MORGAN & CLAYPOOL PUBLISHERS

Geometric Programming for Design and Cost Optimization

with Illustrative Case Study Problems and Solutions

Robert C. Creese

SYNTHESIS LECTURES ON ENGINEERING

Geometric Programming for Design and Cost Optimization

(with Illustrative Case Study Problems and Solutions)

Geometric Programming for Design and Cost Optimization
(with Illustrative Case Study Problems and Solutions)

Robert C. Creese

www.morganclaypool.com

ISBN: 9781608452620 paperback
ISBN: 9781608452637 ebook

DOI 10.2200/S00221ED1V01Y200909ENG010

A Publication in the Morgan & Claypool Publishers series
SYNTHESIS LECTURES ON ENGINEERING

Lecture #10
Series ISSN
Synthesis Lectures on Engineering
Print 1939-5221 Electronic 1939-523X

Synthesis Lectures on Engineering

Geometric Programming for Design and Cost Optimization (with Illustrative Case Study Problems and Solutions)
Robert C. Creese
2009

Style and Ethics of Communication in Science and Engineering
Jay D. Humphrey, Jeffrey W. Holmes
2008

Introduction to Engineering: A Starter's Guide with Hands-On Analog Multimedia Explorations
Lina J. Karam, Naji Mounsef
2008

Introduction to Engineering: A Starter's Guide with Hands-On Digital Multimedia and Robotics Explorations
Lina J. Karam, Naji Mounsef
2008

CAD/CAM of Sculptured Surfaces on Multi-Axis NC Machine: The DG/K-Based Approach
Stephen P. Radzevich
2008

Tensor Properties of Solids, Part Two: Transport Properties of Solids
Richard F. Tinder
2007

Tensor Properties of Solids, Part One: Equilibrium Tensor Properties of Solids
Richard F. Tinder
2007

Essentials of Applied Mathematics for Scientists and Engineers
Robert G. Watts
2007

Project Management for Engineering Design
Charles Lessard, Joseph Lessard
2007

Relativistic Flight Mechanics and Space Travel
Richard F. Tinder
2006

Geometric Programming for Design and Cost Optimization

(with Illustrative Case Study Problems and Solutions)

Robert C. Creese
West Virginia University

SYNTHESIS LECTURES ON ENGINEERING #10

MORGAN & CLAYPOOL PUBLISHERS

ABSTRACT

Geometric programming is used for design and cost optimization and the development of generalized design relationships and cost rations for specific problems. The early pioneers of the process, Zener, Duffin, Peterson, Beightler, and Wilde, played important roles in the development of geometric programming. The theory of geometric programming is presented and 10 examples are presented and solved in detail. The examples illustrate some of the difficulties encountered in typical problems and techniques for overcoming these difficulties. The primal-dual relationships are used to illustrate how to determine the primal variables from the dual solution. These primal-dual relationships can be used to determine additional dual equations when the degrees of difficulty are positive. The goal of this work is to have readers develop more case studies to further the application of this exciting mathematical tool.

KEYWORDS

cost optimization, design optimization, general solutions, posynominials, primal, dual, pioneers, casting design, metal cutting economics, LPG cylinder design

Contents

Preface

The purpose of this text is to introduce manufacturing engineers, design engineers, manufacturing technologists, cost engineers, project managers, industrial consultants and finance managers to the topic of geometric programming. I was fascinated by the topic when first introduced to it over 40 years ago at a National Science Foundation(NSF) short course in Austin, Texas in 1967. The topic was only a day or so of a three week course, but I recognized its potential in the application to riser design in the metal casting industry during the presentation. I was fortunate to have two of the pioneers in Geometric Programming make the presentations, Doug Wilde of Stanford University and Chuck Beightler of the University of Texas, and had them autograph their book "Foundations of Optimization" for me which I fondly cherish even now.

Since I was working on my PhD in pyrometallurgy at the time, I did not have time to work on a separate publication using geometric programming until 1972. I have written several journal papers using geometric programming on metal cutting and riser design, but never had been able to teach a complete course on the topic. Thus, before I retire, I decided to write a brief book on the topic illustrating the basic approach to solving various problems to encourage others to pursue the topic in more depth. I feel that its ability to lead to design and cost relationships in an integrated manner makes this tool essential for engineers, product developers, and project managers be more cost competitive in this global market place.

This book is dedicated to the pioneers of geometric programming such as Clarence Zener, Richard Duffin, Elmor Peterson, Chuck Beightler, Doug Wilde, Don Phillips, and others for developing this topic. This work is also dedicated to my family members, Natalie and Jennifer; Rob, Denie, Robby and Sammy, and Chal and Joyce.

I also want to recognize those who have assisted in the reviewing and editing of this work and they are Dr. M. Adithan, Dean of Faculty and Staff at VIT University in Vellore, India and Dr. Deepak Gupta, Assistant Professor at Southeast Missouri State University, USA.

Robert C. Creese
October 2009

CHAPTER 1

Introduction

1.1 OPTIMIZATION AND GEOMETRIC PROGRAMMING

1.1.1 OPTIMIZATION

Optimization can be defined as the process of determining the best or most effective result utilizing a quantitative measurement system. The measurement unit most commonly used in financial analysis, engineering economics, cost engineering or cost estimating tends to be currency such as US Dollars, Euros, Rupees, Yen, Won, Pounds Sterling, Kroner, Kronor, Mark or specific country currency. The optimization may occur in terms of net cash flows, profits, costs, benefit/cost ratio, etc. Other measurement units may be used, such as units of production or production time, and optimization may occur in terms of maximizing production units, minimizing production time, maximizing profits, or minimizing cost. Design optimization determines the best design that meets the desired design constraints at the desired objective, which typically is the minimum cost. Two of the most important criteria for a successful product are to meet all the functional design requirements and to be economically competitive.

There are numerous techniques of optimization methods such as linear programming, dynamic programming, geometric programming, queuing theory, statistical analysis, risk analysis, Monte Carlo simulation, numerous search techniques, etc. Geometric programming is one of the better tools that can be used to achieve the design requirements and minimal cost objective. The development of geometric programming started in 1961. Geometric programming can be used not only to provide a specific solution to a problem, but it also can, in many instances, give a general solution with specific design relationships. These design relationships based upon the design constants can then be used for the optimal solution without having to resolve the original problem. This fascinating characteristic appears to be unique to geometric programming.

1.1.2 GEOMETRIC PROGRAMMING

Geometric programming is a mathematical technique for optimizing positive polynomials, which are called posynominials. This technique has many similarities to linear programming but has advantages in that:

1. a non-linear objective function can be used;

2. the constraints can be non-linear; and

3. the optimal cost value can be determined with the dual without first determining the specific values of the primal variables.

Geometric programming can lead to generalized design solutions and specific relationships between variables. Thus, a cost relationship can be determined in generalized terms when the degrees of difficulty are low, such as zero or one. This major disadvantage is that the mathematical formulation is much more complex than linear programming, and complex problems are very difficult to solve. It is called geometric programming because it is based upon the arithmetic-geometric inequality where the arithmetic mean is always greater than or equal to the geometric mean. That is:

$$(X_1 + X_2 + \ldots X_n)/n \geq (X_1{}^* X_2{}^* \ldots {}^* X_n)^{(1/n)} . \tag{1.1}$$

Geometric programming was first presented over 50 years but has not received adequate attention similar to that which linear programming has obtained over its history of less than 70 years. Some of the early historical highlights and achievements of geometric programming are presented in the next chapter.

1.2 EVALUATIVE QUESTIONS

1. What is the most common unit of measurement used for optimization?
2. The following series of costs($) were collected: 2, 4, 6, 8, 10.
   (a) What is the arithmetic mean of the series of costs?
   (b) % b. What is the geometric mean of the series of costs?
3. The following series of costs(€) were collected: 20, 50, 100, 500, 600.
   (a) What is the arithmetic mean of the series of costs?
   (b) What is the geometric mean of the series of costs?
4. What is the year recognized as the beginning of geometric programming?

CHAPTER 2

Brief History of Geometric Programming

2.1 PIONEERS OF GEOMETRIC PROGRAMMING

Clarence Zener, Director of Science at Westinghouse Electric in Pittsburgh, Pennsylvania, USA, is credited as being the father of geometric programming. In 1961, he published a paper in the Proceedings of the National Academy of Science on "A Mathematical Aid in Optimizing Engineering Designs," which is considered as the first paper on geometric programming. Clarence Zener is better known in electrical engineering for the Zener diode. He later teamed with Richard J. Duffin and Elmor L. Peterson of the Carnegie Institute of Technology (now Carnegie-Mellon University, USA) to write the first book on geometric programming, named "Geometric Programming" in 1967. A report by Professor Douglas Wilde and graduate student Ury Passey on "Generalized Polynomial Optimization" was published in August 1966. Professor Douglas Wilde of Stanford University and Professor Charles Beightler of the University of Texas included a chapter on geometric programming in their text "Foundations of Optimization." I attended an optimization short course at the University of Texas in August 1967, and that is when I first became interested in geometric programming. I realized at that time that geometric programming could be used for the riser design problem, and I published a paper on it in 1971.

Other early books by these leaders were "Engineering Design by Geometric Programming" by Clarence Zener in 1971, "Applied Geometric Programming" by C.S. Beightler and D.T. Phillips in 1976, and the second edition of "Foundations of Optimization" by C.S. Beightler, D.T. Phillips, and D. Wilde in 1979. Many of the initial applications were in the area of transformer design as Clarence Zener worked for Westinghouse Electric in the area of chemical engineering, which was the area emphasized by Beightler and Wilde. It is also important to note that several graduate students played an important role, namely Elmor Peterson at Carnegie Institute of Technology and Ury Passy and Mordecai Avriel at Stanford University.

2.2 EVALUATIVE QUESTIONS

1. Who is recognized as the father of geometric programming?

2. When was the first book published on geometric programming, and what was the title of the book?

3. Which three universities played an important role in the development of geometric programming?

CHAPTER 3

Theoretical Considerations

3.1 PRIMAL AND DUAL FORMULATION

The mathematics of geometric programming are rather complex; however, the basic equations are presented and followed by an illustrative example. The theory of geometric programming is presented in more detail in some of the references [10, 17, 19, 1] listed at the end of the book. The primal problem is complex, but the dual version is much simpler to solve. The dual is the version typically solved, but the relationships between the primal and dual are needed to determine the specific values of the variables in the primal. The primal problem is formulated as:

$$Y_m(X) = \sum_{T=1}^{T_m} \sigma_{mt} C_{mt} \prod_{n=1}^{N} X_n^{a_{mtn}} \; ; \qquad m = 0, 1, 2, \ldots M \; , \tag{3.1}$$

with $\sigma_{mt} = \pm 1$ and $C_{mt} > 0$
and $Y_m(X) \leq \sigma_m$ for $m = 1, \ldots M$ for the constraints
where C_{mt} = positive constant coefficients in cost and constraint equations
and $Y_m(X)$ = primal objective function
and σ_{mt} = signum function used to indicate sign of term in the equation (either $+1$ or -1).

The dual is the problem formulation that is typically solved to determine the dual variables and value of the objective function. The dual objective function is expressed as:

$$d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\omega_{m0}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \qquad m = 0, 1, \ldots M \text{ and } t = 1, 2, \ldots T_m \; , \tag{3.2}$$

where

σ = signum function ($\pm$ 1)
C_{mt} = constant coefficient
ω_{m0} = dual variables from the linear inequality constraints
ω_{mt} = dual variables of dual constraints
σ_{mt} = signum function for dual constraints,

and by definition:

$$\omega_{00} = 1 \; . \tag{3.3}$$

The dual is formulated from four conditions:
(1) a normality condition:

$$\sum_{T=1}^{T_m} \sigma_{0t}\omega_{0t} = \sigma \quad \text{where} \quad \sigma = \pm 1 \; , \tag{3.4}$$

where

σ_{0t} = signum of objective function terms

ω_{0t} = dual variables for objective function terms.

(2) N orthogonal conditions

$$\sum_{m=0}^{M} \sum_{t=1}^{T} \sigma_{mt} a_{mtn} \omega_{mt} = 0 \ , \tag{3.5}$$

where

σ_{mt} = signum of constraint term

a_{mtn} = exponent of design variable term

ω_{mt} = dual variable of dual constraint.

(3) T non-negativity conditions (dual variables must be positive):

$$\omega_{mt} \geq 0 \qquad m = 0, 1 \ldots M \ \text{ and } \ t = 1, 2 \ldots \ldots T_m \ . \tag{3.6}$$

(4) M linear inequality constraints:

$$\omega_{mo} = \sigma_m \sum_{t=1}^{T_m} \sigma_{mt} \omega_{mt} \geq 0 \ . \tag{3.7}$$

The dual variables, ω_{mt}, are restricted to being positive, which is similar to the linear programming concept of all variables being positive. If the number of independent equations and variables in the dual are equal, the degrees of difficulty are zero. The degrees of difficulty is the difference between the number of dual variables and the number of independent linear equations, and the greater the degrees of difficulty, the more difficult the solution. The degrees of difficulty can be expressed as:

$$D = T - (N + 1) \ , \tag{3.8}$$

where

T = total number of terms (of primal)

N = number of orthogonality conditions plus normality condition (which is equivalent to the number of primal variables).

Once the dual variables are found, the primal variables can be determined from the relationships:

$$C_{ot} \prod_{n=1}^{N} X_n^{a_{mtn}} = \omega_{ot} \sigma Y_o \qquad t = 1, \ldots T_o \ , \tag{3.9}$$

and

$$C_{mt} \prod_{n=1}^{N} X_n^{a_{mtn}} = \omega_{mt} / \omega_{mo} \qquad t = 1, \ldots T_o \ \text{ and } \ m = 1, \ldots M \ . \tag{3.10}$$

The theory may appear to be overwhelming with all the various terms, but a simple example will be presented to illustrate application of the various equations.

3.2 THE OPTIMAL BOX DESIGN PROBLEM

A box manufacturer wants to determine the optimal dimensions for making boxes to sell to customers. The cost for production of the sides is C_1 (\$ 2/sq ft), and the cost for producing the top and bottom is C_2 (\$ 3/sq ft) as more cardboard is used for the top and bottom of the boxes. The volume of the box is to be set a limit "V" (4 ft^3), which can be varied for different customer specifications. If the dimensions of the box are W for the width, H for the box height, and L for the box length, what should the dimensions be based upon the cost values and box volume? The problem is to minimize the box cost for a specific box volume.

The primal objective function is:

$$\text{Minimize:} \qquad \text{Cost}(Y) = C_2WL + C_1H(W+L) \tag{3.11}$$

$$\text{Subject to:} \qquad WLH \geq V \; . \tag{3.12}$$

However, in geometric programming, the inequalities must be written in the form of $\leq$ and the right-hand side must be ± 1. Thus, the primal constraint becomes:

$$\text{Minimize:} \qquad \text{Cost}(Y) = C_1HW + C_1HL + C_2WL \tag{3.13}$$

$$\text{Subject to:} \qquad -WHL/V \leq -1 \; . \tag{3.14}$$

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= 1 \\ \sigma_{02} &= 1 \\ \sigma_{03} &= 1 \\ \sigma_{11} &= -1 \\ \sigma_{1} &= -1 \; . \end{aligned}$$

Thus, the dual formulation is:

Objective Function (using Equations (3.3) and (3.14))	$\omega_{01} + \omega_{02} + \omega_{03} = 1$	(3.15)
L terms (terms (using Equations (3.5), (3.13), and (3.14))	$\omega_{02} + \omega_{03} - \omega_{11} = 0$	(3.16)
H terms (terms (using Equations (3.5), (3.13), and (3.14))	$\omega_{01} + \omega_{02} - \omega_{11} = 0$	(3.17)
W terms (using Equations (3.5), (3.13),) and (3.14))	$\omega_{01} + \omega_{03} - \omega_{11} = 0$.	(3.18)

The degrees of difficulty are equal to:

$$D = T - (N+1) = 4 - (3+1) = 0 \; .$$

Thus, one has the same number of variables as equations, so this can be solved by simultaneous equations as these are linear equations.

