Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains

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Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains

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On the occasion of the centenary of the birth of our teacher
Ya.B. Lopatinskiy

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Introduction

This book investigates the behavior of weak or strong solutions of the boundary value problems for the second order elliptic equations (linear and quasilinear) in the neighborhood of the boundary singularities. The author's main goal is to establish the precise exponent of the solution decrease rate and under the best possible conditions.

Currently there exists a fully developed theory for linear elliptic equations with partial derivatives so that it is now possible to advance toward a nonlinear equations analysis. Considerable success in this direction has been achieved particularly for the second order quasilinear elliptic equations, due to the works of Schauder, Caccioppoli, Leray and others (see [43, 129, 211, 216]). These authors have shaped a method that allows to prove existence theorems given the appropriate *a priori* estimates. This method does not require preliminary construction of the fundamental solution and allows instead an application of some functional analysis theorems rather than using an integral equation theory.

On the one hand, it is quite easy to prove the solvability of boundary value problems for the second order quasilinear equations, given the Hölder coefficients estimate of first derivatives of the solution of the appropriate linear boundary problem, with a constant which depends only on the maximum module of the problem coefficients. Thus, there appeared a necessity of studying linear problems more deeply and giving them more precise estimates. Many mathematicians' efforts were directed towards this. L. Nirenberg [329] obtained the above mentioned estimate for a two-dimensional nonselfadjoint equation, through which it is possible to establish the existence theorem of the Dirichlet problem for the second order quasilinear elliptic equations with minimal assumptions on the smoothness of the equation coefficients. In the case of a multi-dimensional equation such an estimate was obtained by H. Cordes [85], with the assumption that the equation complies with a condition (depending on the euclidean space dimension $N > 2$) that is stronger than uniform ellipticity. On the other hand, attempts of

obtaining the above mentioned *a priori* estimate for the second order general elliptic equations were not successful because such an estimate is simply impossible.

Thus, to prove the classical solvability of boundary value problems for second order quasilinear equations, it is necessary to create methods which produce the needed estimates directly for the non-linear problem itself. Such methods were created. The ideas for this new method can be already found in the works of S. Berstein and by De Giorgi and J. Nash ([**129, 216**]). O. Ladyzhenskaya and N. Ural'tseva improved upon and further developed this method and subsequently published their well-known monograph [**216**], in which the method is formulated and applied to different boundary value problems. Their research has inspired a number of other mathematicians; we note the works [**208**], [**224**] - [**238**], [**367**], [**401**]. All investigations mentioned above refer to boundary value problems in *sufficiently smooth* domains. It should be noted that these investigations represent a major effort by a large number of mathematicians stretching over a period of more than thirty years.

However, many problems of physics and technology lead to the necessity of studying boundary value problems in domains with nonsmooth boundaries. We are referring, in particular, to domains which have a finite number of angular ($N = 2$) or conical ($N > 2$) points, edges and other singularities on the boundary.

The state of the theory of boundary value problems on non-smooth domains, as it was twenty years ago, is described in detail in the well-known survey of V.A. Kondrat'ev and O.A. Oleinik [**177**], in the book of A. Kufner and A.-M. Sandig [**213**] as well as in the monographs of V.G. Maz'ya and his colleagues [**263, 198**]. For this reason we will focus on the developments of this theory since that time.

Among the first studies of the behavior of the solution of the boundary value problem in the neighborhood of an angular boundary point for the Laplace or Poisson equation, we can find the works [**77, 327, 403, 126**]. In the work [**327**] S. Nikol'skiy has established the necessary and sufficient conditions of belonging to the Nikolskiy's space H_p^r of the Dirichlet problem solution for the Laplace equation. E. Volkov [**403**] has described the necessary and sufficient conditions for belonging to the space $C^{k+\alpha}(G)$ (where k is an integer and $\alpha \in (0,1)$) of the Dirichlet problem solution for the Poisson equation $\triangle u = f(x)$ $x \in G$, in the case where G is a rectangle. V. Fufaev [**126**] has considered the Poisson equation $\triangle u = f(x)$ for $x \in G$ in the domain G, where $\partial G \setminus \mathcal{O}$ is an infinitely smooth curve, and in a certain neighborhood of the point $\mathcal{O}$ the boundary ∂G consists of two segments intersecting at an angle ω_0. The smoothness of

the Dirichlet problem solution depends upon ω_0. The smaller the angle ω_0, the smoother the solution is (if $f \in C^\infty(G)$.) There are exceptional values of ω_0, for which there are no obstacles for smoothness. In particular, if $u\big|_{\partial G} = 0$, $f = 0$ in a certain neighborhood of the point $\mathcal{O}$ and if $\frac{\pi}{\omega_0}$ is an integer, then $u \in C^\infty(G)$.

Thus, the violation of the boundary smoothness condition leads to the situation that for the boundary value problem solution there appear singularities in the neighborhood of the boundary irregular point. As we know, in the boundary value problems theory for elliptic equations in smooth domains, the situation is as follows: if the problem data are smooth enough, then the solution is also sufficiently smooth.

Some of the first works studying the general linear boundary value problems for the domains with conical or angular points were V. Kondratiev's fundamental works [**160, 161**] as well as papers of M. Birman & G. Skvortsov [**47**], G. Eskin [**115, 116**], Ya. Lopatinskiy [**242**], V. Maz'ya [**250**]-[**253**], [**255, 294**]. These works examine normal solvability and regularity in the Sobolev weighted spaces of general linear elliptic problems in non-smooth domains under assumptions of *sufficient smoothness* of both the manifold $\partial G \setminus \mathcal{O}$ and the problem coefficients. They consider solutions in special spaces of functions with the derivatives that are summable with some power weight. These spaces clearly show the basic singularity of the solutions of such problems. It has also become clear that the methods used for the analysis of boundary elliptic problems in smooth domains are not applicable because it is impossible to straighten the boundary by using a smooth transformation.

V. Kondratiev [**160, 161**] has studied this problem in L^2 Sobolev spaces, V. Maz'ya and B. Plamenevskiy [**271**]-[**280**] (see also [**263**]-[**269**], [**282**]) have extended the Kondratiev results to L^p Sobolev and other spaces. There are many other works concerning elliptic boundary value problems in non-smooth domains (see the bibliography).

The pioneering works in the study of elliptic boundary value problems in nonsmooth domains for quasilinear equations has been done by V. Maz'ya, I. Krol and B. Plamenevskiy [**256, 258**], [**204**]-[**207**], [**270**].

If we examine a nonlinear elliptic problem, then we would find it necessary to clarify the smoothness conditions for coefficients and for right parts of a linear problem under which the solvability in appropriate functional spaces and the necessary *a priori* estimates for the solution to take place. This clarification is dealt with in chapters 4 and 5. These chapters study the linear elliptic Dirichlet problem for the nondivergent form

equation

(L) $\begin{cases} Lu := a^{ij}(x)D_{ij}u(x) + a^i(x)D_i u(x) + a(x)u(x) = f(x), \text{ in } G, \\ u(x) = \varphi(x) \text{ on } \partial G \end{cases}$

and for the divergent form equation

(DL) $\begin{cases} \frac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j} + a^i(x)u) + b^i(x)u_{x_i} + c(x)u = \\ \qquad = g(x) + \frac{\partial f^j(x)}{\partial x_j}, \quad x \in G; \\ u(x) = \varphi(x), \quad x \in \partial G. \end{cases}$

The question of the smoothness of solutions in the neighborhood of an angular point for the linear nondivergence second order elliptic equation was studied earlier in the works [18]-[22]. There the authors assume that the equation coefficients are Hölder-continuous. Our assumptions concerning the smoothness of the coefficients are the least restrictive possible; leading coefficients of the the equation must be Dini-continuous at the conical point $\mathcal{O}$, whereas lower coefficients can grow and we indicate the exact power growth order. In §4.7 we construct the examples which show that the Dini condition for leading coefficients of the the equation at the conical point as well as the assumption concerning the lower equation coefficients, are essential for the validity of the estimates derived in the chapters 4 and 5. Otherwise in these estimates the exponent λ should be changed to $\lambda - \varepsilon$ with any $\varepsilon > 0$. The fact that the exponent λ in these estimates cannot be increased is shown by the partial solutions of the Laplace equation in the domain with the angular or conical point. In this sense the estimates of chapters 4 and 5 are the best possible.

The estimates obtained in §§4.5, 4.6, 4.9 allow us to formulate new existence theorems for the linear Dirichlet problem solution. These theorems are formulated and proved in §4.10.

The regularity theory of strong solutions for this problem and its solvability in *a smooth domain* are well developed [**129, 217, 208, 211**]. But theory involving nonsmooth domains is in its infancy. Existence theorems obtained in §4.10 play a fundamental role in chapter 7 when we consider the solvability of the quasilinear problem

(QL) $\begin{cases} a_{ij}(x, u, u_x)u_{x_i, x_j} + a(x, u, u_x) = 0, \quad a_{ij} = a_{ji}, \quad x \in G \\ u(x) = \varphi(x), \quad x \in \partial G. \end{cases}$

As mentioned above, to construct the theory of the Dirichlet problem solvability for quasilinear equations, the appropriate *a priori* estimates of a nonlinear task itself are needed. Chapter 7 is dedicated to obtaining such estimates. The local Hölder estimate (near an angular or conical point) of

the first derivatives of the solution is a central part in these estimates. Although the results obtained in §7.2 are completely included in the results of §7.3 (which is quite natural), the special character of the plane case allows us to single it out. In addition, the methods for obtaining estimates differ for the $N = 2$ case from the $N > 2$ case. We are interested in demonstrating the possibility of applying the L. Nirenberg method for domains with an angular point. Thus it becomes possible to establish a basic estimate

$$|u(x)| \leq c_0 |x|^{1+\gamma}$$

with a certain $\gamma > 0$. In the case of a conical point $(N > 2)$ this method is not suitable because it is purely two-dimensional. To obtain the similar estimate in this situation we resort to the barrier technique and apply the comparison principle. Theorems of §7.4 also show that the (QL) problem solutions have the same regularity (at a conical point) as the (L) problem solutions.

There is another observation which is worth pointing out. Known in linear theory, the method of non-smooth domain approximation by a sequence of smooth domains while examining nonlinear problems is not applicable because of the impossibility of the passage to the limit. We avoid this difficulty by introducing a quasi-distance function $r_\varepsilon(x)$. The introduction of such a function allows us to work in the given domain, and then to provide the passage to the limit over $\varepsilon \to +0$ (where $r_\varepsilon(x) \to r = |x|$.) We use the same method on studying the problems (L) in Chapter 4 and (DL) in Chapter 5.

In §7.3.6 we prove the theorems of the solution smoothness rise which are analogous to the linear case. The results of §4.10 (concerning the solvability of the linear problem) and the estimates for solutions of the nonlinear problem given in §§7.2, 7.3 allow us to proceed to the (QL) problem solvability in §7.4.

In summary, for chapters 4 and 7, we have completely constructed the theory of the first boundary problem solvability for second order non-divergent uniform elliptic equations in the domains with conical or angular points.

In Chapters 5, 8 and 9 we consider the theory which deals with equations of divergent type. The history of research development of such equations is richer because it is possible to study weak solutions of these problems which in turn change into the equivalent integral identity with no second generalized derivatives of the sought function. The detailed history of studies of the linear problem can be found in the following surveys [91, 133, 172, 177]. We will dwell on some of them. The exact solution estimates near singularities on the boundary have been obtained in the works [398, 399] under the

condition that leading coefficients of the equation satisfy the Hölder condition, and that lowest coefficients are missing. The rate of the solution decrease in the neighborhood of a boundary point is characterized by the function $\lambda(\varrho)$. (The latter is the least by the modulus eigenvalue of the Laplace-Beltrami operator in a domain Ω_ϱ on the sphere with zero Dirichlet data on $\partial\Omega_\varrho$.) In Chapter 5 we give results of the works [170, 171]. Under certain assumptions about the structure of the boundary of the domain in a neighborhood of the boundary point $\mathcal{O}$ and about the coefficients of the linear equation (DL), one can obtain a power modulus of continuity at $\mathcal{O}$ for a weak solution of the Dirichlet problem which vanishes at that point. Moreover, the exponent is the best possible for domains with the assumed boundary structure in that neighborhood. The assumptions on the coefficients of the equation are essential, as example §5.1.4 shows.

In [21] A.Azzam and V.Kondrat'ev have established the Hölder continuity of the first derivatives of weak solutions of the (DL) problem in the neighborhood of an angular point, under the condition of Hölder continuity of the equation coefficients. In this case the Hölder exponent satisfies the inequality $\alpha < \frac{\pi}{\omega_0} - 1$. In §5.2 we generalize this result for the case of a conical point and we also weaken the coefficient smoothness requirements to Dini-continuity.

In Chapter 6 we study properties of strong and weak solutions of the Dirichlet problem for semi-linear uniform elliptic second order equations in a neighborhood of a conical boundary point.

Let us consider the quasilinear problem for the divergence form equation

$$(DQL) \qquad Q(u, \phi) \equiv \int\limits_G \left\{a_i(x, u, u_x)\phi_{x_i} + a(x, u, u_x)\phi\right\} dx = 0.$$

The regularity theory of weak solutions for this problem and its solvability in *a smooth domain* are well-known [129, 215, 216] (see also [80, 83, 128, 237]). The regularity theory of weak solutions for quasilinear elliptic equations of the arbitrary order and elliptic systems as well as their solvability in *a smooth domain* are investigated in the monographs [183]- [185], [360, 100, 40].

The first investigations of the behavior of solutions for quasilinear elliptic equations in domains with angular and conical points were done by V. Maz'ya, I. Krol and B. Plamenevskiy [206, 207, 204, 205, 270, 281]. V. Maz'ya and I. Krol [204]-[207] have given estimates for the asymptotic behavior near reentrant boundary points of the equation of the type (LPA) solutions. V. Maz'ya and B. Plamenevskiy [270, 281] have constructed the asymptotic solution of the general quasilinear elliptic problem in a neighborhood of an angular or conical point.

Beginning 1981, there has appeared a series of works by P. Tolksdorf [373]-[378], E. Miersemann [301]-[309] and M. Dobrowolski [98, 99], where they have studied the behavior of the weak solutions to the (DQL) (see Chapter 8) in the neighborhood of an angular or conical boundary point. In [301]-[303] it is shown that a weak solution belongs to $W^2 \cap C^{1+\gamma}(\overline{G})$ for a certain $\gamma \in (0,1)$, under the assumption that $m = 2$, $\omega_0 \in (0, \pi)$, $a_i(x, u, z)\, i = 1, \ldots, N$ do not depend on x, u and the function $a(x, u, z)$ does not depend on u, z. Some elaborations and generalizations for a wider class of elliptic equations were made in [307]. In §8.1, chapter 8 of [133], P. Grisvard has considered the problem (DQL) for $N = 2$ in a convex polygon G and $a_i(x, u, z) \equiv a(z)z_i$ $(i = 1, 2)$; $a(x, u, z) \equiv f(x)$ (here $a(z)$ is a positive decreasing function and $a''(z)$ is continuous.) He has proved the existence and uniqueness of the solution from the space $W^{2,m}(G) \cap W_0^{1,m}(G)$, if $f(x) \in C^{1+\alpha}(G)$, $\alpha \in (0, 1)$, $2 < m < \frac{2}{2-\pi/\omega_0}$, where ω_0 is the measure of the largest angle on the polygon boundary. In [373]-[376, 378], P. Tolksdorf has considered the problem (DQL) with $a_i(x, u, z) \equiv a(|z|^2)z_i + b_i(z)$, $i = 1, \ldots, N$ and $a(x, u, z) \equiv f(x)$ under the following conditions

$$\nu(k + t)^{m-2} \le a(t^2) \le \mu(k + t)^{m-2}; \quad (\nu - \frac{1}{2})a(t) \le ta'(t) \le \mu a(t)$$

with some $\nu > 0, \mu > 0, k \in [0, 1]$ and $\forall t > 0$. In addition, it is assumed that

$$\lim_{t \to \infty} \frac{ta'(t)}{a(t)} = \frac{m - 2}{2} > -\frac{1}{2}; \quad \lim_{|z| \to \infty} \left| \frac{\partial b_j(z)}{\partial z_i} \right| \cdot a^{-1}(|z|^2) = 0.$$

He has obtained the upper- and lower-bounded estimates for the rate of the positive weak solution decrease in the neighborhood of the boundary conical point that is characterized by the lowest module eigenvalue of the nonlinear eigenvalue problem $(NEVP)$ (see Chapter 8, §8.2.2). In Chapter 8 we generalize these results for a wider class of equations and analyze arbitrary (not only positive) weak solutions. It is also important to note here that those estimates reinforce the Lipschiz-estimates of the (DQL) problem solution in the neighborhood of the boundary point by O.Ladyzhenskaya and N.Uraltseva [218], in the case when the boundary point is conical. In §8.2 we establish the power weight estimates of weak solutions, similar to the estimates in §7.3. In the latter the weight exponent is the best possible. The estimates of §8.2 allow us to obtain the best possible estimates of the weak solution module and its gradient. Finally, in §8.4 we estimate the second generalized derivatives of weak solutions in the Sobolev weighted space again with the best weight exponent.

In Chapter 9 we investigate the behavior of weak solutions of the first and mixed boundary value problems for the quasilinear second order elliptic equation with the triple degeneracy and singularity in the coefficients in a

neighborhood of the boundary edge. The coefficients of our equation near the edge are close to coefficients of the model equation

$$-\frac{d}{dx_i}\left(r^\tau |u|^q |\nabla u|^{m-2} u_{x_i}\right) + a_0 r^{\tau-m} u|u|^{q+m-2} -$$

$$(ME) \qquad\qquad -\mu r^\tau |u|^{q-1} |\nabla u|^m \operatorname{sgn} u = f(x),$$

$$0 \le \mu < 1, \quad q \ge 0, \quad m > 1, \quad a_0 \ge 0, \quad \tau \ge m - 2.$$

Chapter 10 is devoted to an investigation of the behavior of strong solutions to the Robin boundary value problem for second order elliptic equations (linear and quasilinear) in the neighborhood of a conical boundary point.

CHAPTER 1

Preliminaries

1.1. List of symbols

Let us fix some notations used in the whole book:

- $[l]$: the integral part of l (if l is not integer);
- $\mathbb{R}$ – the set of real numbers;
- $\mathbb{R}_+$ – the set of positive numbers;
- $\mathbb{R}^N$ – the N–dimensional Euclidean space, $N \geq 2$;
- $\mathbb{N}$ – the set of natural numbers;
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ – the set of nonnegative integers;
- $x = (x_1, \ldots, x_N)$ – an element of $\mathbb{R}^N$;
- $\mathcal{O} = (0, \ldots, 0)$;
- $(r, \omega) = (r, \omega_1, \ldots, \omega_{N-1})$ – spherical coordinates in $\mathbb{R}^N$ with pole $\mathcal{O}$ defined by

$$
\begin{aligned}
x_1 &= r \cos \omega_1, \\
x_2 &= r \sin \omega_1 \cos \omega_2, \\
\cdots &\quad \cdots\cdots\cdots\cdots \\
x_{N-1} &= r \sin \omega_1 \sin \omega_2 \cdots \sin \omega_{N-2} \cos \omega_{N-1}, \\
x_N &= r \sin \omega_1 \sin \omega_2 \cdots \sin \omega_{N-2} \sin \omega_{N-1};
\end{aligned}
$$

- S^{N-1} – the unit sphere in $\mathbb{R}^N$;
- $B_r(x_0)$ – the open ball with radius r centred at x_0;
- $\overline{B}_r(x_0)$ – the closed ball with radius r centred at x_0;
- $\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ – the volume of the unit ball in $\mathbb{R}^N$;
- $\sigma_N = N\omega_N$ – the area of the N–dimensional unit sphere;
- $\mathbb{R}^N_+$ – the half-space $\{x : x_N > 0\}$;
- Σ – the hyperplane $\{x : x_N = 0\}$;
- G – a bounded domain in $\mathbb{R}^N$;
- $G' \subset\subset G$ – G' has compact closure in G; G' is strictly contained in G;
- dx – volume element in $\mathbb{R}^N$;
- ds – area element in $\mathbb{R}^{N-1}$;

- $d\sigma$ – area element in $\mathbb{R}^{N-2}$;
- ∂G – the boundary of G, in what follows we shall assume that $\mathcal{O} \in \partial G$;
- $d(x) := \text{dist}(x, \partial G)$;
- $n = (n_1, \ldots, n_N)$ – exterior unit normal vector on ∂G;
- $\overline{G} = G \cup \partial G$ – the closure of G;
- $\text{meas}\, G$ – the Lebesgue measure of G;
- $\text{diam}\, G$ – the diameter of G;
- K – an open cone with vertex in $\mathcal{O}$;
- $\Omega := K \cap S^{N-1}$;
- $\mathcal{C}$: the rotational cone $\{x_1 > r \cos \frac{\omega_0}{2}\}$;
- $\partial \mathcal{C}$: the lateral surface of $\mathcal{C}$: $\{x_1 = r \cos \frac{\omega_0}{2}\}$;
- $\langle \cdot, \cdot \rangle$ – the scalar product of two vectors;
- $D_i u := \frac{\partial u}{\partial x_i}$;
- $\nabla u := (D_1 u, \ldots, D_N u)$;
- $D_{ij} u := \frac{\partial^2 u}{\partial x_i \partial x_j}$;
- $D^2 u$ – the Hessian of u;
- $|\nabla u| := (\sum\limits_{i=1}^{N} (D_i u)^2)^{1/2}$;
- $|D^2 u| := (\sum\limits_{i,j=1}^{N} (D_{ij} u)^2)^{1/2}$;
- $\beta = (\beta_1, \ldots, \beta_N)$, $\beta_i \in \mathbb{N}_0$ – an N–dimensional multi-index;
- $|\beta| := \beta_1 + \cdots + \beta_N$ – the length of the multi-index β;
- $D_x^\beta = D^\beta := \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_N^{\beta_N}}$ – a partial derivative of order $|\beta|$;
- $\frac{\partial u}{\partial n} = \langle \nabla u, n \rangle$ – the exterior normal derivative of u on ∂G;
- δ_i^j – Kronecker's delta;
- $\text{supp}\, u$: the *support* of u, the closure of the set on which $u \neq 0$;
- $c = c(*, \ldots, *)$ – a constant depending only on the quantities appearing in the parentheses. The same letter c will sometimes be used to denote different constants depending on the same set of arguments.

1.2. Elementary inequalities

In this section we review some elementary inequalities (see e.g. [37, 142]) which will be frequently used throughout this book.

LEMMA 1.1. (**Cauchy's Inequality**) *For $a, b \geq 0$ and $\varepsilon > 0$, we have*

$$(1.2.1) \qquad\qquad ab \leq \frac{\varepsilon}{2} a^2 + \frac{1}{2\varepsilon} b^2.$$

LEMMA 1.2. (**Young's Inequality**) *For* $a, b \geq 0$, $\varepsilon > 0$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$(1.2.2) \qquad ab \leq \frac{1}{p}(\varepsilon a)^p + \frac{1}{q}\left(\frac{b}{\varepsilon}\right)^q.$$

LEMMA 1.3. (**Hölder's Inequality**) *Let* a_i, b_i, $i = 1, \ldots, N$, *be non-negative real numbers and* $p, q \in \mathbb{R}$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *we have*

$$(1.2.3) \qquad \sum_{i=1}^{N} a_i b_i \leq \left(\sum_{i=1}^{N} a_i^p\right)^{1/p} \left(\sum_{i=1}^{N} b_i^q\right)^{1/q}.$$

LEMMA 1.4. (Theorem **41** [**142**]). *Let* a, b *be nonnegative real numbers and* $m \geq 1$. *Then*

$$(1.2.4) \qquad ma^{m-1}(a - b) \geq a^m - b^m \geq mb^{m-1}(a - b).$$

LEMMA 1.5. (**Jensen's Inequality**) (Theorem **65** [**142**], Lemma 1 [**357**]).
   Let a_i, $i = 1, \ldots, N$, *be nonnegative real numbers and* $p > 0$. *Then*

$$(1.2.5) \qquad \lambda \sum_{i=1}^{N} a_i^p \leq \left(\sum_{i=1}^{N} a_i\right)^p \leq \Lambda \sum_{i=1}^{N} a_i^p,$$

where $\lambda = \min(1, N^{p-1})$ *and* $\Lambda = \max(1, N^{p-1})$.

LEMMA 1.6. *Let* a, $b \in \mathbb{R}$, $m > 1$. *Then the familiar inequality*

$$(1.2.6) \qquad |b|^m \geq |a|^m + m|a|^{m-2}a(b - a).$$

is valid.

PROOF. By Young's inequality (1.2.2) with $\varepsilon = 1$, $p = m$, $q = \frac{m}{m-1}$, we have

$$m|a|^{m-2}ab \leq m|b| \cdot |a|^{m-1} \leq |b|^m + (m - 1)|a|^m \implies (1.2.6).$$

□

LEMMA 1.7. *For* $m > 1$ *the inequality*

$$(1.2.7) \qquad \int_0^1 |(1 - t)z + tw|^{m-2}dt \geq c(m)(|z| + |w|)^{m-2}$$

holds for some positive constant $c(m)$.

PROOF. This inequality is trivial, if $1 < m \leq 2$, and in this case $c(m) = 1$. Let $m > 2$. If $|z| + |w| = 0$, then this inequality holds with any $c(m)$. Let now $|z| + |w| \neq 0$. Setting

$$\zeta = \frac{z}{|z| + |w|}, \quad \eta = \frac{w}{|z| + |w|} \quad \Longrightarrow \quad |\zeta| + |\eta| = 1$$

we want to prove the inequality

$$\int\limits_0^1 |(1-t)\zeta + t\eta|^{m-2}dt \geq c(m).$$

We consider the function

$$f(\zeta, \eta) = \int\limits_0^1 |(1-t)\zeta + t\eta|^{m-2}dt$$

on the set $\mathfrak{E} = \{(\zeta, \eta) \in \mathbb{R}^2 \big| |\zeta| + |\eta| = 1.\}$ This function is continuous on $\mathfrak{E}$, since $m > 2$. The set $\mathfrak{E}$ is finite-dimensional and bounded, and therefore it is the compact set. By the Weierstrass Theorem, such a function achieves the minimum on $\mathfrak{E}$ in some point $(\zeta_0, \eta_0) \in \mathfrak{E}$. It is clear that $f(\zeta_0, \eta_0) \geq 0$. Suppose that $f(\zeta_0, \eta_0) = 0$. Then we have

$$|(1-t)\zeta_0 + t\eta_0| = 0, \quad \forall t \in [0,1] \quad \Longrightarrow \quad (\zeta_0, \eta_0) = (0,0) \notin \mathfrak{E}.$$

Hence it follows that $f(\zeta_0, \eta_0) > 0$ and therefore there is a positive constant $c(m)$ such that the required inequality is fulfilled. $\square$

1.3. Domains with a conical point

DEFINITION 1.8. Let $G \subset \mathbb{R}^N$ be a bounded domain. We say that G has a conical point in $\mathcal{O}$ if

- $\mathcal{O} \in \partial G$,
- $\partial G \setminus \mathcal{O}$ is smooth,
- G coincides in some neighbourhood of $\mathcal{O}$ with an open cone K,
- $\partial K \cap S^{N-1}$ is smooth,
- K is contained in a circular cone with the opening angle $\omega_0 \in (0, 2\pi)$.

For a domain G which has a conical point at $\mathcal{O} \in \partial G$ we introduce the notations:

- $\Omega := K \cap S^{N-1}$;
- $d\Omega :=$ area element of Ω;
- $G_a^b := G \cap \{(r, \omega) : 0 \leq a < r < b, \omega \in \Omega\}$— a layer in $\mathbb{R}^N$;
- $\Gamma_a^b := \partial G \cap \{(r, \omega) : 0 \leq a < r < b, \omega \in \partial \Omega\}$— the lateral surface of the layer G_a^b;

- $G_d := G \setminus G_0^d$;
- $\Gamma_d := \partial G \setminus \Gamma_0^d$;
- $\Omega_\rho := \overline{G_0^d} \cap \partial B_\varrho(0)$, $\varrho \le d$;
- $G^{(k)} := G_{2-(k+1)d}^{2-kd}$, $k = 0, 1, 2, \ldots$.

Let us recall some well known formulae related to spherical coordinates $(r, \omega_1, \ldots, \omega_{N-1})$ centered at the conical point $\mathcal{O}$:

$$(1.3.1) \qquad dx = r^{N-1} dr \, d\Omega,$$

$$(1.3.2) \qquad d\Omega_\rho = \rho^{N-1} d\Omega,$$

$$(1.3.3) \qquad d\Omega = J(\omega) d\omega$$

denotes the $(N-1)$-dimensional area element of the unit sphere;

$$(1.3.4) \qquad J(\omega) = \sin^{N-2} \omega_1 \sin^{N-3} \omega_2 \cdots \sin \omega_{N-2},$$

$$(1.3.5) \qquad d\omega = d\omega_1 \cdots d\omega_{N-1},$$

$$(1.3.6) \qquad ds = r^{N-2} dr \, d\sigma$$

denotes the $(N-1)$-dimensional area element of the lateral surface of the cone K, where $d\sigma$ denotes the $(N-2)$-dimensional area element on $\partial\Omega$;

$$(1.3.7) \qquad |\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2} |\nabla_\omega u|^2,$$

where $|\nabla_\omega u|$ denotes the projection of the vector ∇u onto the tangent plane to the unit sphere at the point ω

$$(1.3.8) \qquad \nabla_\omega u = \left\{ \frac{1}{\sqrt{q_1}} \frac{\partial u}{\partial \omega_1}, \ldots, \frac{1}{\sqrt{q_{N-1}}} \frac{\partial u}{\partial \omega_{N-1}} \right\},$$

$$(1.3.9) \qquad |\nabla_\omega u|^2 = \sum_{i=1}^{N-1} \frac{1}{q_i} \left(\frac{\partial u}{\partial \omega_i}\right)^2,$$

where $q_1 = 1$, $q_i = (\sin\omega_1 \cdots \sin\omega_{i-1})^2$, $i \ge 2$,

$$(1.3.10) \qquad \Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{N-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_\omega u,$$

$$(1.3.11) \qquad \Delta_\omega u = \frac{1}{J(\omega)} \sum_{i=1}^{N-1} \frac{\partial}{\partial \omega_i} \left(\frac{J(\omega)}{q_i} \cdot \frac{\partial u}{\partial \omega_i}\right) =$$

$$= \sum_{i=1}^{N-1} \frac{1}{q_j \sin^{N-i-1} \omega_i} \frac{\partial}{\partial \omega_i} \left(\sin^{N-i-1} \omega_i \frac{\partial u}{\partial \omega_i}\right)$$

denotes the Beltrami-Laplace operator

$$(1.3.12) \qquad \mathrm{div}_\omega u = \frac{1}{J(\omega)} \sum_{i=1}^{N-1} \frac{\partial}{\partial \omega_i} \left(\frac{J(\omega)}{\sqrt{q_i}} u \right).$$

LEMMA 1.9. *Let* $\alpha \in \mathbb{R}$ *and* $v(x) = r^\alpha u(x)$. *Then*

$$
\begin{aligned}
D_i v &= \alpha r^{\alpha-2} x_i u + r^\alpha D_i u, \\
|\nabla v|^2 &\le c_1 \left(\alpha^2 r^{2\alpha-2} u^2 + r^{2\alpha} |\nabla u|^2 \right), \\
D_{ij} v &= r^\alpha D_{ij} u + \alpha r^{\alpha-2} \left(x_i D_j u + x_j D_i u \right) \\
&\quad + (\alpha^2 - 2\alpha) r^{\alpha-4} x_i x_j u + \alpha r^{\alpha-2} u \delta_i^j, \\
|D^2 v|^2 &\le c_2 \left(r^{2\alpha} |D^2 u|^2 + r^{2\alpha-2} |\nabla u|^2 + r^{2\alpha-4} u^2 \right)
\end{aligned}
$$

with constants $c_1, c_2 > 0$ *depending only on* α *and* N.

LEMMA 1.10. *Let there be a* $d > 0$ *such that* G_0^d *is the convex rotational cone with the vertex at* $\mathcal{O}$ *and the aperture* ω_0, *thus*

$$(1.3.13) \qquad \Gamma_0^d = \left\{ (r, \omega) \middle| x_1^2 = \cot^2 \frac{\omega_0}{2} \sum_{i=2}^N x_i^2; \ |\omega_1| = \frac{\omega_0}{2}, \ \omega_0 \in (0, \pi) \right\}.$$

Then

$$(1.3.14) \qquad x_i \cos(\vec{n}, x_i)|_{\Gamma_0^d} = 0, \ \text{and} \ \cos(\vec{n}, x_1)|_{\Gamma_0^d} = -\sin \frac{\omega_0}{2},$$

PROOF. By virtue of (1.3.13) we can rewrite the equation of Γ_0^d in this way

$$F(x) \equiv x_1^2 - \cot^2 \frac{\omega_0}{2} \sum_{i=2}^N x_i^2 = 0.$$

We use the formula $\cos(\vec{n}, x_i) = \frac{\frac{\partial F}{\partial x_i}}{|\nabla F|}$, $\forall i = 1, \ldots, N$. Because of

$$\frac{\partial F}{\partial x_1} = 2 x_1 \ \text{and} \ \frac{\partial F}{\partial x_i} = -2 \cot^2 \frac{\omega_0}{2} x_i, \ \forall i = 2, \ldots, N,$$

then

$$x_i \cos(\vec{n}, x_i)|_{\Gamma_0^d} = \frac{1}{|\nabla F|} x_i \frac{\partial F}{\partial x_i}\bigg|_{\Gamma_0^d} = \frac{2}{|\nabla F|} \left(x_1^2 - \cot^2 \frac{\omega_0}{2} \sum_{i=2}^N x_i^2 \right)\bigg|_{\Gamma_0^d} = 0.$$

Because of

$$|\nabla F|^2 = \left(\frac{\partial F}{\partial x_1} \right)^2 + \sum_{i=2}^N \left(\frac{\partial F}{\partial x_i} \right)^2 = 4 \left(x_1^2 + \cot^4 \frac{\omega_0}{2} \sum_{i=2}^N x_i^2 \right) \Rightarrow$$

$$|\nabla F|^2 \big|_{\Gamma_0^d} = 4 x_1^2 \left(1 + \frac{\cos^2 \frac{\omega_0}{2}}{\sin^2 \frac{\omega_0}{2}} \right) = \frac{4 x_1^2}{\sin^2 \frac{\omega_0}{2}},$$

we have

$$\cos(\vec{n}, x_1)\Big|_{\Gamma_0^d} = -2x_1 \frac{\sin \frac{\omega_0}{2}}{2x_1} = -\sin \frac{\omega_0}{2}, \quad \text{since} \quad \angle\,(\vec{n}, x_1) > \frac{\pi}{2}.$$

$\square$

1.4. The quasi-distance function r_ε and its properties

Let us assume as in Definition 1.8 that the cone K is contained in a circular cone $\tilde{K}$ with opening angle ω_0. Furthermore, let us suppose that the axis of $\tilde{K}$ coincides with $\{(x_1, 0, \ldots, 0) : x_1 > 0\}$. In this case we define *the quasi-distance* $r_\varepsilon(x)$ as follows. We fix the point $Q = (-1, 0, ..., 0) \in S^{N-1} \setminus \bar{\Omega}$ and consider the unit radius-vector $\vec{l} = \mathcal{O}Q = \{-1, 0, ..., 0\}$. We denote by $\vec{r}$ the radius-vector of the point $x \in \bar{G}$ and introduce the vector $\vec{r_\varepsilon} = \vec{r} - \varepsilon\vec{l}$ for each $\varepsilon > 0$. Since $\varepsilon\vec{l} \notin G_0^d$ for all $\varepsilon \in\,]0, d[$, it follows that $r_\varepsilon(x) = |\vec{r} - \varepsilon\vec{l}| \neq 0$ for all $x \in \bar{G}$. It is easy to see that $r_\varepsilon(x)$ has the following properties:

(1)

LEMMA 1.11. *There exists an $h > 0$ such that*
$r_\varepsilon(x) \geq hr$ *and* $r_\varepsilon(x) \geq h\varepsilon$, $\quad \forall x \in \bar{G}$, *where*

$$h = \begin{cases} 1, & \text{if} \quad 0 < \omega_0 \leq \pi\,, \\ \sin \frac{\omega_0}{2}, & \text{if} \quad \pi < \omega_0 < 2\pi. \end{cases}$$

PROOF. From the definition of $r_\varepsilon(x)$ we know that

$$r_\varepsilon^2 = (x_1 + \varepsilon)^2 + \sum_{i=2}^{N} x_i^2 = (x_1 + \varepsilon)^2 + r^2 - x_1^2 = r^2 + 2\varepsilon x_1 + \varepsilon^2.$$

If $0 < \omega_0 \leq \pi$, we have $x_1 \geq 0$ and therefore we obtain either $r_\varepsilon^2 \geq r^2 \Rightarrow r_\varepsilon \geq r$ or $r_\varepsilon^2 \geq \varepsilon^2 \Rightarrow r_\varepsilon \geq \varepsilon$.
If $x_1 = r\cos\omega \leq 0$ and $|\omega| \in [\frac{\pi}{2}, \frac{\omega_0}{2}]$, we obtain, by the Cauchy inequality either

$$|2\varepsilon r\cos\omega| \leq r^2\cos^2\omega + \varepsilon^2 \Rightarrow 2\varepsilon r\cos\omega \geq -r^2\cos^2\omega - \varepsilon^2 \Rightarrow r_\varepsilon \geq r\cdot\sin\frac{\omega_0}{2}$$

or

$$|2\varepsilon r\cos\omega| \leq \varepsilon^2\cos^2\omega + r^2 \Rightarrow 2\varepsilon r\cos\omega \geq -r^2 - \varepsilon^2\cos^2\omega \Rightarrow r_\varepsilon \geq \varepsilon\cdot\sin\frac{\omega_0}{2}.$$

$\square$

(2)

COROLLARY 1.12.

$$hr \leq r_\varepsilon(x) \leq r + \varepsilon \leq \frac{2}{h}r_\varepsilon(x); \; \forall x \in \bar{G}, \; \forall \varepsilon > 0.$$

(3) *If $x \in G_d$, then $r_\varepsilon(x) \geq \frac{d}{2}$ for all $\varepsilon \in]0, \frac{d}{2}[$.*

(4) $\lim\limits_{\varepsilon \to 0^+} r_\varepsilon(x) = r$, *for all $x \in \bar{G}$.*

(5) $|\nabla r_\varepsilon|^2 = 1$, *and* $\triangle r_\varepsilon = \frac{N-1}{r_\varepsilon}$.

PROOF. Because $\frac{\partial r_\varepsilon}{\partial x_1} = \frac{x_1 + \varepsilon}{r_\varepsilon}$ and $\frac{\partial r_\varepsilon}{\partial x_i} = \frac{x_i}{r_\varepsilon}$ ($i \geq 2$), then

$$|\nabla r_\varepsilon|^2 = \sum_{i=1}^{N} \left(\frac{\partial r_\varepsilon}{\partial x_i}\right)^2 = \frac{(x_1+\varepsilon)^2 + \sum\limits_{i=2}^{N} x_i^2}{r_\varepsilon^2} = 1 \text{ and}$$

$$\triangle r_\varepsilon = \frac{\partial^2 r_\varepsilon}{\partial x_1^2} + \sum_{i=2}^{N} \frac{\partial^2 r_\varepsilon}{\partial x_i^2} = \frac{1}{r_\varepsilon} - \frac{(x_1+\varepsilon)^2}{r_\varepsilon^3} + \sum_{i=2}^{N}\left(\frac{1}{r_\varepsilon} - \frac{x_i^2}{r_\varepsilon^3}\right) = \frac{N}{r_\varepsilon} - \frac{r_\varepsilon^2}{r_\varepsilon^3} =$$

$$= \frac{N-1}{r_\varepsilon}. \qquad \qquad \square$$

1.5. Function spaces

1.5.1. Lebesgue spaces. Let G be a domain in $\mathbb{R}^N$. For $p \geq 1$ we denote by $L^p(G)$ be the space of Lebesgue integrable functions equipped with the norm

$$\|u\|_{L^p(G)} = \left(\int_G |u|^p dx\right)^{1/p}.$$

THEOREM 1.13. **(Fubini's Theorem,** see Theorem 9 §11, Chapter III [101]). *Let $G_1 \subset \mathbb{R}^{m_1}, G_2 \subset \mathbb{R}^{m_2}$ and $f \in L^1(G_1 \times G_2)$. Then for almost all $x \in G_1$ and $y \in G_2$ the integrals*

$$\int_{G_1} f(x,y)dx \quad and \quad \int_{G_2} f(x,y)dy$$

exist. Moreover,

$$\int_{G_1 \times G_2} f(x,y)dxdy = \int_{G_1}\left(\int_{G_2} f(x,y)dy\right)dx = \int_{G_2}\left(\int_{G_1} f(x,y)dx\right)dy.$$

THEOREM 1.14. **(Hölder's Inequality,** see Theorem 189 [142]). *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in L^p(G)$, $v \in L^q(G)$. Then*

$$(1.5.1) \qquad \int_G |uv|dx \leq \|u\|_{L^p(G)}\|v\|_{L^q(G)}.$$

If $p = 1$, then (1.5.1) is valid with $q = \infty$.

COROLLARY 1.15. *Let $1 \leq p \leq q$ and $u \in L^p(G)$, $v \in L^q(G)$. Then*

$$(1.5.2) \qquad \|u\|_{L^p(G)} \leq (\text{meas}\, G)^{1/p - 1/q}\|v\|_{L^q(G)}.$$

COROLLARY 1.16. (**Interpolation inequality**) *Let* $1 < p \leq q \leq r$ *and* $1/q = \lambda/p + (1 - \lambda)/r$. *Then the inequality*

$$\|u\|_{L^q(G)} \leq \|u\|_{L^p(G)}^{\lambda} \|u\|_{L^r(G)}^{1-\lambda}$$

holds for all $u \in L^r(G)$.

THEOREM 1.17. (**Minkowski's Inequality**, see Theorem 198 [**142**]). *Let* $u, v \in L^p(G)$, $p > 1$. *Then* $u + v \in L^p(G)$ *and*

(1.5.3) $$\|u + v\|_{L^p(G)} \leq \|u\|_{L^p(G)} + \|v\|_{L^p(G)}.$$

THEOREM 1.18. (**Clarkson's Inequality**, see §3.2, Chapter I [**363**]). *Let* $u, v \in L^p(G)$. *Then*

$$\left\|\frac{u+v}{2}\right\|_{L^p(G)}^p + \left\|\frac{u-v}{2}\right\|_{L^p(G)}^p \leq \frac{1}{2}\left(\|u\|_{L^p(G)}^p + \|v\|_{L^p(G)}^p\right), \quad 2 \leq p < \infty;$$

$$\left\|\frac{u+v}{2}\right\|_{L^p(G)}^{\frac{p}{p-1}} + \left\|\frac{u-v}{2}\right\|_{L^p(G)}^{\frac{p}{p-1}} \leq \left(\frac{1}{2}\|u\|_{L^p(G)}^p + \frac{1}{2}\|v\|_{L^p(G)}^p\right)^{\frac{1}{p-1}}, \quad 1 \leq p \leq 2.$$

THEOREM 1.19. (**Fatou's Theorem**, see Theorem 19 §6, Chapter III [**101**]). *Let* $f_k \in L^1(G)$, $k \in \mathbb{N}$, *be a sequence of non–negative functions convergent almost everywhere in* G *to the function* f. *Then*

(1.5.4) $$\int_G f \, dx \leq \sup \int_G f_k \, dx.$$

LEMMA 1.20. [**328**, Lemma 1.3.8] *Let* $G_1 \subset \mathbb{R}^{m_1}, G_2 \subset \mathbb{R}^{m_2}$ *and* $f, f_k \in L^p(G_1 \times G_2), k = 1, 2, \ldots$, *with* $1 \leq p \leq \infty$ *and*

$$\lim_{k \to \infty} \|f - f_k\|_{L^p(G_1 \times G_2)} = 0.$$

Then there is a subsequence $\{f_{k_l}\}$ *of* $\{f_k\}$ *such that*

$$\lim_{k_l \to \infty} \|f(y, z) - f_{k_l}(y, z)\|_{L^p(G_2)} = 0$$

holds for almost every $y \in G_1$.

1.5.2. Regularization and Approximation by Smooth Functions. Let us denote by $L_{loc}^p(G)$ the linear space of all measurable functions which are locally p–integrable in G, that is p–integrable on every compact subset of G. Although $L_{loc}^p(G)$ is not a normed spaces, it can be readily topologized.

DEFINITION 1.21. We say that a sequence $\{u_m\}$ converges to u in the sense of $L_{loc}^p(G)$ if $\{u_m\}$ converges to u in $L^p(G')$ for each $G' \subset\subset G$.

Let $r = |x - y|$ for all $x, y \in \mathbb{R}^N$ and h be any positive number. Furthermore, let $\psi_h(r)$ be a non-negative function in $C^\infty(\mathbb{R}^N)$ vanishing outside the ball $B_h(0)$ and satisfying $\int_{\mathbb{R}^N} \psi_h(r)dx = 1$. Such a function is often called a *mollifier*. A typical example is the function $\psi_h(r)$ given by

$$\psi_h(r) = \begin{cases} ch^{-N} \cdot \exp\left(\frac{h^2}{|r|^2 - h^2}\right) & \text{for } r < h, \ c = \text{const} > 0; \\ 0 & \text{for } |r| \geq h, \end{cases}$$

where c is chosen so that $\int \psi_h(r)dx = 1$ and whose graph has the familiar bell shape.

DEFINITION 1.22. For $L^1_{loc}(G)$ and $h > 0$, the *regularization* of u, denoted by u_h is then defined by the convolution

$$(1.5.5) \qquad\qquad u_h(x) = \int_G \psi_h(r)u(y)dy$$

provided $h < dist(x, \partial G)$.

It is clear that u_h belongs to $C^\infty(G')$ for any $G' \subset\subset G$ provided $h < dist(G', \partial G)$. Furthermore, if u belongs to $L^1(G)$ and G is bounded, then u_h belongs to $C^\infty(\mathbb{R}^N)$ for arbitrary $h > 0$. As h tends to zero, the function $\psi_h(r)$ tends to the Dirac delta distribution at the point x. The significant feature of regularization, which we partly explore now, is the sense in which u_h approximates u as h tends to zero. It turns out, roughly stated, that if u lies in a local space, then u_h approximates u in the natural topology of that space.

LEMMA 1.23. *Let $u \in C^0(G)$. Then u_h converges to u uniformly on any subdomain $G' \subset\subset G$.*

PROOF. We have

$$u_h(x) = \int_{|x-y| \leq h} \psi_h(r)u(y)dy = \int_{|z| \leq 1} \psi_1(|z|)|u(x - hz)|dz$$

$\left(\text{putting } z = \frac{x-y}{h}\right)$. Hence if $G' \subset\subset G$ and $2h < dist(G', \partial G)$,

$$\sup_{G'} |u - u_h| \leq \sup_{x \in G'} \int_{|z| \leq 1} \psi_1(|z|)|u(x) - u(x - hz)|dz \leq$$

$$\leq \sup_{x \in G'} \sup_{|z| \leq 1} |u(x) - u(x - hz)|.$$

Since u is uniformly continuous over the set $B_h(G') = \{x \mid dist(x, G') < h\}$, the sequence u_h tends to u uniformly on G'. $\square$

LEMMA 1.24. *Let $u \in L^p_{loc}(G)$ $\left(L^p(G)\right)$, where $1 < p < \infty$. Then u_h converges to u in the sense of $L^p_{loc}(G)$ $\left(L^p(G)\right)$.*

PROOF. Using Hölder's inequality, we obtain from (1.5.5)

$$|u_h(x)|^p \leq \int\limits_{|z| \leq 1} \psi_1(|z|)|u(x - hz)|^p dz$$

so that if $G' \subset\subset G$ and $2h < \text{dist}(G', \partial G)$, then

$$\int\limits_{G'} |u_h(x)|^p dx \leq \int\limits_{G'} \int\limits_{|z| \leq 1} \psi_1(|z|)|u(x - hz)|^p dz dx =$$

$$= \int\limits_{|z| \leq 1} \psi_1(|z|) dz \int\limits_{G'} |u(x - hz)|^p dx \leq \int\limits_{B_h(G')} |u|^p dx,$$

where $B_h(G') = \{x : \text{dist}(x, G') < h\}$. Consequently

$$(1.5.6) \qquad \|u_h\|_{L^p(G')} \leq \|u\|_{L^p(B_h(G'))}.$$

The proof can now be completed by an approximation based on Lemma 1.23. We choose $\varepsilon > 0$ together with a $C^0(G)$ function w satisfying

$$\|u - w\|_{L^p(B_{h'}(G'))} \leq \varepsilon$$

where $2h' < \text{dist}(G', \partial G)$. By virtue of Lemma 1.23, we have for sufficiently small h that $\|w - w_h\|_{L_p(G')} \leq \varepsilon$. Applying the estimate (1.5.6) to the difference $u - w$ we obtain

$$\|u - u_h\|_{L^p(G')} \leq \|u - w\|_{L^p(G')} + \|w - w_h\|_{L^p(G')} + \|u_h - w_h\|_{L^p(G')} \leq$$

$$\leq 2\varepsilon + \|u - w\|_{L^p(B_h(G'))} \leq 3\varepsilon$$

for small enough $h \leq h'$. Hence u_h converges to u in $L^p_{loc}(G)$. The result for $u \in L^p(G)$ can then be obtained by extending u to be zero outside G and applying the result for $L^p_{loc}(\mathbb{R}^N)$. $\square$

LEMMA 1.25. **(On the passage to the limit under the integral symbol)** [361, Theorem III.10] *Let $\chi(x) \in L_\infty(G)$ and let $\chi_h(x)$ be the regularization of χ. Then for any $u \in L^1(G)$*

$$\lim_{h \to 0} \int\limits_G \chi_h(x) u(x) dx = \int\limits_G \chi(x) u(x) dx.$$

1.6. Hölder and Sobolev spaces

1.6.1. Notations and definitions. In this section $G \subset \mathbb{R}^N$ is a bounded domain of the class $C^{0,1}$. Let $x_0 \in \mathbb{R}^N$ be a point and f a function defined on $G \ni x_0$. The function f is *Hölder continuous with exponent* $\alpha \in (0,1)$ *at* x_0 if the quantity

$$[f]_{\alpha;x_0} = \sup_{x \in G} \frac{|f(x) - f(x_0)|}{|x - x_0|^\alpha}$$

is finite. $[f]_{\alpha;x_0}$ is said to be the α-Hölder coefficient of f at x_0 with respect to G.

The function f is *uniformly Hölder continuous with exponent* $\alpha \in (0,1)$ *in* G if the quantity

$$[f]_{\alpha;G} = \sup_{\substack{x,y \in G \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite.

We consider the following spaces.

- $C^l(\overline{G})$: the Banach space of functions having all the derivatives of order at most l (if l is a nonnegative integer) and of order $[l]$ (if l is non-integer) continuous in $\overline{G}$ and whose $[l]$-th order partial derivatives are uniformly Hölder continuous with exponent $l - [l]$ in $\overline{G}$. $|u|_{l;G}$ is the norm of the element $u \in C^l(\overline{G})$; if $l \neq [l]$ then

$$|u|_{l;G} = \sum_{j=0}^{[l]} \sup_G |D^j u| + \sup_{|\alpha|=[l]} \sup_{\substack{x,y \in G \\ x \neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^{l-[l]}}.$$

- $C_0^l(G)$: the set of functions in $C^l(G)$ with compact support in G.
- $W^{k,p}(G)$, $1 \leq p < \infty$: the Sobolev space equipped with the norm

$$\|u\|_{W^{k,p}(G)} = \left(\int_G \sum_{|\beta| \leq k} |D^\beta u|^p dx \right)^{1/p},$$

- $W_0^{k,p}(G)$ is the closure of $C_0^\infty(G)$ with respect to the norm $\| \cdot \|_{W^{k,p}(G)}$.
- $W^{k,p}(G \setminus \mathcal{O}) = W^{k,p}(G \setminus B_\varepsilon(0))$, $\forall \varepsilon > 0$.
- For $p = 2$ we use the notation

$$W^k(G) \equiv W^{k,2}(G), \quad W_0^k(G) \equiv W_0^{k,2}(G).$$

DEFINITION 1.26. Let us say that $u \in W^{k,p}(G)$ satisfies $u \leq 0$ on ∂G in the sense of traces, if its positive part $u^+ = \max\{u, 0\} \in W_0^{k,p}(G)$. If u

is continuous in a neighborhood of ∂G, then u satisfies $u \leq 0$ on ∂G, if the inequality holds in the classical pointwise sense. Other definitions of the inequality at ∂G follow naturally. For example: $u \geq 0$ on ∂G, if $-u \leq 0$ on ∂G; $u = v$ on ∂G, if both $u - v \leq 0$ and $u - v \geq 0$ on ∂G;

$$\sup_{\partial G} u = \inf\{k|\, u \leq k \text{ on } \partial G, \ k \in \mathbb{R}\}; \qquad \inf_{\partial G} u = -\sup_{\partial G}(-u).$$

- For $\Gamma \subseteq \partial G$ and $k \in 1, 2, \ldots$, the space $W^{k-\frac{1}{p},p}(\Gamma)$ consists of traces on Γ of functions from $W^{k,p}(G)$ and is equipped with the norm

$$\|\varphi\|_{W^{k-\frac{1}{p},p}(\Gamma)} = \inf \|\Phi\|_{W^{k,p}(G)},$$

where the infimum is taken over the set of all functions $\Phi \in W^{k,p}(G)$ such that $\Phi = \varphi$ on Γ in the sense of traces. For $p = 2$ we use the notation

$$W^{k-1/2}(\Gamma) \equiv W^{k-\frac{1}{2},2}(\Gamma).$$

THEOREM 1.27. [129, Theorem 7.28] (Interpolation inequality) *Let G be a $C^{1,1}$ domain in $\mathbb{R}^N$ and let $u \in W^{2,p}(G)$ with $p \geq 1$. Then for all $\varepsilon > 0$*

$$\|\nabla u\|_{L^p(G)} \leq \varepsilon \|u\|_{W^{2,p}(G)} + c\varepsilon^{-1}\|u\|_{L^p(G)}$$

with a constant c depending only on the domain G.

THEOREM 1.28. [117, Section 4.3] (Trace Theorem) *Let $1 \leq p < \infty$. There exists a bounded linear operator*

$$T : W^{1,p}(G) \to L^p(\partial G)$$

such that

$$Tu = u \quad on \quad \partial G$$

for all $u \in W^{1,p}(G) \cap C^0(\overline{G})$.

Henceforth, we will write simply u instead of Tu.

THEOREM 1.29. (see e.g. (6.23), (6.24) Chapter I [214] or Lemma 6.36 [237]). *Let ∂G be piecewise smooth and $u \in W^{1,1}(G)$. Then there is a constant $c > 0$ which depends only on G such that*

$$(1.6.1) \qquad \int_{\Gamma} |u|ds \leq c \int_{G} (|u| + |\nabla u|)\, dx, \ \forall \Gamma \subseteq \partial \Omega$$

If $u \in W^{1,2}(G)$, then

$$(1.6.2) \qquad \int_{\partial G} v^2 ds \leq \int_{G} (\delta|\nabla v|^2 + c_\delta v^2)dx, \ \forall v(x) \in W^{1,2}(G), \forall \delta > 0.$$

If $u \in W^{2,2}(G)$, then

$$(1.6.3) \qquad \int\limits_{\partial G} \left(\frac{\partial u}{\partial n}\right)^2 ds \le c \int\limits_{G} \left(2|\nabla u||D^2 u| + |\nabla u|^2\right) dx.$$

1.6.2. Sobolev imbedding theorems. We give the well known *Sobolev inequalities* and *Kondrashov compactness results* also called the *imbedding theorems* (see [**363**], §§1.4.5 - 1.4.6 [**261**], §7.7[**129**]).

THEOREM 1.30. [**412**, Theorem 2.4.1], [**129**, Theorem 7.10] (**Sobolev inequalities**) *Let G be a bounded open domain in $\mathbb{R}^N$ and $p > 1$. Then*

$$(1.6.4) \qquad W_0^{1,p}(G) \hookrightarrow \begin{cases} L^{\frac{Np}{N-p}}(G) & \text{for } p < N, \\ C^0(\overline{G}) & \text{for } p > N. \end{cases}$$

Furthermore, there exists a constant $c = c(N,p)$ such that for all $u \in W_0^{1,p}(G)$ we have

$$(1.6.5) \qquad \|u\|_{L^{Np/(N-p)}(G)} \le c\|\nabla u\|_{L^p(G)}$$

for $p < N$ and

$$(1.6.6) \qquad \sup_G |u| \le c(\operatorname{meas} G)^{1/N - 1/p}\|\nabla u\|_{L^p(G)}$$

for $p > N$.

The following Imbedding Theorems 1.31-1.34 first were proved by Sobolev [**363**] and can be found with complete proofs in [**314**], [**1**, Theorem 5.4], [**210**, Sections 5.7,5.8] and [**261**, Section 1.4]. Let G be a $C^{0,1}$ bounded domain in $\mathbb{R}^N$

THEOREM 1.31. *Let $k \in \mathbb{N}$ and $p \in \mathbb{R}$ with $p \ge 1$, $kp < N$. Then the imbedding*

$$(1.6.7) \qquad W^{k,p}(G) \hookrightarrow L^q(G)$$

is continuous for $1 \le q \le Np/(N - kp)$ and compact for $1 \le q < Np/(N - kp)$. If $kp = N$, then the imbedding (1.6.7) is continuous and compact for any $q \ge 1$.

THEOREM 1.32. *Let $k \in \mathbb{N}_0$, let $m \in \mathbb{N}$ and let $p, q \in \mathbb{R}$ with $p, q \ge 1$. If $kp < N$, then the imbedding*

$$(1.6.8) \qquad W^{m+k,p}(G) \hookrightarrow W^{m,q}(G)$$

is continuous for any $q \in \mathbb{R}$ satisfying $1 \le q \le Np/(N - kp)$. If $k = Np$, then the imbedding (1.6.8) is continuous for any $q \ge 1$.

THEOREM 1.33. *Let $k, m \in \mathbb{N}_0$ and $p > 1$. Then the imbedding*

$$W^{k,p}(G) \hookrightarrow C^{m+\beta}(G)$$

is continuous if

(1.6.9) $(k - m - 1)p < N < (k - m)p$ *and* $0 < \beta \leq k - m - N/p,$

and compact if the inequality in (1.6.9) is sharp. If $(k - m - 1)p = N$, then the imbedding is continuous for any $\beta \in (0, 1)$.

THEOREM 1.34. *Let $u \in W^{k,p}(G)$ with $k \in \mathbb{N}$, $p \in \mathbb{R}$, $kp > N$ and $p > 1$. Then $u \in C^m(G)$ for $0 \leq m < k - N/p$ and there exists a constant c, independent of u, such that*

$$\sup_{x \in G} |D^\alpha u(x)| \leq c\|u\|_{W^{k,p}(G)}$$

for all $|\alpha| < k - N/p$.

THEOREM 1.35. *Let G be a lipschitzian domain and let $T_s \subset \overline{G}$ be piecewise C^k-smooth s-dimensional manifold. Let $k \geq 1$, $p > 1$, $kp < N$, $N - kp < s \leq N$ and $1 \leq q \leq q* = sp/(N - kp)$. Then $W^{k,p}(G) \hookrightarrow L_q(T_s)$ and the inequality*

(1.6.10) $\|u\|_{L^q(T_s)} \leq c\|u\|_{W^{k,p}(G)}$

holds. If $q < q*$, then this imbedding is compact.

1.7. Weighted Sobolev spaces

DEFINITION 1.36. For $k \in \mathbb{N}_0$, $1 < p < \infty$ and $\alpha \in \mathbb{R}$ we define the weighted Sobolev space $V_{p,\alpha}^k(G)$ as the closure of $C_0^\infty(\overline{G} \setminus 0)$ with respect to the norm

$$\|u\|_{V_{p,\alpha}^k(G)} = \left(\int\limits_G \sum_{|\beta| \leq k} r^{\alpha + p(|\beta| - k)} \left| D^\beta u \right|^p dx \right)^{1/p}.$$

For $\Gamma \subseteq \partial G$ and $k \in 1, 2, \ldots$, the space $V_{p,\alpha}^{k-1/p}(\Gamma)$ consists of traces on Γ of functions from $V_{p,\alpha}^k(G)$ and is equipped with the norm

$$\|u\|_{V_{p,\alpha}^{k-1/p}(\Gamma)} = \inf \|v\|_{V_{p,\alpha}^k(G)},$$

where the infimum is taken over the set of all functions $v \in V_{p,\alpha}^k(G)$ such that $v = u$ on Γ.

For $p = 2$ we use the notations

$$\mathring{W}_\alpha^k(G) = V_{2,\alpha}^k(G) \text{ and } \mathring{W}_\alpha^{k-1/2}(\Gamma) = V_{2,\alpha}^{k-1/2}(\Gamma).$$

LEMMA 1.37. **[161, 198]** *Let* $k', k \in \mathbb{N}$ *with* $k' \leq k$ *and*

$$\alpha - pk \leq \alpha' - pk'.$$

Then $V_{p,\alpha}^k(G)$ *is continuously imbedded in* $V_{p,\alpha'}^{k'}(G)$. *Furthermore, the imbedding is compact if* $k' < k$ *and* $\alpha - pk < \alpha' - pk'$.

LEMMA 1.38. **[280, 322]** *Let* $(k - |\gamma|)p > N$, *then for every* $u \in V_{p,\alpha}^k(G)$ *the following inequality is valid*

$$|D^\gamma u(x)| \leq c|x|^{k - |\gamma| - (\alpha + N)/p}\|u\|_{V_{p,\alpha}^k(G)}, \quad \forall x \in G_0^d$$

with a constant c *independent of* u *and some* $d > 0$ *depending only on* G. *In particular*

$$V_{p,\alpha}^k(G) \hookrightarrow C^m(G)$$

for $m < k - (\alpha + N)/p$.

PROOF. Without loss of generality we can assume that G is a cone. We introduce new variables $y = (y_1, \ldots, y_N)$ by $x = yt$ with $t > 0$ and set $v(y) := u(x)$. By the Sobolev Imbedding Theorem 1.34, we have

$$|D_y^\gamma v(y)| \leq c \sum_{|\delta| \leq k} \|D_y^\delta v\|_{L^p(G_1^2)}, \quad \forall y \in G_1^2.$$

Returning back to the variables x

$$t^{|\gamma|}|D_x^\gamma u(x)| \leq c \sum_{|\delta| \leq k} \|t^{|\delta| - N/p}D_x^\delta u\|_{L^p(G_t^{2t})}, \quad \forall x \in G_t^{2t}.$$

Multiplying both sides of this inequality by $t^{N/p - k + \alpha/p}$ we obtain

$$t^{|\gamma| - k + (\alpha + N)/p}|D_x^\gamma u(x)| \leq c \sum_{|\delta| \leq k} \|t^{|\delta| - k + \alpha/p}D_x^\delta u\|_{L^p(G_t^{2t})}, \quad \forall x \in G_t^{2t}.$$

Therefore, because of $t \leq |x| \leq 2t$ in G_t^{2t}, we have

$$|x|^{|\gamma| - k + (\alpha + N)/p}|D_x^\gamma u(x)| \leq c \sum_{|\delta| \leq k} \||x|^{|\delta| - k + \alpha/p}D_x^\delta u\|_{L^p(G_t^{2t})}, \quad \forall x \in G_t^{2t}$$

with a constant c independent of t. Thus the assertion holds. $\qquad\square$

LEMMA 1.39. *Let* $k, m \in \mathbb{N}_0$, $\beta \in \mathbb{R}$ *with*

$$(k - m - 1)p < N < (k - m)p \quad and \quad 0 < \beta \leq k - m - N/p.$$

Then for any $u \in V_{p,\alpha}^k(G)$

$$\sum_{|\gamma| = m} \sup_{x, y \in G, x \neq y} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x - y|^\beta} \leq c|x|^{k - m - \beta - (\alpha + N)/p}\|u\|_{V_{p,\alpha}^k(G)},$$

$$\forall x \in G_0^d$$

with a constant c and some d > 0.

PROOF. The proof is completely analogous to the proof of Lemma 1.38.
□

LEMMA 1.40. [161, Lemma 1.1] *Let* $u \in \mathring{W}_\alpha^{k-1/2}(\Gamma_0^d)$. *Then*

$$\int_{\Gamma_0^d} r^{\alpha-2k+1} u^2(x) ds \le c \|u\|^2_{\mathring{W}_\alpha^{k-1/2}(\Gamma_0^d)}.$$

LEMMA 1.41. *Let* $d > 0$ *and* $\rho \in (0,d)$. *Then the inequality*

$$\int_{\Gamma_0^\rho} r^{3-N} \left(\frac{\partial u}{\partial n}\right)^2 ds \le c_1 \int_{\Gamma_0^\rho} r^{1-N} u^2 ds + c_2 \|u\|^2_{\mathring{W}^2_{4-N}(G_0^\rho)}$$

is valid for all $u \in \mathring{W}^2_{4-N(G_0^\rho)}$ *with constants* c_1, c_2 *independent of* u.

PROOF. Let us first recall that due to Theorem 1.29 we have

$$\int_{\Gamma_0^\rho} \left(\frac{\partial v}{\partial n}\right)^2 ds \le c_3 \int_{G_0^\rho} \left(r|D^2 v|^2 + \frac{1}{r}|\nabla v|^2\right) dx$$

with a constant $c_3 > 0$ depending only on G_0^d. Setting $v = r^{(3-N)/2} u$ we have

$$\frac{\partial v}{\partial n}\bigg|_{\Gamma_0^\rho \backslash \mathcal{O}} = r^{(3-N)/2} \frac{\partial u}{\partial n}\bigg|_{\Gamma_0^\rho \backslash \mathcal{O}} + \frac{3-N}{2} r^{(1-N)/2} u \cdot \frac{\sum_{i=1}^N x_i n_i}{r}.$$

Since $\frac{\sum_{i=1}^N x_i n_i}{r} \le 1$, therefore

$$\int_{\Gamma_0^\rho} r^{3-N} \left(\frac{\partial u}{\partial n}\right)^2 ds \le 2 \int_{\Gamma_0^\rho} \left\{ \left(\frac{\partial v}{\partial n}\right)^2 + \frac{(3-N)^2}{4} r^{1-N} u^2 \langle n, \frac{x}{r}\rangle^2 \right\} ds \le$$

$$\le c_5 \int_{G_0^\rho} \left(r|D^2 v|^2 + r^{-1}|\nabla v|^2\right) ds + \frac{(3-N)^2}{2} \int_{\Gamma_0^\rho} r^{1-N} u^2 ds.$$

The assertion then follows by Lemma 1.9.
□

1.8. Spaces of Dini continuous functions

DEFINITION 1.42. The function $\mathcal{A}$ is called *Dini continuous* at zero if the integral

$$\int\limits_0^d \frac{\mathcal{A}(t)}{t}\,dt$$

is finite for some $d > 0$.

DEFINITION 1.43. The function $\mathcal{A}$ is called an *α-function*, $0 < \alpha < 1$, on $(0, d]$, if $t^{-\alpha}\mathcal{A}(t)$ is monotonously decreasing on $(0, d]$, that is

$$(1.8.1) \qquad \mathcal{A}(t) \le t^\alpha \tau^{-\alpha} \mathcal{A}(\tau), \quad 0 < \tau \le t \le d.$$

In particular, setting $t = c\tau$ for $c > 1$, we have

$$(1.8.2) \qquad \mathcal{A}(c\tau) \le c^\alpha \mathcal{A}(\tau), \quad 0 < \tau \le c^{-1}d.$$

If an α-function $\mathcal{A}$ is Dini continuous at zero then we say that $\mathcal{A}$ is an *α-Dini function*. In that case we define the function

$$\mathcal{B}(t) = \int\limits_0^t \frac{\mathcal{A}(\tau)}{\tau}\,d\tau.$$

Obviously, the function $\mathcal{B}$ is monotonically increasing and continuous on $[0, d]$ and $\mathcal{B}(0) = 0$. Integrating (1.8.1) over $\tau \in (0, t)$ we obtain

$$(1.8.3) \qquad\qquad \mathcal{A}(t) \le \alpha\mathcal{B}(t).$$

Similarly, we derive from (1.8.1) the inequality

$$\int\limits_\delta^d \frac{\mathcal{A}(t)}{t^2}\,dt = \int\limits_\delta^d t^{\alpha-2}\frac{\mathcal{A}(t)}{t^\alpha}\,dt \le \delta^{-\alpha}\mathcal{A}(\delta)\int\limits_\delta^d t^{\alpha-2}\,dt \le (1-\alpha)^{-1}\frac{\mathcal{A}(\delta)}{\delta}$$

and thus

$$(1.8.4) \quad \delta\int\limits_\delta^d \frac{\mathcal{A}(t)}{t^2}\,dt \le (1-\alpha)^{-1}\mathcal{A}(\delta) \le \alpha(1-\alpha)^{-1}\mathcal{B}(\delta),$$

$$\forall \alpha \in (0,1),\ 0 < \delta < d.$$

DEFINITION 1.44. The function $\mathcal{B}$ is called *equivalent* to $\mathcal{A}$, written $\mathcal{A} \sim \mathcal{B}$, if there exist positive constants C_1 and C_2 such that

$$C_1\mathcal{A}(t) \le \mathcal{B}(t) \le C_2\mathcal{A}(t) \quad \forall t \ge 0.$$

THEOREM 1.45. [114] $\mathcal{A} \sim \mathcal{B}$ *if and only if*

$$(1.8.5) \qquad\qquad \liminf_{t \to 0} \mathcal{A}(2t)/\mathcal{A}(t) > 1.$$

PROOF. At first we remark that

$$2\mathcal{B}(h) \geq \mathcal{B}(2h) = \int_0^{2h} \frac{\mathcal{A}(t)}{t} dt \geq \int_h^{2h} \frac{\mathcal{A}(t)}{t} dt \geq \mathcal{A}(h) \ln 2.$$

Therefore we must prove the equivalence of (1.8.5) to the inequality $\mathcal{B}(t) \leq C\mathcal{A}(t)$.

The sufficiency: Let (1.8.5) be satisfied. Then there is a positive θ such that for sufficiently small t the inequality $\frac{\mathcal{A}(2t)}{\mathcal{A}(t)} \geq 1 + \theta$ holds and therefore $\mathcal{A}(2^{-k}t) \leq (1+\theta)^{-k}\mathcal{A}(t)$. Then

$$\mathcal{B}(h) = \int_0^h \frac{\mathcal{A}(t)}{t} dt = \sum_{k=0}^{\infty} \int_{2^{-k-1}h}^{2^{-k}h} \frac{\mathcal{A}(t)}{t} dt \leq \ln 2 \sum_{k=0}^{\infty} \mathcal{A}(2^{-k}h) \leq$$

$$\leq \ln 2 \sum_{k=0}^{\infty} (1+\theta)^{-k}\mathcal{A}(h).$$

The necessity: Let $\liminf\limits_{t \to +0} \mathcal{A}(2t)/\mathcal{A}(t) = 1$. Then there is a sequence t_n such that $\frac{\mathcal{A}(2t_n)}{\mathcal{A}(t_n)} \leq 1 + \frac{1}{n}$ and we have

$$\frac{\mathcal{A}(nt_n)}{\mathcal{A}(t_n)} = \frac{\mathcal{A}(nt_n)}{\mathcal{A}((n-1)t_n)} \cdot \frac{\mathcal{A}((n-1)t_n)}{\mathcal{A}((n-2)t_n)} \cdots \frac{\mathcal{A}(2t_n)}{\mathcal{A}(t_n)} \leq$$

$$\leq \left[\frac{\mathcal{A}(2t_n)}{\mathcal{A}(t_n)}\right]^{n-1} \leq \left(1 + \frac{1}{n}\right)^{n-1} \leq e.$$

Therefore

$$\mathcal{B}(nt_n) \geq \int_{t_n}^{nt_n} \frac{\mathcal{A}(t)}{t} dt \geq \ln n \mathcal{A}(t_n) \geq \frac{1}{e} \ln n \mathcal{A}(nt_n),$$

and

$$\frac{\mathcal{B}(nt_n)}{\mathcal{A}(nt_n)} \geq \frac{1}{e} \ln n, \qquad \lim_{n \to \infty} nt_n = 0.$$

Thus $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are not equivalent. $\qquad\qquad\qquad\Box$

In some cases we shall consider functions $\mathcal{A}(t)$ such that

$$(1.8.6) \qquad\qquad \sup_{0 < \tau \leq 1} \frac{\mathcal{A}(\tau t)}{\mathcal{A}(\tau)} \leq c\mathcal{A}(t), \quad t \in (0, d],$$

with some constant c independent of t.

Examples of α-Dini functions $\mathcal{A}(t)$ which satisfy (1.8.5) and (1.8.6) with $c = 1$ are

$$t^\alpha \text{ for } 0 \le t < \infty;$$

$$t^\alpha \ln(1/t) \text{ for } t \in (0, d], \quad d = \min(e^{-1}, e^{-1/\alpha}), \quad e^{-1} < \alpha < 1.$$

DEFINITION 1.46. The Banach space $C^{0,\mathcal{A}}(G)$ is the set of all bounded and continuous functions u on $G \subset \mathbb{R}^N$ for which

$$[u]_{\mathcal{A};G} = \sup_{x,y \in G, x \ne y} \frac{|u(x) - u(y)|}{\mathcal{A}(|x - y|)} < \infty.$$

It is equipped with the norm

$$\|u\|_{C^{0,\mathcal{A}}(G)} = \|u\|_{C^0(G)} + [u]_{\mathcal{A};G}.$$

If $k \ge 1$, then we denote by $C^{k,\mathcal{A}}(G)$ the subspace of $C^k(G)$ consisting of all functions whose $(k-1)$-th order partial derivatives are uniformly Lipschitz continuous and each k-th order derivative belongs to $C^{0,\mathcal{A}}(G)$. It is a Banach space equipped with the norm

$$\|u\|_{C^{k,\mathcal{A}}(G)} = \|u\|_{C^k(G)} + \sum_{|\beta|=k} [D^\beta u]_{\mathcal{A};G}.$$

Furthermore, let us introduce the following notation

$$[u]_{\mathcal{A},x} = \sup_{y \in G \setminus \{x\}} \frac{|u(x) - u(y)|}{\mathcal{A}(|x - y|)}.$$

LEMMA 1.47. *If $\mathcal{A} \sim \mathcal{B}$, then $[u]_{\mathcal{A}} \sim [u]_{\mathcal{B}}$.*

LEMMA 1.48. [**129**, p. 143, 6.7 (ii)] *Let G be a bounded domain with a Lipschitz boundary ∂G. Then there are two positive constants L and ϱ_1 such that for any $y \in G$ with $\text{dist}(y, \partial G) \le \varrho_1$ and any $0 < \varrho \le \varrho_1$ there exists $x \in B_\varrho(y)$ such that $\overline{B}_{\varrho/L}(x) \subset G$.*

THEOREM 1.49. [**365**, Inequality (10.1)] (**Interpolation inequality**) *Let ∂G be Lipschitz. Then for any $\varepsilon > 0$ there exists a constant $c = c(\varepsilon, G)$ such that for every $u \in C^{1,\mathcal{A}}(G)$ the following inequality holds*

$$\sum_{i=1}^N \|D_i u\|_{C^0(G)} \le \varepsilon \sum_{i=1}^N [D_i u]_{\mathcal{A};G} + c(\varepsilon, G)\|u\|_{C^0(G)}.$$

PROOF. Let L, ϱ be given as in Lemma 1.48 and let $\varepsilon > 0$ be arbitrary. We choose $\varrho > 0$ so small, that $\mathcal{A}(\varrho(1 + 1/L)) \le \varepsilon$. If $\text{dist}(y, \partial G) > \varrho_1$, there are for every $i \in \{1, \dots N\}$ two points $y_1, y_2 \in \partial B_\varrho(y)$ and a $\overline{y} \in B_\varrho(y)$, such that

$$|D_i u(\overline{y})| = \frac{1}{2\varrho}|u(y_1) - u(y_2)| \le \frac{1}{\varrho}\|u\|_{C^0(G)}.$$

Thus

$$|D_i u(y)| \leq |D_i u(\overline{y})| + |D_i u(y) - D_i u(\overline{y})| \leq \frac{1}{\varrho} \|u\|_{C^0(G)} + \mathcal{A}(\varrho)[D_i u]_{\mathcal{A};G}.$$

If $\mathrm{dist}(y, \partial G) \leq \varrho_1$, there are $y_1, y_2 \in \partial B_{\varrho/L}(x)$ and $\overline{y} \in \partial B_{\varrho/L}(x)$ such that

$$|D_i u(\overline{y})| = \frac{L}{2\varrho} |u(y_1) - u(y_2)| \leq \frac{L}{\varrho} \|u\|_{C^0(G)}.$$

Since $|y - \overline{y}| \leq |y - x| + |x - \overline{y}| \leq \varrho(1 + 1/L)$ we conclude

$$|D_i u(y)| \leq |D_i u(\overline{y})| + |D_i u(y) - D_i u(\overline{y})| \leq \frac{L}{\varrho} \|u\|_{C^0(G)} +$$

$$+ \mathcal{A}(\varrho(1 + \frac{1}{L}))[D_i u]_{\mathcal{A};G},$$

which finally implies the statement.   □

DEFINITION 1.50. We shall say that the boundary portion $T \subset \partial G$ is of *class* $C^{1,\mathcal{A}}$ if for each point $x_0 \in T$ there are a ball $B = B(x_0)$, a one-to-one mapping ψ of B onto a ball B', and a constant $K > 0$ such that

(i) $B \cap \partial G \subset T$, $\psi(B \cap G) \subset \mathbb{R}^N_+$;
(ii) $\psi(B \cap \partial G) \subset \Sigma$;
(iii) $\psi \in C^{1,\mathcal{A}}(B)$, $\psi^{-1} \in C^{1,\mathcal{A}}(B')$;
(iv) $\|\psi\|_{C^{1,\mathcal{A}}(B)} \leq K$, $\|\psi^{-1}\|_{C^{1,\mathcal{A}}(B')} \leq K$.

It is not difficult to see that for the diffeomorphism ψ one has

$$(1.8.7)\quad K^{-1}|\psi(x) - \psi(x')| \leq |x - x'| \leq K|\psi(x) - \psi(x')|, \quad \forall x, x' \in B.$$

LEMMA 1.51. [365, Section 7] *Let* $u, v \in C^{0,\mathcal{A}}(G)$. *Then* $u \cdot v \in C^{0,\mathcal{A}}(G)$ *and*

$$\|u \cdot v\|_{C^{0,\mathcal{A}}(G)} \leq \|u\|_{C^{0,\mathcal{A}}(G)} \cdot \|v\|_{C^{0,\mathcal{A}}(G)}.$$

LEMMA 1.52. [365, Section 7] *Let* $\mathcal{A}(t)$ *be an* α-*function on* $[0, d]$ *and let* $u \in C^{0,\mathcal{A}}(B)$. *Furthermore, let* $\psi : B' \to B$ *be Lipschitz continuous the Lipschitz constant* L. *Then* $u \circ \psi \in C^{0,\mathcal{A}}(B')$ *and*

$$(1.8.8)\qquad \|u \circ \psi\|_{C^{0,\mathcal{A}}(B')} \leq \tilde{L}^\alpha \|u\|_{C^{0,\mathcal{A}}(B)}, \text{ where } \tilde{L} = \max(1, L).$$

PROOF. Indeed, if $x, y \in B'$, $|x - y| \leq \frac{d}{L}$, then by (1.8.2)

$$|u(\psi(x)) - u(\psi(y))| \leq \|u\|_{C^{0,\mathcal{A}}(B)} \cdot \mathcal{A}(L|x - y|) \leq$$

$$\leq \|u\|_{C^{0,\mathcal{A}}(B)} \cdot \tilde{L}^\alpha \cdot \mathcal{A}(|x - y|).$$

□

1.9. Some functional analysis

DEFINITION 1.53. Let X, Y be Banach spaces. Then we denote by $\mathcal{L}(X, Y)$ the linear space of all continuous linear mappings $L : X \to Y$.

THEOREM 1.54. [**129**, Theorem 5.2] (**The method of continuity**). *Let X, Y be Banach spaces and $L_0, L_1 \in \mathcal{L}(X, Y)$. Furthermore, let*

$$L_t := (1 - t)L_0 + tL_1 \quad \forall t \in [0, 1]$$

and suppose that there exist a constant c such that

$$\|u\|_X \leq c \|L_t u\|_Y \quad \forall t \in [0, 1].$$

Then L_1 maps X onto Y if and only if L_0 maps X onto Y.

THEOREM 1.55. [**353**] (**Variational principle for the least positive eigenvalue**). *Let H, V be Hilbert spaces with dense and compact imbeddings*

$$V \subset H \subset V'$$

and let $A : V \to V'$ be a continuous operator. We assume that the bilinear form

$$a(u, v) = (Au, v)_H$$

is continuous and V-coercive, that is there are constants c_1 and c_2 such that

$$\begin{aligned} |a(u, v)| &\leq c_1 \|u\|_V \|v\|_V, \\ a(u, u) &\geq c_2 \|u\|_V^2 \end{aligned}$$

for all $u, v \in V$. Then the smallest eigenvalue ϑ of the eigenvalue problem

$$Au + \vartheta u = 0$$

satisfies

$$\vartheta = \inf_{v \in V} \frac{a(v, v)}{\|v\|_H^2}.$$

THEOREM 1.56. [**129**, Theorem 11.3] (**The Leray-Schauder Theorem**). *Let T be a compact mapping of a Banach space $\mathcal{B}$ into itself, and suppose that there exist a constant M such that*

$$\|x\|_{\mathcal{B}} < M$$

for all $x \in \mathcal{B}$ and $\sigma \in [0, 1]$ satisfying $x = \sigma T x$. Then T has a fixed point.

1.10. The Cauchy problem for a differential inequality

THEOREM 1.57. *Let $V(\varrho)$ be a monotonically increasing, nonnegative differentiable function defined on $[0, 2d]$ that satisfies the problem*

(CP) $\begin{cases} V'(\rho) - \mathcal{P}(\varrho)V(\varrho) + \mathcal{N}(\rho)V(2\rho) + \mathcal{Q}(\rho) \geq 0, & 0 < \rho < d, \\ V(d) \leq V_0, \end{cases}$

where $\mathcal{P}(\varrho), \mathcal{N}(\varrho), \mathcal{Q}(\varrho)$ are nonnegative continuous functions defined on $[0, 2d]$ and V_0 is a constant. Then

$$(1.10.1) \quad V(\varrho) \leq \exp\left(\int_\varrho^d \mathcal{B}(\tau)d\tau\right)\left\{V_0 \exp\left(-\int_\varrho^d \mathcal{P}(\tau)d\tau\right) + \right.$$

$$\left. + \int_\varrho^d \mathcal{Q}(\tau)\exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)d\sigma\right)d\tau\right\}$$

with

$$(1.10.2) \qquad \mathcal{B}(\varrho) = \mathcal{N}(\varrho)\exp\left(\int_\varrho^{2\varrho} \mathcal{P}(\sigma)d\sigma\right).$$

PROOF. We define functions

$$(1.10.3) \qquad w(\varrho) = V(\varrho)\exp\left(\int_\varrho^d \mathcal{P}(\sigma)d\sigma\right)$$

$$(1.10.4) \qquad \mathcal{R}(\varrho) = V_0 + \int_\varrho^d \mathcal{Q}(\tau)\exp\left(\int_\tau^d \mathcal{P}(\sigma)d\sigma\right)d\tau$$

Multiplying the differential inequality (CP) by the integrating factor $\exp\left(\int_\varrho^d \mathcal{P}(s)ds\right)$ and integrating from ρ to d we get

$$V(d) - V(\varrho)\exp\left(\int_\varrho^d \mathcal{P}(s)ds\right) + \int_\varrho^d \mathcal{N}(\tau)\exp\left(\int_\tau^d \mathcal{P}(s)ds\right)V(2\tau)d\tau +$$

$$+ \int_\varrho^d \mathcal{Q}(\tau)\exp\left(\int_\tau^d \mathcal{P}(s)ds\right)d\tau \geq 0.$$

Hence it follows that

$$(1.10.5) \qquad w(\varrho) \leq \mathcal{R}(\varrho) + \int\limits_{\varrho}^{d} \mathcal{B}(\tau)w(2\tau)d\tau.$$

Now we have

$$(1.10.6) \qquad \frac{w(\varrho)}{\mathcal{R}(\varrho)} \leq 1 + \int\limits_{\varrho}^{d} \mathcal{B}(\tau)\frac{w(2\tau)}{\mathcal{R}(2\tau)}\frac{\mathcal{R}(2\tau)}{\mathcal{R}(\varrho)}d\tau.$$

Since $\mathcal{R}(2\tau) \leq \mathcal{R}(\varrho)$ for $\tau > \varrho$, setting

$$(1.10.7) \qquad z(\varrho) = \frac{w(\varrho)}{\mathcal{R}(\varrho)}$$

we get

$$(1.10.8) \qquad z(\varrho) \leq 1 + \int\limits_{\varrho}^{d} \mathcal{B}(\tau)z(2\tau)d\tau.$$

Let us define the function

$$\mathcal{Z}(\varrho) = 1 + \int\limits_{\varrho}^{d} \mathcal{B}(\tau)z(2\tau)d\tau.$$

The from (1.10.8) we have

$$(1.10.9) \qquad z(\varrho) \leq \mathcal{Z}(\varrho)$$

and

$$\mathcal{Z}'(\varrho) = -\mathcal{B}(\varrho)z(2\varrho) \geq -\mathcal{B}(\varrho)\mathcal{Z}(2\varrho).$$

Multiplying the obtained differential inequality by the integrating factor $\exp\left(-\int\limits_{\varrho}^{d} \mathcal{B}(s)ds\right)$ and using the equality

$$\frac{d}{d\varrho}\left[\mathcal{Z}(\varrho)\exp\left(-\int\limits_{\varrho}^{d} \mathcal{B}(s)ds\right)\right] = \mathcal{Z}'(\varrho)\exp\left(-\int\limits_{\varrho}^{d} \mathcal{B}(s)ds\right) +$$

$$+ \mathcal{B}(\varrho)\exp\left(-\int\limits_{\varrho}^{d} \mathcal{B}(s)ds\right)\mathcal{Z}(\varrho),$$

we have

$$\frac{d}{d\varrho}\left[\mathcal{Z}(\varrho)\exp\left(-\int\limits_{\varrho}^{d}\mathcal{B}(s)ds\right)\right] \geq \mathcal{B}(\varrho)\exp\left(-\int\limits_{\varrho}^{d}\mathcal{B}(s)ds\right)\left[\mathcal{Z}(\varrho)-\mathcal{Z}(2\varrho)\right].$$

But

$$\mathcal{Z}(2\varrho) = 1 + \int\limits_{2\varrho}^{d}\mathcal{B}(s)z(2s)ds \leq 1 + \int\limits_{\varrho}^{d}\mathcal{B}(s)z(2s)ds = \mathcal{Z}(\varrho),$$

therefore

$$\mathcal{Z}(\varrho) - \mathcal{Z}(2\varrho) \geq 0 \;\Rightarrow\; \frac{d}{d\varrho}\left[\mathcal{Z}(\varrho)\exp\left(-\int\limits_{\varrho}^{d}\mathcal{B}(s)ds\right)\right] \geq 0.$$

Integrating from ϱ to d we have

$$\mathcal{Z}(\varrho)\exp\left(-\int\limits_{\varrho}^{d}\mathcal{B}(s)ds\right) \leq \mathcal{Z}(d) = 1 \;\Rightarrow\; \mathcal{Z}(\varrho) \leq \exp\left(\int\limits_{\varrho}^{d}\mathcal{B}(s)ds\right)$$

Hence, by (1.10.9), we get

$$(1.10.10) \qquad\qquad z(\varrho) \leq \exp\left(\int\limits_{\varrho}^{d}\mathcal{B}(s)ds\right).$$

Now, in virtue of (1.10.3), (1.10.7), and (1.10.10), we finally obtain

$$V(\varrho) \leq \exp\left(-\int\limits_{\varrho}^{d}\mathcal{P}(\sigma)d\sigma\right)\mathcal{R}(\varrho)\exp\left(\int\limits_{\varrho}^{d}\mathcal{B}(\sigma)d\sigma\right)$$

or with regard to (1.10.4) the desired estimate (1.10.1). $\square$

1.11. Additional auxiliary results

1.11.1. Mean Value Theorem.

THEOREM 1.58. *Let* $f \in C^0[a,b]$ *with* $0 \le a < b$. *Then there exist a* $\theta \in (0,1)$ *and a* $\xi \in (0,1)$, *such that*

$$\int_a^b f(x)dx \ge (b-a)f((1-\theta)a + \theta b)$$

and

$$\int_a^b f(x)dx \le (b-a)f((1-\xi)a + \xi b).$$

PROOF. Let us assume that

$$\int_a^b f(x)dx < (b-a)f((1-\theta)a + \theta b)$$

for all $\theta \in (0,1)$. Integrating this inequality with respect to $\theta \in (0,1)$ we obtain the contradiction

$$\int_a^b f(x)dx < (b-a)\int_0^1 f((1-\theta)a + \theta b)d\theta = \int_a^b f(t)dt.$$

The other assertion is proved analogously.   □

1.11.2. Stampacchia's Lemma.

LEMMA 1.59. (See Lemma 3.11 of [316], [366]). *Let* $\varphi : [k_0, \infty) \to \mathbb{R}$ *be a non–negative and non–increasing function which satisfies*

$$(1.11.1) \qquad \varphi(h) \le \frac{C}{(h-k)^\alpha}[\varphi(k)]^\beta \quad for \quad h > k > k_0,$$

where C, α, β *are positive constants with* $\beta > 1$. *Then*

$$\varphi(k_0 + d) = 0,$$

where

$$d^\alpha = C \, |\varphi(k_0)|^{\beta-1} \, 2^{\alpha\beta/(\beta-1)}.$$

PROOF. Define the sequence

$$k_s = k_0 + d - \frac{d}{2^s}, \quad s = 1, 2, \ldots.$$

From (1.11.1) it follows that

$$(1.11.2) \qquad \varphi(k_{s+1}) \leq \frac{C2^{(s+1)\alpha}}{d^\alpha}[\varphi(k_s)]^\beta, \quad s = 1, 2, \ldots.$$

Let us prove by induction that

$$(1.11.3) \qquad \varphi(k_s) \leq \frac{\varphi(k_0)}{2^{-s\mu}}, \text{ where } \mu = \frac{\alpha}{1-\beta} < 0.$$

For $s = 0$ the claim is trivial. Let us suppose that (1.11.3) is valid up to s. By (1.11.2) and the definition of d^α if follows that

$$(1.11.4) \qquad \varphi(k_{s+1}) \leq C\frac{2^{(s+1)\alpha}}{d^\alpha}\frac{[\varphi(k_0]^\beta}{2^{-s\beta\mu}} \leq \frac{\varphi(k_0)}{2^{-(s+1)\mu}}.$$

Since the right hand side of (1.11.4) tends to zero as $s \to \infty$, we obtain

$$0 \leq \varphi(k_0 + d) \leq \varphi(k_s) \to 0.$$

$\square$

1.11.3. Other assertions.

LEMMA 1.60. (see Lemma 2.1 [78]). *Let us consider the function*

$$\eta(x) = \begin{cases} e^{\varkappa x} - 1, & x \geq 0, \\ -e^{-\varkappa x} + 1, & x \leq 0, \end{cases}$$

where $\varkappa > 0$. Let a, b be positive constants, $m > 1$. If $\varkappa > (2b/a) + m$, then we have

$$(1.11.5) \qquad a\eta'(x) - b|\eta(x)| \geq \frac{a}{2}e^{\varkappa x}, \quad \forall x \geq 0$$

and

$$(1.11.6) \qquad \eta(x) \geq \left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \forall x \geq 0.$$

Moreover, there exist a $d \geq 0$ and an $M > 0$ such that

$$(1.11.7) \qquad \eta(x) \leq M\left[\eta\left(\frac{x}{m}\right)\right]^m \text{ and } \eta'(x) \leq M\left[\eta\left(\frac{x}{m}\right)\right]^m, \quad \forall x \geq d;$$

$$(1.11.8) \qquad |\eta(x)| \geq x, \quad \forall x \in \mathbb{R}.$$

PROOF. Formula (1.11.5) is easily verified by direct calculation. By definition, inequality (1.11.6) can be stated as

$$(1.11.9) \qquad \left(e^{\varkappa\frac{x}{m}} - 1\right)^m \leq e^{\varkappa x} - 1, \quad \forall x \geq 0.$$

We set for $x \geq 0$

$$y = e^{\varkappa\frac{x}{m}} \geq 1 \text{ and } f(y) = (y-1)^m + 1 - y^m.$$

Then

$$f'(y) = m(y-1)^{m-1} - my^{m-1} < 0,$$

hence it follows that $f(y)$ is decreasing function, that is $f(y) \leq f(1)$, $\forall y \geq 1$. Because of $f(1) = 0$, we get (1.11.9).

