solutions of the form:

$$T(x,y,z,t) = \sum_{i=0}^{\infty} g_i(x,y,z) T_w^{(i)}(t)$$

The first function $g_0(x,y,z)$ is the spatial distribution of the function f in the steady regime corresponding to the steady conditions imposed which define the reference temperature T_W. The universal functions g_i satisfy the successive equations:

$$Mg_0(x,y,z) = 0;\ Mg_1(x,y,z) = g_0(x,y,z);\ \ldots\ Mg_n(x,y,z) = g_{n-1}(x,y,z);\ \ldots$$

For example, the thermal conduction equation for a cylindrical problem in a heterogenous medium can be written:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho(y,z)c(y,z)}\left[\frac{\partial}{\partial y}\left(\lambda(y,z)\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\lambda(y,z)\frac{\partial T}{\partial y}\right)\right]$$

It corresponds to the operator $M = a(y,z)\dfrac{\partial}{\partial y}\left(\lambda(y,z)\dfrac{\partial T}{\partial y}\right)$:

$$M = \frac{1}{\rho(y,z)c(y,z)}\left[\frac{\partial}{\partial y}\left(\lambda(y,z)\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial y}\left(\lambda(y,z)\frac{\partial}{\partial y}\right)\right]$$

The case where the cylinder is the seat of a *volume heat source* σ_T(t) can be treated in an analogous manner to that of the problem of viscous flow in a cylindrical tube, as studied in section 6.4.2.6:

$$\frac{\partial T}{\partial t} = \sigma_T(t) + a\Delta T; \quad \text{with: } (y,z)\in C: \; T(y,z,t) = 0$$

The equations of the universal functions g have been established in section 6.4.2.6. The parameters P_j of formula [8.69] which characterize the input are here the successive derivatives $\sigma_T^{(i)}(t)$.

8.5.3.2. *Established boundary layer regimes*

The regular perturbation method used for parametric solutions in a finite domain can be extended to the case of boundary layers, on the condition that the problem be cast as a regular problem. Let us reconsider the example of a thermal shock in a semi-infinite medium as described in section 5.4.5.4 (and applied at the end of section 8.3.2.2.2) and verifying the heat conduction equation [8.78] but with the variable wall temperature $T_w(t)$. In the plane (x, t) we have a singular perturbation in the vicinity of $x = 0$ where the solution obtained presents very different scales: for small time values the spatial variations of temperature are rapid, whereas for large time values they are slow. The convergence of solutions close to this base solution towards this solution cannot therefore be uniform. The variable adapted to the boundary layer $\eta = x/2\sqrt{at}$ is defined as in section 5.4.5.4. We first of all perform a change of coordinates $(x,t) \to (\eta,\tau)$, in order that the difference between the two solutions can be characterized in a uniform manner:

$$\eta = x/2\sqrt{at} \qquad \tau = t; \qquad T(x,t) = \theta(\eta,\tau)$$

Transformed heat equation [8.32] can be written:

$$\frac{1}{4}\frac{\partial^2\theta}{\partial\eta^2}+\frac{\eta}{2}\frac{\partial\theta}{\partial\eta}=\tau\frac{\partial\theta}{\partial\tau} \qquad [8.87]$$

This equation is of a form analogous to [8.86] where the differentiation operators are separated into two parts, the time derivative constitutes the perturbation term. The calculation is performed as before and we seek a series development of functions in separated variables:

$$\theta(\eta,\varepsilon\tau)=\sum_{i=0}^{\infty}g_i(\tau)f_i(\eta) \qquad [8.88]$$

Letting $M=\frac{1}{4}\frac{d^2}{d\eta^2}+\frac{\eta}{2}\frac{d}{d\eta}$, equation [8.87] can be written:

$$M\theta=\tau\frac{\partial\theta}{\partial\tau}$$

We obtain the recurrence relations:

$$\begin{array}{ll} Mf_0(\eta)=0 & g_0=T_w(\tau) \\ Mf_1(\eta)=f_0(\eta) & g_1(\tau)=\tau g_0'(\tau) \\ Mf_2(\eta)=f_1(\eta), & g_2(\tau)=\tau g_1'(\tau) \\ \ldots\ldots\ldots\ldots & \\ Mf_i(\eta)=f_{i-1}(\eta) & g_i(\tau)=\tau g_{i-1}'(\tau) \\ \ldots\ldots\ldots\ldots & \end{array} \qquad [8.89]$$

which are associated with the boundary conditions:

$$f_i(\infty)=0 \quad (i=0,1,2,\ldots,\infty); \quad f_0(0)=1; \quad f_i(0)=0 \quad (i=1,2,\ldots,\infty)$$

Taking account of the conditions at infinity, the function $f_0(\eta)$ is (see section 5.4.5.4):

$$f_0(\eta)=1-erf\eta$$

The analytical calculation of the functions f_i is difficult: even if the solution to the homogenous equation $Mf_i(\eta)=0$ can be expressed easily as a function of a multiplicative constant (section 5.4.5.4), the variation of constants method is impractical on account of the complexity of calculations. A numerical solution, which is preferable, makes the easy calculation of the first functions f_i possible. The series of functions f_i is alternated and tends asymptotically to two functions equal to $\pm f_\infty(\eta)$ which can be easily determined from system [8.89]:

$$M f_\infty = -f_\infty \quad \text{with: } f_\infty(0) = f_\infty(\infty) = 0$$

The reader can verify that we have: $f_\infty(\eta) = c\,\eta \exp(-\eta^2)$ (c: constant)