Using Equations (3.15)–(3.18), the values for the dual variables are found to be:

$$\omega_{01}{=}1/3$$
$$\omega_{02}{=}1/3$$
$$\omega_{03}{=}1/3$$
$$\omega_{11}{=}2/3$$

and by definition

$$\omega_{00}{=}1\ .$$

Using the linearity inequality equation expressed by Equation (3.7),

$$\omega_{10} = \omega_{mt} = \sigma_m \sum \sigma_{mt}\,\omega_{mt} = (-1)^*(-1^*2/3) = 2/3 > 0 \quad \text{where } m = 1 \text{ and } t = 1\ .$$

The objective function can be found using Equation (3.8) is:

$$d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\,\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma}, \tag{3.8}$$

$$\begin{aligned}
d(\omega) &= 1\Big[\{(C_1{}^*1)/(1/3)\}^{(1)^*(1/3)}\,{}^*\{(C_1{}^*1)/(1/3)\}^{(1)^*(1/3)}{}^*\{(C2^*1)/(1/3)\}^{(1)^*(1/3)} \\
&\qquad {}^*\ \{((1/V)^*1)/(2/3)\}^{(-1)^*(2/3)}\Big]^1 \\
&= 1\Big[\{(3C_1)^{1/3}\} \quad {}^*\{(3C_1)^{1/3}\} \quad {}^*\{(3C_2)^{1/3}\} \\
&\qquad {}^*\ \{(1/V)^{-2/3}\}\Big] \\
&= 3C_1^{2/3}\,C_2^{1/3}\,V^{2/3} \qquad (3.19)\\
&= 3\,2^{2/3}3^{1/3}4^{2/3} = \$17.31\ .
\end{aligned}$$

Note, the solution has been determined without finding the values for L, W, or H. Also note that the dual expression is expressed in constants, and thus the answer can be found without having to resolve the entire problem as one only needs to use the new constant values. To find the values of L, W, and H, one must use Equations (3.9) and (3.10). Using Equation (3.9), the relationships are:

$$C_1 HW = \omega_{01} Y = Y/3$$
$$C_1 HL = \omega_{02} Y = Y/3$$
$$C_2 WL = \omega_{03} Y = Y/3\ .$$

Combining the first two of these relationships, one obtains

$$W = L\ . \tag{3.20}$$

Combining the last two of these relationships, one obtains

$$H = (C_2/C_1)L \,. \tag{3.21}$$

Since $V = HWL = (C_2/C_1)\,L\,L\,L = (C_2/C_1)L^3$.
Thus,

$$L = [V(C_1/C_2)]^{1/3} \tag{3.22}$$
$$W = L = [V(C_1/C_2)]^{1/3} \tag{3.23}$$
$$H = (C_2/C_1)L = (C_2/C_1)[V(C_1/C_2)]^{1/3} = [V(C_2^2/C_1^2)]^{1/3} \,. \tag{3.24}$$

The specific values for this particular problem would be:

$$L = [4(2/3)]^{1/3} = 1.387 \text{ ft}$$
$$W = L = 1.387 \text{ ft}$$
$$H = [4(3^2/2^2)]^{1/3} = 2.080 \text{ ft} \,.$$

The volume of the box, $LWH = (1.386)(1.386)(2.080) = 4.0 \text{ ft}^3$, which was the minimum volume required for the box. To verify the results, the parameters are used in the primal problem (Equation (3.13)) to make certain the solution obtained is the same.

$$\text{Cost}(Y) = C_1HW + C_1HL + C_2WL \,. \tag{3.13}$$

$$\begin{aligned}
Y_0 &= C_1[V(C_2^2/C_1^2)]^{1/3}[V(C_1/C_2)]^{1/3} + C_1[V(C_2^2/C_1^2)]^{1/3}[V(C_1/C_2)]^{1/3} \\
&\qquad + C_2[V(C_1/C_2)]^{1/3}[V(C_1/C_2)]^{1/3} \\
Y_0 &= C_1^{2/3}(V^{2/3})C_2^{1/3} + C_1^{2/3}(V^{2/3})C_2^{1/3} \\
&\qquad + C_1^{2/3}(V^{2/3})C_2^{1/3} \\
Y_0 &= 3C_1^{2/3}C_2^{1/3}V^{2/3} \,.
\end{aligned} \tag{3.25}$$

The expressions for Equation (3.19) from the primal and Equation (3.25) from the dual are equivalent. The geometric programming solution is in general terms, and thus can be used for any values of C_1, C_2, and V. This ability to obtain general relationships makes the use of geometric programming a very valuable tool for cost engineers.

The dual variables associated with the objective function, ω_{01}, ω_{02} or ω_{03} were equal to the value of 1/3, which implies that each term of the primal contributed equally to the objective function. This can be illustrated by examining the three terms of the primal, individually, in Equation (3.13). The specific values of L, W, and H (1.387 ft, 1.387 ft and 2.080 ft, respectively) were put in the three terms of the primal with the cost parameter:

$$C_1HW = \$2/\text{ft}^2 \times 2.080 \text{ ft} \times 1.387 \text{ ft} = \$5.77$$
$$C_1HL = \$2/\text{ft}^2 \times 2.080 \text{ ft} \times 1.387 \text{ ft} = \$5.77$$
$$C_2WL = \$3/\text{ft}^2 \times 1.386 \text{ ft} \times 1.386 \text{ ft} = \$5.77 \ .$$

The total cost is the sum of the three components, which is \$ 17.31 as determined by the solution of the dual, previously.

3.3 EVALUATIVE QUESTIONS

1. What version of the geometric problem formulation is solved for the objective function and why?

2. What values can the signum function have?

3. How are the primal variables determined?

4. A large box is to be made with the values of $C_1 = 4$ Euros/m^2, $C_2 = 4$ Euros/m^2, and $V = 8$ m^3. What is the cost(Euros) and the values of H, W, and L?

CHAPTER 4

Trash Can Case Study

4.1 INTRODUCTION

Various case studies are used to illustrate the different applications of geometric programming as well as to illustrate the different conditions that must be evaluated in solving the problems. The first case study, the trash can case study, will be easy to solve and have zero degrees of difficulty, and the latter cases will indicate situations where the degrees of difficulty are positive and where the dual variables may be negative. When the dual variables become negative, that indicates that a constraint is non-binding, and thus that constraint can be removed from the solution. The solutions are provided in detail, giving the general solution for the problem in addition to the specific solution. These examples are provided so that the readers can develop solutions to specific problems that they may have and to illustrate the importance of the generalized solution.

4.2 PROBLEM STATEMENT AND GENERAL SOLUTION

Bjorn of Sweden has entered into the trash can manufacturing business and he is making cylindrical trash cans and wants to minimize the material cost. The trash can is an open cylinder and designed to have a specific volume. The objective will be to minimize the total material cost of the can. Figure 4.1 is a sketch of the trash can illustrating the design parameters of radius and height. The bottom and sides can be of different costs as the bottom is typically made of a thicker material. The primal objective function is:

$$\text{Minimize:} \qquad \text{Cost}(Y) = C_1 \pi r^2 + C_2 2\pi r h \tag{4.1}$$

$$\text{Subject to:} \qquad V = \pi r^2 h\ , \tag{4.2}$$

where:

$$\begin{aligned} r &= \text{radius of trash can bottom} \\ h &= \text{height of trash can} \\ V &= \text{volume of trash can} \\ C_1 &= \text{material constant cost of bottom material of trash can} \\ C_2 &= \text{material constant cost of side material of trash can.} \end{aligned}$$

The constraint must be written in the form of an inequality, so

$$V \geq \pi r^2 h\ . \tag{4.3}$$

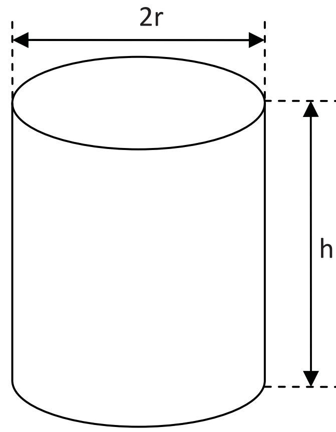

Figure 4.1: Trash can.

And it must be written in the less than equal form, so it becomes

$$-\pi r^2 h/V \leq -1 \ . \tag{4.4}$$

Thus, the primal problem is:

$$\begin{aligned} &\text{Minimize:} && \text{Cost}(Y) = C_1 \pi r^2 + C_2 2\pi r h && (4.5)\\ &\text{Subject to:} && -\pi r^2 h/V \leq -1 \ . && (4.6) \end{aligned}$$

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= \ \ 1\\ \sigma_{02} &= \ \ 1\\ \sigma_{11} &= -1\\ \sigma_{1} \ &= -1 \ . \end{aligned}$$

The dual formulation is:

$$\begin{aligned} &\text{Objective Function} && \omega_{01} + \omega_{02} = 1 && (4.7)\\ &r \text{ terms} && 2\omega_{01} + \omega_{02} - 2\omega_{11} = 0 && (4.8)\\ &h \text{ terms} && \omega_{02} - \omega_{11} = 0 \ . && (4.9) \end{aligned}$$

Using theses equations, the values of the dual variables are found to be:

$$\begin{aligned} \omega_{01} &= 1/3\\ \omega_{02} &= 2/3\\ \omega_{11} &= 2/3 \end{aligned}$$

and by definition

$$\omega_{00} = 1 \ .$$

The degrees of difficulty are equal to:

$$D = T - (N + 1) = 3 - (2 + 1) = 0 \ . \tag{4.10}$$

Using the linearity inequality equation,

$$\omega_{10} = \omega_{mt} = \sigma_m \sum \sigma_{mt} \ \omega_{mt} = (-1)^*(-1^*2/3) = 2/3 > 0 \quad \text{where } m = 1 \text{ and } t = 1 \ .$$

The objective function can be found using the dual expression:

$$Y = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt} \ \omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \tag{4.11}$$

$$= 1 \left[\left[\{(\pi \ C_1{}^*1/(1/3))\}^{(1^*1/3)} \right] \left[\{\pi \ C_2{}^*1/(2/3))\}^{(1^*2/3)} \right] \left[\{(\pi/V)^*((2/3)/(2/3)\}^{(-1^*2/3)} \right] \right]^1$$

$$Y = 3\pi^{1/3} \, C_1^{1/3} \, C_2^{2/3} \, V^{2/3} \ . \tag{4.12}$$

The values for the primal variables can be determined from the relationships between the primal and dual as:

$$C_1 \pi r^2 = \omega_{01} Y = 1/3^* Y \ , \tag{4.13}$$

$$\text{and} \qquad C_2 2\pi r h = \omega_{02} Y = 2/3^* Y \ . \tag{4.14}$$

Dividing these expressions and reducing terms one can obtain:

$$r = (C_2/C_1)^* h \ . \tag{4.15}$$

Setting

$$V = \pi r^2 h \ . \tag{4.16}$$

And using the last two equations one can obtain

$$h = ((V/\pi)(C_1^2/C_2^2))^{1/3} \ , \tag{4.17}$$

$$\text{and} \quad r = ((V/\pi)(C_2/C_1))^{1/3} \ . \tag{4.18}$$

Using (4.17) and (4.18) in Equation (4.5) for the primal, one obtains:

$$Y = C_1 \pi r^2 + C_2 2\pi r h \tag{4.5}$$

$$= 3\pi^{1/3} C_1^{1/3} C_2^{2/3} V^{1/3} \ . \tag{4.19}$$

Note that Equations (4.19) and (4.12) are identical, which is what should happen, as the primal and dual objective functions must be identical.

An important aspect about the dual variables is that they indicate the effect of the terms upon the solution. The values of $\omega_{02} = 2/3$ and $\omega_{01} = 1/3$ indicate that the second term has twice the impact as the first term in the primal. For example, if $C_1 = 9$ \$/sq ft, $C_2 = 16$ \$/sq ft, and $V = 4\pi(12.57)$ cubic feet, then

$$\begin{aligned} h &= ((V/\pi)(C_1^2/C_2^2))^{1/3} &&= ((4\pi/\pi)(9^2/16^2))^{1/3} &&= (4{*}81/256)^{1/3} &&= 1.082 \text{ ft},\\ \text{and} \quad r &= ((V/\pi)(C_2/C_1))^{1/3} &&= ((4\pi/\pi)(16/9))^{1/3} &&= (4{*}16/9)^{1/3} &&= 1.923 \text{ ft}. \end{aligned}$$

Note that:

$$V = \pi r^2 h = 3.1416{*}1.082 \text{ ft}{*}(1.923 \text{ ft})^2 = 12.57 \text{ ft}^3 .$$

And

$$\begin{aligned} Y = C_1\pi r^2 + C_2 2\pi r h &= 9{*}3.14{*}1.923^2 + 16{*}2{*}3.14{*}1.923{*}1.082\\ &= \$104.5 + \$209.0\\ &= \$313.5 . \end{aligned}$$

Note that the contribution of the second term is twice that of the first term, which is what is predicted by the value of the dual variables as $\omega_{01} = 1/3$ and $\omega_{02} = 2/3$. This occurs regardless of the values of the constants used, and this is an important concept for cost analysis. Thus, the cost of the walls ($2\pi r h$) is twice the cost of the base(πr^2), regardless of the values of C_1 and C_2. The values of h and r, as well as the total cost, are dependent upon C_1 and C_2, but the ratio of the cost between the walls and base will remain the same. This information on the ratios of the costs is unique to geometric programming.

4.3 EVALUATIVE QUESTIONS

1. A trash can is designed to hold 3 cubic meters of trash. Determine the cost and the design parameters (radius and height) in meters for if the costs C_1 and C_2 are 20 Swedish Kroner per square meter and 10 Swedish Kroner per square meter, respectively.

2. If the volume is doubled to 6 cubic meters, what are the new dimensions and cost?

3. If the trash can is to have a lid, which will have the same dimensions as the bottom of the trash can, what are the cost and dimensions of the trash can with the lid?

CHAPTER 5

Open Cargo Shipping Box Case Study

5.1 PROBLEM STATEMENT AND GENERAL SOLUTION

This is a classic geometric programming problem as it was the first illustrative problem presented in the first book [10, page 5] on geometric programming. The problem was expanded to determine the dimensions of the box as well as the minimum cost of the shipping box. The problem is stated as such: "Suppose that 400 cubic yards (V) of gravel must be ferried across a river. The gravel is to be shipped in an open cargo box of length x_1, width x_2 and height x_3. The sides and bottom of the box cost \$ 10 per square yard (A_1), and the ends of the box cost \$ 20 per square yard (A_2). The cargo box will have no salvage value and each round trip of the box on the ferry will cost 10 cents (A_3)."

a) What is the minimum total cost of transporting the 400 cubic yards of gravel?

b) What are the dimensions of the cargo box?

c) What is the number of ferry trips to transport the 400 cubic yards of gravel?