Further, the first inequality from (1.11.7) has the form

$$(1.11.10) \qquad\qquad y^m - 1 \leq M(y-1)^m.$$

We consider the function $g(y) = M(y-1)^m - y^m + 1$. Then we have

$$g(y) > M(y-1)^m - y^m \geq 0, \text{ if } y \geq y_0 = \frac{M^{\frac{1}{m}}}{M^{\frac{1}{m}} - 1} > 1,$$

if we choose $M > 1$. Therefore

$$g(y) > 0, \ \forall y \geq y_0 \quad \text{or for} \quad e^{\varkappa \frac{x}{m}} \geq \frac{M^{\frac{1}{m}}}{M^{\frac{1}{m}} - 1} \implies$$

$$x \geq d_1 = \frac{m}{\varkappa} \ln \frac{M^{\frac{1}{m}}}{M^{\frac{1}{m}} - 1},$$

that is the first inequality from (1.11.7) is proved.

Let us now prove the second inequality from (1.11.7). We rewrite it in the form

$$M(y-1)^m \geq \varkappa y^m.$$

Hence it follows:

$$M^{\frac{1}{m}}(y-1) \geq \varkappa^{\frac{1}{m}}(y-1)y \implies y \geq \frac{M^{\frac{1}{m}}}{M^{\frac{1}{m}} - \varkappa^{\frac{1}{m}}},$$

if $M > \varkappa$. The last inequality means that

$$e^{\varkappa \frac{x}{m}} \geq \frac{M^{\frac{1}{m}}}{M^{\frac{1}{m}} - \varkappa^{\frac{1}{m}}} \implies x \geq d_2 = \frac{m}{\varkappa} \ln \frac{M^{\frac{1}{m}}}{M^{\frac{1}{m}} - \varkappa^{\frac{1}{m}}}.$$

Thus, inequalities (1.11.7) are proved, if we take

$$M > \varkappa; \ d = \max(d_1, d_2) = \frac{m}{\varkappa} \ln \frac{M^{\frac{1}{m}}}{M^{\frac{1}{m}} - \varkappa^{\frac{1}{m}}},$$

(since $\varkappa > 1$).

Finally, we prove the inequality (1.11.8). From the definition we have

$$|\eta(x)| = \begin{cases} e^{\varkappa x} - 1, & x \geq 0, \\ e^{-\varkappa x} - 1, & x \leq 0. \end{cases}$$

It is sufficient prove the inequality

$$e^{\varkappa x} - 1 \geq x, \quad x \geq 0.$$

Since $\varkappa > 1$, this result follows from the Taylor formula, □

1.11.4. The distance function. Let G be a domain in $\mathbb{R}^N$ having non-empty boundary ∂G. The *distance function* d is defined by

$$d(x) = \text{dist}(x, \partial G).$$

LEMMA 1.61. *The distance function d is uniformly Lipschitz continuous.*

PROOF. Let $x, y \in \mathbb{R}^N$ and let $y^* \in \partial G$ be such that $|y - y^*| = d(y)$. Then

$$d(x) \leq |x - y^*| \leq |x - y| + d(y)$$

so that by interchanging x and y we have

(1.11.11) $$|d(x) - d(y)| \leq |x - y|.$$

□

1.11.5. Extension Lemma.

LEMMA 1.62. (See Lemma 3.9 [405]). *Let D be a convex bounded set in $\mathbb{R}^N$, $T \subseteq \partial D$, and $f(x) \in C^{1,\mathcal{A}}(\overline{D})$, where $\mathcal{A}(t)$ is a non-decreasing function such that $\lim\limits_{t \to +0} \mathcal{A}(t) = 0$ and $\mathcal{A}(2t) \leq 2\mathcal{A}(t)$. Then there exists a function $F(x)$ with following properties:*

$1°. \quad F(x) \in C^\infty(D);$

$2°. \quad F(x) \in C^{1,\mathcal{A}}(\overline{D});$

$3°. \quad D^\alpha F(x) = D^\alpha f(x), \ x \in T; \ |\alpha| \leq 1;$

$4°. \quad |D^2_{xx}F(x)| \leq Kd^{-1}(x)\mathcal{A}(d(x)),$

where $d(x)$ denotes the distance to T and K depends on N and $\mathcal{A}(t)$ only.

PROOF. We shall use the concept of a partition of unity. Let us consider a finite covering of D by a countable collection $\{D_j\}$ of open sets D_j. Let $\{\zeta_j\}$ be a locally finite partition of unity subordinate to this covering, that is

(i) $\zeta_k \in C_0^\infty(D_j)$ for some $j = j(k)$;

(ii) $\zeta_k \geq 0$, $\sum \zeta_k = 1$ in D;

(iii) at each point of D there is a neighborhood in which only a finite number of the ζ_k are non-zero;

(iv) $\sum\limits_k |D^\alpha \zeta_k(x)| \leq C_\alpha(1 + d^{-\alpha}(x))$, where C_α is independent of T;

(ivv)   there is a constant C independent of k and T such that

$$\text{diam}(\text{supp } \zeta_k) \leq Cd(x).$$

For the proof of the existence of such a partition see, for example, the presentation of Whitney's extension theorem in Hörmander (Lemma 3 [146]). Let $x^* \in T$ be a point satisfying $d(x) = |x - x^*|$ and let x^k be any fixed point in the support of ζ_k. We write the Taylor expansion of $f(x)$ at y as

$$f(x) = P_1(x, y) + R_1(x, y), \text{ where}$$

$$P_1(x, y) = f(y) + \sum_j (x_j - y_j) \frac{\partial f}{\partial x_j}(y)$$

and therefore, by the mean value Lagrange Theorem,

$$R_1(x, y) = (f(x) - f(y)) - \sum_j (x_j - y_j) \frac{\partial f}{\partial x_j}(y) =$$

$$= \sum_j (x_j - y_j) \left(\frac{\partial f}{\partial x_j}(y + \theta(x - y)) - \frac{\partial f}{\partial x_j}(y) \right)$$

for some $\theta \in (0, 1)$. By assumptions about f it follows that

(1.11.12) $|R_1(x, y)| \leq |x - y|\mathcal{A}(\theta|x - y|) \leq |x - y|\mathcal{A}(|x - y|).$

Since

$$D_x R_1(x, y) = D_x f(x) - D_x f(y),$$

we get in the same way

(1.11.13) $|D_x R_1(x, y)| = |D_x f(x) - D_x f(y)| \leq K\mathcal{A}(|x - y|).$

Now we define $F(x)$ by

$$F(x) = \begin{cases} \sum_k \zeta_k(x) P_1(x, x^k) = \sum_k \zeta_k(x) \left(P_1(x, x^k) - P_1(x, x^*) \right) + \\ \qquad\qquad\qquad\qquad\qquad\qquad + P_1(x, x^*), \quad x \in \overline{D} \setminus T; \\ f(x), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x \in T. \end{cases}$$

Then we have

$$D^2 F(x) = \sum_k D^2 \zeta_k(x) \left(P_1(x, x^k) - P_1(x, x^*) \right) +$$

$$+ 2 \sum_k D\zeta_k(x) \left(D_x P_1(x, x^k) - D_x P_1(x, x^*) \right).$$

But it is obvious that

$$P_1(x, x^k) - P_1(x, x^*) = R_1(x, x^*) - R_1(x, x^k)$$

and therefore

$$D^2 F(x) = \sum_k D^2 \zeta_k(x) \left(R_1(x, x^*) - R_1(x, x^k) \right) +$$
$$+ 2 \sum_k D\zeta_k(x) \left(D_x R_1(x, x^*) - D_x R_1(x, x^k) \right).$$

(1.11.14)

If $x \in \operatorname{supp} \zeta_k$, then by (ivv)

(1.11.15) $|x - x^k| \leq |x - x^*| + |x^* - x^k| \leq d(x) + Cd(x) = (1 + C)d(x).$

Therefore we obtain, by (iv) and by (1.11.12)-(1.11.13)

$$|D^2 F(x)| \leq K d^{-1}(x) \mathcal{A}(d(x))$$

and $4°$ is proved.

To prove $2°$, first assume that

(1.11.16) $|x - y| \leq \dfrac{1}{2} d(x).$

By the mean value Lagrange Theorem

$$|D_x F(x) - D_x F(y)| \leq |x - y| \sup |D_{xx}^2 F(z)|,$$

where the supremum is taken over those z for which $|z - x| \leq \frac{1}{2} d(x)$. Then using $4°$, it follows that

$$|D_x F(x) - D_x F(y)| \leq K\mathcal{A}(|x - y|).$$

On the other hand, if $d(x) < 2|x - y|$, we have

$$d(y) \leq d(x) + |x - y| < 4|x - y|$$

and by the definition of $F(x)$ and by (1.11.15):

$$|D_x F(x) - D_x f(x^*)| \leq \sum_k |\zeta_k(x)||D_x R_1(x, x^*) - D_x R_1(x, x^k)| +$$
$$+ \sum_k |D\zeta_k(x)||R_1(x, x^*) - R_1(x, x^k)| \leq K\mathcal{A}(d(x)) \leq$$
$$\leq K\mathcal{A}(|x - y|).$$

Similarly,

$$|D_x F(y) - D_x f(y^*)| \leq K\mathcal{A}(d(y)) \leq cK\mathcal{A}(|x - y|).$$

Since by assumption

$$|D_x f(x^*) - D_x f(y^*)| \leq \mathcal{A}(|x^* - y^*|) \leq K\mathcal{A}(|x - y|),$$

the lemma follows with the triangle inequality. $\square$

1.11.6. Difference quotients. The investigation of differential properties of weak solutions to the boundary value problems may often be deduced through a consideration of their difference quotients.

DEFINITION 1.63. Let $u \in L^m(G)$ and denote by $\mathbf{e_k}$ (k=1,...N) the unit coordinate vector in the x_k direction. The function

$$\triangle^h u(x) = \triangle_k^h u(x) = \frac{u(x + h\mathbf{e_k}) - u(x)}{h}, \quad h \neq 0$$

is said to be a difference quotient of u at x in the direction $\mathbf{e_k}$.

The following lemmas pertain to difference quotients of functions in Sobolev spaces.

LEMMA 1.64. *Let $u \in W^{1,m}(G)$. Then $\triangle^h u \in L^m(G')$ for any $G' \subset\subset G$ satisfying $h < dist(G', \partial G)$, and we have*

$$\|\triangle^h u\|_{L^m(G')} \leq \|D_k u\|_{L^m(G)}.$$

PROOF. At first, we suppose that $u \in C^1(G) \cap W^{1,m}(G)$. Then

$$\triangle^h u(x) = \frac{u(x + h\mathbf{e_k}) - u(x)}{h} =$$

$$= \frac{1}{h} \int_0^h D_k u(x_1, \ldots, x_{k-1}, x_k + \xi, x_{k+1}, \ldots, x_N) d\xi$$

so that by the Hölder inequality

$$|\triangle^h u(x)|^m \leq \frac{1}{h} \int_0^h |D_k u(x_1, \ldots, x_{k-1}, x_k + \xi, x_{k+1}, \ldots, x_N)|^m d\xi,$$

and hence setting $B_h(G') = \{x | \, dist(x, G') < h\}$

$$\int_{G'} |\triangle^h u(x)|^m dx \leq \frac{1}{h} \int_0^h \int_{B_h(G')} |D_k u|^m dx d\xi \leq \int_G |D_k u|^m dx.$$

The extension to arbitrary functions in $W^{1,m}(G)$ follows by a straightforward approximation argument.   □

LEMMA 1.65. *Let $u \in L^m(G)$, $1 < m < \infty$, and suppose there exists a constant K such that $\triangle^h u \in L^m(G')$ and $\|\triangle^h u\|_{L^m(G')} \leq K$ for all $h > 0$ and $G' \subset\subset G$ satisfying $h < dist(G', \partial G)$. Then the weak derivative $D_k u$ exists and satisfies $\|D_k u\|_{L^m(G)} \leq K$.*

PROOF. By the weak compactness of bounded sets in $L^m(G')$, there exist a sequence $\{h_j\}$ tending to zero and a function $v \in L^m(G)$ with $\|v\|_{L^m(G)} \leq K$ satisfying

$$\lim_{h_j \to 0} \int_G \eta \triangle^{h_j} u \, dx = \int_G \eta v \, dx, \quad \forall \eta \in C_0^1(G).$$

Now we have

The summation by parts formula is as follows.

$$(1.11.17) \qquad \int_G \eta \triangle^{h_j} u \, dx = -\int_G u \triangle^{-h_j} \eta \, dx \quad \text{for } h_j < \text{dist}(\text{supp}\,\eta, \partial G).$$

Hence

$$\lim_{h_j \to 0} \int_G u \triangle^{-h_j} \eta \, dx = \int_G u D_k \eta \, dx \Longrightarrow$$

$$\int_G \eta v \, dx = -\int_G u D_k \eta \, dx,$$

whence $v = D_k u$. □

LEMMA 1.66. *Let $u \in W^{1,m}(G)$. Then*

$$\|\triangle_k^{h_j} u(x) - D_k u(x)\|_{L^m(G')} \to 0, \quad k = 1, \ldots, N$$

for any sequence $\{h_j\}$ tending to zero and for every $G' \subset\subset G$. For some subsequence $\{h_{j_l}\}$ functions $\triangle_k^{h_{j_l}} u(x)$ tend to $D_k u(x)$ a.e. in G.

PROOF. For sufficiently small $|h_j|$ and a.e. $x \in G'$ we have

$$\triangle_k^{h_j} u(x) - D_k u(x) = \frac{1}{h_j} \int_0^1 \frac{du(x + th_j \mathbf{e}_k)}{dt} dt - D_k u(x) =$$

$$= \int_0^1 \langle D_k u(x + th_j \mathbf{e}_k) - D_k u(x) \rangle \, dt$$

and therefore

$$\|\triangle_k^{h_j} u(x) - D_k u(x)\|_{L^m(G')} \leq \int_0^1 \|D_k u(x + th_j \mathbf{e}_k) - D_k u(x)\|_{L^m(G')} dt.$$

But the right side in this inequality tends to zero as $h_j \to 0$, because $D_k u(x)$ is continuous in the norm $L^m(G')$. Thus the first part of lemma is proved. The second part follows from properties of the space L^m. □

1.12. Notes

The proof of the Cauchy, Young and Hölder inequalities §1.2 can be found in Chapter 1 [37] or in Chapter II [142]. The formulae (1.3.1)-(1.3.12) are proved in §2, Chapter 1 [310]. For the proof of the Fubini and Fatou Theorems see, for example, Theorem 9 §11 and Theorem 19 §6, Chapter III [101]. The proof of the integral inequalities §1.5 can be found in Chapter VI [142]. The Clarkson inequality is proved in Subsection 2 §3, Chapter I [363]. The material in §1.8 is due to [74, 114]. The simplest version of Theorem 1.57 in §1.10 goes back to G. Peano [331]; the special case was formulated and proved by T. Gronwall [137] and S. Chaplygin [79]. The case $\mathcal{N}(\varrho) \equiv 0$ of this theorem was considered in [170, 171]. The general case belongs [53, 54, 50].

CHAPTER 2

Integral inequalities

2.1. The classical Hardy inequalities

THEOREM 2.1. **(The Hardy inequality,** see Theorem 330 **[142].)**
Let $p > 1$, $s \neq 1$ and

$$F(x) = \begin{cases} \int\limits_0^x f(\xi)d\xi, & \text{if } s > 1; \\[4mm] \int\limits_x^\infty f(\xi)d\xi, & \text{if } s < 1; \end{cases}$$

then

$$(2.1.1) \qquad \int_0^\infty x^{-s} F^p(x)dx \leq \left(\frac{p}{|s-1|} \right)^p \int_0^\infty x^{-s}(xf)^p dx$$

The constant is the best.

We prove the partial case $p = 2$.

THEOREM 2.2. *Let $f \in L^2(0,d)$, $d, \beta > 0$ and $F(x) = \int\limits_0^x y^{\beta - \frac{1}{2}} f(y)dy$.*
Then

$$(2.1.2) \qquad \int\limits_0^d x^{-2\beta - 1} F^2(x)dx \leq \frac{1}{\beta^2} \int\limits_0^d f^2(x)dx.$$

PROOF. Let $0 < \delta < \beta$. Then by Hölder's inequality (1.14)

$$|F(x)|^2 \leq \left| \int\limits_0^x y^\delta f(y) y^{\beta - \delta - \frac{1}{2}} dy \right|^2 \leq$$

$$\leq \int\limits_0^x y^{2\delta} f^2(y) dy \int\limits_0^x y^{2\beta - 2\delta - 1} dy =$$

$$= \frac{1}{2(\beta - \delta)} x^{2(\beta - \delta)} \int\limits_0^x y^{2\delta} f^2(y) dy.$$

Therefore, by the Fubini Theorem 1.13,

$$\int\limits_0^d x^{-2\beta - 1} F^2(x) dx \leq \frac{1}{2(\beta - \delta)} \int\limits_0^d x^{-2\delta - 1} \left(\int\limits_0^x y^{2\delta} f^2(y) dy \right) dx =$$

$$= \frac{1}{2(\beta - \delta)} \int\limits_0^d y^{2\delta} f^2(y) \left(\int\limits_y^d x^{-2\delta - 1} dx \right) dy =$$

$$= \frac{1}{2(\beta - \delta)} \int\limits_0^d y^{2\delta} f^2(y) \frac{y^{-2\delta} - d^{-2\delta}}{2\delta} dy \leq$$

$$\leq \frac{1}{4\delta(\beta - \delta)} \int\limits_0^d f^2(x) dx.$$

Noting, that $\frac{1}{4\delta(\beta - \delta)}$ becomes minimal for $\delta = \frac{1}{2}\beta$, we choose $\delta := \frac{1}{2}\beta$ and obtain the assertion. □

COROLLARY 2.3. *Let $v \in C^0[0, d] \cap W^{1,2}(0, d), d > 0$ with $v(0) = 0$. Then*

$$(2.1.3) \qquad \int\limits_0^d r^{N-5+\alpha} v^2(r) dr \leq \frac{4}{(4 - N - \alpha)^2} \int\limits_0^d r^{N-3+\alpha} \left(\frac{\partial v}{\partial r} \right)^2 dr$$

for $\alpha < 4 - N$ provided that the integral on the right hand side is finite.

PROOF. We apply Hardy's inequality (2.1.2) with $F = v$, $\beta := \frac{4 - N - \alpha}{2}$, noting that $F'(r) = r^{\beta - \frac{1}{2}} f(r)$ and therefore $f^2(r) = r^{1-2\beta} \left(\frac{\partial v}{\partial r} \right)^2$. □

REMARK 2.4. The constant in (2.1.3) is the best possible.

COROLLARY 2.5. *If $u \in C_0^\infty(\mathbb{R}^n)$, $\alpha < 4 - N$ and $u(0) = 0$, then*

$$\int_{\mathbb{R}^N} r^{\alpha-4} u^2(x) dx \le \frac{4}{(4-N-\alpha)^2} \int_{\mathbb{R}^N} r^{\alpha-2} |\nabla u(x)|^2 dx$$

provided that the integral on the right hand side is finite.

PROOF. The assertion follows by integrating both sides of (2.1.3) over Ω for large enough d and applying (1.3.7).   □

COROLLARY 2.6. *If $u \in W_0^{1,2}(G)$, $\alpha < 4 - N$, then*

$$(2.1.4) \qquad \int_G r^{\alpha-4} u^2(x) dx \le \frac{4}{(4-N-\alpha)^2} \int_G r^{\alpha-2} |\nabla u(x)|^2 dx,$$

provided that the integral on the right hand side of (2.1.4) is finite.

PROOF. The claim follows from Corollary 2.5 because $C_0^\infty(G)$ is dense in $W_0^{1,2}(G)$.   □

COROLLARY 2.7. *Let $v \in C^0[\varepsilon, d] \cap W^{1,2}(\varepsilon, d), d > 0$ with $v(\varepsilon) = 0$. Then*

$$(2.1.5) \qquad \int_\varepsilon^d r^{n-5+\alpha} v^2(r) dr \le \frac{4}{(4-n-\alpha)^2} \int_\varepsilon^d r^{n-3+\alpha} \left(\frac{\partial v}{\partial r}\right)^2 dr$$

for $\alpha < 4 - n$.

PROOF. We apply the inequality (2.1.3) to the function $v(r)$ extended by zero into $[0, \varepsilon)$.   □

Note also another generalization of the Hardy inequality:

THEOREM 2.8. *The inequality*

$$(2.1.6) \qquad \int_0^\infty x^{\alpha-p} |f(x)|^p dx \le \frac{p^p}{|\alpha+1-p|^p} \int_0^\infty x^\alpha |f'(x)|^p dx$$

is true if $p > 1$, $\alpha \ne p - 1$ and $f(x)$ is absolutely continuous on $[0, \infty)$ and satisfies the following boundary condition

$$\begin{cases} f(0) = 0 & \text{when } \alpha < p - 1, \\ \lim_{x \to +\infty} f(x) = 0 & \text{when } \alpha > p - 1. \end{cases}$$

2.2. The Poincaré inequality

THEOREM 2.9. **The Poincaré inequality for the domain in $\mathbb{R}^N$** (see e.g. (7.45) [129]).

Let $u \in W^1(G)$ and G is bounded convex domain in $\mathbb{R}^n$. Then

$$(PI\ 1) \qquad \|u - \overline{u}\|_{2;G} \le c(N) \frac{(diamG)^N}{(measS)^{1-1/N}} \|\nabla u\|_{2;G},$$

where

$$\overline{u} = \frac{1}{measS} \int\limits_S u(x)dx,$$

and S is any measurable subset of G.

THEOREM 2.10. **The Poincaré inequality for the domain on the sphere** (see e.g. Theorem 3.21 [145]).

Let $u \in W^1(\Omega)$ and Ω is convex domain on the unit sphere S^{N-1}. Then

$$(PI\ 2) \qquad \|u - u_\Omega\|_{2;\Omega} \le c(N,\Omega)\|\nabla_\omega u\|_{2;\Omega},$$

where

$$u_\Omega = \frac{1}{meas\Omega} \int\limits_\Omega u(x)d\Omega.$$

Also the following lemma is true.

LEMMA 2.11. (see e.g. Lemma 7.16 [129]).   Let $u \in W^{1,1}(G)$ and G is bounded convex domain in $\mathbb{R}^N$. Then

$$(2.2.1) \qquad |u - \overline{u}| \le \frac{(diamG)^N}{N \cdot measS} \int\limits_G |x - y|^{1-N} |\nabla u(y)|dy \quad a.e. \ in \ G,$$

where

$$\overline{u} = \frac{1}{measS} \int\limits_S u(x)dx,$$

and S is any measurable subset of G.

Now we shall prove the version of the Poincaré inequality.

THEOREM 2.12. Let G_0^d be convex domain, $G_0^d \subset G$, G is bounded domain in $\mathbb{R}^N$. Let $u \in L^2(G)$ and $\int\limits_{G_0^d} r^{\alpha-2}|\nabla u|^2 dx < \infty$, $\alpha \le 4 - N$. Then

$$(2.2.2) \qquad \int\limits_{G_0^\varrho} r^{\alpha-4}|u - \widetilde{u}|^2 dx \le c \int\limits_{G_0^\varrho} r^{\alpha-2}|\nabla u|^2 dx, \ \forall \varrho \in (0,d),$$

where

$$(2.2.3) \qquad \widetilde{u} = \frac{1}{meas G^{\varrho}_{\varrho/2}} \int\limits_{G^{\varrho}_{\varrho/2}} u(y) dy$$

and $c > 0$ *depend only on* $N, d, meas\Omega$.

PROOF. Since $\alpha \le 4 - N$ then from our assumption we have $u \in W^1(G)$. By density of $C^{\infty}(G) \cap W^1(G)$ in $W^1(G)$ we can consider $u \in C^1(G)$. We use Lemma 2.11, applying it for the domains $G^{\varrho}_{\varrho/2}$ and $S = G^{\varrho}_{\varrho/2}$. By this Lemma and the Hölder inequality, we have

$$(2.2.4) \quad |u(x) - \widetilde{u}|^2 \le c \left(\int\limits_{G^{\varrho}_{\varrho/2}} |x - y|^{1-N} |\nabla u(y)| dy \right)^2 \le$$

$$\le c \int\limits_{G^{\varrho}_{\varrho/2}} |x - y|^{1-N} |\nabla u(y)|^2 dy \int\limits_{G^{\varrho}_{\varrho/2}} |x - y|^{1-N} dy =$$

$$= \frac{c}{2} \varrho \cdot meas\Omega \int\limits_{G^{\varrho}_{\varrho/2}} |x - y|^{1-N} |\nabla u(y)|^2 dy.$$

From (2.2.4) it follows

$$(2.2.5) \quad \int\limits_{G^{\varrho}_{\varrho/2}} r^{\alpha-4} |u(x) - \widetilde{u}|^2 dx \le$$

$$\le \frac{c}{2} \varrho \cdot meas\Omega \int\limits_{G^{\varrho}_{\varrho/2}} r^{\alpha-4} \left(\int\limits_{G^{\varrho}_{\varrho/2}} |x - y|^{1-N} |\nabla u(y)|^2 dy \right) dx \le$$

$$\le c \cdot meas\Omega \int\limits_{G^{\varrho}_{\varrho/2}} r^{\alpha-3} \left(\int\limits_{G^{\varrho}_{\varrho/2}} |x - y|^{1-N} |\nabla u(y)|^2 dy \right) dx =$$

$$= c \int\limits_{G^{\varrho}_{\varrho/2}} |\nabla u(y)|^2 \left(\int\limits_{G^{\varrho}_{\varrho/2}} |x|^{\alpha-3} |x - y|^{1-N} dx \right) dy \le c \int\limits_{G^{\varrho}_{\varrho/2}} r^{\alpha-2} |\nabla u|^2 dx,$$

since

$$\int\limits_{G^{\varrho}_{\varrho/2}} |x|^{\alpha-3} |x - y|^{1-N} dx \le c\varrho^{\alpha-3} \int\limits_{G^{\varrho}_{\varrho/2}} |x - y|^{1-N} dx = \varrho^{\alpha-3} \cdot \frac{c}{2} \varrho \cdot meas\Omega \le$$

$$\le c\varrho^{\alpha-2}.$$

Replacing in (2.2.5) ϱ by $2^{-k}\varrho$ we can (2.2.5) rewrite so

$$\int\limits_{G^{(k)}} r^{\alpha-4}|u - \widetilde{u}|^2 dx \leq c \int\limits_{G^{(k)}} r^{\alpha-2}|\nabla u|^2 dx, \ \forall \varrho \in (0, d),$$

whence by summing over all $k = 0, 1, \cdots$ we get the required (2.2.2). $\square$

2.3. The Wirtinger inequality: Dirichlet boundary condition

Let $\Omega \subset S^{N-1}$ be bounded domain with smooth boundary $\partial\Omega$. We consider the problem of the eigenvalues for the Laplace-Beltrami operator $\triangle_\omega$ on the unit sphere

(EVD)
$$\begin{cases} \triangle_\omega u + \vartheta u = 0, \ \omega \in \Omega, \\ u\Big|_{\partial\Omega} = 0, \end{cases}$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVD) has a non-zero weak solutions (eigenfunctions). In the following, we denote by ϑ the smallest positive eigenvalue of this problem.

THEOREM 2.13. **(The Wirtinger inequality)** *The following inequality is valid for all $u \in W_0^{1,2}(\Omega)$*

(2.3.1)
$$\int\limits_\Omega u^2(\omega)d\Omega \leq \frac{1}{\vartheta} \int\limits_\Omega |\nabla_\omega u|^2 \ d\Omega.$$

PROOF. Let us consider the eigenvalue problem (EVD) and denote by

$$a(u, v) := \int\limits_\Omega \langle \nabla_\omega u, \nabla_\omega v \rangle d\Omega$$

the bilinear form corresponding to the Laplace-Beltrami operator $\triangle_\omega$. From Theorem 1.55 applied to the spaces $V = W_0^{1,2}(\Omega)$, $H = L^2(\Omega)$ follows, that the smallest positive eigenvalue ϑ of (EVD) satisfies

$$\vartheta = \inf_{v \in W_0^{1,2}(\Omega)} \frac{a(v, v)}{\|v\|_{2,\Omega}^2}$$

Thus for all $u \in W_0^{1,2}(\Omega)$

$$\int\limits_\Omega |\nabla_\omega u|^2 \ d\omega = a(u, u) \geq \vartheta \|v\|_{L^2(\Omega)}^2.$$

$\square$

REMARK 2.14. From the above proof follows that the constant in (2.3.1) is the best possible.

Now let $\theta(r)$ be the least eigenvalue of the Beltrami operator $\triangle_\omega$ on Ω_r with Dirichlet condition on $\partial\Omega_r$. According to the variational principle of eigenvalues (see Theorem 1.55) we have also

THEOREM 2.15.

$$(2.3.2) \qquad \int\limits_{\Omega_r} u^2(\omega)d\Omega_r \le \frac{1}{\theta(r)}\int\limits_{\Omega_r} |\nabla_\omega u|^2\, d\Omega_r, \quad \forall u \in W_0^{1,2}(\Omega_r).$$

2.4. The Wirtinger inequality: Robin boundary condition

2.4.1. The eigenvalue problem. Let $\Omega \subset S^{n-1}$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\vec{\nu}$ be the exterior normal to $\partial\Omega$. Let $\gamma(\omega)$, $\omega \in \partial\Omega$ be a positive bounded piecewise smooth function. We consider the eigenvalue problem for the Laplace-Beltrami operator $\triangle_\omega$ on the unit sphere

$$(EVR) \qquad \begin{cases} \triangle_\omega u + \vartheta u = 0, \ \omega \in \Omega, \\ \left.\frac{\partial u}{\partial \vec{\nu}} + \gamma(\omega)u\right|_{\partial\Omega} = 0, \end{cases}$$

which consists of the determination of all values ϑ (eigenvalues) for which (EVR) has a non-zero weak solutions (eigenfunctions).

DEFINITION 2.16. Function u is called a **weak** solution of the problem (EVR) provided that $u \in W^1(\Omega)$ and satisfies the integral identity

$$(II) \qquad \int\limits_{\Omega} \left\{ \frac{1}{q_i}\frac{\partial u}{\partial \omega_i}\frac{\partial \eta}{\partial \omega_i} - \vartheta u\eta \right\} d\Omega + \int\limits_{\partial\Omega} \gamma(\omega)u\eta d\sigma = 0$$

for all $\eta(x) \in W^1(\Omega)$.

REMARK 2.17. The eigenvalue problem (EVR) was studied in section VI [87] and in §2.5 [363]. We observe that $\vartheta = 0$ is **not** an eigenvalue of (EVR). In fact, setting in (II) $\eta = u$ and $\vartheta = 0$ we have

$$\int\limits_{\Omega} |\nabla_\omega u|^2 d\Omega + \int\limits_{\partial\Omega} \gamma(\omega)|u|^2 d\sigma = 0 \quad \Longrightarrow u \equiv 0.$$

Now, let us introduce the following functionals on $W^1(\Omega)$

$$F[u] = \int\limits_{\Omega} |\nabla_\omega u|^2 d\Omega + \int\limits_{\partial\Omega} \gamma(\omega)u^2 d\sigma, \quad G[u] = \int\limits_{\Omega} u^2 d\Omega,$$

$$H[u] = \int\limits_{\Omega} \left\langle |\nabla_\omega u|^2 - \vartheta u^2 \right\rangle d\Omega + \int\limits_{\partial\Omega} \gamma(\omega)u^2 d\sigma$$

and the corresponding bilinear forms

$$F(u, \eta) = \int\limits_{\Omega} \frac{1}{q_i} \frac{\partial u}{\partial \omega_i} \frac{\partial \eta}{\partial \omega_i} d\Omega + \int\limits_{\partial\Omega} \gamma(\omega) u\eta d\sigma,$$

$$G(u, \eta) = \int\limits_{\Omega} u\eta d\Omega.$$

We define yet the set

$$K = \left\{ u \in W^1(\Omega) \,\middle|\, G[u] = 1 \right\}.$$

Since $K \subset W^1(\Omega)$, $F[u]$ is bounded from below for $u \in K$. The greatest lower bound of $F[u]$ for this family we denote by ϑ :

$$\inf_{u \in K} F[u] = \vartheta$$

We formulate the following statement:

THEOREM 2.18. (See Theorem of Subsection 4 §2.5, p. 123 [**363**]).
Let $\Omega \subset S^{n-1}$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\gamma(\omega)$, $\omega \in \partial\Omega$ be a positive bounded piecewise smooth function. There exist $\vartheta > 0$ and a function $u \in K$ such that

$$F(u, \eta) - \vartheta G(u, \eta) = 0 \text{ for arbitrary } \eta \in W^1(\Omega).$$

In particular $F[u] = \vartheta$. In addition, on Ω, u has continuous derivatives of arbitrary order and satisfies the equation $\triangle_\omega u + \vartheta u = 0$, $\omega \in \Omega$ and the boundary condition of (EVR) in the weak sense (for details see the Remarks 2.19 below).

PROOF. Because of $F[v]$ is bounded from below for $v \in K$, there is $\vartheta = \inf\limits_{v \in K} F[v]$. Consider a sequence $\{v_k\} \subset K$ such that $\lim\limits_{k \to \infty} F[v_k] = \vartheta$ (such sequence exists by the definition of infimum). From $K \subset W^1(\Omega)$ it follows that v_k is bounded in $W^1(\Omega)$ and therefore compact in $L^2(\Omega)$. Choosing a subsequence, we can assume that it is converging in $L^2(\Omega)$. Furthermore,

$$(2.4.1) \qquad \|v_k - v_l\|_{L^2(\Omega)}^2 = G[v_k - v_l] < \varepsilon$$

as soon as $k, l > N(\varepsilon)$. From the obvious equality

$$G\left[\frac{v_k + v_l}{2}\right] = \frac{1}{2}G[v_k] + \frac{1}{2}G[v_l] - G\left[\frac{v_k - v_l}{2}\right]$$

we obtain, using $G[v_k] = G[v_l] = 1$ and $G[\frac{v_k - v_l}{2}] < \frac{\varepsilon}{\vartheta}$, that

$$G\left[\frac{v_k + v_l}{2}\right] > 1 - \frac{\varepsilon}{\vartheta}$$

for big k, l. The functionals $F[v]$ and $G[v]$ are homogeneous quadratic functionals and therefore their ratio $\frac{F[v]}{G[v]}$ does not change under the passage from v to cv ($c = const \neq 0$) and hence

$$\inf_{v \in W^1(\Omega)} \frac{F[v]}{G[v]} = \inf_{v \in K} F[v] = \vartheta.$$

Therefore $F[v] \geq \vartheta G[v]$ for all $v \in W^1(\Omega)$. Since $\frac{v_k + v_l}{2} \in W^1(\Omega)$ together with $v_k, v_l \in K$, then

$$F\left[\frac{v_k + v_l}{2}\right] \geq \vartheta G\left[\frac{v_k + v_l}{2}\right] > \vartheta\left(1 - \frac{\varepsilon}{\vartheta}\right) = \vartheta - \varepsilon, \quad k, l > N(\varepsilon).$$

Then, taking k and l large enough that $F[v_k] < \vartheta + \varepsilon$ and $F[v_l] < \vartheta + \varepsilon$, we obtain

$$F\left[\frac{v_k - v_l}{2}\right] = \frac{1}{2}F[v_k] + \frac{1}{2}F[v_l] - F\left[\frac{v_k + v_l}{2}\right] <$$

$$< \frac{1}{2}(\vartheta + \varepsilon) + \frac{1}{2}(\vartheta + \varepsilon) - (\vartheta - \varepsilon) = 2\varepsilon.$$

Consequently,

(2.4.2) $$F[v_k - v_l] \to 0, \quad k, l \to \infty.$$

From (8.2.10), (8.2.12) it follows that $\|v_k - v_l\|_{W^1(\Omega)} \to 0, \quad k, l \to \infty$. Hence $\{v_k\}$ converges in $W^1(\Omega)$ and as result of the completeness of $W^1(\Omega)$ there exists a limit function $u \in W^1(\Omega)$ such that $\|v_k - u\|_{W^1(\Omega)} \to 0, \quad k \to \infty$. In addition,

$$|F[v_k] - F[u]| = \left|\int_\Omega \left(|\nabla_\omega v_k|^2 - |\nabla_\omega u|^2\right) d\Omega + \int_{\partial\Omega} \gamma(\omega)(v_k^2 - u^2) d\sigma\right| =$$

$$= \left|\int_\Omega (\nabla_\omega v_k - \nabla_\omega u)(\nabla_\omega v_k + \nabla_\omega u) d\Omega + \right.$$

$$\left. + \int_{\partial\Omega} \gamma(\omega)(v_k - u)(v_k + u) d\sigma\right| \leq$$

$$\leq \left(\int_\Omega |\nabla_\omega(v_k - u)|^2 d\Omega\right)^{1/2} \left(\int_\Omega |\nabla_\omega(v_k + u)|^2 d\Omega\right)^{1/2} +$$

$$+ \left(\int_{\partial\Omega} |v_k - u|^2 d\sigma\right)^{1/2} \left(\int_{\partial\Omega} \gamma^2(\omega)|v_k + u|^2 d\sigma\right)^{1/2} \to 0,$$

as $k \to \infty$, since by (1.6.2)

$$\left(\int_{\partial\Omega} |v_k - u|^2 d\sigma \right)^{1/2} \leq C \|v_k - u\|_{W^1(\Omega)} \to 0, \quad k \to \infty,$$

while the terms $\left(\int_{\Omega} |\nabla_\omega (v_k + u)|^2 d\Omega \right)^{1/2}$ and $\left(\int_{\partial\Omega} \gamma^2(\omega) |v_k + u|^2 d\sigma \right)^{1/2}$ are bounded. Therefore we get

$$F[u] = \lim_{k \to \infty} F[v_k] = \vartheta.$$

Analogously one sees that $G[u] = 1$.
Suppose now that η is some function from $W^1(\Omega)$. Consider the ratio

$$\frac{F[u + \mu\eta]}{G[u + \mu\eta]} = \frac{F[u] + 2\mu F(u, \eta) + \mu^2 F[\eta]}{G[u] + 2\mu G(u, \eta) + \mu^2 G[\eta]}.$$

It is a continuously differentiable function of μ on some interval around the point $\mu = 0$. This ratio has a minimum at $\mu = 0$ equal to ϑ and by the Fermat Theorem, we have

$$\left(\frac{F[u + \mu\eta]}{G[u + \mu\eta]} \right)'_{\mu=0} = \frac{2F(u, \eta)G[u] - 2F[u]G(u, \eta)}{G^2[u]} = 0,$$

which by virtue of $F[u] = \vartheta$, $G[u] = 1$ gives

$$F(u, \eta) - \vartheta G(u, \eta) = 0, \quad \forall \eta \in W^1(\Omega).$$

The rest part of our Theorem follows from the smoothness theory for elliptic boundary value problem in smooth domains (details see in §2.5 [363]). □

REMARK 2.19. **Remarks about the eigenvalue problem** (*EVR*) (see Remarks on pp. 121 - 122 [363])

Consider a sequence of domains $\{\Omega'\}$ lying in the interior of Ω and converging to Ω. Let the boundaries $\partial\Omega'$ of these domains be piecewise continuously differentiable. The integral identity (*II*) from Definition 2.16 for $\eta \in W^1(\Omega)$ has the form

$$(2.4.3) \qquad \int_{\Omega} \left\{ \frac{1}{q_i} \frac{\partial u}{\partial \omega_i} \frac{\partial \eta}{\partial \omega_i} - \vartheta u \eta \right\} d\Omega + \int_{\partial\Omega} \gamma(\omega) u \eta d\sigma = 0$$

But

$$\int\limits_{\Omega} \left\{ \frac{1}{q_i} \frac{\partial u}{\partial \omega_i} \frac{\partial \eta}{\partial \omega_i} - \vartheta u \eta \right\} d\Omega = \lim_{\Omega' \to \Omega} \int\limits_{\Omega'} \left\{ \frac{1}{q_i} \frac{\partial u}{\partial \omega_i} \frac{\partial \eta}{\partial \omega_i} - \vartheta u \eta \right\} d\Omega =$$

$$= \lim_{\Omega' \to \Omega} \left[- \int\limits_{\Omega'} \eta(\triangle_\omega u + \vartheta u)\, d\Omega + \int\limits_{\partial\Omega'} \eta \frac{\partial u}{\partial \overrightarrow{\nu}}\, d\sigma \right] = \lim_{\Omega' \to \Omega} \int\limits_{\partial\Omega'} \eta \frac{\partial u}{\partial \overrightarrow{\nu}}\, d\sigma.$$

Thus, the equation (2.4.3) takes the form

$$(2.4.4) \qquad \lim_{\Omega' \to \Omega} \int\limits_{\partial\Omega'} \eta \frac{\partial u}{\partial \overrightarrow{\nu}}\, d\sigma + \int\limits_{\partial\Omega} \gamma(\omega) u \eta\, d\sigma = 0.$$

If in addition $\partial\Omega' \to \partial\Omega$ in the sense that not only the points of $\partial\Omega'$ converge to the points of $\partial\Omega$ but also the normals at these points converge to the corresponding normals of $\partial\Omega$, then

$$\int\limits_{\partial\Omega} \gamma u \eta\, d\sigma = \lim_{\Omega' \to \Omega} \int\limits_{\partial\Omega'} \gamma u \eta\, d\sigma,$$

if we assume that γ is the value on $\partial\Omega$ of some function given on Ω.

Then condition (2.4.4) takes the form

$$(2.4.5) \qquad \lim_{\Omega' \to \Omega} \int\limits_{\partial\Omega'} \left(\frac{\partial u}{\partial \overrightarrow{\nu}} + \gamma u \right) \eta\, d\sigma = 0.$$

Thus, u satisfies the boundary condition of (EVR) "in the weak sense."

Therefore, an *eigenfunction of the problem* (EVR) will be defined to be a function $u(x) \not\equiv 0$ satisfying equation in Ω for some ϑ and the boundary condition in the sense of relation (2.4.5). The number ϑ is called the *eigenvalue corresponding to the eigenfunction u(x)*.

Theorem 2.18 proved implies the existence of an eigenfunction u corresponding to the eigenvalue ϑ in the sense indicated.

2.4.2. The Friedrichs-Wirtinger inequality. Now from the variational principle we obtain

THEOREM 2.20. *Let ϑ be the smallest positive eigenvalue of the problem (EVR). (It exists according to Theorem 2.18.) Let $\Omega \subset S^{n-1}$ be a bounded domain. Let $u \in W^1(\Omega)$ and $\gamma(\omega)$, $\omega \in \partial\Omega$ be a positive bounded piecewise smooth function. Then*

$$(2.4.6) \qquad \vartheta \int\limits_{\Omega} u^2(\omega) d\Omega \leq \int\limits_{\Omega} |\nabla_\omega u(\omega)|^2 d\Omega + \int\limits_{\partial\Omega} \gamma(\omega) u^2(\omega) d\sigma.$$

PROOF. Consider functionals $F[u], G[u], H[u]$ on $W^1(\Omega)$ described above. We will find the minimum of the functional $F[u]$ on the set K. For this we investigate the minimization of the quadratic functional $H[u]$ on all functions $u(\omega)$, for which the integrals exist and which satisfy the boundary condition from (EVR). We use formally the Lagrange multipliers and get the Euler equation from the condition $\delta H[u] = 0$. By the calculation of the first variation δH we have

$$\delta H[u] = -2 \int\limits_{\Omega} (\triangle_\omega u + \vartheta u)\delta u d\Omega + 2 \int\limits_{\partial\Omega} \frac{\partial u}{\partial \overrightarrow{\nu}}\delta u d\sigma + 2 \int\limits_{\partial\Omega} \gamma(\omega) u \delta u d\sigma.$$

Hence we obtain the Euler equation and the boundary condition that are our (EVR). Backwards, let $u(\omega)$ be the solution of (EVR). By Theorem 2.18, $u \in C^2(\Omega)$. Therefore we can multiply both sides of the equation (EVR) by u and integrate over Ω using the Gauss-Ostrogradskiy formula:

$$0 = \int\limits_{\Omega} (u\triangle_\omega u + \vartheta u^2) d\Omega = \vartheta \int\limits_{\Omega} u^2 d\Omega - \int\limits_{\Omega} |\nabla_\omega u|^2 d\Omega +$$

$$+ \int\limits_{\Omega} \frac{\partial}{\partial \omega_i}\left(u \frac{J}{q_i}\frac{\partial u}{\partial \omega_i}\right) d\omega = \vartheta \int\limits_{\Omega} u^2 d\Omega - \int\limits_{\Omega} |\nabla_\omega u|^2 d\Omega + \int\limits_{\partial\Omega} u \frac{\partial u}{\partial \overrightarrow{\nu}} d\sigma =$$

$$= \vartheta \int\limits_{\Omega} u^2 d\Omega - \int\limits_{\Omega} |\nabla_\omega u|^2 d\Omega - \int\limits_{\partial\Omega} \gamma(\omega) u^2 d\sigma \underset{\text{(by } K)}{=} \vartheta - F[u] \Rightarrow$$

$$\vartheta = F[u].$$

Consequently, the required minimum is the least eigenvalue of (EVR). The existence of a function $u \in K$ such that

(2.4.7) $F[u] \le F[v]$ for all $v \in K$

was proved in Theorem 2.18.   □

2.5. Hardy-Friedrichs-Wirtinger type inequalities

2.5.1. The Dirichlet boundary condition. Let $\theta(r)$ be the least eigenvalue of the Beltrami operator $\triangle_\omega$ on Ω_r with Dirichlet condition on $\partial\Omega_r$ and let a neighborhood G_0^d of the boundary point $\mathcal{O}$ satisfy the condition:

(S) $\begin{cases} \theta(r) \ge \theta_0 + \theta_1(r) \ge \theta_2 > 0, \ r \in (0,d), \ \text{where} \\ \theta_0, \theta_2 \text{ are positive constants and} \\ \theta_1(r) \text{ is a function that is Dini-continuous at zero} \\ \lim\limits_{r \to 0} \theta_1(r) = 0, \quad \int\limits_0^d \frac{|\theta_1(r)|}{r} dr < \infty. \end{cases}$

This condition describes our very general assumptions on the structure of the boundary of the domain in a neighborhood of the boundary point $\mathcal{O}$.

THEOREM 2.21. **(Generalized Hardy-Friedrichs-Wirtinger inequality).** *Let* $U(d) = \int_{G_0^d} r^{\alpha-2}|\nabla u|^2 dx$ *is finite and* $u(x) = 0$ *for* $x \in \Gamma_0^d$.
Then

(H-W)
$$\int_{G_0^d} r^{\alpha-4}|u|^2 dx \le H(\lambda, N, \alpha)\left(1 + \frac{|\theta_1(\varrho)|}{\theta_2}\right)\int_{G_0^d} r^{\alpha-2}|\nabla u|^2 dx$$

with some $\varrho \in (0, d)$, *where*

(*)
$$\begin{cases} H(\lambda, N, \alpha) = \left[\frac{(4-N-\alpha)^2}{4} + \lambda(\lambda + N - 2)\right]^{-1}, \\ \lambda = \frac{1}{2}\left(2 - N + \sqrt{(N-2)^2 + 4\theta_0}\right) \end{cases}$$

provided $\alpha \le 4 - N$.

PROOF. Integrating (2.3.2) over $r \in (0, d)$ we get
$$\int_{G_0^d} r^{\alpha-4}|u|^2 dx \le \int_{G_0^d} \frac{r^{\alpha-2}}{\theta(r)}\frac{|\nabla_\omega u|^2}{r^2}dx \le \int_{G_0^d} \frac{r^{\alpha-2}}{\theta_0 + \theta_1(r)}\frac{|\nabla_\omega u|^2}{r^2}dx,$$

but
$$\frac{1}{\theta_0 + \theta_1(r)} = \frac{1}{\theta_0} + \left(\frac{1}{\theta_0 + \theta_1(r)} - \frac{1}{\theta_0}\right) = \frac{1}{\theta_0} - \frac{\theta_1(r)}{\theta_0(\theta_0 + \theta_1(r))} \le$$
$$\le \frac{1}{\theta_0} + \frac{|\theta_1(r)|}{\theta_0\theta_2}$$

because of the condition (S). Hence, applying the mean value theorem with regard to the continuous at zero of the function $\theta_1(r)$ we obtain
$$\theta_0 \int_{G_0^d} r^{\alpha-4}|u|^2 dx \le \int_{G_0^d} r^{\alpha-2}\frac{|\nabla_\omega u|^2}{r^2}dx + \frac{|\theta_1(\varrho)|}{\theta_2}U(d)$$

for some $\varrho \in (0, d)$.

Now we integrate the Hardy inequality over Ω and rewrite the result in the form
$$\left(\frac{4 - N - \alpha}{2}\right)^2 \int_{G_0^d} r^{\alpha-4}|u|^2 dx \le \int_{G_0^d} r^{\alpha-2}u_r^2 dx, \quad \alpha \le 4 - N.$$

Adding two last inequality and applying the formula for $|\nabla u|^2$ and the definition of the value λ we get the required inequality (H-W). $\qquad\square$

COROLLARY 2.22. $\forall \delta > 0 \; \exists d > 0$ *such that*

$$\int\limits_{G_0^d} r^{\alpha-4}u^2 dx \leq \left(H(\lambda, N, \alpha) + \delta\right) \int\limits_{G_0^d} r^{\alpha-2} \left|\nabla u\right|^2 dx,$$

provided the integral on the right hand side is finite and $u(x) = 0$ *for* $x \in \Gamma_0^d$ *in the sense of traces.*

PROOF. Because of the function $\theta_1(\varrho)$ is continuous at zero we establish the statement. $\qquad\square$

For conical domains the following statements are true.

COROLLARY 2.23. *Let* $\int\limits_{G_0^d} r^{\alpha-2}|\nabla u|^2 dx$ *is finite and* u *vanish on* Γ_0^d *in the sense of traces. Then*

$$(2.5.1) \qquad \int\limits_{G_0^d} r^{\alpha-4}u^2(x)dx \leq \frac{4}{(4-N-\alpha)^2} \int\limits_{G_0^d} r^{\alpha-2} \left(\frac{\partial u}{\partial r}\right)^2 dx$$

for $\alpha < 4 - N$ *and*

$$(2.5.2) \qquad \int\limits_{G_0^d} r^{\alpha-4}u^2(x)dx \;\; \leq \;\; \frac{1}{\lambda(\lambda + N - 2)} \int\limits_{G_0^d} r^{\alpha-4} \left|\nabla_\omega u\right|^2 dx$$

for all $\alpha \in \mathbb{R}$.

PROOF. Integrating both sides of Hardy's inequality (2.1.3) over Ω we obtain (2.5.1). The inequality (2.5.2) is derived similarly, by multiplying the generalized Wirtinger inequality (2.3.1) by $r^{\alpha+N-5}$ and integrating over $r \in [0, d]$. $\qquad\square$

THEOREM 2.24. *Let* $u \in W^{1,2}(G_0^d)$ *vanish on* Γ_0^d *in the* $W^{1,2}(G_0^d)$ *sense. Then*

$$(2.5.3) \qquad \int\limits_{G_0^d} r^{\alpha-4}u^2(x)dx \;\; \leq \;\; H(\lambda, N, \alpha) \int\limits_{G_0^d} r^{\alpha-2} \left|\nabla u\right|^2 dx,$$

with $H(\lambda, N, \alpha)$ *from (*) for* $\alpha \leq 4 - N$, *provided the integral on the right hand side is finite.*

PROOF. If $\alpha < 4 - N$, then the assertion follows by adding the inequalities (2.5.1), (2.5.2) and by taking into account the formula 1.3.7. If $\alpha = 4 - N$, then (2.5.3) coincides with (2.5.2). $\qquad\square$

COROLLARY 2.25. *Let* $u \in W^{1,2}(G)$ *with* $u|_{\partial G} = \varphi \in \mathring{W}^{1/2}_{\alpha-2}(\partial\Omega)$. *Then for every* $\delta > 0$ *there exist a constant* $c = c(\delta, \lambda, N, \alpha)$ *such that*

$$(2.5.4) \quad \int\limits_{G^d_0} r^{\alpha-4} u^2(x) dx \leq (1+\delta) H(\lambda, N, \alpha) \int\limits_{G^d_0} r^{\alpha-2} |\nabla u|^2 dx$$

$$+ c(\delta, \lambda, N, \alpha) \|\varphi\|_{\mathring{W}^{1/2}_{\alpha-2}(\Gamma^d_0)},$$

for $\alpha \leq 4 - N$, *provided the integral on the right hand side is finite.*

PROOF. Let $\Phi \in \mathring{W}^1_{\alpha-2}(G^d_0)$ with $\Phi|_{\Gamma^d_0} = \varphi$ on Γ^d_0. Then the difference $u - \Phi$ satisfies the generalized Hardy-Wirtinger inequality

$$\int\limits_{G^d_0} r^{\alpha-4} (u - \Phi)^2 dx \leq H(\lambda, N, \alpha) \int\limits_{G^d_0} r^{\alpha-2} |\nabla u - \nabla \Phi|^2 dx, \quad \alpha \leq 4 - N.$$

Applying Cauchy's inequality twice we obtain

$$\int\limits_{G^d_0} r^{\alpha-4} u^2 dx = \int\limits_{G^d_0} r^{\alpha-4} \left((u - \Phi)^2 + 2u\Phi - \Phi^2 \right) dx$$

$$\leq H(\lambda, N, \alpha) \int\limits_{G^d_0} r^{\alpha-2} \left(|\nabla u|^2 - 2\langle \nabla u, \nabla \Phi \rangle + |\nabla \Phi|^2 \right) dx$$

$$+ \varepsilon \int\limits_{G^d_0} r^{\alpha-4} u^2 dx + \varepsilon^{-1} \int\limits_{G^d_0} r^{\alpha-4} \Phi^2 dx$$

$$\leq H(\lambda, N, \alpha) \int\limits_{G^d_0} r^{\alpha-2} \left((1+\delta_1)|\nabla u|^2 + (1+\delta_1^{-1})|\nabla \Phi|^2) \right) dx$$

$$+ \varepsilon \int\limits_{G^d_0} r^{\alpha-4} u^2 dx + \varepsilon^{-1} \int\limits_{G^d_0} r^{\alpha-4} \Phi^2 dx$$

for all $\varepsilon > 0, \delta_1 > 0$. Thus the claim follows from the definition of the trace norm, if we set $\delta = \frac{\varepsilon + \delta_1}{1 - \varepsilon}$. $\square$

COROLLARY 2.26. *Let* $\varepsilon > 0$ *and* $u \in C^0(\overline{G_\varepsilon}) \cap W^{1,2}(G_\varepsilon)$ *with* $u(x) = 0$ *for* $x \in \Gamma_\varepsilon$. *Then for* $\alpha < 4 - N$ *we have*

$$(2.5.5) \quad \int\limits_{G_\varepsilon} r^{\alpha-4} u^2(x) dx \leq c \int\limits_{G_\varepsilon} r^{\alpha-2} |\nabla u|^2 dx$$

with a constant $c = c(\lambda, N, \alpha)$.

Let us denote by $\zeta : G_0^\rho \to [0,1]$ a cut off function satisfying

$$\zeta(r) = \begin{cases} 1 & \text{for} \quad 0 \le r \le \rho/2 \\ 0 & \text{for} \quad r \ge \rho \end{cases},$$

$$|\zeta'(r)| \le \text{const} \cdot \rho^{-1} \quad \text{for} \quad 0 \le r \le \rho.$$

COROLLARY 2.27. *Let* $u \in C^0(\overline{G_0^d}) \cap W^{1,2}(G_0^d)$ *vanish on* Γ_0^d. *If* $\alpha \le 4 - N$ *and* $\rho \in (0,d]$, *then*

$$(2.5.6) \quad \int_{G_0^\rho} r^{\alpha-4}\zeta^2(r)u^2(x)dx \le H(\lambda, N, \alpha) \int_{G_0^\rho} r^{\alpha-2}\Big((1+\delta)\zeta^2(r)|\nabla u|^2 +$$

$$+ (1 + \delta^{-1})(\zeta')^2(r)u^2(x)\Big)dx$$

for all $\delta > 0$, *provided the integrals on the right hand side are finite.*

PROOF. The assertion follows directly by applying the generalized Hardy-Wirtinger inequality (2.5.3) to the product $\zeta(r)u(x)$ and the Cauchy inequality with $\delta > 0$. $\qquad \square$

LEMMA 2.28. *Let* $U(\rho) = \int_{G_0^\rho} r^{2-N}|\nabla u|^2dx < \infty$, $\rho \in (0,d)$ *and* $\nabla u(\varrho, \cdot) \in L_2(\Omega)$ *for almost all* $\varrho \in (0,d)$. *Then*

$$\int_\Omega \left(\rho u \frac{\partial u}{\partial r} + \frac{N-2}{2}u^2\right)\Bigg|_{r=\varrho} d\Omega \le \frac{\rho}{2\lambda + \theta_1(\varrho)h_1(\varrho)}U'(\rho),$$

where $h_1(\varrho) \le \frac{2}{\sqrt{\theta_0}+\sqrt{\theta_2}}$, $\varrho \in (0,d)$.

PROOF. Writing $U(\rho)$ in spherical coordinates and differentiating by ρ we obtain

$$U'(\rho) = \int_\Omega \left(\rho\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{\rho}|\nabla_\omega u|^2\right)\Bigg|_{r=\varrho} d\Omega.$$

Moreover, by Cauchy's inequality we have for all $\varepsilon > 0$

$$\rho u \frac{\partial u}{\partial r} \le \frac{\varepsilon}{2}u^2 + \frac{1}{2\varepsilon}\rho^2\left(\frac{\partial u}{\partial r}\right)^2.$$

Thus, we obtain by the (2.3.2)

$$\int_\Omega \left(\rho u \frac{\partial u}{\partial r} + \frac{N-2}{2} u^2 \right) d\Omega \leq \frac{\varepsilon + N - 2}{2} \int_\Omega u^2 d\Omega + \frac{\rho^2}{2\varepsilon} \int_\Omega \left(\frac{\partial u}{\partial r} \right)^2 d\Omega \leq$$

$$\leq \frac{\varepsilon + N - 2}{2\theta(\varrho)} \int_\Omega |\nabla_\omega u|^2 d\Omega + \frac{\rho^2}{2\varepsilon} \int_\Omega \left(\frac{\partial u}{\partial r} \right)^2 d\Omega =$$

$$= \frac{\rho^2}{2} \int_\Omega \left\{ \frac{1}{\varepsilon} \left(\frac{\partial u}{\partial r} \right)^2 + \frac{\varepsilon + N - 2}{\varrho^2 \theta(\varrho)} |\nabla_\omega u|^2 \right\} d\Omega.$$

Let us choose ε from the equality

$$\frac{1}{\varepsilon} = \frac{\varepsilon + N - 2}{\theta(\varrho)} \implies \varepsilon = \frac{1}{2} \left(2 - N + \sqrt{(N-2)^2 + 4\theta(\varrho)} \right).$$

Then we get

$$\int_\Omega \left(\rho u \frac{\partial u}{\partial r} + \frac{N-2}{2} u^2 \right) d\Omega \leq \frac{\varrho^2}{2\varepsilon} \int_\Omega |\nabla u|^2 d\Omega,$$

or in virtue of the condition (S)

$$\int_\Omega \left(\rho u \frac{\partial u}{\partial r} + \frac{N-2}{2} u^2 \right) \Bigg|_{r=\varrho} d\Omega \leq \frac{\varrho U'(\varrho)}{2 - N + \sqrt{(N-2)^2 + 4[\theta_0 + \theta_1(\varrho)]}}.$$

Because of (*) by elementary calculation we have

$$\left(2 - N + \sqrt{(N-2)^2 + 4[\theta_0 + \theta_1(\varrho)]} \right) - 2\lambda =$$
$$= \left(2 - N + \sqrt{(N-2)^2 + 4[\theta_0 + \theta_1(\varrho)]} \right) - \left(2 - N + \sqrt{(N-2)^2 + 4\theta_0} \right) =$$
$$= \theta_1(\varrho) h_1(\varrho),$$

where

$$h_1(\varrho) = \frac{4}{\sqrt{(N-2)^2 + 4[\theta_0 + \theta_1(\varrho)]} + \sqrt{(N-2)^2 + 4\theta_0}} \leq \frac{2}{\sqrt{\theta_0} + \sqrt{\theta_2}}.$$

Hence follows the statement of Lemma.   □

COROLLARY 2.29. *Let G_0^d be the conical domain, $\nabla u(\varrho, \cdot) \in L_2(\Omega)$ for almost all $\varrho \in (0, d)$ and $U(\rho) = \int_{G_o^\varrho} r^{2-N} |\nabla u|^2 dx < \infty$, $\rho \in (0, d)$. Then*

$$\int_\Omega \left(\rho u \frac{\partial u}{\partial r} + \frac{N-2}{2} u^2 \right) \Bigg|_{r=\varrho} d\Omega \leq \frac{\rho}{2\lambda} U'(\rho).$$

PROOF. Writing $U(\rho)$ in spherical coordinates and differentiating by ρ we obtain

$$U'(\rho) = \int_\Omega \left(\rho \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{\rho} |\nabla_\omega u|^2 \right) \Bigg|_{r=\varrho} d\Omega.$$

Moreover, by Cauchy's inequality we have for all $\varepsilon > 0$

$$\rho u \frac{\partial u}{\partial r} \leq \frac{\varepsilon}{2} u^2 + \frac{1}{2\varepsilon} \rho^2 \left(\frac{\partial u}{\partial r} \right)^2.$$

Thus, choosing $\varepsilon = \lambda$ we obtain by the generalized Wirtinger's inequality (2.3.1) with (2.5.11):

$$\int_\Omega \left(\rho u \frac{\partial u}{\partial r} + \frac{N-2}{2} u^2 \right) \Bigg|_{r=\varrho} d\Omega \leq \frac{\varepsilon + N - 2}{2} \int_\Omega u^2 d\Omega + \frac{\rho^2}{2\varepsilon} \int_\Omega \left(\frac{\partial u}{\partial r} \right)^2 d\Omega$$

$$\leq \frac{\varepsilon + N - 2}{2\lambda(\lambda + N - 2)} \int_\Omega |\nabla_\omega u|^2 d\Omega + \frac{\rho^2}{2\varepsilon} \int_\Omega \left(\frac{\partial u}{\partial r} \right)^2 d\Omega = \frac{\rho}{2\lambda} U'(\rho).$$

$\square$

Let us assume that the cone K is contained in a circular cone $\tilde{K}$ with the opening angle ω_0 and that the axis of $\tilde{K}$ coincides with $\{(x_1, 0, \ldots, 0)\}$ where $x_1 > 0$. We define the vector $l \in \mathbb{R}^N$ by $l = (-1, 0, \ldots, 0)$.

LEMMA 2.30. *Let $v \in C^0(\overline{G_\varepsilon^d}) \cap W^1(G_\varepsilon^d)$, $v(\varepsilon) = 0$. Then for any $\varepsilon > 0$*

$$(2.5.7) \qquad \int_{G_\varepsilon^d} r^{\alpha-4} v^2 dx \leq H(\lambda, N, \alpha) \int_{G_\varepsilon^d} r^{\alpha-2} |\nabla v|^2 dx,$$

where $H(\lambda, N, \alpha)$ is determined by (*).

PROOF. By Theorem 2.13 the inequality (2.3.1) holds. Multiplying it by $r^{N-5+\alpha}$ and integrating over $r \in (\varepsilon, d)$, by (2.5.11), we obtain (2.5.7) for $\alpha = 4 - N$. If $\alpha < 4 - N$ we consider the inequality (2.1.5) and integrate it over Ω; then we have

$$\frac{1}{4}(4 - N - \alpha)^2 \int_{G_\varepsilon^d} r^{\alpha-4} v^2 dx \leq \int_{G_\varepsilon^d} r^{\alpha-2} v_r^2 dx.$$

Adding this inequality with above for $\alpha = 4 - N$ (see (2.5.2) for G_ε^d) and using the formula $|\nabla u|^2 = \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_\omega u|^2$, we get the desired. $\square$

LEMMA 2.31. *Let* $v \in C^0(\overline{G}) \cap W^1(G)$, $v(0) = 0$. *Then for any* $\varepsilon > 0$

$$(2.5.8) \qquad \int\limits_{G_0^d} r_\varepsilon^{\alpha-4} v^2 dx \le H(\lambda, N, \alpha) \int\limits_{G_0^d} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx,$$

where $H(\lambda, N, \alpha)$ *is determined by (*).*

PROOF. We perform the change of variables $y_i = x_i - \varepsilon l_i$, $i = 1, \dots, N$ and use the inequality (2.5.7); then we get

$$\int\limits_{G_0^d} r_\varepsilon^{\alpha-4} v^2(x) dx = \int\limits_{G_\varepsilon^{d\varepsilon}} |y|^{\alpha-4} v^2(y + \varepsilon l) dy \le$$

$$\le H(\lambda, N, \alpha) \int\limits_{G_\varepsilon^{d\varepsilon}} |y|^{\alpha-2} |\nabla_y v(y + \varepsilon l)|^2 dy = H(\lambda, N, \alpha) \int\limits_{G_0^d} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx.$$

$\square$

LEMMA 2.32. *Let* $u \in W^{1,2}(G_0^d)$ *with* $u(x) = 0$ *for* $x \in \Gamma_0^d$. *Then*

$$(2.5.9) \qquad \int\limits_{G_0^d} r_\varepsilon^{\alpha-2} r^{-2} u^2(x) dx \le \left(\frac{3}{h}\right)^{2-\alpha} \cdot \frac{1}{\lambda(\lambda + N - 2)} \int\limits_{G_0^d}{}' r_\varepsilon^{\alpha-2} |\nabla u|^2 dx$$

for all $\alpha \in \mathbb{R}$, *where*

$$h = \begin{cases} 1, & \text{if} \quad 0 < \omega_0 \le \pi, \\ \sin\frac{\omega_0}{2}, & \text{if} \quad \pi < \omega_0 < 2\pi. \end{cases}$$

PROOF. We consider the Wirtinger inequality (2.3.1). We multiply both sides of this inequality by $(2^{-k}d + \varepsilon)^{\alpha-2} r^{N-3}$ with $\varepsilon > 0$, taking into account that

$$2^{-k-1}d + \varepsilon < r + \varepsilon < 2^{-k}d + \varepsilon \quad \text{in} \quad G^{(k)},$$

and integrate over $r \in (2^{-k-1}d, 2^{-k}d)$

$$\int\limits_{G^{(k)}} r^{-2}(2^{-k}d + \varepsilon)^{\alpha-2} u^2 dx \le \frac{1}{\lambda(\lambda + N - 2)} \int\limits_{G^{(k)}} \left((2^{-k}d + \varepsilon)^{\alpha-2} |\nabla u|^2\right) dx.$$

Since $r_\varepsilon \le r + \varepsilon < 2^{-k}d + \varepsilon$ in $G^{(k)}$ and $\alpha \le 2$, we obtain

$$\int\limits_{G^{(k)}} r^{-2}(2^{-k}d + \varepsilon)^{\alpha-2} u^2 dx \le \frac{1}{\lambda(\lambda + N - 2)} \int\limits_{G^{(k)}} r_\varepsilon^{\alpha-2} |\nabla u|^2 dx.$$

On the other hand, by Lemma 1.11, in $G^{(k)}$

$$2^{-k}d + \varepsilon = 2 \cdot 2^{-k-1}d + \varepsilon < 2r + \varepsilon \leq \frac{3}{h}r_\varepsilon \Rightarrow$$

$$(2^{-k}d + \varepsilon)^{\alpha-2} \geq \left(\frac{3}{h}\right)^{\alpha-2} r_\varepsilon^{\alpha-2}.$$

Hence it follows

$$\left(\frac{3}{h}\right)^{\alpha-2} \int\limits_{G^{(k)}} r^{-2}r_\varepsilon^{\alpha-2}u^2 dx \leq \frac{1}{\lambda(\lambda + N - 2)} \int\limits_{G^{(k)}} r_\varepsilon^{\alpha-2}|\nabla u|^2 dx.$$

Summing up this inequality for $k = 0, 1, 2, \ldots$, we finally obtain the required (2.5.9). $\qquad\square$

THEOREM 2.33. *Let G be a unbounded domain. Let $u \in W^1(G)$ vanish for $|x| > R \ggg 1$. Then*

$$(2.5.10) \quad \int\limits_G r^{\alpha-4}u^2(x)dx \quad \leq \quad H(\lambda, N, \alpha) \int\limits_G r^{\alpha-2}|\nabla u|^2 dx,$$

$$\text{with } H(\lambda, N, \alpha) \quad := \quad [(4 - N - \alpha)^2/4 + \lambda(\lambda + N - 2)]^{-1}$$

for $\alpha \geq 4 - N$, provided the integral on the right hand side of (2.5.10) is finite.

PROOF. Similarly to the Theorem 2.24, if we apply the Theorem 2.8 with $p = 2$ and α replied by $\alpha + N - 3$. $\qquad\square$

2.5.2. The Robin boundary condition. Let ϑ_0 be the smallest positive eigenvalue of the problem

$$(EVR)_0 \qquad \begin{cases} \triangle_\omega \psi + \vartheta_0 \psi = 0, \ \omega \in \Omega, \\ \frac{\partial \psi}{\partial \vec{\nu}} + \gamma_0 \psi \Big|_{\partial\Omega} = 0, \ \gamma_0 > 0. \end{cases}$$

Let us define the value

$$(2.5.11) \qquad \lambda = \frac{2 - N + \sqrt{(N-2)^2 + 4\vartheta_0}}{2}.$$

THEOREM 2.34. **The Hardy-Friedrichs-Wirtinger inequality.**
Let $u \in C^0(\overline{G}) \cap W^1(G)$ and $\gamma(x) \in C^0(\partial G \setminus \mathcal{O})$, $\gamma(x) \geq \gamma_0 > 0$. Then

$$\int_{G_0^d} r^{\alpha-4}u^2 dx \leq H(\lambda, N, \alpha) \left\{ \int_{G_0^d} r^{\alpha-2}|\nabla u|^2 dx + \int_{\Gamma_0^d} r^{\alpha-3}\gamma(x)u^2(x)ds \right\},$$

(2.5.12)

$$H(\lambda, N, \alpha) = \frac{1}{\lambda(\lambda + N - 2) + \frac{1}{4}(4 - N - \alpha)^2}, \quad \alpha \leq 4 - N$$

provided that integrals on the right are finite.

PROOF. By Theorem 2.20 the inequality (2.4.6) holds. Because of the property of the monotonic increase of the eigenvalues together with the increase of $\gamma(x)$ (see for example Theorem 6 §2, chapter VI [87]), from the inequality $\gamma(x) \geq \gamma_0 > 0$ we obtain $\vartheta \geq \vartheta_0$. The latter means the validity of the inequality

(2.5.13) $$\lambda(\lambda + N - 2)\int_\Omega \psi^2 d\Omega \leq \int_\Omega |\nabla_\omega\psi|^2 d\Omega + \int_{\partial\Omega} \gamma(x)\psi^2 d\sigma,$$

$$\forall\psi \in W^1(\Omega), \ \gamma(x) \in C^0(\partial\Omega), \ \gamma(x) \geq \gamma_0 > 0.$$

Multiplying it by $r^{N-5+\alpha}$ and integrating over $r \in (0, d)$ we obtain

(2.5.14) $$\int_{G_0^d} r^{\alpha-4}u^2 dx \leq \frac{1}{\lambda(\lambda + N - 2)}\int_{G_0^d} r^{\alpha-2}\frac{1}{r^2}|\nabla_\omega u|^2 dx +$$

$$+ \frac{1}{\lambda(\lambda + N - 2)}\int_{\Gamma_0^d} r^{\alpha-3}\gamma(x)u^2(x)ds, \ \forall\alpha \leq 4 - N.$$

Hence (2.5.12) follows for $\alpha = 4 - N$. Now, let $\alpha < 4 - N$. We shall show that $u(0) = 0$. In fact, from the representation $u(0) = u(x) - (u(x) - u(0))$ by the Cauchy inequality we have $\frac{1}{2}|u(0)|^2 \leq |u(x)|^2 + |u(x) - u(0)|^2$. Putting $v(x) = u(x) - u(0)$ we obtain

(2.5.15) $$\frac{1}{2}|u(0)|^2\int_{G_0^d} r^{\alpha-4}dx \leq \int_{G_0^d} r^{\alpha-4}u^2(x)dx + \int_{G_0^d} r^{\alpha-4}|v|^2 dx < \infty.$$

(The first integral from the right is finite by (2.5.14) and the second integral is also finite, in virtue of Corollary 2.3.) Since

$$\int_{G_0^d} r^{\alpha-4}dx = \text{meas}\Omega \int_0^d r^{\alpha+N-5}dr = \infty,$$

by $\alpha + N - 4 < 0$, the assumption $u(0) \neq 0$ contradicts (2.5.15). Thus $u(0) = 0$.

Therefore we can use the Corollary 2.3

$$(2.5.16) \qquad \int\limits_{G_0^d} r^{\alpha-4}u^2 dx \leq \frac{4}{|4 - N - \alpha|^2} \int\limits_{G_0^d} r^{\alpha-2}u_r^2 dx.$$

Adding the inequalities (2.5.14), (2.5.16) and using the formula $|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}|\nabla_\omega u|^2$, we get the desired (2.5.12).   □

LEMMA 2.35. *Let G_0^d be the conical domain, $\nabla u(\varrho, \cdot) \in L_2(\Omega)$ for almost all $\varrho \in (0, d)$ and*

$$V(\rho) = \int\limits_{G_o^\varrho} r^{2-N}|\nabla v|^2 dx + \int\limits_{\Gamma_0^\varrho} r^{1-N}\gamma(x)v^2(x)ds < \infty, \quad \varrho \in (0, d).$$

Then

$$\int\limits_\Omega \left(\varrho v \frac{\partial v}{\partial r} + \frac{N-2}{2}v^2\right)\bigg|_{r=\varrho} d\Omega \leq \frac{\varrho}{2\lambda}V'(\varrho).$$

PROOF. Writing $V(\varrho)$ in spherical coordinates

$$V(\varrho) = \int\limits_0^\varrho r^{2-N}\left(\int\limits_\Omega |\nabla v|^2 d\Omega\right)r^{N-1}dr + \int\limits_0^\varrho r^{1-N}\left(\int\limits_{\partial\Omega} \gamma(x)|v|^2 d\sigma\right)r^{N-2}dr$$

$$= \int\limits_0^\varrho r\left(\int\limits_\Omega |\nabla v|^2 d\Omega\right)dr + \int\limits_0^\varrho \frac{1}{r}\left(\int\limits_{\partial\Omega} \gamma(x)|v|^2 d\sigma\right)dr$$

and differentiating with respect to ϱ we obtain

$$V'(\varrho) = \int\limits_\Omega \left(\varrho\left(\frac{\partial v}{\partial r}\right)^2 + \frac{1}{\varrho}|\nabla_\omega v|^2\right)\bigg|_{r=\varrho} d\Omega + \frac{1}{\varrho}\int\limits_{\partial\Omega} \gamma(\varrho,\omega)v^2(\varrho,\omega)d\sigma.$$

Moreover, by Cauchy's inequality, we have for all $\varepsilon > 0$

$$\rho v \frac{\partial v}{\partial r} \leq \frac{\varepsilon}{2}v^2 + \frac{1}{2\varepsilon}\rho^2\left(\frac{\partial v}{\partial r}\right)^2.$$

Thus choosing $\varepsilon = \lambda$ we obtain, by the Friedrichs-Wirtinger inequality (2.5.13),

$$
\int_{\Omega} \left(\varrho v \frac{\partial v}{\partial r} + \frac{N-2}{2} v^2 \right) \bigg|_{r=\varrho} d\Omega \leq \frac{\varrho^2}{2\varepsilon} \int_{\Omega} \left(\frac{\partial v}{\partial r} \right)^2 d\Omega +
$$

$$
+ \frac{\varepsilon + N - 2}{2} \int_{\Omega} v^2 d\Omega \leq \frac{\varrho^2}{2\varepsilon} \int_{\Omega} \left(\frac{\partial v}{\partial r} \right)^2 d\Omega +
$$

$$
+ \frac{\varepsilon + N - 2}{2\lambda(\lambda + N - 2)} \left\{ \int_{\Omega} |\nabla_\omega v|^2 d\Omega + \int_{\partial \Omega} \gamma(x) v^2(x) d\sigma \right\} = \frac{\varrho}{2\lambda} V'(\varrho).
$$

$\square$

LEMMA 2.36. *Let* $v \in C^0(\overline{G_\varepsilon^d}) \cap W^1(G_\varepsilon^d)$, $v(\varepsilon) = 0$ *and* $\gamma(x) \geq \gamma_0 > 0$. *Then for any* $\varepsilon > 0$

$$
(2.5.17) \quad \int_{G_\varepsilon^d} r^{\alpha-4} v^2 dx \leq
$$

$$
\leq H(\lambda, N, \alpha) \left\{ \int_{G_\varepsilon^d} r^{\alpha-2} |\nabla v|^2 dx + \int_{\Gamma_\varepsilon^d} r^{\alpha-3} \gamma(x) v^2(x) ds \right\},
$$

where $H(\lambda, N, \alpha)$ *is determined by (2.5.12).*

PROOF. By Theorem 2.20 the inequality (2.5.13) holds. Multiplying it by $r^{N-5+\alpha}$ and integrating over $r \in (\varepsilon, d)$ we obtain (2.5.17) for $\alpha = 4 - N$. If $\alpha < 4 - N$ we consider the inequality (2.1.5) and integrate it over Ω; then we have

$$
\frac{1}{4}(4 - N - \alpha)^2 \int_{G_\varepsilon^d} r^{\alpha-4} v^2 dx \leq \int_{G_\varepsilon^d} r^{\alpha-2} v_r^2 dx.
$$

Adding this inequality with above for $\alpha = 4 - N$ (see (2.5.14) for G_ε^d) and using the formula $|\nabla u|^2 = \left(\frac{\partial u}{\partial r} \right)^2 + \frac{1}{r^2} |\nabla_\omega u|^2$, we get the desired result. $\square$

LEMMA 2.37. *Let* $v \in C^0(\overline{G}) \cap W^1(G)$, $v(0) = 0$ *and* $\gamma(x) \geq \gamma_0 > 0$. *Then for any* $\varepsilon > 0$

$$(2.5.18) \quad \int_{G_0^d} r_\varepsilon^{\alpha-4} v^2 dx \leq$$

$$\leq H(\lambda, N, \alpha) \left\{ \int_{G_0^d} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx + \int_{\Gamma_0^d} r_\varepsilon^{\alpha-3} \gamma(x) v^2(x) ds \right\},$$

where $H(\lambda, N, \alpha)$ *is determined by (2.5.12).*

PROOF. We perform the change of variables $y_i = x_i - \varepsilon l_i$, $i = 1, \ldots, N$ and use the inequality (2.5.17); we obtain

$$\int_{G_0^d} r_\varepsilon^{\alpha-4} v^2(x) dx = \int_{G_\varepsilon^{d\varepsilon}} |y|^{\alpha-4} v^2(y + \varepsilon l) dy \leq$$

$$\leq H(\lambda, N, \alpha) \left\{ \int_{G_\varepsilon^{d\varepsilon}} |y|^{\alpha-2} |\nabla_y v(y + \varepsilon l)|^2 dy + \right.$$

$$\left. + \int_{\Gamma_\varepsilon^{d\varepsilon}} |y|^{\alpha-3} \gamma(y + \varepsilon l) v^2(y + \varepsilon l) ds \right\} =$$

$$= H(\lambda, N, \alpha) \left\{ \int_{G_0^d} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx + \int_{\Gamma_0^d} r_\varepsilon^{\alpha-3} \gamma(x) v^2(x) ds \right\}.$$

$\square$

REMARK 2.38. The inequalities (2.5.12), (2.5.17) and (2.5.18) are valid in the case $\gamma(x) \equiv \gamma \left(\frac{x}{|x|} \right) \equiv \gamma(\omega)$ with the same constant $H(\lambda, N, \alpha)$ but

$$(2.5.19) \qquad \lambda = \frac{2 - N + \sqrt{(N-2)^2 + 4\vartheta}}{2},$$

where ϑ *is the smallest positive eigenvalue of the problem* (EVR).

2.6. Other auxiliary integral inequalities for $N = 2$

In the following lemmata we assume that $N = 2$ and we denote by ζ a cut-off function defined in the previous section.

LEMMA 2.39. *Let* $u \in W_0^{2,2}(G)$. *Then*

$$\int\limits_{G_0^\rho} r_\varepsilon^{-\alpha} \zeta^2(r) |\nabla u|^4 dx \le c(\max_{x \in G_0^\rho} |u(x)|)^2 \int\limits_{G_0^\rho} \Big(r_\varepsilon^{-\alpha} \zeta^2 |D^2 u|^2 +$$

$$+ \alpha^2 r_\varepsilon^{-2-\alpha} \zeta^2 |\nabla u|^2 + r_\varepsilon^{-\alpha} (\zeta')^2 |\nabla u|^2 \Big) dx$$

for all $\alpha \ge 0$, $\rho \in (0, d)$ *with a sufficiently small* $d > 0$.

PROOF. Taking into account that $\zeta(r)u(x) = 0$ on ∂G_0^ρ, we obtain by partial integration

$$\int\limits_{G_0^\rho} r_\varepsilon^{-\alpha} \zeta^2(r) |\nabla u|^4 dx = \int\limits_{G_0^\rho} r_\varepsilon^{-\alpha} \zeta^2(r) |\nabla u|^2 \langle \nabla u, \nabla u \rangle dx =$$

$$= -\int\limits_{G_0^\rho} u(x) \sum_{i=1}^N D_i \Big(r_\varepsilon^{-\alpha} \zeta^2(r) |\nabla u|^2 D_i u \Big) dx = -\int\limits_{G_0^\rho} u(x) \Big(r_\varepsilon^{-\alpha} \zeta^2 |\nabla u|^2 \Delta u$$

$$+ 2 r_\varepsilon^{-\alpha} \zeta^2 \sum_{i,j=1}^N D_i u D_j u D_{ij} u - \alpha r_\varepsilon^{-1-\alpha} \zeta^2 |\nabla u|^2 \langle \nabla u, \frac{x - \varepsilon l}{r_\varepsilon} \rangle +$$

$$+ 2 r_\varepsilon^{-\alpha} \zeta \zeta' |\nabla u|^2 \langle \nabla u, \frac{x}{r} \rangle \Big) dx.$$

Therefore

$$\int\limits_{G_0^\rho} r_\varepsilon^{-\alpha} \zeta^2(r) |\nabla u|^4 dx \le \int\limits_{G_0^\rho} |u(x)| \Big(\alpha r_\varepsilon^{-1-\alpha} \zeta^2 |\nabla u|^3 +$$

$$+ 4 r_\varepsilon^{-\alpha} \zeta^2 |\nabla u|^2 |D^2 u| + 2 r_\varepsilon^{-\alpha} \zeta |\zeta'| |\nabla u|^3 \Big) dx.$$

Applying the Cauchy inequality with $\sigma > 0$ we get

$$\int\limits_{G_0^\rho} r_\varepsilon^{-\alpha} \zeta^2(r) |\nabla u|^4 dx \le \sup_{x \in G_0^\rho} |u(x)| \int\limits_{G_0^\rho} \Big(\frac{\sigma}{2} r_\varepsilon^{-\alpha} \zeta^2 |\nabla u|^4 + \frac{\gamma^2}{2\sigma} r_\varepsilon^{-\alpha-2} \zeta^2 |\nabla u|^2 |$$

$$+ 3\sigma r_\varepsilon^{-\alpha} \zeta^2 |\nabla u|^4 + \frac{2}{\sigma} r_\varepsilon^{-\alpha} \zeta^2 |D^2 u|^2 + \frac{1}{\sigma} r_\varepsilon^{-\gamma} |\zeta'|^2 |\nabla u|^2 \Big) dx.$$

Choosing $\sigma = (7 \sup_{x \in G_0^\rho} |u(x)|)^{-1}$ we obtain the assertion. $\square$

LEMMA 2.40. *Let* $u \in W_0^{2,2}(G)$. *Then for all* $\varepsilon > 0, \alpha > 0$

$$
\int_G r^2 r_\varepsilon^{\alpha-2} |\nabla u|^4 dx \leq c(\sup_{x \in G} |u(x)|)^2 \int_{G_d} \left(|\nabla u|^2 + |D^2 u|^2 \right) dx
$$

$$
+ 4(\sup_{x \in G_0^d} |u(x)|)^2 \int_{G_0^d} \left((2 + (\alpha - 2))^2 r_\varepsilon^{\alpha-2} |\nabla u|^2 + r_\varepsilon^{\alpha-2} r^2 |D^2 u|^2 \right) dx
$$

with a constant c *depending only on* α *and* d.