However, the recurrence relation $g_i(\tau) = \tau\, g'_{i-1}(\tau)$ does not provide a simple expression for the functions g_i. Furthermore, we note that the perturbation term $\tau\,\partial\theta/\partial\tau$ of equation [8.87], of order $\tau T'_w(t)/T_w(t)$ in relative value, must remain quite small: the temporal derivative of the imposed temperature must decay quickly enough as time increases. From a physical point of view, it seems natural that in order to remain as a small perturbation, the imposed temperature variation decreases with time as the temperature distribution of the base solution T_0 spreads (Figure 8.6). In order to render the perturbation uniform in time, we compress the time scale by performing the temporal variable change:

$$\widehat{t} = \ln\tau \qquad G_i(\hat{t}) = g_i(\tau),$$

which simplifies recurrence relation [8.89] of the functions g_i, which can be written:

$$\begin{array}{lll} G_0(\hat{t}) = T_w(\hat{t}) & G_1(\hat{t}) = T'_w(\hat{t}) & G'_2(\hat{t}) = G'_1(\hat{t}) = T''_w(\hat{t}) \\ \ldots\ldots\ldots\ldots & G_i(\hat{t}) = G'_{i-1}(\hat{t}) = \ldots = T_w^{(i)}(\hat{t}) & \ldots\ldots\ldots\ldots \end{array} \qquad [8.90]$$

The parametric expression of the solutions for the thermal shock is thus:

$$T(x,t) = \sum_{i=0}^{\infty} T_w^{(i)}(\hat{t}) f_i(\eta) \qquad [8.91]$$

The derivatives of the function T_w(t) being taken with respect to the variable $\hat{t} = \ln t$.

The thermal flux density at the wall can be written:

$$q_{Tw} = -\lambda \left.\frac{\partial T}{\partial x}\right|_{x=0} = -\frac{\lambda}{\sqrt{at}} \sum_{i=0}^{\infty} \frac{1}{2} T_w^{(i)}(\hat{t}) f'_i(0)$$

The values of the derivative $f_i'(0)$ are shown in Table 8.2.

i	**0**	**1**	**2**	**3**	**4**
$f_i'(0)/2$	–0.5642	–0.7821	0.3859	–0.3203	0.2985
i	**5**	**6**	**7**	**8**	**9**
$f_i'(0)/2$	–0.2897	0.2857	–0.2838	0.2829	–0.2826

Table 8.2. *Values of the derivative* $f_i'(0)$

The preceding method can also be applied to the expression of solutions in the vicinity of the base solutions of the form $t^n f(\eta)$. In particular, it is easy to verify that the value $n = 1/2$ corresponds to the constant thermal flux which is given at the wall. The solutions where the thermal flux varies gradually can be obtained as before.

8.5.3.3. *Harmonic solutions of the equations for continuous media*

The method described in section 8.5.2.4 can be applied to continuous media by eliminating the temporal variable of the partial differential equation of the problem. Let us take the example of temperature oscillation applied to the surface of abscissa $x = 0$ of a semi-infinite wall. We encounter this problem of oscillatory thermal penetration into rocks or ground which is subjected to daily or annual temperature variations. Assuming a homogenous medium, the temperature $T(x,t)$ satisfies heat equation [8.32] with the following boundary conditions:

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}; \quad T(0,t) = T_0 + \Theta\cos\omega t; \;\; T(\infty,t) = T_0 \qquad [8.92]$$

We seek solutions of the form:

$$T - T_0 = \Theta e^{j\omega t} f(x)$$

The heat equation becomes a differential equation:

$$j\omega f = af''(x)$$

whose characteristic equation ($a\,r^2 - j\omega = 0$) has roots: $r = \pm(1 + j)/\sqrt{2a/\omega}$.

The real solutions sought for the temperature distribution can be written:

$$T - T_0 = \Theta \exp(-x/\delta)\cos(\omega t - x/\delta)$$

The depth $\delta = \sqrt{2\,a/\omega}$ is a space constant for the thermal damping oscillation. The solution represents a temperature oscillation which becomes damped with increasing depth. A numerical application shows that this damping is fast: taking a value of 10^{-6} $m^2.s^{-1}$ for the thermal diffusivity of the ground (corresponding to an average rock), we find that δ is respectively equal to 0.17 meters and to 3.15 meters for daily and annual oscillations of temperature. The oscillation phase is opposite at a depth of $\delta/2$ where amplitude has been reduced by a factor equal to 1.65.

The complex amplitudes method can be applied equally well to problems defined in finite domains (walls, cylinders, etc.).

8.6. Model reduction

8.6.1. *Overview*

The objective of a *knowledge model* is to capture the evolution of a system, the sub-system interactions of which we do not know. It comprises a large number of variables in order to represent all the possible properties. The results involve either a large quantity of numerical results from a computer solution or analytical representations which may be more or less complex. The use of a knowledge model amounts to performing a *numerical experiment,* which is often less expensive than a physical experiment. A model is often too complex for a simple description of a particular category of evolutions. Reduction of this model consists of replacing it with a *reduced model* having a smaller number of variables, and whose objective is to represent the principal phenomena with regard to the objective which is defined (physical understanding of the model mechanisms, elaboration of a simulation program or engineering formulae, control of processes). We are therefore interested in decomposing the system into a small number of components with relatively simple properties and whose interactions can provide a representative description of the system in fixed conditions. Of course, the reduced model is not adapted to the study of other operating conditions in the system.

Model reductions can be performed using various methods. However, *the value of the results of a model or of an experiment always depends on the pertinence of the original hypotheses, on the reasoned use of numerical solutions or physical measurement techniques and on a suitable physical analysis of the phenomena.* Certain methods consist of the performance of a numerical solution of a system by means of a knowledge model, and the coupling of inputs and outputs with representations of an empirical form which can be obtained by least mean square methods; this practice has been used for a long time in order to establish engineering formulae for certain applications. It has evolved into more sophisticated forms through the use of more sophisticated mathematical methods. The use of these methods is only justified as a method of exploiting the calculations resulting from a physical analysis, which evaluates the nature of the approximations that are made and the relative importance of the components in the functioning of a system. We will prefer methods which increase our physical knowledge of the system studied, and we will assume that the variables of the knowledge model have a physical meaning and are not simply numerical values collected from experiments.

8.6.2. *Model reduction of discrete systems*

8.6.2.1. *Principles of reduction of the state representation*

A knowledge model, in the form of a state representation of a system with n state variables or of an equivalent representation (an nth order differential equation in one variable, etc.) makes the detailed description of all possible evolutions of the system. The complexity associated with a large number n of variables leads to it being replaced by a model with s state variables ($s < n$), called a *reduced model*, the objective of which is to represent a particular evolution of the given physical system (including its inputs and outputs) with good accuracy, or more general evolutions with reduced precision. We have already seen some examples in section 8.3.2.2.3.

