Basic Elements of Real Analysis

Murray H. Protter

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(continued after Index)

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Basic Elements of Real Analysis

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Front cover illustration: f_n converges to f, but $\int_0^1 f_n$ does not converge to $\int_0^1 f$. (See p. 167 of text for explanation.)

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To Barbara and Philip

Preface

Some time ago Charles B. Morrey and I wrote *A First Course in Real Analysis*, a book that provides material sufficient for a comprehensive one-year course in analysis for those students who have completed a standard elementary course in calculus. The book has been through two editions, the second of which appeared in 1991; small changes and corrections of misprints have been made in the fifth printing of the second edition, which appeared recently.

However, for many students of mathematics and for those students who intend to study any of the physical sciences and computer science, the need is for a short one-semester course in real analysis rather than a lengthy, detailed, comprehensive treatment. To fill this need the book *Basic Elements of Real Analysis* provides, in a brief and elementary way, the most important topics in the subject.

The first chapter, which deals with the real number system, gives the reader the opportunity to develop facility in proving elementary theorems. Since most students who take this course have spent their efforts in developing manipulative skills, such an introduction presents a welcome change. The last section of this chapter, which establishes the technique of mathematical induction, is especially helpful for those who have not previously been exposed to this important topic.

Chapters 2 through 5 cover the theory of elementary calculus, including differentiation and integration. Many of the theorems that are “stated without proof” in elementary calculus are proved here.

It is important to note that both the Darboux integral and the Riemann integral are described thoroughly in Chapter 5 of this volume. Here we establish the equivalence of these integrals, thus giving the reader insight into what integration is all about.

For topics beyond calculus, the concept of a metric space is crucial. Chapter 6 describes topology in metric spaces as well as the notion of compactness, especially with regard to the Heine–Borel theorem.

The subject of metric spaces leads in a natural way to the calculus of functions in N-dimensional spaces with $N > 2$. Here derivatives of functions of N variables are developed, and the Darboux and Riemann integrals, as described in Chapter 5, are extended in Chapter 7 to N-dimensional space.

Infinite series is the subject of Chapter 8. After a review of the usual tests for convergence and divergence of series, the emphasis shifts to uniform convergence. The reader must master this concept in order to understand the underlying ideas of both power series and Fourier series. Although Fourier series are not included in this text, the reader should find it fairly easy reading once he or she masters uniform convergence. For those interested in studying computer science, not only Fourier series but also the application of Fourier series to wavelet theory is recommended. (See, e.g., *Ten Lectures on Wavelets* by Ingrid Daubechies.)

There are many important functions that are defined by integrals, the integration taken over a finite interval, a half-infinite integral, or one from $-\infty$ to $+\infty$. An example is the well-known Gamma function. In Chapter 9 we develop the necessary techniques for differentiation under the integral sign of such functions (the Leibniz rule). Although desirable, this chapter is optional, since the results are not used later in the text.

Chapter 10 treats the Riemann–Stieltjes integral. After an introduction to functions of bounded variation, we define the R-S integral and show how the usual integration-by-parts formula is a special case of this integral. The generality of the Riemann–Stieltjes integral is further illustrated by the fact that an infinite series can always be considered as a special case of a Riemann–Stieltjes integral.

A subject that is heavily used in both pure and applied mathematics is the Lagrange multiplier rule. In most cases this rule is stated without proof but with applications discussed. However, we establish the rule in Chapter 11 (Theorem 11.4) after developing the facts on the implicit function theorem needed for the proof.

In the twelfth, and last, chapter we discuss vector functions in $\mathbb{R}^N$. We prove the theorems of Green and Stokes and the divergence theorem, not in full generality but of sufficient scope for most applications. The ambitious reader can get a more *general* insight either by referring to the book *A First Course in Real Analysis* or the text *Principles of Mathematical Analysis* by Walter Rudin.

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1 CHAPTER The Real Number System

1.1 Axioms for a Field

In this and the next four chapters we give a reasonably rigorous foundation to the processes of calculus of functions of one variable, a subject usually studied in a first course in calculus. Calculus depends on the properties of the real number system. To give a complete foundation for calculus we would have to develop the real number system from the beginning. Since such a development is lengthy and would divert us from our goal of presenting a course in *analysis*, we suppose that the reader is familiar with the usual properties of real numbers.

In this section we present a set of axioms that form a logical basis for those processes of elementary algebra upon which calculus is based. Any collection of objects satisfying the axioms given below is called a **field**. In particular, the system of real numbers satisfies these axioms, and we indicate how the customary laws of elementary algebra concerning addition, subtraction, multiplication, and division are direct consequences of the axioms for a field.

Throughout the book we use the word *equals* or its symbol $=$ to stand for the words "is the same as." The reader should compare this with other uses for the symbol $=$ such as that in plane geometry when, for example, two line segments are said to be equal if they have the same length.

Axioms of Addition

A-1. Closure property
If a and b are numbers, there is one and only one number, denoted $a + b$, called their ***sum****.*

A-2. Commutative law
For any two numbers a and b, the equality

$$b + a = a + b$$

holds.

A-3. Associative law
For all numbers a, b, and c, the equality

$$(a + b) + c = a + (b + c)$$

holds.

A-4. Existence of a zero
There is one and only one number 0, called ***zero****, such that $a + 0 = a$ for any number a.*

A-5. Existence of a negative
If a is any number, there is one and only one number x such that $a + x = 0$. This number is called the ***negative of*** *a and is denoted by $-a$.*

Theorem 1.1
If a and b are any numbers, then there is one and only one number x such that $a + x = b$. This number x is given by $x = b + (-a)$.

Proof
We must establish two results: (i) that $b + (-a)$ satisfies the equation $a + x = b$ and (ii) that no other number satisfies this equation. To prove (i), suppose that $x = b + (-a)$. Then, using Axioms A-2 through A-4, we see that

$$a + x = a + [b + (-a)] = a + [(-a) + b] = [a + (-a)] + b = 0 + b = b.$$

Therefore, (i) holds. To prove (ii), suppose that x is some number such that $a + x = b$. Adding $(-a)$ to both sides of this equation, we find that

$$(a + x) + (-a) = b + (-a).$$

Now,

$$\begin{aligned}(a + x) + (-a) &= a + [x + (-a)] = a + [(-a) + x] \\ &= [a + (-a)] + x = 0 + x = x.\end{aligned}$$

We conclude that $x = b + (-a)$, and the uniqueness of the solution is established.

Notation. The number $b + (-a)$ is denoted by $b - a$. ■

The next theorem establishes familiar properties of negative numbers.

Theorem 1.2

(i) *If a is a number, then $-(- a) = a$.*
(ii) *If a and b are numbers, then*

$$-(a + b)) = (- a) + (- b).$$

Proof
(i) From the definition of negative, we have

$$(-a) + [-(-a)] = 0, \qquad (-a) + a = a + (-a) = 0.$$

Axiom A-5 states that the negative of $(-a)$ is *unique.* Therefore, $a = -(-a)$. To establish (ii), we know from the definition of negative that

$$(a + b) + [-(a + b)] = 0.$$

Furthermore, using the axioms, we have

$$(a + b) + [(-a) + (-b)] = [a + (-a)] + [b + (-b)] = 0 + 0 = 0.$$

The result follows from the "only one" part of Axiom A-5. ■

Axioms of Multiplication

M-1. Closure property
*If a and b are numbers, there is one and only one number, denoted by ab (or $a \times b$ or $a \cdot b$), called their **product**.*

M-2. Commutative law
For every two numbers a and b, the equality

$$ba = ab$$

holds.

M-3. Associative law
For all numbers a, b, and c, the equality

$$(ab)c = a(bc)$$

holds.

M-4. Existence of a unit
*There is one and only one number u, different from zero, such that $au = a$ for every number a. This number u is called the **unit** and (as is customary) is denoted by* 1.

M-5. Existence of a reciprocal
*For each number a different from zero there is one and only one number x such that $ax = 1$. This number x is called the **reciprocal** of a (or the **inverse** of a) and is denoted by a^{-1} (or $1/a$).*

Axioms M-1 through M-4 are the parallels of Axioms A-1 through A-4 with addition replaced by multiplication. However, M-5 is not the exact analogue of A-5, since the additional condition $a \neq 0$ is required. The reason for this is given below in Theorem 1.3, where it is shown that the result of multiplication of any number by zero is zero. We are familiar with the fact that says that division by zero is excluded.

Special Axiom on *distributivity*

D. Distributive law
For all numbers a, b, and c, the equality

$$a(b + c) = ab + ac$$

holds.

In every logical system there are certain terms that are undefined. For example, in the system of axioms for plane Euclidean geometry, the terms *point* and *line* are undefined. Of course, we have an intuitive idea of the meaning of these two undefined terms, but in the framework of Euclidean geometry it is not possible to define them. In the axioms for algebra given above, the term *number* is undefined. We shall interpret number to mean *real number* (positive, negative, or zero) in the usual sense that we give to it in elementary courses. Actually, the above axioms for a field hold for many systems, of which the collection of real numbers is only one. For example, all the axioms stated so far hold for the system consisting of all *complex numbers.* Also, there are many systems, each consisting of a finite number of elements (*finite fields*), that satisfy all the axioms we have stated until now.

Additional axioms are needed if we insist that the real number system be the *only* collection satisfying all the given axioms. The additional axiom required for this purpose is discussed in Section 1.3.

Theorem 1.3
If a is any number, then $a \cdot 0 = 0$.

Proof
Let b be any number. Then $b + 0 = b$, and therefore $a(b + 0) = ab$. From the distributive law (Axiom D), we find that

$$(ab) + (a \cdot 0) = (ab),$$

so that $a \cdot 0 = 0$ by Axiom A-4. ■

Theorem 1.4
If a and b are numbers and $a \neq 0$, then there is one and only one number x such that $a \cdot x = b$. The number x is given by $x = ba^{-1}$.

The proof of Theorem 1.4 is just like the proof of Theorem 1.1 with addition replaced by multiplication, 0 by 1, and $-a$ by a^{-1}. The details are left to the reader.

Notation. The expression "if and only if," a technical one used frequently in mathematics, requires some explanation. Suppose A and B stand for propositions that may be true or false. To say that A is true *if* B is true means that the truth of B implies the truth of A. The statement A is true *only if* B is true means that the truth of A implies the truth of B. Thus the shorthand statement "A is true if and only if B is true" is equivalent to the *double implication* that the truth of A implies and is implied by the truth of B. As a further shorthand we use the symbol $\Leftrightarrow$ to represent "if and only if," and we write

$$A \Leftrightarrow B$$

for the two implications stated above. The term *necessary and sufficient* is used as a synonym for "if and only if."

We now establish the familiar principle that is the basis for the solution of quadratic and other algebraic equations by factoring.

Theorem 1.5

(i) *We have $ab = 0$ if and only if $a = 0$ or $b = 0$ or both.*
(ii) *We have $a \neq 0$ and $b \neq 0$ if and only if $ab \neq 0$.*

Proof
We must prove two statements in each of parts (i) and (ii). To prove (i), observe that if $a = 0$ or $b = 0$ or both, then it follows from Theorem 1.3 that $ab = 0$. Going the other way in (i), suppose that $ab = 0$. Then there are two cases: either $a = 0$ or $a \neq 0$. If $a = 0$, the result follows. If $a \neq 0$, then we see that

$$b = 1 \cdot b = (a^{-1}a)b = a^{-1}(ab) = a^{-1} \cdot 0 = 0.$$

Hence $b = 0$ and (i) is established. To prove (ii), first suppose $a \neq 0$ and $b \neq 0$. Then $ab \neq 0$, because $a \neq 0$ and $b \neq 0$ is the negation of the statement "$a = 0$ or $b = 0$ or both." Thus (i) applies. For the second part of (ii), suppose $ab \neq 0$. Then $a \neq 0$ and $b \neq 0$, for if one of them were zero, Theorem 1.3 would apply to give $ab = 0$. ■

Theorem 1.6

(i) *If* $a \neq 0$, *then* $a^{-1} \neq 0$ *and* $[(a^{-1})^{-1}] = a$.
(ii) *If* $a \neq 0$ *and* $b \neq 0$, *then* $(a \cdot b)^{-1} = (a^{-1}) \cdot (b^{-1})$.

The proof of this theorem is like the proof of Theorem 1.2 with addition replaced by multiplication, 0 replaced by 1, and $(-a)$, $(-b)$ replaced by a^{-1}, b^{-1}. We leave the details to the reader. Note that if $a \neq 0$, then $a^{-1} \neq 0$ because $aa^{-1} = 1$ and $1 \neq 0$. Then Theorem 1.5(ii) may be used with $b = a^{-1}$.

Using Theorem 1.3 and the distributive law, we easily prove the **laws of signs** stated as Theorem 1.7 below. We emphasize that the numbers a and b may be positive, negative, or zero.

Theorem 1.7
If a *and* b *are any numbers, then*

(i) $a \cdot (-b) = -(a \cdot b)$.
(ii) $(-a) \cdot b = -(a \cdot b)$.
(iii) $(-a) \cdot (-b) = a \cdot b$.

Proof
(i) Since $b + (-b) = 0$, it follows from the distributive law that

$$a[b + (-b)] = a \cdot b + a \cdot (-b) = 0.$$

Also, the negative of $a \cdot b$ has the property that $a \cdot b + [-(a \cdot b)] = 0$. Hence we see from Axiom A-5 that $a \cdot (-b) = -(a \cdot b)$. Part (ii) follows from part (i) by interchanging a and b. The proof of (iii) is left to the reader. ■

We now show that the *laws of fractions*, as given in elementary algebra, follow from the axioms and theorems above.

Notation. We introduce the following standard symbols for $a \cdot b^{-1}$:

$$a \cdot b^{-1} = \frac{a}{b} = a/b = a \div b.$$

These symbols, representing an indicated division, are called **fractions**. The *numerator* and *denominator* of a fraction are defined as usual. A fraction with *denominator* zero has no meaning.

Theorem 1.8

(i) *For every number* a, *the equality* $a/1 = a$ *holds.*
(ii) *If* $a \neq 0$, *then* $a/a = 1$.

Proof

(i) We have $a/1 = (a \cdot 1^{-1}) = (a \cdot 1^{-1}) \cdot 1 = a(1^{-1} \cdot 1) = a \cdot 1 = a$. (ii) If $a \neq 0$, then $a/a = a \cdot a^{-1} = 1$, by definition. ■

Problems

1. Show that in Axiom A-5 it is not necessary to assume that there is only one number x such that $a + x = 0$.
2. If a, b, and c are any numbers, show that
$$a+b+c = a+c+b = b+a+c = b+c+a = c+a+b = c+b+a.$$
3. Prove, on the basis of Axioms A-1 through A-5, that
$$(a + c) + (b + d) = (a + d) + (b + c).$$
4. Prove Theorem 1.4.
5. If a, b, and c are any numbers, show that
$$abc = acb = bac = cab = cba.$$
6. If a, b, and c are any numbers, show that
$$(ac) \cdot (bd) = (ab) \cdot (cd).$$
7. If a, b, and c are any numbers, show that there is one and only one number x such that $x + a = b$.
8. Prove Theorem 1.6.
9. Show that the distributive law may be replaced by the following statement: For all numbers a, b and c, the equality $(b+c)a = ba+ca$ holds.
10. Complete the proof of Theorem 1.7.
11. If a, b, and c are any numbers, show that $a - (b + c) = (a - b) - c$ and that $a - (b - c) = (a - b) + c$. Give reasons for each step of the proof.
12. Show that $a(b + c + d) = ab + ac + ad$, giving reasons for each step.
13. Assuming that $a + b + c + d$ means $(a + b + c) + d$, prove that $a + b + c + d = (a + b) + (c + d)$.
14. Assuming the result of Problem 9, prove that
$$(a + b) \cdot (c + d) = ac + bc + ad + bd.$$

1.2 Natural Numbers and Sequences

Traditionally we build the real number system by a sequence of enlargements. We start with the positive integers and extend that system to include the positive rational numbers (quotients, or ratios, of integers). The system of rational numbers is then enlarged to include all positive real numbers; finally we adjoin the negative numbers and zero to obtain the collection of all real numbers.

The system of axioms in Section 1.1 does not distinguish (or even mention) positive numbers. To establish the relationship between these axioms and the real number system, we begin with a discussion of **natural numbers**. As we know, these are the same as the positive integers.

We can obtain the totality of natural numbers by starting with the number 1 and then forming $1 + 1$, $(1 + 1) + 1$, $[(1 + 1) + 1] + 1$, and so on. We call $1 + 1$ the number 2; then $(1 + 1) + 1$ is called the number 3; in this way the collection of natural numbers is generated. Actually it is possible to give an abstract definition of natural number, one that yields the same collection and is logically more satisfactory. This is done in Section 1.4, where the principle of mathematical induction is established and illustrated. Meanwhile, we shall suppose that the reader is familiar with all the usual properties of natural numbers.

The axioms for a field given in Section 1.1 determine addition and multiplication for any *two* numbers. On the basis of these axioms we were able to *define* the sum and product of three numbers. Before describing the process of defining sums and products for more than three elements, we now recall several definitions and give some notations that will be used throughout the book.

Definitions

The set (or collection) of all real numbers is denoted by $\mathbb{R}^1$. The set of ordered pairs of real numbers is denoted by $\mathbb{R}^2$, the set of ordered triples by $\mathbb{R}^3$, and so on. A **relation** from $\mathbb{R}^1$ to $\mathbb{R}^1$ is a set of ordered pairs of real numbers; that is, a relation from $\mathbb{R}^1$ to $\mathbb{R}^1$ is a set in $\mathbb{R}^2$. The **domain** of this relation is the set in $\mathbb{R}^1$ consisting of all the first elements in the ordered pairs. The **range** of the relation is the set of all the second elements in the ordered pairs. Observe that the range is also a set in $\mathbb{R}^1$.

A **function** f from $\mathbb{R}^1$ into $\mathbb{R}^1$ is a relation in which no two ordered pairs have the same first element. We use the notation $f : \mathbb{R}^1 \to \mathbb{R}^1$ for such a function. The word **mapping** is a synonym for function.

If D is the domain of f and S is its range, we shall also use the notation $f : D \to S$. A function is a relation (set in $\mathbb{R}^2$) such that for each element x in the domain there is precisely one element y in the range such that the pair (x, y) is one of the ordered pairs that constitute the function. Occasionally a function will be indicated by writing $f : x \to y$. Also, for a given function f, the unique number in the range corresponding to an element x in the domain in written $f(x)$. The symbol $x \to f(x)$ is used for this relationship. We assume that the reader is familiar with functional notation.

A **sequence** is a function that has as its domain some or all of the natural numbers. If the domain consists of a finite number of positive integers, we say that the sequence is **finite**. Otherwise, the sequence is called **infinite**. In general, the elements in the domain of a function do not have any particular order. In a sequence, however, there is a natural ordering of the domain induced by the usual order in terms of size that we give to the positive integers. For example, if the domain of a sequence consists of the numbers $1, 2, \ldots, n$, then the elements of the range, that is, the *terms of the sequence*, are usually written in the same order as the natural numbers. If the sequence (function) is denoted by a, then the terms of the sequence are denoted by $a_1, a_2, \ldots, a_n$ or, sometimes by $a(1), a(2), \ldots, a(n)$. The element a_i or $a(i)$ is called the ith *term of the sequence.* If the domain of a sequence a is the set of all natural numbers (so that the sequence is infinite), we denote the sequence by

$$a_1, a_2, \ldots, a_n, \ldots \quad \text{or} \quad \{a_n\}.$$

Definitions

For all integers $n \geq 1$, the **sum** and **product** of the numbers $a_1, a_2, \ldots, a_n$ are defined respectively by b_n and c_n:

$$a_1 + a_2 + \ldots + a_n \equiv b_n \qquad \text{and} \qquad a_1 \cdot a_2 \cdots a_n \equiv c_n.$$

We use the notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n \qquad \text{and} \qquad \prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n.$$

The symbol $\prod$ is in general use as a compact notation for product analogous to the use of $\sum$ for sum. We read $\prod_{i=1}^{n}$ as "the product as i goes from 1 to n."

On the basis of these definitions it is not difficult to establish the next result.

Proposition 1.1

If $a_1, a_2, \ldots, a_n, a_{n+1}$ is any sequence, then

$$\sum_{i=1}^{n+1} a_i = \left(\sum_{i=1}^{n} a_i\right) + a_{n+1} \qquad \textit{and} \qquad \prod_{i=1}^{n+1} a_i = \left(\prod_{i=1}^{n} a_i\right) \cdot a_{n+1}.$$

The following proposition may be proved by using mathematical induction, which is established in Section 1.4.

Proposition 1.2

If $a_1, a_2, \ldots, a_m, a_{m+1}, \ldots, a_{m+n}$ is any sequence, then

$$\sum_{i=1}^{m+n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{m+n} a_i$$

and

$$\prod_{i=1}^{m+n} a_i = \left(\prod_{i=1}^{m} a_i\right) \cdot \left(\prod_{i=m+1}^{m+n} a_i\right).$$

The symbol x^n, for n a natural number, is defined in the customary way as $x \cdot x \cdots x$ with x appearing n times in the product. We assume that the reader is familiar with the laws of exponents and the customary rules for adding, subtracting, and multiplying polynomials. These rules are a simple consequence of the axioms and propositions above.

The decimal system of writing numbers (or the system with any base) depends on a representation theorem that we now state. If n is any natural number, then there is *one and only one* representation for n of the form

$$n = d_0(10)^k + d_1(10)^{k-1} + \cdots + d_{k-1}(10) + d_k$$

in which k is a natural number or zero, each d_i is one of the numbers $0, 1, 2, \ldots, 9$, and $d_0 \neq 0$. The numbers $0, 1, 2, \ldots, 9$ are called **digits** of the decimal system. On the basis of such a representation, the rules of arithmetic follow from the corresponding rules for polynomials in x with $x = 10$.

For completeness, we define the terms *integer*, *rational number*, and *irrational number*.

Definitions

A real number is an **integer** if and only if it is either zero, a natural number, or the negative of a natural number. A real number r is said to be **rational** if and only if there are *integers* p and q, with $q \neq 0$, such that $r = p/q$. A real number that is not rational is called **irrational**.

It is clear that the sum and product of a finite sequence of integers is again an integer, and that the sum, product, or quotient of a finite sequence of rational numbers is a rational number.

The rule for multiplication of fractions is given by an extension of Theorem 1.7 that may be derived by mathematical induction.

We emphasize that the axioms for a field given in Section 1.1 imply only theorems concerned with the operations of addition, subtraction, multiplication, and division. The exact nature of the elements in the field is not described. For example, the axioms do not imply the *existence* of a number whose square is 2. In fact, if we interpret number to be "rational number" and consider no others, then all the axioms for a field are satisfied. The rational number system forms a *field*. An additional axiom is needed if we wish the field to contain irrational numbers such as $\sqrt{2}$.

Problems

1. Suppose $T : \mathbb{R}^1 \to \mathbb{R}^1$ is a relation composed of ordered pairs (x, y). We define the *inverse relation* of T as the set of ordered pairs (x, y) where (y, x) belongs to T. Let a function a be given as a finite sequence $a_1, a_2, \dots, a_n$. Under what conditions will the inverse relation of a be a function?
2. Prove Propositions 1.1 and 1.2 for sequences with 5 terms.
3. Show that if $a_1, a_2, \dots, a_5$ is a sequence of 5 terms, then $\prod_{i=1}^5 a_i = 0$ if and only if at least one term of the sequence is zero.
4. If $a_1, a_2, \dots, a_n, a_{n+1}$ is any sequence, show that $\sum_{i=1}^{n+1} a_i = \sum_{i=1}^{n} a_i + a_{n+1}$.
5. Establish the formula
$$(a+b)^4 = \sum_{i=0}^{4} \frac{4!}{i!(4-i)!}\, a^{4-i} b^i.$$
6. Consider the system with the two elements 0 and 1 and the following rules of addition and multiplication:
$$\begin{aligned} 0+0&=0, & 0+1&=1,\\ 1+0&=1, & 1+1&=0,\\ 0\cdot 0&=0, & 1\cdot 0&=0,\\ 0\cdot 1&=0, & 1\cdot 1&=1. \end{aligned}$$
Show that all the axioms of Section 1.1 are valid; hence this system forms a field.
7. Show that the system consisting of all elements of the form $a + b\sqrt{5}$, where a and b are any rational numbers, satisfies all the axioms for a field if the usual rules for addition and multiplication are used.
8. Is it possible to make addition and multiplication tables so that the four elements 0, 1, 2, 3 form the elements of a field? Prove your statement. [*Hint*: In the multiplication table each row, other than the one consisting of zeros, must contain the symbols 0, 1, 2, 3 in some order.]
9. Consider all numbers of the form $a + b\sqrt{6}$ where a and b are rational. Does this collection satisfy the axioms for a field?
10. Show that the system of complex numbers $a + bi$ with a, b real numbers and $i = \sqrt{-1}$ satisfies all the axioms for a field.

1.3 Inequalities

The axioms for a field describe many number systems. If we wish to describe the real number system to the exclusion of other systems, additional axioms are needed. One of these, an axiom that distinguishes positive from negative numbers, is the Axiom of inequality.

Axiom I

(Axiom of inequality) *Among all the numbers of the system, there is a set called the positive numbers that satisfies the two conditions*: (i) *for any number a exactly one of the following three alternatives holds*: *a is positive or $a = 0$ or $-a$ is positive*; (ii) *any finite sum or product of positive numbers is positive.*

When we add Axiom I to those of Section 1.1, the resulting system of axioms is applicable only to those number systems that have a linear order. For example, the system of complex numbers does not satisfy Axiom I but does satisfy all the axioms for a field. Similarly, it is easy to see that the system described in Problem 6 of Section 1.2 does not satisfy Axiom I. However, both the real number system and the rational number system satisfy all the axioms given thus far.

Definition

A number a is **negative** whenever $-a$ is positive. If a and b are any numbers, we say that $a > b$ (*read*: *a is greater than b*) whenever $a - b$ is positive.

It is convenient to adopt a geometric point of view and to associate a horizontal axis with the totality of real numbers. We select any convenient point for the origin and call points to the right of the origin positive numbers and points to the left negative numbers (Figure 1.1). For every real number there will correspond a point on the line, and conversely, every point will represent a real number. Then the inequality $a < b$ may be read: *a is to the left of b*. This geometric way of looking at inequalities is frequently of help in solving problems.

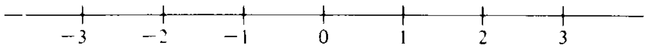

Figure 1.1

It is helpful to introduce the notion of an *interval* of *numbers* or *points.* If a and b are numbers (as shown in Figure 1.2), then the **open interval from a to b** is the collection of all numbers that are both larger than a and smaller than b. That is, an open interval consists of all numbers *between* a and b. A number x is between a and b if both inequalities $a < x$ and $x < b$ are true. A compact way of writing these two inequalities is

$$a < x < b.$$

The **closed interval from a to b** consists of all the points between a and b and in addition the numbers (or points) a and b (Figure 1.3).

Figure 1.2

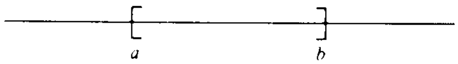

Figure 1.3

Suppose a number x is either equal to a or larger than a, but we don't know which. We write this conveniently as $x \geq a$, which is read: *x is greater than or equal to a*. Similarly, $x \leq b$ is read: *x is less than or equal to b*, and means that x may be either smaller than b or may be b itself. A compact way of designating all points x in the closed interval from a to b is

$$a \leq x \leq b.$$

An interval that contains the endpoint b but not a is said to be **half-open on the left**. Such an interval consists of all points x that satisfy the double inequality

$$a < x \leq b.$$

Similarly, an interval containing a but not b is called **half-open on the right**, and we write

$$a \leq x < b.$$

Parentheses and brackets are used as symbols for intervals in the following way:

(a, b)	for the open interval $a < x < b$,
$[a, b]$	for the closed interval $a \leq x \leq b$,
$(a, b]$	for the interval half-open on the left $a < x \leq b$,
$[a, b)$	for the interval half-open on the right $a \leq x < b$.

We extend the idea of interval to include the unbounded cases. For example, consider the set of *all* numbers larger than 7. This set may be thought of as an interval beginning at 7 and extending to infinity to the right (see Figure 1.4). Of course, infinity is not a number, but we use the symbol $(7, \infty)$ to represent all numbers larger than 7. We also use the double inequality

$$7 < x < \infty$$

to represent this set. Similarly, the symbol $(-\infty, 12)$ stands for all numbers less than 12. We also use the inequalities $-\infty < x < 12$ to represent this set.

Figure 1.4

Definition

The **solution** of an equation or inequality in one unknown, say x, is the collection of all numbers that make the equation or inequality a true statement. Sometimes this set of numbers is called the **solution set**. For example, the inequality

$$3x - 7 < 8$$

has as its solution set all numbers less than 5. To demonstrate this we argue in the following way. If x is a number that satisfies the above inequality, we can add 7 to both sides and obtain a true statement. That is, the inequality

$$3x - 7 + 7 < 8 + 7, \quad \text{or} \quad 3x < 15,$$

holds. Now, dividing both sides by 3, we obtain $x < 5$; therefore, *if* x is a solution, *then* it is less than 5. Strictly speaking, however, we have not *proved* that every number that is less than 5 is a solution. In an actual proof, we begin by supposing that x is any number less than 5; that is, $x < 5$. We multiply both sides of this inequality by 3 and then subtract 7 from both sides to get

$$3x - 7 < 8,$$

the original inequality. Since the hypothesis that x is less than 5 implies the original inequality, we have proved the result. The important thing to notice is that the proof consisted in *reversing* the steps of the original argument that led to the solution $x < 5$ in the first place. So long as each step taken is reversible, the above procedure is completely satisfactory for obtaining solutions of inequalities.

By means of the symbol $\Leftrightarrow$, we can give a solution to this example in compact form. We write

$$3x - 7 < 8 \Leftrightarrow 3x < 15 \qquad \text{(adding 7 to both sides)}$$

and

$$3x < 15 \Leftrightarrow x < 5 \qquad \text{(dividing both sides by 3).}$$

The solution set is the interval $(-\infty, 5)$.

Notation. It is convenient to introduce some terminology and symbols concerning sets. In general, a **set** is a collection of objects. The objects may have any character (number, points, lines, etc.) so long as we know which objects are in a given set and which are not. If S is a set and P is an object in it, we write $P \in S$ and say that *P is an element of S* or that *P belongs to S*. If S_1 and S_2 are two sets, their **union**, denoted by $S_1 \cup S_2$, consists of all objects each of which is in at least one of the two sets. The **intersection** of S_1 and S_2, denoted by $S_1 \cap S_2$, consists of all objects each of which is in both sets.

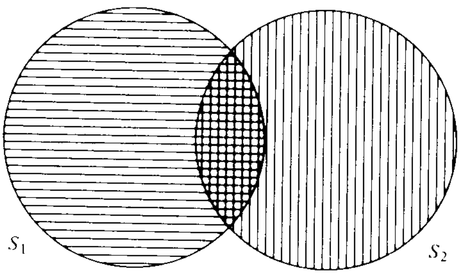

Figure 1.5

Schematically, if S_1 is the horizontally shaded set of points in Figure 1.5 and S_2 is the vertically shaded set, then $S_1 \cup S_2$ consists of the entire shaded area, and $S_1 \cap S_2$ consists of the doubly shaded area. Similarly, we may form the union and intersection of any number of sets. When we write $S_1 \cup S_2 \cup \ldots \cup S_n$ for the union S of the n sets $S_1, S_2, \ldots, S_n$, then S consists of all elements each of which is in at least one of the n sets. We also use the notation $S = \bigcup_{i=1}^{n} S_i$ as shorthand for the union of n sets. The intersection of n sets $S_1, S_2, \ldots, S_n$ is written $S_1 \cap S_2 \cap \ldots \cap S_n$ or, briefly, $\bigcap_{i=1}^{n} S_i$. It may happen that two sets S_1 and S_2 have no elements in common. In such a case their intersection is *empty*, and we use the term **empty set** for the set that is devoid of members.

Most often we will deal with sets each of which is specified by some property or properties of its elements. For example, we may speak of the set of all even integers or the set of all rational numbers between 0 and 1. We employ the special notation

$$\{x : x = 2n \text{ and } n \text{ is an integer}\}$$

to represent the set of all even integers. In this notation the letter x stands for a generic element of the set, and the properties that determine membership in the set are listed to the right of the colon. The notation

$$\{x : x \in (0, 1) \text{ and } x \text{ is rational}\}$$

represents the set of rational numbers in the open interval $(0, 1)$. If a set has only a few elements, it may be specified by listing its members between braces. Thus the notation $\{-2, 0, 1\}$ denotes the set whose elements are the numbers -2, 0, and 1.

The words *and* and *or* have precise meanings when used in connection with sets and their properties. The set consisting of elements that have property A *or* property B is the *union* of the set having property A and the set having property B. Symbolically, we write

$$\begin{aligned}&\{x : x \text{ has property } A \text{ or property } B\}\\ &\quad = \{x : x \text{ has property } A\} \cup \{x : x \text{ has property } B\}.\end{aligned}$$

The set consisting of elements that have both property A *and* property B is the *intersection* of the set having property A with the set having property B. In set notation, we write

$$\begin{aligned}&\{x : x \text{ has property } A \text{ and property } B\}\\ &\quad = \{x : x \text{ has property } A\} \cap \{x : x \text{ has property } B\}.\end{aligned}$$

If A and B are two sets and if every element of A is also an element of B, we say that A *is a* **subset** *of* B, and we write $A \subset B$. The two statements $A \subset B$ and $B \subset A$ imply that $A = B$.

We give two examples that illustrate how set notation is used.

EXAMPLE 1
Solve for x:

$$\frac{3}{x} < 5 \qquad (x \neq 0).$$

Solution. Since we don't know in advance whether x is positive or negative, we cannot multiply by x unless we impose additional conditions. We therefore separate the problem into two cases: (i) x is positive, and (ii) x is negative. The desired solution set can be written as the union of the sets S_1 and S_2 defined by

$$S_1 = \left\{x : \frac{3}{x} < 5 \text{ and } x > 0\right\},$$

$$S_2 = \left\{x : \frac{3}{x} < 5 \text{ and } x < 0\right\}.$$

Now,

$$\begin{aligned} x \in S_1 &\Leftrightarrow 3 < 5x \text{ and } x > 0\\ &\Leftrightarrow x > \frac{3}{5} \text{ and } x > 0\\ &\Leftrightarrow x > \frac{3}{5}.\end{aligned}$$

Similarly,

$$\begin{aligned} x \in S_2 &\Leftrightarrow 3 > 5x \text{ and } x < 0 \\ &\Leftrightarrow x < \frac{3}{5} \text{ and } x < 0 \\ &\Leftrightarrow x < 0. \end{aligned}$$

Thus the solution set is (see Figure 1.6)

$$S_1 \cup S_2 = (\frac{3}{5}, \infty) \cup (-\infty, 0).$$

■

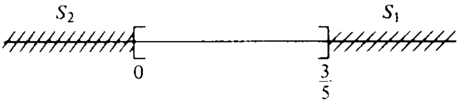

Figure 1.6

Example 2
Solve for x:

$$\frac{2x-3}{x+2} < \frac{1}{3} \qquad (x \neq -2).$$

Solution. As in Example 1, the solution set is the union $S_1 \cup S_2$, where

$$S_1 = \left\{x : \frac{2x-3}{x+2} < \frac{1}{3} \text{ and } x+2 > 0\right\},$$
$$S_2 = \left\{x : \frac{2x-3}{x+2} < \frac{1}{3} \text{ and } x+2 < 0\right\}.$$

For numbers in S_1, we may multiply the inequality by $x + 2$, and since $x + 2$ is positive, the direction of the inequality is preserved. Hence

$$\begin{aligned} x \in S_1 &\Leftrightarrow 3(2x-3) < x+2 \text{ and } x+2 > 0 \\ &\Leftrightarrow 5x < 11 \text{ and } x+2 > 0 \\ &\Leftrightarrow x < \frac{11}{5} \text{ and } x > -2 \\ &\Leftrightarrow x \in (-2, \frac{11}{5}). \end{aligned}$$

For numbers in S_2, multiplication of the inequality by the negative quantity $x+2$ reverses the direction. Therefore,

$$\begin{aligned} x \in S_2 &\Leftrightarrow 3(2x-3) > x+2 \text{ and } x+2 < 0 \\ &\Leftrightarrow 5x > 11 \text{ and } x+2 < 0 \\ &\Leftrightarrow x > \frac{11}{5} \text{ and } x < -2. \end{aligned}$$

Since there are no numbers x satisfying *both* conditions $x > \frac{11}{5}$ and $x < -2$, the set S_2 is empty. The solution set (Figure 1.7) consists of $S_1 = (-2, \frac{11}{5})$. ■

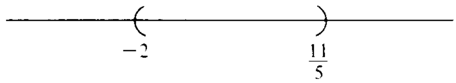

Figure 1.7

We assume that the reader is familiar with the notion of absolute value and elementary manipulations with equations and inequalities involving the absolute value of numbers.

Problems

1. Consider the field consisting of all numbers of the form $a + b\sqrt{7}$ where a and b are rational. Does this field satisfy Axiom I? Justify your answer.
2. Consider the set of all numbers of the form $ai\sqrt{7}$ where a is a real number and $i = \sqrt{-1}$. Show that it is possible to give this set an ordering in such a way that it satisfies Axiom I. Does this set form a field?
3. Show that the set of all complex numbers $a+bi$, with a and b rational, satisfies all the axioms for a field.

In Problems 4 through 7 find in each case the solution set as an interval, and plot.

4. $2x - 2 < 27 + 4x$.
5. $5(x-1)) > 12 - (17 - 3x)$.
6. $(2x+1)/8 < (3x-4)/3$.
7. $(x+10)/6 + 1 - (x/4) > ((4-5x)/6) - 1$.

In Problems 8 through 10, find the solution set of each pair of simultaneous inequalities. Verify in each case that the solution set is the intersection of the solution sets of the separate inequalities.

8. $2x - 3 < 3x - 2$ and $4x - 1 < 2x + 3$.

9. $3x + 5 > x + 1$ and $4x - 3 < x + 6$.
10. $4 - 2x < 1 + 5x$ and $3x + 2 < x - 7$.

In Problems 11 through 14, express each given combination of intervals as an interval. Plot a graph in each case.

11. $[-1, \infty) \cap (-\infty - 2)$. 12. $(-\infty, 2) \cap (-\infty, 4)$.
13. $(-1, 1) \cup (0, 5)$. 14. $(-\infty, 2) \cup (-\infty, 4)$.

In Problems 15 through 19 find the solution set of the given inequality

15. $3x < 2/5$. 16. $(x - 2)/x < 3$.
17. $(x + 2)/(x - 1) < 4$. 18. $x/(2 - x) < 2$.
19. $(x + 2)/(x - 3) < -2$.

1.4 Mathematical Induction

The principle of mathematical induction with which most readers are familiar can be derived as a consequence of the axioms in Section 1.1. Since the definition of natural number is at the basis of the principle of mathematical induction, we shall develop both concepts together.

Definition
A set S of numbers is said to be **inductive** if

(a) $1 \in S$ and
(b) $(x + 1) \in S$ whenever $x \in S$.

Examples of inductive sets are easily found. The set of all real numbers is inductive, as is the set of all rationals. The set of all integers, positive, zero, and negative, is inductive. The collection of real numbers between 0 and 10 is not inductive, since it satisfies (a) but not (b). No finite set of real numbers can be inductive since (b) will be violated at some stage.

Definition
A real number n is said to be a **natural number** if it belongs to *every* inductive set of real numbers. The set of *all* natural numbers will be denoted by the symbol $\mathbb{N}$.

We observe that $\mathbb{N}$ contains the number 1, since by the definition of inductive set, 1 must always be a member of every inductive set. As we know, *the set of natural numbers* $\mathbb{N}$ *is identical with the set of positive integers.*

Theorem 1.9
The set $\mathbb{N}$ *of all natural numbers is an inductive set.*

Proof
We must show that $\mathbb{N}$ has properties (a) and (b) in the definition of inductive set. As we remarked above, (a) holds. Now suppose that k is an element of $\mathbb{N}$. Then k belongs to *every* inductive set S. For each inductive set, if k is an element, so is $k + 1$. Thus $k + 1$ belongs to every inductive set. Therefore $k + 1$ belongs to $\mathbb{N}$. Hence $\mathbb{N}$ has Property (b), and is inductive. ■

The principle of mathematical induction is contained in the next theorem, which asserts that any inductive set of natural numbers must consist of the entire collection $\mathbb{N}$.

Theorem 1.10 (Principle of mathematical induction)
If S is an inductive set of natural numbers, then $S = \mathbb{N}$.

Proof
Since S is an inductive set, we know from the definition of natural number that $\mathbb{N}$ is contained in S. On the other hand, since S consists of natural numbers, it follows that S is contained in $\mathbb{N}$. Therefore $S = \mathbb{N}$. ■

We now illustrate how the principle of mathematical induction is applied in practice. The reader may not recognize Theorem 1.10 as the statement of the familiar principle of mathematical induction. So we shall prove a formula using Theorem 1.10.

Example 1
Show that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \tag{1.1}$$

for every natural number n.

Solution. Let S be the set of natural numbers n for which the formula (1.1) holds. We shall show that S is an inductive set.

(a) Clearly $1 \in S$, since formula (1.1) holds for $n = 1$.
(b) Suppose $k \in S$. Then formula (1.1) holds with $n = k$. Adding $(k+1)$ to both sides, we see that

$$1 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2},$$

which is formula (1.1) for $n = k + 1$. Thus $(k + 1)$ is in S whenever k is.

Combining (a) and (b), we conclude that S is an inductive set of natural numbers and so consists of all natural numbers. Therefore formula (1.1) holds for all natural numbers. ■

Now mathematical induction can be used to establish Proposition 1.2 of Section 1.2. We state the result in the next theorem.

Theorem 1.11
If $a_1, a_2, \ldots, a_m, \ldots, a_{m+n}$ *is any finite sequence, then*

$$\sum_{i=1}^{m+n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{m+n} a_i$$

and

$$\prod_{i=1}^{m+n} a_i = \left(\prod_{i=1}^{m} a_i\right) \cdot \left(\prod_{i=m+1}^{m+n} a_i\right).$$

Since the set of natural numbers $\mathbb{N}$ is identical with the set of positive integers, the fact that a natural number is positive follows from the axiom of inequality and the definition of $\mathbb{N}$.

Theorem 1.12 (The well-ordering principle)
Any nonempty set T *of natural numbers contains a smallest element.*

Proof
Let k be a member of T. We define a set S of natural numbers by the relation

$$S = \{p : p \in T \text{ and } p \leq k\}.$$

The set S contains a portion (not necessarily all) of the set consisting of the natural numbers $\{1, 2, 3, \ldots, k-1, k\}$. Thus since S is finite, it has a smallest element, which we denote by s. We now show that s is the smallest element of T. First, since $s \in S$ and $S \subset T$, then $s \in T$. Suppose t is any element of T different from s. If $t > k$, then the inequality $k \geq s$ implies that $t > s$. On the other hand, if $t \leq k$, then $t \in S$. Since s is the smallest element of S and $s \neq t$, we have $s < t$. Thus s is smaller than any other element of T, and the proof is complete. ∎

Problems
In each of Problems 1 through 6 use the Principle of mathematical induction to establish the given formula.

1. $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$.
2. $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$.
3. $\sum_{i=1}^{n}(2i-1) = n^2$.
4. $\sum_{i=1}^{n} i(i+1) = n(n+1)(n+2)/3$.
5. $\sum_{i=1}^{n} i(i+2) = n(n+1)(2n+7)/6$.
6. $\sum_{i=1}^{n}(1/i(i+1)) = n/(n+1)$.

7. Suppose p, q, and r are natural numbers such that $p + q < p + r$. Show that $q < r$.
8. Suppose p, q, and r are natural numbers such that $p \cdot q < p \cdot r$. Show that $q < r$.
9. (a) Show that the set of positive rational numbers is inductive.
   (b) Is the set $a + b\sqrt{5}$, a and b natural numbers, an inductive set?
   (c) Is the set of all complex numbers an inductive set?

In each of Problems 10 through 15 use the Principle of mathematical induction to establish the given assertion.

10. $\sum_{i=1}^{n}[a + (i-1)d] = n[2a + (n-1)d]/2$, where a and d are any real numbers.
11. $\sum_{i=1}^{n} ar^{i-1} = a(r^n - 1)/(r-1)$, where a and r are real numbers, and $r \neq 1$.
12. $\sum_{i=1}^{n} i(i+1)(i+2) = n(n+1)(n+2)(n+3)/4$.
13. $(1+a)^n \geq 1 + na$ for $a \geq 0$ and n a natural number.
14. $\sum_{i=1}^{n} 3^{2i-1} = 3(9^n - 1)/8$.
15. Prove by induction that

$$(x_1 + x_2 + \ldots + x_k)^2$$

$$= \sum_{i=1}^{k} x_i^2 + 2(x_1x_2 + x_1x_3 + \cdots + x_1x_k + x_2x_3 + x_2x_4 + \cdots$$

$$+ x_2x_k + x_3x_4 + x_3x_5 + \cdots + x_{k-1}x_k).$$

2 CHAPTER Continuity and Limits

2.1 Continuity

Most of the functions we study in elementary calculus are described by formulas. These functions almost always possess derivatives. In fact, a portion of any first course in calculus is devoted to the development of routine methods for computing derivatives. However, not all functions possess derivatives everywhere. For example, the functions $(1 + x^2)/x$, $\cot x$, and $\sin(1/x)$ do not possess derivatives at $x = 0$ no matter how they are defined at $x = 0$.

As we progress in the study of analysis, it is important to enlarge substantially the class of functions we examine. Functions that possess derivatives everywhere form a rather restricted class; extending this class to functions that are differentiable except at a few isolated points does not enlarge it greatly. We wish to investigate significantly larger classes of functions, and to do so we introduce the notion of a continuous function.

Definitions

Suppose that f is a function from a domain D in $\mathbb{R}^1$ to $\mathbb{R}^1$. The function f is **continuous at** a if (i) the point a is in an open interval I contained in D, and (ii) for each positive number ε there is a positive number δ such that

$$|f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad |x - a| < \delta.$$

If f is continuous at each point of a set S, we say that f is **continuous on** S. A function f is called **continuous** if it is continuous at every point of its domain.

The geometric significance of continuity at a point a is indicated in Figure 2.1. We recall that the inequality $|f(x) - f(a)| < \varepsilon$ is equivalent to the double inequality

$$-\varepsilon < f(x) - f(a) < \varepsilon,$$

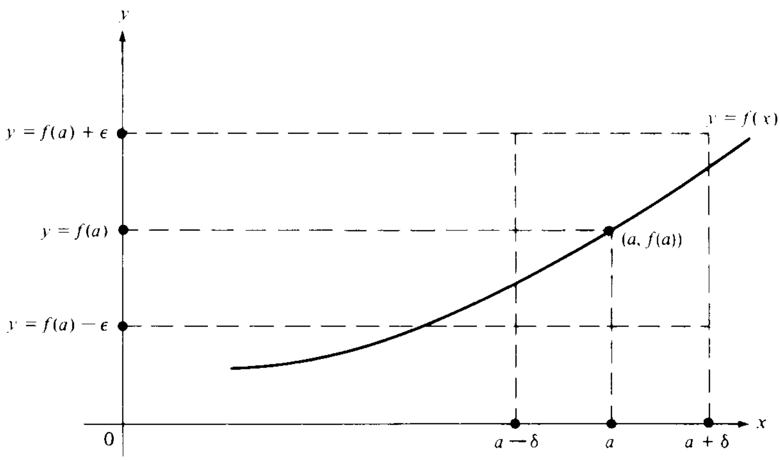

Figure 2.1 The graph of f is in the rectangle for $a - \delta < x < a + \delta$.

or

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon.$$

Similarly, the inequality $|x - a| < \delta$ is equivalent to the two inequalities

$$a - \delta < x < a + \delta.$$

We construct the four lines $x = a - \delta$, $x = a + \delta$, $y = f(a) - \varepsilon$, and $y = f(a) + \varepsilon$, as shown in Figure 2.1. The rectangle determined by these four lines has its center at the point with coordinates $(a, f(a))$. The geometric interpretation of continuity at a point may be given in terms of this rectangle. A function f is continuous at a if for each $\varepsilon > 0$ there is a number $\delta > 0$ such that the graph of f remains within the rectangle for all x in the interval $(a - \delta, a + \delta)$.

It is usually very difficult to verify continuity directly from the definition. Such verification requires that for *every* positive number ε we exhibit a number δ and show that the graph of f lies in the appropriate rectangle. However, if the function f is given by a sufficiently simple expression, it is sometimes possible to obtain an explicit value for the quantity δ corresponding to a given number ε. We describe the method by means of two examples.

EXAMPLE 1
Given the function

$$f : x \to \frac{1}{x+1}, \qquad x \neq -1,$$

and $a = 1$, $\varepsilon = 0.1$, find a number δ such that $|f(x) - f(1)| < 0.1$ for $|x - 1| < \delta$.

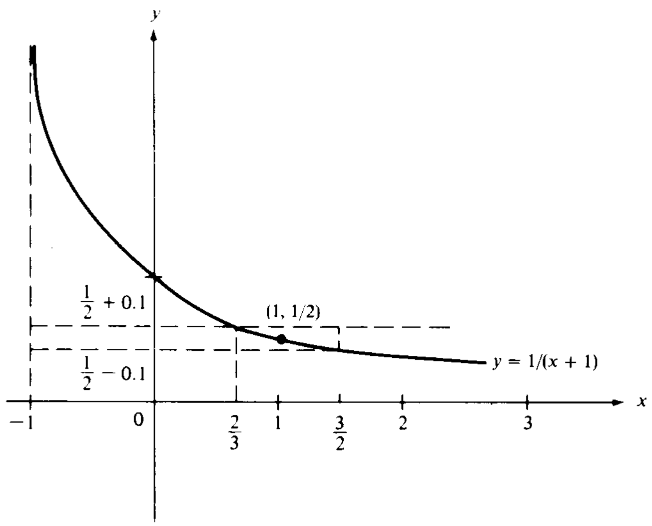

Figure 2.2

Solution. We sketch the graph of f and observe that f is decreasing for $x > -1$ (see Figure 2.2). The equations $f(x)-f(1) = 0.1$, $f(x)-f(1) = -0.1$ can be solved for x. We find

$$\frac{1}{x+1} - \frac{1}{2} = 0.1 \Leftrightarrow x = \frac{2}{3}$$

and

$$\frac{1}{x+1} - \frac{1}{2} = -0.1 \Leftrightarrow x = \frac{3}{2}.$$

Since f is decreasing in the interval $\frac{2}{3} < x < \frac{3}{2}$, it is clear that the graph of f lies in the rectangle formed by the lines $x = \frac{2}{3}$, $x = \frac{3}{2}$, $y = \frac{1}{2} - 0.1$, and $y = \frac{1}{2} + 0.1$. Since the distance from $x = 1$ to $x = \frac{2}{3}$ is smaller than the distance from $x = 1$ to $x = \frac{3}{2}$, we select $\delta = 1 - \frac{2}{3} = \frac{1}{3}$. We make the important general observation that *when a value of δ is obtained for a* *given quantity ε, then any smaller (positive) value for δ may also be used for the same number ε.* ■

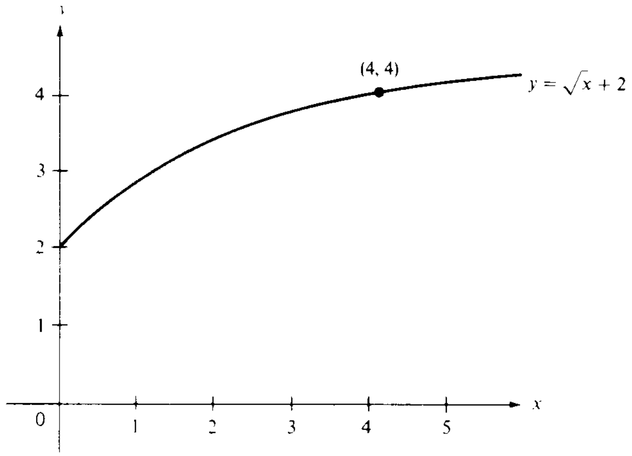

Figure 2.3

EXAMPLE 2
Consider the function

$$f : x \to \begin{cases} \dfrac{x-4}{\sqrt{x}-2}, & x \geq 0, \quad x \neq 4, \\ 4, & x = 4. \end{cases}$$

If $\varepsilon = 0.01$, find a δ such that $|f(x) - f(4)| < 0.01$ for all x such that $|x - 4| < \delta$.

Solution. If $x \neq 4$, then factoring $x - 4$, we get

$$f(x) = \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} = \sqrt{x} + 2 \ .$$

The graph of f is shown in Figure 2.3, and we observe that f is an increasing function. We solve the equations $f(x) - f(4) = 0.01$ and $f(x) - f(4) = -0.01$ and obtain

$$\sqrt{x} + 2 - 4 = 0.01 \Leftrightarrow \sqrt{x} = 2.01 \Leftrightarrow x = 4.0401,$$

$$\sqrt{x} + 2 - 4 = -0.01 \Leftrightarrow \sqrt{x} = 1.99 \Leftrightarrow x = 3.9601 \ .$$

Since f is increasing, it follows that $|f(x) - f(4)| < 0.01$ for $3.9601 < x < 4.0401$. Selecting $\delta = 0.0399$, we find that $|f(x) - f(4)| < \varepsilon$ for $|x - 4| < \delta$. ■

Definition
Suppose that a and L are real numbers and f is a function from a domain D in $\mathbb{R}^1$ to $\mathbb{R}^1$. The number a may or may not be in the domain of f. The function f **tends to** L **as a limit as** x **tends to** a if (i) there is an open interval I containing a that, except possibly for the point a, is contained in D, and (ii) for each positive number ε there is a positive number δ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

If f tends to L as x tends to a, we write

$$f(x) \to L \quad \text{as} \quad x \to a$$

and denote the number L by

$$\lim_{x \to a} f(x).$$

(i) We see that a function f is continuous at a if and only if a is in the domain of f and $f(x) \to f(a)$ as $x \to a$.

(ii) The condition $0 < |x - a| < \delta$ (excluding the possibility $x = a$) is used rather than the condition $|x-a| < \delta$ as in the definition of continuity, since f may not be defined at a itself.

Problems
In Problems 1 through 8 the functions are continuous at the value a given. In each case find a value δ corresponding to the given value of ε such that the definition of continuity is satisfied. Draw a graph.

1. $f(x) = 2x + 5,\ \ a = 1,\ \ \varepsilon = 0.01.$
2. $f(x) = 1 - 3x,\ \ a = 2,\ \ \varepsilon = 0.01.$
3. $f(x) = \sqrt{x},\ \ a = 2,\ \ \varepsilon = 0.01.$
4. $f(x) = \sqrt[3]{x},\ \ a = 1,\ \ \varepsilon = 0.1.$
5. $f(x) = 1 + x^2,\ \ a = 2,\ \ \varepsilon = 0.01.$
6. $f(x) = x^3 - 4,\ \ a = 1,\ \ \varepsilon = 0.5.$
7. $f(x) = x^3 + 3x,\ \ a = -1,\ \ \varepsilon = 0.5.$
8. $f(x) = \sqrt{2x + 1},\ \ a = 4,\ \ \varepsilon = 0.1.$

In Problems 9 through 17 the functions are defined in an interval about the given value of a but not at a. Determine a value δ such that for the given values of L and ε, the statement $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$ is valid. Sketch the graph of the given function.

9. $f(x) = (x^2 - 9)/(x + 3),\ \ a = -3,\ \ L = -6,\ \ \varepsilon = 0.005.$
10. $f(x) = (\sqrt{2x} - 2)/(x - 2),\ \ a = 2,\ \ L = \frac{1}{2},\ \ \varepsilon = 0.01.$
11. $f(x) = (x^2 - 4)/(x - 2),\ \ a = 2,\ \ L = 4,\ \ \varepsilon = 0.01.$
12. $f(x) = (x - 9)/(\sqrt{x} - 3),\ \ a = 9,\ \ L = 6,\ \ \varepsilon = 0.1.$
13. $f(x) = (x^3 - 8)/(x - 2),\ \ a = 2,\ \ L = 12,\ \ \varepsilon = 0.5.$
14. $f(x) = (x^3 + 1)/(x + 1),\ \ a = -1,\ \ L = 3, \varepsilon = 0.1.$

*15. $f(x) = (x-1)(\sqrt[3]{x}-1),\ a = 1,\ L = 3,\ \varepsilon = 0.1.$
16. $f(x) = x\sin(1/x),\ a = 0,\ L = 0,\ \varepsilon = 0.01.$
*17. $f(x) = (\sin x)/x,\ a = 0,\ L = 1,\ \varepsilon = 0.1.$
18. Show that $\lim_{x\to 0}(\sin(1/x))$ does not exist.
*19. Show that $\lim_{x\to 0} x\log|x| = 0$.

2.2 Limits

The basic theorems of calculus depend for their proofs on certain standard theorems on limits. These theorems are usually stated without proof in a first course in calculus. In this section we fill the gap by providing proofs of the customary theorems on limits. These theorems are the basis for the formulas for the derivative of the sum, product, and quotient of functions as well as for the Chain rule.

Theorem 2.1 (Limit of a constant)
If c is a number and $f(x) = c$ for all x on $\mathbb{R}^1$, then for every real number a

$$\lim_{x\to a} f(x) = c.$$

Proof
In the definition of limit, we may choose $\delta = 1$ for every positive ε. Then

$$|f(x) - c| = |c - c| = 0 < \varepsilon \quad \text{for} \quad |x - a| < 1. \qquad \blacksquare$$

Theorem 2.2 (Obvious limit)
If $f(x) = x$ for all x on $\mathbb{R}^1$ and a is any real number, then

$$\lim_{x\to a} f(x) = a.$$

Proof
In the definition of limit, we may choose $\delta = \varepsilon$. Then, since $f(x) - f(a) = x - a$, we clearly have

$$|f(x) - f(a)| = |x - a| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \varepsilon. \qquad \blacksquare$$

Theorem 2.3 (Limit of a sum)
Suppose that

$$\lim_{x\to a} f_1(x) = L_1 \quad \textit{and} \quad \lim_{x\to a} f_2(x) = L_2.$$

Define $g(x) = f_1(x) + f_2(x)$. Then

$$\lim_{x\to a} g(x) = L_1 + L_2.$$

Proof
Let $\varepsilon > 0$ be given. Then, using the quantity $\varepsilon/2$, there are positive numbers δ_1 and δ_2 such that

$$|f_1(x) - L_1| < \frac{\varepsilon}{2} \quad \text{for all } x \text{ satisfying } 0 < |x-a| < \delta_1$$

and

$$|f_2(x) - L_2| < \frac{\varepsilon}{2} \quad \text{for all } x \text{ satisfying } 0 < |x-a| < \delta_2.$$

Define δ as the smaller of δ_1 and δ_2. Then

$$\begin{aligned} |g(x) - (L_1 + L_2)| &= |f_1(x) - L_1 + f_2(x) - L_2| \\ &\le |f_1(x) - L_1| + |f_2(x) - L_2|, \end{aligned}$$

and for $0 < |x-a| < \delta$, it follows that

$$|g(x) - (L_1 + L_2)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The result is established. ∎

Corollary
Suppose that $\lim_{x\to a} f_i(x) = L_i$, $i = 1, 2, \ldots, n$. *Define* $g(x) = \sum_{i=1}^{n} f_i(x)$. *Then*

$$\lim_{x\to a} g(x) = \sum_{i=1}^{n} L_i.$$

The corollary may be established by induction.

Theorem 2.4 (Limit of a product)
Suppose that

$$\lim_{x\to a} f_1(x) = L_1 \quad \textit{and} \quad \lim_{x\to a} f_2(x) = L_2.$$

Define $g(x) = f_1(x) \cdot f_2(x)$. *Then*

$$\lim_{x\to a} g(x) = L_1 L_2.$$

Proof
Suppose that $\varepsilon > 0$ is given. We wish to show that there is a $\delta > 0$ such that

$$|g(x) - L_1 L_2| < \varepsilon \quad \text{whenever} \quad 0 < |x-a| < \delta.$$

If ε_1 and ε_2 are positive numbers (their exact selection will be made later), there are positive numbers δ_1 and δ_2 such that

$$(2.1) \qquad |f_1(x) - L_1| < \varepsilon_1 \quad \text{whenever} \quad 0 < |x-a| < \delta_1$$

and

$$(2.2) \qquad |f_2(x) - L_2| < \varepsilon_2 \quad \text{whenever} \quad 0 < |x - a| < \delta_2.$$

We first show that $|f_1(x)| < |L_1| + \varepsilon_1$ for $0 < |x - a| < \delta_1$. To see this, we write

$$f_1(x) = f_1(x) - L_1 + L_1;$$

hence

$$|f_1(x)| \le |f_1(x) - L_1| + |L_1| < \varepsilon_1 + |L_1|.$$

Define $M = |L_1| + \varepsilon_1$. To establish the result of the theorem, we use the identity

$$g(x) - L_1L_2 = L_2(f_1(x) - L_1) + f_1(x)(f_2(x) - L_2).$$

Now we employ the triangle inequality for absolute values as well as inequalities (2.1) and (2.2) to get

$$\begin{aligned} |g(x) - L_1L_2| &\le |L_2| \cdot |f_1(x) - L_1| + |f_1(x)| \cdot |f_2(x) - L_2| \\ &\le |L_2| \cdot \varepsilon_1 + M \cdot \varepsilon_2. \end{aligned}$$

Select $\varepsilon_1 = (\varepsilon/2)L_2$ and $\varepsilon_2 = (\varepsilon/2)M$. The quantities δ_1 and δ_2 are those that correspond to the values of ε_1 and ε_2 respectively. Then with δ as the smaller of δ_1 and δ_2, it follows that*

$$|g(x) - L_1L_2| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

■

Corollary

Suppose that $f_i(x) \to L_i$ as $x \to a$ for $i = 1,2,\ldots,n$, and suppose that $g(x) = f_1(x)\cdots f_n(x)$. Then $\lim_{x\to a} g(x) = L_1L_2\cdots L_n$.

Theorem 2.5 (Limit of a composite function)

Suppose that f and g are functions on $\mathbb{R}^1$ to $\mathbb{R}^1$. Define the composite function $h(x) = f[g(x)]$. If f is continuous at L and if $g(x) \to L$ as $x \to a$, then

$$\lim_{x\to a} h(x) = f(L).$$

Proof

Since f is continuous at L, we know that for every $\varepsilon > 0$ there is a $\delta_1 > 0$ such that

$$|f(t) - f(L)| < \varepsilon \quad \text{whenever} \quad |t - L| < \delta_1.$$

From the fact that $g(x) \to L$ as $x \to a$, it follows that for every $\varepsilon' > 0$, there is a $\delta > 0$ such that

$$|g(x) - L| < \varepsilon' \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

*This proof assumes that $L_2 \ne 0$. A slight modification of the proof establishes the result if $L_2 = 0$.

In particular, we may select $\varepsilon' = \delta_1$. Then $f[g(x)]$ is defined, and

$$|f[g(x)] - f(L)| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

∎

Theorem 2.6 (Limit of a quotient)
Suppose that

$$\lim_{x\to a} f(x) = L \quad \textit{and} \quad \lim_{x\to a} g(x) = M.$$

Define $h(x) = f(x)/g(x)$. *If* $M \neq 0$, *then*

$$\lim_{x\to a} h(x) = L/M.$$

The proof of Theorem 2.6 is a direct consequence of Theorems 2.4 and 2.5. We leave the details to the reader.

Theorem 2.7 (Limit of inequalities)
Suppose that

$$\lim_{x\to a} f(x) = L \quad \textit{and} \quad \lim_{x\to a} g(x) = M.$$

If $f(x) \le g(x)$ *for all* x *in some interval about* a *(possibly excluding* a *itself), then*

$$L \le M.$$

Proof
We assume that $L > M$ and reach a contradiction. Let us define $\varepsilon = (L - M)/2$; then from the definition of limit there are positive numbers δ_1 and δ_2 such that

$$|f(x) - L| < \varepsilon \quad \text{for all } x \text{ satisfying} \quad 0 < |x - a| < \delta_1$$

and

$$|g(x) - M| < \varepsilon \quad \text{for all } x \text{ satisfying} \quad 0 < |x - a| < \delta_2.$$

We choose a positive number δ that is smaller than δ_1 and δ_2 and furthermore so small that $f(x) \le g(x)$ for $0 < |x - a| < \delta$. In this interval, we have

$$M - \varepsilon < g(x) < M + \varepsilon \quad \text{and} \quad L - \varepsilon < f(x) < L + \varepsilon.$$

Since $M + \varepsilon = L - \varepsilon$, it follows that

$$g(x) < M + \varepsilon = L - \varepsilon < f(x);$$

then $f(x) > g(x)$, a contradiction. ∎

Theorem 2.8 (Sandwiching theorem)
Suppose that f, g, *and* h *are functions defined on the interval* $0 < |x-a| < k$ *for some positive number* k. *If* $f(x) \leq g(x) \leq h(x)$ *on this interval and if*

$$\lim_{x \to a} f(x) = L, \qquad \lim_{x \to a} h(x) = L,$$

then $\lim_{x \to a} g(x) = L$.

Proof
Given any $\varepsilon > 0$, there are positive numbers δ_1 and δ_2 (which we take smaller than k) such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x-a| < \delta_1$$

and

$$|h(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x-a| < \delta_2.$$

In other words, $L - \varepsilon < f(x) < L + \varepsilon$ and $L - \varepsilon < h(x) < L + \varepsilon$ for all x such that $0 < |x-a| < \delta$, where δ is the smaller of δ_1 and δ_2. Therefore,

$$L - \varepsilon < f(x) \leq g(x) \leq h(x) < L + \varepsilon$$

in this interval. We conclude that $|g(x) - L| < \varepsilon$ for $0 < |x-a| < \delta$, and the result is established. ■

Problems

1. Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$. Show that $L = M$. *Hint*: Assume $L \neq M$ and reach a contradiction.
2. Use the Sandwiching theorem to show that $\lim_{x \to 0} x^n = 0$ for every positive integer $n \geq 3$ by selecting $f(x) \equiv 0$, $g(x) = x^n$, $h(x) = x^2$, $-1 < x < 1$.
3. Suppose f and h are continuous at a and $f(x) \leq g(x) \leq h(x)$ for $|x-a| < k$. If $f(a) = h(a)$, show that g is continuous at a.
4. Suppose that $\lim_{x \to a} f_i(x) = L_i$, $\lim_{x \to a} g_i(x) = M_i$, $i = 1, 2, \ldots, n$. Let $a_i, b_i, i = 1, 2, \ldots, n$, be any numbers. Under what conditions is it true that
$$\lim_{x \to a} \frac{a_1 f_1(x) + \ldots + a_n f_n(x)}{b_1 g_1(x) + \ldots + b_n g_n(x)} = \frac{a_1 L_1 + \ldots + a_n L_n}{b_1 M_1 + \ldots + b_n M_n} \quad ?$$
5. Establish the corollary to Theorem 2.3.
6. Prove the corollary to Theorem 2.4.
7. Let f and g be continuous functions from $\mathbb{R}^1$ to $\mathbb{R}^1$. Define
$$F(x) = \max[f(x), g(x)]$$
for each $x \in \mathbb{R}^1$. Show that F is continuous.

2.3 One-Sided Limits

The function $f : x \to \sqrt{x}$ is continuous for all $x > 0$, and since $\sqrt{0} = 0$, it is clear that $f(x) \to f(0)$ as x tends to 0 through positive values. Since f is not defined for negative values of x, the definition of continuity given in Section 2.1 is not fulfilled at $x = 0$. We now extend the definition of continuous function so that a function such as $\sqrt{x}$ will have the natural property of continuity at the endpoint of its domain. For this purpose we need the concept of a one-sided limit.

Definition
Suppose that f is a function from a domain D in $\mathbb{R}^1$ to $\mathbb{R}^1$. The function f **tends to** L **as** x **tends to** a **from the right** if (i) there is an open interval I in D that has a as its left endpoint and (ii) for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < x - a < \delta.$$

If f tends to L as x tends to a from the right, we write

$$f(x) \to L \quad \text{as} \quad x \to a^+,$$

and we denote the number L by

$$\lim_{x \to a^+} f(x).$$

A similar definition is employed for limits from the left. In this case the condition $0 < x - a < \delta$ is replaced by

$$0 < a - x < \delta,$$

and the notation $f(x) \to L$ as $x \to a^-$,

$$\lim_{x \to a^-} f(x) = L,$$

is used.

The condition $0 < |x - a| < \delta$ is equivalent to the two conditions $0 < x - a < \delta$ and $0 < a - x < \delta$. All the theorems on limits that were stated and proved in Section 2.2 used the inequality $0 < |x - a| < \delta$. As a result, all the theorems on limits imply corresponding theorems for limits from the left and limits from the right.

The definition of one-sided limit leads to the definition of one-sided continuity.

Definitions
A function f is **continuous on the right at** a if a is in the domain of f and $f(x) \to f(a)$ as $x \to a^+$. The function f is **continuous on the left at** a if a is in the domain of f and $f(x) \to f(a)$ as $x \to a^-$.

If the domain of a function f is a finite interval, say $a \le x \le b$, then limits and continuity at the endpoints are of the one-sided variety. For example, the function $f : x \to x^2 - 3x + 5$ defined on the interval $2 \le x \le 4$ is continuous on the right at $x = 2$ and continuous on the left at $x = 4$. The following is the general definition of continuity for functions from a set in $\mathbb{R}^1$ to $\mathbb{R}^1$.

Definitions

Let f be a function from a domain in $\mathbb{R}^1$ to $\mathbb{R}^1$. The function f is **continuous at** a **with respect to** D if (i) a is in D, and (ii) for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(2.3) \qquad |f(x) - f(a)| < \varepsilon \quad \text{whenever} \quad x \in D \quad \text{and} \quad |x - a| < \delta.$$

A function f is **continuous on** D if it is continuous with respect to D at every point of D.

The phrase "with respect to D" is usually omitted in the definition of continuity since the context will always make the situation clear.

If a is a point of an open interval I contained in D, then the above definition of continuity coincides with that given in Section 2.1.

Definition

A point a is an **isolated point** of a set D in $\mathbb{R}^1$ if there is an open interval I such that the set $I \cap D$ consists of the single point $\{a\}$.

A function f from a domain D in $\mathbb{R}^1$ to $\mathbb{R}^1$ is continuous at every isolated point of D. To see this, let a be an isolated point of D and suppose I is such that $\{a\} = I \cap D$. We choose $\delta > 0$ so small that the condition $|x - a| < \delta$ implies that x is in I. Then condition (ii) in the definition of continuity is always satisfied, since (ii) holds when $x = a$, the only point in question.

The theorems of Section 2.2 carry over almost without change to the case of one-sided limits. As an illustration we state a one-sided version of Theorem 2.5, the limit of a composite function.

Theorem 2.9

Suppose that f and g are functions on $\mathbb{R}^1$ to $\mathbb{R}^1$. If f is continuous at L and if $g(x) \to L$ as $x \to a^+$, then

$$\lim_{x \to a^+} f[g(x)] = f(L).$$

A similar statement holds if $g(x) \to L$ as $x \to a^-$.

The proof of Theorem 2.9 follows precisely the line of proof of Theorem 2.5. Although we always require $x > a$ when $g(x) \to L$, it is not true that $g(x)$ remains always larger than or always smaller than L. Hence it is necessary to assume that f is continuous at L and not merely continuous on one side.

With the aid of the general definition of continuity it is possible to show that the function $g : x \to \sqrt[n]{x}$, where n is a positive integer, is a continuous function on its domain.

Theorem 2.10
For n an even positive integer, the function $g{:}x \to \sqrt[n]{x}$ is continuous for x on $[0,\infty)$. For n an odd positive integer, g is continuous for x on $(-\infty,\infty)$.

Proof
For $x \geq 0$ and any $\varepsilon > 0$, it follows that

$$|\sqrt[n]{x} - 0| < \varepsilon \quad \text{whenever} \quad |x - 0| < \varepsilon^n.$$

Therefore g is continuous on the right at 0 for every n. If n is odd, then $\sqrt[n]{-x} = -\sqrt[n]{x}$, and g is also continuous on the left at 0. ■

It is sometimes easier to determine one-sided limits than two-sided limits. The next theorem and corollary show that one-sided limits can be used as a tool for finding ordinary limits.

Theorem 2.11 (On one- and two-sided limits)
Suppose that f is a function on $\mathbb{R}^1$ to $\mathbb{R}^1$. Then

$$\lim_{x\to a} f(x) = L \ \text{ if and only if } \ \lim_{x\to a^+} f(x) = L \ \text{ and } \ \lim_{x\to a^-} f(x) = L.$$

Proof
(a) Suppose $f(x) \to L$ as $x \to a$. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that $f(x)$ is defined for $0 < |x - a| < \delta$ and

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta. \tag{2.4}$$

For this value of δ we have

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < x - a < \delta.$$

This last statement implies that $f(x) \to L$ as $x \to a^+$. Similarly, since the condition $0 < |x - a| < \delta$ is implied by the inequality $0 < x - a < \delta$, it follows from inequality (2.4) that $|f(x) - L| < \varepsilon$ whenever $0 < a - x < \delta$. Hence $f(x) \to L$ as $x \to a^-$.

(b) Now assume that both one-sided limits exist. Given any $\varepsilon > 0$, there are numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that $f(x)$ is defined for $0 < x - a < \delta_1$ and $0 < a - x < \delta_2$; moreover, the inequalities

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < x - a < \delta_1,$$
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < a - x < \delta_2$$

hold. If δ is the smaller of δ_1 and δ_2, then

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

■

Corollary

Suppose f is a function on $\mathbb{R}^1$ to $\mathbb{R}^1$. Then f is continuous at a if and only if it is continuous on the left at a and continuous on the right at a.

Problems

In Problems 1 through 12, in each case determine whether or not the function f is continuous at the given value of a. If it is not continuous, decide whether or not the function is continuous on the left or on the right. State reasons for each step in the argument.

1. $f(x) = \begin{cases} x - 4, & -1 < x \le 2, \\ x^2 - 6, & 2 < x < 5, \end{cases} \quad a = 2.$
2. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x^4 - 1}, & 1 < x < 2, \\ x^2 + 3x - 2, & 2 \le x < 5, \end{cases} \quad a = 2.$
3. $f : x \to x/|x|, \; a = 0.$
4. $f : x \to |x - 1|, \; a = 1.$
5. $f : x \to x(1 + (1/x^2))^{\frac{1}{2}}, \; a = 0.$
6. $f(x) = ((x^2 - 27)/(x^2 + 2x + 1))^{\frac{1}{3}}, \; a = 3.$
7. $f(x) = \begin{cases} \left(\dfrac{2x + 5}{3x - 2}\right)^{\frac{1}{2}}, & 1 < x < 2, \\ 2x - 1, & 2 \le x < 4, \end{cases} \quad a = 2.$
8. $f(x) = \begin{cases} \left(\dfrac{x^2 - 4}{x^2 - 3x + 2}\right)^{\frac{1}{2}}, & 1 < x < 2 \\ x^2/2, & 2 \le x < 5, \end{cases} \quad a = 2.$
9. $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \ne 2, \\ 4 & x = 2, \end{cases} \quad a = 2.$
10. $f(x) = \begin{cases} \left(\dfrac{x^2 - 9}{x^2 + 1}\right)^{\frac{1}{2}}, & 3 \le x < 4, \\ x^2 - 9, & 2 < x < 3, \end{cases} \quad a = 3.$
11. $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & x \ne 0 \\ 0 & x = 0, \end{cases} \quad a = 0.$

12. $f(x) = \begin{cases} \cos\left(\dfrac{1}{x}\right), & x \neq 0 \\ 0 & x = 0, \end{cases} \qquad a = 0.$
13. Given $f(x) = \tan x$, $-\pi/2 < x < \pi/2$, $g(x) = (\pi/2) - x$, what conclusion can we draw about
$$\lim_{x \to a^+} f[g(x)]$$
when $a = \pi/4$? Do the same problem when $a = \pi/2$ and when $a = 0$.
14. Suppose $f(x) = \frac{1}{x}$ with domain D the set $\{1, 2, 3, \ldots, n, \ldots\}$. For what values of x is f continuous?

2.4 Limits at Infinity; Infinite Limits

Consider the function
$$f : x \to \frac{x}{x+1}, \qquad x \neq -1,$$
whose graph is shown in Figure 2.4. Intuitively, it is clear that $f(x)$ tends to 1 as x tends to infinity; this statement holds when x goes to infinity through increasing positive values (i.e., x tends to infinity to the right) or when x goes to infinity through decreasing negative values (x tends to infinity to the left). More precisely, the symbol
$$x \to \infty$$
means that x increases without bound through positive values and x decreases without bound through negative values. If x tends to infinity only through increasing positive values, we write
$$x \to +\infty,$$
while if x tends to infinity only through decreasing negative values, we write
$$x \to -\infty.$$
The above conventions are used to define a *limit at infinity.*

Definitions

We say that $f(x) \to L$ **as** $x \to \infty$ if for each $\varepsilon > 0$ there is a number $A > 0$ such that $|f(x) - L| < \varepsilon$ for all x satisfying $|x| > A$.

We say that $f(x) \to L$ **as** $x \to +\infty$ if for each $\varepsilon > 0$ there is an $A > 0$ such that $|f(x) - L| < \varepsilon$ for all x satisfying $x > A$.

We say that $f(x) \to L$ **as** $x \to -\infty$ if for each $\varepsilon > 0$ there is an $A > 0$ such that $|f(x) - L| < \varepsilon$ for all x satisfying $x < -A$.