Figure 5.1 illustrates the parameters of the open cargo shipping box.

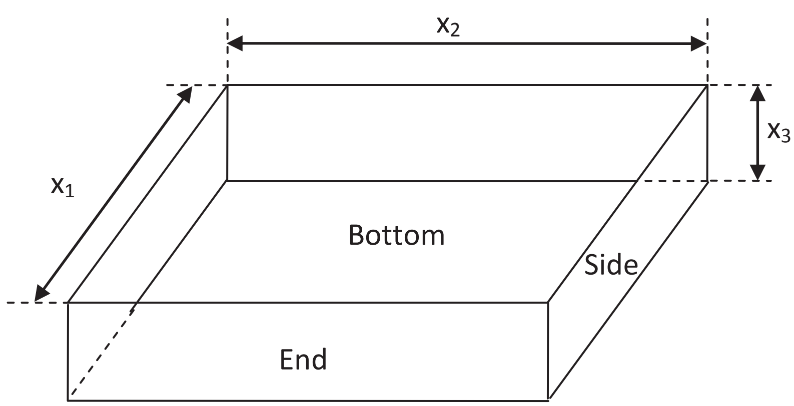

Figure 5.1: Open cargo shipping box.

The first issue is to determine the various cost components to make the objective function. The ferry transportation cost can be determined by:

$$T1 = V^{*}A_3/(x_1{}^{*}x_2{}^{*}x_3) = 400^{*}0.10/(x_1{}^{*}x_2{}^{*}x_3) = 40/(x_1{}^{*}x_2{}^{*}x_3)\ . \tag{5.1}$$

The cost for the ends of the box (2 ends) is determined by:

$$T2 = 2^*(x_2{}^*x_3)^*A_2 = 2^*(x_2{}^*x_3)^*20 = 40^*(x_2{}^*x_3)\ . \tag{5.2}$$

The cost for the sides of the box (2 sides) is determined by:

$$T3 = 2^*(x_1{}^*x_3)^*A_1 = 2^*(x_1{}^*x_3)^*10 = 20^*(x_1{}^*x_3)\ . \tag{5.3}$$

The cost for the bottom of the box is determined by:

$$T4 = (x_1{}^*x_2)^*A_1 = (x_1{}^*x_2)^*10 = 10^*(x_1{}^*x_2)\ . \tag{5.4}$$

The objective function (Y) is the sum of the four components and is:

$$Y = T1 + T2 + T3 + T4 \tag{5.5}$$
$$Y = 40/(x_1{}^*x_2{}^*x_3) + 40^*(x_2{}^*x_3) + 20^*(x_1{}^*x_3) + 10^*(x_1{}^*x_2)\ . \tag{5.6}$$

The primal objective function can be written in terms of generic constants for the cost variables to obtain a generalized solution.

$$Y = C_1/(x_1{}^*x_2{}^*x_3) + C_2{}^*(x_2{}^*x_3) + C_3{}^*(x_1{}^*x_3) + C_4{}^*(x_1{}^*x_2)\ , \tag{5.7}$$

where $C_1 = 40$, $C_2 = 40$, $C_3 = 20$ and $C_4 = 10$.

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= 1 \\ \sigma_{02} &= 1 \\ \sigma_{03} &= 1 \\ \sigma_{04} &= 1\ . \end{aligned}$$

The dual formulation is:

$$\begin{array}{lrcl} \text{Objective Function} & \omega_{01} + \omega_{02} + \omega_{03} + \omega_{04} &=& 1 \quad (5.8) \\ x_1 \text{ terms} & -\omega_{01} \qquad\quad + \omega_{03} + \omega_{04} &=& 0 \quad (5.9) \\ x_2 \text{ terms} & -\omega_{01} + \omega_{02} \qquad\quad + \omega_{04} &=& 0 \quad (5.10) \\ x_3 \text{ terms} & -\omega_{01} + \omega_{02} + \omega_{03} \qquad\quad &=& 0\ . \quad (5.11) \end{array}$$

Using theses equations, the values of the dual variables are found to be:

$$\begin{aligned} \omega_{01} &= 2/5 \\ \omega_{02} &= 1/5 \\ \omega_{03} &= 1/5 \\ \omega_{04} &= 1/5\ , \end{aligned}$$

and by definition

$$\omega_{00} = 1 \ .$$

Thus, the dual variables indicate that the first term of the primal expression is twice as important as the other three terms. The degrees of difficulty are equal to:

$$D = T - (N + 1) = 4 - (3 + 1) = 0 \ . \tag{5.12}$$

The objective function can be found using the dual expression:

$$Y = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{Tm} (C_{mt}\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \tag{5.13}$$

$$\begin{aligned}
&= 1[[\{(C_1{}^*1/(2/5))\}^{(1^*2/5)}][\{C_2{}^*1/(1/5))\}^{(1^*1/5)}][\{C_3{}^*1/(1/5)\}^{(1^*1/5)}][\{C_4{}^*1/(1/5)\}^{(1^*1/5)}]]^1 \\
&= 100^{2/5} \quad * \quad 200^{1/5} \quad * \quad 100^{1/5} \quad * \quad 50^{1/5} \\
&= 100^{2/5} \quad * \quad 1000000^{1/5} \\
&= 100^{2/5} \quad * \quad 100^{3/5} \\
&= \$100 \ .
\end{aligned}$$

Thus, the minimum cost for transporting the 400 cubic yards of gravel across the river is $ 100.

The values for the primal variables can be determined from the relationships between the primal and dual as:

$$C_1/(x_1{}^*x_2{}^*x_3) = \omega_{01}Y = (2/5)Y \tag{5.14}$$
$$C_2{}^*x_2{}^*x_3 = \omega_{02}Y = (1/5)Y \tag{5.15}$$
$$C_3{}^*x_1{}^*x_3 = \omega_{03}Y = (1/5)Y \tag{5.16}$$
$$C_4{}^*x_1{}^*x_2 = \omega_{04}Y = (1/5)Y \ . \tag{5.17}$$

If one combines Equations (5.15) and (5.16), one can obtain the relationship:

$$x_2 = x_1{}^*(C_3/C_2) \ . \tag{5.18}$$

If one combines Equations (5.16), (5.17), and (5.18), one can obtain the relationship:

$$x_3 = x_2{}^*(C_4/C_3) = x_1{}^*(C_3/C_2)^*(C_4/C_3) = x_1{}^*(C_4/C_2) \ . \tag{5.19}$$

If one combines Equations (5.14) and (5.15), one can obtain the relationship:

$$x_1{}^*x_2^{2*}x_3^2 = (1/2)^*(C_1/C_2) \ . \tag{5.20}$$

Using the values for x_2 and x_3 in Equation (5.20), one can obtain:

$$x_1 = [(1/2)^*(C_1C_2^3/(C_3^2C_4^2))]^{1/5} \ . \tag{5.21}$$

Similarly, one can solve for x_2 and x_3 and the equations would be:

$$x_2 = [(1/2)^*(C_1C_3^3/(C_2^2C_4^2))]^{1/5} \tag{5.22}$$

and

$$x_3 = [(1/2)^*(C_1C_4^3/(C_2^2C_3^2))]^{1/5} \ . \tag{5.23}$$

Now using the values of $C_1 = 40$, $C_2 = 40$, $C_3 = 20$ and $C_4 = 10$, the values of x_1, x_2, and x_3 can be determined using Equations (5.21), (5.22), and (5.23) as:

$$\begin{aligned} x_1 &= [(1/2)^*(40^*40^3/(20^2 10^2))]^{1/5} = [32]^{1/5} = 2 \text{ yards} \\ x_2 &= [(1/2)^*(40^*20^3/(40^2 10^2))]^{1/5} = [1]^{1/5} = 1 \text{ yard} \\ x_3 &= [(1/2)^*(40^*10^3/(40^2 20^2))]^{1/5} = [0.03125]^{1/5} = 0.5 \text{ yard} \ . \end{aligned}$$

Thus, the box is 2 yards in length, 1 yard in width, and 0.5 yard in height. The total box volume is the product of the three dimensions, which is 1 cubic yard.

The number of trips the ferry must make is 400 cubic yards/1 cubic yard/trip = 400 trips. If one uses the primal variables in the primal equation, the values are:

$$Y = 40/(x_1{}^*x_2{}^*x_3) + 40^*(x_2 * x_3) + 20^*(x_1{}^*x3) + 10^*(x_1{}^*x2) \ . \tag{5.6}$$

$$Y = 40/(2^*1^*1/2) + 40^*(1^*1/2) + 20^*(2^*1/2) + 10^*(2^*1) \tag{5.24}$$

$$\begin{aligned} Y &= 40 + 20 + 20 + 20 \\ &= \$100 \ . \end{aligned}$$

Note that the primal and dual give the same result for the objective function. Note that the components of the primal solution (40, 20, 20, 20) are in the same ratio as the dual variables (2/5, 1/5, 1/5, 1/5). This ratio will remain constant even as the values of the constants change, and this is important in the ability to determine which of the terms are dominant in the total cost. Thus, the transportation cost is twice the cost of the box bottom, and the box bottom is the same as the cost of the box sides and the same as the cost of the box ends. This indicates the optimal design relationships between the costs of the various box components and the transportation cost associated with the design.

5.2 EVALUATIVE QUESTIONS

1. The ferry cost for a round trip is increased from \$ 0.10 to \$ 3.20. What is the new total cost, the new box dimensions, and the number of ferry trips required to transport the 400 cubic yards of gravel?

2. The ferry cost for a round trip is increased from \$ 3.20 to \$ 213.06. What is the new total cost, the new box dimensions, and the number of ferry trips required to transport the 400 cubic yards of gravel?

3. A cover must be added to the box, and it is made having the same costs as the box bottom. Determine the new total cost, the new box dimensions, and the number of ferry trips required to transport the 400 cubic yards of gravel.

CHAPTER 6

Metal Casting Cylindrical Riser Case Study

6.1 INTRODUCTION

The riser design problem in metal casting is always a concern for foundry engineers. The riser(also called feeders in many parts of the world) is an amount of additional metal added to a metal casting to move the thermal center from the casting into the riser so there will be no solidification shrinkage in the casting. The risers are typically shaped as cylinders as other shapes are difficult for the molding process, and this shape has been successfully used for decades. The riser also has other design conditions such as to supply sufficient feed metal, but thermal design issues are typically the primary concern. There are several papers on riser design using geometric programming concerning side riser, top riser, insulated riser and many other riser design issues in the references [4, 5, 6, 7, 8].

For a riser to be effective, the riser must solidify after the casting in order to provide liquid feed metal to the casting. The object is to have a riser of minimum volume to improve the yield of the casting process which improves the economics of the process. The case study considered is a cylindrical side riser which consists of a cylinder of height H and diameter D. Figure 6.1 indicates the relationship between the casting, the side riser and the parameters of the riser.

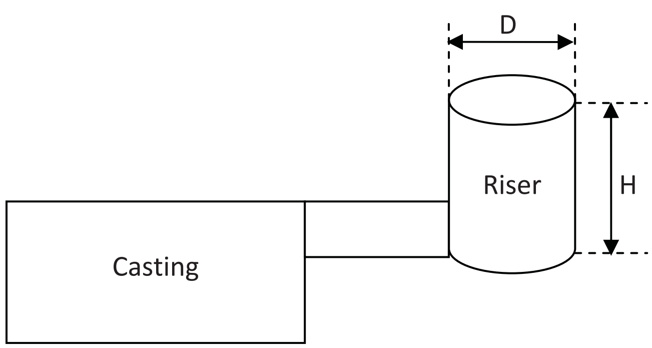

Figure 6.1: The cylindrical riser.

The theoretical basis for riser design is Chvorinov's Rule, which is

$$t = K(V/SA)^2 , \tag{6.1}$$

where

$$t = \text{solidification time (minutes or seconds)}$$
$$K = \text{solidification constant for molding material (minutes/in}^2 \text{ or seconds/cm}^2)$$
$$V = \text{riser volume (in}^3 \text{ or cm}^3)$$
$$SA = \text{cooling surface area of the riser (in}^3 \text{ or cm}^3)\ .$$

The objective is to design the smallest riser such that

$$t_R \geq t_C \tag{6.2}$$

where

$$t_R = \text{solidification time of the riser}$$
$$t_C = \text{solidification time of the casting}\ .$$

This constraint (Equation (6.2)) can be written as:

$$K_R(V_R/SA_R) \geq K_C(V_C/SA_C)\ . \tag{6.3}$$

The riser and the casting are assumed to be molded in the same material so the K_R and K_C are equal and thus the equation can be written as:

$$(V_R/SA_R) \geq (V_C/SA_C)\ . \tag{6.4}$$

The casting has a specified volume and surface area, the right-hand side of the equation can be expressed as a constant $Y = (V_C/SA_C)$, which is called the casting modulus, and Equation (6.4) becomes

$$(V_R/SA_R) \geq Y\ . \tag{6.5}$$

The volume and surface of the cylindrical riser can be written as:

$$V_R = \pi D^2 H/4 \tag{6.6}$$
$$SA_R = \pi DH + 2\pi D^2/4\ . \tag{6.7}$$

The surface area expression neglects the connection area between the casting and the riser as the effect is small. Thus, Equation (6.5) can be rewritten as:

$$(\pi D^2 H/4)/(\pi DH + 2\pi D^2/4) = (DH)/(4H + 2D) \geq Y\ . \tag{6.8}$$

The constraint must be rewritten in the less than equal form with the right-hand side being less than or equal to one, which becomes

$$4YD^{-1} + 2YH^{-1} \leq 1\ . \tag{6.9}$$

6.2 PROBLEM FORMULATION AND GENERAL SOLUTION

The primal form of the side cylindrical riser design problem can be stated as:

Minimize:

$$V = \pi D^2 H/4 \,. \tag{6.10}$$

Subject to:

$$4YD^{-1} + 2YH^{-1} \leq 1 \,. \tag{6.11}$$

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= 1 \\ \sigma_{11} &= 1 \\ \sigma_{12} &= 1 \\ \sigma_1 &= 1 \end{aligned}$$

The dual problem formulation is:

Objective Function	ω_{01}			$= 1$	(6.12)
D terms	$2\omega_{01}$	$-\omega_{11}$		$= 0$	(6.13)
H terms	ω_{01}		$-\omega_{12}$	$= 0$.	(6.14)

Using Equations (6.12) to (6.14), the values of the dual variables were found to be:

$$\begin{aligned} \omega_{01} &= 1 \\ \omega_{11} &= 2 \\ \omega_{12} &= 1 \,. \end{aligned}$$

The degrees of difficulty are equal to:

$$D = T - (N+1) = 3 - (2+1) = 0 \,. \tag{6.15}$$

Using the linearity inequality equation, ω_{10} can be evaluated as:

$$\omega_{10} = \omega_{mt} = \sigma_m \sum \sigma_{mt}\omega_{mt} = (1)^*(1^*2 + 1^*1) = 3 > 0 \text{ where } m = 1 \text{ and } t = 1 \,. \tag{6.16}$$

The objective function can be found using the dual expression:

$$Y = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \tag{6.17}$$

$$\begin{aligned} &= 1[[\{(\pi/4^*1/1)\}^{(1^*1)}][\{(4Y^*3/2)\}^{(1^*2)}][\{(2Y^*3/1)\}^{(1^*1)}]]^1 \\ &= (\pi/4)^*(6Y)^{2*}(6Y) \\ &= (\pi/4)^*(6Y)^3 \,. \end{aligned} \tag{6.18}$$

The values for the primal variables can be determined from the relationships between the primal and dual as:

$$4YD^{-1} = \omega_{11}/\omega_{10} = 2/3 \tag{6.19}$$
$$\text{and } 2YH^{-1} = \omega_{11}/\omega_{10} = 1/3\ . \tag{6.20}$$

The equations for H and D can be determined as:

$$D = 6Y \tag{6.21}$$
$$\text{and} \qquad H = 6Y\ . \tag{6.22}$$

Using (6.21) and (6.22) in Equation (6.6) for the primal, one obtains:

$$V_R = (\pi/4)^*(6Y)^3\ . \tag{6.23}$$

Note that Equations (6.18) and (6.23) are identical, which is what should happen, as the primal and dual objective functions must be identical. The design equations for the riser diameter and the riser height are both six times the casting modulus. This relationship holds for the side cylindrical riser design with negligible effects for the connecting area. This also indicates that the riser height and riser diameter are equal for the side riser. Designs for other riser shapes and with insulating materials using geometric programming are given in the references.