By definition, a differential system of order s cannot be equivalent to a differential system of order n ($n > s$); however, these systems may have an identical behavior for a family of solutions which depend on at most s parameters. The reduction in the number of variables appearing in the differential equations can only come from a regrouping of the equations, allowing the replacement of several variables by a single variable or simplification of the equations, which lose their differential character. Consider knowledge model [8.93]:

$$\frac{dX}{dt} = A.X + B.U \qquad Y = D.X \tag{8.93}$$

Consider the *reduced state vector* X_r with s components ($s < n$) derived from X by a passage matrix R from X to X_r ($X_r = R\ X$); the state vector X_r satisfies the *reduced state representation*:

$$\frac{dX_r}{dt} = A_r.X_r + B_r.U \qquad Y = D_r X_r \tag{8.94}$$

with conditions [8.95] obtained by multiplying the left-hand side of equation [8.93] by R and by comparing with [8.94]:

$$A_r R = RA; \quad B_r = RB; \quad X_r(0) = RX(0) \quad D_r R = D. \qquad [8.95]$$

If conditions [8.95] are exactly satisfied, representation [8.94] is an *exact reduced model* for the ensemble of solutions X of [8.93] such that RX is non-zero. However, in general, with the exception of cases where these solutions are known explicitly, the realization of an exact reduced model is impractical.

Assuming conditions [8.95] are not satisfied, reduced representation [8.94] is an *approximate reduced model*. The reduced vector $X_r = RX$ of the solution X of [8.93], does not exactly satisfy equation [8.94]; if q is the residuum, the error of equation [8.94]:

$$\frac{dX_r}{dt} = A_r.X_r + B_r.U + q \qquad [8.96]$$

Let $\hat{X}_r$ be a solution of [8.94]:

$$\frac{d\hat{X}_r}{dt} = A_r.\hat{X}_r + B_r.U \qquad [8.97]$$

We define the error $e = X_r - \hat{X}_r$ with respect to the exact solution $\hat{X}_r$ assumed to be verified by equation [8.94]; it satisfies the differential equation obtained by subtracting term by term [8.96] and [8.97]:

$$\frac{de}{dt} = A_r e + q = A_r e + \frac{dX_r}{dt} - A_r.X_r - B_r.U$$

We must therefore determine the matrices R, A_r and B_r, such that the error of equation q is minimized. This minimization procedure can be performed on an ensemble of known solutions. We will leave aside the details of this kind of procedure.

It is easy to impose that the reduction be exact for the steady regime. In this case, equations [8.93] and [8.94] can be solved and we obtain:

$$X_r = -A_r^{-1} B_r.U = RX = -RA^{-1}BU$$

and consequently the relation: $B_r = A_r R A^{-1} B$.

In the unsteady regime, the preceding elimination is no longer possible and the vector RX obtained with the solution of [8.93] is no longer equal to the solution $\hat{X}_r$

of [8.97]. The matrix A_r can be determined by minimizing the error of equation q. This minimization can be performed for chosen inputs and for criteria which need to be defined for the weighting to be applied at different instants [PET 91]. This procedure comes down to performing a numerical interpolation on the knowledge model by means of a reduced model.

8.6.2.2. *Physical aspects of the model reduction*

The considerations of the preceding section are essentially of a mathematical nature and they leave a wide choice for the variables (in particular the matrix R), the class of solutions and the objectives of error minimization. They can be applied to a perfect knowledge model from the perspective of thermodynamics (sub-systems in quasi-equilibrium), this needing to be verified for solutions which vary quite slowly in space-time. The reduction in a knowledge model can only have a limited interest if the discretization is too rough in certain domains. The *partitioning of a system into sub-systems is the most important operation* and we will assume that it is suitable. In general, the reduction in the number of variables is associated with a regrouping of components, which leads to a choice of the reduced variables to be retained; this should be done such that the definition of the mean intensive variables of the sub-systems of the reduced model have a reasonable physical meaning in the balance equations (section 1.4.2.5 and section 6.5.2.4).

The *grouping of sub-systems* which are in neighboring or identical states leads to the replacement of many differential equations by a single equation: for example, a material domain which has been segmented into three components and whose temperatures are close can be represented by a single component at a suitable temperature whose thermal energy is the sum of the energies of the components.

The sub-systems *whose extensive quantity contents are small* can often be eliminated or assimilated into neighboring components, leading to the suppression of the corresponding variables. The components whose *extensive quantities are constant or vary little* can become sources of established fluxes: they thus play a role of a simple resistance for the transfer of extensive quantities and their modeling loses its differential character. We have seen two examples in sections 8.3.1.3 and 8.3.1.4.

In general, the number and the size of the sub-systems (or the grid of the system domain) must be adapted to the categories of the solutions studied. Let us take the example of the discretization on a continuous medium of a wall whose surfaces are subjected to a thermal shock (section 8.3.2.2.2). Let us begin by segmenting the wall into 50 equal elements in series (a system analogous to that of section 8.3.1.3). Figure 8.13 shows that the temperature variations in the first instances are rapid in the vicinity of the wall surface, whereas later, they become quite regular. The discretization in equal elements is thus not the best solution: a finer discretization is needed in the vicinity of the wall surfaces if we want to have a suitable precision in the first instants, whereas such a fine discretization is not necessary later in the central part of the wall. A better result with 50 elements is obtained when these are distributed broadly at the center of the wall, and extremely close to the edges of the wall.

8.6.2.3. *The problem of local constraints*

However, many systems present local constraints (in the mathematical sense) on certain intensive variables related to the possible modification of material elements: deteriorations due to excessive temperature or to large[8] instantaneous stresses. In this case, we obviously cannot eliminate from the model the small component which contains, for example, a small thermal energy, but which possesses a critical temperature value (temperature of a thermal measurement probe, a fragile element whose temperature must be limited, etc.). The intensive variable concerned (often the local temperature) is necessary for the regulation of a controller whose job it is to modify or stop the system functioning. Model reduction in such zones is obviously delicate.