PROOF. Taking into account that v vanishes on ∂G, we obtain by partial integration and Cauchy's inequality

$$
\int_G r^2 r_\varepsilon^{\alpha-2} |\nabla u|^4 dx = \int_G r^2 r_\varepsilon^{\alpha-2} |\nabla u|^2 \sum_{i=1}^N D_i u D_i u \, dx =
$$

$$
= - \int_G u \sum_{i=1}^N D_i \left(r^2 r_\varepsilon^{\alpha-2} |\nabla u|^2 D_i u \right) dx = - \int_G u \left(2 r_\varepsilon^{\alpha-2} |\nabla u|^2 \langle \nabla u, x \rangle + \right.
$$

$$
+ (\alpha - 2) r^2 r_\varepsilon^{\alpha-2} |\nabla u|^2 \langle \nabla u, x - \varepsilon l \rangle + 2 r^2 r_\varepsilon^{\alpha-2} \sum_{i,j=1}^N D_i u D_j u D_{ij} u +
$$

$$
+ r^2 r_\varepsilon^{\alpha-2} |\nabla u|^2 \sum_{i=1}^N D_{ii} u \Bigg) dx \leq \int_G |u| \left(r^2 r_\varepsilon^{\alpha-2} |\nabla u|^2 |D^2 v| + \right.
$$

$$
+ 2 r r_\varepsilon^{\alpha-2} |\nabla u|^3 + |\alpha - 2| r^2 r_\varepsilon^{\alpha-3} |\nabla u|^3 \Bigg) dx \leq
$$

$$
\leq \int_G |u| \left(r^2 r_\varepsilon^{\alpha-2} (\frac{1}{2\delta} |D^2 u|^2 + \frac{\delta}{2} |\nabla u|^4) + \frac{3}{2} \delta r^2 r_\varepsilon^{\alpha-2} |\nabla u|^4 + \right.
$$

$$
+ \frac{1}{\delta} r_\varepsilon^{\alpha-2} |\nabla u|^2 + \frac{(\alpha - 2)^2}{2\delta} r^2 r_\varepsilon^{\alpha-4} |\nabla u|^2 \Bigg) dx \leq 2\delta \sup_{x \in G_0^d} |u| \int_{G_0^d} r^2 r_\varepsilon^{\alpha-2} |\nabla u|^4 dx +
$$

$$
+ \frac{1}{2\delta} \sup_{x \in G_0^d} |u| \int_{G_0^d} \left(r^2 r_\varepsilon^{\alpha-2} |D^2 u|^2 + (2 + (\alpha - 2)^2) r_\varepsilon^{\alpha-2} |\nabla u|^2 \right) dx +
$$

$$
+ 2\delta_1 \sup_{x \in G} |u| \int_{G_d} r^2 r_\varepsilon^{\alpha-2} |\nabla u|^4 dx + \frac{1}{2\delta_1} \sup_{x \in G} |u| c(d, \alpha) \int_{G_d} \left(|\nabla u|^2 + |D^2 u|^2 \right) dx
$$

for all $\delta, \delta_1 > 0$. Setting $\delta = 1/(4 \sup_{x \in G_0^d} |u|)$ and $\delta_1 = 1/(4 \sup_{x \in G} |u|)$, we obtain the assertion. $\qquad \square$

LEMMA 2.41. *Let* $u \in W_0^{2,2}(G)$ *and*

$$(2.6.1) \qquad w = \begin{pmatrix} D_2 u D_{12} u - D_1 u D_{22} u \\ D_1 u D_{12} u - D_2 u D_{11} u \end{pmatrix}.$$

Then there exists a constant $K \geq 0$ *such that*

$$(2.6.2) \qquad \int_{\partial G} r^2 r_\varepsilon^{\alpha-2} \langle w, n \rangle d\sigma \leq K \int_{\partial G \backslash (\Gamma_0^d \cup \{\mathcal{O}\})} r^2 r_\varepsilon^{\alpha-2} \left(\frac{\partial u}{\partial n}\right)^2 d\sigma.$$

PROOF. To evaluate the boundary integral we decompose ∂G into $\partial G = \Gamma_0^d \cup \{\mathcal{O}\} \cup \Gamma$ and take into consideration that v vanishes on ∂G. At first we verify that $\langle w, n \rangle = 0$ on Γ_0^d. We write $\Gamma_0^d = \Gamma_{1,0}^d \cup \Gamma_{2,0}^d$ (see the figure)
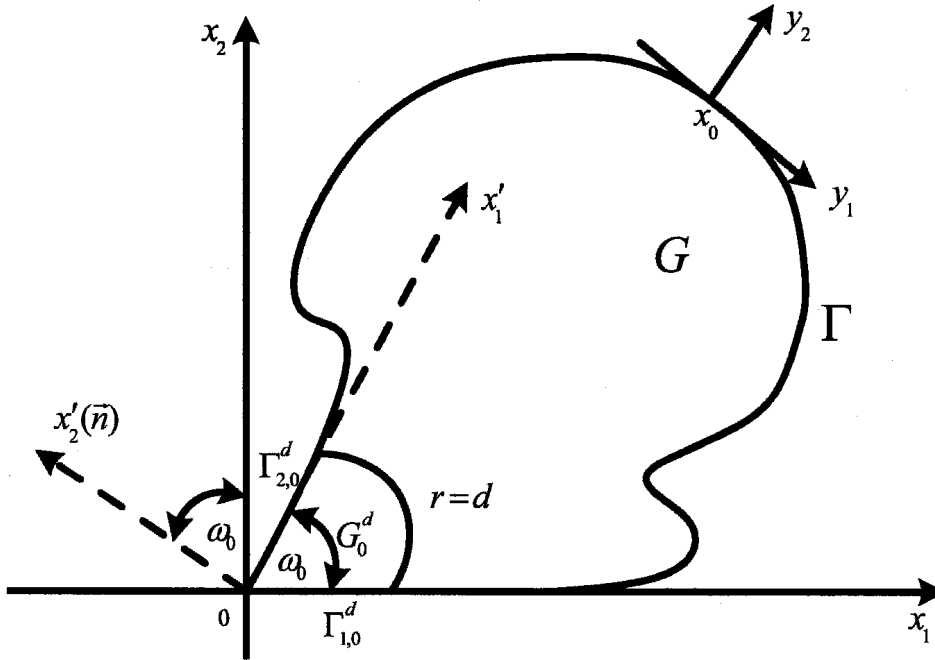
Figure 1

Now we have:

$$\frac{\partial u}{\partial x_1}\Big|_{\Gamma_{1,0}^d} = 0, \quad \frac{\partial^2 u}{\partial x_1^2}\Big|_{\Gamma_{1,0}^d} = 0, \quad n_1\Big|_{\Gamma_{1,0}^d} = 0, \quad n_2\Big|_{\Gamma_{1,0}^d} = -1 \Rightarrow \langle w, n \rangle = 0 \text{ on } \Gamma_{1,0}^d.$$

Further

$$n_1\Big|_{\Gamma_{2,0}^d} = \cos\left(\frac{\pi}{2} + \omega_0\right) = -\sin\omega_0 \text{ and } n_2\Big|_{\Gamma_{2,0}^d} = \cos\omega_0.$$

Let us perform the rotation of axes about the origin $\mathcal{O}$, through an angle ω_0 :

$$\begin{cases} x_1' = x_1 \cos \omega_0 + x_2 \sin \omega_0, \\ x_2' = -x_1 \sin \omega_0 + x_2 \cos \omega_0. \end{cases}$$

Then

$$\begin{cases} \frac{\partial}{\partial x_1} = \cos \omega_0 \frac{\partial}{\partial x_1'} - \sin \omega_0 \frac{\partial}{\partial x_2'}, \\ \frac{\partial}{\partial x_2} = \sin \omega_0 \frac{\partial}{\partial x_1'} + \cos \omega_0 \frac{\partial}{\partial x_2'}. \end{cases}$$

Therefore

$$u_{x_1 x_1} = \cos^2 \omega_0 u_{x_1' x_1'} - \sin(2\omega_0) u_{x_1' x_2'} + \sin^2 \omega_0 u_{x_2' x_2'};$$

$$u_{x_1 x_2} = \frac{1}{2} \sin(2\omega_0) u_{x_1' x_1'} + \cos(2\omega_0) u_{x_1' x_2'} - \frac{1}{2} \sin(2\omega_0) u_{x_2' x_2'};$$

$$u_{x_2 x_2} = \sin^2 \omega_0 u_{x_1' x_1'} + \sin(2\omega_0) u_{x_1' x_2'} + \cos^2 \omega_0 u_{x_2' x_2'};$$

Since

$$\frac{\partial u}{\partial x_1'} \bigg|_{\Gamma_{2,0}^d} = 0, \quad \frac{\partial^2 u}{\partial x_1'^2} \bigg|_{\Gamma_{2,0}^d} = 0,$$

then we obtain

$$\langle w, n \rangle \bigg|_{\Gamma_{2,0}^d} = -\sin \omega_0 \left\{ \cos \omega_0 u_{x_2'} \left(\cos(2\omega_0) u_{x_1' x_2'} - \frac{1}{2} \sin(2\omega_0) u_{x_2' x_2'} \right) + \right.$$

$$+ \sin \omega_0 u_{x_2'} \left(\sin(2\omega_0) u_{x_1' x_2'} + \cos^2 \omega_0 u_{x_2' x_2'} \right) \left. \right\} +$$

$$+ \cos \omega_0 \left\{ -\sin \omega_0 u_{x_2'} \left(\cos(2\omega_0) u_{x_1' x_2'} - \frac{1}{2} \sin(2\omega_0) u_{x_2' x_2'} \right) - \right.$$

$$- \cos \omega_0 u_{x_2'} \left(-\sin(2\omega_0) u_{x_1' x_2'} + \sin^2 \omega_0 u_{x_2' x_2'} \right) \left. \right\} \equiv$$

$$\equiv 0.$$

Now we calculate $\langle w, n \rangle \big|_{\Gamma}$. We suppose that Γ is a smooth curve. The last means that there is a coordinate system (y_1, y_2) centered at $x_0 \in \Gamma$ such that the positive y_2-axis is parallel to the outward normal $\vec{n}$ to Γ at x_0 and the equation of the portion of Γ has the form

$$y_2 = \psi(y_1), \quad \text{where } \psi''(y_1) \leq K, \ K \geq 0$$

and the number K can be choosen independent of x_0. Let us perform the transformation of coordinates

$$y_i = c_{ik} \left(x_k - x_k^0 \right), \quad i = 1, 2,$$

where (c_{ik}) is the orthogonal matrix. In particular, we have

$$n_1 \bigg|_{\Gamma} = c_{21}, \quad n_2 \bigg|_{\Gamma} = c_{22},$$

$$\begin{cases} \dfrac{\partial}{\partial x_1} = c_{11}\dfrac{\partial}{\partial y_1} + c_{21}\dfrac{\partial}{\partial y_2}, \\ \dfrac{\partial}{\partial x_2} = c_{12}\dfrac{\partial}{\partial y_1} + c_{22}\dfrac{\partial}{\partial y_2}. \end{cases}$$

Hence it follows

$$u_{x_1 x_1} = c_{11}^2 u_{y_1 y_1} + 2c_{11}c_{21} u_{y_1 y_2} + c_{21}^2 u_{y_2 y_2};$$

$$u_{x_1 x_2} = c_{11}c_{12} u_{y_1 y_1} + (c_{21}c_{12} + c_{11}c_{22}) u_{y_1 y_2} + c_{21}c_{22} u_{y_2 y_2};$$

$$u_{x_2 x_2} = c_{12}^2 u_{y_1 y_1} + 2c_{12}c_{22} u_{y_1 y_2} + c_{22}^2 u_{y_2 y_2}$$

Because of $u\big|_\Gamma = 0$, we have $u\,(y_1, \psi(y_1)) = 0$ near x_0. If we differentiate this equality, then we get

$$\begin{cases} u_{y_1} + u_{y_2}\psi'(y_1) = 0, \\ u_{y_1 y_1} + 2u_{y_1 y_2}\psi'(y_1) + u_{y_2 y_2}\psi'^2(y_1) + u_{y_2}\psi''(y_1) = 0. \end{cases}$$

But $\psi'(y_1)\big|_{x_0} = 0$, therefore

$$u_{y_1}\Big|_{x_0} = 0, \quad u_{y_1 y_1}\Big|_{x_0} = -\frac{\partial u}{\partial n}\Big|_{x_0} \cdot \psi''(y_1).$$

In addition,

$$u_{x_1}(x_0) = c_{21}\frac{\partial u}{\partial n}\Big|_{x_0}, \quad u_{x_2}(x_0) = c_{22}\frac{\partial u}{\partial n}\Big|_{x_0}$$

Now we can obtain

$$\langle w, n\rangle\Big|_{x_0 \in \Gamma} = c_{21}\frac{\partial u}{\partial n}\left\{ c_{22}\left(-c_{11}c_{12}\frac{\partial u}{\partial n}\psi''(y_1) + (c_{11}c_{22} + c_{21}c_{12})\, u_{y_1 y_2} + \right. \right.$$

$$\left. + c_{21}c_{22} u_{y_2 y_2}\right) -$$

$$- c_{21}\left(-c_{12}^2\frac{\partial u}{\partial n}\psi''(y_1) + 2c_{12}c_{22} u_{y_1 y_2} + c_{22}^2 u_{y_2 y_2}\right)\bigg\} +$$

$$+ c_{22}\frac{\partial u}{\partial n}\left\{ c_{21}\left(-c_{11}c_{12}\frac{\partial u}{\partial n}\psi''(y_1) + (c_{11}c_{22} + c_{21}c_{12})\, u_{y_1 y_2} + \right. \right.$$

$$\left. + c_{21}c_{22} u_{y_2 y_2}\right) -$$

$$- c_{22}\left(-c_{11}^2\frac{\partial u}{\partial n}\psi''(y_1) + 2c_{11}c_{21} u_{y_1 y_2} + c_{21}^2 u_{y_2 y_2}\right)\bigg\} =$$

$$= \psi''(y_1)\left(\frac{\partial u}{\partial n}\right)^2$$

in virtue of $det\,(c_{ik}) = 1$. Thus from above calculations we get the desired (2.6.2). $\square$

LEMMA 2.42. *Let* $u \in W_0^{2,2}(G)$. *Then for all* $\gamma, \varepsilon > 0$ *and all* $\alpha \in \mathbb{R}$

(2.6.3) $$J_{\alpha\varepsilon}[u] \equiv \int\limits_{G} r^2 r_\varepsilon^{\alpha-2}((D_{12}u)^2 - D_{11}u D_{22}u)dx \leq$$

$$\leq \gamma \int\limits_{G_0^d} r^2 r_\varepsilon^{\alpha-2}|D^2 u|^2 dx + c_1(\alpha, \gamma, h) \int\limits_{G_0^d} r_\varepsilon^{\alpha-2}|\nabla u|^2 dx +$$

$$+ c_2 \int\limits_{G_d} \left(|\nabla u|^2 + |D^2 u|^2\right) dx$$

with a constant c_2 *depending only on* $\alpha, \gamma, d, \mathrm{diam}\,G$ *and* $\mathrm{meas}\,G$.

PROOF. Since G is a strictly Lipschitz domain and the set of all $C_0^\infty(G)$ functions is dense in $W_0^{2,2}(G)$, it suffices to prove (2.6.3) for smooth functions.
In order to estimate $J_{\alpha\varepsilon}[u]$ we integrate it by parts, once with respect to x_1 and once with respect to x_2 and add the resulting equations. As a result we obtain

(2.6.4) $$2J_{\alpha\varepsilon}[u] = \int\limits_{\partial G} r^2 r_\varepsilon^{\alpha-2} \langle w, n \rangle d\sigma -$$

$$- \int\limits_{G} r^2 r_\varepsilon^{\alpha-4} \langle 2x + (\alpha-2)(x - \varepsilon l), w \rangle dx$$

with w defined by (2.6.1). The boundary integral in (2.6.4) we evaluate by Lemma 2.41

$$\int\limits_{\partial G} r^2 r_\varepsilon^{\alpha-2} \langle w, n \rangle d\sigma \leq K \int\limits_{\Gamma} r^2 r_\varepsilon^{\alpha-2} \left(\frac{\partial u}{\partial n}\right)^2 d\sigma.$$

By properties of the function r_ε and Theorem 1.29, we obtain

(2.6.5) $$\int\limits_{\partial G} r^2 r_\varepsilon^{\alpha-2} \langle w, n \rangle d\sigma \leq cK \int\limits_{G_d} (|D^2 u|^2 + |\nabla u|^2) dx \quad \forall \varepsilon > 0.$$

The domain integral in (2.6.4) is estimated using (1.2.5) and the Cauchy inequality with $\forall \gamma > 0$:

$$(2.6.6) \quad (2-\alpha) \int_G r_\varepsilon^{\alpha-3} r^2 \langle \frac{x-\varepsilon l}{r_\varepsilon}, w \rangle dx + 2 \int_G r_\varepsilon^{\alpha-2} |\langle x, w \rangle| dx$$

$$\leq \gamma \int_G r^2 r_\varepsilon^{\alpha-2} |D^2 u|^2 dx + c_2(\alpha, \gamma, h) \int_G r_\varepsilon^{\alpha-2} |\nabla u|^2 dx.$$

The desired assertion then follows from (2.6.5) and (2.6.6). $\qquad \square$

2.7. Notes

The classical Hardy inequality was first proved by G. Hardy [**142**]. The various extensions of this inequality as well the proof of Theorem 2.8 can be found in [**362, 108**]. For other versions of the Poincaré inequality, see §2.22 [**108**]. The one-dimensional Wirtinger inequality is given and proved in Chapter VII [**142**]. The variational principle for the Dirichlet boundary condition is given more detail in §4.1 [**108**]. The material in §§2.4.2, 2.5.2 is new. Subsection 2.6 is based on the ideas of [**215**] (see there Lemma 4.5, Chapter II and §8, Chapter III).

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CHAPTER 3

The Laplace operator

3.1. Dini estimates of the generalized Newtonian potential

We shall consider the Dirichlet problem for the Poisson equation

$$
\begin{cases}
\triangle v = g + \sum_{j=1}^{n} D_j f^j, & x \in G, \\
\\
v(x) = 0, x \in \partial G.
\end{cases}
\tag{PE}
$$

Let $\Gamma(x - y)$ be the normalized fundamental solution of Laplace's equation. The following estimates are known (see e.g. (2.12), (2.14)[129])

$$
|\Gamma(x - y)| = \frac{1}{N(N-2)\omega_N}|x - y|^{2-N}, \quad N \geq 3,
$$

$$
|D_i\Gamma(x - y)| \leq \frac{1}{N\omega_N}|x - y|^{1-N},
$$

(3.1.1)

$$
|D_{ij}\Gamma(x - y)| \leq \frac{1}{\omega_N}|x - y|^{-N},
$$

$$
|D^\beta\Gamma(x - y)| \leq C(N,\beta)|x - y|^{2-N-\beta}.
$$

We define the functions

$$
(3.1.2) \quad z(x) = \int_G \Gamma(x - y)g(y)dy \text{ and } w(x) = D_j \int_G \Gamma(x - y)f^j(y)dy,
$$

under the assumption that the functions $g(x)$ and $f^j(x)$, $j = 1, \ldots, N$ are integrable on G. The function $z(x)$ is called the *Newtonian potential* with density function $g(x)$, and $w(x)$ is called the *generalized newtonian potential* with density function div$f$. We now give estimates for these potentials. In the following the D operator is always taken with respect to the x variable.

LEMMA 3.1. *Let* $\partial G \in C^{1,\mathcal{A}}, g \in L^p(G), p > N$ *and* $f^j \in C^{0,\mathcal{A}}(G), j = 1, \ldots, N,$ *where* $\mathcal{A}$ *is an* α-*Dini.*

Then $z \in C^1(\mathbb{R}^N), w \in C^2(G)$ and for any $x \in G$

(3.1.3)
$$D_i z(x) = \int\limits_G D_i \Gamma(x-y) g(y) dy,$$

(3.1.4)
$$D_i w(x) = \int\limits_{G_0} D_{ij} \Gamma(x-y) \big(f^j(y) - f^j(x)\big) dy -$$

$$- f^j(x) \int\limits_{\partial G_0} D_i \Gamma(x-y) \nu_j(y) d_y \sigma$$

(i=1,..., N); here G_0 is any domain containing G for which the Gauss divergence theorem holds and f^j are extended to vanish outside G.

PROOF. By virtue of the estimate (3.1.1) for $D_i \Gamma$, the functions

$$v_i(x) = \int\limits_G D_i \Gamma(x-y) g(y) dy, \ i = 1, \ldots, N$$

are well defined. To show that $v_i = D_i z$, we fix a function $\zeta \in C^1(\mathbb{R})$ satisfying

$$0 \le \zeta \le 1, \ 0 \le \zeta' \le 2, \ \zeta(t) = 0 \text{ for } t \le 1, \ \zeta(t) = 1 \text{ for } t \ge 2$$

and define for $\varepsilon > 0$

$$z_\varepsilon(x) = \int\limits_G \Gamma(x-y) \zeta\left(\frac{|x-y|}{\varepsilon}\right) g(y) dy.$$

Clearly, $z_\varepsilon(x) \in C^1(\mathbb{R}^N)$ and

$$v_i(x) - D_i z_\varepsilon(x) = \int\limits_{|x-y| \le 2\varepsilon} D_i \left\{ \left(1 - \zeta\left(\frac{|x-y|}{\varepsilon}\right)\right) \Gamma(x-y) \right\} g(y) dy$$

so that

$$|v_i(x) - D_i z_\varepsilon(x)| \le \sup |g| \int\limits_{|x-y| \le 2\varepsilon} \left(|D_i \Gamma| + \frac{2}{\varepsilon}|\Gamma|\right) dy \le$$

$$\le \sup |g| \cdot \begin{cases} \frac{2N\varepsilon}{N-2} & \text{for } N > 2, \\ 4\varepsilon(1 + |\lg 2\varepsilon|) & \text{for } N = 2. \end{cases}$$

Consequently, z_ε and $D_i z_\varepsilon$ converge uniformly in compact subsets of $\mathbb{R}^N$ to z and v_i respectively as $\varepsilon \to 0$. Hence, $z \in C^1(\mathbb{R}^N)$ and $D_i z = v_i$.

By virtue of the estimate (3.1.1) for $D_{ij}\Gamma$ and the Dini continuous of f^j, the functions

$$u_i(x) = \int\limits_{G_0} D_{ij}\Gamma(x-y)\big(f^j(y) - f^j(x)\big)dy-$$

$$- f^j(x) \int\limits_{\partial G_0} D_i\Gamma(x-y)\nu_j(y)d_y\sigma, \quad i = 1, \ldots, N$$

are well defined. Let us define for $\varepsilon > 0$

$$v_\varepsilon(x) = \int\limits_{G} D_i\Gamma(x-y)\zeta\Big(\frac{|x-y|}{\varepsilon}\Big)f^j(y)dy.$$

Clearly, $v_\varepsilon(x) \in C^1(G)$ and differentiating, we obtain

$$\sum_{j=1}^{n} D_j v_\varepsilon(x) = \int\limits_{G} D_j\Big(D_i\Gamma(x-y)\zeta\Big(\frac{|x-y|}{\varepsilon}\Big)\Big)f^j(y)dy =$$

$$= f^j(x) \int\limits_{G_0} D_j\Big(D_i\Gamma(x-y)\zeta\Big(\frac{|x-y|}{\varepsilon}\Big)\Big)dy+$$

$$+ \int\limits_{G} D_j\Big(D_i\Gamma(x-y)\zeta\Big(\frac{|x-y|}{\varepsilon}\Big)\Big)\Big(f^j(y) - f^j(x)\Big)dy =$$

$$= \int\limits_{G_0} D_j\Big(D_i\Gamma(x-y)\zeta\Big(\frac{|x-y|}{\varepsilon}\Big)\Big)\Big(f^j(y) - f^j(x)\Big)dy-$$

$$- f^j(x) \int\limits_{\partial G_0} D_i\Gamma(x-y)\nu_j(y)d_y\sigma$$

provided ε is sufficiently small. Hence, by subtraction

$$|u_i(x) - \sum_{j=1}^{n} D_j v_\varepsilon(x)| =$$

$$= \Bigg|\int\limits_{|x-y|\leq 2\varepsilon} D_j\Big\{\Big(1 - \zeta\Big(\frac{|x-y|}{\varepsilon}\Big)\Big)D_i\Gamma(x-y)\Big\}(f^j(y) - f^j(x))dy\Bigg| \leq$$

$$\leq [f^j]_{\mathcal{A};x} \cdot \int\limits_{|x-y|\leq 2\varepsilon} \Big(|D_{ij}\Gamma| + \frac{2}{\varepsilon}|D_i\Gamma|\Big)\mathcal{A}(|x-y|)dy \leq$$

$$\leq C(N,G) \int_0^{2\varepsilon} \frac{A(t)}{t}\,dt \cdot \sum_{j=1}^{N} [f^j]_{\mathcal{A};x}$$

provided $2\varepsilon < dist(x, \partial G)$. Consequently $\sum_{j=1}^{n} D_j v_\varepsilon(x)$ converges to u_i uniformly on compact subsets of G as $\varepsilon \to 0$, and since v_ε converges uniformly to $v_i = D_i z$ in G, we obtain $w \in C^2(G)$ and $u_i = D_i w$. This completes the proof of Lemma 3.1. $\qquad\square$

Let $B_1 = B_R(x_0), B_2 = B_{2R}(x_0)$ be concentric balls in $\mathbb{R}^N$ and $z(x)$, $w(x)$ be Newtonian potentials in B_2.

LEMMA 3.2. *Suppose* $g \in L^p(B_2), p > N/2$, *and* $f^j \in L^\infty(B_2)$, $j = 1, \ldots, N$. *Then*

$$|z|_{0;B_1} \leq c(p) R^{2/p'} \ln^{1/p'}\left(\frac{1}{2R}\right) \|g\|_{p;B_2}, \quad N = 2;$$

(3.1.5)

$$|z|_{0;B_1} \leq c(p, N) R^{2-N+N/p'} \|g\|_{p;B_2}, \quad N \geq 3;$$

(3.1.6)
$$|w|_{0;B_1} \leq 2R \sum_{j=1}^{N} |f^j|_{0;B_2}.$$

PROOF. The estimates follow from inequalities (3.1.1), Hölder's inequality for integrals and Lemma 3.1. $\qquad\square$

LEMMA 3.3. *Let* $g \in L^p(B_2)$, $p > N$, $f^j \in C^{0,\mathcal{A}}(\overline{B}_2)$, $j = 1, \ldots, N$, *where* $\mathcal{A}$ *is an* $\alpha-$ *function Dini continuous at zero. Then* $z, w \in C^{1,\mathcal{B}}(\overline{B}_1)$ *and*

(3.1.7) $$\|z\|_{1,\mathcal{B};B_1} \leq c(p, N, R, \mathcal{A}^{-1}(2R)) \|g\|_{p;B_2},$$

(3.1.8) $$\|w\|_{1,\mathcal{B};B_1} \leq c(p, N, R, \alpha, \mathcal{A}^{-1}(2R), \mathcal{B}(2R)) \sum_{j=1}^{N} \|f^j\|_{0,\mathcal{A};B_2}.$$

PROOF. Let $x, \overline{x} \in B_1$ and $G = B_2$. By formulas (3.1.3), (3.1.4), taking into account (3.1.1) and Hölder's inequality for integrals and setting $|x - y| = t, y - x = t\omega, dy = t^{N-1}dt d\Omega$, we have

$$|D_i z| \leq (N\omega_N)^{-1} \int_{B_2} |x - y|^{1-N} |g(y)| dy \leq$$

(3.1.9) $$\leq (N\omega_N)^{-1} \|g\|_{p;B_2} \left\{ \int_{B_2} |x - y|^{(1-N)p'} dy \right\}^{1/p'} =$$

$$= \frac{p-1}{p-N} (2R)^{(p-N)/(p-1)} \|g\|_{p;B_2};$$

$$|D_i w(x)| \le (N\omega_N)^{-1} R^{1-N} \sum_{j=1}^{N} |f^j(x)| \int_{\partial B_2} d_y \sigma +$$

$$(3.1.10) \qquad + \omega_N^{-1} \sum_{j=1}^{N} [f^j]_{\mathcal{A},x} \cdot \int_{B_2} \frac{\mathcal{A}(x-y)}{|x-y|^N} dy \le 2^{N-1} \sum_{j=1}^{N} |f^j(x)| +$$

$$+ N \sum_{j=1}^{N} [f^j]_{\mathcal{A},x} \cdot \int_0^{2R} \frac{\mathcal{A}(t)}{t} dt \le c(N)\mathcal{B}(2R) \sum_{j=1}^{N} \left(|f^j(x)| + \sum_{j=1}^{N} [f^j]_{\mathcal{A},x} \right).$$

Taking into account (3.1.3) we obtain by subtraction

$$|D_i z(x) - D_i z(\overline{x})| \le \int_{B_2} |D_i \Gamma(x-y) - D_i \Gamma(\overline{x}-y)| \cdot |g(y)| dy.$$

We set $\delta = |x - \overline{x}|, \xi = \frac{1}{2}(x - \overline{x})$ and represent $B_2 = B_\delta(\xi) \cup \{B_2 \setminus B_\delta(\xi)\}$. Then

$$\int_{B_\delta(\xi)} |D_i \Gamma(x-y) - D_i \Gamma(\overline{x}-y)| \cdot |g(y)| dy \le$$

$$\le \int_{B_\delta(\xi)} |D_i \Gamma(x-y)| \cdot |g(y)| dy + \int_{B_\delta(\xi)} |D_i \Gamma(\overline{x}-y)| \cdot |g(y)| dy \le$$

$$\le (N\omega_N)^{-1} \left\{ \int_{B_\delta(\xi)} |x-y|^{1-N} |g(y)| dy + \int_{B_\delta(\xi)} |\overline{x}-y|^{1-N} |g(y)| dy \right\} \le$$

$$(3.1.11) \qquad \le 2(N\omega_N)^{-1} \int_{B_{3\delta/2}(x)} |x-y|^{1-N} |g(y)| dy \le$$

$$\le 2(N\omega_N)^{-1} \|g\|_{p;B_2} \left(\int_{B_{3\delta/2}(x)} |x-y|^{(1-N)p'} dy \right)^{1/p'} \le$$

$$\le 2(N\omega_N)^{-1/p} \|g\|_{p;B_2} \left(\frac{3\delta}{2} \right)^{1-N/p} \{N + (1-N)p'\}^{-1/p'} \le$$

$$\le \frac{2(N\omega_N)^{-1/p}(2R)^{1-N/p}}{\{N + (1-N)p'\}^{-1/p'}} \cdot \frac{\mathcal{A}(|x-\overline{x}|)}{\mathcal{A}(2R)} \|g\|_{p;B_2}.$$

(Here we take into account that $\delta^\alpha \le (2R)^\alpha \mathcal{A}(\delta)/\mathcal{A}(2R)$ for all $\alpha > 0$ by (1.8.1), since $\delta \le 2R$.) Similarly,

$$\int\limits_{B_2 \setminus B_\delta(\xi)} |D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y)| \cdot |g(y)| dy \le$$

$$\le |x - \overline{x}| \int\limits_{B_2 \setminus B_\delta(\xi)} |DD_i\Gamma(\widetilde{x}-y)| \cdot |g(y)| dy$$

(for some $\widetilde{x}$ between x and $\overline{x}$)

(3.1.12)

$$\le \delta\omega_N^{-1} \int\limits_{|y-\xi|\ge\delta} |\widetilde{x}-y|^{-N}|g(y)| dy \le 2^N\delta\omega_N^{-1} \int\limits_{|y-\xi|\ge\delta} |\xi-y|^{-N}|g(y)| dy$$

(since $|y - \xi| \le 2|y - \widetilde{x}|$)

(3.1.13)

$$\le 2^N\delta\omega_N^{-1}\|g\|_{p;B_2} \Big(\int\limits_{|y-\xi|\ge\delta} |\xi-y|^{-np'} dy \Big)^{1/p'} \le$$

$$\le 2^N\delta^{1-N/p}\omega_N^{-1/p}(p-1)^{1/p'}\|g\|_{p;B_2}$$

$$\le 2^N(2R)^{1-N/p}\omega_N^{-1/p}(p-1)^{1/p'} \cdot \frac{\mathcal{A}(|x-\overline{x}|)}{\mathcal{A}(2R)}\|g\|_{p;B_2}.$$

From (3.1.11) and (3.1.12), taking into account (1.8.3), we obtain

(3.1.14) $|D_iz(x) - D_iz(\overline{x})| \le c(N,p,R)\mathcal{A}^{-1}(2R)\|g\|_{p;B_2}\mathcal{A}(|x-\overline{x}|)$

$$\le c(N,p,R)\mathcal{A}^{-1}(2R)\|g\|_{p;B_2}\mathcal{B}(|x-\overline{x}|), \quad \forall x,\overline{x} \in B_1.$$

The first from the required estimates (3.1.7) follows from the inequalities (3.1.5) and (3.1.14).

Now we derive the estimate (3.1.8). By (3.1.4) for $x,\overline{x} \in B_1$ we have

$$D_iw(\overline{x}) - D_iw(x) = \sum_{j=1}^{N}\Big(f^j(x)\mathcal{J}_{1j} + \big(f^j(x) - f^j(\overline{x})\big)\mathcal{J}_{2j} \Big)$$

(3.1.15) $+\mathcal{J}_3 + \mathcal{J}_4 + \sum_{j=1}^{N}\big(f^j(x) - f^j(\overline{x})\big)\mathcal{J}_{5j} + \mathcal{J}_6,$

where

$$\mathcal{J}_{1j} = \int\limits_{\partial B_2} \big(D_i\Gamma(x-y) - D_i\Gamma(\overline{x}-y)\big)\nu_j(y)d_y\sigma,$$

$$\mathcal{J}_{2j} = \int\limits_{\partial B_2} D_i\Gamma(\overline{x}-y)\nu_j(y)d_y\sigma,$$

$$\mathcal{J}_3 = \int\limits_{B_\delta(\xi)} D_{ij}\Gamma(x-y)\big(f^j(x)-f^j(y)\big)dy,$$

$$\mathcal{J}_4 = \int\limits_{B_\delta(\xi)} D_{ij}\Gamma(\overline{x}-y)\big(f^j(y)-f^j(\overline{x})\big)dy,$$

$$\mathcal{J}_{5j} = \int\limits_{B_2\backslash B_\delta(\xi)} D_{ij}\Gamma(x-y)dy,$$

$$\mathcal{J}_6 = \int\limits_{B_2\backslash B_\delta(\xi)} \big(D_{ij}\Gamma(x-y)-D_{ij}\Gamma(\overline{x}-y)\big)\big(f^j(\overline{x})-f^j(y)\big)dy$$

We estimate these integrals thusly

$$|\mathcal{J}_{1j}| \le |x-\overline{x}| \int\limits_{\partial B_2} |DD_i\Gamma(\widetilde{x}-y)|d_y\sigma$$

(for some point $\widetilde{x}$ between x and $\overline{x}$)

$$\le |x-\overline{x}|N\omega_N^{-1} \int\limits_{\partial B_2} |\widetilde{x}-y)|^{-N}d_y\sigma \le N^2 2^{N-1}|x-\overline{x}|R^{-1}$$

(since $|\widetilde{x}-y)| \ge R$ for $y \in \partial B_2$)

$$\le N^2 2^{N-1}\mathcal{A}(|x-\overline{x}|)R^{-1}\delta/\mathcal{A}(\delta) \le N^2 2^N \mathcal{A}(|x-\overline{x}|)/\mathcal{A}(2R)$$

(since $\delta = |x-\overline{x}| \le 2R$ and $\delta/\mathcal{A}(\delta) \le 2R/\mathcal{A}(2R)$ by (1.8.1))

$$\le N^2 2^N \alpha\mathcal{B}(\delta)/\mathcal{A}(2R) \quad \text{(by (1.8.3))}.$$

Next,

$$|\mathcal{J}_{2j}| \le 2^{N-1},$$

$$|\mathcal{J}_3| \le \omega_N^{-1}[f^j]_{\mathcal{A},x} \int\limits_{B_\delta(\xi)} |x-y)|^{-N}\mathcal{A}(|x-y|)dy$$

$$\le \omega_N^{-1}[f^j]_{\mathcal{A},x} \int\limits_{B_{3\delta/2}(x)} |x-y)|^{-N}\mathcal{A}(|x-y|)dy$$

$$= N[f^j]_{\mathcal{A},x} \int\limits_0^{3\delta/2} \frac{\mathcal{A}(t)}{t}dt \le N\Big(\frac{3}{2}\Big)^\alpha [f^j]_{\mathcal{A},x}\mathcal{B}(\delta) \quad \text{(by (1.8.2))}.$$

By analogy with the estimate for $\mathcal{J}_3$ we obtain

$$|\mathcal{J}_4| \le N\Big(\frac{3}{2}\Big)^\alpha [f^j]_{\mathcal{A},\overline{x}}\mathcal{B}(\delta).$$

By (3.1.1) it is obvious

$$|\mathcal{J}_{5j}| \le 2^N.$$

At last

$$|\mathcal{J}_6| \le |x - \overline{x}| \int\limits_{B_2 \backslash B_\delta(\xi)} |DD_{ij}\Gamma(\widetilde{x} - y)| \cdot |f^j(\overline{x}) - f^j(y)| dy$$

(for some point $\widetilde{x}$ between x and $\overline{x}$)

$$\le |x - \overline{x}| c(N) \int\limits_{|y - \xi| \ge \delta} |\widetilde{x} - y)|^{-N-1} \cdot |f^j(\overline{x}) - f^j(y)| dy$$

$$\le c(N)\delta[f^j]_{\mathcal{A},\overline{x}} \int\limits_{|y - \xi| \ge \delta} |\widetilde{x} - y)|^{-N-1}\mathcal{A}(|\overline{x} - y|) dy$$

$$\le c(N)\delta[f^j]_{\mathcal{A},\overline{x}} \int\limits_{|y - \xi| \ge \delta} |\xi - y)|^{-N-1}\mathcal{A}\left(\frac{3}{2}|\xi - y|\right) dy$$

(since $|\overline{x} - y| \le \dfrac{3}{2}|\xi - y| \le 3|\widetilde{x} - y)|$)

$$\le c(N)\omega_N\delta\left(\frac{3}{2}\right)^\alpha [f^j]_{\mathcal{A},\overline{x}} \int\limits_\delta^R t^{-2}\mathcal{A}(t) dt$$

(since $\mathcal{A}\left(\dfrac{3}{2}t\right) \le \left(\dfrac{3}{2}\right)^\alpha \mathcal{A}(t)$ by (1.8.2))

$$\le c(N)\omega_N \frac{\alpha}{1 - \alpha}\left(\frac{3}{2}\right)^\alpha [f^j]_{\mathcal{A},\overline{x}}\mathcal{B}(\delta) \qquad \text{by (1.8.4)}.$$

Now from (3.1.15) and the above estimates we obtain

$$|D_i w(\overline{x}) - D_i w(x)| \le c(N, \alpha) \sum_{j=1}^N \Big(|f^j(x)|\mathcal{A}^{-1}(2R) +$$

(3.1.16) $$+ [f^j]_{\mathcal{A},x} + [f^j]_{\mathcal{A},\overline{x}}\Big)\mathcal{B}(|x - \overline{x}|) \qquad \forall x, \overline{x} \in B_1.$$

Finally, from (3.1.10) and (3.1.16) it follows that $w(x) \in C^{1,\mathcal{B}}(B_1)$ and the estimate (3.1.8) holds. Lemma 3.3 is proved. $\square$

Now we can assert a $C^{1,\mathcal{B}}$ interior estimate.

LEMMA 3.4. *Let G be a domain in $\mathbb{R}^N$, and let $v(x) \in C^{1,\mathcal{B}}(G)$ be a generalized solution of Poisson's equation (PE) with $g \in L^{\frac{N}{1-\alpha}}(G)$,
$f^j \in C^{0,\mathcal{A}}(\overline{G})$, where $\mathcal{A}$ is an α-Dini function. Then for any two concentric*

balls $B_1 = B_R(x_0), B_2 = B_{2R}(x_0) \subset\subset G$ *we have*

$$(3.1.17) \qquad \|v\|_{1,\mathcal{B};B_1} \leq c\Big(|v|_{0;B_2} + \|g\|_{\frac{N}{1-\alpha};B_2} + \sum_{j=1}^N \|f^j\|_{0,\mathcal{A};B_2}\Big),$$

where $c = c\big(N, R, \alpha, \mathcal{A}^{-1}(2R), \mathcal{B}(2R)\big).$

PROOF. It is easily shown that the Newtonian potential, given by

$$V(x) = \int_G \Gamma(x-y)g(y)dy + \int_G D_j\Gamma(x-y)f^j(y)dy$$

is a weak solution of the equation from (PE). We can write

$$(3.1.18) \qquad v(x) = V(x) + \widetilde{v}(x), \quad x \in B_2,$$

where $\widetilde{v}(x)$ is harmonic in B_2. By Lemma 3.3, we have

$$(3.1.19) \qquad \|V\|_{1,\mathcal{B};B_1} \leq c\Big(\|g\|_{\frac{N}{1-\alpha};B_2} + \sum_{j=1}^N \|f^j\|_{0,\mathcal{A};B_2}\Big),$$

where $c = c\big(N, R, \alpha, \mathcal{A}^{-1}(2R), \mathcal{B}(2R)\big)$. By Theorem 2.10 [129] we obtain

$$(3.1.20) \quad \|\widetilde{v}\|_{1,\mathcal{B};B_1} \leq |\widetilde{v}|_{1,B_1} + \sum_{i=1}^N \sup_{\substack{x,y\in B_1 \\ x\neq y}} \frac{|D_i\widetilde{v}(x) - D_i\widetilde{v}(y)|}{\mathcal{B}(|x-y|)} \leq$$

$$\leq |\widetilde{v}|_{1,B_1} + \sup_{x\in B_1} |D^2\widetilde{v}| \cdot \sup_{\substack{x,y\in B_1 \\ x\neq y}} \frac{|x-y|}{\mathcal{B}(|x-y|)} \leq$$

$$\leq c_1\big(R, \mathcal{A}^{-1}(2R)\big)|\widetilde{v}|_{2,B_1} \leq c_2\big(R, \mathcal{A}^{-1}(2R)\big)|\widetilde{v}|_{0,B_2} \leq c_2\big(|v|_{0,B_2} + |V|_{0,B_2}\big) \leq$$

$$\leq c_3\big(|v|_{0,B_2} + \|g\|_{\frac{N}{1-\alpha};B_2} + \sum_{j=1}^N \|f^j\|_{0,B_2}\big)$$

in virtue of Lemma 3.2. From (3.1.18)-(3.1.20) it follows the desired estimate (3.1.17). $\qquad\qquad\square$

Corresponding boundary estimates can be derived in a similar way. Let us first derive the appropriate extension of the estimate for the generalized newtonian potential $w(x)$ with density function $\operatorname{div} f$.

LEMMA 3.5. *Let* $f^j \in C^{0,\mathcal{A}}(\overline{B_2^+})$ $(j = 1,\dots,N)$. *Then* $w \in C^{1,\mathcal{B}}(\overline{B_1^+})$ *and*

$$(3.1.21) \qquad \|w\|_{1,\mathcal{B};B_1^+} \leq c\big(p, N, R, \alpha, \mathcal{A}^{-1}(2R), \mathcal{B}(2R)\big) \sum_{j=1}^N \|f^j\|_{0,\mathcal{A};B_2^+}.$$

PROOF. We assume that B_2 intersects Σ since otherwise the result is already contained in Lemma 3.3. The representation (3.1.4) holds for $D_i w(x)$ with $G_0 = B_2^+$. If either i or $j \neq N$, then the portion of the boundary integral

$$\int_{\partial B_2^+ \cap \Sigma} D_i \Gamma(x-y)\nu_j(y)d_y\sigma = \int_{\partial B_2^+ \cap \Sigma} D_j \Gamma(x-y)\nu_i(y)d_y\sigma = 0$$

since ν_i or $\nu_j = 0$ on Σ. The estimates in Lemma 3.3 for $D_i w(x)$ (i or $j \neq N$) then proceed exactly as before with B_2 replaced by B_2^+, $B_\delta(\xi)$ replaced by $B_\delta(\xi) \cap B_2^+$ and ∂B_2 replaced by $\partial B_2^+ \setminus \Sigma$. Finally $D_{NN}w$ can be estimated from the equation of the problem (PE) and the estimates on $D_{kk}w$ for $k = 1, \ldots, N-1$.   □

THEOREM 3.6. *Let* $v(x) \in C^0(\overline{B_2^+})$ *be a generalized solution of equation* (PE) *in* B_2^+ *with* $g \in L^{\frac{N}{1-\alpha}}(B_2^+)$, $f^j \in C^{0,\mathcal{A}}(\overline{B_2^+})$ $(j = 1, \ldots, N)$, *where* $\mathcal{A}$ *is an* $\alpha-$ *function satisfying the Dini condition at zero, and let* $v = 0$ *on* $B_2 \cap \Sigma$. *Then* $v \in C^{1,\mathcal{B}}(\overline{B_1^+})$, *and*

$$(3.1.22) \qquad \|v\|_{1,\mathcal{B};B_1^+} \leq c\left(|v|_{0;B_2^+} + \|g\|_{\frac{N}{1-\alpha};B_2^+} + \sum_{j=1}^{N} \|f^j\|_{0,\mathcal{A};B_2^+}\right),$$

where $c = c\big(N, R, \alpha, \mathcal{A}^{-1}(2R), \mathcal{B}(2R)\big)$.

PROOF. We use the method of reflection. Let $x' = (x_1, \ldots, x_{N-1})$, $x^* = (x', -x_N)$ and define

$$f_*^i(x) = \begin{cases} f^i(x) & \text{if } x_N \geq 0, \\ f^i(x^*) & \text{if } x_N \leq 0 \end{cases} \qquad (i = 1, \ldots, N).$$

We assume that B_2 intersects Σ; otherwise Lemma 3.4 implies (3.1.22). We set $B_2^- = \{x \in \mathbb{R}^N | x^* \in B_2^+\}$ and $D = B_2^+ \cup B_2^- \cup (B_2 \cap \Sigma)$. Then $f_*^i(x) \in C^{0,\mathcal{A}}(\overline{D})$ and

$$\|f_*^i\|_{0,\mathcal{A};D} \leq 2\|f^i\|_{0,\mathcal{A};B_2^+} \qquad (i = 1, \ldots, N).$$

Let

$$G(x,y) = \Gamma(x-y) - \Gamma(x-y^*) = \Gamma(x-y) - \Gamma(x^*-y)$$

denote the Green's function of the half-space $\mathbb{R}_+^N$, and consider

$$\vec{w}(x) = -\int_{B_2^+} D_y G(x,y)\vec{f}(y)dy, \qquad D_y = (D_{y_1}, \ldots, D_{y_N}).$$

For each $i = 1, \ldots, N$ let $w_i(x)$ denote the component of $\overrightarrow{w}(x)$ given by

$$w_i(x) = \int\limits_{B_2^+} D_{y_i}\Gamma(x-y)f^i(y)dy + \int\limits_{B_2^+} D_{y_i}\Gamma(x^*-y)f^i(y)dy.$$

We can see that $\overrightarrow{w}(x)$ and $w_i(x)$ vanish on $B_2 \cap \Sigma$. Noting that

$$\int\limits_{B_2^+} \Gamma(x^*-y)f^i(y)dy = \int\limits_{B_2^-} \Gamma(x-y)f_*^i(y)dy, \quad (i = 1, \ldots, N-1),$$

we obtain

$$(3.1.23) \quad w_i(x) = D_i\Big[2\int\limits_{B_2^+} \Gamma(x-y)f^i(y)dy - \int\limits_{D} \Gamma(x-y)f_*^i(y)dy\Big],$$

$$(i = 1, \ldots, N-1).$$

And when $i = N$, since

$$\int\limits_{B_2^+} D_{y_N}\Gamma(x^*-y)f^N(y)dy = \int\limits_{B_2^-} D_{y_N}\Gamma(x-y)f_*^N(y)dy,$$

we have

$$(3.1.24) \qquad w_N(x) = D_N\int\limits_{D} \Gamma(x-y)f_*^N(y)dy.$$

Letting

$$w_i^*(x) = -D_i\int\limits_{D} \Gamma(x-y)f_*^i(y)dy, \quad (i = 1, \ldots, N),$$

we have by Lemma 3.3

$$\|w^*\|_{1,\mathcal{B};B_1^+} \le c(p, N, R, \alpha, \mathcal{A}^{-1}(2R), \mathcal{B}(2R))\sum_{j=1}^{N} \|f_*^j\|_{0,\mathcal{A};D} \le$$

$$\le 2c(p, N, R, \alpha, \mathcal{A}^{-1}(2R), \mathcal{B}(2R))\sum_{j=1}^{N} \|f^j\|_{0,\mathcal{A};B_2^+}.$$

Combining this with Lemma 3.5, we obtain

$$(3.1.25) \qquad \|w\|_{1,\mathcal{B};B_1^+} \le c(p, N, R, \alpha, \mathcal{A}^{-1}(2R), \mathcal{B}(2R))\sum_{j=1}^{N} \|f^j\|_{0,\mathcal{A};B_2^+}.$$

Now let $\widetilde{v}(x) = v(x) - V(x)$, where $V(x)$ is the Newtonian potential from Lemma 3.4. Then $\widetilde{v}(x)$ is harmonic in B_2^+ and $\widetilde{v}(x) = 0$ on Σ. By Schwarz reflection principle $\widetilde{v}(x)$ may be extended to a harmonic function

in B_2 and hence the estimate (3.1.22) follows from the interior derivative estimate for harmonic functions by Theorem 2.10 [129] (see the proof of Lemma 3.4).   □

3.2. The equation with constant coefficients. Green's function

Let

$$\mathcal{L}_0 \equiv a_0^{ij} D_{ij}, \quad a_0^{ij} = a_0^{ji}$$

be a differential operator with constant coefficients a_0^{ij} satisfying

$$\nu|\xi|^2 \leq a_0^{ij}\xi_i\xi_j \leq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^N$$

for positive constants ν, μ and let $det\left(a_0^{ij}\right) = 1$.

DEFINITION 3.7. *The Green's function of the first kind of the operator $\mathcal{L}_0$ for the domain G is the function $G(x,y)$ satisfying the following properties*

- $\mathcal{L}_0 G(x,y) = \delta(x-y)$, $x \in G$, *where* $\delta(x-y)$ *is the Dirac function;*
- $G(x,y) = 0$, $x \in \partial G$.

For the properties, the existence and the construction of Green's functions in detail see, for example, §5.1 [43], chapter I [313]. We note the following statements

LEMMA 3.8. *Let $G(x,y)$ be the Green function of $\mathcal{L}_0$ in $\mathbb{R}_+^N$, $N \geq 3$. Then $G(x,y)$ satisfies the following inequalities:*

$$G(x,y) \leq \begin{cases} |x-y|^{2-N}, \\ Cy_N|x-y|^{1-N}, \\ Cx_Ny_N|x-y|^{-N}; \end{cases}$$

$$|D_iG(x,y)| \leq \begin{cases} C|x-y|^{1-N}, \\ Cy_N|x-y|^{-N}; \end{cases}$$

$$|D_{ij}G(x,y)| \leq \begin{cases} C|x-y|^{-N}, \\ Cy_N|x-y|^{-1-N}, \end{cases}$$

where C depends only on ν, μ, N.

PROOF. Let A be the matrix (a_0^{ij}) and T be a constant matrix which defines a nonsingular linear transformation $x' = xT$ from $\mathbb{R}^N$ onto $\mathbb{R}^N$. Letting $\widetilde{v}(x') = v(xT)$ one verifies easily that

$$a_0^{ij} D_{ij} v(x) = \widetilde{a}_0^{ij} D_{ij}\widetilde{v}(x'),$$

where $\widetilde{A} = T^t A T$, $T^t = T$ transpose. For suitable orthogonal matrix T, $\widetilde{A}$ is a diagonal matrix Λ whose diagonal elements are the eigenvalues $\lambda_1, \ldots, \lambda_N$

of A. If $Q = T\Lambda^{-1/2}$, where $\Lambda^{-1/2} = [\lambda_i^{-1/2}\delta_i^j]$, then the transformation $x' = xQ$ takes $\mathcal{L}_0 v = \Delta'\widetilde{v}(x')$, that is $\mathcal{L}_0$ is transformed into the Laplace operator. By a further rotation we may assume that Q takes the half-space $x_N > 0$ onto the half-space $x'_N > 0$.

Since the orthogonal matrix T preserves length, we have

$$\Lambda^{-1/2}|x| \leq |x'| = |xQ| \leq \lambda^{-1/2}|x|;$$
$$\lambda = \min\{\lambda_1, \ldots, \lambda_N\} = \nu; \quad \Lambda = \max\{\lambda_1, \ldots, \lambda_N\} = \mu.$$

It follows that $\widetilde{G}(x', y') = G(xQ, yQ)$ is the Green function of the Laplace operator in the half-space $x'_N > 0$.

The corresponding inequalities for $\widetilde{G}(x', y')$ are well known, since we know $\widetilde{G}(x', y')$ explicitly (see, for example, §2.4 [129] or §§8, 10 Chapter I [313]). Here C depends on N only. Now required inequalities follow easily, since the dilation of distance is bounded above and below with μ and ν. □

In the same way we can prove the next Lemma. (Here we use the explicit form of the Green function for a ball, see, for example, §2.5 [129],- and a homothety.)

LEMMA 3.9. *Let $G(x, y)$ be the Green function of $\mathcal{L}_0$ for the ball $\overline{B}_\varrho(0)$. Then $G(x, y)$ satisfies the following inequalities*

$$G(x, y) \leq C|x - y|^{2-N}, \quad |\nabla_x G(x, y)| \leq C|x - y|^{1-N}, \quad \text{for } x, y \in \overline{B}_\varrho(0);$$
$$\left|\frac{\partial}{\partial y_i}\nabla_x G(x, y)\right| \leq C\varrho^{-N}, \quad \left|\frac{\partial^2}{\partial y_i \partial y_j}\nabla_x G(x, y)\right| \leq C\varrho^{-N-1},$$
$$\text{for } y \in \overline{B}_{\varrho/2}(0), \ |x| = \varrho, \ N \geq 3,$$

where C depends only on ν, μ, N.

Finally, we note the Green representation formula

$$(3.2.1) \qquad u(y) = \int_{\partial G} u(x)\frac{\partial G(x, y)}{\partial \nu_x}ds_x + \int_G G(x, y)\mathcal{L}_0 u \, dx,$$

where $G(x, y)$ is the Green function of the operator $\mathcal{L}_0$ in G and $\frac{\partial}{\partial \nu}$ denotes the conormal derivative, that is the derivative with direction cosines $a_0^{ij}n_j$, $i = 1, \cdots, N$. It is well known that this formula is valid in a Dini-Liapunov region (see Chapter I [313]).

Now we establish a necessary preliminary result that extends Lemma 3.4 and Theorem 3.6 from Poisson's equation to other elliptic equations with constant coefficients. We state these extensions in the following theorem:

THEOREM 3.10. *In the equation*

$$(ECC) \qquad \mathcal{L}_0 u \equiv a_0^{ij} D_{ij} u = g(x) + \frac{\partial f^j(x)}{\partial x_j}, \quad a_0^{ij} = a_0^{ji}, \ x \in G$$

let $A = (a_0^{ij})$ be a constant matrix such that

$$\nu|\xi|^2 \leq a_0^{ij} \xi_i \xi_j \leq \mu|\xi|^2, \quad \forall \xi \in \mathbb{R}^N$$

for positive constants ν, μ.

(a) *Let G be a domain in $\mathbb{R}^N$, and let $v(x) \in C^{1,\mathcal{B}}(G)$ be a generalized solution of equation (ECC) with $g \in L^{\frac{N}{1-\alpha}}(G)$, $f^j \in C^{0,\mathcal{A}}(\overline{G}), (j = 1, \ldots, N)$ where $\mathcal{A}$ is an α-Dini function. Then for any two concentric balls $B_1 = B_R(x_0), B_2 = B_{2R}(x_0) \subset\subset G$ we have*

$$(3.2.2) \qquad \|v\|_{1,\mathcal{B};B_1} \leq c\left(|v|_{0;B_2} + \|g\|_{\frac{N}{1-\alpha};B_2} + \sum_{j=1}^N \|f^j\|_{0,\mathcal{A};B_2} \right),$$

where $c = c(N, R, \alpha, \nu, \mu, \mathcal{A}^{-1}(2R), \mathcal{B}(2R))$.

(b) *Let $v(x) \in C^0(\overline{B_2^+})$ be a generalized solution of equation $\mathcal{L}_0 v = g(x) + \frac{\partial f^j(x)}{\partial x_j}$ in B_2^+ with $g \in L^{\frac{N}{1-\alpha}}(B_2^+)$, $f^j \in C^{0,\mathcal{A}}(\overline{B_2^+})$, $(j = 1, \ldots, N)$, where $\mathcal{A}$ is an $\alpha-$ function satisfying the Dini condition at zero, and let $v = 0$ on $B_2 \cap \Sigma$. Then $v \in C^{1,\mathcal{B}}(\overline{B_1^+})$, and*

$$(3.2.3) \qquad \|v\|_{1,\mathcal{B};B_1^+} \leq c\left(|v|_{0;B_2^+} + \|g\|_{\frac{N}{1-\alpha};B_2^+} + \sum_{j=1}^N \|f^j\|_{0,\mathcal{A};B_2^+} \right),$$

where $c = c(N, R, \alpha, \nu, \mu, \mathcal{A}^{-1}(2R), \mathcal{B}(2R))$.

PROOF. Let T be a constant matrix which defines a nonsingular linear transformation $y = xT$ from $\mathbb{R}^N$ onto $\mathbb{R}^N$. Letting $\tilde{v}(y) = v(xT)$ one verifies easily that

$$a_0^{ij} D_{ij} v(x) = \tilde{a}_0^{ij} D_{ij} \tilde{v}(y),$$

where $\tilde{A} = T^t A T$, $T^t = T$ transpose. For suitable orthogonal matrix T, $\tilde{A}$ is a diagonal matrix Λ whose diagonal elements are the eigenvalues $\lambda_1, \ldots, \lambda_N$ of A. If $Q = T\Lambda^{-1/2}$, where $\Lambda^{-1/2} = [\lambda_i^{-1/2} \delta_i^j]$, then the transformation $y = xQ$ takes $\mathcal{L}_0 v = g(x) + \frac{\partial f^j(x)}{\partial x_j}$ into the Poisson equation $\Delta \tilde{v}(y) = \tilde{g}(y) + q_j^k \frac{\partial \tilde{f}^j(y)}{\partial y_k}$. By a further rotation we may assume that Q takes the half-space $x_N > 0$ onto the half-space $y_N > 0$.

Since the orthogonal matrix T preserves length, we have

$$\Lambda^{-1/2}|x| \leq |y| = |xQ| \leq \lambda^{-1/2}|x|;$$

$$\lambda = \min\{\lambda_1, \ldots, \lambda_N\} = \nu \text{ and } \Lambda = \max\{\lambda_1, \ldots, \lambda_N\} = \mu.$$

It follows that if $\widetilde{B}(y_0)$ is the image of $B(x_0)$ under the transformation $y = xQ$ then the norms $\| \bullet \|_{k,\mathcal{A}}$ defined on B and $\widetilde{B}$ are equivalent, that is these norms are related by the inequality

$$c^{-1}\|v\|_{k,\mathcal{A};B} \le \|\widetilde{v}\|_{k,\mathcal{A};\widetilde{B}} \le c\|v\|_{k,\mathcal{A};B}; \ k = 0, \ 1,$$

where $c = c(k, N, \nu, \mu)$.

Similarly if $\widetilde{B}^+(y_0)$ with boundary portion $\widetilde{\sigma}$ on $y_N = 0$ is the image of $B^+(x_0)$ with a boundary portion σ on Σ, the norms $\| \bullet \|_{k,\mathcal{A}}$ defined on B^+ and $\widetilde{B}^+$ are equivalent, i.e. these norms are related by the inequality

$$c^{-1}\|v\|_{k,\mathcal{A};B^+ \cup \sigma} \le \|\widetilde{v}\|_{k,\mathcal{A};\widetilde{B}^+ \cup \widetilde{\sigma}} \le c\|v\|_{k,\mathcal{A};B^+ \cup \sigma}; \ k = 0, \ 1,$$

where $c = c(k, N, \nu, \mu)$.

To prove part (a) of our Theorem we apply Lemma 3.4 in $\widetilde{B}(y_0)$ to obtain

$$\|v\|_{1,\mathcal{B};B_1} \le c\|\widetilde{v}\|_{1,\mathcal{B};\widetilde{B}_1} \le C\left(|\widetilde{v}|_{0;\widetilde{B}_2} + \|\widetilde{g}\|_{\frac{N}{1-\alpha};\widetilde{B}_2} + \sum_{j=1}^{N} \|\widetilde{f^j}\|_{0,\mathcal{A};\widetilde{B}_2} \right) \le$$

$$\le C\left(|v|_{0;B_2} + \|g\|_{\frac{N}{1-\alpha};B_2} + \sum_{j=1}^{N} \|f^j\|_{0,\mathcal{A};B_2} \right),$$

which is the desired conclusion (3.2.2).

Part (b) of our Theorem is proved in the same way, using Theorem 3.6. $\square$

3.3. The Laplace operator in weighted Sobolev spaces

Let G be a conical domain. We consider the Dirichlet problem for the Poisson equation

(DPE) $\begin{cases} \Delta u = f & \text{in } G, \\ u = \varphi & \text{on } \partial G. \end{cases}$

It is known from the classical paper by Kondrate'v [161] that the behavior of solutions of (DPE) is controlled by the eigenvalues of the eigenvalue problem (EVD) for the Laplace-Beltrami operator Δ_ω.

THEOREM 3.11. (See Theorem 4.1 [275], Theorem 2.6.5 [199]).
Let $p \in (1, \infty), k \in \mathbb{N}$ with $k \ge 2$ and $\alpha \in \mathbb{R}$. Let λ be defined by (2.5.11) with the smallest positive eigenvalue ϑ of (EVD). Then the Dirichlet problem (DPE) has a unique solution $u \in V_{p,\alpha}^k(G)$ for all $f \in V_{p,\alpha}^{k-2}(G)$, $\varphi \in V_{p,\alpha}^{k-1/p}(\partial G)$ if and only if

$$-\lambda + 2 - N < k - (\alpha + N)/p < \lambda.$$

In this case the following a–priori estimate is valid

$$\|u\|_{V_{p,\alpha}^k(G)} \le c \left\{ \|f\|_{V_{p,\alpha}^{k-2}(G)} + \|\varphi\|_{V_{p,\alpha}^{k-1/p}(\partial G)} \right\}.$$

3.4. Notes

Section 3.1 is a modification of Chapter 4 [**129**]: we replaced Hölder-continuity by Dini-continuity.

Discussions of boundary value problems for the Laplacian in nonsmooth domains can be found in a number of works (see e.g. [**2, 9, 91, 89, 113, 116, 126, 127, 133, 135, 161, 177, 198, 199, 213, 242, 243, 249, 278, 248, 322, 327, 332, 347, 350, 356, 398, 403, 407, 408, 409**]). Theorem 3.11 was established for the first time in the work [**161**] for $p = 2$. V.G. Maz'ya and B.A. Plamenevsky [**275**] extended this result to the case $1 < p < \infty$. For details we refer to [**199**] (in particular, see there Notes 1.5, 2.7).

Other boundary value problems for the Laplace equation or for general second order elliptic equations and systems with constant coefficients in nonsmooth domain have been studied in many works: W. Zajączkowski and V. Solonnikov [**409**] - Neumann problem in a domain with edges; P. Grisvard [**133**], M. Dauge [**92**], N. Wigley [**407, 408**] - Neumann and mixed problem on curvilinear polyhedra; L. Stupelis - Neumann problem in a plane angle, N. Grachev and V. Maz'ya [**131, 132**] - Neumann problem in a polyhedral cone; Y. Saito - the limiting equation for Neumann Laplacians on shrinking domains [**351**]; V. Maz'ya and J. Rossmann [**291**]-[**293**] - the mixed problem in a polyhedral domain. J. Banasiak [**30**] investigated the elliptic transmission problem for Laplacian in plane domains with curvilinear polygons as its boundaries. New elliptic regularity results for polyhedral Laplace interface problems for anisotropic materials are established by V. Maz'ya, J. Elschner, J. Rehberg and G. Schmidt [**262**]. Some unilateral boundary value problems (e.g., Signorini transmission problems with mixed boundary conditions) in polygonal and polyhedral domains are studied in [**82**].

CHAPTER 4

Strong solutions of the Dirichlet problem for linear equations

4.1. The Dirichlet problem in general domains

Let $G \subset \mathbb{R}^N$ be a bounded domain. We consider the following Dirichlet problem

$$(L) \quad \begin{cases} Lu := a^{ij}(x)D_{ij}u(x) + a^i(x)D_i u(x) + a(x)u(x) = f(x) \text{ in } G, \\ u(x) = \varphi(x) \text{ on } \partial G, \end{cases}$$

where the coefficients $a^{ij}(x) = a^{ji}(x)$ and satisfy the uniform ellipticity condition

$$\nu|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^N,\ x \in \overline{G}$$

with the ellipticity constants $\nu, \mu > 0$.

Let us recall some well known facts about $W^{2,p}(G)$ solutions of this problem.

THEOREM 4.1. (Unique solvability) [129, Theorem 9.30 and the remark in the end of §9.5].

Let G satisfy an exterior cone condition at every boundary point and let be given $p \ge N$. Let

- $a^{ij} \in C^0(G) \cap L^\infty(G), \quad a^i \in L^q(G), a \in L^p(G), \quad i,j = 1,\ldots,N,$
  where $q > N$, if $p = N$, and $q = N$, if $p > N$;
- $a(x) \le 0\ \forall x \in G:$
- $f \in L^p(G), \varphi \in C^0(\partial G).$

Then the boundary value problem (L) has a unique solution $u \in W^{2,p}_{loc}(G) \cap C^0(\overline{G}).$

THEOREM 4.2. [129, Theorem 9.1] (Alexandrov's Maximum Principle) *Let $u \in W^{2,N}_{loc}(G) \cap C^0(\overline{G})$ satisfy the boundary value problem (L). Furthermore let*

- $\left(\sum_{i=1}^{N} |a^i|^2\right)^{1/2}, f \in L^N(G),$

- $a(x) \leq 0 \ \forall x \in G$.

Then

$$\sup_G u \leq \sup_{\partial G} u^+ + c \|f\|_{L^N(G)},$$

where c depends only on $N, \nu, \operatorname{diam} G$ and $\left\| \left(\sum_{i=1}^{N} |a^i|^2 \right)^{1/2} \right\|_{L^N(G)}$.

THEOREM 4.3. **The E. Hopf strong maximum principle** *(see Theorems 9.6, 3.5 [129]).*

Let L be elliptic in the domain G and $a^i(x), i = 1, \ldots, N$; $a(x) \in L^\infty_{loc}(G), a(x) \leq 0$. If $u \in W^{2,N}_{loc}(G)$ satisfies $L[u] \geq 0 (\leq 0)$ in G, then u cannot achieve a nonnegative maximum (non-positive minimum) in G unless it is a constant.

Applying the Alexandrov Maximum Principle to the difference of two functions we obtain the following comparison principle.

THEOREM 4.4. **(Comparison principle)** *Let L be elliptic in G, let* $\left(\sum_{i=1}^{N} |a^i|^2 \right)^{1/2}, f \in L^N(G), \forall x \in G : a(x) \leq 0$ and $u, v \in W^{2,N}_{loc}(G) \cap C^0(\overline{G})$ *with $Lu \geq Lv$ in G and $u \leq v$ on ∂G. Then $u \leq v$ throughout G.*

THEOREM 4.5. **[129, Theorem 9.26], [383] (Local maximum principle)** *Let G be a bounded domain with subdomains T, G' such that $T \subset G' \subset \overline{G}$ and suppose that $a^i \in L^q(G), q > N$ and $a \in L^N(G)$. Let $u \in W^{2,N}(G) \cap C^0(\overline{G})$ satisfy $Lu \geq f$ in G and $u \leq 0$ on $T \cap \partial G$ where $f \in L^N(G')$. Then for any $p > 0$, we have*

$$\sup_T u \leq c \left\{ \|f\|_{L^N(G')} + \|u\|_{L^p(G')} \right\},$$

where the constant c depends only on $N, \mu, \nu, p, \|a^i\|_{q,G'}, \|a\|_{N,G'}, T, G', G$.

THEOREM 4.6. **[129, Theorem 9.13] (L^p-estimate)** *Let G be a bounded domain in $\mathbb{R}^N$ and $T \subset \partial G$ be of the class $C^{1,1}$. Furthermore, let $u \in W^{2,p}(G), 1 < p < \infty$, be a strong solution of (L) with $f \in L^p(G)$ and $u = 0$ on T in the sense of $W^{1,p}(G)$. We assume that*

- $a^{ij} \in C^0(G \cup T)$,
- $a^i \in L^q(G)$, where $q > N$ if $p \leq N$ and $q = p$ if $p > N$,
- $a \in L^r(G)$, where $r > N/2$ if $p \leq N/2$ and $r = p$ if $p > N/2$.

Then, for any domain $G' \subset\subset G \cup T$,

$$(4.1.1) \qquad \|u\|_{W^{2,p}(G')} \leq c \left(\|u\|_{L^p(G)} + \|f\|_{L^p(G)} \right),$$

where c depends only on N, p, ν, μ, T, G', G, the moduli of continuity of the coefficients a^{ij} on G' and on $\left\| \left(\sum_{i=1}^{N} |a^i|^2 \right)^{1/2} \right\|_{L^q(G)}$, $\|a\|_{L^r(G)}$.

THEOREM 4.7. [4, Theorem 15.2] *Let G be a bounded domain of class C^k with $k \in \mathbb{N}$, $k \geq 2$ and suppose that the coefficients of the operator L belong to $C^{k-2}(G)$ and have $C^{k-2}-$ norms bounded by K. Let u be a solution of (L) with $f \in W^{k-2,p}(G)$ and $\varphi \in W^{k-1/p,p}(\partial G)$. Then $u \in W^{k,p}(G)$ and the following estimate is valid*

$$\|u\|_{W^{k,p}(G)} \leq c \left\{ \|f\|_{W^{k-2,p}(G)} + \|\varphi\|_{W^{k-1/p,p}(\partial G)} + \|u\|_{L^p(G)} \right\},$$

where c depends only on ν, μ, K, k, p, the domain G, and the modulus of continuity of the leading coefficients of L.

By use of a suitable cut–off function we obtain the following localized version of the above theorem.

THEOREM 4.8. [4, Theorem 15.3] *Let G be a bounded domain of class C^k with subdomains T, G' such that $T \subset G' \subset \overline{G}$. We suppose that the coefficients of the operator L belong to $C^{k-2}(G)$ with $k \in \mathbb{N}$, $k \geq 2$. Let u be a solution of (L) with $f \in W^{k-2,p}(G')$ and $\varphi \in W^{k-1/p,p}(\partial G' \cap \partial G)$. Then $u \in W^{k,p}(T)$ and the following estimate is valid*

$$\|u\|_{W^{k,p}(T)} \leq c \left\{ \|f\|_{W^{k-2,p}(G')} + \|\varphi\|_{W^{k-1/p,p}(\partial G' \cap \partial G)} + \|u\|_{L^p(G')} \right\}.$$

The stronger result is valid for the case $N = 2$; it is the Bernstein estimate (see in detail §19 Chapter III, the inequality (19.20) [216]).

THEOREM 4.9. *Let $G \subset \mathbb{R}^2$ be a bounded domain and $G' \subset\subset \overline{G} \setminus \mathcal{O}$ be any subdomain with a $W^{2,p}$, $p > 2$ boundary portion $T = (\partial G' \cap \partial G) \subset \partial G \setminus \mathcal{O}$. Let $u \in W^2(G)$ be a strong solution of the equation $a^{ij}(x) D_{ij} u(x) = f(x)$ in G' with $u = 0$ on T in the sense of $W^1(G)$. Let the equation satisfy the uniform ellipticity condition with the ellipticity constants ν, μ. Then, for any subdomain $G'' \subset\subset G' \cup T$ we have*

$$\|u\|_{W^2(G'')}^2 \leq C \int_{G'} \left(u^2 + f^2 \right) dx,$$

where C depends on ν, μ, p, T, G'', G'.

Finally, we cite one theorem about local gradient bound for uniformly elliptic equations with two variables in general form.

THEOREM 4.10. [215, Theorem 17.4], [216, Theorem 19.4].
Let $G \subset \mathbb{R}^2$ be a bounded domain and $G' \subset\subset \overline{G} \setminus \mathcal{O}$ be any subdomain with a $W^{2,p}$, $p > 2$ boundary portion $T = (\partial G' \cap \partial G) \subset \partial G \setminus \mathcal{O}$. Let $u \in W^2(G')$

be a strong solution of the problem (L) in G', where L is uniformly elliptic and satisfy

$$\|a^i(x),\, a(x),\, f(x)\|_{L^p(G')} \leq \mu_1 \text{ and } \|\varphi(x)\|_{W^{2,p}(G')} \leq \mu_1.$$

Then for any subdomain $G'' \subset\subset G' \cup T$ there is a constant $M_1 > 0$ depending only on $\nu, \mu, \mu_1, p, \|u\|_{2,G'}, \|\varphi\|_{W^{2,p}(G')}$ and G', G'', T such that

$$\sup_{G''} |\nabla u| \leq M_1.$$

4.2. The Dirichlet problem in a conical domain

In the following part of this chapter we denote by $G \subset \mathbb{R}^N$ a bounded domain with a conical point in $\mathcal{O}$ as described in Section 1.3.

DEFINITION 4.11. A *(strong) solution* of the Dirichlet problem (L) in G is a function $u \in W^2(G_\varepsilon) \cap C^0(\overline{G})$, $\forall \varepsilon > 0$ which satisfies the equations $Lu = f$ for almost all $x \in G$ and the boundary condition $u = \varphi$ for all $x \in \partial G$.

In the following we assume that the coefficients $a^{ij}(x), a^i(x)$ and $a(x)$ satisfy the following conditions

    (a) the uniform ellipticity condition

$$\nu|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \ x \in \overline{G}$$

    with some $\nu, \mu > 0$,

  (aa) $a^{ij}(0) = \delta_i^j$,

 (aaa) $a^{ij} \in C^0(\overline{G})$, $a^i \in L^p(G)$, $a \in L^{p/2}(G)$, $p > N$,

    (b) there exists a monotonically increasing nonnegative function $\mathcal{A}$ such that

$$\left(\sum_{i,j=1}^{N} |a^{ij}(x) - a^{ij}(y)|^2\right)^{1/2} \leq \mathcal{A}(|x-y|) \quad and$$

$$|x|\left(\sum_{i=1}^{N} a^{i^2}(x)\right)^{1/2} + |x|^2|a(x)| \leq \mathcal{A}(|x|)$$

    for $x, y \in \overline{G}$.

REMARK 4.12. *The Assumption (b) guarantees that the coefficients a^i and a are bounded on $G \setminus B_\varepsilon(0)$ for every $\varepsilon > 0$.*

4.2.1. Estimates in weighted Sobolev spaces.

THEOREM 4.13. *Let u be a solution of (L) and let λ be the smallest positive eigenvalue of (EVD). Suppose that*

$$(4.2.1) \qquad\qquad \lim_{r\to+0}\mathcal{A}(r) = 0$$

and that $f \in \overset{\circ}{W}{}^0_\alpha(G)$, $\varphi \in \overset{\circ}{W}{}^{3/2}_\alpha(\partial G) \cap C^0(\partial G)$, where

$$(4.2.2) \qquad\qquad 4 - N - 2\lambda < \alpha \le 2.$$

Then $u \in \overset{\circ}{W}{}^2_\alpha(G)$ and

$$\|u\|_{\overset{\circ}{W}{}^2_\alpha(G)} \le c\left(\|u\|_{L^2(G)} + \|f\|_{\overset{\circ}{W}{}^0_\alpha(G)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_\alpha(\partial G)}\right),$$

where $c > 0$ depends only on ν, μ, α, λ, N, $\max\limits_{x\in\overline{G}}\mathcal{A}(|x|)$, G. Furthermore, if $N < 4$, there exists real constant c_2 independent of u such that

$$(4.2.3) \qquad\qquad |u(x)| \le c_2|x|^{(4-N-\alpha)/2}, \quad \forall x \in G^d_0$$

for some $d > 0$.

PROOF. Let $\Phi \in \overset{\circ}{W}{}^2_\alpha(G) \cap C^0(\overline{G})$ be an arbitrary extension of the boundary function φ into G. The function $v = u - \Phi$ then satisfies the homogeneous Dirichlet problem

$$(L)_0 \qquad \begin{cases} a^{ij}(x)D_{ij}v(x) + a^i(x)D_iv(x) + a(x)v(x) = F(x) & \text{in } G, \\ v(x) = 0 & \text{on } \partial G, \end{cases}$$

where

$$(4.2.4) \qquad F(x) = f(x) - \left(a^{ij}(x)D_{ij}\Phi(x) + a^i(x)D_i\Phi(x) + a(x)\Phi(x)\right).$$

Since $a^{ij}(0) = \delta^j_i$, we have

$$(4.2.5) \quad \Delta v(x) = F(x) - \left(a^{ij}(x) - a^{ij}(0)\right)D_{ij}v(x) -$$

$$- a^i(x)D_iv(x) - a(x)v(x) \quad \text{in } G.$$

Case I: $4 - N \le \alpha \le 2$.
Integrating by parts we show that

$$\begin{aligned}
\int\limits_{G_\varepsilon} r^{\alpha-2}v\Delta v\,dx &= -\varepsilon^{\alpha-2}\int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r}d\Omega_\varepsilon - \int\limits_{G_\varepsilon}\langle\nabla v, \nabla r^{\alpha-2}v\rangle dx = \\
&= -\varepsilon^{\alpha-2}\int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r}d\Omega_\varepsilon - \int\limits_{G_\varepsilon} r^{\alpha-2}|\nabla v|^2dx+
\end{aligned}$$

$$+ \quad (2-\alpha) \int\limits_{G_\varepsilon} r^{\alpha-4} v \langle x, \nabla v \rangle dx.$$

Integrating again by parts we obtain

$$\int\limits_{G_\varepsilon} r^{\alpha-4} v \langle x, \nabla v \rangle dx \; = \; \frac{1}{2} \int\limits_{G_\varepsilon} \langle r^{\alpha-4} x, \nabla v^2 \rangle dx$$

$$= \; -\frac{1}{2} \varepsilon^{\alpha-3} \int\limits_{\Omega_\varepsilon} v^2 d\Omega_\varepsilon - \frac{1}{2} \int\limits_{G_\varepsilon} v^2 \sum_{i=1}^{N} D_i(r^{\alpha-4} x_i) dx$$

$$= \; -\frac{1}{2} \varepsilon^{\alpha-3} \int\limits_{\Omega_\varepsilon} v^2 d\Omega_\varepsilon - \frac{N+\alpha-4}{2} \int\limits_{G_\varepsilon} r^{\alpha-4} v^2 dx$$

because

$$\sum_{i=1}^{N} D_i(r^{\alpha-4} x_i) = N r^{\alpha-4} + (\alpha-4) r^{\alpha-5} \sum_{i=1}^{N} \frac{x_i^2}{r} = (N+\alpha-4) r^{\alpha-4}.$$

Thus, multiplying both sides of $(L)_0$ by $r^{\alpha-2} v(x)$ and integrating over G_ε, we obtain

$$(4.2.6) \quad \varepsilon^{\alpha-2} \int\limits_{\Omega_\varepsilon} v \frac{\partial v}{\partial r} d\Omega_\varepsilon + \int\limits_{G_\varepsilon} r^{\alpha-2} |\nabla v|^2 dx + \frac{2-\alpha}{2} \varepsilon^{\alpha-3} \int\limits_{\Omega_\varepsilon} v^2 d\Omega_\varepsilon +$$

$$+ \frac{2-\alpha}{2} (N+\alpha-4) \int\limits_{G_\varepsilon} r^{\alpha-4} v^2 dx =$$

$$= \int\limits_{G_\varepsilon} r^{\alpha-2} v \Big(-F(x) + \big(a^{ij}(x) - a^{ij}(0) \big) D_{ij} v(x) +$$

$$+ a^i(x) D_i v(x) + a(x) v(x) \Big) dx.$$

Let us estimate in the above equation the integrals over Ω_ε. To this end we consider the function

$$M(\varepsilon) = \max_{x \in \Omega_\varepsilon} |v(x)|.$$

Since $v \in C^0(\overline{G})$ and $v = 0$ on ∂G, we have

$$\lim_{\varepsilon \to +0} M(\varepsilon) = 0.$$

LEMMA 4.14.

$$(4.2.7) \qquad \lim_{\varepsilon \to +0} \varepsilon^{\alpha-2} \int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r} d\Omega_\varepsilon = 0, \ \forall \alpha \in [4 - N, 2].$$

PROOF. We consider the set $G_\varepsilon^{2\varepsilon}$ and we have $\Omega_\varepsilon \subset \partial G_\varepsilon^{2\varepsilon}$. Now we use the inequality (1.6.1)

$$\int\limits_{\Omega_\varepsilon} |w| d\Omega_\varepsilon \le c \int\limits_{G_\varepsilon^{2\varepsilon}} (|w| + |\nabla w|) dx.$$

Setting $w = v\frac{\partial v}{\partial r}$ we find

$$|w| + |\nabla w| \le c(r^2 v_{xx}^2 + |\nabla v|^2 + r^{-2}v^2).$$

Therefore we get

$$(4.2.8) \qquad \int\limits_{\Omega_\varepsilon} \left| v\frac{\partial v}{\partial r} \right| d\Omega_\varepsilon \le c \int\limits_{G_\varepsilon^{2\varepsilon}} (r^2 v_{xx}^2 + |\nabla v|^2 + r^{-2}v^2) dx.$$

Let us now consider the sets $G_{\varepsilon/2}^{5/2\varepsilon}$ and $G_\varepsilon^{2\varepsilon} \subset G_{\varepsilon/2}^{5/2\varepsilon}$ and new variables x' defined by $x = \varepsilon x'$. Then the function $w(x') = v(\varepsilon x')$ satisfies in $G_{1/4}^2$ the equation

$$(4.2.9) \qquad a^{ij}(\varepsilon x')\frac{\partial^2 w}{\partial x_i' \partial x_j'} + \varepsilon a^i(\varepsilon x')\frac{\partial w}{\partial x_i'} + \varepsilon^2 a(\varepsilon x')w = \varepsilon^2 F(\varepsilon x').$$

Applying the L^2-estimate (4.1.1) for the solution w in $G_{1/4}^2$ we get

$$(4.2.10) \qquad \int\limits_{G_1^2} \left(|D'^2 w|^2 + |\nabla' w|^2 \right) dx' \le c \int\limits_{G_{1/2}^{5/2}} \left(\varepsilon^4 F^2(\varepsilon x') + w^2 \right) dx',$$

where $c > 0$ depends only on $\nu, \mu, G, \max\limits_{x' \in G_{1/2}^{5/2}} \mathcal{A}(|x'|)$; and

$$|D'^2 w|^2 = \sum_{i,j=1}^N \left| \frac{\partial^2 w}{\partial x_i' \partial x_j'} \right|^2, \quad |\nabla' w|^2 = \sum_{i=1}^N \left| \frac{\partial w}{\partial x_i'} \right|^2.$$

Returning to the variable x, we obtain

$$(4.2.11) \qquad \int\limits_{G_\varepsilon^{2\varepsilon}} \left(r^2 |D^2 v|^2 + |\nabla v|^2 + r^{-2}v^2 \right) dx \le c \int\limits_{G_{\varepsilon/2}^{5/2\varepsilon}} \left(r^2 F^2 + r^{-2}v^2 \right) dx.$$

By the Mean Value Theorem 1.58 with regard to $v \in C^0(\overline{G})$ we have

$$
\int\limits_{G^{5/2\varepsilon}_{\varepsilon/2}} r^{-2}v^2 dx = \int\limits_{\varepsilon/2}^{5/2\varepsilon} r^{N-3} \int\limits_{\Omega} v^2(r,\omega)d\Omega dr
$$

(4.2.12)

$$
\leq 2\varepsilon(\theta_1\varepsilon)^{N-3} \int\limits_{\Omega} v^2(\theta_1\varepsilon,\omega)d\Omega
$$

$$
\leq 2\varepsilon^{N-2}\theta_1^{N-3}M^2(\theta_1\varepsilon)\,\mathrm{meas}\,\Omega
$$

for some $\frac{1}{2} < \theta_1 < \frac{5}{2}$.
From (4.2.8), (4.2.11) and (4.2.12) we obtain

$$
(4.2.13) \quad \int\limits_{\Omega_\varepsilon} \left| v\frac{\partial v}{\partial r} \right| d\Omega_\varepsilon \leq c_1\varepsilon^{N-2}M^2(\varepsilon) + c_2 \int\limits_{G^{5/2\varepsilon}_{\varepsilon/2}} r^2 F^2 dx \leq
$$

$$
\leq c_1\varepsilon^{N-2}M^2(\varepsilon) + c_3\varepsilon^{2-\alpha} \int\limits_{G^{5/2\varepsilon}_{\varepsilon/2}} r^\alpha F^2 dx, \ \forall \alpha \leq 2.
$$

Hence we obtain as well

$$
(4.2.14) \quad \varepsilon^{\alpha-2} \int\limits_{\Omega_\varepsilon} \left| v\frac{\partial v}{\partial r} \right| d\Omega_\varepsilon \leq c_1\varepsilon^{\alpha+N-4}M^2(\varepsilon) + c_3 \int\limits_{G^{5/2\varepsilon}_{\varepsilon/2}} r^\alpha F^2 dx, \ \forall \alpha \leq 2.
$$

By the assumption (b) and hypotheses of our Theorem we have that $F \in \overset{\circ}{W}{}^0_\alpha(G)$, hence

$$
(4.2.15) \quad \lim_{\varepsilon\to+0} \int\limits_{G^{5/2\varepsilon}_{\varepsilon/2}} r^\alpha F^2 dx = 0
$$

and thus from (4.2.14) with regard to that $v(0) = 0$ we deduce the validity of statement (4.2.7) of our lemma. $\qquad\square$

Further, we get by the Cauchy inequality

$$
\int\limits_{G_\varepsilon} r^{\alpha-2}v(x)F(x)dx = \int\limits_{G_\varepsilon} (r^{\alpha/2-2}v(x))(r^{\alpha/2}F(x))dx
$$

(4.2.16)

$$
\leq \frac{\delta}{2}\int\limits_{G_\varepsilon} r^{\alpha-4}v^2 dx + \frac{1}{2\delta}\int\limits_{G_\varepsilon} r^\alpha F^2(x)dx
$$

for arbitrary $\delta > 0$. Applying assumption (b) together with the Hölder and the Cauchy inequality

$$(4.2.17) \quad r^{\alpha-2}v\left(\left(a^{ij}(x) - a^{ij}(0)\right)D_{ij}v(x) + a^i(x)D_iv(x) + a(x)v(x)\right)$$

$$\leq \mathcal{A}(r)\left((r^{\frac{\alpha}{2}}|D^2v|)(r^{\frac{\alpha}{2}-2}v) + r^{\alpha-2}|\nabla v|(r^{-1}v) + r^{\alpha-4}v^2\right)$$

$$\leq \mathcal{A}(r)\left(r^\alpha|D^2v|^2 + r^{\alpha-2}|\nabla v|^2 + 2r^{\alpha-4}v^2\right).$$

Finally, from (4.2.6)-(4.2.17) we obtain

$$(4.2.18) \quad \int\limits_{G_\varepsilon} r^{\alpha-2}|\nabla v|^2 dx + \frac{2-\alpha}{2}(N+\alpha-4)\int\limits_{G_\varepsilon} r^{\alpha-4}v^2 dx$$

$$\leq \varepsilon^{\alpha-2}\int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r}d\Omega_\varepsilon + \frac{\delta}{2}\int\limits_{G_\varepsilon} r^{\alpha-4}v^2 dx +$$

$$+ \frac{1}{2\delta}\int\limits_{G_\varepsilon} r^\alpha F^2(x)dx + \int\limits_{G_\varepsilon} \mathcal{A}(|x|)\left(r^\alpha|D^2v|^2 + r^{\alpha-2}|\nabla v|^2 + 2r^{\alpha-4}v^2\right)dx$$

for all $\delta > 0$.