6.3 EXAMPLE PROBLEM

A rectangular plate casting with dimensions $L = W = 10$ cm and $H = 4$ cm is to be produced, and a cylindrical side riser is to be used. The optimal dimensions for the side riser can be obtained from the casting modulus Y and Equations (6.21) and (6.22). The casting modulus is obtained by:

$$\begin{aligned} Y = (V_C/SA_C) &= (10\text{ cm } \times 10\text{ cm } \times 4\text{ cm})/[2(10\text{ cm } \times 10\text{ cm}) \\ &\quad + 2(10\text{ cm } \times 4\text{ cm}) + 2(10\text{ cm } \times 4\text{ cm})] \\ &= 400\text{ cm}^3/360\text{ cm}^2 = 1.111\text{ cm}\ . \end{aligned}$$

Thus,

$$\begin{aligned} H &= 6Y = 6 \times 1.111\text{ cm } = 6.67\text{ cm} \\ D &= 6Y = 6 \times 1.111\text{ cm } = 6.67\text{ cm}\ . \end{aligned}$$

The volume of the riser can be obtained from Equation (6.23) as:

$$V_R = (\pi/4)(6Y)^3 = 233\text{ cm}^3\ .$$

Thus, once the modulus of the casting is determined, the riser height, diameter, and volume can be determined using Equations (6.21)–(6.23).

6.4 EVALUATIVE QUESTIONS

1. A side riser is to be designed for a metal casting, which has a surface area of 40 cm^2 and a volume of 120 cm^3. The hot metal cost is 100 Rupees per kg, and the metal density is 3.0 gm/cm^3.

   (a) What are the dimensions in centimeters for the side riser (H and D)?

   (b) What is the volume of the side riser (cm^3)?

   (c) What is the metal cost of the side riser(Rupees)?
   item What is the metal cost of the casting(Rupees)?

2. Instead of a side riser, a top riser is to be used; that is, the riser is placed on the top surface of the riser. The cooling surface area for the top riser is:

$$SA_R = \pi DH + \pi D^2/4 \ .$$

   Show that for the top riser that $D = 6Y$ and $H = 3Y$.

CHAPTER 7

Process Furnace Design Case Study

7.1 PROBLEM STATEMENT AND SOLUTION

An economic process model was developed [20, 16] for an industrial metallurgical application. The annual cost for a furnace operation in which the slag-metal reaction was a critical factor of the process was considered, and a modified version of the problem is presented. The object was to minimize the annual cost, and the primal equation representing the model was:

$$Y = C_1/(L^{2*}D^*T^2) + C_2{}^*L^*D + C_3L^*D^*T^4\ . \tag{7.1}$$

The model was subject to the constraint that:

$$D \leq L\ .$$

The constraint must be set in geometric programming for which would be:

$$(D/L) \leq 1\ , \tag{7.2}$$

where

$$\begin{aligned} D &= \text{Depth of the furnace (ft)} \\ L &= \text{Characteristic Length of the furnace (ft)} \\ T &= \text{Furnace Temperature (K)}\ . \end{aligned}$$

For the specific example problem, the values of the constants were:

$$\begin{aligned} C_1 &= 10^{13}(\$ - \text{ft}^3 - \text{K}^2) \\ C_2 &= 100(\$/\text{ft}^2) \\ C_3 &= 5^*10^{-11}(\text{ft}^{-2} - \text{K}^{-4})\ . \end{aligned}$$

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= 1 \\ \sigma_{02} &= 1 \\ \sigma_{03} &= 1 \\ \sigma_{11} &= 1 \\ \sigma_1 &= 1\ . \end{aligned}$$

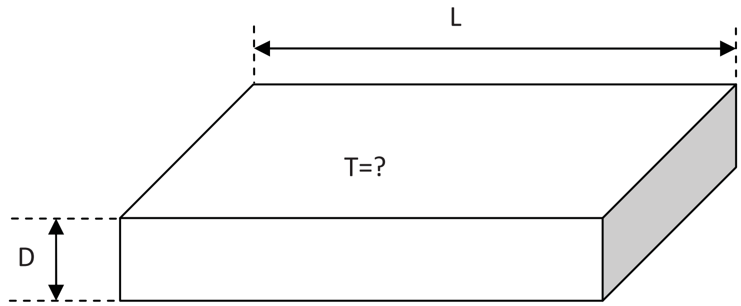

Figure 7.1: Process furnace.

The dual problem formulation is:

$$\begin{array}{llr} \text{Objective Function} & \omega_{01}+\omega_{02}+\omega_{03} = 1 & (7.3)\\ L \text{ terms} & -2\omega_{01}+\omega_{02}+\omega_{03}-\omega_{11} = 0 & (7.4)\\ D \text{ terms} & -\omega_{01}+\omega_{02}+\omega_{03}+\omega_{11} = 0 & (7.5)\\ T \text{ terms} & -2\omega_{01}+4\omega_{03} = 0\,. & (7.6) \end{array}$$

Using Equations (7.3) to (7.6), the values of the dual variables were found to be:

$$\omega_{01} = 0.4 \quad (7.7)$$
$$\omega_{02} = 0.4 \quad (7.8)$$
$$\omega_{03} = 0.2 \quad (7.9)$$
$$\omega_{11} = -0.2\,. \quad (7.10)$$

The dual variables cannot be negative, and the negative value implies that the constraint is not binding, that is it is a loose constraint. Thus, the problem must be reformulated without the constraint, and the dual variable is forced to zero, that is $\omega_{11} = 0$ and the equations resolved. The new dual becomes:

$$\begin{array}{llr} \text{Objective Function} & \omega_{01}+\omega_{02}+\omega_{03} = 1 & (7.11)\\ L \text{ terms} & -2\omega_{01}+\omega_{02}+\omega_{03} = 0 & (7.12)\\ D \text{ terms} & -\omega_{01}+\omega_{02}+\omega_{03} = 0 & (7.13)\\ T \text{ terms} & -2\omega_{01}+4\omega_{03} = 0\,. & (7.14) \end{array}$$

Now the problem is that it has 4 equations to solve for three variables. If one examines Equations (7.12) and (7.13), one observes that Equation (7.12) is dominant over Equation (7.13), and thus Equation (7.13) will be removed from the dual formulation. The new dual formulation is:

$$\begin{array}{llr} \text{Objective Function} & \omega_{01} + \omega_{02} + \omega_{03} = 1 & (7.15) \\ L \text{ terms} & -2\omega_{01} + \omega_{02} + \omega_{03} = 0 & (7.16) \\ T \text{ terms} & -2\omega_{01} + 4\omega_{03} = 0 \,. & (7.17) \end{array}$$

The new solution for the dual becomes:

$$\omega_{01} = 1/3$$
$$\omega_{02} = 1/2$$
$$\omega_{03} = 1/6$$

and by definition

$$\omega_{00} = 0.0 \,.$$

The dual variables indicate that the second term is the most important, followed by the first term and then the third term. The degrees of difficulty are now equal to:

$$D = T - (N + 1) = 3 - (2 + 1) = 0 \,.$$

The objective function can be found using the dual expression:

$$Y = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \tag{7.18}$$

$$= 1[[\{(C_1{}^*1/(1/3)1)\}^{(1/3^*1)}][\{(C_2{}^*1/(1/2)\}^{(1/2^*2)}][\{(C_3/(1/6))\}^{(1/6^*1)}]]^1$$

$$= 1[[\{(1^*10^{13}{}^*1/(1/3)1)\}^{(1/3^*1)}][\{(100^*1/(1/2)\}^{(1/2^*2)}][\{(5^*10^{-11}/(1/6))\}^{(1/6^*1)}]]^1$$

$$= \$11,370 \,.$$

This can be expressed in a general form in terms of the constants as:

$$Y = (3C_1)^{1/3}(2C_2)^{1/2}(6C_3)^{1/6} \,. \tag{7.19}$$

The values for the primal variables can be determined from the relationships between the primal and dual, which are:

$$C_1{}^*L^{-2*}D^{-1*}T^{-2} = \omega_{01}Y \tag{7.20}$$
$$C_2{}^*L^*D = \omega_{02}Y \tag{7.21}$$
and
$$C_3{}^*L^*D^*T^4 = \omega_{03}Y \,. \tag{7.22}$$

The fully general expressions are somewhat difficult, but the variables can be expressed in terms of the constants and objective function as:

$$T = [(\omega_{03}/\omega_{02})^{*}(C_2/C_3)]^{1/4} \tag{7.23}$$

$$L = [(C_1{}^{*}(C_2{}^{*}C_3)^{(1/2)})]/[\omega_{01}{}^{*}(\omega_{02}{}^{*}\omega_{03})^{(1/2)*}Y^2] \tag{7.24}$$

$$D = [(\omega_{01}{}^{*}\omega_{02}^{(3/2)*}\omega_{03}^{(1/2)}Y^3)/(C_1{}^{*}C_2^{(3/2)*}C_3^{(1/2)}] \,. \tag{7.25}$$

The expressions developed for the variables in terms of the constants in a reduced form were:

$$T = (1/3\ C_2/C_3)^{1/4} \tag{7.26}$$

$$L = 1.3743 * (C_1^{1/3}C_2^{-1/2}C_3^{1/6}) \tag{7.27}$$

$$D = 1 \,. \tag{7.28}$$

Using the values of $Y = 11369$, $C_1 = 10^{13}$, $C_2 = 100$, $C_3 = 5^{*}10^{-11}$, $\omega_{01} = 1/3$, $\omega_{02} = 1/2$, and $\omega_{03} = 1/6$, the values for the variables are:

$$T = 903K$$

$$L = 56.85 \text{ ft}$$

and

$$D = 1.00 \text{ ft} \,.$$

Using the values of the variables in the primal equation, the objective function is:

$$\begin{aligned} Y &= C_1/(L^{2*}D^{*}T^2) + C_2{}^{*}L^{*}D + C_3{}^{*}L^{*}D^{*}T^4 \\ &= 10^{13}/(56.85^{2*}1^{*}903^2) + 100^{*}56.85^{*}1 + 5^{*}10^{-11*}56.85^{*}1^{*}(903^4) \\ &= 3,795 + 5,685 + 1,890 \\ &= \$11,370 \,. \end{aligned}$$

The values of the objective function for the primal and dual are identical, which implies that the values for the primal variables have been correctly obtained. Note that the costs terms are in the same ratio as the dual variables; the third term is the smallest, the first term is twice the third term, and the second term is three times the third term.

This problem was presented to indicate the difficulties in that when the constraint is loose, the problem must be restated with the loose constraint removed and a new solution obtained for the dual variables. The constraint is loose as $D = 1$ ft is much lower than $L = 56.85$ ft. The other item of interest was that equations dominated by other equations can prevent a solution and must be removed. The removal of the dominated equation was necessary to obtain a solution and may be the cause of D being unity.

7.2 EVALUATIVE QUESTIONS

1. The problem constraint was given as $D \leq L$, but the designer decided that was incorrect and reversed the constraint to $L \leq D$. Resolve the problem and determine the dual and primal variables as well as the objective function.

2. Resolve the problem making the initial assumption that $L = D$ and reformulate the primal and dual problems and find the variables and objective function.

CHAPTER 8

Gas Transmission Pipeline Case Study

8.1 PROBLEM STATEMENT AND SOLUTION

The energy crisis is with us today, and one of the problems is in the transmission of energy. A gas transmission model was developed [20, 14] to minimize the total transmission cost of gas in a new gas transmission pipeline. The problem is more difficult than the previous case studies as several of the exponents are not integers. The primal expression for the cost developed was:

$$C = C_1{}^* L^{1/2*} V/(F^{0.387*} D^{2/3}) + C_2{}^* D^* V + C_3/(L^* F) + C_4{}^* F/L \; . \tag{8.1}$$

Subject to:

$$(V/L) \geq F \; .$$

The constraint must be restated in the geometric form as:

$$-(V/(LF)) \leq -1 \; , \tag{8.2}$$

where

L = Pipe length between compressors (feet)
D = Diameter of Pipe (in)
V = Volume Flow Rate (ft^3/sec)
F = Compressor Pressure Ratio Factor.

Figure 8.1 is a sketch of the problem indicating the variables and is not drawn to scale.

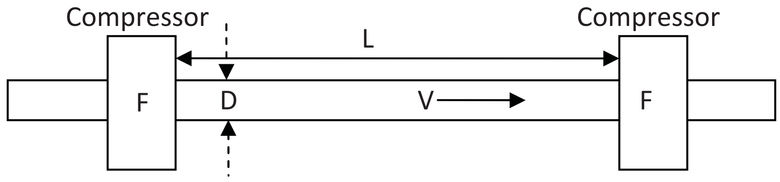

Figure 8.1: Gas transmission pipeline.