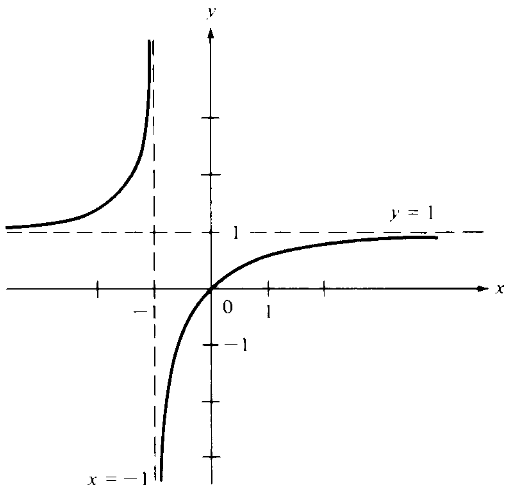

Figure 2.4 $y = x/(x+1)$.

Theorem 2.12 (Obvious limit)
If $f(x) = 1/x$ for all $x \neq 0$, then

$$\lim_{x\to\infty} f(x) = 0, \qquad \lim_{x\to+\infty} f(x) = 0, \qquad \lim_{x\to-\infty} f(x) = 0.$$

The next theorem extends Theorem 2.5 on composite functions.

Theorem 2.13 (Limit of a composite function)
Suppose that f and g are functions on $\mathbb{R}^1$ to $\mathbb{R}^1$. If f is continuous at L and $g(x) \to L$ as $x \to +\infty$, then

$$\lim_{x\to+\infty} f[g(x)] = f(L).$$

The proofs of Theorems 2.12 and 2.13 follow the pattern of the proofs of Theorems 2.2 and 2.5, respectively.

Referring to Figure 2.4 and the function $f : x \to x/(x+1)$, $x \neq -1$, we see that f increases without bound as x tends to –1 from the left. Also, f decreases without bound as x tends to –1 from the right. We say that f has an *infinite limit* as x tends to –1; a more precise statement is given in the next definition.

Definitions
A function f **becomes infinite as** $x \to a$ if for each number $A > 0$ there is a number $\delta > 0$ such that $|f(x)| > A$ for all x satisfying $0 < |x - a| < \delta$. We write $f(x) \to \infty$ as $x \to a$ for this limit. We also use the notation

$$\lim_{x\to a} f(x) = \infty,$$

although we must remember that *infinity is not a number and the usual rules of algebra do not apply.*

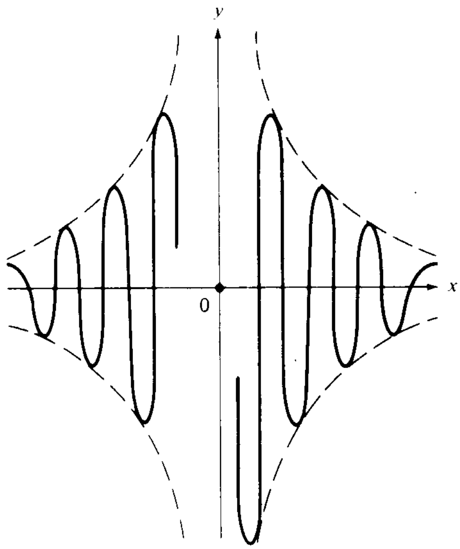

Figure 2.5 $y = (1/x)\cos(1/x)$.

If f is defined only on one side of a number a rather than in a deleted interval containing a, we may define a one-sided infinite limit. A function f **becomes infinite as** $x \to a^+$ if for each number $A > 0$ there is a number $\delta > 0$ such that $|f(x)| > A$ for all x satisfying $0 < x - a < \delta$. An *infinite limit from the left* is defined similarly. Various other possibilities may occur. For example, the function $f : x \to 1/(x-1)^2$ has the property that $f \to +\infty$ as $x \to 1$. In such a case f has a *positive infinite limit as* $x \to 1$. Similarly, a function may have a *negative infinite limit*, and it may have both positive and negative one-sided infinite limits.

Example 1
Given the function

$$f : x \to \frac{1}{x} \cos \frac{1}{x} ,$$

decide whether or not f tends to a limit as x tends to 0.

Solution. For the values $x_n = 1/(2n\pi)$, $n = 1, 2, \ldots$, we have $f(x_n) = 2n\pi$, while for $x'_n = 1/((\pi/2)+2n\pi)$, we have $f(x'_n) = 0$. Hence there are certain values, x_n, tending to zero as $n \to \infty$ at which f grows without bound, while on other values, x'_n, also tending to zero, the function f always has the value zero. Therefore, f has no limit as $x \to 0$. See Figure 2.5. ■

Example 2

Given the function

$$f : x \to \frac{\sqrt{x^2 + 2x + 4}}{2x + 3} ,$$

evaluate $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Give a reason for each step.

Solution. For $x > 0$, division of numerator and denominator by x yields

$$f : x \to \frac{\sqrt{x^2 + 2x + 4}}{2x + 3} = \frac{\sqrt{1 + 2\left(\frac{1}{x}\right) + 4\left(\frac{1}{x^2}\right)}}{2 + 3\left(\frac{1}{x}\right)} .$$

We now employ the theorems on limit of a constant, obvious limit, and the limits of a sum and product to obtain

$$\begin{aligned} &\lim_{x\to+\infty} \left[1 + 2 \cdot \frac{1}{x} + 4\left(\frac{1}{x^2}\right)\right] \\ &= \lim_{x\to+\infty} 1 + \lim_{x\to+\infty} 2 \cdot \lim_{x\to+\infty} \frac{1}{x} + \lim_{x\to+\infty} 4 \cdot \lim_{x\to+\infty} \left(\frac{1}{x^2}\right) \\ &= 1 + 2 \cdot 0 + 4 \cdot 0 \\ &= 1; \end{aligned}$$

also,

$$\lim_{x\to+\infty} \left(2 + 3\left(\frac{1}{x}\right)\right) = 2.$$

Now, using the theorem on the limit of composite functions, we find

$$\lim_{x\to+\infty} \sqrt{1 + 2\left(\frac{1}{x}\right) + 4\left(\frac{1}{x^2}\right)} = \sqrt{1} = 1.$$

Therefore, $\lim_{x\to+\infty} f(x) = \frac{1}{2}$ by the theorem on the limit of a quotient. For $x < 0$, care must be exercised in rewriting the algebraic expression for f. Division by x in the numerator and denominator yields

$$\begin{aligned} f(x) = \frac{\sqrt{x^2 + 2x + 4}}{2x + 3} &= \frac{-\sqrt{\frac{1}{x^2}}}{\frac{1}{x}} \; \frac{\sqrt{x^2 + 2x + 4}}{2x + 3} \\ &= -\frac{\sqrt{1 + 2\left(\frac{1}{x}\right) + 4\left(\frac{1}{x^2}\right)}}{2 + 3\left(\frac{1}{x}\right)} . \end{aligned}$$

Now, proceeding as in the case for positive values of x, we obtain

$$\lim_{x\to-\infty} f(x) = -\frac{1}{2} .$$

■

The theorems on the limits of sums, products, and quotients of functions require special treatment when one or more of the functions has an infinite limit or when the limit of a function appearing in a denominator is zero.

In the case of the limit of the sum of two functions, the appropriate theorem states that if

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = \infty,$$

then

$$\lim_{x\to a}[f(x) + g(x)] = \infty.$$

If both f and g tend to infinity as x tends to a, no conclusion can be drawn without a more detailed examination of the functions f and g. We must bear in mind that ∞ cannot be treated as a number. However, we do have the "rules"

$$+\infty + (+\infty) = +\infty \qquad \text{and} \qquad -\infty + (-\infty) = -\infty.$$

In the case of the limit of a product, if $f(x) \to L$, $L \neq 0$, and $g(x) \to \infty$, as $x \to a$, then $f(x)g(x) \to \infty$ as $x \to a$. However, if $L = 0$, no conclusion can be drawn without a closer investigation of the functions f and g.

Problems

In each of Problems 1 through 10 evaluate the limit or conclude that the function tends to ∞, $+\infty$, or $-\infty$.

1. $\lim_{x\to\infty}(x^2 - 2x + 3)/(x^3 + 4)$.
2. $\lim_{x\to\infty}(2x^2 + 3x + 4)/(x^2 - 2x + 3)$.
3. $\lim_{x\to\infty}(x^4 - 2x^2 + 6)/(x^2 + 7)$.
4. $\lim_{x\to+\infty}(x - \sqrt{x^2 - a^2})$.
5. $\lim_{x\to-\infty}(x - \sqrt{x^2 - a^2})$.
6. $\lim_{x\to 1^+}(x - 1)/\sqrt{x^2 - 1}$.
7. $\lim_{x\to 1^-}\sqrt{1 - x^2}/(1 - x)$.
8. $\lim_{x\to+\infty}(x^2 + 1)/x^{\frac{3}{2}}$.
9. $\lim_{x\to 2^-}\sqrt{4 - x^2}/\sqrt{6 - 5x + x^2}$.
10. $\lim_{x\to+\infty}(\sqrt{x^2 + 2x} - x)$.

11. Suppose $f(x) \to +\infty$ and $g(x) \to -\infty$ as $x \to +\infty$. Find examples of functions f and g with these properties and such that
    (a) $\lim_{x\to+\infty}[f(x) + g(x)] = +\infty$.
    (b) $\lim_{x\to+\infty}[f(x) + g(x)] = -\infty$.
    (c) $\lim_{x\to+\infty}[f(x) + g(x)] = A$, A an arbitrary real number.
12. Find the values of p, if any, for which the following limits exist.
    (a) $\lim_{x\to 0^+} x^p \sin(1/x)$.
    (b) $\lim_{x\to+\infty} x^p \sin(1/x)$.
    (c) $\lim_{x\to-\infty} |x|^p \sin(1/x)$.

2.5 Limits of Sequences

An infinite sequence may or may not tend to a limit. When an infinite sequence does tend to a limit, the rules of operation are similar to those given in Section 2.2 for limits of functions.

Definition
Given the infinite sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$, we say that $\{x_n\}$ **tends to** L **as** n **tends to infinity** if and only if for each $\varepsilon > 0$ there is a natural number N such that $|x_n - L| < \varepsilon$ for all $n > N$. We also write $x_n \to L$ as $n \to \infty$, and we say that the sequence $\{x_n\}$ has L as a limit.* The notation

$$\lim_{n\to\infty} x_n = L$$

will be used. If the sequence $\{x_n\}$ increases without bound as n tends to infinity, the symbol $x_n \to +\infty$ as $n \to \infty$ is used; similarly if it decreases without bound, we write $x_n \to -\infty$ as $n \to \infty$.

The theorems on limit of a constant, limit of equal functions, limit of a product, limit of a quotient, limit of inequalities, and the Sandwiching theorem have corresponding statements for sequences.

A variation of the theorem on composite functions, Theorem 2.5, leads to the next result, which we state without proof.

Theorem 2.13
Suppose that f is continuous at a and that $x_n \to a$ as $n \to \infty$. Then there is an integer N such that $f(x_n)$ is defined for all integers $n > N$; furthermore, $f(x_n) \to f(a)$ as $n \to \infty$.

EXAMPLE
Given the function

$$f : x \to \begin{cases} \dfrac{1}{x} \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

and the sequences

$$x_n = \frac{1}{n\pi}, \qquad n = 1, 2, \ldots,$$
$$y_n = \frac{1}{(2n + \frac{1}{2})\pi}, \quad n = 1, 2, \ldots,$$
$$z_n = \frac{1}{(2n + \frac{3}{2})\pi}, \quad n = 1, 2, \ldots,$$

*Strictly speaking, we should write $n \to +\infty$ instead of $n \to \infty$. However, for the natural numbers it is a long-established custom to write ∞ instead of $+\infty$; we shall continue to observe this custom.

find the limits of $f(x_n)$, $f(y_n)$, and $f(z_n)$ as $n \to \infty$.

Solution. Since $\sin n\pi = 0$, $n = 1, 2, \ldots$, we have $f(x_n) \to 0$ as $n \to \infty$. Also,

$$f(y_n) = (2n + \frac{1}{2})\pi \sin(2n + \frac{1}{2})\pi = (2n + \frac{1}{2})\pi \to +\infty \quad \text{as} \quad n \to \infty,$$

and

$$f(z_n) = (2n + \frac{3}{2})\pi \sin(2n + \frac{3}{2})\pi = -(2n + \frac{3}{2})\pi \to -\infty \quad \text{as} \quad n \to \infty.$$

Observing that x_n, y_n, and z_n all tend to 0 as $n \to \infty$, we note that $f(x)$ cannot tend to a limit as $x \to 0$. ■

The next axiom, when added to those of Chapter 1, serves to describe the real number system uniquely.

Axiom C (Axiom of continuity)
Suppose that an infinite sequence $x_1, x_2, \ldots, x_n, \ldots$ is such that $x_{n+1} \geq x_n$ for all n, and there is a number M such that $x_n \leq M$ for all n. Then there is a number $L \leq M$ such that $x_n \to L$ as $n \to \infty$ and $x_n \leq L$ for all n.

The situation is shown in Figure 2.6. If we plot the numbers x_n on a horizontal line, the corresponding points move steadily to the right as n increases. It is geometrically evident that they must cluster at some value such as L that cannot exceed the number M.

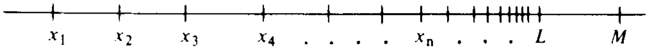

Figure 2.6 x_n tends to L as $n \to \infty$.

By means of Axiom C we establish the following simple result.

Theorem 2.14

(a) *There is no real number M that exceeds all positive integers.*
(b) *If $x_n \geq n$ for every positive integer n, then $x_n \to +\infty$ as $n \to \infty$.*
(c) $\lim_{n\to\infty} (1/n) = 0$.

Proof
(a) Let $x_n = n$ and suppose there is a number M such that $M \geq x_n$ for every n. By Axiom C there is a number L such that $x_n \to L$ as $n \to \infty$ and $x_n \leq L$ for all n. Choose $\varepsilon = 1$ in the definition of limit. Then for a sufficiently large integer N, it follows that $L - 1 < x_n < L + 1$ for all $n > N$. But $x_{N+1} = N + 1$, and so $L - 1 < N + 1 < L + 1$. Hence $N + 2 = x_{N+2} > L$, contrary to the statement $x_n \leq L$ for all n.

(b) Let M be any positive number. From part (a) there is an integer N such that $N > M$; hence $x_n > M$ for all $n \geq N$.
(c) Let $\varepsilon > 0$ be given. There is an integer N such that $N > 1/\varepsilon$. Then if $n > N$, clearly $1/n < \varepsilon$. The result follows. ■

Problems

1. Given the function $f : x \to x \cos x$
   (a) Find a sequence of numbers $\{x_n\}$ such that $x_n \to +\infty$ and $f(x_n) \to 0$.
   (b) Find a sequence of numbers $\{y_n\}$ such that $y_n \to +\infty$ and $f(y_n) \to +\infty$.
   (c) Find a sequence of numbers $\{z_n\}$ such that $z_n \to +\infty$ and $f(z_n) \to -\infty$.
2. Prove that if $x_n \to a$ and $y_n \to b$ as $n \to \infty$, then $x_n + y_n \to a + b$ as $n \to \infty$.
3. If a is any number greater than 1 and n is a positive integer greater than 1, show that $a^n > 1 + n(a - 1)$.
4. If $a > 1$, show that $a^n \to +\infty$ as $n \to \infty$.
5. Prove Theorem 2.13.
6. If $-1 < a < 1$, show that $a^n \to 0$ as $n \to \infty$.
7. Suppose that $-1 < a < 1$. Define

$$s_n = b(1 + a + a^2 + \ldots + a^{n-1}).$$

   Show that $\lim_{n\to\infty} s_n = b/(1 - a)$.

In Problems 8 through 12 evaluate each of the limits or conclude that the given expression tends to ∞, $+\infty$, or $-\infty$.

8. $\lim_{n\to\infty}(n^2 + 2n - 1)/(n^2 - 3n + 2)$.
9. $\lim_{n\to\infty}(n^3 + 4n - 1)/(2n^2 + n + 5)$.
10. $\lim_{n\to\infty}(n + (1/n))/(2n^2 - 3n)$.
11. $\lim_{n\to\infty} \sqrt{n^3 + 2n - 1}/\sqrt[3]{n^2 + 4n - 2}$.
12. $\lim_{n\to\infty}[3 + \sin(n)]n$.
13. Suppose that $x_n \leq y_n \leq z_n$ for each n and suppose that $x_n \to L$ and $z_n \to L$ as $n \to \infty$. Then $y_n \to L$ as $n \to \infty$.

3 CHAPTER Basic Properties of Functions on $\mathbb{R}^1$

3.1 The Intermediate-Value Theorem

The proofs of many theorems of calculus require a knowledge of the basic properties of continuous functions. In this section we establish the Intermediate-value theorem, an essential tool used in the proofs of the Fundamental theorem of calculus and the Mean-value theorem.

Theorem 3.1 (Nested intervals theorem)
Suppose that

$$I_n = \{x : a_n \le x \le b_n\}, \qquad n = 1, 2, \dots,$$

is a sequence of closed intervals such that $I_{n+1} \subset I_n$ *for each* n. *If* $\lim_{n\to\infty}(b_n - a_n) = 0$, *then there is one and only one number* x_0 *that is in every* I_n.

Proof
By hypothesis, we have $a_n \le a_{n+1}$ and $b_n \le b_{n+1}$. Since $a_n < b_n$ for every n, the sequence $\{a_n\}$ is nondecreasing and bounded from above by b_1 (see Figure 3.1). Similarly, the sequence b_n is nonincreasing and bounded from below by a_1. Using Axiom C, we conclude that there are numbers x_0 and x_0' such that $a_n \to x_0$, $a_n \le x_0$, and $b_n \to x_0'$, $b_n \ge x_0'$, as $n \to \infty$. Using the fact that $b_n - a_n \to 0$, $n \to \infty$, we find that $x_0 = x_0'$ and

$$a_n \le x_0 \le b_n$$

for every n. Thus x_0 is in every I_n. ∎

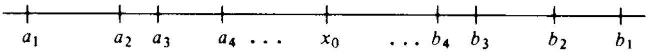

Figure 3.1 Nested intervals.

The hypothesis that each I_n is *closed* is essential. The sequence of half-open intervals

$$I_n = \{x : 0 < x \leq \frac{1}{n}\}$$

has the property that $I_{n+1} \subset I_n$ for every n. Since no I_n contains 0, any number x_0 in all the I_n must be positive. But then by choosing $N > 1/x_0$, we see that x_0 cannot belong to $I_N = \{x : 0 < x \leq 1/N\}$. Although the intervals I_n are nested, they have no point in common.

Before proving the main theorem, we prove the following result, which we shall use often in this chapter.

Theorem 3.2
Suppose f is continuous on $[a,b]$, $x_n \in [a,b]$ for each n, and $x_n \to x_0$. Then $x_0 \in [a,b]$ and $f(x_n) \to f(x_0)$.

Proof
Since $x_n \geq a$ for each n, it follows from the theorem on the limit of inequalities for sequences that $x_0 \geq a$. Similarly, $x_0 \leq b$ so $x_0 \in [a, b]$. If $a < x_0 < b$, then $f(x_n) \to f(x_0)$ on account of the composite function theorem (Theorem 2.5). The same conclusion holds if $x_0 = a$ or b. A detailed proof of the last sentence is left to the reader. ■

We now establish the main theorem of this section.

Theorem 3.3 (Intermediate-value theorem)
Suppose f is continuous on $[a,b]$, $c \in \mathbb{R}^1$, $f(a) < c$, and $f(b) > c$. Then there is at least one number x_0 on $[a,b]$ such that $f(x_0) = c$.

Note that there may be more than one such x_0, as indicated in Figure 3.2.

Proof
Define $a_1 = a$ and $b_1 = b$. Then observe that $f((a_1 + b_1)/2)$ is either equal to c, greater than c, or less than c. If it equals c, choose $x_0 = (a_1 + b_1)/2$ and the result is proved. If $f((a_1 + b_1)/2) > c$, then define $a_2 = a_1$ and $b_2 = (a_1 + b_1)/2$. If $f((a_1 + b_1)/2) < c$, then define $a_2 = ((a_1 + b_1)/2)$ and $b_2 = b_1$.

In each of the last two cases we have $f(a_2) < c$ and $f(b_2) > c$. Again compute $f((a_2 + b_2)/2)$. If this value equals c, the result is proved. If

$f((a_2 + b_2)/2) > c$ set $a_3 = a_2$ and $b_3 = (a_2 + b_2)/2$. If $f((a_2 + b_2)/2) < c$, set $a_3 = (a_2 + b_2)/2$ and $b_3 = b_2$.

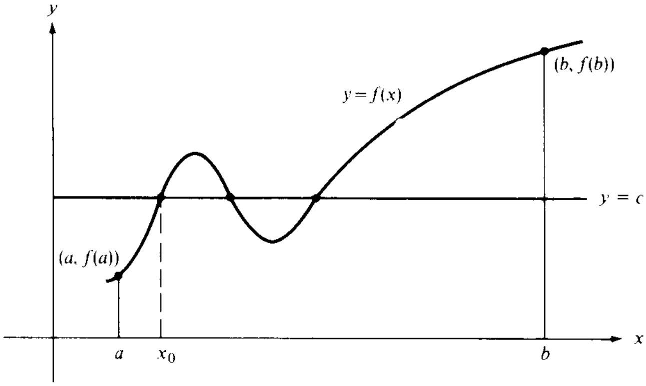

Figure 3.2 Illustrating the Intermediate-value theorem.

Continuing in this way, either we find a solution in a finite number of steps or we find a sequence $\{[a_n, b_n]\}$ of closed intervals each of which is one of the two halves of the preceding one, and for which we have

$$\begin{aligned} &b_n - a_n = (b_1 - a_1)/2^{n-1}, \\ &f(a_n) < c, \quad f(b_n) > c \qquad \text{for each } n. \end{aligned} \tag{3.1}$$

From the Nested intervals theorem it follows that there is a unique point x_0 in all these intervals and that

$$\lim_{n\to\infty} a_n = x_0 \qquad \text{and} \qquad \lim_{n\to\infty} b_n = x_0.$$

From Theorem 3.2, we conclude that

$$f(a_n) \to f(x_0) \qquad \text{and} \qquad f(b_n) \to f(x_0).$$

From inequalities (3.1) and the limit of inequalities it follows that $f(x_0) \le c$ and $f(x_0) \ge c$, so that $f(x_0) = c$. ■

Problems

1. Given the function

$$f : x \to \begin{cases} 1 & \text{if } x \text{ is rational}, \\ 0 & \text{if } x \text{ is irrational}. \end{cases}$$

   (a) Show that f is not continuous at any x_0.
   (b) Let g be a function with domain all of $\mathbb{R}^1$. If $g(x) = 1$ if x is rational and if g is continuous for all x, show that $g(x) = 1$ for $x \in \mathbb{R}^1$.

2. Let $f : x \to a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$ be a polynomial function of odd degree.
   (a) Use the Intermediate-value theorem to show that the equation $f(x) = 0$ has at least one root.
   (b) Show that the range of f is $\mathbb{R}^1$.
3. Let $f : x \to a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$ be a polynomial function of even degree. If $a_m a_0 < 0$, show that the equation $f(x) = 0$ has at least two real roots.
4. Prove the last sentence of Theorem 3.2.
5. A function f defined on an interval $I = \{x : a \leq x \leq b\}$ is called *increasing* if $f(x_1) > f(x_2)$ whenever $x_1 > x_2$ where $x_1, x_2 \in I$. Suppose that f has the *intermediate-value property*, that is, for each number c between $f(a)$ and $f(b)$ there is an $x_0 \in I$ such that $f(x_0) = c$. Show that a function f that is increasing and has the intermediate-value property must be continuous.
6. Give an example of a function that is continuous on $[0, 1]$ except at $x = \frac{1}{2}$ and for which the intermediate-value theorem does not hold.
7. Give an example of a nonconstant continuous function for which Theorem 3.3 holds and such that there are infinitely many values x_0 where $f(x_0) = c$.

3.2 Least Upper Bound; Greatest Lower Bound

In this section we prove an important principle about real numbers that is often used as an axiom in place of Axiom C.

Definitions
A set S of real numbers has the **upper bound** M if $x \leq M$ for every number x in S; we also say that the set S is *bounded above* by M. The set S has the **lower bound** m if $x \geq m$ for every number x in S; we also say that S is *bounded below* by m. A set S is **bounded** if S has an upper and a lower bound. Suppose that f is a function on $\mathbb{R}^1$ whose domain D contains the set S. We denote by $f|_S$ the **restriction of f to the set** S; that is, $f|_S$ has domain S and $f|_S(x) = f(x)$ for all x in S. A function f is bounded above, bounded below, or bounded if the set R consisting of the range of f satisfies the corresponding condition.

Suppose we define

$$f(x) = \begin{cases} \dfrac{1}{2x}, & 0 < x \leq 1, \\ 1, & x = 0. \end{cases}$$

Then f is not bounded on $S = \{x : 0 \le x \le 1\}$. See Figure 3.3.

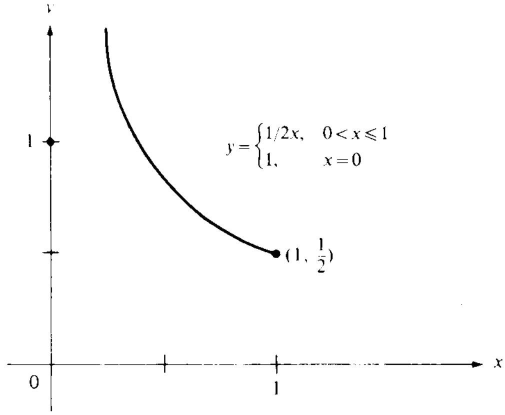

Figure 3.3

We now prove the fundamental principle on upper and lower bounds (see Figure 3.4).

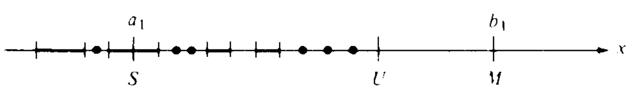

Figure 3.4 U is the least upper bound for S.

Theorem 3.4
If a nonempty set $S \subset \mathbb{R}^1$ has an upper bound M, then it has a least upper bound U; that is, there is a unique number U that is an upper bound for S but is such that no number $U' < U$ is an upper bound for S. If a nonempty set has a lower bound m, it has a greatest lower bound L.

Proof
The second statement follows by applying the first to the set $S' = \{x : -x \in S\}$. We prove the first statement using the Nested intervals theorem. If $M \in S$, then we may take $U = M$, since in this case every x in S is less than or equal to U, and if $U' < U$, then U' is not an upper bound, since $U \in S$ and $U > U'$. If M is not in S, let $b_1 = M$ and choose a_1 as any point of S. Now, either $(a_1 + b_1)/2$ is greater than every x in S, or there is some x in S greater than or equal to $(a_1 + b_1)/2$. In the first case, if we define $a_2 = a_1$ and $b_2 = (a_1 + b_1)/2$, then $a_2 \in S$ and b_2 is greater than every x in S. In the second case, choose for a_2 one of the numbers in S that is greater than or equal to $(a_1 + b_1)/2$ and set $b_2 = b_1$; then we again have $a_2 \in S$ and b_2 greater than every x in S. Continuing in this way, we define

an infinite sequence $\{[a_n, b_n]\}$ of closed intervals such that

$$(3.2) \qquad [a_{n+1}, b_{n+1}] \subset [a_n, b_n] \qquad \text{and} \qquad (b_{n+1} - a_{n+1}) \le \tfrac{1}{2}(b_n - a_n)$$

for each n, and so for each n,

$$(3.3) \quad b_n - a_n \le (b_1 - a_1)/2^{n-1}, \qquad a_n \in S, \qquad b_n > \text{every number in } S.$$

From the Nested intervals theorem, it follows that there is a unique number U in all these intervals and

$$\lim_{n\to\infty} b_n = U, \qquad \lim_{n\to\infty} a_n = U.$$

Let x be any number in S. Then $x < b_n$ for each n, so that $x \le U$ by the limit of inequalities. Thus U is an upper bound for S. But now let $U' < U$ and let $\varepsilon = U - U'$. Then, since $a_n \to U$, it follows that there is an N such that $U' = U - \varepsilon < a_n \le U$ for all $n > N$. But since all the $a_n \in S$, it is clear that U' is not an upper bound. Therefore, U is unique. ■

Definitions
The number U in Theorem 3.4, the **least upper bound**, is also called the **supremum** of S and is denoted by

$$\text{l.u.b. } S \qquad \text{or} \qquad \sup S.$$

The number L of Theorem 3.4, the **greatest lower bound**, is also called the **infimum** of S and is denoted by

$$\text{g.l.b. } S \qquad \text{or} \qquad \inf S.$$

Corollary
If x_0 is the largest number in S, that is, if $x_0 \in S$ and x_0 is larger than every other number in S, then $x_0 = \sup S$. If S is not empty and $U = \sup S$, with U not in S, then there is a sequence $\{x_n\}$ such that each x_n is in S and $x_n \to U$. Also, if $\varepsilon > 0$ is given, there is an x in S such that $x > U - \varepsilon$. Corresponding results hold for $\inf S$.

These results follow from the definitions and from the proof of Theorem 3.4.

With the help of this corollary it is possible to show that the Axiom of Continuity is a consequence of Theorem 3.4.

Definitions
Let f have an interval I of $\mathbb{R}^1$ as its domain and a set in $\mathbb{R}^1$ as its range. We say that f is **increasing on** I if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$. The function f is **nondecreasing on** I if $f(x_2) \ge f(x_1)$ whenever $x_2 > x_1$. The function f is **decreasing on** I if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$. The function f is **nonincreasing on** I if $f(x_2) \le f(x_1)$ whenever $x_2 > x_1$. A

function that has any one of these four properties is called **monotone on** I.

Monotone functions are not necessarily continuous, as the following *step function* exhibits:

$$f : x \to n, \qquad n-1 \le x < n, \qquad n = 1, 2, \ldots .$$

See Figure 3.5. Also, monotone functions may not be bounded. The function $f : x \to 1/(1-x)$ is monotone on the interval $I = \{x : 0 \le x < 1\}$ but is not bounded there.

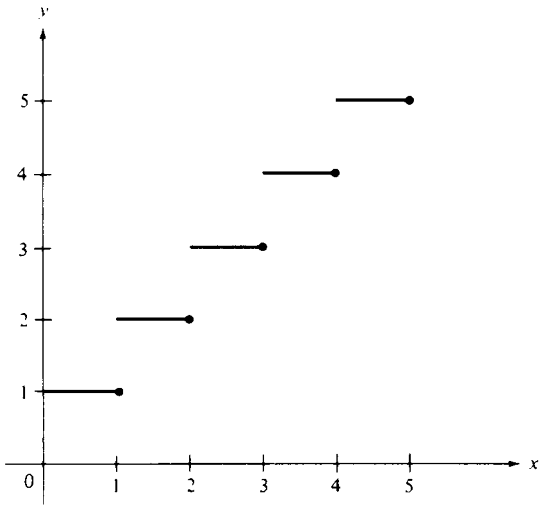

Figure 3.5 A step function.

If a function f defined on an interval I is continuous on I, then an extension of the Intermediate-value theorem shows that the range of f is an interval. In order to establish this result, we use the following characterization of an interval on $\mathbb{R}^1$.

Theorem 3.5
A set S in $\mathbb{R}^1$ is an interval $\Leftrightarrow$ (i) *S contains more than one point and* (ii) *for every $x_1, x_2 \in S$ the number x is in S whenever $x \in (x_1, x_2)$.*

Proof
If S is an interval, then clearly the two properties hold. Therefore, we suppose that (i) and (ii) hold and show that S is an interval. We consider two cases.

Case 1: *S is bounded.* Define $a = \inf S$ and $b = \sup S$. Let x be an element of the open interval (a, b). We shall show that $x \in S$.

Since $a = \inf S$, we use the corollary to Theorem 3.4 to assert that there is an $x_1 \in S$ with $x_1 < x$. Also, since $b = \sup S$, there is an x_2 in S with $x_2 > x$. Hence $x \in (x_1, x_2)$, which by (ii) of the hypothesis shows that $x \in S$. We conclude that every element of (a, b) is in S. Thus S is either a closed interval, an open interval, or a half-open interval; its endpoints are a, b.

Case 2: *S is unbounded in one direction.* Assume that S is unbounded above and bounded below, the case of S bounded above and unbounded below being similar. Let $a = \inf S$. Then $S \subset [a, \infty)$. Let x be any number such that $x > a$. We shall show that $x \in S$. As in Case 1, there is a number $x_1 \in S$ such that $x_1 < x$. Since S has no upper bound there is an $x_2 \in S$ such that $x_2 > x$. Using (ii) of the hypothesis, we conclude that $x \in S$. Therefore, $S = [a, \infty)$ or $S = (a, \infty)$. ■

We now establish a stronger form of the Intermediate-value theorem.

Theorem 3.6

Suppose that the domain of f is an interval I and f is continuous on I. Assume that f is not constant on I. Then the range of f, denoted by J, is an interval.

Proof

We shall show that J has properties (i) and (ii) of Theorem 3.5 and therefore is an interval. Since f is not constant, its range must have more than one point, and property (i) is established. Now let $y_1, y_2 \in J$. Then there are numbers $x_1, x_2 \in I$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. We may assume that $x_1 < x_2$. The function f is continuous on $[x_1, x_2]$, and so we may apply the Intermediate-value theorem. If c is any number between y_1 and y_2, there is an $x_0 \in [x_1, x_2]$ such that $f(x_0) = c$. Thus $c \in J$, and we have established property (ii). The set J is an interval. ■

If f is continuous on $I = \{x : a \leq x \leq b\}$, it is not necessarily the case that J is determined by $f(a)$ and $f(b)$. Figure 3.6 shows that J may exceed the interval $[f(a), f(b)]$. In fact, I may be bounded and J unbounded, as is illustrated by the function $f : x \to 1/x$ with $I = \{x : 0 < x < 1\}$. The range is $J = \{y : 1 < y < \infty\}$. The function $f : x \to 1/(1 + x^2)$ with $I = \{x : 0 < x < \infty\}$ and $J = \{y : 0 < y < 1\}$ is an example of a continuous function with an unbounded domain and a bounded range.

Consider the restriction to $[0, \infty)$ of the function $f : x \to x^n$ where n is a positive integer. By Theorem 3.6, the range of f is an interval, which must be $[0, \infty)$, since $f(0) = 0$, f is increasing, and $f(x) \to +\infty$ as $x \to +\infty$. Hence, for each $x \geq 0$, there is at least one number $y \geq 0$ such that $y^n = x$. If $y' > y$, then $(y')^n > y^n = x$. Also, if $0 \leq y' < y$, then $(y')^n < x$. Thus the solution y is unique. We denote it by $x^{1/n}$. *Every positive number has an nth root for every positive integer n.* The function $x \to x^{1/n}$, $x \geq 0$, can be shown to be continuous on $[0, \infty)$ by the methods of Chapter 2.

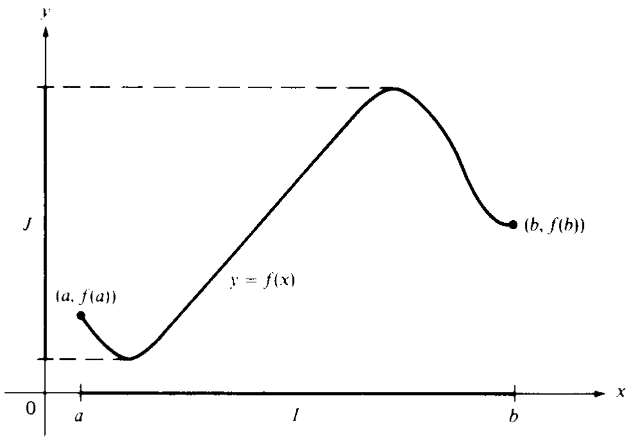

Figure 3.6

Problems

In Problems 1 through 8 find l.u.b. S and g.l.b. S. State whether or not these numbers are in S.

1. $S = \{x : 0 < x \leq 3\}$.
2. $S = \{x : x^2 - 3 < 0\}$.
3. $S = \{x : x^2 - 2x - 3 < 0\}$.
4. $S = \{y : y = x/(x+1),\ x \geq 0\}$.
5. $S = \{s_n : s_n = \sum_{i=1}^{n}(1/2^i),\ n=1, 2, \ldots\}$.
6. $S = \{s_n : s_n = 1 + \sum_{i=1}^{n}((-1)^i/i!),\ n=1, 2, \ldots\}$.
7. $S = \{x : 0 < x < 5,\ \cos x = 0\}$.
8. $S = \{x : -10 < x < 10,\ \sin x = \frac{1}{2}\}$.
9. Given that $S = \{s_n : s_n = 1 + \sum_{i=1}^{n}(1/i!),\ n=1, 2, \ldots\}$, show that S has 3 as an upper bound.
10. Suppose that $B_1 =$ l.u.b. S_1, $B_2 =$ l.u.b. S_2, $b_1 =$ g.l.b. S_1, $b_2 =$ g.l.b. S_2. If $S_1 \subset S_2$, show that $B_1 \leq B_2$ and $b_2 \leq b_1$.
11. Suppose that $S_1, S_2, \ldots, S_n$ are sets in $\mathbb{R}^1$ and that $S = S_1 \cup S_2 \cup \ldots \cup S_n$. Define $B_i = \sup S_i$ and $b_i = \inf S_i$, $i = 1, 2, \ldots, n$.
    (a) Show that $\sup S = \max(B_1, B_2, \ldots, B_n)$ and $\inf S = \min(b_1, b_2, \ldots, b_n)$.
    (b) If S is the union of an infinite collection of $\{S_i\}$, find the relation between $\inf S$, $\sup S$, and the $\{b_i\}$ and $\{B_i\}$.
12. Suppose that $S_1, S_2, \ldots, S_n$ are sets in $\mathbb{R}^1$ and that $S = S_1 \cap S_2 \cap \ldots \cap S_n$. If $S \neq \emptyset$, find a formula relating $\sup S$ and $\inf S$ in terms of the $\{b_i\}$, $\{B_i\}$ as defined in Problem 11.
13. Use the corollary to Theorem 3.4 to show that Axiom C is a consequence of Theorem 3.4. [*Hint*: Let B be the least upper bound of the *set* of numbers $\{x_n\}$, which as a *sequence* are increasing. Then show that $x_n \to B$.]

14. Let S_1, S_2 be sets in $\mathbb{R}^1$. Define $S = \{x : x = x_1 + x_2,\ x_1 \in S_1,\ x_2 \in S_2\}$. Find l.u.b. S, g.l.b. S in terms of l.u.b. S_i, g.l.b. S_i, $i = 1, 2$. In particular, if $-S_1 = \{x : -x \in S_1\}$, show that l.u.b. $(-S_1) =$ g.l.b. S_1.

3.3 The Bolzano–Weierstrass Theorem

Suppose that

$$x_1, x_2, \ldots, x_n, \ldots \tag{3.4}$$

is a sequence of numbers. Then the sequences $x_1, x_3, x_5, \ldots$ and $x_2, x_5, x_8, x_{11}, \ldots$ are examples of **subsequences** of (3.4). More generally, suppose that $k_1, k_2, k_3, \ldots, k_n, \ldots$ is an *increasing* sequence of positive integers. Then we say that

$$x_{k_1}, x_{k_2}, \ldots, x_{k_n}, \ldots$$

is a **subsequence** of (3.4). The choice $k_1 = 1, k_2 = 3, k_3 = 5, k_4 = 7, \ldots$ is an example of a subsequence of (3.4) above. To avoid double subscripts, which are cumbersome, we will frequently write $y_1 = x_{k_1}, y_2 = x_{k_2}, \ldots, y_n = x_{k_n}, \ldots$, in which case

$$y_1, y_2, \ldots, y_n, \ldots$$

is a subsequence of (3.4).

We easily prove by induction that if $k_1, k_2, k_3, \ldots, k_n, \ldots$ is an increasing sequence of positive integers, then $k_n \geq n$ for all n.

The sequence

$$x_1 = 0,\ x_2 = \tfrac{1}{2},\ x_3 = -\tfrac{2}{3}, \ldots, x_n = (-1)^n(1 - \tfrac{1}{n}), \ldots \tag{3.5}$$

has the subsequences

$$0, -\tfrac{2}{3}, -\tfrac{4}{5}, -\tfrac{6}{7}, \ldots,$$

$$\tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{6}, \tfrac{7}{8}, \ldots,$$

which are obtained from (3.5) by taking the odd-numbered terms and the even-numbered terms, respectively. Each of these subsequences is convergent, but the original sequence (3.5) is not. The notion of a convergent subsequence of a given sequence occurs frequently in problems in analysis. The Bolzano–Weierstrass theorem is basic in that it establishes the existence of such convergent subsequences under very simple hypotheses on the given sequence.

Theorem 3.7 (Bolzano–Weierstrass Theorem)
Any bounded infinite sequence of real numbers contains a convergent subsequence.

Proof

We shall use the Nested intervals theorem (Theorem 3.1). Let $\{x_n\}$ be a given bounded sequence. Then it is contained in some closed interval $I = \{x : a \le x \le b\}$. Divide I into two equal subintervals by the midpoint $(a+b)/2$. Then either the left subinterval contains an infinite number of the $\{x_n\}$ or the right subinterval does (or both). Denote by $I_1 = \{x : a_1 \le x \le b_1\}$ the closed subinterval of I that contains infinitely many $\{x_n\}$. (If both subintervals do, choose either one.) Next, divide I_1 into two equal parts by its midpoint. Either the right subinterval or the left subinterval of I_1 contains infinitely many $\{x_n\}$. Denote by I_2 the closed subinterval that does. Continue this process, obtaining the sequence

$$I_n = \{x : a_n \le x \le b_n\}, \qquad n = 1, 2, \ldots,$$

with the property that each I_n contains x_p for infinitely many values of p. Since $b_n - a_n = (b-a)/2^n \to 0$ as $n \to \infty$, we may apply the Nested intervals theorem to obtain a unique number x_0 contained in every I_n.

We now construct a subsequence of $\{x_p\}$ converging to x_0. Choose x_{k_1} to be any member of $\{x_p\}$ in I_1 and denote x_{k_1} by y_1. Next choose x_{k_2} to be any member of $\{x_p\}$ such that x_{k_2} is in I_2 and such that $k_2 > k_1$. We can do this because I_2 has infinitely many of the $\{x_p\}$. Set $x_{k_2} = y_2$. Next, choose x_{k_3} as any member of $\{x_p\}$ in I_3 and such that $k_3 > k_2$. We can do this because I_3 also has infinitely may of the $\{x_p\}$. Set $x_{k_3} = y_3$. We continue, and by induction obtain the subsequence $y_1, y_2, \ldots, y_n, \ldots$. By the method of selection we have

$$a_n \le y_n \le b_n, \qquad n = 1, 2, \ldots.$$

Since $a_n \to x_0$, $b_n \to x_0$, as $n \to \infty$, we can apply the Sandwiching theorem (Theorem 2.8) to conclude that $y_n \to x_0$ as $n \to \infty$. ∎

Problems

In Problems 1 through 7 decide whether or not the given sequence converges to a limit. If it does not, find, in each case, at least one convergent subsequence. We suppose $n = 1, 2, 3, \ldots$

1. $x_n = (-1)^n(1 - (1/n))$.
2. $x_n = 1 + ((-1)^n/n)$.
3. $x_n = (-1)^n(2 - 2^{-n})$.
4. $x_n = \sum_{i=1}^{n}((-1)^i/2^i)$.
5. $x_n = \sum_{j=1}^{n}(1/j!)$.
6. $x_n = \sin(n\pi/2) + \cos n\pi$.
7. $x_n = (\sin(n\pi/3))(1 - (1/n))$.
8. The sequence

$$1\tfrac{1}{2}, 2\tfrac{1}{2}, 3\tfrac{1}{2}, 1\tfrac{1}{3}, 2\tfrac{1}{3}, 3\tfrac{1}{3}, 1\tfrac{1}{4}, 2\tfrac{1}{4}, 3\tfrac{1}{4}, \ldots$$

has subsequences that converge to the numbers 1, 2, and 3.

(a) Write a sequence that has subsequences that converge to N different numbers where N is any positive integer.
(b) Write a sequence that has subsequences that converge to infinitely many different numbers.

3.4 The Boundedness and Extreme-Value Theorems

In this section we establish additional basic properties of continuous functions from $\mathbb{R}^1$ to $\mathbb{R}^1$. The Boundedness and Extreme-value theorems proved below are essential in the proofs of the basic theorems in differential calculus. The Boundedness theorem shows that a function that is continuous on a closed interval must have a bounded range. The Extreme-value theorem adds additional precise information about such functions. It states that the supremum and the infimum of the values in the range are also in the range.

Theorem 3.8 (Boundedness Theorem)
Suppose that the domain of f is the closed interval $I = \{x: a \leq x \leq b\}$, and f is continuous on I. Then the range of f is bounded.

Proof
We shall assume that the range is unbounded and reach a contradiction. Suppose that for each positive integer n, there is an $x_n \in I$ such that $|f(x_n)| > n$. The sequence $\{x_n\}$ is bounded, and by the Bolzano–Weierstrass theorem, there is a convergent subsequence $y_1, y_2, \ldots, y_n, \ldots$, where $y_n = x_{k_n}$, that converges to an element $x_0 \in I$. Since f is continuous on I, we have $f(y_n) \to f(x_0)$ as $n \to \infty$. Choosing $\varepsilon = 1$, we know that there is an N_1 such that for $n > N_1$, we have

$$|f(y_n) - f(x_0)| < 1 \qquad \text{whenever} \qquad n > N_1.$$

For these n it follows that

$$|f(y_n)| < |f(x_0)| + 1 \qquad \text{for all} \quad n > N_1.$$

On the other hand,

$$|f(y_n)| = |f(x_{k_n})| > k_n \geq n \qquad \text{for all} \quad n.$$

Combining these results, we obtain

$$n < |f(x_0)| + 1 \qquad \text{for all} \quad n > N_1.$$

Clearly, we may choose n larger than $|f(x_0)| + 1$, which is a contradiction. ■

In Theorem 3.8, it is essential that the domain of f is closed. The function $f : x \to 1/(1-x)$ is continuous on the half-open interval $I = \{x : 0 \le x < 1\}$, but is not bounded there.

Theorem 3.9 (Extreme-value theorem)
Suppose that f has for its domain the closed interval $I = \{x{:}a \le x \le b\}$, and that f is continuous on I. Then there are numbers x_0 and x_1 on I such that $f(x_0) \le f(x) \le f(x_1)$ for all $x \in I$. That is, f takes on its maximum and minimum values on I.

Proof
Theorem 3.8 states that the range of f is a bounded set. Define

$$M = \sup f(x) \;\text{ for }\; x \in I, \qquad m = \inf f(x) \;\text{ for }\; x \in I.$$

We shall show that there are numbers $x_0, x_1 \in I$ such that $f(x_0) = m$, $f(x_1) = M$. We prove the existence of x_1, the proof for x_0 being similar. Suppose that M is not in the range of f; we shall reach a contradiction. The function F with domain I defined by

$$F : x \to \frac{1}{M - f(x)}$$

is continuous on I and therefore (by Theorem 3.8) has a bounded range. Define $\overline{M} = \sup F(x)$ for $x \in I$. Then $\overline{M} > 0$ and

$$\frac{1}{M - f(x)} \le \overline{M} \qquad \text{or} \qquad f(x) \le M - \frac{1}{\overline{M}} \quad \text{for } \; x \in I.$$

This last inequality contradicts the statement that $M = \sup f(x)$ for $x \in I$, and hence M must be in the range of f. There is an $x_1 \in I$ such that $f(x_1) = M$. ■

The conclusion of Theorem 3.9 is false if the interval I is not closed. The function $f : x \to x^2$ is continuous on the *half-open* interval $I = \{x : 0 \le x < 1\}$ but does not achieve a maximum value there. Note that f is also continuous on the *closed* interval $I_1 = \{x : 0 \le x \le 1\}$, and its maximum on this interval occurs at $x = 1$; Theorem 3.9 applies in this situation.

3.5 Uniform Continuity

In the definition of continuity of a function f at a point x_0, it is necessary to obtain a number δ for each positive number ε prescribed in advance (see Section 2.1). The number δ, which naturally depends on ε, also depends on the particular value x_0. Under certain circumstances it may happen that the same value δ may be chosen for all points x in the domain. Then we say that f is uniformly continuous.

Definition

A function f with domain S is said to be **uniformly continuous** on S if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon \qquad \text{whenever} \qquad |x_1 - x_2| < \delta$$

and x_1, x_2 are in S. The important condition of uniform continuity states that the same value of δ holds for *all* x_1, x_2 in S.

Among the properties of continuous functions used in proving the basic theorems of integral calculus, that of uniform continuity plays a central role. The principal result of this section (Theorem 3.10 below) shows that under rather simple hypotheses continuous functions are uniformly continuous.

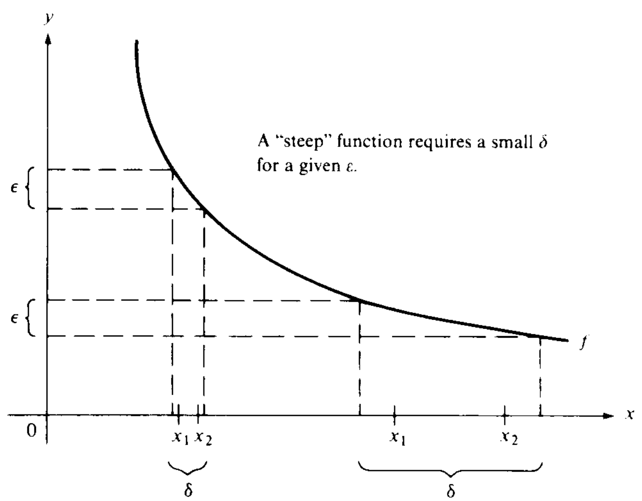

Figure 3.7

A function may be continuous on a set S without being uniformly continuous. As Figure 3.7 shows, once an ε is given, the value of δ required in the definition of ordinary continuity varies according to the location of x_1 and x_2; the "steeper" the function, the smaller the value of δ required.

As an example of a continuous function that is not uniformly continuous, consider

$$f : x \to \frac{1}{x}$$

defined on the set $S = \{x : 0 < x \leq 1\}$. It is clear that f is continuous for each x in S. However, with ε any positive number, say 1, we shall show there is no number δ such that

$$|f(x_1) - f(x_2)| < 1 \qquad \text{whenever} \qquad |x_1 - x_2| < \delta$$

for all x_1, x_2 in S. To see this we choose $x_1 = 1/n$ and $x_2 = 1/(n+1)$ for a positive integer n. Then $|f(x_1) - f(x_2)| = |n - (n+1)| = 1$; also, we have $|x_1 - x_2| = 1/(n(n+1))$. If n is very large, then x_1 and x_2 are close together. Therefore, if a δ is given, simply choose n so large that x_1 and x_2 are closer together than δ. The condition of uniform continuity is violated, since $f(x_1)$ and $f(x_2)$ may not be "close."

An example of a uniformly continuous function on a set S is given by

$$f : x \to x^2 \tag{3.6}$$

on the domain $S = \{x : 0 \le x \le 1\}$. To see this, suppose $\varepsilon > 0$ is given. We must find a $\delta > 0$ such that

$$|x_1^2 - x_2^2| < \varepsilon \qquad \text{whenever} \qquad |x_1 - x_2| < \delta$$

for all x_1, x_2 in S. To accomplish this, choose $\delta = \varepsilon/2$. Then, because $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$, we have

$$|x_1^2 - x_2^2| = |x_1 + x_2| \cdot |x_1 - x_2| < 2 \cdot \tfrac{1}{2}\varepsilon = \varepsilon.$$

This inequality holds for all x_1, x_2 on $[0, 1]$ such that $|x_1 - x_2| < \delta$.

The same function (3.6) above defined on the domain $S_1 = \{x : 0 \le x < \infty\}$ is not uniformly continuous there. To see this, suppose $\varepsilon > 0$ is given. Then for any $\delta > 0$, choose x_1, x_2 such that $x_1 - x_2 = \delta/2$ and $x_1 + x_2 = 4\varepsilon/\delta$. Then we have

$$|x_1^2 - x_2^2| = |x_1 + x_2| \cdot |x_1 - x_2| = \frac{4\varepsilon}{\delta} \cdot \frac{1}{2}\delta = 2\varepsilon.$$

The condition of uniform continuity is violated for this x_1, x_2.

An important criterion for determining when a function is uniformly continuous is established in the next result.

Theorem 3.10 (Uniform continuity Theorem)
If f is continuous on the closed interval $I = \{x : a \le x \le b\}$, then f is uniformly continuous on I.

Proof
We shall suppose that f is not uniformly continuous on I and reach a contradiction. If f is not uniformly continuous, there is an $\varepsilon_0 > 0$ for which there is *no* $\delta > 0$ with the property that $|f(x_1) - f(x_2)| < \varepsilon_0$ for all pairs $x_1, x_2 \in I$ and such that $|x_1 - x_2| < \delta$. Then for each positive integer n, there is a pair x_n', x_n'' on I such that

$$|x_n' - x_n''| < \frac{1}{n} \qquad \text{and} \qquad |f(x_n') - f(x_n'')| \ge \varepsilon_0. \tag{3.7}$$

From the Bolzano–Weierstrass theorem, it follows that there is a subsequence of $\{x'_n\}$, which we denote $\{x'_{k_n}\}$, convergent to some number x_0 in I. Then, since $|x'_{k_n} - x''_{k_n}| < 1/n$, we see that $x''_{k_n} \to x_0$ as $n \to \infty$. Using the fact that f is continuous on I, we have

$$f(x'_{k_n}) \to f(x_0), \qquad f(x''_{k_n}) \to f(x_0).$$

That is, there are positive integers N_1 and N_2 such that

$$|f(x'_{k_n}) \to f(x_0)| < \tfrac{1}{2}\varepsilon_0 \qquad \text{for all} \qquad n > N_1$$

and

$$|f(x''_{k_n}) \to f(x_0)| < \tfrac{1}{2}\varepsilon_0 \qquad \text{for all} \qquad n > N_2.$$

Hence for all n larger than both N_1 and N_2 we find

$$|f(x'_{k_n}) - f(x''_{k_n}| \le |f(x'_{k_n}) - f(x_0)| + |f(x_0) - f(x''_{k_n})| < \tfrac{1}{2}\varepsilon_0 + \tfrac{1}{2}\varepsilon_0 = \varepsilon_0.$$

This last inequality contradicts (3.7), and the result is established. ∎

As the example of the function $f : x \to 1/x$ shows, the requirement that I be a closed interval is essential. Also, the illustration of the function $f : x \to x^2$ on the set $S_1 = \{x : 0 \le x < \infty\}$ shows that Theorem 3.10 does not apply if the interval is unbounded. It may happen that continuous functions are uniformly continuous on unbounded sets. But these must be decided on a case-by-case basis. See Problems 3 and 4 at the end of the section.

Problems

1. Suppose that f is a continuous, increasing function on a closed interval $I = \{x : a \le x \le b\}$. Show that the range of f is the interval $[f(a), f(b)]$.
2. Suppose that f is uniformly continuous on the closed intervals I_1 and I_2. Show that f is uniformly continuous on $S = I_1 \cup I_2$.
3. Show that the function $f : x \to 1/x$ is uniformly continuous on $S = \{x : 1 \le x < \infty\}$.
4. Show that the function $f : x \to \sin x$ is uniformly continuous on $S = \{x : -\infty < x < \infty\}$. [*Hint*: $\sin A - \sin B = 2\sin((A+B)/2)\cos((A+B)/2)$.]
5. Suppose that f is continuous on $I = \{x : a < x < b\}$. If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exist, show that by defining $f_0(a) = \lim_{x\to a^+} f(x)$, $f_0(b) = \lim_{x\to b^-} f(x)$, and $f_0(x) = f(x)$ for $x \in I$, the function f_0 defined on $I_1 = \{x : a \le x \le b\}$ is uniformly continuous.
6. Consider the function $f : x \to \sin(1/x)$ defined on $I = \{x : 0 < x \le 1\}$. Decide whether or not f is uniformly continuous on I.
7. Show directly from the definition that the function $f : x \to \sqrt{x}$ is uniformly continuous on $I_1 = \{x : 0 \le x \le 1\}$.

8. Suppose that f is uniformly continuous on the half-open interval $I = \{x : 0 < x \le 1\}$. Is it true that f is bounded on I?
9. Given the general polynomial
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$
show that f is uniformly continuous on $0 \le x \le 1$.
10. Show that the function
$$f(x) = \begin{cases} x\sin(1/x), & 0 < x \le 1, \\ 0, & x = 0, \end{cases}$$
is uniformly continuous on $I = \{x : 0 \le x \le 1\}$.

3.6 The Cauchy Criterion

We recall the definition of a convergent sequence $x_1, x_2, \ldots, x_n, \ldots$. A sequence **converges to a limit** L if for every $\varepsilon > 0$ there is a positive integer N such that

$$|x_n - L| < \varepsilon \quad \text{whenever} \quad n > N. \tag{3.8}$$

Suppose we are given a sequence and wish to examine the possibility of convergence. Usually the number L is not given, so that condition (3.8) above cannot be verified directly. For this reason it is important to have a technique for deciding convergence that doesn't employ the limit L of the sequence. Such a criterion, presented below, was given first by Cauchy.

Definition
An infinite sequence $\{x_n\}$ is called a **Cauchy sequence** if and only if for each $\varepsilon > 0$, there is a positive integer N such that

$$|x_n - x_m| < \varepsilon \quad \text{for all } m > N \text{ and all } n > N.$$

Theorem 3.11 (Cauchy criterion for convergence)
A necessary and sufficient condition for convergence of a sequence $\{x_n\}$ is that it be a Cauchy sequence.

Proof
We first show that if $\{x_n\}$ is convergent, then it is a Cauchy sequence. Suppose L is the limit of $\{x_n\}$, and let $\varepsilon > 0$ be given. Then from the definition of convergence there is an N such that

$$|x_n - L| < \tfrac{1}{2}\varepsilon \quad \text{for all } n > N.$$

Let x_m be any element of $\{x_n\}$ with $m > N$. We have

$$|x_n - x_m| = |x_n - L + L - x_m| \le |x_n - L| + |L - x_m| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence.

Now assume that $\{x_n\}$ is a Cauchy sequence; we wish to show that it is convergent. We first prove that $\{x_n\}$ is a *bounded* sequence. From the definition of Cauchy sequence with $\varepsilon = 1$, there is an integer N_0 such that

$$|x_n - x_m| < 1 \quad \text{for all } n > N_0 \text{ and } m > N_0.$$

Choosing $m = N_0 + 1$, we find that $|x_n - x_{N_0+1}| < 1$ if $n > N_0$. Also,

$$|x_n| = |x_n - x_{N_0+1} + x_{N_0+1}| \le |x_n - x_{N_0+1}| + |x_{N_0+1}| < 1 + |x_{N_0+1}|.$$

Keep N_0 fixed and observe that all $|x_n|$ beyond x_{N_0} are bounded by $1 + |x_{N_0+1}|$, a fixed number. Now examine the *finite* sequence of numbers

$$|x_1|, |x_2|, \ldots, |x_{N_0}|, |x_{N_0+1}| + 1$$

and denote by M the largest of these. Therefore, $|x_n| \le M$ for *all* positive integers n, and so $\{x_n\}$ is a bounded sequence.

We now apply the Bolzano–Weierstrass theorem (Theorem 3.7) and conclude that there is a subsequence $\{x_{k_n}\}$ of $\{x_n\}$ that converges to some limit L. We shall show that the sequence $\{x_n\}$ itself converges to L. Let $\varepsilon > 0$ be given. Since $\{x_n\}$ is a Cauchy sequence, there is an N_1 such that

$$|x_n - x_m| < \tfrac{1}{2}\varepsilon \quad \text{for all } n > N_1 \text{ and } m > N_1.$$

Also, since $\{x_{k_n}\}$ converges to L, there is an N_2 such that

$$|x_{k_n} - L| < \tfrac{1}{2}\varepsilon \quad \text{for all } n > N_2.$$

Let N be the larger of N_1 and N_2, and consider any integer n larger than N. We recall from the definition of subsequence of a sequence that $k_n \ge n$ for every n. Therefore,

$$|x_n - L| = |x_n - x_{k_n} + x_{k_n} - L| \le |x_n - x_{k_n}| + |x_{k_n} - L| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Since this inequality holds for all $n > N$, the sequence $\{x_n\}$ converges to L. ■

As an example, we show that the sequence $x_n = (\cos n\pi)/n$, $n = 1, 2, \ldots$, is a Cauchy sequence. Let $\varepsilon > 0$ be given. Choose N to be any integer larger than $2/\varepsilon$. Then we have

$$|x_n - x_m| = \left| \frac{\cos n\pi}{n} - \frac{\cos m\pi}{m} \right| \le \frac{m|\cos n\pi| + n|\cos m\pi|}{mn} = \frac{m+n}{mn}.$$

If $m > n$, we may write

$$|x_n - x_m| \le \frac{m+n}{mn} < \frac{2m}{mn} = \frac{2}{n}.$$

However, because $n > N$, we have $n > 2/\varepsilon$, and so $|x_n - x_m| < \varepsilon$, and the sequence is a Cauchy sequence.

3.7 The Heine–Borel Theorem

In this section we establish a theorem that is useful in the further development of differentiation and integration. The Heine–Borel theorem shows that "covering" a set with a collection of special sets is particularly helpful in verifying uniform continuity.

Consider, for example, the collection of intervals

$$I_n = \left\{x : \frac{1}{n} < x < 1\right\}, \qquad n = 2, 3, 4, \ldots. \tag{3.9}$$

Any collection of intervals such as (3.9) is called a **family of intervals**. The symbol F is usually used to denote the totality of intervals in the family. Each interval is called a **member** (or **element**) of the family. Care must be exercised to distinguish the points of a particular interval from the interval itself. In the above example, the interval $J = \{x : \frac{1}{3} < x < \frac{2}{3}\}$ is *not* a member of F. However, every point of J is in every interval I_n for $n > 2$. The points $0 < x < 1$ are not themselves members of F although each point of $0 < x < 1$ is in at least one I_n.

Definitions

A family F of intervals is said to **cover the set** S **in** $\mathbb{R}^1$ if each point of S is in at least one of the intervals of F. A family F_1 is a **subfamily** of F if each member of F_1 is a member of F.

For example, the intervals

$$I'_n = \left\{x : \frac{1}{2n} < x < 1\right\}, \qquad n = 1, 2, \ldots \tag{3.10}$$

form a subfamily of family (3.9). Family (3.9) covers the set $S = \{x : 0 < x < 1\}$, since every number x such that $0 < x < 1$ is in I_n for all n larger than $1/x$. Thus every x is in at least one I_n. Family (3.10) also covers S. More generally, we may consider a collection of sets $A_1, A_2 \ldots A_n, \ldots$, which we call a **family of sets**. We use the same symbol F to denote such a family. A family F **covers a set** S if and only if every point of S is a point in at least one member of F.

The family F,

$$J_n = \{x : n < x < n + 2\}, \qquad n = 0, \pm 1, \pm 2, \ldots,$$

covers all of $\mathbb{R}^1$. It is simple to verify that if any interval is removed from F, the resulting family fails to cover $\mathbb{R}^1$. For example, if J_1 is removed, then no member of the remaining intervals of F contains the number 2.

We shall be interested in families of open intervals that contain infinitely many members and that cover a set S. We shall examine those sets S that have the property that they are covered by a **finite subfamily**, i.e., a subfamily with only a finite number of members. For example, the family F of intervals $K_n = \{x : 1/(n + 2) < x < 1/n\}$, $n = 1, 2, \ldots$, covers the set $S = \{x : 0 < x < \frac{1}{2}\}$, as is easily verified. It may also be verified that no finite subfamily of $F = \{K_n\}$ covers S. However, if we consider the set $S_1 = \{x : \varepsilon < x < \frac{1}{2}\}$ for any $\varepsilon > 0$, it is clear that S_1 is covered by the finite subfamily $\{K_1, K_2, \ldots, K_n\}$ for any n such that $n + 2 > 1/\varepsilon$.

We may define families of intervals indirectly. For example, suppose that $f : x \to 1/x$ with domain $D = \{x : 0 < x \leq 1\}$ is given. We obtain a family F of intervals by considering all solution sets of the inequality

$$|f(x) - f(a)| < \frac{1}{3} \leftrightarrow -\frac{1}{3} < \frac{1}{x} - \frac{1}{a} < \frac{1}{3} \tag{3.11}$$

for every $a \in D$. For each value a, the interval

$$L_a = \left\{x : \frac{3a}{3+a} < x < \frac{3a}{3-a}\right\} \tag{3.12}$$

is the solution of Inequality (3.11). This family $F = \{L_a\}$ covers the set $D = \{x : 0 < x \leq 1\}$, but no finite subfamily covers D.

Theorem 3.12 (Heine–Borel theorem)
Suppose that a family F of open intervals covers the closed interval $I = \{x : a \leq x \leq b\}$. Then a finite subfamily of F covers I.

Proof
We shall suppose that an infinite number of members of F are required to cover I and reach a contradiction. Divide I into two equal parts at the midpoint. Then an infinite number of members of F are required to cover either the left subinterval or the right subinterval of I. Denote by $I_1 = \{x : a_1 \leq x \leq b_1\}$ the particular subinterval needing this infinity of members of F. We proceed by dividing I_1 into two equal parts, and denote by $I_2 = \{x : a_2 \leq x \leq b_2\}$ that half of I_1 that requires an infinite number of members of F. Repeating the argument, we obtain a sequence of closed intervals $I_n = \{x : a_n \leq x \leq b_n\}$, $n = 1, 2, \ldots$, each of which requires an infinite number of intervals of F in order to be covered. Since $b_n - a_n = (b - a)/2^n \to 0$ as $n \to \infty$, the Nested intervals theorem (Theorem 3.1) states that there is a unique number $x_0 \in I$ that is in every I_n. However, since F covers I, there is a member of F, say $J = \{x : \alpha < x < \beta\}$, such that $x_0 \in J$. By choosing N sufficiently large, we can find an I_N contained in J. But this contradicts the fact that infinitely many intervals of F are required to cover I_N, since we found that the one interval J covers I_N. ■

In the Heine–Borel theorem, the hypothesis that $I = \{x : a \leq x \leq b\}$ is a *closed* interval is crucial. The open interval $J_1 = \{x : 0 < x < 1\}$ is covered by the family F of intervals (3.9), but no finite subfamily of F covers J_1.

Problems

In each of Problems 1 through 6 decide whether or not the infinite sequence is a Cauchy sequence. If it is not a Cauchy sequence, find at least one subsequence that is a Cauchy sequence. In each case $n = 1, 2, \ldots$.

1. $x_n = \sum_{j=1}^{n} (-1)^j (1/(2j))$.
2. $x_n = \sum_{j=1}^{n} (1/j!)$.
3. $x_n = \sum_{i=1}^{n} ((-1)^j/j!)$.
4. $x_n = 1 + (-1)^n + (1/n)$.
5. $x_n = \sin(n\pi/3) + (1/n)$.
6. $x_n = (1 + (-1)^n)n + (1/n)$.
7. Suppose that $s_n = \sum_{j=1}^{n} u_j$ and $S_n = \sum_{j=1}^{n} |u_j|$, $n = 1, 2, \ldots$. If $S_n \to S$ as $n \to \infty$, show that s_n tends to a limit as $n \to \infty$.
8. Show that Theorem 3.1, the Nested intervals theorem, may be proved as a direct consequence of the Cauchy criterion for convergence (Theorem 3.11). [*Hint*: Suppose $I_n = \{x : a_n \le x \le b_n\}$ is a nested sequence. Then show that $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences. Hence they tend to a limit. Since $b_n - a_n \to 0$, the limits must be the same. Finally, the Sandwiching theorem shows that the limit is in every I_n.]
9. Suppose that f has a domain that contains $I = \{x : a \le x \le b\}$, and suppose that $f(x) \to L$ as $x \to b^-$. Prove the following: For each $\varepsilon > 0$ there is a $\delta > 0$ such that
(3.13)
$$|f(x) - f(y)| < \varepsilon \text{ for all } x, y \text{ with } b - \delta < x < b,\ b - \delta < y < b.$$
10. Prove the converse of the result in Problem 9. That is, suppose that for every $\varepsilon > 0$ there is a $\delta > 0$ such that condition (3.13) is satisfied. Prove that there is a number L such that $f(x) \to L$ as $x \to b^-$.
11. Show that the result of Problem 9 holds if I is replaced by $J = \{a < x < \infty\}$ and limits are considered as $x \to +\infty$. Show that the result in Problem 10 does not hold.
12. Show that if f is uniformly continuous on $I = \{x : a < x < b\}$, then $f(x)$ tends to a limit as $x \to b^-$ and as $x \to a^+$. Hence, show that if $f_0(a)$ and $f_0(b)$ are these limits and $f_0(x) = f(x)$ on I, then the extended function f_0 is uniformly continuous on $J = \{x : a \le x \le b\}$. Use the result of Problem 10.
13.
   (a) Show that the family F of all intervals of the form $I_n = \{x : 1/(n+2) < x < 1/n\}$ covers the interval $J = \{x : 0 < x < \frac{1}{2}\}$.
   (b) Show that no finite subfamily of F covers J.
14. Let G be the family of intervals obtained by adjoining the interval $\{x : \frac{-1}{6} < x < \frac{1}{6}\}$ to the family F of Problem 13. Exhibit a finite subfamily of G that covers $K = \{x : 0 \le x \le \frac{1}{2}\}$.

15. Let F_1 be the family of all intervals $I_n = \{x : 1/2^n < x < 2\}$, $n = 1, 2, \ldots$. Show that F_1 covers the interval $J = \{x : 0 < x < 1\}$. Does any finite subfamily of F_1 cover J? Prove your answer.
16. For $f(x) = 1/x$ defined on $D = \{x : 0 < x \le 1\}$, consider the inequality $|f(x) - f(a)| < \frac{1}{3}$ for $a \in D$. Show that the solution set of this inequality, denoted by L_a, is given by equation (3.12). Show that the family $F = \{L_a\}$ covers D. Does any finite subfamily of F cover D? Prove your answer.
17. Consider $f : x \to 1/x$ defined on $E = \{x : 1 \le x < \infty\}$. Let I_a be the interval of values of x for which $|f(x) - f(a)| < \frac{1}{3}$. Find I_a for each $a \in E$. Show that the family $F = \{I_a\}$, $a \in E$, covers E. Is there a finite subfamily of F that covers E? Prove your answer.

CHAPTER 4 Elementary Theory of Differentiation

4.1 The Derivative in $\mathbb{R}^1$

In a first course in calculus we learn a number of theorems on differentiation, but usually the proofs are not provided. In this section we review those elementary theorems and provide the missing proofs for the basic results.

Definition
Let f be a function on $\mathbb{R}^1$ to $\mathbb{R}^1$. The **derivative** f' is defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad . \tag{4.1}$$

That is, f' is the function whose domain consists of all x in $\mathbb{R}^1$ for which the limit on the right side of equation (4.1) exists. Of course the range of f' is in $\mathbb{R}^1$.

The following seven elementary theorems on the derivative are simple consequences of the theorems on limits in Chapter 2.

Theorem 4.1
If f is a constant, then $f'(x) = 0$ for all x.

Theorem 4.2
Suppose that f is defined on an open interval I, that c is a real number, and that g is defined by the equation $g(x) = cf(x)$. If $f'(x)$ exists, then $g'(x)$ does, and $g'(x) = cf'(x)$.

Theorem 4.3
Suppose that f and g are defined on an open interval I and that F is defined by the equation $F(x) = f(x) + g(x)$. If $f'(x)$ and $g'(x)$ exist, then $F'(x)$ does, and $F'(x) = f'(x) + g'(x)$.

Theorem 4.4
If $f'(a)$ exists for some number a, then f is continuous at a.

Proof
We write

$$f(a+h) - f(a) = \left(\frac{f(a+h) - f(a)}{h} \right) \cdot h.$$

The term in parentheses on the right approaches $f'(a)$, a finite number, as $h \to 0$. The second part, namely h, tends to zero. Therefore $f(a+h) - f(a)$ tends to zero as $h \to 0$. That is, f is continuous at a. ■

Theorem 4.5
Suppose that u and v are defined on an open interval I and that f is defined by the equation $f(x) = u(x) \cdot v(x)$. If $u'(x)$ and $v'(x)$ exist, then $f'(x)$ does, and $f'(x) = u(x)v'(x) + u'(x)v(x)$.

Proof
We define $\Delta f = f(x+h) - f(x)$, $\Delta u = u(x+h) - u(x)$, $\Delta v = v(x+h) - v(x)$. Then a computation yields

$$\Delta f = u\Delta v + v\Delta u + \Delta u \cdot \Delta v.$$

Divide by h:

$$\frac{\Delta f}{h} = u\frac{\Delta v}{h} + v\frac{\Delta u}{h} + \Delta u \cdot \frac{\Delta v}{h}.$$

As $h \to 0$, we find $\frac{\Delta u}{h} \to u'(x)$, $\frac{\Delta v}{h} \to v'(x)$ and $\Delta u \to 0$. Hence

$$f'(x) = u(x)v'(x) + v(x)u'(x).$$

■

Theorem 4.6
Suppose that u and v are defined on an open interval I, that $v(x) \neq 0$ for all $x \in I$, and that f is defined by the equation $f(x) = u(x)/v(x)$. If $u'(x)$ and $v'(x)$ exist, then $f'(x)$ does, and

$$f'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}.$$

Proof

We have

$$\Delta f \equiv f(x+h) - f(x) = \frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}.$$

With $\Delta u = u(x+h) - u(x)$, $\Delta v = v(x+h) - v(x)$, we find

$$\Delta f = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v} = \frac{v(u+\Delta u) - u(v+\Delta v)}{v(v+\Delta v)} = \frac{v\Delta u - u\Delta v}{v(v+\Delta v)}.$$

Then

$$\lim_{h\to 0} \frac{\Delta f}{h} = f'(x) = \frac{vu'(x) - uv'(x)}{v^2}.$$

■

We leave the proof of the following theorem to the reader.

Theorem 4.7

Given $f : x \to x^n$ and n is an integer, then

$$f'(x) = nx^{n-1}.$$

(*We assume that $x \neq 0$ for $n < 0$.*)

We now establish the Chain rule (Theorem 4.9 below) which, together with Theorems 4.1 through 4.7, forms the basis for calculating the derivatives of most elementary functions. The Chain rule is a consequence of the following lemma, which states that every differentiable function is approximated by a linear function whose slope is the derivative.

Theorem 4.8 (Fundamental lemma of differentiation)

Suppose that f has a derivative at x_0. Then there is a function η defined in an interval about 0 such that

$$f(x_0 + h) - f(x_0) = [f'(x_0) + \eta(h)] \cdot h. \tag{4.2}$$

Also, η is continuous at 0 with $\eta(0) = 0$.

Proof

We define η by the formula

$$\eta(h) = \begin{cases} \dfrac{1}{h}[f(x_0+h) - f(x_0)] - f'(x_0), & h \neq 0, \\ 0, & h = 0. \end{cases}$$

Since f has a derivative at x_0, we see that $\eta(h) \to 0$ as $h \to 0$. Hence η is continuous at 0. Formula (4.2) is a restatement of the definition of η. ■

Theorem 4.9 (Chain rule)
Suppose that g and u are functions on $\mathbb{R}^1$ and $f(x) = g[u(x)]$. Suppose that u has a derivative at x_0 and that g has a derivative at $u(x_0)$. Then $f'(x_0)$ exists and

$$f'(x_0) = g'[u(x_0)] \cdot u'(x_0).$$

Proof
We use the notation $\Delta f = f(x_0 + h) - f(x_0)$, $\Delta u = u(x_0 + h) - u(x_0)$, and we obtain

$$\Delta f = g[u(x_0 + h)] - g[u(x_0)] = g(u + \Delta u) - g(u). \tag{4.3}$$

We apply Theorem 4.8 to the right side of equation (4.3), getting

$$\Delta f = [g'(u) + \eta(\Delta u)]\Delta u.$$

Dividing by h and letting h tend to zero, we obtain (since $\Delta u \to 0$ as $h \to 0$)

$$\lim_{h\to 0} \frac{\Delta f}{h} = f'(x_0) = \lim_{h\to 0}[g'(u) + \eta(\Delta u)] \cdot \lim_{h\to 0} \frac{\Delta u}{h} = g'[u(x_0)] \cdot u'(x_0). \qquad \blacksquare$$

The Extreme-value theorem (Theorem 3.9) shows that a function which is continuous on a closed interval I takes on its maximum and minimum values there. If the maximum value occurs at an interior point of I, and if the function possesses a derivative at that point, the following result gives a method for locating maximum (and minimum) values of such functions.

Theorem 4.10
Suppose that f is continuous on an interval I and that f takes on its maximum value at x_0, an interior point of I. If $f'(x_0)$ exists, then

$$f'(x_0) = 0.$$

The proof follows immediately from the definition of derivative. A similar result holds for functions that take on their minimum value at an interior point. See Figure 4.1.

The next two theorems, Rolle's theorem and the Mean-value theorem, form the groundwork for further developments in the theory of differentiation.

Theorem 4.11 (Rolle's theorem)
Suppose that f is continuous on the closed interval $I = \{x : a \le x \le b\}$, and that f has a derivative at each point of $I_1 = \{x : a < x < b\}$. If $f(a) = f(b) = 0$, then there is a number $x_0 \in I_1$ such that $f'(x_0) = 0$.