Let us now estimate the last integral in (4.2.18). Due to the assumption (4.2.1) we have

$$(4.2.19) \qquad \forall \delta > 0 \quad \exists d > 0 \text{ such that } \mathcal{A}(r) < \delta \text{ for all } 0 < r < d.$$

Let $4\varepsilon < d$. From (4.2.11) and (4.2.12) follows that

$$\int\limits_{G_\varepsilon^{3\varepsilon}} r^\alpha|D^2v|^2 dx \leq c_4\varepsilon^{\alpha+N-4} + c_3\int\limits_{G_{\varepsilon/2}^{7/2\varepsilon}} r^\alpha F^2 dx,$$

and consequently

$$\int\limits_{G_\varepsilon} \mathcal{A}(r)r^\alpha|D^2v|^2 dx = \int\limits_{G_\varepsilon^{3\varepsilon}} \mathcal{A}(r)r^\alpha|D^2v|^2 dx + \int\limits_{G_{3\varepsilon}^d} \mathcal{A}(r)r^\alpha|D^2v|^2 dx +$$

$$+ \int\limits_{G_d} \mathcal{A}(r)r^\alpha|D^2v|^2 dx \leq c_4\mathcal{A}(3\varepsilon)\varepsilon^{\alpha+N-4} +$$

$$+ c_3\mathcal{A}(3\varepsilon)\int\limits_{G_{\varepsilon/2}^{7/2\varepsilon}} r^\alpha F^2 dx + \delta\int\limits_{G_\varepsilon^{2d}} (r^\alpha F^2(x) + r^{\alpha-4}v^2)dx$$

$$+ c_5\max_{r\in[d,\text{diam }G]}\mathcal{A}(r)\int\limits_{G_d} |D^2v|^2 dx$$

for all $\delta > 0$ and $0 < \varepsilon < d/4$. Here, c_5 depends only on α, d and $\operatorname{diam} G$.

Applying all this estimates to the inequality (4.2.18) we obtain

$$
(4.2.20) \quad \int\limits_{G_\varepsilon} r^{\alpha-2}|\nabla v|^2 dx + \frac{2-\alpha}{2}(N+\alpha-4)\int\limits_{G_\varepsilon} r^{\alpha-4}v^2 dx
$$

$$
\leq \varepsilon^{\alpha-2}\int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r}d\Omega_\varepsilon + c_6 \mathcal{A}(3\varepsilon)\left(\varepsilon^{\alpha-N-4} + \int\limits_{G_{\varepsilon/2}^{7/2\varepsilon}} r^\alpha F^2 dx\right) +
$$

$$
+ \delta \int\limits_{G_\varepsilon}(r^{\alpha-2}|\nabla v|^2 + r^{\alpha-4}v^2)dx + c_7\int\limits_{G_d}|D^2 v|^2 dx + c_8\int\limits_G r^\alpha F^2(x)dx
$$

for all $\delta > 0$ and $0 < \varepsilon < d/4$.

Finally, we apply Theorem 4.6 to the solution v of $(L)_0$ in G_d

$$
\int\limits_{G_d}|D^2 v|^2 dx \;\leq\; c_9\int\limits_{G_{d/2}}\left(v^2 + f^2 + |a^{ij}D_{ij}\Phi + a^i D_i\Phi + a\Phi|^2\right)dx
$$

$$
(4.2.21) \qquad\qquad \leq\; c_9\int\limits_{G_{d/2}}(v^2 + f^2)dx + c_{10}\|\varphi\|^2_{W^{3/2,2}(\partial G\setminus\Gamma_0^{d/2})}.
$$

Furthermore, if $(2-\alpha)(N+\alpha-4)=0$, then we apply the inequality (2.5.7).

Now, let $\delta > 0$ be small enough and $d > 0$ chosen according to (4.2.19). Then we obtain from (4.2.20) and (4.2.21) the estimate

$$
\int\limits_{G_\varepsilon}\left(r^\alpha|D^2 v|^2 + r^{\alpha-2}|\nabla v|^2 + r^{\alpha-4}v^2\right)dx \leq \varepsilon^{\alpha-2}\int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r}d\Omega_\varepsilon +
$$

$$
+ c_{11}\mathcal{A}(3\varepsilon)\left(\varepsilon^{\alpha+N-4} + \int\limits_{G_{\varepsilon/2}^{7/2\varepsilon}} r^\alpha F^2 dx\right) + c_{12}\left(\|v\|^2_{L^2(G)} + \|f\|^2_{\overset{\circ}{W}{}^0_\alpha(G)} + \right.
$$

$$
\left. + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_\alpha(\partial G)}\right)
$$

where the constants c_{11} and c_{12} do not depend on ε. Letting $\varepsilon \to +0$, applying Lemma 4.14 and noting that

$$
\|u\|_{\overset{\circ}{W}{}^2_\alpha(G)} \leq \|v\|_{\overset{\circ}{W}{}^2_\alpha(G)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_\alpha(\partial G)}
$$

we obtain the assertion of our theorem in the case I.

Case II: $4 - N - 2\lambda < \alpha < 4 - N$.

Due to the embedding theorem (see Lemma 1.37) we have

$$f \in \mathring{W}^0_{4-N}(G), \quad \varphi \in \mathring{W}^{3/2}_{4-N}(\partial G) \cap C^0(\partial G).$$

Therefore, by case I, $u \in \mathring{W}^2_{4-N}(G)$ and

$$\int\limits_G \left(r^{4-N}|D^2 u|^2 + r^{2-N}|\nabla u|^2 + r^{-N} u^2 \right) dx \leq \text{const.}$$

According to (4.2.10) with $\varrho = 2^{-k}d$, $k = 0, 1, 2, \ldots$, we have

$$\int\limits_{G^{3/2}_{1/2}} \left(|D'^2 w|^2 + |\nabla' w|^2 \right) dx' \leq c_{13} \int\limits_{G^2_{1/4}} \left(2^{-4k}d^4 F^2(x'2^{-k}d) + w^2 \right) dx'.$$

Multiplying both sides of this inequality by $(2^{-k}d+\varepsilon)^{\alpha-2}$ with $\varepsilon > 0$, taking into account that

$$2^{-k-1}d + \varepsilon < r + \varepsilon < 2^{-k}d + \varepsilon \quad \text{in} \quad G^{(k)}$$

and returning to the variables x we obtain

$$\int\limits_{G^{(k)}} r^2(r + \varepsilon)^{\alpha-2}|D^2 v| dx \leq c_{13} \int\limits_{G^{(k-1)} \cup G^k \cup G^{(k+1)}} \left(r^2(r + \varepsilon)^{\alpha-2} F^2 + \right.$$
$$\left. + r^{-2}(r + \varepsilon)^{\alpha-2} v^2 \right) dx.$$

Since $r_\varepsilon \leq r + \varepsilon \leq \frac{2}{h} r_\varepsilon$ in $\overline{G}$ with h defined as in Lemma 1.11, we obtain

$$(4.2.22) \quad \int\limits_{G^{(k)}} r^2 r_\varepsilon^{\alpha-2}|D^2 v| dx \leq c_{14} \int\limits_{G^{(k-1)} \cup G^k \cup G^{(k+1)}} \left(r^2 r_\varepsilon^{\alpha-2} F^2 + \right.$$
$$\left. + r^{-2} r_\varepsilon^{\alpha-2} v^2 \right) dx.$$

Summing up the inequalities (4.2.22) for $k = 0, 1, 2, \ldots$, we finally obtain

$$(4.2.23) \quad \int\limits_{G^d_0} r^2 r_\varepsilon^{\alpha-2}|D^2 v| dx \leq c_{14} \int\limits_{G^{2d}_0} \left(r^\alpha F^2 + r^{-2} r_\varepsilon^{\alpha-2} v^2 \right) dx,$$

since $\alpha \leq 2$ and $r_\varepsilon \geq hr$.

Let us return back to the equation $(L)_0$. Multiplying its both sides by $r_\varepsilon^{\alpha-2}v$ and integrating by parts twice we obtain (compare with case I)

$$(4.2.24) \quad \int\limits_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx = \frac{2-\alpha}{2}(4-N-\alpha)\int\limits_G r_\varepsilon^{\alpha-4}v^2 dx+$$

$$+ \int\limits_G r_\varepsilon^{\alpha-2}v\left(\left(a^{ij}(x)-a^{ij}(0)\right)D_{ij}v(x)+a^i(x)D_iv(x)+a(x)v(x)\right)dx-$$

$$-\int\limits_G r_\varepsilon^{\alpha-2}vF(x)dx.$$

By assumption (b) we obtain with the help of the Cauchy and the Hölder inequalities and the properties of the quasi–distance r_ε

$$r_\varepsilon^{\alpha-2}v\left(\sum_{i,j=1}^N\left(a^{ij}(x)-a^{ij}(0)\right)D_{ij}v(x)+\sum_{i=1}^N a^i(x)D_iv(x)+a(x)v(x)\right)$$

$$\leq c(h)\mathcal{A}(r)\left(r_\varepsilon^{\alpha-2}r^2|D^2v|^2+r_\varepsilon^{\alpha-2}|\nabla v|^2+r_\varepsilon^{\alpha-2}r^{-2}v^2\right)$$

and

$$r_\varepsilon^{\alpha-2}vF(x)\leq\frac{\delta}{2}r_\varepsilon^{\alpha-2}r^{-2}v^2+c(\delta,h)r^\alpha F^2,\quad\forall\delta>0.$$

Decomposing G into $G=G_0^d\cup G_d$, we then obtain from (4.2.24)

$$\int\limits_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx = c(h)\mathcal{A}(d)\int\limits_{G_0^d}\left(r_\varepsilon^{\alpha-2}r^2|D^2v|^2+r_\varepsilon^{\alpha-2}|\nabla v|^2+\right.$$

$$\left.+r_\varepsilon^{\alpha-2}r^{-2}v^2\right)dx + \frac{2-\alpha}{2}(4-N-\alpha)\int\limits_G r_\varepsilon^{\alpha-4}v^2 dx +$$

$$+\frac{\delta}{2}\int\limits_{G_0^d}r_\varepsilon^{\alpha-2}r^{-2}v^2 dx + c_{15}\int\limits_{G_d}\left(|D^2v|^2+v^2\right)dx$$

$$+c(\delta,h)\int\limits_G r^\alpha F^2(x)dx =: J_1+J_2+J_3+J_4+J_5$$

with an arbitrary $\delta>0$. Let us further estimate the right hand side of this inequality.
By the inequality (2.5.8),

$$J_1\leq\frac{2-\alpha}{2}(4-N-\alpha)H(\lambda,N,\alpha)\int\limits_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx.$$

Thus

$$C(\lambda, N, \alpha) \int\limits_G r_\varepsilon^{\alpha-2} |\nabla v|^2 dx \leq J_2 + J_3 + J_4 + J_5,$$

where

$$C(\lambda, N, \alpha) = 1 - \frac{2-\alpha}{2}(4 - N - \alpha) H(\lambda, N, \alpha).$$

The integrals J_2, J_3, J_4 and J_5 can be estimated using (4.2.23), (4.2.21) and Lemma 2.32 . In this way we obtain

$$
\begin{aligned}
C(\lambda, N, \alpha) \int\limits_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx \quad &\leq \quad c_{16}\left[A(d) + \delta\right] \int\limits_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx \\
&+ \quad c_{17}\left(\|v\|_{L^2(G)} + \|f\|_{\mathring{W}^0_\alpha(G)} + \|\varphi\|_{\mathring{W}^{3/2}_\alpha(\partial G)} \right)
\end{aligned}
$$

where $C(\lambda, N, \alpha) > 0$ due to assumption (4.2.2). Choosing $\delta > 0$ and $d > 0$ small enough and passing to the limits as $\varepsilon \to 0$, by the Fatou Theorem we obtain the assertion, if we recall (4.2.23).

The estimate (4.2.3) follows directly from Lemma 1.38.   □

REMARK 4.15. **On the belonging of weak solutions to $W^2(G)$.** Suppose that all assumptions of Theorem 4.13 are fulfilled with

$$a^{ij}(x) = \delta^j_i, \ x \in G, \ \forall i,j = 1, \ldots, N; \quad f \in L^2(G), \ \varphi \in \mathring{W}^{3/2}(\partial G) \cap C^0(\partial G).$$

We want study the regularity of a weak solution $u \in W^1(G)$. The following statement is valid.

PROPOSITION 4.16. A weak solution $u \in W^1(G)$ belongs to $W^2(G)$, if either

- $N \geq 4$;

or

- $N = 2$ and $0 < \omega_0 < \pi$;

or

- $N = 3$ and the domain G is *convex*;

or

- $N = 3$ and $\Omega \subset \Omega_0 = \{(\vartheta, \varphi) \big| 0 < |\vartheta| < \vartheta_0; \ 0 < \varphi < 2\pi\}$, where ϑ_0 is the smallest positive root of the Legendre function $P_{\frac{1}{2}}(\cos\vartheta)$.

PROOF. We apply Theorem 4.13 with $\alpha = 0$. Since $\lambda > 0$, then for $N \geq 4$, $\alpha = 0$ the assumption (4.2.2) of Theorem 4.13 is fulfilled and therefore we have

$$(4.2.25) \qquad\qquad u \in \mathring{W}^2_0(G) \implies u \in W^2(G).$$

If $N = 2$ and $0 < \omega_0 < \pi$, then the assumption (4.2.2) with $\alpha = 0$ of Theorem 4.13 is fulfilled too, since in this case we have that $\lambda > 1$. Therefore we have again (4.2.25).

Let now $N = 3$. If G is a convex domain, then it is well known (see e.g. Theorem 3 §2 chapter VI [87]) that $\lambda > 1$. Then the assumption (4.2.2) with $\alpha = 0$ of Theorem 4.13 is fulfilled and therefore (4.2.25) is valid. Let $G \subset \mathbb{R}^3$ be any domain and denote by $\Omega_0 \subset S^2$ the domain, in which the problem (EVD) is solvable for $\lambda = \frac{1}{2}$, thus

$$\begin{cases} \triangle_\omega \psi + \frac{1}{2}\left(1 + \frac{1}{2}\right)\psi = 0, & \omega \in \Omega_0, \\ \psi\big|_{\partial\Omega_0} = 0. \end{cases}$$

Now the assumption (4.2.2) with $\alpha = 0$ of Theorem 4.13 is fulfilled, if $\lambda > \frac{1}{2}$. Again in virtue of the monotony Theorem 3 §2 chapter VI [87] we have $\Omega \subset \Omega_0$. Let us reduce to the eigenvalue problem above. We shall look for the particular solution in the form $\psi = \psi(\vartheta)$. Then $\psi(\vartheta)$ is a solution of the Sturm-Liouville problem

$$\begin{cases} \frac{1}{\sin\vartheta} \cdot \frac{d}{d\vartheta}\left(\sin\vartheta \frac{d\psi}{d\vartheta}\right) + \frac{3}{4}\psi = 0, & |\vartheta| \le \vartheta_0, \\ \psi(-\vartheta_0) = \psi(\vartheta_0) = 0. \end{cases}$$

A solution of the equation of this problem is the Legendre function of first genus $\psi(\vartheta) = P_{1/2}(\cos\vartheta)$. This function has precisely one root on the interval $(0, \pi)$ (see e.g. example 39, page 158 [404]); we denote it by ϑ_0. $\quad\square$

THEOREM 4.17. *Let $u(x)$ be a strong solution of problem (L) and assumptions (a) and (b) are satisfied with $\mathcal{A}(r)$ Dini continuous at zero. Suppose*

$$\varphi(x) \in \overset{o}{W}{}^{3/2}_{4-N-2\lambda}(\partial G),$$

(4.2.26)

$$\iint\limits_{G} r^{4-N-2\lambda}\mathcal{H}^{-1}(r)f^2(x)dx + \int\limits_{\partial G} r^{1-N-2\lambda}\mathcal{H}^{-1}(r)\varphi^2(x)d\sigma < \infty,$$

where $\mathcal{H}(r)$ is a Dini-continuous at zero, monotone increasing function, λ is the smallest positive eigenvalue of problem (EVD) with (2.5.11).

Then $u(x) \in \overset{\circ}{W}{}^2_{4-N}(G)$ and

$$(4.2.27) \quad \|u\|^2_{\overset{\circ}{W}{}^2_{4-N}(G_0^\varrho)} \leq C\varrho^{2\lambda}\left(\|u\|^2_{2,G} + \iint\limits_G r^{4-N-2\lambda}\mathcal{H}^{-1}(r)f^2(x)dx+\right.$$

$$\left. + \int\limits_{\partial G} r^{1-N-2\lambda}\mathcal{H}^{-1}(r)\varphi^2(x)d\sigma + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N-2\lambda}(\partial G)}\right), \quad 0 < \varrho < d,$$

where the constant $C > 0$ depends only on $\nu, \mu, d, \mathcal{A}(d), \mathcal{H}(d), N, \lambda,$ meas G, and on the quantities $\int\limits_0^d \frac{\mathcal{A}(r)}{r}dr, \quad \int\limits_0^d \frac{\mathcal{H}(r)}{r}dr.$

PROOF. Since $u \in \overset{\circ}{W}{}^2_{4-N}(G)$ due to Theorem 4.13, it remains to prove (4.2.27). Let

$$U(\varrho) := \int\limits_{G_0^\varrho} r^{2-N}|\nabla u|^2 dx.$$

We write the equation (L) in the form

$$\Delta u(x) = f(x) - \left(a^{ij}(x) - a^{ij}(0)\right)D_{ij}u(x) - a^i(x)D_iu(x) - a(x)u(x),$$

multiply both sides by $r^{2-N}u$ and integrate over G_0^ϱ, $\varrho \in (0, d)$. As a result we obtain

$$(4.2.28) \quad U(\varrho) = \int\limits_{\Gamma_0^\varrho} r^{2-N}\varphi(x)\frac{\partial u}{\partial n}d\sigma + \int\limits_\Omega \left(\varrho u\frac{\partial u}{\partial r} + \frac{N-2}{2}u^2\right)d\Omega$$

$$+ \int\limits_{G_0^\varrho} r^{2-N}u(x)\left(\left(a^{ij}(x) - a^{ij}(0)\right)D_{ij}u(x)+\right.$$

$$\left. + a^i(x)D_iu(x) + a(x)u(x) - f(x)\right)dx.$$

We will estimate each integral on the right hand side of this equation from above. From Lemma 1.41 and Lemma 1.40 it follows by Cauchy's inequality

$$\int_{\Gamma_0^\varrho} r^{2-N} \varphi(x) \frac{\partial u}{\partial n} d\sigma = \int_{\Gamma_0^\varrho} \left(\sqrt{\mathcal{H}(r)} r^{(3-N)/2} \frac{\partial u}{\partial n} \right) \cdot \left(\mathcal{H}^{-1/2}(r) r^{(1-N)/2} \varphi(x) \right) d\sigma$$

$$(4.2.29) \qquad \leq \frac{\mathcal{H}(\varrho)}{2} \int_{\Gamma_0^\varrho} r^{3-N} \left(\frac{\partial u}{\partial n} \right)^2 d\sigma + \frac{1}{2} \int_{\Gamma_0^\varrho} \mathcal{H}^{-1}(r) r^{1-N} \varphi^2(x) d\sigma$$

$$\leq c_1 \mathcal{H}(\varrho) \|u\|^2_{\overset{\circ}{W}{}^2_{4-N}(G_0^\varrho)} + c_1 \int_{\Gamma_0^\varrho} \mathcal{H}^{-1}(r) r^{1-N} \varphi^2(x) d\sigma.$$

Moreover, as in the proof of Theorem 4.13 we have

$$\int_{G_0^\varrho} r^{2-N} u(x) \left(\left(a^{ij}(x) - a^{ij}(0) \right) D_{ij} u(x) + a^i(x) D_i u(x) + a(x) u(x) \right) dx \leq$$

$$(4.2.30) \qquad \leq \mathcal{A}(\varrho) \int_{G_0^\varrho} \left(r^{4-N} |D^2 u|^2 + r^{2-N} |\nabla u|^2 + 2 r^{-N} u^2 \right) dx$$

and

$$\int_{G_0^\varrho} r^{2-N} u(x) f(x) dx = \int_{G_0^\varrho} \left(\sqrt{\mathcal{H}(r)} r^{-N/2} u(x) \right) \cdot \left(\mathcal{H}^{-1/2}(r) r^{2-N/2} f(x) \right) dx$$

$$(4.2.31) \qquad \leq \frac{\mathcal{H}(\varrho)}{2} \int_{G_0^\varrho} r^{-N} u^2(x) dx + \frac{1}{2} \int_{G_0^\varrho} \mathcal{H}^{-1}(r) r^{4-N} f^2(x) dx.$$

Therefore, using (4.2.29)–(4.2.31) and Corollary 2.29, (2.5.8) from Corollary 2.25 we obtain from (4.2.28) the inequality

$$U(\varrho) \leq \frac{\varrho}{2\lambda} U'(\varrho) + \varepsilon(\varrho) \int_{G_0^\varrho} r^{4-N} |D^2 u|^2 dx + \delta(\varrho) U(\varrho) + \mathcal{F}(\varrho),$$

where

$$(4.2.32) \quad \begin{cases} \varepsilon(\varrho) = \mathcal{A}(\varrho) + c_1 \mathcal{H}(\varrho), \\ \delta(\varrho) = c_2(\lambda, N)(\mathcal{A}(\varrho) + \mathcal{H}(\varrho)), \\ \mathcal{F}(\varrho) = c_1 \int_{\Gamma_0^\varrho} \mathcal{H}^{-1}(r) r^{1-N} \varphi^2(x) d\sigma + \\ \qquad + \frac{1}{2} \int_{G_0^\varrho} \mathcal{H}^{-1}(r) r^{4-N} f^2(x) dx + \\ \qquad + c_2(\lambda, N)\left(\mathcal{A}(\varrho) + \mathcal{H}(\varrho) \right) \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma_0^\varrho)}. \end{cases}$$

Let us now estimate $\int\limits_{G_0^{\varrho}} r^{4-N}|D^2u|^2dx$. To this end we consider again the estimate (4.2.11) with ε replaced by $2^{-k}\varrho$. Summing up this inequalities for $k = 0, 1, \ldots$, we obtain

$$\int\limits_{G_0^{\varrho}} r^{4-N}|D^2u|^2dx \le c_3 \int\limits_{G_0^{2\varrho}} \left(r^{4-N}F^2(x) + r^{-N}u^2\right) dx + c_4\|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma_0^{2\varrho})}.$$

Inserting the definition (4.2.4) of F and applying (2.5.2) we then obtain

$$(4.2.33) \quad \int\limits_{G_0^{\varrho}} r^{4-N}|D^2u|^2dx \le c_5 \bigg(U(2\varrho) + \|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G_0^{2\varrho})} +$$

$$+ \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma_0^{2\varrho})} \bigg), \quad 0 < \varrho < d$$

and therefore

$$(4.2.34) \quad U(\varrho) \le \frac{\varrho}{2\lambda}U'(\varrho) + c_5\varepsilon(\varrho)U(2\varrho) + \delta(\varrho)U(\varrho) + \mathcal{F}(\varrho) +$$

$$+ c_5\varepsilon(\varrho) \left(\|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G_0^{2\varrho})} + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma_0^{2\varrho})} \right).$$

Moreover we have the initial condition (see the proof of Theorem 4.13)

$$U(d) = \int\limits_{G_0^d} r^{2-N}|\nabla u|^2dx \le c \left(\|u\|^2_{L^2(G)} + \|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\partial G)} \right)$$

$$\equiv V_0.$$

From (4.2.34) we obtain the differential inequality (CP) from §1.10 with

$$(4.2.35) \quad \begin{cases} \mathcal{P}(\varrho) = \frac{2\lambda}{\varrho}\left(1 - \delta(\varrho)\right); \\ \mathcal{N}(\varrho) = \frac{2\lambda}{\varrho}c_5\varepsilon(\varrho); \\ \mathcal{Q}(\varrho) = \frac{2\lambda}{\varrho}\mathcal{F}(\varrho) + c_6\frac{\varepsilon(\varrho)}{\varrho} \left(\|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G_0^{2\varrho})} + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma_0^{2\varrho})} \right). \end{cases}$$

Now we apply Theorem 1.57. For this we have

$$\int\limits_{\varrho}^{2\varrho} \mathcal{P}(s)ds = 2\lambda \ln 2 - 2\lambda \int\limits_{\varrho}^{2\varrho} \frac{\delta(s)}{s}ds \le 2\lambda \ln 2 \Rightarrow$$

$$(4.2.36) \quad \exp\left(\int\limits_{\varrho}^{2\varrho} \mathcal{P}(s)ds\right) = 2^{2\lambda}\exp\left(-2\lambda \int\limits_{\varrho}^{2\varrho} \frac{\delta(s)}{s}ds\right) \le 2^{2\lambda}.$$

Furthermore,

$$\mathcal{B}(\varrho) = \mathcal{N}(\varrho)\exp\left(\int\limits_{\varrho}^{2\varrho}\mathcal{P}(s)ds\right) \le 2^{2\lambda}\frac{2\lambda}{\varrho}c_5\varepsilon(\varrho) \Rightarrow$$

$$(4.2.37)\qquad \int\limits_{\varrho}^{d}\mathcal{B}(\tau)d\tau \le 2\lambda2^{2\lambda}c_5\int\limits_{0}^{d}\frac{\varepsilon(\tau)}{\tau}d\tau.$$

In addition,

$$\int\limits_{\varrho}^{d}\mathcal{P}(s)ds = 2\lambda\ln\frac{d}{\varrho} - 2\lambda\int\limits_{\varrho}^{d}\frac{\delta(s)}{s}ds \Rightarrow$$

$$\exp\left(-\int\limits_{\varrho}^{d}\mathcal{P}(s)ds\right) \le \left(\frac{\varrho}{d}\right)^{2\lambda}\exp\left(2\lambda\int\limits_{0}^{d}\frac{\delta(s)}{s}ds\right) \le c_7\left(\frac{\varrho}{d}\right)^{2\lambda}.$$

$(4.2.38)$

$$\exp\left(-\int\limits_{\varrho}^{\tau}\mathcal{P}(s)ds\right) \le \left(\frac{\varrho}{\tau}\right)^{2\lambda}\exp\left(2\lambda\int\limits_{0}^{d}\frac{\delta(s)}{s}ds\right) \le c_7\left(\frac{\varrho}{\tau}\right)^{2\lambda},$$

$$0 < \varrho < \tau < d.$$

Now by Theorem 1.57 from (1.10.1) by virtue of (4.2.38), and (4.2.37) we obtain

$$(4.2.39)\qquad U(\varrho) \le c_8\varrho^{2\lambda}\left\{V_0 + \int\limits_{\varrho}^{d}\tau^{-2\lambda}\mathcal{Q}(\tau)d\tau\right\},$$

where c_8 is a positive constant depending only on $N, \lambda, \int\limits_{0}^{d}\frac{\mathcal{A}(s)+\mathcal{H}(s)}{s}ds$. We now have to estimate $\int\limits_{\varrho}^{d}\tau^{-2\lambda}\mathcal{Q}(\tau)d\tau$. For this we recall (4.2.35) and therefore we obtain

$$(4.2.40)\qquad \int\limits_{\varrho}^{d}\tau^{-2\lambda}\mathcal{Q}(\tau)d\tau \le 2\lambda\int\limits_{\varrho}^{d}\tau^{-2\lambda-1}\mathcal{F}(\tau)d\tau +$$

$$+ c_6\varepsilon(d)\int\limits_{\varrho}^{d}\tau^{-2\lambda-1}\left(\|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\tau}_0)} + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma^{2\tau}_0)}\right)d\tau$$

Now, by changing the order of integration in virtue of the Fubini Theorem in the integral

$$\int\limits_{\varrho}^{d} \tau^{-2\lambda-1}\Big(\int\limits_{0}^{\tau} r^{\alpha}\mathcal{K}(r)dr\Big)d\tau = \int\limits_{0}^{\varrho} r^{\alpha}\mathcal{K}(r)\Big(\int\limits_{\varrho}^{d} \tau^{-2\lambda-1}d\tau\Big)dr +$$

$$+ \int\limits_{\varrho}^{d} r^{\alpha}\mathcal{K}(r)\Big(\int\limits_{r}^{d} \tau^{-2\lambda-1}d\tau\Big)dr = \frac{\varrho^{-2\lambda} - d^{-2\lambda}}{2\lambda}\int\limits_{0}^{\varrho} r^{\alpha}\mathcal{K}(r)dr +$$

$$+ \frac{1}{2\lambda}\int\limits_{\varrho}^{d} r^{\alpha}\mathcal{K}(r)\big(r^{-2\lambda} - d^{-2\lambda}\big)dr \leq \frac{1}{2\lambda}\int\limits_{0}^{\varrho} r^{\alpha}\varrho^{-2\lambda}\mathcal{K}(r)dr +$$

$$+ \frac{1}{2\lambda}\int\limits_{\varrho}^{d} r^{\alpha-2\lambda}\mathcal{K}(r)dr \leq \frac{1}{2\lambda}\int\limits_{0}^{d} r^{\alpha-2\lambda}\mathcal{K}(r)dr,$$

we find

1) $$\int\limits_{\varrho}^{d} \tau^{-2\lambda-1}\Big(\int\limits_{G_0^\tau} r^{4-N}\mathcal{H}^{-1}(r)f^2(x)dx\Big)d\tau \leq$$

$$\leq \frac{1}{2\lambda}\int\limits_{G_0^d} r^{4-N-2\lambda}\mathcal{H}^{-1}(r)f^2(x)dx.$$

2) $$\int\limits_{\varrho}^{d} \tau^{-2\lambda-1}\Big(\int\limits_{\Gamma_0^\tau} r^{1-N}\mathcal{H}^{-1}(r)\varphi^2(x)d\sigma\Big)d\tau \leq$$

$$\leq \frac{1}{2\lambda}\int\limits_{\Gamma_0^d} r^{1-N-2\lambda}\mathcal{H}^{-1}(r)\varphi^2(x)d\sigma.$$

In the same way we find

3) $$\int\limits_{\varrho}^{d} \tau^{-2\lambda-1}\|\varphi\|^2_{\overset{o}{W}^{3/2}_{4-N}(\Gamma_0^{2\tau})} d\tau \leq \frac{1}{2\lambda}\|\varphi\|^2_{\overset{o}{W}^{3/2}_{4-N-2\lambda}(\Gamma_0^{2d})}.$$

4) $$\int\limits_{\varrho}^{d} \tau^{-2\lambda-1}\|f\|^2_{\overset{o}{W}^{0}_{4-N}(G_0^{2\tau})} d\tau \leq \frac{1}{2\lambda}\|f\|^2_{\overset{o}{W}^{0}_{4-N-2\lambda}(G_0^{2d})}$$

Hence and from (4.2.39) and (4.2.35) it follows that

$$(4.2.41) \quad U(\varrho) \le C\varrho^{2\lambda}\left(\|u\|_{2,G}^2 + \int\limits_G r^{4-N-2\lambda}\mathcal{H}^{-1}(r)f^2(x)dx + \right.$$

$$\left. + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N-2\lambda}(\partial G)}^2 + \int\limits_{\partial G} r^{1-N-2\lambda}\mathcal{H}^{-1}(r)\varphi^2(x)d\sigma \right), \quad 0 < \varrho < d.$$

At last we apply (4.2.33) and deduce from (4.2.41) the validity of (4.2.27). $\quad\square$

THEOREM 4.18. *Let $u(x)$ be a strong solution of problem (L) and assumptions (a) and (b) are satisfied with $\mathcal{A}(r)$ Dini continuous at zero. Suppose*

$$f \in \overset{\circ}{W}{}^{0}_{4-N}(G) \quad \varphi(x) \in \overset{\circ}{W}{}^{3/2}_{4-N}(\partial G) \cap C^0(\partial G)$$

and there exist real numbers $s > 0$, $k_s \ge 0$ such that

$$(4.2.42) \qquad k_s =: \sup_{\varrho > 0}\varrho^{-s}\left(\|f\|_{\overset{\circ}{W}{}^{0}_{4-N}(G_0^\varrho)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma_0^\varrho)} \right).$$

Then there are $d \in (0, \frac{1}{e})$ and a constant $C > 0$ depends only on ν, $\mu, d, \mathcal{A}(d), N, s, \lambda, \text{meas} G$, and on the quantity $\int\limits_0^d \frac{\mathcal{A}(r)}{r}dr$ such that $\forall \varrho \in (0, d)$

$$(4.2.43) \quad \|u\|_{\overset{\circ}{W}{}^{2}_{4-N}(G_0^\varrho)} \le C\left(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}{}^{0}_{4-N}(G)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(\partial G)} + k_s \right) \times$$

$$\times \begin{cases} \varrho^\lambda, & \text{if } s > \lambda, \\ \varrho^\lambda \ln^{3/2}\left(\frac{1}{\varrho}\right), & \text{if } s = \lambda, \\ \varrho^s, & \text{if } s < \lambda. \end{cases}$$

PROOF. We consider the function $v = u - \Phi$ as a solution of homogeneous problem $(L)_0$ in the form (4.2.5) with (4.2.4). Multiplying both sides of (4.2.5) by $r^{2-N}v$ and integrating over G_0^ϱ, we obtain

$$(4.2.44) \quad \int\limits_{G_0^\varrho} r^{2-N}v\triangle v dx = -\int\limits_{G_0^\varrho}\left(r^{2-N}\left(a^{ij}(x) - a^{ij}(0)\right)vv_{x_ix_j} + \right.$$

$$\left. + r^{2-N}a^iv_{x_i}v + r^{2-N}a(x)v^2 \right)dx + \int\limits_{G_0^\varrho} r^{2-N}vF(x)dx$$

Integrating by parts twice we show that

$$(4.2.45) \quad \int\limits_{G_0^\varrho} r^{2-N} v \triangle v dx = \int\limits_{\Omega} \left(\varrho v \frac{\partial v}{\partial r} + \frac{N-2}{2} v^2 \right) d\Omega - \int\limits_{G_0^\varrho} r^{2-N} |\nabla v|^2 dx$$

We define

$$V(\varrho) := \int\limits_{G_0^\varrho} r^{2-N} |\nabla v|^2 dx.$$

Because of (b), Corollary 2.29, (2.5.3) and the Cauchy inequality we obtain for $\forall \delta > 0$

$$(4.2.46) \quad V(\varrho) \le \frac{\varrho}{2\lambda} V'(\varrho) + c\mathcal{A}(\varrho) \int\limits_{G_0^\varrho} r^{4-N} v_{xx}^2 dx +$$

$$+ c\mathcal{A}(\varrho) V(\varrho) + \frac{\delta}{2} V(\varrho) + \frac{1}{2\delta} \left(\|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G_0^{2\varrho})} + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma_0^{2\varrho})} \right).$$

If we take into account (4.2.42), by (4.2.33), we get

$$(4.2.47) \quad V(\varrho) \le \frac{\varrho}{2\lambda} V'(\varrho) + c_1 \mathcal{A}(\varrho) V(2\varrho) + c_2 \big(\mathcal{A}(\varrho) + \delta \big) V(\varrho) +$$

$$+ c_3 \frac{1}{\delta} k_s^2 \varrho^{2s}, \; \forall \delta > 0, \; 0 < \varrho < d.$$

1) $\underline{s > \lambda}$

Choosing $2\lambda c_2 \delta = \varrho^\varepsilon$, $\forall \varepsilon > 0$ we obtain from (4.2.47) the problem (CP) §1.10 with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} - 2\lambda c_2 \frac{\mathcal{A}(\varrho)}{\varrho} - \varrho^{\varepsilon-1}, \quad \mathcal{N}(\varrho) = 2\lambda c_1 \frac{\mathcal{A}(\varrho)}{\varrho} \text{ and}$$

$$\mathcal{Q}(\varrho) = k_s^2 c_4 \varrho^{2s-1-\varepsilon}.$$

Now we have

$$\int\limits_\varrho^d \mathcal{P}(\tau) d\tau = 2\lambda \ln \frac{d}{\varrho} - 2\lambda c_2 \int\limits_\varrho^d \frac{\mathcal{A}(\tau)}{\tau} d\tau - \frac{d^\varepsilon - \varrho^\varepsilon}{\varepsilon} \Rightarrow$$

$$\exp \left(\int\limits_\varrho^{2\varrho} \mathcal{P}(\tau) d\tau \right) \le 2^{2\lambda}, \quad \int\limits_\varrho^d \mathcal{B}(\tau) d\tau \le 2^{2\lambda+1} \lambda c_1 \int\limits_0^d \frac{\mathcal{A}(\tau)}{\tau} d\tau$$

and

$$\exp\left(-\int_{\varrho}^{d}\mathcal{P}(\tau)d\tau\right) \le \left(\frac{\varrho}{d}\right)^{2\lambda}\exp\left(2\lambda c_2\int_{0}^{d}\frac{\mathcal{A}(\tau)}{\tau}d\tau\right)\exp\left(\varepsilon^{-1}d^{\varepsilon}\right) = c_5\left(\frac{\varrho}{d}\right)^{2\lambda}$$

if we recall (1.10.2).

In this case we have as well

$$\int_{\varrho}^{d}\mathcal{Q}(\tau)\exp\left(-\int_{\varrho}^{\tau}\mathcal{P}(\sigma)d\sigma\right)d\tau \le k_s^2 c_4 c_5 \varrho^{2\lambda}\int_{\varrho}^{d}\tau^{2s-2\lambda-\varepsilon-1}d\tau \le k_s^2 c_6 \varrho^{2\lambda},$$

since $s > \lambda$.

Now we apply Theorem 1.57. From (1.10.1) by virtue of deduced inequalities and with regard to (4.2.33), we obtain the first statement of (4.2.43).

2) $s < \lambda$

In this case we have from (4.2.47) the problem (CP) §1.10 with

$$\mathcal{P}(\varrho) = \frac{2\lambda(1-\delta)}{\varrho} - 2\lambda c_2\frac{\mathcal{A}(\varrho)}{\varrho}; \quad \mathcal{N}(\varrho) = 2\lambda c_1\frac{\mathcal{A}(\varrho)}{\varrho};$$

$$\mathcal{Q}(\varrho) = k_s^2 c_8 \delta^{-1}\varrho^{2s-1}, \ \forall\delta > 0.$$

Now similarly to the case 1) we have

$$\exp\left(\int_{\varrho}^{2\varrho}\mathcal{P}(\tau)d\tau\right) \le 2^{2\lambda(1-\delta)}, \quad \int_{\varrho}^{d}\mathcal{B}(\tau)d\tau \le 2^{2\lambda+1}c_1\int_{0}^{d}\frac{\mathcal{A}(\tau)}{\tau}d\tau \quad \text{and}$$

$$\exp\left(-\int_{\varrho}^{d}\mathcal{P}(\tau)d\tau\right) \le \left(\frac{\varrho}{d}\right)^{2\lambda(1-\delta)}\exp\left(2\lambda c_2\int_{0}^{d}\frac{\mathcal{A}(\tau)}{\tau}d\tau\right) = c_9\left(\frac{\varrho}{d}\right)^{2\lambda(1-\delta)},$$

if we recall (1.10.2).

In this case we have as well

$$\int_{\varrho}^{d}\mathcal{Q}(\tau)\exp\left(-\int_{\varrho}^{\tau}\mathcal{P}(\sigma)d\sigma\right)d\tau \le k_s^2 c_{10}\delta^{-1}\varrho^{2\lambda(1-\delta)}\int_{\varrho}^{d}\tau^{2s-2\lambda(1-\delta)-1}d\tau \le$$

$$\le k_s^2 c_{11}\varrho^{2s},$$

if we choose $\delta \in (0, \frac{\lambda-s}{\lambda})$.

Now we apply Theorem 1.57. From (1.10.1) by virtue of deduced inequalities we obtain

$$V(\varrho) \le c_{12}\big(V_0 \varrho^{2\lambda(1-\delta)} + k_s^2 \varrho^{2s}\big) \le c_{13}(V_0 + k_s^2)\varrho^{2s},$$

because of the choice of δ. Taking into account of (4.2.33) we deduced the third statement of (4.2.43).

3) $s = \lambda$

As in the proof of Theorem 4.17 we consider the function $U(\varrho)$ satisfying the equation (4.2.28). We will estimate each integral on the right hand side of this equation from above. From Lemma 1.41 and Lemma 1.40 it follows by the Hölder inequality for integrals

$$\int\limits_{\Gamma_0^\varrho} r^{2-N}\varphi(x)\frac{\partial u}{\partial n}d\sigma \;=\; \int\limits_{\Gamma_0^\varrho}\Big(r^{(3-N)/2}\frac{\partial u}{\partial n}\Big)\cdot\Big(r^{(1-N)/2}\varphi(x)\Big)d\sigma \;\le$$

$$(4.2.48) \qquad \le \; \Bigg(\int\limits_{\Gamma_0^\varrho} r^{3-N}\Big(\frac{\partial u}{\partial n}\Big)^2 d\sigma\Bigg)^{1/2}\cdot\Bigg(\int\limits_{\Gamma_0^\varrho} r^{1-N}\varphi^2(x)d\sigma\Bigg)^{1/2} \;\le$$

$$\le\; c_1\|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(G_0^\varrho)}\|u\|_{\overset{\circ}{W}{}^{2}_{4-N}(G_0^\varrho)} + c_2\|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(G_0^\varrho)} \;\le$$

$$\le\; c_1 k_s \varrho^\lambda \|u\|_{\overset{\circ}{W}{}^{2}_{4-N}(G_0^\varrho)} + c_2 k_s^2 \varrho^{2\lambda}$$

in virtue of the assumption (4.2.42). In the same way

$$(4.2.49) \qquad \int\limits_{G_0^\varrho} r^{2-N}u(x)f(x)dx = \int\limits_{G_0^\varrho}\Big(r^{-N/2}u(x)\Big)\cdot\Big(r^{2-N/2}f(x)\Big)dx \;\le$$

$$\le\; cU^{1/2}(\varrho)\|f\|_{\overset{\circ}{W}{}^{0}_{4-N}(G_0^\varrho)} \;\le\; ck_s\varrho^\lambda U^{1/2}(\varrho).$$

Moreover, as in the proof of Theorem 4.13 we have

$$(4.2.50) \qquad \int\limits_{G_0^\varrho} r^{2-N}u(x)\Big(\big(a^{ij}(x) - a^{ij}(0)\big)D_{ij}u(x) + a^i(x)D_i u(x) +$$

$$+ a(x)u(x)\Big)dx \le \mathcal{A}(\varrho)\|u\|^2_{\overset{\circ}{W}{}^{2}_{4-N}(G_0^\varrho)}.$$

Therefore, using (4.2.48)–(4.2.50) and Corollary 2.29 we obtain from (4.2.28) the inequality

$$U(\varrho) \le \frac{\varrho}{2\lambda} U'(\varrho) + \mathcal{A}(\varrho) \|u\|^2_{\overset{\circ}{W}{}^2_{4-N}(G_0^\varrho)} + c_1 k_s \varrho^\lambda \|u\|_{\overset{\circ}{W}{}^2_{4-N}(G_0^\varrho)} +$$
$$+ k_s \varrho^\lambda U^{1/2}(\varrho) + c_2 k_s^2 \varrho^{2\lambda}.$$

Now we apply the inequality (4.2.33). As a result we obtain

$$(4.2.51) \quad U(\varrho) \le \frac{\varrho}{2\lambda} U'(\varrho) + \big(\mathcal{A}(\varrho) + \delta(\varrho)\big) U(2\varrho) + \mathcal{A}(\varrho) U(\varrho) +$$
$$+ c_3 k_s^2 \delta^{-1}(\varrho) \varrho^{2\lambda}, \quad \forall \delta(\varrho) > 0.$$

Moreover we have the initial condition (see the proof of Theorem 4.13)

$$U(d) = \int_{G_0^d} r^{2-N} |\nabla u|^2 dx \le c \left(\|u\|^2_{L^2(G)} + \|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{4-N}(\partial G)} \right)$$
$$\equiv V_0.$$

From (4.2.47) we obtain the differential inequality (CP) from §1.10 with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} - 2\lambda \frac{\mathcal{A}(\varrho)}{\varrho}, \quad \mathcal{N}(\varrho) = 2\lambda \frac{\mathcal{A}(\varrho) + \delta(\varrho)}{\varrho} \quad \text{and}$$
$$\mathcal{Q}(\varrho) = 2c_3 \lambda k_s^2 \delta^{-1}(\varrho) \varrho^{2\lambda-1}, \quad \forall \delta(\varrho) > 0.$$

We choose

$$\delta(\varrho) = \frac{1}{\lambda 2^{2\lambda+1} \ln\left(\frac{ed}{\varrho}\right)}, \quad 0 < \varrho < d,$$

where e is the Euler number. Since according to the assumption of theorem $\mathcal{A}(\varrho)$ is Dini-continuous at zero, then we have

$$\exp\left(\int_\varrho^{2\varrho} \mathcal{P}(\tau) d\tau \right) \le 2^{2\lambda},$$

$$\exp\left(\int_\varrho^d \mathcal{B}(\tau) d\tau \right) \le \exp\left(C(\lambda) \int_0^d \frac{\mathcal{A}(\tau)}{\tau} d\tau \right) \ln\left(\frac{ed}{\varrho}\right), \quad \text{and}$$

$$-\int_\varrho^d \mathcal{P}(\tau) d\tau \le \ln\left(\frac{\varrho}{d}\right)^{2\lambda} + 2\lambda \int_0^d \frac{\mathcal{A}(\tau)}{\tau} d\tau \Rightarrow$$

$$\exp\left(-\int_\varrho^d \mathcal{P}(\tau) d\tau \right) \le \left(\frac{\varrho}{d}\right)^{2\lambda} \exp\left(C(\lambda) \int_0^d \frac{\mathcal{A}(\tau)}{\tau} d\tau \right)$$

if we recall (1.10.2). In this case we have as well

$$\int\limits_{\varrho}^{d} \mathcal{Q}(\tau) \exp\left(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma) d\sigma\right) d\tau \le k_s^2 C(\lambda) \varrho^{2\lambda} \int\limits_{\varrho}^{d} \ln\left(\frac{ed}{\tau}\right) \frac{d\tau}{\tau} \le$$

$$\le k_s^2 C(\lambda) \varrho^{2\lambda} \ln^2\left(\frac{ed}{\varrho}\right).$$

Now we apply Theorem 1.57. From (1.10.1) by virtue of deduced inequalities we obtain

$$(4.2.52) \qquad U(\varrho) \le C(V_0 + k_s^2) \varrho^{2\lambda} \ln^3 \frac{1}{\varrho}, \quad 0 < \varrho < d < \frac{1}{e}.$$

Taking into account of (4.2.33) we deduce the second statement of (4.2.43). □

Both the following theorems and examples from Section 4.2.5 show that assumptions about the smoothness of the coefficients of (L), that is *Dini continuity* at zero of the function $\mathcal{A}(r)$ from hypothesis (b) Theorems 4.17 and 4.18 are essential for their validity.

THEOREM 4.19. *Let $u(x)$ be a strong solution of problem (L) and assumptions (a) and (b) are satisfied with $\mathcal{A}(r)$, which is a continuous at zero function, but not Dini continuous at zero. Suppose*

$$f(x) \in \overset{\circ}{W}{}_{4-N}^{0}(G), \quad \varphi(x) \in \overset{\circ}{W}{}_{4-N}^{3/2}(\partial G) \cap C^0(\partial G)$$

and there exist real numbers $s > 0$, $k_s \ge 0$ such that

$$(4.2.53) \qquad k_s =: \sup_{\varrho > 0} \varrho^{-s} \left(\|f\|_{\overset{\circ}{W}{}_{4-N}^{0}(G_0^\varrho)} + \|\varphi\|_{\overset{\circ}{W}{}_{4-N}^{3/2}(\Gamma_0^\varrho)} \right).$$

Then for $\forall \varepsilon > 0$ there are $d \in (0,1)$ and a constant $C_\varepsilon > 0$ depends only on $\nu, \mu, d, s, N, \varepsilon, \lambda, \text{meas } G$ such that $\forall \varrho \in (0, d)$

$$(4.2.54) \quad \|u\|_{\overset{\circ}{W}{}_{4-N}^{2}(G_0^\varrho)} \le C_\varepsilon \left(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}{}_{4-N}^{0}(G)} + \|\varphi\|_{\overset{\circ}{W}{}_{4-N}^{3/2}(\partial G)} + \right.$$

$$\left. + k_s \right) \cdot \begin{cases} \varrho^{\lambda - \varepsilon}, & \text{if } s > \lambda, \\ \varrho^{s - \varepsilon}, & \text{if } s \le \lambda. \end{cases}$$

PROOF. As above in Theorem 4.18 we find (4.2.47), through the Cauchy inequality, we get the problem (CP) §1.10 with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho}\left(1 - \frac{\delta}{2} - C_8 \mathcal{A}(\varrho)\right), \quad \forall \delta > 0; \quad \mathcal{N}(\varrho) = 2\lambda C_8 \frac{\mathcal{A}(\varrho)}{\varrho} \text{ and}$$

$$\mathcal{Q}(\varrho) = k_s^2 C_{20} \varrho^{2s-1}.$$

Therefore we have

$$-\int_\varrho^d \mathcal{P}(\tau)d\tau = 2\lambda(1 - \frac{\delta}{2})\ln\frac{\varrho}{d} + 2\lambda C_8 \int_\varrho^d \frac{\mathcal{A}(\tau)}{\tau}d\tau.$$

Now we apply the mean value theorem for integrals

$$\int_\varrho^d \frac{\mathcal{A}(\tau)}{\tau}d\tau \le \mathcal{A}(d)\ln\frac{d}{\varrho}$$

and choose $d > 0$ by continuity of $\mathcal{A}(r)$ so that $2C_8\mathcal{A}(d) < \delta$. Thus we obtain

$$\exp\left(-\int_\varrho^d \mathcal{P}(\tau)d\tau\right) \le \left(\frac{\varrho}{d}\right)^{2\lambda(1-\delta)}, \quad \forall\delta > 0$$

Similarly we have

$$\exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)d\sigma\right) \le \left(\frac{\varrho}{\tau}\right)^{2\lambda(1-\delta)}, \quad \forall\delta > 0.$$

Further it is obvious that

$$\int_\varrho^{2\varrho} \mathcal{P}(\tau)d\tau \le 2\lambda\ln 2$$

and with regard to (1.10.2)

$$\int_\varrho^d \mathcal{B}(\tau)d\tau \le 2\lambda 2^{2\lambda}C_8 \int_\varrho^d \frac{\mathcal{A}(\tau)}{\tau}d\tau \le 2\lambda 2^{2\lambda}C_8\mathcal{A}(d)\ln\frac{d}{\varrho} \le \delta\lambda 2^{2\lambda}\ln\frac{d}{\varrho} \Rightarrow$$

$$\exp\left(\int_\varrho^d \mathcal{B}(\tau)d\tau\right) \le \left(\frac{\varrho}{d}\right)^{-\delta\lambda 2^{2\lambda}}, \quad \forall\delta > 0.$$

Hence by (1.10.1) of Theorem 1.57 we deduce

$$(4.2.55)\quad U(\varrho) \le \left(\frac{\varrho}{d}\right)^{-\delta\lambda 2^{2\lambda}}\left\{V_0\left(\frac{\varrho}{d}\right)^{2\lambda(1-\delta)} + \right.$$

$$\left. + \int_\varrho^d \mathcal{Q}(\tau)\exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)d\sigma\right)d\tau\right\}, \quad \forall\delta > 0.$$

Now we estimate the last integral

$$\int\limits_{\varrho}^{d} \mathcal{Q}(\tau)\exp\left(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)d\sigma\right)d\tau\Bigg\} \leq k_s^2 C_{20}\varrho^{2\lambda(1-\delta)} \int\limits_{\varrho}^{d} \tau^{2s-2\lambda(1-\delta)-1}d\tau =$$

(4.2.56)

$$= k_s^2 C_{20}\varrho^{2\lambda(1-\delta)}\frac{d^{2s-2\lambda(1-\delta)} - \varrho^{2s-2\lambda(1-\delta)}}{2s - 2\lambda(1-\delta)} \leq k_s^2 C_{21}\begin{cases}\varrho^{2\lambda(1-\delta)}, & \text{if } s \geq \lambda \\ \varrho^{2s}, & \text{if } s < \lambda.\end{cases}$$

In this connection we choose $\delta > 0$ so that $\delta \neq \frac{\lambda-s}{\lambda}$.

From (4.2.55) and (4.2.56) and because of (4.2.33), the desired estimate (4.2.54) follows.   □

We can now correct Theorem 4.19 in the case $s = \lambda$, if $\mathcal{A}(r) \sim \frac{1}{\ln\frac{1}{r}}$.

THEOREM 4.20. *Let $u(x)$ be a strong solution of problem (L) and assumptions (a) and (b) are satisfied with $\mathcal{A}(r) \sim \frac{1}{\ln\frac{1}{r}}$, $\mathcal{A}(0) = 0$. Suppose that*

$$f(x) \in \overset{\circ}{W}{}^{0}_{4-N}(G) \text{ and } \varphi(x) \in \overset{\circ}{W}{}^{3/2}_{4-N}(\partial G)$$

and that there exist real number $k_\lambda \geq 0$ such that

(4.2.57) $$k_\lambda =: \sup_{\varrho>0}\varrho^{-\lambda}\left(\|f\|_{\overset{\circ}{W}{}^{0}_{4-N}(G_0^\varrho)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma_0^\varrho)}\right).$$

Then there are $d \in (0, \frac{1}{e})$ and constants $C > 0$, $c > 0$ depends only on $\nu, \mu, d, N, \lambda, measG$ such that

(4.2.58) $$\|u\|_{\overset{\circ}{W}{}^{2}_{4-N}(G_0^\varrho)} \leq C\left(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}{}^{0}_{4-N}(G)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(\partial G)} +\right.$$

$$\left. + k_\lambda\right)\cdot\varrho^\lambda\ln^{c+1}\frac{1}{\varrho}, \quad 0 < \varrho < d.$$

PROOF. As above in Theorem 4.18 we obtain the problem (CP) §1.10 with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} - 2\lambda c_5\frac{\mathcal{A}(\varrho)}{\varrho}, \quad \mathcal{N}(\varrho) = 2\lambda c_5\frac{\mathcal{A}(\varrho)}{\varrho} + \lambda\frac{\delta(\varrho)}{\varrho} \text{ and}$$

$$\mathcal{Q}(\varrho) = c(\lambda)k_\lambda^2\left(1 + \delta^{-1}(\varrho)\right)\varrho^{2\lambda-1}.$$

We choose

$$\delta(\varrho) = \frac{1}{2\lambda\ln\left(\frac{ed}{\varrho}\right)}, \quad 0 < \varrho < d,$$

where e is the Euler number. Since according to the assumption of theorem $\mathcal{A}(\varrho) \sim \delta(\varrho)$ for suitable small $d > 0$, we then have:

$$\exp\left(\int_{\varrho}^{2\varrho} \mathcal{P}(\tau)d\tau\right) \leq 2^{2\lambda}, \quad \exp\left(\int_{\varrho}^{d} \mathcal{B}(\tau)d\tau\right) \leq C(d,\lambda) \ln^{c(\lambda)}\left(\frac{ed}{\varrho}\right) \text{ and}$$

$$-\int_{\varrho}^{d} \mathcal{P}(\tau)d\tau \leq \ln\left(\frac{\varrho}{d}\right)^{2\lambda} + 2\lambda c_5 \int_{\varrho}^{d} \frac{d\tau}{\tau \ln\left(\frac{ed}{\tau}\right)} = \ln\left(\frac{\varrho}{d}\right)^{2\lambda} + 2\lambda c_5 \ln\ln\left(\frac{ed}{\varrho}\right) \Rightarrow$$

$$\exp\left(-\int_{\varrho}^{d} \mathcal{P}(\tau)d\tau\right) \leq \left(\frac{\varrho}{d}\right)^{2\lambda} \ln^c\left(\frac{ed}{\varrho}\right),$$

if we recall (1.10.2). In this case we have as well

$$\int_{\varrho}^{d} \mathcal{Q}(\tau) \exp\left(-\int_{\varrho}^{\tau} \mathcal{P}(\sigma)d\sigma\right)d\tau \leq k_\lambda^2 C(\lambda)\varrho^{2\lambda} \int_{\varrho}^{d} \frac{1+\delta^{-1}(\tau)}{\tau} \ln^c\left(\frac{e\tau}{\varrho}\right)d\tau \leq$$

$$\leq k_\lambda^2 C(\lambda)\varrho^{2\lambda} \ln^{c+2}\left(\frac{ed}{\varrho}\right).$$

Now we apply Theorem 1.57. From (1.10.1) by virtue of deduced inequalities we obtain

$$(4.2.59) \qquad U(\varrho) \leq C_{25}(V_0 + k_\lambda^2)\varrho^{2\lambda} \ln^{2c+2} \frac{1}{\varrho}, \quad 0 < \varrho < d < \frac{1}{e}.$$

From (4.2.59) and because of (4.2.33) the desired estimate (4.2.58) follows.

$\square$

4.2.2. The power modulus of continuity.

THEOREM 4.21. *Let $u \in W^{2,N}(G) \cap C^0(\overline{G})$ be a strong solution of problem (L) and assumptions (a) and (b) are satisfied with $\mathcal{A}(r)$ Dini continuous at zero. Suppose, in addition,*

$$a^i \in L^p(G), \ p > N; \ a \in L^N(G), \ f \in L^N(G) \cap \overset{\circ}{W}{}^0_{4-N}(G),$$

$$\varphi(x) \in \overset{\circ}{W}{}^{3/2}_{4-N}(\partial G) \cap V^{2-1/N}_{N,0}(\partial G) \cap C^\lambda(\partial G)$$

and there exist real numbers $s > 0$, $k_s \geq 0$, $k \geq 0$ such that

$$(4.2.60) \qquad k_s =: \sup_{\varrho>0} \varrho^{-s}\left(\|f\|_{\overset{\circ}{W}{}^0_{4-N}(G^\varrho_0)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(\Gamma^\varrho_0)}\right),$$

$$(4.2.61) \qquad k =: \sup_{\varrho>0} \varrho^{1-s}\left(\|f\|_{N;G^{2\varrho}_{\varrho/4}} + \|\varphi\|_{V^{2-1/N}_{N,0}(\Gamma^{2\varrho}_{\varrho/4})}\right).$$

Then there are $d \in (0, \frac{1}{e})$ and a constant $C > 0$ depends only on $\nu, \mu, d, s,$

$N, \lambda, \text{meas } G$ and on the quantity $\int\limits_0^d \frac{\mathcal{A}(r)}{r} dr$ such that $\forall x \in G_0^d$

$$(4.2.62) \quad |u(x)| \le C \Big(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}{}^{0}_{4-N}(G)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(\partial G)} + $$

$$+ |\varphi|_{C^\lambda(\partial G)} + k_s + k \Big) \cdot \begin{cases} |x|^\lambda, & \text{if } s > \lambda, \\ |x|^\lambda \ln^{3/2}\left(\frac{1}{|x|}\right), & \text{if } s = \lambda, \\ |x|^s, & \text{if } s < \lambda. \end{cases}$$

PROOF. Let the functions Φ, v and F be defined as in the proof of Theorem 4.13. We remark that $\Phi(0) = 0$ due to Lemma 1.38.

Let us introduce the function

$$(4.2.63) \qquad \psi(\varrho) = \begin{cases} \varrho^\lambda, & \text{if } s > \lambda, \\ \varrho^\lambda \ln^{3/2}\left(\frac{1}{\varrho}\right), & \text{if } s = \lambda, \\ \varrho^s, & \text{if } s < \lambda, \end{cases}$$

for $0 < \varrho < d$ and consider two sets $G^{2\varrho}_{\varrho/4}$ and $G^\varrho_{\varrho/2} \subset G^{2\varrho}_{\varrho/4}$, $\varrho > 0$. We make the transformation $x = \varrho x'$; $v(\varrho x') = \psi(\varrho) w(x')$. The function $w(x')$ satisfies the problem

$$\begin{cases} a^{ij}(\varrho x') w_{x'_i x'_j} + \varrho a^i(\varrho x') w_{x'_i} + \varrho^2 a(\varrho x') w = \frac{\varrho^2}{\psi(\varrho)} F(\varrho x'), & x' \in G^2_{1/4} \\ w(x') = 0, & x' \in \Gamma^2_{1/4}, \end{cases}$$

where

$$\frac{\varrho^2}{\psi(\varrho)} F(\varrho x') = \frac{\varrho^2}{\psi(\varrho)} f(\varrho x') - \frac{1}{\psi(\varrho)} \Big(a^{ij}(\varrho x') \Phi_{x'_i x'_j} + \varrho a^i(\varrho x') \Phi_{x'_i} + $$

$$+ \varrho^2 a(\varrho x') \Phi(\varrho x') \Big) \le \frac{\varrho^2}{\psi(\varrho)} |f| + \frac{\mu}{\psi(\varrho)} |\Phi_{x'x'}| + \frac{\mathcal{A}(\varrho)}{\psi(\varrho)} (|\nabla' \Phi| + |\Phi|).$$

Let us now note that

$$\int\limits_{G^2_{1/4}} \left(\left| \varrho \Big(\sum_{i=1}^N |a^i(\varrho x')|^2 \Big)^{1/2} \right|^p + \big(\varrho^2 |a(\varrho x')| \big)^N \right) dx' \le$$

$$\le c(N, p) \int\limits_{G^{2\varrho}_{\varrho/4}} \frac{\mathcal{A}^p(r) + \mathcal{A}^N(r)}{r^N} dx \le c(N, p) \mathcal{A}^{N-1}(d) \text{ meas } \Omega \int\limits_0^{2d} \frac{\mathcal{A}(r)}{r} dr$$

and

$$\frac{\varrho^2}{\psi(\varrho)}\|F(\varrho x')\|_{L^N(G^2_{1/4})} \leq c(\mu, \mathcal{A}(d))\frac{\varrho}{\psi(\varrho)}\left(\|f\|_{L^N(G^{2\varrho}_{\varrho/4})} + \|\varphi\|_{V^{2-1/N}_{N,0}(\Gamma^{2\varrho}_{\varrho/4})}\right)$$

$$(4.2.64) \qquad\qquad \leq kc(\mu, \mathcal{A}(d))\frac{\varrho^s}{\psi(\varrho)} \leq k \cdot const(\mu, \mathcal{A}(d), s, \lambda, d),$$

because of (4.2.61) and (4.2.63). We apply now Theorem 4.5 (Local Maximum Principle). Because of the proven estimates we have

$$(4.2.65) \quad \sup_{G^1_{1/2}} |w(x')| \leq C(N, \nu, \mu)\left\{\left(\iint_{G^2_{1/4}} w^2 dx'\right)^{\frac{1}{2}} + \right.$$

$$\left. + \frac{\varrho^2}{\psi(\varrho)}\left(\iint_{G^2_{1/4}} |F|^N dx'\right)^{\frac{1}{N}}\right\}.$$

Returning back to the variable x and the function $v(x)$ by Theorem 4.18 with (4.2.63), we obtain:

$$(4.2.66) \quad \iint_{G^2_{1/4}} w^2 dx' = \frac{1}{\psi^2(\varrho)}\iint_{G^{2\varrho}_{\varrho/4}} r^{-N}v^2 dx \leq$$

$$\leq C\left(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(\partial G)} + k_s\right)^2.$$

Because of $\varphi \in C^\lambda(\partial G)$ we then obtain

$$|u(x)| \leq |v(x)| + |\Phi(x)| \leq |v| + |\Phi(x) - \Phi(0)| \leq |v| + |x|^\lambda|\varphi|_{\lambda,\partial G}$$

Hence and from (4.2.64), (4.2.65) and (4.2.66) it follows

$$\sup_{G^\varrho_{\varrho/2}} |u(x)| \leq C\left(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\varphi\|_{\overset{\circ}{W}{}^{3/2}_{4-N}(\partial G)} + \right.$$

$$\left. + |\varphi|_{\lambda,\partial G} + k_s + k\right)\psi(\varrho).$$

Putting now $|x| = \frac{2}{3}\varrho$ we obtain finally the desired estimate (4.2.62). $\qquad\square$

THEOREM 4.22. *Let $u(x)$ be a strong solution of problem (L) and assumptions (a) and (b) are satisfied with $\mathcal{A}(r)$, which is a continuous at zero function, but not Dini continuous at zero. Suppose, in addition,*

$$a^i \in L^p(G), \ p > N; \ a \in L^N(G), \ f \in L^N(G) \cap \overset{\circ}{W}{}^{0}_{4-N}(G),$$

$$\varphi \in \overset{\circ}{W}{}^{3/2}_{4-N}(\partial G) \cap V^{2-1/N}_{N,0}(\partial G) \cap C^\lambda(\partial G)$$

and that there exist real numbers $s > 0$, $k_s \geq 0$, $k \geq 0$ such that

$$(4.2.67) \qquad k_s =: \sup_{\varrho > 0} \varrho^{-s} \left(\|f\|_{\overset{\circ}{W}^{0}_{4-N}(G_0^\varrho)} + \|\varphi\|_{\overset{\circ}{W}^{3/2}_{4-N}(\Gamma_0^\varrho)} \right);$$

$$(4.2.68) \qquad k =: \sup_{\varrho > 0} \varrho^{1-s} \left(\|f\|_{N;G_{\varrho/4}^{2\varrho}} + \|\varphi\|_{V^{2-1/N}_{N,0}(\Gamma^{2\varrho}_{\varrho/4})} \right).$$

Then $\forall \varepsilon > 0$ there are $d \in (0,1)$ and a constant $C_\varepsilon > 0$ depends only on $\nu, \mu, d, s, \varepsilon, N, \lambda,$ meas G and on $\mathcal{A}(diam G)$ such that $\forall x \in G_0^d$

$$(4.2.69) \quad |u(x)| \leq C_\varepsilon \Big(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}^{0}_{4-N}(G)} + \|\varphi\|_{\overset{\circ}{W}^{3/2}_{4-N}(\partial G)} +$$

$$+ |\varphi|_{\lambda, \partial G} + k_s + k \Big) \cdot \begin{cases} |x|^{\lambda - \varepsilon}, & \text{if } s > \lambda, \\ |x|^{s-\varepsilon}, & \text{if } s \leq \lambda. \end{cases}$$

PROOF. We repeat verbatim the proof of Theorem 4.21 by taking

$$\psi(\varrho) = \begin{cases} \varrho^{\lambda - \varepsilon}, & \text{if } s > \lambda, \\ \varrho^{s-\varepsilon}, & \text{if } s \leq \lambda, \end{cases}$$

and applying Theorem 4.19. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

THEOREM 4.23. *Let* $u(x)$ *be a strong solution of problem* (L) *and assumptions* (a) *and* (b) *are satisfied with* $\mathcal{A}(r) \sim \frac{1}{\ln \frac{1}{r}}$, $\mathcal{A}(0) = 0$. *Suppose, in addition,*

$$a^i \in L^p(G),\ p > N;\ a \in L^N(G),\ f \in L^N(G) \cap \overset{\circ}{W}^{0}_{4-N}(G),$$

$$\varphi(x) \in \overset{\circ}{W}^{3/2}_{4-N}(\partial G) \cap V^{2-1/N}_{N,0}(\partial G) \cap C^\lambda(\partial G)$$

and that there exist real numbers $k_\lambda \geq 0$, $k \geq 0$ *such that*

$$(4.2.70) \qquad k_\lambda =: \sup_{\varrho > 0} \varrho^{-\lambda} \left(\|f\|_{\overset{\circ}{W}^{0}_{4-N}(G_0^\varrho)} + \|\varphi\|_{\overset{\circ}{W}^{3/2}_{4-N}(\Gamma_0^\varrho)} \right),$$

$$(4.2.71) \qquad k =: \sup_{\varrho > 0} \varrho^{1-\lambda} \left(\|f\|_{N;G^{2\varrho}_{\varrho/4}} + \|\varphi\|_{V^{2-1/N}_{N,0}(\Gamma^{2\varrho}_{\varrho/4})} \right).$$

Then there are $d \in (0, \frac{1}{e})$ *and constants* $C > 0, c > 0$ *depends only on* $\nu, \mu, d, N, \lambda,$ *meas* G *and on* $\mathcal{A}(diam G)$, *such that* $\forall x \in G_0^d$

$$(4.2.72) \quad |u(x)| \leq C \Big(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}^{0}_{4-N}(G)} + \|\varphi\|_{\overset{\circ}{W}^{3/2}_{4-N}(\partial G)} +$$

$$+ |\varphi|_{\lambda, \partial G} + k_\lambda + k \Big) \cdot |x|^\lambda \ln^{c+1} \frac{1}{|x|}.$$

PROOF. We repeat verbatim the proof of Theorem 4.21 by taking

$$\psi(\varrho) = \varrho^\lambda \ln^{c+1} \frac{1}{\varrho}$$

and applying Theorem 4.20.   □

4.2.3. L^p-estimates.

In this and the next sections we establish the exact smoothness of strong solutions of (L).

Let u be a strong solution of (L), where $p > N$, and let

$$f \in L^p(G), \quad \varphi \in W^{2-1/p,p}(\partial G).$$

Let us consider the sets $G_{\varrho/4}^{2\varrho}$ and $G_{\varrho/2}^\varrho \subset G_{\varrho/4}^{2\varrho}$ and new variables x' defined by $x = \varrho x'$. Then the function $z(x') = v(\varrho x') = v(x)$ satisfies in $G_{1/4}^2$ the problem

$$(4.2.73) \quad \begin{cases} a^{ij}(\varrho x')\frac{\partial^2 z}{\partial x_i' \partial x_j'} + \varrho a^i(\varrho x')\frac{\partial z}{\partial x_i'} + \varrho^2 a(\varrho x')z = \varrho^2 f(\varrho x') - \\ \quad - \left(a^{ij}(\varrho x')\frac{\partial^2 \Phi}{\partial x_i' \partial x_j'} + \varrho a^i(\varrho x')\frac{\partial \Phi}{\partial x_i'} + \varrho^2 a(\varrho x')\Phi \right), \quad x' \in G_{1/4}^2 \\ w(x') = 0, \quad x' \in \Gamma_{1/4}^2, \end{cases}$$

where the functions Φ, v be defined as in the proof of Theorem 4.13.

By the Sobolev Imbedding Theorems 1.33 and 1.34 we have

$$\sup_{\substack{x',y' \in G_{1/2}^1 \\ x' \neq y'}} \frac{|z(x') - z(y')|}{|x' - y'|^\beta} \leq c\|z\|_{W^{2,N}(G_{1/2}^1)}, \quad \forall \beta \in (0,1) \quad \text{and}$$

$$(4.2.74)$$

$$\sup_{x' \in G_{1/2}^1} |\nabla' z(x')| + \sup_{\substack{x',y' \in G_{1/2}^1 \\ x' \neq y'}} \frac{|\nabla' z(x') - \nabla' z(y')|}{|x' - y'|^{1-N/p}} \leq c\|z\|_{W^{2,p}(G_{1/2}^1)}, \quad p > N.$$

By the local L^p- *a priori* estimate (Theorem 4.6) for solutions of (4.2.73) we obtain

$$(4.2.75) \quad \|z\|_{W^{2,p}(G_{1/2}^1)} \leq c(N,\nu,\mu,A(2))\Big\{ \|z\|_{L^p(G_{1/4}^2)} + \varrho^2\|f\|_{L^p(G_{1/4}^2)} + $$

$$+ \|\varphi\|_{W^{2-1/p,p}(\partial G_{1/4}^2)} \Big\}.$$

Returning to the variables x, from (4.2.74), (4.2.75) it follows that

$$(4.2.76) \quad \sup_{\substack{x,y \in G^{\varrho}_{\varrho/2} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x-y|^{\beta}} \leq c\varrho^{-\beta} \big\{ \varrho^{-1} \|v\|_{L^{N}(G^{2\varrho}_{\varrho/4})} + \|f\|_{V^{0}_{N,N}(G^{2\varrho}_{\varrho/4})} +$$

$$+ \|\varphi\|_{V^{2-1/N}_{N,N}(\Gamma^{2\varrho}_{\varrho/4})} \big\}, \quad \forall \beta \in (0,1),$$

$$(4.2.77) \quad \sup_{G^{\varrho}_{\varrho/2}} |\nabla v| \leq c\varrho^{-1} \big\{ \varrho^{-N/p} \|v\|_{L^{p}(G^{2\varrho}_{\varrho/4})} + \|f\|_{V^{0}_{p,2p-N}(G^{2\varrho}_{\varrho/4})} +$$

$$+ \|\varphi\|_{V^{2-1/p}_{p,2p-N}(\Gamma^{2\varrho}_{\varrho/4})} \big\}$$

and

$$(4.2.78) \quad \sup_{\substack{x,y \in G^{\varrho}_{\varrho/2} \\ x \neq y}} \frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1-N/p}} \leq c\varrho^{\frac{N}{p}-2} \big\{ \varrho^{-N/p} \|v\|_{L^{p}(G^{2\varrho}_{\varrho/4})} +$$

$$+ \|f\|_{V^{0}_{p,2p-N}(G^{2\varrho}_{\varrho/4})} + \|\varphi\|_{V^{2-1/p}_{p,2p-N}(\Gamma^{2\varrho}_{\varrho/4})} \big\}.$$

Moreover, if we rewrite the inequality (4.2.75) in the equivalent form

$$\int_{G^{1}_{1/2}} \left(|D'^{2}z|^{p} + |\nabla'z|^{p} + |z|^{p} \right) dx' \leq c \int_{G^{2}_{1/4}} \left(|z|^{p} + r^{2p}|f|^{p} + \right.$$

$$\left. + |D'^{2}\Phi|^{p} + |\nabla'\Phi|^{p} + |\Phi|^{p} \right) dx',$$

multiply both sides of this inequality by $\varrho^{\alpha-2p}$ and return to the variables x, then we obtain

$$\int_{G^{\varrho}_{\varrho/2}} \left(r^{\alpha}|D^{2}v|^{p} + r^{\alpha-p}|\nabla v|^{p} + r^{\alpha-2p}|v|^{p} \right) dx \leq$$

$$\leq c \int_{G^{2\varrho}_{\varrho/4}} \left(r^{\alpha-2p}|v|^{p} + r^{\alpha}|f|^{p} + r^{\alpha}|D^{2}\Phi|^{p} + r^{\alpha-p}|\nabla\Phi|^{p} + r^{\alpha-2p}|\Phi|^{p} \right) dx$$

and consequently

$$(4.2.79) \quad \|v\|_{V^{2}_{p,\alpha}(G^{\varrho}_{\varrho/2})} \leq c \big\{ \|v\|_{V^{0}_{p,\alpha-2p}(G^{2\varrho}_{\varrho/4})} + \|f\|_{V^{0}_{p,\alpha}(G^{2\varrho}_{\varrho/4})} +$$

$$+ \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\Gamma^{2\varrho}_{\varrho/4})} \big\}.$$

THEOREM 4.24. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.21 be satisfied. Furthermore, we suppose that*

$$f \in V^{0}_{p,\alpha}(G), \quad \varphi \in V^{2-1/p}_{p,\alpha}(\partial G), \quad p > N$$

with

$$\alpha > \begin{cases} (2-\lambda)p - N, & if \quad s > \lambda, \\ (2-s)p - N, & if \quad s \le \lambda \end{cases}$$

and there is some constant $k_2 \ge 0$ such that

$$(4.2.80) \qquad k_2 =: \sup_{\varrho > 0} \frac{\|f\|_{V^0_{p,\alpha}(G^\varrho_{\varrho/2})} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\Gamma^\varrho_{\varrho/2})}}{\chi(\varrho)},$$

where

$$(4.2.81) \qquad \chi(\varrho) \equiv \begin{cases} \varrho^{\lambda - 2 + \frac{\alpha+N}{p}}, & if \quad s > \lambda, \\ \varrho^{\lambda - 2 + \frac{\alpha+N}{p}} \ln^{3/2} \frac{1}{\varrho}, & if \quad s = \lambda, \\ \varrho^{s - 2 + \frac{\alpha+N}{p}}, & if \quad s < \lambda \end{cases}$$

for all sufficiently small $\varrho > 0$.

Then $u \in V^2_{p,\alpha}(G)$ and the estimate

$$(4.2.82) \qquad \|u\|_{V^2_{p,\alpha}(G^\varrho_0)} \le c\chi(\varrho)$$

holds with c independent of u.

PROOF. The statement of theorem follows from (4.2.79), since

$$\|v\|_{V^0_{p,\alpha-2p}(G^{2\varrho}_{\varrho/4})} = \left(\int_{G^{2\varrho}_{\varrho/4}} r^{\alpha-2p}|v|^p dx \right)^{1/p} \le \text{\scriptsize in virtue of (4.2.62), (4.2.63)}$$

$$\le c\psi(\varrho) \left(\int_{\varrho/4}^{2\varrho} r^{\alpha-2p+N-1} dr \right)^{1/p} \le c\varrho^{-2+\frac{\alpha+N}{p}} \psi(\varrho) = c\chi(\varrho).$$

Hence and from (4.2.79), (4.2.80) replacing ϱ by $2^{-k}\varrho$, we have

$$\|u\|_{V^2_{p,\alpha-2p}(G^{(k)})} \le c\chi(2^{-k}\varrho).$$

By summing these inequalities over all $k = 0, 1, \cdots$ we obtain our desired assertion. $\square$

In similar way we prove following theorems.

THEOREM 4.25. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.22 be satisfied. Furthermore, we suppose that*

$$f \in V^0_{p,\alpha}(G), \quad \varphi \in V^{2-1/p}_{p,\alpha}(\partial G), \quad p > N$$

with

$$\alpha > \begin{cases} (2-\lambda)p - N, & if \quad s > \lambda, \\ (2-s)p - N, & if \quad s \le \lambda \end{cases}$$

and there is some constant $k_2 \geq 0$ such that

$$(4.2.83) \quad \|f\|_{V^0_{p,\alpha}(G^\varrho_{\varrho/2})} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\Gamma^\varrho_{\varrho/2})} \leq k_2 \begin{cases} \varrho^{\lambda-2-\varepsilon+\frac{\alpha+N}{p}}, & \text{if } s > \lambda, \\ \varrho^{s-2-\varepsilon+\frac{\alpha+N}{p}}, & \text{if } s \leq \lambda \end{cases}$$

for all sufficiently small $\varrho > 0$ and $\forall \varepsilon > 0$.

Then $u \in V^2_{p,\alpha}(G)$ and the estimate

$$(4.2.84) \quad \|u\|_{V^2_{p,\alpha}(G^\varrho_0)} \leq c_\varepsilon \begin{cases} \varrho^{\lambda-2-\varepsilon+\frac{\alpha+N}{p}}, & \text{if } s > \lambda, \\ \varrho^{s-2-\varepsilon+\frac{\alpha+N}{p}}, & \text{if } s \leq \lambda \end{cases}$$

holds with c_ε independent of u.

THEOREM 4.26. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.23 be satisfied. Furthermore, we suppose that*

$$f \in V^0_{p,\alpha}(G), \quad \varphi \in V^{2-1/p}_{p,\alpha}(\partial G), \quad p > N, \quad \alpha > (2-\lambda)p - N$$

and there is some constant $k_2 \geq 0$ such that

$$(4.2.85) \quad \|f\|_{V^0_{p,\alpha}(G^\varrho_{\varrho/2})} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\Gamma^\varrho_{\varrho/2})} \leq k_2 \varrho^{\lambda-2+\frac{\alpha+N}{p}} \ln^{c+1} \frac{1}{\varrho}$$

for all sufficiently small $\varrho > 0$, where c is defined by Theorem 4.23.

Then $u \in V^2_{p,\alpha}(G)$ and the estimate

$$(4.2.86) \quad \|u\|_{V^2_{p,\alpha}(G^\varrho_0)} \leq C \varrho^{\lambda-2+\frac{\alpha+N}{p}} \ln^{c+1} \frac{1}{\varrho}$$

holds with C independent of u.

4.2.4. C^λ-estimates. Let known be that

$$|v(x)| \leq c_0 \psi(|x|), \quad x \in G^d_0.$$

Then we have

$$(4.2.87) \quad \varrho^{-1} \|v\|_{L^N(G^{2\varrho}_{\varrho/4})} \leq c_1 \psi(\varrho),$$

$$(4.2.88) \quad \varrho^{-\frac{N}{p}} \|v\|_{L^p(G^{2\varrho}_{\varrho/4})} \leq c_2 \psi(\varrho),$$

$$(4.2.89) \quad \varrho^{-2} \|v\|_{L^p(G^{2\varrho}_{\varrho/4})} \leq c_3 \varrho^{\frac{N}{p}-2} \psi(\varrho).$$

THEOREM 4.27. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.21 be satisfied. Let $\lambda = 1$. Then*

$$(4.2.90) \quad u \in \begin{cases} C^\beta(\overline{G}), & \forall \beta \in (0,1) & \text{if } s \geq 1 \\ C^s(\overline{G}), & & \text{if } 0 < s < 1. \end{cases}$$

PROOF. From (4.2.76) and (4.2.87), (4.2.61) it follows

$$(4.2.91) \qquad \sup_{\substack{x,y\in G^{\varrho}_{\varrho/2} \\ x\neq y}} \frac{|v(x) - v(y)|}{|x - y|^{\beta}} \leq c\varrho^{-\beta}\psi(\varrho), \quad \forall \beta \in (0,1),$$

where in our case $\psi(\varrho)$ is defined by (4.2.63). By Theorem 4.21 it follows that

$$(4.2.92) \qquad \sup_{\substack{x,y\in G^{\varrho}_{\varrho/2} \\ x\neq y}} \frac{|v(x) - v(y)|}{|x - y|^{\beta}} \leq c \begin{cases} \varrho^{1-\beta}, & \text{if } s > 1, \\ \varrho^{1-\beta-\varepsilon}, & \text{if } s = 1, \\ \varrho^{s-\beta}, & \text{if } s < 1, \end{cases}$$

$\forall \varepsilon > 0, \quad \forall \beta \in (0,1)$.

By definition of the set $G^{\varrho}_{\varrho/2}$ we have $|x - y| \leq 2\varrho$ and therefore from (4.2.92) it follows that

$$(4.2.93) \quad |v(x) - v(y)| \leq c|x - y|^{\beta} \begin{cases} \varrho^{1-\beta}, & \text{if } s > 1, \\ \varrho^{1-\beta-\varepsilon}, & \text{if } s = 1, \leq \\ \varrho^{s-\beta}, & \text{if } s < 1, \end{cases}$$

$$\leq c \begin{cases} |x - y|^{\beta}, & \text{if } s \geq 1, \\ |x - y|^{s}, & \text{if } s < 1 \end{cases} \forall \beta \in (0,1), \quad \forall x,y \in G^{\varrho}_{\varrho/2}.$$

If $|x - y| \geq \varrho = |x|$, then from Theorem 4.21 it follows that

$$(4.2.94) \quad \frac{|v(x) - v(y)|}{|x - y|^{\beta}} \leq 2|v(x)||x - y|^{-\beta} \leq 2c\psi(\varrho)\varrho^{-\beta} \leq$$

$$\leq c \begin{cases} \varrho^{1-\beta}, & \text{if } s > 1, \\ \varrho^{1-\beta-\varepsilon}, & \text{if } s = 1, \leq const, \\ \varrho^{s-\beta}, & \text{if } s < 1, \end{cases}$$

if we choose $\beta = s$ for $0 < s < 1$. Together with $\varphi \in C^{\lambda}$ we prove our theorem. $\qquad \square$

By repeating verbatim the proof of the previous theorem we obtain the next theorems.

THEOREM 4.28. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.22 be satisfied. Let $\lambda = 1$. Then*

$$(4.2.95) \qquad u \in \begin{cases} C^{\beta}(\overline{G}), & \forall \beta \in (0,1) \quad \text{if } s \geq 1, \\ C^{s-\varepsilon}(\overline{G}), & \forall \varepsilon > 0, \quad \text{if } 0 < s < 1. \end{cases}$$

PROOF. In this case we have (4.2.91) with $\psi(\varrho)$ from Theorem 4.22, that is

$$(4.2.96) \qquad \sup_{\substack{x,y \in G^{\varrho}_{\varrho/2} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^{\beta}} \leq c \begin{cases} \varrho^{1-\beta-\varepsilon}, & \text{if } s \geq 1, \\ \varrho^{s-\beta-\varepsilon}, & \text{if } s < 1, \end{cases}$$

$\forall \varepsilon > 0, \quad \forall \beta \in (0,1)$.

Hence follows our statement, if we choose $\varepsilon = 1 - \beta$ for $s \geq 1$ and $\beta = s - \varepsilon$ for $0 < s < 1$. $\qquad \square$

THEOREM 4.29. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.23 be satisfied. Let $\lambda = 1$.* *Then*

$$(4.2.97) \qquad u \in C^{1-\varepsilon}(\overline{G}), \quad \forall \varepsilon > 0.$$

PROOF. In this case we have (4.2.91) with $\psi(\varrho)$ from Theorem 4.23, that is

$$(4.2.98) \qquad \sup_{\substack{x,y \in G^{\varrho}_{\varrho/2} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^{\beta}} \leq c \varrho^{1-\beta-\varepsilon}, \quad \forall \varepsilon > 0, \quad \forall \beta \in (0,1).$$

Hence the desired statement follows, if we choose $\beta = 1 - \varepsilon$. $\qquad \square$

THEOREM 4.30. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.21 be satisfied. Let $0 < \lambda < 1$.* *Then*

$$(4.2.99) \qquad u \in \begin{cases} C^{\lambda}(\overline{G}), & \text{if } s > \lambda, \\ C^{\lambda-\varepsilon}(\overline{G}), & \forall \varepsilon > 0, \quad \text{if } s = \lambda, \\ C^{s}(\overline{G}), & \text{if } 0 < s < \lambda. \end{cases}$$

PROOF. By Theorem 4.21 from (4.2.91) it follows

$$(4.2.100) \qquad \sup_{\substack{x,y \in G^{\varrho}_{\varrho/2} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^{\beta}} \leq c \begin{cases} \varrho^{\lambda-\beta}, & \text{if } s > \lambda, \\ \varrho^{\lambda-\beta-\varepsilon}, & \text{if } s = \lambda, \\ \varrho^{s-\beta}, & \text{if } s < \lambda, \end{cases}$$

$\forall \varepsilon > 0, \quad \forall \beta \in (0,1)$.

Putting

$$\beta = \begin{cases} \lambda, & \text{if } s > \lambda, \\ \lambda - \varepsilon, & \text{if } s = \lambda, \\ s, & \text{if } 0 < s < \lambda, \end{cases}$$

we obtain the required assertion. $\qquad \square$

THEOREM 4.31. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.22 be satisfied. Let $0 < \lambda < 1$. Then*

$$(4.2.101) \qquad u \in \begin{cases} C^{\lambda-\varepsilon}(\overline{G}), & \forall \varepsilon > 0, \quad \text{if } s \geq \lambda, \\ C^{s-\varepsilon}(\overline{G}), & \text{if } 0 < s < \lambda. \end{cases}$$

PROOF. By Theorem 4.22 from (4.2.91) it follows

$$(4.2.102) \qquad \sup_{\substack{x,y \in G^{\varrho}_{\varrho/2} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^{\beta}} \leq c \begin{cases} \varrho^{\lambda-\beta-\varepsilon}, & \text{if } s \geq \lambda, \\ \varrho^{s-\beta-\varepsilon}, & \text{if } s < \lambda, \end{cases}$$

$\forall \varepsilon > 0, \quad \forall \beta \in (0,1)$.
Putting

$$\beta = \begin{cases} \lambda - \varepsilon, & \text{if } s \geq \lambda, \\ s - \varepsilon, & \text{if } 0 < s < \lambda, \end{cases}$$

we obtain the required assertion. □

THEOREM 4.32. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.23 be satisfied. Let $0 < \lambda < 1$. Then*

$$(4.2.103) \qquad u \in C^{\lambda-\varepsilon}(\overline{G}), \quad \forall \varepsilon > 0.$$

PROOF. By Theorem 4.23 from (4.2.91) it follows

$$(4.2.104) \qquad \sup_{\substack{x,y \in G^{\varrho}_{\varrho/2} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^{\beta}} \leq c\varrho^{\lambda-\beta-\varepsilon}$$

$\forall \varepsilon > 0, \quad \forall \beta \in (0,1)$.
Putting

$$\beta = \lambda - \varepsilon,$$

we obtain the required assertion. □

Now, let we will fulfill **Assumption** (bb):
There exists some constant $k \geq 0$ such that

$$k =: \sup_{\varrho > 0} \frac{\|f\|_{V^0_{p,2p-N}(G^{2\varrho}_{\varrho/4})} + \|\varphi\|_{V^{2-1/p}_{p,2p-N}(\Gamma^{2\varrho}_{\varrho/4})}}{\psi(\varrho)}, \quad p > N.$$

Then from (4.2.88), (4.2.77), and (4.2.78) we obtain

(4.2.105)
$$\sup_{G^{\varrho}_{\varrho/2}} |\nabla v| \le c\varrho^{-1}\psi(\varrho),$$

(4.2.106)
$$\sup_{\substack{x,y\in G^{\varrho}_{\varrho/2} \\ x\ne y}} \frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1-N/p}} \le c\varrho^{\frac{N}{p}-2}\psi(\varrho).$$

THEOREM 4.33. *Let u be a strong solution of the boundary value problem* (L) *and let the assumptions of Theorem 4.21 and* (bb) *with $\psi(\varrho)$ defined by* (4.2.63) *be satisfied. Then it is true for the next estimate*

(4.2.107)
$$|\nabla u(x)| \le c \begin{cases} |x|^{\lambda-1}, & \text{if } s > \lambda, \\ |x|^{\lambda-1}\ln^{3/2}\frac{1}{|x|}, & \text{if } s = \lambda, \\ |x|^{s-1}, & \text{if } s < \lambda. \end{cases}$$

Moreover,

- 1) *if $\lambda \ge 2 - \frac{N}{p}$, then*

$$u \in \begin{cases} C^{2-\frac{N}{p}}(\overline{G}), & \text{if } s > \lambda, \\ C^{2-\frac{N}{p}-\varepsilon}(\overline{G}), & \forall \varepsilon > 0, \quad \text{if } s = \lambda, \\ C^{s-\lambda+2-\frac{N}{p}}(\overline{G}), & \text{if } \lambda - 1 + \frac{N}{p} \le s < \lambda. \end{cases}$$

- 2) *if $1 < \lambda \le 2 - \frac{N}{p}$, then*

$$u \in \begin{cases} C^{\lambda}(\overline{G}), & \text{if } s > \lambda, \\ C^{\lambda-\varepsilon}(\overline{G}), & \forall \varepsilon > 0, \quad \text{if } s = \lambda, \\ C^{s}(\overline{G}), & \text{if } 1 \le s < \lambda. \end{cases}$$

PROOF. From (4.2.105), (4.2.106) with (4.2.63) we obtain

(4.2.108)
$$\sup_{G^{\varrho}_{\varrho/2}} |\nabla v| \le c \begin{cases} \varrho^{\lambda-1}, & \text{if } s > \lambda, \\ \varrho^{\lambda-1}\ln^{3/2}\frac{1}{\varrho}, & \text{if } s = \lambda, \\ \varrho^{s-1}, & \text{if } s < \lambda, \end{cases}$$

(4.2.109)

$$\sup_{\substack{x,y\in G^{\varrho}_{\varrho/2} \\ x\ne y}} \frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1-N/p}} \le c \begin{cases} \varrho^{\frac{N}{p}-2+\lambda}, & \text{if } s > \lambda, \\ \varrho^{\frac{N}{p}-2+\lambda-\varepsilon}, & \forall \varepsilon > 0, \quad \text{if } s = \lambda, \\ \varrho^{\frac{N}{p}-2+s}, & \text{if } s < \lambda. \end{cases}$$

Putting $|x| = \frac{2}{3}\varrho$ we obtain from (4.2.108) the (4.2.107).