8.6.2.4. *Modal reduction of time-invariant linear systems*

8.6.2.4.1. Introduction

Modal reduction is essentially concerned with time-invariant linear systems whose transitional solution is the sum of eigenmodes of the system of equations (see the examples of discrete or continuous systems in section 8.3). The solution of a linear system can be expressed in an explicit modal form using initial conditions and an established solution (section 8.2.2.2); the reduction here amounts to a simplification operation, but the question regarding the pertinence of the different modes arises for discretized systems. We are then interested in writing a reduced state representation which will allow the numerical simulation of the system in complex conditions.

8.6.2.4.2. Modal reduction of continuous solution of continuous media

The modal solution of the heat equation involves replacing the continuous variable by a denumerable set of mode coefficients in the explicit expression of solution [8.38] in the form of a series. The reduction problem therefore consists of simplifying and/or truncating this series. A consequence of this truncation is the introduction of a discontinuity at the time origin, which amounts to assuming that

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[8] For example, cavitation (vaporization and chiefly sudden condensation) when local pressure in a liquid flow is less than the saturation vapor pressure, high temperature on a wall in high supersonic or hypersonic air flow, production of pollutants inside engines or chemical reactors due to local bad conditions of temperature or concentrations, etc.

fast energy transfers in the higher order modes occur instantaneously (section 8.2.2.2). Various authors avoid this discontinuity by adding a fictitious mode or by attributing to the last mode retained the total remaining amplitude. Let us illustrate this problem by reconsidering the example of thermal shock on the wall edges (section 8.3.2.2.2).

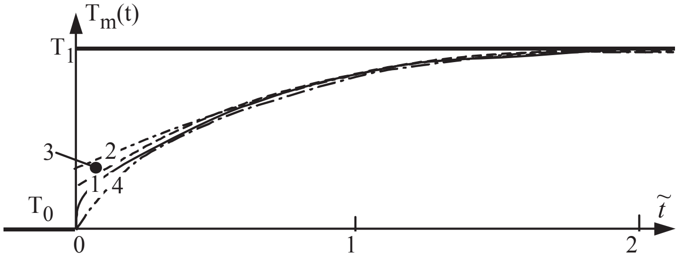

Figure 8.17. *Mean temperature on a wall during a thermal shock on its two faces (section 8.3.2.2.2). The draughts are indicative*

Let us consider the mean temperature T_m (formula [8.41]), which translates the thermal energy evolution over the course of the heating processes caused by the thermal shock (curve 1, Figure 8.17). We have seen (Table 8.1) that the first mode contains 81% of the process energy; in only conserving this first mode, we obtain a representation for the temperature T_m (curve 2, Figure 8.17), which contains a discontinuity at the origin equal to $0.19\left(T_1 - T_0\right)$; considering the first two modes of the series representing T_m (curve 3, Figure 8.17) this discontinuity is reduced to $0.10\left(T_1 - T_0\right)$).

We can remove this discontinuity by including the total remaining amplitude (0.19 in relative units) in the highest mode retained (curve 4, Figure 8.17) in accordance with the formula:

$$T_m(t) = T_1 + \left(T_0 - T_1\right)[0.81e^{-\frac{\pi^2}{4}\tilde{t}} + 0.19e^{-\frac{9\pi^2}{4}t}]$$

However, the second mode is now too large; we can therefore look to determine a time constant of the second mode in order to achieve a better representation of the function T_m (t), for example by a mean square error minimization method. It is also possible to add a fictitious mode. It is clear that a good solution is not achieved and the values obtained by empirical adjustments are not physically pertinent.

In effect, in the first instants of the evolution, only zones in the vicinity of edges are concerned with the heat transfer. Further from these, the sum of the series terms [8.41] is zero: *the modal representation is not adapted to the representation of a thermal boundary layer problem* (section 8.3.2.2.2). We can obviously note that the thermal boundary layer is independent of the wall thickness ℓ that can take any value: defining modes by means of an arbitrary length is indeed an irrational method which cannot lead to a judicious mathematical representation.

A good reduced model of thermal conduction in a wall must also take into account the modal aspect as the evolution of the unsteady boundary layer. There is no other (or nearly no other) way to obtain such a reduced model than *a composite representation matching the modal representation and the thermal shock solution*: we have presented this method in section 8.3.2.2.3, where a good precision was obtained for the mean temperature [8.46], using only the first mode. This method also has the advantage of giving precise values for the thermal flux density at the edges [8.47] at any instant.

8.6.2.4.3. Modal reduction of discrete models

In section 8.6.2.2 we considered the model with 50 elements of a continuous wall subjected to a thermal shock, leading to a linear system with 50 variables, and which thus comprises 50 eigenvalues and eigenmodes. We will consider that a half period of a sinusoid requires at least ten intervals in order to be represented by a constant function in each element. The interval under study cannot therefore comprise more than five arches: we can only represent the first three even modes and the first two odd modes (see Figure 8.12). The 45 other modes are increasingly noisy as the order is increased (the 50th mode corresponds to a change of sign of the eigenfunction between each of the 50 intervals). Their physical existence is increasingly problematic and it is not useful to consider them despite the fact that they constitute exact solutions of the model.

A discretization into sub-systems should comprise a sufficiently large number of elements, but only a few modes are actually useful. The modal solution is obviously the most interesting because it provides a structured knowledge which highlights the system properties. However, the discretization of a linear system proceeding from the calculation of its modes requires more elements than a discretization, taking into account physical aspects and particularly the level of unbalance between two neighboring sub-systems: if we consider the preceding example of the wall on the interval [-1,+1], it is necessary to calculate the modes to be retained over the entire interval, whose form (Figure 8.12) requires discretization of the interval [-1,+1] in equal segments, as opposed to a numerical resolution for which a discretization, narrower near the wall faces and wider in the central part, is more fitted to the form of the solution (Figure 8.13). This more rational method of discretization has the advantage of reducing the number of variables.