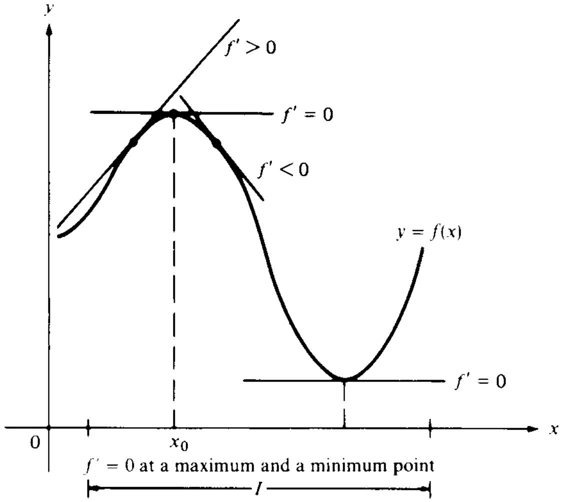

Figure 4.1

Proof

Unless $f(x) \equiv 0$, in which case the result is trivial, f must be positive or negative somewhere. Suppose it is positive. Then according to the Extreme-value theorem, f achieves its maximum at some interior point, say x_0. Now we apply Theorem 4.10 to conclude that $f'(x_0) = 0$. If f is negative on I_1, we apply the same theorems to the minimum point. ■

Theorem 4.12 (Mean-value theorem)

Suppose that f is continuous on the closed interval $I = \{x{:}a \le x \le b\}$ and that f has a derivative at each point of $I_1 = \{x{:}a < x < b\}$. Then there exists a number $\xi \in I_1$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Proof

We construct the function

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a).$$

We have

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

By substitution, we see that $F(a) = F(b) = 0$. Thus F satisfies the hypotheses of Rolle's theorem, and there is a number $\xi \in I_1$ such that $F'(\xi) = 0$.

Hence

$$0 = F'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{b - a},$$

and the result is established. See Figure 4.2. ∎

The next theorem, a direct consequence of the Mean-value theorem, is useful in the construction of graphs of functions. The proof is left to the reader; see Problem 10 at the end of this section.

Theorem 4.13
Suppose that f is continuous on the closed interval I and has a derivative at each point of I_1, the interior of I.

(a). *If f' is positive on I_1, then f is increasing on I.*
(b). *If f' is negative on I_1, then f is decreasing on I.*

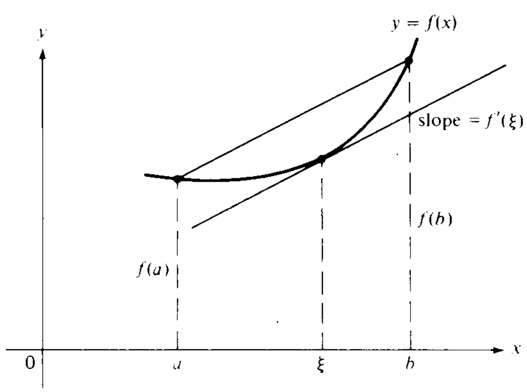

Figure 4.2 Illustrating the Mean-value theorem.

See Figure 4.1.

L'Hôpital's rule, so useful in the evaluation of indeterminate forms, has its theoretical basis in the following three theorems.

Theorem 4.14 (Generalized Mean-value theorem)
Suppose that f and F are continuous functions defined on $I = \{x : a \le x \le b\}$. Suppose that f' and F' exist on $I_1 = \{x : a < x < b\}$, the interior of I, and that $F'(x) \neq 0$ for $x \in I_1$. Then

(i) $F(b) - F(a) \neq 0$, *and*
(ii) *there is a number $\xi \in I_1$ such that*

$$\text{(4.4)} \qquad \frac{f(b) - f(a)}{F(b) - F(a)} = \frac{f'(\xi)}{F'(\xi)}.$$

Proof

According to the Mean-value theorem applied to F, there is a number $\eta \in I_1$ such that $F(b) - F(a) = F'(\eta)(b - a)$. Since $F'(\eta) \neq 0$, we conclude that $F(b) - F(a) \neq 0$, and (i) is proved. To prove (ii), define the function ϕ on I by the formula

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{F(b) - F(a)}[F(x) - F(a)].$$

A simple calculation shows that $\phi(a) = \phi(b) = 0$; thus Rolle's theorem can be applied to ϕ. The number $\xi \in I_1$ such that $\phi'(\xi) = 0$ yields

$$0 = \phi'(\xi) = f'(\xi) - \frac{f(b) - f(a)}{F(b) - F(a)} F'(\xi),$$

which is equation (4.4). ■

Theorem 4.15 (L'Hôpital's rule for **0/0**)
Suppose that f and F are continuous functions on $I = \{x : a < x < b\}$ and that f' and F' exist on I with $F' \neq 0$ on I. If

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} F(x) = 0 \textit{ and } \lim_{x \to a^+} \frac{f'(x)}{F'(x)} = L,$$

then

$$\lim_{x \to a^+} \frac{f(x)}{F(x)} = L.$$

Proof

We extend the definitions of f and F to the half-open interval $I_1 = \{a \leq x < b\}$ by setting $f(a) = 0$, $F(a) = 0$. Since $F'(x) \neq 0$ on I and $F(a) = 0$, we see from Theorem 4.12 that $F(x) \neq 0$ on I. Let $\epsilon > 0$ be given. Then there is a $\delta > 0$ such that

$$\left| \frac{f'(x)}{F'(x)} - L \right| < \epsilon \text{ for all } x \text{ such that } a < x < a + \delta.$$

Now we write

$$\left| \frac{f(x)}{F(x)} - L \right| = \left| \frac{f(x) - f(a)}{F(x) - F(a)} - L \right|,$$

and apply the Generalized Mean-value theorem to obtain

$$\left| \frac{f(x)}{F(x)} - L \right| = \left| \frac{f'(\xi)}{F'(\xi)} - L \right| < \epsilon,$$

where ξ is such that $a < \xi < x < a + \delta$. Since ϵ is arbitrary, the result follows. ■

Corollary 1

L'Hôpital's rule holds for limits from the left as well as limits from the right. If the two-sided limits are assumed to exist in the hypotheses of Theorem 4.15, *then we may conclude the existence of the two-sided limit of f/F.*

Corollary 2

A theorem similar to Theorem 4.15 *holds for limits as $x \to +\infty$, $-\infty$, or ∞.*

Proof of Corollary 2.

Assume that

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} F(x) = 0,\ \lim_{x\to+\infty} \frac{f'(x)}{F'(x)} = L,\ F'(x) \neq 0 \text{ on } I.$$

We let $z = 1/x$ and define the functions g and G by the formulas

$$g(z) = f\left(\frac{1}{z}\right),\ G(z) = F\left(\frac{1}{z}\right).$$

Then

$$g'(z) = -\frac{1}{z^2} f'\left(\frac{1}{z}\right),\ G'(z) = -\frac{1}{z^2} F'\left(\frac{1}{z}\right),$$

so that $g'(z)/G'(z) \to L$, $g(z) \to 0$, and $G(z) \to 0$ as $z \to 0^+$. Hence we apply Theorem 4.15 to g/G, and the result is established. The argument when $x \to -\infty$ or ∞ is similar. ■

Theorem 4.16 (L'Hôpital's rule for **∞/∞**)

Suppose that f' and F' exist on $I = \{x{:}a < x < b\}$. If

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} F(x) = \infty,\ \textit{and}\ \lim_{x\to a^+} \frac{f'(x)}{F'(x)} = L,$$

and if $F'(x) \neq 0$ on I, then

$$\lim_{x\to a^+} \frac{f(x)}{F(x)} = L.$$

Theorem 4.16 holds for limits from the left as well as limits from the right. The proof is more complicated than the proof of Theorem 4.15 although the same in principle. We omit the details.

Indeterminate forms such as $0 \cdot \infty$, 1^∞, and $\infty - \infty$ may frequently be reduced by algebraic manipulation and by use of the logarithm function to $0/0$ or ∞/∞. Then we may apply Theorem 4.15 or 4.16. Sometimes $f'(x)/F'(x)$ as well as $f(x)/F(x)$ is indeterminate. In such cases, we can apply Theorem 4.15 or 4.16 to f'/F' and see whether or not f''/F'' tends to a limit. This process may be applied any number of times.

EXAMPLE
Evaluate

$$\lim_{x\to 0} \frac{1+x-e^x}{2x^2}.$$

Solution. Set $f : x \to 1 + x - e^x$ and $F : x \to 2x^2$. Then $\lim_{x\to 0}(f(x)/F(x))$ yields the indeterminate form 0/0. We try to apply Theorem 4.15 and find that $\lim_{x\to 0}(f'(x)/F'(x))$ also gives 0/0. A second attempt at application of the theorem shows that

$$\lim_{x\to 0} \frac{f''(x)}{F''(x)} = \lim_{x\to 0} \frac{-e^x}{4} = -\frac{1}{4}.$$

Hence $\lim_{x\to 0}(f'(x)/F'(x)) = -\frac{1}{4}$, so that $\lim_{x\to 0}(f(x)/F(x)) = -\frac{1}{4}$. ■

Problems

1. Use Theorem 2.3 on the limit of a sum to write out a proof of Theorem 4.3 on the derivative of the sum of two functions.
2. Use the identity

$$\begin{aligned}&u(x+h)v(x+h) - u(x)v(x)\\ &\quad = u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)\end{aligned}$$

   and Theorems 2.3 and 2.4 to prove the formula for the derivative of a product (Theorem 4.5).
3. The **Leibniz rule** for the nth derivative of a product is given by

$$\begin{aligned}\frac{d^n}{dx^n}(f(x)g(x)) &= f^{(n)}(x)g(x) + \frac{n}{1!}f^{(n-1)}(x)g'(x)\\ &\quad + \frac{n(n-1)}{2!}f^{(n-2)}(x)g^{(2)}(x) + \ldots + f(x)g^{(n)}(x),\end{aligned}$$

   where the coefficients are the same as the coefficients in the binomial expansion

$$\begin{aligned}(A+B)^n &= A^n + \frac{n}{1!}A^{n-1}B + \frac{n(n-1)}{2!}A^{n-2}B^2\\ &\quad + \frac{n(n-1)(n-2)}{3!}A^{n-3}B^3 + \ldots + B^n.\end{aligned}$$

   Use mathematical induction to establish the Leibniz rule.
4. State hypotheses that assure the validity of the formula (the extended Chain rule)

$$F'(x_0) = f'\{u[v(x_0)]\}u'[v(x_0)]v'(x_0)$$

where $F(x) = f\{u[v(x)]\}$. Prove the result.

5. We define the **one-sided derivative from the right** and **the left** by the formulas (respectively)

$$D^+f(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h)-f(x_0)}{h},$$

$$D^-f(x_0) = \lim_{h\to 0^-} \frac{f(x_0+h)-f(x_0)}{h}.$$

Show that a function f has a derivative at x_0 if and only if $D^+f(x_0)$ and $D^-f(x_0)$ exist and are equal.

6. Prove Theorem 4.7 for n a positive integer. Then use Theorem 4.6 to establish the result for n a negative integer.

7. (a) Suppose $f : x \to x^3$ and $x_0 = 2$ in the Fundamental lemma of differentiation. Show that $\eta(h) = 6h + h^2$.

   (b) Find the explicit value of $\eta(h)$ if $f : x \to x^{-3}$, $x_0 = 1$.

8. (Partial **converse of the Chain rule**.) Suppose that f, g, and u are related such that $f(x) = g[u(x)]$, u is continuous at x_0, $f'(x_0)$ exists, and $g'[u(x_0)]$ exists and is not zero. Then prove that $u'(x_0)$ is defined and $f'(x_0) = g'[u(x_0)]u'(x_0)$. [*Hint*: Proceed as in the proof of the Chain rule (Theorem 4.9) to obtain the formula

$$\frac{\Delta u}{h} = \frac{\Delta f/h}{[g'(u) + \eta(\Delta u)]}.$$

Since (by hypothesis) u is continuous at x_0, it follows that $\lim_{h\to 0} \Delta u = 0$. Hence $\eta(\Delta u) \to 0$ as $h \to 0$. Complete the proof.]

9. Suppose that f is continuous on an open interval I containing x_0, suppose that f' is defined on I except possibly at x_0, and suppose that $f'(x) \to L$ as $x \to x_0$. Prove that $f'(x_0) = L$.

10. Prove Theorem 4.13. [*Hint*: Use the Mean-value theorem.]

11. Discuss the differentiability at $x = 0$ of the function

$$f : x \to \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

12. Discuss the differentiability at $x = 0$ of the function

$$f : x \to \begin{cases} x^n \sin \dfrac{1}{x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where n is an integer larger than 1. For what values of k, does the kth derivative exist at $x = 0$? (See Problem 11.)

13. If f is differentiable at x_0, prove that

$$\lim_{h\to 0} \frac{f(x_0+\alpha h) - f(x_0 - \beta h)}{h} = (\alpha + \beta) f'(x_0).$$

14. Prove Corollary 1 to Theorem 4.15.
15. Evaluate the following limit:
$$\lim_{x \to 0} \frac{\tan x - x}{x^3}.$$
16. Evaluate the following limit:
$$\lim_{x \to +\infty} \frac{x^3}{e^x}.$$
17. Evaluate
$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x.$$
[*Hint*: Take logarithms.]
18. If the second derivative f'' exists at a value x_0, show that
$$\lim_{h \to 0} \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} = f''(x_0).$$

4.2 Inverse Functions in $\mathbb{R}^1$

We have seen in Section 1.2 that a **relation from $\mathbb{R}^1$ to $\mathbb{R}^1$** is a set of ordered pairs of real numbers; that is, a relation is a set in $\mathbb{R}^2$. A function from $\mathbb{R}^1$ to $\mathbb{R}^1$ is a particular case of a relation, one in which no two ordered pairs have the same first element.

Definition
Suppose that S is a relation from $\mathbb{R}^1$ to $\mathbb{R}^1$. We define the **inverse relation** of S as the set of pairs (x, y) such that (y, x) is in S.

If S is the solution set in $\mathbb{R}^2$ of an equation such as $f(x, y) = 0$, the inverse of S is the solution set in $\mathbb{R}^2$ of the equation $f(y, x) = 0$. Suppose that f is a function, and we consider the ordered pairs in $\mathbb{R}^2$ that form the **graph** of the equation $y = f(x)$. Then the **inverse** of f is the graph of the equation $x = f(y)$. The simplest examples of functions show that the inverse of a function is a relation, but not necessarily a function. For example, the function $f : x \to x^2$, with graph $y = x^2$, has for its inverse the relation whose graph is given by $y^2 = x$. This inverse relation is not a function.

Theorem 4.17 (Inverse function theorem)
Suppose that f is a continuous, increasing function that has an interval I for domain and has range J. (*See Figure* 4.3.) *Then,*

(a) *J is an interval.*

(b) *The inverse relation g of f is a function with domain J, and g is continuous and increasing on J.*

(c) *We have*

$$g[f(x)] = x \text{ for } x \in I \text{ and } f[g(x)] = x \text{ for } x \in J. \tag{4.5}$$

A similar result holds if f is decreasing on I.

Proof

From the Intermediate-value theorem (Theorem 3.3) we know at once that J is an interval. We next establish the formulas in part (c). Let x_0 be any point of J. Then there is a number y_0 in I such that $x_0 = f(y_0)$. Since f is increasing, y_0 is unique. Hence g is a function with domain J, and the formulas in part (c) hold. To show that g is increasing, suppose that $x_1 < x_2$ with x_1, x_2 in J. Then it follows that $g(x_1) < g(x_2)$, for otherwise we would have $f[g(x_1)] \geq f[g(x_2)]$ or $x_1 \geq x_2$.

We now show that g is continuous. Let x_0 be an interior point of J and suppose $y_0 = g(x_0)$ or, equivalently, $x_0 = f(y_0)$. Let x_1' and x_2' be points of J such that $x_1' < x_0 < x_2'$. Then the points $y_1' = g(x_1')$ and $y_2' = g(x_2')$ of I are such that $y_1' < y_0 < y_2'$. Hence y_0 is an interior point of I. Now let $\epsilon > 0$ be given and chosen so small that $y_0 - \epsilon$ and $y_0 + \epsilon$ are points of I. We define $x_1 = f(y_0 - \epsilon)$ and $x_2 = f(y_0 + \epsilon)$. See Figure 4.3. Since g is increasing,

$$y_0 - \epsilon = g(x_1) \leq g(x) \leq g(x_2) = y_0 + \epsilon \text{ for all } x \text{ such that } x_1 \leq x \leq x_2.$$

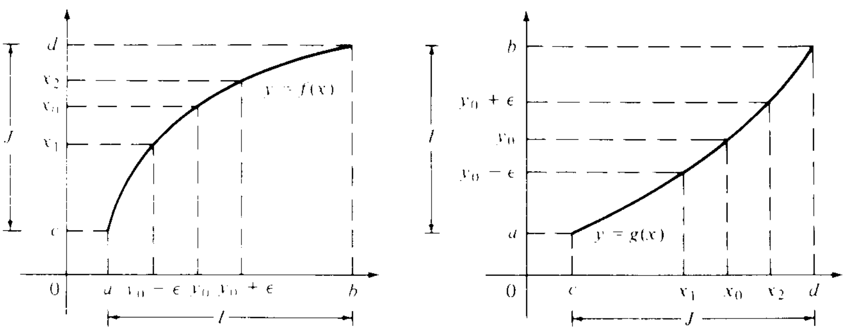

Figure 4.3 f and g are inverse functions.

Because $y_0 = g(x_0)$, the above inequalities may be written

$$g(x_0) - \epsilon \leq g(x) \leq g(x_0) + \epsilon \text{ for } x_1 \leq x \leq x_2.$$

Choosing δ as the minimum of the distances $x_2 - x_0$ and $x_0 - x_1$, we obtain

$$|g(x) - g(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta.$$

That is, g is continuous at x_0, an arbitrary interior point of J. A slight modification of the above argument shows that if the range J contains its endpoints, then g is continuous on the right at c and continuous on the left at d. ■

The proof of Theorem 4.17 for functions that are decreasing is completely analogous. In this case the inverse function g is also decreasing. If the function f in the Inverse function theorem has a derivative, then the following result on the differentiability of the inverse function g holds.

Theorem 4.18 (Inverse differentiation theorem)
Suppose that f satisfies the hypotheses of Theorem 4.17. Assume that x_0 is a point of J such that $f'[g(x_0)]$ is defined and is different from zero. Then $g'(x_0)$ exists and

$$g'(x_0) = \frac{1}{f'[g(x_0)]}. \tag{4.6}$$

Proof
From Theorem 4.17 we have formula (4.5),

$$f[g(x)] = x.$$

Using the Chain rule, we conclude that $g'(x)$ exists and

$$f'[g(x_0)]g'(x_0) = 1.$$

Since $f' \neq 0$ by hypothesis, the result follows. ■

It is most often the case that a function f is not always increasing or always decreasing. In such situations the inverse relation of f may be analyzed as follows: First find the intervals $I_1, I_2, \ldots$ on each of which f is always increasing or always decreasing. Denote by f_i the function f restricted to the interval I_i, $i = 1, 2, \ldots$. Then the inverse of each f_i is a function which we denote by g_i. These inverses may be analyzed separately and differentiation applied to the formulas $f_i[g_i(x)] = x$, $i = 1, 2, \ldots$.

For example, suppose $f : x \to x^2$ is defined on $I = \{x : -\infty < x < \infty\}$. Then f is increasing on the interval $I_1 = \{x : 0 \leq x < \infty\}$ and decreasing on $I_2 = \{x : -\infty < x \leq 0\}$. The restriction of f to I_1, denoted by f_1, has the inverse $g_1 : x \to \sqrt{x}$ with domain $J_1 = \{x : 0 \leq x < \infty\}$. The restriction of f to I_2, denoted by f_2, has the inverse $g_2 : x \to -\sqrt{x}$, also with domain J_1. Equations (4.5) become in these two cases

$$\begin{aligned} (\sqrt{x})^2 &= x \quad \text{for } x \in J_1, & \sqrt{x^2} &= x \quad \text{for } x \in I_1, \\ (-\sqrt{x})^2 &= x \quad \text{for } x \in J_1, & -\sqrt{x^2} &= x \quad \text{for } x \in I_2. \end{aligned}$$

The inverse relation of f is $g_1 \cup g_2$.

Problems

In each of Problems 1 through 10 a function f is given. Find the intervals $I_1, I_2, \ldots$ on which f is either increasing or decreasing, and find the corresponding intervals $J_1, J_2, \ldots$ on which the inverses are defined. Plot a graph of f and the inverse functions $g_1, g_2, \ldots$ corresponding to $J_1, J_2, \ldots$. Find expressions for each g_i when possible.

1. $f : x \to x^2 + 2x + 2$.
2. $f : x \to (x^2/2) + 3x - 4$.
3. $f : x \to 4x - x^2$.
4. $f : x \to 2 - x - (x^2/2)$.
5. $f : x \to 2x/(x+2)$.
6. $f : x \to (1+x)/(1-x)$.
7. $f : x \to (4x)/(x^2+1)$.
8. $f : x \to (x-1)^3$.
9. $f : x \to x^3 + 3x$.
10. $f : x \to (2x^3/3) + x^2 - 4x + 1$.
11. Suppose that f and g are increasing on an interval I and that $f(x) > g(x)$ for all $x \in I$. Denote the inverses of f and g by F and G and their domains by J_1 and J_2, respectively. Prove that $F(x) < G(x)$ for each $x \in J_1 \cap J_2$.

In each of Problems 12 through 16 a function f is given that is increasing or decreasing on I. Hence there is an inverse function g. Compute f' and g' and verify formula (4.6) of the Inverse differentiation theorem.

12. $f : x \to 4x/(x^2+1)$, $I = \{x : \frac{1}{2} < x < \infty\}$.
13. $f : x \to (x-1)^3$, $I = \{x : 1 < x < \infty\}$.
14. $f : x \to x^3 + 3x$, $I = \{x : -\infty < x < \infty\}$.
15. $f : x \to \sin x$, $I = \{x : \pi/2 < x < \pi\}$.
16. $f : x \to e^{3x}$, $I = \{x : -\infty < x < \infty\}$.
17. In the Inverse function theorem show that if the range J contains its endpoints, then g is continuous on the right at c and continuous on the left at d.

CHAPTER 5 Elementary Theory of Integration

5.1 The Darboux Integral for Functions on $\mathbb{R}^N$

The reader is undoubtedly familiar with the idea of the integral and with methods of performing integrations. In this section we define the integral precisely and prove the basic theorems for the processes of integration.

Let f be a bounded function whose domain is a closed interval $I = \{x : a \le x \le b\}$. We subdivide I by introducing points $t_1, t_2, \ldots, t_{n-1}$ that are interior to I. Setting $a = t_0$ and $b = t_n$ and ordering the points so that $t_0 < t_1 < t_2 < \ldots < t_n$, we denote by $I_1, I_2, \ldots, I_n$ the intervals $I_i = \{x : t_{i-1} \le x \le t_i\}$. Two successive intervals have exactly one point in common. (See Figure 5.1.) We call such a decomposition of I into subintervals a **subdivision** of I, and we use the symbol Δ to indicate such a subdivision.

Since f is bounded on I, it has a least upper bound (l.u.b.), denoted by M. The greatest lower bound (g.l.b.) of f on I is denoted by m. Similarly, M_i and m_i denote the l.u.b. and g.l.b., respectively, of f on I_i. The length of the interval I_i is $t_i - t_{i-1}$; it is denoted by $l(I_i)$.

Definitions

The **upper Darboux sum of f with respect to the subdivision Δ**, denoted $S^+(f, \Delta)$, is defined by

$$S^+(f, \Delta) = \sum_{i=1}^{n} M_i l(I_i). \tag{5.1}$$

Similarly, the **lower Darboux sum**, denoted $S_-(f, \Delta)$, is defined by

$$S_-(f, \Delta) = \sum_{i=1}^{n} m_i l(I_i). \tag{5.2}$$

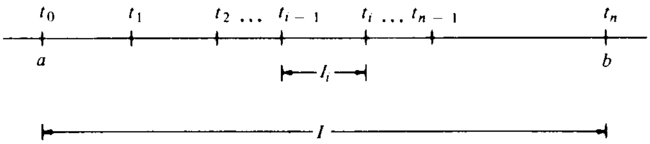

Figure 5.1 A subdivision of I.

Suppose Δ is a subdivision $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$, with the corresponding intervals denoted by $I_1, I_2, \ldots, I_n$. We obtain a new subdivision by introducing additional subdivision points between the various $\{t_i\}$. This new subdivision will have subintervals $I_1', I_2', \ldots, I_m'$ each of which is a part (or all) of one of the subintervals of Δ. We denote this new subdivision by Δ' and call it a **refinement of** $\boldsymbol{\Delta}$.

Suppose that Δ_1 with subintervals $I_1, I_2, \ldots, I_n$ and that Δ_2 with subintervals $J_1, J_2, \ldots, J_m$ are two subdivisions of an interval I. We get a new subdivision of I by taking *all* the endpoints of the subintervals of both Δ_1 and Δ_2, arranging them in order of increasing size, and then labeling each subinterval having as its endpoints two successive subdivision points. Such a subdivision is called the **common refinement of** $\boldsymbol{\Delta_1}$ **and** $\boldsymbol{\Delta_2}$, and each subinterval of this new subdivision is of the form $I_i \cap J_k$, $i = 1, 2, \ldots, n$, $k = 1, 2, \ldots, m$. Each $I_i \cap J_k$ is either empty or entirely contained in a unique subinterval (I_i) of Δ_i and a unique subinterval (J_k) of Δ_2. An illustration of such a common refinement is exhibited in Figure 5.2.

Theorem 5.1
Suppose that f is a bounded function with domain $I = \{x{:}a \le x \le b\}$. Let Δ be a subdivision of I and suppose that M and m are the least upper bound and greatest lower bound of f on I, respectively. Then

(a) $$m(b-a) \le S_-(f, \Delta) \le S^+(f, \Delta) \le M(b-a).$$

(b) *If Δ' is a refinement of Δ, then*

$$S_-(f, \Delta) \le S_-(f, \Delta') \le S^+(f, \Delta') \le S^+(f, \Delta).$$

(c) *If Δ_1 and Δ_2 are any two subdivisions of I, then*

$$S_-(f, \Delta_1) \le S^+(f, \Delta_2).$$

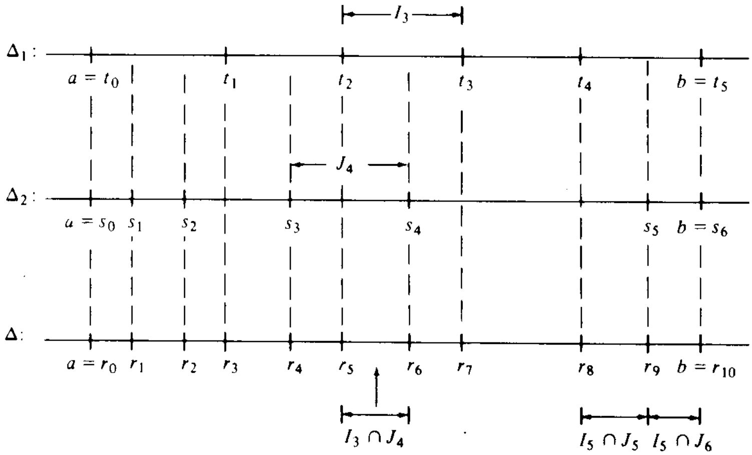

Figure 5.2 A common refinement.

Proof

(a) From the definition of least upper bound and greatest lower bound, we have

$$m \le m_i \le M_i \le M, \quad i = 1, 2, \dots, n.$$

Also, $b - a = \sum_{i=1}^{n} l(I_i)$. Thus the inequalities in part (a) are an immediate consequence of the definition of S^+ and S_- as given in equations (5.1) and (5.2):

$$m(b-a) = \sum_{i=1}^{n} ml(I_i) \le \sum_{i=1}^{n} m_i l(I_i) = S_-(f, \Delta)$$

$$\le S^+(f, \Delta) = \sum_{i=1}^{n} M_i l(I_i) \le M(b-a).$$

(b) To prove part (b), let Δ_i be the subdivision of I_i that consists of all those intervals of Δ' that lie in I_i. Since each interval of Δ' is in a unique Δ_i we have, applying part (a) to each Δ_i,

$$S_-(f, \Delta) = \sum_{i=1}^{n} m_i l(I_i) \le \sum_{i=1}^{n} S_-(f, \Delta_i) = S_-(f, \Delta')$$

$$\le S^+(f, \Delta') = \sum_{i=1}^{n} S^+(f, \Delta_i) \le \sum_{i=1}^{n} M_i l(I_i) = S^+(f, \Delta).$$

(c) To prove part (c), let Δ be the common refinement of Δ_1 and Δ_2. Then, using the result in part (b), we obtain

$$S_-(f, \Delta_1) \le S_-(f, \Delta) \le S^+(f, \Delta) \le S^+(f, \Delta_2).$$

■

Definitions

If f is a function on $\mathbb{R}^1$ that is defined and bounded on $I = \{x : a \le x \le b\}$, we define its **upper** and **lower Darboux integrals** by

$$\overline{\int_a^b} f(x)dx = \inf S^+(f, \Delta) \text{ for all subdivisions } \Delta \text{ of } I,$$

$$\underline{\int_a^b} f(x)dx = \sup S_-(f, \Delta) \text{ for all subdivisions } \Delta \text{ of } I.$$

If

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx,$$

we say that f is **Darboux integrable**, or just **integrable**, **on I**, and we designate the common value by

$$\int_a^b f(x)dx.$$

The following results are all direct consequences of the above definitions.

Theorem 5.2

If $m \le f(x) \le M$ for all $x \in I = \{x{:}a \le x \le b\}$, then

$$m(b-a) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le M(b-a).$$

Proof

Let Δ_1 and Δ_2 be subdivisions of I. Then from parts (a) and (c) of Theorem 5.1, it follows that

$$m(b-a) \le S_-(f, \Delta_1) \le S^+(f, \Delta_2) \le M(b-a).$$

We keep Δ_1 fixed and let Δ_2 vary over all possible subdivisions. Thus $S^+(f, \Delta_2)$ is always larger than or equal to $S_-(f, \Delta_1)$, and so its greatest lower bound is always larger than or equal to $S_-(f, \Delta_1)$. We conclude that

$$m(b-a) \le S_-(f, \Delta_1) \le \overline{\int_a^b} f(x)dx \le M(b-a).$$

Now letting Δ_1 vary over all possible subdivisions, we see that the least upper bound of $S_-(f, \Delta_1)$ cannot exceed $\overline{\int_a^b} f(x)dx$. The conclusion of the theorem follows. ■

The next theorem states simple facts about upper and lower Darboux integrals. The results all follow from the definitions of inf, sup, and upper Darboux and lower Darboux integrals.

Theorem 5.3
Assuming that all functions below are bounded, we have the following formulas:
(a) *If* $g(x) = kf(x)$ *for all* $x \in I = \{x{:}a \le x \le b\}$ *and* k *is a positive number, then*

(i) $\underline{\int_a^b} g(x)dx = k\underline{\int_a^b} f(x)$ *and* $\overline{\int_a^b} g(x)dx = k\overline{\int_a^b} f(x)dx$.

If $k < 0$, *then*

(ii) $\underline{\int_a^b} g(x)dx = k\overline{\int_a^b} f(x)dx$ *and* $\overline{\int_a^b} g(x)dx = k\underline{\int_a^b} f(x)dx$.

(b) *If* $h(x) = f_1(x) + f_2(x)$ *for all* $x \in I$, *then*

(i) $\underline{\int_a^b} h(x)dx \ge \underline{\int_a^b} f_1(x)dx + \underline{\int_a^b} f_2(x)dx$.

(ii) $\overline{\int_a^b} h(x)dx \le \overline{\int_a^b} f_1(x)dx + \overline{\int_a^b} f_2(x)dx$.

(c) *If* $f_1(x) \le f_2(x)$ *for all* $x \in I$, *then*

(i) $\underline{\int_a^b} f_1(x)dx \le \underline{\int_a^b} f_2(x)dx$ *and*

(ii) $\overline{\int_a^b} f_1(x)dx \le \overline{\int_a^b} f_2(x)dx$.

(d) *If* $a < b < c$, *then*

(i) $\underline{\int_a^c} f(x)dx = \underline{\int_a^b} f(x)dx + \underline{\int_b^c} f(x)dx$.

(ii) $\overline{\int_a^c} f(x)dx = \overline{\int_a^b} f(x)dx + \overline{\int_b^c} f(x)dx$.

Corollary
If the functions considered in Theorem 5.3 *are Darboux integrable, the following formulas hold*:
(a) *If* $g(x) = kf(x)$ *and* k *is any constant, then*

$$\int_a^b g(x)dx = k\int_a^b f(x)dx.$$

(b) *If* $h(x) = f_1(x) + f_2(x)$, *then*

$$\int_a^b h(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx.$$

(c) *If* $f_1(x) \le f_2(x)$ *for all* $x \in I$, *then*

$$\int_a^b f_1(x)dx \le \int_a^b f_2(x)dx.$$

(d) *Suppose* f *is Darboux integrable on* $I_1 = \{x{:}a \le x \le b\}$ *and on* $I_2 = \{x{:}b \le x \le c\}$. *Then it is Darboux integrable on* $I = \{x{:}a \le x \le c\}$, *and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx. \tag{5.3}$$

It is useful to define

$$\int_a^a f(x)dx = 0 \text{ and } \int_b^a f(x)dx = -\int_a^b f(x)dx.$$

Then formula (5.3) above is valid whether or not b is between a and c so long as all the integrals exist.

It is important to be able to decide when a particular function is integrable. The next theorem gives a necessary and sufficient condition for integrability.

Theorem 5.4
Suppose that f *is bounded on an interval* $I = \{x{:}a \le x \le b\}$. *Then* f *is integrable* $\Leftrightarrow$ *for every* $\epsilon > 0$ *there is a subdivision* Δ *of* I *such that*

$$S^+(f, \Delta) - S_-(f, \Delta) < \epsilon. \tag{5.4}$$

Proof
Suppose that condition (5.4) holds. Then from the definitions of upper and lower Darboux integrals, we have

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le S^+(f, \Delta) - S_-(f, \Delta) < \epsilon.$$

Since this inequality holds for every $\epsilon > 0$ and the left side of the inequality is independent of ϵ, it follows that

$$\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx = 0.$$

Hence f is integrable.

Now assume that f is integrable; we wish to establish inequality (5.4). For any $\epsilon > 0$ there are subdivisions Δ_1 and Δ_2 such that

$$S^+(f, \Delta_1) < \overline{\int_a^b} f(x)dx + \frac{1}{2}\epsilon \text{ and } S_-(f, \Delta_2) > \underline{\int_a^b} f(x)dx - \frac{\epsilon}{2}. \tag{5.5}$$

We choose Δ as the common refinement of Δ_1 and Δ_2. Then (from Theorem 5.1)

$$S^+(f, \Delta) - S_-(f, \Delta) \le S^+(f, \Delta_1) - S_-(f, \Delta_2).$$

Substituting inequalities (5.5) into the above expression and using the fact that $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$, we obtain $S^+(f, \Delta) - S_-(f, \Delta) < \epsilon$, as required. ■

Corollary

If f is continuous on the closed interval $I = \{x{:}a \leq x \leq b\}$, then f is integrable.

Proof

The function f is uniformly continuous, and hence for any $\epsilon > 0$ there is a δ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b-a} \text{ whenever } |x - y| < \delta.$$

Choose a subdivision Δ of I such that no subinterval of Δ has length larger than δ. This allows us to establish formula (5.4), and the result follows from Theorem 5.4. ■

Theorem 5.5

(**Mean-value theorem for integrals**). *Suppose that f is continuous on $I = \{x{:}a \leq x \leq b\}$. Then there is a number ξ in I such that*

$$\int_a^b f(x)dx = f(\xi)(b - a).$$

Proof

According to Theorem 5.2, we have

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a), \tag{5.6}$$

where m and M are the minimum and maximum of f on I, respectively. From the Extreme-value theorem (Theorem 3.12) there are numbers x_0 and $x_1 \in I$ such that $f(x_0) = m$ and $f(x_1) = M$. From inequalities (5.6) it follows that

$$\int_a^b f(x)dx = A(b - a),$$

where A is a number such that $f(x_0) \leq A \leq f(x_1)$. Then the intermediate-value theorem (Theorem 3.3) shows that there is a number $\xi \in I$ such that $f(\xi) = A$. ■

We now present two forms of the Fundamental theorem of calculus, a result that shows that differentiation and integration are inverse processes.

Theorem 5.6
(**Fundamental theorem of calculus—first form**). *Suppose that* f *is continuous on* $I = \{x{:}a \le x \le b\}$, *and that* F *is defined by*

$$F(x) = \int_a^x f(t)dt.$$

Then F *is continuous on* I, *and* $F'(x) = f(x)$ *for each* $x \in I_1 = \{x{:}a < x < b\}$.

Proof
Since f is integrable on every subinterval of I, we have

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt,$$

or

$$F(x+h) = F(x) + \int_x^{x+h} f(t)dt.$$

We apply the Mean-value theorem for integrals (Theorem 5.5) to the last term on the right, getting

$$\frac{F(x+h) - F(x)}{h} = f(\xi),$$

where ξ is some number between x and $x+h$. Now, let $\epsilon > 0$ be given. There is a $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all y on I_1 (taking $\xi = y$) with $|y - x| < \delta$. Thus, if $0 < |h| < \delta$, we find that

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = |f(\xi) - f(x)| < \epsilon,$$

since $|\xi - x| \le |h| < \delta$. Since this is true for each $\epsilon > 0$, the result follows. ■

Theorem 5.7
(**Fundamental theorem of calculus—second form**). *Suppose that* f *and* F *are continuous on* $I = \{x{:}a \le x \le b\}$ *and that* $F'(x) = f(x)$ *for each* $x \in I_1 = \{x{:}a < x < b\}$. *Then*

$$\int_a^b f(x)dx = F(b) - F(a). \tag{5.7}$$

Proof
We consider a subdivision Δ of I. We write

$$F(b) - F(a) = \sum_{i=1}^{n} \{F(x_i) - F(x_{i-1})\}.$$

We apply the Mean-Value theorem to each term in the sum. Then as the mesh size tends to zero we get (5.7). ■

Problems

1. Compute $S^+(f, \Delta)$ and $S_-(f, \Delta)$ for the function $f : x \to x^2$ defined on $I = \{x : 0 \le x \le 1\}$ where Δ is the subdivision of I into 5 subintervals of equal size.
2. (a) Given the function $f : x \to x^3$ defined on $I = \{x : 0 \le x \le 1\}$, suppose Δ is a subdivision and Δ' is a refinement of Δ that adds one more point. Show that

$$S^+(f, \Delta') < S^+(f, \Delta) \text{ and } S_-(f, \Delta') > S_-(f, \Delta).$$

   (b) Give an example of a function f defined on I such that

$$S^+(f, \Delta') = S^+(f, \Delta) \text{ and } S_-(f, \Delta') = S_-(f, \Delta)$$

   for the two subdivisions in part (a).

   (c) If f is a strictly increasing continuous function on I, show that $S^+(f, \Delta') < S^+(f, \Delta)$, where Δ' is any refinement of Δ.
3. If $g(x) = kf(x)$ for all $x \in I = \{x : a \le x \le b\}$, show that

$$\underline{\int_a^b} g(x)dx = k \overline{\int_a^b} f(x)dx \text{ if } k < 0.$$

4. (a) If $h(x) = f_1(x) + f_2(x)$ for $x \in I = \{x : a \le x \le b\}$, show that

$$\underline{\int_a^b} h(x)dx \ge \underline{\int_a^b} f_1(x)dx + \underline{\int_a^b} f_2(x)dx,$$

$$\overline{\int_a^b} h(x)dx \le \overline{\int_a^b} f_1(x)dx + \overline{\int_a^b} f_2(x)dx.$$

   (b) If f_1 and f_2 are Darboux integrable, show that

$$\int_a^b h(x)dx = \int_a^b f_1(x)dx + \int_a^b f_2(x)dx.$$

5. If $f_1(x) \le f_2(x)$ for $x \in I = \{x : a \le x \le b\}$, show that

$$\underline{\int_a^b} f_1(x)dx \le \underline{\int_a^b} f_2(x)dx \text{ and } \overline{\int_a^b} f_1(x)dx \le \overline{\int_a^b} f_2(x)dx.$$

   If f_1 and f_2 are Darboux integrable, conclude that

$$\int_a^b f_1(x)dx \le \int_a^b f_2(x)dx.$$

6. Show that

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

   whether or not b is between a and c so long as all three integrals exist.

7. If f is increasing on the interval $I = \{x : a \le x \le b\}$, show that f is integrable. [*Hint*: Use the formula

$$S^+(f, \Delta) - S_-(f, \Delta) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] l(I_i).]$$

Replace each $l(I_i)$ by the length of the *longest* subinterval, thereby obtaining an inequality, and then note that the terms "telescope."

8. Suppose that f is a bounded function on $I = \{x : a \le x \le b\}$. Let $M = \sup f(x)$ and $m = \inf f(x)$ for $x \in I$. Also, define $M^* = \sup |f(x)|$ and $m^* = \inf |f(x)|$ for $x \in I$.
   (a) Show that $M^* - m^* \le M - m$.
   (b) If f and g are nonnegative bounded functions on I, and $N = \sup g(x)$, $n = \inf g(x)$ for $x \in I$, show that

$$\sup f(x)g(x) - \inf f(x)g(x) \le MN - mn.$$

9. (a) Suppose that f is bounded and integrable on $I = \{x : a \le x \le b\}$. Prove that $|f|$ is integrable on I. [*Hint*: See Problem 8a.]
   (b) Show that $|\int_a^b f(x)\, dx| \le \int_a^b |f(x)|\, dx$.
10. Suppose that f and g are nonnegative, bounded, and integrable on $I = \{x : a \le x \le b\}$. Prove that fg is integrable on I. [*Hint*: See Problem 8b.]
11. Suppose that f is defined on $I = \{x : 0 \le x \le 1\}$ by the formula

$$f(x) = \begin{cases} \dfrac{1}{2^n} & \text{if } x = \dfrac{j}{2^n} \text{ where } j \text{ is an odd integer} \\ & \text{and } 0 < j < 2^n,\ n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Determine whether or not f is integrable and prove your result.

12. Given the function f defined on $I = \{x : 0 \le x \le 1\}$ by the formula

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $\underline{\int_0^1} f(x)dx = 0$ and $\overline{\int_0^1} f(x)dx = 1$.

13. Suppose that f and g are positive and continuous on $I = \{x : a \le x \le b\}$. Prove that there is a number $\xi \in I$ such that

$$\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx.$$

[*Hint*: Use the Intermediate-value theorem and see Problem 5.]

14. Suppose that f is continuous on $I = \{x : a \le x \le b\}$ except at an interior point c. If f is also bounded on I, prove that f is integrable on I. Show that the value of f at c does not affect the value of $\int_a^b f(x)dx$.

Conclude that the value of f at any *finite* number of points cannot affect the value of the integral of f.

15. Suppose that f is continuous, nonnegative, and not identically zero on $I = \{x : a \le x \le b\}$. Prove that $\int_a^b f(x)dx > 0$. Is the result true if f is not continuous but only integrable on I?
16. Suppose that f is continuous on $I = \{x : a \le x \le b\}$ and that $\int_a^b f(x)g(x)dx = 0$ for every function g continuous on I. Prove that $f(x) \equiv 0$ on I.
17. Suppose that f is continuous on $I = \{x : a \le x \le b\}$ and that $\int_a^b f(x)dx = 0$. If f is nonnegative on I show that $f \equiv 0$.

5.2 The Riemann Integral

In addition to the development of the integral by the method of Darboux, there is a technique due to Riemann that starts with a direct approximation of the integral by a sum. The main result of this section shows that the two definitions of integral are equivalent.

Definitions

Suppose that Δ is a subdivision of $I = \{x : a \le x \le b\}$ with subintervals $I_1, I_2, \ldots, I_n$. We call the **mesh** of the subdivision Δ the length of the largest subinterval among $l(I_1), l(I_2), \ldots, l(I_n)$. The mesh is denoted by $\|\Delta\|$. Suppose that f is defined on I and that Δ is a subdivision. In each subinterval of Δ we choose a point $x_i \in I_i$. The quantity

$$\sum_{i=1}^{n} f(x_i)l(I_i)$$

is called a **Riemann sum**.

For nonnegative functions it is intuitively clear that a Riemann sum with very small mesh gives a good approximation to the area under the curve. (See Figure 5.3.)

Definitions

Suppose that f is defined on $I = \{x : a \le x \le b\}$. Then f is **Riemann integrable on** I if there is a number A with the following property: For every $\epsilon > 0$ there is a $\delta > 0$ such that for every subdivision Δ with mesh smaller than δ, the inequality

$$\left| \sum_{i=1}^{n} f(x_i)l(I_i) - A \right| < \epsilon \tag{5.8}$$

holds for every possible choice of x_i in I_i. The quantity A is called the **Riemann integral of** f.

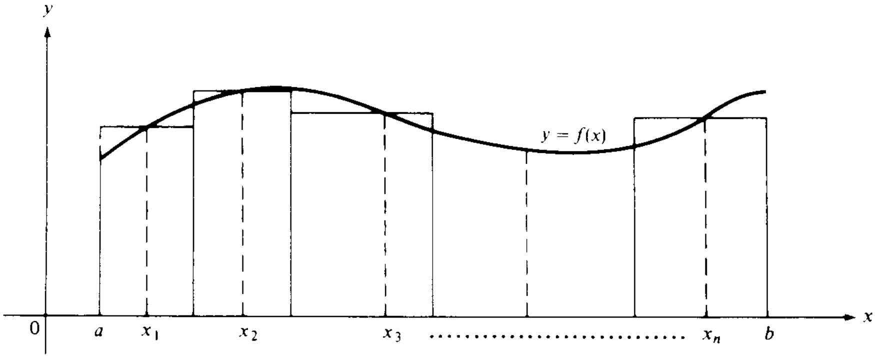

Figure 5.3 A Riemann sum.

Theorem 5.8
If f is Riemann integrable on $I = \{x{:}a \le x \le b\}$, then f is bounded on I.

Proof
In the definition of Riemann integral, choose $\epsilon = 1$ and let Δ be a subdivision such that inequality (5.8) holds. We have

$$\left|\sum_{i=1}^{n} f(x_i)l(I_i) - A\right| < 1 \text{ and } \left|\sum_{i=1}^{n} f(x_i')l(I_i) - A\right| < 1,$$

where $x_1, x_2, \ldots, x_n$ and $x_1', x_2', \ldots, x_n'$ are any points of I such that x_i and x_i' are in I_i. Therefore,

$$\left|\sum_{i=1}^{n} f(x_i)l(I_i) - \sum_{i=1}^{n} f(x_i')l(I_i)\right| < 2.$$

Now select $x_i = x_i'$ for $i = 2, 3, \ldots, n$. Then the above inequality becomes

$$|f(x_1) - f(x_1')|l(I_1) < 2 \Leftrightarrow |f(x_1) - f(x_1')| < \frac{2}{l(I_1)}.$$

Using the general inequality $|\alpha| - |\beta| < |\alpha - \beta|$, we obtain

$$|f(x_1)| < \frac{2}{l(I_1)} + |f(x_1')|.$$

Fix x_1' and observe that the above inequality is valid for every $x_1 \in I_1$. Hence f is bounded on I_1. Now the same argument can be made for $I_2, I_3, \ldots, I_n$. Therefore, f is bounded on I. ■

Theorem 5.9

If f is Riemann integrable on $I = \{x{:}a \le x \le b\}$, then f is Darboux integrable on I. Letting A denote the Riemann integral, we have

$$A = \int_a^b f(x)dx.$$

Proof

In Formula (5.8), we may replace ϵ by $\epsilon/4$ for a subdivision Δ with sufficiently small mesh:

$$\left|\sum_{i=1}^{n} f(x_i)l(I_i) - A\right| < \frac{\epsilon}{4}. \tag{5.9}$$

Let M_i and m_i denote the least upper bound and greatest lower bound, respectively, of f in I_i. Then there are points x_i' and x_i'' such that

$$f(x_i') > M_i - \frac{\epsilon}{4(b-a)} \text{ and } f(x_i'') < m_i + \frac{\epsilon}{4(b-a)}.$$

These inequalities result just from the definition of l.u.b. and g.l.b. Then we have

$$\begin{aligned} S^+(f,\Delta) &= \sum_{i=1}^{n} M_i l(I_i) < \sum_{i=1}^{n}\left[f(x_i') + \frac{\epsilon}{4(b-a)}\right] l(I_i) \\ &= \sum_{i=1}^{n} f(x_i')l(I_i) + \frac{1}{4}\epsilon. \end{aligned}$$

Hence, using inequality (5.9), we obtain

$$S^+(f,\Delta) < A + \frac{\epsilon}{4} + \frac{\epsilon}{4} = A + \frac{\epsilon}{2}. \tag{5.10}$$

Similarly,

$$\begin{aligned} S_-(f,\Delta) &= \sum_{i=1}^{n} m_i l(I_i) > \sum_{i=1}^{n}\left[f(x_i'') - \frac{\epsilon}{4(b-a)}\right] l(I_i) \\ &= \sum_{i=1}^{n} f(x_i'')l(I_i) - \frac{1}{4}\epsilon, \end{aligned}$$

and

$$S_-(f,\Delta) > A - \frac{\epsilon}{2}. \tag{5.11}$$

Subtraction of inequality (5.11) from inequality (5.10) yields

$$S^+(f,\Delta) - S_-(f,\Delta) < \epsilon,$$

thus f is Darboux integrable. Furthermore, these inequalities show that

$$A = \int_a^b f(x)dx.$$

■

We omit the proof of the theorem that shows that every Darboux integrable function is Riemann integrable. Thus we state the following result:

Theorem 5.10
A function f is Riemann integrable on $[a,b]$ if and only if it is Darboux integrable on $[a,b]$.

In light of Theorem 5.10 we shall now drop the terms *Darboux* and *Riemann*, and just refer to functions as *integrable*.

One of the most useful methods for performing integrations when the integrand is not in standard form is the method known as *substitution*. For example, we may sometimes be able to show that a complicated integral has the form

$$\int_a^b f[u(x)]u'(x)dx.$$

Then, if we set $u = u(x)$, $du = u'(x)dx$, the above integral becomes

$$\int_{u(a)}^{u(b)} f(u)du,$$

and this integral may be one we recognize as a standard integration. In an elementary course in calculus these substitutions are usually made without concern for their validity. The theoretical foundation for such processes is based on the next result.

Theorem 5.11
Suppose that f is continuous on an open interval I. Let u and u' be continuous on an open interval J, and assume that the range of u is contained in I. If $a,b \in J$, then

$$\int_a^b f[u(x)]u'(x)dx = \int_{u(a)}^{u(b)} f(u)du. \tag{5.12}$$

Proof
Let $c \in I$ and define

$$F(u) = \int_c^u f(t)dt. \tag{5.13}$$

From the Fundamental theorem of calculus (Theorem 5.6) we have

$$F'(u) = f(u).$$

Defining $G(x) = F[u(x)]$, we employ the Chain rule to obtain

$$G'(x) = F'[u(x)]u'(x) = f[u(x)]u'(x).$$

Since all functions under consideration are continuous, it follows that

$$\int_a^b f[u(x)]u'(x)dx = \int_a^b G'(x)dx = G(b) - G(a) = F[u(b)] - F[u(a)].$$

From equation (5.13) above, we see that

$$F[u(b)] - F[u(a)] = \int_c^{u(b)} f(t)dt - \int_c^{u(a)} f(t)dt = \int_{u(a)}^{u(b)} f(t)dt,$$

and the theorem is proved. ■

Problems

1. (a) Suppose that F is continuous on $I = \{x : a \le x \le b\}$. Show that all upper and lower Darboux sums are Riemann sums.
   (b) Suppose that f is increasing on $I = \{x : a \le x \le b\}$. Show that all upper and lower Darboux sums are Riemann sums.
   (c) Give an example of a bounded function f defined on I in which a Darboux sum is not a Riemann sum.
2. Suppose that u, u', v, and v' are continuous on $I = \{x : a \le x \le b\}$. Establish the formula for **integration by parts**:

$$\int_a^b u(x)v'(x)dx = u(b)v(b) - u(a)v(a) - \int_a^b v(x)u'(x)dx.$$

3. Suppose u, v, w, u', v', and w' are continuous on $I = \{x : a \le x \le b\}$. Establish the extended integration by parts formula:

$$\int_a^b u(x)v(x)w'(x)dx = u(b)v(b)w(b) - u(a)v(a)w(a) - \int_a^b u(x)v'(x)w(x)dx - \int_a^b u'(x)v(x)w(x)dx.$$

4. Give an example of a function f defined on $I = \{x : 0 \le x \le 1\}$ such that $|f|$ is integrable but f is not.
5. Show, from the definition, that the Riemann integral is unique. [*Hint*: Assume that there are two different numbers A and A' satisfying inequality (5.8), and then reach a contradiction.]

6. State conditions on u, v, and f such that the following formula is valid:

$$\int_a^b f[u\{v(x)\}]u'[v(x)]v'(x)dx = \int_{u[v(a)]}^{u[v(b)]} f(u)du.$$

7. Given f on $I = \{x : a \leq x \leq b\}$, if $\int_a^b f^3(x)dx$ exists, does it follow that $\int_a^b f(x)dx$ exists?

5.3 The Logarithm and Exponential Functions

The reader is undoubtedly familiar with the logarithm function and the exponential function. In this section, we define these functions precisely and develop their principal properties.

Definitions

The **natural logarithm function**, denoted by log, is defined by the formula

$$\log x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

Theorem 5.12

Let $f{:}x \to \log x$ be defined for $x > 0$ and suppose a and b are positive numbers. The following statements hold:

(i) $\log(ab) = \log a + \log b$.
(ii) $\log 1 = 0$.
(iii) $\log(a^r) = r\log a$ *for every rational number* r.
(iv) $f'(x) = 1/x$.
(v) $\log x \to +\infty$ *as* $x \to +\infty$.
(vi) *The range of* f *is all of* $\mathbb{R}^1$.

Proof

To prove (i), we write

$$\log(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt.$$

Changing variables in the last integral on the right by letting $u = t/a$, we see that

$$\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{u}\,du.$$

Hence $\log(ab) = \log a + \log b$.

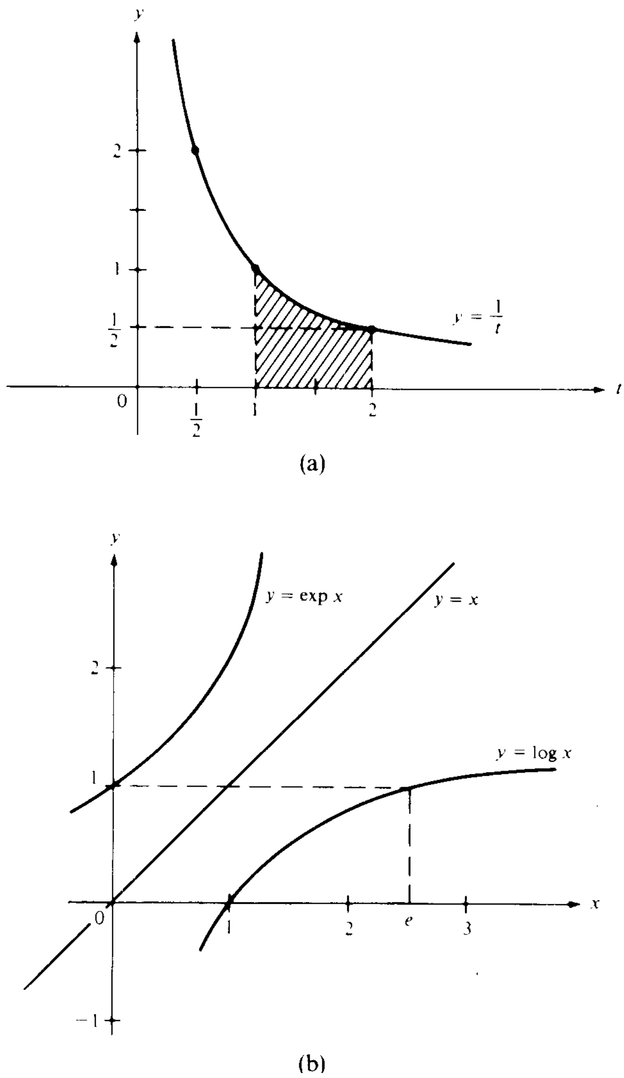

Figure 5.4 The logarithm and its inverse.

To verify (ii) simply set $b = 1$ in the above formula.

To establish (iii) we proceed step by step. If r is a positive integer, we get the result from (i) with $a = b$ and mathematical induction. For negative integers, write $a^{-n} = 1/a^n$. Finally, if $r = p/q$ where p and q are integers, set $u = a^{1/q}$, and thus $u^q = a$. Hence $q \log u = \log a$. Since

$$a^r = a^{p/q} = u^p,$$

we have

$$\log(a^r) = \log(u^p) = p \log u = \frac{p}{q} \log a = r \log a.$$

Statement (iv) is simply a statement of the Fundamental theorem of calculus.

To prove (v), note that if $x > 2^n$, n a positive integer, then

$$\log x > \log(2^n) = n \log 2 \geq \frac{1}{2} n,$$

the last inequality being obvious from Figure 5.4(a). That the range of f is all of $\mathbb{R}^1$ (vi) follows from the Intermediate-value theorem. ■

Since the logarithm function is strictly increasing, its inverse is a function; therefore, the following definition makes sense.

Definition
The inverse of the logarithm function is called the **exponential function** and is denoted by exp.

Theorem 5.13
If $f : x \to \exp x$ is the exponential function, the following statements hold:

(i) *f is continuous and increasing for all $x \in \mathbb{R}^1$; the range of f is $I = \{x : 0 < x < +\infty\}$.*
(ii) $f'(x) = \exp x$ *for all x.*
(iii) $\exp(x + y) = (\exp x) \cdot (\exp y)$.
(iv) $\exp(rx) = (\exp x)^r$, *for r rational.*
(v) $f(x) \to +\infty$ *as* $x \to +\infty$.
(vi) $f(x) \to 0$ *as* $x \to -\infty$.
(vii) $\log(\exp x) = x$ *for all* $x \in \mathbb{R}^1$ *and* $\exp(\log x) = x$ *for all* $x > 0$.
(viii) *If $a > 0$ and r is rational, then* $\exp(r \log a) = a^r$.

Proof
Items (i) and (vii) are immediate consequences of the Inverse function theorem.

To establish (ii), first set $y = \exp x$. By the Chain rule applied to $\log y = x$, it follows that

$$\frac{1}{y} y' = 1 \text{ or } y' = y = \exp x.$$

To prove (iii), set $y_1 = \exp x_1$ and $y_2 = \exp x_2$. Then $x_1 = \log y_1$, $x_2 = \log y_2$, and

$$x_1 + x_2 = \log y_1 + \log y_2 = \log(y_1 y_2).$$

Hence

$$\exp(x_1 + x_2) = y_1 y_2 = (\exp x_1)(\exp x_2).$$

The formula in (iv) is obtained by induction (as in the proof of part (iii) of Theorem 5.12). The proofs of (v) and (vi) follow from the corresponding results for the logarithm function. See Figure 5.4, in which we note that since the exponential function is the inverse of the logarithm, it is the reflection of the logarithm function with respect to the line $y = x$.

Item (viii) is simply proved: $\exp(r \log a) = \exp(\log(a^r)) = a^r$. ■

Expressions of the form

$$a^x, \ a > 0,$$

for x *rational* have been defined by elementary means. If $x = p/q$, then we merely take the pth power of a and then take the qth root of the result. However, the definition of quantities such as

$$3^{\sqrt{2}}, \ (\sqrt{7})^{\pi}$$

cannot be given in such an elementary way. For this purpose, we use the following technique.

Definition

For $a > 0$, and $x \in \mathbb{R}^1$, we define

$$a^x = \exp(x \log a).$$

Observe that when x is rational this formula coincides with (viii) of Theorem 5.13, so that the definition is consistent with the basic idea of "raising to a power."

Definitions

If $b > 0$ and $b \neq 1$, the function $\log_b$ (*called* **logarithm to the base** b) is defined as the inverse of the function $f : x \to b^x$. When $b = 10$ we call the function the **common logarithm**.

The number e is defined as $\exp 1$. In the graph of Figure 5.4(b) we can estimate the value of e as a number slightly larger than 2.7. The following statements about e are mostly self-evident:

(i) $\log_e x = \log x$ for all $x > 0$.
(ii) $\exp x = e^x$ for all x.
(iii) Given $f : x \to x^n$ for an arbitrary real number n, then $f'(x) = nx^{n-1}$.
(iv) $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

Problems

1. Given $f : x \to a^x$, show that $f'(x) = a^x \log a$. (Theorem 5.13, Part (ii).)
2. Given $f : x \to x^n$ with n any real number; show that $f'(x) = nx^{n-1}$.
3. Show that $2 < e < 4$.

4. Prove that $\lim_{x\to 0^+}(1+x)^{1/x} = e$.
5. If $x > -1$, show that $\log(1+x) \le x$.
6. Prove that $e^x \ge 1 + x$ for all x.
7. Show that $F : x \to [1 + (1/x)]^x$ is an increasing function.
8. Given $f : x \to \log x$, from the definition of derivative, we know that

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{x}.$$

Use this fact to deduce that $\lim_{h\to 0}(1+h)^{1/h} = e$.

6 CHAPTER Elementary Theory of Metric Spaces

6.1 The Schwarz and Triangle Inequalities; Metric Spaces

In Chapters 2 through 5 we developed many properties of functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ with the purpose of proving the basic theorems in differential and integral calculus of one variable. The next step in analysis is the establishment of the basic facts needed in proving the theorems of calculus in two and more variables. One way would be to prove extensions of the theorems of Chapters 2–5 for functions from $\mathbb{R}^2$ into $\mathbb{R}^1$, then for functions from $\mathbb{R}^3$ into $\mathbb{R}^1$, and so forth. However, all these results can be encompassed in one general theory obtained by introducing the concept of a metric space and by considering functions with domain in one metric space and with range in a second metric space. In this chapter we introduce the fundamentals of this theory, and in the following chapters we apply the results to differentiation and integration in Euclidean space in any number of dimensions.

We establish a simple version of the Schwarz inequality, one of the most useful inequalities in analysis.

Theorem 6.1 (Schwarz inequality)
Let $x = (x_1,x_2,\ldots,x_N)$ and $y = (y_1,y_2,\ldots,y_N)$ be elements of $\mathbb{R}^N$. Then

$$\left|\sum_{i=1}^{N} x_i y_i\right| \le \left(\sum_{i=1}^{N} x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{N} y_i^2\right)^{\frac{1}{2}} . \tag{6.1}$$

The equality sign holds if and only if either all the x_i are zero or there is a number λ such that $y_i = \lambda x_i$ for $i = 1,2,\ldots,N$.

Proof

If every x_i is zero, then the equality sign in expression (6.1) holds. Assume that there is at least one x_i not zero. We form the function

$$f(\lambda) = \sum_{i=1}^{N}(y_i - \lambda x_i)^2,$$

which is nonnegative for all values of λ. We set

$$A = \sum_{i=1}^{N} x_i^2, \qquad B = \sum_{i=1}^{N} x_i y_i, \qquad C = \sum_{i=1}^{N} y_i^2.$$

Then clearly,

$$f(\lambda) = A\lambda^2 - 2B\lambda + C \geq 0, \qquad A > 0.$$

From elementary calculus it follows that the nonnegative minimum value of the quadratic function $f(\lambda)$ is

$$\frac{AC - B^2}{A}.$$

The statement $AC - B^2 \geq 0$ is equivalent to expression (6.1). The equality sign in expression (6.1) holds if $f(\lambda) = 0$ for some value of λ, say λ_1; in this case $y_i - \lambda_1 x_i = 0$ for every i. ■

We recall that in the Euclidean plane the length of any side of a triangle is less than the sum of the lengths of the other two sides. A generalization of this fact is known as the *Triangle inequality.* It is proved by means of a simple application of the Schwarz inequality.

Theorem 6.2 (Triangle inequality)

Let $x = (x_1,x_2,\ldots,x_N)$ and $y = (y_1,y_2,\ldots,y_N)$ be elements of $\mathbb{R}^N$. Then

$$\sqrt{\sum_{i=1}^{N}(x_i + y_i)^2} \leq \sqrt{\sum_{i=1}^{N} x_i^2} + \sqrt{\sum_{i=1}^{N} y_i^2}. \tag{6.2}$$

The equality sign in expression (6.2) holds if and only if either all the x_i are zero or there is a nonnegative number λ such that $y_i = \lambda x_i$ for $i = 1,2,\ldots,N$.

Proof

We have

$$\sum_{i=1}^{N}(x_i + y_i)^2 = \sum_{i=1}^{N}(x_i^2 + 2x_i y_i + y_i^2) = \sum_{i=1}^{N} x_i^2 + 2\sum_{i=1}^{N} x_i y_i + \sum_{i=1}^{N} y_i^2.$$

We apply the Schwarz inequality to the middle term on the right, obtaining

$$\sum_{i=1}^{N}(x_i + y_i)^2 \le \sum_{i=1}^{N} x_i^2 + 2\sqrt{\sum_{i=1}^{N} x_i^2}\sqrt{\sum_{i=1}^{N} y_i^2} + \sum_{i=1}^{N} y_i^2 ,$$

thus

$$\sum_{i=1}^{N}(x_i + y_i)^2 \le \left(\sqrt{\sum_{i=1}^{N} x_i^2} + \sqrt{\sum_{i=1}^{N} y_i^2}\right)^2 .$$

Since the left side of this inequality is nonnegative, we may take the square root to obtain expression (6.2).

If all x_i are zero, then equality in expression (6.2) holds. Otherwise, equality holds if and only if it holds in the application of Theorem 6.1. The nonnegativity of λ is required to ensure that $2\sum_{i=1}^{N} x_i y_i$ has all nonnegative terms when we set $y_i = \lambda x_i$. ■

Corollary

Let $x = (x_1,x_2,\ldots,x_N)$, $y = (y_1,y_2,\ldots,y_N)$, and $z = (z_1,z_2,\ldots,z_N)$ be elements of $\mathbb{R}^N$. Then

$$\sqrt{\sum_{i=1}^{N}(x_i - z_i)^2} \le \sqrt{\sum_{i=1}^{N}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{N}(y_i - z_i)^2} . \tag{6.3}$$

The equality sign in expression (6.3) *holds if and only if there is a number r with $0 \le r \le 1$ such that $y_i = rx_i + (1-r)z_i$ for $i = 1,2,\ldots,N$.*

Proof

Setting $a_i = x_i - y_i$ and $b_i = y_i - z_i$ for $i = 1, 2, \ldots, N$, we see that inequality (6.3) reduces to inequality (6.2) for the elements $a = (a_1, \ldots, a_N)$ and $b = (b_1, b_2, \ldots, b_N)$. The number r is $\lambda_1/(1+\lambda_1)$ in Theorem 6.1. ■

Remark

The corollary to Theorem 6.2 is the familiar assertion that the sum of the lengths of any two sides of a triangle exceeds the length of the third side in Euclidean N-dimensional space.

In this chapter we shall be concerned with *sets* or *collections* of elements, which may be chosen in any manner whatsoever. A set will be considered fully described whenever we can determine whether or not any given element is a member of the set.

Definition

Let S and T be sets. The **Cartesian product of** S **and** T, denoted by $S \times T$, is the set of all ordered pairs (p, q) in which $p \in S$ and $q \in T$. The

Cartesian product of any finite number of sets $S_1, S_2, \ldots, S_N$ is the set of ordered N-tuples $(p_1, p_2, \ldots, p_N)$ in which $p_i \in S_i$ for $i = 1, 2, \ldots, N$. We write $S_1 \times S_2 \times \cdots \times S_N$.

Examples

(1) The space $\mathbb{R}^N$ is the Cartesian product $\mathbb{R}^1 \times \mathbb{R}^1 \times \cdots \times \mathbb{R}^1$ (N factors).
(2) The Cartesian product of $I_1 = \{x : a \leq x \leq b\}$ and $I_2 = \{x : c \leq x \leq d\}$ is the rectangle $T = \{(x, y) : a \leq x \leq b, c \leq x \leq d\}$. That is,

$$T = I_1 \times I_2.$$

Definition

Let S be a set and suppose d is a function with domain consisting of all pairs of points of S and with range in $\mathbb{R}^1$. That is, d is a function from $S \times S$ into $\mathbb{R}^1$. We say that S and the function d form a **metric space** when the function d satisfies the following conditions

(i) $d(x, y) \geq 0$ for all $(x, y) \in S \times S$; and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(y, x) = d(x, y)$ for all $(x, y) \in S \times S$.
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $(x, y, z) \in S$. (Triangle inequality.)

The function d satisfying conditions (i), (ii), and (iii) is called the **metric**, or **distance**, **function** in S. Hence a metric space consists of the pair (S, d).

Examples of metric spaces

(1) In the space $\mathbb{R}^N$, choose

$$d(x, y) = \sqrt{\sum_{i=1}^{N}(x_i - y_i)^2},$$

where $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$. The function d is a metric. Conditions (i) and (ii) are obvious, while (iii) is precisely the content of the corollary to Theorem 6.2. The pair $(\mathbb{R}^N, d)$ is a metric space. This metric, known as the **Euclidean metric**, is the familiar one employed in two- and three-dimensional Euclidean geometry.
(2) In the space $\mathbb{R}^N$, choose

$$d_1(x, y) = \max_{1 \leq i \leq N} |x_i - y_i|,$$

where $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$. The reader can verify that d_1 is a metric. Therefore, $(\mathbb{R}^N, d_1)$ is a metric space. We observe that this metric space is different from the space $(\mathbb{R}^N, d)$ exhibited in the first example.
(3) Let C be the collection of all continuous functions that have $I = \{x : 0 \leq x \leq 1\}$ for domain and have range in $\mathbb{R}^1$. For any two elements f, g in

C, define

$$d(f, g) = \max_{1 \le i \le N} |f(x) - g(x)|.$$

It is not difficult to verify that d is a metric. Hence (C, d) is a metric space.

Examples 1 and 2 above show that a given set may become a metric space in a variety of ways. Let S be a given set and suppose that (S, d_1) and (S, d_2) are metric spaces. We define the metrics d_1 and d_2 as **equivalent** if there are positive constants c and C such that

$$cd_1(x, y) \le d_2(x, y) \le Cd_1(x, y)$$

for all x, y in S. It is not difficult to show that the metrics in Examples 1 and 2 are equivalent.

Problems

1. Show that $(\mathbb{R}^N, d_1)$ is a metric space where

$$d_1(x, y) = \max_{1 \le i \le N} |x_i - y_i|,$$

$x = (x_1, x_2, \ldots, x_N)$, $y = (y_1, y_2, \ldots, y_N)$.

2. Suppose that (S, d) is a metric space. Show that (S, d') is a metric space, where

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)} .$$

[*Hint*: Show first that $d(x, z) = \lambda[d(x, y) + d(y, z)]$ for some λ with $0 \le \lambda \le 1$.]

3. Given the metric spaces $(\mathbb{R}^N, d)$ and $(\mathbb{R}^N, d')$,

$$d(x, y) = \sqrt{\sum_{i=1}^{N} (x_i - y_i)^2}$$

and d' is defined as in Problem 2. Decide whether or not d and d' are equivalent.

4. Show that the metrics d and d_1 in Examples (1) and (2) above of metric spaces are equivalent.

5. Show that $(\mathbb{R}^N, d_2)$ is a metric space, where

$$d_2(x, y) = \sum_{i=1}^{N} |x_i - y_i|,$$

$x = (x_1, x_2, \ldots, x_N)$, $y = (y_1, y_2, \ldots, y_N)$. Is d_2 equivalent to the metric d_1 given in Problem 1?

6. Let $(x_1, x_2), (x_1', x_2')$ be points of $\mathbb{R}^2$. Show that $(\mathbb{R}^2, d_3)$ is a metric space, where

$$d_3((x_1, x_2), (x_1', x_2')) = \begin{cases} |x_2| + |x_2'| + |x_1 - x_1'| & \text{if } x_1 \neq x_1', \\ |x_2 - x_2'| & \text{if } x_1 = x_1'. \end{cases}$$

7. For $x, y \in \mathbb{R}^1$, define $d_4(x, y) = |x - 3y|$. Is $(\mathbb{R}^1, d_4)$ a metric space?
8. Let C be the collection of continuous functions that have $I = \{x : 0 \leq x \leq 1\}$ for domain and have range in $\mathbb{R}^1$. Show that (C, d) is a metric space, where

$$d(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|$$

for $f, g \in C$.
9. Let C be the same collection described in Problem 8. Define

$$\bar{d}(f, g) = \int_0^1 |f(x) - g(x)|\, dx$$

for $f, g \in C$. Show that $(C, \bar{d})$ is a metric space.
10. A sequence $x_1, x_2, \ldots, x_n, \ldots$ is **bounded** if there is a number m such that $|x_i| \leq m$ for all i. Let M denote the collection of all bounded sequences, and define

$$d(x, y) = \sup_{1 \leq i \leq \infty} |x_i - y_i|.$$

Show that (M, d) is a metric space.
11. Let C be the subset of $\mathbb{R}^2$ consisting of pairs $(\cos\theta, \sin\theta)$ for $0 \leq \theta < 2\pi$. Define

$$d^*(p_1, p_2) = |\theta_1 - \theta_2|,$$

where $p_1 = (\cos\theta_1, \sin\theta_1)$, $p_2 = (\cos\theta_2, \sin\theta_2)$. Show that (C, d^*) is a metric space. Is d^* equivalent to the metric d_1 of Problem 1 applied to the subset C of $\mathbb{R}^2$?
12. Let S be a set and d a function from $S \times S$ into $\mathbb{R}^1$ with the properties:
    (i) $d(x, y) = 0$ if and only if $x = y$.
    (ii) $d(x, z) \leq d(x, y) + d(z, y)$ for all $x, y, z \in S$.
    (iii) Show that d is a metric and that (S, d) is a metric space.

6.2 Topology in Metric Spaces

We now develop some of the basic properties of metric spaces. From an intuitive viewpoint we frequently think of a metric space as Euclidean space of one, two, or three dimensions. However, the definitions and theorems apply to all metric spaces, many of which are far removed from ordinary Euclidean space. For this reason we designate the properties as the topology of the space.

For convenience we will use the letter S to denote a metric space with the understanding that a metric d is attached to S.

Definitions

Let $p_1, p_2, \ldots, p_n, \ldots$ denote a sequence of elements of a metric space S. We use the symbol $\{p_n\}$ to denote such a sequence. Suppose $p_0 \in S$. We say that p_n **tends to** p_0 **as** n **tends to infinity** if $d(p_n, p_0) \to 0$ as $n \to \infty$. The notations $p_n \to p_0$ and $\lim_{n\to\infty} p_n = p_0$ will be used.

Let p_0 be an element of S, a metric space, and suppose r is a positive number. The **open ball with center at** p_0 **and radius** r is the set $B(p_0, r)$ given by

$$B(p_0, r) = \{p \in S : d(p, p_0) < r\}.$$

The **closed ball with center at** p_0 **and radius** r is the set $\overline{B(p_0, r)}$ given by

$$\overline{B(p_0, r)} = \{p \in S : d(p, p_0) \leq r\}.$$

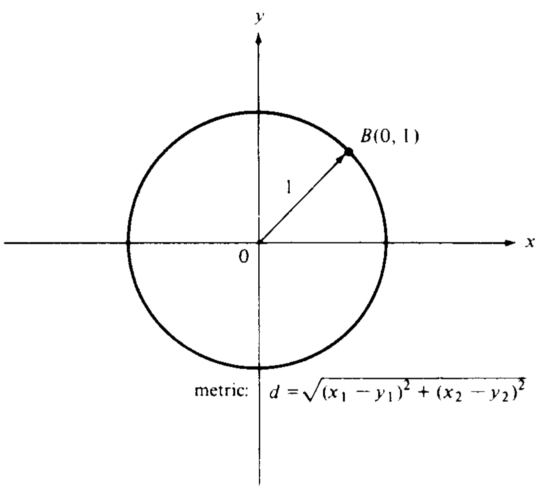

Figure 6.1 Ball with center at (0, 0) and radius 1. The circle forming the boundary of $B(0, 1)$ is not part of the open ball.

Figure 6.1 shows an open ball with center at (0, 0) and radius 1 in the space $\mathbb{R}^2$ with metric $d(x, y) = (\sum_{i=1}^{2}(x_i - y_i)^2)^{\frac{1}{2}}$. Figure 6.2 shows an open ball with center at (0, 0) and radius 1 in the space $\mathbb{R}^2$ with metric $d(x, y) = (\max_{i=1,2} |x_i - y_i|)$.

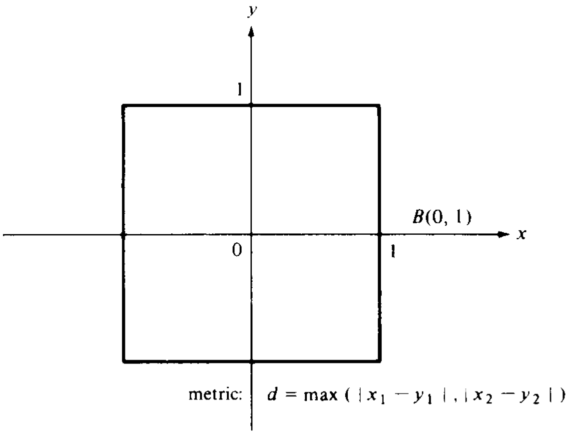

Figure 6.2 An open ball with center at (0, 0) and radius 1. The four sides of the square forming the boundary of $B(0, 1)$ are not part of the open ball.