Now we set

$$\kappa = \begin{cases} 0, & \text{if } s > \lambda, \\ -\varepsilon, & \text{if } s = \lambda, \\ s - \lambda, & \text{if } \lambda - 1 + \frac{N}{p} \leq s < \lambda. \end{cases}$$

Let us consider the first case $\lambda \geq 2 - \frac{N}{p}$. If $x, y \in G_{\varrho/2}^{\varrho}$, then $|x-y| \leq 2\varrho$ and therefore $\varrho^{\kappa} \leq c|x-y|^{\kappa}$, since $\kappa \leq 0$. Then from (4.2.109) we get

$$|\nabla v(x) - \nabla v(y)| \leq c \begin{cases} |x-y|^{1-N/p}, & \text{if } s > \lambda, \\ |x-y|^{1-N/p-\varepsilon}, & \forall \varepsilon > 0, \text{ if } s = \lambda, \\ |x-y|^{1-N/p+s-\lambda}, & \text{if } s < \lambda. \end{cases}$$

If $x, y \in \overline{G}$ and $|x-y| \geq \varrho = |x|$, then by (4.2.108) we get

$$\frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1-N/p+\kappa}} \leq 2|\nabla v||x-y|^{N/p-1-\kappa} \leq c\varrho^{\lambda-1+\kappa}\varrho^{\frac{N}{p}-1-\kappa} =$$

$$= c\varrho^{\frac{N}{p}-2+\lambda} \leq const;$$

we have taken into account that in the considered case $1 - N/p + \kappa \geq 0$. Thus case 1) of our theorem is proved.

Now we consider the second case $1 < \lambda \leq 2 - \frac{N}{p}$. If $x, y \in G_{\varrho/2}^{\varrho}$, then $|x - y| \leq 2\varrho$ and therefore $\varrho^{\kappa} \leq c|x - y|^{\kappa}$, since $\kappa \leq 0$. Then from (4.2.109) we get

$$|\nabla v(x) - \nabla v(y)| \leq c|x-y|^{1-N/p}\varrho^{\frac{N}{p}-2+\lambda+\kappa} \leq c|x-y|^{\lambda-1+\kappa}.$$

If $x, y \in \overline{G}$ and $|x - y| \geq \varrho = |x|$, then by (4.2.108) we get

$$\frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{\lambda-1+\kappa}} \leq 2|\nabla v||x-y|^{1-\lambda-\kappa} \leq c\varrho^{\lambda-1+\kappa}|x-y|^{1-\lambda-\kappa} \leq const;$$

we have taken into account that in the considered case $1 - \lambda - \kappa \leq 0$. Thus case 2) of our theorem is proved as well. □

THEOREM 4.34. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.22 and (bb) with $\psi(\varrho)$ defined by Theorem 4.22 be satisfied. Then it is true for the next estimate*

$$(4.2.110) \qquad |\nabla u(x)| \leq c \begin{cases} |x|^{\lambda-1-\varepsilon}, & \text{if } s \geq \lambda, \\ |x|^{s-1-\varepsilon}, & \text{if } s < \lambda. \end{cases} \forall \varepsilon > 0.$$

Moreover,

- **1)** *if* $\lambda \geq 2 - \frac{N}{p}$, *then*

$$u \in \begin{cases} C^{2-\frac{N}{p}-\varepsilon}(\overline{G}), & if \quad s \geq \lambda, \\ C^{s-\lambda+2-\frac{N}{p}-\varepsilon}(\overline{G}), & if \quad \lambda - 1 + \frac{N}{p} < s < \lambda, \end{cases} \quad \forall \varepsilon > 0.$$

- **2)** *if* $1 < \lambda \leq 2 - \frac{N}{p}$, *then*

$$u \in \begin{cases} C^{\lambda-\varepsilon}(\overline{G}), & if \quad s \geq \lambda, \\ C^{s-\varepsilon}(\overline{G}), & if \quad 1 < s < \lambda, \end{cases} \quad \forall \varepsilon > 0.$$

PROOF. From (4.2.105), (4.2.106) with $\psi(\varrho)$ from Theorem 4.22 we obtain

$$(4.2.111) \qquad \sup_{G^\varrho_{\varrho/2}} |\nabla v| \leq c \begin{cases} \varrho^{\lambda-1-\varepsilon}, & if \quad s \geq \lambda, \\ \varrho^{s-1-\varepsilon}, & if \quad s < \lambda, \end{cases}$$

$$(4.2.112) \qquad \sup_{\substack{x,y \in G^\varrho_{\varrho/2} \\ x \neq y}} \frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1-N/p}} \leq c \begin{cases} \varrho^{\frac{N}{p}-2+\lambda-\varepsilon}, & if \quad s = \lambda, \\ \varrho^{\frac{N}{p}-2+s-\varepsilon}, & if \quad s < \lambda. \end{cases}$$

$\forall \varepsilon > 0$. Putting $|x| = \frac{2}{3}\varrho$ we obtain from (4.2.111) the estimate (4.2.110).

Let us consider the first case $\lambda \geq 2 - \frac{N}{p}$. If $x, y \in G^\varrho_{\varrho/2}$, then $|x-y| \leq 2\varrho$ and therefore $\varrho^{-\varepsilon} \leq c|x-y|^{-\varepsilon}$. Then from (4.2.112) we get

$$|\nabla v(x) - \nabla v(y)| \leq c \begin{cases} |x-y|^{1-N/p-\varepsilon}, & if \quad s \geq \lambda, \\ |x-y|^{1-N/p+s-\lambda-\varepsilon}, & if \quad s < \lambda. \end{cases} \quad \forall \varepsilon > 0.$$

If $x, y \in \overline{G}$ and $|x-y| \geq \varrho = |x|$, then by (4.2.111) we get

- 1) for $s \geq \lambda$

$$\frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1-N/p-\varepsilon}} \leq 2|\nabla v||x-y|^{N/p-1+\varepsilon} \leq c\varrho^{\lambda-1-\varepsilon}|x-y|^{N/p-1+\varepsilon} \leq$$

$$\leq c|x-y|^{\frac{N}{p}-2+\lambda} \leq const.$$

- 2) for $N/p - 1 + \lambda < s < \lambda$

$$\frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1-N/p-\varepsilon+s-\lambda}} \leq 2|\nabla v||x-y|^{N/p-1+\varepsilon-s+\lambda} \leq$$

$$\leq c\varrho^{s-1-\varepsilon}|x-y|^{N/p-1+\varepsilon+\lambda-s} \leq c\varrho^{\frac{N}{p}-2+\lambda} \leq const.$$

Thus case 1) of our theorem is proved.

Now we consider the second case $1 < \lambda \leq 2 - \frac{N}{p}$. For this we define

$$\kappa = \begin{cases} -\varepsilon, & \text{if } s \geq \lambda, \\ s - \lambda - \varepsilon, & \text{if } 1 < s < \lambda. \end{cases}$$

If $x, y \in G^{\varrho}_{\varrho/2}$, then $|x-y| \leq 2\varrho$ and therefore $\varrho^{\kappa} \leq c|x-y|^{\kappa}$, since $\kappa < 0$. Then from (4.2.112) we get

$$|\nabla v(x) - \nabla v(y)| \leq c|x - y|^{1-N/p} \varrho^{\frac{N}{p}-2+\lambda+\kappa} \leq c|x - y|^{\lambda-1+\kappa}.$$

If $x, y \in \overline{G}$ and $|x - y| \geq \varrho = |x|$, then by (4.2.111) we get

$$\frac{|\nabla v(x) - \nabla v(y)|}{|x - y|^{\lambda-1+\kappa}} \leq 2|\nabla v||x - y|^{1-\lambda-\kappa} \leq c\varrho^{\lambda-1+\kappa}|x - y|^{1-\lambda-\kappa} \leq const.$$

We have taken into account that in the considered case $1 - \lambda - \kappa < 0$. Thus case 2) of our theorem is proved as well.   □

At last, in the same way we prove

THEOREM 4.35. *Let u be a strong solution of the boundary value problem (L) and let the assumptions of Theorem 4.23 and (bb) with $\psi(\varrho)$ defined by Theorem 4.23 be satisfied. Then the next estimate is true*

$$(4.2.113) \qquad\qquad |\nabla u(x)| \leq C|x|^{\lambda-1} \ln^{c+1} \frac{1}{|x|}.$$

Moreover,

- 1) *if $\lambda \geq 2 - \frac{N}{p}$, then $u \in C^{2-\frac{N}{p}-\varepsilon}(\overline{G})$, $\forall \varepsilon > 0$;*
- 2) *if $1 < \lambda \leq 2 - \frac{N}{p}$, then $u \in C^{\lambda-\varepsilon}(\overline{G})$, $\forall \varepsilon > 0$.*

4.2.5. Examples. Let us present some examples which demonstrate that the assumptions on the coefficients of the operator L are essential for the validity of Theorems from Section 4.2.2.

Let $N = 2$, let the domain G lie inside the sector

$$G^{\infty}_0 = \{(r, \omega) | 0 < r < \infty, 0 < \omega < \omega_0, \; 0 < \omega_0 \leq 2\pi\}$$

and suppose that $\mathcal{O} \in \partial G$ and in some neighborhood G^d_0 of $\mathcal{O}$ the boundary ∂G coincides with the sides $\omega = 0$ and $\omega = \omega_0$ of the sector G^{∞}_0. In our case the least eigenvalue of (EVD) is $\lambda = \frac{\pi}{\omega_0}$.

EXAMPLE 4.36. Let us consider the function

$$u(r, \omega) = r^{\lambda} \left(\ln \frac{1}{r}\right)^{(\lambda-1)/(\lambda+1)} \sin(\lambda\omega), \quad \lambda = \frac{\pi}{\omega_0}$$

in $G_0^d := \{x \in \mathbb{R}^2 : 0 < r < d,\ 0 < \omega < \omega_0\}$. It satisfies the equation

$$\sum_{i,j=1}^{2} a^{ij}(x)D_{ij}u = 0 \quad \text{in} \quad G_0^d$$

with

$$
\begin{aligned}
a^{11}(x) &= 1 - \frac{2}{\lambda+1}\frac{x_2^2}{r^2\ln(1/r)}, \\
a^{12}(x) = a^{21}(x) &= \frac{2}{\lambda+1}\frac{x_1 x_2}{r^2\ln(1/r)}, \\
a^{22}(x) &= 1 - \frac{2}{\lambda+1}\frac{x_1^2}{r^2\ln(1/r)}, \\
a^{ij}(0) &= \delta_i^j
\end{aligned}
$$

and the boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_0^d.$$

If $d < e^{-2}$, then the equation is uniformly elliptic with ellipticity constants

$$\nu = 1 - \frac{2}{\ln(1/d)} \quad \text{and} \quad \mu = 1.$$

Furthermore,

$$\mathcal{A}(r) = \frac{2}{(\lambda+1)\ln(1/r)}, \quad \int_0^d \frac{\mathcal{A}(r)}{r}dr = +\infty,$$

that is the leading coefficients of the equation are continuous but not Dini continuous at zero. From the explicit form of the solution u we have

$$(4.2.114) \qquad |u(x)| \le c|x|^{\lambda-\varepsilon}, \quad \|u\|_{\overset{\circ}{W}_2^2(G_0^\varrho)} \le c\varrho^{\lambda-\varepsilon}$$

for all $\varepsilon > 0$. This example shows that it is not possible to replace $\lambda - \varepsilon$ in (4.2.114) by λ without additional assumptions regarding the continuity modulus of the leading coefficients of the equation at zero.

EXAMPLE 4.37. Let G_0^d be defined as in the previous example and let

$$u(x) = r^\lambda \ln(\frac{1}{r})\sin(\lambda\omega), \quad \lambda = \frac{\pi}{\omega_0}.$$

The function u satisfies

$$\begin{cases} \Delta u + \frac{2\lambda}{r^2\ln(1/r)}u = 0 & \text{in } G_0^d, \\ u = 0 & \text{on } \Gamma_0^d. \end{cases}$$

Here

$$\mathcal{A}(r) = \frac{2\lambda}{\ln(1/r)}, \quad \int\limits_0^d \frac{\mathcal{A}(r)}{r} dr = +\infty.$$

Thus the assumptions about the lower order coefficients are essential, too.

EXAMPLE 4.38. The function

$$u(x) = r^\lambda \ln(\frac{1}{r}) \sin(\lambda\omega), \quad \lambda = \frac{\pi}{\omega_0}$$

satisfies

$$\begin{cases} \Delta u = f := -2\lambda r^{\lambda-2} \sin(\lambda\omega) & \text{in } G_0^d, \\ u = 0 & \text{on } \Gamma_0^d. \end{cases}$$

Here, all assumptions on the coefficients are satisfied but

$$\|f\|_{\overset{\circ}{W}{}^0_2(G_0^\varrho)} \leq c\varrho^s$$

with $s = \lambda$. This verifies the importance of conditions of our theorems.

4.2.6. Higher regularity results. Now we begin the study of the higher regularity of the strong solutions of the problem (L). This smoothness depends on the value λ.

THEOREM 4.39. *Let* $p, \alpha \in \mathbb{R}, k \in \mathbb{N}$ *satisfy* $p \geq 1, k \geq 2$. *Let* $u \in W^{2,N}(G) \cap C^0(\overline{G})$ *be a strong solution of the boundary value problem* (L) *and assumptions of Theorem 4.21 with* $s > \lambda$ *are satisfied. Suppose, in addition, that there are derivatives* $D^l a^{ij}, D^l a^i, D^l a, |l| \leq k-2$ *and numbers* $\mu_l \geq 0$ *such that*

$$\left(\sum_{i,j=1}^N |D^l a^{ij}(x)|^2 \right)^{1/2} + |x|^{l+1} \left(\sum_{i=1}^N |D^l a^i(x)|^2 \right)^{1/2} + |x|^{l+2} |D^l a(x)| \leq \mu_l;$$

$$x \in \overline{G}; \ |l| = 1, 2, \ldots, k-2.$$

If $f \in V_{p,\alpha}^{k-2}(G), \varphi \in V_{p,\alpha}^{k-1/p}(\partial G)$ *and*

$$(4.2.115) \quad \|f\|_{V_{p,\alpha}^{k-2}(G_{\varrho/4}^{2\varrho})} + \|\varphi\|_{V_{p,\alpha}^{k-1/p}(\Gamma_{\varrho/4}^{2\varrho})} \leq k_1 \varrho^{\lambda-k+\frac{\alpha+N}{p}},$$

where

$$(4.2.116) \quad \alpha > p(k-\lambda) - N,$$

then there are numbers $c > 0, d > 0$ *such that* $u \in V_{p,\alpha}^k(G_0^d)$ *and the following estimate is valid*

$$(4.2.117) \quad \|u\|_{V_{p,\alpha}^k(G_0^\varrho)} \leq c\varrho^{\lambda-k+\frac{\alpha+N}{p}}, \ \varrho \in (0,d).$$

PROOF. Let us consider two sets $G^{2\varrho}_{\varrho/4}$ and $G^{\varrho}_{\varrho/2} \subset G^{2\varrho}_{\varrho/4}$, $\varrho > 0$. We make transformation $x = \varrho x'$; $u(\varrho x') = \varrho^{\lambda} w(x')$. The function $w(x')$ satisfies the problem

$$(L)' \quad \begin{cases} a^{ij}(\varrho x') w_{x'_i x'_j} + \varrho a^i(\varrho x') w_{x'_i} + \varrho^2 a(\varrho x') w = \varrho^{2-\lambda} f(\varrho x'), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x' \in G^2_{1/4}; \\ w(x') = \varrho^{-\lambda} \varphi(\varrho x'), \quad x' \in \Gamma^2_{1/4}. \end{cases}$$

By Theorem 4.7 we have

$$(4.2.118) \quad \|w\|_{W^{k,p}(G^1_{1/2})} \leq C_k \Big(\|w\|_{L^p(G^2_{1/4})} + \varrho^{2-\lambda} \|f\|_{L^p(G^2_{1/4})} + $$
$$+ \varrho^{-\lambda} \|\varphi\|_{W^{k-1/p,p}(\Gamma^2_{1/4})} \Big),$$

where C_k does not depend on w and depends only on G, N, p, ν, μ and $\max_{x \in \overline{G^{2\varrho}_{\varrho/4}}} \mathcal{A}(|x|)$. Returning to the variables x, u, multiplying both sides of this inequality by $\varrho^{\frac{\alpha+N}{p} - k}$ and noting that $\varrho/4 \leq r \leq 2\varrho$ in $G^{2\varrho}_{\varrho/4}$, we obtain

$$(4.2.119) \quad \|u\|_{V^k_{p,\alpha}(G^{\varrho}_{\varrho/2})} \leq C \Big\{ \|f\|_{V^{k-2}_{p,\alpha}(G^{2\varrho}_{\varrho/4})} + \|\varphi\|_{V^{k-1/p}_{p,\alpha}(\Gamma^{2\varrho}_{\varrho/4})} + $$
$$+ \|u\|_{V^0_{p,\alpha-kp}(G^{2\varrho}_{\varrho/4})} \Big\}.$$

By Theorem 4.21 we have $|u(x)| \leq c_0 |x|^{\lambda}$ therefore

$$\|u\|^p_{V^0_{p,\alpha-kp}(G^{2\varrho}_{\varrho/4})} = \int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{\alpha-kp} |u(x)|^p dx \leq c_0^p \int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{\alpha-kp+\lambda p} dx \leq$$
$$\leq c_0^p \text{meas}\Omega \cdot \varrho^{\alpha+N+p(\lambda-k)}.$$

Hence and from (4.2.120) with regard to (4.2.115) it follows that

$$(4.2.120) \quad \|u\|_{V^k_{p,\alpha}(G^{\varrho}_{\varrho/2})} \leq C\varrho^{\lambda-k+\frac{\alpha+N}{p}}, \quad \varrho \in (0,d).$$

Replacing ϱ in the above inequality by $2^{-m}\varrho$ and summing up the resulting inequalities for every $m = 0, 1, 2, \ldots$, we obtain

$$\|u\|_{V^k_{p,\alpha}(G^{\varrho}_0)} \leq C\varrho^{\lambda-k+\frac{\alpha+N}{p}} \sum_{m=0}^{\infty} 2^{-m(\lambda-k+\frac{\alpha+N}{p})}.$$

By (4.2.116), the numerical series from the right converges. Thus the estimate (4.2.117) is proved. □

THEOREM 4.40. *Let u be a strong solution of (L). Suppose that the conditions of Theorem 4.21 with $s > \lambda$ and Theorem 4.39 are satisfied. Let, in addition,*

$$(4.2.121) \qquad k - 1 < \lambda \le k - \frac{N}{p}, \quad k \ge 2, p > N.$$

Then $u \in C^\lambda(\overline{G_0^d})$ and there are nonnegative numbers C_l such that

$$(4.2.122) \qquad |D^l u(x)| \le C_l |x|^{\lambda - |l|} \quad \forall x \in \overline{G_0^d}; \quad |l| = 0, 1, \ldots, k - 1$$

for some $d > 0$. If $\lambda = k - 1$, $p = N$, then $u \in C^{\lambda - \varepsilon}(\overline{G_0^d})$, $\forall \varepsilon > 0$.

PROOF. We consider the function $w(x')$ as a solution of the problem $(L)'$ in the domain $G_{1/4}^2$. By the Sobolev Imbedding Theorem 1.33,

$$W^{k,p}(G) \hookrightarrow C^{k-1+\beta}(G), \quad 0 < \beta \le 1 - \frac{N}{p}$$

and, in addition,

$$(4.2.123) \quad \sum_{|l| \le k-1} \sup_{x' \in G_{1/2}^1} |D_{x'}^l w(x')| +$$

$$+ \sup_{\substack{x', y' \in G_{1/2}^1 \\ x' \ne y'}} \frac{|D_{x'}^{k-1} w(x') - D_{y'}^{k-1} w(y')|}{|x' - y'|^{1 - N/p}} \le c \|w\|_{W^{k,p}(G_{1/2}^1)}$$

with a constant c independent of u and defined only by N, p and the domain G. Returning to the variables x, u, we have for $\forall \varrho \in (0, d)$

$$\sup_{x \in G_{\varrho/2}^\varrho} |D^l u(x)| \le C_l \varrho^{k - |l| - \frac{N + \alpha}{p}} \|u\|_{V_{p,\alpha}^k(G_{\varrho/2}^\varrho)}, \quad |l| = 0, 1, \ldots, k - 1$$

$(4.2.124)$

$$\sup_{\substack{x, y \in G_{\varrho/2}^\varrho \\ x \ne y}} \frac{|D^{k-1} u(x) - D^{k-1} u(y)|}{|x - y|^{1 - N/p}} \le c \varrho^{-\frac{\alpha}{p}} \|u\|_{V_{p,\alpha}^k(G_{\varrho/2}^\varrho)}.$$

Since $\varrho/2 \le r = |x| \le \varrho$ for $x \in G_{\varrho/2}^\varrho$, by $(4.2.120)$, from $4.2.124)$ it follows

$$(4.2.125) \qquad |D^l u(x)| \le C_l |x|^{\lambda - |l|}, \quad |l| = 0, 1, \ldots, k - 1; \quad x \in G_0^d;$$

$$(4.2.126) \qquad \sup_{\substack{x, y \in G_{\varrho/2}^\varrho \\ x \ne y}} \frac{|D^{k-1} u(x) - D^{k-1} u(y)|}{|x - y|^{1 - N/p}} \le c \varrho^{\lambda - k + \frac{N}{p}}.$$

Now from $(4.2.126)$ for $\tau = \lambda - k + \frac{N}{p} \le 0$ we have

$$(4.2.127) \quad |D^{k-1} u(x) - D^{k-1} u(y)| \le c \varrho^\tau |x - y|^{\lambda - k + 1 - \tau} \quad \forall x, y \in G_{\varrho/2}^\varrho.$$

Since $\tau \leq 0$, we have

$$|x - y|^\tau \geq (2\varrho)^\tau \quad \forall x, y \in G^\varrho_{\varrho/2}$$

and therefore from (4.2.127) it follows that

$$(4.2.128) \quad |D^{k-1}u(x) - D^{k-1}u(y)| \leq c2^{-\tau}|x - y|^{\lambda-k+1} \quad \forall x, y \in G^\varrho_{\varrho/2}.$$

The inequality (4.2.128) together with the (4.2.125) leads to the assertion $u \in C^\lambda(\overline{G^d_0})$, if the (4.2.121) is fulfilled.

Let now $\lambda = k - 1$, $p = N$. Then, by the Sobolev Imbedding Theorem 1.33, we have

$$(4.2.129) \quad \sup_{\substack{x', y' \in G^1_{1/2} \\ x' \neq y'}} \frac{|D^{k-2}_{x'}w(x') - D^{k-2}_{y'}w(y')|}{|x' - y'|^\beta} \leq c\|w\|_{W^{k,p}(G^1_{1/2})},$$

$$\forall \beta \in (0,1); \ k \geq 2.$$

Returning to the variables x, u and considering the inequality (4.2.120), we have for $\forall \varrho \in (0, d)$

$$\sup_{\substack{x,y \in G^\varrho_{\varrho/2} \\ x \neq y}} \frac{|D^{k-2}u(x) - D^{k-2}u(y)|}{|x - y|^\beta} \leq c\varrho^{2-\beta-\frac{\alpha+N}{p}}\|u\|_{V^k_{p,\alpha}(G^\varrho_{\varrho/2})} \leq$$

$$(4.2.130) \quad \leq c\varrho^{\lambda-k+2-\beta} = c\varrho^{1-\beta}, \quad \forall \beta \in (0,1), \ k \geq 2.$$

The inequality (4.2.130) for $\beta = 1 - \varepsilon$, $\forall \varepsilon > 0$ together with the (4.2.125) for $|l| = 0, 1, \ldots, k - 2$ means $u \in C^{\lambda-\varepsilon}(\overline{G^d_0})$, $\forall \varepsilon > 0$. Thus the assertion follows. $\square$

4.3. Smoothness in a Dini-Liapunov region

In this Section we shall study strong solutions $u \in W^{2,p}_{loc}(G) \cap W^{1,p}(G)$, $p > N$ of (L) in a Dini-Liapunov region G. We follow some results in K.-O. Widman [405], [406].

DEFINITION 4.41. *A Dini-Liapunov surface is a closed, bounded* $(N - 1)-$ *dimensional surface S satisfying the following conditions:*

- *At every point of S there is a uniquely defined tangent (hyper-) plane, and thus also a normal.*
- *There exists a Dini function $\mathcal{A}(r)$ such that if θ is the angle between two normals, and r is the distance between their foot points, then the inequality $\theta \leq \mathcal{A}(r)$ holds.*
- *There is a constant $\varrho > 0$ such that if Ω_ϱ is a sphere with radius ϱ and center $x_0 \in S$, then a line parallel to the normal at x_0 meets S at most once inside Ω_ϱ.*

A Dini-Liapunov surface is called a Liapunov surface, if $\mathcal{A}(r) = cr^\gamma$, $\gamma \in (0,1)$. Dini-Liapunov and Liapunov regions are regions the boundary of which are Dini-Liapunov and Liapunov surfaces respectively.

For the properties of Liapunov regions see Günter [140]. In particular we note that **a Dini-Liapunov domain belongs to** $C^{1,\mathcal{A}}$.

Since some minor complications arise from the logarithmic singularity of the fundamental solutions of the elliptic equation with constant coefficients in the case $N = 2$, we will concentrate on domains in $\mathbb{R}^N$ with $N \geq 3$.

We note that it is well known, that the first derivatives of u are continuous functions which are locally absolutely continuous on all straight lines parallel to one of the coordinate axis except those issuing from a set of $(N-1)-$ dimensional Lebesgue measure zero on the orthogonal hyperplane.

Further we will always suppose that the following assumptions on the equation (L) are as follows

- (a) *and* (b) *with det* $\left(a^{ij}\right) = 1$, *which is no further restriction.*

(c) *There exists a α-Dini function $\mathcal{A}$ such that*

$$\left(\sum_{i,j=1}^{N} |a^{ij}(x) - a^{ij}(y)|^2\right)^{1/2} \leq \mathcal{A}(|x - y|), \quad \forall x, y \in \overline{G}.$$

(cc)

$$\left(\sum_{i=1}^{N} |a^i(x)|^2\right)^{1/2} + |a(x)| + |f(x)| \leq K d^{\lambda-2}(x),$$

where $\lambda \in (1,2)$ and by $d(x)$ is denoted the distance from x to ∂G.

THEOREM 4.42. *Let G be a bounded Liapunov domain in $\mathbb{R}^N$ with a C^λ, $1 < \lambda < 2$ boundary portion $T \subset \partial G$. Let $u(x)$ be a strong solution of the problem (L) with $\varphi(x) \in C^\lambda(\partial G)$. Suppose the coefficients of the equation in (L) satisfy assumptions (a) - (cc).*

Then $u \in C^\lambda(G')$ for any domain $G' \subset\subset G \cup T$ and

$$(4.3.1) \quad |u|_{\lambda;G'\cup T} \leq c(N,T,G,\nu,\mu,K,k,d')\left(|u|_{0;G} + \|f\|_{p;G} + |\varphi|_{\lambda;\partial G}\right),$$

where $d' = dist(G', \partial G \setminus T)$, $k = \max_{i,j=1,\ldots,N}\{\|a^{ij}, \|_{0,\mathcal{A};G}\}$, $N < p < \frac{N}{2-\lambda}$.

PROOF. *Step 1.*

By the definition of Liapunov surfaces, there is a sphere S_ϱ of radius $\varrho > 0$ and center $x_0 \in T$ such that a line parallel to the normal at x_0 intersects T at most once inside S_ϱ. We can choose $\varrho > 0$ so small that any two normals issuing from points of T inside S_ϱ form an angle less than $\frac{\pi}{4}$, say. It will be no restriction to assume that $x_0 = \mathcal{O}$ and that the positive x_N– axis is along the inner normal of T at x_0. Then, inside S_ϱ, the surface T is described by

$$x_N = h(x') \in C^\lambda(|x'| < \frac{1}{2}\varrho + \varepsilon); \quad x' = (x_1, \ldots, x_{N-1}).$$

Now we use Extension Lemma 1.62 to extend the function $x_N - h(x')$ from T into G. We denote this extension by $H(x)$. Since $\frac{\partial H}{\partial x_N} = 1$ on T we can consider the connected region G' that is a connected component of the set

$$\{x \,|\, |x'| < \frac{1}{2}\varrho, \; \frac{\partial H}{\partial x_N} > \frac{1}{2}, \; H > 0\}$$

which has T as a portion of its boundary. By Extension Lemma 1.62 H has the following properties in G' :

1°. $H(x) \in C^\infty(G')$;

2°. $H(x) \in C^\lambda(\overline{G'})$;

3°. $K_1[x_N - h(x')] \leq H(x) \leq K_2[x_N - h(x')], \implies$
     $d(x) \geq K_3 H(x), \quad K_1, K_2, K_3 > 0, \quad$ (see also §2 [**235**]);

4°. $|D^2_{xx} H(x)| \leq K d^{\lambda-2}(x)$;

5°. $H(x)$ is strictly monotonic considered as a function of x_N

     for each x', $|x'| < \frac{1}{2}\varrho$.

From 3° follows

COROLLARY 4.43.

$$d(x) \geq \frac{1}{2} K_3 |x|, \quad x \in G'.$$

PROOF. In fact, we have

$$d(x) \geq K_3 H(x) = K_3 \left(H(x) - H(x_0) \right) = K_3 |\nabla H| \cdot |x| \geq \frac{1}{2} K_3 |x|.$$

$\square$

Similarly, let $\Phi(x)$ be an extension of the boundary function $\varphi(x)$ from T into G. By Extension Lemma 1.62 Φ has the following properties in G' :

$1°.$ $\Phi(x) \in C^\infty(G');$

$2°.$ $\Phi(x) \in C^\lambda(\overline{G'})$

$3°.$ $|D^2_{xx}\Phi(x)| \leq Kd^{\lambda-2}(x).$

Now we flatten the boundary portion T. Let us consider the diffeomorphism ψ that is given in the following way

$$\begin{cases} y_k &= x_k; \quad k = 1, \dots, N-1, \\ y_N &= H(x). \end{cases}$$

The mapping $y = \psi(x)$, $x \in \overline{G'}$, is one-one and maps $\overline{G'}$ onto a region $\overline{D'}$ which contains the set $\{y \mid |y'| < \frac{1}{2}\varrho, 0 < y_N < \tau\}$ for some $\tau > 0$, in such a way that T and $\{|y'| < \frac{1}{2}\varrho\}$ correspond.

Let us consider the problem (L) for the function $v = u - \Phi$. The function v then satisfies the homogeneous Dirichlet problem

$(L)_0$ $$\begin{cases} a^{ij}(x)D_{ij}v(x) + a^i(x)D_iv(x) + a(x)v(x) = F(x) \quad \text{in } G, \\ v(x) = 0 \quad \text{on } \partial G, \end{cases}$$

where

$(4.3.2)$ $$F(x) = f(x) - \left(a^{ij}(x)D_{ij}\Phi(x) + a^i(x)D_i\Phi(x) + a(x)\Phi(x) \right).$$

Under the mapping $y = \psi(x)$, let $\widetilde{v}(y) = v(x)$. Since

$$v_{x_i} = \frac{\partial\psi_k}{\partial x_i}\widetilde{v}_{y_k} \quad \text{and} \quad v_{x_ix_j} = \frac{\partial\psi_k}{\partial x_i}\frac{\partial\psi_m}{\partial x_j}\widetilde{v}_{y_ky_m} + \frac{\partial^2\psi_k}{\partial x_i\partial x_j}\widetilde{v}_{y_k},$$

it follows from $(L)_0$ that $\widetilde{v}(y)$ is a strong solution in D' of the problem

$\widetilde{(L)}_0$ $$\begin{cases} \widetilde{a}^{ij}(y)D_{ij}\widetilde{v}(y) + \widetilde{a}^i(y)D_i\widetilde{v}(y) + \widetilde{a}(y)\widetilde{v}(y) = \widetilde{F}(y) \quad \text{in } D', \\ \widetilde{v}(y') = 0 \quad \text{on } |y'| \leq \frac{1}{2}\varrho, \end{cases}$$

where

$$\widetilde{F}(y) = \widetilde{f}(y) - \left(\widetilde{a}^{ij}(y)D_{ij}\widetilde{\Phi}(y) + \widetilde{a}^i(y)D_i\widetilde{\Phi}(y) + \widetilde{a}(y)\widetilde{\Phi}(y) \right),$$

$(4.3.3)$ $$\widetilde{a}^{ij}(y) = a^{km}(x)\frac{\partial\psi_i}{\partial x_k}\frac{\partial\psi_j}{\partial x_m}, \quad \widetilde{a}^i(y) = a^k(x)\frac{\partial\psi_i}{\partial x_k},$$

$$\widetilde{a}(y) = a(x), \quad \widetilde{f}(y) = f(x), \quad \widetilde{\Phi}(y) = \Phi(x),$$

$$x = \psi^{-1}(y).$$

It is not difficult to observe that the conditions on coefficients of the equation and on the portion T are invariant under maps of class $C^{1,\mathcal{A}}$. Further by the ellipticity condition we have

$$\widetilde{a}^{ij}(y)\xi_i\xi_j = a^{km}(x)\frac{\partial(\xi_i y_i)}{\partial x_k}\frac{\partial(\xi_j \partial y_j)}{\partial x_m} \geq$$

$$\geq \nu \sum_{k=1}^{N}\left(\frac{\partial}{\partial x_k}\left(\sum_{i=1}^{N}\xi_i y_i\right)\right)^2 =$$

$$= \nu \sum_{k=1}^{N}\left(\sum_{i=1}^{N}\xi_i\frac{\partial y_i}{\partial x_k}\right)^2 = \nu \sum_{k=1}^{N}\left(\xi_k + \xi_N\frac{\partial y_N}{\partial x_k}\right)^2 =$$

$$= \nu\left(\xi^2 + 2\xi_N^2 - 2\xi_N\sum_{k=1}^{N-1}\xi_k\frac{\partial h}{\partial x_k} + \xi_N^2\left[1 + \sum_{k=1}^{N-1}\left(\frac{\partial h}{\partial x_k}\right)^2\right]\right).$$

But by the Cauchy inequality with $\forall \varepsilon > 0$ we have

$$2\xi_N\frac{\partial h}{\partial x_k}\xi_k \leq \varepsilon\xi_N^2\left(\frac{\partial h}{\partial x_k}\right)^2 + \frac{1}{\varepsilon}\xi_k^2$$

therefore from the previous inequality it follows that

$$\widetilde{a}^{ij}(y)\xi_i\xi_j \geq \nu\left\{\left(1 - \frac{1}{\varepsilon}\right)\xi'^2 + (1 - \varepsilon)\xi_N^2\sum_{k=1}^{N-1}\left(\frac{\partial h}{\partial x_k}\right)^2 + 4\xi_N^2\right\} =$$

$$(4.3.4) \qquad = \nu\left\{\left(1 - \frac{1}{\varepsilon}\right)\xi'^2 + \xi_N^2\left[4 + (1 - \varepsilon)|\nabla h|^2\right]\right\} \geq$$

$$\geq \nu\left\{\left(1 - \frac{1}{\varepsilon}\right)\xi'^2 + \xi_N^2\left[4 + (1 - \varepsilon)K^2\right]\right\}, \quad \forall \varepsilon > 1.$$

Now we show that there is $\varepsilon > 1$ such that

$$1 - \frac{1}{\varepsilon} = 4 + (1 - \varepsilon)K^2$$

For this we solve the equation

$$K^2\varepsilon^2 - (3 + K^2)\varepsilon - 1 = 0$$

and obtain

$$\varepsilon = \frac{1}{2} + \frac{3}{2K^2} + \sqrt{\frac{1}{4} + \frac{10}{4K^2} + \frac{9}{4K^4}}.$$

Hence we see that $\varepsilon > 1$ and we also have

$$1 - \frac{1}{\varepsilon} = \frac{8}{K^2 + 5 + \sqrt{K^4 + 10K^2 + 9}}.$$

Thus from (4.3.4) it follows finally that

$$\widetilde{a}^{ij}(y)\xi_i\xi_j \geq \nu c(K)\xi^2,$$

(4.3.5)

$$c(K) = \frac{8}{K^2 + 5 + \sqrt{K^4 + 10K^2 + 9}}.$$

Now we rewrite the problem $\widetilde{(L)}_0$ in the form

$$\begin{cases} \widetilde{\mathcal{L}}_0\widetilde{v} \equiv \widetilde{a}_0^{ij} D_{ij}\widetilde{v} = \widetilde{\mathcal{G}}(y), & y \in D' \\ \widetilde{v}(y') = 0 & \text{on } |y'| \leq \tfrac{1}{2}\varrho, \end{cases}$$

where

$$\widetilde{a}_0^{ij} = \widetilde{a}^{ij}(0),$$

(4.3.6) $\quad \widetilde{\mathcal{G}}(y) = \widetilde{f}(y) - \left(\widetilde{a}^{ij}(y)D_{ij}\widetilde{\Phi}(y) + \widetilde{a}^i(y)D_i\widetilde{\Phi}(y) + \widetilde{a}(y)\widetilde{\Phi}(y)\right) -$

$$- \left(\left(\widetilde{a}^{ij}(y) - \widetilde{a}^{ij}(0)\right)D_{ij}\widetilde{v}(y) + \widetilde{a}^i(y)D_i\widetilde{v}(y) + \widetilde{a}(y)\widetilde{v}(y)\right),$$

and we can apply to this problem Theorem 3.10

(4.3.7) $\quad |\widetilde{v}|_{\lambda,D'''} \leq c\left(|\widetilde{v}|_{0,D''} + \|\widetilde{\mathcal{G}}\|_{p,D''}\right), \quad N < p < \dfrac{N}{2-\lambda},$

$$\forall D''' \subset D'' \subset \overline{D'}.$$

Noting that $dx = |J|dy$, where $J = \frac{\mathcal{D}(\psi_1,\ldots,\psi_N)}{\mathcal{D}(x_1,\ldots,x_N)}$ is jacobian of the transformation $\psi(x)$ and $J = \frac{\partial H}{\partial x_N} \geq \frac{1}{2}$, further, from assumptions (a), (b), (c) - (cc) and (4.3.3), (4.3.6) and (4.3.7) with regard for above properties of $H(x), \Phi(x)$ it follows that

$$|v|_{\lambda,G'''} \leq c_1|v|_{0,G''} + c(p,\mu,K)\left\{\int_{G''}\left\langle \mathcal{A}^p(d(x))|v_{xx}|^p + |f|^p + \right.\right.$$

(4.3.8)

$$+ \left. d^{p(\lambda-2)}(x)(|\nabla v|^p + |v|^p + |\varphi|_\lambda + 1)\right\rangle dx \Bigg\}^{\frac{1}{p}},$$

$$N < p < \frac{N}{2-\lambda}, \forall G''' \subset G'' \subset \overline{G'}.$$

(Here $G''' = \psi^{-1}(D''')$, $G'' = \psi^{-1}(D'')$.)

Now we apply L^p-estimate (Theorem 4.6) to the solution of $\widetilde{(L)}_0$

(4.3.9) $\qquad \displaystyle\int_{D''} |\widetilde{v}_{yy}|^p dy \leq c \int_{D'} \left(|\widetilde{v}|^p + |\widetilde{F}|^p\right) dy,$

where c depends on $N, \nu, \mu, K, \lambda, \mathcal{A}, D', |\psi|_\lambda, |\psi^{-1}|_\lambda$ with K from the assumption (cc). Considering the property $4°$ of H, from (4.3.9) it follows

$$
\begin{aligned}
(4.3.10) \qquad \int_{G''} |v_{xx}|^p dx &\leq c_1(|H|_1) \int_{D''} |\tilde{v}_{yy}|^p dy + \\
&+ c_2(|\psi^{-1}|_1) \int_{G''} d^{p(\lambda-2)}(x)|\nabla v|^p dx \leq \\
&\leq cc_1 \int_{D'} \left(|\tilde{v}|^p + |\tilde{F}|^p \right) dy + \\
&+ c_2 \int_{G''} d^{p(\lambda-2)}(x)|\nabla v|^p dx \leq \\
&\leq c \int_{G'} \Big\{ d^{p(\lambda-2)}(x) \left(|\nabla v|^p + |v|^p + |\varphi|_\lambda + 1 \right) + \\
&+ |f|^p \Big\} dx
\end{aligned}
$$

in virtue of the property $3°$ of Φ and the assumption (cc).

Thus from (4.3.8) and (4.3.10) we obtain

$$
(4.3.11) \quad |v|_{\lambda,G''} \leq c_1 |v|_{0,G'} + c \Bigg\{ \int_{G'} d^{p(\lambda-2)}(x) \Big(|\nabla v|^p + |v|^p + |\varphi|_\lambda +
$$

$$
+ 1 \Big) + |f|^p dx \Bigg\}^{\frac{1}{p}}, \quad N < p < \frac{N}{2-\lambda}, \ \forall G'' \subset G' \subset G \cup T.
$$

Step 2.

Let $x_0 \in G$, $x_0^* \in \partial G$ be arbitrary points. Put $d = \frac{1}{4}d(x_0)$. We rewrite the equation $(L)_0$ in the form

$$
(L)_0^* \qquad a^{ij}(x_0^*)D_{ij}v = \mathcal{F} + \left(a^{ij}(x_0^*) - a^{ij}(x) \right)D_{ij}v,
$$

where, by the assumption (cc) and the properties of Φ,

$$
(4.3.12) \qquad |\mathcal{F}| \leq c(\mu, K)d^{\lambda-2}(x)(1 + |v| + |\nabla v|).
$$

Let $\mathfrak{G}(x, y)$ be the Green function of the operator $a^{ij}(x_0^*)D_{ij}$ in the ball $B_\varrho(0)$. Then according to the Green representation formula (3.2.1), almost

everywhere

$$(4.3.13) \quad v(y) = \int\limits_{\partial B_\varrho(x_0)} v(x) \frac{\partial \mathfrak{G}(x,y)}{\partial \nu_x} ds_x +$$

$$+ \int\limits_{B_\varrho(x_0)} \mathfrak{G}(x,y) \left\{ \mathcal{F} + \left(a^{ij}(x_0^*) - a^{ij}(x)\right) D_{ij} v \right\} dx \equiv$$

$$\equiv J_1(y) + J_2(y), \quad \varrho \in [2d, 3d].$$

REMARK 4.44. We observe that the Green representation is valid because v and $D_i v$ are absolutely continuous on almost every line parallel to one of the coordinate axis, and thus partial integration is allowed.

Now using Lemma 3.9 with the Hölder inequality

$$|D_k J_1(y)|^p \leq \left[C\varrho^{-N} \int\limits_{\partial B_\varrho(x_0)} |v| ds_x \right]^p \leq Cd^{1-p-N} \int\limits_{\partial B_\varrho(x_0)} |v|^p ds_x$$

from which follows

$$(4.3.14) \quad \int\limits_{B_d(x_0)} d^{p(\lambda-2)}(y) |D_k J_1(y)|^p dy \leq Cd^{p(\lambda-3)+1} \int\limits_{\partial B_\varrho(x_0)} |v|^p ds_x,$$

if we take into account that

$$d(y) \leq |d(y) - d(x_0)| + d(x_0) \leq d + 4d = 5d,$$
$$d(y) \geq d(x_0) - |y - x_0| \geq 4d - d = 3d$$
and therefore
$$(4.3.15) \qquad\qquad 3d \leq d(y) \leq 5d.$$

Similarly, by Lemma 3.9 and the Hölder inequality,

$$|D_k J_2(y)|^p = \left| \int\limits_{B_\varrho(x_0)} D_k \mathfrak{G}(x,y) \left\{ \mathcal{F} + \left(a^{ij}(x_0^*) - a^{ij}(x)\right) D_{ij} v \right\} dx \right|^p dx \leq$$

$$\leq C \left(\int\limits_{B_{3d}(x_0)} |x-y|^{1-N} \left| \mathcal{F} + \left(a^{ij}(x_0^*) - a^{ij}(x)\right) D_{ij}v \right| dx \right)^p dx =$$

$$= C \Bigg(\int\limits_{B_{3d}(x_0)} |x-y|^{\frac{p-1}{p}(\alpha-N)} \times$$

$$\times |x-y|^{1-N-\frac{p-1}{p}(\alpha-N)} \left| \mathcal{F} + \left(a^{ij}(x_0^*) - a^{ij}(x)\right) D_{ij}v \right| dx \Bigg)^p dx \leq$$

$$\leq C \left(\int\limits_{B_{3d}(x_0)} |x-y|^{\alpha-N} dx \right)^{p-1} \times$$

$$\times \int\limits_{B_{3d}(x_0)} |x-y|^{p(1-\alpha)+\alpha-N} \left| \mathcal{F} + \left(a^{ij}(x_0^*) - a^{ij}(x)\right) D_{ij}v \right|^p dx,$$

$$\forall \alpha \in (0,1)$$

or

$$\int\limits_{B_d(x_0)} d^{p(\lambda-2)}(y) |D_k J_2(y)|^p dy \leq$$

$$\leq C d^{p(\lambda-1)} \int\limits_{B_{3d}(x_0)} \left| \mathcal{F} + \left(a^{ij}(x_0^*) - a^{ij}(x)\right) D_{ij}v \right|^p dx.$$

Hence and from (4.3.13), (4.3.14) we have

$$(4.3.16) \quad \int\limits_{B_d(x_0)} d^{p(\lambda-2)}(y) |\nabla v(y)|^p dy \leq C d^{p(\lambda-3)+1} \int\limits_{\partial B_\varrho(x_0)} |v|^p ds_x +$$

$$+ C d^{p(\lambda-1)} \int\limits_{B_{3d}(x_0)} \left| \mathcal{F} + \left(a^{ij}(x_0^*) - a^{ij}(x)\right) D_{ij}v \right|^p dx.$$

Now we take into account that $d(x_0) \leq d(x) + |x - x_0|$ and therefore in $B_{3d}(x_0)$ hold

$$d = \frac{1}{4} d(x_0) \leq \frac{1}{4} d(x) + \frac{3}{4} d \implies d \leq d(x).$$

Therefore, integrating (4.3.16) with respect to ϱ from $2d$ to $3d$, we get the inequality

$$(4.3.17) \quad \int\limits_{B_d(x_0)} d^{p(\lambda-2)}(x)|\nabla v(x)|^p dx \leq$$

$$\leq C \int\limits_{B_{3d}(x_0)} \left\{ d^{p(\lambda-3)}(x)|v|^p + d^{p(\lambda-1)}(x)\left(|\mathcal{F}|^p + \right.\right.$$

$$\left.\left. + |(a^{ij}(x) - a^{ij}(x_0^*))D_{ij}v|^p\right)\right\} dx.$$

Finally, from the inequalities (4.3.10), (4.3.12) and (4.3.17) it follows that

$$\int\limits_{B_d(x_0)} d^{p(\lambda-2)}(x)|\nabla v(x)|^p dx \leq c_1 \int\limits_{B_{4d}(x_0)} d^{p(2\lambda-3)}(x)|\nabla v(x)|^p dx +$$

$$(4.3.18) \qquad + c_2 \int\limits_{B_{4d}(x_0)} \left\{ d^{p(\lambda-3)}(x)|v|^p + d^{p(2\lambda-3)}(x) \right\} dx.$$

Step 3.

It is well known (see, for example, §2.2.2 [199]), that the smallest positive eigenvalue ϑ of the problem (EVD) for $(N-1)$-dimensional sphere or half-sphere is equal to $N-1$ and therefore, by the formula (2.5.11), the corresponding value $\lambda = 1$. To the problem $\widetilde{(L)}_0$ we apply Theorem 4.21 in $(N-1)$-dimensional sphere with $s = \lambda > 1$. As a result we obtain

$$|v(x)| \leq c_0 d(x), \quad x \in B_{4d}(x_0).$$

Therefore from (4.3.18) we get

$$\int\limits_{B_d(x_0)} d^{p(\lambda-2)}(x)|\nabla v(x)|^p dx \leq c_1 \int\limits_{B_{4d}(x_0)} d^{p(2\lambda-3)}(x)|\nabla v(x)|^p dx +$$

$$(4.3.19) \qquad + c_2 \int\limits_{B_{4d}(x_0)} d^{p(\lambda-2)}(x) dx.$$

Now consider the region G'_t defined by

$$G'_t = \{x \in G' \,|\, d(x) > t\},$$

where $d(x)$ is the boundary distance function of G while $d_t(x)$ will be that of G_t. We apply the following *lemma on the covering.*

LEMMA 4.45. (See Lemma 3.1 [405], §1.2.1 [261]). *Let G be any bounded open domain in $\mathbb{R}^N$ and let $\{B\}$ be the set of balls $B = B_{\frac{1}{4}d(x)}(x)$ with center x and radius $\frac{1}{4}d(x)$, $d(x)$ being the distance from x to ∂G. Then*

there exists a denumerable sequence of balls $B^{(k)} = B_{\frac{1}{4}d(x^{(k)})}(x^{(k)})$, $\{B^{(k)}\}_1^\infty$ *with the properties*

- $\bigcup B^{(k)} = G$;
- *every point of* G *is inside at most* $C(N)$ *of balls* $\{B^{'(k)}\}_1^\infty$, $B^{'(k)} = B_{\frac{3}{4}d(x^{(k)})}(x^{(k)})$ *and* $C(N)$ *depends only on* N.

Let us choose a covering $\{B^{(k)}\}_1^\infty$ of G_t'. Assuming the centers of the balls in the covering to be $\{x^{(k)}\}_1^\infty$, define $x^{(k)*}$ as one the points satisfying $x^{(k)*} \in \partial G \cap \partial G'$, $|x^{(k)} - x^{(k)*}| = d(x^{(k)})$. Then apply the estimate (4.3.19) for each k with $x_0 = x^{(k)}$ and $x_0^* = x^{(k)*}$. Since

$$C_1 d_k \le d_t(x) \le C_2 d_k \text{ for } |x - x^{(k)}| \le 4d_k \text{ where } d_k = \frac{1}{4}d_t(x^{(k)}),$$

we get

$$\int_{B_{d_k}(x^{(k)})} d_t^{p(\lambda-2)}(x)|\nabla v(x)|^p dx \le c_1 \int_{B_{4d_k}(x^{(k)})} d_t^{p(2\lambda-3)}|\nabla v|^p dx +$$
$$+ c_2 \int_{B_{4d_k}(x^{(k)})} d_t^{p(\lambda-2)}(x) dx.$$

Now, summing these inequalities over all k, we have

$$(4.3.20) \quad \int_{G_t'} d_t^{p(\lambda-2)}(x)|\nabla v(x)|^p dx \le c_1 \int_{G_t'} d_t^{p(2\lambda-3)}|\nabla v|^p dx +$$
$$+ c_2 \int_{G_t'} d_t^{p(\lambda-2)}(x) dx.$$

Since c_1 does not depend on t and $\lambda > 1$, we can find some t' which is independent of t and is such that

$$c_1 d^{p(\lambda-1)}(x) < \frac{1}{2},$$

if $d(x) < t'$. Then, if $t < \frac{1}{2}t'$,

$$c_1 \int_{G_t'} d_t^{p(2\lambda-3)}|\nabla v|^p dx \le \frac{1}{2} \int_{G_t' \cap \{d(x) \le t'\}} d_t^{p(\lambda-2)}(x)|\nabla v(x)|^p dx +$$
$$+ c_1 \int_{G_t' \cap \{d(x) > t'\}} d_t^{p(2\lambda-3)}|\nabla v|^p dx.$$

Hence and from (4.3.20) it follows

$$(4.3.21) \quad \int_{G'_t} d_t^{p(\lambda-2)}(x)|\nabla v(x)|^p dx \leq C \int_{G'_t \cap \{d(x)>t'\}} d_t^{p(2\lambda-3)}|\nabla v|^p dx + \\ + C \int_{G'_t} d_t^{p(\lambda-2)}(x) dx.$$

It should be noted that the second integral does not depend on t, but depends on $|\nabla v|$ and t'; in fact, since $t < \frac{1}{2}t'$ we have $G'_t \cap \{d(x) > t'\} = G'_{t'}$ and $d_t(x) = d(x) - t > t' - \frac{1}{2}t' = \frac{1}{2}t'$ on this set. We note that $G'_{t'} \subset G'_t \subset G'$.

Finally, from (4.3.21) we get

$$(4.3.22) \quad \int_{G'_t} d_t^{p(\lambda-2)}(x)|\nabla v(x)|^p dx \leq C(\lambda, p, \mathrm{diam}G) \int_{G'_{t'}} |\nabla v(x)|^p dx + \\ + C \int_{G'_t} d_t^{p(\lambda-2)}(x) dx,$$

since $\lambda > 1$.

Now we apply the L^p-estimate (Theorem 4.6) to the solution of $(L)_0$

$$(4.3.23) \quad \int_{G'_{t'}} |\nabla v(x)|^p dx \leq c \int_{G'_t} \left(|v|^p + |f|^p + K d_t^{p(\lambda-2)} \right) dx,$$

where c depends on $N, \nu, \mu, |\varphi|_\lambda, \lambda, \mathcal{A}, G'$ with K from the property $3°$ of Φ. Then from (4.3.22), (4.3.23) we have

$$(4.3.24) \quad \int_{G'_t} d_t^{p(\lambda-2)}(x)|\nabla v(x)|^p dx \leq c \int_{G'_t} \left(|v|^p + |f|^p + d_t^{p(\lambda-2)} \right) dx$$

Step 4.

Let $x_0 \in T$ and $N < p < \frac{N}{2-\lambda}$, $1 < \lambda < 2$. Then, by Corollary 4.43, we get

$$\int_{B_d(x_0)} d^{p(\lambda-2)}(x) dx \leq c \int_0^d r^{p(\lambda-2)+N-1} dr \leq \mathrm{const}.$$

Performing a covering of G' by the spheres with centers $x_0 \in T$ hence we get that

$$(4.3.25) \quad \int_{G'} d^{p(\lambda-2)}(x) dx \leq C < \infty.$$

Similarly, by setting $\rho(x) = d(x) - t$, we obtain

$$(4.3.26) \qquad \int\limits_{G'_t} d_t^{p(\lambda-2)}(x)dx = \int\limits_{G'\cap\{\rho(x)>0\}} \rho^{p(\lambda-2)}(x)dx \le C < \infty.$$

Hence, if we put $t = t_k$ and let $k \to \infty$, (4.3.24) and (4.3.26) imply that (4.3.11) is finite, with Fatou's lemma. The theorem is proved. $\square$

4.4. Unique solvability results

In this section we investigate the existence of solutions in weighted Sobolev spaces for the boundary value problem (L) under minimal assumptions on the smoothness of the coefficients. Let λ be the smallest positive eigenvalue of (EVD) with (2.5.11).

THEOREM 4.46. *Let $p \in (1, \infty)$, $\alpha, \beta \in \mathbb{R}$ with*

$$-\lambda + 2 - N < 2 - (\beta + N)/p \le 2 - (\alpha + N)/p < \lambda.$$

Furthermore, let us assume that

$$(4.4.1) \qquad |x|^{(\alpha-\beta)/p}\mathcal{A}(|x|) \to 0 \quad as \quad |x| \to 0$$

and suppose that assumptions (a) - (b) are fulfilled. If $u \in V^2_{p,\beta}(G)$ is a solution of the boundary value problem (L) with $f \in V^0_{p,\alpha}(G)$, $\varphi \in V^{2-1/p}_{p,\alpha}(\partial G)$ then $u \in V^2_{p,\alpha}(G)$ and the following a priori estimates are valid

$$(4.4.2) \qquad \|u\|_{V^2_{p,\alpha}(G)} \le c\left\{ \|f\|_{V^0_{p,\alpha}(G)} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\partial G)} + \|u\|_{V^0_{p,\alpha}(G)} \right\}$$

with a constant $c > 0$ which depends only on $\nu, \mu, \alpha, N, \mathcal{A}(\operatorname{diam} G)$ and the moduli of continuity of a^{ij}.

PROOF. We write the equation $Lu = f$ in the form

$$(4.4.3) \qquad \begin{aligned} \Delta u(x) = f(x) - \\ - \left(\left(a^{ij}(x) - \delta^j_i\right) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + a^i(x)\frac{\partial u}{\partial x_i}(x) + a(x)u(x) \right). \end{aligned}$$

Due to Theorem 3.11 we then have

$$(4.4.4) \qquad \|u\|_{V^2_{p,\alpha}(G)} \le c_2\left\{ \|\Delta u\|_{V^0_{p,\alpha}(G)} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\partial G)} \right\}.$$

Estimating the $V^0_{p,\alpha}$-norm of the right hand side of (4.4.3) we obtain from the condition (b)

$$\|\Delta u\|^p_{V^0_{p,\alpha}(G)} \le c_3 \left(\|f\|^p_{V^0_{p,\alpha}(G)} + \int_G \mathcal{A}^p(|x|) r^\alpha \Big(|D^2 u|^p + \right.$$
$$\left. + r^{-p}|\nabla u|^p + r^{-2p}|u|^p \Big) dx \right)$$

with c_3 depending only on p and N.

Decomposing the domain G into $G = G^d_0 \cup G_d$ we then obtain

$$(4.4.5) \quad \|\Delta u\|_{V^0_{p,\alpha}(G)} \le c_4 \left(\|f\|_{V^0_{p,\alpha}(G)} + \sup_{x \in G} \mathcal{A}(|x|)\|u\|_{W^{2,p}(G_d)} + \right.$$
$$\left. + \sup_{x \in (0,d)} |x|^{(\alpha-\beta)/p} \mathcal{A}(|x|)\|u\|_{V^2_{p,\beta}(G^d_0)} \right)$$

with c_4 depending only on N, p and d. Since all terms on the right hand side of (4.4.5) are finite, we conclude that $u \in V^2_{p,\alpha}(G)$.

Furthermore, from the local L^p a priori estimates (see Theorem 4.6) applied to the solution u of (L) we have

$$(4.4.6) \quad \|u\|_{W^{2,p}(G_d)} \le c_5 \Big(\|f\|_{L^p(G_{d/2})} + \|\varphi\|_{W^{2-1/p,p}(\Gamma_{d/2})} + $$
$$+ \|u\|_{L^p(G_{d/2})} \Big) \le c_6 \Big(\|f\|_{V^0_{p,\alpha}(G)} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\partial G)} + $$
$$+ \|u\|_{V^0_{p,\alpha}(G)} \Big)$$

with c_6 depending only on $N, p, \nu, \mu, G, d, \alpha$, the moduli of continuity of the coefficients a^{ij} on G_d and on

$$\left\| \left(\sum_{i=1}^N |a^i|^2 \right)^{1/2} \right\|_{L^N(G)}, \quad \|a\|_{L^p(G)}.$$

Combining the estimates (4.4.4)–(4.4.6) and taking the continuity of the imbedding

$$V^2_{p,\alpha}(G) \hookrightarrow V^2_{p,\beta}(G)$$

into account we arrive at

$$\|u\|_{V^2_{p,\alpha}(G)} \le c_7 \sup_{|x| \in (0,d)} |x|^{(\alpha-\beta)/p} \mathcal{A}(|x|)\|u\|_{V^2_{p,\alpha}(G^d_0)}$$
$$+ c_8 \Big(\|f\|_{V^0_{p,\alpha}(G)} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\partial G)} + \|u\|_{V^0_{p,\alpha}(G)} \Big).$$

Choosing d small enough and applying the condition (4.4.1) we obtain

$$(4.4.7) \qquad \|u\|_{V^2_{p,\alpha}(G)} \leq c_9 \left(\|f\|_{V^0_{p,\alpha}(G)} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\partial G)} + \|u\|_{V^0_{p,\alpha}(G)} \right).$$

$\square$

THEOREM 4.47. *Let* $p \in (1,+\infty), \alpha \in \mathbb{R}$ *with*

$$-\lambda + 2 - N < 2 - \frac{\alpha + N}{p} < \lambda$$

and let $u \in V^2_{p,\alpha}(G)$ *be a strong solution of* (L) *with* $f \in V^0_{p,\alpha}(G)$ *and* $\varphi \in V^{2-1/p}_{p,\alpha}(\partial G)$. *If* u *is the only solution in the space* $V^2_{p,\alpha}(G)$ *then following a priori estimate is valid*

$$(4.4.8) \qquad \|u\|_{V^2_{p,\alpha}(G)} \leq c \left(\|f\|_{V^0_{p,\alpha}(G)} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\partial G)} \right).$$

PROOF. Due to Theorem 4.46 we have

$$\|u\|_{V^2_{p,\alpha}(G)} \leq c \left(\|Lu\|_{V^0_{p,\alpha}(G)} + \|u\|_{V^{2-1/p}_{p,\alpha}(\partial G)} + \|u\|_{V^0_{p,\alpha}(G)} \right).$$

Let us suppose that (4.4.8) is not valid. Then there exists a sequence $\{u_j\}_{j=1}^{\infty} \subset V^2_{p,\alpha}(G)$ such that

$$\|u_j\|_{V^2_{p,\alpha}(G)} \geq j \left(\|Lu_j\|_{V^0_{p,\alpha}(G)} + \|u_j\|_{V^{2-1/p}_{p,\alpha}(\partial G)} + \|u_j\|_{V^0_{p,\alpha}(G)} \right).$$

After the normalization $\|u_j\|_{V^2_{p,\alpha}(G)} = 1$ we obtain

$$\|Lu_j\|_{V^0_{p,\alpha}(G)} + \|u_j\|_{V^{2-1/p}_{p,\alpha}(\partial G)} + \|u_j\|_{V^0_{p,\alpha}(G)} \leq \frac{1}{j}.$$

Since the imbedding $V^2_{p,\alpha}(G) \hookrightarrow V^0_{p,\alpha}(G)$ is compact, there exists a subsequence $\{u_{j'}\}_{j'=1}^{\infty}$ such that

$$u_{j'} \to u^* \text{ in } V^0_{p,\alpha}(G) \text{ for some } u^* \in V^0_{p,\alpha}(G).$$

Moreover, we have

$$\|u_{i'} - u_{j'}\|_{V^2_{p,\alpha}(G)} \leq c \Big(\|Lu_{i'} - Lu_{j'}\|_{V^0_{p,\alpha}(G)} +$$

$$+ \|u_{i'} - u_{j'}\|_{V^{2-1/p}_{p,\alpha}(\partial G)} + \|u_{i'} - u_{j'}\|_{V^0_{p,\alpha}(G)} \Big).$$

Thus $\{u_{j'}\}_{j'=1}^{\infty}$ is a Cauchy sequence in $V^2_{p,\alpha}(G)$. Consequently u^* belongs to $V^2_{p,\alpha}(G)$ and is a nontrivial solution of the boundary value problem (L) with $f \equiv 0$, $\varphi \equiv 0$, in contradiction to the unique solvability assumption. $\square$

THEOREM 4.48. *Let $p \geq N, \alpha \in \mathbb{R}$ and suppose that assumptions (a) and (b) are fulfilled and that $a(x) \leq 0$ for all $x \in G$. Then the Dirichlet problem (L) has a unique solution $u \in V_{p,\alpha}^2(G)$ for all $f \in V_{p,\alpha}^0(G) \cap L^p(G)$, $\varphi \in V_{p,\alpha}^{2-1/p}(\partial G)$ if and only if*

$$0 < 2 - (\alpha + N)/p < \lambda.$$

In this case the following a priori estimate is valid

$$(4.4.9) \qquad \|u\|_{V_{p,\alpha}^2(G)} \leq c \left\{ \|f\|_{V_{p,\alpha}^0(G)} + \|\varphi\|_{V_{p,\alpha}^{2-1/p}(\partial G)} \right\}.$$

PROOF. We prove the existence of a solution by the method of continuity (see Theorem 1.54). We consider the family of boundary value problems depending on the parameter $t \in [0,1]$

$$(L)^t \qquad \begin{cases} L^t u := tLu + (1-t)\Delta u = f & \text{in } G, \\ u = \varphi & \text{on } \partial G. \end{cases}$$

The operator L^t is uniformly elliptic with the ellipticity constants

$$\mu_t = \max\{1, \mu\}, \quad \nu_t = \max\{1, \nu\}$$

and is continuous if considered between the Banach spaces

$$L^t : V_{p,\alpha}^2(G) \to V_{p,\alpha}^0(G) \times V_{p,\alpha}^{2-1/p}(\partial G).$$

Let us denote by u_t a solution of the boundary value problem $(L)^t$ for $t \in [0,1]$. We will show that

$$(4.4.10) \qquad \|u_t\|_{V_{p,\alpha}^2(G)} \leq c_1 \left\{ \|f\|_{V_{p,\alpha}^0(G)} + \|\varphi\|_{V_{p,\alpha}^{2-1/p}(\partial G)} \right\} \quad \forall t \in [0,1]$$

with a constant c_1 independent of t, u_t and f, φ. To this end we write the equation $L_t u_t = f$ in the form

$$(4.4.11) \quad \Delta u_t(x) = f(x) - t\Big(\big(a^{ij}(x) - a^{ij}(0)\big) D_{ij} u_t(x) + a^i(x) D_i u_t(x) + \\ + a(x) u_t(x) \Big).$$

Due to Theorem 3.11 we then have

$$(4.4.12) \qquad \|u_t\|_{V_{p,\alpha}^2(G)} \leq c_2 \left\{ \|\Delta u_t\|_{V_{p,\alpha}^0(G)} + \|\varphi\|_{V_{p,\alpha}^{2-1/p}(\partial G)} \right\}.$$

Estimating the $V_{p,\alpha}^0$–norm of the right hand side of (4.4.11) we obtain from the condition (b)

$$\|\Delta u_t\|_{V_{p,\alpha}^0(G)}^p \leq c_3 \Big(\|f\|_{V_{p,\alpha}^0(G)}^p + \int\limits_G \mathcal{A}^p(|x|) r^\alpha \big(|D^2 u_t|^p + r^{-p} |\nabla u_t|^p + \\ + r^{-2p} |u_t|^p \big) dx \Big)$$

with c_3 depending only on p and N.

Decomposing the domain G into $G = G_0^d \cup G_d$ we then obtain

$$(4.4.13) \quad \|\Delta u_t\|_{V_{p,\alpha}^0(G)} \le c_4 \Big(\|f\|_{V_{p,\alpha}^0(G)} + \mathcal{A}(d)\|u_t\|_{V_{p,\alpha}^2(G_0^d)} +$$

$$+ \sup_{x \in G} \mathcal{A}(|x|)\|u_t\|_{W^{2,p}(G_d)} \Big)$$

with c_4 depending only on N, p and d. Furthermore, from the L_p-estimate (see Theorem 4.6) applied to the solution u_t of $(L)^t$ we have

$$(4.4.14) \quad \|u_t\|_{W^{2,p}(G_d)} \le c_5 \Big(\|f\|_{L^p(G_{d/2})} + \|\varphi\|_{W^{2-1/p,p}(\Gamma_{d/2})} +$$

$$+ \|u_t\|_{L^p(G_{d/2})} \Big) \le c_6 \Big(\|f\|_{V_{p,\alpha}^0(G)} + \|\varphi\|_{V_{p,\alpha}^{2-1/p}(\partial G)} +$$

$$+ \|u_t\|_{V_{p,\alpha-2p-1}^0(G)} \Big)$$

with c_5 depending only on N, p, ν, μ, G, d, the continuity moduli of the coefficients a^{ij} on G_d and on

$$\Big\| \Big(\sum_{i=1}^N |a^i|^2 \Big)^{1/2} \Big\|_{L^p(G)}, \quad \|a\|_{L^{p/2}(G)}, \quad p > N.$$

Combining the estimates (4.4.12)–(4.4.14) we arrive at

$$\|u_t\|_{V_{p,\alpha}^2(G)} \le c_2 c_4 \mathcal{A}(d)\|u_t\|_{V_{p,\alpha}^2(G_0^d)} + c_7 \Big(\|f\|_{V_{p,\alpha}^0(G)} + \|\varphi\|_{V_{p,\alpha}^{2-1/p}(\partial G)} +$$

$$+ \|u_t\|_{V_{p,\alpha-2p-1}^0(G)} \Big).$$

If we choose d small enough, then

$$c_2 c_4 \mathcal{A}(d) \le 1/2$$

due to the continuity of the function $\mathcal{A}$. Therefore,

$$(4.4.15) \quad \|u_t\|_{V_{p,\alpha}^2(G)} \le 2c_7 \Big(\|f\|_{V_{p,\alpha}^0(G)} + \|\varphi\|_{V_{p,\alpha}^{2-1/p}(\partial G)} +$$

$$+ \|u_t\|_{V_{p,\alpha-2p-1}^0(G)} \Big).$$

We remark that according to Lemma 1.38 we have $V_{p,\alpha}^2(G) \hookrightarrow C^0(\overline{G})$ and $\varphi \in C^0(\partial G)$ for $0 < 2 - (\alpha + N)/p$. Thus the boundary value problem (L) can have at most one solution in the space $V_{p,\alpha}^2(G)$ due to Theorem 4.1. Due to Lemma 1.37 the imbedding

$$V_{p,\alpha}^2(G) \hookrightarrow V_{p,\alpha-2p-1}^0(G)$$

is compact and we can apply the standard compactness argument (see Theorem 4.47) in order to get rid of the $\|u_t\|_{V^0_{p,\alpha-2p-1}(G)}$ term on the right hand side of (4.4.15). Thus

$$\|u_t\|_{V^2_{p,0}(G)} \leq c_{11} \left(\|f\|_{V^0_{p,\alpha}(G)} + \|\varphi\|_{V^{2-1/p}_{p,\alpha}(\partial G)} \right).$$

Since the boundary value problem $(L)^t$ is uniquely solvable for $t = 0$ due to Theorem 3.11 we conclude from Theorem 1.54 that $(L)^t$ is uniquely solvable for $t = 1$, too. $\square$

THEOREM 4.49. *Let* $\Gamma_d \in C^{1,1}$ *with some* $d > 0$. *Let* $\lambda \in (1,2)$ *and the numbers are given* $q \geq \frac{N}{2-\lambda}$, $N \leq p < q$ *and* $\alpha \in \mathbb{R}$ *satisfying the inequality*

$$0 < 2 - (\alpha + N)/p < \lambda.$$

Suppose that assumptions (a) and (b) are fulfilled with $\mathcal{A}(r)$ *Dini continuous at zero and, in addition,*

(d) $a \in L^N(G)$ *and* $a(x) \leq 0$ *for all* $x \in G$;

(dd) $f \in V^0_{q,\alpha}(G) \cap L^q(G)$, $\varphi \in V^{2-1/q}_{q,\alpha}(\partial G) \cap W^{2-1/q,q}(\partial G)$ *and there exist real numbers* $s > \lambda$, $k_1 \geq 0, k_2 \geq 0, k_3 \geq 0$ *such that*

$$k_1 =: \sup_{\varrho>0} \varrho^{-s} \left(\|f\|_{\overset{\circ}{W}^0_{4-N}(G^\varrho_0)} + \|\varphi\|_{\overset{\circ}{W}^{3/2}_{4-N}(\Gamma^\varrho_0)} \right) +$$

$$+ \sup_{\varrho>0} \varrho^{1-s} \left(\|f\|_{N;G^{2\varrho}_{\varrho/4}} + \|\varphi\|_{V^{2-1/N}_{N,0}(\Gamma^{2\varrho}_{\varrho/4})} \right),$$

$$k_2 =: \sup_{\varrho>0} \varrho^{2-\lambda-\frac{\alpha+N}{q}} \left(\|f\|_{V^0_{q,\alpha}(G^{2\varrho}_{\varrho/4})} + \|\varphi\|_{V^{2-1/q}_{q,\alpha}(\Gamma^{2\varrho}_{\varrho/4})} \right),$$

(cc) $\left(\sum_{i=1}^{N} |a^i(x)|^2 \right)^{1/2} + |a(x)| + |f(x)| \leq k_3 d^{\lambda-2}(x), \quad x \in G_\varepsilon, \forall \varepsilon > 0,$

where $d(x)$ *is the distance from* x *to* ∂G.

Then the problem (L) *has a unique solution*

$$u \in W^{2,q}_{loc}(G) \cap V^2_{p,\alpha}(G) \cap C^\lambda(\overline{G})$$

and the following a priori estimate is valid

(4.4.16) $\|u\|_{C^\lambda(\overline{G})} \leq K$

with the constant K *independent of* u *and defined only by* $N, q, \nu, \mu, \lambda, s, k_1$, $k_2, k_3, \|f\|_{L^q(G)}, \max_{x \in \overline{G}} \mathcal{A}(|x|), \|\varphi\|_{V^{2-1/q}_{q,\alpha}(\partial G)}, \int_0^d \frac{\mathcal{A}(t)}{t} dt$ *and the domain* G.

PROOF. In virtue of Theorem 4.1 the problem (L) has a unique solution $u \in W^{2,q}_{loc}(G) \cap C^0(\overline{G})$. Using the Hölder inequality with $s = \frac{q}{p} > 1$, $s' = \frac{q}{q-p}$

we obtain

$$\int\limits_G r^\alpha |f|^p dx = \int\limits_G r^{\alpha/s} |f|^p \cdot r^{\alpha/s'} dx \leq \left(\int\limits_G r^\alpha |f|^q dx \right)^{p/q} \cdot \left(\int\limits_G r^\alpha dx \right)^{(q-p)/q}$$

$$\leq C \|f\|^p_{V^0_{q,\alpha}(G)},$$

since $\alpha + N > p(2 - \lambda) > 0$. Now it is easy to verify that all assumptions of Theorem 4.48 are fulfilled and therefore according to this theorem $u \in V^2_{p,\alpha}(G)$ and the estimate (4.4.9) is true.

Now let us prove $u \in C^\lambda(\overline{G})$ and the estimate (4.4.16). For this we apply the local estimates of §§4.4, 4.6, 4.9. We consider the partition of unity

$$1 = \sum_k \zeta_k(x), \text{ where } \zeta_k(x) \in C_0^\infty(G^j), \quad \bigcup_j G^j = G.$$

Let $\Phi \in V^2_{q,\alpha}(G) \cap C^0(\overline{G})$ be an arbitrary extension of the boundary function φ into G. The function $v = u - \Phi$ then satisfies the homogeneous Dirichlet problem

$$(L)_0 \qquad \begin{cases} Lv = F & \text{in } G, \\ v = 0 & \text{on } \partial G. \end{cases}$$

with $F(x)$ determined by (4.2.4). Setting $v_k(x) = \zeta_k(x)v(x)$ we have

$$(4.4.17) \qquad \begin{aligned} Lv_k(x) = F_k(x) &\equiv \zeta_k(x)F(x) + 2a^{ij}(x)\zeta_{kx_j}v_{x_i} + \Big(a^{ij}(x)\zeta_{kx_ix_j} + \\ &+ a^i(x)\zeta_{kx_i}\Big)v(x). \end{aligned}$$

At first we consider such $\zeta_k(x)$ the support of which intersects with the d-vicinity of the origin $\mathcal{O}$. The assumptions of our theorem guarantee the fulfilment of all conditions of Theorems 4.21, 4.33 and therefore we have

$$(4.4.18) \qquad \begin{aligned} |F_k(x)| &\leq c_k \left(|F(x)| + |\nabla v(x)| + |x|^{-1}|v(x)| \right) \leq \\ &\leq c_k \left(|F(x)| + |x|^{\lambda-1} \right) \leq c_k \Big(|f(x)| + |x|^{\lambda-1} + \\ &+ |\Phi_{xx}| + \frac{A(|x|)}{|x|}|\nabla\Phi| + \frac{A(|x|)}{|x|^2}|\Phi| \Big), \quad x \in G_0^d, \end{aligned}$$

if we recall (4.2.4). Now we verify that we can apply to the solutions of (4.4.17) Theorems 4.21, 4.33, too. In fact, by (4.4.18) and the assumption

(dd), we obtain

$$\int\limits_{G_0^\varrho} r^{4-N} F_k^2(x) dx \le 2c_k^2 \int\limits_{G_0^\varrho} \left(r^{4-N} F^2(x) + r^{2\lambda+2-N} \right) dx \le 2c_k^2 k_1^2 \varrho^{2s} +$$

$$+ 2c_k^2 \frac{meas\,\Omega}{2\lambda+2} \varrho^{2\lambda+2} \le C_k \varrho^{2\widetilde{s}} \quad \widetilde{s} > \lambda, \ \varrho \in (0, d).$$

Similarly

$$\|F_k\|_{N;G_{\varrho/2}^\varrho} \le c_k \left(\|f\|_{N;G_{\varrho/4}^{2\varrho}} + \|\varphi\|_{V_{N,0}^{2-1/N}(\Gamma_{\varrho/4}^{2\varrho})} \right) +$$

$$+ \left(\int\limits_{G_0^\varrho} r^{N(\lambda-1)} dx \right)^{1/N} \le c_k k_1 \varrho^{s-1} + c_k \varrho^\lambda.$$

Hence (4.2.61) follows with $s > \lambda$ and $F_k \in L^N(G)$ since $s > \lambda > 1$. Thus we verify the conditions of Theorem 4.21.

Further,

$$\int\limits_{G_{\varrho/2}^\varrho} r^{2q-N} |F_k(x)|^q dx \le c_k^q \int\limits_{G_{\varrho/2}^\varrho} \left(r^{2q-N} |f(x)|^q + r^{2q-N} |\Phi_{xx}|^N + r^{q-N} |\nabla\Phi|^q \right.$$

$$\left. + r^{-N} |\Phi|^q + r^{q(\lambda+1)-N} \right) dx \le$$

$$\le \widetilde{c}_k^q \varrho^{2q-N-\alpha} \int\limits_{G_{\varrho/2}^\varrho} \left(r^\alpha |f|^q + r^\alpha |\Phi_{xx}|^q + r^{\alpha-q} |\nabla\Phi|^q + \right.$$

$$\left. + r^{\alpha-2q} |\Phi|^q \right) dx + C_k \varrho^{q(\lambda+1)} \le$$

$$\le c_k k_2^q \varrho^{q\lambda}, \quad \varrho \in (0, d)$$

because of the assumption (dd). Thus we verified the assumption (bb) and therefore all conditions of Theorem 4.33 are fulfilled.

Finally, on the basis of the Alexandrov Maximum Principle (see Theorem 4.2) we have

$$M_0 = \sup_G |u| \le \sup_{\partial G} |\varphi| + c \|f\|_{L^N(G)}.$$

Thus, by Theorems 4.21, 4.33, we get $v_k(x) \in C^\lambda(\overline{G_0^d})$ and

$$(4.4.19) \qquad \|v_k\|_{C^\lambda(\overline{G_0^d})} \le K_k.$$

Now let us consider such $\zeta_k(x)$ the support of which intersects with the Γ_d with some $d > 0$. In this case we can apply the Widman local estimates

(see §4.9) near the smooth piece of the boundary of G. In particular, by Theorem 4.42 with regard to the assumption (cc), we obtain

$$(4.4.20) \qquad \qquad \|F_k(x)\| \le C_k d^{\lambda-1}(x).$$

The inequality (4.4.20) and the assumption (cc) allow to apply Theorem 4.42 to the equation (4.4.17), too. Therefore we can conclude that

$$(4.4.21) \qquad \qquad \|v_k(x)\|_{C^\lambda(\overline{G_d^{jk}})} \le C_{j_k}.$$

Finally, if the support of $\zeta_k(x)$ belongs strictly to the angular domain G since $u \in W_{loc}^{2,q}(G)$, $q \ge \frac{N}{2-\lambda}$, by the Sobolev Imbedding Theorem, we have that $v_k(x) \in C^\lambda(G'_k)$, $\forall G'_k \subset\subset G$ and in virtue of Theorem 4.7 for $k = 2$ the estimate

$$(4.4.22) \qquad \|v_k(x)\|_{C^\lambda(G'_k)} \le C\|v_k\|_{W^{2,q}(G'_k)} \le C_k$$

holds.

Since $v(x) = \sum_k v_k(x)$ and this sum is finite, from the estimates (4.4.19), (4.4.21), (4.4.22) it follows that $v \in C^\lambda(\overline{G})$ and the validity of (4.4.16). Thus our Theorem is proved. $\qquad \qquad \qquad \square$

Since the Widman results (§4.9) are true for the Liapunov domains, in this way the following theorem is proved.

THEOREM 4.50. *Let $\Gamma_d \in C^\lambda$ with some $d > 0$. Let the assumptions of Theorem 4.49 be fulfilled. Then the problem (L) has a unique solution $u \in W_{loc}^{2,q}(G) \cap C^\lambda(\overline{G})$ and the estimate (4.4.16) holds.*

4.5. Notes

The behavior of the problem (L)-solutions near a conical point was studied in the case of the Hölder continuity coefficients in [16] - [19], [398, 399]. Our presentation of the results of this chapter follows [53, 56, 57, 58, 63, 66]. These results were generalized in [369, 50] on linear elliptic equations whose coefficients may degenerate near a conical boundary point. Theorem 4.48 was known earlier in two cases: either when the problem (L) equation is the Poisson equation [400] or when G is a cone, but the lowest equation coefficients are smoother (Theorem 2.2 [189]). Theorems 4.49 and 4.50 are new because without our new estimates from §§4.5, 4.6 as well as the Widman estimates from §4.9 they could not be proved. Moreover, in these theorems we weaken the smoothness requirement on the surface $\partial G \setminus \mathcal{O}$. In Theorem 4.49 these requirements allow a locally smooth piece of surface to "straighten". In Theorem 4.50 the surface $\partial G \setminus \mathcal{O}$ can be the Liapunov surface because in such a domain the Widman results (§4.9) are correct, and we use them in the neighborhood of a smooth piece of $\partial G \setminus \mathcal{O}$.

Other boundary value problems (the Neumann problem, mixed problem) for general elliptic second order equations in nonsmooth domains have been studied by A. Azzam [20], A. Azzam and E. Kreyszig [22, 23], G. Lieberman [230] and V. Chernetskiy [81].

CHAPTER 5

The Dirichlet problem for elliptic linear divergent equations in a nonsmooth domain

5.1. The best possible Hölder exponents for weak solutions

5.1.1. Introduction. In this Section, the behavior of weak solutions of the Dirichlet problem for a second order elliptic equation in a neighborhood of a boundary point is studied. Under certain assumptions on the structure of the domain boundary in a neighborhood of the boundary point $\mathcal{O}$ and on the equation coefficients, one obtains a power modulus of continuity at $\mathcal{O}$ for a generalized solution of the Dirichlet problem vanishing at that point. Moreover, the exponent is the best possible for domains with the assumed boundary structure in that neighborhood. The assumptions on the equation coefficients are essential, as the example in §5.1.4 shows.

Next, it is shown, with the help of the previous results on the continuity modulus at boundary points of the domain, that a weak solution of the Dirichlet problem in a domain G belongs to a Hölder space C^λ in the closed domain $\overline{G}$, the exponent λ being determined by the structure of the domain boundary and being the best possible for the class of domains in question.

We consider weak solutions of the Dirichlet problem for the linear uniformly elliptic second order equation of the divergent form

(DL)
$$\begin{cases} \frac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j} + a^i(x)u) + b^i(x)u_{x_i} + c(x)u = \\ \qquad = g(x) + \frac{\partial f^j(x)}{\partial x_j}, \quad x \in G; \\ u(x) = \varphi(x), \quad x \in \partial G \end{cases}$$

(summation over repeated indices from 1 to N is understood.)

At first, we describe our very general assumptions on the structure of the domain boundary in a neighborhood of the boundary point $\mathcal{O}$. Namely, we denote by $\theta(r)$ the least eigenvalue of the Beltrami operator $\triangle_\omega$ on Ω_r with the Dirichlet condition on $\partial\Omega_r$. According to the variational theory of

eigenvalues (for the analog of the Wirtinger inequality see (2.3.2) Theorem 2.15), we have

$$(5.1.1) \qquad \int\limits_{\Omega_r} u^2(\omega)d\Omega_r \le \frac{1}{\theta(r)} \int\limits_{\Omega_r} |\nabla_\omega u|^2 \, d\Omega_r, \quad \forall u \in W_0^{1,2}(\Omega_r).$$

Assumption I.

$$\theta(r) \ge \theta_0 + \theta_1(r) \ge \theta_2 > 0, \text{ where}$$

$$\theta_0, \theta_2 \text{ are positive constants and}$$

$$\theta_1(r) \text{ is a Dini continuous at zero function}$$

$$\lim_{r \to 0} \theta_1(r) = 0, \quad \int\limits_0^d \frac{|\theta_1(r)|}{r} dr < \infty.$$

Assumption II.

- *(i)*   Uniform ellipticity condition

$$\nu |\xi|^2 \le \sum_{i,j=1}^N a^{ij}(x)\xi_i\xi_j \le \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \ x \in \overline{G}$$

     with some $\nu, \mu > 0$.
- *(ii)* $a^{ij}(0) = \delta_i^j$.
- *(iii)* $a^{ij}(x) \in C^0(\overline{G}), \ (i,j = 1,\ldots,N); \quad a^i(x), b^i(x) \in L^p(G), \ (i = 1,\ldots,N)$ and $c(x) \in L^{p/2}(G), \ p > N.$
- *(v)*   There exists a monotonically increasing nonnegative function $\mathcal{A}$
     such that

$$\left(\sum_{i,j=1}^N |a^{ij}(x) - a^{ij}(0)|^2 \right)^{1/2} + |x| \left(\sum_{i=1}^N |a^i(x)|^2 + \sum_{i=1}^N |b^i(x)|^2 \right)^{1/2} +$$

$$+ |x|^2 |c(x)| \le \mathcal{A}(|x|) \ \forall x \in \overline{G}.$$

- *(iv)* $g(x), f^i(x) \ (i = 1,\ldots,N) \in L^2(G), \ \varphi(x) \in W^{\frac{1}{2}}(\partial G).$

DEFINITION 5.1. The function $u(x)$ is called a *weak* solution of the problem (DL) provided that $u(x) - \Phi(x) \in W_0^1(G)$ and satisfies the integral

identity

$$\int\limits_{G} \left\{ a^{ij}(x)u_{x_j}\eta_{x_i} + a^i(x)u\eta_{x_i} - b^i(x)u_{x_i}\eta - c(x)u\eta \right\} dx =$$

(II)
$$= \int\limits_{G} \left\{ f^i(x)\eta_{x_i} - g(x)\eta \right\} dx$$

for all $\eta(x) \in W_0^1(G)$.