Once a discretization is chosen for our knowledge model of a time-invariant linear system, we have to deal with a lot of modes that have no physical significance, as we have seen at the beginning of this section. The best reduction method thus consists of using the properties of the modal solution, where only first modes, corresponding to the actual evolution of the system, are retained. Returning to the definition of an exact reduced state representation (section 8.6.2.1), let us assume that the reduced state vector $X_{\mathrm{r}} = R\ X$ with s components ($\mathrm{s} < n$), derived from X by means of the reduction matrix R, satisfies reduced state representation [8.98]:

$$\frac{dX_r}{dt} = A_r.X_r + B_r.U \tag{8.98}$$

With condition [8.99] obtained by multiplying the left-hand side of equation [8.93] by R and comparing with [8.98]:

$$A_r R = RA; \quad B_r = RB; \quad X_r(0) = RX(0). \tag{8.99}$$

we verify that an eigenmode of [8.94] is also an eigenmode of complete state representation [8.93]:

$$(A - \Lambda_i I)\Theta_i = 0 \quad \Rightarrow \quad (RA - \Lambda_i RI)\Theta_i = (A_r - \Lambda_i I)R\Theta_i = 0$$

The s eigenvalues Λ_i of the reduced model are thus eigenvalues of the complete model, the corresponding eigenvectors being equal to $R\Theta_i$[9], the choice of which modes to retain in the reduced model results from the definition of matrix R. The reduction matrix R is thus such that the reduced state vector is constituted of the s retained modes. Taking eigenvalues as a basis (modal basis), the matrix A is the diagonal eigenvalue matrix Λ_i. We assume that the s eigenvalues to be retained have been placed in the first s elements of the diagonal. In the modal basis, the reduction matrix R, of dimension $s*n$ contains 1 on the diagonal, in correspondence with the first diagonal element. The reader can verify that the reduced square matrix A_r contains the s eigenvalues which were retained.

The same result can in principle be obtained in another manner. We have seen in section 8.2.2.2 that the suppression of a mode amounts to requiring that its coefficient be zero in the modal development at each instant: this results in an instantaneous linear relationship between the state vector components (formula

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[9] Eigenvectors which have not been retained belong to the kernel of *R*.

[8.19]). The suppression of *n-s* modes of the model comes down to writing *n-s* relations between the variables, which allows us to eliminate *n-s* state variables. We thus obtain a representation by *s* state variables of the system which is reduced. This method does not require us to calculate the modal basis. While it is in principal arbitrary in the preceding methods, the choice of physical variables which are not eliminated should nonetheless be performed so as to be conducive to physical interpretation. The reader will also note that the principle of modal reduction itself leads to a situation where *discontinuous data leads to a discontinuity of the variables of the reduced representation.*

8.6.2.5. *Reduction of input-output representations*

8.6.2.5.1. Introduction

Simple input-output representations are useful:

– for obtaining a global knowledge of the behavior of a system in view of the implementation of control;

– for the dimensioning of components during the realization of a system;

– for the representation of sub-systems in models of complex systems.

The properties of these reduced representations differ according to their utilization: command or engineering formulae.

8.6.2.5.2. Command models

The essential characteristics to know are thus relatively global and it is sufficient to know one or several "response times" corresponding to the inputs on which we act. The output variables allow us to define values of actions to be effected. We will here consider a model of the form [8.59] of low order: for example, for a single input single output system, we often consider a second order differential equation which is satisfied by the output variable and whose input is the right-hand side. We will here only consider an open-loop system (without any retroaction by the command); the system operation in close-loop depends on the command structure and is not the object of this work (see [OBI 00]).

Second order differential equations represent accumulation (of mass or heat for example), damping and stiffness mechanisms, which are themselves represented, respectively, by second order terms, first order terms and zero order terms (the function). It is thus possible to represent aperiodic or stable oscillatory systems. The coefficients of such differential equations can be determined by imposing an impulse excitation as an input, and then by "best" identifying the response of the system to an expression of the form $ae^{-\alpha t}\cos(\omega t + \varphi)$. This is equivalent to performing a spectral study of the system (section 8.5.2.4).

8.6.2.5.3. Engineering formulae

We will designate under this category the input-output relations derived from knowledge models and which are simple enough to be immediately useful for example for quick dimensional assessment of components of a system in view of a realization or for the representation of a sub-system in a more complex system. These formulae are the result of reduced state representations whose pertinence has been previously verified. They can also be obtained by simplification of an analytical solution when one exists, for example by truncating a series (example of section 8.6.2.4.2) or when using composite representations (section 8.3.2.2.3).

8.6.2.5.4. Established regimes

The parametric expression of established regimes, obtained in section 8.5.2 in the form of a series, can be simplified by truncation of this. Let us recall that the representation in the form of a series is only of interest if it is possible to limit this to a small number of terms. If this is not the case, this indicates that thermal boundary layers exist in the domain, and it is therefore preferable to find a direct expression which represents the boundary layer and to describe a composite solution by matched asymptotic expansion (see section 8.3.2.2.3).

8.7. Application in fluid mechanics and transfer in flows

The evolution equations of a discrete or continuous system as a time function are parabolic (or irreversible). In the presence of flow, the evolution variable is the time only when Lagrangian variables are used. In Eulerian variables, the evolution speed of a quantity g is no longer represented by its temporal derivative $\partial g/\partial t$ but by the material derivative dg/dt. This representation does not change the parabolic character of the balance equations for extensive quantities with Euler variables expressed along curvilinear abscissa of trajectories (or characteristic curves), which is a parabolic variable equivalent to the time with Lagrange variables (see interpretation of section 5.2.1) upstream then becoming the equivalent of the past. In steady flow, the time variable disappears and the evolution variable becomes the coordinate of the particle trajectories: systems studied thus appear as dynamic systems along the trajectories. The same is true of boundary layer equations, or more generally of the evolution of fluid properties along trajectories. These preceding ideas are thus applicable to problems encountered in flows.