Definition
Let A be a set in a metric space S and suppose p_0 is an element of A. We say that p_0 is an **isolated point** of A if for some positive number r there is an open ball $B(p_0, r)$ such that

$$B(p_0, r) \cap A = \{p_0\},$$

where $\{p_0\}$ is the set consisting of the single element p_0.

For example, in $\mathbb{R}^1$ with the Euclidean metric the set $A = \{x : 1 \leq x \leq 2,\ x = 3, 4\}$ has the elements 3 and 4 as isolated points of A. None of the points in the interval $1 \leq x \leq 2$ is an isolated point of A.

Definition
A point p_0 is a **limit point of a set** A if every open ball $B(p_0, r)$ contains a point p of A that is distinct from p_0. Note that p_0 may or may not be an element of A.

For example, in $\mathbb{R}^1$ with the usual Euclidean metric defined by $d(x_1, x_2) = |x_1 - x_2|$, the set $C = \{x : 1 \leq x < 3\}$ has $x = 3$ as a limit point. In fact, every member of C is a limit point of C.

A set A in a metric space is **closed** if A contains all of its limit points.

A set A in a metric space S is **open** if each point p_0 in A is the center of an open ball $B(p_0, r)$ that is contained in A. That is, $B(p_0, r) \subset A$. It is important to notice that the radius r may change from point to point in A.

Theorem 6.3
A point p_0 is a limit point of a set A if and only if there is a sequence $\{p_n\}$ with $p_n \in A$ and $p_n \neq p_0$ for every n and such that $p_n \to p_0$ as $n \to \infty$.

Proof

(a) If a sequence $\{p_n\}$ of the theorem exists, then clearly every open ball with p_0 as center will have points of the sequence (all $\neq p_0$). Thus p_0 is a limit point.

(b) Suppose p_0 is a limit point of A. We construct a sequence $\{p_n\}$ with the desired properties. According to the definition of limit point, the open ball $B(p_0, \frac{1}{2})$ has a point of A. Let p_1 be such a point. Define $r_2 = d(p_0, p_1)/2$ and construct the ball $B(p_0, r_2)$. There are points of A in this ball (different from p_0), and we denote one of them by p_2. Next, define $r_3 = d(p_0, p_2)/2$ and choose $p_3 \in A$ in the ball $B(p_0, r_3)$. Continuing this process, we see that $r_n < 1/2^n \to 0$ as $n \to \infty$. Since $d(p_0, p_n) < r_n$, it follows that $\{p_n\}$ is a sequence tending to p_0, and the proof is complete. ■

Theorem 6.4

(a) *An open ball is an open set.*
(b) *A closed ball is a closed set.*

Proof

(a) Let $B(p_0, r)$ be an open ball. Suppose $q \in B(p_0, r)$. We must show that there is an open ball with center at q that is entirely in $B(p_0, r)$. Let $r_1 = d(p_0, q)$. We consider a ball of radius $\bar{r} = (r - r_1)/2$ with center at q. See Figure 6.3. Let q' be a typical point of $B(q, \bar{r})$. We shall show that $q' \in B(p_0, r)$. We can accomplish this by showing that $d(p_0, q') < r$. We have

$$d(p_0, q') \leq d(p_0, q) + d(q, q') < r_1 + \bar{r} = r_1 + \tfrac{1}{2}(r - r_1)) = \tfrac{1}{2}(r + r_1) < r.$$

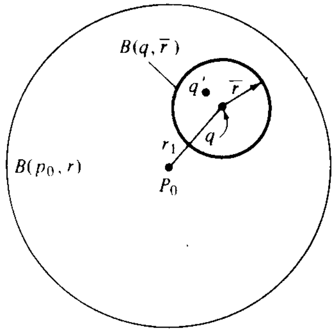

Figure 6.3 An open ball is an open set.

(b) To show that $\overline{B(p_0, r)}$ is a closed set we must show that every limit point of $\overline{B(p_0, r)}$ belongs to $\overline{B(p_0, r)}$. If q is such a limit point, there is a sequence $p_1, p_2, \ldots, p_n, \ldots$ in $\overline{B(p_0, r)}$ such that $d(p_n, q) \to 0$. However, for each n, we have

$$d(p_n, p_0) \leq r.$$

Therefore

$$d(q, p_0) \leq d(q, p_n) + d(p_n, p_0) \leq d(q, p_n) + r.$$

Letting $n \to \infty$, the theorem on the limit of inequalities shows that $d(q, p_0) \leq r$. That is, $q \in \overline{B(p_0, r)}$. ■

Let S be a space of elements, by which we mean that S is the totality of points (or elements) under consideration. A set of points is a collection of elements in S. We shall often deal with a set of sets, that is, a set whose elements are sets in S. We speak of such a set as a **family of sets** and denote it by a script letter such as $\mathcal{F}$. If the number of sets in the family is finite, we use the term *finite family* of sets and use subscripts to identify the members. For example, $\{A_1, A_2, \ldots, A_n\}$, where each A_i is a set in S, is a finite family of sets.

Definitions

Let $\mathcal{F}$ be a family of sets, a typical member of $\mathcal{F}$ being denoted by the letter A. That is, A is a set of points in a space S. We define the **union of the sets in** $\mathcal{F}$, denoted by $\bigcup_{A\in\mathcal{F}} A$, by the formula

$$\bigcup_{A\in\mathcal{F}} A = \{p : p \in S \text{ and } p \text{ is in at least one set of } \mathcal{F}\}.$$

We define the **intersection of the sets in** $\mathcal{F}$, denoted by $\bigcap_{A\in\mathcal{F}} A$, by the formula

$$\bigcap_{A\in\mathcal{F}} A = \{p : p \in S \text{ and } p \text{ is in every set of } \mathcal{F}\}.$$

If A and B are sets in S, we define their **difference** $B - A$ by the formula

$$B - A = \{p : p \text{ is in } B \text{ and } p \text{ is not in } A\}.$$

If A is any set in S, we define the **complement of** A, denoted by $C(A)$, as the set $S - A$.

Theorem 6.5

(a) *The union of any family $\mathcal{F}$ of open sets in a metric space is open.*
(b) *Let $A_1, A_2, \ldots, A_n$ be a finite family of open sets. Then the intersection $\bigcap_{i=1}^n A_i$ is open.*

Proof

(a) Suppose that $p \in \bigcup_{A\in\mathcal{F}} A$. Then p is in at least one set A in $\mathcal{F}$. Since A is open, there is an open ball with center at p and radius r, denoted by $B(p, r)$, that is entirely in A. Hence $B(p, r) \subset \bigcup_{A\in\mathcal{F}} A$. We have just

shown that any point of $\bigcup_{A \in \mathcal{F}} A$ is the center of an open ball that is also in $\bigcup_{A \in \mathcal{F}} A$. Thus the union of any family of open sets is open.
(b) Suppose that $p \in \bigcap_{i=1}^{n} A_i$. Then for each i there is an open ball $B(p, r_i)$ that is entirely in A_i. Define $\bar{r} = \min(r_1, r_2, \ldots, r_n)$. The open ball $B(p, \bar{r})$ is in every A_i and hence is in $\bigcap_{i=1}^{n} A_i$. Thus the set $\bigcap_{i=1}^{n} A_i$ is open. ■

Remark
The result of Part (b) of Theorem 6.5 is false if the word finite is dropped from the hypothesis. To see this, consider $\mathbb{R}^2$ with the Euclidean metric and define the infinite family of open sets A_k, $k = 1,2,\ldots,$ by the formula

$$A_k = \{(x, y) : 0 \le x^2 + y^2 < \frac{1}{k}\}.$$

Each A_k is an open ball (and therefore an open set), but the intersection $\bigcap_{k=1}^{\infty} A_k$ of this family consists of the single point $(0,0)$, clearly not an open set.

Theorem 6.6
(a) *Let A be any set in a metric space S. Then A is closed $\Leftrightarrow$ $C(A)$ is open.*
(b) *The space S is both open and closed.*
(c) *The null set is both open and closed.*

Proof
We first assume that A is closed and prove that $C(A)$ is open. Let $p \in C(A)$ and suppose that there is no open ball about p lying entirely in $C(A)$. We shall reach a contradiction. If there is no such ball, then every ball about p contains points of A (necessarily different from p, since $p \in C(A)$). This fact asserts that p is a limit point of A. But since A is assumed closed, we conclude that p is in A, which contradicts the fact that $p \in C(A)$. Thus there is an open ball about p in $C(A)$, and so $C(A)$ is open.

The proof that A is closed if $C(A)$ is open is similar.

Part (b) is a direct consequence of the definitions of open and closed sets. Part (c) follows immediately from parts (a) and (b), since the null set is the complement of S. ■

Theorem 6.7
(a) *The intersection of any family $\mathcal{F}$ of closed sets is closed.*
(b) *Let $A_1, A_2, \ldots, A_n$ be a finite family of closed sets. Then the union $\bigcup_{i=1}^{n} A_i$ is closed.*

The proof of Theorem 6.7 is an immediate consequence of the definition of a closed set and the definition of the intersection of sets.

It is important to observe that Part (b) of Theorem 6.7 is false if the word *finite* is dropped from the hypothesis. It is not difficult to construct an infinite family of closed sets such that the union of the family is not a closed set.

Definition

Let $\{p_n\}$ be an infinite sequence of points. An infinite sequence of points $\{q_n\}$ is called a **subsequence** of $\{p_n\}$ if there is an *increasing* sequence of positive integers $k_1, k_2, \ldots, k_n, \ldots$ such that for every n,

$$q_n = p_{k_n}.$$

To illustrate subsequences, observe that the sequence

$$p_1, p_2, \ldots, p_n, \ldots$$

has subsequences such as

$$p_1, p_3, p_5, \ldots,$$

$$p_5, p_{10}, p_{15}, \ldots,$$

$$p_1, p_4, p_7, \ldots.$$

It is clear that the *subscripts* $\{k_n\}$ *of a subsequence have the property that* $k_n \geq n$ *for every* n. This fact may be proved by induction.

Theorem 6.8

Suppose that $\{p_n\}$ *is an infinite sequence in a metric space* S, *and that* $\{q_n\}$ *is a subsequence. If* $p_n \to p_0$ *as* $n \to \infty$, *then* $q_n \to p_0$ *as* $n \to \infty$.

(*Every subsequence of a convergent sequence converges to the same limit.*)

Proof

Let $\varepsilon > 0$ be given. From the definition of convergence, there is a positive integer N such that $d(p_n, p_0) < \varepsilon$ for all $n > N$. However, for each n, we have $q_n = p_{k_n}$ with $k_n \geq n$. Thus

$$d(q_n, p_0) = d(p_{k_n}, p_0) < \varepsilon \quad \text{for all} \quad n > N.$$

Therefore $q_n \to p_0$ as $n \to \infty$. ■

Definitions

A point P is an **interior point** of a set A in a metric space if there is an $r > 0$ such that $B(p, r)$ is contained in A. The **interior of a set** A, denoted $A^{(0)}$, is the set of all interior points of A. For any set A, the **derived set** of A, denoted A' is the collection of limit points of A. The **closure** of A, denoted $\bar{A}$, is defined by $\bar{A} = A \cup A'$. The **boundary of a set** A, denoted ∂A, is defined by $\partial A = \bar{A} - A^{(0)}$.

Example

Let A be the open ball in $\mathbb{R}^2$ (with the Euclidean metric) given by $A = \{(x, y) : 0 \leq x^2 + y^2 < 1\}$. Then every point of A is an interior point, so that $A = A^{(0)}$. More generally, the property that characterizes any open set is the fact that every point is an interior point. The boundary of A is the circle $\partial A = \{(x, y) : x^2 + y^2 = 1\}$. The closure of A is the closed ball $\bar{A} = \{(x, y) : 0 \le x^2 + y^2 \le 1\}$. We also see that A', the derived set of A, is identical with $\bar{A}$.

Problems

1. Suppose that $p_n \to p_0$ and $q_n \to q_0$ in a metric space S. If $d(p_n, q_n) < a$ for all n, show that $d(p_0, q_0) \le a$.
2. If $p_n \to p_0$, $p_n \to q_0$ in a metric space S, show that $p_0 = q_0$.
3. Given $\mathbb{R}^2$ with the metric $d(x, y) = |y_1 - x_1| + |y_2 - x_2|$, $x=(x_1, x_2)$, $y=(y_1, y_2)$. Describe (and sketch) the ball with center at $(0, 0)$ and radius 1.
4. Given $\mathbb{R}^1$ with the metric $d(x, y) = |x - y|$, find an example of a set that is neither open nor closed.
5. Given $\mathbb{R}^1$ with the metric $d(x, y) = |x - y|$, show that a finite set consists only of isolated points. Is it true that a set consisting only of isolated points must be finite?
6. Give an example of a set in $\mathbb{R}^1$ with exactly four limit points.
7. In $\mathbb{R}^1$ with the Euclidean metric, define $A = \{x : 0 \le x \le 1$ and x is a rational number$\}$. Describe the set $\bar{A}$.
8. Given $\mathbb{R}^2$ with the Euclidean metric, show that the set $S = \{(x, y) : 0 < x^2 + y^2 < 1\}$ is open. Describe the sets $S^{(0)}$, S', ∂S, $\bar{S}$, $C(S)$.
9. If S is any space and $\mathcal{F}$ is a family of sets, show that
$$C\left[\bigcap_{A\in\mathcal{F}} A\right] = \bigcup_{A\in\mathcal{F}} C(A).$$
10. Let A, B, and C be arbitrary sets in a space S. Show that
    (a) $C(A - B) = B \cup C(A)$.
    (b) $A - (A - B) = A \cap B$.
    (c) $A \cap (B - C) = (A \cap B) - (A \cap C)$.
    (d) $A \cup (B - A) = A \cup B$.
    (e) $(A - C) \cup (B - C) = (A \cup B) - C$.
11. In $\mathbb{R}^2$ with the Euclidean metric, find an infinite collection of open sets $\{A_n\}$ such that $\bigcap_n A_n$ is the closed ball $\overline{B(0, 1)}$.
12. In a metric space S, show that a set A is closed if $C(A)$ is open.
13. Show that in a metric space the intersection of any family $\mathcal{F}$ of closed sets is closed.
14. Let $A_1, A_2, \ldots, A_n$ be a finite family of closed sets in a metric space. Show that the union is a closed set.
15. Let $A_1, A_2, \ldots, A_n, \ldots$ be sets in a metric space. Define $B = \bigcup A_i$. Show that $\bar{B} \supset \bigcup \bar{A}_i$ and give an example to show that $\bar{B}$ may not equal $\bigcup \bar{A}_i$.

16. In $\mathbb{R}^1$ with the Euclidean metric, give an example of a sequence $\{p_n\}$ with two subsequences converging to different limits. Give an example of a sequence that has infinitely many subsequences converging to different limits.
17. Using only the definition of closed set prove that the intersection of any family of closed sets is closed.
18. Suppose that f is continuous on an interval $I = \{x : a \le x \le b\}$ with $f(x) > 0$ on I. Let $S = \{(x, y) \in \mathbb{R}^2 : a \le x \le b,\ 0 \le y \le f(x)\}$ (Euclidean metric).
    (a) Show that S is closed.
    (b) Find S' and $\bar{S}$.
    (c) Find $S^{(0)}$ and prove the result.
    (d) Find ∂S.
19. If A is a set in a metric space, show that A' and $\bar{A}$ are closed sets.
20. If A is a set in a metric space, show that ∂A is closed and $A^{(0)}$ is open.
21. If A and B are sets in a metric space, show that if $A \subset B$ and B is closed, then $\bar{A} \subset B$.
22. If A and B are sets in a metric space, show that if $B \subset A$ and B is open, then $B \subset A^{(0)}$.
23. If A is a set in a metric space, show that

$$A \text{ is open} \quad \Leftrightarrow \quad C(A) \text{ is closed.}$$

6.3 Countable and Uncountable Sets

If A and B are sets with a finite number of elements, we can easily compare their sizes by pairing the elements of A with those of B. If after we complete the pairing process there remain unpaired members of one of the sets, say of A, then A is said to be larger than B. The same situation holds for sets with infinitely many elements. While many pairs of infinite sets can be matched in a one-to-one way, it is a remarkable fact that there are infinite sets that cannot be paired with each other. The problem of the relative sizes of sets with infinitely many members has been studied extensively, and in this section we shall describe only the basic properties of the sizes of finite and infinite sets. These properties and the more sophisticated developments of the theory of infinite sets turn out to be useful tools in the development of analysis. The theorems we consider below pertain to sets in R^N.

Definitions

A set S is **denumerable** if S can be put into one-to-one correspondence with the positive integers. A set S is **countable** if S is either finite or denumerable.

Theorem 6.9
(a) *Any nonempty subset of the natural numbers is countable.*
(b) *Any subset of a countable set is countable.*

Proof
To establish Part (a), let S be the given nonempty subset. Since S is a subset of the positive integers, it must have a smallest element. We denote it by k_1. If $S = \{k_1\}$, then S has exactly one element. If the set $S - \{k_1\}$ is not empty, it has a smallest element, which we denote by k_2. We continue the process. If for some integer n, the set $S - \{k_1, k_2, \dots, k_n\}$ is empty, then S is finite and we are done. Otherwise, the sequence $k_1, k_2, \dots, k_n, \dots$ is infinite. We shall show that $S = \{k_1, k_2, \dots, k_n, \dots\}$. Suppose, on the contrary, that there is an element of S not among the k_n. Then there is a smallest such element, which we denote by p. Now, k_1 is the smallest element of S, and so $k_1 \leq p$. Let T be those elements of $k_1, k_2, \dots, k_n, \dots$ with the property that $k_i \leq p$. Then T is a finite set, and there is an integer i such that $k_i \leq p < k_{i+1}$ for some i. Since p is not equal to any of the k_j, it follows that $k_i < p < k_{i+1}$. By construction, the element k_{i+1} is the smallest element of S that is larger than $k_1, k_2, \dots, k_j$. Since p is in S, we have a contradiction of the inequality $k_i < p < k_{i+1}$. Thus $S = \{k_1, k_2, \dots, k_n, \dots\}$.

Part (b) is a direct consequence of part (a). ∎

Theorem 6.10
The union of a countable family of countable sets is countable.

Proof
Let $S_1, S_2, \dots, S_m, \dots$ be a countable family of countable sets. For each fixed m, the set S_m can be put into one-to-one correspondence with some or all of the pairs $(m, 1), (m, 2), \dots, (m, n), \dots$. Hence the union S can be put into one-to-one correspondence with a subset of the totality of ordered pairs (m, n) with $m, n = 1, 2, \dots$. However, the totality of ordered pairs can be arranged in a single sequence. We first write the ordered pairs as shown in Figure 6.4.

$$\begin{array}{lllll}
(1,1) & (1,2) & (1,3) & (1,4) & \dots \\
(2,1) & (2,2) & (2,3) & (2,4) & \dots \\
(3,1) & (3,2) & (3,3) & (3,4) & \dots \\
(4,1) & (4,2) & (4,3) & (4,4) & \dots \\
\;\dots & & & & \\
\;\dots & & & & \\
\;\dots & & & &
\end{array}$$

Then we write the terms in diagonal order as follows:

$$(1,1), (2,1), (1,2), (3,1), (2,2), (1,3), (4,1), \dots .$$

This sequence is in one-to-one correspondence with the positive integers. The reader may verify that the ordered pair (p, q) in the above ordering corresponds to the natural number r given by

$$r = \frac{(p+q-1)(p+q-2)}{2} + q. \qquad \blacksquare$$

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
\vdots & & & &
\end{array}$$

Figure 6.4

Theorem 6.10 shows that the set of all integers (positive, negative, and zero) is denumerable. Since the rational numbers can be put into one-to-one correspondence with a subset of ordered pairs (p, q) of integers (p is the numerator, q the denominator), we see that the rational numbers are denumerable.

The next theorem shows that the set of all real numbers is essentially larger than the set of all rational numbers. Any attempt to establish a one-to-one correspondence between these two sets must fail.

Theorem 6.11
$\mathbb{R}^1$ is not countable.

Proof
It is sufficient to show that the interval $I = \{x : 0 < x < 1\}$ is not countable. Suppose I is countable. Then we may arrange the members of I in a sequence $x_1, x_2, \ldots, x_n, \ldots$. We shall show there is a number $x \in I$ not in this sequence, thus contradicting the fact that I is countable. Each x_i has a unique proper decimal development, as shown in Figure 6.5. Each d_{ij} is a digit between 0 and 9.

$$\begin{array}{l}
x_1 = 0.d_{11}\, d_{12}\, d_{13} \ldots \\
x_2 = 0.d_{21}\, d_{22}\, d_{23} \ldots \\
x_3 = 0.d_{31}\, d_{32}\, d_{33} \ldots \\
\vdots \qquad \vdots\ \vdots \quad \vdots \quad \vdots
\end{array}$$

Figure 6.5 Decimal expansions of $x_1, x_2, \ldots, x_n, \ldots$.

We now define a number x in I by the following decimal development:

$$x = 0.a_1 a_2 a_3 \ldots. \tag{6.4}$$

Each digit a_i is given by

$$a_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4, \\ 5 & \text{if } d_{ii} = 4. \end{cases}$$

Then expression (6.4) is a proper decimal expansion of the number x, and it differs from x_n in the nth place. Hence $x \neq x_n$ for every n, and I is not countable. ■

Definitions

An **open cell** is $\mathbb{R}^N$ Is a set of the form

$$\{(x_1, x_2, \ldots, x_n) : a_i < x_i < b_i,\ i=1, 2, \ldots, N\}.$$

We suppose that $a_i < b_i$ for every i. Note that an open cell is a straightforward generalization of an open interval in $\mathbb{R}^1$. Similarly, a **closed cell** is a set of the form

$$\{(x_1, x_2, \ldots, x_n) : a_i \leq x_i \leq b_i,\ i=1, 2, \ldots, N\}.$$

Again, we assume that $a_i < b_i$ for every i.

Theorem 6.12

An open set in $\mathbb{R}^1$ is the union of a countable family of disjoint open intervals.

Proof

Let G be the given open set and suppose $x_0 \in G$. Since G is open, there is an open interval J that contains x_0 and such that $J \subset G$. Define $I(x_0)$ as the union of all such open intervals J. Then $I(x_0)$ is open and contained in G. Furthermore, since the union of a set of open intervals having a common point is an open interval, $I(x_0)$ is an open interval. Let x_1 and x_2 be distinct points of G. We shall show that $I(x_1)$ and $I(x_2)$ are disjoint or that $I(x_1) \equiv I(x_2)$. Suppose that $I(x_1) \cap I(x_2)$ is not empty. Then $I(x_1) \cup I(x_2)$ is an interval in G that contains both x_1 and x_2. Hence $I(x_1) \cup I(x_2)$ is contained in both $I(x_1)$ and $I(x_2)$. Therefore, $I(x_1) \equiv I(x_2)$. Thus G is composed of the union of disjoint open intervals of the type $I(x_0)$, and by Theorem 6.10 the union is countable. ■

Problems

1. Show that the totality of **rational points** in $\mathbb{R}^2$ (that is, points both of whose coordinates are rational numbers) is denumerable.
2. Show that the totality of rational points in $\mathbb{R}^N$ is denumerable.
3. Define the set S as follows: The element x is in S if x is an infinite sequence of the form $(r_1, r_2, \ldots, r_n, 0, 0, \ldots, 0, \ldots)$. That is, from

some n on, the sequence consists entirely of zeros and the nonzero entries are rational numbers. Show that S is denumerable.

4. Show that any family of pairwise disjoint open intervals in $\mathbb{R}^1$ is countable. [*Hint*: Set up a one-to-one correspondence between the disjoint intervals and a subset of the rational numbers.]
5. Show that any family of disjoint open sets in $\mathbb{R}^N$ is countable.
6. Show that every infinite set contains a denumerable subset.
7. Let G be an open set in $\mathbb{R}^1$. Show that G can be represented as the union of open intervals with rational endpoints.

6.4 Compact Sets and the Heine–Borel Theorem

In Theorem 3.7 we established the Bolzano–Weierstrass theorem, which states that *any bounded infinite sequence in* $\mathbb{R}^1$ *contains a convergent subsequence.* In this section we investigate the possibility of extending this theorem to metric spaces. We shall give an example below which shows that the theorem does not hold if the bounded sequence in $\mathbb{R}^1$ is replaced by a bounded sequence in a metric space S. However, the important notion of compactness, which we now define, leads to a modified extension of the Bolzano-Weierstrass theorem.

Definition

A set A in a metric space S is **compact** if each sequence of points $\{p_n\}$ in A contains a subsequence, say $\{q_n\}$, that converges to a point p_0 in A.

We see immediately from the definition of compactness and from the Bolzano–Weierstrass theorem that every closed, bounded set in $\mathbb{R}^1$ is compact. Also, it is important to notice that the subsequence $\{q_n\}$ not only converges to p_0 but that p_0 is in A.

Definition

A set A in a metric space S is **bounded** if A is contained in some ball $B(p, r)$ with $r > 0$.

Theorem 6.13

A compact set A in a metric space S is bounded and closed.

Proof

Assume that A is compact and not bounded. We shall reach a contradiction. Define the sequence $\{p_n\}$ as follows: Choose $p_1 \in A$. Since A is not bounded, there is an element p_2 in A such that $d(p_1, p_2) > 1$. Continuing, for each n, there is an element p_n in A with $d(p_1, p_n) > n - 1$. Since A is compact, we can find a subsequence $\{q_n\}$ of $\{p_n\}$ such that q_n converges to an element $p_0 \in A$. From the definition of convergence, there is an integer N such that $d(q_n, p_0) < 1$ for all $n \geq N$. But

$$d(q_n, p_1) \leq d(q_n, p_0) + d(p_0, p_1) < 1 + d(p_0, p_1). \tag{6.5}$$

Since q_n is actually at least the n th member of the sequence $\{p_n\}$, it follows that $d(q_n, p_1) > n - 1$. For n sufficiently large, this inequality contradicts inequality (6.5). Hence A is bounded.

Now assume that A is not closed. Then there is a limit point p_0 of A that is not in A. According to Theorem 6.4, there is a sequence $\{p_n\}$ of elements in A such that $p_n \to p_0$ as $n \to \infty$. But any subsequence of a convergent sequence converges to the same element. Hence the definition of compactness is contradicted unless $p_0 \in A$. ■

It is natural to ask whether or not the converse of Theorem 6.13 holds. It is not difficult to give an example of a sequence of bounded elements in a metric space that has no convergent subsequence. Thus any set containing such a sequence, even if it is bounded and closed, cannot be compact.

For some metric spaces the property of being bounded and closed is equivalent to compactness. The next theorem establishes this equivalence for the spaces $\mathbb{R}^N$ with a Euclidean or equivalent metric.

Theorem 6.14

A bounded, closed set in $\mathbb{R}^N$ is compact.

Proof

Let A be bounded and closed. Then A lies in a cell

$$C = \{(x_1, x_2, \ldots, x_N) : a_i \leq x_i \leq b_i, \quad i = 1, 2, \ldots, N\}.$$

Let $\{p_n\}$ be any sequence in A. Writing $p_k = (p_k^1, p_k^2, \ldots, p_k^N)$, we have $a_i \leq p_k^i \leq b_i$, $i = 1, 2, \ldots, N$, $k = 1, 2, \ldots, n, \ldots$. Starting with $i = 1$, it is clear that $\{p_k^1\}$ is a bounded sequence of real numbers. By the Bolzano–Weierstrass theorem, $\{p_k^1\}$ has a convergent subsequence. Denote this convergent subsequence by $\{p_{k'}^1\}$. Next let $i = 2$ and consider the subsequence $\{p_{k'}^2\}$ of $\{p_k^2\}$. Since $\{p_{k'}^2\}$ is bounded, it has a convergent subsequence, which we denote by $\{p_{k''}^2\}$. Observe that $\{p_{k''}^1\}$ being a subsequence of the convergent sequence $\{p_{k'}^1\}$ is also convergent. Proceeding to $i = 3$, we choose a subsequence of $\{p_{k''}^3\}$ that converges, and so on until $i = N$. We finally obtain a subsequence of $\{p_k\}$, which we denote by $\{q_k\}$, such that every component of q_k consists of a convergent subsequence of real numbers. Since A is closed, q_k converges to an element c of $\mathbb{R}^N$ with c in A. The set A is compact. ■

Definition

A sequence $\{p_n\}$ in a metric space S is a **Cauchy sequence** if for every $\varepsilon > 0$ there is a positive integer N such that $d(p_n, p_m) < \varepsilon$ whenever $n, m > N$.

We know that a Cauchy sequence in $\mathbb{R}^1$ converges to a limit (Section 3.6). It is easy to see that a Cauchy sequence $\{p_n\}$ in $\mathbb{R}^N$ also converges to a limit. Writing $p_n = (p_n^1, p_n^2, \ldots, p_n^N)$, we observe that $\{p_k^i\}$, $k = 1, 2, \ldots$, is a Cauchy sequence of real numbers for each $i = 1, 2, \ldots, N$. Hence $p_k^i \to c^i$, $i = 1, 2, \ldots, N$, and the element $c = (c^1, c^2, \ldots, c^N)$ is the limit of the sequence $\{p_k\}$. However, we cannot conclude that in a general metric space a Cauchy sequence always converges to an element in the space. In fact, the statement is false. To see this, consider the metric space (S, d) consisting of all rational numbers with the metric $d(a, b) = |a - b|$. Since a Cauchy sequence of rational numbers may converge to an irrational number, it follows that a Cauchy sequence in S may not converge to a limit in S.

Let H be a set in a space S and suppose that $\mathcal{F}$ is a family of sets in S. The family $\mathcal{F}$ **covers** H if every point of H is a point in at least one member of $\mathcal{F}$. Coverings of compact sets by a finite number of open balls have a natural extension to coverings by any family of open sets. The next theorem, an important and useful one in analysis, gives one form of this result. It is a direct extension of Theorem 3.12.

Theorem 6.15 (Heine–Borel theorem)

Let A be a compact set in a metric space S. Suppose that $\mathcal{F} = \{\mathcal{G}_\alpha\}$ is a family of open sets that covers A. Then a finite subfamily of $\mathcal{F}$ covers A.

Proof

Suppose no such subfamily exists. We shall reach a contradiction. Since A is compact, there is a finite number of balls, say of radius $\frac{1}{2}$, that cover A. Suppose the centers of these balls are the points $p_{11}, p_{12}, \ldots, p_{1k_1}$ in A, and the balls are denoted by $B(p_{1i}, \frac{1}{2})$, $i = 1, 2, \ldots, k_1$. We have $\bigcup_{i=1}^{k_1} B(p_{1i}, \frac{1}{2}) \supset A$. Then one of the sets

$$A \cap \overline{B(p_{11}, \frac{1}{2})}, \quad A \cap \overline{B(p_{12}, \frac{1}{2})}, \quad \ldots, \quad A \cap \overline{B(p_{1k_1}, \frac{1}{2})},$$

is such that an infinite number of open sets of $\mathcal{F}$ are required to cover it. (Otherwise, the theorem would be proved.) Call this set $A \cap \overline{B(p_1, \frac{1}{2})}$ and observe that it is a compact subset of A. Now we can find a finite number of balls $B(p_{2i}, \frac{1}{4})$, $i = 1, 2, \ldots, k_2$, such that

$$A \cap \overline{B(p_1, \tfrac{1}{2})} \subset \bigcup_{i=1}^{k_2} B(p_{2i}, \frac{1}{4}).$$

One of the balls $B(p_{21}, \frac{1}{4}), \ldots, B(p_{2k_2}, \frac{1}{4})$ is such that an infinite number of open sets of $\mathcal{F}$ are required to cover the portion of A contained in that ball. Denote that ball by $B(p_2, \frac{1}{4})$. Continuing in this way we obtain a sequence of closed balls

$$\overline{B(p_1, \tfrac{1}{2})}, \overline{B(p_2, \tfrac{1}{4})}, \ldots, \overline{B(p_n, \tfrac{1}{2^n})}, \ldots$$

such that $p_n \in A \cap B(p_{n-1}, 1/2^{n-1})$, and an infinite number of open sets of $\mathcal{F}$ are required to cover $A \cap \overline{B(p_n, 1/2^n)}$. Since $d(p_{n-1}, p_n) \leq 1/2^{n-1}$ for each n, the sequence $\{p_n\}$ is a Cauchy sequence and hence converges to some element p_0. Now, since A is compact, we have $p_0 \in A$. Therefore, p_0 lies in some open set G_0 of $\mathcal{F}$. Since $p_n \to p_0$, there is a sufficiently large value of n such that $\overline{B(p_n, 1/2^n)} \subset G_0$. However, this contradicts the fact that an infinite number of members of $\mathcal{F}$ are required to cover $\overline{B(p_n, 1/2^n)} \cap A$. ■

Remarks

The compactness of A is essential in establishing the Heine–Borel theorem. It is not enough to assume, for example, that A is bounded. A simple example is given by the set in $\mathbb{R}^1$ defined by $A = \{x : 0 < x < 1\}$. We cover A with the family $\mathcal{F}$ of open intervals defined by $I_a = \{x : a/2 < x < 1\}$ for each $a \in A$. It is not difficult to verify that no finite subcollection of $\{I_a\}$ can cover A.

Problems

1. Show that the union of a compact set and a finite set is compact.
2. (a) Show that the intersection of any number of compact sets is compact.
   (b) Show that the union of any finite number of compact sets is compact.
   (c) Show that the union of an infinite number of compact sets may not be compact.
3. Consider the space $\mathbb{R}^2$ with the Euclidean metric. We define $A = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$, and we define $B = A - \{p(1, 1)\}$. Show that B can be covered by an infinite family $\mathcal{F}$ of open rectangles in such a way that no finite subfamily of $\mathcal{F}$ covers B.
4. Suppose that $\mathcal{F}$ is a family of open sets in $\mathbb{R}^2$ that covers the rectangle $R = \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$. Show that R can be divided into a finite number of rectangles by lines parallel to the sides of R such that no two rectangles have common interior points and such that each closed rectangle is contained in one open set of $\mathcal{F}$.
5. Let A be the space of sequences $\mathbf{x} = \{x_1, x_2, \ldots, x_n, \ldots\}$ in which only a finite number of the x_i are different from zero. In A define

$d(\mathbf{x}, \mathbf{y})$ by the formula

$$d(\mathbf{x}, \mathbf{y}) = \max_{1 \le i < \infty} |x_i - y_i|.$$

(a) Show that A is a metric space.
(b) Find a closed, bounded set in A that is not compact.

6. Suppose that $\mathcal{F}$ is a family of open sets in $\mathbb{R}^2$ that covers the circle $C = \{(x, y) : x^2 + y^2 = 1\}$. Show that there is a $\rho > 0$ such that $\mathcal{F}$ covers the set
$$A = \{(x, y) : (1 - \rho)^2 \le x^2 + y^2 \le (1 + \rho)^2\}.$$
7. Show that in a metric space a closed subset of a compact set is compact.
8. Give an example of a collection of points in $\mathbb{R}^1$ that form a compact set and whose limit points form a countable set.
9. Show that in a metric space a convergent sequence is a Cauchy sequence.
10. Let $\{p_n\}$ be a Cauchy sequence in a compact set A in a metric space. Show that there is a point $p_0 \in A$ such that $p_n \to p_0$ as $n \to \infty$.
11. Let $S_1, \ldots, S_n, \ldots$ be a sequence of closed sets in the compact set A such that $S_n \supset S_{n+1}$, $n = 1, 2, \ldots$. Show that $\bigcap_{n=1}^{\infty} S_n$ is not empty.
12. In $\mathbb{R}^3$ with the Euclidean metric consider all points on the surface $x_1^2 + x_2^2 - x_3^2 = 1$. Is this set compact?

6.5 Functions on Compact Sets

In Chapter 3, we discussed the elementary properties of functions defined on $\mathbb{R}^1$ with range in $\mathbb{R}^1$. We were also concerned with functions whose domain consisted of a part of $\mathbb{R}^1$, usually an interval. In this section we develop properties of functions with domain all or part of an arbitrary metric space S and with range in $\mathbb{R}^1$. That is, we consider real-valued functions. We first take up the fundamental notions of limits and continuity.

Definition
Suppose that f is a function with domain A, a subset of a metric space S, and with range in $\mathbb{R}^1$; we write $f : A \to \mathbb{R}^1$. We say that $f(p)$ **tends to l as p tends to p_0 through points of** A if (i) p_0 is a limit point of A, and (ii) for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(p) - l| < \varepsilon \quad \text{for all } p \text{ in } A$$

with the property that $0 < d(p, p_0) < \delta$. We write

$$f(p) \to l \text{ as } p \to p_0, \quad p \in A.$$

We shall also use the notation

$$\lim_{\substack{p\to p_0\\ p\in A}} f(p) = l.$$

Observe that the above definition does not require $f(p_0)$ to be defined, nor does p_0 have to belong to the set A. However, when both of these conditions hold, we are able to define continuity for real-valued functions.

Definitions

Let A be a subset of a metric space S, and suppose $f : A \to \mathbb{R}^1$ is given. Let $p_0 \in A$. We say that f **is continuous with respect to A at** p_0 if (i) $f(p_0)$ is defined, and (ii) either p_0 is an isolated point of A or p_0 is a limit point of A and

$$f(p) \to f(p_0) \text{ as } p \to p_0, \quad p \in A.$$

We say that f **is continuous on** A if f is continuous with respect to A at every point of A.

If the domain of f is the entire metric space S, then we say that f is continuous at p_0, omitting the phrase "with respect to A." The definitions of limit and continuity for functions with domain in one metric space S_1 and range in another metric space S_2 are extensions of those given above.

The basic properties of functions defined on $\mathbb{R}^1$ discussed in Chapter 3 do not always have direct analogues for functions defined on a subset of a metric space. For example, the Intermediate-value theorem (Theorem 3.3) for functions defined on an interval of $\mathbb{R}^1$ does not carry over to real-valued functions defined on all or part of a metric space. However, when the domain of a real-valued function is a *compact* subset of a metric space, many of the theorems of Chapter 3 have natural extensions. For example, the following theorem is the analogue of the boundedness theorem (Theorem 3.8).

Theorem 6.16

Let A be a compact subset of a metric space S. Suppose that $f{:}A \to \mathbb{R}^1$ is continuous on A. Then the range of f is bounded.

Proof

We assume that the range is unbounded and reach a contradiction. Suppose that for each positive integer n, there is a $p_n \in A$ such that $|f(p_n)| > n$. Since A is compact, the sequence $\{p_n\} \subset A$ must have a convergent subsequence, say $\{q_n\}$, and $q_n \to \bar{p}$ with $\bar{p} \in A$. Since f is continuous on A, we have $f(q_n) \to f(\bar{p})$ as $n \to \infty$. Choosing $\varepsilon = 1$ in the definition of continuity on A and observing that $d(q_n, \bar{p}) \to 0$ as $n \to \infty$, we can state that there is an N_1 such that for $n > N_1$, we have

$$|f(q_n) - f(\bar{p})| < 1 \quad \text{whenever} \quad n > N_1.$$

We choose N_1 so large that $|f(\bar{p})| < N_1$. Now for $n > N_1$ we may write

$$|f(q_n)| = |f(q_n) - f(\bar{p}) + f(\bar{p})| \leq |f(q_n) - f(\bar{p})| + |f(\bar{p})| < 1 + |f(\bar{p})|.$$

Since q_n is at least the nth member of the sequence $\{p_n\}$, it follows that $|f(q_n)| > n$ for each n. Therefore,

$$n < |f(q_n)| < 1 + |f(\bar{p})| < 1 + N_1,$$

a contradiction for n sufficiently large. Hence the range of f is bounded. ∎

Note the similarity of the above proof to that of Theorem 3.8. Also, observe the essential manner in which the compactness of A is employed. The result clearly does not hold if A is not compact.

Problems

In the following problems a set A in a metric space S is given. All functions are real-valued (range in $\mathbb{R}^1$) and have domain A.

1. If c is a number and $f(p) = c$ for all $p \in A$, then show that for any limit point p_0 of A, we have
$$\lim_{\substack{p \to p_0 \\ p \in A}} f(p) = c$$
(theorem on limit of a constant).
2. Suppose f and g are such that $f(p) = g(p)$ for all $p \in A - \{p_0\}$, where p_0 is a limit point of A, and suppose that $f(p) \to l$ as $p \to p_0$, $p \in A$. Show that $g(p) \to l$ as $p \to p_0$, $p \in A$ (limit of equal functions).
3. Suppose that $f_1(p) \to l_1$ as $p \to p_0$, $p \in A$, and $f_2(p) \to l_2$ as $p \to p_0$, $p \in A$. Show that $f_1(p) + f_2(p) \to l_1 + l_2$ as $p \to p_0$, $p \in A$.
4. Under the hypotheses of Problem 3, show that $f_1(p) \cdot f_2(p) \to l_1 \cdot l_2$ as $p \to p_0$, $p \in A$.
5. Assume that the hypotheses of Problem 3 hold and that $l_2 \neq 0$. Show that $f_1(p)/f_2(p) \to l_1/l_2$ as $p \to p_0$, $p \in A$.
6. Suppose that $f(p) \leq g(p)$ for all $p \in A$, that $f(p) \to L$ as $p \to p_0$, $p \in A$, and that $g(p) \to M$ as $p \to p_0$, $p \in A$. Show that $L \leq M$. Show that the same result holds if we assume $f(p) \leq g(p)$ in some open ball containing p_0.

7 CHAPTER Differentiation and Integration in $\mathbb{R}^N$

7.1 Partial Derivatives and the Chain Rule

There are two principal extensions to $\mathbb{R}^N$ of the theory of differentiation of real-valued functions on $\mathbb{R}^1$. In this section, we develop the natural generalization of ordinary differentiation discussed in Chapter 4 to partial differentation of functions from $\mathbb{R}^N$ to $\mathbb{R}^1$. In Section 7.3, we extend the ordinary derivative to the total derivative.

We shall use letters x, y, z, etc. to denote elements in $\mathbb{R}^N$. The components of an element x are designated by $(x_1, x_2, \ldots, x_N)$, and as usual, the Euclidean distance, given by the formula

$$d(x, y) = \sqrt{\sum_{i=1}^{N}(x_i - y_i)^2},$$

will be used. We also write $d(x, y) = |x - y|$ and $d(x, 0) = |x|$.

Definition

Let f be a function with domain an open set in $\mathbb{R}^N$ and range in $\mathbb{R}^1$. We define the N functions $f_{,i}$ with $i = 1, 2, \ldots, N$ by the formulas

$$f_{,i}(x_1, x_2, \ldots, x_N) = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_N) - f(x_1, \ldots, x_i, \ldots, x_N)}{h}$$

whenever the limit exists. The functions $f_{,1}, f_{,2}, \ldots, f_{,N}$ are called the **first partial derivatives** of f.

We assume that the reader is familiar with the elementary processes of differentiation. The partial derivative with respect to the ith variable is computed by holding all other variables constant and calculating the ordinary derivative with respect to x_i. That is, to compute $f_{,i}$ at the value $a = (a_1, a_2, \ldots, a_N)$, form the function φ (from $\mathbb{R}^1$ to $\mathbb{R}^1$) by setting

$$\varphi(x_i) = f(a_1, a_2, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_N), \tag{7.1}$$

and observe that

$$f_{,i}(a) = \varphi'(a_i). \tag{7.2}$$

There are many notations for partial differentiation. In addition to the one above, the most common ones are

$$D_i f, \qquad \frac{\partial f}{\partial x_i}, \qquad f_{x_i}.$$

Good notation is important in studying partial derivatives, since the many letters and subscripts that occur can lead to confusion. The two "best" symbols, especially in discussing partial derivatives of higher order, are $f_{,i}$ and $D_i f$, and we shall employ these most of the time.

We recall that the equations of a line segment in $\mathbb{R}^N$ connecting the points $a = (a_1, \ldots, a_N)$ and $a + h = (a_1 + h_1, \ldots, a_N + h_N)$ are given parametrically by the formulas (the parameter is t)

$$x_1 = a_1 + th_1, \quad x_2 = a_2 + th_2, \quad \ldots, \quad x_N = a_N + th_N, \quad 0 \le t \le 1.$$

The next theorem is the extension to functions from $\mathbb{R}^N$ to $\mathbb{R}^1$ of the Mean-value theorem (Theorem 4.12).

Theorem 7.1 (Extended Mean-value theorem)
Let τ_i be the line segment in $\mathbb{R}^N$ connecting the points $(a_1,a_2,\ldots,a_i,\ldots,a_N)$ and $(a_1,a_2,\ldots,a_i+h_i,\ldots,a_N)$. Suppose f is a function from $\mathbb{R}^N$ into $\mathbb{R}^1$ with domain containing τ_i, and suppose that the domain of $f_{,i}$ contains τ_i. Then there is a real number ξ_i on the closed interval in $\mathbb{R}^1$ with endpoints a_i and $a_i + h_i$ such that
(7.3)
$$f(a_1, \ldots, a_i + h_i, \ldots, a_N) - f(a_1, \ldots, a_i, \ldots, a_N) = h_i f_{,i}(a_1, \ldots, \xi_i, \ldots, a_N).$$

Proof
If $h_i = 0$, then equation (7.3) holds. If $h_i \neq 0$, we use the notation of expression (7.1) to write the left side of equation (7.3) in the form

$$\varphi(a_i + h_i) - \varphi(a_i).$$

For this function on $\mathbb{R}^1$, we apply the Mean-value theorem (Theorem 4.12) to conclude that

$$\varphi(a_i + h_i) - \varphi(a_i) = h_i\varphi'(\xi_i).$$

This formula is a restatement of equation (7.3). ■

Theorem 4.8, the Fundamental lemma of differentiation, has the following generalization for functions from $\mathbb{R}^N$ into $\mathbb{R}^1$. We state the theorem in the general case and prove it for $N = 2$.

Theorem 7.2 (Fundamental lemma of differentiation)
Suppose that the functions f and $f_{,1},f_{,2},\ldots,f_{,N}$ all have a domain in $\mathbb{R}^N$ that contains an open ball about the point $a = (a_1,a_2,\ldots,a_N)$. Suppose all the $f_{,i}$ $i = 1,2,\ldots,N$, are continuous at a. Then

(a) *f is continuous at a;*

(b) *there are functions $\epsilon_1(x), \epsilon_2(x),\ldots,\epsilon_N(x)$, continuous at $x = 0$, such that $\epsilon_1(0) = \epsilon_2(0) = \ldots = \epsilon_N(0) = 0$ and*

$$f(a + h) - f(a) = \sum_{i=1}^{N}[f_{,i}(a) + \epsilon_i(h)]h_i \tag{7.4}$$

for $h = (h_1,h_2,\ldots,h_N)$ in some ball $B(0,r)$ in $\mathbb{R}^N$ of radius r and center at $h = 0$.

Proof
For $N = 2$. We employ the identity

(7.5)
$$f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) = [f(a_1 + h_1, a_2) - f(a_1, a_2)] + [f(a_1 + h_1, a_2 + h_2) - f(a_1 + h_1, a_2)].$$

Since $f, f_{,1}$, and $f_{,2}$ are defined in an open ball about $a = (a_1, a_2)$, this identity is valid for h_1, h_2 in a sufficiently small ball (of radius, say, r) about $h_1 = h_2 = 0$. Applying Theorem 7.1 to the right side of equation (7.5), we find that there are numbers $\xi_1(h_1, h_2), \xi_2(h_1, h_2)$ on the closed intervals from a_1 to $a_1 + h_1$ and from a_2 to $a_2 + h_2$, respectively, such that

$$f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) = f_{,1}(\xi_1, a_2)h_1 + f_{,2}(a_1 + h_1, \xi_2)h_2. \tag{7.6}$$

Equation (7.6) is valid for h_1, h_2 in the ball of radius r about $h_1 = h_2 = 0$.

Now define

$$\begin{aligned}\epsilon_1(h_1, h_2) &= f_{,1}(\xi_1, a_2) - f_{,1}(a_1, a_2),\\ \epsilon_2(h_1, h_2) &= f_{,2}(a_1 + h_1, \xi_2) - f_{,2}(a_1, a_2).\end{aligned} \tag{7.7}$$

We wish to show that ϵ_1 and ϵ_2 are continuous at (0, 0). Let $\epsilon > 0$ be given. Since $f_{,1}$ and $f_{,2}$ are continuous at (a_1, a_2), there is a $\delta > 0$ such that

$$|f_{,1}(x_1, x_2) - f_{,1}(a_1, a_2)| < \epsilon \text{ and } |f_{,2}(x_1, x_2) - f_{,2}(a_1, a_2)| < \epsilon, \tag{7.8}$$

for (x_1, x_2) in a ball of radius δ with center at (a_1, a_2). Comparing expressions (7.7) and (7.8), we see that if h_1 and h_2 are sufficiently small (so that (ξ_1, ξ_2) is close to (a_1, a_2)), then

$$|\epsilon_1(h_1, h_2)| < \epsilon \text{ and } |\epsilon_2(h_1, h_2)| < \epsilon.$$

Moreover, $\epsilon_1(0, 0) = 0$ and $\epsilon_2(0, 0) = 0$. Substituting the values of ϵ_1 and ϵ_2 from equations (7.7) into equation (7.6) we obtain part (b) of the theorem. The continuity of f at (a_1, a_2) follows directly from Equation (7.4).

The proof for $N > 2$ is similar. ■

The chain rule for ordinary differentiation (Theorem 4.9) can be extended to provide a rule for taking partial derivatives of composite functions. We establish the result for a function $f : \mathbb{R}^N \to \mathbb{R}^1$ when it is composed with N functions $g^1, g^2, \ldots, g^N$ each of which is a mapping from $\mathbb{R}^M$ into $\mathbb{R}^1$. The integer M may be different from N. If $y = (y_1, y_2, \ldots, y_N)$ and $x = (x_1, x_2, \ldots, x_M)$ are elements of $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively, then in customary terms, we wish to calculate the derivative of $H(x)$ (a function from $\mathbb{R}^M$ into $\mathbb{R}^1$) with respect to x_j, where $f = f(y_1, y_2, \ldots, y_N)$ and

$$H(x) = f[g^1(x), g^2(x), \ldots, g^N(x)]. \tag{7.9}$$

Theorem 7.3 (Chain rule)
Suppose that each of the functions $g^1, g^2, \ldots, g^N$ is a mapping from $\mathbb{R}^M$ into $\mathbb{R}^1$. For a fixed integer j between 1 and M, assume that $g^1_{,j}, g^2_{,j}, \ldots, g^N_{,j}$ are defined at some point $b = (b_1, b_2, \ldots, b_M)$. Suppose that f and its partial derivatives

$$f_{,1}, f_{,2}, \ldots, f_{,N}$$

are continuous at the point $a = (g^1(b), g^2(b), \ldots, g^N(b))$. Form the function H as in equation (7.9). Then the partial derivative of H with respect to x_j is given by

$$H_{,j}(b) = \sum_{i=1}^{N} f_{,i}[g^1(b), g^2(b), \ldots, g^N(b)]g^i_{,j}(b).$$

Proof
Define the following functions from $\mathbb{R}^1$ into $\mathbb{R}^1$.

$$\varphi(x_j) = H(b_1, \ldots, b_{j-1}, x_j, b_{j+1}, \ldots, b_M),$$

$$\psi^i(x_j) = g^i(b_1, \ldots, b_{j-1}, x_j, b_{j+1}, \ldots, b_M), \quad i = 1, 2, \ldots, N.$$

Then, according to equation (7.9), we have

$$\varphi(x_j) = f[\psi^1(x_j), \psi^2(x_j), \ldots, \psi^N(x_j)].$$

With h denoting a real number, define

$$\Delta\varphi = \varphi(b_j + h) - \varphi(b_j),$$
$$\Delta\psi^i = \psi^i(b_j + h) - \psi^i(b_j), \quad i = 1, 2, \ldots, N.$$

Since each ψ^i is continuous at b_j, it follows that $\Delta\psi^i \to 0$ as $h \to 0$. We apply equation (7.4) from Theorem 7.2 to the function φ and use $\Delta\psi^i$ instead of h_i in that equation. We obtain

$$\Delta\varphi = \sum_{i=1}^{N} \{f_{,i}[\psi^1(b), \ldots, \psi^N(b)] + \epsilon_i\}\Delta\psi^i. \tag{7.10}$$

In this formula, we have $\epsilon_i = \epsilon_i(\Delta\psi^1, \ldots, \Delta\psi^N)$ and $\epsilon_i \to 0$ as $\Delta\psi^k \to 0$, $k = 1, 2, \ldots, N$.

Now write equation (7.10) in the form

$$\frac{\Delta\varphi}{h} = \sum_{i=1}^{N} \{f_{,i}[g^1(b), \ldots, g^N(b)] + \epsilon_i\} \frac{\Delta\psi^i}{h},$$

valid for $|h|$ sufficiently small. Letting h tend to 0, we get the statement of the Chain rule. ■

Problems

1. Let D be a ball in $\mathbb{R}^N$ and suppose $f : D \to \mathbb{R}^1$ has the property that $f_{,1} = f_{,2} = \cdots = f_{,N} = 0$ for all $x \in D$. Show that $f \equiv$ constant in D.
2. Let $f : \mathbb{R}^N \to \mathbb{R}^1$ be given and suppose that $g^1, g^2, \ldots, g^N$ are N functions from $\mathbb{R}^M$ into $\mathbb{R}^1$. Let $h^1, h^2, \ldots, h^M$ be M functions from $\mathbb{R}^p$ into $\mathbb{R}^1$. Give a formula for the Chain rule for $H_{,i}(x)$, where
$$H(x) = f\{g^1[h^1(x), \ldots, h^M(x)], \quad g^2[h^1(x), \ldots, h^M(x)], \ldots\}.$$
3. Write a proof of Theorem 7.2 for $N = 3$.
4. Use the Chain rule to compute $H_{,2}(x)$, where H is given by (7.9) and $f(x) = x_1^2 + x_2^3 - 3x_3$, $g^1(x) = \sin 2x_1 + x_2x_3$, $g^2(x) = \tan x_2 + 3x_3$, $g^3(x) = x_1x_2x_3$.
5. Use the Chain rule to compute $H_{,3}(x)$, where H is given by (7.9) and $f(x) = 2x_1x_2 + x_3^2 - x_4^2$, $g^1(x) = \log(x_1 + x_2) - x_3^2$, $g^2(x) = x_1^2 + x_4^2$, $g^3(x) = x_1^2x_3 + x_4$, $g^4(x) = \cos(x_1 + x_3) - 2x_4$.
6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by
$$f(x_1, x_2) = \begin{cases} \dfrac{x_1x_2}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0, 0), \\ 0, & x_1 = x_2 = 0. \end{cases}$$
   (a) Show that f is not continuous at $x_1 = x_2 = 0$.
   (b) Show that $f_{,1}$ and $f_{,2}$ exist at $x_1 = x_2 = 0$. Why does this fact not contradict Theorem 7.2?

7. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^1$ such that $f_{,1}$ and $f_{,2}$ exist and are bounded in a region about $x_1 = x_2 = 0$. Show that f is continuous at $(0, 0)$.
8. Given the function $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$f(x_1, x_2) = \begin{cases} \dfrac{x_2^3}{x_1^2 + x_2^2}, & (x_1, x_2) \neq (0, 0), \\ 0, & x_1 = x_2 = 0, \end{cases}$$

show that $f_{,1}$ and $f_{,2}$ are bounded near $(0, 0)$ and therefore (Problem 6) that f is continuous at $(0, 0)$.

7.2 Taylor's Theorem; Maxima and Minima

Definitions

Let f be a function from $\mathbb{R}^N$ into $\mathbb{R}^1$. We define the **second partial derivative** $f_{,i,j}$ as the first partial derivative of $f_{,i}$ with respect to x_j. We define the **third partial derivative** $f_{,i,j,k}$ as the first partial derivative of $f_{,i,j}$. Fourth, fifth, and higher derivatives are defined similarly.

In computing second partial derivatives it is natural to ask whether the order of computation affects the result. That is, is it always true that $f_{,i,j} = f_{,j,i}$ for $i \neq j$? There are simple examples that show that the order of computation may lead to different results. (See Problem 3 at the end of this section.) The next theorem, stated without proof, gives a sufficient condition that validates the interchange of order of partial differentiation.

Theorem 7.4

Let $f : \mathbb{R}^N \to \mathbb{R}^1$ be given and suppose that f, $f_{,i}$, $f_{,i,j}$, and $f_{,j,i}$ are all continuous at a point a. Then

$$f_{,i,j}(a) = f_{,j,i}(a).$$

Definitions

A **multi-index** α is an element $(\alpha_1, \alpha_2, \dots, \alpha_N)$ in $\mathbb{R}^N$ where each α_i is a nonnegative integer. The **order** of a multi-index, denoted by $|\alpha|$, is the nonnegative integer $\alpha_1 + \alpha_2 + \dots + \alpha_N$. We extend the factorial symbol to multi-indices by defining $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_N!$. If β is another multi-index $(\beta_1, \beta_2, \dots, \beta_N)$, we define

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \cdots, \alpha_N + \beta_N).$$

Let $x = (x_1, x_2, \ldots, x_N)$ be any element of $\mathbb{R}^N$. Then the **monomial** x^α is defined by the formula

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}.$$

Clearly, the degree of x^α is $|\alpha|$. Any polynomial in $\mathbb{R}^N$ is a function f of the form

$$f(x) = \sum_{|\alpha| \le n} c_\alpha x^\alpha \tag{7.11}$$

in which α is a multi-index, the c_α are constants, and the sum is taken over all multi-indices with order less than or equal to n, the degree of the polynomial.

Lemma 7.1 (Binomial theorem)
Suppose that $x, y \in \mathbb{R}^1$ and n is a positive integer. Then

$$(x + y)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^{n-j} y^j.$$

The proof is easily established by induction on n, and we leave the details to the reader.

The Multinomial theorem, an extension to several variables of the binomial theorem, is essential for the development of the Taylor expansion for functions of several variables. We state the result without proof, although the formula is not difficult to establish by induction. (Fix the integer n and use induction on N.)

Lemma 7.2 (Multinomial theorem)
Suppose that $x = (x_1, x_2, \ldots, x_N)$ is an element of $\mathbb{R}^N$ and that n is any positive integer. Then

$$(x_1 + x_2 + \ldots + x_N)^n = \sum_{|\alpha| = n} \frac{n!}{\alpha!} x^\alpha \equiv \sum_{\alpha_1 + \ldots + \alpha_N = n} \frac{n!}{\alpha_1! \cdots \alpha_N!} x_1^{\alpha_1} \cdots x_N^{\alpha_N}.$$

Let G be an open set in $\mathbb{R}^N$ and let $f : G \to \mathbb{R}^1$ be a function with continuous second derivatives in G. We know that in the computation of second derivatives the order of differentiation may be interchanged. That is, $f_{,i,j} = f_{,j,i}$ for all i and j. We may also write $f_{,i,j} = D_j[D_i f]$. With any polynomial in $\mathbb{R}^N$ of the form

$$P(\xi_1, \xi_2, \ldots, \xi_N) = \sum_{|\alpha| \le n} c_\alpha \xi^\alpha \tag{7.12}$$

we associate the operator

$$P(D_1, D_2, \ldots, D_N) = \sum_{|\alpha| \le n} c_\alpha D^\alpha. \tag{7.13}$$

If α is the multi-index $(\alpha_1, \alpha_2, \ldots, \alpha_N)$, then D^α is the operator given by $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_N^{\alpha_N}$. That is, $D^\alpha f$ means that f is first differentiated with respect to x_N exactly α_N times; then it is differentiated α_{N-1} times with respect to x_{N-1}, and so on until all differentiations of f are completed. The **order** of the operator (7.13) is the degree of the polynomial (7.12).

By induction it is easy to see that every operator consisting of a linear combination of maps is of the form (7.13). The differentiations may be performed in any order. The polynomial $P(\xi_1, \xi_2, \ldots, \xi_N)$ in (7.12) is called the **auxiliary polynomial** of the operator (7.13).

Definitions

Let $f : \mathbb{R}^N \to \mathbb{R}^1$ be a given function, and suppose that $a \in \mathbb{R}^N$ and $b \in \mathbb{R}^N$ with $|b| = 1$. The **directional derivative of *f* in the direction *b* at the point *a***, denoted by $d_b f(a)$, is the number defined by

$$d_b f(a) = \lim_{t \to 0} \frac{f(a + tb) - f(a)}{t}. \tag{7.14}$$

Note that the difference quotient in the definition is taken by subtracting $f(a)$ from the value of f taken on the line segment in $\mathbb{R}^N$ joining a and $a + b$. The **second directional derivative of *f* in the direction *b* at the point *a*** is simply $d_b[d_b f](a)$, and it is denoted by $(d_b)^2 f(a)$. The nth directional derivative, $(d_b)^n f(a)$, is defined similarly.

Lemma 7.3

Suppose that $f : \mathbb{R}^N \to \mathbb{R}^1$ and all its partial derivatives up to and including order n are continuous in a ball $B(a,r)$. Then

$$(d_b)^n f(a) = \sum_{|\alpha| = n} \frac{n!}{\alpha!} b^\alpha D^\alpha f(a). \tag{7.15}$$

Symbolically, the nth directional derivative is written

$$(d_b)^n = \sum_{|\alpha| = n} \frac{n!}{\alpha!} b^\alpha D^\alpha.$$

Proof

For $n = 1$, we set $\phi(t) = f(a + tb)$. Then $d_b f(a) = \phi'(0)$. We use the Chain rule to compute ϕ', and denoting $b = (b_1, b_2, \ldots, b_N)$, we find

$$d_b f(a) = b_1 D_1 f(a) + \ldots + b_N D_N f(a).$$

That is, $d_b = b_1 D_1 + \ldots + b_N D_N$. By induction, we obtain

$$(d_b^n) f = (b_1 D_1 + \ldots + b_N D_N)^n f.$$

Using the Multinomial theorem (Lemma 7.2), we get equation (7.15). ∎

Theorem 7.4 (Taylor's theorem for functions from $\mathbb{R}^1$ to $\mathbb{R}^1$)
Suppose that $f:\mathbb{R}^1$ to $\mathbb{R}^1$ and all derivatives of f up to and including order $n+1$ are continuous on an interval $I = \{x:|x-a| < r\}$. Then for each x on I, there is a number ξ in the open interval between a and x such that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k + \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-a)^{n+1}. \tag{7.16}$$

The proof of Theorem 7.4 makes use of Rolle's theorem (Theorem 4.11) and is deferred until Chapter 8. See Theorem 8.24. Observe that for $n = 0$, formula (7.16) is the Mean-value theorem. (See Problem 6 at the end of this section and the hint given there.) The last term in (7.16) is called the **remainder**.

We now use Theorem 7.4 to establish Taylor's theorem for functions from a domain in $\mathbb{R}^N$ to $\mathbb{R}^1$.

Theorem 7.5 (Taylor's theorem with remainder)
Suppose that $f:\mathbb{R}^N \to \mathbb{R}^1$ and all its partial derivatives up to and including order $n+1$ are continuous on a ball $B(a,r)$. Then for each x in $B(a,r)$, there is a point ξ on the straight line segment from a to x such that

$$f(x) = \sum_{|\alpha|\le n} \frac{1}{\alpha!} D^\alpha f(a)(x-a)^\alpha + \sum_{|\alpha|=n+1} \frac{1}{\alpha!} D^\alpha f(\xi)(x-a)^\alpha. \tag{7.17}$$

Proof
If $x = a$, the result is obvious. If $x \neq a$, define $b = (b_1, b_2, \ldots, b_N)$ by

$$b_i = \frac{x_i - a_i}{|x-a|}.$$

We note that $|b| = 1$ and define $\phi(t) = f(a+tb)$. We observe that $\phi(0) = f(a)$ and $\phi(|x-a|) = f(a + |x-a| \cdot b) = f(x)$. By induction, it follows that ϕ has continuous derivatives up to and including order $n+1$, since f does. Now we apply Taylor's theorem (Theorem 7.4) to ϕ, a function of one variable. Then

$$\phi(t) = \sum_{j=0}^{n} \frac{1}{j!} \phi^{(j)}(0)t^j + \frac{1}{(n+1)!} \phi^{(n+1)}(\tau)t^{n+1}, \tag{7.18}$$

where τ is between 0 and t. From Lemma 7.3, the formula for $(d_b)^j$, it follows that

$$\phi^{(j)}(t) = (d_b)^j f(a+tb) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} b^\alpha D^\alpha f(a+tb). \tag{7.19}$$

We set $t = |x-a|$ in equation (7.18), getting

$$\phi(|x-a|) = f(x) = \sum_{j=0}^{n} \frac{1}{j!} \phi^{(j)}(0)|x-a|^j + \frac{1}{(n+1)!} \phi^{(n+1)}(\tau)|x-a|^{n+1}.$$

Inserting equation (7.19) into this expression, we find

$$f(x) = \sum_{j=0}^{n} \frac{1}{j!}\left[\sum_{|\alpha|=j} \frac{j!}{\alpha!} b^\alpha D^\alpha f(a)|x-a|^j\right] + \frac{1}{(n+1)!}\left[\sum_{|\alpha|=n+1} \frac{(n+1)!}{\alpha!} b^\alpha D^\alpha(\xi)|x-a|^{n+1}\right].$$

By definition we have $x_i - a_i = b_i|x-a|$, and so $(x-a)^\alpha = b^\alpha|x-a|^{|\alpha|}$. Employing this fact in the above expression for $f(x)$, we obtain equation (7.17). ■

The last terms in formula (7.17) are known as the **remainder**.

For functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ the second derivative test is one of the most useful for identifying the maximum and minimum points on the graph of the function. We recall that a function f with two derivatives and with $f'(a) = 0$ has a relative maximum at a if its second derivative at a is negative. It has a relative minimum at a if $f''(a) > 0$. If $f''(a) = 0$, the test fails. With the aid of Taylor's theorem for functions from $\mathbb{R}^N$ into $\mathbb{R}^1$ we can establish the corresponding result for functions of N variables.

Definitions

Let $f : \mathbb{R}^N \to \mathbb{R}^1$ be given. The function f has a **local maximum at the value** a if there is a ball $B(a,r)$ such that $f(x) - f(a) \le 0$ for $x \in B(a,r)$. The function f has a **strict local maximum** at a if $f(x) - f(a) < 0$ for $x \in B(a,r)$ except for $x = a$. The corresponding definitions for **local minimum** and **strict local minimum** reverse the inequality sign. If f has partial derivatives at a, we say that f has a **critical point at the value** a if $D_i f(a) = 0$, $i = 1, 2, \ldots, N$.

Theorem 7.6 (Second derivative test)

Suppose that $f : \mathbb{R}^N \to \mathbb{R}^1$ and its partial derivatives up to and including order 2 are continuous in a ball $B(a,r)$. Suppose that f has a critical point at a. For $h = (h_1, h_2, \ldots, h_N)$, define $\Delta f(a,h) = f(a+h) - f(a)$; also define

$$Q(h) = \sum_{|\alpha|=2} \frac{1}{\alpha!} D^\alpha f(a) h^\alpha = \frac{1}{2!}\sum_{i,j=1}^{N} D_i D_j f(a) h_i h_j. \tag{7.20}$$

(a) *If $Q(h) > 0$ for $h \neq 0$, then f has a strict local minimum at a.*

(b) *If $Q(h) < 0$ for $h \neq 0$, then f has a strict local maximum at a.*

(c) *If $Q(h)$ has a positive maximum and a negative minimum, then $\Delta f(a,h)$ changes sign in any ball $B(a,\rho)$ such that $\rho < r$.*

Proof

We establish part (a), the proofs of parts (b) and (c) being similar. Taylor's theorem with remainder (Theorem 7.5) for $n = 1$ and $x = a + h$ is

$$f(a+h) = f(a) + \sum_{|\alpha|=1} D^\alpha f(a)h^\alpha + \sum_{|\alpha|=2} \frac{1}{\alpha!} D^\alpha f(\xi)h^\alpha, \tag{7.21}$$

where ξ is on the straight line segment connecting a with $a + h$. Since f has a critical point at a, the first sum in Equation (7.21) is zero, and the second sum may be written

$$\Delta f(a,h) = \sum_{|\alpha|=2} \frac{1}{\alpha!} D^\alpha f(a)h^\alpha + \sum_{|\alpha|=2} \frac{1}{\alpha!} [D^\alpha f(\xi) - D^\alpha f(a)]h^\alpha. \tag{7.22}$$

Setting $|h|^2 = h_1^2 + h_2^2 + \ldots + h_N^2$ and

$$\epsilon(h) = \sum_{|\alpha|=2} \frac{1}{\alpha!} [D^\alpha f(\xi) - D^\alpha f(a)] \frac{h^\alpha}{|h|^2},$$

we find from equation (7.22) that

$$\Delta f(a,h) = Q(h) + |h|^2 \epsilon(h). \tag{7.23}$$

Because the second partial derivatives of f are continuous near a, it follows that $\epsilon(h) \to 0$ as $h \to 0$. Also,

$$Q(h) = |h|^2 \sum_{i,j=1}^{N} D_i D_j f(a) \frac{h_i}{|h|} \cdot \frac{h_j}{|h|} \equiv |h|^2 Q_1(h).$$

The expression $Q_1(h)$ is continuous for h on the unit sphere in $\mathbb{R}^N$. According to the hypothesis in part (a), Q_1, a continuous function, must have a positive minimum on the unit sphere, which is a closed set. Denote this minimum by m. Hence,

$$Q(h) \geq |h|^2 m \text{ for all } h.$$

Now choose $|h|$ so small that $|\epsilon(h)| < m/2$. Inserting the inequalities for $Q(h)$ and $|\epsilon(h)|$ into equation (7.23), we find

$$\Delta f(a,h) > \frac{1}{2} |h|^2 m$$

for $|h|$ sufficiently small and $h \neq 0$. We conclude that f has a strict local minimum at a. ■

Problems

1. Given $f : \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$f(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}, \quad (x_1, x_2, x_3) \neq (0,0,0).$$

Show that f satisfies the equation

$$f_{,1,1} + f_{,2,2} + f_{,3,3} = 0.$$

2. Given $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by $f(x_1, x_2) = x_1^4 - 2x_1^3 x_2 - x_1 x_2$ and given $L_1(D) = 2D_1 - 3D_2$, $L_2(D) = D_1 D_2$, show that $(L_1 L_2)(f) = (L_2 L_1)(f)$.
3. Given $f : \mathbb{R}^2 \to \mathbb{R}^1$ defined by

$$f(x_1, x_2) = \begin{cases} x_1 x_2 \dfrac{x_1^2 - x_2^2}{x_1^2 + x_2^2}, & x_1^2 + x_2^2 > 0, \\ 0, & x_1 = x_2 = 0, \end{cases}$$

show that $f_{,1,2}(0, 0) = -1$ and $f_{,2,1}(0, 0) = 1$.
4. Prove the Binomial theorem (Lemma 7.1).
5. Establish the Taylor expansion for functions from $\mathbb{R}^1$ into $\mathbb{R}^1$ (Theorem 7.4). [*Hint*: Make use of the function

$$\varphi(t) = f(x) - f(t) - \frac{f'(t)(x-t)}{1!} - \dots - \frac{f^{(n)}(t)(x-t)^n}{n!}$$
$$- R_n(a, x) \frac{(x-t)^{n+1}}{(x-a)^{n+1}},$$

where $R_n(a, x)$ is the remainder in the Taylor expansion of $f(x)$. Note that $\varphi(a) = \varphi(x) = 0$.]
6. Find the relative maxima and minima of the function $f : \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$f(x_1, x_2) = x_1^3 + 3x_1 x_2^2 - 3x_1^2 - 3x_2^2 + 4.$$

7. Write out explicitly all the terms of the Taylor expansion for a function $f : \mathbb{R}^3 \to \mathbb{R}^1$ for the case $n = 2$.
8. Find the critical points of the function $f : \mathbb{R}^4 \to \mathbb{R}^1$ given by

$$f(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 - x_4^2 - 2x_1x_2 + 4x_1x_3 + 3x_1x_4$$
$$- 2x_2x_4 + 4x_1 - 5x_2 + 7.$$

In each of Problems 9 through 11, determine whether $Q : \mathbb{R}^3 \to \mathbb{R}^1$ is positive definite, negative definite, or neither.

9. $Q(x_1, x_2, x_3) = x_1^2 + 5x_2^2 + 3x_3^2 - 4x_1x_2 + 2x_1x_3 - 2x_2x_3$.
10. $Q(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 6x_2x_3$.
11. $Q(x_1, x_2, x_3) = -x_1^2 - 2x_2^2 - 4x_3^2 - 2x_1x_2 - 2x_1x_3$.

7.3 The Derivative in $\mathbb{R}^N$

Each partial derivative of a function $f : \mathbb{R}^N \to \mathbb{R}^1$ is a mapping from $\mathbb{R}^N$ into $\mathbb{R}^1$. This generalization of the ordinary derivative, useful for many purposes, singles out a particular direction in which the differentiation is performed. We now take up another extension of the ordinary derivative, one in which the difference quotient tends to a limit as $x \to a$ regardless of the direction of approach.

Let A be an open subset of $\mathbb{R}^N$, and suppose that f and g are functions from A into $\mathbb{R}^1$. We denote by $d(x, y)$ the Euclidean distance between two points x, y in $\mathbb{R}^N$.

Definition

The continuous functions f and g are **tangent at a point** $a \in A$ if

$$\lim_{\substack{x \to a \\ x \neq a}} \frac{|f(x) - g(x)|}{d(x, a)} = 0.$$

We note that if two functions are tangent at a point a, then necessarily $f(a) = g(a)$. Also, if f, g, and h are functions with f and g tangent at a and with g and h tangent at a, then we verify simply that f and h are tangent at a. To see this, observe that

$$\frac{|f(x) - h(x)|}{d(x, a)} \leq \frac{|f(x) - g(x)|}{d(x, a)} + \frac{|g(x) - h(x)|}{d(x, a)}.$$

As $x \to a$, the right side tends to 0, and so must the left side.

Let $L : \mathbb{R}^N \to \mathbb{R}^1$ be a linear function; that is, L is of the form

$$L(x) = b_0 + \sum_{k=1}^{N} b_k x_k,$$

where $b_0, b_1, \ldots, b_N$ are real numbers. It may happen that a function f is tangent to a linear function at a point a. If so, this linear function is unique, as we now show.

Theorem 7.7

Suppose that L_1 and L_2 are linear functions tangent to a function f at a point a. Then $L_1 \equiv L_2$.

Proof

It is convenient to write the linear functions L_1 and L_2 in the form

$$L_1(x) = c_0 + \sum_{k=1}^{N} c_k(x_k - a_k), \quad L_2(x) = c_0' + \sum_{k=1}^{N} c_k'(x_k - a_k).$$

From the discussion above, it follows that $L_1(a) = L_2(a) = f(a)$. Hence, $c_0 = c_0'$. Also, for every $\epsilon > 0$, we have

$$|L_1(x) - L_2(x)| \leq \epsilon d(x, a) \tag{7.24}$$

for all x sufficiently close to a. For $z \in \mathbb{R}^N$, we use the notation $\|z\| = d(z, 0)$. Now, with δ a real number, choose

$$x - a = \frac{\delta z}{\|z\|}.$$

Then, for sufficiently small $|\delta|$, we find from inequality (7.24) that

$$\left|\sum_{k=1}^{N}(c_k - c_k')\frac{\delta z_k}{\|z\|}\right| \leq \epsilon|\delta|\frac{\|z\|}{\|z\|} = \epsilon|\delta|.$$

Therefore,

$$\left|\sum_{k=1}^{N}(c_k - c_k')\frac{z_k}{\|z\|}\right| \leq \epsilon,$$

and since this inequality holds for all positive ϵ, we must have $c_k = c_k'$, $k = 1, 2, \ldots, N$. ■

Definitions

Suppose that $f : A \to \mathbb{R}^1$ is given, where A is an open set in $\mathbb{R}^N$ containing the point a. The function f is **differentiable at** a if there is a linear function $L(x) = f(a) + \sum_{k=1}^{N} c_k(x_k - a_k)$ that is tangent to f at a. A function f is **differentiable on a set** A in $\mathbb{R}^N$ if it is differentiable at each point of A. The function L is called the **derivative**, or **total derivative**, **of** $\boldsymbol{f}$ **at** $\boldsymbol{a}$. We use the symbol $f'(a)$ for the derivative of f at the point a in $\mathbb{R}^N$.

Theorem 7.8

If f is differentiable at a point a, then f is continuous at a.

The proof is left to the reader. (Problem 5 at the end of this section.)

As we saw in Section 7.1, a function f may have partial derivatives without being continuous. An example of such a function is given in Problem 6 at the end of Section 7.1. Theorem 7.8 suggests that differentiability is a stronger condition than the existence of partial derivatives. The next theorem verifies this point.

Theorem 7.9

If f is differentiable at a point a, then all first partial derivatives of f exist at a.

Proof

Let L be the total derivative of f at a. We write

$$L(x) = f(a) + \sum_{k=1}^{N} c_k(x_k - a_k).$$

From the definition of derivative, it follows that

$$\lim_{\substack{x \to a \\ x \neq a}} \frac{\left| f(x) - f(a) - \sum_{k=1}^{N} c_k(x_k - a_k) \right|}{d(x, a)} = 0. \tag{7.25}$$

For x we choose the element $a + h$, where $h = (0, 0, \ldots, 0, h_i, 0, \ldots, 0)$. Then equation (7.25) becomes

$$\lim_{\substack{h_i \to 0 \\ h_i \neq 0}} \frac{|f(a + h) - f(a) - c_i h_i|}{|h_i|} = 0;$$

we may therefore write

$$\left| \frac{f(a + h) - f(a)}{h_i} - c_i \right| = \epsilon(h_i), \tag{7.26}$$

where $\epsilon(h_i) \to 0$ as $h_i \to 0$.