LEMMA 5.2. *Let $u(x)$ be a weak solution of (DL). For*

$$\forall v(x) \in V := \left\{ v \in W^1(G_0^\varrho) \mid v(x) = 0, \ x \in \Gamma_0^\varrho \right\}$$

the equality

$$\int\limits_{G_0^\varrho} \left\{ \left(a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x) \right)v_{x_i} + \left(g(x) - b^i(x)u_{x_i} - c(x)u \right)v \right\} dx =$$

(5.1.2)
$$= \int\limits_{\Omega_\varrho} \left(a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x) \right)v(x)\cos(r,x_i)d\Omega_\varrho$$

holds for a.e. $\varrho \in (0,d)$.

PROOF. By $u(x) \in W_0^1(G)$ and because of

$$\int\limits_{G_0^\varrho} |\nabla u|^2 dx = \int\limits_0^\varrho d\varrho \int\limits_{\Omega_\varrho} |\nabla u(r,\omega)|^2 d\Omega_r,$$

from the Fubini Theorem follows that the function

(5.1.3)
$$V(r) = \int\limits_{\Omega_r} |\nabla u(r,\omega)|^2 d\Omega_r$$

is determined and finite for almost every $r \in (0,d)$. We consider the function

(5.1.4)
$$J(\varrho) \equiv \int\limits_{\Omega_\varrho} \left(a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x) \right)v(x)\cos(r,x_i)d\Omega_\varrho$$

for almost every $\varrho \in (0,d)$ for all $v \in V$. By virtue of ellipticity condition and assumptions on the equation coefficients we have

$$a^{ij}(x)u_{x_j}\cos(r,x_i) \leq \mu|\nabla u| \text{ and } a^i(x)u\cos(r,x_i) \leq r^{-1}\mathcal{A}(r)|u|,$$

therefore using the Cauchy inequality, we get

(5.1.5)
$$J(\varrho) \leq \left(1 + \mu\varrho^{N-1} + \mathcal{A}(\varrho)\varrho^{N-2} \right) \int\limits_{\Omega} (|\nabla u|^2 + u^2 + v^2)d\Omega.$$

Since the integral (5.1.3) is finite for almost every $r \in (0,d)$ from (5.1.5) follows that the function $J(\varrho)$ is determined and finite for almost every $\varrho \in (0,d)$.

Now let $\chi_\varrho(x)$ be the characteristic function of the set G_0^ϱ and $(\chi_\varrho)_h$ be the regularization of χ (see §1.5.2, chapter 1)

$$(5.1.6) \qquad (\chi_\varrho)_h(x) = \int\limits_G \psi_h(|x-y|)\chi_\varrho(y)dy$$

where $\psi_h(|x-y|)$ is the mollifier. It is well known that the regularization is an infinite-differentiable function in the whole of the space and

$$\frac{\partial(\chi_\varrho)_h(x)}{\partial x_i} = \int\limits_G \chi_\varrho(y)\frac{\partial\psi_h(|x-y|)}{\partial x_i}dy =$$

$$(5.1.7) \qquad = -\int\limits_G \chi_\varrho(y)\frac{\partial\psi_h(|x-y|)}{\partial y_i}dy \quad (i=1,\ldots,N).$$

Let us take a function $v \in W_0^1(G)$, and set $\eta(x) = (\chi_\varrho)_h(x)v(x)$ in the integral identity (II). It is easily seen that such a function $\eta(x)$ is admissible and moreover,

$$\frac{\partial\eta}{\partial x_i} = (\chi_\varrho)_h(x)\frac{\partial v}{\partial x_i} - v(x)\int\limits_G \chi_\varrho(y)\frac{\partial\psi_h(|x-y|)}{\partial y_i}dy.$$

Denoting by

$$\mathfrak{A}(x) \equiv \left(a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x)\right)v_{x_i} + \left(g(x) - b^i(x)u_{x_i} - c(x)u\right)v(x)$$

from (II) follows that

$$(5.1.8) \quad \int\limits_G \mathfrak{A}(x)(\chi_\varrho)_h(x)dx = \int\limits_G \left(a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x)\right)v(x)\times$$

$$\times \left\{\int\limits_G \chi_\varrho(y)\frac{\partial\psi_h(|x-y|)}{\partial y_i}dy\right\}dx = \text{(by the Fubuini Theorem)}$$

$$= \int\limits_{G} \chi_{\varrho}(y) \left\{ \int\limits_{G} (a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x))v(x)\frac{\partial\psi_h(|x-y|)}{\partial y_i}dx \right\}dy =$$

(by the Theorem about differentiability of the integral)

$$= \int\limits_{G} \chi_{\varrho}(y) \left\{ \frac{\partial}{\partial y_i} \int\limits_{G} (a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x))v(x)\psi_h(|x-y|)dx \right\}dy =$$

(by definition of the regularization)

$$= \int\limits_{G} \chi_{\varrho}(y) \left\{ \frac{\partial}{\partial y_i}\left((a^{ij}u_{x_j} + a^i u - f^i)v \right)_h(y) \right\}dy =$$

$$= \int\limits_{G_0^\varrho} \frac{\partial}{\partial y_i}\left((a^{ij}u_{x_j} + a^i u - f^i)v \right)_h(y)dy =$$

$$= \int\limits_{\partial G_0^\varrho} \left((a^{ij}u_{x_j} + a^i u - f^i)v \right)_h(y) \cos(\overrightarrow{n}, y_i)d_y\sigma =$$

$$= \int\limits_{\Omega_\varrho} (a^{ij}u_{x_j} + a^i u - f^i)v(x)\cos(r, x_i)d\Omega_\varrho +$$

$$+ \int\limits_{\partial G_0^\varrho} \left\{ \left((a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x))v(x) \right)_h(y) - \right.$$

$$\left. - \left(a^{ij}(y)u_{x_j}(y) + a^i(y)u(y) - f^i(y) \right)v(y) \right\} \cos(\overrightarrow{n}, y_i)d_y\sigma$$

in virtue of $\partial G_0^\varrho = \Gamma_0^\varrho \cup \Omega_\varrho$ and $v(x)\big|_{\Gamma_0^\varrho} = 0$.

Now we show that $\mathfrak{A}(x) \in L^1(G)$. First of all because of the assumptions on coefficients,

$$|\mathfrak{A}(x)| \leq \left(\mu|\nabla u| + |x|^{-1}\mathcal{A}(|x|)|u| + |\overrightarrow{f}| \right)|\nabla v| +$$
$$+ \left(|g| + |x|^{-1}\mathcal{A}(|x|)|\nabla u| + |x|^{-2}\mathcal{A}(|x|)|u| \right)|v|.$$

Using the Cauchy inequality, we have

$$|\mathfrak{A}(x)| \leq c(\mu)\left(|\nabla u|^2 + |\nabla v|^2 + v^2 + g^2 + |\overrightarrow{f}|^2 \right) +$$
$$+ \mathcal{A}(|x|)\left(|\nabla u|^2 + |\nabla v|^2 + |x|^{-2}(u^2 + v^2) \right).$$

Further, we apply the inequality (2.5.2) (see Corollary 2.23)

$$\int_{G_0^\varrho} |x|^{-2} u^2(x) dx \leq C \int_{G_0^\varrho} |\nabla u|^2 dx.$$

Because of this inequality and the above bounds, it is obvious that $\mathfrak{A}(x) \in L^1(G)$.

Now in virtue of Lemma 1.21 we can then obtain

$$(5.1.9) \quad \lim_{h \to 0} \int_G \mathfrak{A}(x)(\chi_\varrho)_h(x) dx = \int_G \mathfrak{A}(x)\chi_\varrho(x) dx = \int_{G_0^\varrho} \mathfrak{A}(x) dx =$$

$$= \int_{G_0^\varrho} \Big(\big(a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x)\big) v_{x_i} + \big(g(x) - b^i(x)u_{x_i} - c(x)u\big)v(x)\Big) dx$$

Next, setting $\mathcal{A}_i(x) \equiv a^{ij}(x)u_{x_j} + a^i(x)u - f^i(x)$ we have $\mathcal{A}_i(x)v(x) \in L^1(G)$ $(i = 1, \dots, N)$ and in virtue of Lemma 1.20

$$(5.1.10) \quad \lim_{h \to 0} \big\| \big(\mathcal{A}_i(x)v(x)\big)_h - \mathcal{A}_i(x)v(x)\big) \big\|_{L^1(G_0^\varrho)} = 0, \ (i = 1, \dots, N).$$

Representing $G_0^\varrho = (0, \varrho) \times \Omega_\varrho$, because of Lemma 1.16, we obtain from (5.1.10) that for some subsequence $\{h_n\}$

$$(5.1.11) \quad \lim_{h_n \to 0} \big\| \big(\mathcal{A}_i(x)v(x)\big)_{h_n} - \mathcal{A}_i(x)v(x)\big) \big\|_{L^1(\Omega_\varrho)} = 0 \ \text{ a.e. } \varrho \in (0, d)$$

$$(i = 1, \dots, N).$$

Similarly, representing $G_0^\varrho = \Gamma_0^\varrho \times (-\omega_0, \omega_0)$, because of the same Lemma 1.16, we obtain from (5.1.10) that for some subsequence $\{h_m\}$

$$(5.1.12) \quad \lim_{h_m \to 0} \big\| \big(\mathcal{A}_i(x)v(x)\big)_{h_m n} - \mathcal{A}_i(x)v(x)\big) \big\|_{L^1(\Gamma_0^\varrho)} = 0$$

$$\text{a.e. } \omega_1 \in (-\omega_0, \omega_0), \ (i = 1, \dots, N).$$

Thus, performing in (5.1.8) the passage to the limit over $h \to 0$ by (5.1.9)-(5.1.12) we get the required equality. Lemma 5.2 is proved. $\square$

5.1.2. The estimate of the weighted Dirichlet integral. Setting $v = u - \Phi$ we obtain that $v(x)$ satisfies the integral identity

$$\int_G \big\{ a^{ij}(x)v_{x_j}\eta_{x_i} + a^i(x)v\eta_{x_i} - b^i(x)v_{x_i}\eta - c(x)v\eta \big\} dx =$$

$$= \int_G \big\{ \mathcal{F}^i(x)\eta_{x_i} - \mathcal{G}(x)\eta \big\} dx \qquad (II)_0$$

for all $\eta(x) \in W_0^{1,2}(G)$, where

(5.1.13)
$$\mathcal{F}^i(x) = f^i(x) - a^{ij}(x)D_j\Phi - a^i(x)\Phi(x) \quad (i = 1, \ldots, N),$$
$$\mathcal{G}(x) = g(x) - b^i(x)D_i\Phi - c(x)\Phi(x).$$

At first, we will obtain a global estimate for the weighted Dirichlet integral.

THEOREM 5.3. *Let $u(x)$ be a weak solution of the problem (DL) and suppose that assumptions I, II are satisfied with a function $\mathcal{A}(r)$ that is continuous at zero . Let us assume, in addition, that*

(5.1.14)
$$g \in \overset{\circ}{W}_\alpha^0(G), \ f \in \overset{\circ}{W}_{\alpha-2}^0(G), \ \varphi \in \overset{\circ}{W}_{\alpha-2}^{1/2}(\partial G),$$

where

(*)
$$\begin{cases} 4 - N - 2\lambda < \alpha \le 2 \\ \lambda = \frac{1}{2}\left(2 - N + \sqrt{(N-2)^2 + 4\theta_0}\right) \end{cases}$$

Then we have $u(x) \in \overset{\circ}{W}_{\alpha-2}^1(G)$ and

(5.1.15)
$$\|u\|_{\overset{\circ}{W}_{\alpha-2}^1(G)} \le C\Big\{\|u\|_{W^{1,2}(G)} + \|g\|_{\overset{\circ}{W}_\alpha^0(G)} +$$
$$+ \|f\|_{\overset{\circ}{W}_{\alpha-2}^0(G)} + \|\varphi\|_{\overset{\circ}{W}_{\alpha-2}^{1/2}(\partial G)}\Big\},$$

where $C > 0$ is the constant dependent only on $\alpha, \lambda, \omega_0, N, \mu, G$ and independent of u.

PROOF. Replacing u by $v = u - \Phi$ and setting $\eta(x) = r_\varepsilon^{\alpha-2}v(x)$, with regard to

$$\eta_{x_i} = r_\varepsilon^{\alpha-2}v_{x_i} + (\alpha - 2)r_\varepsilon^{\alpha-3}\frac{x_i - \varepsilon l_i}{r_\varepsilon}v(x)$$

we obtain

(5.1.16)
$$\int_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx = \frac{2-\alpha}{2}\int_G r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)(v^2)_{x_i}dx +$$

$$+ (2-\alpha)\int_G \left((a^{ij}(x) - a^{ij}(0))v_{x_j} + a^i(x)v + \mathcal{F}^i(x)\right)r_\varepsilon^{\alpha-3}\frac{x_i - \varepsilon l_i}{r_\varepsilon}v(x)dx -$$

$$- \int_G \left((a^{ij}(x) - a^{ij}(0))v_{x_j} + a^i(x)v + \mathcal{F}^i(x)\right)r_\varepsilon^{\alpha-2}v_{x_i}dx +$$

$$+ \int_G \left(b^i(x)v_{x_i} + c(x)v - \mathcal{G}(x)\right)r_\varepsilon^{\alpha-2}v(x)dx$$

We transform the first integral on the right

$$\int\limits_G r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)(v^2)_{x_i} dx = -\int\limits_G v^2 \frac{\partial}{\partial x_i}\left(r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\right) dx$$

because of $v \in W_0^{1,2}(G)$. By elementary calculation

$$\frac{\partial}{\partial x_i}\left(r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)\right) = N r_\varepsilon^{\alpha-4} + (\alpha - 4)(x_i - \varepsilon l_i) r_\varepsilon^{\alpha-5}\frac{x_i - \varepsilon l_i}{r_\varepsilon} =$$
$$= (N + \alpha - 4)r_\varepsilon^{\alpha-4},$$

we obtain

$$(5.1.17) \quad \frac{2-\alpha}{2}\int\limits_G r_\varepsilon^{\alpha-4}(x_i - \varepsilon l_i)(v^2)_{x_i} dx =$$
$$= \frac{(2-\alpha)(4 - N - \alpha)}{2}\int\limits_G r_\varepsilon^{\alpha-4}v^2 dx.$$

We estimate the other integrals on the right by using our assumptions and (5.1.13)

$$(5.1.18) \quad \begin{aligned} \left|\left(a^{ij}(x) - a^{ij}(0)\right)v_{x_j} + a^i(x)v + \mathcal{F}^i(x)\right| &\le \\ \le \mathcal{A}(r)|\nabla v| + \mathcal{A}(r)r^{-1}(|v| + |\Phi|) + \mu|\nabla\Phi| + |f|; \\ \left|b^i(x)v_{x_i} + c(x)v - \mathcal{G}(x)\right| &\le \\ \le \mathcal{A}(r)r^{-1}(|\nabla v| + |\nabla\Phi|) + \mathcal{A}(r)r^{-2}(|v| + |\Phi|) + |g|. \end{aligned}$$

Now from (5.1.16), (5.1.17) it follows that

$$(5.1.19) \quad \int\limits_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx \le \frac{(2-\alpha)(4 - N - \alpha)}{2}\int\limits_G r_\varepsilon^{\alpha-4}v^2 dx +$$

$$+ c(N,\alpha)\int\limits_G \left\{ r_\varepsilon^{\alpha-2}\mathcal{A}(r)\Big\langle r^{-1}|\nabla v|(|v| + |\Phi|) + r^{-1}|v||\nabla\Phi| + r^{-2}(v^2 + |v||\Phi|) + \right.$$

$$+ |\nabla v|^2 + \Big\rangle + r_\varepsilon^{\alpha-2}(\mu|\nabla\Phi||\nabla v| + |v||g|) + r_\varepsilon^{\alpha-3}\mathcal{A}(r)(|v||\nabla v| +$$

$$\left. + r^{-1}v^2 + r^{-1}|v||\Phi|) + \mu r_\varepsilon^{\alpha-3}|v||\nabla\Phi| \right\} dx.$$

Further, we estimate the following using the Cauchy inequality with $\forall \delta > 0$

$$r^{-1}|\nabla v||v| \le \frac{1}{2}|\nabla v|^2 + \frac{1}{2}r^{-2}|v|^2;$$

$$r^{-1}|\nabla v||\Phi| \le \frac{1}{2}|\nabla v|^2 + \frac{1}{2}r^{-2}|\Phi|^2;$$

$$r^{-1}|\nabla \Phi||v| \le \frac{1}{2}|\nabla \Phi|^2 + \frac{1}{2}r^{-2}|v|^2;$$

$$r^{-2}|v||\Phi| \le \frac{1}{2}r^{-2}|v|^2 + \frac{1}{2}r^{-2}|\Phi|^2;$$

(5.1.20)
$$\mu|\nabla v||\nabla \Phi| \le \frac{\delta}{2}|\nabla v|^2 + \frac{\mu^2}{2\delta}|\nabla \Phi|^2;$$

$$|g||v| = (r^{-1}|v|)(r|g|) \le \frac{\delta}{2}r^{-2}|v|^2 + \frac{1}{2\delta}r^2|g|^2;$$

$$r_\varepsilon^{-1}|\nabla v||v| \le \frac{1}{2}|\nabla v|^2 + \frac{1}{2}r_\varepsilon^{-2}|v|^2;$$

$$|v||\Phi| \le \frac{1}{2}|v|^2 + \frac{1}{2}|\Phi|^2;$$

$$\mu r_\varepsilon^{-1}|\nabla \Phi||v| \le \frac{\delta}{2}r_\varepsilon^{-2}|v|^2 + \frac{\mu^2}{2\delta}|\nabla \Phi|^2.$$

As a result from (5.1.19) we obtain

(5.1.21)
$$\int\limits_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx \le \frac{(2-\alpha)(4-N-\alpha)}{2}\int\limits_G r_\varepsilon^{\alpha-4}v^2 dx +$$

$$+ c(N,\alpha,\mu)\int\limits_G \Big\{ r_\varepsilon^{\alpha-2}\mathcal{A}(r)|\nabla v|^2 + r_\varepsilon^{\alpha-2}r^{-2}\mathcal{A}(r)|v|^2 +$$

$$+ r_\varepsilon^{\alpha-2}\mathcal{A}(r)(|\nabla \Phi|^2 + r^{-2}|\Phi|^2) + r_\varepsilon^{\alpha-4}\mathcal{A}(r)|v|^2 + r_\varepsilon^{\alpha-3}r^{-1}\mathcal{A}(r)|v|^2 +$$

$$+ r_\varepsilon^{\alpha-3}r^{-1}\mathcal{A}(r)|\Phi|^2 + r_\varepsilon^{\alpha-2}|\nabla \Phi|^2 + \frac{\delta}{2}r_\varepsilon^{\alpha-2}|\nabla v|^2 +$$

$$+ \frac{\delta}{2}r_\varepsilon^{\alpha-2}r^{-2}|v|^2 + r_\varepsilon^{\alpha-2}r^2|g|^2 \Big\}dx.$$

Now we apply the inequality (2.5.8) to the first integral from the right side; because of the condition (*) of our theorem we have

$$C(\lambda, N, \alpha) = 1 - \frac{(2-\alpha)(4-N-\alpha)}{2}H(\lambda, N, \alpha) > 0.$$

Therefore we can write the inequality (5.1.21) in the following way

$$C(\lambda, N, \alpha) \int\limits_{G} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx \le c_0 \left[A(d) + \delta\right] \int\limits_{G} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx +$$

$$+ c_1(N, \alpha, \mu) \int\limits_{G} A(r) \left\{ r_\varepsilon^{\alpha-2} r^{-2} |v|^2 + r_\varepsilon^{\alpha-4} |v|^2 \right\} dx + \delta \int\limits_{G} \left(r_\varepsilon^{\alpha-2} r^{-2} |v|^2 \right) dx +$$

$$+ c_2(N, \alpha, \mu, \omega_0) \int\limits_{G} \left(r^{\alpha-4} |\Phi|^2 + r^{\alpha-2} |\nabla \Phi|^2 + r^\alpha |g|^2 \right) dx, \ \forall \delta > 0.$$

(Here we use property 1) of the function $r_\varepsilon(x)$.) We apply now Lemmas 2.30 and 2.31 and choose $\delta > 0$ from the condition

$$\left(1 + \frac{1}{\lambda(\lambda + N - 2)}\right)\delta = \frac{1}{2}C(\lambda, N, \alpha).$$

As a result we obtain

$$\int\limits_{G} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx \le c(N, \alpha, \mu\lambda, \omega_0) \int\limits_{G} \left\{ A(r) r_\varepsilon^{\alpha-2} |\nabla v|^2 + |v|^2 + \right.$$

$$\left. + r^{\alpha-2}(|\nabla \Phi|^2 + r^{-2}|\Phi|^2) + r^\alpha |g|^2 \right\} dx.$$

We now write the representation $G = G_0^d \cup G_d$ and choose $d > 0$ so small that

$$A(d) c(N, \alpha, \mu\lambda, \omega_0) < 1.$$

(This is possible because of the continuity at zero of $A(r)$.)

Thus, finally we obtain

$$\int\limits_{G} r_\varepsilon^{\alpha-2} |\nabla u|^2 dx \le c(N, \alpha, \mu\lambda, \omega_0) \int\limits_{G} \left\{ u^2 + |\nabla u|^2 + r^{\alpha-2} |\nabla \Phi|^2 + \right.$$

$$\left. + r^{\alpha-4} |\Phi|^2 + r^\alpha |g|^2 \right) dx, \ \forall \varepsilon > 0.$$

Passaging to the limit when $\varepsilon \to +0$ by the Fatou Theorem we have the required estimate (5.1.15).   □

We pass now to the derivation of the local estimate for the weighted Dirichlet integral. For this together with *Assumptions I* and *II* we make the following

Assumptions III.

- *(ivv) the function $A(r)$ satisfies the Dini condition at zero;*

- *(w)* $\int\limits_{G} r^{4-Nn-2\lambda}\mathcal{H}^{-1}(r)g^2(x)dx < \infty;$

$$\int\limits_{G} r^{4-N-2\lambda}\mathcal{H}^{-1}(r)\varphi^2(x)dx < \infty;$$

$$\int\limits_{G} r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\left(\sum_{i=1}^{N}|f^i(x)|^2dx + |\nabla\Phi|^2\right) < \infty,$$

where $\mathcal{H}(r)$ is a continuous, monotone increasing, Dini continuous at zero function.

THEOREM 5.4. *Let $u(x)$ be a weak solution of (DL) and suppose that assumptions I, II, III are satisfied. Then there exist positive constants d, C_1, independent of u, g, f_i, φ such that*

$$\int\limits_{G_0^\varrho} r^{2-N}|\nabla u|^2dx \le C_1\varrho^{2\lambda}\int\limits_{G_0^{2d}}\left\{|u(x)|^2 + |\nabla u|^2 + g^2(x) + \sum_{i=1}^{N}|f^i(x)|^2+\right.$$

$$+|\Phi|^2 + |\nabla\Phi|^2 + r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{-N-2\lambda}\mathcal{H}^{-1}(r)|\Phi|^2+$$

$$(5.1.22)\qquad r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2\bigg\}dx,$$

$$\rho \in (0, d).$$

PROOF. By the above proved Theorem 5.3 we have that $u(x) \in \mathring{W}^1_{\alpha-2}(G)$. Therefore we can apply Lemma 5.2 and take the function $r^{2-n}\big(u(x)-\Phi(x)\big)$ as $v(x)$ in the equal (5.1.2). Now replacing u by $v = u-\Phi$ as a result we obtain

$$\int\limits_{G_0^\varrho}\Big\{\big(a^{ij}(x)v_{x_j} + a^i(x)v - \mathcal{F}^i(x)\big)\big(r^{2-N}v_{x_i} + (2-N)r^{-N}x_iv\big)+$$

$$+ \big(\mathcal{G}(x) - b^i(x)v_{x_i} - c(x)v\big)r^{2-N}v\Big\}dx =$$

$$= \varrho\int\limits_{\Omega}\big(a^{ij}(x)v_{x_j} + a^i(x)v - \mathcal{F}^i(x)\big)v(x)\cos(r, x_i)d\Omega.$$

Hence we have

$$(5.1.23)\quad \int\limits_{G_0^\varrho} r^{2-N}|\nabla v|^2dx = \frac{N-2}{2}\int\limits_{G_0^\varrho} r^{-N}x_i\frac{\partial v^2}{\partial x_i}dx + \varrho\int\limits_{\Omega} v\frac{\partial v}{\partial r}d\Omega+$$

$$+ \int\limits_{G_0^\varrho}\Big\{\big(a^{ij}(x) - a^{ij}(0)\big)\big((N-2)r^{-N}vx_iv_{x_j} - r^{2-N}v_{x_i}v_{x_j}\big)+$$

$$+ (N-2)r^{-N}x_i a^i(x)v^2 + r^{2-N}v\big(b^i(x)v_{x_i} + c(x)v - \mathcal{G}\big) +$$

$$+ r^{2-N}va^i(x)v_{x_i} + r^{2-N}\mathcal{F}^i(x)v_{x_i} + (2-N)r^{-N}vx_i\mathcal{F}^i(x)\Big\}dx +$$

$$+ \varrho \int\limits_{\Omega} \big\{ \big(a^{ij}(x) - a^{ij}(0)\big)vv_{x_j} + a^i(x)v^2 - v\mathcal{F}^i(x)\big\} \cos(r, x_i)d\Omega.$$

The first integral from the right we transform in the following way.

$$\int\limits_{G_0^\varrho} r^{-N}x_i\frac{\partial v^2}{\partial x_i}dx = \int\limits_{\Omega_\varrho} r^{-N}v^2 x_i \cos(r, x_i)d\Omega_\varrho -$$

$$- \int\limits_{G_0^\varrho} v^2\Big(Nr^{-N} - Nx_i r^{-N-1}\frac{x_i}{r}\Big)dx = \int\limits_{\Omega} v^2 d\Omega.$$

Therefore we can rewrite (5.1.23) in this way

$$(5.1.24) \quad \int\limits_{G_0^\varrho} r^{2-N}|\nabla v|^2 dx = \int\limits_{\Omega} \Big(\varrho v\frac{\partial v}{\partial r} + \frac{N-2}{2}v^2\Big)d\Omega +$$

$$+ \int\limits_{G_0^\varrho} \Big\{ \big(a^{ij}(x) - a^{ij}(0)\big)\big((N-2)r^{-N}vx_i v_{x_j} - r^{2-N}v_{x_i}v_{x_j}\big) +$$

$$+ (N-2)r^{-N}x_i a^i(x)v^2 + r^{2-N}v\big(b^i(x)v_{x_i} + c(x)v - \mathcal{G}\big) +$$

$$+ r^{2-N}va^i(x)v_{x_i} + r^{2-N}\mathcal{F}^i(x)v_{x_i} + (2-N)r^{-N}vx_i\mathcal{F}^i(x)\Big\}dx +$$

$$+ \varrho \int\limits_{\Omega} \big\{ \big(a^{ij}(x) - a^{ij}(0)\big)vv_{x_j} + a^i(x)v^2 - v\mathcal{F}^i(x)\big\} \cos(r, x_i)d\Omega.$$

We set $V(\rho) = \int\limits_{G_o^\varrho} r^{2-N}|\nabla v|^2 dx$ and estimate every integral from the right side. The first integral is estimated by Lemma 2.28. We estimate other integrals from the right side by using our assumptions and (5.1.18), (5.1.20) as well as

$$r^{2-N}|v||g| = \big(\sqrt{\mathcal{H}(r)}r^{-\frac{N}{2}}|v|\big)\big(\sqrt{\mathcal{H}^{-1}(r)}r^{2-\frac{N}{2}}|g|\big) \leq \frac{1}{2}\mathcal{H}(r)r^{-N}|v|^2 +$$

$$+ \frac{1}{2}\mathcal{H}^{-1}(r)r^{4-N}|g|^2$$

and

$$r^{2-N}|v_{x_i}\mathcal{F}^i(x)| \le \frac{1}{2}\mathcal{H}(r)r^{2-N}|\nabla v|^2 + \frac{1}{2}\mathcal{H}^{-1}(r)r^{2-N}|\mathcal{F}|^2 \le$$

$$\le \frac{1}{2}\mathcal{H}(r)r^{2-N}|\nabla v|^2 + \frac{1}{2}\mathcal{H}^{-1}(r)r^{2-N}\left(|f|^2 + \mu^2|\nabla\Phi|^2 + r^{-2}\mathcal{A}^2(r)|\Phi|^2\right).$$

Then we obtain

$$(5.1.25) \quad V(\varrho) \le \frac{\rho}{2\lambda + \theta_1(\varrho)h_1(\varrho)}V'(\rho) + c(N,\mu)\int\limits_{G_0^\varrho}\left\{\mathcal{A}(r)r^{2-N}|\nabla v|^2 + \right.$$

$$+ \mathcal{A}(r)r^{-N}v^2 + \mathcal{A}(r)r^{2-N}|\nabla\Phi|^2 + \mathcal{A}(r)r^{-N}\Phi^2 + \mathcal{H}(r)r^{2-N}|\nabla v|^2 +$$

$$+ \mathcal{H}(r)r^{-N}v^2 + \mathcal{H}^{-1}(r)r^{2-N}\left(r^2g^2 + |f|^2 + \mu^2|\nabla\Phi|^2 + r^{-2}\mathcal{A}^2(r)|\Phi|^2\right)\Big\}dx +$$

$$+ \varrho^{2-N}\int\limits_{\Omega_\varrho}\left\{\mathcal{A}(\varrho)|v||\nabla v| + \varrho^{-1}\mathcal{A}(\varrho)v^2 + |v|(|f| + \mu|\nabla\Phi| + \varrho^{-1}\mathcal{A}(\varrho)|\Phi|)\right\}d\Omega_\varrho,$$

where $0 \le h_1(\varrho) \le \frac{2}{\sqrt{\theta_0}+\sqrt{\theta_2}}$. To estimate the last integral from the right

side we apply the Cauchy inequality and the inequality (H-W)

$$(5.1.26) \qquad \varrho^{2-N}\int\limits_{\Omega_\varrho}\left\{\mathcal{A}(\varrho)|v||\nabla v| + \varrho^{-1}\mathcal{A}(\varrho)v^2 + |v|(|f| + \mu|\nabla\Phi| + \right.$$

$$+ \varrho^{-1}\mathcal{A}(\varrho)|\Phi|)\Big\}d\Omega_\varrho \le \left(\mathcal{A}(\varrho) + \mathcal{H}(\varrho)\right)\int\limits_\Omega\left(\varrho^2|\nabla v|^2 + v^2 + \Phi^2\right)d\Omega +$$

$$+ \varrho^2\mathcal{H}^{-1}(\varrho)\int\limits_\Omega\left(|\nabla\Phi|^2 + f^2\right)d\Omega \le c(N,\lambda,\theta_2)\left(\mathcal{A}(\varrho) + \mathcal{H}(\varrho)\right)\varrho V'(\varrho) + F_1(\varrho),$$

where

$$(5.1.27) \qquad F_1(\varrho) = \varrho^2\mathcal{H}^{-1}(\varrho)\int\limits_\Omega\left(|\nabla\Phi|^2 + f^2\right)d\Omega + \mathcal{A}(\varrho)\int\limits_\Omega|\Phi|^2 d\Omega$$

Thus, from (5.1.25) - (5.1.27) we obtain

$$V(\varrho) \le \frac{\rho}{2\lambda + \theta_1(\varrho)h_1(\varrho)}V'(\rho) + c_1(N,\mu,\lambda,\theta_2)\left(\mathcal{A}(\varrho) + \mathcal{H}(\varrho)\right)V(\varrho) +$$

$$+ c_2(N,\mu,\lambda,\theta_2)\left(\mathcal{A}(\varrho) + \mathcal{H}(\varrho)\right)\varrho V'(\varrho) + F_1(\varrho) + F_2(\varrho),$$

where

$$F_2(\varrho) = \int\limits_0^\varrho \mathcal{K}(r) dr$$

(5.1.28) $$\mathcal{K}(r) = \int\limits_{\Omega_r} \left\{ \mathcal{H}^{-1}(r) r^{4-N} g^2 + \mathcal{H}^{-1}(r) r^{2-N} |f|^2 + \right.$$

$$\left. + \mathcal{H}^{-1}(r) r^{2-N} |\nabla \Phi|^2 + r^{-N} \mathcal{H}^{-1}(r) |\Phi|^2 \right\} d\Omega_r.$$

Finally, setting

$$\mathcal{P}(\varrho) = \frac{2\lambda + \theta_1(\varrho) h_1(\varrho)}{\varrho} \cdot \frac{1 - c_1 \big(\mathcal{A}(\varrho) + \mathcal{H}(\varrho)\big)}{1 + \big(2\lambda + \theta_1(\varrho) h_1(\varrho)\big) c_2 \big(\mathcal{A}(\varrho) + \mathcal{H}(\varrho)\big)},$$

(5.1.29)

$$\mathcal{Q}(\varrho) = \frac{2\lambda + \theta_1(\varrho) h_1(\varrho)}{\varrho} \cdot \frac{F_1(\varrho) + F_2(\varrho)}{1 + \big(2\lambda + \theta_1(\varrho) h_1(\varrho)\big) c_2 \big(\mathcal{A}(\varrho) + \mathcal{H}(\varrho)\big)},$$

we get the differential inequality (CP) §1.10 with $\mathcal{N}(\varrho) = \mathcal{B}(\varrho) \equiv 0$

(5.1.30) $$V'(\varrho) \geq \mathcal{P}(\varrho) V(\varrho) - \mathcal{Q}(\varrho), \quad \varrho \in (0, d).$$

It is easy to verify that

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} + \frac{\delta(\varrho)}{\varrho},$$

where $\delta(\varrho)$ satisfies the Dini condition at zero. Therefore we have

$$\int\limits_\varrho^d \mathcal{P}(s) ds = \ln\left(\frac{d}{\varrho}\right)^{2\lambda} + \int\limits_\varrho^d \frac{\delta(s)}{s} ds,$$

From this it follows that

(5.1.31) $$\left(\frac{d}{\varrho}\right)^{2\lambda} \leq \exp\left(\int\limits_\varrho^d \mathcal{P}(s) ds\right) \leq \left(\frac{d}{\varrho}\right)^{2\lambda} \int\limits_0^d \frac{\delta(s)}{s} ds, \quad \forall \varrho \in (0, d).$$

Now because of Theorem 1.52 we obtain

(5.1.32) $$V(\varrho) \leq V(d) \exp\left(-\int\limits_\varrho^d \mathcal{P}(s) ds\right) + \int\limits_\varrho^d \mathcal{Q}(\tau) \exp\left(-\int\limits_\varrho^\tau \mathcal{P}(s) ds\right) d\tau,$$

and in virtue of (5.1.31) hence we have

$$(5.1.33) \qquad V(\varrho) \le C\varrho^{2\lambda}\left(V(d) + \int\limits_{\varrho}^{d} \tau^{-2\lambda}\mathcal{Q}(\tau)d\tau\right),$$

where $C > 0$ is a constant independent of v.

Now we estimate the last integral. Because of (5.1.29) we get

$$\int\limits_{\varrho}^{d} \tau^{-2\lambda}\mathcal{Q}(\tau)d\tau \le c_3 \int\limits_{\varrho}^{d} \tau^{-2\lambda-1}F_1(\tau)d\tau + c_4 \int\limits_{\varrho}^{d} \tau^{-2\lambda-1}F_2(\tau)d\tau$$

From (5.1.27) it follows that

$$(5.1.34) \quad \int\limits_{\varrho}^{d} \tau^{-2\lambda-1}F_1(\tau)d\tau \le \int\limits_{\varrho}^{d} \tau^{-2\lambda+1}\mathcal{H}^{-1}(\tau)\int\limits_{\Omega}\left(|\nabla\Phi|^2 + f^2\right)d\Omega d\tau +$$

$$+ \int\limits_{\varrho}^{d} \tau^{-2\lambda-1}\mathcal{A}(\tau)\int\limits_{\Omega}|\Phi|^2 d\Omega d\tau \le$$

$$\le \int\limits_{G_0^d}\left\{r^{2-2\lambda-N}\mathcal{H}^{-1}(r)\left(f^2 + |\nabla\Phi|^2\right) + r^{-2\lambda-N}\mathcal{H}^{-1}(r)\Phi^2\right\}dx.$$

Further, because of (5.1.28) we change the order of integration and obtain

$$\int\limits_{\varrho}^{d} \tau^{-2\lambda-1}\left(\int\limits_{0}^{\tau}\mathcal{K}(r)dr\right)d\tau = \int\limits_{0}^{\varrho}\mathcal{K}(r)dr\int\limits_{\varrho}^{d}\tau^{-2\lambda-1}d\tau +$$

$$+ \int\limits_{\varrho}^{d}\mathcal{K}(r)dr\int\limits_{r}^{d}\tau^{-2\lambda-1}d\tau = \frac{1}{2\lambda}\int\limits_{0}^{\varrho}\mathcal{K}(r)\left(\varrho^{-2\lambda} - d^{-2\lambda}\right)dr +$$

$$+ \frac{1}{2\lambda}\int\limits_{\varrho}^{d}\mathcal{K}(r)\left(r^{-2\lambda} - d^{-2\lambda}\right)dr \le \frac{1}{2\lambda}\int\limits_{0}^{d}r^{-2\lambda}\mathcal{K}(r)dr.$$

Hence in virtue of (5.1.28) it follows that

$$
(5.1.35) \quad \int_{\varrho}^{d} \tau^{-2\lambda-1} F_2(\tau) d\tau \le \frac{1}{2\lambda} \int_{G_0^d} \Big\{ r^{4-2\lambda-N} \mathcal{H}^{-1}(r) g^2(x) +
$$

$$
+ r^{-2\lambda-N} \mathcal{H}^{-1}(r) \Phi^2 + r^{2-2\lambda-N} \mathcal{H}^{-1}(r) \big(f^2 + |\nabla \Phi|^2 \big) \Big\} dx.
$$

From (5.1.33) - (5.1.35) together with Theorem 5.3 follows the required (5.1.22). Theorem 5.4 is proved.   □

THEOREM 5.5. *Let $u(x)$ be a weak solution of the problem (DL) with $\varphi \equiv 0$ and suppose that assumptions I and II are satisfied with a function $\mathcal{A}(r)$ that is continuous at zero, but not Dini continuous. Let us assume, in addition, that*

$$
(5.1.36) \qquad g \in \mathring{W}_{4-N-2\lambda}^{0}(G), \ f^i \in \mathring{W}_{2-N-2\lambda}^{0}(G) \ i = 1, \dots, N.
$$

Then for every $\varepsilon > 0$ there exist positive constants d, c_ε, independent of u, g, f_i such that

$$
(5.1.37) \quad \int_{G_0^\varrho} r^{2-N} |\nabla u|^2 dx \le c_\varepsilon \varrho^{2\lambda(1-\varepsilon)} \int_{G} \Big\{ |u(x)|^2 + |\nabla u|^2 +
$$

$$
+ r^{4-N-2\lambda} g^2(x) + r^{2-N-2\lambda} f^2(x) \} dx,
$$

$$
\rho \in (0, d), \ \forall \varepsilon > 0.
$$

PROOF. Similar to (5.1.24) we get from (DL)

$$
(5.1.38) \quad \int_{G_0^\varrho} r^{2-N} |\nabla u|^2 dx = \int_{\Omega} \Big(\varrho u \frac{\partial u}{\partial r} + \frac{N-2}{2} u^2 \Big) d\Omega +
$$

$$
+ \int_{G_0^\varrho} \Big\{ \big(a^{ij}(x) - a^{ij}(0) \big) \big((N-2) r^{-N} u x_i u_{x_j} - r^{2-N} u_{x_i} u_{x_j} \big) +
$$

$$
+ (N-2) r^{-N} x_i a^i(x) u^2 + r^{2-N} u \big(b^i(x) u_{x_i} + c(x) u - g(x) \big) +
$$

$$
+ r^{2-N} u a^i(x) u_{x_i} + r^{2-N} f^i(x) u_{x_i} + (2-N) r^{-N} u x_i f^i(x) \Big\} dx +
$$

$$
+ \varrho \int_{\Omega} \big\{ \big(a^{ij}(x) - a^{ij}(0) \big) u u_{x_j} + a^i(x) u^2 - u f^i(x) \big\} \cos(r, x_i) d\Omega.
$$

We set $U(\rho) = \int_{G_0^\varrho} r^{2-N} |\nabla u|^2 dx$ and estimate every integral from the right side. The first integral is estimated by Lemma 2.28. The other integrals from

the right side we estimate by using our assumptions and (5.1.18) as well

$$r^{2-N}|u||g| \le \frac{\delta}{2}r^{-N}u^2 + \frac{1}{2\delta}r^{4-N}|g|^2,$$

$$r^{2-N}|u_{x_i}f^i(x)| \le \frac{\delta}{2}r^{2-N}|\nabla u|^2 + \frac{1}{2\delta}r^{2-N}|f|^2,$$

$$\forall \delta > 0.$$

Thus we obtain

$$(5.1.39) \quad U(\varrho) \le c(N,\mu) \int\limits_{G_0^\varrho} \left\{ \left(\mathcal{A}(r)+\delta\right)\left(r^{-N}u^2 + r^{2-N}|\nabla u|^2\right) + \right.$$

$$\left. + \frac{1}{2\delta}\left(r^{2-N}|f|^2 + r^{4-N}|g|^2\right) \right\} dx + \frac{\rho}{2\lambda}U'(\rho) +$$

$$+ \varrho^{2-N} \int\limits_{\Omega_\varrho} \left\{ \mathcal{A}(\varrho)|u||\nabla u| + \varrho^{-1}\mathcal{A}(\varrho)u^2 + |u||f| \right\} d\Omega_\varrho, \ \forall \delta > 0.$$

As above in (5.1.26), to estimate the last integral from the right side we apply the Cauchy inequality and the Wirtinger inequality

$$(5.1.40) \qquad \varrho^{2-N} \int\limits_{\Omega_\varrho} \left(\mathcal{A}(\varrho)|u||\nabla u| + \varrho^{-1}\mathcal{A}(\varrho)u^2 + |u||f|\right) d\Omega_\varrho \le$$

$$\le \mathcal{A}(\varrho) \int\limits_\Omega \left(\varrho^2|\nabla u|^2 + u^2\right)\Big|_{r=\varrho} d\Omega + \frac{\delta}{2}\int\limits_\Omega u^2\Big|_{r=\varrho} d\Omega + \frac{\varrho^2}{2\delta}\int\limits_\Omega f^2 d\Omega \le$$

$$\le c\big(\mathcal{A}(\varrho)+\delta\big)\varrho U'(\varrho) + \frac{\varrho^2}{2\delta}\int\limits_\Omega f^2 d\Omega, \ \forall \delta > 0.$$

Thus, from (5.1.39) and (5.1.40) we obtain

$$(5.1.41) \quad U(\varrho) \le \left(\frac{\rho}{2\lambda} + \delta\varrho\right)U'(\rho) + \left(c\mathcal{A}(\varrho) + \frac{\delta}{2}\right)U(\varrho) + F_1(\varrho) + F_2(\varrho),$$

$$\forall \delta > 0, \ \varrho \in (0,d),$$

where

$$(5.1.42) \qquad F_1(\varrho) = \frac{\varrho^2}{2\delta}\int\limits_\Omega f^2 d\Omega, \quad F_2(\varrho) = \frac{1}{2\delta}\int\limits_0^\varrho \mathcal{K}(r)dr$$

and

$$\mathcal{K}(r) = \int\limits_{\Omega_r} \left\{ r^{4-N}|g|^2 + r^{2-N}|f|^2 \right\} d\Omega_r.$$

Finally, since $\mathcal{A}(r)$ is continuous at zero and $\mathcal{A}(\varrho) \leq \mathcal{A}(d)$, $\varrho \in (0,d)$, we can choose $\forall \delta > 0$ such $d > 0$ that $c\mathcal{A}(d) < \frac{\delta}{2}$. Therefore we can rewrite (5.1.41) in this way

$$(5.1.43) \quad U(\varrho) \leq \frac{\rho}{2\lambda}(1+\delta)U'(\rho) + \delta U(\varrho) + F_1(\varrho) + F_2(\varrho),$$

$$\forall \delta > 0, \ \varrho \in (0,d).$$

Setting now

$$(5.1.44) \quad \mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} \cdot \frac{1-\delta}{1+\delta}, \quad \mathcal{B}(\varrho) \equiv 0 \text{ and}$$

$$\mathcal{Q}(\varrho) = \frac{2\lambda}{1+\delta} \cdot \frac{F_1(\varrho) + F_2(\varrho)}{\varrho},$$

as a result we get the differential inequality (CP) §1.10. Now, putting $\varepsilon = \frac{2\delta}{1+\delta}$ by calculating, we have

$$(5.1.45) \quad \exp\left(-\int_\varrho^d \mathcal{P}(s)ds\right) = \left(\frac{\varrho}{d}\right)^{2\lambda(1-\varepsilon)}, \quad \forall \varrho \in (0,d).$$

Now, because of Theorem 1.52, we obtain

$$(5.1.46) \quad U(\varrho) \leq c\varrho^{2\lambda(1-\varepsilon)}\left(U(d) + \int_\varrho^d \tau^{-2\lambda(1-\varepsilon)}\mathcal{Q}(\tau)d\tau\right),$$

where $c > 0$ is a constant independent of u.

Now we estimate the last integral. Because of (5.1.44) we get

$$(5.1.47) \quad \int_\varrho^d \tau^{-2\lambda(1-\varepsilon)}\mathcal{Q}(\tau)d\tau = \frac{2\lambda}{1+\delta}\int_\varrho^d \tau^{-2\lambda(1-\varepsilon)-1}F_1(\tau)d\tau +$$

$$+ \frac{2\lambda}{1+\delta}\int_\varrho^d \tau^{-2\lambda(1-\varepsilon)-1}F_2(\tau)d\tau.$$

From (5.1.42) it follows that

$$(5.1.48) \quad \int_\varrho^d \tau^{-2\lambda(1-\varepsilon)-1}F_1(\tau)d\tau = \frac{1}{2\delta}\int_\varrho^d \tau^{-2\lambda(1-\varepsilon)+1}\int_\Omega f^2 d\Omega d\tau =$$

$$= \frac{1}{2\delta}\int_{G_\varrho^d} r^{2-N-2\lambda(1-\varepsilon)}|f|^2 dx \leq c\int_{G_0^d} r^{2-2\lambda-N}f^2 dx.$$

Further because of (5.1.42) we change the order of integration and obtain

$$
\int_{\varrho}^{d} \tau^{-2\lambda(1-\varepsilon)-1}\Big(\int_{0}^{\tau}\mathcal{K}(r)dr\Big)d\tau = \int_{0}^{\varrho}\mathcal{K}(r)dr\int_{\varrho}^{d}\tau^{-2\lambda(1-\varepsilon)-1}d\tau+
$$

$$
+\int_{\varrho}^{d}\mathcal{K}(r)dr\int_{r}^{d}\tau^{-2\lambda(1-\varepsilon)-1}d\tau = \Big(\int_{0}^{\varrho}\mathcal{K}(r)\big(\varrho^{-2\lambda(1-\varepsilon)}-d^{-2\lambda(1-\varepsilon)}\big)dr+
$$

$$
+\int_{\varrho}^{d}\mathcal{K}(r)\big(r^{-2\lambda(1-\varepsilon)}-d^{-2\lambda(1-\varepsilon)}\big)dr\Big)\cdot\frac{1}{2\lambda(1-\varepsilon)}\le c_{\varepsilon}\int_{0}^{d}r^{-2\lambda}\mathcal{K}(r)dr.
$$

Hence because of (5.1.42) it follows that

$$
(5.1.49)\quad \int_{\varrho}^{d}\tau^{-2\lambda(1-\varepsilon)-1}F_{2}(\tau)d\tau \le c_{\varepsilon}\int_{G_{0}^{d}}\Big\{r^{4-2\lambda-N}g^{2}(x)+r^{2-2\lambda-N}f^{2}\Big\}dx.
$$

From (5.1.46)-(5.1.49), together with Theorem 5.3, follows the required (5.1.37). Theorem 5.5 is proved.   □

5.1.3. Local bound of a weak solution.

We pass now to the establishing of the local (near the singular boundary point) bound for a weak solution of the problem (DL).

THEOREM 5.6. *Let* $u(x)$ *be a weak solution of the problem* (DL). *Suppose that assumptions I, II and III are satisfied. Let us assume, in addition, that* $g(x) \in L^{p}(G)$ *for some* $p > N/2$, $f^{i}(x) \in L^{q}(G)$, $(i = 1, \dots, N)$ *for some* $q > N$, $\Phi \in W^{1,s}(G)$, $s = \max(2p, q) > N$ *and*

$$
(5.1.50)\quad
\begin{aligned}
&\int_{G}r^{2p-N-p\lambda}|g(x)|^{p}dx < \infty;\quad \sum_{i=1}^{N}\int_{G}r^{q-N-q\lambda}|f^{i}(x)|^{q}dx < \infty,\\
&\int_{G}\big(r^{-N-s\lambda}|\Phi|^{s}+r^{s-N-s\lambda}|\nabla\Phi|^{s}\big)dx < \infty.
\end{aligned}
$$

Then there exist positive constants d, c, independent of u, g, f_i, φ such that

$$|u(x)| \le c|x|^\lambda \left(\left\{ \int\limits_G \left(|u|^2 + |\nabla u|^2 + g^2(x) + \right. \right. \right.$$

$$+r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 +$$

$$(5.1.51) \qquad +r^{-N-2\lambda}\mathcal{H}^{-1}(r)|\Phi|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \bigg) dx \bigg\}^{1/2} +$$

$$+\left\{ \int\limits_G r^{2p-N-p\lambda}|g(x)|^p dx \right\}^{1/p} + \left\{ \int\limits_G \sum_1^N r^{q-N-q\lambda}|f^i(x)|^q dx \right\}^{1/q} +$$

$$+\left(\int\limits_G \left(r^{-N-s\lambda}|\Phi|^s + r^{s-N-s\lambda}|\nabla\Phi|^s \right) dx \right)^{2/s} \bigg\} \bigg), \quad x \in G_0^d.$$

PROOF. At first we refer to well-known local estimate at the boundary (see e.g. §8.10 [**129**]).

LEMMA 5.7. *Let the (i) and (iii) of assumptions II are satisfied and suppose that $\mathcal{F}^i(x) \in L^q(G), (i = 1, \ldots, N); \ \mathcal{G}(x) \in L^p(G))$ for some $q > N, \ p > \frac{N}{2}$.*

Then if $v(x) \in W_0^1(G)$ is a solution of the problem $(II)_0$, we have

$$(5.1.52) \quad \sup_{G''} |v(x)|^2 \le C\left\{ \int\limits_{G'} v^2 dx + \left(\int\limits_{G'} |\mathcal{G}|^p dx \right)^{2/p} + \right.$$

$$+ \sum_{i=1}^{N} \left(\int\limits_{G'} |\mathcal{F}^i|^q dx \right)^{2/q} \bigg\}, \quad \forall G'' \subset G' \subset G,$$

where $C = const(N, \nu, \mu, q, p, dist(G'', G'))$.

We make the change of variables $x = \varrho x'$. Then the function $v(x') = u(\varrho x') - \Phi(\varrho x')$ satisfies the following problem

$$(5.1.53) \quad \begin{cases} \frac{\partial}{\partial x'_i}(a^{ij}(\varrho x')v_{x'_j} + \varrho a^i(\varrho x')v) + \varrho b^i(\varrho x')v_{x'_i} + \varrho^2 c(\varrho x')v = \\ \qquad = \varrho^2 \mathcal{G}(\varrho x') + \varrho \frac{\partial \mathcal{F}^j(\varrho x')}{\partial x'_j}, \quad x' \in G_{1/4}^2, \\ v(x') = 0, \quad x' \in \Gamma_{1/4}^2 \end{cases}$$

in the domain $G^2_{1/4}$, where

$$\mathcal{G}(\varrho x') = g(\varrho x') - \varrho^{-1} b^i(\varrho x') \Phi_{x'_i} - c(\varrho x') \Phi;$$
$$\mathcal{F}^i(\varrho x') = f^i(\varrho x') - \varrho^{-1} a^{ij}(\varrho x') \Phi_{x'_j} - a^i(\varrho x') \Phi$$

and, therefore, because of the (i) and (v) of assumptions II,

$$(5.1.54) \qquad \begin{aligned} \varrho^2 |\mathcal{G}(\varrho x')| &\leq \varrho^2 |g(\varrho x')| + c(|\nabla' \Phi| + |\Phi|), \\ \varrho |\mathcal{F}^i(\varrho x')| &\leq \varrho |f^i(\varrho x')| + c(|\nabla' \Phi| + |\Phi|). \end{aligned}$$

From an estimate of the type (5.1.52) for (5.1.53) and the domains $G'' = G^1_{1/2}$ and $G' = G^2_{1/4}$ we obtain

$$\sup_{G^1_{1/2}} |v(x')|^2 \leq C \Big\{ \int_{G^2_{1/4}} v^2 dx' + \varrho^4 \Big(\int_{G^2_{1/4}} |\mathcal{G}|^p dx' \Big)^{2/p} +$$
$$+ \varrho^2 \sum_{i=1}^N \Big(\int_{G^2_{1/4}} |\mathcal{F}^i|^q dx' \Big)^{2/q} \Big\}.$$

Hence, by (5.1.54) and the Hölder inequality, we have

$$\sup_{G^1_{1/2}} |v(x')|^2 \leq C \Big\{ \int_{G^2_{1/4}} v^2 dx' + \varrho^4 \Big(\int_{G^2_{1/4}} |g|^p dx' \Big)^{2/p} +$$
$$+ \varrho^2 \sum_{i=1}^N \Big(\int_{G^2_{1/4}} |f^i|^q dx' \Big)^{2/q} + \Big(\int_{G^2_{1/4}} \big(|\Phi|^s + |\nabla' \Phi|^s \big) dx' \Big)^{2/s} \Big\}.$$

Now, returning again to the variables x, we find that

$$\sup_{G^\varrho_{\varrho/2}} |v(x)|^2 \leq C \Big\{ \int_{G^{2\varrho}_{\varrho/4}} r^{-N} v^2 dx + \Big(\int_{G^{2\varrho}_{\varrho/4}} r^{2p-N} |g|^p dx \Big)^{2/p} +$$
$$+ \sum_{i=1}^N \Big(\int_{G^{2\varrho}_{\varrho/4}} r^{q-N} |f^i|^q dx \Big)^{2/q} + \Big(\int_{G^{2\varrho}_{\varrho/4}} \big(r^{-N} |\Phi|^s + r^{s-N} |\nabla \Phi|^s \big) dx \Big)^{2/s} \Big\}.$$

We apply the inequality (H-W) to the first integral from the right side

$$(5.1.55) \quad \sup_{G^{\varrho}_{\varrho/2}} |v(x)|^2 \le C\Big\{ \int_{G^{2\varrho}_{\varrho/4}} r^{2-N}|\nabla v|^2 dx + \Big(\int_{G^{2\varrho}_{\varrho/4}} r^{2p-N}|g|^p dx \Big)^{2/p} + $$

$$+ \sum_{i=1}^{N} \Big(\int_{G^{2\varrho}_{\varrho/4}} r^{q-N}|f^i|^q dx \Big)^{2/q} + \Big(\int_{G^{2\varrho}_{\varrho/4}} \big(r^{-N}|\Phi|^s + r^{s-N}|\nabla\Phi|^s \big) dx \Big)^{2/s} \Big\}.$$

Now, because of the bound (5.1.22) from Theorem 5.4 as well as the Sobolev imbedding theorem,

$$\max_{\overline{G}^{\varrho}_{\varrho/2}} |\Phi| \le C(N,s)\|\nabla\Phi\|_{L^s(G^{\varrho}_{\varrho/2})}, \quad s > N.$$

From (5.1.55) it follows that

$$|u(x)|^2 \le C_1 \varrho^{2\lambda} \int_{G^{2d}_0} \Big\{ |u(x)|^2 + |\nabla u|^2 + g^2(x) + \sum_{i=1}^{N} |f^i(x)|^2 +$$

$$+ |\Phi|^2 + |\nabla\Phi|^2 + r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{-N-2\lambda}\mathcal{H}^{-1}(r)|\Phi|^2 +$$

$$+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 \Big\} dx +$$

$$+ C_2 \varrho^{2\lambda}\Big\{ \Big(\int_{G^{2d}_0} r^{2p-N-p\lambda}|g|^p dx \Big)^{2/p} + \sum_{i=1}^{N}\Big(\int_{G^{2d}_0} r^{q-N-q\lambda}|f^i|^q dx \Big)^{2/q} +$$

$$+ \Big(\int_{G^{2d}_0} \big(r^{-N-s\lambda}|\Phi|^s + r^{s-N-s\lambda}|\nabla\Phi|^s \big) dx \Big)^{2/s} \Big\}, \quad \forall x \in G^{\varrho}_{\varrho/2}.$$

Setting now $|x| = \frac{2}{3}\varrho$ hence we obtain the required estimate (5.1.51). Thus Theorem 5.6 is proved. $\square$

In a similar way, using the bound (5.1.37) and Theorem 5.5 instead of (5.1.22) and Theorem 5.4, we get the following.

THEOREM 5.8. *Let $u(x)$ be a weak solution of the problem (DL) with $\varphi \equiv 0$. Suppose that assumptions I and II are satisfied with a function $\mathcal{A}(r)$ that is continuous at zero, but not Dini continuous. Let us assume, in addition, that $g(x) \in L^p(G)$ for some $p > N/2$, $f^i(x) \in L^q(G)$,*

$(i = 1, \ldots, N)$ *for some $q > N$ and*

(5.1.56) $\displaystyle\int_G r^{2p-N-p\lambda}|g(x)|^p dx < \infty \text{ and } \sum_{i=1}^N \int_G r^{q-N-q\lambda}|f^i(x)|^q dx < \infty.$

Then for every $\varepsilon > 0$ there exist positive constants d, c_ε, independent of u, g, f_i such that

$$|u(x)| \le c_\varepsilon |x|^{\lambda-\varepsilon} \left(\left\{ \int_G \left(|u(x)|^2 + |\nabla u|^2 + r^{4-N-2\lambda} g^2(x) + \right. \right.\right.$$

(5.1.57) $$\left. + r^{2-N-2\lambda}|f|^2(x) \right) dx \right\}^{1/2} + \left\{ \int_G r^{2p-N-p\lambda}|g(x)|^p dx \right\}^{1/p} +$$

$$\left. + \left\{ \int_G \sum_1^N r^{q-N-q\lambda}|f^i(x)|^q dx \right\}^{1/q} \right), \quad x \in G_0^d, \ \forall \varepsilon > 0.$$

5.1.4. Example. We provide an example to show that the assumption (v) is essential for the validity of the estimates (5.1.22) and (5.1.51).

Let $N = 2$, let the domain G lie inside the sector

$$G_0^\infty = \{(r,\omega) | 0 < r < \infty, 0 < \omega < \omega_0, \ 0 < \omega \le 2\pi\}$$

and suppose that $\mathcal{O} \in \partial G$ and in some neighborhood G_0^d of $\mathcal{O}$ the boundary ∂G coincides with the sides $\omega = 0$ and $\omega = \omega_0$ of the sector G_0^∞. In our case the least eigenvalue of (EVD) is $\lambda = \frac{\pi}{\omega_0}$. We consider Example 4.36 of Section 4.2.5 and rewrite it in the form (DL)

$$a^{11}(x) = 1 - \frac{2}{\lambda+1} \cdot \frac{x_2^2}{r^2 \ln(1/r)},$$

$$a^{12}(x) = a^{21}(x) = \frac{2}{\lambda+1} \cdot \frac{x_1 x_2}{r^2 \ln(1/r)},$$

(5.1.58) $$a^{22}(x) = 1 - \frac{2}{\lambda+1} \cdot \frac{x_1^2}{r^2 \ln(1/r)},$$

$$a^{ij}(0) = \delta_i^j, \ i,j = 1,2;$$

$$b^1(x) = -\frac{1}{r}\mathcal{A}(r)\cos\omega, \quad b^2(x) = -\frac{1}{r}\mathcal{A}(r)\sin\omega,$$

$$a^1(x) = a^2(x) = c(x) = g(x) = f^1(x) = f^2(x) = \varphi(x) \equiv 0,$$

where

$$\mathcal{A}(r) = \frac{2}{(\lambda+1)\ln(1/r)}, \quad \Longrightarrow \int_0^d \frac{\mathcal{A}(r)}{r} dr = +\infty.$$

Clearly, the (DL) equation with (5.1.58) is uniformly elliptic in G_0^d for $0 < d < e^{-2}$ with the ellipticity constants

$$\nu = 1 - \frac{2}{\ln(1/d)} \quad \text{and} \quad \mu = 1.$$

The (DL) equation with (5.1.58) has a particular solution of the form

$$u(r, \omega) = r^\lambda \left(\ln \frac{1}{r} \right)^{(\lambda-1)/(\lambda+1)} \sin(\lambda\omega), \quad \lambda = \frac{\pi}{\omega_0},$$

that satisfies the boundary conditions

$$u = 0 \quad \text{on} \quad \Gamma_0^d.$$

This solution is continuous in G, and easy to verify that it belongs to $W^1(G)$. Clearly, this solution does not satisfy (5.1.51) and therefore not (5.1.22), since (5.1.22) implies (5.1.51).

5.1.5. Hölder continuity of weak solutions. We shall now assume that $a^{ij}(x)$, $i, j = 1, \ldots, N$ are continuous in G and satisfy a Dini condition on ∂G, that is there exists a continuous function $\mathcal{A}(t)$ such that

$$|a^{ij}(x) - a^{ij}(y)| \leq \mathcal{A}(|x - y|)$$

for any points $x \in \partial G$ and $y \in G$, with $\int\limits_0^1 \frac{\mathcal{A}(t)}{t} dt < \infty$.   Let $\mathcal{O}$ be any point on ∂G. We place the origin at $\mathcal{O}$ and perform a linear change of independent variables such that $\tilde{a}^{ij}(\mathcal{O}) = \delta_i^j$, where $\tilde{a}^{ij}(\mathcal{O})$ is the coefficient of $\frac{\partial^2 u}{\partial x_i' \partial x_j'}$ in the equation of (DL), written in terms of the new variables x'. As in the Introduction, we define a function $\theta(r)$ for the point $\mathcal{O}$ and shall suppose that Assumptions I are satisfied for all points $\mathcal{O} \in \partial G$, where $\theta_0, \theta_1, \theta_2$ do not depend on $\mathcal{O}$. For the point $\mathcal{O}$ we construct integrals in the variables x' of the form (5.1.50) and (w) from Assumptions III and assume that they are bounded by constants independent of $\mathcal{O}$.

THEOREM 5.9. *Let $u(x)$ be a weak solution of the problem (DL). Suppose that Assumptions I, II and III (indicated above) are satisfied. Let us assume, in addition, that $g(x), f^i(x) \in L^p(G)$, $(i = 1, \ldots, N)$; $\Phi(x) \in W^{1,2p}$ for some $p \geq \frac{N}{1-\lambda}$, $\lambda < 1$, where λ is defined by (5.3.1). Suppose that (5.1.50) is fulfilled.*

Then $u \in C^\lambda(\overline{G})$. If $\lambda = 1$ and $g, f^i \in L^\infty(G)$, $(i = 1, \ldots, N)$, then $u \in C^{\lambda-\varepsilon}(\overline{G})$ for $\forall \varepsilon > 0$.

PROOF. We consider an arbitrary pair of points $\overline{x}, \overline{y} \in G$. Let

$$\max(d(\overline{x}), d(\overline{y})) < 2|\overline{x} - \overline{y}|.$$

By virtue of (5.1.51) of Theorem 5.6, in this case we have

$$\frac{|u(\overline{x}) - u(\overline{y})|}{|\overline{x} - \overline{y}|^\lambda} \le \frac{2|u(\overline{x})|}{d^\lambda(\overline{x})} + \frac{2|u(\overline{y})|}{d^\lambda(\overline{y})} \le C_1,$$

where C_1 is the positive constant.

Consider the case $2|\overline{x} - \overline{y}| < d(\overline{x}) = \varrho$. We make a change of variables $x - \overline{x} = \varrho x'$. Then the function $v(x') = u(\overline{x} + \varrho x') - \Phi(\overline{x} + \varrho x')$ satisfies in the domain G_0^2 the problem

$$\begin{cases} \frac{\partial}{\partial x_i'}(a^{ij}(\overline{x} + \varrho x')v_{x_j'} + \varrho a^i(\overline{x} + \varrho x')v) + \varrho b^i(\overline{x} + \varrho x')v_{x_i'} + \varrho^2 c(\overline{x} + \varrho x')v = \\ \qquad = \varrho^2 \mathcal{G}(\overline{x} + \varrho x') + \varrho \frac{\partial \mathcal{F}^j(\overline{x} + \varrho x')}{\partial x_j'}, \quad x' \in G_0^2; \\ v(x') = 0, \quad x' \in \Gamma_0^2, \end{cases}$$

where

$$(5.1.59) \quad \begin{aligned} \mathcal{G}(\overline{x} + \varrho x') &= g(\overline{x} + \varrho x') - \varrho^{-1} b^i(\overline{x} + \varrho x')\Phi_{x_i'} - c(\overline{x} + \varrho x')\Phi; \\ \mathcal{F}^i(\overline{x} + \varrho x') &= f^i(\overline{x} + \varrho x') - \varrho^{-1} a^{ij}(\overline{x} + \varrho x')\Phi_{x_j'} - a^i(\overline{x} + \varrho x')\Phi. \end{aligned}$$

This problem satisfies the ellipticity condition (i) with the same constants ν, μ and its coefficients are uniformly bounded in virtue of the condition (v), since G is a bounded domain. On the basis of Theorem 15.3' in [4], we have

$$(5.1.60) \quad \int\limits_{G_0^1} |\nabla' v|^p dx' \le C_2 \int\limits_{G_0^2} \left(|v|^p + \varrho^{2p}|\mathcal{G}|^p + \varrho^p \sum_{i=1}^N |\mathcal{F}^i|^p \right) dx',$$

where the constant C_2 does not depend on v. Because of conditions (i), (v), from (5.1.60) and (5.1.59) it follows that

$$(5.1.61) \quad \int\limits_{G_0^1} |\nabla' v|^p dx' \le C_3 \int\limits_{G_0^2} \left(|v|^p + \varrho^{2p}|g|^p + \varrho^p \sum_{i=1}^N |f^i|^p + \right.$$

$$\left. + |\nabla'\Phi|^p + |\Phi|^p \right) dx',$$

where the constant C_3 does not depend on v. Since according to Theorem 5.6 the function $u(x)$ is bounded in G, and by our assumptions about g, f^i, Φ, it follows from (5.1.61) that $v \in W^{1,p}(G_0^1)$, where $p \ge \frac{N}{1-\lambda}$. From the Sobolev Imbedding Theorem 1.33 it follows that $u \in C^\lambda(G)$, if $\lambda < 1$. We

therefore have

$$|u(\overline{x}) - u(\overline{y})|^p = |v(0) - v(\widetilde{y})|^p \le C_4 |\widetilde{y}|^{p\lambda} \int\limits_{G_0^2} \left(|v|^p + \varrho^{2p}|g|^p + \right.$$

$$+ \varrho^p \sum_{i=1}^{N} |f^i|^p + |\nabla'\Phi|^p + |\Phi|^p \Big) dx' \le$$

(5.1.62)

$$\le C_4 \varrho^{-p\lambda} |\overline{x} - \overline{y}|^{p\lambda} \cdot \int\limits_{G \cap \{|x - \overline{x}| < \varrho\}} \left(|v|^p + \varrho^{2p}|g|^p + \right.$$

$$+ \varrho^p \sum_{i=1}^{N} |f^i|^p + |\nabla'\Phi|^p + |\Phi|^p \Big) \varrho^{-N} dx$$

and

$$|v(x)| \le C_5 |x - x^*|^\lambda \le C_5 (2\varrho)^\lambda \quad \text{for} \quad x \in G \cap B_\varrho(x_0),$$

where $C_4, C_5 = const$, $\widetilde{y} = \varrho^{-1}(\overline{y} - \overline{x})$ and x^* is a point of ∂G such that $d(x) = |x - x^*|$. From (5.1.62) we have

$$|u(\overline{x}) - u(\overline{y})| \le C_6 |\overline{x} - \overline{y}|^\lambda, \quad C_6 = const.$$

If $\lambda = 1$, then according to the Sobolev Imbedding Theorem 1.33 $v(x') \in C^{1-\varepsilon}$, where $\varepsilon = const > 0$, and therefore $u(x) \in C^{1-\varepsilon}$. This proves our theorem.   □

5.1.6. Weak solutions of an elliptic inequality. In this subsection we consider the properties of weak solutions of an elliptic inequality

(IDL)
$$\begin{cases} \frac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j} + a^i(x)u) + b^i(x)u_{x_i} + c(x)u \le \\ \qquad\qquad \le g(x) + \frac{\partial f^j(x)}{\partial x_j}, \quad x \in G \subset K; \\ u(x) = 0, \quad x \in \partial G \setminus \mathcal{O}. \end{cases}$$

DEFINITION 5.10. The function $u(x)$ is called a **weak** solution of the problem (IDL) provided that $u(x) \in W^1(G_\varepsilon)$, $\forall \varepsilon > 0$ and satisfies the integral inequality

$$\int\limits_{G} \left\{ a^{ij}(x)u_{x_j}\eta_{x_i} + a^i(x)u\eta_{x_i} - b^i(x)u_{x_i}\eta - c(x)u\eta \right\} dx \le$$

(II*)
$$\le \int\limits_{G} \left\{ f^i(x)\eta_{x_i} - g(x)\eta \right\} dx$$

whatever $\eta \ge 0$ may be, $\eta(x) \in W^1(G)$ and has a support compact in G.

THEOREM 5.11. *Let $u(x)$ be a weak solution of (IDL) in G, let $G \subset K$ be a bounded domain, and suppose that Assumptions II are satisfied. Let us assume, in addition,*

- *$u > 0$ in G,*
- *$\int_G r^{\alpha-2}|\nabla u|^2 dx < \infty$, $2 \le \alpha < N + 2\lambda$,*
- *$g \in \mathring{W}^0_\alpha(G)$, $f \in \mathring{W}^0_{\alpha-2}(G)$.*

There exists $\delta > 0$ such that $\mathcal{A}(|x|) \le \delta$, $x \in K$, where δ depends only on α, K, then

$$(5.1.63) \qquad \int_G \left(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2 \right) dx \le c \int_G \left(r^{\alpha-2}|f|^2 + r^\alpha g^2 \right) dx,$$

where $c > 0$ is independent of u, g, f^i or G.

PROOF. We may redefine the functions u, η beyond G as having a zero value. Let us assume that $a^{ij} \equiv \delta^j_i$ beyond G. Then from the inequality (II*) it follows

$$(5.1.64) \qquad \int_K u_{x_i} \eta_{x_i} dx \le \int_K \Bigg\{ \left(a^{ij}(0) - a^{ij}(x) \right) u_{x_j} \eta_{x_i} - a^i(x) u \eta_{x_i} +$$

$$+ b^i(x) u_{x_i} \eta + c(x) u \eta + f^i(x) \eta_{x_i} - g(x) \eta \Bigg\} dx.$$

Let us set $\delta = \max\limits_{\overline{G}} \mathcal{A}(|x|)$ and let us consider a function
$\vartheta(t) \in C^\infty(\mathbb{R}^1)$, $\vartheta(t) \ge 0$,

$$\vartheta(t) \equiv \begin{cases} 0 & \text{for } t < 1, \\ 1 & \text{for } t > 2. \end{cases}$$

Now let us consider the function

$$\eta(x) = r^{\alpha-2}\vartheta_\varepsilon(r)u(x) \text{ where } \vartheta_\varepsilon(r) = \vartheta\left(\frac{r}{\varepsilon}\right).$$

The function $\eta(x)$ can be taken as a probe function in (5.1.64), because $u\big|_{\Gamma_\varepsilon} = 0$. By calculating, we obtain

$$\eta_{x_i} = r^{\alpha-2}\vartheta_\varepsilon(r)u_{x_i} + (\alpha - 2)r^{\alpha-4}\vartheta_\varepsilon(r)x_i u(x) + \frac{1}{\varepsilon}\vartheta'\left(\frac{r}{\varepsilon}\right)r^{\alpha-3}x_i u(x).$$

Now from (5.1.64) with this probe function it follows that

(5.1.65) $$\int\limits_{K} \Big\langle r^{\alpha-2}\vartheta_\varepsilon(r)|\nabla u|^2 + (\alpha-2)r^{\alpha-4}\vartheta_\varepsilon(r)x_i u u_{x_i} +$$

$$+ \frac{1}{\varepsilon}r^{\alpha-3}\vartheta'\Big(\frac{r}{\varepsilon}\Big)x_i u u_{x_i}\Big\rangle dx \leq \int\limits_{K}\Big\{\big(a^{ij}(0) - a^{ij}(x)\big)\big[r^{\alpha-2}\vartheta_\varepsilon(r)u_{x_i}u_{x_j} +$$

$$+ (\alpha-2)r^{\alpha-4}\vartheta_\varepsilon(r)x_i u u_{x_j} + \frac{1}{\varepsilon}r^{\alpha-3}\vartheta'\Big(\frac{r}{\varepsilon}\Big)x_i u u_{x_j}\big] -$$

$$- a^i(x)\big[r^{\alpha-2}\vartheta_\varepsilon(r)u u_{x_i} + (\alpha-2)r^{\alpha-4}\vartheta_\varepsilon(r)x_i u^2 + \frac{1}{\varepsilon}r^{\alpha-3}\vartheta'\Big(\frac{r}{\varepsilon}\Big)x_i u^2\big] +$$

$$+ b^i(x)r^{\alpha-2}\vartheta_\varepsilon(r)u u_{x_i} + c(x)r^{\alpha-2}\vartheta_\varepsilon(r)u^2 + r^{\alpha-2}\vartheta_\varepsilon(r)u_{x_i}f^i(x) +$$

$$+(\alpha-2)r^{\alpha-4}\vartheta_\varepsilon(r)x_i u f^i(x) + \frac{1}{\varepsilon}r^{\alpha-3}\vartheta'\Big(\frac{r}{\varepsilon}\Big)x_i u f^i(x) - r^{\alpha-2}\vartheta_\varepsilon(r)u g(x)\Big\}dx.$$

If we observe that

$$\vartheta_\varepsilon(r) = \vartheta\Big(\frac{r}{\varepsilon}\Big) = \begin{cases} 0 & \text{for } r < \varepsilon, \\ 1 & \text{for } r > 2\varepsilon, \end{cases} \Rightarrow \vartheta'_\varepsilon(r)\begin{cases} = 0 & \text{for } r < \varepsilon \text{ and } r > 2\varepsilon, \\ \neq 0 & \text{for } \varepsilon < r < 2\varepsilon, \end{cases}$$

then we obtain

1)

$$\int\limits_{K} r^{\alpha-2}\vartheta_\varepsilon(r)|\nabla u|^2 dx = \int\limits_{G_\varepsilon} r^{\alpha-2}\vartheta_\varepsilon(r)|\nabla u|^2 dx;$$

2)

$$(\alpha-2)\int\limits_{K} r^{\alpha-4}\vartheta_\varepsilon(r)x_i u u_{x_i} = \frac{\alpha-2}{2}\int\limits_{G_\varepsilon} r^{\alpha-4}\vartheta_\varepsilon(r)x_i \frac{\partial u^2}{\partial x_i} =$$

$$= \frac{2-\alpha}{2\varepsilon}\int\limits_{G_\varepsilon^{2\varepsilon}} r^{\alpha-3}\vartheta'\Big(\frac{r}{\varepsilon}\Big)u^2 dx + \frac{(2-\alpha)(N+\alpha-4)}{2}\int\limits_{G_\varepsilon} r^{\alpha-4}\vartheta_\varepsilon(r)u^2 dx;$$

3)

$$\frac{1}{\varepsilon}\int\limits_{K} r^{\alpha-3}\vartheta'\Big(\frac{r}{\varepsilon}\Big)x_i u u_{x_i} = -\frac{1}{2\varepsilon^2}\int\limits_{G_\varepsilon^{2\varepsilon}} r^{\alpha-2}\vartheta''\Big(\frac{r}{\varepsilon}\Big)u^2 dx -$$

$$- \frac{N+\alpha-3}{2\varepsilon}\int\limits_{G_\varepsilon^{2\varepsilon}} r^{\alpha-3}\vartheta'\Big(\frac{r}{\varepsilon}\Big)u^2 dx.$$

(Here we have integrated by parts.)

Further we estimate the integrals on the right hand side of (5.1.65) using the Cauchy inequality and taking into account Assumptions II. As a result we get

$$\int\limits_{G_\varepsilon} r^{\alpha-2}\vartheta\left(\frac{r}{\varepsilon}\right)|\nabla u|^2 dx \le \frac{(2-\alpha)(4-N-\alpha)}{2}\int\limits_{G_\varepsilon} r^{\alpha-4}\vartheta\left(\frac{r}{\varepsilon}\right)u^2 dx+$$

$$+c_1\int\limits_{G_\varepsilon^{2\varepsilon}}\left(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2\right)dx+$$

$$+c_2\int\limits_{G_\varepsilon}\mathcal{A}(r)\vartheta\left(\frac{r}{\varepsilon}\right)\left(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2\right)dx$$

$$+c_3\int\limits_{G_\varepsilon}\vartheta\left(\frac{r}{\varepsilon}\right)\left[\sigma\left(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2\right) + \frac{1}{\sigma}\left(r^{\alpha-2}|f|^2 + r^\alpha g^2\right)\right]dx+$$

$$(5.1.66)\qquad +c_4\int\limits_{G_\varepsilon^{2\varepsilon}}\vartheta'\left(\frac{r}{\varepsilon}\right)\left(r^{\alpha-2}|f|^2 + r^{\alpha-4}u^2\right)dx, \ \forall\sigma > 0.$$

Since all necessary integrals exist (by the assumptions of our theorem), we may let ε tend to zero. Then we obtain

$$(5.1.67)\qquad \int\limits_G r^{\alpha-2}|\nabla u|^2 dx \le \frac{(2-\alpha)(4-N-\alpha)}{2}\int\limits_G r^{\alpha-4}u^2 dx+$$

$$+ (c_2\delta + c_3\sigma)\int\limits_G\left(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2\right)dx+$$

$$+ \frac{c_3}{\sigma}\left(r^{\alpha-2}|f|^2 + r^\alpha g^2\right)\Big]dx, \ \forall\sigma > 0.$$

(Here we took into account that $\mathcal{A}(r) \le \delta$ in G by definition of δ.)

Now we apply the Hardy-Wirtinger inequality (see Theorem 2.33) for unbounded cone that is true at $\alpha \ge 4 - N$. Since by the condition of our theorem

$$2 \le \alpha < N + 2\lambda,$$

then it is easy to verify that

$$\frac{(2-\alpha)(4-N-\alpha)}{2}H(\lambda, N, \alpha) < 1,$$

where $H(\lambda, N, \alpha)$ is from (2.5.10). Therefore from (5.1.67) it follows that

$$(5.1.68) \quad C(\lambda, N, \alpha) \int\limits_G r^{\alpha-2}|\nabla u|^2 dx \le (c_2\delta + c_3\sigma) \int\limits_G \Big(r^{\alpha-2}|\nabla u|^2 +$$

$$+ r^{\alpha-4}u^2\Big) dx + \frac{c_3}{\sigma}\big(r^{\alpha-2}|f|^2 + r^\alpha g^2\big)\Big] dx, \ \forall \sigma > 0$$

with

$$C(\lambda, N, \alpha) = 1 - \frac{(2-\alpha)(4-N-\alpha)}{2} H(\lambda, N, \alpha) > 0.$$

Now we require that

$$(5.1.69) \qquad\qquad \delta = \frac{C(\lambda, N, \alpha)}{2c_2}$$

and choose a constant σ so that $c_3\sigma = \frac{1}{4}C(\lambda, N, \alpha)$. Then from (5.1.68) we obtain the required inequality (5.1.63). $\square$

5.2. Dini continuity of the first derivatives of weak solutions

We consider weak solutions to the Dirichlet problem (DL) in a bounded domain $G \subset \mathbb{R}^N$ with boundary ∂G that is a Dini-Lapunov surface containing the origin $\mathcal{O}$ as a conical point. The last means that $\partial G \setminus \mathcal{O}$ is a smooth manifold but near $\mathcal{O}$ the domain G is diffeomorfic to a cone.

5.2.1. Local Dini continuity near a boundary smooth portion.

THEOREM 5.12. *Let $\mathcal{A}$ be an $\alpha-$ Dini function $(0 < \alpha < 1)$ satisfying the condition (1.8.5). Let G be a domain in $\mathbb{R}^N$ with a $C^{1,\mathcal{A}}$ boundary portion $T \subset \partial G$. Let $u(x) \in W^1(G)$ be a weak solution of the problem (DL) with $\varphi(x) \in C^{1,\mathcal{A}}(\partial G)$ Suppose the coefficients of the equation in (DL) satisfy the conditions*

$$a^{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2, \quad \forall x \in \overline{G}, \ \xi \in \mathbb{R}^N; \ \nu = const > 0;$$

$$a^{ij}, a^i, f^i \in C^{0,\mathcal{A}}(\overline{G}) \quad (i, j = 1, \ldots, N),$$

$$b^i, c \in L^\infty(G), \quad g \in L^{N/(1-\alpha)}(G).$$

Then $u \in C^{1,\mathcal{B}}(G \cup T)$ and for every $G' \subset\subset G \cup T$

$$(5.2.1) \quad \|u\|_{1,\mathcal{B};G'} \le c(N, T, \nu, k, d')\Big(|u|_{0;G} + \|g\|_{\frac{N}{1-\alpha};G}$$

$$+ \sum_{i=1}^N \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G}\Big),$$

where $d' = dist(G', \partial G \setminus T)$ and $k = \max\limits_{i,j=1,\ldots,N}\big\{\|a^{ij}, a^i\|_{0,\mathcal{A};G}, |b^i, c|_{0;G}\big\}.$

Proof. At first we flatten the boundary portion T. By the definition of a $C^{1,\mathcal{A}}$ domain, at each point $x_0 \in T$ there is a neighborhood B of x_0 and a $C^{1,\mathcal{A}}$ diffeomorphism ψ that flatten the boundary in B. Let $B_\varrho(x_0) \subset\subset B$ and set $\gamma = B_\varrho(x_0) \cap G$, $\tilde{\gamma} = \psi(\gamma)$; $\tau = B_\varrho(x_0) \cap T \subset \partial\gamma$ and $\tilde{\tau} = \psi(\tau) \subset \partial\tilde{\gamma}$ ($\tilde{\tau}$ is a hyperplane portion of $\partial\tilde{\gamma}$).Under the mapping $y = \psi(x)$, let $\tilde{v}(y) = v(x)$, $\tilde{\eta}(y) = \eta(x)$. Since

$$v_{x_i} = \frac{\partial\psi_k}{\partial x_i} v_{y_k}, \quad dx = |J|dy,$$

where $J = \frac{\mathcal{D}(\psi_1,...,\psi_N)}{\mathcal{D}(x_1,...,x_N)}$ is a jacobian of the transformation $\psi(x)$, it follows from $(II)_0$ that

$$\int_{\tilde{\gamma}} \left\{ \left\langle \tilde{a}^{ij}(y)\tilde{v}_{y_j} + \tilde{a}^i(y)\tilde{v} - \tilde{\mathcal{F}}^i(y) \right\rangle \tilde{\eta}_{y_i} + \right.$$

$$\left. + \left\langle \tilde{\mathcal{G}}(y) - \tilde{b}^i(y)\tilde{v}_{y_i} - \tilde{c}(y)\tilde{v}(y) \right\rangle \tilde{\eta}(y) \right\} |J|dy = 0 \qquad (\widetilde{II})_0$$

for all $\tilde{\eta}(y) \in W_0^{1,2}(\tilde{\gamma})$, where

$$\tilde{a}^{ij}(y) = a^{km}(x)\frac{\partial\psi_i}{\partial x_k}\frac{\partial\psi_j}{\partial x_m}, \qquad \tilde{a}^i(y) = a^k(x)\frac{\partial\psi_i}{\partial x_k},$$

$$\tilde{b}^i(y) = b^k(x)\frac{\partial\psi_i}{\partial x_k}, \qquad \tilde{c}(y) = c(x),$$

$$\tilde{\mathcal{F}}^i(y) = \mathcal{F}^k(x)\frac{\partial\psi_i}{\partial x_k}, \qquad \tilde{\mathcal{G}}(y) = \mathcal{G}(x).$$

It is not difficult to observe that conditions on coefficients of the equation and on the portion T are invariant under maps of class $C^{1,\mathcal{A}}$. Indeed, let us consider the diffeomorphism ψ that is given in the following way

$$\begin{cases} y_k & = x_k - x_k^0; \quad k = 1,\ldots,N-1 \\ y_N & = x_N - h(x'), \quad x' = (x_1,\ldots,x_{N-1}) \end{cases}$$

where $x_N = h(x')$ is the equation of the surface τ and $h \in C^{1,\mathcal{A}}(\tau)$. In virtue of the property (iv) of ψ it is easy to see that $|\nabla h| \leq K$. We have also that $|J| = 1$. Further by the ellipticity condition

$$\tilde{a}^{ij}(y)\xi_i\xi_j = a^{km}(x)\frac{\partial(\xi_i y_i)}{\partial x_k}\frac{\partial(\xi_j y_j)}{\partial x_m} \geq \nu\sum_{k=1}^N \left(\frac{\partial}{\partial x_k}\left(\sum_{i=1}^N \xi_i y_i\right)\right)^2 =$$

$$= \nu \sum_{k=1}^{N} \left(\sum_{i=1}^{N} \xi_i \frac{\partial y_i}{\partial x_k} \right)^2 = \nu \sum_{k=1}^{N} \left(\xi_k + \xi_N \frac{\partial y_N}{\partial x_k} \right)^2 =$$

$$= \nu \left(\xi^2 + 2\xi_N^2 - 2\xi_N \sum_{k=1}^{N-1} \xi_k \frac{\partial h}{\partial x_k} + \xi_N^2 \left[1 + \sum_{k=1}^{N-1} \left(\frac{\partial h}{\partial x_k} \right)^2 \right] \right).$$

But by the Cauchy inequality with $\forall \varepsilon > 0$

$$2\xi_N \frac{\partial h}{\partial x_k} \xi_k \leq \varepsilon \xi_N^2 \left(\frac{\partial h}{\partial x_k} \right)^2 + \frac{1}{\varepsilon} \xi_k^2,$$

therefore from the previous inequality it follows

$$\widetilde{a}^{ij}(y)\xi_i\xi_j \geq \nu \left\{ \left(1 - \frac{1}{\varepsilon} \right) \xi'^2 + (1 - \varepsilon)\xi_N^2 \sum_{k=1}^{N-1} \left(\frac{\partial h}{\partial x_k} \right)^2 + 4\xi_N^2 \right\} =$$

(5.2.2)

$$= \nu \left\{ \left(1 - \frac{1}{\varepsilon} \right) \xi'^2 + \xi_N^2 \left[4 + (1 - \varepsilon)|\nabla h|^2 \right] \right\} \geq$$

$$\geq \nu \left\{ \left(1 - \frac{1}{\varepsilon} \right) \xi'^2 + \xi_N^2 \left[4 + (1 - \varepsilon)K^2 \right] \right\}, \quad \forall \varepsilon > 1.$$

Now we show that there is $\varepsilon > 1$ such that

$$1 - \frac{1}{\varepsilon} = 4 + (1 - \varepsilon)K^2$$

For this we solve the equation

$$K^2 \varepsilon^2 - (3 + K^2)\varepsilon - 1 = 0$$

and obtain

$$\varepsilon = \frac{1}{2} + \frac{3}{2K^2} + \sqrt{\frac{1}{4} + \frac{10}{4K^2} + \frac{9}{4K^4}}.$$

Hence we see that $\varepsilon > 1$ and we have also

$$1 - \frac{1}{\varepsilon} = \frac{8}{K^2 + 5 + \sqrt{K^4 + 10K^2 + 9}}.$$

Thus from (5.2.2) follows finally

$$\widetilde{a}^{ij}(y)\xi_i\xi_j \geq \nu c(K)\xi^2,$$

(5.2.3)

$$c(K) = \frac{8}{K^2 + 5 + \sqrt{K^4 + 10K^2 + 9}}.$$

Therefore after the preliminary flattening of the portion T by means of a diffeomorphism $\psi \in C^{1,\mathcal{A}}$ it is sufficient prove the theorem in the case $T \subset \Sigma$. We use the perturbation method. We freeze the leading coefficients $a^{ij}(x)$ at $x_0 \in G \cup T$ by setting $a^{ij}(x_0) = a_0^{ij}$ and rewrite the equation (DL)

in the form of the Poisson equation (PE) for the function $v(x) = u(x) - \varphi(x)$ with

(5.2.4) $\qquad \mathcal{G}(x) = g(x) - b^i(x)(D_i v + D_i \varphi) - c(x)(v(x) + \varphi(x)),$

(5.2.5) $\qquad \begin{aligned} \mathcal{F}^i(x) &= \left(a^{ij}(x_0) - a^{ij}(x)\right)D_j v - a^{ij}(x)D_j\varphi \\ &\quad - a^i(x)\left(v(x) + \varphi(x)\right) + f^i(x), \quad (i = 1, \dots, N). \end{aligned}$

Now we can apply Theorem 3.6 and thus we obtain the desired assertion of our theorem. In this connection we use following estimates for functions (5.2.4) and (5.2.5):

(5.2.6) $\quad \|G\|_{\frac{N}{1-\alpha};B_2^+} \le \|g\|_{\frac{N}{1-\alpha};B_2^+} + k\Bigg(\sum_{i=1}^{N} |D_i v|_{0;B_2^+} + |v|_{0;B_2^+} +$

$$+ \sum_{i=1}^{N} |D_i\varphi|_{0;B_2^+} + |\varphi|_{0;B_2^+}\Bigg) \le \|g\|_{\frac{N}{1-\alpha};B_2^+} +$$

$$+ k\Bigg(\varepsilon \sum_{i=1}^{N} |D_i v|_{0,\mathcal{A};B_2^+} + c_\varepsilon |v|_{0;B_2^+} + |\varphi|_{1;B_2^+}\Bigg) \text{ (by (1.11.6)),}$$

(5.2.7) $\quad \sum_{i=1}^{N} \|\mathcal{F}^i\|_{0,\mathcal{A};B_2^+} \le nk\mathcal{A}(2R)\|\nabla v\|_{0,\mathcal{A};B_2^+} + k\sum_{i=1}^{N} |D_i v|_{0;B_2^+} +$

$$+ c(k)\big(|v|_{0;B_2^+} + \|\varphi\|_{1,\mathcal{A};B_2^+}\big) + \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};B_2^+}.$$

Taking into account once more the interpolation inequality (Theorem 1.49) and the condition (1.8.5) that ensures the equivalence $[\dots]_\mathcal{A} \sim [\dots]_\mathcal{B}$, from (5.2.6) and (5.2.7) we finally obtain the inequality

(5.2.8) $\quad \|G\|_{\frac{N}{1-\alpha};B_2^+} + \sum_{i=1}^{N} \|\mathcal{F}^i\|_{0,\mathcal{A};B_2^+} \le$

$$\le k\big(\varepsilon + N\mathcal{A}(2R)\big)\|v\|_{1,\mathcal{B};B_2^+} + c_\varepsilon(k)\big(|v|_{0;B_2^+} + \|\varphi\|_{1,\mathcal{A};B_2^+}\big) +$$

$$+ \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};B_2^+} + \|g\|_{\frac{N}{1-\alpha};B_2^+} \ \forall\varepsilon > 0.$$

Since $\mathcal{A}(t)$ is the continuous function, choosing $\varepsilon, R > 0$ sufficiently small we obtain the desired assertion and the estimate (5.2.1) in a standard way from (3.2.3), and (5.2.7) and (5.2.8). $\qquad \square$

5.2.2. Dini-continuity near a conical point. We consider the problem *(DL)* under the following **assumptions**:

(i) ∂G is a Dini-Lyapunov surface and contains the conical point $\mathcal{O}$;

(ii) the uniform ellipticity holds

$$\nu\xi^2 \le a^{ij}(x)\xi_i\xi_j \le \mu\xi^2, \quad \forall x \in G \text{ and } \xi\mathbb{R}^N,$$

$$\nu,\mu = const > 0; a^{ij}(0) = \delta_i^j, \ (i,j = 1,\ldots,N);$$

(iii) $a^{ij}(x), a^i(x) \in C^{0,\mathcal{A}}(G)$, $(i,j = 1,\ldots,N)$, where $\mathcal{A}(t)$ is an $\alpha-$ Dini function on $(0,d], \alpha \in (0,1)$, satisfying the conditions *(1.8.5)* and *(1.8.6)* and also

$$(5.2.9) \qquad \sup_{0<\varrho<1} \frac{\varrho^{\lambda-1}}{\mathcal{A}(\varrho)} \le const,$$

$$|x|\left(\sum_{i=1}^N |b^i(x)|^2\right)^{1/2} + |x|^2|c(x)| \le \mathcal{A}(|x|);$$

(iv) $g(x) \in L^{\frac{N}{1-\alpha}}(G)$, $\varphi(x) \in C^{1,\mathcal{A}}(\partial G)$, $f^j(x) \in C^{0,\mathcal{A}}(\overline{G})$, $j = 1,\ldots,N$;

(v) $\displaystyle\int_G r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x)dx < \infty;$

$$\int_G r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\left(\sum_{i=1}^N |f^i|^2 + |\nabla\Phi|^2 + r^{-2}\Phi^2\right)dx < \infty,$$

where $\mathcal{H}(t)$ is a continuous monotone increasing function satisfying the Dini condition at $t = 0$.