The methods evoked in this chapter are encountered when dealing with the solution of flow and transfer problems in boundary layers ([SCH 99], [YIH 77]). However, these are generally non-linear and computations cannot be effected in as complete a manner as in this chapter. Writing balance equations, ultimately with approximations which may be more or less global, leads to state representations where time is replaced by a boundary layer coordinate. The inputs are therefore often evolution laws for the velocity of a perfect fluid as a function of the spatial coordinate, the initial conditions having been provided in the initial section of a boundary layer or a pipe. It is thus possible to obtain approximate external representations in forms which are more or less explicit.

In principle, the procedure is of the same nature as for linear problems: an elementary analytical solution makes it possible to study situations which are more or less similar, and to identify the parameters associated with the category of solutions studied. However, non-linearity does not enable the easy use of series developments of eigenfunctions or of integral transformations. On the other hand, singular or regular perturbation methods remain useful for writing the equations as a cascade of successive approximations (section 6.4.2.3). The differential equations obtained must often be solved, either by numerical means or by some global methods (section 6.3.1.2). In other words, the general methodology of the present chapter is applicable to problems of flow and associated transfer by means of an adaptation of the calculation methods. The preceding discussions of linear dynamic systems often allow us to organize knowledge gained from these non-linear systems in a more rational way.

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Appendix 1

Laplace Transform

A1.1. Definition

The Laplace transform of a real variable function $f(t)$, considered for $t \geq 0$ ($f(t) = 0$ for $t < 0$), is defined by the relation

$$L_f(p) = \int_0^{\infty} f(t) e^{-pt} dt$$

where p is a complex valued variable. It is particularly well adapted to causal signals.

A1.2. Properties

– The Laplace transform is a linear application on integrable functions.

– The Laplace transform of the derivative of a function is written:

$$L_{f'}(p) = \int_0^{\infty} f'(t) e^{-pt} dt = p\int_0^{\infty} f(t) e^{-pt} dt - f(0) = pL_f(p) - f(0)$$

The derivative is here taken in the sense of distributions: any discontinuity of the function leads to a Dirac distribution at that point for the derivative.

– The Laplace transform of a primitive is written:

$$g(t) = \int_0^t f(u)du \quad \Rightarrow \quad L_g(p) = \frac{1}{p} L_f(p)$$

– The Laplace transform of a delayed temporal signal can be written:

if: $g(t) = f(t - \tau) \quad t \geq \tau, \quad g(t) = 0 \quad t < \tau$

$$L_g(p) = \int_0^\infty g(t)e^{-pt}dt = \int_\tau^\infty f(t)e^{-pt}dt = \int_0^\infty f(x)e^{-p(x+\tau)}dx = L_f(p)e^{-p\tau}$$

Conversely, if: $g(t) = f(t)e^{-a\,t}$

$$L_g(p) = \int_0^\infty f(t)e^{-a\,t}e^{-pt}dt = \int_0^\infty f(t)e^{-(p+a)t}dt = L_f(p+a)$$

– The Laplace transform of a convolution product of two functions f and g is equal to the product of the Laplace transforms of these functions:

$$L_{f*g}(p) = L_f(p)\,L_g(p)$$

A1.3. Some Laplace transforms

– The Laplace transform of a Dirac function is equal to 1:

$$L_\delta(p) = \int_0^\infty \delta(t)e^{-pt}dt = \int_{-\infty}^\infty \delta(t)e^{-pt}dt = 1$$

δ being zero for negative values of t.

– *The unit step H(t)*, equal to 1 for t positive, is the primitive of the Dirac distribution: $\int_0^\infty \delta(t)dt$; its transform is: $L_H(p) = 1/p$.

– The *unit ramp t* is the primitive $\int_0^t H(u)du$ of the unit step H(f); its transform is: $L_t(p) = 1/p^2$.

In general, we obtain: $L_{t^n}(p) = 1/p^{n+1}$.

– $f(0_+) = \lim\limits_{t \to 0} f(t) = \lim\limits_{p \to \infty} pL_f(p)$

– $f(\infty) = \lim_{t \to \infty} f(t) = \lim_{p \to 0} pL_f(p)$

– Consider the *limited development* $P_m(t)$ to order m for small values of time t. We obtain:

$$P_m(t) = \sum_{i=0}^{m} a_i t^i \quad \Rightarrow \quad L_{P_m}(p) = \sum_{i=0}^{m} \frac{a_i}{p^{i+1}}$$

The lowest order terms in p^{-1} represent the Laplace transform which corresponds to small values of time t.

For a limited development, and for large values of t, ($t \to \infty$) we obtain:

$$Q_m(t) = \sum_{i=n-m}^{n} a_i t^i \quad \Rightarrow \quad L_{Q_m}(p) = \sum_{i=n-m}^{n} \frac{a_i}{p^{i+1}}$$

The highest order terms in p^{-1} represent the Laplace transform which corresponds to large values of t.

– The Laplace transforms of *exponential and harmonic functions* are:

$$L_{\exp(at)}(p) = \int_0^\infty e^{(a-p)t}\,dt = \frac{1}{p-a} \quad ; \qquad L_{\exp(j\omega t)}(p) = \frac{1}{p - j\omega}$$

$$L_{\cos(\omega t)}(p) = \frac{p}{p^2 + \omega^2} \quad ; \qquad L_{\sin(\omega t)}(p) = \frac{\omega}{p^2 + \omega^2}$$

A1.4. Application to the solution of constant coefficient differential equations

Consider the system: $\dfrac{dX}{dt} = A.X + BU \qquad X(0) = X_0$.