We recognize the left side of equation (7.26) as the expression used in the definition of $f_{,i}(a)$. In fact, we conclude that

$$c_i = f_{,i}(a), \quad i = 1, 2, \ldots, N. \qquad \blacksquare$$

A partial converse of Theorem 7.9 is also true.

Theorem 7.10
Suppose that $f_{,i}$, $i = 1, 2, \ldots, N$, are continuous at a point a. Then f is differentiable at a.

This result is most easily established by means of the Taylor expansion with remainder (Theorem 7.5) with $n = 0$. We leave the details for the reader.

Definitions
Suppose that f possesses all first partial derivatives at a point a in $\mathbb{R}^N$. The **gradient of** f is the element in $\mathbb{R}^N$ whose components are

$$(f_{,1}(a), f_{,2}(a), \ldots, f_{,N}(a)).$$

We denote the gradient of f by ∇f or grad f.

Suppose that A is a subset of $\mathbb{R}^N$ and $f : A \to \mathbb{R}^1$ is differentiable on A. Let $h = (h_1, h_2, \ldots, h_N)$ be an element of $\mathbb{R}^N$. We define the **total differential** df as that function from $A \times \mathbb{R}^N \to \mathbb{R}^1$ given by the formula

$$df(x, h) = \sum_{k=1}^{N} f_{,k}(x) h_k. \tag{7.27}$$

Using the inner or dot product notation for elements in $\mathbb{R}^N$, we may also write

$$df(x, h) = \nabla f(x) \cdot h.$$

Equation (7.27) shows that the total differential (also called the **differential**) of a function f is linear in h. The differential vanishes when $x = a$ and $h = 0$. The Chain rule (Theorem 7.3) takes a natural form for differentiable functions, as the following theorem shows.

Theorem 7.11 (Chain rule)
Suppose that each of the functions $g^1, g^2, \ldots, g^N$ is a mapping from $\mathbb{R}^M$ into $\mathbb{R}^1$ and that each g^i is differentiable at a point $b = (b_1, b_2, \ldots, b_M)$. Suppose that $f:\mathbb{R}^N \to \mathbb{R}^1$ is differentiable at the point $a = (g^1(b), g^2(b), \ldots, g^N(b))$. Form the function

$$H(x) = f[g^1(x), g^2(x), \ldots, g^N(x)].$$

Then H is differentiable at b and

$$dH(b, h) = \sum_{i=1}^{N} f_{,i}[g(b)]dg^i(b, h).$$

The proof is similar to the proof of Theorem 7.3.

Problems

1. Let f and g be functions from $\mathbb{R}^N$ into $\mathbb{R}^1$. Show that if f and g are differentiable at a point a, then $f + g$ is differentiable at a. Also, αf is differentiable at a for every real number α.
2. Let $f : \mathbb{R}^3 \to \mathbb{R}^1$ be given by
$$f(x_1, x_2, x_3) = x_1x_3e^{x_1x_2} + x_2x_3e^{x_1x_3} + x_1x_2e^{x_2x_3}.$$
Find ∇f and $df(x, h)$.
3. Let f and g be functions from $\mathbb{R}^N$ into $\mathbb{R}^1$. Show that if f and g are differentiable, and α and β are numbers, then
$$\nabla(\alpha f + \beta g) = \alpha\nabla f + \beta\nabla g,$$
$$\nabla(fg) = f\nabla g + g\nabla f.$$
4. Let $f : \mathbb{R}^N \to \mathbb{R}^1$ and $g : \mathbb{R}^1 \to \mathbb{R}^1$ be given. Assume that the range of f is contained in the domain of g. Show that in such a case,
$$\nabla[g(f)] = g'\nabla f.$$
5. Show that if f is differentiable at a point a, then it is continuous at a. (Theorem 7.8.)
6. Suppose that all first partial derivatives of a function $f : \mathbb{R}^N \to \mathbb{R}^1$ are continuous at a point a. Show that f is differentiable at a. (Theorem 7.10.)
7. Given $f : \mathbb{R}^N \to \mathbb{R}^1$ and an element $b \in \mathbb{R}^N$ with $\|b\| = 1$, the **directional derivative of f in the direction b at the point a**,

denoted by $d_b f$, is the function from $\mathbb{R}^N$ into $\mathbb{R}^1$ defined by

$$d_b f = \lim_{t \to 0} \frac{f(a+tb) - f(a)}{t}.$$

Suppose that all first partial derivatives of f are continuous at a point a. Show that the directional derivative of f in every direction exists at a and that

$$(d_b f)(a) = \nabla f(a) \cdot b.$$

8. Suppose that A is an open set in $\mathbb{R}^N$. Let $f : \mathbb{R}^N \to \mathbb{R}^1$ be differentiable at each point of A. Assume that the derivative of f, a mapping from A into $\mathbb{R}^1$, is also differentiable at each point of A. Then show that all second partial derivatives of f exist at each point of A and that

$$D_i D_j f = D_j D_i f$$

for all $i, j = 1, 2, \ldots, N$.
9. Prove Theorem 7.11.
10. Suppose f and g are functions from $\mathbb{R}^N$ into $\mathbb{R}^M$. We denote by d_N and d_M Euclidean distance in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively. We say that ***f* and *g* are tangent at a point *a* ∈ $\mathbb{R}^N$** if

$$\lim_{x \to a} \frac{d_M(f(x), g(x))}{d_N(x, a)} = 0.$$

Show that if f is tangent to a linear function $L : \mathbb{R}^N \to \mathbb{R}^M$ at a point a, then there can be only one such. [*Hint*: Write L as a system of linear equations and follow the proof of Theorem 7.7.]
11. (a) Let $f : \mathbb{R}^N \to \mathbb{R}^M$ be given. Using the result of Problem 10, define the **derivative of *f* at a point *a***.
(b) Let the components of f be denoted by $f^1, f^2, \ldots, f^M$. If f is differentiable at a, show that all partial derivatives of $f^1, f^2, \ldots, f^M$ exist at the point a.
12. Let $L : \mathbb{R}^N \to \mathbb{R}^M$ be a linear function. Show that $L = (L_1, L_2, \ldots, L_M)$, where $L_j = c_0^j + \sum_{k=1}^N c_k^j x_k$, $j = 1, 2, \ldots, M$.

7.4 The Darboux Integral in $\mathbb{R}^N$

The development of the theory of integration in $\mathbb{R}^N$ for $N \geq 2$ parallels the one-dimensional case given in Section 5.1. For functions defined on an interval I in $\mathbb{R}^1$, we formed upper and lower sums by dividing I into a number of subintervals. In $\mathbb{R}^N$ we begin with a domain F, i.e., a bounded region that has volume, and in order to form upper and lower sums, we divide F into a number of subdomains. These subdomains are the generalizations of the subintervals in $\mathbb{R}^1$, and the limits of the upper and lower sums as the number of subdomains tends to infinity yield upper and lower integrals. The subdomains can be thought of as hypercubes in $\mathbb{R}^N$ with the volume of each hypercube just N times the length of a side.

Definition

Let F be a bounded set in $\mathbb{R}^N$ that is a domain. A **subdivision** of F is a finite collection of domains $\{F_1, F_2, \ldots, F_n\}$ no two of which have common interior points and the union of which is F. That is,

$$F = F_1 \cup F_2 \cup \ldots \cup F_n, \qquad \mathrm{Int}(F_i) \cap \mathrm{Int}(F_j) = \emptyset \text{ for } i \neq j.$$

We denote such a subdivision by the single letter Δ.

Let D be a set in $\mathbb{R}^N$ containing F and suppose that $f : D \to \mathbb{R}^1$ is a bounded function on F. Let Δ be a subdivision of F and set

$$M_i = \sup f \text{ on } F_i, \qquad m_i = \inf f \text{ on } F_i.$$

Definitions

The **upper sum** of f with respect to the subdivision Δ is defined by the formula

$$S^+(f, \Delta) = \sum_{i=1}^{n} M_i V(F_i),$$

where $V(F_i)$ is the volume in $\mathbb{R}^N$ of the domain F_i. Similarly, the **lower sum** of f is defined by

$$S_-(f, \Delta) = \sum_{i=1}^{n} m_i V(F_i).$$

Let Δ be a subdivision of a domain F. Then Δ', another subdivision of F, is called a **refinement** of Δ if every domain of Δ' is entirely contained in *one* of the domains of Δ. Suppose that $\Delta_1 = \{F_1, F_2, \ldots, F_n\}$ and $\Delta_2 = \{G_1, G_2, \ldots, G_m\}$ are two subdivisions of F. We say that Δ', the subdivision consisting of all nonempty domains of the form $F_i \cap G_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, is the **common refinement** of Δ_1 and Δ_2. Note that Δ' is a refinement of both Δ_1 and Δ_2.

Theorem 7.12

Let F be a domain in $\mathbb{R}^N$ and suppose that $f : F \to \mathbb{R}^1$ is bounded on F.

(a) *If $m \leq f(x) \leq M$ for all $x \in F$ and if Δ is any subdivision of F, then*

$$mV(F) \leq S_-(f, \Delta) \leq S^+(f, \Delta) \leq MV(F).$$

(b) *If Δ' is a refinement of Δ, then*

$$S_-(f, \Delta) \leq S_-(f, \Delta') \text{ and } S^+(f, \Delta') \leq S^+(f, \Delta).$$

(c) *If* Δ_1 *and* Δ_2 *are any two subdivisions of* F, *then*

$$S_-(f, \Delta_1) \leq S^+(f, \Delta_2).$$

Proof

(a) The proof of part (a) is identical to the proof of part (a) in Theorem 5.1, the same theorem for functions from $\mathbb{R}^1$ into $\mathbb{R}^1$.

(b) Let $\Delta' = \{F_1', \ldots, F_m'\}$ be a refinement of $\Delta = \{F_1, F_2, \ldots, F_n\}$. We denote by $F_1', F_2', \ldots, F_k'$ the domains of Δ' contained in F_1. Then, using the symbols

$$m_1 = \inf f \text{ in } F_1 \text{ and } m_1' = \inf f \text{ in } F_1', \qquad i = 1, 2, \ldots, k,$$

we have immediately $m_1 \leq m_i'$, $i = 1, 2, \ldots, k$, since each F_i' is a subset of F_1.

Because $V(F_1) = V(F_1') + \ldots + V(F_k')$, we have

$$m_1 V(F_1) \leq m_1'(F_1') + \ldots + m_k' V(F_k'). \tag{7.28}$$

The same type of inequality as (7.28) holds for $F_2, F_3, \ldots, F_n$. Summing these inequalities, we get $S_-(F, \Delta) \leq S_-(f, \Delta')$. The proof that $S^+(f, \Delta') \leq S^+(f, \Delta)$ is similar.

(c) If Δ_1 and Δ_2 are two subdivisions, let Δ' denote the common refinement. Then, from parts (a) and (b), it follows that

$$S_-(f, \Delta_1) \leq S_-(f, \Delta') \leq S^+(f, \Delta') \leq S^+(f, \Delta_2). \quad \blacksquare$$

Definitions

Let F be a domain in $\mathbb{R}^N$ and suppose that $f : F \to \mathbb{R}^1$ is bounded on F. The **upper integral** of f is defined by

$$\overline{\int_F} f \, dV = \text{g.l.b. } S^+(f, \Delta),$$

where the greatest lower bound is taken over all subdivisions Δ of F. The **lower integral** of f is

$$\underline{\int_F} f \, dV = \text{l.u.b. } S_-(f, \Delta),$$

where the least upper bound is taken over all possible subdivisions Δ of F. If

$$\overline{\int_F} f \, dV = \underline{\int_F} f \, dV,$$

then we say that f is **Darboux integrable**, or just **integrable**, on F, and we designate the common value by

$$\int_F f \, dV.$$

When we wish to emphasize that the integral is N-dimensional, we write

$$\int_F f\, dV_N.$$

The following elementary results for integrals in $\mathbb{R}^N$ are the direct analogues of the corresponding theorems given in Chapter 5 for functions from $\mathbb{R}^1$ into R^1. The proofs are the same except for the necessary alterations from intervals in $\mathbb{R}^1$ to domains in $\mathbb{R}^n$.

Theorem 7.13
Let F be a domain in $\mathbb{R}^N$. Let f, f_1, f_2 be functions from F into $\mathbb{R}^1$ that are bounded.

(a) *If $m \le f(x) \le M$ for $x \in F$, then*

$$mV(F) \le \underline{\int_F} f\, dV \le \overline{\int_F} f\, dV \le MV(F).$$

(b) *The following inequalities hold:*

$$\underline{\int_F} (f_1 + f_2) dV \ge \underline{\int_F} f_1 dV + \underline{\int_F} f_2 dV;$$

$$\overline{\int_F} (f_1 + f_2) dV \le \overline{\int_F} f_1 dV + \overline{\int_F} f_2 dV.$$

(c) *If $f_1(x) \le f_2(x)$ for all $x \in F$, then*

$$\underline{\int_F} f_1 dV \le \underline{\int_F} f_2 dV, \qquad \overline{\int_F} f_1 dV \le \overline{\int_F} f_2 dV.$$

Theorem 7.14
Let F be a domain in $\mathbb{R}^N$ and suppose that $f{:}F \to \mathbb{R}^1$ is bounded.

(a) *f is Darboux integrable on F if and only if for each $\varepsilon > 0$ there is a subdivision Δ of F such that*

$$S^+(f, \Delta) - S_-(f, \Delta) < \varepsilon.$$

(b) *If f is uniformly continuous on F, then F is Darboux integrable on F.*
(c) *If f is Darboux integrable on F, then $|f|$ is also.*
(d) *If f_1, f_2 are each Darboux integrable on F, then $f_1 \cdot f_2$ is also.*
(e) *If f is Darboux integrable on F, and H is a domain contained in F, then f is Darboux integrable on H.*

The proof of Theorem 7.14 follows the lines of the analogous theorem in $\mathbb{R}^1$.

For positive functions from an interval I in $\mathbb{R}^1$ to $\mathbb{R}^1$, the Darboux integral gives a formula for finding the area under a curve. If F is a domain in $\mathbb{R}^N$, $N \ge 2$, and if $f : F \to \mathbb{R}^1$ is a nonnegative function, then the

Darboux integral of f gives the $(N+1)$-dimensional volume "under the hypersurface f."

Problems

1. Let F be a domain in $\mathbb{R}^N$. Give an example of a function $f : F \to \mathbb{R}^N$ such that f is bounded on F and $\underline{\int_F} f\,dV \neq \overline{\int_F} f\,dV$.
2. Let F be a domain in $\mathbb{R}^N$ and suppose that $f : F \to \mathbb{R}^1$ is continuous on F. Let Δ be a subdivision of F. Suppose that $S_-(f, \Delta) = S_-(f, \Delta')$ for every refinement Δ' of Δ. What can be concluded about the function f?
3. Suppose that F is a domain in $\mathbb{R}^N$ and that f is integrable over F. Show that
$$\left|\int_F f(x)dV\right| \leq \int_F |f(x)|dV.$$
4. Let F be a domain in $\mathbb{R}^N$ and suppose that $f : F \to \mathbb{R}^1$ is uniformly continuous on F. Show that f is Darboux integrable on F (Theorem 7.14, part (b)).
5. Let F be a domain in $\mathbb{R}^N$ with $V(F) > 0$ and suppose that $f : F \to \mathbb{R}^1$ is continuous on F. Suppose that for every continuous function $g : F \to \mathbb{R}^1$ we have $\int_F fg\,dV = 0$. Prove that $f \equiv 0$ on F.

7.5 The Riemann Integral in $\mathbb{R}^N$

The method of extending the Riemann integral from $\mathbb{R}^1$ to $\mathbb{R}^N$ is similar to the extension of the Darboux integral described in Section 7.4.

Definitions

Let A be a set in a metric space S with metric d. We define the **diameter** of A as the $\sup d(x, y)$, where the supremum is taken over all x, y in A. The notation diam A is used for the diameter of A.

Suppose that F is a domain in $\mathbb{R}^N$ and that $\Delta = \{F_1, F_2, \ldots, F_n\}$ is a subdivision of F. The **mesh** *of* Δ, denoted by $\|\Delta\|$, is the maximum of the diameters of $F_1, F_2, \ldots, F_n$.

Definition

Let f be a function from F, a domain in $\mathbb{R}^N$ into $\mathbb{R}^1$. Then f is **Riemann integrable** on F if there is a number L with the following property: For each $\varepsilon > 0$ there is a $\delta > 0$ such that if Δ is any subdivision of F with $\|\Delta\| < \delta$, and $x^i \in F_i$, $i = 1, 2, \ldots, n$, then

$$\left|\sum_{i=1}^{n} f(x^i)V(F_i) - L\right| < \varepsilon.$$

This inequality must hold no matter how the x^i are chosen in the F_i. The number L is called the **Riemann integral of f over F**, and we use the notation

$$\int_F f\, dV$$

for this value.

We recall that in $\mathbb{R}^1$ the Darboux and Riemann integrals are the same. The next two theorems state that the same result holds for integrals in $\mathbb{R}^N$.

Theorem 7.15
Let F be a domain in $\mathbb{R}^N$ and suppose that $f{:}F \to \mathbb{R}^1$ is Riemann integrable on F. Then f is Darboux integrable on F, and the two integrals are equal.

The converse of Theorem 7.15 is contained in the following result.

Theorem 7.16
Let F be a domain in $\mathbb{R}^N$ and suppose that $f{:}F \to \mathbb{R}^1$ is bounded on F. Then

(a) *for each $\varepsilon > 0$ there is an $\varepsilon > 0$ such that*

$$S^+(f, \Delta) < \overline{\int_F} f\, dV + \varepsilon \text{ and } S^-(f, \Delta) > \underline{\int_F} f\, dV - \varepsilon \tag{7.29}$$

for every subdivision of mesh $\|\Delta\| < \delta$;

(b) *if f is Darboux integrable on F, then it is Riemann integrable on F, and the integrals are equal.*

Proof
Observe that (b) is an immediate consequence of (a), since each Riemann sum is between $S_-(f, \Delta)$ and $S^+(f, \Delta)$. Hence, if $\overline{\int_F} f\, dV = \underline{\int_F} f\, dV$, this value is also the value of the Riemann integral.

We shall prove only the first inequality in (7.29), because the proof of the second is similar.

Since f is bounded, there is a number M such that $|f(x)| \le M$ for all $x \in F$. Let $\varepsilon > 0$ be given. According to the definition of upper Darboux integral, there is a subdivision $\Delta_0 = \{F_1, F_2, \ldots, F_m\}$ such that

$$S^+(f, \Delta_0) < \overline{\int_F} f\, dV + \frac{\varepsilon}{2}. \tag{7.30}$$

Define $F_i^{(0)}$ to be the set of interior points of F_i. It may happen that $F_i^{(0)}$ is empty for some values of i. For each $F_i^{(0)}$ that is not empty, we select a closed domain G_i contained in $F_i^{(0)}$ such that

$$V(F_i - G_i) < \frac{\varepsilon}{4Mm}, \qquad i = 1, 2, \ldots, m.$$

It is not difficult to verify that such closed domains $G_1, G_2, \ldots, G_m$ can always be found. For example, each G_i may be chosen as the union of closed hypercubes interior to F_i for a sufficiently small grid size.

Since each set G_i is closed and is contained in $F_i^{(0)}$, there is a positive number δ such that every ball $B(x, \delta)$ with x in some G_i has the property that $B(x, \delta)$ is contained in the corresponding set $F_i^{(0)}$. Let Δ be any subdivision with mesh less than δ. We shall show that the first inequality in (7.29) holds for this subdivision.

We separate the domains of Δ into two classes: $J_1, J_2, \ldots, J_n$ are those domains Δ_1 of Δ containing points of some G_i; $K_1, K_2, \ldots, K_q$ are the remaining domains of Δ, denoted by Δ_0.

Denote by Δ' the common refinement of Δ_1 and Δ_0. Because of the manner in which we chose Δ, each J_i is contained entirely in some $F_k^{(0)}$. Therefore, $J_1, J_2, \ldots, J_n$ are domains in the refinement Δ'. The remaining domains of Δ' are composed of the sets $K_i \cap F_j$, $i = 1, 2, \ldots, q$; $j = 1, 2, \ldots, m$. We have the inequality

$$\sum_{k=1}^{q} V(K_k) < \sum_{i=1}^{m} V(F_i - G_i) < m \cdot \frac{\varepsilon}{4Mm} = \frac{\varepsilon}{4M}.$$

We introduce the notation

$$M_i = \sup_{x \in J_i} f(x), \qquad M_i' = \sup_{x \in K_i} f(x), \qquad M_{ij} = \sup_{x \in K_i \cap F_j} f(x).$$

Using the definitions of $S^+(f, \Delta)$ and $S^+(f, \Delta')$, we obtain

$$S^+(f, \Delta) = \sum_{i=1}^{n} M_i V(J_i) + \sum_{i=1}^{q} M_i' V(K_i),$$

$$S^+(f, \Delta') = \sum_{i=1}^{n} M_i V(J_i) + \sum_{i=1}^{q} \sum_{j=1}^{m} M_{ij} V(K_i \cap F_j).$$

Now it is clear that $V(K_i) = \sum_{j=1}^{m} V(K_i \cap F_j)$. Therefore, by subtraction, it follows that

$$\begin{aligned} S^+(f, \Delta) - S^+(f, \Delta') &= \sum_{i=1}^{q} \sum_{j=1}^{m} (M_i' - M_{ij}) V(K_i \cap F_j) \\ &\le 2M \sum_{i=1}^{q} \sum_{j=1}^{m} V(K_i \cap F_j) \qquad (7.31) \\ &\le 2M \sum_{i=1}^{q} V(K_i) < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}. \end{aligned}$$

According to part (b) of Theorem 7.12, we have

$$S^+(f, \Delta') \le S^+(f, \Delta_0).$$

Combining this fact with inequalities (7.30) and (7.31), we conclude that

$$S^+(f, \Delta) < \overline{\int_F} f \, dV + \varepsilon.$$

which is the first inequality in part (a) of the theorem. ■

Theorem 7.16 leads to the following result (stated without proof) on interchanging the order of integration in multiple integrals.

Theorem 7.17
Suppose that F is a domain in $\mathbb{R}^M$ and that G_x is a domain in $\mathbb{R}^N$ for each $x \in F$. Define $B = \{(x,y): x \in F,\ y \in G_x \text{ for each such } x\}$. Let $f: B \to \mathbb{R}^1$ be integrable over B and suppose that f is integrable over G_x for each $x \in F$. Then the function

$$\phi(x) = \int_{G_x} f(x, y) dV_N$$

is integrable over F, and

$$\int_B f(x, y) dV_{M+N} = \int_F \left[\int_{G_x} f(x, y) dV_N \right] dV_M.$$

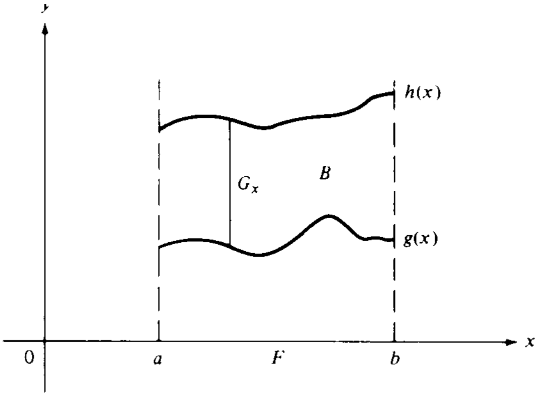

Figure 7.1 $B = \{(x, y) : a \le x \le b,\ y \in G_x\}$.

Figure 7.1 shows a simple illustration of the theorem for functions on $\mathbb{R}^2$. Let $B = \{(x, y) : a \le x \le b,\ g(x) \le y \le h(x)\}$. Then $F = \{x : a \le x \le b\}$ and $G_x = \{y : g(x) \le y \le h(x) \text{ for each } x\}$. The theorem states that

$$\int_B f(x, y) dA = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y) dy \right] dx.$$

Problems

1. Prove that the Riemann integral of a function in $\mathbb{R}^N$ is unique.
2. Suppose that F is a domain in $\mathbb{R}^N$ for $N \geq 2$ and that $f : F \to \mathbb{R}^1$ is Riemann integrable on F. Show that f need not be bounded on F.
3. Let F be a domain in $\mathbb{R}^N$. Show that if $f : F \to \mathbb{R}^1$ is Riemann integrable, then it is Darboux integrable (Theorem 7.15).
4. Suppose f is defined in the square $S = \{(x, y) : 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ by the formula

$$f(x, y) = \begin{cases} 1 & \text{if } x \text{ is irrational,} \\ 4y^3 & \text{if } x \text{ is rational.} \end{cases}$$

   (a) Show that $\int_0^1 (\int_0^1 f(x, y)dy)dx$ exists and has the value 1.
   (b) Show that $\int_S f(x, y)dV$ does not exist.

8 CHAPTER Infinite Series

8.1 Tests for Convergence and Divergence

It is customary to use expressions such as

$$u_1 + u_2 + \cdots + u_n + \cdots \quad \text{and} \quad \sum_{n=1}^{\infty} u_n \tag{8.1}$$

to represent infinite series. The u_i are called the **terms** of the series, and the quantities

$$s_n = u_1 + u_2 + \cdots + u_n, \qquad n = 1, 2, \ldots,$$

are called the **partial sums** of the series. The symbols in (8.1) not only define an infinite series but also are used as an expression for the sum of the series when it converges. To avoid this ambiguity we define an infinite series in terms of ordered pairs.

Definitions
An **infinite series** is an ordered pair $(\{u_n\}, \{s_n\})$ of infinite sequences in which $s_n = u_1 + u_2 + \ldots + u_n$ for each n. The u_n are called the **terms** of the series and the s_n are called the **partial sums**. If there is a number s such that $s_n \to s$ as $n \to \infty$, we say that the series is **convergent** and that the **sum** of the series is s. If the s_n do not tend to a limit, we say that the series is **divergent**.

It is clear that an infinite series is uniquely determined by the sequence $\{u_n\}$ of its terms. There is almost never any confusion in using the symbols in (8.1) for an infinite series. While the definition in terms of ordered pairs is satisfactory from the logical point of view, it does require a cumbersome notation. Rather than have unwieldy proofs that may obscure their essential features, we shall use the standard notation $u_1 + u_2 + \cdots + u_n + \cdots$ and $\sum_{n=1}^{\infty} u_n$ to denote infinite series. The context will always show whether the expression represents the series itself or the sum of the terms.

Theorem 8.1
If the series $\sum_{n=1}^{\infty} u_n$ converges, then $u_n \to 0$ as $n \to \infty$.

Proof
For all $n > 1$, we have $u_n = s_n - s_{n-1}$. If s denotes the sum of the series, then $s_n \to s$ and $s_{n-1} \to s$ as $n \to \infty$. Hence $u_n \to s - s = 0$ as $n \to \infty$. ■

Let

$$u_1 + u_2 + \cdots + u_n + \cdots$$

be a given series. Then a new series may be obtained by deleting a finite number of terms. It is clear that *the new series will be convergent if and only if the original series is.*

Theorem 8.2
Let $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ be given series and let $c \neq 0$ be a constant.

(a) *If $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ are convergent, then $\sum_{n=1}^{\infty} (u_n + v_n)$, $\sum_{n=1}^{\infty} (u_n - v_n)$, and $\sum_{n=1}^{\infty} cu_n$ are convergent series. Also,*

$$\sum_{n=1}^{\infty}(u_n \pm v_n) = \sum_{n=1}^{\infty} u_n \pm \sum_{n=1}^{\infty} v_n, \qquad \sum_{n=1}^{\infty} cu_n = c\sum_{n=1}^{\infty} u_n.$$

(b) *If $\sum_{n=1}^{\infty} u_n$ diverges, then $\sum_{n=1}^{\infty} cu_n$ diverges.*

Proof
For each positive integer n, we have

$$\sum_{k=1}^{n}(u_k \pm v_k) = \sum_{k=1}^{n} u_k \pm \sum_{k=1}^{n} v_k, \qquad \sum_{k=1}^{n}(cu_k) = c\sum_{k=1}^{n} u_k.$$

Then part (a) follows from the theorems on limits. (See Section 2.5.) To prove part (b), we have only to observe that if $\sum_{k=1}^{n} cu_n$ converges, then so does $\sum_{n=1}^{n}(1/c)(cu_n) = \sum_{n=1}^{n} u_n$. ■

A series of the form

$$a + ar + ar^2 + \ldots + ar^n + \cdots$$

is called a **geometric series**. The number r is the **common ratio**.

Theorem 8.3
A geometric series with $a \neq 0$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. In the convergent case, we have

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} .$$

Proof
We easily verify that for each positive integer n,

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a}{1-r} - \frac{ar^n}{1-r} .$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$. Hence $s_n \to a/(1-r)$. If $|r| \geq 1$, then u_n does not tend to zero. According to Theorem 8.1, the series cannot converge. ■

Theorem 8.4 (Comparison test)
Suppose that $u_n \geq 0$ for all n.

(a) *If $u_n \leq a_n$ for all n and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} u_n$ converges and $\sum_{n=1}^{\infty} u_n \leq \sum_{n=1}^{\infty} a_n$.*

(b) *Let $a_n \geq 0$ for all n. If $\sum_{n=1}^{\infty} a_n$ diverges and $u_n \geq a_n$ for all n, then $\sum_{n=1}^{\infty} u_n$ diverges.*

The proof is left to the reader. (See Problem 11 at the end of this section.)

Let f be continuous on $[a, +\infty)$. We define

$$\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx \tag{8.2}$$

when the limit on the right exists. The term **improper integral** is used when the range of integration is infinite. We say that the improper integral **converges** when the limit in (8.2) exists; otherwise the integral **diverges**.

Theorem 8.5 (Integral test)
Suppose that f is continuous, nonnegative, and nonincreasing on $[1, +\infty)$. Suppose that $\sum_{n=1}^{\infty} u_n$ is a series with $u_n = f(n)$, $n = 1, 2, \ldots$. Then

(a) *$\sum_{n=1}^{\infty} u_n$ converges if $\int_1^{\infty} f(x)dx$ converges; and*

(b) *$\sum_{n=1}^{\infty} u_n$ diverges if $\int_1^{\infty} f(x)dx$ diverges.*

Proof
Since f is positive and nonincreasing, we have for $n \geq 2$,

$$\sum_{j=2}^{n} u_j \leq \int_1^n f(x)dx \leq \sum_{j=1}^{n-1} u_j \,.$$

We define

$$F(X) = \int_1^X f(x)dx.$$

If this integral converges, then $F(X)$ is a nondecreasing function that tends to a limit, and so $F(n)$ is a bounded nondecreasing sequence. Thus setting $s_n = \sum_{j=2}^{n} u_j$, we see that $s_n \leq F(n)$, and s_n tends to a limit. Part (a) is now established. If the integral diverges, then $F(X) \to +\infty$ as $X \to \infty$. Therefore, $F(n) \to +\infty$. Since $F(n) \leq \sum_{j=1}^{n-1} u_j$, we conclude that the series diverges. ■

Corollary to Theorem 8.5
The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \,,$$

known as the p-series, converges if $p > 1$ *and diverges if* $p \leq 1$.

The proof of the corollary is an immediate consequence of the integral test with $f(x) = 1/x^p$. For $0 < p \leq 1$, the terms of the p-series tend to zero as $n \to \infty$ although the series diverges. This fact shows that the converse of Theorem 8.1 is false. The hypothesis that $u_n \to 0$ as $n \to \infty$ does *not* imply the convergence of $\sum_{n=1}^{\infty} u_n$.

Example
Test the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)\log(n+2)}$$

for convergence or divergence.

Solution. We define $f(x) = 1/[(x+2)\log(n+2)]$ and observe that f is positive and nonincreasing with $f(n) = 1/[(n+2)\log(n+2)]$. We have

$$\int_1^a \frac{dx}{(x+2)\log(x+2)} = \int_3^{a+2} \frac{du}{u \log u}$$

$$= \int_3^{a+2} \frac{d(\log u)}{\log u} = \log[\log(a+2)] - \log\log 3.$$

Since $\log[\log(a+2)] \to +\infty$ as $a \to +\infty$, the integral diverges. Therefore the series diverges. ∎

Problems

In each of Problems 1 through 10 test for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$.
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$.
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$.
5. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^3}$.
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$.
7. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{n^3}$.
8. $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n^{3/2}}$.
9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$.
10. $\displaystyle\sum_{n=1}^{\infty} \frac{n^p}{n!}$, $p > 0$ constant
11. Prove Theorem 8.4(*a*). [*Hint*: Use the fact that a bounded, nondecreasing sequence tends to a limit (Axiom of continuity).]
12. Prove Theorem 8.4(*b*).
13. For what values of p does the series $\sum_{n=1}^{\infty} \log n/n^p$ converge?
14. For what values of p does the series $\sum_{n=2}^{\infty} (\log n)^p/n$ converge?
15. Prove the corollary to Theorem 8.5.
16. Prove the **Limit comparison theorem**:

    Suppose that $a_n \geq 0$, $b_n \geq 0$, $n = 1, 2, \ldots$, and that

    $$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0.$$

    Then either $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge. [*Hint*: For sufficiently large n, we have $\frac{1}{2}L < a_n/b_n < \frac{3}{2}L$. Now use the comparison test (Theorem 8.4) and the fact that the early terms of a series do not affect convergence.]
17. Use the result of Problem 16 to test for convergence:

    $$\sum_{n=1}^{\infty} \frac{2n^2+n+2}{5n^3+3n}.$$

    Take

    $$a_n = \frac{2n^2+n+2}{5n^3+3n}, \qquad b_n = \frac{1}{n}.$$
18. Use the result of Problem 16 to test for convergence:

    $$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2+5}}.$$

Take

$$a_n = \frac{1}{\sqrt[3]{n^2+5}}, \qquad b_n = \frac{1}{n^{2/3}}.$$

8.2 Series of Positive and Negative Terms; Power Series

When all the terms of a series are nonnegative, the Comparison test (Theorem 8.4) is a useful tool for testing the convergence or divergence of an infinite series. (See also the Limit comparison test in Problem 16 at the end of Section 8.1.) Now we show that the same test may be used when a series has both positive and negative terms.

Definition
A series $\sum_{n=1}^{\infty} u_n$ that is such that $\sum_{n=1}^{\infty} |u_n|$ converges is said to be **absolutely convergent**. However, if $\sum_{n=1}^{\infty} u_n$ converges and $\sum_{n=1}^{\infty} |u_n|$ diverges, then the series $\sum_{n=1}^{\infty} u_n$ is said to be **conditionally convergent**.

The next theorem shows that if a series is absolutely convergent, then the series itself converges.

Theorem 8.6
If $\sum_{n=1}^{\infty} |u_n|$ converges, then $\sum_{n=1}^{\infty} u_n$ converges and

$$\left|\sum_{n=1}^{\infty} u_n\right| \le \sum_{n=1}^{\infty} |u_n|.$$

Proof
For $n = 1, 2, \ldots$, we define

$$v_n = \frac{|u_n| + u_n}{2}, \qquad w_n = \frac{|u_n| - u_n}{2}.$$

Then we have

$$u_n = v_n - w_n, \qquad |u_n| = v_n + w_n$$

and

$$0 \le v_n \le |u_n|, \qquad 0 \le w_n \le |u_n|.$$

Both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge by the Comparison test (Theorem 8.4). Therefore, $\sum_{n=1}^{\infty}(v_n - w_n) = \sum_{n=1}^{\infty} u_n$ converges. Also,

$$\left|\sum_{n=1}^{\infty} u_n\right| = \left|\sum_{n=1}^{\infty}(v_n - w_n)\right| \le \sum_{n=1}^{\infty}(v_n + w_n) = \sum_{n=1}^{\infty} |u_n|.$$

■

A series may be conditionally convergent and not absolutely convergent. The next theorem shows that $\sum_{n=1}^{\infty}(-1)^n(1/n)$ is convergent. However, the series $\sum_{n=1}^{\infty} 1/n$, a p-series with $p = 1$, is divergent; hence $\sum_{n=1}^{\infty}(-1)^n(1/n)$ is a *conditionally convergent* series.

Theorem 8.7 (Alternating series theorem)
Suppose that the numbers u_n, $n = 1,2,\ldots$, satisfy the following conditions:

(i) *The u_n are alternately positive and negative.*
(ii) $|u_{n+1}| < |u_n|$ *for every n.*
(iii) $\lim_{n\to\infty} u_n = 0$.

Then $\sum_{n=1}^{\infty} u_n$ is convergent. Furthermore, if the sum is denoted by s, then s lies between the partial sums s_n and s_{n+1} for each n.

Proof
Assume that u_1 is positive. If not, consider the series beginning with u_2, since discarding one term does not affect convergence. With $u_1 > 0$, we have, clearly,

$$u_{2n-1} > 0 \text{ and } u_{2n} < 0 \text{ for all } n.$$

We now write

$$s_{2n} = (u_1 + u_2) + (u_3 + u_4) + \ldots + (u_{2n-1} + u_{2n}).$$

Since by (ii) above $|u_{2k}| < u_{2k-1}$ for each k, each term in parentheses is positive, and so s_{2n} increases with n. Also,

$$s_{2n} = u_1 + (u_2 + u_3) + (u_4 + u_5) + \ldots + (u_{2n-2} + u_{2n-1}) + u_{2n}.$$

The terms in parentheses are negative, and $u_{2n} < 0$. Therefore, $s_{2n} < u_1$ for all n. Hence s_{2n} is a bounded, increasing sequence and therefore convergent to a number, say s; also, $s_{2n} \le s$ for each n. By observing that $s_{2n-1} = s_{2n} - u_{2n}$, we have $s_{2n-1} > s_{2n}$ for all n. In particular, $s_{2n-1} > s_2 = u_1 + u_2$, and so s_{2n-1} is bounded from below. Also,

$$s_{2n+1} = s_{2n-1} + (u_{2n} + u_{2n+1}) < s_{2n-1}.$$

Therefore, s_{2n+1} is a decreasing sequence that tends to a limit. Since $u_{2n} \to 0$ as $n \to \infty$, we see that s_{2n} and s_{2n+1} tend to the same limit s. Because s_{2n-1} is decreasing, it follows that $s_{2n-1} \ge s$ for every n. Thus $s \le s_p$ for odd p and $s \ge s_p$ for even p. ■

The least upper bound and greatest lower bound of a set were discussed in Chapter 3. Now we introduce a related concept, one that is useful in the study of infinite series. Let $a_1, a_2, a_3, \ldots, a_n, \ldots$ be a bounded sequence of real numbers. A number M is called the limit superior of the sequence $\{a_n\}$ and denoted by $\limsup_{n\to\infty} a_n$ if the following two conditions hold:

(a). There is a subsequence of $\{a_n\}$ that converges to M.

(b). For every $\varepsilon > 0$ there are at most a finite number of a_n such that $a_n \geq M + \varepsilon$.

The limit inferior of $\{a_n\}$, denoted by $\liminf_{n\to\infty} a_n$, is equal to a number m if the following conditions are satisfied:

(a′) There is a subsequence of $\{a_n\}$ that converges to m.
(b′) For every $\varepsilon > 0$ there are at most a finite number of terms of the sequence such that $a_n \leq m - \varepsilon$.

We use the notations

$$M = \limsup_{n\to\infty} a_n, \text{ or } M = \overline{\lim_{n\to\infty}} a_n;$$

$$m = \liminf_{n\to\infty} a_n, \text{ or } m = \underline{\lim_{n\to\infty}}\, a_n.$$

Examples

(1) Let $a_n = 1 + \frac{(-1)^n}{n}$, $n = 1, 2, \ldots$. It is clear that $a_n \to 1$ as $n \to \infty$, and so $M = m = 1$. On the other hand, we note that the least upper bound of $\{a_n\}$ is $\frac{3}{2}$ and the greatest lower bound is 0.
(2) The sequence $a_n = (-1)^n$, $n = 1, 2, \ldots$, has the values $M = 1$, $m = -1$. The l.u.b. is 1 and the g.l.b. is -1.

In the above development we assumed that M and m are real numbers. It is a minor modification to add the cases where M or m are either $+\infty$ or $-\infty$; we leave the details to the reader.

The next test is one of the most useful for deciding absolute convergence of series.

Theorem 8.8 (Ratio test)
Suppose that $u_n \neq 0$, $n = 1,2,\ldots$, and that

$$\overline{\lim_{n\to\infty}} \left|\frac{u_{n+1}}{u_n}\right| = \rho \text{ or } \left|\frac{u_{n+1}}{u_n}\right| \to +\infty \text{ as } n \to \infty.$$

Then

(i) *if $\rho < 1$, the series $\sum_{n=1}^{\infty} u_n$ converges absolutely;*
(ii) *if $\rho > 1$ or $|u_{n+1}/u_n| \to +\infty$ as $n \to \infty$, the series diverges;*
(iii) *if $\rho = 1$, the test gives no information.*

Proof
(i) Suppose that $\rho < 1$. Choose any ρ' such that $\rho < \rho' < 1$. Then there exists an integer N such that $|u_{n+1}/u_n| < \rho'$ for all $n \geq N$. That is,

$$|u_{n+1}| < \rho' |u_n| \text{ for } n \geq N.$$

By induction, we have

$$|u_n| \leq (\rho')^{n-N} |u_N| \text{ for all } n \geq N.$$

The series $\sum_{n=N}^{\infty} |u_n|$ converges by the Comparison test, using the geometric series $\sum_{n=0}^{\infty}(\rho')^n$. The original series converges absolutely, since the addition of the finite sum $\sum_{n=1}^{N-1} |u_n|$ does not affect convergence.
(ii) If $\rho > 1$ or $|u_{n+1}/u_n| \to +\infty$, then there is an integer N such that $|u_{n+1}/u_n| > 1$ for all $n \geq N$. Then $|u_n| > |u_N|$ for all $n \geq N$. Hence u_n does not approach zero as $n \to \infty$. By Theorem 8.1 the series cannot converge.
(iii) The p-series for *all* values of p yields the limit $\rho = 1$. Since the p-series converges for $p > 1$ and diverges for $p \leq 1$, the ratio test can yield no information on convergence when $\rho = 1$. ∎

The next theorem provides a useful test for many series.

Theorem 8.9 (Root test)
Let $\sum_{n=1}^{\infty} u_n$ be a series with either

$$\overline{\lim_{n\to\infty}}(|u_n|)^{1/n} = \rho \text{ or } \overline{\lim_{n\to\infty}}(|u_n|)^{1/n} = +\infty.$$

Then

(i) *if $\rho < 1$, the series $\sum_{n=1}^{\infty} u_n$ converges absolutely;*
(ii) *if $\rho > 1$ or if $\overline{\lim}_{n\to\infty}|u_n|^{1/n} \to +\infty$, the series diverges;*
(iii) *if $\rho = 1$, the test gives no information.*

Proof
(i) Suppose $\rho < 1$. Choose $\varepsilon > 0$ so small that $\rho + \varepsilon < 1$ as well. It follows that $|u_n|^{1/n} < \rho + \varepsilon$ for all $n \geq N$ if N is sufficiently large. Therefore, $|u_n| < (\rho + \varepsilon)^n$ for all $n \geq N$. We observe that

$$\sum_{n=1}^{\infty}(\rho + \varepsilon)^n$$

is a convergent geometric series, since $\rho + \varepsilon < 1$. The Comparison test (Theorem 8.4) shows that $\sum_{n=1}^{\infty} |u_n|$ converges. Hence (i) is established.

(ii) Suppose $\rho > 1$ or $|u_n|^{1/n} \to +\infty$. Choose $\varepsilon > 0$ so small that $(\rho - \varepsilon) > 1$. Therefore, in case (ii), $\rho - \varepsilon < |u_n|^{1/n}$ for all sufficiently large n. We conclude that

$$\lim_{n\to\infty} |u_n| \neq 0;$$

and hence both $\sum_{n=1}^{\infty} |u_n|$ and $\sum_{n=1}^{\infty} u_n$ are divergent series. ∎

A **power series** is a series of the form

$$c_0 + c_1(x - a) + c_2(x - a)^2 + \ldots + c_n(x - a)^n + \cdots,$$

in which a and c_i, $i = 0, 1, 2, \ldots$, are constants. If a particular value is given to x, then the above expression is an infinite series of numbers that can be examined for convergence or divergence. For those values of x in $\mathbb{R}^1$ that yield a convergent power series, a function is defined whose range is the actual sum of the series. Denoting this function by f, we write

$$f : x \to \sum_{n=0}^{\infty} c_0(x-a)^n.$$

It will be established that most of the elementary functions such as the trigonometric, logarithmic, and exponential functions have power series expansions. In fact, power series may be used for the definition of many of the functions studied thus far. For example, the function $\log x$ may be defined by a power series rather than by an integral as in Section 5.3. If a power series definition is used, the various properties of functions, such as those given in Theorems 5.12 and 5.13 are usually more difficult to establish.

We first state a lemma and then prove two theorems that establish the basic properties of power series.

Lemma 8.1
If the series $\sum_{n=1}^{\infty} u_n$ converges, then there is a number M such that $|u_n| \leq M$ for all n.

The proof is left to the reader. (See Problem 24 at the end of the section and the hint given there.)

Theorem 8.10
If the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $x = x_1$ where $x_1 \neq a$, then the series converges absolutely for all x such that $|x-a| < |x_1 - a|$. Furthermore, there is a number M such that

$$(8.3) \quad |c_n(x-a)^n| \leq M\left(\frac{|x-a|}{|x_1-a|}\right)^n \text{ for } |x-a| \leq |x_1 - a| \text{ and for all } n.$$

Proof
By Lemma 8.1 there is a number M such that

$$|c_n(x_1-a)^n| \leq M \text{ for all } n.$$

Then (8.3) follows, since

$$|c_n(x-a)^n| = |c_n(x_1-a)^n| \cdot \left|\frac{(x-a)^n}{(x_1-a)^n}\right| \leq M\,\frac{|x-a|^n}{|x_1-a|^n}.$$

We deduce the convergence of the series at $|x-a|$ by comparison with the geometric series the terms of which are the right side of the inequality in (8.3). ■

Theorem 8.11
Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ *be a given power series. Then*

(i) *the series converges only for* $x = a$*; or*
(ii) *the series converges for all values of* x*; or*
(iii) *there is a number* R *such that the series converges for* $|x-a| < R$ *and diverges for* $|x-a| > R$*.*

Proof
There are simple examples of series that show that (i) and (ii) may happen. To prove (iii), suppose there is a number $x_1 \neq a$ for which the series converges and a number $x_2 \neq a$ for which it diverges. By Theorem 8.10 we must have $|x_1 - a| \leq |x_2 - a|$, for if $|x_2 - a| < |x_1 - a|$, the series would converge for $x = x_2$. Define the set S of real numbers

$$S = \{\rho : \text{ the series converges for } |x-a| < \rho\},$$

and define $R = \sup S$. Now suppose $|x' - a| < R$. Then there is a $\rho \in S$ such that $|x' - a| < \rho < R$. By Theorem 8.10, the series converges for $x = x'$. Hence the series converges for all x such that $|x - a| < R$. Now suppose that $|x'' - a| = \rho' > R$. If the series converges for x'', then $\rho' \in S$, and we contradict the fact that $R = \sup S$. Therefore, the series diverges for $|x - a| > R$, completing the proof. ■

Example 3
Find the values of x for which the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^n}{2^n n^2}$$

converges.

Solution. We apply the ratio test:

$$\left| \frac{u_{n+1}}{u_n} \right| = \frac{1}{2} |x-1| \frac{n^2}{(n+1)^2}$$

and

$$\overline{\lim_{n\to\infty}} \left| \frac{u_{n+1}}{u_n} \right| = \frac{1}{2} |x-1|.$$

Therefore, the series converges for $\frac{1}{2}|x-1| < 1$ or for $-1 < x < 3$. By noting that for $x = -1$ and $x = 3$ the series is a p-series with $p = 2$, we conclude that the series converges for all x in the interval $-1 \leq x \leq 3$ and diverges for all other values of x. ■

Problems

In each of Problems 1 through 16, test the series for convergence or divergence. If the series is convergent, determine whether it is absolutely or conditionally convergent.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(10)^n}{n!}$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n!}{(10)^n}$.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)}{n^2+1}$.

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n-1/2)}$.

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}(4/3)^n}{n^4}$.

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}n!}{1\cdot 3\cdot 5\ldots(2n-1)}$.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}(n+1)}{\sqrt[n]{n}}$.

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}\cdot 2\cdot 4\cdot 6\ldots 2n}{1\cdot 4\cdot 7\ldots(3n-2)}$.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}(2n^2-3n+2)}{n^3}$.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)\log(n+1)}$.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\log(n+1)}{n+1}$.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\log n}{n^2}$.

13. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^{n}}{n^3}$.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$.

15. $\displaystyle\sum_{n=1}^{\infty}(-1)^n 3^{-n}$.

16. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$.

In each of Problems 17 through 22, find all the values of x for which the given power series converges.

17. $\displaystyle\sum_{n=0}^{\infty}(n+1)x^n$.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{\sqrt[n]{n}}$.

19. $\displaystyle\sum_{n=0}^{\infty} \frac{(3/2)^n x^n}{n+1}$.

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+2)^n}{2^n}$.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n!(x-3)^n}{1\cdot 3\ldots(2n-1)}$.

22. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n^2+2n+1)x^n}{2^n(n+1)^3}$.

23. Find the interval of convergence of the binomial series

$$1+\sum_{n=1}^{\infty}\frac{m(m-1)\cdots(m-n+1)}{n!}x^n, \qquad m \text{ a constant.}$$

24. Prove Lemma 8.1. [*Hint*: Use Theorem 8.1 and the fact that a finite number of terms has (at least) one that is largest in absolute value.]

8.3 Uniform Convergence

Let $\{f_n\}$ be a sequence of functions with each function having a domain containing an interval I of $\mathbb{R}^1$ and with range in $\mathbb{R}^1$. The convergence of such a sequence may be examined at each value x in I. The concept of uniform convergence, one that determines the nature of the convergence of the sequence for all x in I, has many applications in analysis. Of special interest is Theorem 8.12, which states that if all the f_n are continuous, then the limit function must be also.

Definition
We say the sequence $\{f_n\}$ **converges uniformly on the interval** I to the function f if for each $\varepsilon > 0$ there is a number N *independent of* x such that

(8.4) $$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in I \text{ and all } n > N.$$

Uniform convergence differs from ordinary pointwise convergence in that the integer N *does not depend on* x, although naturally it depends on ε.

The geometric meaning of uniform convergence is illustrated in Figure 8.1. Condition (8.4) states that if ε is any positive number, then for $n > N$ the graph of $y = f_n(x)$ lies entirely below the graph of $f(x) + \varepsilon$ and entirely above the graph of $f(x) - \varepsilon$.

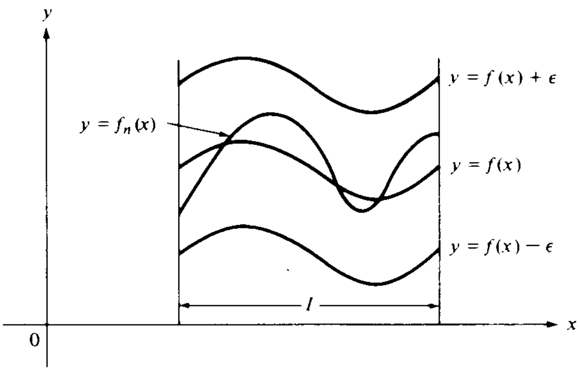

Figure 8.1 Illustrating uniform convergence.

It can happen that a sequence $\{f_n(x)\}$ converges to $f(x)$ for each x on an interval I but that the convergence is not uniform. For example, consider the functions

$$f_n : x \to \frac{2nx}{1 + n^2x^2}, \qquad I = \{x : 0 \le x \le 1\}.$$

The graphs of f_n for $n = 1, 2, 3, 4$ are shown in Figure 8.2. If $x \neq 0$, we write

$$f_n(x) = \frac{2x/n}{x^2 + (1/n^2)}$$

and observe that $f_n(x) \to 0$ for each $x > 0$. Moreover, $f_n(0) = 0$ for all n. Hence, setting $f(x) \equiv 0$ on I, we conclude that $f_n(x) \to f(x)$ for all x on I.

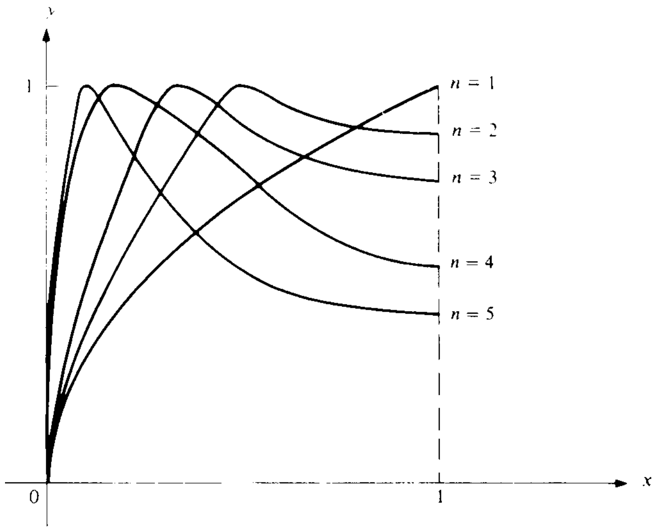

Figure 8.2 Illustrating nonuniform convergence to $f(x) \equiv 0$.

Taking the derivative of f_n, we find

$$f_n'(x) = \frac{2n(1 - n^2x^2)}{(1 + n^2x^2)^2}.$$

Therefore, f_n has a maximum at $x = 1/n$ with $f_n(1/n) = 1$. Thus if $\varepsilon > 1$, there is no number N such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$ and *all* x on I; in particular, $|f_n(1/n) - f(1/n)| = 1$ for all n.

The definition of uniform convergence is seldom a practical method for deciding whether or not a specific sequence converges uniformly. The importance of uniform convergence with regard to continuous functions is illustrated in the next theorem.

Theorem 8.12
Suppose that f_n, $n = 1,2,\ldots$, is a sequence of continuous functions on an interval I and that $\{f_n\}$ converges uniformly to f on I. Then f is continuous on I.

Proof

Suppose $\varepsilon > 0$ is given. Then there is an N such that

$$(8.5) \qquad |f_n(x) - f(x)| < \frac{\varepsilon}{3} \text{ for all } x \text{ on } I \text{ and all } n > N.$$

Let x_0 be any point in I. Since f_{N+1} is continuous on I, there is a $\delta > 0$ such that

$$(8.6) \qquad |f_{N+1}(x) - f_{N+1}(x_0)| < \frac{\varepsilon}{3} \text{ for all } x \in I \text{ such that } |x - x_0| < \delta.$$

Also, by means of (8.5) and (8.6), we obtain

$$\begin{aligned} |f(x) - f(x_0)| &\le |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which is valid for all x on I such that $|x - x_0| < \delta$. Thus f is continuous at x_0, an arbitrary point of I. ■

The next result shows that a uniformly convergent sequence of continuous functions may be integrated term by term.

Theorem 8.13 (Integration of uniformly convergent sequences)
Suppose that each f_n, $n = 1,2,\ldots$, is continuous on the bounded interval I and that $\{f_n\}$ converges uniformly to f on I. Let $c \in I$ and define

$$F_n(x) = \int_c^x f_n(t)dt.$$

Then f is continuous on I, and F_n converges uniformly to the function

$$F(x) = \int_c^x f(t)dt.$$

Proof

That f is continuous on I follows from Theorem 8.12. Let L be the length of I. For any $\varepsilon > 0$ it follows from the uniform convergence of $\{f_n\}$ that there is an N such that

$$|f_n(t) - f(t)| < \frac{\varepsilon}{L} \text{ for all } n > N \text{ and all } t \text{ on } I.$$

We conclude that

$$|F_n(x) - F(x)| = \left| \int_c^x [f_n(t) - f(t)]dt \right| \le \left| \int_c^x |f_n(t) - f(t)|dt \right| \le \frac{\varepsilon}{L} |x - c| \le \varepsilon$$

for all $n > N$ and all $x \in I$. Hence $\{F_n\}$ converges uniformly on I. ■

The next theorem illustrates when we can draw conclusions about the term-by-term differentiation of convergent sequences.

Theorem 8.14
Suppose that $\{f_n\}$ *is a sequence of functions each having one continuous derivative on an open interval* I. *Suppose that* $f_n(x)$ *converges to* $f(x)$ *for each* x *on* I *and that the sequence* f_n' *converges uniformly to* g *on* I. *Then* g *is continuous on* I; *and* $f'(x) = g(x)$ *for all* x *on* I.

Proof
That g is continuous on I follows from Theorem 8.12. Let c be any point on I. For each n and each x on I, we have

$$\int_c^x f_n'(t)dt = f_n(x) - f_n(c).$$

Since $\{f_n'\}$ converges uniformly to g and $f_n(x)$ converges to $f(x)$ for each x on I, we may apply Theorem 8.13 to get

$$\int_c^x g(t)dt = f(x) - f(c). \tag{8.7}$$

The result follows by differentiating (8.7). ■

Many of the results on uniform convergence of sequences of functions from $\mathbb{R}^1$ to $\mathbb{R}^1$ generalize directly to functions defined on a set A in a metric space S and with range in $\mathbb{R}^1$.

Definition
Let A be a set in a metric space S and suppose that $f_n : S \to \mathbb{R}^1$, $n = 1, 2, \ldots$, is a sequence of functions. The sequence $\{f_n\}$ **converges uniformly** on A *to a function* $f : A \to \mathbb{R}^1$ if for every $\varepsilon > 0$ there is a number N such that

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in A \text{ and all } n > N.$$

The next theorem is an extension of Theorem 8.12.

Theorem 8.15
Suppose that each f_n *is continuous on a set* A *in a metric space* S *where* $f_n{:}S \to \mathbb{R}^1$. *If* $\{f_n\}$ *converges uniformly to* f *on* A, *then* f *is continuous on* A.

The proof is similar to the proof of Theorem 8.12.

Theorem 8.13 has a generalization to functions defined in N-dimensional Euclidean space.

Theorem 8.16
Let F *be a closed region in* $\mathbb{R}^N$ *and suppose that* $f_n{:}F \to \mathbb{R}^1$, $n = 1,2,\ldots$, *is a sequence of continuous functions that converges uniformly to* f *on* F. *Then* f *is continuous on* F, *and*

$$\int_F f dV_N = \lim_{n\to\infty} \int_F f_n dV_N.$$

The proof is similar to the proof of Theorem 8.13.

Let $\{f_n\}$ be a sequence defined on a bounded open set G in $\mathbb{R}^N$ with range in $\mathbb{R}^1$. Writing $f_n(x_1, x_2, \ldots, x_N)$ for the value of f_n at $(x_1, x_2, \ldots, x_N)$, we recall that

$$f_{n,k} \text{ and } \frac{\partial}{\partial x_k} f_n(x_1, x_2, \ldots, x_N)$$

are symbols for the partial derivative of f_n with respect to x_k.

Theorem 8.17

Let k be an integer such that $1 \le k \le N$. Suppose that for each n, the functions $f_n{:}G \to \mathbb{R}^1$ and $f_{n,k}$ are continuous on G, a bounded open set in $\mathbb{R}^N$. Suppose that $\{f_n(x)\}$ converges to $f(x)$ for each $x \in G$ and that $\{f_{n,k}\}$ converges uniformly to a function g on G. Then g is continuous on G, and

$$f_{,k}(x) = g(x) \text{ for all } x \in G.$$

The proof is almost identical to the proof of Theorem 8.14.

If a sequence of continuous functions $\{f_n\}$ converges at every point to a continuous function f, it is not necessarily true that

$$\int f_n dV \to \int f dV \text{ as } n \to \infty.$$

Simple convergence at every point is not sufficient, as the following example shows. We form the sequence (for $n = 2, 3, 4, \ldots$)

$$f_n : x \to \begin{cases} n^2 x, & 0 \le x \le \dfrac{1}{n}, \\ -n^2 x^2 + 2n, & \dfrac{1}{n} \le x \le \dfrac{2}{n}, \\ 0, & \dfrac{2}{n} \le x \le 1. \end{cases}$$

It is easy to see from Figure 8.3 that $f_n(x) \to f(x) \equiv 0$ for each $x \in I = \{x : 0 \le x \le 1\}$. Also since the area of each triangle under the curve $f_n(x)$ is exactly 1, we have

$$1 = \int_0^1 f_n(x)dx \not\to \int_0^1 f(x)dx = 0.$$

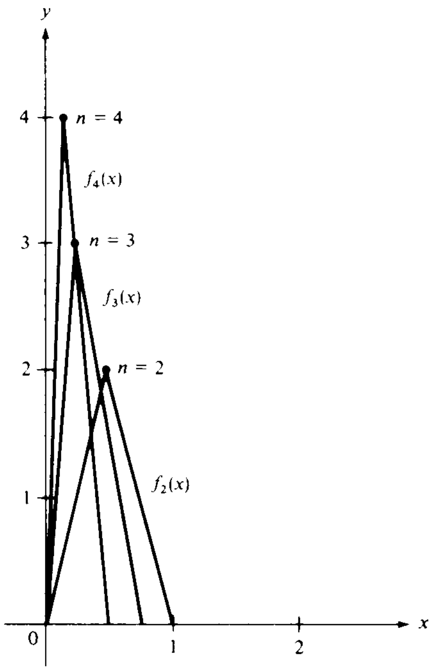

Figure 8.3 f_n converges to f, but $\int_0^1 f_n$ does not converge to $\int_0^1 f$.

Problems

In each of Problems 1 through 10 show that the sequence $\{f_n(x)\}$ converges to $f(x)$ for each $x \in I$ and determine whether or not the convergence is uniform.

1. $f_n : x \to \frac{2x}{1+nx}$, $f(x) \equiv 0$, $I = \{x : 0 \le x \le 1\}$.
2. $f_n : x \to \frac{\cos nx}{\sqrt{n}}$, $f(x) \equiv 0$, $I = \{x : 0 \le x \le 1\}$.
3. $f_n : x \to \frac{n^3 x}{1+n^4 x}$, $f(x) \equiv 0$, $I = \{x : 0 \le x \le 1\}$.
4. $f_n : x \to \frac{n^3 x}{1+n^4 x^2}$, $f(x) \equiv 0$, $I = \{x : a \le x < \infty, a > 0\}$.
5. $f_n : x \to \frac{nx^2}{1+nx}$, $f(x) = x$, $I = \{x : 0 \le x \le 1\}$.
6. $f_n : x \to \frac{1}{\sqrt{x}} + \frac{1}{n}\cos(1/nx)$, $f(x) = 1/\sqrt{x}$, $I = \{x : 0 < x \le 2\}$.
7. $f_n : x \to \frac{\sin nx}{2nx}$, $f(x) \equiv 0$, $I = \{x : 0 < x < \infty\}$.
8. $f_n : x \to x^n(1-x)\sqrt{n}$, $f(x) \equiv 0$, $I = \{x : 0 \le x \le 1\}$.
9. $f_n : x \to \frac{1-x^n}{1-x}$, $f(x) = \frac{1}{1-x}$, $I = \left\{x : -\frac{1}{2} \le x \le \frac{1}{2}\right\}$.
10. $f_n : x \to nxe^{-nx^2}$, $f(x) \equiv 0$, $I = \{x : 0 \le x \le 1\}$.
11. Show that the sequence $f_n : x \to x^n$ converges for each $x \in I = \{x : 0 \le x \le 1\}$ but that the convergence is not uniform.
12. Given that $f_n(x) = (n+2)(n+1)x^n(1-x)$ and that $f(x) \equiv 0$ for x on $I = \{x : 0 \le x \le 1\}$, show that $f_n(x) \to f(x)$ as $n \to \infty$ for each $x \in I$. Determine whether or not $\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$ as $n \to \infty$.

13. Give an example of a sequence of functions $\{f_n\}$ defined on the set $A = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ such that f_n converges to a function f at each point $(x, y) \in A$ but $\int_A f_n(x, y)dV \not\to \int_A f(x, y)\, dV$.
14. Suppose that $\{f_n\}$ converges uniformly to f and $\{g_n\}$ converges uniformly to g on a set A in a metric space S. Show that $\{f_n + g_n\}$ converges uniformly to $f + g$.
15. (a) Suppose $\{f_n\}$ and $\{g_n\}$ are bounded sequences each of which converges uniformly on a set A in a metric space S to functions f and g, respectively. Show that the sequence $\{f_n g_n\}$ converges uniformly to fg on A.
    (b) Give an example of sequences $\{f_n\}$ and $\{g_n\}$ that converge uniformly but are such that $\{f_n g_n\}$ does not converge uniformly.

8.4 Uniform Convergence of Series; Power Series

Let $u_k(x)$, $k = 1, 2, \ldots$, be functions defined on a set A in a metric space S with range in $\mathbb{R}^1$.

Definition
The infinite series $\sum_{k=1}^{\infty} u_k(x)$ has the partial sums $s_n(x) = \sum_{k=1}^{n} u_k(x)$. The series is said to **converge uniformly on a set A to a function s** if the sequence of partial sums $\{s_n\}$ converges uniformly to s on the set A.

The above definition shows that theorems on uniform convergence of infinite series may be reduced to corresponding results for uniform convergence of sequences.

Theorem 8.18 (Analogue of Theorem 8.12)
Suppose that u_n, $n = 1,2, \ldots$, are continuous on a set A in a metric space S and that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on A to a function $s(x)$. Then s is continuous on A.

The proof of Theorem 8.18 as well as the proofs of the next two theorems reduce to the proofs of the corresponding results for sequences.

Theorem 8.19 (Term-by-term integration of infinite series)
Let $u_n(x)$, $n = 1,2, \ldots$, be functions whose domain is a bounded interval I in $\mathbb{R}^1$ with range in $\mathbb{R}^1$. Suppose that each u_n is integrable on I and that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on I to $s(x)$. Then s is integrable on I. If c is in I, and U_n, S are defined by

$$U_n(x) = \int_c^x u_n(t)dt, \qquad S(x) = \int_c^x s(t)dt,$$

then $\sum_{n=1}^{\infty} U_n(x)$ *converges uniformly to* $S(x)$ *on* I.

Theorem 8.20 (Term-by-term differentiation of infinite series)
Let k *be an integer with* $1 \le k \le N$. *Suppose that* u_n *and* $u_{n,k}$, $n = 1,2,\ldots$, *are continuous functions defined on an open set* G *in* $\mathbb{R}^N$ *(range in* $\mathbb{R}^1$*). Suppose that the series* $\sum_{n=1}^{\infty} u_n(x)$ *converges for each* $x \in G$ *to* $s(x)$ *and that the series* $\sum_{n=1}^{\infty} u_{n,k}(x)$ *converges uniformly on* G *to* $t(x)$. *Then*

$$s_{,k}(x) = t(x) \text{ for all } x \text{ in } G.$$

The following theorem gives a useful indirect test for uniform convergence. It is important to observe that the test can be applied without any knowledge of the sum of the series.

Theorem 8.21 (Weierstrass *M*-test)
Let $u_n(x)$, $n = 1,2,\ldots$, *be defined on a set* A *in a metric space* S *with range in* $\mathbb{R}^1$. *Suppose that* $|u_n(x)| \le M_n$ *for all* n *and for all* $x \in A$. *Suppose that the series of constants* $\sum_{n=1}^{\infty} M_n$ *converges. Then* $\sum_{n=1}^{\infty} u_n(x)$ *and* $\sum_{n=1}^{\infty} |u_n(x)|$ *converge uniformly on* A.

Proof
By the Comparison test (Theorem 8.4) we know that $\sum_{n=1}^{\infty} |u_n(x)|$ converges for each x. Set

$$t_n(x) = \sum_{k=1}^{n} |u_k(x)| \text{ and } t(x) = \sum_{k=1}^{\infty} |u_k(x)|.$$

From Theorem 8.6 it follows that $\sum_{n=1}^{\infty} u_n(x)$ converges. Set

$$s_n(x) = \sum_{k=1}^{n} u_k(x), \qquad s(x) = \sum_{k=1}^{\infty} u_k(x).$$

Then

$$\begin{aligned} |s_n(x) - s(x)| &= \left| \sum_{k=n+1}^{\infty} u_k(x) \right| \\ &\le \sum_{k=n+1}^{\infty} |u_k(x)| = |t(x) - t_n(x)| \\ &\le \sum_{k=n+1}^{\infty} M_k. \end{aligned}$$

We define $S = \sum_{k=1}^{\infty} M_k$, $S_n = \sum_{k=1}^{n} M_k$. Then $\sum_{k=n+1}^{\infty} M_k = S - S_n$; since $S - S_n \to 0$ as $n \to \infty$ independently of x, we conclude that the convergence of $\{s_n\}$ and $\{t_n\}$ are uniform. ■

The next theorem on the uniform convergence of power series is a direct consequence of the Weierstrass M-test.

Theorem 8.22

Suppose that the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges for $x = x_1$ with $x_1 \neq a$. Then the series converges uniformly on $I = \{x{:}a - h \leq x \leq a + h\}$ for each $h < |x_1 - a|$. Also, there is a number M such that

$$|c_n(x-a)^n| \leq M \cdot \left(\frac{h}{|x_1 - a|}\right)^n \tag{8.8}$$

for

$$|x - a| \leq h < |x_1 - a|.$$

Proof

Inequality (8.8) is the inequality stated in Theorem 8.10. The series

$$\sum_{n=0}^{\infty} M(\frac{h}{|x_1 - a|})^n$$

is a geometric series of constants that converges. Therefore, the uniform convergence of $\sum_{n=0}^{\infty} c_n(x-a)^n$ follows from the Weierstrass M-test. ■

Example 1

Given the series

$$\sum_{n=0}^{\infty}(n+1)x^n, \tag{8.9}$$

find all values of h such that the series converges uniformly on $I = \{x : |x| \leq h\}$.

Solution. For $|x| \leq h$, we have $|(n+1)x^n| \leq (n+1)h^n$. By the ratio test, the series $\sum_{n=0}^{\infty}(n+1)h^n$ converges when $h < 1$. Therefore, the series (8.9) converges uniformly on $I = \{x : |x| \leq h\}$ if $h < 1$. The series (8.9) does not converge for $x = \pm 1$, and hence there is uniform convergence if and only if $h < 1$. ■

Example 2

Given the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}, \tag{8.10}$$

find all values of h such that the series converges uniformly on $I = \{x : |x| \leq h\}$.

Solution. For $|x| \leq h$, we have $|x^n/n^2| \leq h^n/n^2$. The p-series $\sum_{n=1}^{\infty} 1/n^2$ converges, and by the Comparison test, the series $\sum_{n=1}^{\infty} h^n/n^2$ converges if $h \leq 1$. By the Ratio test, the series (8.10) diverges if $x > 1$. We conclude that the series (8.10) converges uniformly on $I = \{x : |x| \leq 1\}$. ■

Theorem 8.23
Suppose that the series

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \tag{8.11}$$

converges for $|x-a| < R$ *with* $R > 0$. *Then* f *and* f' *are continuous on* $|x-a| < R$, *and*

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} \text{ for } |x-a| < R. \tag{8.12}$$

Proof
Choose x_1 such that $|x_1 - a| < R$ and then choose h such that $0 < h < |x_1 - a|$. Then the series (8.11) converges uniformly on $I = \{x : |x-a| \leq h\}$, and there is a number M such that

$$|c_n(x-a)^n| \leq M\left(\frac{h}{|x_1 - a|}\right)^n \quad \text{for } |x-a| \leq h.$$

According to Theorem 8.18, the function f is continuous on I. Also, we obtain

$$|nc_n(x-a)^{n-1}| \leq n|c_n|h^{n-1} \leq \frac{nM}{h}\left(\frac{h}{|x_1 - a|}\right)^n \equiv u_n.$$

From the ratio test, the series $\sum_{n=1}^{\infty} u_n$ converges, so that the series (8.12) converges uniformly on I. Hence the function f' is continuous on I. Since h may be chosen as any positive number less than R, we conclude that f and f' are continuous for $|x-a| < R$. ■

With the aid of Theorem 8.23, we obtain the following theorem on term-by-term differentiation and integration of power series.

Theorem 8.24
Let f *be given by*

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ for } |x-a| < R \text{ with } R > 0. \tag{8.13}$$

(i) *Then* f *possesses derivatives of all orders. For each positive integer* m, *the derivative* $f^{(m)}(x)$ *is given for* $|x-a| < R$ *by the term-by-term differentiation of* (8.13) m *times.*
(ii) *If* F *is defined for* $|x-a| < R$ *by*

$$F(x) = \int_a^x f(t)dt,$$

then F is given by the series obtained by term-by-term integration of the series (8.13).

(iii) *The constants c_n are given by*

$$c_n = \frac{f^{(n)}(a)}{n!}. \tag{8.14}$$

Proof

The proof of (i) is obtained by induction. We obtain (ii) by application of Theorem 8.19. To establish (iii), differentiate the series (8.13) n times and set $x = a$ in the resulting expression for $f^{(n)}(x)$. ■

Combining expressions (8.13) and (8.14), we see that any function f defined by a power series with $R > 0$ has the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \tag{8.15}$$

A function that has continuous derivatives of all orders in a neighborhood of some point is said to be **infinitely differentiable**. It may happen that a function f is infinitely differentiable at some point a but is not representable by a power series such as (8.15). An example of such an f is given by

$$f : x \to \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

By differentiation, we find that for $x \neq 0$

$$\begin{aligned} f'(x) &= 2x^{-3}e^{-1/x^2}, \\ f''(x) &= x^{-6}(4 - 6x^2)e^{-1/x^2}, \\ &\vdots \\ f^{(n)}(x) &= x^{-3n}P_n(x)e^{-1/x^2}, \end{aligned}$$

where P_n is a polynomial. By l'Hôpital's rule, it is not hard to verify that $f^{(n)}(x) \to 0$ as $x \to 0$ for $n = 1, 2, \ldots$. Therefore, we know that $f^{(n)}(0) = 0$ for every n, and so f has continuous derivatives of all orders in a neighborhood of 0. The series (8.15) for f is identically zero, but the function f is not; therefore the power series with $a = 0$ does not represent the function.

One of the tools for establishing the validity of power series expansions for infinite series is the next result, known as Taylor's theorem with remainder.

Theorem 8.25 (Taylor's theorem with remainder)
Suppose that f and its first n derivatives are continuous on an interval containing $I = \{x{:}a \le x \le b\}$. Suppose that $f^{(n+1)}(x)$ exists for each x between a and b. Then there is a ξ with $a < \xi < b$ such that

$$f(b) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(b-a)^j + R_n,$$

where

$$R_n = \frac{f^{(n+1)}(\xi)(b-a)^{n+1}}{(n+1)!}. \tag{8.16}$$

Proof
We define R_n by the equation

$$f(b) = \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(b-a)^j + R_n;$$

that is,

$$R_n \equiv f(b) - \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(b-a)^j.$$

We wish to find the form R_n takes. For this purpose define ϕ for x on I by the formula

$$\phi(x) = f(b) - \sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!}(b-x)^j - R_n\frac{(b-x)^{n+1}}{(b-a)^{n+1}}.$$

Then ϕ is continuous on I, and $\phi'(x)$ exists for each x between a and b. A simple calculation shows that $\phi(a) = \phi(b) = 0$. By Rolle's theorem there is a number ξ with $a < \xi < b$ such that $\phi'(\xi) = 0$. We compute

$$\phi'(x) = -f'(x) + \sum_{j=1}^{n} \frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1} - \sum_{j=1}^{n} \frac{f^{(j+1)}(x)}{j!}(b-x)^j + (n+1)R_n\frac{(b-x)^n}{(b-a)^{n+1}}.$$

Replacing j by $j-1$ in the second sum above, we find

$$\begin{aligned}\phi'(x) &= -f'(x) + f'(x) + \sum_{j=2}^{n} \frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1} - \sum_{j=2}^{n} \frac{f^{(j)}(x)}{(j-1)!}(b-x)^{j-1}\\ &\quad - \frac{f^{(n+1)}(x)}{n!}(b-x)^n + (n+1)R_n\frac{(b-x)^n}{(b-a)^{n+1}}\\ &= -\frac{(n+1)(b-x)^n}{(b-a)^{n+1}}\left[\frac{f^{(n+1)}(x)(b-a)^{n+1}}{(n+1)!} - R_n\right].\end{aligned}$$

The formula for R_n given in (8.16) is obtained by using the fact that $\phi'(\xi) = 0$ and setting $x = \xi$ in the above expression. ■

Using Theorem 8.25 we can now establish the validity of the Taylor expansion for many of the functions studied in elementary calculus.

Theorem 8.26
For any values of a and x, the expansion

$$e^x = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} \tag{8.17}$$

is valid.

Proof
We apply Taylor's theorem to $f(x) = e^x$. We have $f^{(n)}(x) = e^x$ for $n = 1, 2, \ldots$, and, setting $b = x$ in (8.16), we find

$$e^x = e^a \sum_{j=0}^{n} \frac{(x-a)^j}{j!} + R_n(a, x), \qquad R_n = \frac{e^{\xi}(x-a)^{n+1}}{(n+1)!},$$

where ξ is between a and x. If $x > a$, then $a < \xi < x$, and so $e^{\xi} < e^x$; if $x < a$, then $x < \xi < a$ and $e^{\xi} < e^a$. Hence

$$R_n(a, x) \le C(a, x) \frac{|x-a|^{n+1}}{(n+1)!}, \text{ where } C(a, x) = \begin{cases} e^x & \text{if } x \ge a, \\ e^a & \text{if } x \le a. \end{cases} \tag{8.18}$$

Note that $C(a, x)$ is *independent of* n. By the Ratio test, the series in (8.17) converges for all x. The form of R_n in (8.18) shows that $R_n \to 0$ as $n \to \infty$. Therefore, the series (8.17) converges to e^x for each x and a. ■

Theorem 8.27
For all a and x the following expansions are valid:

$$\sin x = \sum_{n=0}^{\infty} \frac{\sin(a + n\pi/2)}{n!}(x-a)^n,$$

$$\cos x = \sum_{n=0}^{\infty} \frac{\cos(a + n\pi/2)}{n!}(x-a)^n.$$

The proof is left to the reader.