For the specific problem, the values of the constants were:

$$\begin{aligned} C_1 &= 4.55{}^*10^5 \\ C_2 &= 3.69{}^*10^4 \\ C_3 &= 6.57{}^*10^5 \\ C_4 &= 7.72{}^*10^5 \ . \end{aligned}$$

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= \ \ 1 \\ \sigma_{02} &= \ \ 1 \\ \sigma_{03} &= \ \ 1 \\ \sigma_{04} &= \ \ 1 \\ \sigma_{11} &= -1 \\ \sigma_{1} &= -1 \ . \end{aligned}$$

The dual problem formulation is:

Objective Function	$\omega_{01} + \omega_{02} + \omega_{03} + \omega_{04}$	$= 1$	(8.3)
L terms	$0.5\omega_{01} - \omega_{03} - \omega_{04} + \omega_{11}$	$= 0$	(8.4)
F terms	$-0.387\omega_{01} - \omega_{03} + \omega_{04} + \omega_{11}$	$= 0$	(8.5)
V terms	$\omega_{01} + \omega_{02} - \omega_{11}$	$= 0$	(8.6)
D terms	$-0.667\omega_{01} + \omega_{02}$	$= 0$.	(8.7)

Using Equations (8.3) to (8.7), the values of the dual variables were found to be:

$$\begin{aligned} \omega_{01} &= 0.26087 \\ \omega_{02} &= 0.17391 \\ \omega_{03} &= 0.44952 \\ \omega_{04} &= 0.11570 \\ \omega_{11} &= 0.43478 \end{aligned}$$

and by definition

$$\omega_{00} = 1 \ ,$$

and

$$\omega_{10} = \omega_{mt} = \sigma_m \sum \sigma_{mt}\omega_{mt} = (-1)^*(-1^*0.43478) = 0.43478 \text{ where } m = 1 \text{ and } t = 1 \ .$$

The objective function can be found using the dual expression:

$$Y = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \tag{8.8}$$

$$\begin{aligned} &= 1[[\{(4.55^*10^5{}^*1/0.26087)\}^{(1^*0.26087)}]^*[\{(3.69^*10^4{}^*1/0.17391)\}^{(1^*0.17391)}]^* \\ &\quad [\{(6.57^*10^5{}^*1/0.44952)\}^{(1^*0.44952)}]^*[\{(7.72^*10^5{}^*1/0.11570)\}^{(1^*0.11570)}]^* \\ &\quad [\{(1^*0.43478/0.43478)\}^{(-1^*0.43478)}]]^1 \\ &= \$1.3043^*10^6/\ \text{yr}\,. \end{aligned}$$

The degrees of difficulty are equal to:

$$D = T - (N + 1) = 5 - (4 + 1) = 0\,. \tag{8.9}$$

The values for the primal variables can be determined from the relationships between the primal and dual which are:

$$C_1{}^*L^{1/2*}V/(F^{0.387*}D^{2/3}) = \omega_{01}Y \tag{8.10}$$

$$C_2{}^*D^*V = \omega_{02}Y \tag{8.11}$$

$$C_3/(L^*F) = \omega_{03}Y \tag{8.12}$$

$$C_4{}^*F/L = \omega_{04}Y \tag{8.13}$$

$$V/(F^*L) = \omega_{11}/\omega_{10} = 1\,. \tag{8.14}$$

The fully general expressions are somewhat difficult, but the variables can be expressed in terms of the constants and objective function as:

$$F = [(C_3\omega_{04})/(C_4\omega_{03})]^{1/2} \tag{8.15}$$

$$V = C_3/(\omega_{03}{}^*Y) \tag{8.16}$$

$$L = [(C_3C_4)/(\omega_{03}\omega_{04})]/Y \tag{8.17}$$

$$D = [(\omega_{02}{}^*\omega_{03})/(C_2{}^*C3)]^*Y^2\,. \tag{8.18}$$

Using the values of $Y = 1.3043^*10^6$, $C_1 = 4.55^*10^5$, $C_2 = 3.69^*10^4$, $C_3 = 6.57^*10^5$, $C_4 = 7.72^*10^5$, $\omega_{01} = 0.26087$, $\omega_{02} = 0.17391$, $\omega_{03} = 0.44952$, $\omega_{04} = 0.11570$, and $\omega_{11} = 0.43478$ one obtains:

$$\begin{aligned} F &= [(C_3{}^*\omega_{04})/(C_4{}^*\omega_{03})]^{1/2} = [(6.57^*10^5{}^*0.11570)/(7.72^*10^5{}^*0.44952)]^{1/2} = 0.468 \\ V &= C_3/(\omega_{03}{}^*Y) = 6.57^*10^5/(0.44952 * 1.3043^*10^6) = 1.1205\ \text{ft}^3/\text{sec} \\ L &= [(C_3{}^*C_4)/(\omega_{03}{}^*\omega_{04})]^{1/2}/Y = [(6.57^*10^5{}^*7.72^*10^5)/(0.44952^*0.1157)]^{1/2}/1.3043^*10^6 = 2.3943\ \text{ft} \\ D &= [(\omega_{02}{}^*\omega_{03})/(C_2{}^*C_3)]^*Y^2 = [(0.17391^*0.44952)(3.69^*10^4 6.57^*10^5)]^*(1.3043^*10^6)^2 = 5.4857\ \text{in}\,. \end{aligned}$$

The primal expression can now be solved using the primal variables and the contribution of each of the terms can be observed.

$$\begin{aligned}
C &= C_1{}^*L^{1/2*}V/(F^{0.387*}D^{2/3}) + C_2{}^*D^*V + C_3/(L^*F) + C_4{}^*F/L \\
&= 4.55^*10^{5*}2.3943^{1/2*}1.1205/(0.468^{0.387*}5.4857^{2/3}) + 3.69^*10^{4*}5.4857^*1.1205+ \\
&\quad 6.57^*10^5/(2.3943^*0.468) + 7.72^*10^{5*}0.468/2.3943 \\
&= 3.4029^*10^5 + 2.26814^*10^5 + 5.8633^*10^5 + 1.5090^*10^5 \\
&= \$1,304,300\ .
\end{aligned}$$

The third term is slightly higher than the others, but all terms are of the same magnitude. Since the constraint is binding, that is $V = L^*F$, and the results indicate that holds as:

$$1.1205 = 2.3943^*0.468 = 1.1205\ .$$

The values of the dual variables were more complex for this problem than the previous problems, but the values of these dual variables still have the same relationship to the terms of the primal cost function. The first dual variable, ω_{01} was 0.26087, and the relation between the first cost term of the primal to the total cost is $3.4029^*10^5/1.3043^*10^6 = 0.2609$. The reader should show that the other dual variables have the same relationships between the terms of the primal cost function and the total primal cost.

8.2 EVALUATIVE QUESTIONS

1. Resolve the problem with the values of $C_1 = 6^*10^5$, $C_2 = 5^*10^4$, $C_3 = 7^*10^5$, and $C_4 = 8^*10^5$. Determine the effect upon the dual variables, the objective function, and the primal variables. Also examine the percentage of each of the primal terms in the objective function and in the original objective function.

2. The constraint is a binding constraint. If the constraint is removed, the objective function should be lower. What problem(s) occurs when the constraint is removed that causes concern?

CHAPTER 9

Journal Bearing Design Case Study

9.1 INTRODUCTION

An interesting problem with one degree of difficulty is a journal bearing design problem presented by Beightler, Lo, and Bylander [2]. The objective was to minimize the cost(P), and the variables were the half-length of the bearing(L) and the radius of the journal(R). The objective function presented and the constants in the problem are those presented in the original paper, and the derivations of the constants were not detailed. The solution presented is based upon deriving an additional equation whereas the original problem was solved by reducing the degree of difficulty and determining upper and lower bounds to the solution. The solution presented solves the problem, directly using the additional equation and without needing to use search techniques.

9.2 PRIMAL AND DUAL FORMULATION OF JOURNAL BEARING DESIGN

The generalized primal problem was:

$$\text{Minimize } P: \quad = C_{01} R^3 L^{-2} + C_{02} R^{-1} + C_{03} R L^{-3} \tag{9.1}$$

$$\text{Subject to:} \quad C_{11}{}^* R^{-1*} L^3 \leq 1\ , \tag{9.2}$$

where

P = Cost ($)
R = radius of the journal (in)
L = half-length of the bearing (in)
C_{01} = 0.44 (for example problem)
C_{02} = 10 (for example problem)
C_{03} = 0.592 (for example problem)
and C_{11} = 8.62 (for example problem).

Figure 9.1 is a sketch illustrating the variables for the problem.

From the coefficients and signs, the signum values for the dual from Equations (9.1) and (9.2) are:

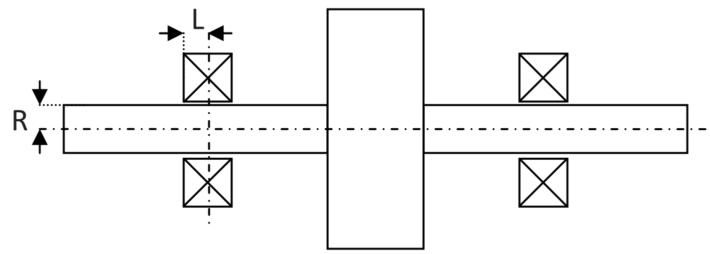

Figure 9.1: Journal bearing parameters.

$$\sigma_{01} = 1$$
$$\sigma_{02} = 1$$
$$\sigma_{03} = 1$$
$$\sigma_{11} = 1$$
$$\sigma_{1} = 1 \,.$$

The dual problem formulation is:

$$\begin{array}{lrcl} \text{Objective Function} & \omega_{01} + \omega_{02} + \omega_{03} & = 1 & (9.3) \\ R \text{ terms} & 3\omega_{01} - \omega_{02} + \omega_{03} - \omega_{11} & = 0 & (9.4) \\ L \text{ terms} & -2\omega_{01} - 3\omega_{03} + 3\omega_{11} & = 0\,. & (9.5) \end{array}$$

From the constraint equation there is only one term, so:

$$\omega_{10} = \omega_{11} \,. \tag{9.6}$$

This adds one additional equation but also one additional term, so the degrees of difficulty are equal to:

$$D = T - (N + 1) = 4 - (2 + 1) = 1 \geq 0 \,. \tag{9.7}$$

The dual has more variables than equations, and thus another equation is needed to solve for the dual variables. The relationships between the primal and dual variables will be used to determine an additional equation, and the equation typically is non-linear. The relationships between the primal and dual which are:

$$C_{01} R^3 L^{-2} = \omega_{01} P \tag{9.8}$$
$$C_{02} R^{-1} = \omega_{02} P \tag{9.9}$$
$$C_{03} R L^{-3} = \omega_{03} P \tag{9.10}$$
$$C_{11} R^{-1} L^3 = (\omega_{11}/\omega_{10}) \,. \tag{9.11}$$

Since $\omega_{10} = \omega_{11}$, Equation (9.11) can be used to relate the primal variables, that is:

$$R = C_{11} L^3 \,. \tag{9.12}$$

Using Equation (9.12) in Equation (9.9), one obtains

$$\begin{aligned} P &= C_{02}/(\omega_{02}R) \\ &= [C_{02}/(C_{11}\omega_{02}L^3)] . \end{aligned} \tag{9.13}$$

Using Equation (9.10) with Equations (9.12) and (9.13), one obtains after reducing terms:

$$\begin{aligned} L^3 &= (C_{03}R)/(\omega_{03}P) \\ &= [(C_{02}/(C_{03}C_{11}))^*(\omega_{03}/\omega_{02})] . \end{aligned} \tag{9.14}$$

Now using Equation (9.7) and the values for R and L, one obtains

$$(C_{01}C_{11}L^{10})/C_{02} = \omega_{01}/\omega_{02} . \tag{9.15}$$

Now using Equation (9.14) in Equation (9.15) and reducing it, one can obtain:

$$(C_{01}C_{02}^{7/3})/(C_{03}^{10/3}C_{11}^{8/3}) = (\omega_{01}\omega_{02}^{7/3}/\omega_{03}^{10/3}) , \tag{9.16}$$

or the form of

$$(C_{01}^3C_{02}^7)/(C_{03}^{10}C_{11}^8) = (\omega_{01}^3\omega_{02}^7/\omega_{03}^{10}) . \tag{9.17}$$

Now using Equations (9.3) to (9.5) to solve for the dual variables in terms of ω_{02}, one obtains

$$\omega_{01} = (3/7)\omega_{02} \tag{9.18}$$

$$\omega_{03} = 1 - (10/7)^*\omega_{02} \tag{9.19}$$

$$\omega_{11} = 1 - (8/7)^*\omega_{02} . \tag{9.20}$$

Using Equations (9.18) to (9.20) in Equation (9.17), one can obtain

$$\begin{aligned} & (3/7\omega_{02})^3(\omega_{02})^7/[1 - ((10/7)^*\omega_{02})] = (C_{01}^3C_{02}^7)/(C_{03}^{10}C_{11}^8) = A \\ \text{or} \quad & (\omega_{02}/(1 - ((10/7)\omega_{02}) = [A^*(7/3)^3]^{1/10} = B \\ \text{or} \quad & \omega_{02} = (7B/(7 + 10B)) . \end{aligned} \tag{9.21}$$

Thus, the remaining dual variables can be solved for as:

$$\omega_{01} = 3B/(7 + 10B) \tag{9.22}$$

$$\omega_{03} = 7/(7 + 10B) \tag{9.23}$$

$$\omega_{11} = (7 + 2B)/(7 + 10B) . \tag{9.24}$$

Using Equations (9.10) and (9.12)

$$\begin{aligned} P &= (C_{03}/\omega_{03})RL^{-3} \\ &= (C_{03}/\omega_{03})(C_{11}L^3)L^{-3} \\ &= C_{11}C_{03}/\omega_{03} \\ &= (C_{11}C_{03})(1 + (10/7)B) \\ &= (C_{11}C_{03})(1 + (10/7)[(7/3)^3A]^{1/10}) \\ &= (C_{11}C_{03})(1 + (10/7)(7/3)^{3/10}[(C_{01}^3C_{02}^7)/(C_{03}^{10}C_{11}^8)]^{1/10} \\ &= C_{11}C_{03} + C_{11}C_{03}(10/7)(7/3)^{3/10}[(C_{01}^3C_{02}^7)/(C_{03}^{10}C_{11}^8)]^{1/10} \\ &= C_{11}C_{03} + (10/7)[((7/3)(C_{01})]^{3/10*}(C_{02})^{7/10*}(C_{11})^{2/10} . \end{aligned} \tag{9.25}$$

Using Equation (9.9) to solve for R, one obtains:

$$
\begin{aligned}
R &= C_{02}/(\omega_{02}P) \\
&= C_{02}/[(7B/(7+10B))^*(C_{11}C_{03})(1+(10/7)B) \\
&= C_{02}/[(7B/(7+10B))^*(C_{11}C_{03})(7+(10B)7) \\
&= (C_{02}/(C_{11}C_{03}))/B \\
&= (C_{02}/(C_{11}C_{03}))/[(7/3)^{3*}C_{01}^3C_{02}^7/(C_{03}^{10}C_{11}^8)]^{1/10} \\
&= [(3/7)^*(C_{02}/C_{01})]^{3/10}C_{11}^{-2/10}\ .
\end{aligned}
\tag{9.26}
$$

Using Equation (9.10) to solve for L, one obtains:

$$
\begin{aligned}
L &= [C_{03}R/(\omega_{03}P)]^{1/3} \\
&= [C_{03}{}^*[(3/7)^*(C_{02}/C_{01})]^{3/10}C_{11}^{-2/10}]/[\omega_{03}{}^*C_{11}C_{03}/\omega_{03}] \\
&= [(3/7)(C_{02}/C_{01})]^{1/10*}C_{11}^{-4/10}\ .
\end{aligned}
\tag{9.27}
$$

The equations for P, R, and L are general equations but are rather complex equations compared to the previous problems illustrated. The solution was based upon determining an additional equation from the primal-dual relationships, which was highly non-linear and resulted in rather complex expressions for the variables. The additional equation along with the dual variables were used in the equations relating the primal and dual to determine the final expressions for the variables. This frequently happens when the degrees of difficulty are greater than zero.