THEOREM 5.13. *Let $u(x)$ be the generalized solution of (DL) and suppose assumptions (i)-(v) are satisfied. Then there exist $d > 0$ and a constant $c > 0$ independent of $u(x)$ and defined only by parameters and norms of the given functions appearing in assumptions (i)-(v) such that*

$$(5.2.10) \quad |u(x)| \le c|x|\mathcal{A}(|x|)\left(\|g\|_{\frac{N}{1-\alpha};G} + \sum_{i=1}^N \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G}+\right.$$

$$+\left\{\int_G \left(r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^N |f^i(x)|^2+\right.\right.$$

$$\left.\left.+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 + |u|^2 + |\nabla u|^2\right)dx\right\}^{1/2}\right), \ \forall x \in G_0^d,$$

$$(5.2.11) \quad |\nabla u(x)| \le c\mathcal{A}(|x|) \left(\|g\|_{\frac{N}{1-\alpha};G} + \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} + \right.$$

$$+ \left\{ \int_G \left(r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{2-N-2\lambda}\mathcal{H}^{-1}(r) \sum_{i=1}^{N} |f^i(x)|^2 + \right. \right.$$

$$\left. \left. + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 + |u|^2 + |\nabla u|^2 \right) dx \right\}^{1/2} \right), \quad \forall x \in G_0^d.$$

PROOF. We use the method of layers, that is we move away from the conical point of $\varrho > 0$ and work in $G_{\varrho/4}^{2\varrho}$. After the change of variables $x = \rho x'$, the layer $G_{\varrho/4}^{2\varrho}$ takes the position of a fixed domain $G_{1/4}^2$ with smooth boundary.

Step 1. We consider a solution $u(x)$ in the domain G_0^{2d} with some positive $d \ll 1$; then $u(x)$ is a weak solution in G_0^{2d} of the problem

$$\begin{cases} \frac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j} + a^i(x)u) + b^i(x)u_{x_i} + c(x)u = \\ \quad = g(x) + \frac{\partial f^j(x)}{\partial x_j}, \quad x \in G_0^{2d}, \\ u(x) = \varphi(x), \quad x \in \Gamma_0^{2d} \subset \partial G_0^{2d}. \end{cases}$$

We make the change of variables $x = \varrho x'$ and $v(x') = \varrho^{-1}\mathcal{A}^{-1}(\varrho)u(\varrho x')$, $\varrho \in (0, d), 0 < d \ll 1$. Then the function $v(x')$ satisfies in the domain $G_{1/4}^2$ the problem

$$\begin{cases} \frac{\partial}{\partial x_i'}(a^{ij}(\varrho x')v_{x_j'} + \varrho a^i(\varrho x')v) + \varrho b^i(\varrho x')v_{x_i'} + \varrho^2 c(\varrho x')v = \\ \quad = \varrho\mathcal{A}^{-1}(\varrho)g(\varrho x') + \mathcal{A}^{-1}(\varrho)\frac{\partial f^j(\varrho x')}{\partial x_j'}, \quad x' \in G_{1/4}^2; \\ v(x') = \varrho^{-1}\mathcal{A}^{-1}(\varrho)\varphi(\varrho x'), \quad x' \in \Gamma_{1/4}^2. \end{cases}$$

To solve this problem we use Theorem 5.12 about the local Dini continuity of the first derivatives for weak solutions of the problem *(DL)*. We check the possibility of using this theorem. Since under assumption (ii), $\mathcal{A}(t)$ is monotone increasing function, $\varrho \in (0, d), 0 < d \ll 1$, from the inequality $\varrho^{-1}|x - y| \ge |x - y|$ it follows that

$$\mathcal{A}(|x' - y'|) = \mathcal{A}(\varrho^{-1}|x - y|) \ge \mathcal{A}(|x - y|)$$

and by (iii) we have

$$\sum_{i,j} \|a^{i,j}(\varrho x')\|_{0,\mathcal{A};G^2_{1/4}} + \varrho \sum_{i} \|a^{i}(\varrho x')\|_{0,\mathcal{A};G^2_{1/4}} \le$$

$$\le \sum_{i,j} \|a^{i,j}(x)\|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} + d \sum_{i} \|a^{i}(x)\|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} < \infty.$$

Further, let $\Phi(x)$ be a regularity preserving extension of the boundary function $\varphi(x)$ into a domain G^d_ε with any $\varepsilon > 0$. (Such an extension exists; see e.g. the Lemma 6.38 [129].) Since $\varphi(x) \in C^{1,\mathcal{A}}(\partial G)$ we have

$$\|\Phi\|_{1,\mathcal{A};G^{2\varrho}_{\varrho/4}} \le c(G)\|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}} \le const.$$

By definition of the norm in $C^{1,\mathcal{A}}$ we obtain

$$(5.2.12) \qquad \sup_{\substack{x,y \in G^{2\varrho}_{\varrho/4} \\ x \ne y}} \frac{|\nabla\Phi(x) - \nabla\Phi(y)|}{\mathcal{A}(|x-y|)} \le \|\Phi\|_{1,\mathcal{A};G^{2\varrho}_{\varrho/4}} \le c(G)\|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}.$$

Now we show that by (v) and by the smoothness of $\varphi(x)$

$$(5.2.13) \qquad |\varphi(x)| \le c|x|\mathcal{A}(|x|), \quad |\nabla\Phi(x)| \le c\mathcal{A}(|x|), \quad \forall x \in G^{2\varrho}_{\varrho/4}.$$

Indeed from the equality

$$\varphi(x) - \varphi(0) = \int_0^1 \frac{d}{dt}\Phi(\tau x)d\tau = x_i \int_0^1 \frac{\partial\Phi(\tau x)}{\partial \tau x_i}d\tau$$

by Hölder's inequality we have

$$(5.2.14) \qquad\qquad |\varphi(x) - \varphi(0)| \le r|\nabla\Phi|.$$

From (iv) it follows that

$$(5.2.15) \quad \int_{G^\varrho_0} (r^{2-N}|\nabla\Phi|^2 + r^{-N}|\varphi|^2)dx = \int_{G^\varrho_0} (r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 +$$

$$+ (r^{-N-2\lambda}\mathcal{H}^{-1}(r)|\varphi|^2)(r^{2\lambda}\mathcal{H}(r))dx \le const \cdot \varrho^{2\lambda}\mathcal{H}(\varrho).$$

Since $|\varphi(0)| \le |\varphi(x)| + |\varphi(x) - \varphi(0)|$, by (5.2.14) we obtain

$$|\varphi(0)| \le |\varphi(x)| + r|\nabla\Phi|.$$

Squaring both sides of last inequality, multiplying by r^{-N} and integrating over G^ϱ_0 we obtain

$$(5.2.16) \qquad |\varphi(0)|^2 \int_{G^\varrho_0} (r^{-N}dx \le 2 \int_{G^\varrho_0} (r^{2-N}|\nabla\Phi|^2 + r^{-N}|\varphi|^2)dx < \infty$$

by (5.2.15). Since $\int_{G_0^\varrho} r^{-N} dx = mes\Omega \int_0^\varrho \frac{dr}{r} = \infty$, the assumption $\varphi(0) \neq 0$ contradicts (5.2.16). Thus $\varphi(0) = 0$. Then from (5.2.12) we have

$$|\nabla\Phi(x) - \nabla\Phi(y)| \leq const\mathcal{A}(|x-y|)\|\varphi\|_{1,\mathcal{A};\Gamma_{\varrho/4}^{2\varrho}}, \quad \forall x,y \in G_{\varrho/4}^{2\varrho},$$

$$|\nabla\Phi(y)| \leq |\nabla\Phi(x) - \nabla\Phi(y)| + |\nabla\Phi(x)| \leq$$

$$\leq c\mathcal{A}(|x-y|)\|\varphi\|_{1,\mathcal{A};\Gamma_{\varrho/4}^{2\varrho}} + |\nabla\Phi(x)|.$$

Hence considering y to be fixed in $G_{\varrho/4}^{2\varrho}$ and x as variable, we get

$$|\nabla\Phi(y)|^2 \int_{G_{\varrho/4}^{2\varrho}} r^{2-N} dx \leq 2c^2 \|\varphi\|_{1,\mathcal{A};\Gamma_{\varrho/4}^{2\varrho}} \int_{G_{\varrho/4}^{2\varrho}} r^{2-N}\mathcal{A}^2(|x-y|)dx +$$

$$+ 2\int_{G_{\varrho/4}^{2\varrho}} r^{2-N} |\nabla\Phi(x)|^2 dx$$

or by (5.2.15)

$$\varrho^2 |\nabla\Phi(y)|^2 \leq c(mes\Omega, k_1)\left(\varrho^2\mathcal{A}^2(\varrho) + \varrho^{2\lambda}\mathcal{H}(\varrho)\right), \quad \forall y \in G_{\varrho/4}^{2\varrho}.$$

Hence the assumption (5.2.9) yields the second inequality of (10.2.85). Now the first inequality of (10.2.85) follows from (5.2.14) and $\varphi(0) = 0$. Thus (10.2.85) is proved.

Now we obtain

$$(5.2.17) \quad \varrho^{-1}\mathcal{A}^{-1}(\varrho)\|\varphi(\varrho x')\|_{1,\mathcal{A};\Gamma_{1/4}^2} \leq c\varrho^{-1}\mathcal{A}^{-1}(\varrho)\|\Phi(\varrho x')\|_{1,\mathcal{A};G_{1/4}^2} =$$

$$= c\varrho^{-1}\mathcal{A}^{-1}(\varrho)\left\{ \sup_{x' \in G_{1/4}^2} |\Phi(\varrho x')| + \sup_{x' \in G_{1/4}^2} |\nabla'\Phi(\varrho x')| + \right.$$

$$+ \sup_{\substack{x',y' \in G_{1/4}^2 \\ x' \neq y'}} \frac{|\nabla'\Phi(\varrho x') - \nabla'\Phi(\varrho y')|}{\mathcal{A}(|x'-y'|)} \right\} \leq c\mathcal{A}^{-1}(\varrho) \sup_{\substack{x,y \in G_{\varrho/4}^{2\varrho} \\ x \neq y}} \frac{|\nabla\Phi(x) - \nabla\Phi(y)|}{\mathcal{A}(\varrho^{-1}|x-y|)} +$$

$$+ c_1 = c_1 + c\mathcal{A}^{-1}(\varrho) \sup_{0 < t < 4\varrho} \frac{\mathcal{A}(t)}{\mathcal{A}(\varrho^{-1}t)} \cdot [\nabla\Phi]_{0,\mathcal{A};G_{\varrho/4}^{2\varrho}} \leq const, \quad \forall \varrho \in (0,d),$$

by (10.2.85), since by (1.8.6)

$$\sup_{0 < t < 4\varrho} \frac{\mathcal{A}(t)}{\mathcal{A}(\varrho^{-1}t)} = \sup_{0 < \tau < 4} \frac{\mathcal{A}(\tau\varrho)}{\mathcal{A}(\tau)} \leq c\mathcal{A}(\varrho).$$

In the same way we have

$$(5.2.18) \quad \mathcal{A}^{-1}(\varrho)\|f^j(\varrho x')\|_{0,\mathcal{A};G^2_{1/4}} = \mathcal{A}^{-1}(\varrho)\Big(|f^j(x)|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} +$$
$$+ \sup_{\substack{x,y\in G^{2\varrho}_{\varrho/4} \\ x\neq y}} \frac{|f^j(x) - f^j(y)|}{\mathcal{A}(\varrho^{-1}|x-y|)}\Big).$$

Since $f^j \in C^{0,\mathcal{A}}(\overline{G})$, we get

$$(5.2.19) \qquad |f^j(x) - f^j(y)| \leq \widetilde{c}_j\mathcal{A}(|x-y|), \; \forall x,y \in G^{2\varrho}_{\varrho/4}$$

and

$$(5.2.20) \quad \int\limits_{G^\varrho_0} r^{2-N}|f^j(x)|^2 dx = \int\limits_{G^\varrho_0} \big(r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|f^j(x)|^2\big)\big(r^{2\lambda}\mathcal{H}(r)\big)dx \leq$$

$$\leq const \cdot \varrho^{2\lambda}\mathcal{H}(\varrho)$$

by (v). Now let y be fixed in $G^{2\varrho}_{\varrho/4}$. Then

$$|f^j(y)| \leq |f^j(x)| + |f^j(x) - f^j(y)| \leq |f^j(x)| + \widetilde{c}_j\mathcal{A}(|x-y|)$$

Hence

$$|f^j(y)|^2 \int\limits_{G^{2\varrho}_{\varrho/4}} r^{2-N}dx \leq 2\widetilde{c}_j^2 \int\limits_{G^{2\varrho}_{\varrho/4}} r^{2-N}\mathcal{A}^2(|x-y|)dx + 2\int\limits_{G^{2\varrho}_{\varrho/4}} r^{2-N}|f^j(x)|^2 dx.$$

Calculations and (5.2.20) give

$$\varrho^2|f^j(y)|^2 \leq c(\widetilde{c}_j, k_1, mes\Omega)\big(\varrho^2\mathcal{A}^2(\varrho) + \varrho^{2\lambda}\mathcal{H}(\varrho)\big) \; \forall y \in G^{2\varrho}_{\varrho/4}.$$

Hence by the assumption (5.2.9) it follows that

$$(5.2.21) \qquad |f^j(x)| \leq c_j\mathcal{A}(\varrho) \; \forall x \in G^{2\varrho}_{\varrho/4}, \quad j = 1,\ldots,N.$$

Further, in the same way as in the proof of (5.2.17),

$$(5.2.22) \quad \sup_{\substack{x,y\in G^{2\varrho}_{\varrho/4} \\ x\neq y}} \frac{|f^j(x) - f^j(y)|}{\mathcal{A}(\varrho^{-1}|x-y|)} \leq [f^j]_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} \sup_{0<t<4\varrho} \frac{\mathcal{A}(t)}{\mathcal{A}(\varrho^{-1}t)} \leq$$

$$\leq c\mathcal{A}(\varrho)[f^j]_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}}.$$

Now from (5.2.18), (5.2.21) and (5.2.22) we obtain

$$(5.2.23) \qquad \mathcal{A}^{-1}(\varrho)\sum_{j=1}^{N}\|f^j(\varrho x')\|_{0,\mathcal{A};G^2_{1/4}} \leq const.$$

It remains to verify the finiteness of $\varrho\mathcal{A}^{-1}(\varrho)\|g(\varrho x')\|_{\frac{N}{1-\alpha};G^2_{1/4}}$. We have

$$\varrho\mathcal{A}^{-1}(\varrho)\Big(\int\limits_{G^2_{1/4}} |g(\varrho x')|^{\frac{N}{1-\alpha}}dx'\Big)^{\frac{1-\alpha}{N}} =$$

$$= \varrho^{\alpha}\mathcal{A}^{-1}(\varrho)\Big(\int\limits_{G^{2\varrho}_{\varrho/4}} |g(x)|^{\frac{N}{1-\alpha}}dx\Big)^{\frac{1-\alpha}{N}} \le$$

$$\le d^{\alpha}\mathcal{A}^{-1}(d)\Big(\int\limits_{G^{2\varrho}_{\varrho/4}} |g(x)|^{\frac{N}{1-\alpha}}dx\Big)^{\frac{1-\alpha}{N}} \le const \quad \forall\varrho \in (0,d)$$

by the condition (1.8.1). Thus the conditions of Theorem 5.12 are satisfied. By this theorem we have

$$\|v\|_{1,\mathcal{B};G^1_{1/2}} \le \Big(|v|_{0;G^2_{1/4}} + \varrho^{-1}\mathcal{A}^{-1}(\varrho)\|\varphi(\varrho x')\|_{1,\mathcal{A};\Gamma^2_{1/4}}+$$

$$(5.2.24) \quad +\varrho\mathcal{A}^{-1}(\varrho)\|g(\varrho x')\|_{\frac{N}{1-\alpha};G^2_{1/4}} + \mathcal{A}^{-1}(\varrho)\sum_{j=1}^{N}\|f^j(\varrho x')\|_{0,\mathcal{A};G^2_{1/4}}\Big)\times$$

$$\times c\Big(N,\nu,G, \max_{i,j=1,\ldots,N}(\|a^{i,j}(\varrho x')\|_{0,\mathcal{A};G^2_{1/4}}, \varrho\|a^i(\varrho x')\|_{0,\mathcal{A};G^2_{1/4}}),\mathcal{A}(2\varrho)\Big),$$

$$\forall\varrho \in (0,d).$$

Step 2.

To estimate $|v|_{0;G^2_{1/4}}$ we use the local estimate at the boundary of the maximum of the modulus of a solution (Theorem 8.25 [129]). We check the assumptions of this theorem. To this end, we set

$$z(x') = v(x') - \varrho^{-1}\mathcal{A}^{-1}(\varrho)\Phi(\varrho x')$$

and write the problem for the function $z(x')$

$$\begin{cases} \frac{\partial}{\partial x'_i}(a^{ij}(\varrho x')z_{x'_j} + \varrho a^i(\varrho x')z) + \varrho b^i(\varrho x')z_{x'_i} + \varrho^2 c(\varrho x')z = \\ \quad = G(x') + \frac{\partial F^j(x')}{\partial x'_j}, \quad x' \in G^2_{1/4}, \\ z(x') = 0, \quad x' \in \Gamma^2_{1/4}. \end{cases}$$

where

$$(5.2.25) \quad \begin{aligned} G(x') &\equiv \varrho\mathcal{A}^{-1}(\varrho)g(\varrho x') - \mathcal{A}^{-1}(\varrho)b^i(\varrho x')\Phi_{x'_i}(\varrho x')- \\ &\quad - \varrho\mathcal{A}^{-1}(\varrho)c(\varrho x')\Phi(\varrho x') \end{aligned}$$

and

$$(5.2.26) \quad F^i(x') \equiv \mathcal{A}^{-1}(\varrho) f^i(\varrho x') - \varrho^{-1} \mathcal{A}^{-1}(\varrho) a^{ij}(\varrho x') \Phi_{x'_j}(\varrho x') - \\ - \mathcal{A}^{-1}(\varrho) a^i(\varrho x') \Phi(\varrho x'), \quad (i = 1, \dots, N).$$

At first we verify the necessary smoothness of coefficients. (See the remark in the end of §8.10 [129].) Let $q > N$. We have:

$$(5.2.27) \quad \int\limits_{G^2_{1/4}} |\varrho a^i(\varrho x')|^q dx' = \varrho^{q-N} \int\limits_{G^{2\varrho}_{\varrho/4}} |a^i(x)|^q dx \le \\ \le c(G) d^q \|a^i\|^q_{0,\mathcal{A};G}, \quad \forall \varrho \in (0, d).$$

By (iii) we also obtain

$$\int\limits_{G^2_{1/4}} |\varrho b^i(\varrho x')|^q dx' = \varrho^{q-N} \int\limits_{G^{2\varrho}_{\varrho/4}} |b^i(x)|^q dx \le 4^q \varrho^{-N} \int\limits_{G^{2\varrho}_{\varrho/4}} |r b^i(x)|^q dx \le$$

$$(5.2.28) \quad \le 4^q \varrho^{-N} \int\limits_{G^{2\varrho}_{\varrho/4}} \mathcal{A}^q(r) dx \le 2^{N+2q} \int\limits_{G^{2\varrho}_{\varrho/4}} r^{-N} \mathcal{A}^q(r) dx =$$

$$= 2^{N+2q} mes\Omega \int\limits_{\varrho/4}^{2\varrho} \frac{\mathcal{A}^q(r)}{r} dr \le 2^{N+2q} mes\Omega \cdot \mathcal{A}^{q-1}(2d) \int\limits_0^{2d} \frac{\mathcal{A}(r)}{r} dr,$$

$$\int\limits_{G^2_{1/4}} |\varrho^2 c(\varrho x')|^{q/2} dx' = \varrho^{q-N} \int\limits_{G^{2\varrho}_{\varrho/4}} |c(x)|^{q/2} dx \le 4^q \varrho^{-N} \int\limits_{G^{2\varrho}_{\varrho/4}} |r^2 c(x)|^{q/2} dx \le$$

$$(5.2.29) \quad \le 2^{N+2q} \int\limits_{G^{2\varrho}_{\varrho/4}} r^{-N} \mathcal{A}^{q/2}(r) dx \le 2^{N+2q} mes\Omega \cdot \mathcal{A}^{\frac{q-2}{2}}(2d) \int\limits_0^{2d} \frac{\mathcal{A}(r)}{r} dr,$$

$$\forall \varrho \in (0, d).$$

In the same way from (5.2.25) we get

$$\varrho \mathcal{A}^{-1}(\varrho) \|G(x')\|_{q/2; G^2_{1/4}} = \varrho \mathcal{A}^{-1}(\varrho) \Big(\int\limits_{G^{2\varrho}_{\varrho/4}} \{ |g(x)|^{q/2} +$$

$$(5.2.30) \quad + (\sum_{i=1}^N |b^i(x)|)^{q/2} |\nabla \Phi|^{q/2} + |c(x)|^{q/2} |\Phi(x)|^{q/2} \} \varrho^{-N} dx \Big)^{2/q}.$$

By (iv) setting $q = N/(1 - \alpha) > N$ and applying Holder's inequality for the integrals we obtain

$$\varrho \mathcal{A}^{-1}(\varrho) \Big(\int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{-N} |g(x)|^{q/2} dx \Big)^{2/q} \le$$

$$(5.2.31) \qquad \le c\rho^\alpha \mathcal{A}^{-1}(\varrho) \Big(\int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{-N/2} |g(x)|^{q/2} dx \Big)^{2/q} \le$$

$$\le c\rho^\alpha \mathcal{A}^{-1}(\varrho) \|g\|_{q;G^{2\varrho}_{\varrho/4}} (mes\Omega \ln 8)^{1/q} \le c(d, \alpha, q, mes\Omega, \mathcal{A}(d)) \|g\|_{q;G^{2\varrho}_{\varrho/4}},$$

since by (1.8.1), $\varrho^\alpha \mathcal{A}^{-1}(\varrho) \le d^\alpha \mathcal{A}^{-1}(d) \quad \forall \varrho \in (0, d)$. Similarly

$$\varrho \mathcal{A}^{-1}(\varrho) \Big(\int\limits_{G^{2\varrho}_{\varrho/4}} r^{-N} \{ (\sum_{i=1}^N |b^i(x)|)^{q/2} |\nabla \Phi|^{q/2} + |c(x)|^{q/2} |\Phi(x)|^{q/2} \} dx \Big)^{2/q} \le$$

$$(5.2.32) \qquad \le c(mes\Omega)^{2/q} \|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}} \mathcal{A}^{\frac{q-2}{2}}(\varrho) \int\limits_{\varrho/4}^{2\varrho} \frac{\mathcal{A}(r)}{r} dr.$$

From (5.2.30) and (5.2.32) we obtain

$$\|G(x')\|_{q/2;G^2_{1/4}} \le c\Big(d, \alpha, q, mes\Omega, \mathcal{A}(d), \int\limits_{\varrho/4}^{2\varrho} \frac{\mathcal{A}(r)}{r} dr \Big) \times$$

$$(5.2.33) \qquad \times \Big(\|g\|_{q;G^{2\varrho}_{\varrho/4}} + \|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}} \Big), \quad q = \frac{N}{1-\alpha} > N.$$

Finally, in the same way from (5.2.26) we have

$$\sum_{i=1}^N \int\limits_{G^2_{1/4}} |F^i(x')|^q dx' \le c\Big(N, q, G, \max_{j=1,\dots,N} (\sum_{i=1}^N \|a^{i,j}\|^q_{0,\mathcal{A};G}, \sum_{i=1}^N \|a^i\|^q_{0,\mathcal{A};G}) \Big)$$

$$(5.2.34) \qquad \times \int\limits_{G^{2\varrho}_{\varrho/4}} r^{-N} \mathcal{A}^{-q}(r) \Big(\sum_{i=1}^N |f^i(x)|^q + |\nabla \Phi|^q + |\Phi(x)|^q \Big) dx.$$

It follows from (10.2.85) as $\varrho \to +0$ that $|\nabla \Phi(0)| = 0$. Therefore

$$|\nabla \Phi(x)| = |\nabla \Phi(x) - \nabla \Phi(0)| \le \mathcal{A}(|x|) \|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}, \quad \forall x \in G^{2\varrho}_{\varrho/4},$$

and hence we have

$$|\Phi(x)| \le r |\nabla \Phi(x)| \le |x| \mathcal{A}(|x|) \|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}, \quad \forall x \in G^{2\varrho}_{\varrho/4}.$$

Similarly it follows from (5.2.21) as $\varrho \to +0$ that $f^j(0) = 0$, $\forall j = 1, \ldots, N$. Therefore we have $\forall x \in G^{2\varrho}_{\varrho/4}$

$$|f^j(x)| = |f^j(x) - f^j(0)| \le \mathcal{A}(r)[f^j]_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}}.$$

Consequently, estimating the right side of (5.2.34) and taking into account the inequalities obtained, we have

$$\sum_{i=1}^{N} \|F^i\|_{q;G^2_{1/4}} \le c\bigg(N, q, G, \max_{j=1,\ldots,N}\big(\sum_{i=1}^{N}\|a^{i,j}\|_{0,\mathcal{A};G}, \sum_{i=1}^{N}\|a^i\|_{0,\mathcal{A};G}\big)\bigg) \times$$

$$(5.2.35) \qquad \times mes\Omega \cdot \bigg(\sum_{i=1}^{N}\|f^i\|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} + \|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}\bigg).$$

So all conditions of Theorem 8.25 [129] are satisfied. By this theorem we get

$$\sup_{x' \in G^1_{1/2}} |z(x')| \le c\bigg(\|z\|_{2;G^2_{1/4}} + \|G\|_{\frac{N}{2(1-\alpha)};G^2_{1/4}} + \sum_{i=1}^{N}\|F^i\|_{\frac{N}{1-\alpha};G^2_{1/4}}\bigg) \le$$

$$(5.2.36) \quad \le c\bigg(\|z\|_{2;G^2_{1/4}} + \|g\|_{\frac{N}{1-\alpha};G^{2\varrho}_{\varrho/4}} + \sum_{i=1}^{N}\|f^i\|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} + \|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}\bigg).$$

Setting $w(x) = u(x) - \varphi(x)$ we have for $w(x)$ the problem

$$\begin{cases} \frac{\partial}{\partial x_i}(a^{ij}(x)w_{x_j} + a^i(x)w) + b^i(x)w_{x_i} + c(x)w = \\ \qquad = G(x) + \frac{\partial F^j(x)}{\partial x_j}, \quad x \in G^{2d}_0; \\ w(x) = 0, \quad x \in \Gamma^{2d}_0 \subset \partial G^{2d}_0, \end{cases}$$

where

$$G(x) = g(x) - b^i(x)\Phi_{x_i} - c(x)\Phi(x),$$
$$F^i(x) = f^i(x) - a^{ij}(x)\Phi_{x_j} - a^i(x)\Phi(x).$$

Moreover by assumptions (i) and (ii)

$$|a^{ij}(x) - \delta^j_i| \le \|a^{i,j}\|_{0,\mathcal{A};G}\mathcal{A}(|x|), \quad x \in G.$$

In virtue of the estimate (5.1.22) of Theorem 5.4 there is a constant $c > 0$ independent on w, G, F^i such that

$$(5.2.37) \qquad \int\limits_{G_0^\varrho} r^{2-N} |\nabla w|^2 dx \leq c\varrho^{2\lambda} \int\limits_{G_0^{2d}} \Big\{ |w(x)|^2 + |\nabla w|^2 + G^2(x) +$$

$$+ \sum_{i=1}^N |F^i(x)|^2 + r^{4-N-2\lambda} \mathcal{H}^{-1}(r) G^2(x) + r^{2-N-2\lambda} \mathcal{H}^{-1}(r) \sum_{i=1}^N |F^i(x)|^2 \Big\} dx,$$

$$\rho \in (0, d).$$

Our assumptions guarantee that the integral on the right side is finite. Since $z(x') = \varrho^{-1} \mathcal{A}^{-1}(\varrho) w(\varrho x')$ we obtain from (5.2.37)

$$(5.2.38) \qquad \int\limits_{G_{1/4}^2} |\nabla' z|^2 dx' \leq 2^{N-2} \varrho^{-2} \mathcal{A}^{-2}(\varrho) \int\limits_{G_{\varrho/4}^{2\varrho}} r^{2-N} |\nabla w|^2 dx \leq$$

$$\leq c\varrho^{2\lambda-2} \mathcal{A}^{-2}(\varrho) \int\limits_G \Big\{ |w(x)|^2 + |\nabla w|^2 + G^2(x) + \sum_{i=1}^N |F^i(x)|^2 +$$

$$+ r^{4-N-2\lambda} \mathcal{H}^{-1}(r) G^2(x) + r^{2-N-2\lambda} \mathcal{H}^{-1}(r) \sum_{i=1}^N |F^i(x)|^2 \Big\} dx, \ \rho \in (0, d).$$

By assumptions (i)-(iv) we have

$$|G(x)|^2 \leq c\{ |g|^2 + \mathcal{A}^2(r)(r^{-2}|\nabla\Phi|^2 + r^{-4}\Phi^2) \},$$

$$(5.2.39) \quad \sum_{i=1}^N |F^i(x)|^2 \leq c\Big\{ \sum_{i=1}^N |f^i(x)|^2 +$$

$$+ \max_{i,j=1,\dots,N} (\|a^{i,j}\|_{0,\mathcal{A};G}, \|a^i\|_{0,\mathcal{A};G})(|\nabla\Phi|^2 + \Phi^2) \Big\}.$$

Applying now the Friedrichs inequality and taking into account (5.2.9), we obtain from (5.2.38) and (5.2.39)

$$(5.2.40) \qquad \|z\|_{2;G_{1/4}^2}^2 \leq c\|\nabla' z\|_{2;G_{1/4}^2}^2 \leq c\varrho^{2\lambda-2} \mathcal{A}^{-2}(\varrho) \int\limits_G \Big\{ |w(x)|^2 +$$

$$+ |\nabla w|^2 + g^2(x) + \sum_{i=1}^N |f^i(x)|^2 + |\nabla\Phi|^2 + \Phi^2 + r^{4-N-2\lambda} \mathcal{H}^{-1}(r) g^2(x) +$$

$$+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 +$$

$$+ r^{-2}\mathcal{A}^2(r)|\nabla\Phi|^2\Big\}dx \le const\Big\{ \|g\|^2_{\frac{N}{1-\alpha};G} + \sum_{i=1}^{N}\|f^i\|^2_{0,\mathcal{A};G} + \|\varphi\|^2_{1,\mathcal{A};\partial G} +$$

$$+ \int_G \Big(r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 +$$

$$+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 + |w|^2 + |\nabla w|^2\Big)dx\Big\}$$

by assumptions (iii)-(v). By the definition of $z(x')$, inequalities (5.2.36) and (5.2.40), and assumptions (i)-(v) we finally obtain

(5.2.41)
$$|v|_{0;G^2_{1/4}} \le |z|_{0;G^2_{1/4}} + \varrho^{-1}\mathcal{A}^{-1}(\varrho)|\varphi|_{0;\Gamma^2_{1/4}} \le$$

$$\le c\Big(\|g\|_{\frac{N}{1-\alpha};G} + \sum_{i=1}^{N}\|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} +$$

$$+ \Big\{ \int_G \Big(r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 +$$

$$+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 + |w|^2 + |\nabla w|^2\Big)dx\Big\}^{1/2}\Big).$$

Step 3.

Returning to the variables $x, u(x)$, we now obtain from inequalities (5.2.24) and (5.2.41) the inequality

(5.2.42)
$$\varrho^{-1}\mathcal{A}^{-1}(\varrho) \sup_{x\in G^{\varrho}_{\varrho/2}} |u(x)| + \mathcal{A}^{-1}(\varrho) \sup_{x\in G^{\varrho}_{\varrho/2}} |\nabla u(x)| +$$

$$+ \sup_{\substack{x,y\in G^{\varrho}_{\varrho/2} \\ x\ne y}} \frac{|\nabla u(x) - \nabla u(y)|}{\mathcal{A}(\varrho)\mathcal{B}|x-y|)} \le c\Big(\|g\|_{\frac{N}{1-\alpha};G} + \sum_{i=1}^{N}\|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} +$$

$$+ \Big\{ \int_G \Big(r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 +$$

$$+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 + |u|^2 + |\nabla u|^2\Big)dx\Big\}^{1/2}\Big).$$

Setting $|x| = 2\varrho/3$ we deduce from (5.2.42) the validity of (5.2.10), (5.2.11). This completes the proof of Theorem 5.13.   □

REMARK 5.14. As an example of $\mathcal{A}(r)$, that satisfies all the conditions of Theorem 5.13, besides the function r^α, one may take $\mathcal{A}(r) = r^\alpha \ln(1/r)$, provided $\lambda \geq 1 + \alpha$. In the case of $\mathcal{A}(r) = r^\alpha$ the result of [21] follows from Theorem 5.13 for a single equation and the estimate (5.2.10) coincides with (5.1.51).

5.2.3. Global regularity and solvability.

THEOREM 5.15. *Let $\mathcal{A}$ be an α-Dini function $(0 < \alpha < 1)$ that satisfies the conditions (1.8.5), (1.8.6) and (5.2.9). Let $\overline{G} \setminus \{\mathcal{O}\}$ be a domain of class $C^{1,\mathcal{A}}$ and $\mathcal{O} \in \partial G$ be a conical point of G. Suppose that the assumptions (i)-(iv) are valid and*

$$(vi) \qquad \int\limits_G \big(c(x)\eta - a^i(x)D_i\eta\big)dx \leq 0, \quad \forall \eta \geq 0, \, \eta \in C_0^1(G).$$

Then the generalized problem (DL) has a unique solution $u \in C^{1,\mathcal{A}}(\overline{G})$ and we have the estimate

$$\|u\|_{1,\mathcal{A};G} \leq c\Bigg(\|g\|_{\frac{N}{1-\alpha};G} + \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} +$$

$$(5.2.43) \qquad + \Bigg\{ \int\limits_G \Bigg(r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 +$$

$$+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \Bigg)dx \Bigg\}^{1/2} \Bigg).$$

PROOF. The inequality (5.2.42) implies that

$$|\nabla u(x) - \nabla u(y)| \leq c\mathcal{B}(|x-y|)\Bigg(\|g\|_{\frac{N}{1-\alpha};G} + \|\varphi\|_{1,\mathcal{A};\partial G} +$$

$$(5.2.44) \qquad + \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};G} + \Bigg\{ \int\limits_G \Bigg(|u|^2 + |\nabla u|^2 + r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) +$$

$$+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \Bigg)dx \Bigg\}^{1/2} \Bigg)$$

$$\forall x,y \in G_{\varrho/2}^\varrho, \, \varrho \in (0,d).$$

From (5.2.42), (5.2.44) we now infer that $u \in C^{1,\mathcal{B}}(\overline{G_0^d})$. Indeed, let $x,y \in \overline{G_0^d}$ and $\varrho \in (0,d)$. If $x,y \in G_{\varrho/2}^\varrho$ then (5.2.44) holds. If $|x-y| > \varrho = |x|$

then by (5.2.42) we obtain

$$
\frac{|\nabla u(x) - \nabla u(y)|}{\mathcal{B}(|x-y|)} \leq 2c\mathcal{A}(|x|)\mathcal{B}^{-1}(|x|)\bigg(\|g\|_{\frac{N}{1-\alpha};G} + \|\varphi\|_{1,\mathcal{A};\partial G} +
$$

$$
+ \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};G} + \bigg\{ \int\limits_{G} \bigg(|u|^2 + |\nabla u|^2 + r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) +
$$

$$
+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N} |f^i(x)|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \bigg) dx \bigg\}^{1/2} \bigg) \leq
$$

$$
\leq 2c\alpha\bigg(\|g\|_{\frac{N}{1-\alpha};G} + \|\varphi\|_{1,\mathcal{A};\partial G} + \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};G} +
$$

$$
+ \bigg\{ \int\limits_{G} \bigg(|u|^2 + |\nabla u|^2 + r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) +
$$

$$
+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N} |f^i(x)|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \bigg) dx \bigg\}^{1/2} \bigg)
$$

in view of (1.8.3). Because of the condition (1.8.5) for the equivalence of $\mathcal{A}$ and $\mathcal{B}$, we derive $u \in C^{1,\mathcal{B}}(\overline{G_0^d})$ and the estimate

$$
\|u\|_{1,\mathcal{A};G_0^d} \leq c\bigg(\|g\|_{\frac{N}{1-\alpha};G} + \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} +
$$

$$
\text{(5.2.45)} \qquad\qquad + \bigg\{ \int\limits_{G} \bigg(|u|^2 + |\nabla u|^2 + r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) +
$$

$$
+ r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N} |f^i(x)|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \bigg) dx \bigg\}^{1/2} \bigg)
$$

following from the above arguments.

By means of a partition of unity, from the bounds (5.2.1) of Theorem 5.12 and (5.2.45), we derive

$$
\|u\|_{1,\mathcal{A};G} \leq c\bigg(\|g\|_{\frac{N}{1-\alpha};G} + \sum_{i=1}^{N} \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} +
$$

$$+|u|_{0;G} + \left\{ \int\limits_G \left(|u|^2 + |\nabla u|^2 + r^{4-N-2\lambda}\mathcal{H}^{-1}(r)g^2(x) + \right. \right.$$

(5.2.46)

$$\left. \left. + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{N}|f^i(x)|^2 + r^{2-N-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \right)dx \right\}^{1/2} \right).$$

By the assumption (vi) that guarantees the uniqueness of the solution for the problem *(DL)*, we have the bound (see Corollary 8.7 [**129**])

$$\int\limits_G (|u|^2 + |\nabla u|^2)dx \le C \int\limits_G \left(g^2 + \sum_{i=1}^{N}|f^i(x)|^2 + |\nabla\Phi|^2 + \Phi^2 \right)dx,$$

which together with the global boundedness of weak solution (Theorem 8.16 [**129**]) and the bound (5.2.46), leads to the desired estimate (5.2.43).

Finally, the global estimate (5.2.43) leads to the assertion on the unique solvability in $C^{1,\mathcal{A}}(\overline{G})$. This is proved by an approximation argument (see e.g. the proof of Theorem 8.34 [**129**]). □

REMARK 5.16. The conclusion of Theorem 5.15 is best possible. This is shown for the function $\mathcal{A}(r) = r^\alpha, \lambda \ge 1 + \alpha, \alpha \in (0,1)$ in [**171**]. (See also examples in Section 4.7 of the Chapter 4.)

5.3. Notes

The best possible Hölder exponents for weak solutions was first obtained in [**170, 171**]. There the method of non-smooth domain approximation by the sequence of smooth domains was used. We apply here the quasi-distance function $r_\varepsilon(x)$. The introduction of such function allows us to work in the given domain, and then to provide the passage to the limit over $\varepsilon \to +0$ (where $r_\varepsilon(x) \to r = |x|$).

The L^p-regularity of the (DL) in the cone was studied in [**84**], and in the domains with angles - in [**249**]. Finally, let us point out two further works. In [**8**] Alkhutov and Kondrat'ev proved the single-valued solvability in the space $W_0^{1,p}(G)$ of the (DL) in arbitrary convex bounded domain G assuming only the continuity in G of the leading coefficients .

Hölder estimates for the first derivatives of generalized solutions to the problem *(DL)* are well known in the case, if the leading coefficients $a^{ij}(x)$ of the equation are Hölder continuous (see e.g. 8.11 [**129**] for smooth domains and [**21**] for the domain with angular point). Here we derive Dini estimates for the first derivatives of generalized solutions of the problem *(DL)* in a domain with conical boundary point under *minimal* condition

on the smoothness of the leading coefficients (Dini continuity). The presentation of Section 5.2 follows [64]. It should be noted that the interior Dini-continuity of the first and second derivatives of generalized solutions to the problem *(DL)* was investigated in [74] and [224] under the condition of Dini continuity of the first derivatives of the leading coefficients.

Recently, V. Kozlov and V. Maz'ya [195, 196] derived an asymptotic formula near a point $\mathcal{O}$ at the smooth boundary of a new type for weak solutions of the Dirichlet problem for elliptic equations of arbitrary order. We formulate an idea of their results for the linear uniformly elliptic second order equation

$$\begin{cases} \frac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j} = g(x), & x \in G; \\ u(x) = 0, & x \in \partial G, \end{cases}$$

where $G \subset \mathbb{R}^N$ is a bounded domain with smooth boundary ∂G. It is assumed that $a^{ij}(x)$ are measurable and bounded complex-valued functions, $u(x)$ has a finite Dirichlet integral and $g = 0$ in a certain neighborhood G_0^d of the origin $\mathcal{O}$. In addition, let there exist a constant symmetric matrix $A_0 = \left(a_0^{ij}\right)$ with positive definite real part such that the function

$$\mathcal{A}(r) := \sup_{x \in G_0^r} \|A(x) - A_0\|$$

is sufficiently small for $r < d$, where $A = \left(a^{ij}\right)$. Let us define the function

$$Q(x) = \frac{< (A(x) - A)n, n >}{\sigma_N(\det A)^{1/2} < A^{-1}x, x >^{N/2}} -$$

$$- \frac{N < A^{-1}(A(x) - A)n, x >< n, x >< A^{-1}x, x >^{-1}}{\sigma_N(\det A)^{1/2} < A^{-1}x, x >^{N/2}},$$

where $< a, b > = \sum_{k=1}^N a_k b_k$ and n is the exterior unit normal at $\mathcal{O}$. The following asymptotic formula holds

(5.3.1)
$$u(x) = \exp\left(- \int_{G_0^d \backslash G_0^{|x|}} Q(y)dy + O\left(\int_{|x|}^d \frac{\mathcal{A}^2(r)}{r}dr\right)\right) \times$$

$$\times \left(Cd(x) + O\left(|x|^{2-\varepsilon} \int_{|x|}^d \frac{\mathcal{A}(r)}{r^{2-\varepsilon}}dr\right)\right) + O\left(|x|^{2-\varepsilon}\right),$$

where $C = const$ and ε is a small positive number. The sharp two-sided estimate for the Hölder exponent of u at the origin may be derived from (5.3.1).

They establish also the following criterion. Under the condition $\int\limits_{0}^{d} \frac{A^2(r)}{r} dr < \infty$, all solutions u are Lipschitz at the origin if and only if

$$(5.3.2) \qquad \liminf_{r\to+0} \int\limits_{G_0^d \setminus G_0^r} \Re Q(x) dx > -\infty.$$

Needless to say, this new one-sided restriction (5.3.2) is weaker that the classical Dini condition at the origin.

We point also to the work [262]. This work investigates L^q-regularity of weak solutions of the Dirichlet problem for linear elliptic second order equation in the divergent form with piecewise constant leading coefficients in a Lipschitz polyhedron.

Other boundary value problems (the Neumann problem, mixed problem) for elliptic variational equations in smooth, convex, or nonsmooth domains have been studied by V. Adolfsson and D. Jerison [2, 3]. They have investigated L^p-integrability of the second order derivatives for the Neumann problem in convex domains. J. Banasiak [27] - [29], J. Banasiak and G.F. Roach [31, 32] have considered the mixed boundary value problem of Dirichlet oblique-derivative type in plane domains with piecewise differentiable boundary. K. Gröger [136] has established a $W^{1,p}$-estimate for solutions to mixed boundary value problems, P. Shi and S. Wright [358] have investigated the higher integrability of the gradient in linear elasticity, M.K.V. Murthy and G. Stampacchia [317] have considered a variational inequality with mixed boundary conditions, W. Zajączkowski and V. Solonnikov [409] have investigated the Neumann problem in a domain with edges.

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CHAPTER 6

The Dirichlet problem for semilinear equations in a conical domain

6.1. The behavior of strong solutions for nondivergent equations near a conical point

In this section we study the properties of strong solutions of the Dirichlet problem for nondivergent semilinear uniformly elliptic second order equations in a neighborhood of a conical boundary point

$$(SL) \quad \begin{cases} Lu := a^{ij}(x)D_{ij}u(x) + a^i(x)D_iu(x) + a(x)u(x) = \\ \quad = g(u) + f(x) \quad \text{in} \quad G, \\ g(u) = a_0(x)u|u|^{q-1}, \ q > 0; \\ u(x) = 0 \quad \text{on} \quad \partial G \setminus \mathcal{O}. \end{cases}$$

Let $G \subset \mathbb{R}^N$ be a bounded domain with a conical point in $\mathcal{O}$ as described in Section 1.3 of chapter 1. We shall assume that G_0^d is *a convex* cone for small $d > 0$.

DEFINITION 6.1. By a *strong* solution of the Dirichlet problem (SL) in G we mean a function $u \in W^2(G) \cap C^0(\overline{G} \setminus \mathcal{O})$ which satisfies the equation of (SL) for almost all $x \in G$ and the boundary condition for all $x \in \partial G \setminus \mathcal{O}$.

In the following we will always use these **assumptions**

a) *the uniform ellipticity condition*

$$\nu|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \ x \in \overline{G}$$

with some $\nu, \mu > 0$; $\quad a^{ij}(0) = \delta_i^j$;
aa) $a^{ij} \in C^0(\overline{G})$, $a^i \in L^p(G)$ *and* $a \in L^{p/2}(G)$ *with some* $p > N$;

aaa) *there exists a monotonically increasing nonnegative continuous at zero function* $\mathcal{A}(r)$, $\mathcal{A}(0) = 0$ *such that for* $x, y \in \overline{G}$

$$\left(\sum_{i,j=1}^{N} |a^{ij}(x) - a^{ij}(y)|^2 \right)^{1/2} \leq \mathcal{A}(|x - y|),$$

$$|x| \left(\sum_{i=1}^{N} a^{i^2}(x) \right)^{1/2} + |x|^2 |a(x)| \leq \mathcal{A}(|x|);$$

b) $a_0(x)$ *is a nonnegative measurable in G function;*

c) *there exist real numbers $k_1 \geq 0$ and $\beta > -1$ such that*

$$|f(x)| \leq k_1 |x|^{\beta}.$$

6.1.1. The weighted integral estimates $(0 < q \leq 1)$. Now we prove certain weighted integral estimates of strong solutions of (SL). Here the function $a_0(x)$ can be unbounded.

THEOREM 6.2. *Let u be a strong solution of (SL) and the conditions a), aa), aaa) and b) are satisfied. Suppose that $a_0(x) \in V^0_{\frac{2}{1-q}; \frac{4}{1-q}-N}(G)$,*

$f(x) \in \overset{\circ}{W}^0_{4-N}(G)$, $0 < q < 1$.

Then $u(x) \in \overset{\circ}{W}^2_{4-N}(G)$ and there is a positive constant c, determined by $\nu, \mu, q, N, \max_{x \in \overline{G}} \mathcal{A}(|x|), G$ such that

$$\int\limits_G \left(r^{4-N} |D^2 u|^2 + r^{2-N} |Du|^2 + r^{-N} |u|^2 + a_0(x) r^{2-N} |u|^{1+q} \right) dx \leq$$

(6.1.1)

$$\leq c \int\limits_G \left(u^2 + r^{4-N} f^2(x) + 2 + a_0^{2/(1-q)}(x) r^{4/(1-q)-N} \right) dx.$$

PROOF. We multiply both parts of the equation of (SL) by $r_\varepsilon^{2-N} u(x)$ and integrate over the domain (G). Similar to the theorem 4.13 proof from

Chapter 4 we have

$$\int\limits_{G} r_\varepsilon^{2-N}|\nabla u|^2 dx + \int\limits_{G} a_0(x)r_\varepsilon^{2-N}|u|^{1+q}dx \le$$

$$(6.1.2) \quad \le c(h)\mathcal{A}(d) \int\limits_{G_0^d} \left(r_\varepsilon^{2-N}r^2|D^2u|^2 + r_\varepsilon^{2-N}|\nabla u|^2 + r_\varepsilon^{2-N}r^{-2}u^2 \right) dx +$$

$$+ c(d) \int\limits_{G_d} \left(|D^2u|^2 + u^2 \right) dx + \frac{1}{2}\int\limits_{G} r_\varepsilon^{4-N}f^2(x)dx, \quad \forall \varepsilon > 0, d > 0.$$

By the layers method based on the local L^2-estimate, we derive the inequality (see the derivation of (4.2.23)

$$(6.1.3) \quad \int\limits_{G_0^d} r_\varepsilon^{2-N}r^2|D^2u|^2 dx \le c \int\limits_{G_0^{2d}} \left(r_\varepsilon^{2-N}r^{-2}|u|^2 + r_\varepsilon^{4-N}a_0^2(x)|u|^{2q} + \right.$$

$$\left. + r_\varepsilon^{4-N}f^2(x) \right) dx, \quad \forall \varepsilon > 0, d > 0,$$

where c is a constant depending only on $\nu, \mu, q, N, \max\limits_{x \in \overline{G}}\mathcal{A}(|x|), G$. Taking into account that $q < 1$ by Young's inequality we have

$$(6.1.4) \quad r_\varepsilon^{4-N}a_0^2(x)|u|^{2q} = \left(r_\varepsilon^{-Nq}|u|^{2q} \right)\left(r_\varepsilon^{4-N+Nq}a_0^2(x) \right) \le \sigma r_\varepsilon^{-N}|u|^2 +$$

$$+ c(\sigma, q)a_0^{2/(1-q)}(x)r_\varepsilon^{4/(1-q)-N}, \quad \forall \sigma > 0.$$

The estimate (6.1.1) we seek follows from (6.1.2) - (6.1.4) under proper small $d > 0$ with the help of the same arguments as during the completion of the Theorem 4.13 proof.   □

THEOREM 6.3. *Let u be a strong solution of (SL) and the conditions a)-c) with $\mathcal{A}(r)$ that is Dini-continuous at zero are satisfied. In addition, suppose $f(x) \in L_N(G) \cap \overset{\circ}{W}{}^0_{4-N}(G)$, $a_0(x) \in V^0_{\frac{2}{1-q};\frac{4}{1-q}-N}(G)$ and there is a constant $k_2 \ge 0$ such that*

$$(6.1.5) \quad \|a_0\|^{1/(1-q)}_{V^0_{2/(1-q),4/(1-q)-N}(G_0^\varrho)} \le k_2\varrho^{2+\beta}, \quad \varrho \in (0,d).$$

Then there are positive constants c and $d \in (0, e^{-1})$ such that for $\varrho \in (0, d)$

$$\|u\|_{\overset{\circ}{W}^2_{4-N}(G_0^\varrho)} \leq c\Bigg(\|u\|_{L^2(G)} + \|f\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|a_0\|^{1/(1-q)}_{V^0_{\frac{2}{1-q};\frac{4}{1-q}-N}(G)} + k_1 +$$

$$(6.1.6) \qquad + k_2\Bigg) \begin{cases} \varrho^\lambda, & \text{if } \beta + 2 > \lambda, \\ \varrho^\lambda \ln^{3/2}\left(\frac{1}{\varrho}\right), & \text{if } \beta + 2 = \lambda, \\ \varrho^{\beta+2}, & \text{if } \beta + 2 < \lambda. \end{cases}$$

PROOF. At first, because of Theorem 6.2, we have $u(x) \in \overset{\circ}{W}^2_{4-N}(G)$. Now we introduce the function

$$U(\varrho) = \int\limits_{G_0^\varrho} r^{2-N} |Du|^2 dx, \quad \varrho \in (0, d)$$

and multiply both parts of (SL) by $r^{2-N} u(x)$ and integrate the obtained equality over the domain G_0^ϱ, $\varrho \in (0, d)$. As a result, similarly to Theorem 4.18, we obtain

$$U(\varrho) + \int\limits_{G_0^\varrho} a_0(x) r^{2-N} |u|^{1+q} dx \leq \frac{\varrho}{2\lambda} U'(\varrho) + c\mathcal{A}(\varrho) U(2\varrho) +$$

$$(6.1.7) \qquad + c\mathcal{A}(\varrho) \int\limits_{G_0^{2\varrho}} \left(a_0^{2/(1-q)}(x) r^{4/(1-q)-N} + r^{4-N} f^2(x) \right) dx +$$

$$+ c\mathcal{A}(\varrho) U(\varrho) + \int\limits_{G_0^\varrho} r^{2-N} |u| |f(x)| dx.$$

For this we used the inequalities (6.1.3), (6.1.4) with $\varepsilon = 0, \sigma = 1$.

From hypothesis (c) we have

$$(6.1.8) \qquad \int\limits_{G_0^{2\varrho}} r^{4-N} f^2(x) dx \leq \frac{k_1^2 meas \Omega}{2(\beta+2)} (2\varrho)^{4+2\beta}$$

and apply as well the Cauchy and Poincaré inequalities

$$(6.1.9) \quad \int\limits_{G_0^\varrho} r^{2-N} |u| |f(x)| dx \leq k_1 \int\limits_{G_0^\varrho} r^{\beta+2-N} |u| dx =$$

$$+ k_1 \int\limits_{G_0^\varrho} \left(r^{-N/2} |u| \right) r^{\beta+2-N/2} dx \leq c\delta U(\varrho) + c\delta^{-1} k_1^2 \varrho^{2\beta+4}.$$

From (6.1.5) and (6.1.7)-(6.1.9) finally we obtain the differential inequality

$$(6.1.10) \quad U(\varrho) \le \frac{\varrho}{2\lambda} U'(\varrho) + c_1 \mathcal{A}(\varrho) U(2\varrho) + c_2 (\mathcal{A}(\varrho) + \delta) U(\varrho) +$$

$$+ c_3 \delta^{-1} (k_1^2 + k_2^2) \varrho^{2\beta+4}, \quad \forall \delta > 0, \ 0 < \varrho < d.$$

Moreover, by Theorem 6.2, we have the initial condition

$$(6.1.11) \quad U_0 \equiv U(d) = \int\limits_{G_0^d} r^{2-N} |\nabla u|^2 dx \le$$

$$\le c \left(\|u\|_{L^2(G)}^2 + \|f\|_{\overset{\circ}{W}_{4-N}^{0}(G)}^2 + \|a_0\|_{V_{\frac{2}{1-q};\frac{4}{1-q}-N}^{0}(G)}^{2/(1-q)} \right).$$

The differential inequality (6.1.10) with initial condition is the same type as (4.2.47) with $s = \beta + 2$ or (4.2.51), if $\beta + 2 = \lambda$. Repeating verbatim the investigation of these inequalities in the proof of Theorem 4.18 we obtain

$$U(\varrho) \le c \left(\|u\|_{L^2(G)}^2 + \|f\|_{\overset{\circ}{W}_{4-N}^{0}(G)}^2 + \|a_0\|_{V_{\frac{2}{1-q};\frac{4}{1-q}-N}^{0}(G)}^{2/(1-q)} + k_1^2 + k_2^2 \right) \times$$

$$(6.1.12) \qquad \times \begin{cases} \varrho^{2\lambda}, & \text{if } \beta + 2 > \lambda, \\ \varrho^{\lambda} \ln^3 \left(\frac{1}{\varrho}\right), & \text{if } \beta + 2 = \lambda, \\ \varrho^{2(\beta+2)}, & \text{if } \beta + 2 < \lambda. \end{cases}$$

From (6.1.3) and (6.1.4) passing to the limits as $\varepsilon \to 0$, we obtain

$$(6.1.13) \quad \int\limits_{G_0^\varrho} r^{4-N} |D^2 u|^2 dx \le c \int\limits_{G_0^{2\varrho}} \left(r^{-N} |u|^2 + a_0^{2/(1-q)}(x) r^{4/(1-q)-N} + \right.$$

$$\left. + r^{4-N} f^2(x) \right) dx, \quad 0 < \varrho < d.$$

Now taking into account the inequality (H-W) from (6.1.12) and (6.1.13) the desired (6.1.6) follows. $\qquad\square$

THEOREM 6.4. Let $u \in W^{2,N}(G)$ be a strong solution of (SL) and the conditions a)-c) with $\mathcal{A}(r)$ that is Dini continuous at zero are satisfied. In addition, suppose

$$a(x) \le 0, \ f(x) \in L_N(G) \cap \overset{\circ}{W}_{4-N}^{0}(G), \ a_0(x) \in V_{N/(1-q);2qN/(1-q)}^{0}(G).$$

Then there is a positive constant c such that

$$(6.1.14) \qquad \|u\|_{V_{N,0}^2(G)} \le c \left(\|a_0\|_{V_{N/(1-q);2qN/(1-q)}^{0}(G)}^{1/(1-q)} + \|f\|_{L^N(G)} \right).$$

PROOF. By Theorem 4.48, there exists the unique solution $u \in V_{N,0}^2(G)$ of the linear problem

$$\begin{cases} Lu = F(x), & x \in G, \\ u(x) = 0, & x \in \partial G \setminus \mathcal{O} \end{cases}$$

provided $\lambda > 1$, $F \in L_N(G)$ and

(6.1.15) $$\|u\|_{V_{N,0}^2(G)} \leq c\|F\|_{N,G},$$

where $c > 0$ depends only on $\nu, \mu, N, \max\limits_{x \in \overline{G}} \mathcal{A}(|x|), \|a^i\|_{p,G}, \|a\|_{p/2,G}, p > N$ and the domain G. The condition $\lambda > 1$ is fulfilled by the convexity of G_0^d. From (6.1.15) with $F(x) = f(x) + a_0(x)u|u|^{q-1}$ using the inequality (1.2.5) we obtain:

(6.1.16) $$\int\limits_G \left(|D^2u|^N + r^{-N}|Du|^N + r^{-2N}|u|^N\right) dx \leq$$

$$\leq 2^{N-1} \int\limits_G \left(|a_0(x)|^N|u|^{qN} + |f(x)|^N\right) dx.$$

Using Young's inequality and taking into account $q \in (0,1)$, we have

(6.1.17) $$|a_0(x)|^N|u|^{qN} = \left(r^{-2qN}|u|^{qN}\right)\left(r^{2qN}|a_0(x)|^N\right) \leq$$

$$\leq \varepsilon r^{-2N}|u|^N + \varepsilon^{q/(q-1)}r^{2qN/(1-q)}|a_0(x)|^{N/(1-q)}, \quad \forall \varepsilon > 0.$$

By the choice $\varepsilon = 2^{-N}$ from (6.1.16) and (6.1.17) the desired (6.1.14) follows. $\square$

6.1.2. The estimate of the solution modulus $(0 < q \leq 1)$. Now we want to deduce the estimate of our solution modulus in the case $(0 < q \leq 1)$. To that end we introduce the function

(6.1.18) $$\psi(\varrho) = \begin{cases} \varrho^\lambda & \text{if,} \quad \lambda < \beta + 2; \\ \varrho^\lambda \ln^{3/2} \frac{1}{\varrho} & \text{if,} \quad \lambda = \beta + 2; \\ \varrho^{\beta+2} & \text{if,} \quad \lambda > \beta + 2. \end{cases}$$

THEOREM 6.5. *Let $u(x) \in W^{2,N}(G)$ be a strong solution of (SL) and the conditions a) - c) with $\mathcal{A}(r)$ that is Dini continuous at zero are satisfied. In addition, suppose $a(x) \leq 0$, $a(x) \in L^N(G)$, $f(x) \in L_N(G) \cap \mathring{W}_{4-N}^0(G)$, $a_0(x) \in L^{N/(1-q)}(G) \cap V_{\frac{2}{1-q}; \frac{4}{1-q}-N}^0(G)$ together with (6.1.5) and there exists a nonnegative constant k_0 such that*

(6.1.19) $$\|a_0(x)\|_{L^{\frac{N}{1-q}}(G_{\varrho/4}^{2\varrho})} \leq k_0\varrho^{1-2q}\psi^q(\varrho).$$

Then there are positive constants c_0, d independent of u such that the following estimates are held:

1) $|u(x)| \leq c_0 |x|^{\frac{2}{1-q}}$, $x \in G_0^d$,   *if* $\lambda > \beta + 2$, $0 < q < 1 - \dfrac{2}{\lambda}$;

2) $|u(x)| \leq c_0 |x|^{\lambda}$, $x \in G_0^d$,   *if* $\lambda < \beta + 2$, $1 - \dfrac{2}{\lambda} \leq q \leq 1$;

3) $|u(x)| \leq c_0 |x|^{\beta+2}$, $x \in G_0^d$,   *if* $\lambda > \beta + 2$, $1 - \dfrac{2}{\lambda} \leq q \leq 1$;

4) $|u(x)| \leq c_0 |x|^{\lambda} \ln \dfrac{1}{|x|}$, $x \in G_0^d$,   *if* $\lambda > \beta + 2$, $1 - \dfrac{2}{\lambda} \leq q \leq 1$.

PROOF. Let us perform the variables substitution $x = \varrho x'$, $u(\varrho x') = \psi(\varrho) v(x')$ in the problem (SL). Let G' be the image of the domain G under transformation of coordinates $x_i = \varrho x'_i$; $i = 1, \cdots, N$. As a result we infer that $v(x')$ is a solution of the problem

$$(SL)' \quad \begin{cases} a^{ij}(\varrho x') v_{x'_i x'_j} + \varrho a^i(\varrho x') v_{x'_i} + \varrho^2 a(\varrho x') v = \frac{\varrho^2}{\psi(\varrho)} f(\varrho x') + \\ \qquad\qquad + \varrho^2 \psi^{q-1}(\varrho) a_0(\varrho x') v |v|^{q-1}, \ x' \in G', \\ v(x') = 0, \quad x' \in \partial G'. \end{cases}$$

We apply now Theorem 4.5 (Local Maximum Principle)

$$(6.1.20) \quad \sup_{x' \in G_{1/2}^1} |v(x')| \leq c \Bigg\{ \left(\int\limits_{G_{1/4}^2} v^2(x') dx' \right)^{1/2} + \frac{\varrho^2}{\psi(\varrho)} \|f\|_{L^N(G_{1/4}^2)} +$$

$$+ \varrho^2 \psi^{q-1}(\varrho) \left(\int\limits_{G_{1/4}^2} |a_0(\varrho x')|^N |v|^{qN}) dx' \right)^{1/N} \Bigg\}.$$

By the inequality (6.1.17), we have

$$(6.1.21) \quad |a_0(x)|^N |v|^{qN} \leq \varepsilon^{qN/(q-1)} |x'|^{2qN/(1-q)} |a_0(x)|^{N/(1-q)} +$$

$$+ \varepsilon^N |x'|^{-2N} |v|^N, \ \forall \varepsilon > 0.$$

From (6.1.20) and (6.1.21), we get

$$
\sup_{x' \in G^1_{1/2}} |v(x')| \le c \left(\int\limits_{G^2_{1/4}} v^2(x') dx' \right)^{1/2} + c \frac{\varrho^2}{\psi(\varrho)} \|f\|_{L^N(G^2_{1/4})} +
$$

$$
(6.1.22) \qquad + \varrho^2 \psi^{q-1}(\varrho) \left\{ \varepsilon \left(\int\limits_{G^2_{1/4}} |x'|^{-2N} |v|^N dx' \right)^{1/N} + \right.
$$

$$
\left. + c\varepsilon^{\frac{q}{q-1}} \left(\int\limits_{G^2_{1/4}} |x'|^{\frac{2qN}{1-q}} |a_0|^{\frac{N}{1-q}} dx' \right)^{1/N} \right\}, \quad \forall \varepsilon > 0.
$$

Now we estimate each term on the right in (6.1.22)

- $\left(\int\limits_{G^2_{1/4}} v^2(x') dx' \right)^{1/2} \le 2^{n/2} \psi^{-1}(\varrho) \left(\int\limits_{G^{2\varrho}_{\varrho/4}} r^{-N} u^2(x) dx \right)^{1/2} \le C,$

  in virtue of (6.1.6);

- $\varrho^2 \psi^{-1}(\varrho) \|f\|_{L^N(G^2_{1/4})} \le 2\varrho \psi^{-1}(\varrho) \left(\int\limits_{G^{2\varrho}_{\varrho/4}} |f|^N dx \right)^{1/N}$

  $\le ck_1 \varrho^{\beta+2} \psi^{-1}(\varrho) \le const;$ here we apply hypothesis c) and the definition (6.1.18) of $\psi(\varrho)$;

- $\left(\int\limits_{G^2_{1/4}} |x'|^{-2N} |v|^N dx' \right)^{1/N} \le 2^4 \left(\int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{-N} |v|^N dx \right)^{1/N} \le$

  $\le 2^6 \varrho \left(\int\limits_{G^{2\varrho}_{\varrho/4}} r^{-2N} |v|^N dx \right)^{1/N} \le 2^6 \frac{\varrho}{\psi(\varrho)} \left(\int\limits_{G} r^{-2N} |u|^N dx \right)^{1/N};$

- $\left(\int\limits_{G^2_{1/4}} |x'|^{\frac{2qN}{1-q}} |a_0|^{\frac{N}{1-q}} dx' \right)^{1/N} \le c(q,N) \varrho^{-1} \left(\int\limits_{G^{2\varrho}_{\varrho/4}} |a_0|^{\frac{N}{1-q}} dx \right)^{1/N} \quad .$

In (6.1.22) we choose $\varepsilon = \frac{\psi(\varrho)}{\varrho}$; then, because of (6.1.16) from two latter estimates, we obtain

$$(6.1.23) \qquad \varepsilon \left(\int\limits_{G^2_{1/4}} |x'|^{-2N} |v|^N dx' \right)^{1/N} \leq const$$

and

$$(6.1.24) \quad \varepsilon^{\frac{q}{q-1}} \left(\int\limits_{G^2_{1/4}} |x'|^{\frac{2qN}{1-q}} |a_0|^{\frac{N}{1-q}} dx' \right)^{1/N} \leq$$

$$\leq c \left(\frac{\varrho}{\psi(\varrho)} \right)^{\frac{q}{1-q}} \frac{1}{\varrho} \left(\int\limits_{G^{2\varrho}_{\varrho/4}} |a_0|^{\frac{N}{1-q}} dx \right)^{1/N} \leq const,$$

in virtue of hypothesis (6.1.19). The obtained estimates result for (6.1.22)

$$(6.1.25) \qquad \sup_{x' \in G^1_{1/2}} |v(x')| \leq c\big(1 + \varrho^2 \psi^{q-1}(\varrho)\big).$$

Now we show that for all interesting cases of our theorem,

$$(6.1.26) \qquad \varrho^2 \psi^{q-1}(\varrho) < \infty, \forall \varrho > 0$$

is true.

1) $\underline{\beta + 2 < \lambda} \quad \Rightarrow \quad \psi(\varrho) = \varrho^{\beta+2}.$ In this case we have

$$\varrho^2 \psi^{q-1}(\varrho) = \varrho^{q(\beta+2)-\beta} < \infty, \forall \varrho > 0,$$

if $\beta + 2 \leq \frac{2}{1-q}$. Choosing the best exponent $\beta + 2 = \frac{2}{1-q} < \lambda$ we get the first statement of our theorem. In fact, from (6.1.25) and (6.1.26) we have

$$|v(x')| \leq M'_0 = const \quad \forall x' \in G^1_{1/2}.$$

Returning to former variables hence it follows

$$|u(x)| \leq M'_0 \psi(\varrho) = M'_0 \varrho^{\frac{2}{1-q}}, \quad \forall x \in G^\varrho_{\varrho/2}, \quad \varrho \in (0, d).$$

Setting $|x| = \frac{2}{3}\varrho$ hence follows the required assertion.

2) $\beta + 2 > \lambda \quad \Rightarrow \quad \psi(\varrho) = \varrho^\lambda$. In this case we have

$$\varrho^2 \psi^{q-1}(\varrho) = \varrho^{2+\lambda(q-1)} < \infty, \forall \varrho > 0,$$

if $1 - \frac{2}{\lambda} \le q \le 1$. Repeating verbatim above stated arguments as in the first case, we get the second statement of our theorem.

3) $\beta + 2 < \lambda \quad \Rightarrow \quad \psi(\varrho) = \varrho^{\beta+2}$. In this case we have

$$\varrho^2 \psi^{q-1}(\varrho) = \varrho^{q(\beta+2)-\beta} = \varrho^{2q}\varrho^{\beta(q-1)} \le \varrho^{2q}\varrho^{(\lambda-2)(q-1)} < \infty, \forall \varrho > 0,$$

if $1 - \frac{2}{\lambda} \le q \le 1$. Repeating verbatim above stated arguments as in the first case we get the third statement of our theorem.

4) $\beta + 2 = \lambda \quad \Rightarrow \quad \psi(\varrho) = \varrho^\lambda \ln^{3/2} \frac{1}{\varrho}$. In this case we have

$$\varrho^2 \psi^{q-1}(\varrho) = \varrho^{2+\lambda(q-1)} \ln^{\frac{3}{2}(q-1)} \frac{1}{\varrho} < \infty, \forall \varrho > 0,$$

if $1 - \frac{2}{\lambda} \le q \le 1$. Repeating verbatim above stated arguments as in the first case, we get the fourth statement of our theorem.   □

Now we go on to the deduce some corollaries from Theorem 6.5.

LEMMA 6.6. *Let* $a_{(}x) \ge a_0 = const > 0$. *and hypotheses a) and aaa) are satisfied. There are positive numbers* η, ϱ, *determined only by* $\nu, \mu, q,$ a_0, N *such that, if* $u(x)$ *is a strong solution of the equation or* (SL) *with* $f(x) \equiv 0$ *and* $0 < q < 1$ *in the ball* $B_\varrho(0)$ *and* $|u(x)| < \eta$, $x \in \partial B_\varrho(0)$, *then* $u(0) = 0$.

PROOF. Let $s > \frac{2}{1-q}$. We set $R(x) = |x|^s$. Then

$$LR(x) - a_0(x)R^q(x) = sr^{s-2}\Big\{\sum_i a^{ii}(x) + (s-2)\frac{a^{ij}(x)x_ix_j}{r^2} + x_ia^i(x)+$$

$$+ \frac{1}{s}a(x)r^2\Big\} - a_0(x)r^{sq} \le$$

$$\le sr^{s-2}\Big(\mu(s+N-2) + \mathcal{A}(r)\Big) - a_0r^{sq}.$$

By the continuity of $\mathcal{A}(r)$ at zero, there exists $d > 0$ such that $\mathcal{A}(r) < 1$ as soon as $r < d$. Therefore we obtain

$$LR(x) - a_0(x)R^q(x) \le sr^{s-2}\Big(\mu(s+N-2) + 1\Big) - a_0r^{sq} < 0,$$

provided $r < d$ and $r^{s-2-sq} < \dfrac{a_0}{s\left(1+\mu(s+N-2)\right)}$. So

$$LR(x) - a_0(x)R^q(x) < 0 \text{ provided } \quad \varrho = \min\left\{d; \dfrac{a_0}{s\left(1 + \mu(s + N - 2)\right)}\right\}.$$

By the Maximum Principle (see below Theorem 6.8), $|u| < R$ provided that $\eta < \varrho^s$, hence $u(0) = 0$. $\qquad \square$

THEOREM 6.7. *Let* $u(x) \in W^{2,N}(G)$ *be a strong solution of* (SL) *and the conditions a)-c) with* $\mathcal{A}(r)$ *that is Dini continuous at zero are satisfied. Let* $\lambda > \beta + 2$, $\quad 0 < q < 1 - \frac{2}{\lambda}$. *In addition, suppose that* $f(x) \equiv 0$,
$a_0(x) \quad \in \quad L^{N/(1-q)}(G) \cap V^0_{\frac{2}{1-q};\frac{4}{1-q}-N}(G) \quad$ *together with* $\quad$ *(6.1.5),*
$a(x) \le 0$, $a(x) \in L^N(G)$, $a_0(x) \ge a_0 = \text{const} > 0$, *and there exists a nonnegative constant* k_0 *such that*

$$(6.1.27) \qquad\qquad \|a_0(x)\|_{L^{\frac{N}{1-q}}(G^{2\varrho}_{\varrho/4})} \le k_0 \varrho^{1+\beta q}.$$

Then there is a positive constant d *independent of* u *such that* $u(x) \equiv 0$, $x \in G^d_0$.

PROOF. Let $c_0, d > 0$ are chosen according to Theorem 6.4 and such that $G^d_0 \subset G$. Let $x_0 \in G^d_0$. We make the transform $x - x_0 = \varrho x'$, $u(x) = \varrho^{\frac{2}{1-q}} v(x')$. The function $v(x')$ is a solution of $(SL)'$ with $f \equiv 0$ and, by Lemma 6.6, we have $v(0) = 0$ provided $|v(x')| < \eta$ for $|x'| = R$ with some positive R, η. Hence $u(x_0) = 0$ provided $|u(x)| < \eta \varrho^{\frac{2}{1-q}}$ for $|x - x_0| = R\varrho$. But the latter condition is satisfied in virtue of assertion 1) of Theorem 6.5, if we set $\eta = c_0$, $R = 2$. Thus we get $u(x_0) = 0$. Since any $x_0 \in G^d_0$ we obtain the assertion of our Theorem. $\qquad \square$

6.1.3. The estimate of the solution modulus ($q > 1$). Let us recall the well known Comparison Principle.

THEOREM 6.8. **(Comparison Principle)** *Suppose* $D \in \mathbb{R}^N$ *is a bounded domain, L is elliptic in D, $a(x) \le 0$ in D. Let us define the function $g(x, u)$ with the properties*

$$g(x, u_1) \ge g(x, u_2) \text{ for } u_1 \ge u_2.$$

Let $u, v \in W^{2,N}_{loc}(D) \cap C^0(\overline{D})$ *satisfy the inequalities*

$$Lu \le g(x, u), \quad Lv \ge g(x, v) \text{ in } D.$$

Then

$$u \ge v \text{ on } \partial D \quad \Rightarrow \quad u \ge v \text{ throughout } D.$$

PROOF. Let $w = u - v$. We have

$$Lw = Lu - Lv \leq g(x, u) - g(x, v) \leq 0$$

on $D^- = \{x \in D \mid w(x) < 0\}$ and $w \geq 0$ on ∂D. From the Alexandrov maximum principle (Theorem 4.2) we get

$$w(x) \geq \inf_{D^-} w(x) = \inf_{\partial D^-} w(x) = 0, \ \forall x \in D.$$

$\square$

Now we consider the case $q > 1$. For this at first we study the $C^0 \cap W^{2,N}_{loc}$ — solutions of differential inequality in $\mathbb{R}^N$

(DI) $\qquad\qquad$ $\mathrm{sign}\, u \cdot Lu - a_0(x)|u|^q \geq -k,$

In this connection we suppose

(*) L is the uniformly elliptic operator with the ellipticity constants ν, μ,

$(\nu \leq \mu)$ and with bounded coefficients

$$\left(\sum_{i=1}^{N} |a^i(x)|^2 \right)^{1/2} + |a(x)| \leq m, \qquad a(x) \leq 0,$$

$$a_0(x) \geq a_0 > kN^{q-1}, \ \forall x \in G,$$

where m, a_0, k are nonnegative constants.

We derive as a preliminary the next statements.

LEMMA 6.9. *Let L satisfy (*). There are a bounded domain $D \subset \mathbb{R}^N$ containing the origin $\mathcal{O}$ and a positive function $U(x)$ defined in D such that*

(6.1.28) $\qquad$ $\begin{cases} LU - a_0 U^q \leq -k, \ x \in D, \\ U(0) = 1, \quad \lim\limits_{x \to \partial D} U(x) = \infty. \end{cases}$

PROOF. We first set $U(x) = \sum\limits_{i=1}^{N} y(x_i)$, where $y(t)$ is a positive solution of the Cauchy problem

(6.1.29) $\qquad$ $\begin{cases} \mu y''(t) + m|y'(t)| - a_0 y^q = -\frac{k}{N}, \\ y(0) = \frac{1}{N}, \ y'(0) = 0. \end{cases}$

By setting $y' = p(y)$ we get

(6.1.30) $\qquad\qquad$ $t = \displaystyle\int\limits_{1/N}^{y} \frac{d\eta}{p(\eta)}.$

The function $p(y)$ is a solution of the Cauchy problem

$$(6.1.31) \qquad \begin{cases} \mu p p' + m|p| - a_0 y^q = -\frac{k}{N}, \\ p\left(\frac{1}{N}\right) = 0. \end{cases}$$

Now we apply the Hardy theorem (see, for example, Theorem 3 §5, chapter V [38]). By virtue of this theorem, any positive solution of (6.1.31) fulfills the asymptotic relation

$$(6.1.32) \qquad p(\eta) \sim \eta^\kappa \quad \text{as } \eta \to +\infty, \ \kappa \in \mathbb{R}.$$

Now we calculate the quantity κ. From (6.1.31) we infer
$\mu \eta^\kappa p'(\eta) + m \eta^\kappa \sim a_0 \eta^q$ as $\eta \to +\infty$ or

$$(6.1.33) \qquad p'(\eta) \sim \frac{1}{\mu}(a_0 \eta^{q-\kappa} - m) \quad \text{as } \eta \to +\infty.$$

Integrating the relation (6.1.33) with regard to (6.1.32) we find

$$(6.1.34) \qquad p(\eta) \sim \frac{1}{\mu}(a_0 \frac{\eta^{q-\kappa+1}}{q-\kappa+1} - m\eta) \quad \text{as } \eta \to +\infty.$$

From (6.1.32) and (6.1.34) it follows that $\kappa = q - \kappa + 1 \geq 1$, or $\kappa = \frac{q+1}{2}$, $q \geq 1 \Rightarrow$

$$(6.1.35) \qquad \int\limits_{1/N}^{\infty} \frac{d\eta}{p(\eta)} \sim \int\limits_{1/N}^{\infty} \frac{d\eta}{\eta^{\frac{q+1}{2}}} < \infty, \text{ if } q > 1.$$

From (6.1.30) and (6.1.35) it follows that

$$y(t) \to \infty \text{ as } t \to \int\limits_{1/N}^{\infty} \frac{d\eta}{p(\eta)} < \infty.$$

Now we remark that because of (*)

$$y''(0) = \frac{1}{\mu}\left(\frac{a_0}{N^q} - \frac{k}{N}\right) > 0$$

and, consequently, by the continuity of y'', we have $y''(t) > 0$ in a certain neighborhood of zero. Therefore, returning now to $U(x)$ we have

$$U_{x_i} = y'(x_i), \quad U_{x_i x_j} = \delta_i^j y''(x_i),$$

$$LU \equiv \sum_{i=1}^{N} a^{ii}(x)y''(x_i) + \sum_{i=1}^{N} a^i(x)y'(x_i) + a(x)\sum_{i=1}^{N} y(x_i) \leq$$

$$\leq \mu \sum_{i=1}^{N} y''(x_i) + m\sum_{i=1}^{N} |y'(x_i)| = \sum_{i=1}^{N}\left(a_0 y^q(x_i) - \frac{k}{N}\right) \leq a_0 U^q - k,$$

if we recall (6.1.29) and use the Jensen inequality (1.2.5). Thus we proved (6.1.28) as well as our Lemma.   □

LEMMA 6.10. *There exists $R_0 > 0$ such that in the ball $B_{R_0}(0)$ there is no solution of the inequality (DI), satisfying the condition $|u(0)| > 1$.*

PROOF. For R_0 we take any number R such that $B_R(0) \supset\supset D$, where D is the domain constructed in Lemma 6.9. Let u be a positive solution of (DI) in $B_{R_0}(0)$ with $u(0) > 1$. We define in D the function $w = u - U$, where U is the function constructed in Lemma 6.9. By Lemma 6.9, the function w has the following properties: $\lim_{x \to \partial D} w(x) = -\infty$, $w(0) > 0$. We set $D_+ = \{x \in D \mid w(x) > 0\}$. Since $\mathcal{O} \in D_+$ we have $D_+ \neq \emptyset$. Now we apply the comparison principle (Theorem 6.8) to w in D

$$\begin{cases} Lu \geq a_0 u^q - k \equiv g(u) & \text{in } D, \\ LU \leq a_0 U^q - k = g(U) & \text{in } D, \\ u < U & \text{on } \partial D. \end{cases}$$

From the comparison principle it follows that $w < 0$ in D, and hence $w < 0$ in D_+ . We get a contradiction with the definition of D_+.   □

LEMMA 6.11. *If $u(x)$ is a strong solution of the inequality (DI) in $B_R(x_0)$ such that $|u(x_0)| > h$, then*

$$(6.1.36) \qquad\qquad R \leq R_0 h^{\frac{1-q}{2}},$$

where R_0 depends only on ν, μ, q, a_0, N.

PROOF. We make the change of variables $x - x_0 = h^{\frac{1-q}{2}} x'$ and $u = hv$. The function v satisfies (DI), and $|v(0)| > 1$. Hence, by Lemma 6.10, $v(x')$ is defined in a ball of radius not exceeding R_0, that is in the ball $|x'| < R h^{\frac{q-1}{2}} \leq R_0 \quad \Rightarrow \quad R \leq R_0 h^{\frac{1-q}{2}}$.   □

COROLLARY 6.12. *Let G be a bounded domain containing the origin $\mathcal{O}$. Let $u(x)$ be a strong solution of inequality (DI) in $G \setminus \mathcal{O}$. Then*

$$(6.1.37) \qquad\qquad |u(x)| \leq c|x|^{\frac{2}{1-q}},$$

where $c > 0$ is a constant depending on ν, μ, q, a_0, N.

Now we are estimating the modulus of a strong solution of (SL). At first we derive an auxiliary estimate.

LEMMA 6.13. *Let $u(x)$ be a strong solution of (SL) and the conditions a)-c) are satisfied. In addition, suppose $a_0(x) \geq a_0 = const > 0$. Then there are $d > 0, c_0 > 0$ such that the inequality*

$$(6.1.38) \qquad\qquad |u(x)| \leq c_0 |x|^{\frac{2}{1-q}}, \quad x \in G_0^d$$

holds.

PROOF. Let us perform the substitution of variables $x = \varrho x'$, $u(\varrho x') = hv(x')$, $h > 0$ in the problem (SL). The function $v(x')$ is a solution in the domain $G_{1/2}^1$ of the problem

$$(6.1.39) \quad \begin{cases} L'v \equiv a^{ij}(\varrho x')v_{x_i'x_j'} + \varrho a^i(\varrho x')v_{x_i'} + \varrho^2 a(\varrho x')v = \\ \quad = \varrho^2 h^{q-1} a_0(\varrho x')v|v|^{q-1} + \varrho^2 h^{-1} f(\varrho x'), \quad x' \in G_{1/2}^1, \\ v(x') = 0, \quad x' \in \Gamma_{1/2}^1. \end{cases}$$

Now we choose $h > 0$ so

$$(6.1.40) \qquad\qquad \varrho^2 h^{q-1} = 1$$

Because of $a_0(x) \geq a_0 > 0$ and the assumption c), from (6.1.39) and (6.1.40) it follows that

$$(6.1.41) \qquad \text{sign}v \cdot L'v - a_0|v|^q \geq \varrho^{\frac{2q}{q-1}} f(\varrho x')\text{sign}v \geq -k_1\varrho^{\beta+\frac{2q}{q-1}}$$

But $\beta > -1$, $q > 1$, therefore $\beta + \frac{2q}{q-1} > \frac{q+1}{q-1} > 0$. Hence for $0 < \varrho \leq d < 1$, we have $\varrho^{\beta+\frac{2q}{q-1}} \leq \varrho^{\frac{q+1}{q-1}} \leq d^{\frac{q+1}{q-1}}$. Now from (6.1.41) we obtain

$$(6.1.42) \qquad \text{sign}v \cdot L'v - a_0|v|^q \geq -k_1 d^{\frac{q+1}{q-1}}.$$

By setting $k = k_1 d^{\frac{q+1}{q-1}}$, we see from (6.1.42) that for a small positive d, namely

$$(6.1.43) \qquad\qquad 0 < d < \left(\frac{a_0}{k_1 N^{q-1}}\right)^{\frac{q-1}{q+1}},$$

the following inequalities

$$(6.1.44) \qquad \text{sign}v \cdot L'v - a_0|v|^q \geq -k, \quad a_0 > kN^{q-1}$$

hold. This allows us to apply Corollary 6.12 and we obtain

$$|v(x')| \leq M_0', \quad x' \in G_{1/2}^1,$$

where $M_0' > 0$ is a constant depending only on $\nu, \mu, q, a_0, N, \sup\limits_{x \in G} \mathcal{A}(|x|)$. Returning to the former variables we get

$$(6.1.45) \qquad |u(x)| \leq M_0' \varrho^{\frac{2}{1-q}}, \quad x \in G_{\varrho/2}^\varrho, \quad \varrho \in (0, d).$$

Taking $|x| = \frac{2\varrho}{3}$ finally we arrive to the desired inequality (6.1.38). $\qquad\square$

LEMMA 6.14. *Let L be a linear elliptic operator with the conditions a)-aaa). Then for $\forall \gamma \in (-\lambda - N + 1, \lambda - 1)$ there exist a number $d > 0$ and the function $w \in C^0(\overline{G_0^d}) \cap C^2(G_0^d)$ with the following properties*

$$(6.1.46) \qquad Lw \leq -\frac{(\lambda + N + \gamma - 1)(\lambda - \gamma - 1)}{2\lambda(\lambda + N - 2)} r^{\gamma - 1}, \quad x \in G_0^d,$$

$$(6.1.47) \qquad 0 \leq w(x) \leq c|x|^{1+\gamma}, \quad x \in G_0^d,$$

$$(6.1.48) \qquad \begin{cases} w(x) > 0, & x \in \Gamma_0^d, \\ w(x) \geq \frac{\varrho^{\gamma+1}}{\lambda(\lambda + N - 2)}, & x \in \Omega_\varrho, \ 0 < \varrho \leq d, \end{cases}$$

where c depends only on $\lambda, \gamma, N, \Omega$.