Theorem 8.28
The following expansions are valid for $|x| < 1$:

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n},$$

$$\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n-1}}{2n-1}.$$

The proof is left to the reader.

In many applications of Taylor's theorem with remainder, it is important to obtain specific bounds on the remainder term. The theorem which follows gives the remainder R_n in an integral form and is often useful in obtaining precise estimates.

Theorem 8.29 (Taylor's theorem with integral form of the remainder)
Suppose that f and its derivatives of order up to $n+1$ are continuous on an interval I containing a. Then for each $x \in I$,

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(a)(x-a)^j}{j!} + R_n(a, x), \tag{8.19}$$

where

$$R_n(a, x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt. \tag{8.20}$$

Proof
For each $x \in I$, by integrating a derivative we have

$$f(x) = f(a) + \int_a^x f'(t)dt.$$

We integrate by parts in the above integral by setting

$$u = f'(t) \qquad du = f''(t)dt,$$
$$v = -(x-t), \qquad dv = dt.$$

We obtain

$$\begin{aligned} f(x) &= f(a) - [(x-t)f'(t)]_a^x + \int_a^x (x-t)f''(t)dt \\ &= f(a) + f'(a)(x-a) + \int_a^x (x-t)f''(t)dt. \end{aligned}$$

We repeat the integration by parts in the integral above by setting

$$u = f''(t), \qquad du = f'''(t)dt,$$
$$v = -\frac{(x-t)^2}{2}, \qquad dv = (x-t)dt.$$

The result is

$$f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + \int_a^x f'''(t)\frac{(x-t)^2}{2!}\,dt.$$

We repeat the process and apply mathematical induction in the general case to obtain Formulas (8.19) and (8.20). ■

The "binomial formula" usually stated without proof in elementary courses is a direct corollary to Taylor's theorem.

Theorem 8.30 (Binomial series theorem)
For each $m \in \mathbb{R}^1$, the following formula holds:

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n \text{ for } |x| < 1. \tag{8.21}$$

Proof
We apply Theorem 8.29 with $a = 0$ and $f(x) = (1+x)^m$. The result is

$$(1+x)^m = 1 + \sum_{n=1}^{k} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n + R_k(0, x),$$

where

$$R_k(0, x) = \int_0^x \frac{(x-t)^k}{k!} m(m-1)\cdots(m-k)(1+t)^{m-k-1}dt.$$

We wish to show that $R_k \to 0$ as $k \to \infty$ if $|x| < 1$. For this purpose we define

$$C_m(x) = \begin{cases} (1+x)^{m-1} & \text{if } m \geq 1,\ x \geq 0, \text{ or } m \leq 1,\ x \leq 0, \\ 1 & \text{if } m \leq 1,\ x \geq 0, \text{ or } m \geq 1,\ x \leq 0. \end{cases}$$

We notice at once that $(1+t)^{m-1} \leq C_m(x)$ for all t between 0 and x. Therefore,

$$|R_k(0, x)| \leq C_m(x)\frac{|m(m-1)\cdots(m-k)|}{k!} \int_0^x \left|\frac{x-t}{1+t}\right|^k dt.$$

We define

$$u_k(x) = C_m(x)\frac{|m(m-1)\cdots(m-k)|}{k!}|x|^{k+1},$$

and set $t = xs$ in the above integral. Hence

$$|R_k(0, x)| \leq u_k(x)\int_0^1 \frac{(1-s)^k}{(1+xs)^k}\,ds.$$

This last integral is bounded by 1 for each x such that $-1 \leq x \leq 1$; hence $|R_k(0, x)| \leq u_k(x)$. By the Ratio test it is not difficult to verify that the series $\sum_{n=0}^{\infty} u_k(x)$ converges for $|x| < 1$. Therefore, $u_k(x) \to 0$ as $k \to \infty$, and so $R_k(0,x) \to 0$ for $|x| < 1$ and all m. ■

Problems

In each of Problems 1 through 10, determine the values of h for which the given series converges uniformly on the interval I.

1. $\sum_{n=1}^{\infty} \frac{x^n}{n}$, $I = \{x : |x| \le h\}$.
2. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt[n]{n}}$, $I = \{x : |x| \le h\}$.
3. $\sum_{n=1}^{\infty} n(x-1)^n$, $I = \{x : |x-1| \le h\}$.
4. $\sum_{n=0}^{\infty} \frac{(5x)^n}{n!}$, $I = \{x : |x| \le h\}$.
5. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{3^n(n+1)}$, $I = \{x : |x| \le h\}$.
6. $\sum_{n=0}^{\infty} \frac{(n!)^2(x-1)^n}{(2n)!}$, $I = \{x : |x-1| \le h\}$.
7. $\sum_{n=1}^{\infty} \frac{x^n}{(n+1)\log(n+1)}$, $I = \{x : |x| \le h\}$.
8. $\sum_{n=1}^{\infty} \frac{(\log n)2^n x^n}{3^n n\sqrt{n}}$, $I = \{x : |x| \le h\}$.
9. $\sum_{n=1}^{\infty} x^n(1-x)$, $I = \{x : |x| \le h\}$.
10. $\sum_{n=1}^{\infty} \frac{x^2}{(1+nx^2)\sqrt{n}}$, $I = \{x : |x| \le h\}$.
11. Given that $\sum_{n=1}^{\infty} |a_n|$ converges, show that $\sum_{n=1}^{\infty} a_n \cos nx$ converges uniformly for all x.
12. Given that $\sum_{n=1}^{\infty} n|b_n|$ converges, let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$. Show that $f'(x) = \sum_{n=1}^{\infty} nb_n \cos nx$ and that both series converge uniformly for all x.
13. Show that if $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly for $|x-a| < h$, then it converges uniformly for $|x-a| \le h$.

In each of Problems 14 through 21 find the Taylor expansion about $a = 0$ and prove its validity.

14. $f : x \to \arcsin x$
15. $f : x \to (1+x^2)^{-\frac{1}{2}}$
16. $f : x \to \arctan(x^2)$
17. $f : x \to (1-x)^{-3}$
18. $f : x \to (1-x)^{-2}$
19. $f : x \to (1-x^2)^{-3}$
20. $f : (1-x^2)^{-\frac{1}{2}}$
21. $f : x \to \arcsin(x^3)$

9 CHAPTER The Derivative of an Integral

9.1 The Derivative of a Function Defined by an Integral. The Leibniz Rule

In this chapter we develop rules for deciding when it is possible to interchange the processes of differentiation and integration. We establish the required theorems for bounded functions defined over a finite interval in this section and treat integrals over an infinite range in Sections 9.2 and 9.3.

Let f be a function with domain a rectangle $R = \{(x, t) : a \le x \le b,\ c \le t \le d\}$ in $\mathbb{R}^2$ and with range in $\mathbb{R}^1$. Let I be the interval $\{x : a \le x \le b\}$ and form the function $\phi : I \to \mathbb{R}^1$ by the formula

$$\phi(x) = \int_c^d f(x, t)dt\ . \tag{9.1}$$

We now seek conditions under which we can obtain the derivative ϕ' by differentiation of the integrand in (9.1). The basic formula is given in the following result.

Theorem 9.1 (Leibniz's rule)
Suppose that f and $f_{,1}$ are continuous on the rectangle R and that ϕ is defined by (9.1)*. Then*

$$\phi'(x) = \int_c^d f_{,1}(x,t)dt\,, \qquad a < x < b. \tag{9.2}$$

Proof
Form the difference quotient

$$\frac{\phi(x+h)-\phi(x)}{h} = \frac{1}{h}\int_c^d [f(x+h,t)-f(x,t)]dt.$$

Observing that

$$f(x+h,t)-f(x,t) = \int_x^{x+h} f_{,1}(z,t)dz,$$

we have

$$\frac{\phi(x+h)-\phi(x)}{h} = \frac{1}{h}\int_c^d \int_x^{x+h} f_{,1}(z,t)dz\, dt. \tag{9.3}$$

Since $f_{,1}$ is continuous on the closed, bounded set R, it is uniformly continuous there. Hence, if $\varepsilon > 0$ is given, there is a $\delta > 0$ such that

$$|f_{,1}(z,t) - f_{,1}(x,t)| < \frac{\varepsilon}{d-c}$$

for all t such that $c \le t \le d$ and all z such that $|z-x| < \delta$. We now use the artifice

$$\int_c^d f_{,1}(x,t)dt = \frac{1}{h}\int_c^d \int_x^{x+h} f_{,1}(x,t)dz\, dt, \tag{9.4}$$

which is valid because z is absent in the integrand on the right. Subtracting (9.4) from (9.3), we obtain

$$\left|\frac{\phi(x+h)-\phi(x)}{h} - \int_c^d f_{,1}(x,t)dt\right|$$
$$= \left|\int_c^d \left\{\frac{1}{h}\int_x^{x+h} [f_{,1}(z,t)f_{,1}(x,t)]dz\right\} dt\right|. \tag{9.5}$$

Now, if $|h|$ is so small that $|z-x| < \delta$ in the integrand on the right side of (9.5), it follows that

$$\left|\frac{\phi(x+h)-\phi(x)}{h} - \int_c^d f_{,1}(x,t)dt\right|$$
$$\le \int_c^d \left|\frac{1}{h}\int_x^{x+h} \frac{\varepsilon}{d-c} dz\right| dt = \frac{\varepsilon}{d-c}\cdot(d-c) = \varepsilon.$$

Since ε is arbitrary, the left side of the above inequality tends to 0 as $h \to 0$. Formula (9.2) is the result. ■

If the function f in (9.1) can be integrated explicitly with respect to t, then finding the derivative of ϕ is a straightforward computation. However, there are situations in which f cannot be integrated, but $f_{,1}$ can. We give an example illustrating this point.

EXAMPLE 1

Define $f : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$f(x, t) \to \begin{cases} (\sin xt)/t & \text{for } t \neq 0, \\ x & \text{for } t = 0. \end{cases}$$

Find ϕ', where $\phi(x) = \int_0^{\pi/2} f(x, t)dt$.

Solution. We have

$$\lim_{t \to 0} \frac{\sin xt}{t} = x \lim_{t \to 0} \frac{\sin(xt)}{(xt)} = x \cdot 1 = x,$$

and so f is continuous on $A = \{(x, t) : -\infty < x < \infty,\ 0 \leq t \leq \pi/2\}$. Also,

$$f_{,1}(x, t) = \begin{cases} (\cos xt)/t & \text{for } t \neq 0, \\ 1 & \text{for } t = 0. \end{cases}$$

Hence $f_{,1}$ is continuous on A. We apply Leibniz's rule and obtain

$$\phi'(x) = \int_0^{\pi/2} \cos\ xt\ dt = \frac{\sin(\pi/2)x}{x}, \quad x \neq 0,$$

and so $\phi'(0) = \pi/2$. Observe that the expression for $\phi(x)$ cannot be integrated. ■

We now take up an important extension of Leibniz's rule. Suppose that f is defined as before and, setting $I = \{x : a \leq x \leq b\}$ and $J = \{t : c \leq t \leq d\}$, let h_0 and h_1 be two given functions with domain on I and range on J. Suppose that $\phi : I \to \mathbb{R}^1$ is defined by

$$\phi(x) = \int_{h_0(x)}^{h_1(x)} f(x, t)\, dt.$$

We now develop a formula for ϕ'. To do so we consider a function $F : \mathbb{R}^3 \to \mathbb{R}^1$ defined by

$$F(x, y, z) = \int_y^z f(x, t)dt. \tag{9.6}$$

Theorem 9.2

Suppose that f and $f_{,1}$ are continuous on $R = \{(x,t):a \le x \le b,\ c \le t \le d\}$ and that F is defined by (9.6) with x on I and y,z on J. Then

$$F_{,1}(x,y,z) = \int_y^z f_{,1}(x,t)dt, \qquad F_{,2} = -f(x,y), \qquad F_{,3} = f(x,z). \tag{9.7}$$

Proof

The first formula in (9.7) is Theorem 9.1. The second and third formulas hold because of the Fundamental theorem of calculus. ■

Theorem 9.3 (General Leibniz rule)

Suppose that f and $f_{,1}$ are continuous on $R = \{(x,t):a \le x \le b,\ c \le t \le d\}$ and that h_0 and h_1 both have a continuous first derivative on I with range on J. If $\phi:I \to \mathbb{R}^1$ is defined by

$$\phi(x) = \int_{h_0(x)}^{h_1(x)} f(x,t)dt,$$

then

$$\phi'(x) = f[x, h_1(x)]h_1'(x) - f[x, h_0(x)]h_0'(x) + \int_{h_0(x)}^{h_1(x)} f_{,1}(x,t)dt. \tag{9.8}$$

Proof

Referring to F defined in (9.6), we see that $\phi(x) = F(x, h_0(x), h_1(x))$. We apply the chain rule for finding ϕ' and obtain

$$\phi'(x) = F_{,1} + F_{,2}h_0'(x) + F_{,3}h_1'(x).$$

Now, inserting the values of $F_{,1}$ and $F_{,2}$ and $F_{,3}$ from (9.7) into the above formula with $y = h_0(x)$ and $z = h_1(x)$, we get the General Leibniz rule (9.8). ■

EXAMPLE 2

Given $\phi : x \to \int_0^{x^2} \arctan\left(\frac{t}{x^2}\right) dt$, find ϕ'.

Solution. We have

$$\frac{\partial}{\partial x}\left(\arctan\left(\frac{t}{x^2}\right)\right) = -\frac{2tx}{t^2 + x^4}.$$

Using the General Leibniz rule (9.8), we obtain

$$\phi'(x) = (\arctan 1)(2x) - \int_0^{x^2} \frac{2tx}{t^2 + x^4}\, dt.$$

Setting $t = x^2u$ in the integral on the right, we get

$$\phi'(x) = \frac{\pi x}{2} - x\int_0^1 \frac{2u\, du}{1 + u^2} = x\left(\frac{\pi}{2} - \log 2\right).$$

■

Problems

In each of Problems 1 through 10, find ϕ'.

1. $\phi : x \to \displaystyle\int_0^1 \frac{\sin xt}{1+t}\,dt.$
2. $\phi : x \to \displaystyle\int_1^2 \frac{e^{-t}}{1+xt}\,dt.$
3.

$$\phi : x \to \int_0^1 f(x,t)dt \text{ where}$$

$$f : (x,t) \to \begin{cases} (t^x - 1)/\log t & \text{for } t \neq 0, 1, \\ 0 & \text{for } t = 0, \\ x & \text{for } t = 1. \end{cases}$$

4. $\phi : x \to \displaystyle\int_1^{x^2} \cos(t^2)dt.$
5. $\phi : x \to \displaystyle\int_{x^2}^{x} \sin(xt)dt.$
6. $\phi : x \to \displaystyle\int_{x^2}^{e^x} \tan(xt)dt.$
7. $\phi : x \to \displaystyle\int_{\cos x}^{1+x^2} \frac{e^{-t}}{1+xt}\,dt.$
8. $\phi : x \to \displaystyle\int_{\pi/2}^{\pi} \frac{\cos xt}{t}\,dt.$
9. $\phi : x \to \displaystyle\int_{x^2}^{x} \frac{\sin xt}{t}\,dt.$
10. $\phi : x \to \displaystyle\int_{x^m}^{x^n} \frac{dt}{x+t}\,.$

In each of Problems 11 through 13, compute the indicated partial derivative.

11. $\phi : (x,y) \to \displaystyle\int_y^{x^2} \frac{1}{t} e^{xt}\,dt;$ compute $\phi_{,1}$.
12. $\phi : (x,z) \to \displaystyle\int_{z^3}^{x^2} f(x,t)dt$ where $f : (x,t) \to \begin{cases} \dfrac{1}{t}\sin^2(xt), & t \neq 0, \\ 0, & t = 0. \end{cases}$
    Compute $\phi_{,1}$.
13. $\phi : (x,y) \to \displaystyle\int_{x^2+y^2}^{x^2-y^2} (t^2 + 2x^2 - y^2)dt;$ compute $\phi_{,2}$.

14. Show that if m and n are positive integers, then

$$\int_0^1 t^n (\log t)^m \, dt = (-1)^m \frac{m!}{(n+1)^{m+1}} \, .$$

[*Hint*: Differentiate $\int_0^1 x^n dx$ with respect to n and use induction.]

15. Given

$$F : (x, y) \to \int_{h_0(x,y)}^{h_1(x,y)} f(x, y, t) dt,$$

find formulas for $F_{,1}$, $F_{,2}$.

9.2 Convergence and Divergence of Improper Integrals

Suppose that a real-valued function f is defined on the half-open interval $I = \{x : a \leq x < b\}$ and that for each $c \in I$, the integral

$$\int_a^c f(x) dx$$

exists. We are interested in functions f that are unbounded in a neighborhood of b. For example, the function $f : x \to (1 - x)^{-1}$ with domain $J = \{x : 0 \leq x < 1\}$ is unbounded and has the integral

$$\int_0^c (1 - x)^{-1} dx = -\log(1 - c) \quad \text{for } \; c \in J.$$

As c tends to 1, the value of the integral tends to $+\infty$. On the other hand, for the unbounded function $g : x \to (1 - x)^{-\frac{1}{2}}$ defined on J, we obtain

$$\int_0^c (1 - x)^{-1/2} dx = 2 - 2\sqrt{1 - c} \quad \text{for } \; c \in J.$$

Observe that this integral has the finite limiting value 2 as $c \to 1$. With these examples in mind we define an improper integral.

Definitions

Suppose that f is integrable for each number c in the half-open interval $I = \{x : a \leq x < b\}$. The integral

$$\int_a^b f(x) dx$$

is **convergent** if

$$\lim_{c \to b^-} \int_a^c f(x) dx$$

exists. If the limit does not exist, the integral is **divergent**. If f is bounded on an interval $J = \{x : a \le x \le b\}$ except in a neighborhood of an interior point $d \in J$, then $\int_a^b f(x)dx$ is **convergent** if both limits

$$\lim_{c_1 \to d^-} \int_a^{c_1} f(x)dx \quad \text{and} \quad \lim_{c_2 \to d^+} \int_{c_2}^{b} f(x)dx$$

exist. Otherwise, the integral is **divergent**.

EXAMPLE 1
Show that the integrals $\int_a^b (b-x)^{-p}dx$ and $\int_a^b (x-a)^{-p}dx$ converge for $p < 1$ and diverge for $p \ge 1$.

Solution. For $a < c < b$ and $p \ne 1$ we have

$$\int_a^c (b-x)^{-p}dx = -\frac{(b-x)^{1-p}}{1-p}\Bigg|_a^c = \frac{(b-a)^{1-p}}{1-p} - \frac{(b-c)^{1-p}}{1-p} \ .$$

For $p < 1$, the expression on the right tends to $(b-a)^{1-p}/(1-p)$ as $c \to b^-$. For $p > 1$, there is no limit. The case $p = 1$ yields $\log(b-a) - \log(b-c)$, which has no limit as $c \to b^-$. ■

When the integral of an unbounded function is convergent, we say that the **improper integral** *exists.*

In analogy with the convergence of infinite series, it is important to establish criteria that determine when improper integrals exist. The following result, the Comparison test, is a basic tool in determining when integrals converge and diverge.

Theorem 9.4 (Comparison Test)
Suppose that f is continuous on the half-open interval $I = \{x{:}a \le x < b\}$ and that $0 \le |f(x)| \le g(x)$ for all $x \in I$. If $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ converges and

$$\left|\int_a^b f(x)dx\right| \le \int_a^b g(x)\,dx.$$

Proof
First suppose that $f(x) \ge 0$ on I and define

$$F : x \to \int_a^x f(t)\,dt, \qquad G : x \to \int_a^x g(t)\,dt.$$

Then F and G are nondecreasing on I, and by hypothesis, $G(x)$ tends to a limit, say M, as $x \to b^-$. Since $F(x) \le G(x) \le M$ on I, we find from the Axiom of continuity that $F(x)$ tends to a limit as $x \to b^-$.

If f is not always nonnegative, define

$$f_1(x) = \frac{|f(x)| + f(x)}{2}, \quad f_2(x) = \frac{|f(x)| - f(x)}{2}.$$

Then f_1 and f_2 are continuous on I and nonnegative there. Moreover,

$$f_1(x) + f_2(x) = |f(x)| \le g(x), \qquad f_1(x) - f_2(x) = f(x).$$

From the proof for nonnegative functions, the integrals $\int_a^b f_1(x)dx$, $\int_a^b f_2(x)dx$ exist; using the theorem on the limit of a sum, we see that $\int_a^b |f(x)|dx$ and $\int_a^b f(x)dx$ exist. Finally,

$$\begin{aligned}\left|\int_a^b f(x)dx\right| &= \left|\int_a^b f_1(x)\,dx - \int_a^b f_2(x)dx\right| \\ &\le \int_a^b f_1(x)\,dx + \int_a^b f_2(x)dx \le \int_a^b g(x)\,dx.\end{aligned}$$

■

Example 2

Test for convergence or divergence:

$$\int_0^1 \frac{x^\beta}{\sqrt{1-x^2}}\,dx, \qquad \beta > 0.$$

Solution. We shall compare $f : x \to x^\beta/\sqrt{1-x^2}$ with $g : x \to 1/\sqrt{1-x}$. We have

$$\frac{x^\beta}{\sqrt{1-x^2}} = \frac{x^\beta}{\sqrt{(1-x)(1+x)}} = \frac{x^\beta}{\sqrt{1+x}}g(x).$$

Since $x^\beta \le \sqrt{1+x}$ for $0 \le x \le 1$, it follows that $|f(x)| \le g(x)$. By Example 1 we know that $\int_0^1 1/\sqrt{1-x}dx$ is convergent, and so the original integral converges. ■

We now take up the convergence of integrals in which the integrand is bounded but where the interval of integration is unbounded.

Definitions

Let f be defined on $I = \{x : a \le x < +\infty\}$ and suppose that $\int_a^c f(x)dx$ exists for each $c \in I$. Define

$$\int_a^{+\infty} f(x)dx = \lim_{c\to+\infty}\int_a^c f(x)dx$$

whenever the limit on the right exists. In such cases, we say that **the integral converges**; when the limit does not exist, we say that **the integral**

diverges. If f is defined on $J = \{x : -\infty < x < +\infty\}$, we may consider expressions of the form

$$\int_{-\infty}^{+\infty} f(x)\,dx,$$

which are determined in terms of two limits, one tending to $+\infty$ and the other to $-\infty$. Let d be any point in J. Define

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{c_1 \to -\infty} \int_{c_1}^{d} f(x)dx + \lim_{c_2 \to +\infty} \int_{d}^{c_2} f(x)dx \tag{9.9}$$

whenever both limits on the right exist. It is a simple matter to see that if f is integrable for every finite interval of J, then the values of the limits in (9.9) do not depend on the choice of the point d.

To illustrate the convergence and divergence properties of integrals when the path of integration is infinite, we show that

$$\int_{a}^{+\infty} x^{-p}dx, \qquad a > 0,$$

converges for $p > 1$ and diverges for $p \le 1$. To see this, observe that $(p \neq 1)$

$$\int_{a}^{c} x^{-p}dx = \frac{1}{1-p}[c^{1-p} - a^{1-p}].$$

For $p > 1$, the right side tends to $(1/(p-1))a^{1-p}$ as $c \to +\infty$, while for $p < 1$ there is no limit. By the same argument, the case $p = 1$ yields divergence. Similarly, the integral

$$\int_{-\infty}^{b} |x|^{-p}dx, \qquad b < 0,$$

converges for $p > 1$ and diverges for $p \le 1$. In analogy with the Comparison test for integrals over a finite path of integration, we state the following result.

Corollary

For continuous functions f and g defined on the interval $I = \{x{:}a \le x < +\infty\}$, Theorem 9.4 is valid with respect to integrals $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$.

Example 3

Test for convergence:

$$\int_{1}^{+\infty} \frac{\sqrt{x}}{1+x^{3/2}}\,dx\,.$$

Solution. For $x \ge 1$, observe that

$$\frac{\sqrt{x}}{1+x^{3/2}} \ge \frac{\sqrt{x}}{2x^{3/2}} = \frac{1}{2x}\,.$$

However, $\int_1^{+\infty}(1/(2x))dx$ diverges, and so the integral is divergent. ■

The convergence of integrals with unbounded integrands and over an infinite interval may be treated by combining Theorem 9.4 and the corollary. The next example illustrates the method.

Example 4

Test for convergence:

$$\int_0^{+\infty} \frac{e^{-x}}{\sqrt{x}}\,dx\,.$$

Solution. Because the integrand is unbounded near $x = 0$ we decompose the problem into two parts:

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx \quad \text{and} \quad \int_1^{+\infty} \frac{e^{-x}}{\sqrt{x}}\,dx.$$

In the first integral we have $e^{-x}/\sqrt{x} \le 1/\sqrt{x}$, and the integral converges by the Comparison test. In the second integral we have $e^{-x}/\sqrt{x} \le e^{-x}$, and once again the Comparison test yields the result, since $\int_1^{+\infty} e^{-x}dx$ is convergent. Hence, the original integral converges. ■

We consider the integral

$$\int_0^{+\infty} t^{x-1}e^{-t}dt,$$

which we shall show is convergent for $x > 0$. Since the integrand is unbounded near zero when x is between 0 and 1, the integral may be split into two parts:

$$\int_0^{+\infty} t^{x-1}e^{-t}dt = \int_0^1 t^{x-1}e^{-t}dt + \int_1^{+\infty} t^{x-1}e^{-t}dt \equiv I_1 + I_2.$$

In the first integral on the right, we use the inequality

$$t^{x-1}e^{-t} \le t^{x-1} \quad \text{for } x > 0 \quad \text{and} \quad 0 < t \le 1.$$

The integral $\int_0^1 t^{x-1}dt$ converges for $x > 0$, and so I_1 does also. As for I_2, an estimate for the integrand is obtained first by writing

$$t^{x-1}e^{-t} = t^{x+1}e^{-t}t^{-2}$$

and then estimating the function

$$f : t \to t^{x+1}e^{-t}.$$

We find $f'(t) = t^x e^{-t}(x + 1 - t)$, and f has a maximum when $t = x + 1$. This maximum value is $f(x + 1) = (x + 1)^{x+1} e^{-(x+1)}$. Therefore,

$$\begin{aligned} I_2 &= \int_1^{+\infty} t^{x-1} e^{-t} dt = \int_1^{+\infty} (t^{x+1} e^{-t}) t^{-2} dt \\ &\leq (x + 1)^{x+1} e^{-(x+1)} \int_1^{+\infty} \frac{1}{t^2}\, dt \\ &= (x + 1)^{x+1} e^{-(x+1)}. \end{aligned}$$

Hence I_2 is convergent for each fixed $x > -1$.

Definition

For $x > 0$ the **Gamma function**, denoted by $\Gamma(x)$, is defined by the formula

$$\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt.$$

The recursion formula

$$\Gamma(x + 1) = x\Gamma(x), \tag{9.10}$$

one of the most important properties of the Gamma function, is derived by means of an integration by parts. To see this, we write

$$\int_0^c t^x e^{-t} dt = [t^x(-e^{-t})]|_0^c + x \int_0^c t^{x-1} e^{-t} dt.$$

Now, letting $c \to +\infty$, we obtain (9.10).

It is easy to verify that $\Gamma(1) = 1$, and consequently that $\Gamma(n+1) = n!$ for positive integers n. Note that the Gamma function is a smooth extension to the positive real numbers of the factorial function, which is defined only for the natural numbers.

Problems

In each of Problems 1 through 12 test for convergence or divergence.

1. $\displaystyle\int_1^{+\infty} \frac{dx}{(x + 2)\sqrt{x}}$.
2. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1 - x^4}}$.
3. $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{1 + x^3}}$.
4. $\displaystyle\int_0^{+\infty} \frac{x\,dx}{\sqrt{1 + x^4}}$.
5. $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt{1 - x^2}}$.
6. $\displaystyle\int_0^{\pi/2} \frac{\sqrt{x}}{\sin x}\, dx$.
7. $\displaystyle\int_1^3 \frac{\sqrt{x}}{\log x}\, dx$.
8. $\displaystyle\int_0^{+\infty} \frac{(\arctan x)^2}{(1 + x^2)}\, dx$.
9. $\displaystyle\int_0^{+\infty} x^2 e^{-x^2} dx$.
10. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x - x^2}}$.
11. $\displaystyle\int_0^{+\infty} e^{-x} \sin x\, dx$.
12. $\displaystyle\int_0^{\pi} \frac{\sin x}{x\sqrt{x}}\, dx$.

13. Show that $\int_2^{+\infty} x^{-1}(\log x)^{-p}dx$ converges for $p > 1$ and diverges for $p \leq 1$.
14. Define $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, $0 < x < \infty$, $0 < y < \infty$.
    (a) Show that $B(x, y) = B(y, x)$.
    (b) Find $B(2, 2)$ and $B(4, 3)$.

CHAPTER 10 The Riemann–Stieltjes Integral

10.1 Functions of Bounded Variation

In this section we introduce the notion of the *variation* of a function. This quantity is useful for problems in physics, engineering, probability theory, Fourier series, and many other subjects. However, our main purpose here is to show how this concept can enlarge enormously the class of functions that are integrable.

Definitions
Let $I = \{x : a \le x \le b\}$ be an interval and $f : I \to \mathbb{R}^1$ a given function. The **variation of f over I**, denoted by $V_a^b f$, is the quantity

$$V_a^b f = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

where the supremum is taken over all possible subdivisions $a = x_0 < x_1 < \ldots < x_n = b$ of I. If the number $V_a^b f$ is finite, we say that f *is of* **bounded variation on I**. If f is not of bounded variation, we write $V_a^b f = +\infty$.

We first establish several of the fundamental properties of the variation of a function. If the function f is kept fixed, then $V_a^b f$ depends only on the interval $I = \{x : a \le x \le b\}$. The following theorem shows that the variation is *additive* on intervals.

Theorem 10.1

Let $f : I \to \mathbb{R}^1$ be given and suppose that $c \in I$. Then

$$V_a^b f = V_a^c f + V_c^b f. \tag{10.1}$$

Proof

(a) We first show that $V_a^b f \le V_a^c f + V_c^b f$. We may suppose that both terms on the right are finite. Let $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ be any subdivision. We form the subdivision Δ' by introducing the additional point c, which falls between x_{k-1} and x_k, say. Then

$$\sum_{i=1}^{k-1} |f(x_i) - f(x_{i-1})| + |f(c) - f(x_{k-1})| \le V_a^c f,$$

$$\sum_{i=k+1}^{n} |f(x_i) - f(x_{i-1})| + |f(x_k) - f(c)| \le V_c^b f.$$

From the inequality $|f(x_k) - f(x_{k-1})| \le |f(x_k) - f(c)| + |f(c) - f(x_{k-1})|$, it follows that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le V_a^c f + V_c^b f.$$

Since the subdivision Δ is arbitrary, we obtain $V_a^b f \le V_a^c f + V_c^b f$.

(b) We now show that $V_a^b f \ge V_a^c f + V_c^b f$. If $V_a^b f = +\infty$, then the inequality clearly holds. Hence we may assume that all three quantities are finite. Now let $\varepsilon > 0$ be given. From the definition of supremum, there is a subdivision

$$\Delta_1 : a = x_0 < x_1 < \ldots < x_k = c$$

such that

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| > V_a^c f - \frac{1}{2}\varepsilon.$$

Similarly, there is a subdivision $\Delta_2 : c = x_k < x_{k+1} \ldots < x_n = b$ such that

$$\sum_{i=k+1}^{n} |f(x_i) - f(x_{i-1})| > V_c^b f - \frac{1}{2}\varepsilon.$$

Therefore,

$$V_a^b f \ge \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| > V_a^c f + V_c^b f - \varepsilon.$$

Since ε is arbitrary, the result follows. ∎

Example 1

Suppose that f is a nondecreasing function on $I = \{x : a \le x \le b\}$. Show that $V_a^x f = f(x) - f(a)$ for $x \in I$.

Solution. Let x be in I and let Δ be a subdivision $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = x$. We have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] = f(x) - f(a),$$

the first equality holding because f is nondecreasing, and the second because all the terms except the first and last cancel. Since the above equalities hold for every subdivision, they hold also for the supremum. Hence $V_a^x f = f(x) - f(a)$. ■

By the same argument it is clear that if f is nonincreasing on I, then $V_a^x f = f(a) - f(x)$.

Theorem 10.2

Suppose that f is continuous on $I = \{x : a \le x \le b\}$ and f' is bounded on I. Then f is of bounded variation.

Proof

If $|f'| \le M$ on I, we may apply the Mean-value theorem to obtain $|f(x_k) - f(x_{k-1})| \le M|x_k - x_{k-1}|$ for any two points x_{k-1}, x_k of I. Therefore, for any subdivision Δ,

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \le M \sum_{k=1}^{n} |x_k - x_{k-1}| = M(b - a).$$

Hence $V_a^b f \le M(b - a)$. ■

The theorem above gives us an easy sufficient condition for determining when a function is of bounded variation. Since functions may be of bounded variation without being continuous, the set of functions with finite variation is much larger than the set having a bounded first derivative.

The next theorem exhibits an important relationship between monotone functions and functions of bounded variation. It shows that *every* function of bounded variation is the difference of two nondecreasing functions.

Theorem 10.3

Suppose that f is of bounded variation on $I = \{x : a \le x \le b\}$. Then there are nondecreasing functions g and h on I such that

$$f(x) = g(x) - h(x) \quad \text{for} \quad x \in I, \tag{10.2}$$

and $V_a^x f = g(x) + h(x) - f(a)$ *for* $x \in I$. *Moreover, if* f *is continuous on the left at any point* $c \in I$, *then* g *and* h *are also. Similarly,* g *and* h *are continuous on the right wherever* f *is.*

Proof

Choose

$$g(x) = \frac{1}{2}[f(a) + V_a^x f + f(x)], \qquad h(x) = \frac{1}{2}[f(a) + V_a^x f - f(x)]. \quad ((10.3))$$

To show that g is nondecreasing, observe that $2g(x_2) - 2g(x_1) = V_{x_1}^{x_2} f + f(x_2) - f(x_1)$. Since we have $V_{x_1}^{x_2} f \geq |f(x_2) - f(x_1)|$, it follows that $g(x_2) \geq g(x_1)$. The proof for h is similar.

Suppose that f is continuous on the left or right at c. Then so is φ, where $\varphi(x) = V_a^x f$. The continuity result follows directly from Theorem 10.2 and the explicit formulas for g and h. ■

The decomposition of f given by (10.2) and (10.3) is not unique. If ψ is *any* nondecreasing function on I, we also have the decomposition $f = (g + \psi) - (h + \psi)$. In fact, if ψ is strictly increasing, then this decomposition of f is the difference of two strictly increasing functions.

The next result gives a sufficient condition for determining when a function is of bounded variation, and also a method for computing its variation.

Theorem 10.4

Suppose that f *and* f' *are continuous on an interval* $I = \{x : a \leq x \leq b\}$. *Then* f *is of bounded variation on* I, *and*

$$V_a^b f = \int_a^b |f'(x)| dx.$$

Proof

The first part of the theorem was established in Theorem 10.2. Let Δ : $a = x_0 < x_1 < \ldots < x_n = b$ be any subdivision. Then by the Mean-value theorem, there are numbers ξ_i such that $x_{i-1} \leq \xi_i \leq x_i$ with

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} |f'(\xi_i)| \cdot |x_i - x_{i-1}|.$$

Let $\varepsilon > 0$ be given. From the definition of integral, there is a $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} |f'(\xi_i)||x_i - x_{i-1}| - \int_a^b |f'(x)| dx \right| < \frac{1}{2}\varepsilon$$

for *every* subdivision with mesh less than δ. Now we use the definition of bounded variation to assert that for the above ε, there is a subdivision

$\Delta_0 : a = z_0 < z_1 < \ldots < z_m = b$ such that

$$V_a^b f \geq \sum_{i=1}^{m} |f(z_i) - f(z_{i-1})| > V_a^b f - \frac{1}{2}\varepsilon.$$

Let $\Delta_1 : a = x_0' < x_1' < \ldots < x_p' = b$ be the common refinement of Δ and Δ_0. Then Δ_1 has mesh less than δ. Hence

$$V_a^b f \geq \sum_{i=1}^{p} |f(x_i') - f(x_{i-1}')| \geq \sum_{i=1}^{m} |f(z_i) - f(z_{i-1})| > V_a^b f - \frac{1}{2}\varepsilon$$

and

$$\left| \sum_{i=1}^{p} |f(x_i') - f(x_{i-1}')| - \int_a^b |f'(x)| dx \right| < \frac{1}{2}\varepsilon.$$

Combining these inequalities, we obtain

$$\left| V_a^b f - \int_a^b |f'(x)| dx \right| < \varepsilon.$$

Since ε is arbitrary, the result follows. ■

A function f may be of bounded variation without having a bounded derivative. For example, the function $f : x \to x^{2/3}$ on $I = \{x : 0 \leq x \leq 1\}$ is continuous and increasing on I. Hence it is of bounded variation. However, f' is unbounded at the origin.

Problems

1. Suppose that f is of bounded variation on $I = \{x : a \leq x \leq b\}$. Show that f is bounded on I. In fact, show that $|f(x)| \leq |f(a)| + V_a^b f$.
2. Suppose that f and g are of bounded variation on $I = \{x : a \leq x \leq b\}$. Show that $f - g$ and fg are functions of bounded variation.
3. Given $f(x) = \sin^2 x$ for $x \in I = \{x : 0 \leq x \leq \pi\}$, find $V_0^{\pi} f$.
4. Given $f(x) = x^3 - 3x + 4$ for $x \in I = \{x : 0 \leq x \leq 2\}$, find $V_0^2 f$.
5. Given

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < 1, \\ \frac{1}{2} & \text{for } 1 \leq x < 2, \\ 2 & \text{for } 2 \leq x \leq 3, \end{cases}$$

find $V_0^3 f$.
6. Let $I_i = \{x : i - 1 \leq x < i\}$, $i = 1, 2, \ldots, n$. Let $f(x) = c_i$ for $x \in I_i$ and $f(n) = c_n$, where each c_i is a constant. Find $V_0^n f$.
7. Show that the function

$$f : x \to \begin{cases} x \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

is not of bounded variation on $I = \{x : 0 \le x \le 1\}$. However, prove that

$$g : x \to \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases}$$

is of bounded variation on I.

8. Let $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ be a subdivision of $I = \{x : a \le x \le b\}$ and suppose f is defined on I. Then $\sum_{i=1}^{n}[(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1})^2]^{\frac{1}{2}}$ is the length of the inscribed polygonal arc of f. We define the **length of f on I**, denoted by $L_a^b f$, by the formula

$$L_a^b f = \sup \sum_{i=1}^{n}[(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2]^{\frac{1}{2}},$$

where the supremum is taken over all possible subdivisions.

   (a) Show that for any function f, the inequalities

   $$V_a^b f + (b-a) \ge L_a^b f \ge [(V_a^b f)^2 + (b-a)^2]^{\frac{1}{2}}$$

   hold. Hence conclude that a function is of bounded variation if and only if it has finite length.

   (b) Show that if $a < c < b$, then

   $$L_a^b f = L_a^c f + L_c^b f.$$

9. Show that if f is continuous and has a finite number of maxima and minima on an interval $I = \{x : a \le x \le b\}$, then f is of bounded variation on I. Conclude that every polynomial function is of bounded variation on every finite interval.

10. Suppose that f is of bounded variation on $I = \{x : a \le x \le b\}$. If $|f(x)| \ge c > 0$ for all $x \in I$, where c is a constant, show that $g(x) = 1/f(x)$ is of bounded variation on I.

10.2 The Riemann–Stieltjes Integral

We introduce a generalization of the Riemann integral, one in which a function f is integrated with respect to a second function g. If $g(x) = x$, then the generalized integral reduces to the Riemann integral. This new integral, called the Riemann–Stieltjes integral, has many applications not only in various branches of mathematics, but in physics and engineering as well. By choosing the function g appropriately we shall see that the Riemann–Stieltjes integral allows us to represent discrete as well as continuous processes in terms of integrals. This possibility yields applications to probability theory and statistics.

Definitions

Let f and g be functions from $I = \{x : a \le x \le b\}$ into $\mathbb{R}^1$. Suppose that there is a number A such that for each $\varepsilon > 0$ there is a $\delta > 0$ for which

$$\left|\sum_{i=1}^{n} f(\zeta_i)[g(x_i) - g(x_{i-1})] - A\right| < \varepsilon \tag{10.4}$$

for every subdivision Δ of mesh size less than δ and for every sequence $\{\zeta_i\}$ with $x_{i-1} \le \zeta_i \le x_i$, $i = 1, 2, \ldots, n$. Then we say that f **is integrable with respect to g on I**. We also say that the integral exists in the **Riemann–Stieltjes** sense. The number A is called the R–S **integral of f with respect to g**, and we write

$$A = \int_a^b f\,dg = \int_a^b f(x)dg(x).$$

As in the case of a Riemann integral, it is a simple matter to show that when the number A exists, it is unique. Furthermore, when $g(x) = x$, the sum in (10.4) is a Riemann sum, and the R–S integral reduces to the Riemann integral.

It is important to observe that the R–S integral may exist when g is not continuous. For example, with $I = \{x : 0 \le x \le 1\}$ let $f(x) \equiv 1$ on I,

$$g(x) = \begin{cases} 0 & \text{for } 0 \le x < \dfrac{1}{2}, \\ 1 & \text{for } \dfrac{1}{2} \le x \le 1. \end{cases}$$

The quantity $\sum_{i=1}^n f(\zeta_i)[g(x_i) - g(x_{i-1})]$ reduces to $\sum_{i=1}^n [g(x_i) - g(x_{i-1})]$. However, all these terms are zero except for the subinterval that contains $x = \frac{1}{2}$. In any case, the terms of the sum "telescope," so that its value is $g(1) - g(0) = 1$. Therefore, for every subdivision, the Riemann–Stieltjes sum has the value 1, and this is the value of the R–S integral.

The R–S integral may not exist if f has a single point of discontinuity, provided that the function g is also discontinuous at the same point. For example, with $I = \{x : 0 \le x \le 1\}$ define

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < \dfrac{1}{2}, \\ 2 & \text{for } \dfrac{1}{2} \le x \le 1, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{for } 0 \le x < \dfrac{1}{2}, \\ 1 & \text{for } \dfrac{1}{2} \le x \le 1. \end{cases}$$

The quantity $\sum_{i=1}^n f(\zeta_i)[g(x_i) - g(x_{i-1})]$ reduces to the single term that corresponds to the subinterval containing $x = \frac{1}{2}$. That is, we have one term of the form

$$f(\zeta_k)[g(x_k) - g(x_{k-1})],$$

where $x_{k-1} \le \frac{1}{2} \le x_k$ and $x_{k-1} \le \zeta_k \le x_k$. Then $g(x_k) - g(x_{k-1}) = 1$, but the value of $f(\zeta_k)$ will be 1 or 2, depending on whether ζ_k is chosen less than $\frac{1}{2}$ or greater than or equal to $\frac{1}{2}$. Since these two choices may be made regardless of the mesh of the subdivision, the R–S integral does not exist.

The R–S integral has properties of additivity and homogeneity similar to those of the Riemann integral. These are stated in the next theorem.

Theorem 10.5

(a) *Suppose that* $\int_a^b f\,dg_1$ *and* $\int_a^b f\,dg_2$ *both exist. Define* $g = g_1 + g_2$. *Then* f *is integrable with respect to* g, *and*

$$\int_a^b f\,dg = \int_a^b f\,dg_1 + \int_a^b f\,dg_2.$$

(b) *Suppose that* $\int_a^b f_1\,dg$ *and* $\int_a^b f_2\,dg$ *both exist. Define* $f = f_1 + f_2$. *Then* f *is integrable with respect to* g, *and*

$$\int_a^b f\,dg = \int_a^b f_1\,dg + \int_a^b f_2\,dg.$$

The next theorem shows that if g is smooth, then the R–S integral is reducible to an ordinary Riemann integral. This reduction is useful for the calculation of Riemann–Stieltjes integrals. We show later that the R–S integral exists for much larger classes of functions.

Theorem 10.6

Suppose that f, g, *and* g' *are continuous on the interval* $I = \{x{:}a \le x \le b\}$. *Then* $\int_a^b f\,dg$ *exists, and*

$$\int_a^b f\,dg = \int_a^b f(x)g'(x)\,dx. \tag{10.5}$$

Proof

Let $\varepsilon > 0$ be given. We wish to show that

$$\left|\sum_{i=1}^{n} f(\zeta_i)[g(x_i) - g(x_{i-1})] - \int_a^b f(x)g'(x)dx\right| < \varepsilon \tag{10.6}$$

provided that the mesh of the subdivision is sufficiently small. We apply the Mean-value theorem to the Riemann–Stieltjes sum in (10.6), getting

$$\sum_{i=1}^{n} f(\zeta_i)[g(x_i) - g(x_{i-1})] = \sum_{i=1}^{n} f(\zeta_i)g'(\eta_i)(x_i - x_{i-1}), \tag{10.7}$$

where $x_{i-1} \le \eta_i \le x_i$. The sum on the right would be a Riemann sum if η_i were equal to ζ_i. We show that for subdivisions with sufficiently small mesh this sum is close to a Riemann sum. Let M denote the maximum of $|f(x)|$ on I. Since g' is continuous on I, it is uniformly continuous there.

Hence there is a $\delta > 0$ such that for $|\zeta_i - \eta_i| < \delta$ it follows that

$$(10.8) \qquad |g'(\zeta_i) - g'(\eta_i)| < \frac{\varepsilon}{2M(b-a)}.$$

From the definition of Riemann integral there is a subdivision with mesh so small (and less than δ) that

$$(10.9) \qquad \left| \sum_{i=1}^{N} f(\zeta_i) g'(\zeta_i)(x_i - x_{i-1}) - \int_a^b f(x) g'(x) dx \right| < \frac{1}{2}\varepsilon.$$

By means of (10.8), we have

(10.10)

$$\left| \sum_{i=1}^{N} f(\zeta_i)[g'(\eta_i) - g'(\zeta_i)](x_i - x_{i-1}) \right| < \sum_{i=1}^{N} M \left| \frac{\varepsilon}{2M(b-a)} (x_i - x_{i-1}) \right| = \frac{1}{2}\varepsilon.$$

From (10.9) and (10.10), for any ζ_i, η_i such that $x_{i-1} \le \zeta_i \le x_i$ and $x_{i-1} \le \eta_i \le x_i$, we get the inequality

$$\left| \sum_{i=1}^{N} f(\zeta_i) g'(\eta_i)(x_i - x_{i-1}) - \int_a^b f(x) g'(x) dx \right| < \varepsilon.$$

Now taking (10.6) and (10.7) into account, we get the desired result. ■

The next theorem shows how to change variables in R–S integrals, a result that is useful when actually performing integrations.

Theorem 10.7

Suppose that f is integrable with respect to g on $I = \{x : a \le x \le b\}$. Let $x = x(u)$ be a continuous, increasing function on $J = \{u : c \le u \le d\}$ with $x(c) = a$ and $x(d) = b$. Define

$$F(u) = f[x(u)] \text{ and } G(u) = g[x(u)].$$

Then F is integrable with respect to G on J, and

$$(10.11) \qquad \int_c^d F(u) dG(u) = \int_a^b f(x) dg(x).$$

Proof

Let $\varepsilon > 0$ be given. From the definition of the R–S integral, there is a $\delta > 0$ such that

$$\left| \sum_{i=1}^{n} f(\zeta_i)[g(x_i) - g(x_{i-1})] - \int_a^b f(x) dg(x) \right| < \varepsilon$$

for all subdivisions $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ with mesh less than δ and any $\{\zeta_i\}$ in which $x_{i-1} \le \zeta_i \le x_i$. Since x is (uniformly) continuous on J, there is a δ_1 such that $|x(u')-x(u'')| < \delta$ whenever $|u'-u''| < \delta_1$. Consider the subdivision $\Delta_1 : c = u_0 < u_1 < \ldots < u_n = d$ of J with mesh less than δ_1. Let η_i be such that $u_{i-1} \le \eta_i \le u_i$ for $i = 1, 2, \ldots, n$. Since $x(u)$ is increasing on J, set $x(u_i) = x_i$ and $x(\eta_i) = \zeta_i$. Then the subdivision Δ of I has mesh less than δ, and $x_{i-1} \le \zeta_i \le x_i$ for $i = 1, 2, \ldots, n$. Therefore

$$\sum_{i=1}^{n} F(\eta_i)[G(u_i) - G(u_{i-1})] = \sum_{i=1}^{n} f(\zeta_i)[g(x_i) - g(x_{i-1})].$$

Taking into account the definition of the R–S integral, (10.4), we see that the integrals in (10.11) are equal. ■

The next result is a generalization of the customary integration by parts formula. Formula (10.12) below is most useful for the actual computation of R–S integrals. It also shows that if either of the integrals $\int_a^b f\,dg$ or $\int_a^b g\,df$ exists, then the other one does.

Theorem 10.8 (Integration by parts)
If $\int_a^b f\,dg$ exists, then so does $\int_a^b g\,df$ and

$$\int_a^b g\,df = g(b)f(b) - g(a)f(a) - \int_a^b f\,dg. \tag{10.12}$$

Proof
Let $\varepsilon > 0$ be given. From the definition of R–S integral, there is a $\delta' > 0$ such that

$$\left|\sum_{i=1}^{m} f(\zeta_i')[g(x_i') - g(x_{i-1}')] - \int_a^b f\,dg\right| < \varepsilon \tag{10.13}$$

for any subdivision $\Delta' : a = x_0' < x_1' < \ldots < x_m' = b$ of mesh less than δ' and any ζ_i' with $x_{i-1}' \le \zeta_1' \le x_i'$. Let $\delta = \frac{1}{2}\delta'$ and choose a subdivision $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ of mesh less than δ and points ζ_i such that $x_{i-1} \le \zeta_i \le x_i$, $i = 1, 2, \ldots, n$. We further select $\zeta_0 = a$ and $\zeta_{n+1} = b$. Then we observe that $a = \zeta_0 \le \zeta_1 \le \ldots \le \zeta_n \le \zeta_{n+1} = b$ is a subdivision with mesh size less than δ', and furthermore, $\zeta_{i-1} \le x_{i-1} \le \zeta_i$, $i = 1, 2, \ldots, n+1$.

Therefore,

$$
\begin{aligned}
\sum_{i=1}^{n} g(\zeta_i)[f(x_i) - f(x_{i-1})] &= \sum_{i=1}^{n} g(\zeta_i)f(x_i) - \sum_{i=1}^{n} g(\zeta_i)f(x_{i-1}) \\
&= \sum_{i=2}^{n+1} g(\zeta_{i-1})f(x_{i-1}) + g(a)f(a) - g(a)f(a) \\
&\quad - \sum_{i=1}^{n} g(\zeta_i)f(x_{i-1}) - g(b)f(b) + g(b)f(b) \\
&= \sum_{i=1}^{n+1} g(\zeta_{i-1})f(x_{i-1}) - g(a)f(a) \\
&\quad - \sum_{i=1}^{n+1} g(\zeta_i)f(x_{i-1}) + g(b)f(b),
\end{aligned}
$$

or

(10.14)

$$\sum_{i=1}^{n} g(\zeta_i)[f(x_i) - f(x_{i-1})] = g(b)f(b) - g(a)f(a) - \sum_{i=1}^{n+1} f(x_{i-1})[g(\zeta_i) - g(\zeta_{i-1})].$$

The sum on the right is a R–S sum that satisfies (10.13). Hence the right side of (10.14) differs from the right side of (10.12) by less than ε. But the left side of (10.14) is the R–S sum for the left side of (10.12). Since ε is arbitrary, the result follows. ■

We shall show that if f is continuous and g of bounded variation on an interval $I = \{x : a \leq x \leq b\}$, then $\int_a^b f\, dg$ exists. Theorem (10.8) shows that if the hypotheses on f and g are reversed, the R–S integral will still exist.

We recall that any function of bounded variation may be represented as the difference of two monotone functions. Therefore, if the existence of the R–S integral $\int_a^b f\, dg$ when f is continuous and g is nondecreasing is established, it follows that the R–S integral exists when f is continuous and g is of bounded variation. To establish the existence of the R–S integral for as large a class as possible, we shall employ the method of Darboux described in Chapter 5. We begin by defining upper and lower Darboux–Stieltjes sums and integrals.

Definitions

Suppose that f and g are defined on $I = \{x : a \leq x \leq b\}$ and g is nondecreasing on I. Let $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ be a subdivision of I. Define $\Delta_i g = g(x_i) - g(x_{i-1})$, $i = 1, 2, \ldots, n$. Note that $\Delta_i g \geq 0$ for all i; also set

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x), \qquad m_i = \inf_{x_{i-1} \le x \le x_i} f(x).$$

Define the **upper** and **lower Darboux–Stieltjes sums**

$$S^+(f, g, \Delta) = \sum_{i=1}^{n} M_i \Delta_i g, \qquad S_-(f, g, \Delta) = \sum_{i=1}^{n} m_i \Delta_i g.$$

Since g is nondecreasing, observe that the numbers $I_i = \Delta_i g = g(x_i) - g(x_{i-1})$ are nonnegative and that $\sum_{i=1}^{n} \Delta_i g = g(b) - g(a)$. Thus the set $\{I_i\}$ is a subdivision $\bar{\Delta}$ of the interval $g(b) - g(a)$, and if Δ' is a refinement of Δ, then there is induced a corresponding refinement $\bar{\Delta}'$ of $\bar{\Delta}$.

The next result is completely analogous to the corresponding theorem on Darboux sums.

Theorem 10.9
Suppose that f and g are defined on $I = \{x : a \le x \le b\}$ and g is nondecreasing on I.

(a) *If $m \le f(x) \le M$ on I and Δ is any subdivision, then*

$$m[g(b) - g(a)] \le S_-(f, g, \Delta) \le S^+(f, g, \Delta) \le M[g(b) - g(a)].$$

(b) *If Δ' is a refinement of Δ, then*

$$S_-(f, g, \Delta') \ge S_-(f, g, \Delta) \qquad \text{and} \qquad S^+(f, g, \Delta') \le S^+(f, g, \Delta).$$

(c) *If Δ_1 and Δ_2 are any subdivisions of I, then*

$$S_-(f, g, \Delta_1) \le S^+(f, g, \Delta_2).$$

Definitions
The **upper** and **lower Darboux–Stieltjes integrals** are given by

$$\overline{\int_a^b} f\, dg = \inf S^+(f, g, \Delta) \text{ for all subdivisions } \Delta \text{ of } I.$$

$$\underline{\int_a^b} f\, dg = \sup S_-(f, g, \Delta) \text{ for all subdivisions } \Delta \text{ of } I.$$

These integrals are defined for all functions f on I and all functions g that are nondecreasing on I. If

$$\overline{\int_a^b} f\, dg = \underline{\int_a^b} f\, dg \tag{10.15}$$

we say that f is **integrable with respect to *g* on *I***. The *integral* is the common value (10.15).

The following elementary properties of upper and lower Darboux–Stieltjes integrals are similar to the corresponding ones for upper and lower Darboux integrals given in Chapter 5.

Theorem 10.10

Suppose that f, f_1, and f_2 are bounded on the interval $I = \{x: a \le x \le b\}$ and that g is nondecreasing on I.

(a) *If $m \le f(x) \le M$ on I, then*

$$m[g(b) - g(a)] \le \underline{\int_a^b} f\, dg \le \overline{\int_a^b} f\, dg \le M[g(b) - g(a)].$$

(b) *If $f_1(x) \le f_2(x)$ for $x \in I$, then*

$$\underline{\int_a^b} f_1\, dg \le \underline{\int_a^b} f_2\, dg \qquad \textit{and} \qquad \overline{\int_a^b} f_1\, dg \le \overline{\int_a^b} f_2\, dg.$$

(c) *Suppose that $a < c < b$. Then*

$$\underline{\int_a^c} f\, dg + \underline{\int_c^b} f\, dg = \underline{\int_a^b} f\, dg, \qquad \overline{\int_a^c} f\, dg + \overline{\int_c^b} f\, dg = \overline{\int_a^b} f\, dg.$$

The proof of Theorem 10.10 is similar to the proof of the corresponding results for Riemann integrals.

Theorem 10.11

Suppose that f is continuous on $I = \{x: a \le x \le b\}$ and that g is nondecreasing on I. Then the Riemann–Stieltjes integral $\int_a^b f\, dg$ exists.

Proof

Since f is uniformly continuous on I, for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2[1 + g(b) - g(a)]}$$

whenever $|x - y| < \delta$. Let $\Delta : a = x_0 < x_1 < \ldots < x_n = b$ be a subdivision of I with mesh less than δ. Define $I_i = \{x : x_{i-1} \le x \le x_i\}$. Since f is continuous on each I_i, there are numbers η_i and ζ_i on I_i such that $m_i = f(\eta_i)$ and $M_i = f(\zeta_i)$. Then we have

$$\begin{aligned} S^+(f, g, \Delta) - S_-(f, g, \Delta) &= \sum_{i=1}^{n} [f(\zeta_i) - f(\eta_i)]\Delta_i g \\ &\le \sum_{i=1}^{n} \frac{\varepsilon}{2[1 + g(b) - g(a)]} \Delta_i g < \frac{1}{2}\varepsilon. \end{aligned}$$

Since ε is arbitrary it follows that the Darboux–Stieltjes integral, denoted by (D–S) $\int_a^b f\, dg$, exists. Now let ξ_i be any point in I_i. From the fact that the Darboux–Stieltjes integral is between S^+ and S_-, we obtain

$$\begin{aligned}(\text{D–S})\int_a^b f\,dg - \varepsilon &< S_-(f, g, \Delta)\\ &= \sum_{i=1}^n f(\eta_i)\Delta_i g \le \sum_{i=1}^n f(\xi_i)\Delta_i g\\ &\le \sum_{i=1}^n f(\zeta_i)\Delta_i g = S^+(f, g, \Delta)\\ &< (\text{D–S})\int_a^b f\,dg + \varepsilon.\end{aligned}$$

Since $\sum_{i=1}^n f(\xi_i)\Delta_i g$ tends to (D–S) $\int_a^b f\,dg$ as $\varepsilon \to 0$, the (R–S) integral exists and, in fact, is equal to the (D–S) integral. ∎

Corollary

Suppose that f is continuous on $I = \{x{:}a \le x \le b\}$ and that g is of bounded variation on I. Then the Riemann–Stieltjes integral $\int_a^b f\,dg$ exists.

Problems

1. (a) Let $a < c_1 < c_2 < \dots < c_n < b$ be any points of $I = \{x : a \le x \le b\}$. Suppose that

$$g(x) = \begin{cases} 1 & \text{for } x \ne c_i,\ i = 1, 2, \dots, n,\\ d_i & \text{for } x = c_i,\ i = 1, 2, \dots, n,\end{cases}$$

   where d_i is a constant for each i.

   Suppose that f is continuous on I. Find an expression for $\int_a^b f\,dg$.

   (b) Work part (a) if $a = c_1 < c_2 < \dots < c_n = b$. [*Hint*: First do part (a) with $a = c_1 < c_2 = b$.]
2. Let $g(x) = \sin x$ for $0 \le x \le \pi$. Find the value of $\int_0^\pi x\,dg$.
3. Let $g(x) = e^{|x|}$ for $-1 \le x \le 1$. Find the value of $\int_{-1}^1 x\,dg$.
4. Let $g(x) = k$ for $k - 1 < x \le k$, $k = 1, 2, 3, \dots$. Find the value of $\int_1^4 x\,dg$.
5. Show that with g as in Problem 4, $\int_1^5 g\,dg$ does not exist.
6. Show that if f and g have a common point of discontinuity on an interval $I = \{x : a \le x \le b\}$, then $\int_a^b f\,dg$ cannot exist.
7. Suppose that $f \equiv c$ on $I = \{x : a \le x \le b\}$ where c is any constant. If g is of bounded variation on I, use integration by parts (Theorem 10.8) to show that $\int_a^b f\,dg = c[g(b) - g(a)]$.
8. Let f be continuous and g of bounded variation on $I = \{x : a \le x < \infty\}$. Define $\int_a^\infty f\,dg = \lim_{x\to\infty} \int_a^x f\,dg$ when the limit exists.

Show that if f is bounded on I and $g(x) = 1/k^2$ for $k - 1 \leq x < k$, $k = 1, 2, \ldots$, then

$$\int_1^\infty f\, dg = -\sum_{k=1}^\infty f(k+1)\left[\frac{1}{(k+1)^2} - \frac{1}{(k+2)^2}\right].$$

9. Use the function $g(x) = b_k$ for $k - 1 < x \leq k$, $k = 1, 2, \ldots, b_k$ constant, and the definition of integral over a half-infinite interval, as in Problem 8, to show that *any infinite series* $\sum_{k=1}^\infty a_k$ *may be represented as a Riemann–Stieltjes integral*.

11 CHAPTER The Implicit Function Theorem. Lagrange Multipliers

11.1 The Implicit Function Theorem

Suppose we are given a relation in $\mathbb{R}^2$ of the form

$$F(x, y) = 0. \tag{11.1}$$

Then to each value of x there may correspond one or more values of y that satisfy (11.1)—or there may be no values of y that do so. If $I = \{x : x_0 - h < x < x_0 + h\}$ is an interval such that for each $x \in I$ there is exactly one value of y satisfying (11.1), then we say that $F(x, y) = 0$ defines y as a function of x **implicitly** on I. Denoting this function by f, we have $F[x, f(x)] = 0$ for x on I.

An implicit function theorem determines conditions under which a relation such as (11.1) defines y as a function of x or x as a function of y. The solution is a local one in the sense that the size of the interval I may be much smaller than the domain of the relation F. Figure 11.1 shows the graph of a relation such as (11.1). We see that F defines y as a function of x in a region about P, but not beyond the point Q. Furthermore, the relation does not yield y as a function of x in any region containing the point Q in its interior.

The simplest example of an implicit function theorem states that if F is smooth and if P is a point at which $F_{,2}$ (that is, $\partial F/\partial y$) does not vanish, then it is possible to express y as a function of x in a region containing this point. More precisely, we have the following result.

Theorem 11.1
Suppose that F, $F_{,1}$, and $F_{,2}$ are continuous on an open set A in $\mathbb{R}^2$ containing the point $P(x_0,y_0)$, and suppose that

$$F(x_0, y_0) = 0, \qquad F_{,2}(x_0, y_0) \neq 0.$$

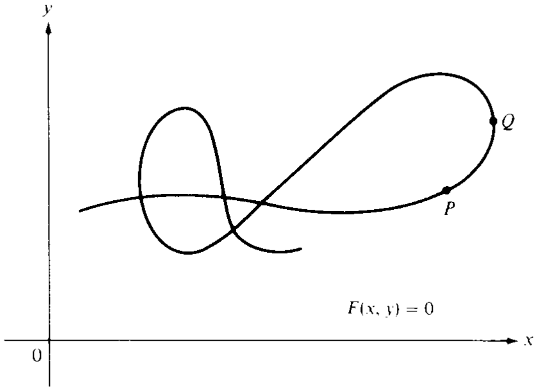

Figure 11.1

(a) *Then there are positive numbers h and k that determine a rectangle R contained in A given by*

$$R = \{(x,y):|x - x_0| < h, |y - y_0| < k\},$$

such that for each x in $I = \{x:|x - x_0| < h\}$ there is a unique number y in $J = \{y:|y - y_0| < k\}$ that satisfies the equation $F(x,y) = 0$. The totality of the points (x,y) forms a function f whose domain contains I and whose range is in J.

(b) *The function f and its derivative f' are continuous on I.*

Proof
(a) We assume $F_{,2}(x_0, y_0) > 0$; if not we replace F by $-F$ and repeat the argument. Since $F_{,2}$ is continuous, there is a (sufficiently small) square $S = \{(x, y) : |x - x_0| \leq k, |y - y_0| \leq k\}$ that is contained in A and on which $F_{,2}$ is positive. For each fixed value of x such that $|x - x_0| < k$ we see that $F(x, y)$, considered as a function of y, is an increasing function. Since $F(x_0, y_0) = 0$, it is clear that

$$F(x_0, y_0 + k) > 0 \qquad \text{and} \qquad F(x_0, y_0 - k) < 0.$$

Because F is continuous on S, there is a (sufficiently small) number h such that $F(x, y_0 + k) > 0$ on $I = \{x : |x - x_0| < h\}$ and $F(x, y_0 - k) < 0$ on I. We fix a value of x in I and examine solutions of $F(x, y) = 0$ in the rectangle R. Since $F(x, y_0 - k)$ is negative and $F(x, y_0 + k)$ is positive, there is a value $\bar{y}$ in R such that $F(x, \bar{y}) = 0$. Also, because $F_{,2} > 0$, there is precisely one such value. The correspondence $x \to \bar{y}$ is the function we seek, and we denote it by f.

(b) To show that f is continuous at x_0 let $\varepsilon > 0$ be given and suppose that ε is smaller than k. Then we may construct a square S_ε with side 2ε and center at (x_0, y_0) as in the proof of part (a). There is a value $h' < h$ such that f is a function on $I' = \{x : |x - x_0| < h'\}$. Therefore,

$$|f(x) - f(x_0)| < \varepsilon \qquad \text{whenever} \qquad |x - x_0| < h',$$

and f is continuous at x_0. At any point $x_1 \in I$, we construct a square S_1 with center at $(x_1, f(x_1))$ and repeat the above argument.

To show that f' exists and is continuous we use the Fundamental lemma on differentiation (Theorem 7.2). Let $x \in I$ and choose a number ρ such that $x + \rho \in I$. Then

$$F(x + \rho, f(x + \rho)) = 0 \qquad \text{and} \qquad F(x, f(x)) = 0.$$

Writing $f(x + \rho) = f + \Delta f$ and using Theorem 7.2, we obtain

$$[F_{,1}(x, f) + \varepsilon_1(\rho, \Delta f)]\rho + [F_{,2}(x, f) + \varepsilon_2(\rho, \Delta f)]\Delta f = 0, \tag{11.2}$$

where ε_1 and ε_2 tend to zero as $\rho, \Delta f \to 0$. From the continuity of f, which we have established, it follows that $\Delta f \to 0$ as $\rho \to 0$. From (11.2) it is clear that

$$\frac{\Delta f}{\rho} = \frac{f(x + \rho) - f(x)}{\rho} = -\frac{F_{,1}(x, f) + \varepsilon_1(\rho, \Delta f)}{F_{,2}(x, f) + \varepsilon_2(\rho, \Delta f)}.$$

Since the right side tends to a limit as $\rho \to 0$, we see that

$$f'(x) = -\frac{F_{,1}(x, f)}{F_{,2}(x, f)}. \tag{11.3}$$

By hypothesis the right side of (11.3) is continuous, and so f' is also. ■

The implicit function theorem has a number of generalizations and applications. If F is a function from $\mathbb{R}^{N+1}$ to $\mathbb{R}^1$, we may consider whether or not the relation $F(x_1, x_2, \dots, x_N, y) = 0$ defines y as a function from $\mathbb{R}^N$ into $\mathbb{R}^1$. That is, we state conditions that show that $y = f(x_1, x_2, \dots, x_N)$. The proof of the following theorem is a straightforward extension of the proof of Theorem 11.1.

Theorem 11.2

Suppose that $F, F_{,1}, F_{,2}, \dots, F_{,N+1}$ are continuous on an open set A in $\mathbb{R}^{N+1}$ containing the point $P(x_1^0, x_2^0, \dots, x_N^0, y^0)$. We use the notation $x = (x_1, x_2, \dots, x_N)$, $x^0 = (x_1^0, x_2^0, \dots, x_N^0)$ and suppose that

$$F(x^0, y^0) = 0, \qquad F_{,N+1}(x^0, y^0) \neq 0.$$

(a) *Then there are positive numbers h and k that determine a cell R contained in A given by*

$$R = \{(x,y): |x_i - x_i^0| < h, \qquad i = 1,2,\ldots,N, \quad |y - y^0| < k\},$$

such that for each x in the N-dimensional hypercube

$$I_N = \{x: |x_i - x_i^0| < h, \qquad i = 1,2,\ldots,N\}$$

there is a unique number y in the interval

$$J = \{y: |y - y^0| < k\}$$

that satisfies the equation $F(x,y) = 0$. That is, y is a function of x that may be written $y = f(x)$. The domain of f contains I_N and its range is in J.

(b) *The function f and its partial derivatives $f_{,1}, f_{,2}, \ldots, f_{,N}$ are continuous on I_N.*

A special case of Theorem 11.1 is the Inverse function theorem that was established in Chapter 4 (Theorems 4.17 and 4.18). If f is a function from $\mathbb{R}^1$ to $\mathbb{R}^1$, written $y = f(x)$, we wish to decide when it is true that x may be expressed as a function of y. Set

$$F(x, y) = y - f(x) = 0,$$

and in order for Theorem 11.1 to apply, f' must be continuous and $F_{,1} = -f'(x) \neq 0$. We state the result in the following corollary to Theorem 11.1.

Corollary (Inverse function theorem)
Suppose that f is defined on an open set A in $\mathbb{R}^1$ with values in $\mathbb{R}^1$. Also, assume that f' is continuous on A and that $f(x_0) = y_0$, $f'(x_0) \neq 0$. Then there is an interval I containing y_0 such that the inverse function of f, denoted f^{-1}, exists on I and has a continuous derivative there. Furthermore, the derivative $(f^{-1})'$ is given by the formula

$$(f^{-1}(y))' = \frac{1}{f'(x)}, \tag{11.4}$$

where $y = f(x)$.

Since $f^{-1}(f(x)) = x$, we can use the Chain rule to obtain (11.4). However, (11.4) is also a consequence of formula (11.3), with $F(x, y) = y - f(x)$, and we obtain

$$(f^{-1}(y))' = -\frac{F_{,2}}{F_{,1}} = -\frac{1}{-f'(x)}.$$

Observe that in Theorems 4.17 and 4.18 the inverse mapping is one-to-one over the entire interval in which f' does not vanish.

Example

Given the relation

$$F(x,y) = y^3 + 2x^2y - x^4 + 2x + 4y = 0, \tag{11.5}$$

show that this relation defines y as a function of x for all values of x in $\mathbb{R}^1$.

Solution. We have

$$F_{,2} = 3y^2 + 2x^2 + 4,$$

and so $F_{,2} > 0$ for all x, y. Hence for each fixed x, the function F is an increasing function of y. Furthermore, from (11.5) it follows that $F(x,y) \to -\infty$ as $y \to -\infty$, and $F(x,y) \to +\infty$ as $y \to +\infty$. Since F is continuous, for each fixed x there is exactly one value of y such that $F(x,y) = 0$. Applying Theorem 11.1, we conclude that there is a function f on $\mathbb{R}^1$ that is continuous and differentiable such that $F[x, f(x)] = 0$ for all x. ■

Problems

In each of Problems 1 through 4 show that the relation $F(x,y) = 0$ yields y as a function of x in an interval I about x_0 where $F(x_0, y_0) = 0$. Denote the function by f and compute f'.

1. $F(x,y) \equiv y^3 + y - x^2 = 0$; $(x_0, y_0) = (0, 0)$.
2. $F(x,y) \equiv x^{\frac{2}{3}} + y^{\frac{2}{3}} - 4 = 0$; $(x_0, y_0) = (1, 3\sqrt{3})$.
3. $F(x,y) \equiv xy + 2\ln x + 3\ln y - 1 = 0$; $(x_0, y_0) = (1, 1)$.
4. $F(x,y) \equiv \sin x + 2\cos y - \frac{1}{2} = 0$; $(x_0, y_0) = (\pi/6, 3\pi/2)$.
5. Give an example of a relation $F(x,y) = 0$ such that $F(x_0, y_0) = 0$ and $F_{,2}(x_0, y_0) = 0$ at a point $O = (x_0, y_0)$, and yet y is expressible as a function of x in an interval about x_0.

In each of Problems 6 through 9 show that the relation $F(x_1, x_2, y) = 0$ yields y as a function of (x_1, x_2) in a neighborhood of the given point $P(x_1^0, x_2^0, y^0)$. Denoting this function by f, compute $f_{,1}$ and $f_{,2}$ at P.

6. $F(x_1, x_2, y) \equiv x_1^3 + x_2^3 + y^3 - 3x_1x_2y - 4 = 0$; $P(x_1^0, x_2^0, y^0) = (1, 1, 2)$.
7. $F(x_1, x_2, y) \equiv e^y - y^2 - x_1^2 - x_2^2 = 0$; $P(x_1^0, x_2^0, y^0) = (1, 0, 0)$.
8. $F(x_1, x_2, y) \equiv x_1 + x_2 - y - \cos(x_1x_2y) = 0$; $P(x_1^0, x_2^0, y^0) = (0, 0, -1)$.
9. $F(x_1, x_2, y) \equiv x_1 + x_2 + y - e^{x_1x_2y} = 0$; $P(x_1^0, x_2^0, y^0) = (0, \frac{1}{2}, \frac{1}{2})$.
10. Prove Theorem 11.2.
11. Suppose that $F(x,y,z) = 0$ is such that the functions $z = f(x,y)$, $x = g(y,z)$, and $y = h(z,x)$ all exist by the Implicit function theorem. Show that

$$f_{,1} \cdot g_{,1} \cdot h_{,1} = -1.$$

This formula is frequently written

$$\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} = -1.$$

11.2 Lagrange Multipliers

Let D be a region in $\mathbb{R}^m$ and suppose that $f : D \to \mathbb{R}^1$ is a C^1 function. At any local maximum or minimum of $f(x) = f(x_1, x_2, \ldots, x_m)$ we know that $f_{,i} = 0$, $i = 1, 2, \ldots, m$. Solving these m equations in m unknowns yields the *critical points* of f. Then there are second derivative tests to determine which among these points are maxima, minima, or points of inflection. In many applications, especially in physics and engineering, we seek not the pure maxima and minima of f, but the maximum or minimum values of f when additional conditions are imposed. These are usually called *constraints*. That is, the constraints are given by a set of equations such as

(11.6)
$$\varphi^1(x_1, x_2, \ldots, x_m) = 0, \varphi^2(x_1, x_2, \ldots, x_m) = 0, \ldots, \varphi^k(x_1, x_2, \ldots, x_m) = 0.$$

Equations (11.6) are called *side conditions*. There cannot be too many side conditions. For example, if k were larger than m, then (11.6) may not have any solutions at all. Therefore, we suppose generally that k is less than m, and it is easy to see that the imposition of conditions (11.6) will change the location of the critical points of f.

The method of *Lagrange multipliers* employs a simple technique for determining the maxima and minima of f when equations (11.6) hold. The method is especially useful when it is difficult or not possible to find solutions of the system (11.6).

The Lagrange multiplier rule is frequently explained but seldom proved. In Theorem 11.4 below we establish the validity of this rule, which we now describe. We introduce k *new* variables (or parameters), denoted by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, and we form the function of $m + k$ variables

$$F(x, \lambda) = F(x_1, \ldots, x_m, \lambda_1, \ldots, \lambda_k) \equiv f(x) + \sum_{j=1}^{k} \lambda_j \varphi^j(x).$$

For this function F we compute the critical points when x is in D and λ in $\mathbb{R}^k$ *without* constraints. That is, we find solutions to the $m + k$ equations formed by all the first derivatives of $F(x, \lambda)$:

$$\text{(11.7)} \qquad \begin{cases} F_{,i} = 0, & i = 1, 2, \ldots, m, \\ F_{,j} = 0, & j = 1, 2, \ldots, k. \end{cases}$$

We shall show that the critical points of the desired solution are among the solutions of the system given by (11.7).

Suppose that f takes on its minimum at x^0, a point in the set D_0 consisting of all points x in D where the side conditions (11.6) hold. Suppose there is a function $g = (g^1, g^2, \dots, g^m)$ from $I = \{t : -t_0 < t < t_0\}$ into $\mathbb{R}^m$ that is of class C^1 and has the properties

(11.8)
$$g(0) = x^0 \qquad \text{and} \qquad \phi^j[g(t)] = 0 \qquad \text{for} \qquad j = 1, 2, \dots, k;\ t \in I.$$

Then the function $\Phi : I \to \mathbb{R}^m$ defined by

$$\Phi(t) = f[g(t)] \tag{11.9}$$

takes on its minimum at $t = 0$. Differentiating (11.8) and (11.9) with respect to t and setting $t = 0$, we get

$$\sum_{i=1}^{m} \phi^j_{,i}(x^0)\frac{dg^i(0)}{dt} = 0 \qquad \text{and} \qquad \sum_{i=1}^{m} f_{,i}(x_0)\frac{dg^i(0)}{dt} = 0. \tag{11.10}$$

Now let $h = (h_1, h_2, \dots, h_m)$ be any vector in the m-dimensional vector space V_m that is orthogonal to the k vectors $(\phi^j_{,1}(x^0), \phi^j_{,2}(x^0), \dots, \phi^j_{,m}(x^0))$, $j = 1, 2, \dots, k$. That is, suppose that

$$\sum_{i=1}^{m} \phi^j_{,i}(x^0)h_i = 0,$$

or, in vector notation,

$$\nabla\phi^j(x_0) \cdot h = 0, \qquad j = 1, 2, \dots, k.$$

From the Implicit function theorem, it follows that we may solve (11.6) for $x_1, \dots, x_k$ in terms of $x_{k+1}, \dots, x_m$, getting

$$x_i = \mu^i(x_{k+1}, \dots, x_m), \qquad i = 1, 2, \dots, k.$$

If we set $x^0 = (x_1^0, \dots, x_m^0)$ and define

$$g^i(t) = \begin{cases} \mu^i(x^0_{k+1} + th_{k+1}, \dots, x^0_m + th_m), & i = 1, 2, \dots, k, \\ x^0_i + th_i, & i = k+1, \dots, m, \end{cases}$$

then $g = (g^1(t), \dots, g^m(t))$ satisfies conditions (11.8) and (11.10). We have thereby proved the following result.