For this particular example problem where $C_{01} = 0.44$, $C_{02} = 10$, $C_{03} = 0.592$ and $C_{11} = 8.62$, the value for A and B are:

$$A = (C_{01}^3C_{02}^7)/(C_{03}^{10}C_{11}^8) = [(0.44)^3(10)^7]/[(0.592)^{10}(8.62)^8] = 5.285 \tag{9.28}$$

$$B = [A(7/3)^3]^{1/10} = [5.285(7/3)^3]^{1/10} = 1.523\ . \tag{9.29}$$

Now using the equations for the dual variables, Equations (9.21) to (9.24), one obtains

$$
\begin{aligned}
\omega_{02} &= 7B/(7+10B) = 0.480 \\
\omega_{01} &= 3B/(7+10B) = 0.205 \\
\omega_{03} &= 7/(7+10B) = 0.315 \\
\omega_{11} &= (7+2B)/(7+10B) = 0.452\ .
\end{aligned}
$$

From Equation (9.25) the value of P can be found as:

$$
\begin{aligned}
P &= C_{11}C_{03} + (10/7)[((7/3)(C_{01})]^{3/10*}(C_{02})^{7/10*}(C_{11})^{2/10} \\
&= (8.62)(0.592) + (10/7)[((7/3)(0.44)]^{0.3}(10)^{0.7}(8.62)^{0.2} \\
&= 5.10 + 11.10 \\
&= \$16.2\ .
\end{aligned}
$$

The primal variables can be determined from Equations (9.26) and (9.27) as:

$$\begin{aligned}
R &= [(3/7)(C_{02}/C_{01})]^{3/10} C_{11}^{-2/10} \\
&= [(3/7)(10/0.44]^{3/10} 8.62^{-2/10} \\
&= 1.29 \text{ in} \\
L &= [(3/7)(C_{02}/C_{01})]^{1/10} C_{11}^{-4/10} \\
&= [(3/7)(10/0.44)]^{1/10} 8.62^{-4/10} \\
&= 0.530 \text{ in} .
\end{aligned}$$

Now, if the values of R and L are used in Equation (9.1) for evaluating the primal, one obtains:

$$\begin{aligned}
P &= C_{01} R^3 L^{-2} + C_{02} R^{-1} + C_{03} R L^{-3} \\
&= 0.44(1.29)^3(0.53)^{-2} + 10(1.29)^{-1} + 0.592(1.29)(0.53)^{-3} \\
&= 3.363 + 7.752 + 5.130 \\
&= \$16.2 .
\end{aligned}$$

As in the previous case studies, the value of the primal and dual objective functions are equivalent.

It was difficult to determine the additional equation, and other methods can be used. One method that is often used is the constrained derivative approach. This method has the dual equations rearranged in terms of one unknown dual variable, and these are substituted into the dual objective function. The objective function is set into logarithmic form and differentiated with respect to the unknown dual variable, set to zero, and then solved for the unknown dual variable. The solved dual variable is used in the dual equations to obtain the values of the other dual variables.

9.3 EVALUATIVE QUESTIONS

1. Use the values of $C_{01} = 0.54$, $C_{02} = 10$, $C_{03} = 0.65$ and $C_{11} = 9.00$, determine the values of A and B, of the dual variables and the value of the objective function. Also determine the values of L and R, and use these to determine P.

2. Determine the sensitivity of the objective function and the primal variables of R and L by changing one of the constants by 20% (such as C_{01}).

3. There are other approaches for determining the additional equation other than the algebraic approach used. Select one method, and use it to derive the additional equation.

CHAPTER 10

Metal Casting Hemispherical Top Cylindrical Side Riser Case Study

10.1 INTRODUCTION

The sphere is the shape which will give the longest solidification time, so a riser with a hemispheric shaped top should be a more efficient riser than a cylindrical riser and thus be more economical. However, the problem of designing this type of riser is more complex and has two degrees of difficulty. The case study considered is a side riser with a hemispherical top and a cylindrical bottom for ease of molding and to provide a better connection to the casting. The top hemispherical cap has the same diameter(D) as the cylinder.

10.2 PROBLEM FORMULATION

The volume of the riser is the sum of the cylinder part and the hemisphere part and can be written as:

$$V = \text{cylindrical part} + \text{Hemispherical part}$$

$$V = \pi D^2 H/4 + \pi D^3/12 \tag{10.1}$$

$$SA = \pi D^2/4 + \pi DH + \pi D^2/2$$

$$= 3/4\pi D^2 + \pi DH\ . \tag{10.2}$$

The constraint for riser design is Chvorinov's Rule, which is

$$t = K(V/SA)^2\ , \tag{10.3}$$

where

t = solidification time (minutes or seconds)

K = solidification constant for molding material (minutes/in^2 or seconds/cm^2)

V = volume (in^3 or cm^3)

SA = cooling surface area (in^2 or cm^2).

An illustration of the hemispherical top side riser is shown in Figure 10.1 where the radius of the hemisphere is the same as the radius of the cylinder which is the same as the diameter divided by two.

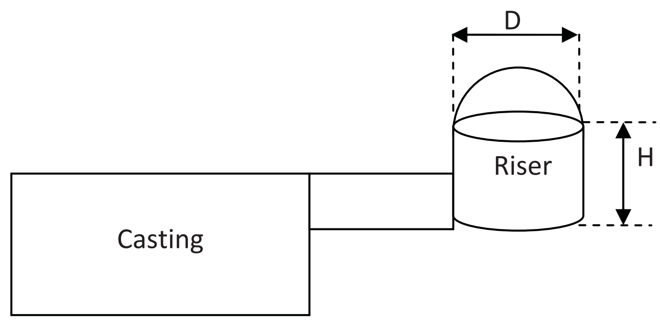

Figure 10.1: Hemispherical top side riser design.

This results in the relation

$$(V/SA) \geq M_c = K\ , \tag{10.4}$$

where

M_c = the modulus of the casting (a constant K for a particular casting).

Thus, using the cooling surface area (SA) and volume (V) expressions, Equation (10.4) can be rewritten as:

$$\begin{aligned} V/SA &= (\pi D^2 H/4 + \pi D^3/12)/(3/4\pi D^2 + \pi DH) && \geq K \\ &= (\pi D^2/12)^*(3H + D)/[(\pi D/4)^*(3D + 4H) && \geq K \\ &= D^*(3H + D)/[3^*(3D + 4H)] && \geq K\ . \end{aligned} \tag{10.5}$$

Rearranging the equation in the less than equal form results in:

$$4KD^{-1} + 3KH^{-1} - (1/3)DH^{-1} \leq 1\ . \tag{10.6}$$

Thus, the primal form of the problem can be stated as:

$$\text{Min } V = \pi D^2 H/4 + \pi D^3/12\ . \tag{10.7}$$

Subject to:

$$4KD^{-1} + 3KH^{-1} - (1/3)DH^{-1} \leq 1\ . \tag{10.8}$$

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= \ \ 1 \\ \sigma_{02} &= \ \ 1 \\ \sigma_{11} &= \ \ 1 \\ \sigma_{12} &= \ \ 1 \\ \sigma_{13} &= -1 \\ \sigma_{1} &= \ \ 1\ . \end{aligned}$$

The dual problem formulation is:

$$\begin{array}{llr}\text{Objective Function} & \omega_{01} + \omega_{02} = 1 & (10.9)\\ D \text{ terms} & 2\omega_{01} + 3\omega_{02} - \omega_{11} - \omega_{13} = 0 & (10.10)\\ H \text{ terms} & \omega_{01} - \omega_{12} + \omega_{13} = 0\ . & (10.11)\end{array}$$

The degrees of difficulty are equal to:

$$D = T - (N+1) = 5 - (2+1) = 2\ . \tag{10.12}$$

Using the linearity inequality equation,

$$\begin{aligned}\omega_{10} &= \omega_{mt} = \sigma_m \sum \sigma_{mt}\omega_{mt} = (1)^*(1^*\omega_{11} + 1^*\omega_{12} + (-1)^*\omega_{13})\\ \omega_{10} &= \omega_{11} + \omega_{12} - \omega_{13}\ .\end{aligned} \tag{10.13}$$

The dual variables cannot be determined directly as the degrees of difficulty are 2; that is, there are two more variables than there are equations. The objective function can be found using the dual expression:

$$V = d(\omega) = \sigma\left[\prod_{m=0}^{M}\prod_{t=1}^{T_m}(C_{mt}\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}}\right]^{\sigma} \tag{10.14}$$

$$\begin{aligned}V = 1[[\{(\pi/4)^*1/\omega_{01})\}(1^*\omega_{01})]^*[\{(\pi/12^*\omega_{02})\}^{(1^*\omega_{02})}]^*[\{(4K^*\omega_{10}/\omega_{11})\}^{(1^*\omega_{10})}]^*\\ [\{(3K^*\omega_{10}/\omega_{12})\}^{(1^*\omega_{10})}]^*[\{((1/3)^*\omega_{10}/\omega_{13})\}^{(-1^*\omega_{10})}]]^1\ .\end{aligned} \tag{10.15}$$

The relationships between the primal and dual variables can be written as

$$(\pi/4)D^2H = \omega_{01}V \tag{10.16}$$
$$(\pi/12)D^3 = \omega_{02}V \tag{10.17}$$
$$(4K)D^{-1} = \omega_{11}/\omega_{10} \tag{10.18}$$
$$(3K)H^{-1} = \omega_{12}/\omega_{10} \tag{10.19}$$
$$(1/3)DH^{-1} = \omega_{13}/\omega_{10}\ . \tag{10.20}$$

If one takes Equation (10.16) and divides by Equation (10.17), one obtains,

$$3H/D = \omega_{01}/\omega_{02}\ . \tag{10.21}$$

If one takes the inverse of Equation 10.20, one obtains,

$$3H/D = \omega_{10}/\omega_{13}\ . \tag{10.22}$$

Now comparing Equations (10.21) and (10.22), one can obtain an equation between the dual variables as:

$$\omega_{01}/\omega_{02} = \omega_{10}/\omega_{13}\ . \tag{10.23}$$

If one takes Equation (10.18) and divides by Equation (10.19), one obtains:

$$(4/3)H/D = \omega_{11}/\omega_{12}\ . \tag{10.24}$$

Now comparing Equations (10.21) and (10.24), one can obtain an additional equation between the dual variables as:

$$\omega_{01}/\omega_{02} = (9/4)\omega_{11}/\omega_{12}\ . \tag{10.25}$$

Now there are six equations with only the six dual variables, and they are Equations (10.9) to (10.11), (10.13), (10.23), and (10.25). The procedure used was to solve for all of the variables in terms of ω_{02} and then obtain the specific value of ω_{02}.

From Equation (10.9), one obtains:

$$\omega_{01} = 1 - \omega_{02}\ . \tag{10.26}$$

If one adds Equations (10.10) and (10.11), one obtains:

$$3\omega_{01} + 3\omega_{02} - \omega_{11} - \omega_{12} = 0\ .$$

Which can be reduced to:

$$\omega_{11} + \omega_{12} = 3\ . \tag{10.27}$$

Using Equations (10.13) and (10.27), one obtains:

$$\begin{aligned}\omega_{10} &= \omega_{11} + \omega_{12} - \omega_{13}\\ \omega_{10} &= 3 - \omega_{13}\ .\end{aligned} \tag{10.28}$$

Now using Equations (10.28) and (10.23), one obtains:

$$\omega_{01}/\omega_{02} = \omega_{10}/\omega_{13} = (3 - \omega_{13})/\omega_{13} = (1 - \omega_{02})/\omega_{02}\ .$$

Solving for ω_{13} one obtains:

$$\omega_{13} = 3\omega_{02}\ . \tag{10.29}$$

From Equations (10.28) and (10.29):

$$\begin{aligned}\omega_{10} &= 3 - \omega_{13} = 3 - 3\omega_{02}\\ \omega_{10} &= 3(1 - \omega_{02})\ .\end{aligned} \tag{10.30}$$

Using Equations (10.26) and (10.29) in Equation (10.11), one obtains:

$$\begin{aligned}\omega_{12} &= \omega_{10} + \omega_{13}\\ &= (1 - \omega_{02}) + 3\omega_{02}\\ &= 1 + 2\omega_{02}\ .\end{aligned} \tag{10.31}$$

Using Equations (10.10), (10.26), and (10.29), one has:

$$\begin{aligned}\omega_{11} &= 2\omega_{01} + 3\omega_{02} - \omega_{13} \\ &= 2(1-\omega_{02}) + 3\omega_{02} - 3\omega_{02} \\ &= 2(1-\omega_{02})\ .\end{aligned} \tag{10.32}$$

Now using Equation (10.25), one can solve for ω_{02} using the values for ω_{01}, ω_{11}, and ω_{12}

$$\begin{aligned}\omega_{01}/\omega_{02} &= (9/4)\omega_{11}/\omega_{12} \\ (1-\omega_{02})/\omega_{02} &= (9/4)2(1-\omega_{02})/(1+2\omega_{02})\ .\end{aligned}$$

And solving for ω_{02} results in:

$$\omega_{02} = 0.4\ .$$

Therefore,

$\omega_{01} = 0.6$
$\omega_{11} = 1.2$
$\omega_{12} = 1.8$
$\omega_{13} = 1.2$
$\omega_{10} = 1.8$.

Now using the dual variables in Equation (10.15) to find the minimum volume, one obtains:

$$\begin{aligned}V &= 1[[\{(\pi/4{}^*1/\omega_{01})\}^{(1*\omega_{01})}]^*[\{(\pi/12{}^*\omega_{02})\}^{(1^*\omega_{02})}]^*[\{(4K^*\omega_{10}/\omega_{11})\}^{(1^*\omega_{11})}]^* \\ &\quad [\{(3K^*\omega_{10}/\omega_{12})\}^{(1^*\omega_{12})}]^*[\{((1/3)^*\omega_{10}/\omega_{13})\}^{(-1^*\omega_{13})}]]^1 \\ &= 1[[\{(\pi/4)^*1/0.6)\}^{(1^*0.6)}]^*[\{(\pi/12^*0.4)\}^{(1^*0.4)}]^*[\{(4K^*1.8/1.2)\}^{(1^*1.2)}]^* \\ &\quad [\{(3K^*1.8/1.8)\}^{(1^*1.8)}]^*[\{((1/3)^*1.8/1.2)\}^{(-1^*1.2)}]]^1 \\ &= 1[[\{((5/12)\pi)\}^{(0.6)}]^*[\{((5/24)\pi)\}^{(0.4)}]^*[\{(6K)\}^{(1.2)}]^*[\{(3K)\}^{(1.8)}]^*[\{(1/2)\}^{(-1.2)}]]^1 \\ &= 1[[\{((2^*5/24)\pi)\}^{(0.6)}]^*[\{((5/24)\pi)\}^{(0.4)}]^*[\{(2^*3K)\}^{(1.2)}]^*[\{(3K)\}^{(1.8)}]^*[\{(2)\}^{(1.2)}]]^1 \\ &= 2^{0.6}[(5/24)\pi]^{(0.6+0.4)*}2^{1.2*}(3K)^{(1.2+1.8)}2^{1.2} \\ &= (5\pi/24)^{1*}2^{(0.6+1.2+1.2)*}(3K)^3 \\ V &= (5\pi/24)^*(6K)^3\ .\end{aligned} \tag{10.33}$$

The values for H and D can be found from Equations (10.18) and (10.19)

$$\begin{aligned}D &= 4K^*(\omega_{10}/\omega_{11}) \\ &= 4k^*(1.8/1.2) \\ D &= 6K\ .\end{aligned} \tag{10.34}$$

And

$$\begin{aligned}H &= 3K^*(\omega_{10}/\omega_{12}) \\ &= 3K^*(1.8/1.8) \\ H &= 3K\ .\end{aligned} \tag{10.35}$$

Now the primal can be evaluated using Equation (10.7) with the values for H and D from Equations (10.34) and (10.35).

$$
\begin{aligned}
V &= \pi D^2 H/4 + \pi D^3/12 \\
V &= \pi(6K)^2(3K)/4 + \pi(6K)^3/12 \\
&= \pi 27K^3 + \pi 18K^3 \\
&= 45\pi K^3 && (10.36) \\
&= (5\pi/24)^*(6K)^3\ . && (10.37)
\end{aligned}
$$

The values for the primal and dual are equivalent, which is required for the solution. The expression of Equation (10.37) is the preferred expression for foundry as $6K$ is the value for the diameter for the simple cylindrical risers where K is the modulus of the casing. This example illustrates that it is possible to solve problems with two degrees of difficulty in some instances, but there are numerous mathematical operations that must be performed.