The Laplace transform $L_X(p)$ of vector $X(t)$ satisfies the equation:

$$(pI - A)L_X(p) = L_{BU}(p) + X_0$$

From this we deduce:

$$L_X(p) = (pI - A)^{-1}\left[L_{BU}(p) + X_0\right]$$

The matrix $\left(pI - A\right)^{-1}$ is comprised of elements which are rational fractions of the frequency p which can be developed in a complete series in p. We can write this in the form:

$$\left(pI - A\right)^{-1} = \sum_{i=0}^{\infty} p^i D_i$$

The transformed linear system can thus be written:

$$L_X(p) = \sum_{i=0}^{\infty} p^i D_i L_{BU}(p) + \left(pI - A\right)^{-1} X_0 = \sum_{i=0}^{\infty} D_i L_{BU^{(i)}}(p) + \left(pI - A\right)^{-1} X_0$$

The second term represents the transient response to the initial conditions. The steady response is thus:

$$X = \sum_{i=0}^{\infty} BD_i U^{(i)}\left(t\right)$$

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Appendix 2

Hilbert Transform

The amplitude a and the frequency ν of an oscillatory harmonic motion appear naturally if the signal is written as the real part of a complex number in the trigonometric form $ae^{j\varphi(t)}$, in which the time derivative of the instantaneous phase $\varphi(t) = 2\pi\nu t + \phi$ is equal to $2\pi\nu$. In the same way, we can associate a complex-valued function $\alpha_x(t)$ (called an analytical signal) with any real signal $x(t)$ by adding to it an imaginary-valued function of time $jy(t)$. This complex signal can be written in the trigonometric form:

$$\alpha_x(t) = x(t) + jy(t) = a(t).e^{j\varphi(t)} \quad \text{with : } x(t) = \mathrm{Re}[a(t)e^{j\varphi(t)}]$$

The modulus $a(t)$ and the argument $\varphi(t)$ of this complex number can be defined as the amplitude and the instantaneous phase of the signal $x(t)$; we thus define the instantaneous frequency as the derivative $\varphi'(t)/2\pi$.

These definitions are only meaningful if the point whose affix is the complex function turns nearly regularly in the positive direction around the origin, in a way quite similar to a complex exponential function. This is the case for amplitude- or phase-modulated harmonic signals, or for the sounds of musical instruments. A narrow-band signal $x(t)$ appears on an oscilloscope as a carrier of the neighbouring frequency ν_o, whose amplitude $[a(\mathrm{t})]$ varies slowly and whose phase $[\varphi(\mathrm{t})]$ is fluctuating.

Given the real part of a complex number, it cannot be determined uniquely, because we only know the product $a\cos\varphi(t)$. As the signals to be represented are quite close to harmonic signals, it is natural to apply the Fourier transform and to generalize the properties of harmonic signals. However, we note that the definition of an analytic signal $e^{j2\pi\nu t}$ for a sinusoidal function $\cos 2\pi\nu t$ consists of eliminating the negative frequencies of the Fourier transform of this analytical signal, by folding over the negative frequencies onto the positive frequency axis. This operation can be applied to any signal $x(t)$ and it allows the analytical signal $\alpha_x(t)$ to be defined. Let $F_x(\nu)$ be the Fourier transform of the signal $x(t)$. The Fourier transform of the analytical signal $\alpha_x(t)$ associated with $x(t)$ is defined by:

$$\begin{cases} F_{\alpha x}(\nu) \equiv 0 & \text{for } \nu < 0 \\ F_{\alpha x}(\nu) \equiv 2F_x(\nu) & \text{for } \nu > 0 \end{cases}$$

$$\text{i.e. } F_{ax}(v) = F_x(v) + Sign(v).F_x(v) \quad \text{with: } \begin{cases} x > 0 & Sign(x) = +1 \\ x < 0 & Sign(x) = -1 \end{cases} \qquad [A2.1]$$

As the inverse Fourier transform $F^{-1}(Sign(\nu).)$ of the sign function is equal to $j/\pi t$ ([BEE 03], [PRI 91]), the analytical signal $\alpha_x(t)$ is thus obtained by performing the inverse Fourier transform of expression [A2.1]

$$\alpha_x(t) = F^{-1}[F_{\alpha x}(\nu)] = x(t) + jH_x(t)$$

where $H_x(t)$ designates the Hilbert transform of the signal $x(t)$:

$$H(t) = -jF^{-1}(Sign(\nu).F_x(\nu)) = \frac{1}{\pi t} * x(t) = \frac{1}{\pi} VP\int_{-\infty}^{\infty} x(u).\frac{du}{t-u}$$

The Hilbert transform is limited to positive time, contrary to the Fourier transform. The Hilbert "filter" is a linear filter whose impulse response is $1/\pi t$; it is thus not causal. The transfer function of this filter is $H_H(\nu) = F_{(1/\pi t)}(\nu) = -j.sign(\nu)$. The Hilbert filter transforms a cosine to a sine; it is a *perfect quadrature filter*. It is unfortunately not achievable as it is non-causal; we can only create quadratures which function in a limited band of frequencies.

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Appendix 3

Cepstral Analysis

A3.1. Introduction

The cepstrum is an integral transform that can be calculated from a spectrum and contributes to its analysis. For example, in a vibration problem, it makes it possible to separate the impulse response from the excitation forces. The analysis of vibration signals from rotating machinery is important. The different operating conditions of a machine, as well as defects, can be observed in the spectral domain ([STR 96]).

It is also used in speech processing, the vocal signal coming from the convolution of the excitation (source) and the impulse response of the vocal passage.

Another application is the extraction of incident sound from a signal containing both the incident sound and its reflection (echo suppression). We will quickly cover this application, which enables the demonstration of the use of logarithms and thus of the cepstrum.