Theorem 11.3

Suppose that $f, \phi^1, \phi^2, \dots, \phi^k$ are C^1 functions on an open set D in $\mathbb{R}^m$ containing a point x^0, that the vectors $\nabla\phi^1(x^0), \dots, \nabla\phi^k(x^0)$ are linearly independent, and that f takes on its minimum among all points of D_0 at x^0, where D_0 is the subset of D on which the side conditions (11.6) *hold. If h is any vector in V_m orthogonal to $\nabla\phi^1(x^0), \dots, \nabla\phi^k(x^0)$, then*

$$\nabla f(x^0) \cdot h = 0.$$

The next result, concerning a simple fact about vectors in V_m, is needed in the proof of the Lagrange multiplier rule.

Lemma 11.1

Let $b^1, b^2, \ldots, b^k$ be linearly independent vectors in the vector space V_m. Suppose that a is a vector in V_m with the property that a is orthogonal to any vector h that is orthogonal to all the b^i. Then there are numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$a = \sum_{i=1}^{k} \lambda_i b^i.$$

That is, a is in the subspace spanned by $b^1, b^2, \ldots, b^k$.

Proof

Let B be the subspace of V_m spanned by $b^1, b^2, \ldots, b^k$. Then there are vectors $c^{k+1}, c^{k+2}, \ldots, c^m$, such that the set $b^1, \ldots, b^k, c^{k+1}, \ldots, c^m$ form a linearly independent basis of vectors in V_m. Let h be any vector orthogonal to all the b^i; then h will have components $h_1, \ldots, h_m$ in terms of the above basis with $h_1 = h_2 = \ldots = h_k = 0$. The vector a with components $(a_1, \ldots, a_m)$ and with the property $a \cdot h = 0$ for *all* such h must have $a_{i+1} = a_{i+2} = \ldots = a_m = 0$. Therefore, $a = \sum_{i=1}^{k} a_i b^i$. We set $a_i = \lambda_i$ to obtain the result. ■

Theorem 11.4 (Lagrange multiplier rule)

Suppose that $f, \phi^1, \phi^2, \ldots, \phi^k$ and x^0 satisfy the hypotheses of Theorem 11.3. Define

$$F(x, \lambda) = f(x) - \sum_{i=1}^{k} \lambda_i \phi^i(x).$$

Then there are numbers $\lambda_1^0, \lambda_2^0, \ldots, \lambda_k^0$ such that

$$F_{,x_i}(x^0, \lambda^0) = 0, \qquad i = 1, 2, \ldots, m,$$

and

$$F_{,\lambda_j}(x^0, \lambda^0) = 0, \qquad j = 1, 2, \ldots, k. \tag{11.11}$$

Proof

Equations (11.11) are

$$\nabla f(x_0) = \sum_{l=1}^{k} \lambda_l^0 \nabla \phi^l(x^0) \qquad \text{and} \qquad \phi^j(x^0) = 0, \qquad j = 1, 2, \ldots, k.$$

We set $a = \nabla f(x^0)$ and $b^j = \nabla \phi^j(x^0)$. Then Theorem 11.3 and Lemma 11.1 combine to yield the result. ■

This theorem shows that the minimum (or maximum) of f subject to the side conditions $\phi^1 = \phi^2 = \ldots = \phi^k = 0$ is among the minima (or maxima) of the function F without any constraints.

Example

Find the maximum of the function $x_1 + 3x_2 - 2x_3$ on the sphere $x_1^2 + x_2^2 + x_3^2 = 14$.

Solution. Let $F(x_1, x_2, x_3, \lambda) = x_1 + 3x_2 - 2x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 14)$. Then $F_{,1} = 1 + 2\lambda x_1$, $F_{,2} = 3 + 2\lambda x_2$, $F_{,3} = -2 + 2\lambda x_3$, $F_{,4} = x_1^2 + x_2^2 + x_3^2 - 14$. Setting $F_{,i} = 0$, $i = 1, \dots, 4$, we obtain

$$x_1 = -\frac{1}{2\lambda}, \qquad x_2 = -\frac{3}{2\lambda}, \qquad x_3 = \frac{1}{\lambda}, \qquad 14 = \frac{14}{4\lambda^2}.$$

The solutions are $(x_1, x_2, x_3, \lambda) = (1, 3, -2, -\frac{1}{2})$ or $(-1, -3, 2, \frac{1}{2})$. The first solution gives the maximum value of 14. ■

Problems

In each of Problems 1 through 10 find the solution by the Lagrange multiplier rule.

1. Find the minimum value of $x_1^2 + 3x_2^2 + 2x_3^2$ subject to the condition $2x_1 + 3x_2 + 4x_3 - 15 = 0$.
2. Find the minimum value of $2x_1^2 + x_2^2 + 2x_3^2$ subject to the condition $2x_1 + 3x_2 - 2x_3 - 13 = 0$.
3. Find the minimum value of $x_1^2 + x_2^2 + x_3^2$ subject to the conditions $2x_1 + 2x_2 + x_3 + 9 = 0$ and $2x_1 - x_2 - 2x_3 - 18 = 0$.
4. Find the minimum value of $4x_1^2 + 2x_2^2 + 3x_3^2$ subject to the conditions $x_1 + 2x_2 + 3x_3 - 9 = 0$ and $4x_1 - 2x_2 + x_3 + 19 = 0$.
5. Find the minimum value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$ subject to the condition $2x_1 + x_2 - x_3 - 2x_4 - 5 = 0$.
6. Find the minimum value of $x_1^2 + x_2^2 + x_3^2 + x_4^2$ subject to the conditions $x_1 - x_2 + x_3 + x_4 - 4 = 0$ and $x_1 + x_2 - x_3 + x_4 + 6 = 0$.
7. Find the points on the curve $4x_1^2 + 4x_1x_2 + x_2^2 = 25$ that are nearest to the origin.
8. Find the points on the curve $7x_1^2 + 6x_1x_2 + 2x_2^2 = 25$ that are nearest to the origin.
9. Find the points on the curve $x_1^4 + y_1^4 + 3x_1y_1 = 2$ that are farthest from the origin.
10. Let $b_1, b_2, \dots, b_k$ be positive numbers. Find the maximum value of $\sum_{i=1}^{k} b_i x_i$ subject to the side condition $\sum_{i=1}^{k} x_i^2 = 1$.

12 CHAPTER Vector Functions on $\mathbb{R}^N$; The Theorems of Green and Stokes

12.1 Vector Functions on $\mathbb{R}^N$

In this section we develop the basic properties of differential and integral calculus for functions whose domain is a subset of $\mathbb{R}^M$ and whose range is in a vector space V_N. We shall emphasize the coordinate-free character of the definitions and results, but we will introduce coordinate systems and use them in proofs and computations whenever convenient.

Definition
A **vector function** from a domain D in $\mathbb{R}^M$ into V_N is a mapping $\mathbf{f} : D \to V_N$. If the image is V_1 (which we identify with $\mathbb{R}^1$ in the natural way), then $\mathbf{f}$ is called a **scalar function**. Vector or scalar functions are often called **vector** or **scalar fields**. The term field is used because we frequently visualize a vector function as superimposed on $\mathbb{R}^M$. At each point of the domain D of $\mathbb{R}^M$ a representative directed line segment of $\mathbf{f}$ is drawn with base at that point. We obtain a **field** of vectors.

Let $\mathbf{f} : D \to V_N$ be a vector function where D is a domain in $\mathbb{R}^M$. An **orthonormal basis** for V_N is a set of N unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ such that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$, $i, j = 1, 2, \ldots, N$. Then for each point $P \in D$, we may represent $\mathbf{f}$ by the formula

$$\mathbf{f}(P) = \sum_{i=1}^{N} f_i(P)\mathbf{e}_i, \tag{12.1}$$

where $f_i : D \to \mathbb{R}^1$, $i = 1, 2, \ldots, N$, are scalar functions from D into $\mathbb{R}^1$. We can consider vector fields with domain in any space such as V_M or E_M (Euclidean space). Equation (12.1) illustrates what is meant in that case. We shall ordinarily assume that D is in $\mathbb{R}^M$. The function $\mathbf{f}$ is **continuous at $\boldsymbol{P}$** if each f_i, $i = 1, 2, \ldots, N$, is continuous at P. We recall that the basic properties of functions from $\mathbb{R}^M$ into $\mathbb{R}^1$ were developed in Chapters 6 and 7. Properties developed there that involve linear processes are easily transferred to functions $\mathbf{f}$ from $\mathbb{R}^M$ into V_N.

For simplicity we shall usually consider functions $\mathbf{f}$ from $\mathbb{R}^N$ into V_N, although many of the results developed below are valid with only modest changes if the dimension of the domain of $\mathbf{f}$ is different from that of its range.

The definite integral of a vector function, defined below, is similar to the integral of a scalar function on $\mathbb{R}^N$ as given in Chapter 7.

Definition

Let D be a domain in $\mathbb{R}^N$ and suppose that $\mathbf{u} : D \to V_N$ is a vector function. Let F be a subdomain in D. The function $\mathbf{u}$ is **integrable over** F if there is a vector $\mathbf{L}$ in V_N with the following property: For every $\varepsilon > 0$, there is a $\delta > 0$ such that if $\Delta = \{F_1, F_2, \ldots, F_n\}$ is any subdivision of F with mesh less than δ, and $p_1, p_2, \ldots, p_n$ is any finite set of points with p_i in F_i, then

$$\left| \sum_{i=1}^{n} \mathbf{u}(p_i)V(F_i) - \mathbf{L} \right| < \varepsilon,$$

where $V(F_i)$ is the volume of F_i. We call $\mathbf{L}$ the **integral of u over** F and we write

$$\mathbf{L} = \int_F \mathbf{u}\, dV.$$

If, for example, $\mathbf{u}$ is given in the form (12.1), so that

$$\mathbf{u}(P) = \sum_{i=1}^{N} u_i(P)\mathbf{e}_i, \tag{12.2}$$

then it is clear that $\mathbf{u}$ is integrable over F if and only if each scalar function u_i is integrable over F. Therefore, if $\mathbf{L}$ exists, it must be unique. In fact, if $\mathbf{u}$ is integrable and given by (12.2), we have

$$\int_F \mathbf{u}\, dV = \sum_{i=1}^{N} \left(\int_F u_i dV \right) \mathbf{e}_i. \tag{12.3}$$

Each coefficient on the right in (12.3) is an integral of a function from $\mathbb{R}^N$ into $\mathbb{R}^1$ as defined in Chapter 7.

Let $\mathbf{v}$ be a vector in V_N and denote by $\overrightarrow{p_0p}$ a directed line segment in $\mathbb{R}^N$ that represents $\mathbf{v}$. We use the notation $\mathbf{v}(\overrightarrow{p_0p})$ for this vector.

Definitions

Let p_0 be a point in $\mathbb{R}^N$ and let $\mathbf{a}$ be a unit vector in V_N. Suppose that $\mathbf{w} : D \to V_N$ is a vector function with domain D in $\mathbb{R}^N$. We define the **directional derivative of w in the direction a** at p_0, denoted by $D_{\mathbf{a}}\mathbf{w}(p_0)$, by the formula

$$D_{\mathbf{a}}\mathbf{w}(p_0) = \lim_{h\to 0} \frac{\mathbf{w}(p) - \mathbf{w}(p_0)}{h},$$

where the point $p \in D$ is chosen such that $\mathbf{v}(\overrightarrow{p_0p}) = h\mathbf{a}$. The vector function $\mathbf{w}$ is **continuously differentiable** *in* D if $\mathbf{w}$ and $D_{\mathbf{a}}\mathbf{w}$ are continuous on D for every unit vector $\mathbf{a}$ in V_N. We write $\mathbf{w} \in C^1(D)$.

In order to derive formulas for computing directional derivatives, we require a coordinate system in $\mathbb{R}^N$. When we write $\mathbf{w}(p)$, $p \in \mathbb{R}^N$, we indicate the coordinate-free character of the function. If a Cartesian coordinate system $(x_1, x_2, \ldots, x_N)$ is introduced, then we write $\mathbf{w}(x_1, x_2, \ldots, x_N)$ or $\mathbf{w}(x)$. If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ is an orthonormal basis in V_N, we use coordinates to write

$$\mathbf{w}(x) = \sum_{i=1}^{N} w_i(x)\mathbf{e}_i,$$

where each of the scalar functions $w_i(x)$ is now expressed in terms of a coordinate system.

The following theorem establishes a formula for obtaining the directional derivative of scalar and vector fields.

Theorem 12.1

(i) *Let* $\mathbf{a}$ *be a unit vector in* V_N. *Suppose that* $w{:}D \to \mathbb{R}^1$ *is a continuously differentiable scalar field with domain* $D \subset \mathbb{R}^N$. *If* $\mathbf{a}$ *has the representation* $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \ldots + a_N\mathbf{e}_N$, *then*

$$D_{\mathbf{a}}\mathbf{w} = \sum_{i=1}^{N} \frac{\partial w}{\partial x_i} a_i. \tag{12.4}$$

(ii) *If* $\mathbf{w}{:}D \to V_N$ *is a vector field and* $\mathbf{w} = \sum_{i=1}^{N} w_i\mathbf{e}_i$, *then*

$$D_{\mathbf{a}}\mathbf{w} = \sum_{i=1}^{N} D_{\mathbf{a}}w_i\mathbf{e}_i = \sum_{i,j=1}^{N} w_{i,j}a_j\mathbf{e}_i. \tag{12.5}$$

Proof

(i) If $\mathbf{a}$ is a unit vector in one of the coordinate directions, say x_j, then (12.4) holds, since $D_{\mathbf{a}}w$ is the partial derivative with respect to x_j. In the general case, we fix a point $x^0 = (x_1^0, \ldots, x_N^0)$ and define

$$\varphi(t) = w(x_1^0 + a_1 t, \ldots, x_N^0 + a_N t).$$

Then $D_{\mathbf{a}}w(x^0)$ is obtained by computing $\varphi'(0)$ according to the Chain rule, Theorem 7.3. The proof of (ii) is an immediate consequence of formula (12.4) and the representation of $\mathbf{w}$ in the form $\sum w_i\mathbf{e}_i$. ■

The next theorem shows that the directional derivative of a scalar function may be expressed in terms of the scalar product of the given direction $\mathbf{a}$ with a uniquely determined vector field. Observe that the result is independent of the coordinate system, although coordinates are used in the proof.

Theorem 12.2

Let $f{:}D \to \mathbb{R}^1$ be a continuously differentiable scalar field with domain D in $\mathbb{R}^N$. Then there is a unique vector field $\mathbf{w}{:}D \to V_N$ such that for each unit vector $\mathbf{a} \in V_N$ and for each $p \in D$, the directional derivative of f is given by the scalar product of $\mathbf{a}$ and $\mathbf{w}$:

$$D_{\mathbf{a}}f(p) = \mathbf{a} \cdot \mathbf{w}(p). \tag{12.6}$$

If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ is an orthonormal set of unit vectors in V_N, then $\mathbf{w}(x)$ may be computed by the formula

$$\mathbf{w}(x) = \sum_{j=1}^{N} f_{,j}(x)\mathbf{e}_j \text{ for each } x \in D. \tag{12.7}$$

Proof

From Formula (12.4), we obtain

$$D_{\mathbf{a}}f(x) = \sum_{j=1}^{N} f_{,j}(x)a_j. \tag{12.8}$$

We now define $\mathbf{w}(x)$ by (12.7), and therefore the scalar product of $\mathbf{w}$ and $\mathbf{a}$ yields (12.8). To show uniqueness, assume that $\mathbf{w}'$ is another vector field satisfying (12.6). Then $(\mathbf{w} - \mathbf{w}') \cdot \mathbf{a} = 0$ for all unit vectors $\mathbf{a}$. If $\mathbf{w} - \mathbf{w}' \neq 0$, choose $\mathbf{a}$ to be the unit vector in the direction of $\mathbf{w} - \mathbf{w}'$, in which case $(\mathbf{w} - \mathbf{w}') \cdot \mathbf{a} = |\mathbf{w} - \mathbf{w}'| \neq 0$, a contradiction. ■

Definition

The vector $\mathbf{w}$ defined by (12.6) in Theorem 12.2 is called the **gradient** *of* f and is denoted by grad f or ∇f.

If f is a scalar field from $\mathbb{R}^N$ into $\mathbb{R}^1$, then in general $f(p) =$ constant represents a hypersurface in $\mathbb{R}^N$. At any point p, a vector $\mathbf{a}$ tangent to this hypersurface at p has the property that $\sum_{i=1}^N f_{,i}(p) \cdot a_i = 0$, where $\mathbf{a} = \sum_{i=1}^N a_i\mathbf{e}_i$. We conclude that ∇f *is orthogonal to the hypersurface* $f(p) =$ constant *at each point p on the hypersurface.*

EXAMPLE 1

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal set of vectors in $V_3(\mathbb{R}^3)$. Given the scalar and vector functions

$$f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3,$$

$$\mathbf{u}(x_1, x_2, x_3) = (x_1^2 - x_2 + x_3)\mathbf{e}_1 + (2x_2 - 3x_3)\mathbf{e}_2 + (x_1 + x_3)\mathbf{e}_3$$

and the vector $\mathbf{a} = \lambda\mathbf{e}_1 + \mu\mathbf{e}_2 + \nu\mathbf{e}_3$, $\lambda^2 + \mu^2 + \nu^2 = 1$, find ∇f and $D_{\mathbf{a}}\mathbf{u}$ in terms of $x_1, x_2, x_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Solution. According to (12.7), we have

$$\nabla f = 3(x_1^2 - x_2x_3)\mathbf{e}_1 + 3(x_2^2 - x_1x_3)\mathbf{e}_2 + 3(x_3^2 - x_1x_2)\mathbf{e}_3.$$

Employing (12.5), we obtain

$$D_{\mathbf{a}}\mathbf{w} = (2x_1\lambda - \mu + \nu)\mathbf{e}_1 + (2\mu - 3\nu)\mathbf{e}_2 + (\lambda + \nu)\mathbf{e}_3.$$

■

Theorem 12.3

Suppose that f, g, and u are C^1 scalar fields with domain $D \subset \mathbb{R}^N$. Let $h:\mathbb{R}^1 \to \mathbb{R}^1$ be a C^1 function with the range of u in the domain of h. Then

$$\nabla(f + g) = \nabla f + \nabla g, \qquad \nabla(fg) = f\nabla g + g\nabla f,$$

$$\nabla\left(\frac{f}{g}\right) = \frac{1}{g^2}(g\nabla f - f\nabla g) \text{ if } g \neq 0, \qquad \nabla h(u) = h'(u)\nabla u.$$

Proof

We prove the second formula, the remaining proofs being similar. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ be an orthonormal set in V_N. Then from the formula for $\nabla(fg)$ given by (12.7) it follows that

$$\nabla(fg) = \sum_{j=1}^{N}(fg)_{,j}\mathbf{e}_j.$$

Performing the differentiations, we obtain

$$\nabla(fg) = \sum_{j=1}^{N}(fg_{,j} + f_{,j}g)\mathbf{e}_j = f\nabla g + g\nabla f.$$

■

The operator ∇ carries a continuously differentiable scalar field from $\mathbb{R}^N$ to $\mathbb{R}^1$ into a continuous vector field from $\mathbb{R}^N$ to V_N. In a Cartesian coordinate system, we may write ∇ symbolically according to the formula

$$\nabla = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\mathbf{e}_j.$$

However, the operator ∇ has a significance independent of the coordinate system. Suppose that $\mathbf{w}$ is a vector field from $\mathbb{R}^N$ into V_N that in coordinates may be written $\mathbf{w} = \sum_{j=1}^{N} w_j\mathbf{e}_j$. We define the operator $\nabla \cdot \mathbf{w}$ by the formula

$$\nabla \cdot \mathbf{w} = \sum_{j=1}^{N} \frac{\partial w_j}{\partial x_j},$$

and we shall show that this operator, called the **divergence operator**, is independent of the coordinate system (Theorem 12.4 below).

Definitions

The **support** of a scalar function f with domain D is the closure of the set of points p in D where $f(p) \neq 0$. If the support of a function is a compact subset of D, we say that f has **compact support in** D. If f is a C^n function for some nonnegative integer n with compact support, we use the symbol $f \in C_0^n(D)$ to indicate this fact.

Lemma 12.1

Let D be an open domain in $\mathbb{R}^N$ and suppose that $f \in C_0^1(D)$. Then

$$\int_D f_{,j}(x)dV = 0, \qquad j = 1, 2, \ldots, N.$$

Proof

Let S be the set of compact support of f. Then $f = 0$ and $\nabla f = 0$ on $D - S$. Let R be any hypercube that contains $\bar{D}$ in its interior. We extend the definition of f to be zero on $R - D$. Then integrating with respect to x_j first and the remaining $N - 1$ variables next, we obtain (in obvious notation)

$$\int_D f_{,j}(x)dV_N = \int_R f_{,j}(x)dV_N = \int_{R'} \left[\int_{a_j}^{b_j} f_{,j}(x)dx_j\right] dV_{N-1}$$
$$= \int_{R'} [f(b_j, x') - f(a_j, x')]dV_{N-1} = 0.$$

■

The next lemma, useful in many branches of analysis, shows that a continuous function f must vanish identically if the integral of the product of f and all arbitrary smooth functions with compact support is always zero.

Lemma 12.2

Let D be an open domain in $\mathbb{R}^N$ and suppose that the scalar function f is continuous on D. If

(12.9) $$\int_D fg dV_N = 0 \text{ for all } g \in C_0^1(D),$$

then $f \equiv 0$ on D.

Proof

We prove the result by contradiction. Suppose there is an $x^0 \in D$ such that $f(x^0) \neq 0$. We may assume $f(x^0) > 0$; otherwise, we consider $-f$. Since f is continuous and D is open, there is a ball in D of radius $3r_0$ and center x^0 on which $f > 0$. Denoting distance from x^0 by r, we define the function

$$g(x) = \begin{cases} 1 & \text{for } 0 \le r \le r_0, \\ \dfrac{1}{r_0^3}(2r - r_0)(r - 2r_0)^2 & \text{for } r_0 \le r \le 2r_0, \\ 0 & \text{for } r \ge 2r_0. \end{cases}$$

It is easily verified that $g(x) \in C_0^1(D)$ with $g(x) \ge 0$. We have

$$\int_D fg dV_N = \int_{B(x^0,3r_0)} fg dV \ge \int_{B(x^0,r_0)} fg dV > 0,$$

which contradicts (12.9). Hence $f \equiv 0$ on D. ■

We now show that the divergence operator is independent of the coordinate system.

Theorem 12.4

Let D be an open domain in $\mathbb{R}^N$ and suppose that $\mathbf{w} \in C^1(D)$ is a vector field. Then there is a unique scalar field v, continuous on D, such that for all $u \in C_0^1(D)$, we have

$$\int_D (uv + \nabla u \cdot \mathbf{w}) dV = 0. \tag{12.10}$$

Furthermore, if $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ is an orthonormal basis and if for each point $x \in D$, $\mathbf{w}$ is given by

$$\mathbf{w}(x) = \sum_{j=1}^{N} w_j(x)\mathbf{e}_j,$$

then

$$v(x) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j} w_j(x). \tag{12.11}$$

Proof

Suppose that $\mathbf{w}$ is given and that $\mathbf{e}_1, \ldots, \mathbf{e}_N$ is an orthonormal basis in some coordinate system. We *define* v by (12.11). Let u be any function in the class $C_0^1(D)$. Then, according to Lemma 12.1,

$$\int_D (uv + \nabla u \cdot \mathbf{w})dV = \int_D \sum_{j=1}^{N} \left(u \frac{\partial}{\partial x_j} w_j + \frac{\partial u}{\partial x_j} w_j \right) dV$$

$$= \sum_{j=1}^{N} \int_D \frac{\partial}{\partial x_j}(uw_j)dV = 0.$$

To show that v is unique, suppose that v' is another scalar field that satisfies (12.10). By subtraction we get

$$\int_D (v - v')udV = 0 \text{ for all } u \in C_0^1(D).$$

Thus $v' \equiv v$ according to Lemma 12.2. ■

Definition

The scalar field v determined by (12.10) in Theorem 12.4 and defined in any coordinate system by (12.11) is called the **divergence of w**. We use the notation $v = \operatorname{div} \mathbf{w}$ or $v = \nabla \cdot \mathbf{w}$. We note that v is determined in (12.10) without reference to a coordinate system, although (12.11) is used for actual computations.

Theorem 12.5

Let D be a domain in $\mathbb{R}^N$ and suppose that $\mathbf{w}, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ are C^1 vector fields on D. Let f be a C^1 scalar field on D and suppose that $c_1, c_2, \ldots, c_n$ are real numbers. Then

(a) $\operatorname{div}\left(\sum_{j=1}^n c_j \mathbf{w}_j\right) = \sum_{j=1}^n c_j \operatorname{div} \mathbf{w}_j$.
(b) $\operatorname{div}(f\mathbf{w}) = f \operatorname{div} \mathbf{w} + \nabla f \cdot \mathbf{w}$.

The result is obtained by a straightforward computation using the rules of calculus.

The vector, or cross, product of two vectors in a three-dimensional vector space allows us to introduce a new differential operator acting on smooth vector fields. We shall define the operator without reference to a coordinate system; however coordinates are used in all the customary computations. If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is any orthonormal basis for V_3, we may construct the *formal operator*

(12.12)

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \mathbf{e}_1 + \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \mathbf{e}_2 + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathbf{e}_3.$$

We note that if the symbol ∇ is replaced by a vector v with components v_1, v_2, v_3 and if the partial derivatives in (12.12) are replaced by these components, then we obtain the usual formula for the vector product of two vectors.

Theorem 12.6

Let D be any set in $\mathbb{R}^3$ and let $\mathbf{u} \in C^1(D)$ be a vector field into V_3. Then there is a unique continuous vector field $\mathbf{w}$ from D into V_3 such that

$$\text{div}(\mathbf{u} \times \mathbf{a}) = \mathbf{w} \cdot \mathbf{a} \tag{12.13}$$

for every constant vector $\mathbf{a}$. If $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ is any orthonormal basis, then the vector $\mathbf{w}$ is given by (12.12).

Proof

With the orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ given, we write $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$. The formula for the vector product yields

$$\mathbf{u} \times \mathbf{a} = (u_2a_3 - u_3a_2)\mathbf{e}_1 + (u_3a_1 - u_1a_3)\mathbf{e}_2 + (u_1a_2 - u_2a_1)\mathbf{e}_3.$$

Using (12.11) for the divergence formula, we obtain

$$\text{div}(\mathbf{u} \times \mathbf{a}) = a_3\frac{\partial u_2}{\partial x_1} - a_2\frac{\partial u_3}{\partial x_1} + a_1\frac{\partial u_3}{\partial x_2} - a_3\frac{\partial u_1}{\partial x_2} + a_2\frac{\partial u_1}{\partial x_3} - a_1\frac{\partial u_2}{\partial x_3}.$$

If we denote by w_1, w_2, w_3 the coefficients in the right side of (12.12), then

$$\text{div}(\mathbf{u} \times \mathbf{a}) = w_1a_1 + w_2a_2 + w_3a_3.$$

That is, $\text{div}(\mathbf{u}\times\mathbf{a}) = \mathbf{w}\cdot\mathbf{a}$ where $\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$. The vector $\mathbf{w}$ is unique since if $\mathbf{w}'$ were another such vector we would have $\mathbf{w} \cdot \mathbf{a} = \mathbf{w}' \cdot \mathbf{a}$ for all unit vectors $\mathbf{a}$. This fact implies that $\mathbf{w} = \mathbf{w}'$. ■

Definition

The vector $\mathbf{w}$ in Theorem 12.6 is called the **curl of u** and is denoted by curl $\mathbf{u}$ and $\nabla \times \mathbf{u}$.

The elementary properties of the curl operator are given in the next theorem.

Theorem 12.7

Let D be a domain in $\mathbb{R}^3$ and suppose that $\mathbf{u},\mathbf{v},\mathbf{u}_1,\dots,\mathbf{u}_n$ are C^1 vector fields from D into V_3. Let f be a C^1 scalar field on D and $c_1,\dots,c_n$ real numbers. Then

(a) $\text{curl}\,(\sum_{j=1}^n c_j\mathbf{u}_j) = \sum_{j=1}^n c_j(\text{curl}\,\mathbf{u}_j)$.
(b) $\text{curl}\,(f\mathbf{u}) = f\,\text{curl}\,\mathbf{u} + \nabla f \times \mathbf{u}$.
(c) $\text{div}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}$.
(d) $\text{curl}\,\nabla f = \mathbf{0}$ *provided that* $\nabla f \in C^1(D)$.
(e) $\text{div}\,\text{curl}\,\mathbf{v} = 0$ *provided that* $\mathbf{v} \in C^2(D)$.

Proof

We shall establish Part (b), the remaining proofs being similar. Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis and set $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$. Then

$$\begin{aligned}\operatorname{curl}(f\mathbf{u}) = &\left(\frac{\partial(fu_3)}{\partial x_2} - \frac{\partial(fu_2)}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial(fu_1)}{\partial x_3} - \frac{\partial(fu_3)}{\partial x_1}\right)\mathbf{e}_2 \\ &+ \left(\frac{\partial(fu_2)}{\partial x_1} - \frac{\partial(fu_1)}{\partial x_2}\right)\mathbf{e}_3.\end{aligned}$$

Therefore,

$$\begin{aligned}\operatorname{curl}(f\mathbf{u}) &= f \operatorname{curl}\mathbf{u} + \left(u_3\frac{\partial f}{\partial x_2} - u_2\frac{\partial f}{\partial x_3}\right)\mathbf{e}_1 \\ &+ \left(u_1\frac{\partial f}{\partial x_3} - u_3\frac{\partial f}{\partial x_1}\right)\mathbf{e}_2 + \left(u_2\frac{\partial f}{\partial x_1} - u_1\frac{\partial f}{\partial x_2}\right)\mathbf{e}_3.\end{aligned} \tag{12.14}$$

Recalling that $\nabla f = (\partial f/\partial x_1)\mathbf{e}_1 + (\partial f/\partial x_2)\mathbf{e}_2 + (\partial f/\partial x_3)\mathbf{e}_3$ and using the formula for the vector product as given by (12.12), we see that (12.14) is precisely

$$\operatorname{curl}(f\mathbf{u}) = f \operatorname{curl}\mathbf{u} + \nabla f \times \mathbf{u}.$$

■

Part (d) in Theorem 12.7 states that if a vector $\mathbf{u}$ is the gradient of a scalar function f, that is, if $\mathbf{u} = \nabla f$, then $\operatorname{curl}\mathbf{u} = \mathbf{0}$. It is appropriate to ask whether or not the condition $\operatorname{curl}\mathbf{u} = \mathbf{0}$ implies that $\mathbf{u}$ is the gradient of a scalar function f. Under certain conditions this statement is valid. In any Cartesian coordinate system, the condition $\mathbf{u} = \nabla f$ becomes

$$u_1 = \frac{\partial f}{\partial x_1}, \qquad u_2 = \frac{\partial f}{\partial x_2}, \qquad u_3 = \frac{\partial f}{\partial x_3},$$

or

$$df = u_1 dx_1 + u_2 dx_2 + u_3 dx_3. \tag{12.15}$$

Then the statement $\operatorname{curl}\mathbf{u} = \mathbf{0}$ asserts that the right side of (12.15) is an exact differential.

Example 2

Let the vector field $\mathbf{u}$ be given (in coordinates) by

$$\mathbf{u} = 2x_1x_2x_3\mathbf{e}_1 + (x_1^2x_3 + x_2)\mathbf{e}_2 + (x_1^2x_2 + 3x_3^2)\mathbf{e}_3.$$

Verify that $\operatorname{curl}\mathbf{u} = \mathbf{0}$ and find the function f such that $\nabla f = \mathbf{u}$.

Solution. Computing $\nabla \times \mathbf{u}$ by formula (12.12), we get

$$\operatorname{curl} \mathbf{u} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ 2x_1x_2x_3 & x_1^2x_3 + x_2 & x_1^2x_2 + 3x_3^2 \end{vmatrix}$$

$$= (x_1^2 - x_1^2)\mathbf{e}_1 + (2x_1x_2 - 2x_1x_2)\mathbf{e}_2 + (2x_1x_3 - 2x_1x_3)\mathbf{e}_3 = \mathbf{0}.$$

We seek the function f such that

$$\frac{\partial f}{\partial x_1} = 2x_1x_2x_3, \qquad \frac{\partial f}{\partial x_2} = x_1^2x_3 + x_2, \qquad \frac{\partial f}{\partial x_3} = x_1^2x_2 + 3x_3^2.$$

Integrating the first equation, we obtain

$$f(x_1, x_2, x_3) = x_1^2x_2x_3 + C(x_2, x_3).$$

Differentiating this expression with respect to x_2 and x_3, we obtain

$$\frac{\partial f}{\partial x_2} = x_1^2x_3 + \frac{\partial C}{\partial x_2} = x_1^2x_3 + x_2 \qquad \text{and} \qquad \frac{\partial f}{\partial x_3} = x_1^2x_2 + \frac{\partial C}{\partial x_3} = x_1^2x_2 + 3x_3^2.$$

Thus it follows that

$$\frac{\partial C}{\partial x_2} = x_2, \qquad \frac{\partial C}{\partial x_3} = 3x_3^2, \qquad \text{or} \qquad C = \frac{1}{2}x_2^2 + K(x_3),$$

where $K'(x_3) = 3x_3^2$. Therefore, $C(x_2, x_3) = \frac{1}{2}x_2^2 + x_3^3 + K_1$ with K_1 a constant. Hence

$$f(x_1, x_2, x_3) = x_1^2x_2x_3 + \frac{1}{2}x_2^2 + x_3^3 + K_1.$$

■

Problems

1. Suppose that $\mathbf{u} : \mathbb{R}^2 \to V_2$ is given by $\mathbf{u}(x_1, x_2) = (x_1^2 - x_2^2)\mathbf{e}_1 + 2x_1x_2\mathbf{e}_2$. Find $\int_F \mathbf{u}dv$, where $F = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$.

In each of Problems 2 through 5 express $\nabla f(x)$ in terms of (x_1, x_2, x_3) and $(\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_3)$ and compute $D_{\mathbf{a}}f(\bar{x})$, where $\mathbf{a}$ is the given unit vector and $\bar{x}$ is the given point.

2. $f(x) = 2x_1^2 + x_2^2 - x_1x_3 - x_3^2$, $\mathbf{a} = \frac{1}{3}(2\mathbf{e}_1 - 2\mathbf{e}_2 - \mathbf{e}_3)$, $\bar{x} = (1, -1, 2)$.
3. $f(x) = x_1^2 + x_1x_2 - x_2^2 + x_3^2$, $\mathbf{a} = \frac{1}{7}(3\mathbf{e}_1 + 2\mathbf{e}_2 - 6\mathbf{e}_3)$, $\bar{x} = (2, 1, -1)$.
4. $f(x) = e^{x_1}\cos x_2 + e^{x_2}\cos x_3$, $\mathbf{a} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3)$, $\bar{x} = (1, \pi, -1/2)$.
5. $f(x) = x_1^2\log(1 + x_2^2) - x_3^3$, $\mathbf{a} = \frac{1}{\sqrt{10}}(3\mathbf{e}_1 + \mathbf{e}_3)$, $\bar{x} = (1, 0, -2)$.

In each of Problems 6 and 7 express $D_{\mathbf{a}}\mathbf{w}(x)$ in terms of (x_1, x_2, x_3) and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. Find the value at $\bar{x}$ as given.

6. $\mathbf{w}(x) = x_2x_3\mathbf{e}_1 + x_1x_3\mathbf{e}_2 + x_1x_2\mathbf{e}_3$, $\mathbf{a} = \frac{1}{3}(\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3)$, $\bar{x} = (1, 2, -1)$.

7. $\mathbf{w}(x) = (x_1 - 2x_2)\mathbf{e}_1 + x_2x_3\mathbf{e}_2 - (x_2^2 - x_3^2)\mathbf{e}_3$, $\mathbf{a} = \frac{1}{\sqrt{14}}(3\mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3)$, $\bar{x} = (2, -1, 3)$.
8. Prove Theorem 12.5.

In each of Problems 9 through 13 find the value of div $\mathbf{v}(x)$ at the given point $\bar{x}$.

9. $\mathbf{v}(x) = x_1x_2\mathbf{e}_1 + x_3^2\mathbf{e}_2 - x_1^2\mathbf{e}_3$, $\quad \bar{x} = (1, 0, 1)$.
10. $\mathbf{v}(x) = (x_1^2 - x_2x_3)\mathbf{e}_1 + (x_2^2 - x_1x_3)\mathbf{e}_2 + (x_3^2 - x_1x_2)\mathbf{e}_3$, $\quad \bar{x} = (2, -1, 1)$.
11. $\mathbf{v}(x) = \nabla u$, $\quad u(x) = 3x_1x_2^2 - x_2^3 + x_3$, $\quad \bar{x} = (-1, 1, 2)$.
12. $\mathbf{v}(x) = r^{-n}\mathbf{r}$, $\quad \mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, $\quad r = |\mathbf{r}|$, $\quad \bar{x} = (2, 1, -2)$.
13. $\mathbf{v} = \mathbf{a} \times \mathbf{r}$, $\quad \mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, $\quad \mathbf{a}$ = constant vector, $\bar{x} = (x_1^0, x_2^0, x_3^0)$.

In each of Problems 14 through 16, find curl $\mathbf{v}$ in terms of (x_1, x_2, x_3) and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. If curl $\mathbf{v} = \mathbf{0}$, find the function f such that $\nabla f = \mathbf{v}$.

14. $\mathbf{v}(x) = (x_1^2 + x_2^2 + x_3^2)^{-1}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3)$.
15. $\mathbf{v}(x) = (x_1^2 + x_2x_3)\mathbf{e}_1 + (x_2^2 + x_1x_3)\mathbf{e}_2 + (x_3^2 + x_1x_2)\mathbf{e}_3$.
16. $\mathbf{v}(x) = e^{x_1}(\sin x_2 \cos x_3\mathbf{e}_1 + \sin x_2 \sin x_3\mathbf{e}_2 + \cos x_2\mathbf{e}_3)$.

12.2 Line Integrals in $\mathbb{R}^N$

Let $I = \{t : a \le t \le b\}$ be an interval and $\mathbf{f}$ a vector function with domain I and range D, a subset of V_N. We consider in $\mathbb{R}^N$ the directed line segments having base at the origin 0 that represent the vectors $\mathbf{f}$ in D. The heads of these directed line segments trace out a curve in $\mathbb{R}^N$, which we denote by C. If $\mathbf{f}$ is continuous and the curve C has finite length, then we say that C is a *rectifiable path*.

Let $\mathbf{g}$ be a continuous vector function from C into V_N. If $\mathbf{f}$, as defined above, is rectifiable, then we can determine the *Riemann–Stieltjes* integral of $\mathbf{g}$ with respect to $\mathbf{f}$. To do so we introduce a coordinate system with an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ in V_N, although the formula we shall obtain will be independent of the coordinate system. We write

$$\begin{aligned} &\mathbf{g}(x) = g_1(x)\mathbf{e}_1 + \ldots + g_N(x)\mathbf{e}_N, \\ &\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + \ldots + f_N(t)\mathbf{e}_N, \qquad x \in C,\ t \in I. \end{aligned} \tag{12.16}$$

If $\mathbf{f}$ is rectifiable, then the functions $f_i : I \to \mathbb{R}^1$, $i = 1, 2, \ldots, N$, are continuous and of bounded variation. Also, if $\mathbf{g}$ is continuous, then the functions $g_i : C \to \mathbb{R}^N$, $i = 1, 2, \ldots, N$, are continuous. Using the notation $g_i[\mathbf{f}(t)]$ for $g_i[f_1(t), \ldots, f_N(t)]$, we observe that the following Riemann–Stieltjes integrals exist:

$$\int_a^b g_i[\mathbf{f}(t)]df_i(t), \qquad i = 1, 2, \ldots, N. \tag{12.17}$$

We now establish the basic theorem for the existence of a Riemann–Stieltjes integral of a vector function $\mathbf{g}$ with respect to a vector function $\mathbf{f}$.

Theorem 12.8
Suppose that $\mathbf{f}: I \to V_N$ is a vector function and that the range C in $\mathbb{R}^N$ as defined above by the vector $\mathbf{r}(t) = \mathbf{v}(\overrightarrow{OP})$ is a rectifiable path. Let $\mathbf{g}$ be a continuous vector field from C into V_N. Then there is a number L with the following property: For every $\varepsilon > 0$, there is a $\delta > 0$ such that for any subdivision $\Delta: a = t_0 < t_1 < \ldots < t_n = b$ with mesh less than δ and any choices of ξ_i with $t_{i=1} \le \xi_i \le t_i$, it follows that

$$\left| \sum_{i=1}^{n} \mathbf{g}[\mathbf{f}(\xi_i)] \cdot [\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})] - L \right| < \varepsilon. \tag{12.18}$$

The number L is unique.

Proof
We introduce an orthonormal set $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ in V_N and write $\mathbf{g}$ and $\mathbf{f}$ in the form (12.16). Then we set

$$L = \sum_{i=1}^{N} \int_a^b g_i[\mathbf{f}(t)] df_i(t),$$

and each of the integrals exists. Replacing each integral by its Riemann sum, we obtain inequality (12.18). ■

Definition
We write $L = \int_a^b \mathbf{g}[\mathbf{f}(t)] \cdot d\mathbf{f}(t)$ and we call this number the **Riemann–Stieltjes integral of g with respect to f**.

EXAMPLE 1
Given $\mathbf{f}$ and $\mathbf{g}$ defined by

$$\mathbf{f}(t) = t^2\mathbf{e}_1 + 2t\mathbf{e}_2 - t\mathbf{e}_3, \qquad \mathbf{g}(x) = (x_1^2 + x_3)\mathbf{e}_1 + x_1x_3\mathbf{e}_2 + x_1x_2\mathbf{e}_3,$$

find $\int_0^1 \mathbf{g} \cdot d\mathbf{f}$.

Solution. We have

$$\begin{aligned}\int_0^1 \mathbf{g}[\mathbf{f}(t)] \cdot d\mathbf{f}(t) &= \int_0^1 [(t^4 - t)\mathbf{e}_1 + t^2(-t)\mathbf{e}_2 + t^2(2t)\mathbf{e}_3] \\ &\qquad \cdot [d(t^2\mathbf{e}_1 + 2t\mathbf{e}_2 - t\mathbf{e}_3)] \\ &= \int_0^1 [(t^4 - t) \cdot 2t + t^2(-t) \cdot 2 + t^2(2t)(-1)]dt = -\frac{4}{3}.\end{aligned}$$

■

Definitions

Let $\mathbf{f} : I \to V_N$ define an arc C in $\mathbb{R}^N$ by means of the vector $\mathbf{r}(t) = \mathbf{v}(\overrightarrow{OP})$, where $\overrightarrow{OP}$ is a representation of $\mathbf{f}$. It is intuitively clear that $\mathbf{f}$ may traverse the arc C in one direction or its opposite direction. When we associate one of the two possible directions to the arc C, we say that we have a **directed arc** and denote it by $\overrightarrow{C}_1$. The arc directed in the opposite direction, say $\overrightarrow{C}_2$, satisfies the relation $\overrightarrow{C}_2 = -\overrightarrow{C}_1$.

The Riemann–Stieltjes integral along a directed arc $\overrightarrow{C}$ with radius vector $\mathbf{r}$ is defined by the formula

$$\int_{\overrightarrow{C}} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{g}[\mathbf{f}(t)] \cdot d\mathbf{f}(t), \tag{12.19}$$

where $\mathbf{f}$ is a parametric representation of $\overrightarrow{C}$. It is not difficult to see that the integral along a directed arc as in (12.19) depends only on $\mathbf{g}$ and $\overrightarrow{C}$ and not on the particular parametric representation of $\overrightarrow{C}$. Also, it follows at once that

$$\int_{-\overrightarrow{C}} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{r} = -\int_{\overrightarrow{C}} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{r}.$$

A directed arc $\overrightarrow{C}$ may be decomposed into the union of directed subarcs. For example, if

$$\overrightarrow{C} = \overrightarrow{C}_1 + \overrightarrow{C}_2 + \ldots + \overrightarrow{C}_n,$$

then each subarc $\overrightarrow{C}_k$ would be determined by a function f_k. The following result is an immediate consequence of the basic properties of Riemann–Stieltjes integrals.

Theorem 12.9

(a) *Suppose that $\overrightarrow{C}$ is a rectifiable arc and that $\overrightarrow{C} = \overrightarrow{C}_1 + \overrightarrow{C}_2 + \ldots + \overrightarrow{C}_n$. If $\mathbf{g}$ is continuous on $\overrightarrow{C}$, then*

$$\int_{\overrightarrow{C}} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{r} = \sum_{k=1}^{n} \int_{\overrightarrow{C}_k} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{r}. \tag{12.20}$$

(b) *Suppose that $\overrightarrow{C}$ has a piecewise smooth representation $\mathbf{f}$ and that $\mathbf{g}$ is continuous on $|\overrightarrow{C}|$. Then*

$$\int_{\overrightarrow{C}} \mathbf{g}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{g}[\mathbf{f}(t)] \cdot \mathbf{f}'(t)dt. \tag{12.21}$$

Remark

If $\mathbf{f}$ *is piecewise smooth, then the integral on the right in* (12.21) *may be evaluated by first decomposing* $\overrightarrow{C}$ *into subarcs, each of which has a smooth representation, and then using* (12.20) *to add up the integrals evaluated by a smooth* $\mathbf{f}$ *on the individual subarcs. We give an illustration.*

EXAMPLE 2

In V_3, let $\mathbf{g} = 2x_1\mathbf{e}_1 - 3x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ and define the arc $\overrightarrow{C}$ as the union $\overrightarrow{C}_1 + \overrightarrow{C}_2$, where $\overrightarrow{C}_1$ is the directed line segment from (1, 0, 1) to (2, 0, 1), and $\overrightarrow{C}_2$ is the directed line segment from (2, 0, 1) to (2, 0, 4) (see Figure 12.1). Find the value of $\int_{\overrightarrow{C}} \mathbf{g}(\mathbf{f}) \cdot d\mathbf{r}$.

Solution. $\overrightarrow{C}_1 = \{\mathbf{r} : \mathbf{r} = x_1\mathbf{e}_1 + \mathbf{e}_3,\ 1 \le x_1 \le 2\}$ and $\overrightarrow{C}_2 = \{\mathbf{r} : \mathbf{r} = 2\mathbf{e}_1 + x_3\mathbf{e}_3,\ 1 \le x_3 \le 4\}$. On $\overrightarrow{C}_1$, we have

$$d\mathbf{r} = (dx_1)\mathbf{e}_1$$

and on $\overrightarrow{C}_2$,

$$d\mathbf{r} = (dx_3)\mathbf{e}_3.$$

Therefore,

$$\int_{\overrightarrow{C}_1} \mathbf{g} \cdot d\mathbf{r} = \int_1^2 2x_1 dx_1 = 3 \text{ and } \int_{\overrightarrow{C}_2} \mathbf{g} \cdot d\mathbf{r} = \int_1^4 x_3 dx_3 = \frac{15}{2}.$$

We conclude that

$$\int_{\overrightarrow{C}} \mathbf{g} \cdot d\mathbf{r} = \frac{21}{2}.$$

■

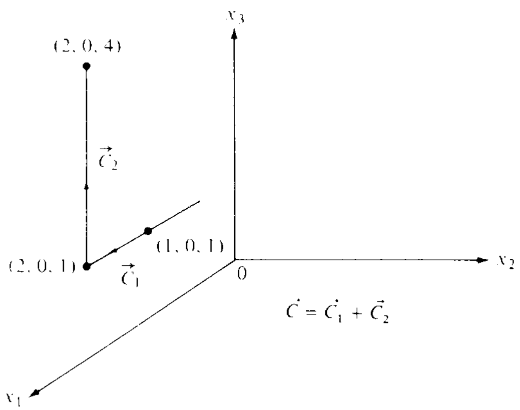

Figure 12.1

If a function $\mathbf{g}$ is continuous on a domain D in $\mathbb{R}^N$, then $\int_{\overrightarrow{C}} \mathbf{g} \cdot d\mathbf{r}$ will depend on the path $\overrightarrow{C}$ chosen in D. However, there are certain situations in which the value of the integral will be the same for all paths $\overrightarrow{C}$ in D that have the same endpoints. Under such circumstances we say that *the* **integral is independent of the path**. The next result illustrates this fact.

Theorem 12.10
Suppose that u is a continuously differentiable scalar field on a domain D in $\mathbb{R}^N$ and that p,q are points of D. Let $\overrightarrow{C}$ be any smooth path with representation $\mathbf{f}$ such that $\mathbf{f}(t) \in D$ for all t on $I = \{t{:}a \leq t \leq b\}$, the domain of $\mathbf{f}$. Suppose that $\mathbf{f}(a) = p$, $\mathbf{f}(b) = q$. Then

$$\int_{\overrightarrow{C}} \nabla u \cdot d\mathbf{r} = u(q) - u(p).$$

Proof
Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N$ be an orthonormal basis. Define

$$G(t) = u[\mathbf{f}(t)] \text{ for } t \in I,$$

with $\mathbf{f}(t) = f_1(t)\mathbf{e}_1 + \ldots + f_N(t)\mathbf{e}_N$. Using the Chain rule we obtain

$$G'(t) = \sum_{i=1}^{N} u_{,i}[\mathbf{f}(t)]f_i'(t) = \nabla u[\mathbf{f}(t)] \cdot \mathbf{f}'(t).$$

Hence

$$\int_{\overrightarrow{C}} \nabla u \cdot d\mathbf{r} = \int_a^b \nabla u[\mathbf{f}(t)] \cdot \mathbf{f}'(t)dt = \int_a^b G'(t)dt = G(b) - G(a)$$
$$= u(q) - u(p).$$

■

Next we establish a converse of Theorem 12.10.

Theorem 12.11
Let v be a continuous vector field with domain D in $\mathbb{R}^N$ and range in V_N. Suppose that for every smooth arc $\overrightarrow{C}$ lying entirely in D, the value of

$$\int_{\overrightarrow{C}} \mathbf{v} \cdot d\mathbf{r}$$

is independent of the path. Then there is a continuously differentiable scalar field u on D such that $\nabla u(p) = \mathbf{v}(p)$ for all $p \in D$.

Proof

Let p_0 be a fixed point of D and suppose that $\overrightarrow{C}$ is any smooth path in D from p_0 to a point p. Define

$$u(p) = \int_{\overrightarrow{C}} \mathbf{v} \cdot d\mathbf{r},$$

which, because of the hypothesis on $\mathbf{v}$, does not depend on $\overrightarrow{C}$. Let p_1 be any point of D, and $\overrightarrow{C}_0$ an arc from p_0 to p_1. Extend $\overrightarrow{C}_0$ at p_1 by adding a straight line segment $\overrightarrow{L}$ that begins at p_1 in such a way that the extended arc $\overrightarrow{C}_0 + \overrightarrow{L}$ is smooth. Denote by $\mathbf{a}$ the unit vector in the direction $\overrightarrow{L}$. We introduce a coordinate system in $\mathbb{R}^N$ and designate the coordinates of p_1 by x^0. Then any point q on $\overrightarrow{L}$ will have coordinates $x^0 + t\mathbf{a}$ for $t \in \mathbb{R}^1$. Thus, if $h > 0$, we obtain

$$\frac{1}{h}[u(x^0 + h\mathbf{a}) - u(x^0)] = \frac{1}{h}\int_{x^0}^{x^0+h\mathbf{a}} \mathbf{v}(t) \cdot \mathbf{a}dt.$$

Therefore,

$$\lim_{h\to 0^+} \frac{1}{h}[u(x^0 + h\mathbf{a}) - u(x^0)] = \mathbf{v}(x^0) \cdot \mathbf{a}$$

$$\begin{aligned}\lim_{h\to 0^-} \frac{1}{h}[u(x^0 + h\mathbf{a}) - u(x^0)] &= -\lim_{k\to 0} \frac{1}{k}[u(x^0 - k\mathbf{a}) - u(x^0)] \\ &= -\mathbf{v}(x^0) \cdot (-\mathbf{a}) = \mathbf{v}(x^0) \cdot \mathbf{a}.\end{aligned}$$

This procedure can be carried out for any arc $\overrightarrow{C}_0$ passing through p_1 with $\mathbf{a}$ a unit vector in any direction. From Theorem 12.2 it follows that $\nabla u = \mathbf{v}$ at x^0, any point of D. ■

Suppose that u is a smooth scalar field in $\mathbb{R}^3$. We saw earlier that if $\mathbf{v} = \nabla u$, then it follows that curl $\mathbf{v} = \mathbf{0}$. On the other hand, if $\mathbf{v}$ is a given vector field such that curl $\mathbf{v} = \mathbf{0}$ in a domain D, it is not necessarily true that $\mathbf{v}$ is the gradient of a scalar field in D. In fact, the following example shows that the integral of $\mathbf{v}$ may not be path-independent, in which case $\mathbf{v}$ cannot be the gradient of a scalar field. To see this, we set

$$\mathbf{v} = (x_1^2 + x_2^2)^{-1}(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2 + 0 \cdot \mathbf{e}_3)$$

with $D = \{(x_1, x_2, x_3) : \frac{1}{2} \le x_1^2 + x_2^2 \le \frac{3}{2},\ -1 \le x_3 \le 1\}$. We choose for $\overrightarrow{C}$ the path $\mathbf{r}(t) = (\cos t)\mathbf{e}_1 + (\sin t) \cdot \mathbf{e}_2 + 0 \cdot \mathbf{e}_3$, $0 \le t \le 2\pi$. Then

$$\int_{\overrightarrow{C}} v \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi.$$

However, if the integral were independent of the path, its value should be zero, since the initial and terminal points of $\overrightarrow{C}$ are the same. The difficulty arises because the path $\overrightarrow{C}$ encloses a singularity of $\mathbf{v}$ on the line $x_1 = x_2 = 0$, whereas the cylindrical domain D does not contain this line.

If a smooth vector field $\mathbf{v}$ is defined in a domain that is not merely connected but also **simply connected**, then it follows that if curl $\mathbf{v} = \mathbf{0}$, then $\mathbf{v}$ is the gradient of a scalar field u. We say (informally) that a domain D is **simply connected** if any two paths situated in D that have the same starting and ending points can be deformed continuously one into the other without leaving D. More precisely, we state without proof the following result.

Theorem 12.12
Suppose that $\mathbf{v}$ *is a continuously differentiable vector field on a simply connected domain* D *in* $\mathbb{R}^3$ *and that* curl $\mathbf{v} = \mathbf{0}$ *on* D. *Then there is a continuously differentiable scalar field* u *on* D *such that* $\mathbf{v} = \nabla u$.

Theorem 12.12 can be extended to functions $\mathbf{v}$ from a domain D in $\mathbb{R}^N$ to V_N. If curl $\mathbf{v} = \mathbf{0}$ is replaced by the condition

$$v_{i,j} - v_{j,i} = 0, \qquad i, j = 1, 2, \ldots, N,$$

where $\mathbf{v} = v_1\mathbf{e}_1 + \ldots + v_N\mathbf{e}_N$, then $\mathbf{v}$ is the gradient of a scalar field u provided that the domain D is simply connected.

Problems

In each of Problems 1 through 8 assume that $\mathbf{g}$ is a continuous vector field from a domain D in $\mathbb{R}^N$ into V_N for the appropriate value of N. The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$ form an orthonormal basis for a coordinate system. Compute $\int_{\overrightarrow{C}} \mathbf{g} \cdot d\mathbf{r}$.

1. $\mathbf{g} = x_1x_3\mathbf{e}_1 - x_2\mathbf{e}_2 + x_1\mathbf{e}_3$; $\overrightarrow{C}$ is the directed line segment from $(0, 0, 0)$ to $(1, 1, 1)$.
2. $\mathbf{g} = x_1x_3\mathbf{e}_1 - x_2\mathbf{e}_2 + x_1\mathbf{e}_3$; $\overrightarrow{C}$ is the directed arc given by $\mathbf{f}(t) = t\mathbf{e}_1 + t^2\mathbf{e}_2 + t^3\mathbf{e}_3$, $0 \le t \le 1$, from 0 to 1.
3. $\mathbf{g} = -x_1\mathbf{e}_1 + x_2\mathbf{e}_2 - x_3\mathbf{e}_3$; $\overrightarrow{C}$ is the helix given by $\mathbf{f}(t) = (\cos t)\mathbf{e}_1 + (\sin t)\mathbf{e}_2 + (t/\pi)\mathbf{e}_3$, $0 \le t \le 2\pi$, from 0 to 2π.
4. $\mathbf{g} = -x_1\mathbf{e}_1 + x_2\mathbf{e}_2 - x_3\mathbf{e}_3$; $\overrightarrow{C}$ is the directed line segment from $(1, 0, 0)$ to $(1, 0, 2)$.
5. $\mathbf{g} = x_1^2\mathbf{e}_1 + x_1x_2\mathbf{e}_2 + 0 \cdot \mathbf{e}_3$; $\overrightarrow{C} = \{(x_1, x_2, x_3) : x_2 = x_1^2, x_3 = 0, 0 \le x_1 \le 1$, from $(0, 0, 0)$ to $(1, 1, 0)\}$.
6. $\mathbf{g} = 2x_2x_3\mathbf{e}_2 + (x_3^2 - x_2^2)\mathbf{e}_3$; $\overrightarrow{C}$ is the shorter circular arc given by $x_1 = 0$, $x_2^2 + x_3^2 = 4$, from $(0, 2, 0)$ to $(0, 0, 2)$.
7. $\mathbf{g} = x_1x_4\mathbf{e}_1 + x_2\mathbf{e}_2 - x_2x_4\mathbf{e}_3 + x_3\mathbf{e}_4$; $\overrightarrow{C}$ is the directed straight line segment from $(0, 0, 0, 0)$ to (a_1, a_2, a_3, a_4).

8. $\mathbf{g} = \sum_{i=1}^N x_i^2 \mathbf{e}_i$; $\overrightarrow{C}$ is the directed line segment given by $\mathbf{f}(t) = \sum_{i=1}^N \alpha_i t \mathbf{e}_i$, $0 \le t \le 1$ from 0 to 1; $\{\alpha_i\}$ are constants.
9. Given the scalar function $u(x) = x_1^2 + 2x_2^2 - x_3^2 + x_4$ in $\mathbb{R}^4$, verify that $\int_{\overrightarrow{C}} \nabla u \cdot d\mathbf{r}$ is independent of the path by computing the value of the integral along $\overrightarrow{C}_1$, the straight line segment from $(0, 0, 0, 0)$ to $(1, 1, 1, 1)$, and then along $\overrightarrow{C}_2$, the straight segment from $(0, 0, 0, 0)$ to $(1, 0, 0, 0)$ followed by the straight segment from $(1, 0, 0, 0)$ to $(1, 1, 1, 1)$. Show that the two values are the same.
10. Given the scalar function $u(x) = \sum_{i=1}^N x_i^2$ in $\mathbb{R}^N$, let $\overrightarrow{C}_1$ be the line segment from $(0, 0, \dots, 0)$ to $(1, 1, \dots, 1)$ and $\overrightarrow{C}_2$ the path $\mathbf{f}(t) = t^2\mathbf{e}_1 + \sum_{i=2}^N t\mathbf{e}_i$, $0 \le t \le 1$. Verify that $\int_{\overrightarrow{C}} \nabla u \cdot d\mathbf{r}$ is independent of the path by computing this integral along $\overrightarrow{C}_1$ and $\overrightarrow{C}_2$ and showing that the values are the same.

12.3 Green's Theorem in the Plane

Green's theorem is an extension to the plane of the Fundamental theorem of calculus. We recall that this Fundamental theorem states that if I is an interval in $\mathbb{R}^1$ and $f : I \to \mathbb{R}^1$ is a continuously differentiable function then for any points a and b in I we have the formula

$$\int_a^b f'(x)dx = f(b) - f(a). \tag{12.22}$$

In the extension of this theorem to the plane we suppose that F is a domain in $\mathbb{R}^2$ and C is its boundary. Then Green's theorem is a formula that connects the line integral of a vector function over C with the double integral of the derivative of the function taken over the domain F.

Let D be a region in $\mathbb{R}^2$ with a boundary that is a simple closed curve. We denote the unoriented boundary by ∂D and the boundary oriented in a counterclockwise direction by $\overrightarrow{\partial D}$. The symbol $-\overrightarrow{\partial D}$ is used for the clockwise orientation.

We shall first establish Green's theorem for regions in $\mathbb{R}^2$ that have a special simple shape. Then we shall show how the result for these special regions can be used to yield the same formula for general regions in the plane.

Definitions

Suppose D is a region in $\mathbb{R}^2$ and $\mathbf{v} : D \to V_2$ a vector function. We introduce a coordinate system in V_2 with orthonormal basis $\mathbf{e}_1$ and $\mathbf{e}_2$. There are two possible orientations for such a basis: $\mathbf{e}_2$ is obtained from $\mathbf{e}_1$ by a

90° rotation in the counterclockwise direction, or $\mathbf{e}_2$ is obtained from $\mathbf{e}_1$ by a 90° rotation in the clockwise direction. We call the plane $\mathbb{R}^2$ **oriented** when one of these systems is introduced in $V_2(\mathbb{R}^2)$, and we use the notation $\overrightarrow{\mathbb{R}}^2$ and $-\overrightarrow{\mathbb{R}}^2$ for the orientations. Generally, we shall consider the oriented plane $\overrightarrow{\mathbb{R}}^2$. In coordinates, we write $\mathbf{v} = P(x_1, x_2)\mathbf{e}_1 + Q(x_1, x_2)\mathbf{e}_2$, where P and Q are functions from D into $\mathbb{R}^1$. The **scalar curl** of $\mathbf{v}$ is the function $Q_{,1} - P_{,2}$; we write

$$\operatorname{curl} \mathbf{v} = Q_{,1} - P_{,2}.$$

We recall that in V_3 the curl of a vector function, is a vector function whereas the above curl is a scalar function. To justify the above terminology, observe that if $\mathbf{v}$ is a vector function from $\mathbb{R}^3$ to V_3 in the special form

$$\mathbf{v} = P\mathbf{e}_1 + Q\mathbf{e}_2 + 0 \cdot \mathbf{e}_3,$$

and $P = P(x_1, x_2)$, $Q = Q(x_1, x_2)$, then

$$\operatorname{curl} \mathbf{v} = (Q_{,1} - P_{,2})\mathbf{e}_3.$$

Lemma 12.3

Let f be a continuously differentiable function with domain $I = \{x_1 : a \le x_1 \le b\}$ and range in $\mathbb{R}^1$. Let c be a constant with $f(x_1) \ge c$ for $x_1 \in I$. Define $D = \{(x_1, x_2) : a \le x_1 \le b, c \le x_2 \le f(x_1)\}$. (See Figure 12.2.) Suppose that G is any region in $\overrightarrow{\mathbb{R}}^2$ with $\bar{D} \subset G$ and let $\mathbf{v} : G \to V_2(\mathbb{R}^2)$ be a continuously differentiable vector function. Then

$$\int_D \operatorname{curl} \mathbf{v}\, dA = \int_{\overrightarrow{\partial D}} \mathbf{v} \cdot d\mathbf{r}, \tag{12.23}$$

where $\overrightarrow{\partial D}$ is oriented in the counterclockwise sense.

Proof

Writing $\mathbf{v}(x_1, x_2) = P(x_1, x_2)\mathbf{e}_1 + Q(x_1, x_2)\mathbf{e}_2$, we see that it is sufficient to prove (12.23) for the functions $P(x_1, x_2)\mathbf{e}_1$ and $Q(x_1, x_2)\mathbf{e}_2$ separately. Letting $\mathbf{v}_1 = P\mathbf{e}_1$, we have

$$\int_D \operatorname{curl} \mathbf{v}_1 dA = \int_D P_{,2}(x_1, x_2) dA = -\int_a^b \{P(x_1, f(x_1)) - P(x_1, c)\} dx_1.$$

We write $\overrightarrow{\partial D} = \overrightarrow{C}_1 + \overrightarrow{C}_2 + \overrightarrow{C}_3 + \overrightarrow{C}_4$ as shown in Figure 12.2. Then

$$\int_D \operatorname{curl} \mathbf{v}_1 dA = \int_{\overrightarrow{C}_3} P dx_1 + \int_{\overrightarrow{C}_1} P dx_1.$$

Since x_1 is constant along $\overrightarrow{C}_2$ and $\overrightarrow{C}_4$, it follows that $\int_{\overrightarrow{C}_2} P dx_1 = \int_{\overrightarrow{C}_4} P dx_1 = 0$. Therefore,

$$\text{(12.24)} \qquad \int_D \operatorname{curl} \mathbf{v}_1 dA = \int_{\sum_{i=1}^4 \overrightarrow{C}_i} P dx_1 = \int_{\overrightarrow{\partial D}} P dx_1 .$$

Next, let $\mathbf{v}_2 = Q\mathbf{e}_2$ and define

$$U(x_1, x_2) = \int_c^{x_2} Q(x_1, \eta) d\eta.$$

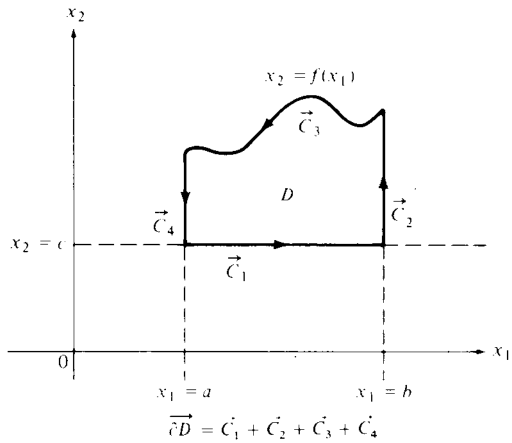

Figure 12.2

Then

$$U_{,1}(x_1, x_2) = \int_c^{x_2} Q_{,1}(x_1, \eta) d\eta, \qquad U_{,2}(x_1, x_2) = Q(x_1, x_2),$$

and

$$U_{,1,2} = U_{,2,1} = Q_{,1} .$$

Since U is smooth in G, it follows that

$$\text{(12.25)} \qquad \int_{\overrightarrow{\partial D}} (U_{,1} dx_1 + U_{,2} dx_2) = 0 = \int_{\overrightarrow{\partial D}} U_{,1} dx_1 + \int_{\overrightarrow{\partial D}} Q dx_2,$$

and using formula (12.24) with $U_{,1}$ in place of P, we obtain

$$\int_D \operatorname{curl} \mathbf{v}_2 dA = \int_D U_{,1,2} dA = -\int_{\overrightarrow{\partial D}} U_{,1} dx_1 .$$

Taking (12.25) into account we conclude that

$$\text{(12.26)} \qquad \int_D \operatorname{curl} \mathbf{v}_2 dA = \int_{\overrightarrow{\partial D}} Q dx_2 .$$

Adding (12.24) and (12.26), we get the result. ∎

Definitions
Let C_1 and C_2 be two smooth arcs that have as their only point in common an endpoint of each of them. This point, denoted by P, is called a **corner of C_1 and C_2** if each of the arcs has a (one-sided) tangent line at P and if the two tangent lines make a *positive angle* (see Figure 12.3). A **piecewise smooth simple closed curve** is a simple closed curve that is made up of a finite number of smooth arcs that are joined at corners.

In Lemma 12.3 we may assume that f is a piecewise continuously differentiable (i.e., piecewise smooth) function on I consisting of a finite number of smooth arcs joined at corners. The proof is the same.

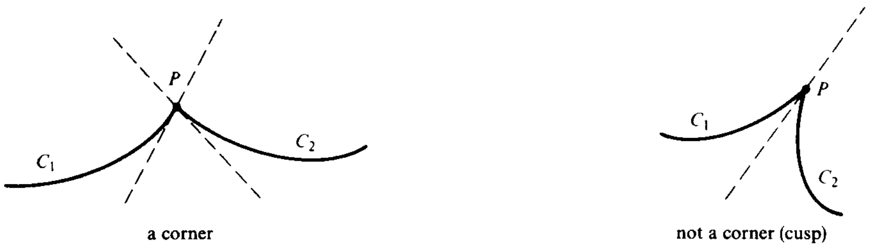

Figure 12.3

Definition
A region D in $\mathbb{R}^2$ is said to be **regular** if (i) D is bounded, (ii) ∂D consists of a finite number of piecewise smooth simple closed curves, and (iii) at each point P of ∂D a Cartesian coordinate system can be constructed with P as origin with the following properties: For sufficiently small a and b, a rectangle

$$R = \{(x_1, x_2) : -a \leq x_1 \leq a, -b \leq x_2 \leq b\}$$

can be found such that the part of ∂D in R has the form $x_2 = f(x_1)$ for x_1 on $I = \{x_1 : -a \leq x_1 \leq a\}$ and with range on $J = \{x_2 : -b < x_2 < b\}$, where f is a piecewise smooth function (see Figure 12.4).

Of course, the values of a, b, and the function f will change with the point P. If a rectangle R is determined for a point P, then it is clear that any rectangle with the same value for b and a smaller value for a is also adequate. See Figure 12.5 for examples of regions that are not regular.

If D is a regular region in $\overrightarrow{\mathbb{R}}^2$, a tangent vector can be drawn at any point of ∂D that is not a corner. Let $\overrightarrow{T}$ be the tangent vector at a point $P \in \partial D$ and construct a coordinate system in $\overrightarrow{\mathbb{R}}^2$ such that $\mathbf{e}_1$ is parallel to $\overrightarrow{T}$. Then ∂D may be oriented at P in two ways: The vector $\mathbf{e}_2$ may point into D, or the vector $\mathbf{e}_2$ may point outward from D. In the first case we say that ∂D *is oriented at P so that D is to the left*; in the second case, we say that *D is to the right*. It can be shown that the oriented piecewise smooth simple closed curve $\overrightarrow{\partial D}$ has the property that D is always on the left or always on the right for all points at which a tangent vector can be drawn.

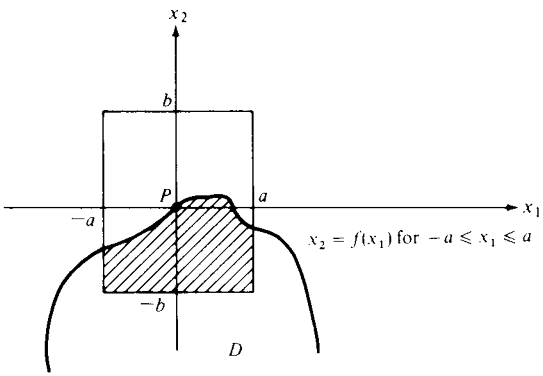

Figure 12.4

If D is a regular region in $\mathbb{R}^2$, then it can be shown that D can be decomposed into a finite number of regions of the type shown in Figure 12.2. Of course, the regions will be facing various directions, but when the integrations are performed, all those that occur twice along a straight line segment will cancel each other. As a result, we have the following theorem.

Theorem 12.13 (Green's theorem in the plane)
If P and Q are smooth in a regular region D in $\mathbb{R}^2$, then

$$\int_D \left(\frac{\partial Q}{\partial x_1} - \frac{\partial P}{\partial x_2} \right) dA = \oint_{\partial D} (P dx_1 + Q dx_2), \tag{12.27}$$

where the line integral is taken in a counterclockwise direction.

We provide two examples.

Example 1
Given the disk $K = \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$, use Green's theorem to evaluate

$$\int_{\overrightarrow{\partial K}} [(2x_1 - x_2^3)dx_1 + (x_1^3 + 3x_2^2)dx_2].$$

Solution. Setting $P = 2x_1 - x_2^3$, $Q = x_1^3 + 3x_2^2$, we see from (12.27) that

$$\oint_{\partial K} (Pdx_1 + Qdx_2) = \int_K 3(x_1^2 + x_2^2)dA = 3\int_0^{2\pi}\int_0^1 r^2 \cdot r dr d\theta = \frac{3}{2}\pi. \qquad \blacksquare$$

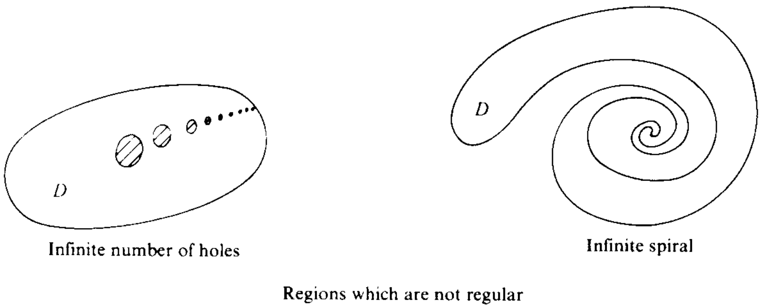

Figure 12.5

EXAMPLE 2

Let $K = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ be the unit disk and let D be the region outside K that is bounded on the left by the parabola $x_2^2 = 2(x_1 + 2)$ and on the right by the line $x_1 = 2$ (see Figure 12.6). Use Green's theorem to evaluate

$$\int_{\vec{C}_1} \left(-\frac{x_2}{x_1^2 + x_2^2}\, dx_1 + \frac{x_1}{x_1^2 + x_2^2}\, dx_2 \right),$$

where $\vec{C}_1$ is the outer boundary of D.

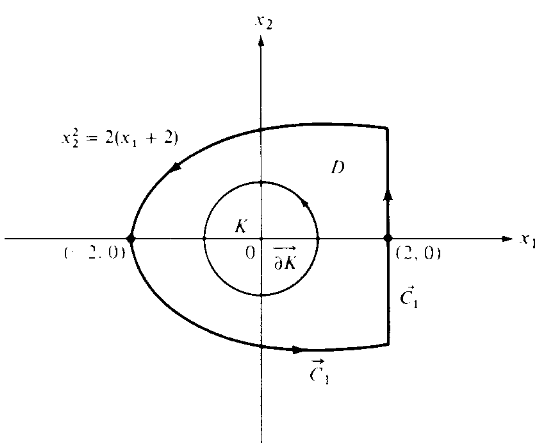

Figure 12.6

Solution. We write $P = -x_2/(x_1^2 + x_2^2)$, $Q = x_1/(x_1^2 + x_2^2)$ and note that $Q_{,1} - P_{,2} = 0$ in D. Hence with $\overrightarrow{\partial K}$ oriented in the counterclockwise sense, Green's theorem implies that

$$0 = \int_{\overrightarrow{\partial G}} (Pdx_1 + Qdx_2) = \int_{\overrightarrow{C}_1} (Pdx_1 + Qdx_2) - \int_{\overrightarrow{\partial K}} (Pdx_1 + Qdx_2).$$

Using the representation $x_1 = \cos\theta$, $x_2 = \sin\theta$, $-\pi \le \theta \le \pi$, for the last integral on the right, we obtain

$$\int_{\overrightarrow{C}_1} (Pdx_1 + Qdx_2) = \int_{-\pi}^{\pi} (\sin^2\theta + \cos^2\theta) d\theta = 2\pi.$$

■

A precise statement of Green's theorem without reference to a particular coordinate system is provided in the next statement.

Theorem 12.14

Let D be a regular region in $\overrightarrow{\mathbb{R}}^2$. Suppose that G is a region in $\overrightarrow{\mathbb{R}}^2$ with $\bar{D} \subset G$, and let $\mathbf{v}: G \to V_2(\overrightarrow{\mathbb{R}}^2)$ be a continuously differentiable vector function on G. Assume that $\overrightarrow{\partial D}$ is oriented so that D is on the left and that $\overrightarrow{\partial D} = \overrightarrow{C}_1 + \overrightarrow{C}_2 + \ldots + \overrightarrow{C}_k$, where each $\overrightarrow{C}_i$ is a smooth arc. Then

$$\int_D \operatorname{curl} \mathbf{v} dA = \sum_{i=1}^{k} \int_{\overrightarrow{C}_i} \mathbf{v} \cdot d\mathbf{r} = \int_{\overrightarrow{\partial D}} \mathbf{v} \cdot d\mathbf{r}. \tag{12.28}$$

Example 3

Let D be a regular region with area A. Let $\mathbf{v} = -\frac{1}{2}(x_2\mathbf{e}_1 - x_1\mathbf{e}_2)$. Show that

$$A = \int_{\overrightarrow{\partial D}} \mathbf{v} \cdot d\mathbf{r}.$$

Solution. We apply Green's theorem as in (12.28) and find that

$$\int_{\overrightarrow{\partial D}} \mathbf{v} \cdot d\mathbf{r} = \int_D \operatorname{curl} \mathbf{v} dA = \int_D \left(\frac{1}{2} + \frac{1}{2}\right) dA = A.$$

■

Example 3 shows that the area of any regular region may be expressed as an integral over the boundary of that region.

Problems

In each of Problems 1 through 8 verify Green's theorem.

1. $P(x_1, x_2) = -x_2$, $Q(x_1, x_2) = x_1$; $D = \{(x_1, x_2) : 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$.