10.3 EVALUATIVE QUESTIONS

1. A side riser with a hemispheric top is to be designed for a casting which has a surface area of 40 cm^2 and a volume of 120 cm^3. The hot metal cost is 100 Rupees per kg and the metal density is 3.0 gm/cm^3. Compare these results with Problem 1 in Section 6.3.
   (a) What are the dimensions of the hemispherical side riser(H and D)?
   (b) What is the volume of the hemispherical side riser(cm^3)?
   (c) What is the metal cost of the hemispherical side riser(Rupees)?
   (d) What is the metal cost of the casting(Rupees)?
2. Two castings of equal volume but of different dimensions are to be cast. If one is a 3 inch cube and the other is a plate of $1 \times 3 \times 9$ inches and a top riser is to be used, what are the dimensions (H and D) of the risers for the two cases?

CHAPTER 11

Liquefied Petroleum Gas(LPG) Cylinders Case Study

11.1 INTRODUCTION

This case study problem deals with the design of liquefied petroleum gas cylinders, more commonly known as propane gas cylinders in the USA. This is a very interesting problem as it has two general solutions as well as one degree of difficulty. The two general solutions occur depending upon the relationship between the constants. What happens in this particular problem is that one of the two constraints can be either binding or loose, depending upon the value of the constants.

11.2 PROBLEM FORMULATION

The problem [13] was to minimize the drawing force(Z) to produce the tank by deep drawing, and two constraints were considered so the tank would have a minimum volume and the height/diameter ratio would be less than one. The formulation of the primal problem was:

$$Z(\min) = K_1 hd + K_2 d^2 \,. \tag{11.1}$$

Subject to the two constraints:

$$\pi d^2 h/4 \geq V_{\min} \,, \tag{11.2}$$

or in the proper geometric programming form as

$$(4V_{\min}/\pi) d^{-2} h^{-1} \leq 1 \,, \tag{11.3}$$

and

$$h/d \leq 1 \,, \tag{11.4}$$

where

$$K_1 = \pi PYC/F \tag{11.5}$$

$$K_2 = ((C - E)\pi PY/2F \,, \tag{11.6}$$

where

Z = drawing force
P = internal gas pressure
Y = material yield strength
F = hoop stress
C = constant = 1.04
E = constant = 0.65 .

If all the constants are combined, the primal form can be written as:

$$Z = K_1 hd + K_2 d^2 \ . \tag{11.1}$$

Subject to:

$$K_3 h^{-1} d^{-2} \leq 1 \ , \tag{11.7}$$

and

$$K_4 hd^{-1} \leq 1 \ , \tag{11.8}$$

where

$$K_3 = (4V_{\min}/\pi) \ , \tag{11.9}$$

and

$$K_4 = 1 \qquad \text{(This could be taken as the minimum } d/h \text{ ratio)} \ . \tag{11.10}$$

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= 1 \\ \sigma_{02} &= 1 \\ \sigma_{11} &= 1 \\ \sigma_{21} &= 1 \\ \sigma_1 &= 1 \\ \sigma_2 &= 1 \ . \end{aligned}$$

The dual problem formulation is:

$$\begin{array}{ll} \text{Objective Function} & \omega_{01} + \omega_{02} = 1 \qquad (11.11) \\ h \text{ terms} & \omega_{01} - \omega_{11} + \omega_{21} = 0 \qquad (11.12) \\ d \text{ terms} & \omega_{01} + 2\omega_{02} - 2\omega_{11} - \omega_{21} = 0 \ . \qquad (11.13) \end{array}$$

The degrees of difficulty are equal to:

$$D = T - (N + 1) = 4 - (2 + 1) = 1 \ . \tag{11.14}$$

From the constraint equations which have only one term it is apparent that:

$$\omega_{10} = \omega_{11} \ , \tag{11.15}$$

and

$$\omega_{20} = \omega_{21} \ . \tag{11.16}$$

An additional equation is needed, and one must examine the primal dual relationships to find the additional relationship, which are:

$$K_1 hd = \omega_{01} Z \tag{11.17}$$
$$K_2 d^2 = \omega_{02} Z \tag{11.18}$$
$$K_3 h^{-1} d^{-2} = \omega_{11}/\omega_{10} = 1 \tag{11.19}$$
$$K_4 hd^{-1} = \omega_{21}/\omega_{20} = 1 \ . \tag{11.20}$$

If one adds Equations (11.12) and (11.13), one can solve for ω_{11} by:

$$2\omega_{01} + 2\omega_{02} - 3\omega_{11} = 0 \tag{11.21}$$
$$2(\omega_{01} + \omega_{02)} - 3\omega_{11} = 0 \tag{11.22}$$
$$2 - 3\omega_{11} = 0 \tag{11.23}$$
$$\omega_{11} = 2/3 \ . \tag{11.24}$$

If one takes Equation (11.17) and divides it by Equations (11.18) and (11.20), one obtains:

$$K_1 hd/(K_2 d^{2*} K_4 hd^{-1}) = K_1/(K_2 K_4) = \omega_{01} Z/(\omega_{02} Z^* 1) = \omega_{01}/\omega_{02} \ . \tag{11.25}$$

This can be solved for ω_{01} in terms of ω_{02} and the constants:

$$\omega_{01} = \omega_{02}(K_1/(K_2 K_4)) \ . \tag{11.26}$$

Now using Equations (11.26) and (11.11), one can solve for ω_{01} and ω_{02} and obtain:

$$\omega_{01} = K_1/(K_1 + K_2 K_4) \tag{11.27}$$
$$\omega_{02} = K_2 K_4/(K_1 + K_2 K_4) \ . \tag{11.28}$$

Now using Equation (11.12) and substituting the values for ω_{01} and ω_{11}, one obtains:

$$\omega_{21} = \omega_{11} - \omega_{01} = (2K_2 K_4 - K_1)/[3(K_1 + K_2 K_4)] \ . \tag{11.29}$$

Now ω_{21} must be ≥ 0, so that implies that:

$$2K_2 K_4 - K_1 \geq 0 \ , \tag{11.30}$$

or

$$K_2 K_4 / K_1 \geq 1/2 \ . \tag{11.31}$$

There are two sets of solutions, depending upon whether Equation (11.31) holds as illustrated in Figure 11.1.

Now, the solutions can be found for the two cases. If the answer is "No," then the objective function can be evaluated using the dual expression:

$$Z = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} , \tag{11.32}$$

where by definition

$$\omega_{00} = 1 . \tag{11.33}$$

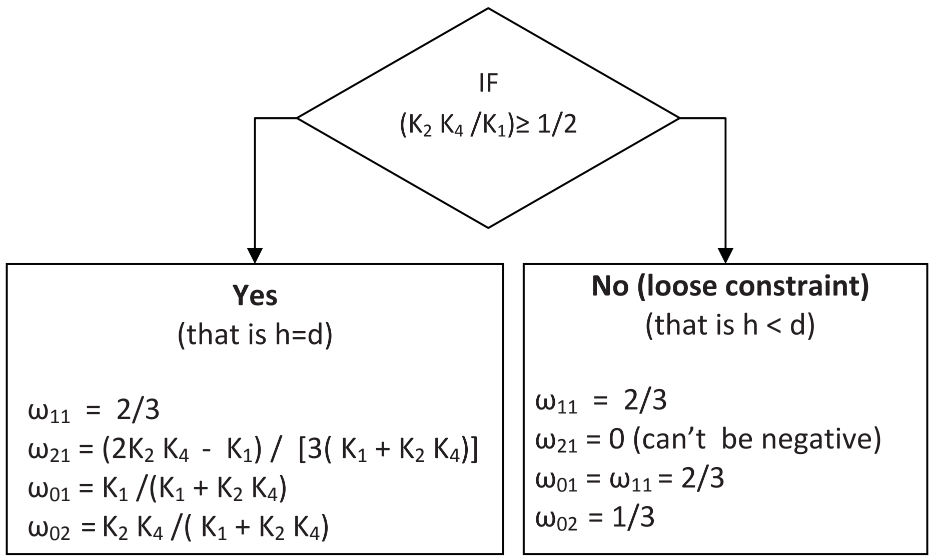

Figure 11.1: Values of dual variables based upon making $\omega_{21} \geq 0$.

$$Z = [K_1{}^*1/(2/3)]^{1^*2/3}[K_2{}^*1/(1/3)]^{1^*1/3}[K_3{}^*(2/3)/(2/3)]^{1^*2/3} , \tag{11.34}$$

which can be reduced to:

$$Z = [3/2^{2/3}]K_1^{2/3}K_2^{1/3}K_3^{2/3} . \tag{11.35}$$

Now, the primal variables can be determined from the primal dual relationships. If one uses Equation (11.18) and solves for d^2 and then for d, one obtains:

$$d^2 = \omega_{02}Z/K_2 = (1/3)^*\{[3/2^{2/3}]K_1^{2/3}K_2^{1/3}K_3^{2/3}\}/K_2 . \tag{11.36}$$

Solving for d and reducing terms results in:

$$d = (K_1K_3/2K_2)^{1/3} . \tag{11.37}$$

Similarly, if one uses Equation (11.17) to solve for h, one obtains:

$$h = \omega_{01} Z/K_1 d = \{(2/3)^*\{[3/2^{2/3}]K_1^{2/3} K_2^{1/3} K_3^{2/3}\}\}/[K_1{}^*(K_1 K_3/2K_2)^{1/3}] \,. \tag{11.38}$$

Reducing terms, one obtains:

$$h = 2^{2/3} K_1^{-2/3} K_2^{2/3} K_3^{1/3} \,. \tag{11.39}$$

Now substituting the primal variables into the primal objective function, one has:

$$\begin{aligned} Z &= K_1 hd + K_2 d^2 \,, && (11.1)\\ &= K_1{}^* 2^{2/3} K_1^{-2/3} K_2^{2/3} K_3^{1/3*}(K_1 K_3/2K_2)^{1/3} + K_2{}^*(K_1 K_3/2K_2)^{2/3} \\ &= 2K_1^{2/3} K_2^{1/3} K_3^{2/3}/2^{2/3} + K_1^{2/3} K_2^{1/3} K_3^{2/3}/2^{2/3} \\ &= 3K_1^{2/3} K_2^{1/3} K_3^{2/3}/2^{2/3} \\ &= [3/2^{2/3}]^* K_1^{2/3} K_2^{1/3} K_3^{2/3} \,. && (11.40) \end{aligned}$$

The equations for the primal and dual, Equations (11.35) and (11.40), give the same results. Thus, one has a general solution for the objective function and the two primal variables when the "No" route was taken.

The "Yes" route has the dual variables as functions of the constants and thus is the more complex route. The objective function can be evaluated using the dual expression:

$$Z = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \,, \tag{11.32}$$

where by definition

$$\omega_{00} = 1 \,. \tag{11.33}$$

The dual variables are:

$$\begin{aligned} \omega_{11} &= 2/3 && (11.41)\\ \omega_{21} &= (2K_2K_4 - K_1)/[3(K_1 + K_2K_4)] && (11.42)\\ \omega_{01} &= K_1/(K_1 + K_2K_4) && (11.43)\\ \omega_{02} &= K_2K_4/(K_1 + K_2K_4) \,. && (11.44) \end{aligned}$$

Using these dual variables and the signum values the dual objective function is:

$$\begin{aligned} Z &= (K_1\omega_{00}/\omega_{01})^{1^*\omega_{01}} (K_2\omega_{00}/\omega_{02})^{1^*\omega_{02}} (K_3)^{1^*\omega_{11}} (K_4)^{1^*\omega_{21}} \\ &= (K_1 + K_2K_4)^{\omega_{01}} [(K_1 + K_2K_4)/K_4]^{\omega_{02}} (K_3)^{2/3} (K_4)^{\omega_{11}-\omega_{01}} \\ &= [(K_1 + K_2K_4)/K_4]^{\omega_{01}} [(K_1 + K_2K_4)/K_4]^{\omega_{02}} (K_3K_4)^{2/3} \\ &= [(K_1 + K_2K_4)/K_4](K_3K_4)^{2/3} \\ &= (K_3)^{2/3}[(K_1K_4^{-1/3} + K_2K_4^{2/3})] \,. && (11.45) \end{aligned}$$

The primal variables can be obtained from the relationships between the primal and dual variables. Using Equation (11.20), one can obtain a relation between h and d which is:

$$h = d/K_4 . \tag{11.46}$$

If one combines Equations (11.19) and (11.20) one obtains:

$$(K_3 h^{-1} d^{-2})(K_4 h d^{-1}) = 1^* 1 ,$$

which yields:

$$K_3 K_4 d^{-3} = 1 ,$$

or

$$d = (K_3 K_4)^{1/3} . \tag{11.47}$$

Now from Equations (11.46) and (11.47) one obtains:

$$h = d/K_4 = K_3^{1/3} K_4^{-2/3} . \tag{11.48}$$

Now, using the primal equation for the objective function with the primal variables, one has:

$$\begin{aligned} Z &= K_1 hd + K_2 d^2 , && (11.1)\\ &= K_1 (K_3^{1/3} K_4^{-2/3})(K_3 K_4)^{1/3} + K_2 (K_3 K_4)^{2/3} \\ &= K_1 (K_3^{2/3} K_4^{-1/3}) + K_2 (K_3 K_4)^{2/3} \\ &= K_3^{2/3} (K_1 K_4^{-1/3} + K_2 K_4^{2/3}) . && (11.49) \end{aligned}$$

Note that the general objective function is the same for both the primal and dual solutions. This problem illustrates that it is possible to solve a problem with more than one degree of difficulty and have two solutions, depending upon the specific values of the constants in the problem.

11.3 EVALUATIVE QUESTIONS

1. A tank is to be designed with a minimum volume of 17,500,000 mm^3 and the values for parameters are:

   P = 0.2535 kg/mm^2
   F = 32.33 kg/mm^2
   Y = 25 kg/mm^2
   C = 1.04
   E = 0.65 .

   Determine the amount of the drawing force (kg) and the height (mm) and diameter (mm) of the tank.

2. A new procedure was developed for another older machine which changed the expression for K_2. The new expression was:

$$K_2 = (2C - E)\pi PY/(2F) . \tag{11.50}$$

Using the same data as in Evaluative Question 1, determine the amount of the drawing force (kg) and the height (mm) and diameter (mm) of the new tank.

3. A tank is to be designed with a minimum volume of 11,000 in^3 and the values for parameters are:
   P = 360 lb/in^2
   F = 46,000 lb/in^2
   Y = 35,500 lb/in^2
   C = 1.04
   E = 0.65 .

   Determine the amount of the drawing force (lb and tons) and the height (in) and diameter (in) of the tank.