A3.2. Definitions

Let $x(t)$ be a time signal and $F_x(\nu)$ its Fourier transform; by definition the *complex cepstrum* $C_x(t)$ is the inverse Fourier transform, denoted F^{-1}, of the logarithm of function $F_x(\nu)$ ([BOU 98], [NOR 03], [STR 96])

$$C_x(\tau) = F^{-1}(\ln\{F_x(\nu)\})$$

if $F_x(\nu) = |F_x(\nu)|\ e^{j\varphi(\nu)}$, then: $\ln\{F_x(\nu)\} = \ln\{|F_x(\nu)|\} + j\ \varphi(\nu)$

The *power cepstrum* may be defined in several ways, for example, as the square of the modulus of the complex cepstrum of the signal:

$$C_{px} = \left| F^{-1}(\ln\{F_x(\nu)\}) \right|^2 .$$

or by taking the inverse Fourier transform of logarithm of the modulus square:

$$C_{px} = F^{-1}\left(\ln\left\{ \left| F_x(\nu) \right|^2 \right\} \right)$$

A3.3. Example of echo suppression

Let $x(t)$ be a sound signal comprising the superposition of an original sound $s(t)$ and an unwanted echo $s_r(t)$:

$$x(t) = s(t) + s_r(t)$$

The reflected sound is attenuated and dephased compared with the original signal

$$s_r(t) = a\ s(t - t_0)$$

The Fourier transform of $x(t)$ can be written:

$$F_x(\nu) = F_s(\nu)\left(1 + a\ e^{-2\pi\ j\ \nu\ t_0}\right)$$

and the square of the modulus of this transform is:

$$\left| F_x(\nu) \right|^2 = \left| F_s(\nu) \right|^2 \left(1 + a^2 + 2a\ \cos 2\pi\, \nu\, t_0\right)$$

If we take the logarithm of this quantity, the echo phenomenon is seen in frequency space by the addition of a periodic term, of period $1/t_0$:

$$\ln\left\{ \left| F_x(\nu) \right|^2 \right\} = \ln\left\{ \left| F_s(\nu) \right|^2 \right\} + \ln\left(1 + a^2 + 2a\ \cos 2\pi \nu t_0\right)$$

We now take the inverse Fourier transform of this expression ([ALL 04], ([MAD 98], [NOR 03]): the first two terms will be, respectively, the power cepstrums $x(t)$ and $x(t)$; the third term will be the inverse Fourier transform of the periodic function shown above, which will be comprised of Dirac functions of different amplitude, separated by t_0

$$C_{px}(\tau) = C_{ps}(\tau) + F^{-1}\left\{\ln\left(1 + a^2 + 2a\cos 2\pi\nu t_0\right)\right\}$$

We thus find ourselves in a pseudo-temporal space, whose variable is known as “quefrency”, in which we perform a “liftering” of the Dirac signals (the terms “cepstrum”, “quefrency” and “liftering” are respectively anagrams of the words spectrum, frequency and filtering). The echo signal is thus eliminated by the suppression of the Dirac signals in this space. The initial signal is thus reconstructed without its echo by an inverse process ([NOR 03]); however, we note that phase information has been lost with this procedure of cepstrum power.

A3.4. General case

We consider a source signal $x(t)$ going through a passive linear system whose impulse response is $h(t)$. The output signal $y(t)$ is the convolution product of x by h:

$$y(t) = h(t) * x(t)$$

The convolution product becomes a simple product after Fourier transform:

$$F_y(\nu) = H(\nu)F_x(\nu)$$

The cepstrum transforms the product to a sum:

$$C_y(\tau) = C_h(\tau) + C_x(\tau)$$

In order to perform the deconvolution, in other words the separation of the source from that of the medium, some assumptions must be made regarding the class of functions which comprise either the source or the medium:

– in the case of speech, we assume that the vocal excitation is comprised of a periodic impulse t_0 (this property is of course only applied to the stable parts of sounds) and that the cepstral contribution of the vocal passage is found in the low quefrencies; we can thus obtain a smoothed spectrum which only contains information on the vocal passage. The inverse Fourier transform of the spectrum gives the estimated impulse response of the passage and is used for voice synthesis ([JUR 08], [MAD 98]).

– in mechanics, free oscillatory periods of a medium are generally small compared to the periods of the excitations: the cepstrum of the impulse response of the structure is in the left part, while that of the excitation is in the right part of the cepstrum graph of *structure response to excitation* ([DES 00], [NOR 03], [ALL 04]).

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Appendix 4

Eigenfunctions of an Operator

A4.1. Eigenfunctions of an operator

Consider a linear operator L applied to a function $f(x)$ which satisfies homogenous boundary conditions at the extremities of the interval defined by $[a,b]$. The eigenvalues λ_i of this operator and the associated eigenfunctions f_i satisfy the relation:

$$L\,f_i = \lambda\,f_i$$

A4.2. Self-adjoint operator

A4.2.1. *Eigenfunctions*

Let us define the scalar product $< f|g >$, for the foregoing functions *f*, by the relation:

$$< f|g >= \int_a^b f(x).g(x)dx$$

The operator L is the self-adjoint if it satisfies the relation

$$< f|Lg >=< Lf|g >= \int_a^b Lf(x).g(x)dx = \int_a^b f(x).Lg(x)dx$$

Under some general conditions:

– the eigenvalues of the self-adjoint operator L are positive and form an infinite denumerable set $\lambda_i \left(i = 1, 2, ..., n, ...\right)$. With each eigenvalue λ_i we can associate an eigenfunction f_i;

– the eigenfunctions f_i form a basis for the class of functions f.

A4.2.2. *Expression of a function f using an eigenfunction basis-set*

Consider the following series development of f on the eigenfunction-basis:

$$f(x) = \sum_{j=1}^{\infty} c_j f_j(x)$$

Let us calculate the scalar product $< f | f_i >$:

$$< f(x) | f_i(x) > = \sum_{j=1}^{\infty} c_i < f_j(x) | f_i(x) > \qquad [A4.1]$$

Now, the eigenfunctions are orthogonal for the scalar product. In effect, with the assumption of a self-adjoint operator, we obtain for $\lambda_i \neq \lambda_j$:

$$< f_i | L f_j > - < L f_i | f_j > = \left(\lambda_j - \lambda_i\right) < f_i | f_j > = 0$$

We also obtain:

$$< f_i | L f_i > = \lambda_i < f_i | f_i > = \lambda_i \left\| f_i \right\|^2$$

Substituting previous results into [A4.1], we obtain the value of the coefficient c_i:

$$< f(x) | f_i(x) > = \sum_{j=1}^{\infty} c_i < f_j(x) | f_i(x) > = c_i \left\| f_i(x) \right\|^2$$

or:

$$c_i = \frac{< f(x) | f_i(x) >}{\left\| f_i(x) \right\|^2}$$

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