2. $P(x_1, x_2) = x_1x_2$, $Q(x_1, x_2) = -2x_1x_2$; $D = \{(x_1, x_2 : 1 \leq x_1 \leq 2, 0 \leq x_2 \leq 3\}$.
3. $P(x_1, x_2) = 2x_1 - 3x_2$, $Q(x_1, x_2) = 3x_1 + 2x_2$, $D = \{(x_1, x_2) : 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 1\}$.
4. $P(x_1, x_2) = 2x_1 - x_2$, $Q(x_1, x_2) = x_1 + 2x_2$; D is the region outside the unit disk, above the curve $x_2 = x_1^2 - 2$, and below the line $x_2 = 2$.
5. $P = x_1^2 - x_2^2$, $Q = 2x_1x_2$; D is the triangle with vertices at $(0, 0)$, $(2, 0)$, and $(1, 1)$.
6. $P = -x_2$, $Q = 0$; D is the region inside the circle $x_1^2 + x_2^2 = 4$ and outside the circles $x_1^2 + (x_2 - 1)^2 = 1/4$ and $x_1^2 + (x_2 + 1)^2 = \frac{1}{4}$.
7. $\mathbf{v} = (x_1^2 + x_2^2)^{-1}(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2)$; $D = \{(x_1, x_2) : 1 < x_1^2 + x_2^2 < 4\}$.
8. $P = 4x_1 - 2x_2$, $Q = 2x_1 + 6x_2$; D is the interior of the ellipse $x_1 = 2\cos\theta$, $x_2 = \sin\theta$, $-\pi \leq \theta \leq \pi$.

In Problems 9 through 12 compute $\int_{\overrightarrow{\partial D}} \mathbf{v} \cdot d\mathbf{r}$ by means of Green's theorem.

9. $\mathbf{v}(x_1, x_2) = (4x_1e^{x_2} + 3x_1^2x_2 + x_2^3)\mathbf{e}_1 + (2x_1^2e^{x_2} - \cos x_2)\mathbf{e}_2$; $D = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 4\}$.
10. $\mathbf{v}(x_1, x_2) = \arctan(x_2/x_1)\mathbf{e}_1 + \frac{1}{2}\log(x_1^2 + x_2^2)\mathbf{e}_2$; $D = \{(x_1, x_2) : 1 \leq x_1 \leq 3, -2 \leq x_2 \leq 2\}$.
11. $\mathbf{v}(x_1, x_2) = -x_2\mathbf{e}_1 + x_1\mathbf{e}_2$; $D = \{(x_1, x_2) : (x_1 - 1)^2 + x_2^2 < 1\}$.
12. $\mathbf{v}(x_1, x_2) = -3x_1^2x_2\mathbf{e}_1 + 3x_1x_2^2\mathbf{e}_2$; $D = \{(x_1, x_2) : -a \leq x_1 \leq a, 0 \leq x_2 \leq \sqrt{a^2 - x_1^2}\}$.

12.4 Area of a Surface in $\mathbb{R}^3$

Let a surface S in $\mathbb{R}^3$ be given by

$$S = \{(x_1, x_2, x_3) : x_3 = f(x_1, x_2), (x_1, x_2) \in D \cup \partial D\}, \tag{12.29}$$

where D is a bounded region in $\mathbb{R}^2$. We know that if f has continuous first derivatives, the area of the surface, $A(S)$, may be computed by the formula developed in calculus:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2}\, dA.$$

Definitions

Let G be an open set in $\mathbb{R}^3$ and let $\mathbf{f} : G \to \mathbb{R}^3$ be a C^1 function. That is, $\mathbf{f}$ has components f^1, f^2, f^3, and $f^i : G \to \mathbb{R}^1$ are C^1 functions for $i = 1, 2, 3$. The **Jacobian** of $\mathbf{f}$ is the 3×3 matrix having the first partial derivative $f^i_{,j}$ as the entry in the ith row and jth column, $i, j = 1, 2, 3$. We also use the terms **Jacobian matrix** and **gradient**, and we denote this matrix by $\nabla\mathbf{f}$.

Definitions

Let D be a region in $\mathbb{R}^2$. A **smooth surface element** is the graph of a system of equations of the form

(12.30)
$$x_1 = x_1(s, t), \qquad x_2 = x_2(s, t), \qquad x_3 = x_3(s, t), \qquad (s, t) \in D \cup \partial D$$

in which x_1, x_2, x_3 are C^1 functions with domain a region G in $\mathbb{R}^2$ containing $D \cup \partial D$. We suppose that D is a region whose boundary consists of a finite number of piecewise smooth simple closed curves. In vector notation, equations (12.30) may be written

$$\mathbf{v}(\overrightarrow{OQ}) = \mathbf{r}(s, t), \qquad (s, t) \in D \cup \partial D, \qquad O \text{ given.}$$

Let σ be a smooth surface element given by

$$\sigma : \mathbf{v}(\overrightarrow{OQ}) = \mathbf{r}(s, t), \qquad (s, t) \in D \cup \partial D, \tag{12.31}$$

with $\mathbf{r}$ and D satisfying the conditions stated in the definition of a smooth surface element. If t is held constant, say equal to t_0, then the graph of (12.31) is a smooth curve on σ. Therefore, $\mathbf{r}_s(s, t_0)$, the partial derivative of $\mathbf{r}$ with respect to s, is a vector tangent to this curve on the surface. Similarly, the vector $\mathbf{r}_t(s_0, t)$ is tangent to the curve on σ obtained when we set $s = s_0$. The vectors $\mathbf{r}_s(s_0, t_0)$ and $\mathbf{r}_t(s_0, t_0)$ lie in the plane tangent to the surface element σ at the point $P_0 = \mathbf{r}(s_0, t_0)$.

Definition

Let σ be a smooth surface element given by (12.31). We subdivide D into a number of subregions $D_1, D_2, \ldots, D_n$ and define the mesh size $\|\Delta\|$ as the maximum diameter of all the D_i. The **area of the surface element** σ, denoted by $A(\sigma)$, is defined by the formula

$$A(\sigma) = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} A(D_i) |\mathbf{r}_s(s_i, t_i) \times \mathbf{r}_t(s_i, t_i)|,$$

where $A(D_i)$ is the area of D_i, and (s_i, t_i) is any point of D_i, and where the limit exists in the same manner as that determined in the definition of a definite integral. From this definition of area we obtain at once the formula

$$A(\sigma) = \iint_D |\mathbf{r}_s(s, t) \times \mathbf{r}_t(s, t)| dA. \tag{12.32}$$

If the representation of σ is of the form $x_3 = f(x_1, x_2)$ discussed at the beginning of the section, we may set $x_1 = s$, $x_2 = t$, $x_3 = f(s, t)$ and find that

$$J\left(\frac{x_2, x_3}{s, t}\right) = -f_s, \qquad J\left(\frac{x_3, x_1}{s, t}\right) = -f_t, \qquad J\left(\frac{x_1, x_2}{s, t}\right) = 1.$$

Then (12.32) becomes the familiar formula

$$A(\sigma) = \iint_D \sqrt{1 + f_s^2 + f_t^2}\, dA. \tag{12.33}$$

When a surface σ is described by a single equation such as $x_3 = f(x_1, x_2)$, we say that σ is given in **nonparametric form**.

Unfortunately, most surfaces cannot be described in nonparametric form; thus (12.33) cannot be used in general for the computation of surface area. In fact, it can be shown that a simple surface such as a sphere cannot be part of a single smooth surface element.

If S is a piecewise continuous surface, we can define area by decomposition. First, if σ is a smooth surface element, a set F is a **domain in** σ if and only if F is the image under (12.31) of a domain E in the plane region $D \cup \partial D$. The area of F is defined by formula (12.32).

Definitions

Let S be a piecewise continuous surface. A set F contained in S is a **domain** if and only if $F = F_1 \cup F_2 \cup \ldots \cup F_k$, where each F_i is a domain contained in a single smooth surface element σ_i of S, and no two F_i have common interior points. The **area** of F is defined by the formula

$$A(F) = A(F_1) + \ldots + A(F_k).$$

We now discuss integration of a function f defined on a surface F. Suppose that F is a closed domain on a piecewise smooth surface. We write $F = F_1 \cup F_2 \cup \ldots \cup F_k$, where each F_i is a domain contained in one smooth surface element. Then each F_i is the image of a domain E_i in $\mathbb{R}^2$, $i = 1, 2, \ldots, k$, under the map

$$\mathbf{v}(\overrightarrow{OQ}) = \mathbf{r}_i(s, t) \text{ for } (s, t) \in D_i \cup \partial D_i \tag{12.34}$$

with $E_i \subset (D_i \cup \partial D_i)$. Let $f : F \to \mathbb{R}^1$ be a continuous scalar field. We define

$$\iint_F f dS = \sum_{i=1}^{k} \iint_{F_i} f dS, \tag{12.35}$$

where each term on the right side of (12.35) is defined by the formula

$$\iint_{F_i} f dS = \iint_{E_i} f[\mathbf{r}_i(s, t)] \cdot |\mathbf{r}_{is} \times \mathbf{r}_{it}| dA. \tag{12.36}$$

The result is independent of the particular subdivision $\{F_i\}$ and the particular parametric representation (12.34).

If a surface element σ has a nonparametric representation $x_3 = \varphi(x_1, x_2)$, then (12.36) becomes

$$\iint_{F_i} f dS = \iint_{E_i} f[x_1, x_2, \varphi(x_1, x_2)] \sqrt{1 + \left(\frac{\partial \varphi}{\partial x_1}\right)^2 + \left(\frac{\partial \varphi}{\partial x_2}\right)^2}\, dA. \tag{12.37}$$

The evaluation of the integral in (12.37) follows the usual rule for evaluation of ordinary double and iterated integrals. We show the technique and give an example.

Example

Find the value of $\iint_F x_3^2 dS$, where F is the part of the lateral surface of the cylinder $x_1^2 + x_2^2 = 4$ between the planes $x_3 = 0$ and $x_3 = x_2 + 3$ (see Figure 12.7).

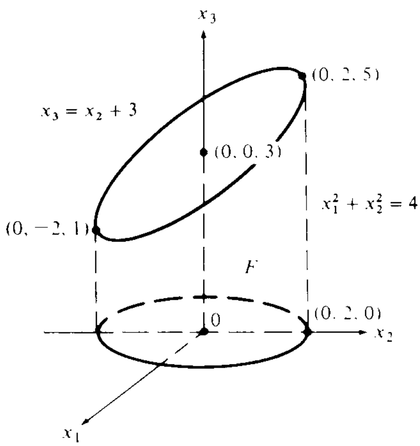

Figure 12.7

Solution. When we transform to the cylindrical coordinate system $x_1 = r\sin\theta$, $x_2 = r\cos\theta$, $x_3 = z$, then F lies on the surface $r = 2$. We choose θ and z as parametric coordinates on F and set

$$G = \{(\theta, z) : -\pi \le \theta \le \pi, 0 \le z \le 3 + 2\cos\theta\},$$

$$F = \{(x_1, x_2, x_3) : x_1 = 2\sin\theta, x_2 = 2\cos\theta, x_3 = z, (\theta, z) \in G\}.$$

The element of surface area dS is given by $dS = |\mathbf{r}_\theta \times \mathbf{r}_z| dA_{\theta z}$ (see Figure 12.8), and since

$$\begin{aligned}\mathbf{r}_\theta \times \mathbf{r}_z &= J\left(\frac{x_2, x_3}{\theta, z}\right)\mathbf{e}_1 + J\left(\frac{x_3, x_1}{\theta, z}\right)\mathbf{e}_2 + J\left(\frac{x_1, x_2}{\theta, z}\right)\mathbf{e}_3 \\ &= -(2\sin\theta)\mathbf{e}_1 - (2\cos\theta)\mathbf{e}_2 + 0\cdot\mathbf{e}_3,\end{aligned}$$

we have

$$dS = 2dA_{\theta z}.$$

Therefore,

$$\text{(12.38)} \quad \begin{aligned} \iint_F x_3^2 dS &= 2\iint_G z^2 dA_{\theta z} = 2\int_{-\pi}^{\pi}\int_0^{3+2\cos\theta} z^2 dz d\theta \\ &= \frac{2}{3}\int_{-\pi}^{\pi}(3+2\cos\theta)^3 d\theta = 60\pi. \end{aligned}$$

The surface F is not a single smooth surface element, since the transformation from G to F shows that the points $(-\pi, z)$ and (π, z) of G are carried into the same points of F. The condition that the transformation be one-to-one, necessary for a smooth surface element, is therefore violated. However, if we divide G into G_1 and G_2 as shown in Figure 12.8, the image of each is a smooth surface element. The evaluation of the integral (12.38) is unchanged. ■

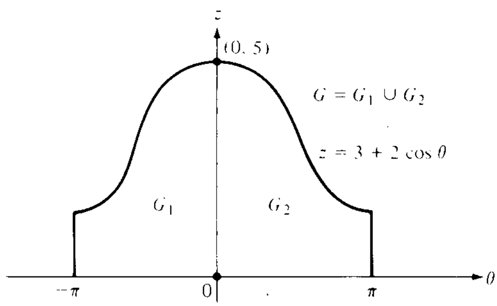

Figure 12.8

Problems

In each of Problems 1 through 9, find the value of

$$\iint_S f(x_1, x_2, x_3) dS.$$

1. $f(x_1, x_2, x_3) = x_1$, $S = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$.
2. $f(x_1, x_2, x_3) = x_1^2$, $S = \{(x_1, x_2, x_3) : x_3 = x_1, x_1^2 + x_2^2 \leq 1\}$.
3. $f(x_1, x_2, x_3) = x_1^2$, $S = \{(x_1, x_2, x_3) : x_3^2 = x_1^2 + x_2^2, 1 \leq x_3 \leq 2\}$.
4. $f(x_1, x_2, x_3) = x_1^2$, S is the part of the cylinder $x_3 = x_1^2/2$ cut out by the planes $x_2 = 0$, $x_1 = 2$, and $x_1 = x_2$.
5. $f(x_1, x_2, x_2) = x_1x_3$, $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, 0 \leq x_3 \leq x_1 + 2\}$.
6. $f(x_1, x_2, x_3) = x_1$, S is the part of the cylinder $x_1^2 - 2x_1 + x_2^2 = 0$ between the two nappes of the cone $x_1^2 + x_2^2 = x_3^2$.
7. $f(x_1, x_2, x_3) = 1$; using polar coordinates (r, θ) in the (x_1, x_2)-plane, S is the part of the vertical cylinder erected on the spiral $r = \theta$, $0 \leq \theta \leq \pi/2$, bounded below by the (x_1, x_2)-plane and above by the cone $x_1^2 + x_2^2 = x_3^2$.

8. $f(x_1, x_2, x_3) = x_1^2 + x_2^2 - 2x_3^2$, $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = a^2\}$.
9. $f(x_1, x_2, x_3) = x_1^2$, $S = \partial R$, where $R = \{(x_1, x_2, x_3) : x_3^2 \geq x_1^2 + x_2^2, 1 \leq x_3 \leq 2\}$ (see Problem 3).

The electrostatic potential $E(Q)$ at a point Q due to a distribution of electric charge (with charge density ρ) on a surface S is given by

$$E(Q) = \iint_S \frac{\rho(P)dS}{d_{PQ}},$$

where d_{PQ} is the distance from a point Q in $\mathbb{R}^3 - S$ to a point $P \in S$.

In Problems 10 through 13 find $E(Q)$ at the point given, assuming that ρ is constant.

10. $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 1, 0 \leq x_3 \leq 1\}$; $Q = (0, 0, 0)$.
11. $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = a^2\}$; $Q = (0, 0, c)$. Case 1: $c > a > 0$; Case 2: $a > c > 0$.
12. $S = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = a^2, x_3 \geq 0\}$; $Q = (0, 0, c)$; $0 < c < a$.
13. S is the surface of Problem 3; $Q = (0, 0, 0)$.

12.5 The Stokes Theorem

Suppose that S is a smooth surface element represented parametrically by

$$\mathbf{v}(\overrightarrow{OP}) = \mathbf{r}(s, t) \text{ with } (s, t) \in D \cup \partial D, \tag{12.39}$$

where D is a region in the (s, t) plane with a piecewise smooth boundary ∂D. From the definition of a smooth surface element, we know that $\mathbf{r}_s \times \mathbf{r}_t \neq \mathbf{0}$ for (s, t) in a region G containing $D \cup \partial D$.

Definition

For a smooth surface element S, the **unit normal function to** S is defined by the formula

$$\mathbf{n} = \frac{\mathbf{r}_s \times \mathbf{r}_t}{|\mathbf{r}_s \times \mathbf{r}_t|}, \qquad (s, t) \in D.$$

Whenever S is a smooth surface element, the vector $\mathbf{n}$ is a continuous function of s and t. Using Jacobian notation, $\mathbf{n}$ may be written in coordinates:

$$\mathbf{n} = |\mathbf{r}_s \times \mathbf{r}_t|^{-1} \left[J\left(\frac{x_2, x_3}{s, t}\right)\mathbf{e}_1 + J\left(\frac{x_3, x_1}{s, t}\right)\mathbf{e}_2 + J\left(\frac{x_1, x_2}{s, t}\right)\mathbf{e}_3 \right]. \tag{12.40}$$

It is a fact that the unit normal function is unique up to a change in sign. That is, if $\mathbf{n}'$ is another unit normal function for the same smooth surface element, then either $\mathbf{n}' = \mathbf{n}$ or $\mathbf{n}' = -\mathbf{n}$.

Definitions
A smooth surface S is **orientable** if there exists a continuous unit normal function defined over all of S. Such a unit normal function is called an **orientation of** $\boldsymbol{S}$.

Since the unit normal function to S at a point P is either $\mathbf{n}(P)$ or $-\mathbf{n}(P)$, each orientable surface possesses exactly two orientations, each of which is the negative of the other. An **oriented surface** is the pair $(S, \mathbf{n})$, where $\mathbf{n}$ is one of the two orientations of S. We denote such an oriented surface by $\overrightarrow{S}$. Suppose that F is a smooth surface element of the oriented surface $\overrightarrow{S}$. The function $\mathbf{n}$ when restricted to F provides an orientation for F, so that $(F, \mathbf{n}) = \overrightarrow{F}$ is an oriented surface element. We say that the orientation of F agrees with the orientation of S if the parametric representations of F and S yield unit normal functions on F that are identical.

If S is a smooth surface element represented parametrically by

$$\mathbf{v}(\overrightarrow{OP}) = \mathbf{r}(s, t), \qquad (s, t) \in D \cup \partial D,$$

then the boundary ∂S is the image of ∂D. Since $\mathbb{R}^2$ is oriented, we say that ∂D is *positively directed* if it is oriented so that D is on the left as ∂D is traversed. We write $\overrightarrow{\partial D}$ for this orientation and $-\overrightarrow{\partial D}$ when the boundary is oppositely directed.

Let $\overrightarrow{S}$ be a smooth oriented surface in $\mathbb{R}^3$ with boundary $\overrightarrow{\partial S}$ and suppose that $\mathbf{r}$ is a vector function defined on S and ∂S. Stokes's theorem is a generalization to surfaces of Green's theorem. We recall that Green's theorem establishes a relation between the integral of the derivative of a function in a domain D in $\mathbb{R}^2$ and the integral of the same function over ∂D (Theorem 12.13). The theorem of Stokes establishes an equality between the integral of curl $\mathbf{v} \cdot \mathbf{n}$ over a surface $\overrightarrow{S}$ and the integral of $\mathbf{v}$ over the boundary of $\overrightarrow{S}$. The principal result is given in Theorem 12.15.

Let S be represented parametrically by

$$\mathbf{r}(s, t) = x_1(s, t)\mathbf{e}_1 + x_2(s, t)\mathbf{e}_2 + x_3(s, t)\mathbf{e}_3 \text{ for } (s, t) \in D,$$

where D is a bounded region in the (s, t)-plane. Then the positive unit normal function is

$$\mathbf{n}(s, t) = |\mathbf{r}_s \times \mathbf{r}_t|^{-1}\left[J\left(\frac{x_2, x_3}{s, t}\right)\mathbf{e}_1 + J\left(\frac{x_3, x_1}{s, t}\right)\mathbf{e}_2 + J\left(\frac{x_1, x_2}{s, t}\right)\mathbf{e}_3\right].$$

Suppose that $\mathbf{v}$ is a continuous vector field defined on S and given in coordinates by

$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3.$$

Since S is a smooth surface, the scalar product $\mathbf{v} \cdot \mathbf{n}$ is continuous on S, and a direct computation yields

$$\mathbf{v} \cdot \mathbf{n} = |\mathbf{r}_s \times \mathbf{r}_t|^{-1} \left[v_1 J \left(\frac{x_2, x_3}{s, t} \right) + v_2 J \left(\frac{x_3, x_1}{s, t} \right) + v_3 J \left(\frac{x_1, x_2}{s, t} \right) \right].$$

Therefore, it is possible to define the surface integral

$$\iint_S \mathbf{v} \cdot \mathbf{n} dS. \tag{12.41}$$

The surface element dS can be computed in terms of the parameters (s, t) by the formula

$$dS = |\mathbf{r}_s \times \mathbf{r}_t| dA_{st},$$

where dA_{st} is the element of area in the plane region D. Hence

(12.42)

$$\iint_S \mathbf{v} \cdot \mathbf{n} dS = \iint_D \left[v_1 J \left(\frac{x_2, x_3}{s, t} \right) + v_2 J \left(\frac{x_3, x_1}{s, t} \right) + v_3 J \left(\frac{x_1, x_2}{s, t} \right) \right] dA_{st}.$$

If S is piecewise smooth rather than smooth, it can be represented as the union of a finite number of smooth surface elements $S_1, S_2, \ldots, S_m$. Then it is natural to define

$$\iint_S \mathbf{v} \cdot \mathbf{n} dS = \sum_{i=1}^{m} \iint_{S_i} \mathbf{v} \cdot \mathbf{n}_i dS_i,$$

where each integral on the right may be evaluated according to (12.42). In a similar way we can define the line integral over the boundary $\overrightarrow{\partial S}$ of a piecewise smooth oriented surface $\overrightarrow{S}$. If $\overrightarrow{\partial S}$ is made up of a finite number of smooth positively directed arcs $\overrightarrow{C}_1, \overrightarrow{C}_2, \ldots, \overrightarrow{C}_n$, and if $\mathbf{v}$ is a continuous vector field defined in a region of $\mathbb{R}^3$ that contains $\overrightarrow{\partial S}$, we define

$$\int_{\overrightarrow{\partial S}} \mathbf{v} \cdot d\mathbf{r} = \sum_{i=1}^{n} \int_{\overrightarrow{C}_i} \mathbf{v} \cdot d\mathbf{r}.$$

The following lemma is used in the proof of the Stokes theorem.

Lemma 12.4
Suppose that $\overrightarrow{S}$ is a smooth oriented surface element in $\mathbb{R}^3$ with a parametric representation

$$\mathbf{r}(s, t) = x_1(s, t)\mathbf{e}_1 + x_2(s, t)\mathbf{e}_2 + x_3(s, t)\mathbf{e}_3 \text{ for } (s, t) \in D \cup \partial D.$$

Let $\mathbf{v}$ be a continuous vector field defined on an open set G containing $\overrightarrow{\partial S}$ given by

$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3$$

for $(x_1, x_2, x_3) \in G$. *Then*

(12.43)

$$\int_{\overrightarrow{\partial S}} \mathbf{v} \cdot d\mathbf{r}$$

$$= \int_{\overrightarrow{\partial D}} \left[\left(v_1 \frac{\partial x_1}{\partial s} + v_2 \frac{\partial x_2}{\partial s} + v_3 \frac{\partial x_3}{\partial s} \right) ds + \left(v_1 \frac{\partial x_1}{\partial t} + v_2 \frac{\partial x_2}{\partial t} + v_3 \frac{\partial x_3}{\partial t} \right) dt \right].$$

Proof

We establish the result when $\overrightarrow{\partial D}$ consists of a single piecewise smooth simple closed curve. The extension to several such curves is clear. Let $\overrightarrow{\partial D}$ be given parametrically by the equations

$$s = s(\tau), \qquad t = t(\tau), \qquad a \le \tau \le b.$$

Then the equations

$$x_1 = x_1[s(\tau), t(\tau)], \qquad x_2 = x_2[s(\tau), t(\tau)],$$

$$x_3 = x_3[s(\tau), t(\tau)], \qquad a \le \tau \le b,$$

give a parametric representation of $\overrightarrow{\partial S}$. Using the Chain rule we obtain

$$\begin{aligned} d\mathbf{r} &= (dx_1)\mathbf{e}_1 + (dx_2)\mathbf{e}_2 + (dx_3)\mathbf{e}_3 \\ &= \left[\frac{\partial x_1}{\partial s}\frac{ds}{d\tau} + \frac{\partial x_1}{\partial t}\frac{dt}{d\tau} \right] d\tau \mathbf{e}_1 + \left[\frac{\partial x_2}{\partial s}\frac{ds}{d\tau} + \frac{\partial x_2}{\partial t}\frac{dt}{d\tau} \right] d\tau \mathbf{e}_2 \\ &\quad + \left[\frac{\partial x_3}{\partial s}\frac{ds}{d\tau} + \frac{\partial x_3}{\partial t}\frac{dt}{d\tau} \right] d\tau \mathbf{e}_3. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\overrightarrow{\partial S}} \mathbf{v} \cdot d\mathbf{r} = \int_a^b &\left[\left(v_1 \frac{\partial x_1}{\partial s} + v_2 \frac{\partial x_2}{\partial s} + v_3 \frac{\partial x_3}{\partial s} \right) \frac{ds}{d\tau} \right. \\ &\left. + \left(v_1 \frac{\partial x_1}{\partial t} + v_2 \frac{\partial x_2}{\partial t} + v_3 \frac{\partial x_3}{\partial t} \right) \frac{dt}{d\tau} \right] d\tau, \end{aligned}$$

and (12.43) follows at once. ∎

Let $\mathbf{v}$ be a smooth vector field defined in a region G of $\mathbb{R}^3$ given in coordinates by

(12.44) $\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3.$

We recall that curl $\mathbf{v}$ is a vector field on G given by

(12.45)

$$\operatorname{curl} \mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) \mathbf{e}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \mathbf{e}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \mathbf{e}_3.$$

Theorem 12.15 (The Stokes theorem)

Suppose that $\overrightarrow{S}$ is a bounded, closed, oriented, piecewise smooth surface and that $\mathbf{v}$ is a smooth vector field on a region in $\mathbb{R}^3$ containing $\overrightarrow{S} \cup \overrightarrow{\partial S}$. Then

$$\iint_{\overrightarrow{S}} (\text{curl}\,\mathbf{v}) \cdot \mathbf{n} dS = \int_{\partial S} \mathbf{v} \cdot d\mathbf{r}. \tag{12.46}$$

Proof

We first suppose that $\overrightarrow{S}$ is a smooth oriented surface element and that the components of $\mathbf{r}$ have continuous second derivatives. From equation (12.43) in Lemma 12.4, it follows that

$$\begin{aligned}\int_{\overrightarrow{\partial S}} \mathbf{v} \cdot d\mathbf{r} = \int_{\overrightarrow{\partial D}} \Bigg[& \left(v_1 \frac{\partial x_1}{\partial s} + v_2 \frac{\partial x_2}{\partial s} + v_3 \frac{\partial x_3}{\partial s} \right) ds \\ & + \left(v_1 \frac{\partial x_1}{\partial t} + v_2 \frac{\partial x_2}{\partial t} + v_3 \frac{\partial x_3}{\partial t} \right) dt \Bigg].\end{aligned}$$

We apply Green's theorem to the integral on the right and use the Chain rule to obtain

$$\begin{aligned}\iint_{\overrightarrow{\partial S}} \mathbf{v} \cdot d\mathbf{r} &= \iint_D \Bigg\{ \frac{\partial}{\partial s} \left[v_1 \frac{\partial x_1}{\partial t} + v_2 \frac{\partial x_2}{\partial t} + v_3 \frac{\partial x_3}{\partial t} \right] \\ &\qquad - \frac{\partial}{\partial t} \left[v_1 \frac{\partial x_1}{\partial s} + v_2 \frac{\partial x_2}{\partial s} + v_3 \frac{\partial x_3}{\partial s} \right] \Bigg\} dA_{st} \\ &= \iint_D \Bigg\{ \left(\frac{\partial v_1}{\partial x_1} \frac{\partial x_1}{\partial s} + \frac{\partial v_1}{\partial x_2} \frac{\partial x_2}{\partial s} + \frac{\partial v_1}{\partial x_3} \frac{\partial x_3}{\partial s} \right) \frac{\partial x_1}{\partial t} \\ &\qquad + \left(\frac{\partial v_2}{\partial x_1} \frac{\partial x_1}{\partial s} + \frac{\partial v_2}{\partial x_2} \frac{\partial x_2}{\partial s} + \frac{\partial v_2}{\partial x_3} \frac{\partial x_3}{\partial s} \right) \frac{\partial x_2}{\partial t} \\ &\qquad + \left(\frac{\partial v_3}{\partial x_1} \frac{\partial x_1}{\partial s} + \frac{\partial v_3}{\partial x_2} \frac{\partial x_2}{\partial s} + \frac{\partial v_3}{\partial x_3} \frac{\partial x_3}{\partial s} \right) \frac{\partial x_3}{\partial t} \\ &\qquad - \left(\frac{\partial v_1}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial v_1}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial v_1}{\partial x_3} \frac{\partial x_3}{\partial t} \right) \frac{\partial x_1}{\partial s} \\ &\qquad - \left(\frac{\partial v_2}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial v_2}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial v_2}{\partial x_3} \frac{\partial x_3}{\partial t} \right) \frac{\partial x_2}{\partial s} \\ &\qquad - \left(\frac{\partial v_3}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial v_3}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial v_3}{\partial x_3} \frac{\partial x_3}{\partial t} \right) \frac{\partial x_3}{\partial s} \Bigg\} dA_{st}.\end{aligned}$$

All the terms containing $\partial^2 x_1/\partial s\partial t$, $\partial^2 x_2/\partial s\partial t$, $\partial^2 x_3/\partial s\partial t$ in the above expression cancel. Collecting the terms on the right, we obtain

$$\int_{\overrightarrow{\partial S}} \mathbf{v}\cdot d\mathbf{r} = \iint_D \left\{ \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right) J\left(\frac{x_2, x_3}{s, t}\right) + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right) J\left(\frac{x_3, x_1}{s, t}\right) \right.$$
$$\left. + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) J\left(\frac{x_1, x_2}{s, t}\right) \right\} dA_{st}.$$

This expression is equivalent to (12.46). ■

Corollary

Suppose that S is a bounded, closed, oriented, piecewise smooth surface without boundary and that $\mathbf{v}$ is a smooth vector field defined on an open set containing S. Then

$$\iint_{\overrightarrow{S}} (\text{curl}\,\mathbf{v})\cdot \mathbf{n}\,dS = 0. \tag{12.47}$$

In the following example we show how an integral over a surface in $\mathbb{R}^3$ may be calculated by reducing it to an ordinary double integral in the plane.

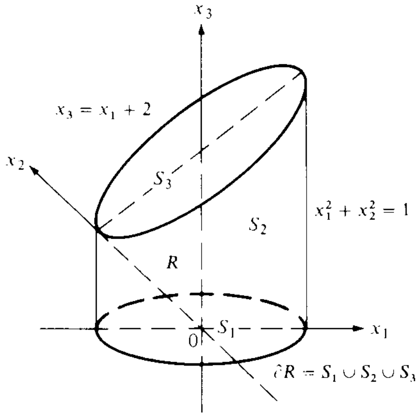

Figure 12.9

Example 1

Let R be the region in $\mathbb{R}^3$ defined by $R = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \le 1, 0 \le x_3 \le x_1 + 2\}$. Let S be the boundary of R. Define $\mathbf{v}(x_1, x_2, x_3) = 2x_1\mathbf{e}_1 - 3x_2\mathbf{e}_2 + x_3\mathbf{e}_3$. Find the value of

$$\iint_S \mathbf{v}\cdot\mathbf{n}\,dS, \tag{12.48}$$

where $\mathbf{n}$ is the unit normal directed outward from S (see Figure 12.9).

Solution. S is a piecewise smooth surface, and we divide it into three smooth surface elements as shown in Figure 12.9. The normal to S_1 is $-\mathbf{e}_3$, and therefore $\mathbf{v}\cdot\mathbf{n} = -x_3$ on S_1. Since $x_3 = 0$ on S_1, the value of (12.48) over S_1 is 0. Since $x_3 = x_1 + 2$ on S_3, it follows that

$$\mathbf{n} = \frac{1}{\sqrt{2}}(-\mathbf{e}_1 + \mathbf{e}_3)$$

on S_3. Hence on S_3,

$$\mathbf{v}\cdot\mathbf{n} = \frac{1}{\sqrt{2}}(-2x_1 + x_3) = \frac{1}{\sqrt{2}}(-x_1 + 2),$$

$$dS = \sqrt{2}dA_{st},$$

where dA_{st} is the element of area in the disk $D = \{(s,t) : s^2 + t^2 < 1\}$. Using the parameters $x_1 = s$, $x_2 = t$, we obtain

$$\iint_{S_3} \mathbf{v}\cdot\mathbf{n}dS = \iint_D (-s+2)dA_{st} = 2\pi.$$

To evaluate (12.48) over S_2 choose cylindrical coordinates:

$$x_1 = \cos s, \qquad x_2 = \sin s, \qquad x_3 = t, \qquad (s,t) \in D_1,$$

where $D_1 = \{(s,t) : -\pi \le s \le \pi, 0 \le t \le 2 + \cos s\}$. The outward normal on S_2 is $\mathbf{n} = (\cos s)\mathbf{e}_1 + (\sin s)\mathbf{e}_2$ and $\mathbf{v}\cdot\mathbf{n} = 2\cos^2 s - 3\sin^2 s$. Therefore,

$$\begin{aligned}\iint_{S_2} \mathbf{v}\cdot\mathbf{n}dS &= \iint_{D_1} (2\cos^2 s - 3\sin^2 s)dA_{st}\\ &= \int_{-\pi}^{\pi}\int_0^{2+\cos s} (2 - 5\sin^2 s)dtds = -2\pi.\end{aligned}$$

Finally,

$$\iint_S \mathbf{v}\cdot\mathbf{n}dS = \iint_{S_1\cup S_2\cup S_3} \mathbf{v}\cdot\mathbf{n}dS = 0 + 2\pi - 2\pi = 0.$$

■

Example 2

Verify Stokes's theorem given that

$$\mathbf{v} = x_2\mathbf{e}_1 + x_3\mathbf{e}_2 + x_1\mathbf{e}_3$$

and S_2 is the lateral surface in Example 1 with $\mathbf{n}$ pointing outward.

Solution. The boundary of $\overrightarrow{S}_2$ consists of the circle

$$C_1 = \{(x_1,x_2,x_3) : x_1^2 + x_2^2 = 1, x_3 = 0\}$$

and the ellipse

$$C_2 = \{(x_1,x_2,x_3) : x_1^2 + x_2^2 = 1, x_3 = x_1 + 2\}.$$

The curves $\overrightarrow{C}_1$, $\overrightarrow{C}_2$ are oriented as shown in Figure 12.10. We select cylindrical coordinates to describe $\overrightarrow{S}_2$:

$$\overrightarrow{S}_2 = \{(x_1, x_2, x_3) : x_1 = \cos s, \quad x_2 = \sin s, \quad x_3 = t\}, \quad (s, t) \in D_1,$$

where $D_1 = \{(s, t) : -\pi \le s \le \pi, 0 \le t \le 2 + \cos s\}$. We decompose $\overrightarrow{S}_2$ into two smooth surface elements, one part corresponding to $x_2 \ge 0$ and the other to $x_2 \le 0$. Then D_1 is divided into two parts E_1 and E_2 corresponding to $s \ge 0$ and $s \le 0$ as shown in Figure 12.11. This subdivision is required because the representation of $\overrightarrow{S}$ by D_1 is not one-to-one ($s = \pi$ and $s = -\pi$ correspond to the same curve on $\overrightarrow{S}_2$). A computation yields

$$J\left(\frac{x_2, x_3}{s, t}\right) = \cos s, \qquad J\left(\frac{x_3, x_1}{s, t}\right) = \sin s, \qquad J\left(\frac{x_1, x_2}{s, t}\right) = 0,$$

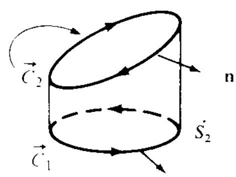

Figure 12.10

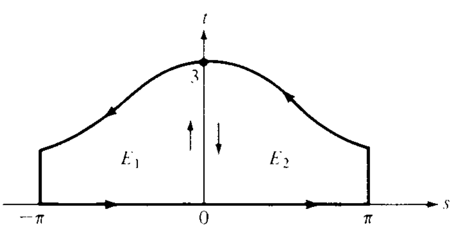

Figure 12.11

and $\mathbf{n} = (\cos s)\mathbf{e}_1 + (\sin s)\mathbf{e}_2$. Also, $\operatorname{curl} \mathbf{v} = -\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, $dS = dA_{st}$. Therefore,

$$(12.49) \qquad \iint_{\overrightarrow{S}} (\operatorname{curl} \mathbf{v}) \cdot \mathbf{n} dS = \int_{-\pi}^{\pi} \int_0^{2+\cos\theta} (-\cos\theta - \sin\theta) dt ds = -\pi.$$

The boundary integrals are

$$\int_{\overrightarrow{\partial S}} \mathbf{v} \cdot d\mathbf{r} = \int_{\overrightarrow{C}_1} \mathbf{v} \cdot d\mathbf{r} - \int_{-\overrightarrow{C}_2} \mathbf{v} \cdot d\mathbf{r}.$$

Using cylindrical coordinates on C_1 and C_2, we obtain

$$\mathbf{v} = (\sin s)\mathbf{e}_1 + (\cos s)\mathbf{e}_3, \qquad d\mathbf{r} = [(-\sin s)\mathbf{e}_1 + (\cos s)\mathbf{e}_2]ds \text{ on } \overrightarrow{C}_1;$$

and on $\overrightarrow{C}_2$,

$$\mathbf{v} = (\sin s)\mathbf{e}_1 + (2+\cos s)\mathbf{e}_2 + (\cos s)\mathbf{e}_3,$$
$$d\mathbf{r} = [(-\sin s)\mathbf{e}_1 + (\cos s)\mathbf{e}_2 + (-\sin s)\mathbf{e}_3]ds.$$

Taking scalar products, we obtain

$$(12.50)\quad \int_{\overrightarrow{C}_1} \mathbf{v}\cdot d\mathbf{r} = \int_{-\pi}^{\pi} (-\sin^2 s)ds = -\pi,$$

$$(12.51)\quad \int_{\overrightarrow{C}_2} \mathbf{v}\cdot d\mathbf{r} = \int_{-\pi}^{\pi} (-\sin^2\theta + 2\cos\theta + \cos^2\theta - \sin\theta\cos\theta)d\theta$$
$$= 0.$$

Stokes's theorem is verified by comparing (12.49) with (12.50) and (12.51). ■

Problems

In each of Problems 1 through 6 compute

$$\iint_{\overrightarrow{S}} \mathbf{v}\cdot\mathbf{n}dS.$$

1. $\mathbf{v} = (x_1+1)\mathbf{e}_1 - (2x_2+1)\mathbf{e}_2 + x_3\mathbf{e}_3$; $\overrightarrow{S}$ is the triangular region with vertices at (1, 0, 0), (0, 1, 0), (0, 0, 1), and $\mathbf{n}$ is pointing away from the origin.
2. $\mathbf{v} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = 2x_3, (x_1-1)^2 + x_2^2 \le 1\}$, oriented so that $\mathbf{n}\cdot\mathbf{e}_3 > 0$.
3. $\mathbf{v} = x_1^2\mathbf{e}_1 + x_2^2\mathbf{e}_2 + x_3^2\mathbf{e}_2$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 = x_3^2; 1 \le x_3 \le 2\}$, $\mathbf{n}\cdot\mathbf{e}_3 > 0$.
4. $\mathbf{v} = x_1x_2\mathbf{e}_1 + x_1x_3\mathbf{e}_2 + x_2x_3\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_2^2 = 2 - x_1; x_3^3 \le x_2 \le x_3^{\frac{1}{2}}\}$.
5. $\mathbf{v} = x_2^2\mathbf{e}_1 + x_3\mathbf{e}_2 - x_1\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_2^2 = 1 - x_1, 0 \le x_3 \le x_1; x_1 \ge 0\}$, $\mathbf{n}\cdot\mathbf{e}_1 > 0$.
6. $\mathbf{v} = 2x_1\mathbf{e}_1 - x_2\mathbf{e}_2 + 3x_3\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_3^2 = x_1; x_2^2 \le 1 - x_1; x_2 \ge 0\}$, $\mathbf{n}\cdot\mathbf{e}_1 > 0$.

In each of Problems 7 through 12, verify Stokes's theorem.

7. $\mathbf{v} = x_3\mathbf{e}_1 + x_1\mathbf{e}_2 + x_2\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_3 = 1 - x_1^2 - x_2^2, x_3 \ge 0\}$, $\mathbf{n}\cdot\mathbf{e}_3 > 0$.
8. $\mathbf{v} = x_2^2\mathbf{e}_1 + x_1x_2\mathbf{e}_2 - 2x_1x_3\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \ge 0\}$, $\mathbf{n}\cdot\mathbf{e}_3 > 0$.

9. $\mathbf{v} = -x_2x_3\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1^2 + x_2^2 + x_3^2 = 4, x_1^2 + x_2^2 \geq 1\}$, $\mathbf{n}$ pointing outward from the sphere.
10. $\mathbf{v} = -x_3\mathbf{e}_2 + x_2\mathbf{e}_3$; $\overrightarrow{S}$ is the surface of the cylinder given in cylindrical coordinates by $r = \theta$, $0 \leq \theta \leq \pi/2$, which is bounded below by the plane $x_3 = 0$ and above by the surface of the cone $x_1^2 + x_2^2 = x_3^2$; $\mathbf{n} \cdot \mathbf{e}_1 > 0$ for $\theta > 0$.
11. $\mathbf{v} = x_2\mathbf{e}_1 + x_3\mathbf{e}_2 + x_1\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_3^2 = 4 - x_1, x_1 \geq x_2^2\}$, $\mathbf{n} \cdot \mathbf{e}_1 > 0$.
12. $\mathbf{v} = x_3\mathbf{e}_1 - x_1\mathbf{e}_3$; $\overrightarrow{S}$ is the surface of the cylinder given in cylindrical coordinates by $r = 2 + \cos\theta$ above the plane $x_3 = 0$ and exterior to the cone $x_3^2 = x_1^2 + x_2^2$; $\mathbf{n}$ is pointing outward from the cylindrical surface.

In each of Problems 13 through 15 use Stokes's theorem to compute $\int_{\partial \overrightarrow{S}} \mathbf{v} \cdot d\mathbf{r}$.

13. $\mathbf{v} = r^{-3}\mathbf{r}$ where $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ and $r = |\mathbf{r}|$; $\overrightarrow{S}$ is the surface $\overrightarrow{S}_2$ of Example 2.
14. $\mathbf{v} = (e^{x_1} \sin x_2)\mathbf{e}_1 + (e^{x_1} \cos x_2 - x_3)\mathbf{e}_2 + x_2\mathbf{e}_3$; $\overrightarrow{S}$ is the surface in Problem 3.
15. $\mathbf{v} = (x_1^2 + x_3)\mathbf{e}_1 + (x_1 + x_2^2)\mathbf{e}_2 + (x_2 + x_3^2)\mathbf{e}_3$; $\overrightarrow{S} = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1, x_3 \geq (x_1^2 + x_2^2)^{\frac{1}{2}}\}$; $\mathbf{n}$ points outward from the spherical surface.
16. Show that if $\overrightarrow{S}$ is given by $x_3 = f(x_1, x_2)$ for $(x_1, x_2) \in D = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$, if f is smooth, and if $\mathbf{v} = (1 - x_1^2 - x_2^2)\mathbf{w}(x_1, x_2, x_3)$ where $\mathbf{w}$ is any smooth vector field defined on an open set containing S, then
$$\iint_{\overrightarrow{S}} (\text{curl}\, \mathbf{v}) \cdot \mathbf{n} dS = 0.$$
17. Suppose that $\mathbf{v} = r^{-3}(x_2\mathbf{e}_1 + x_3\mathbf{e}_2 + x_1\mathbf{e}_3)$, where $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ and $r = |\mathbf{r}|$, and $\overrightarrow{S}$ is the sphere $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ with $\mathbf{n}$ pointing outward. Show that
$$\iint_S (\text{curl}\, \mathbf{v}) \cdot \mathbf{n} dS = 0.$$

12.6 The Divergence Theorem

Green's theorem establishes a relation between the line integral of a function over the boundary of a plane region and the double integral of the derivative of the same function over the region itself. Stokes's theorem extends this result to two-dimensional surfaces in three-space. In this section we establish another kind of generalization of the Fundamental theorem of calculus known as the **Divergence theorem**. This theorem determines the relationship between an integral of the derivative of a function over a three-dimensional region in $\mathbb{R}^3$ and the integral of the function itself over the boundary of that region. All three theorems (Green, Stokes, Divergence) are special cases of a general formula that connects an integral over a set of points in $\mathbb{R}^N$ with another integral over the boundary of that set of points. The integrand in the first integral is a certain derivative of the integrand in the boundary integral.

Let $\mathbf{v} = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3$ be a vector field defined for (x_1, x_2, x_3) in a region E in $\mathbb{R}^3$. We recall that div $\mathbf{v}$ is a scalar field given in coordinates by the formula

$$\operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}.$$

The Divergence theorem consists in proving the formula

$$\iiint_E \operatorname{div} \mathbf{v} dV = \iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS, \tag{12.52}$$

where ∂E is oriented by choosing $\mathbf{n}$ as the exterior normal to ∂E. We first establish (12.52) in several special cases and then show that the formula holds generally, provided that the boundary of E is not too irregular and that $\mathbf{v}$ is smooth.

Lemma 12.5

Let D be a domain in the (x_1,x_2)-plane with smooth boundary. Let $f: D \cup \partial D \to \mathbb{R}^1$ be a piecewise smooth function and define

$$E = \{(x_1, x_2, x_3) : (x_1, x_2) \in D, c < x_3 < f(x_1, x_2)\}$$

for some constant c. Suppose that $\mathbf{v} = u(x_1,x_2,x_3)\mathbf{e}_3$ is such that u and $\partial u/\partial x_3$ are continuous on an open set in $\mathbb{R}^3$ containing $E \cup \partial E$. Then

$$\iint_E \operatorname{div} \mathbf{v} dV = \iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ is the outward unit normal of ∂E.

Proof

Since div $\mathbf{v} = \partial u/\partial x_3$, it follows that

$$\iiint_E \operatorname{div} \mathbf{v} dV = \iiint_E \frac{\partial u}{\partial x_3} dV = \iint_D \int_c^{f(x_1,x_2)} \frac{\partial u}{\partial x_3} dx_3 dA_{x_1x_2}.$$

Performing the integration with respect to x_3, we obtain

$$\iiint_E \operatorname{div} \mathbf{v} dV = \iint_D \{u[x_1, x_2, f(x_1, x_2)] - u(x_1, x_2, c)\} dA_{x_1x_2}. \tag{12.53}$$

Let $\partial E = S_1 \cup S_2 \cup S_3$, where S_1 is the domain in the plane $x_3 = c$ that is congruent to D, S_2 is the lateral cylindrical surface of ∂E, and S_3 is the part of ∂E corresponding to $x_3 = f(x_1, x_2)$. We wish to show that (12.53) is equal to

$$\iint_{S_1 \cup S_2 \cup S_3} \mathbf{v} \cdot \mathbf{n} dS.$$

Along S_2, the unit normal $\mathbf{n}$ is parallel to the (x_1, x_2)-plane, and so $\mathbf{n} \cdot \mathbf{e}_3 = 0$. Therefore, $\mathbf{v} \cdot \mathbf{n} = 0$ on S_2 and

$$\iint_{S_2} \mathbf{v} \cdot \mathbf{n} dS = 0. \tag{12.54}$$

The outer normal along S_1 is clearly $-\mathbf{e}_3$, and therefore

$$\iint_{S_1} \mathbf{v} \cdot \mathbf{n} dS = -\iint_D u(x_1, x_2, c) dA_{x_1 x_2}. \tag{12.55}$$

As for S_3, the unit normal function is given by

$$\mathbf{n} = \frac{-\frac{\partial f}{\partial x_1}\mathbf{e}_1 - \frac{\partial f}{\partial x_2}\mathbf{e}_2 + \mathbf{e}_3}{\left[1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2\right]^{\frac{1}{2}}}.$$

Also, for $(x_1, x_2, x_3) \in S_3$,

$$dS = \left[1 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2\right]^{\frac{1}{2}} dA_{x_1 x_2}.$$

Therefore, since $\mathbf{v} = u\mathbf{e}_3$, we have

$$\iint_{S_3} \mathbf{v} \cdot \mathbf{n} dS = \iint_D u[x_1, x_2, f(x_1, x_2)] dA_{x_1 x_2}. \tag{12.56}$$

The result follows when (12.53) is compared with (12.54), (12.55), and (12.56). ■

Lemma 12.6
Suppose that the hypotheses of Lemma 12.5 *hold, except that* $\mathbf{v}$ *has the form*

$$\mathbf{v} = u_1(x_1, x_2, x_3)\mathbf{e}_1 + u_2(x_1, x_2, x_3)\mathbf{e}_2$$

with u_1,u_2 *smooth functions on an open set containing* $E \cup \partial E$*. Then*

$$\iiint_E \operatorname{div} \mathbf{v} dV = \iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ *is the outward unit normal of* ∂E*.*

Proof

Let $U_1(x_1, x_2, x_3)$, $U_2(x_1, x_2, x_3)$ be defined by

$$U_1(x_1, x_2, x_3) = -\int_c^{x_3} u_1(x_1, x_2, t)dt, \qquad U_2(x_1, x_2, x_3) = \int_c^{x_3} u_2(x_1, x_2, t)dt.$$

In addition, we define

$$\mathbf{w} = U_2\mathbf{e}_1 + U_1\mathbf{e}_2, \qquad U_3 = -\frac{\partial U_1}{\partial x_1} + \frac{\partial U_2}{\partial x_2}, \qquad \mathbf{u} = -U_3\mathbf{e}_3.$$

Then $\mathbf{w}$ is a smooth vector field and U_3, $\partial U_3/\partial x_3$ are continuous, and hence

$$\begin{aligned} \operatorname{curl} \mathbf{w} &= -\frac{\partial U_1}{\partial x_3}\mathbf{e}_1 + \frac{\partial U_2}{\partial x_3}\mathbf{e}_2 + \left(\frac{\partial U_1}{\partial x_1} - \frac{\partial U_2}{\partial x_2}\right)\mathbf{e}_3 = \mathbf{v} + \mathbf{u}, \\ \operatorname{div} \mathbf{v} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \frac{\partial U_3}{\partial x_3} = -\operatorname{div} \mathbf{u}. \end{aligned} \tag{12.57}$$

Since ∂E is a piecewise smooth surface without boundary, the corollary to Stokes's theorem is applicable. Hence

$$\iint_{\partial E} (\operatorname{curl} \mathbf{w}) \cdot \mathbf{n} dS = 0 = \iint_{\partial E} (\mathbf{v} + \mathbf{u}) \cdot \mathbf{n} dS.$$

Therefore, using Lemma 12.5 for the function $\mathbf{u} = -U_3\mathbf{e}_3$, we obtain

$$\iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS = -\iint_{\partial E} \mathbf{u} \cdot \mathbf{n} dS = -\iiint_E \operatorname{div} \mathbf{u} dV. \tag{12.58}$$

The result of the lemma follows by inserting (12.57) into (12.58). ■

The Divergence theorem will now be established for a wide class of regions that are called "regular." Intuitively, a region in $\mathbb{R}^3$ is regular if its boundary can be subdivided into small pieces in such a way that each piece has a smooth representation of the form $x_3 = f(x_1, x_2)$ if a suitable Cartesian coordinate system is introduced.

Definition

A region E in $\mathbb{R}^3$ is **regular** if:

(i) ∂E consists of a finite number of smooth surfaces, each without boundary;

(ii) at each point P of ∂E a Cartesian coordinate system is introduced with P as origin. There is a cylindrical domain $\Gamma = \{(x_1, x_2, x_3) : (x_1, x_2) \in D, -\infty < x_3 < \infty\}$, with D a region in the plane $x_3 = 0$ containing the origin, that has the property that $\Gamma \cap \partial E$ is a surface that can be represented in the form $x_3 = f(x_1, x_2)$ for $(x_1, x_2) \in D \cup \partial D$. Furthermore, f is smooth;

(iii) the set

$$\Gamma_1 = \{(x_1, x_2, x_3) : (x_1, x_2) \in D, -c < x_3 < f(x_1, x_2)\}$$

for some positive constant c (depending on P) is contained entirely in E.

If E is a regular region, then each point P of ∂E is interior to a smooth surface element except for those points on a finite number of arcs on ∂E that have zero surface area. If a line in the direction of the unit normal to ∂E is drawn through a point P interior to a smooth surface element on ∂E, then a segment of the line on one side of P will be in E, while a segment on the other side will be exterior to E. Finally, we observe that $\mathbf{n}$ varies continuously when P is in a smooth surface element of ∂E.

Theorem 12.16 (The Divergence theorem)
Suppose that E is a closed, bounded, regular region in $\mathbb{R}^3$ and that $\mathbf{v}$ is a continuously differentiable vector field on an open set G containing $E \cup \partial E$. Then

$$\iiint_E \operatorname{div} \mathbf{v} dV = \iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS,$$

where $\mathbf{n}$ is the unit normal function pointing outward from E.

Proof
With each point $P \in \partial E$ associate a coordinate system and a cylindrical domain Γ as in the definition of a regular region. Let Γ_P be the bounded portion of Γ such that $-c < x_3 < c$ (where c is the constant in the definition of regular region; c depends on P). With each interior point P of E, introduce a Cartesian coordinate system and a cube Γ_P with P as origin, with the sides of Γ_P parallel to the axes and with $\bar{\Gamma}_P$ entirely in E. Since $E \cup \partial E$ is compact, a finite number $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ cover $E \cup \partial E$. As described in the proof of Green's theorem, the function $\mathbf{v}$ can be decomposed into a finite number of functions $\mathbf{v}_i$, $i = 1, 2, \ldots, n$, such that each $\mathbf{v}_i$ is continuously differentiable on G and vanishes on $G - F_i$. Also, $\mathbf{v} = \sum_{i=1}^n \mathbf{v}_i$. Therefore, it is sufficient to establish the result for each $\mathbf{v}_i$. Suppose first that P_i is an interior point of E. Then $\bar{\Gamma}_i \subset E$, and $\mathbf{v}_i$ vanishes on $\partial\Gamma_i$. Hence

$$\iiint_E \operatorname{div} \mathbf{v}_i dV = \iiint_{\Gamma_i} \operatorname{div} \mathbf{v}_i dV = 0$$

because of Lemma 12.6. Also, because $\mathbf{v}_i = 0$ on ∂E, we have

$$\iint_{\partial E} \mathbf{v}_i \cdot \mathbf{n} dS = 0.$$

Thus the result is established for all such Γ_i. Suppose now that $P_i \in \partial E$. Define $E_i = \Gamma_i \cap E$. Then $v_i = 0$ on the three sets (i) $E - E_i$; (ii) $\partial E - \gamma_i$, where $\gamma_i = \partial E \cap \Gamma_i$; (iii) $\partial E_i - \gamma_i$. Now, since E_i is a region of the type described in Lemmas 12.5 and 12.6, we obtain

$$\iiint_E \operatorname{div} \mathbf{v}_i dV = \iiint_{E_i} \operatorname{div} \mathbf{v}_i dV = \iint_{\partial E_i} \mathbf{v}_i \cdot \mathbf{n} dS$$
$$= \iint_{\gamma_i} \mathbf{v}_i \cdot \mathbf{n} dS = \iint_{\partial E} \mathbf{v}_i \cdot \mathbf{n} dS.$$

The result is established for all $\mathbf{v}_i$ and hence for $\mathbf{v}$. ■

Example 1
Let E be the region given by

$$E = \{(x_1, x_2, x_3) : 1 \leq x_1^2 + x_2^2 + x_3^2 \leq 9\}.$$

Let $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ and $\mathbf{v} = r^{-3}\mathbf{r}$, where $r = |\mathbf{r}|$. Verify the Divergence theorem.

Solution. Let $\overrightarrow{S}_1 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$ and $\overrightarrow{S}_2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 9\}$ be the boundary spheres of E with $\mathbf{n}$ pointing outward from the origin $\mathbf{0}$. Then

$$\iint_{\overrightarrow{\partial E}} \mathbf{v} \cdot \mathbf{n} dS = \iint_{\overrightarrow{S}_2} \mathbf{v} \cdot \mathbf{n} dS - \iint_{\overrightarrow{S}_1} \mathbf{v} \cdot \mathbf{n} dS.$$

A computation shows that $\operatorname{div} \mathbf{v} = 0$ and so $\iiint_E \operatorname{div} \mathbf{v} dV = 0$. On both $\overrightarrow{S}_1$ and $\overrightarrow{S}_2$ the normal $\mathbf{n}$ is in the radial direction and hence $\mathbf{n} = r^{-1}\mathbf{r}$. Therefore,

$$\iint_{\overrightarrow{S}_2} \mathbf{v} \cdot \mathbf{n} dS - \iint_{\overrightarrow{S}_1} \mathbf{v} \cdot \mathbf{n} dS = \frac{1}{9} A(S_2) - A(S_1) = 0.$$

■

Example 2
Let $E = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < 1, 0 < x_3 < x_1 + 2\}$ and define $\mathbf{v} = \frac{1}{2}(x_1^2 + x_2^2)\mathbf{e}_1 + \frac{1}{2}(x_2^2 + x_3x_1^2)\mathbf{e}_2 + \frac{1}{2}(x_3^2 + x_1^2x_2)\mathbf{e}_3$. Use the Divergence theorem to evaluate

$$\iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS.$$

Solution. A computation shows that $\operatorname{div} \mathbf{v} = x_1 + x_2 + x_3$. We denote the unit disk by F and obtain

$$\iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS = \iiint_E (x_1 + x_2 + x_3) dV$$
$$= \iint_F \left[(x_1 + x_2)x_3 + \frac{1}{2} x_3^2\right]_0^{x_1+2} dA_{x_1x_2}$$

$$= \frac{1}{2} \iint_F [2x_2(x_1 + 2) + 2x_1^2 + 4x_1 + (x_1^2 + 4x_1 + 4)]dA_{x_1x_2}$$
$$= \frac{1}{2} \iint_F (3x_1^2 + 4)dA_{x_1x_2}$$
$$= \frac{1}{2} \int_0^1 \int_0^{2\pi} (3r^2 \cos^2 \theta + 4])rdrd\theta = \frac{19}{8}\pi. \quad \blacksquare$$

Problems

In each of Problems 1 through 10 verify the Divergence theorem by computing $\iiint_E \operatorname{div} \mathbf{v} dV$ and $\iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS$ separately.

1. $\mathbf{v} = x_1x_2\mathbf{e}_1 + x_2x_3\mathbf{e}_2 + x_3x_1\mathbf{e}_3$; E is the tetrahedron with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.
2. $\mathbf{v} = x_1^2\mathbf{e}_1 - x_2^2\mathbf{e}_2 + x_3^2\mathbf{e}_3$; $E = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < 4, 0 < x_3 < 2\}$.
3. $\mathbf{v} = 2x_1\mathbf{e}_1 + 3x_2\mathbf{e}_2 - 4x_3\mathbf{e}_3$; $E = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < 4\}$.
4. $\mathbf{v} = x_1^2\mathbf{e}_1 + x_2^2\mathbf{e}_2 + x_3^2\mathbf{e}_3$; $E = \{(x_1, x_2, x_3) : x_2^2 < 2 - x_1, 0 < x_3 < x_1\}$.
5. $\mathbf{v} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$; $E = E_1 \cap E_2$ where $E_1 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 > 1\}$, $E_2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < 4\}$.
6. $\mathbf{v} = x_1\mathbf{e}_1 - 2x_2\mathbf{e}_2 + 3x_3\mathbf{e}_3$; $E = E_1 \cap E_2$, where $E_1 = \{(x_1, x_2, x_3) : x_2^2 < x_1\}$, $E_2 = \{(x_1, x_2, x_3) : x_3^2 < 4 - x_1\}$.
7. $\mathbf{v} = r^{-3}(x_3\mathbf{e}_1 + x_1\mathbf{e}_2 + x_2\mathbf{e}_3)$, $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$; $E = \{(x_1, x_2, x_3) : 1 < x_1^2 + x_2^2 + x_3^2 < 4\}$.
8. $\mathbf{v} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$; $E = E_1 \cap E_2$ where $E_1 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < 4\}$, $E_2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 - x_3^2 > 1\}$.
9. $\mathbf{v} = 2x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$; $E = E_1 \cap E_2$ where $E_1 = \{(x_1, x_2, x_3) : x_3 \leq x_1^2 + x_2^2\}$, $E_2 = \{(x_1, x_2, x_3) : 2x_1 \leq x_3\}$.
10. $\mathbf{v} = 3x_1\mathbf{e}_1 - 2x_2\mathbf{e}_2 + x_3\mathbf{e}_3$; $E = E_1 \cap E_2 \cap E_3$ where $E_1 = \{(x_1, x_2, x_3) : x_2 \geq 0\}$, $E_2 = \{(x_1, x_2, x_3) : x_1^2 + x_3^2 \leq 4\}$; $E_3 = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 3\}$.

In each of Problems 11 through 13, use the Divergence theorem to evaluate $\iint_{\partial E} \mathbf{v} \cdot \mathbf{n} dS$.

11. $\mathbf{v} = x_2e^{x_3}\mathbf{e}_1 + (x_2 - 2x_3e^{x_1})\mathbf{e}_2 + (x_1e^{x_2} - x_3)\mathbf{e}_3$; $E = \{(x_1, x_2, x_3) : [(x_1^2 + x_2^2)^{\frac{1}{2}} - 2]^2 + x_3^2 < 1\}$.
12. $\mathbf{v} = x_1^3\mathbf{e}_1 + x_2^3\mathbf{e}_2 + x_3^3\mathbf{e}_3$; $E = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < 1\}$.
13. $\mathbf{v} = x_1^3\mathbf{e}_1 + x_2^3\mathbf{e}_2 + x_3\mathbf{e}_3$; $E = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 < 1, 0 < x_3 < x_1 + 2\}$.

Answers to Odd-Numbered Problems

Section 1.2

1. The inverse of T is a function if the a_i are all distinct.
9. No.

Section 1.3

1. Yes. Each member $a + b\sqrt{7}$, with a, b rational, corresponds to a unique point on the line.
5. $(0, \infty)$
7. $(-4, \infty)$
9. $(-2, 3)$
11. $[-1, 2)$
13. $(-1, 5)$
15. $(-\infty, 0) \cup \left(\frac{15}{2}, +\infty\right)$
17. $(-\infty, 1) \cup (2, +\infty)$
19. $\left(\frac{4}{3}, 3\right)$

Section 1.4

9. (b) Yes. (c) Yes.

Section 2.1

1. $\delta = 0.005$
3. $\delta = 0.02$

5. $\delta = 0.002$
7. $\delta = 0.06$
9. $\delta = 0.005$
11. $\delta = 0.01$
13. $\delta = 0.07$
15. $\delta = 0.09$
17. $\delta = 0.7$

Section 2.3

1. Yes
3. No. Neither
5. No. Neither
7. No. On the right
9. No. Neither
11. No. Neither

Section 2.4

1. 0
3. $+\infty$
5. 0
7. $+\infty$
9. 2

Section 2.5

1. (a) $x_n = \left(n + \frac{1}{2}\right)\pi$ (b) $y_n = 2n\pi$ (c) $z_n = (2n+1)\pi$
3. *Hint*: Write $a^n = (1 + (a-1))^n$ and use the binomial theorem.
9. ∞
11. $+\infty$

Section 3.1

3. If $a_0 < 0$, then $f(x) \to +\infty$ as $x \to \pm\infty$; if $a_0 > 0$, then $f(x) \to -\infty$ as $x \to \pm\infty$.
7. $f(x) = x\sin(1/x)$, $c = 0$.

Section 3.2

1. 3, Yes; 0, No
3. 3, No; -1, No
5. 1, No; $\frac{1}{2}$, Yes
7. $3\pi/2$ Yes; $\pi/2$ Yes
11. (b) $\inf S = \inf(b_i)$, provided that the right-hand expression exists. Similarly, $\sup S = \sup\{B_i\}$.

Section 3.3

1. No; $\{x_{2n}\}$
3. No; $\{x_{2n}\}$
5. Yes
7. $No; \{x_{3n}\}$

Section 3.7

1. Yes
3. Yes
5. No; $\{x_{3n}\}$

13. (b) A finite subfamily has a smallest interval I_N. The point $x = 1/(N+3)$ is not covered.
15. No.
17. Finite subfamilies that cover E are $I_{\frac{4}{3}}$ and $I_{\frac{3}{2}}$.

Section 4.1

7(b). $\eta(h) = 3 - \frac{3+3h+h^2}{(1+h)^3}$

15. $\frac{1}{3}$
17. e

Section 4.2

1. $I_1 = (-\infty, -1]$, $I_2 = [-1, +\infty)$, $J_1 = [1, +\infty)$, $g_1(x) = -\sqrt{x-1}$, $g_2(x) = \sqrt{x-1} - 1$
3. $I_1 = (-\infty, 2]$, $I_2 = [2, +\infty)$, $J_1 = (-\infty, 4]$, $J_2 = (-\infty, 4]$, $g_1(x) = 2 - \sqrt{4-x}$, $g_2(x) = 2 + \sqrt{4-x}$
5. $I_1 = (-\infty, -2)$, $I_2 = (-2, +\infty)$, $J_1 = (2, +\infty)$, $J_2 = (-\infty, 2)$, $g_1(x) = 2x/(2-x)$, $g_2(x) = 2x/(2-x)$
7. $I_1 = (-\infty, -1]$, $I_2 = [-1, +1]$, $I_3 = [1, +\infty)$, $J_1 = [-2, 0)$, $J_2 = [-2, 2]$, $J_3 = (0, 2]$, $g_1(x) = (2 + \sqrt{4 - x^2})/x$, $g_2(x) = (2 - \sqrt{4 - x^2})/x$, $g_3(x) = (2 + \sqrt{4 - x^2})/x$
9. $I_1 = (-\infty, +\infty)$, $J_1 = (-\infty, +\infty)$
13. $f'(x) = 3(x-1)^2$, $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$
15. $f'(x) = \cos x$, $g'(x) = 1/\sqrt{1-x^2}$

Section 5.1

1. $S^+(f, \Delta) = \frac{11}{25}$, $S_-(f, \Delta) = \frac{6}{25}$
7. Note that $\sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] - f(x_n) - f(x_0) = f(b) - f(a)$.
17. Use the result in Problem 16.

Section 5.2

1. (c) Choose $f(x) = x$, $a \le x < b$, $f(b) = b - 1$.

7. No. Let $f(x) = 1/x$, $a = 1$, $b = +\infty$.

Section 6.1

3. Not equivalent
5. Equivalent
7. No

Section 6.2

3. The square with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.
5. No. The set $\{1/n\}$, $n = 1, 2, \ldots$, is an infinite set of isolated points.
7. $\bar{A} = \{x : 0 \le x \le 1\}$.
11. The sets $A_n = \{(x, y) : 0 \le x^2 + y^2 < 1/n\}$, $n = 1, 2, \ldots$.
15. Define (in $\mathbb{R}^1$), $A_i = \{x : 1/i < x \le 1\}$. Then $\cup A_i = \{x : 0 < x \le 1\}$; $\bar{B} = \{x : 0 \le x \le 1\} \ne \cup \bar{A}_i = \{x : 0 < x \le 1\}$.

Section 6.3

1. Arrange the rational points as in Figure 6.4.

Section 6.4

5. (b) Choose $x^n = (x_1^n, x_2^n, \ldots, x_k^n, \ldots)$ such that $x_k^n = 1$ if $k = n$, and $x_k^n = 0$ otherwise.

Section 7.1

5. $H_{,3}(x) = -4x_3(x_1^2 + x_4^2) + 2x_1^2(x_1^2 x_3 + x_4) + 2[\cos(x_1 + x_3) - 2x_4] \sin(x_1 + x_3)$.

Section 7.2

1. $f_{,1,1} = \frac{2x_1^2 - x_2^2 - x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}}$; f is symmetric in x_1, x_2, x_3.
7. $f(x) = f(a) + D_1 f(a)(x_1 - a_1) + D_2 f(a)(x_2 - a_2) + D_3 f(a)(x_3 - a_3) + \frac{1}{2} D_1^2 f(a)(x_1 - a_1)^2 + D_1 D_2 f(a)(x_1 - a_1)(x_2 - a_2) + D_1 D_3 f(a)(x_1 - a_1)(x_3 - a_3) + \frac{1}{2} D_2^2 f(a)(x_2 - a_2)^2 + D_2 D_3 f(a)(x_2 - a_2)(x_3 - a_3) + \frac{1}{2} D_3^2 f(a)(x_3 - a_3)^2 + \frac{1}{6} D_1^3 f(\xi)(x_1 - a_1)^3 + \frac{1}{2} D_1^2 D_2 f(\xi)(x_1 - a_1)^2(x_2 - a_2) + \frac{1}{2} D_1^2 D_3 f(\xi)(x_1 - a_1)^2(x_3 - a_3) + \frac{1}{2} D_1 D_2^2 f(\xi)(x_1 - a_1)(x_2 - a_2)^2 + D_1 D_2 D_3 f(\xi)(x_1 - a_1)(x_2 - a_2)(x_3 - a_3) + D_1 D_3^2 f(\xi)(x_1 - a_1)(x_3 - a_3)^2 + \frac{1}{6} D_2^3 f(\xi)(x_2 - a_2)^3 + \frac{1}{2} D_2^2 D_3 f(\xi)(x_2 - a_2)^2(x_3 - a_3) + \frac{1}{2} D_2 D_3^2 f(\xi)(x_2 - a_2)(x_3 - a_3)^2 + \frac{1}{6} D_3^3 f(\xi)(x_3 - a_3)^3$
9. Positive definite.
11. Negative definite.

Section 7.3

11. (a) Definition: The derivative of f at a is the linear function $L : \mathbb{R}^N \to \mathbb{R}^M$ such that

$$\lim_{x \to a} \frac{d_M(f(x), L(x))}{d_N(x, a)} = 0.$$

Section 8.1

1. Convergent
3. Convergent
5. Convergent
7. Convergent
9. Convergent
13. $p > 1$
17. Divergent

Section 8.2

1. Absolutely convergent
3. Conditionally convergent
5. Divergent
7. Divergent
9. Conditionally convergent
11. Conditionally convergent
13. Divergent
15. Convergent
17. $-1 < x < 1$
19. $-\frac{2}{3} \le x < \frac{2}{3}$
21. $1 < x < 5$
23. $-1 < x < +1$

Section 8.3

1. Uniform
3. Uniform
5. Uniform
7. Not uniform
9. Uniform

Section 8.4

1. $h < 1$
3. $h < 1$
5. $h < 3/2$
7. $h < 1$
9. $h < 1$

15. $1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{1}{2}-n+1)}{n!} x^{2n}$, $|x| < 1$
17. $1 + \sum_{n=1}^{\infty} \frac{(-1)^n(-3)(-4)\dots(-3-n+1)x^n}{n!}$, $|x| < 1$
19. $1 + \sum_{n=1}^{\infty} \frac{(-1)^n(-3)(-4)\dots(-3-n+1)x^{2n}}{n!}$, $|x| < 1$
21. $x^3 + \sum_{n=1}^{\infty} \frac{(-1)^n(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{1}{2}-n+1)}{n!(2n+1)} x^{6n+3}$, $|x| < 1$

Section 9.1

1. $\sin x - \int_0^1 \frac{\cos xt}{1+t}\, dt$
3. $\frac{1}{(x+1)}$, $x \neq -1$
5. $2\sin(x^2) - 3x\sin(x^3) + \frac{\cos(x^2)-\cos(x^3)}{x^2}$, $x \neq 0$, $\phi'(0) = 0$.
7. $\frac{2xe^{-1-x^2}}{1+x+x^3} + \frac{\sin xe^{-\cos x}}{1+x\cos x} - \frac{e^{-1-x^2}}{x^2(1+x+x^3)} + \frac{e^{-\cos x}}{x^2(1+x\cos x)} - \frac{1+x}{x^2}\int_{\cos x}^{1+x^2} \frac{e^{-t}}{1+xt}\, dt$, $x \neq 0$.
9. $\frac{2\sin(x^2)-3\sin(x^3)}{x}$.
11. $\frac{3e^{x^3}-e^{xy}}{x}$.
13. $-4x^4y - 4y^5 - 8x^2y + 8y^3$.
15.

$$\begin{aligned} F_{,1} &= f[x, y, h_1(x,y)]h_{1,1}(x,y) - f[x, y, h_0(x,y)]h_{0,1}(x,y) \\ &\quad + \int_{h_0(x,y)}^{h_1(x,y)} f_{,1}(x, y, t)dt, \\ F_{,2} &= f[x, y, h_1(x,y)]h_{1,2}(x,y) - f[x, y, h_0(x,y)]h_{0,2}(x,y) \\ &\quad + \int_{h_0(x,y)}^{h_1(x,y)} f_{,2}(x, y, t)dt. \end{aligned}$$

Section 9.2

1. Convergent
3. Convergent
5. Convergent
7. Divergent
9. Convergent
11. Convergent

Section 10.1

3. 4
5. 2

Section 10.2

1. (a) $\int_a^b f dg = 0$ (b) $\int_a^b f dg = (d_n - 1)f(b) + (1 - d_1)f(a)$.

3. 2

Section 11.1

1. $f' = 2x/(3y^2 + 1)$.
3. $f' = -\frac{xy^2+2y}{x^2y+3x}$.
5. Example: $F(x, y) = y^3 - x$, $(x_0, y_0) = (0, 0)$.
7. $f_{,1}(1, 0) = 2$, $f_{,2}(1, 0) = 0$.
9. $f_{,1}\left(0, \frac{1}{2}\right) = -\frac{3}{4}$, $f_{,2}\left(0, \frac{1}{2}\right) = -1$.

Section 11.2

1. 15
3. 45
5. 5/2
7. $(2, 1)$; $(-2, -1)$
9. $(\sqrt{2}, -\sqrt{2})$; $(-\sqrt{2}, \sqrt{2})$

Section 12.1

1. $\int_F \mathbf{u} dV = 0$.
3. $(2x_1 + x_2)\mathbf{e}_1 + (x_1 - 2x_2)\mathbf{e}_2 + 2x_3\mathbf{e}_3$ $D_{\mathbf{a}}f(\bar{x}) = 27/7$.
5. $2x_1 \log(1 + x_2^2)\mathbf{e}_1 + \frac{2x_1^2x_2}{1+x_2^2}\mathbf{e}_2 - 2x_3\mathbf{e}_3$; $D_{\mathbf{a}}f(\bar{x}) = \frac{4}{\sqrt{10}}$.
7. $D_{\mathbf{a}}\mathbf{w}(x) = -\frac{1}{\sqrt{14}}\mathbf{e}_1 + \frac{-x_2+2x_3}{\sqrt{14}}\mathbf{e}_2 + \frac{4x_2-2x_3}{\sqrt{14}}\mathbf{e}_3$; $D_{\mathbf{a}}\mathbf{w}(\bar{x}) = \frac{1}{\sqrt{14}}(-\mathbf{e}_1 + 7\mathbf{e}_2 - 2\mathbf{e}_3)$.
9. $\operatorname{div} \mathbf{v}(\bar{x}) = 0$.
11. $\operatorname{div} \mathbf{v}(\bar{x}) = -12$.
13. $\operatorname{div} \mathbf{v}(\bar{x}) = -\mathbf{a} \operatorname{curl} \mathbf{r}(\bar{x})$.
15. $\operatorname{curl} \mathbf{v} = \mathbf{0}$; $f(x) = x_1x_2x_3 + \frac{1}{3}(x_1^3 + x_2^3 + x_3^3)$.

Section 12.2

1. $\int_{\overrightarrow{C}} \mathbf{g} \cdot d\mathbf{r} = \frac{1}{3}$.
3. $\int_{\overrightarrow{C}} \mathbf{g} \cdot d\mathbf{r} = -2$.
5. $\int_{\overrightarrow{C}} \mathbf{g} \cdot d\mathbf{r} = \frac{11}{15}$.
7. $\int_{\overrightarrow{C}} \mathbf{g} \cdot d\mathbf{r} = \frac{a_1^2a_4}{3} + \frac{a_2^2}{2} - \frac{a_2a_3a_4}{3} + \frac{a_3a_4}{2}$.
9. $\int_{\overrightarrow{C}} \nabla\mathbf{u} \cdot d\mathbf{r} = 3$.

Section 12.3

9. -24π
11. 2π

Section 12.4

1. $\frac{\sqrt{3}}{6}$
3. $\frac{15\sqrt{2\pi}}{4}$
5. 2π
7. $\pi^4/192$
9. $(17 + 15\sqrt{2})\pi/4$
11. Case 1: $4\pi a^2\rho/c$; Case 2: $4\pi a\rho$.
13. $2\pi\rho$.

Section 12.5

1. 0
3. $\frac{15\pi}{2}$
5. $\frac{4}{15}$
13. 0
15. $\frac{\pi}{2}$

Section 12.6

1. $\frac{1}{8}$
3. $\frac{32\pi}{3}$
5. $12\sqrt{3}\pi$
7. 0
9. 2π
11. 0
13. 5π

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