CHAPTER 12

Material Removal/Metal Cutting Economics Case Study

12.1 INTRODUCTION

Material removal economics, also known as metal cutting economics or machining economics, is an example of a problem which has non-integer exponents, and this makes the problem challenging. This problem has been presented previously [9, 11], but this version is slightly different from, and easier than, those presented earlier. The material removal economics problem is based upon the Taylor Tool Life Equation, which was developed by Frederick W. Taylor over 100 years ago. There are several versions of the equation, and the form selected is one of the modified versions which includes cutting speed and feed rate. The equation selected was:

$$T V^{1/n} f^{1/m} = C \ , \tag{12.1}$$

where
- T = tool life (minutes)
- V = cutting speed (ft/min or m/min)
- F = feed rate (inches/rev or mm/rev)
- 1/n = cutting speed exponent
- 1/m = feed rate exponent
- C = Taylor's Modified Tool Life Constant.

The object is to minimize the total cost for machining, operator, tool cost and tool changing cost.

12.2 PROBLEM FORMULATION

An expression for the machining cost, operator cost, tool cost and tool changing cost was developed [9] and the resulting expression was:

$$C_u = K_{00} + K_{01} f^{-1} V^{-1} + K_{02} f^{(1/m-1)} V^{(1/n-1)} \ , \tag{12.2}$$

where
- C_u = total unit cost
- K_{00} = $(R_o + R_m)t_l$
- K_{01} = $(R_o + R_m)B$
- K_{02} = $[(R_o + R_m)t_{ch} + C_t]QBC^{-1}$,

and

R_o = operator rate ($/min)
R_m = machine rate ($/min)
t_l = machine loading & unloading time (min)
t_{ch} = tool changing time (min)
B = cutting path surface factor of tool (in-ft, or mm-m)
Q = fraction of cutting path that tool is cutting material
C_t = tool cost ($/cutting edge)
C = Taylor's Modified Tool Life Constant.

The first part of the objective function expression represents the loading and unloading costs, the second part represents the cutting costs, and the third part represents the tool and tool changing costs. The loading and unloading costs are not a function of the feed and cutting speed. Since K_{00} is a constant, the primal problem can be formulated as solving for the variable cost, C_u(var) as:

$$C_u(\text{var}) = K_{01} f^{-1} V^{-1} + K_{02} f^{(1/m-1)} V^{1/n-1)} . \tag{12.3}$$

Subject to a maximum feed constraint written as:

$$K_{11} f \leq 1 , \tag{12.4}$$

where

$$K_{11} = 1/f_{\max} .$$

From the coefficients and signs, the signum values for the dual are:

$$\begin{aligned} \sigma_{01} &= 1 \\ \sigma_{02} &= 1 \\ \sigma_{11} &= 1 \\ \sigma_1 &= 1 . \end{aligned}$$

The dual problem formulation is:

$$\begin{array}{lll} \text{Objective Function} & \omega_{01} + \omega_{02} & = 1 & (12.5) \\ f \text{ terms} & -\omega_{01} + (1/m - 1)\omega_{02} + \omega_{11} & = 0 & (12.6) \\ V \text{ terms} & -\omega_{01} + (1/n - 1)\omega_{02} & = 0 . & (12.7) \end{array}$$

The degrees of difficulty are equal to:

$$D = T - (N + 1) = 3 - (2 + 1) = 0 . \tag{12.8}$$

From the constraint equations, which have only one term, it is apparent that:

$$\omega_{10} = \omega_{11} . \tag{12.9}$$

Since there are zero degrees of difficulty, the dual parameters can be solve directly. Thus, if one adds Equations (12.5) and (12.7), one can solve for ω_{02} directly and obtain:

$$\omega_{02} = n \ . \tag{12.10}$$

Then, from Equation (12.5), one obtains;

$$\omega_{01} = 1 - n \ . \tag{12.11}$$

Finally, by using the values for ω_{01} and ω_{02}, one can determine ω_{11} as:

$$\omega_{11} = 1 - n/m \ . \tag{12.12}$$

The objective function can be evaluated using the dual expression:

$$C_u(\text{var}) = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\omega_{mo}/\omega_{mt})^{\sigma_{mt}\omega_{mt}} \right]^{\sigma} \tag{12.13}$$

$$\begin{aligned} C_u(\text{var}) &= 1\{[(K_{01}\omega_{00}/\omega_{01})^{\omega_{01}}][(K_{02}\omega_{00}/\omega_{02})^{\omega_{02}}][(K_{11}\omega_{10}/\omega_{11})^{\omega_{11}}]\}^1 && (12.14)\\ &= [(K_{01}1/(1-n))^{(1-n)}][(K_{02}1/n)^n][(K_{11}(1-n/m)/(1-n/m))^{(1-n/m)}] \\ &= [K_{01}/(1-n)][(K_{02}/K_{01})((1-n)/n)]^n[(K_{11})^{(1-n/m)}] \\ &= (K_{11})^{1-n/m} K_{01}^{1-n} K_{02}^{n} (1-n)^{n-1}/n^n \ . && (12.15) \end{aligned}$$

The primal variables, V and f, can be evaluated from the primal-dual relationships.

$$K_{01} f^{-1} V^{-1} = \omega_{01} C_u(\text{var}) \tag{12.16}$$

$$K_{02} f^{1/m-1} V^{1/n-1} = \omega_{02} C_u(\text{var}) \tag{12.17}$$

$$K_{11} f = 1 \ . \tag{12.18}$$

From Equation (12.18), is seen that:

$$f = 1/K_{11} \ . \tag{12.19}$$

If one divides Equation (12.17) by Equation (12.16), one obtains:

$$f^{1/m} V^{1/n} = (K_{01}/K_{02})(n/(1-n)) \ . \tag{12.20}$$

Using Equation (12.19) in (12.20) and solving for V, one obtains

$$V = [(n/(1-n)]^n (K_{01}/K_{02})^n K_{11}^{n/m} \ . \tag{12.21}$$

Now, if one uses the values of f and V from Equations (12.19) and (12.21) in the primal Equation (12.3) and also using Equation (12.20), one obtains:

$$\begin{aligned} C_u(\text{var}) &= K_{01} f^{-1} V^{-1} + K_{02} f^{(1/m-1)} V^{(1/n-1)} \ , && (12.3)\\ &= f^{-1} V^{-1} [K_{01} + K_{02} f^{1/m} V^{1/n}] \\ &= K_{11}[(n/(1-n))^{-n} (K_{01}/K_{02})^{-n} K_{11}^{-n/m} [K_{01} + K_{02}(n/(1-n)(K_{01}/K_{02})] \\ &= K_{11}^{1-n/m} (n/(1-n))^{-n} (K_{01}/K_{02})^{-n} [K_{01} + K_{01}(n/(1-n)] \\ &= K_{11}^{1-n/m} K_{01}^{1-n} K_{02}^{n} (1-n)^{n-1}/n^n \ . && (12.22) \end{aligned}$$

The variable unit cost expressions, C_u (var), are identical for both the primal and dual formulations. The expressions for the primal variables and the variable unit cost are more complex than the expressions obtained in the previous models because of the non-integer exponents.

Elaborate research work has been done with the material removal problems, and the dissertation by Pingfang Tsai [15] has solutions for problems with an additional variable, the depth of cut, and additional constraints on horsepower and depth of cut. With the additional constraints, there is the possibility of loose constraints, and a flow chart has been developed for the different solutions depending upon which constraints are loose.

12.3 EVALUATIVE QUESTIONS

1. A cylindrical bar, 6 inches long and 1 inch in diameter is to be finished turned on a lathe. The maximum feed to be used to control the surface finish is 0.005 in/rev. Find the total cost to machine the part, the variable cost to machine the part, the feed rate, the cutting speed, and the tool life in minutes. Use both the primal and dual equations to determine the variable unit cost. The data are:

   R_o = 0.60 \$/min
   R_m = 0.40 \$/min
   C_t = \$ 2.00/edge
   t_l = 1.5 min
   D = 1 inch
   L = 6 inches
   1/m = 1.25 (m = 0.80)
   1/n = 4.00 (n = 0.25)
   C = 5.0 x 10^8 min
   Q = 1.0 (for turning).

   Using these values, one can obtain:

   K_{00} = 1.50
   K_{01} = 1.57 in-ft
   K_{02} = 8.8 x 10^{-9}
   (solution f = 0.005 in/rev, V = 459 ft/min, C_u (var) = 0.91, and T = 8.5 min).

2. A cylindrical bar, 0.15 meters long and 25 mm in diameter is to be finished turned on a lathe. The maximum feed to be used to control the surface finish is 0.125 mm/rev. Find the total cost to machine the part, the variable cost to machine the part, the feed rate, the cutting speed, and the tool life in minutes. The data are:

R_o = 0.60 \$/min
R_m = 0.40 \$/min
C_t = \$ 2.00/edge
t_l = 1.5 min
D = 25 mm
L = 0.15 m
1/m = 1.25 (m = 0.80)
1/n = 4.00 (n = 0.25)
C = 2.46 x 10^8 min
Q = 1.0 (for turning).

Using these values, one can obtain:

K_{00} = 1.50
K_{01} = 11.78 mm-m
K_{02} = 1.34 x 10^{-7}
(solution f = 0.125 mm/rev, $V = 140$ m/min, C_u (var)=0.90, and $T = 8.6$ min).

CHAPTER 13

Summary and Future Directions

13.1 SUMMARY

The object of this text is to generate interest in geometric programming amongst manufacturing engineers, design engineers, manufacturing technologists, cost engineers, project managers, industrial consultants and finance managers by illustrating the procedure for solving certain industrial and practical problems. The various case studies were selected to illustrate a variety of applications as well as a set of different types of problems from diverse fields. Table 13.1 is a summary of the case studies presented in this text, giving the type of problem, degrees of difficulty, and other details.

The metal removal economics example also had variable exponents in the general solution. The problems were worked in detail so general solutions could be obtained and also to show that the dual and primal solutions were identical. The problems were selected to illustrate a variety of types and also to show the use of the primal-dual relationships to determine the equations for the primal variables. It is by showing the various types of applications in detailed examples that others can follow the procedure and develop new applications.

13.2 FUTURE DIRECTIONS

The author is hopeful that others will communicate with him additional examples to illustrate new applications that can be included in future editions. New applications will attract new practitioners to this fascinating area of geometric programming. It is believed that the scope of geometric programming will expand with new applications.

The author would like to include some software for different applications in the future and would welcome contributions.

13.3 DEVELOPMENT OF NEW DESIGN RELATIONSHIPS

There are many different types of problems that can be solved by geometric programming, and one of the significant advantages of the method is that it is possible in many applications to develop general design relationships. The general design relationships can save considerable time and effort in instances where the constants are changed.

Although, geometric programming was first presented nearly 50 years ago, the applications have been rather sparse compared to that of linear programming. One goal is that as researchers take advantage of the potential to develop design relationships where new applications will rapidly occur. The development of new design relationships can significantly reduce the development time

Table 13.1: Summary of Case Study Problems

Chapter	Case Study	Degrees of Difficulty	Number of Variables	Variable Description	Number of Solutions	Special Characteristics for Chapters 7 and 8
3	Optimal Box Design	0	3	Length, Width, Height	1	
4	Trash Can Design	0	2	Height, Diameter	1	
5	Open Cargo Shipping Box	0	3	Length, Width, Height	1	Classic Problem
6	Metal Casting Cylindrical Riser Design	0	2	Height, Diameter	1	
7	Process Furnace Design	0	3	Temperature, Length, Height	1	Dominant Equation Negative Dual Variable
8	Gas Transmission Pipe Line	0	4	Length, Diameter, Flow Length, Pressure Ratio Factor	1	Fractional Exponents Four Variables
9	Journal Bearing Design	1	2	Journal Radius, Bearing Half-length	1	
10	Hemispherical Riser Design	1	2	Height, Diameter	1	
11	Liquefied Petroleum Gas (LPG) Cylinder	1	2	Height, Diameter	2	Multiple Solutions
12	Material Removal/Metal Cutting Economics	0	2	Feed Rate, Cutting Speed	1	Fractional Exponents

and cost for new products, and this is essential for companies to remain competitive in the global economy.

Bibliography

[1] C.S. Beightler, D.T. Phillips, and D.J. Wilde, *Foundations of Optimization,* 2nd Edition, Prentice-Hall, Englewood Cliffs, New Jersey, 1979.

[2] C.S. Beightler, T. Lo, and H.G. Bylander, "Optimal Design by Geometric Programming," ASME, *Journal of Engineering for Industry*, 1970, pp. 191–196.

[3] R.C. Creese, "A Primal-Dual Solution Procedure for Geometric Programming," ASME, *Journal of Mechanical Design,* 1979. (also as Paper No. 79-DET-78).

[4] R.C. Creese, "Optimal Riser Design by Geometric Programming," *AFS Cast Metals Research Journal,* Vol. 7, 1971, pp. 118–121.

[5] R.C. Creese, "Dimensioning of Risers for Long Freezing Range Alloys by Geometric Programming," *AFS Cast Metals Research Journal,* Vol. 7, 1971, pp. 182–184.

[6] R.C. Creese, "Generalized Riser Design by Geometric Programming," *AFS Transactions,* Vol. 87, 1979, pp. 661–664.

[7] R.C. Creese, "An Evaluation of Cylindrical Riser Designs with Insulating Materials," *AFS Transactions,* Vol. 87, 1979, pp. 665–669.

[8] R.C. Creese, "Cylindrical Top Riser Design Relationships for Evaluating Insulating Materials," *AFS Transactions,* Vol. 89, 1981, pp. 345–348.

[9] R.C. Creese and P. Tsai, "Generalized Solution for Constrained Metal Cutting Economics Problem," *1985 Annual International Industrial Conference Proceedings*, Institute of Industrial Engineers, pp. 113–117.

[10] R.J. Duffin, E.L. Peterson, and C. Zener, *Geometric Programming*, John Wiley and Sons, New York, 1967.

[11] D.S. Ermer, "Optimization of the Constrained Machining Economics Problem by Geometric Programming," *Journal of Engineering for Industry*, Transactions of the ASME, November 1971, pp. 1067–1072.

[12] U. Passey and D.J. Wilde, "Generalized Polynomial Optimization," Stanford Chemical Engineering Report, August 1966.

[13] R. Pericherla, *Design and Manufacture of Liquefied Petroleum Gas Cylinders*, MS Thesis, West Virginia University, 1992, Morgantown, WV, USA.

[14] T.K. Sherwood, *A Course in Process Design*, MIT Press, Cambridge, Mass. (1963).

[15] P. Tsai, *An Optimization Algorithm and Economic Analysis for a Constrained Machining Model*, PhD Dissertation, West Virginia University, Morgantown, WV, pp. 214.

[16] W.H. Ray and J. Szekely, *Process Optimization with Applications in Metallurgy and Chemical Engineering*, John Wiley and Sons, Inc., New York (1973).

[17] D.J. Wilde and C.S. Beighler, *Foundations of Optimization,* Prentice-Hall, Englewood Cliffs, New Jersey, 1967.

[18] C. Zener, "A Mathematical Aid in Optimizing Engineering Design," *Proceedings of the National Academy of Science,* Vol. 47, 1961, pp. 537.

[19] C. Zener, *Engineering Design by Geometric Programming*, John Wiley and Sons, New York, 1971. DOI: 10.1073/pnas.47.4.537

[20] `http://www.mpri.lsu.edu/textbook/Chapter3.htm` (Chapter 3 Geometric Programming) (visited May 2009).

Index