PROOF. Let us consider in the domain $\Omega \subset S^{N-1}$ the auxiliary Dirichlet problem for the Beltrami-Laplace operator

$$\begin{cases} \triangle_\omega \psi + (1 + \gamma)(N - 1 + \gamma)\psi = -1, & \omega \in \Omega, \\ \psi(\omega) = 0, & \omega \in \partial\Omega. \end{cases}$$

It is well known (see Subsection 3 §2, chapter 7 [203]) that this problem has the unique solution having the properties

$$\psi \in C^2(\Omega) \cap C^0(\overline{\Omega}), \ \psi > 0 \text{ in } \Omega, \ \|\psi\|_{C^2(\Omega)} \leq c(\gamma, N, \Omega)$$

provided the inequality

$$(6.1.49) \qquad (1 + \gamma)(N - 1 + \gamma) < \lambda(\lambda + N - 2)$$

is satisfied. The solutions of the latter inequality are the numbers

$$\gamma \in (-\lambda - N + 1, \lambda - 1).$$

Now we define the function

$$(6.1.50) \qquad w(x) = |x|^{1+\gamma} \left(\psi(\omega) + \frac{1}{\lambda(\lambda + N - 2)}\right).$$

By direct calculation we get

$$(6.1.51) \qquad \triangle w = -\frac{(\lambda + N + \gamma - 1)(\lambda - \gamma - 1)}{\lambda(\lambda + N - 2)}|x|^{\gamma - 1}.$$

Now, by the assumptions a)-aaa), we have

$$Lw = \triangle w + \left(a^{ij}(x) - a^{ij}(0)\right)D_{ij}w(x) + a^i(x)D_i w(x) + a(x)w(x) \leq$$

$$\leq \triangle w + c\mathcal{A}(r)(|D^2 w| + r^{-1}|Dw| + r^{-2}|w|) \leq$$

$$\leq r^{\gamma - 1}\left(c\mathcal{A}(r) - \frac{(\lambda + N + \gamma - 1)(\lambda - \gamma - 1)}{\lambda(\lambda + N - 2)}\right),$$

where $c > 0$ depend only on $\lambda, \gamma, N, \Omega$. By the continuity of $\mathcal{A}(r)$ at zero, we find $d > 0$ such that

$$\mathcal{A}(r) \leq \frac{(\lambda + N + \gamma - 1)(\lambda - \gamma - 1)}{2c\lambda(\lambda + N - 2)}, \quad r \in (0, d).$$

By this (6.1.46) is proved. The other properties of w are trivial. $\quad\square$

DEFINITION 6.15. The above constructed function w we shall call the *barrier* function.

THEOREM 6.16. *Let $u(x)$ be a strong solution of (SL) and the conditions a)-c) are satisfied. In addition, suppose $0 < a_0 \leq a(x) \leq a_1$, where a_0, a_1 are positive constants.*

Then for $\forall \varepsilon > 0$ there are positive constants c_ε, d independent of u such that the following estimate holds

(6.1.52) $$|u(x)| \leq c_\varepsilon |x|^{\lambda - \varepsilon}, \quad x \in G_0^d,$$

$$\text{if} \quad \beta + 2 \geq \lambda > 1, \quad q > 1 + \frac{2}{\lambda + N - 2}.$$

PROOF. Since $|a_0(x)| \leq a_1$ then from (SL), in virtue of (6.1.38) and the assumption c), it follows

(6.1.53) $$Lu \geq -a_1 |u|^q - k_1 r^\beta \geq -c_0^q a_1 r^{\frac{2q}{1-q}} - k_1 r^\beta.$$

Set

(6.1.54) $$\gamma - 1 = \frac{2q}{1 - q} \in (-\lambda - N, \lambda - 2).$$

It is easily seen that such a number γ satisfies Lemma 6.14 about the barrier function. Let

(6.1.55) $$B \geq \frac{2\lambda(\lambda + N - 2)(k_1 + a_1 c_0^q)}{(\lambda + N - 1 + \gamma)(\lambda - 1 - \gamma)}.$$

Now from (6.1.53), (6.1.46), (6.1.54) taking into account $\beta \geq \lambda - 2 > \frac{2q}{1-q}$ it follows that

(6.1.56) $$L(Bw) \leq Lu, \quad x \in G_\varepsilon^d, \quad \forall \varepsilon > 0.$$

Moreover, from the properties of the barrier function it follows that

(6.1.57) $$u(x) = 0 < w(x), \quad x \in \Gamma_\varepsilon^d, \quad \forall \varepsilon > 0,$$

(6.1.58) $$Bw(x) \geq \frac{B}{\lambda(\lambda + N - 2)} |x|^{\frac{2}{1-q}} \geq c_0 |x|^{\frac{2}{1-q}} \geq u(x),$$

$$x \in \Omega_\varrho, \ 0 < \varrho \leq d, \text{ if } B \geq c_0 \lambda(\lambda + N - 2).$$

Thus, if the number $B > 0$ satisfies (6.1.55), (6.1.58), then it is proved that

$$\begin{cases} L(Bw) \le Lu & \text{in } G_\varepsilon^d, \\ u(x) \le Bw(x) & \text{on } \partial G_\varepsilon^d. \end{cases}$$

By the comparison principle (Theorem 4.4), hence we obtain

$$u(x) \le Bw(x), \quad x \in \overline{G_\varepsilon^d}, \ \forall \varepsilon \in (0, d).$$

Similarly $u(x)$ is estimated from below. Thus we get

$$|u(x)| \le c|x|^{1+\gamma}, \quad x \in \overline{G_0^d} \setminus \mathcal{O},$$

where γ satisfies (6.1.54). In particular, we can choose $1 + \gamma = \lambda - \varepsilon$, $\forall \varepsilon > 0$, which gives us the estimate sought for. Our theorem is proved. $\square$

6.2. The behavior of weak solutions for divergence equations near a conical point

Here we study the properties of weak solutions of the Dirichlet problem for the divergence semilinear uniformly elliptic second order equation in a neighborhood of conical boundary point

$$(DSL) \quad \begin{cases} \mathcal{L}u := \frac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j} + a^i(x)u) + b^i(x)u_{x_i} + c(x)u = \\ \quad = a_0(x)u|u|^{q-1}, \ q > 0, \quad x \in G_\varepsilon; \\ u(x) = 0 \quad \text{on} \quad \Gamma_\varepsilon, \ \forall \varepsilon > 0. \end{cases}$$

DEFINITION 6.17. The function $u(x) \in W^1(G_\varepsilon) \cap L^\infty(G_\varepsilon)$ is called a *weak* solution of the problem (DSL) provided that it satisfies the integral identity

$$\int\limits_G \Big\{ a^{ij}(x)u_{x_j}\eta_{x_i} + a^i(x)u\eta_{x_i} - b^i(x)u_{x_i}\eta - c(x)u\eta + $$

$$\text{(III)} \qquad\qquad + a_0(x)u|u|^{q-1}\eta \Big\} dx = 0$$

for all $\eta(x) \in W^1(G)$, which has a support compact in G.

In the following we will always make the following **assumptions**

   i) $G \subset K$ *is a bounded domain;*

   a) *the uniform ellipticity condition*

$$\nu|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \ x \in \overline{G}$$

   with some $\nu, \mu > 0$; $a^{ij}(0) = \delta_i^j$;

  aa) $a^{ij}(x) \in C^0(\overline{G})$, $(i, j = 1, \ldots, N)$; $a^i(x), b^i(x) \in L_p(G)$, $(i = 1, \ldots, N)$; $c(x) \in L_{p/2}(G)$, $p > N$;

aaa) *there exists a monotonically increasing nonnegative continuous at
zero function $\mathcal{A}(r)$, $\mathcal{A}(0) = 0$ such that for all $x \in \overline{G}$*

$$\left(\sum_{i,j=1}^{N} |a^{ij}(x) - a^{ij}(0)|^2 \right)^{1/2} + |x| \left(\sum_{i=1}^{N} a^{i^2}(x) \right)^{1/2} + |x|^2 |a(x)| \leq \mathcal{A}(|x|);$$

b) $0 \leq a_0(x) \leq a_0 = $ *const is a nonnegative measurable in G function;*
c) *for all $\eta(x) \in W^1(G)$ which has a support compact in G*

$$\int\limits_{G} (c(x)\eta - a^i(x)D_i\eta)\,dx \leq 0.$$

Now we derive a bound of the weak solution of (DSL) modulus. Let λ be
the smallest positive eigenvalue of $(EVPI)$ with (2.5.11).

THEOREM 6.18. *Let u be a weak solution of (DSL) and the conditions
i) and a)-c) are satisfied. Suppose that*

$$\int\limits_{G} r^{\alpha} |Du|^2 dx < \infty \text{ at some } \alpha \in [2, 2\lambda + N).$$

*Then $\forall \varepsilon > 0$, there is a positive constant c_ε, determined only by ν, μ, q, N,
$\max\limits_{x \in \overline{G}} \mathcal{A}(|x|), G$ such that*

(6.2.1) $$|u(x)| \leq c_\varepsilon |x|^{\lambda - \varepsilon}.$$

PROOF. Let $v \in W^1(G_0^d)$ be a weak solution of the linear problem

$$\begin{cases} \mathcal{L}v = 0 & \text{in } G_0^d, \\ v\big|_{\Omega_d} = u_+, \quad v\big|_{\Gamma_0^d} = 0, \end{cases}$$

where u_+ is the positive part of u. The constant $d > 0$ we choose so that
$G_0^d \subset G$. Such v exists and is unique. By Theorem 5.8, we obtain

(6.2.2) $$|v(x)| \leq c_\varepsilon |x|^{\lambda - \varepsilon}.$$

Let us show that

(6.2.3) $$u(x) \leq v(x).$$

Suppose the contrary is true, that is we have $u(x) > v(x)$ in a domain
$D \subset G_0^d$. Then

(6.2.4) $$\begin{cases} \mathcal{L}(u - v) \geq 0 & \text{in } D, \\ \int\limits_{G} r^{\alpha} |D(u - v)|^2 dx < \infty \; \forall \alpha \in [2, 2\lambda + N) \end{cases}$$

is satisfied. In fact, in D we obtain

$$\mathcal{L}(u - v) = a_0(x)u|u|^{q-1} > a_0(x)v|v|^{q-1} \geq 0,$$

since, by weak maximum principle, $v \geq 0$ in G_0^d. Moreover, by Theorem 5.5, it is easily seen that

$$\int\limits_{G_0^d} r^\alpha |Dv|^2 dx < \infty \ \forall \alpha \in [2, 2\lambda + N)$$

and therefore (6.2.4) is verified. From (6.2.4) and Theorem 5.11 it follows that $u = v$. Thus, u satisfies (6.2.1) too.   □

Theorem 6.18 is a simple extension of well known results of the linear equation theory to (DL). It should be noted that we cannot take $u > 0$ in (6.2.1) without additional restrictions. The following theorem is only valid for solutions of nonlinear equations. Note that the behavior of $u(x)$ in the neighborhood of the vertex of the cone is not restricted *a priori* in the theorem, which is mandatory in the theory of linear problems. It is usually required in linear problems that either the Dirichlet integral be limited or the solution be continuous.

THEOREM 6.19. *If $a_0(x) \geq a_0 = const > 0$, $x \in G, q > 1$,*

$$(6.2.5) \qquad \qquad \frac{2}{1-q} > 2 - N - \lambda,$$

then inequality (6.2.1) is satisfied.

PROOF. We state the assertion established in [172]. Let $q > 1$, $a_0(x) \geq a_0 > 0$, and $u(x)$ be a solution of (DSL) in some domain $G \ni \mathcal{O}$, which is inside the unit sphere $|x| < 1$ and vanishes in that part of ∂G which is strictly inside the sphere. Then

$$(6.2.6) \qquad \qquad |u(0)| \leq C_1, \int\limits_{|x|<1/2,\ x \in G} |\nabla u|^2 dx \leq C_2,$$

where the constants C_1 and C_2 are only dependent on the elliptic constants of (DSL) [assumption a)] and on a_0 and q. If we change the variables so that $x = \rho x'$, $u = hv$, and $\rho^{-2} = h^{q-1}$, which retains the structure of (DSL), then we obtain the following assertion from (6.2.6).

Let $u(x)$ be a solution of (DSL) in domain $G_{\varrho/2}^{2\varrho}$ and vanishes in $\Gamma_{\varrho/2}^{2\varrho}$. Then

$$(6.2.7) \qquad \qquad \int\limits_{G_{3\varrho/4}^{3\varrho/2}} |\nabla u|^2 dx \leq C_2 \varrho^{2 \cdot \frac{1 \pm q}{1-q} + N}.$$

According to (6.2.7)

$$(6.2.8) \qquad \int_G r^\alpha |\nabla u|^2 dx < \infty,$$

whenever

$$(6.2.9) \qquad \alpha + 2 \cdot \frac{1+q}{1-q} + N > 0$$

Since, in view of the condition of Theorem 6.19,

$$2 \cdot \frac{1+q}{1-q} + N = 2\left(\frac{2}{1-q} - 1 + N\right) > 2 - N - 2\lambda,$$

we can choose an $\alpha < 2\lambda + N - 2$ which satisfies (6.2.9). In this case (6.2.8) is satisfied and we can use Theorem 6.18. This is the proof of the inequality (6.2.1). □

THEOREM 6.20. *If* $0 < q < 1,$ $a_0(x) \geq a_0 = const > 0,$
$\frac{2}{1-q} < \lambda,$ $u(x) \in W_2^1(G),$ *then* $u(x) \equiv 0$ *in some neighborhood of the vertex of the cone* K.

PROOF. The following statement was proved in [173]. Let $G \ni \mathcal{O}$ be in the unit sphere, let $u(x)$ be the solution of (DSL), and let $u(x) = 0$ in that part of ∂G which is strictly inside the unit sphere. There exists a $B = const > 0$ which depends only on q, ν, μ and on a_0. If $|u(x)| \leq B$ at $|x| = 1, x \in \overline{G}$, then $u(0) = 0$. The constant B does not depend on either u or the structure of domain G. Thus, using the transform $x = \varrho x',$ $u = hv,$ $\varrho^{-2} = h^{q-1}$, we readily obtain the following statement.

Let $u(x)$ be the solution of (DSL) in the part of the domain $G \ni \mathcal{O}$ lying inside the sphere $|x| < 2\varrho$, and vanishes in that part of ∂G, inside the sphere. If

$$(6.2.10) \qquad |u(x)| \leq B\varrho^{\frac{2}{1-q}}$$

for $|x| = 2\varrho,$ $x \in \overline{G}$, then $u(x) \equiv 0$ for $|x| < \varrho$.

If the conditions of Theorem 6.20 are satisfied, we will obtain (6.2.1), by applying Theorem 6.18. Inequality (6.2.1) yields (6.2.10) at small ϱ. Hence, $u(x) \equiv 0$ if $|x| < \varrho$. □

Note that if the condition (6.2.5) of Theorem 6.19 is not satisfied, then (DSL) has unbounded solutions in the neighborhood of $x = 0$. We will now prove this fact under the assumption $a^i(x) = b^i(x) = c(x) \equiv 0$. We state some assertions about the characteristics of the solutions of linear elliptic equations in conic domains for which we shall use [161].

1. Let K be a cone in $\mathbb{R}^n$. We consider the boundary value problem

(6.2.11)
$$\begin{cases} \sum\limits_{i,j=1}^{N} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = \sum\limits_{i=1}^{N} \frac{\partial f^i}{\partial x_i} + f_0(x), & x \in K, \\ u(x) = 0, & x \in \partial K. \end{cases}$$

Let β be such that $\frac{\beta^2 - (N-2)^2}{4}$ is not an eigenvalue of (EVD). There exists a $\delta > 0$, which depends on K and β such that if

$$|a^{ij}(x) - \delta_i^j| \le \delta, \; x \in K,$$

(6.2.12)

$$\int\limits_K |x|^{\beta+2} f_0^2 dx + \int\limits_K \sum_{i=1}^{N} |x|^\beta |f^i|^2 dx < \infty,$$

then there exists a unique solution of (6.2.11) such that

(6.2.13)
$$\int\limits_K |x|^\beta |\nabla u|^2 dx + \int\limits_K |x|^{\beta-2} u^2 dx \le C \int\limits_K |x|^{\beta+2} f_0^2 dx +$$

$$+ C \int\limits_K \sum_{i=1}^{N} |x|^\beta |f^i|^2 dx.$$

This statement is proved for $a^{ij}(x) \equiv \delta_i^j$ in **[161]**. It follows from the Banach theorem on the invertibility of the sum of an invertible operator and the operator which is small by the norm.

2. (See **[161]**). Suppose that K is a cone in $\mathbb{R}^N$, $\lim\limits_{x \to 0} a^{ij}(x) = \delta_i^j$, the numbers β_1 and β_2 are such that the interval

$$\left[\frac{\beta_1^2 - (N-2)^2}{4}, \frac{\beta_2^2 - (N-2)^2}{4} \right]$$

has no points from spectrum of (EVD), and $u(x)$ is a solution of (6.2.11),

$$\sum_{i=1}^{N} \int\limits_K |x|^{\beta_1} |f^i|^2 dx + \sum_{i=1}^{N} \int\limits_K |x|^{\beta_2} |f^i|^2 dx +$$

$$+ \int\limits_K |x|^{\beta_1+2} f_0^2 dx + \int\limits_K |x|^{\beta_2+2} f_0^2 dx < \infty,$$

$$\int\limits_K |x|^{\beta_2} |\nabla u|^2 dx + \int\limits_K |x|^{\beta_2-2} u^2 dx < \infty.$$

Then

$$(6.2.14) \qquad \int_{K_0^1} |x|^{\beta_1} |\nabla u|^2 dx + \int_{K_0^1} |x|^{\beta_1-2} u^2 dx < \infty.$$

Let us consider (6.2.11) where $f^i \equiv 0$, $i = 0, 1, ..., N$. We will show that if δ in (6.2.12) is small, $a^{ij}(x) \equiv \delta_i^j$ at $|x| > R_1$, then (6.2.11) has a non-trivial solution. Let $\Gamma_0(x) = |x|^{2-N-\lambda}\Phi(\omega)$, where $\Phi(\omega) > 0$ in K is the eigenfunction of (EVD) corresponding to ϑ. We seek $\Gamma(x)$, the solution of (6.2.11), in the form of $\Gamma(x) = \Gamma_0(x) - V(x)$, where $V(x)$ is a solution of

$$(6.2.15) \qquad \begin{cases} \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a^{ij}(x)\frac{\partial V}{\partial x_j}\right) = \sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a^{ij}(x) - \delta_i^j)\frac{\partial \Gamma_0}{\partial x_j} = \\ \qquad\qquad\qquad\qquad\qquad = \sum_{i=1}^N \frac{\partial}{\partial x_i} F^i(x), \quad x \in K, \\ V(x) = 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \partial K. \end{cases}$$

Note that

$$(6.2.16) \qquad \sum_{i=1}^N \int_K |F^i|^2 |x|^\beta dx < \infty,$$

if $\beta > N - 2 + 2\lambda$. We fix β so that $N - 2 + 2\lambda < \beta < -N + 2 - 2\lambda_-^1(K)$, where $\lambda_-^1(K) = \frac{1}{2}\left(2 - N - \sqrt{(N-2)^2 + 4\vartheta_2}\right)$, ϑ_2 is the smallest eigenvalue of (EVD) which is larger than ϑ. It follows from (6.2.13) that according to the condition of (6.2.12) there exists a $V(x)$, a solution of (6.2.15) such that

$$(6.2.17) \qquad \int_K |x|^{\beta-2} V^2 dx + \int_K |x|^\beta |\nabla V|^2 dx \le \sum_{i=1}^N \int_K |x|^\beta |F^i|^2 dx.$$

We will now discuss some characteristics of $V(x)$ and $\Gamma(x)$. It follows from the classical estimates of the solutions of the elliptic differential equations that

$$(6.2.18) \qquad V^2(x) \le C\lambda^{-N} \int_{K_{\lambda/2}^{2\lambda}} V^2 dx, \quad \text{if } |x| = \lambda.$$

From this and (6.2.17) we have

$$V^2(x) \le C_1 |x|^{-N+2-\beta} = o(|x|^{4-N-2\lambda}) \text{ as } x \to \infty.$$

Besides, $|\Gamma_0(x)| \le C|x|^{2-N-\lambda} \to 0$ as $|x| \to \infty$. Hence, $\Gamma \to 0$ as $x \to \infty$. Since $\Gamma_0(x) = \Phi(\omega)|x|^{2-N-\lambda}$ and $V(x) = o(|x|^{2-N-\lambda})$, then $\Gamma \not\equiv 0$. Note

that

$$(6.2.19) \qquad \int_{K_0^1} |x|^\alpha |\nabla \Gamma|^2 dx + \int_{K_0^1} |x|^{\alpha-2} \Gamma^2 dx = \infty$$

at any α such that $\alpha < N - 2 + 2\lambda$. Otherwise we would have $|\Gamma(x)| \le C_\varepsilon |x|^{\lambda-\varepsilon}$, that is, $\Gamma(x) \to 0$ as $x \to 0$. This is impossible, in view of the maximum principle. Finally, from (6.2.14) we have

$$(6.2.20) \qquad \int_{K_0^1} \Gamma^2 |x|^{s-2} dx + \int_{K_0^1} |x|^s |\nabla \Gamma|^2 dx < \infty$$

regardless of $s > N - 2 + 2\lambda$. According to (6.2.20) and (6.2.18) we also have that $|\Gamma(x)| \le C_\varepsilon |x|^{2-N-\lambda-\varepsilon}$ for any $\varepsilon > 0$.

Using $\Gamma(x)$, we construct an unbounded solution of (DSL) provided $a^i(x) = b^i(x) = c(x) \equiv 0$. Suppose that d_1 is so small that for $x \in K_0^{d_1}$ (6.2.12) is satisfied for some $\beta > N - 2 + 2\lambda$. We change $a^{ij}(x)$ at $|x| > d_1$, making them equal to δ_i^j. We constructed $\Gamma(x)$, a solution of (6.2.11) at $f^i \equiv 0$, that is unbounded in the neighborhood of $x = 0$ and satisfies (6.2.20). Suppose that $\Gamma(x) \to +\infty$ along a sequence $x_m \to 0$.

Let $\Gamma_k(x)$ be a solution of (DSL) in the domain $G_k : x \in K, 2^{-k} < |x| < d_k$, $k = 1, 2, \ldots$ such that

$$(6.2.21) \qquad u\Big|_{\partial K \cap \overline{G}_k} = 0, \quad u\Big|_{|x|=d_1} = \Gamma, \quad u\Big|_{|x|=2^{-k}} = \Gamma.$$

Then (DSL) has a solution satisfying (6.2.21) and it is unique. It can be constructed by the variational method. Let us consider $z(x) = -\Gamma(x) + \Gamma_k(x)$. This function is a solution of

$$(6.2.22) \qquad \begin{cases} L(z) = a_0(x)|\Gamma_k|^{q-1}\Gamma_k = a_0(x)\frac{|\Gamma_k|^{q-1}\Gamma_k - |\Gamma_k|^{q-1}\Gamma}{\Gamma_k - \Gamma}(\Gamma_k - \Gamma) + \\ \qquad\qquad +a_0(x)|\Gamma_k|^{q-1}\Gamma, \qquad x \in K_{2^{-k}}^{d_1}, \\ z\Big|_{\partial K_{2^{-k}}^{d_1}} = 0. \end{cases}$$

It follows from Theorem 5.11 that for any $\alpha < 2\lambda + N - 2$,

$$\int_{K_{2^{-k}}^{d_1}} |x|^\alpha |\nabla z|^2 dx + \int_{K_{2^{-k}}^{d_1}} |x|^{\alpha-2} z^2 dx \le$$

$$(6.2.23)$$

$$\le C \int_{K_0^{d_1}} |x|^{\alpha+2} |\Gamma(x)|^{2q} dx \le CC_\varepsilon \int_{K_0^{d_1}} |x|^{\alpha+2q(2-N-\lambda)+2-\varepsilon} dx \le A,$$

if

(6.2.24) $\alpha + 2 + 2q(2 - N - \lambda) - \varepsilon + N > 0.$

We can choose α such that (6.2.24) is valid and $\alpha < N - 2 + 2\lambda$ since in this case $2 > (1 - q)(2 - N - \lambda)$. The constant A on the right-hand side of (6.2.23) is not dependent on k. Let us consider the boundary value problem in K

(6.2.25)
$$\begin{cases} L(Z) = |a_0^*(x)||\Gamma(x)|^q = f_0(x), & x \in K, \\ Z = 0, & x \in \partial K, \end{cases}$$

where

$$a_0^*(x) = \begin{cases} a_0(x) & \text{at } |x| < d_1, \\ 0 & \text{for } |x| > d_1. \end{cases}$$

If α satisfied (6.2.24) and $\alpha < N - 2 + 2\lambda$, then (6.2.22) has a solution such that (6.2.25) holds and

(6.2.26) $\displaystyle\int_K |x|^\alpha |\nabla Z|^2 dx + \int_K |x|^{\alpha-2} Z^2 dx < \infty.$

It follows from (6.2.26) and (6.2.18) that $Z(x) \to 0$ as $|x| \to \infty$. In view of Theorem 5.11, $Z(x) < 0$ in K. From this and (6.2.22) we have

(6.2.27) $|Z(x)| \le |Z(x)|.$

This implies that $q_k(x) = a_0(x)\dfrac{|\Gamma_k|^{q-1}\Gamma_k - |\Gamma|^{q-1}\Gamma}{\Gamma_k - \Gamma}$ is uniformly bounded with respect to k in each domain $K_{d_0}^{d_1}$, $d_0 > 0$. Hence, the functions $Z(x)$ form a sequence which contains a subsequence compact in the sense of the topology of inform convergence in each subdomain $K_{d_0}^{d_1}$. Let $Z_0(x)$ be its limit. It follows from (6.2.24) that $Z_0(x)$ satisfies (6.2.26). Thus, $u(x) = \Gamma(x) - z_0(x)$ is the solution of (DSL) with $a^i(x) = b^i(x) = c(x) \equiv 0$ in K_{d_1}. According to (6.2.19) and (6.2.23)

$$\int_K |x|^{q-2} u^2 dx = \infty \text{ for any } \alpha < N - 2 + 2\lambda,$$

satisfying (6.2.24). It implies that $u(x)$ is the solution of (DSL) with $a^i(x) = b^i(x) = c(x) \equiv 0$ which is unbounded in any neighborhood of the origin.

6.3. Notes

The properties of the (SL) solutions in a neighborhood of an isolated singular point were studied in [172, 173]. Positive solutions of singular value problems for the semilinear equations in smooth domains were investigated also in [174, 175, 176]. The solutions smoothness of some superlinear elliptic equations was investigated by S. Pohozhaev [337, 339, 340].

The results of Section 6.1 was established in [62] and of Section 6.2 - in [165].

We point out other problems which are not investigated here. M. Marcus and L. Veron have studied [245, 246, 247] the uniqueness and expansion properties of the positive solutions of the equation $\Delta u + hu - ku^p = 0$ in nonsmooth domain G, subject to the condition $u(x) \to \infty$, when $dist(x, \partial G) \to 0$, where h, k are continuous functions in $\overline{G}$, $k > 0$ and $p > 1$. They have proved that the solution is unique, when ∂G has the local graph property. They have obtained the asymptotic behavior of solutions, when ∂G has a singularity of conical or wedge-like type; if ∂G has a re-entrant cuspidal singularity then the rate of blow-up may not be of the same order as in the previous and more regular cases.

Many other problems for elliptic semilinear equations was studied by L. Veron together with colleagues in works [45, 46, 119, 120, 130, 181, 342, 392, 394, 397, 343] as well as by other authors [33, 36, 51, 107, 110, 111, 180, 330, 370].

Semilinear degenerate elliptic equations and axially symmetric problems were considered by J. Below and H. Kaul [39].

CHAPTER 7

Strong solutions of the Dirichlet problem for nondivergence quasilinear equations

7.1. The Dirichlet problem in smooth domains

Let $G \subset R^N$ be a bounded domain with a smooth boundary ∂G. We consider the Dirichlet problem

$$(QL) \quad \begin{cases} a_{ij}(x, u, u_x) u_{x_i, x_j} + a(x, u, u_x) = 0, & a_{ij} = a_{ji}, \quad x \in G; \\ u(x) = \varphi(x), \quad x \in \partial G. \end{cases}$$

(Summation from 1 to N is assumed over repeated indices.) The value $M_0 = \max_{x \in \overline{G}} |u(x)|$ is assumed to be known.

REMARK 7.1. For the finding of M_0 see for example §10.2 [**129**].

Let us define the set $\mathfrak{M} = \left\{ (x, u, z) \big| x \in \overline{G}, u \in \mathbb{R}, z \in \mathbb{R}^N \right\}$. With regard to the equation of the problem (QL) we assume that on the set $\mathfrak{M}$ the following *conditions* are satisfied

(A) *Caratheodory for the functions* $a(x, u, z), a_{ij}(x, u, z) \in CAR$, $(i, j = 1, \ldots, N)$; *that is*

    (i) *for* $\forall u, z$ *the functions* $a(x, u, z), a_{ij}(x, u, z)$ $(i, j = 1, \ldots, N)$ *are measurable on* G *as the functions of variable* x,

    (ii) *for almost all* $x \in G$ *functions* $a(x, u, z), a_{ij}(x, u, z)$ $(i, j = 1, \ldots, N)$ *are continuous with respect to* u, z;

(B) *the uniform ellipticity ; that is there exist positive constants* ν, μ *independent of* u, z *and such that*

$$\nu \xi^2 \leq a_{ij}(x, u, z) \xi_i \xi_j \leq \mu \xi^2, \quad \forall \xi \in \mathbb{R}^N;$$

(C) *there exist a number* μ_1 *and functions* $b(x), f(x) \in L_{q,loc}(\overline{G})$, $q \geq N$ *independent of* u, z *such that*

$$|a(x, u, z)| \leq \mu_1 |z|^2 + b(x)|z| + f(x).$$

Let us recall some well known facts about $W^{2,p}(G)$, $p \geq N$ solutions of this problem.

DEFINITION 7.2. *A bounded open set $T \subset \partial G$ is said to be of type (A) if there exist two positive constants ϱ_0 and θ_0 such that for every ball $B_r(x_0)$, $x_0 \in T$ with radius $r \leq \varrho_0$ and every connected component $G_{r,i}$ of the intersection $B_r(x_0) \cap G$ the inequality $\mathrm{meas}\, G_{r,i} \leq (1-\theta_0)\mathrm{meas}\, B_r$ holds.*

THEOREM 7.3. (See §2 [217].) *Let $u \in W_{loc}^{2,N}(\overline{G}) \cap C^0(\overline{G})$ be a strong solution of (QL) and suppose that assumptions (A)-(C) are satisfied. Let G be of type (A) and $\varphi \in C^{\beta}(\overline{G}), \beta \in (0,1)$. Then $u \in C^{\alpha}(\overline{G})$, $\alpha \in (0,1)$ and $|u(x) - \varphi(x)|_{\alpha,\overline{G}} \leq M_{\alpha}$, where α is determined by $N, \nu, \mu, \beta, \theta_0, G$ and M_{α} depends on the same values and also on $\mu_1, M_0, \|b\|_N, \|f\|_N, |\varphi|_{\beta,\partial G}$.*

THEOREM 7.4. (See Theorem 2.1 [219].) *Let $u \in W_{loc}^{2,N}(\overline{G}) \cap C^0(\overline{G})$ be a strong solution of (QL) and suppose that assumptions (A) - (C) are satisfied. Let $T \subset \partial G$ be a piece of class $W^{2,q}$, $q > N$. Then there is a constant $c > 0$ depending only on $N, \nu, \mu, \mu_1, q, \|b\|_q, M_0$ and the domain G such that, if $\varphi\big|_T = 0$, then $|\nabla u|_{0,T} \leq c(1 + \|f\|_q)$.*

Yet let us introduce a set:

$$\mathfrak{M}^{(u)} \equiv \left\{ (x,u,z) \big| x \in G,\ u = u(x),\ z = \nabla u(x) \right\}.$$

We assume, in addition, that in the neighborhood of the set $\mathfrak{M}^{(u)}$ the following condition is fulfilled

(D) *the functions $a_{ij}(x,u,z)$, $(i,j = 1,\ldots,N)$ have weak first order derivatives over all its own arguments and there exist the non-negative constants μ_0, μ_2, μ_3, k_2 and the functions $g(x), h(x) \in L_{q,loc}(\overline{G} \setminus \mathcal{O})$, $q > N$ independent of u, z such that*

$$\sum_{i,j,k=1}^{N} \left| \frac{\partial a_{ij}(x,u,z)}{\partial z_k} - \frac{\partial a_{ik}(x,u,z)}{\partial z_j} \right| \leq \mu_0 \left(1 + |z|^2\right)^{-1/2};$$

$$\sum_{i,j=1}^{N} \left| \sum_{k=1}^{N} \left(\frac{\partial a_{ij}(x,u,z)}{\partial u} z_k^2 - \frac{\partial a_{kj}(x,u,z)}{\partial u} z_k z_i + \right. \right.$$

$$\left. \left. + \frac{\partial a_{ij}(x,u,z)}{\partial x_k} z_k - \frac{\partial a_{kj}(x,u,z)}{\partial x_k} z_i \right) \right| \leq \left(1 + |z|^2\right)^{1/2} \left(\mu_2 |z| + g(x)\right);$$

$$\|g(x)\|_{q,G_{\varrho/2}^{\varrho}} \leq k_2 \varrho^{N/q - 1 + \gamma}, \quad \varrho \in (0, d^*);$$

$$\left\{ \sum_{i,j=1}^{N} \left(\left| \frac{\partial a_{ij}(x,u,z)}{\partial u} \right|^2 + \sum_{k=1}^{N} \left| \frac{\partial a_{ij}(x,u,z)}{\partial x_k} \right|^2 \right) \right\}^{1/2} \leq h(x);$$

$$\left\{ \sum_{i,j,k=1}^{N} \left| \frac{\partial a_{ij}(x,u,z)}{\partial z_k} \right|^2 \right\}^{1/2} \leq \mu_3,$$

where γ is a number from the estimate (7.3.1).

THEOREM 7.5. (See Theorems 4.1, 4.3 [**217**].) *Let G be a bounded domain in $\mathbb{R}^N$ with a $W^{2,q}-$ boundary portion $T \subset \partial G$. Let $u \in C^0(\overline{G}) \cap C^1(G') \cap W^{2,q}_{loc}(\overline{G} \setminus \mathcal{O})$, $q > N$ be a strong solution of (QL) and suppose that assumptions $(A) - (D)$ are satisfied. Let $\varphi \in C^{1+\alpha}(\partial G)$, $\alpha \in (0,1)$. Then there are the constants $M_1 > 0, \gamma \in (0,1)$ depending only on $N, \nu, \mu, \mu_0, \mu_1, \mu_2, \mu_3, q, \alpha, \|f\|_q, \|b\|_q, \|g\|_q, \|h\|_q, \|\varphi\|_{C^{1+\alpha}(\partial G)}, M_0$ and the domain G such that for $\forall G' \subset\subset (G \cup T)$ the inequality*

$$\|u\|_{C^{1+\gamma}(\overline{G'})} \leq M_1$$

holds.

7.2. The estimate of the Nirenberg type

7.2.1. Introduction. Until recently the problem of the solution smoothness to the boundary value problems for the second order quasilinear elliptic equations of nondivergence form remained open. An exception is Nirenberg's paper [**329**], in which this problem was investigated for equations with two independent variables in a bounded plane domain with a smooth boundary. In the last decade, thanks to the efforts of many mathematicians, first of all O.A. Ladyzhenskaya and N.N.Ural'tseva (see their survey [**217**], 1986), this problem has arrived at a definitive solution for equations in multidimensional domains bounded by a sufficiently smooth boundary. As concerns the equations in domains with a piecewise smooth boundary, only the investigation [**90**] of I. I. Danilyuk is known. (We emphasize that here we are talking of elliptic nonlinear and nondivergence equations.) There the solvability of the Dirichlet problem is proved for a two-dimensional equation in the Sobolev space $W^{2,p}$ for $p > 2$ and sufficiently close to 2.

In the present section we investigate the behavior of solutions of the Dirichlet problem for a uniformly elliptic quasilinear equation of second order of nondivergence form near a corner point of the boundary of a bounded

plane domain. It is here assumed that the coefficients of the equation satisfy minimal conditions of smoothness and coordinated growth (no higher than quadratic) modulo the gradient of the unknown function. We first extend to domains whose boundaries contain a corner point and to equations with an unbounded right side the method of Nirenberg [329] for estimating the Hölder constant of the first derivatives of solutions. The weighted L_2-estimate of the second derivatives of a solution obtained in this manner (we call it the Nirenberg estimate) and the Sobolev imbedding theorems make it possible to estimate the maximum of the modulus of a solution and its gradient and thus establish a power rate of decay (temporarily with a small positive exponent) of a solution in a neighborhood of a corner point. Using the "weak" smoothness of a solution established in §7.2.4, in §7.2.5 we refine the Nirenberg estimate and obtain a weighted integral estimate with best-possible exponent of the weight. While for the Nirenberg estimate boundedness of the leading coefficients of the equation is sufficient, it is now necessary to require their continuity. The estimate of §7.2.5 makes it possible to obtain sharp estimates of the modulus of a solution and of its gradient as well as weighted L^p-estimates of the second derivatives, and to prove Hölder continuity of the first derivatives of a solution with the best-possible Hölder exponent.

7.2.2. Formulation of the problem and the main result.

Let $G \subset R^2$ be a bounded domain with boundary ∂G which is assumed to be a Jordan curve smooth everywhere except at a point $\mathcal{O} \in \partial G$; in some neighborhood of the point $\mathcal{O}$ the boundary ∂G consists of two segments intersecting at an angle $\omega_0 \in (0, \pi)$. We place the origin of a rectangular coordinate system (x_1, x_2) at the point $\mathcal{O}$. Let (r, ω) be a polar coordinate system with pole at $\mathcal{O}$. We direct the abscissa of the rectangular coordinate system along the ray $\omega = 0$ on which one the segments of ∂G lies, and we situate the ordinate axis so that the second segment of ∂G lying on the ray $\omega = \omega_0$ lies in the upper half-plane $x_2 > 0$. For any numbers $d > a \geq 0$ we set

$$G_a^d = G \cap \{(r, \omega) |\ a < r < d,\ 0 < \omega < \omega_0\}$$

(we henceforth assume that d is a sufficiently small positive number);

$$\Gamma_{1,a}^d = \{(r, 0) |\ a < r < d\}; \quad \Gamma_{2,a}^d = \{(r, \omega_0) |\ a < r < d\};$$

$$\Gamma_a^d = \Gamma_{1,a}^d \cup \Gamma_{d,a}^d; \quad S_d = \{(d, \omega) |\ 0 \leq \omega \leq \omega_0\}.$$

DEFINITION 7.6. *A strong solution of problem* (QL) *is a function* $u \in W^2(G)$ *satisfying the equation of the problem for almost all* $x \in G$ *and the boundary condition* $u - \Phi \in W_0^2(G)$ *with any* $\Phi \in W^2(G)$ *such that* $\Phi(x) = \varphi(x)$, $x \in \partial G$.

The main result of this section is the proof of the following theorem.

THEOREM 7.7. *Suppose* $u \in W^2(G)$ *is a solution of problem* (QL) *with* $a_{ij}(0,0,0) = \delta_i^j = $ *the Kronecker symbol* $(i, j, = 1, 2)$, *conditions* $(A) - (C)$ *are satisfied, and the following quantities are known*

(7.2.1) $M_0 = \max_{x \in \overline{G}} |u(x)|$ *and* $M_1 = \operatorname*{ess\,sup}_{x \in G} |\nabla u(x)|$.

Suppose the functions $a_{ij}(x, u, u_x)$ $(i, j = 1, 2)$ *are Dini continuous at the point* $(0, 0, 0)$,

$$b^2, f \in V_{p,\alpha}^0(G), \quad \varphi \in C^{\pi/\omega_0}(\partial G) \cap \mathring{W}_0^{3/2}(\partial G) \cap V_{p,\alpha}^{2-1/p}(\partial G),$$
$$p > 2 \quad \alpha > p(2 - \pi/\omega_0) - 2, \quad 0 < \omega_0 < \pi,$$

and there exist numbers $k_1, k_2 > 0$ *and* $s > \pi/\omega_0$ *such that for all* $\rho \in (0, d)$ *the following inequalities hold*

(7.2.2) $\|b^2\|_{2, G_0^\rho} + \|f\|_{2, G_0^\rho} + \|\varphi\|_{\mathring{W}_0^{3/2}(\Gamma_0^\rho)} \leq k_1 \rho^{s-1}$

(7.2.3) $\|b^2\|_{V_{p,\alpha}^0(G_{\rho/2}^\rho)} + \|f\|_{V_{p,\alpha}^0(G_{\rho/2}^\rho)} + \|\varphi\|_{V_{p,\alpha}^{2-1/p}(\Gamma_{\rho/2}^\rho)} \leq k_2 \rho^{\pi/\omega_0 - 2 + \frac{\alpha+2}{p}}$.

Then the following assertions are true

   1) $u \in \mathring{W}_2^2(G)$, *and*

(7.2.4) $\|u\|_{\mathring{W}_2^2(G_0^\rho)} \leq c \rho^{\pi/\omega_0}, \quad 0 < \rho < d;$

   2) *for* $0 < \rho < d$

(7.2.5) $|u(x)| \leq c_1 |x|^{\pi/\omega_0}, \quad x \in G_0^\rho,$

(7.2.6) $|\nabla u(x)| \leq c_2 |x|^{\pi/\omega_0 - 1}, \quad x \in G_0^\rho;$

   3) $u \in V_{p,\alpha}^2(G)$, *and*

(7.2.7) $\|u\|_{V_{p,\alpha}^2(G_0^\rho)} \leq c_3 \rho^{\pi/\omega_0 - 2 + \frac{\alpha+2}{p}}, \quad 0 < \rho < d;$

   4) *if* $p \geq \frac{2}{2 - \pi/\omega_0}$ *with* $\pi/2 < \omega_0 < \pi$, *then* $u \in C^{\pi/\omega_0}(\overline{G})$.

7.2.3. The Nirenberg estimate. Let $\Phi(x)$ be any extension of the boundary function $\varphi(x)$ into the domain G. The change of function $v(x) = u(x) - \Phi(x)$ reduces the inhomogeneous problem (QL) to the homogenous problem

$(QL)_0$ $\begin{cases} a_{ij}(x, v + \Phi, \; v_x + \Phi(x)) \cdot v_{x_i x_j} = F(x, v, v_x), & x \in G, \\ v(x) = 0, & x \in \partial G, \end{cases}$

$$(7.2.8) \quad F(x, v, v_x) \equiv -a_{ij}(x, v + \Phi, \ v_x + \Phi(x) \cdot \Phi_{x_i x_j} -$$
$$- a(x, v + \Phi, \ v_x + \Phi(x)),$$

where by assumptions (b) and (c) the following condition is satisfied

$$(7.2.9) \quad |F(x, v, p)| \le 2\mu_1 |p|^2 + b(x) \cdot |p| + 2\mu |\Phi_{xx}|$$
$$+ 2\mu_1 \cdot |\nabla \Phi|^2 + b(x) \cdot |\nabla \Phi| + f(x).$$

By a *solution of problem* $(QL)_0$ we mean a function $v \in \mathring{W}_0^2(G)$ satisfying the equation of the problem almost everywhere in G. By Theorem 7.3, such solution is Hölder continuous in G, and there exists $\gamma_0 \in (0, 1)$, depending on ν^{-1}, μ, and ω_0 such that

$$(7.2.10) \quad |v(x)| \le c_0 \cdot |x|^{\gamma_0}$$

with a positive constant c_0 depending on $\nu^{-1}, \mu, \mu_1, \omega_0, M_0, \|b^2\|_{2,G}, \|f\|_{2,G}$, and $\|\varphi\|_{W^{3/2}}(\partial G)$.

THEOREM 7.8. (cf. [329]; see also Chapter IX, §6 [215]). *Suppose* $u(x) \in W^2(G)$ *is a solution of problem* (QL), *assumptions* $(A) - (C)$ *are satisfied, and the quantities* $(7.2.1)$ *are known. Then there exists a constant* γ, *determined by the quantities* $\gamma, \mu, \mu_1, \omega_0, c_0, \gamma_0, d$, *and satisfying the inequality*

$$(7.2.11) \quad 0 < \gamma < 2\min(\gamma_0; \pi/\omega_0 - 1) = \gamma^*$$

such that if $f, b^2 \in \mathring{W}_{-\gamma^*}^0(G)$ *and* $\varphi \in C^1(\partial G) \cap \mathring{W}_{-\gamma^*}^{3/2}(\partial G)$, *then* $u \in \mathring{W}_{-\gamma}^2(G)$, *and*

$$(7.2.12) \quad \|u\|_{\mathring{W}_{-\gamma}^2(G_0^{\rho/2})} \le c(d), \quad 0 < \rho \le d.$$

PROOF. First of all, we consider the expression

$$\mathfrak{V} \equiv a_{ij} v_{x_i, x_j} \left(\frac{v_{x_1 x_1}}{a_{22}} + \frac{v_{x_2 x_2}}{a_{11}} \right)$$

and write it in the form

$$\mathfrak{V} = \left(\frac{a_{11}}{a_{22}} v_{x_1 x_1}^2 + \frac{2a_{12}}{a_{22}} v_{x_1 x_1} v_{x_1 x_2} + v_{x_1 x_2}^2 \right) +$$
$$+ \left(\frac{a_{22}}{a_{11}} v_{x_2 x_2}^2 + \frac{2a_{12}}{a_{11}} v_{x_2 x_2} v_{x_1 x_2} + v_{x_1 x_2}^2 \right) +$$
$$+ 2 \left(v_{x_1 x_1} v_{x_2 x_2} - v_{x_1 x_2}^2 \right).$$

Because of $(QL)_0$, the uniform ellipticity condition (B), hence it follows

(7.2.13) $\dfrac{\nu}{\mu}\left(v_{x_1x_1}^2 + 2v_{x_1x_2}^2 + v_{x_2x_2}^2\right) \le$

$$\le 2\left(v_{x_1x_2}^2 - v_{x_1x_1}v_{x_2x_2}\right) + F\left(\dfrac{v_{x_1x_1}}{a_{22}} + \dfrac{v_{x_2x_2}}{a_{11}}\right).$$

Now, let $\zeta(r)$ be a cut-off function for the domain G_0^ϱ, $\varrho \in (0,d)$

(7.2.14) $$\zeta(r) = \begin{cases} 1 & 0 \le r \le \varrho/2, \\ 0 & r \ge \varrho, \end{cases}$$

$$0 \le \zeta(r) \le 1, \quad |\zeta'(r)| \le c\varrho^{-1}, \ 0 \le r \le \varrho.$$

Multiplying both sides of (7.2.13) by $r_\varepsilon^{-\gamma}\zeta^2(r)$ and integrating over G_0^ϱ, we have

(7.2.15) $$\dfrac{\nu}{2\mu} \cdot \int_{G_0^\rho} r_\varepsilon^{-\gamma} v_{xx}^2 \zeta^2(r)\,dx \le J_\varepsilon^{(1)}(\rho) + J_\varepsilon^{(2)}(\rho),$$

where

(7.2.16) $$J_\varepsilon^{(1)}(\rho) = \int_{G_0^\rho} r_\varepsilon^{-\gamma}\zeta^2(r)(v_{x_1x_2}^2 - v_{x_1x_2}v_{x_1x_2})\,dx$$

and

(7.2.17) $$J_\varepsilon^{(2)}(\rho) = \dfrac{1}{2\nu}\int_{G_0^\rho} r_\varepsilon^{-\gamma}\zeta^2(r)\left(|v_{x_1x_1}| + |v_{x_2x_2}|\right) \cdot |F(x,v,v_x)|\,dx.$$

Repeating the computations made in the proof of Lemma 2.41, we obtain

(7.2.18) $$J_\varepsilon^{(1)}(\rho) = J_\varepsilon^{(11)}(\rho) + J_\varepsilon^{(12)}(\rho) + J_\varepsilon^{(13)}(\rho),$$

(7.2.19) $$J_\varepsilon^{(11)}(\rho) = \dfrac{\gamma}{2}\int_{G_0^\rho} r_\varepsilon^{-\gamma-2} \cdot \zeta^2(r)x_i w_i(x)\,dx,$$

(7.2.20) $$J_\varepsilon^{(12)}(\rho) = \varepsilon\dfrac{\gamma}{2} \cdot \int_{G_0^\rho} r_\varepsilon^{-\gamma-2} \cdot \zeta^2(r)w_2(x)\,dx$$

and

(7.2.21) $$J_\varepsilon^{(13)}(\rho) = -\int_{G_0^\rho} r^{-\gamma} \cdot \zeta\zeta'(r)\dfrac{x_i}{r}w_i(x)\,dx,$$

where the $w_i(x)$ are defined by (2.6.1), by virtue of which

$$(7.2.22) \qquad x_i w_i(x) = v_{x_2} \cdot \frac{\partial v_{x_1}}{\partial \omega} - v_{x_1} \cdot \frac{\partial v_{x_2}}{\partial \omega},$$

and therefore (7.2.19) can be rewritten in the form

$$(7.2.23) \quad J_\varepsilon^{(11)}(\rho) = \frac{\gamma}{2} \cdot \int_0^\rho r_\varepsilon^{-\gamma-2} \cdot r \zeta^2(r) dr \int_0^{\omega_0} \left(v_{x_2} \cdot \frac{\partial v_{x_1}}{\partial \omega} - v_{x_1} \cdot \frac{\partial v_{x_2}}{\partial \omega} \right) d\omega.$$

To estimate the integral J_ε^{11} we perform the transformation of coordinates. From the rectangular system (x_1, x_2) we go over to an affine system (y_1, y_2) namely we place the axis OY_1 along the axis OX_1 (along the ray $\omega = 0$,) and we direct the axis OY_2 along the ray $\omega = \omega_0$. We then have

$$v_{x_2} \frac{\partial v_{x_1}}{\partial \omega} - v_{x_1} \frac{\partial v_{x_2}}{\partial \omega} = \frac{1}{\sin \omega_0} \left(v_{y_2} \frac{\partial v_{y_1}}{\partial \omega} - v_{y_1} \frac{\partial v_{y_2}}{\partial \omega} \right)$$

Further, by the boundary condition $v(x) = 0$, $x \in \partial G$, we have

$$v_{y_1}(r, 0) = 0, \quad v_{y_2}(r, \omega_0) = 0,$$

and hence

$$(7.2.24) \quad J_\varepsilon^{(11)}(\rho) = \frac{\gamma}{2 \sin \omega_0} \cdot \int_0^\rho r_\varepsilon^{-\gamma-2} r \zeta^2(r) dr \cdot \int_0^{\omega_0} \left\{ [v_{y_2}(r, \omega) - \right.$$

$$\left. - v_{y_2}(r, \omega_0)] \frac{\partial v_{y_1}}{\partial \omega} - [v_{y_1}(r, \omega) - v_{y_1}(r, 0)] \cdot \frac{\partial v_{y_2}}{\partial \omega} \right\} d\omega.$$

By the Hölder inequality for integrals

$$(7.2.25)$$
$$|v_{y_1}(r, \omega) - v_{y_1}(r, 0)|^2 = \left| \int_0^\omega \frac{\partial v_{y_1}(r, \theta)}{\partial \theta} d\theta \right|^2 \leq$$

$$\leq \omega \cdot \int_0^\omega \left| \frac{\partial v_{y_1}(r, \theta)}{\partial \theta} \right|^2 d\theta \leq \omega r^2 \cdot \int_0^\omega |\nabla v_{y_1}(r, \theta)|^2 d\theta,$$

and, similarly,

$$(7.2.26) \qquad |v_{y_2}(r, \omega_0) - v_{y_2}(r, \omega)|^2 \leq (\omega_0 - \omega) \cdot r^2 \cdot \int_0^{\omega_0} |\nabla v_{y_2}(r, \theta)|^2 d\theta.$$

We estimate the integrals in (7.2.24) by Cauchy's inequality and consider the estimates (7.2.25) and (7.2.26). As a result, we obtain

$$
J_\varepsilon^{(11)}(\rho) \leq \frac{\gamma}{4\sin\omega_0} \cdot \int\limits_0^\rho r_\varepsilon^{-\gamma-2} r\zeta^2(r) \left\{ \int\limits_0^{\omega_0} \left[\left(\frac{\partial v_{y_1}}{\partial\omega} \right)^2 + \left(\frac{\partial v_{y_2}}{\partial\omega} \right)^2 \right] d\omega +
$$

$$
+ r^2 \cdot \int\limits_0^{\omega_0} \left[\omega \cdot \int\limits_0^\omega |\nabla v_{y_1}(r,\theta)|^2 d\theta + (\omega_0 - \omega) \int\limits_\omega^{\omega_0} |\nabla v_{y_2}(r,\theta)|^2 d\theta \right] d\omega \right\} dr \leq
$$

$$
(7.2.27) \qquad \leq \frac{2+\omega_0^2}{8\sin\omega_0} \gamma \cdot \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma} \zeta^2(r) v_{xx}^2 dx
$$

by property 2) of the function $r_\varepsilon(x)$. By property 3) of this function and Cauchy's inequality we can estimate the integral (7.2.20) as follows

$$
(7.2.28) \qquad J_\varepsilon^{(12)}(\rho) \leq \frac{\gamma}{2} \int\limits_{G_0^\rho} [r_\varepsilon^{-\gamma} \zeta^2(r) v_{xx}^2 + r_\varepsilon^{-\gamma-2} \cdot \zeta^2(r) |\nabla v|^2] dx
$$

Finally, applying Cauchy's inequality with $\forall \delta > 0$ and considering (2.6.1), we estimate the integral (7.2.21)

$$
(7.2.29) \qquad J_\varepsilon^{(13)}(\rho) \leq \int\limits_{G_0^\rho} \left[4\delta r_\varepsilon^{-\gamma} \cdot \zeta^2(r) v_{xx}^2 + \frac{1}{\delta} r_\varepsilon^{-\gamma} \zeta'^2(r) \cdot |\nabla v|^2 \right] dx.
$$

Thus, from the representation (7.2.18) and estimates (7.2.27)-(7.2.29), we obtain

$$
(7.2.30) \quad J_\varepsilon^{(1)}(\rho) \leq \left(4\delta + \frac{\gamma}{2} + \gamma \frac{2+\omega_0^2}{8\sin\omega_0} \right) \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma} \zeta^2(r) v_{xx}^2 dx +
$$

$$
+ \frac{\gamma}{2} \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-2} \zeta^2(r) |\nabla v|^2 dx + \frac{1}{\delta} \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma} \zeta'^2(r) |\nabla v|^2 dx, \quad \forall \delta > 0.
$$

We now turn to the estimation of the integral (7.2.17). Using Cauchy's inequality, considering the condition (7.2.9), and applying Lemma 2.39 together with the inequality (7.2.10), we obtain

$$J_\varepsilon^{(2)}(\rho) \le \frac{1}{2\mu} \left[\delta + c\delta^{-1}(1 + 8\mu_1^2)c_0^2 d^{2\gamma_0}\right] \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma} \zeta^2(r) v_{xx}^2 dx +$$

$$+ \frac{c}{\delta\mu}(4\mu_1^2 + \frac{1}{2})c_0^2 d^{2\gamma_0} \int\limits_{G_0^\rho} \left(r_\varepsilon^{-\gamma-2} \cdot \zeta^2(r) + r_\varepsilon^{-\gamma} \cdot \zeta'^2(r)\right) |\nabla v|^2 dx +$$

(7.2.31)

$$+ c(\delta^{-1}, \mu, \mu_1, M_0) \int\limits_{G_0^\rho} \left[r_\varepsilon^{-\gamma}\zeta^2(r)\left(b^4(x) + f^2(x) + \Phi_{xx}^2\right) + \right.$$

$$+ \left(r_\varepsilon^{-\gamma-2} \cdot \zeta^2(r) + r_\varepsilon^{-\gamma} \cdot \zeta'^2(r)\right) |\nabla\Phi|^2 \Big] dx, \quad \forall \delta > 0, \ 0 < \rho \le d.$$

LEMMA 7.9. *Under the conditions of Theorem 7.8 we have*

$$\int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-2} \cdot \zeta^2(r) |\nabla v|^2 dx \le c(\mu, \mu_1, c_0, \gamma_0, \gamma, \omega_0, d) \times$$

(7.2.32) $$\times \left\{\rho^{2\gamma_0 - \gamma - 2} + \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma}\zeta^2(r) v_{xx}^2 dx + \int\limits_{G_0^\rho} \left[r_\varepsilon^{-\gamma}\zeta^2(r)\left(b^4(x) + f^2(x) + \right.\right.\right.$$

$$\left.\left.\left. + \Phi_{xx}^2\right) + r_\varepsilon^{-\gamma-2} \cdot \zeta^2(r)|\nabla\Phi|^2\right] dx, \right\} \quad 0 < \rho \le d.$$

PROOF. We multiply the equality $(QL)_0$ by $r_\varepsilon^{-\gamma-2}\zeta^2(r)v(x)$ and integrate it over the domain G_0^ρ. Integrating by parts, we have

$$\int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-2}\zeta^2(r)|\nabla v|^2 dx = \frac{(2+\gamma)^2}{2} \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-4} \cdot \zeta^2(r) v^2 dx -$$

$$-(\gamma+2) \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-4} \cdot \zeta\zeta'(r)\frac{x_i}{r}(x_i - \varepsilon l_i)v^2 dx - 2 \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-2} \cdot \zeta\zeta'(r)\frac{x_i}{r}vv_{x_i} dx +$$

$$+ \int\limits_{G_0^\rho} [a_{ij}(x, v + \Phi, v_x + \Phi_x) - a_{ij}(0, 0, 0)] v_{x_i x_j} v r_\varepsilon^{-\gamma-2}\zeta^2(r) dx +$$

(7.2.33) $$+ \int\limits_{G_0^\rho} \Big[a_{ij}(x, v + \Phi, v_x + \Phi_x)\Phi_{x_i x_j} +$$

$$+ a(x, v + \Phi, v_x + \Phi_x)\Big] r_\varepsilon^{-\gamma-2}\zeta^2(r)v(x) dx.$$

We estimate the integrals on the right using Cauchy's inequality, assumptions $(B), (C)$, and the estimate (7.2.10) for the Hölder continuity of $v(x)$. As a result, from (7.2.33) we obtain

$$
\left[\left(\frac{\pi}{\omega_0}\right)^2 - \left(\frac{2+\gamma}{2}\right)^2\right] \cdot \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-2}\zeta^2(r)|\nabla v|^2 dx \le c_1(c_0,\gamma,\mu_1)(\delta + d^{\gamma_0}) \times
$$

$$
\times \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-2}\zeta^2(r)|\nabla v|^2 dx + c_2(\mu,\gamma,\omega_0,\delta^{-1}) \cdot \int\limits_{G_0^\rho} \big[r_\varepsilon^{-\gamma} \cdot \zeta^2(r)v_{xx}^2 +
$$

(7.2.34) $\qquad\qquad + r_\varepsilon^{-\gamma-2}\zeta'^2(r)v^2\big] dx + c_3(\mu,\mu_1,\gamma_0,\omega_0,d,\gamma,\delta^{-1}) \times$

$$
\times \int\limits_{G_0^\rho} \big[r_\varepsilon^{-\gamma} \cdot \zeta^2(r)(\Phi_{xx}^2 + b^4(x) + f^2(x)) + r_\varepsilon^{-\gamma-2} \cdot \zeta^2(r) \cdot |\nabla\Phi|^2\big] dx,
$$

$$
0 < \rho \le d, \quad \forall \delta > 0.
$$

Further, by property 2) of the function $r_\varepsilon(x)$ and the properties of the function $\zeta(r)$ with consideration of the inequality (7.2.11), we have

$$
\int\limits_{G_0^\rho} r_\varepsilon^{-\gamma-2}\zeta'^2(r)v^2 dx \le (cc_0)^2 \cdot \rho^{-2} \cdot \int\limits_{G_0^\rho} r^{2\gamma_0-\gamma-2} dx =
$$

(7.2.35)

$$
= \frac{\omega_0 \cdot (cc_0)^2}{2\gamma_0 - \gamma} \cdot \rho^{2\gamma_0-\gamma-2}.
$$

Since by (7.2.11) the left side of the (7.2.34) contains a strictly positive constant factor, choosing the quantities $\delta, d > 0$ sufficiently small, we obtain the desired inequality (7.2.32).   □

Returning to the inequality (7.2.15), on the basis of (7.2.30)-(7.2.32) and the choice of the quantities $\gamma, \delta, d > 0$ as sufficiently small, we obtain

(7.2.36) $\quad \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma}\zeta^2(r)v_{xx}^2 dx \le c(\nu,\mu,\mu_1,\gamma_0,\gamma,c_0,\omega_0,M_0,d)\Big\{ \rho^{2\gamma_0-\gamma-2} +$

$$
+ \int\limits_{G_0^\rho} r_\varepsilon^{-\gamma}\zeta'^2(r)|\nabla v|^2 dx + \int\limits_{G_0^\rho} \big[r_\varepsilon^{-\gamma}\zeta^2(r)(b^4(x) + f^2(x) + \Phi_{xx}^2) +
$$

$$
+ \big(r_\varepsilon^{-\gamma-2}\zeta^2(r) + r_\varepsilon^{-\gamma}\zeta'^2(r)\big) \cdot |\nabla\Phi|^2\big] dx \Big\}, \quad 0 < \rho \le d.
$$

Finally, noting that by the hypotheses of theorem the quantities (7.2.1) are known, in analogy to (7.2.35) we have

$$\int_{G_0^\rho} r_\varepsilon^{-\gamma} \zeta'^2(r) |\nabla u|^2 dx \le c^2 M_1^2 \rho^{-2} \cdot \int_{G_0^\rho} r^{-\gamma} dx = \frac{\omega_0^2 c^2 M_1^2}{2-\gamma} \varrho^{-\gamma}.$$

Therefore, by the properties of the functions r_ε and $\zeta(r)$, the inequality (7.2.36) gives

$$\int_{G_0^\rho} r_\varepsilon^{-\gamma} \zeta^2(r) u_{xx}^2 dx \le c(\nu, \mu, \mu_1, M_0, \gamma_0, \gamma, c_0, d, \omega_0) \times$$

$$\times \left\{ M_1^2 \rho^{-\gamma} + \rho^{2\gamma_0 - \gamma - 2} + \int_{G_0^\rho} [r^{-\gamma^*} \zeta^2(r)(b^4(x) + f^2(x) + \Phi_{xx}^2) + \right.$$

$$\left. + r^{-\gamma^* - 2} \cdot \zeta^2(r) |\nabla \Phi|^2] dx \right\},$$

where $0 < \rho \le d$. Since the right side does not depend on ε, passing to the limit as $\varepsilon \to +0$, we finally obtain

$$\int_{G_0^{\rho/2}} r^{-\gamma} u_{xx}^2 dx \le \int_{G_0^d} r^{-\gamma} \zeta^2(r) u_{xx}^2 dx \le c(\nu, \mu, \mu_1, M_0, M_1, \gamma_0, \gamma, c_0, d, \omega_0) \times$$

$$\times \left\{ d^{-\gamma} + d^{2\gamma_0 - \gamma - 2} + \|b^2\|^2_{\overset{\circ}{W}{}^0_{-\gamma^*}(G_0^d)} + \|f\|^2_{\overset{\circ}{W}{}^0_{-\gamma^*}(G_0^d)} + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_{-\gamma^*}(\Gamma_0^d)} \right\}.$$

The assertion of the theorem and the estimate (7.2.12) follow from this estimate.   □

COROLLARY 7.10. *Suppose the hypotheses of Theorem 7.8 are satisfied except for the finiteness of* M_1. *Then*

$$\int_{G_0^{\rho/2}} u_{xx}^2 dx \le c(\nu, \mu, \mu_1, M_0, \gamma_0, \gamma, c_0, d, \omega_0) \times$$

(7.2.37)

$$\times \left\{ \rho^{-2} \int_{G_0^\rho} |\nabla u|^2 dx + \int_{G_0^\rho} [b^4(x) + f^2(x)] dx + \|\varphi\|^2_{\overset{\circ}{W}{}^{3/2}_0(\Gamma_0^\rho)} \right\}.$$

PROOF. This follows from (7.2.15), the Hardy-Wirtinger inequality, and estimates (7.2.30), (7.2.31), and (7.2.34) for $\gamma = 0$ and sufficiently small $\delta, d > 0$ with consideration of the properties of the functions r_ε and $\xi(r)$.   □

7.2.4. The behavior of the solution near a corner point (weak smoothness).
In this Subsection we establish power decay of a solution of the homogeneous problem $(QL)_0$ near a corner point.

THEOREM 7.11. *Suppose the conditions of Theorem 7.8 are satisfied, and let $\gamma > 0$ be the number determined by this theorem. Suppose, moreover, that*

$$f, b^2 \in V^0_{\rho,-\gamma}(G), \quad \varphi \in C^1(\partial G) \cap V^{2-1/\rho}_{p,-\gamma}(\partial G), \quad p > 2$$

and

$$(7.2.38) \quad ||f||_{V^0_{p,-\gamma}(G^\rho_{\rho/2})} + ||b^2||_{V^0_{p,-\gamma}(G^\rho_{\rho/2})} + ||\varphi||_{V^{2-1/\rho}_{p,-\gamma}(\Gamma^\rho_{\rho/2})} \leq k_1\rho^{2/\rho-1}.$$

Then

$$(7.2.39) \quad |v(x)| \leq c_1 \cdot |x|^{1+\gamma/2}$$

$$(7.2.40) \quad |\nabla v(x)| \leq c_2 \cdot |x|^{\gamma/2}.$$

PROOF. The inequality (7.2.39) follows from the imbedding theorem (Lemma 1.38) because of Theorem 7.8.

To estimate the modulus of the gradient of the solution in the ring $G^1_{1/2}$ we consider the function

$$(7.2.41) \quad z(x') = v(\rho x') \cdot \rho^{-1-\gamma/2},$$

assuming that $v \equiv 0$ outside of G. In $G^1_{1/2}$ this function satisfies

$$\tilde{a}_{ij}(x')z_{x'_j x'_i} = \widetilde{F}(x'),$$

$$(7.2.42) \qquad \hat{a}_{ij}(x') \equiv a_{ij}(\rho x', u(\rho x'), \rho^{-1}u_{x'}(\rho x')),$$

$$\widetilde{F}(x') \equiv -\rho^{1-\gamma/2} \cdot a(\rho x', u(\rho x'), \rho^{-1}u_{x'}(\rho x')) - \rho^{-1-\gamma/2}\tilde{a}_{ij}(x')\Phi_{x'_j x'_i}(\rho x'),$$

where by assumptions $(B), (C)$

$$(7.2.43) \quad |\widetilde{F}(x')| \leq (\mu_1 + \frac{1}{2})\rho^{-1-\gamma/2} \cdot |\nabla'u|^2 + \rho^{1-\gamma/2}(f + b^2) +$$
$$+ \mu\rho^{-1-\gamma/2}|\Phi_{x'x'}|.$$

To the equation (7.2.42) we apply Theorem 4.10 regarding the boundedness of the modulus of the gradient of a solution inside the domain and near a smooth portion of the boundary

$$(7.2.44) \qquad \operatorname*{ess\,sup}_{G^1_{1/2}} |\nabla'z| \leq M'_1$$

where M_1' is determined only by $\nu, \mu, \mu_1, \omega_0$, and the integrals

$$\int\limits_{G_{1/2}^1} z^2 dx', \quad \left(\int\limits_{G_{1/2}^1} |\widetilde{F}(x')|^p dx'\right)^{1/\rho}, \quad p > 2.$$

To verify the finiteness of these integrals, we have

$$\int\limits_{G_{1/2}^1} z^2 dx' \le \int\limits_{G_{\rho/2}^\rho} r^{-\gamma-4} \cdot v^2 dx \le \int\limits_{G_0^d} r^{-\gamma-4} v^2 dx \le c(d)$$

by Theorem 7.8. Further, by (7.2.43) and the assumption (7.2.38) of the theorem we have

$$\left(\int\limits_{G_{1/2}^1} |\widetilde{F}(x')|^p dx'\right)^{1/p} \le c(\mu, \mu_1, p)\left\{\int\limits_{G_{\rho/2}^\rho} r^{p(1-\frac{\gamma}{2})-2} \cdot \left[|\nabla u|^{2p} + |\Phi_{xx}|^p +\right.\right.$$

$$\left.\left. + b^{2p} + f^p\right]dx\right\}^{1/p} \le c(\mu, \mu_1, p)\left\{\frac{\omega_0 M_1^2 d^{1-\gamma/2}}{[p \cdot (1-\gamma/2)]^{1/p}} + k_1\right\}.$$

Returning to the function $v(x)$, from (7.2.41) and (7.2.44), we obtain

$$|\nabla v(x)| \le M_1' \rho^{\gamma/2}, \quad x \in G_{\frac{\rho}{2}}^\rho \cap G_0^d.$$

Setting $|x| = 2\rho/3$, from this we obtain (7.2.40). Theorem 7.11 is proved.
$\square$

7.2.5. The weighted integral estimate.

We can refine the Nirenberg estimate on the basis of the weak smoothness of a solution established in Subsection 7.2.4. This refinement is possible due to the requirement of continuity of the leading coefficients of the equation.

THEOREM 7.12. *Suppose $u \in W^2(G)$ is a solution of problem (QL) and the assumptions of Theorem 7.11 are satisfied. Suppose the functions $a_{ij}(x, u, z)$ $(i, j = 1, 2)$ are continuous at the point $(0, 0, 0)$. If, in addition, $b^2, f \in \overset{\circ}{W}_\alpha^0(G), \varphi \in \overset{\circ}{W}_\alpha^{3/2}(\partial G)$, and*

(K) $2 - 2\pi/\omega_0 < \alpha \le 2,$

then $u \in \overset{\circ}{W}_\alpha^2(G)$, and

$$(7.2.45) \quad \|u\|_{\overset{\circ}{W}_\alpha^2(G)} \le c \cdot \left(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}_\alpha^0(G)} + \|b^2\|_{\overset{\circ}{W}_\alpha^0(G)} +\right.$$

$$\left. + \|\varphi\|_{\overset{\circ}{W}_\alpha^{3/2}(\partial G)}\right),$$

where $c > 0$ is a constant depending only on $\nu^{-1}, \mu, \mu_1, \alpha, \omega_0, M_0, M_1, \gamma_0, c_0$, meas G and diam G and also on k_1, p, c_1 and c_2 of (7.2.38)-(7.2.40).

PROOF. Let $r_\varepsilon(x)$ be the function defined in §1.4 of Chapter 1. We multiply both sides of $(QL)_0$ by $r_\varepsilon^{\alpha-2}(x)v(x)$ and integrate over the domain G, using the condition (B) and integration by parts

$$\int_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx = \frac{(2-\alpha)^2}{2}\int_G r_\varepsilon^{\alpha-4}v^2 dx+$$

(7.2.46)

$$+\int_G r_\varepsilon^{\alpha-2}(x)v(x)\big\{[a_{ij}(x,u,u_x) - a_{ij}(0,0,0)]v_{x_i x_j} - F(x,v,v_x)\big\}dx.$$

We decompose G into two subdomains G_0^d and G_d, in each of which we obtain an upper bound for integral on the right side of (7.2.46).

Estimates in G_0^d. By the continuity of $a_{ij}(x,u,z)$ at the point $(0,0,0)$ assumed in the theorem, for any $\delta > 0$ there exists $d_0(\delta) > 0$ such that

(7.2.47) $$\left(\sum_{i,j=1}^2 |a_{ij}(x,u,u_x) - a_{ij}(0,0,0)|^2\right)^{1/2} < \delta,$$

provided that

(7.2.48) $$|x| + |u(x)| + |\nabla u(x)| < d_0.$$

The smoothness of the boundary function $\varphi(x)$ assumed in Theorem 7.11 makes it possible to conclude by Lemma 1.38 that $\varphi(0) = 0$ and $|\nabla\Phi(0)| = 0$. Therefore, by (7.2.39) and (7.2.40) of Theorem 7.11 we have, for any $x \in G_0^d$,

$$|x| + |u(x)| + |\nabla u(x)| \le |x| + |v(x)| + |\nabla v(x)| + |\Phi(x) - \varphi(0)|+$$

$$+|\nabla\Phi(x) - \nabla\Phi(0)| \le d + c_1 d^{1+\gamma/2} + c_2 d^{\gamma/2} + \frac{1}{2}d_0,$$

and hence (7.2.48) is ensured because of the sufficient smallness of $d > 0$. With the Cauchy inequality we now estimate the integral

$$\int_{G_0^d} r_\varepsilon^{\alpha-2}(x)v(x)\left[a_{ij}(x,u,u_x) - a_{ij}(0,0,0)\right]v_{x_i x_j}dx \le$$

(7.2.49)

$$\le \frac{\delta}{2}\int_{G_0^d}\left(r^2 r_\varepsilon^{\alpha-2}v_{xx}^2 + r_\varepsilon^{\alpha-2}\frac{v^2}{r^2}\right)dx, \quad \forall\delta > 0.$$

Further, by the condition (7.2.9) with the help of Cauchy's inequality and (7.2.39) we obtain

$$\int\limits_{G_0^d} r_\varepsilon^{\alpha-2}(x)v(x)F(x,v,v_x)dx \le \left(\frac{1}{2}+2\mu_1\right)c_1 d^{1+\gamma/2}\int\limits_{G_0^d} r_\varepsilon^{\alpha-2}|\nabla v|^2 dx +$$

$$+\left(\frac{1}{2}+2\mu_1\right)M_0 d^{\alpha-2}\int\limits_{G_d}|\nabla v|^2 dx + \frac{3}{2}\delta\int\limits_{G} r_\varepsilon^{\alpha-2}\frac{v^2}{r^2}dx+$$

(7.2.50)

$$+\frac{1}{2\delta}\int\limits_{G} r^2 r_\varepsilon^{\alpha-2}\left(b^4(x)+f^2(x)\right)dx+$$

$$+c(\mu,\mu_1,M_0,\delta^{-1})\int\limits_{G}(r^2 r_\varepsilon^{\alpha-2}\Phi_{xx}^2 + r_\varepsilon^{\alpha-2}|\nabla\Phi|^2)dx, \ \forall\delta>0.$$

Estimates in G_d. By condition (B) and properties of the function $r_\varepsilon(x)$ we have

$$(7.2.51)\quad \int\limits_{G_d} r_\varepsilon^{\alpha-2}v(x)[a_{ij}(x,u,u_x)-a_{ij}(0,0,0)]v_{x_i x_j}dx \le$$

$$\le \frac{\mu+1}{2}d^{\alpha-2}\int\limits_{G_d}\left(v_{xx}^2 + v^2\right)dx.$$

On the basis of (7.2.49)-(7.2.51) and the inequality (2.5.8) from (7.2.46) we now obtain for $\forall\delta>0$

$$(7.2.52)\quad \frac{(\pi/\omega_0)^2-((2-\alpha)/2)^2}{(\pi/\omega_0)^2+((2-\alpha)/2)^2}\cdot\int\limits_{G} r_\varepsilon^{\alpha-2}\cdot|\nabla v|^2 dx \le$$

$$\le\left[2\delta\cdot H(\alpha,\omega_0)+(\frac{1}{2}+2\mu_1)c_1 d^{1+\gamma/2}\right]\int\limits_{G} r_\varepsilon^{\alpha-2}|\nabla v|^2 dx+$$

$$+\frac{\delta}{2}\cdot\int\limits_{G_0^d} r^2 r_\varepsilon^{\alpha-2}\cdot v_{xx}^2 dx + c(\mu,\mu_1,M_0,\delta^{-1})\cdot\int\limits_{G}(r_\varepsilon^{\alpha-2}r^2\Phi_{xx}^2 + r_\varepsilon^{\alpha-2}\cdot|\nabla\Phi|^2)dx+$$

$$+\frac{1}{2\delta}\int\limits_{G} r^2 r_\varepsilon^{\alpha-2}\left(b^4(x)+f^2(x)\right)dx + c(\mu,\mu_1,M_0)d^{\alpha-2}\int\limits_{G_d}(v_{xx}^2+v^2)dx.$$

To estimate the integral with second derivatives in (7.2.52) we apply the method of S. N. Bernstein (see, for example, [216], Chapter III, §19).

We rewrite the problem $(QL)_0$ in the form

$$(7.2.53) \qquad \begin{cases} \Delta v = \mathcal{F}, & x \in G, \\ v(x) = 0, & x \in \partial G, \end{cases}$$

where

$$(7.2.54) \quad \mathcal{F} \equiv -\big[a_{ij}(x, v + \Phi, v_x + \Phi_x) - a_{ij}(0,0,0)\big]v_{x_i x_j} + F(x, v, v_x).$$

We multiply (7.2.53) first by $r^2 r_\varepsilon^{\alpha-2} \cdot v_{x_1 x_1}$ and next by $r^2 r_\varepsilon^{\alpha-2} \cdot v_{x_2 x_2}$, and add the equalities thus obtained. Next we integrate the result over G

$$(7.2.55) \qquad \int_G r^2 r_\varepsilon^{\alpha-2} \cdot v_{xx}^2 dx = \int_G r^2 r_\varepsilon^{\alpha-2} \cdot \Delta v \cdot \mathcal{F} dx + 2 J_{\alpha,\varepsilon}[v],$$

where the last term is defined by (2.6.3) and Lemma 2.42 holds for it. We estimate the first term on the right in (7.2.55) on the basis of (7.2.53), (7.2.47) and conditions (B), (7.2.9)

$$(7.2.56) \qquad \int_G r^2 r_\varepsilon^{\alpha-2} \cdot \Delta v \cdot \mathcal{F} dx \leq \delta_1 \int_G r^2 r_\varepsilon^{\alpha-2} \cdot v_{xx}^2 dx +$$

$$+ c_1 \int_G r^2 r_\varepsilon^{\alpha-2}\big(|\nabla v|^4 + b^4(x) + f^2(x) + \Phi_{xx}^2 + |\nabla\Phi|^4\big) +$$

$$+ c(d,\alpha)(1 + \mu) \int_{G_d} v_{xx}^2 dx, \ \forall \delta_1 > 0.$$

From (7.2.55), (7.2.56) on the basis of Lemmas 2.39, 2.40, 2.42 and with consideration of (7.2.10) and since $d, \delta_1 > 0$ were chosen sufficiently small, we obtain

$$\int_G r^2 r_\varepsilon^{\alpha-2} v_{xx}^2 dx \leq c(\alpha, M_0, \mu, meas G, diam G) \int_{G_d} (v_{xx}^2 + v^2) dx +$$

$$(7.2.57) \qquad + c(M_0, \mu, \mu_1) \int_G \big[r^\alpha\big(b^4(x) + f^2(x) + \Phi_{xx}^2\big) +$$

$$+ r^{\alpha-2}|\nabla\Phi|^2\big] dx + c(\alpha, M_0) \int_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx.$$

The next lemma follows from Theorem 4.9 and Lemma 2.39 with $\alpha = 0$.

LEMMA 7.13.

$$(7.2.58) \quad \int_{G_d} v_{xx}^2 dx \le c(\nu^{-1}, \mu, \mu_1, c_0, \gamma_0) \times$$

$$\times \int_{G_d} [v^2(x) + b^4(x) + f^2(x) + \Phi_{xx}^2 + |\nabla\Phi|^2 + \Phi^2] dx.$$

From (7.2.52), (7.2.57), (7.2.58), and the condition (K) of the theorem, by choosing δ, and d sufficiently small we finally obtain

$$\int_G (r^2 r_\varepsilon^{\alpha-2} u_{xx}^2 + r_\varepsilon^{\alpha-2} \cdot |\nabla u|^2 + r_\varepsilon^{\alpha-4} \cdot u^2) dx \le$$

$$\le c(\alpha, \omega_0, \nu^{-1}, \mu, \mu_1, M_0, c_0, c_1, \gamma_0, \gamma, \text{diam}G, \text{meas}G) \times$$

$$\times \int_G [u^2 + r^\alpha (b^4(x) + f^2(x) + \Phi_{xx}^2) + r^{\alpha-2} \cdot |\nabla\Phi|^2 + r^{\alpha-4} \cdot \Phi^2] dx \quad \forall \varepsilon > 0.$$

Passing to the limit as $\varepsilon \to +0$, we establish Theorem 7.12 and (7.2.45). $\square$

7.2.6. Proof of Theorem 7.7.

PROOF. That u belongs to the space $\mathring{W}_2^2(G)$ follows from Theorem 7.12 for $\alpha = 2$ so that to prove assertion 1) we need to prove (7.2.4). For this we multiply both sides of $(QL)_0$ by $v(x)$ and integrate over the domain G_0^ρ, $0 < \rho < d$. Setting

$$(7.2.59) \qquad V(\rho) = \int_{G_0^\rho} |\nabla v|^2 dx$$

we obtain

$$(7.2.60) \quad V(\rho) = \rho \int_0^{\omega_0} v \frac{\partial v}{\partial r} d\omega + \int_{G_0^\rho} \{v(x)[a_{ij}(x, u, u_x) - a_{ij}(0, 0, 0)]v_{x_i x_j} -$$

$$- v(x) \cdot F(x, v, v_x)\} dx.$$

We shall obtain an upper bound for each integral on the right. For the first integral we have Corollary 2.29

$$(7.2.61) \qquad \rho \int_0^{\omega_0} v \frac{\partial v}{\partial r} d\omega \le \frac{\rho \omega_0}{2\pi} \cdot V'(\rho).$$

Condition (7.2.3) of the theorem ensures that (7.2.38) is satisfied, and hence estimates (7.2.39) and (7.2.3) of Theorem 7.11 are valid. On the basis of these estimates and the assumed Dini continuity of the functions

$a_{ij}(x, u, u_x)$ at $(0, 0, 0)$ and the smoothness of the boundary function $\varphi(x)$ it is not hard to establish the existence of a positive, monotonically increasing function $\delta(r)$, continuous on $[0, d]$, which satisfies a Dini condition at zero and is such that

$$\left(\sum_{i,j=1}^{2} |a_{ij}(x, u(x), u_x(x)) - a_{ij}(0, 0, 0)|^2 \right)^{1/2} \leq \delta(\rho), \quad |x| < \rho.$$

In fact, by Dini continuity, we have

$$\left(\sum_{i,j=1}^{2} |a_{ij}(x, u(x), u_x(x)) - a_{ij}(0, 0, 0)|^2 \right)^{1/2} \leq \mathcal{A}(|x| + |u(x)| + |\nabla u(x)|),$$

where $\mathcal{A}(t)$ satisfies the Dini condition at zero, that is $\int_0^d \frac{\mathcal{A}(t)}{t} < \infty$. But from estimates (7.2.39) and (7.2.3) it follows

$$|x| + |u(x)| + |\nabla u(x)| \leq |x| + c_1 |x|^{1+\gamma/2} + c_2 |x|^{\gamma/2} \leq c|x|^{\gamma/2}.$$

Hence we obtain

$$\mathcal{A}(|x| + |u(x)| + |\nabla u(x)|) \leq \mathcal{A}(cr^{\gamma/2}) \equiv \delta(r),$$

where

$$\int_0^1 \frac{\delta(r)}{r} dr = \int_0^1 \frac{\mathcal{A}(cr^{\gamma/2})}{r} dr = \frac{2}{\gamma} \int_0^c \frac{\mathcal{A}(t)}{t} dt < \infty.$$

Therefore, by the Cauchy inequality we obtain

$$\int_{G_0^\rho} v(x) \cdot [a_{ij}(x, u, u_x) - a_{ij}(0, 0, 0)] v_{x_i x_j} dx \leq$$

(7.2.62)
$$\leq \frac{1}{2} \delta(\rho) \cdot \int_{g_0^\rho} (r^2 \cdot v_{xx}^2 + r^{-2} \cdot v^2) dx.$$

Finally, on the basis of (7.2.8) and by the condition (7.2.9) and the Hölder continuity of the function v (inequality(7.2.10)), we obtain

$$\int\limits_{G_0^\rho} v(x) \cdot F(x, v, x_x) dx \le$$

(7.2.63)
$$\le \left(\frac{1}{2} + 2\mu_1\right) c_0 \rho^{\gamma_0} V(\rho) + (1 + \mu)\rho^\delta \cdot \int\limits_{G_0^\rho} r^{-2} v^2 dx$$

$$+ c(\mu, \mu_1, M_0)\rho^{-\delta} \cdot \int\limits_{G_0^\rho} \left[r^2 \left(\Phi_{xx}^2 + b^4(x) + f^2(x)\right) + |\nabla\Phi|^2 \right] dx, \quad \forall \delta > 0.$$

From (7.2.60) on the basis of (7.2.61)-(7.2.63), the Hardy-Wirtinger inequality for $\alpha = 2$, and the estimate (7.2.37) of Corollary 7.10 it now follows that the function $V(\rho)$ satisfies the differential inequality (CP) from §1.10 of the Preliminaries, in which

$$\mathcal{P}(\rho) = \frac{2\pi}{\omega_0 \rho} - 2c_0 \frac{\pi}{\omega_0}\left(\frac{1}{2} + 2\mu_1\right)\rho^{\gamma_0 - 1} + \frac{\omega_0}{\pi}\rho^{-1}\delta(\rho) + 2\frac{\omega_0}{\pi}(1 + \mu)\rho^{\delta-1},$$

$$\mathcal{N}(\rho) = \frac{\pi}{\omega_0}\rho^{-1}\delta(\rho)c(\nu, \mu, \mu_1, M_0, c_0, \gamma_0, \omega_0),$$

(7.2.64)
$$\mathcal{Q}(\rho) = \frac{2\pi}{\omega_0}k\rho^{2s-1-\delta} \quad k = k_1^2 \cdot c(\mu, \mu_1, M_0) \cdot (1 + 2^{2s}),$$

$$V_0 = V(d) \le M_1^2 \cdot \text{meas } G$$

$$\forall \delta \in \left(0, 2s - 2\frac{\pi}{\omega_0}\right).$$

(Here k_1 and s are defined in condition (7.2.2).) According to Theorem 1.57 the estimate (1.10.1) holds, which together with (7.2.37) leads to the desired estimate (7.2.4).

The estimate (7.2.5) now follows from the imbedding theorem (Lemma 1.38) and from (7.2.4).

To prove the remaining assertions of Theorem 7.7 we apply the method of rings and arguments analogous to those in the proof of Theorem 7.11. We perform the coordinate transformation $x = \rho x'$. In $G_{1/2}^1$ the function $z(x') = \rho^{-\lambda}v(\rho x')$ satisfies (7.2.42) and (7.2.43) with γ replaced by $2(\lambda-1)$.

By the Sobolev-Kondrashov theorems on imbedding of function spaces we have

(7.2.65)
$$\left(\int\limits_{G_{1/2}^1} |\nabla' z|^p dx'\right)^{1/p} \le c \cdot \left[\int\limits_{G_{1/2}^1} (z_{x'x'}^2 + z^2) dx'\right]^{1/2}, \quad \forall p > 2$$

and

$$(7.2.66) \quad \sup_{x' \in G_{5/8}^{7/8}} |\nabla' z| + \sup_{\substack{x',y' \in G_{5/8}^{7/8} \\ x \neq y'}} \frac{|\nabla' z(x') - \nabla' z(y')|}{|x' - y'|^{1-2/p}} \leq$$

$$\leq c\|z\|_{W^{2,p}(G_{5/8}^{7/8})}, \quad \forall p > 2.$$

We consider (7.2.42) as a linear equation whose leading coefficients are Dini continuous. By Theorem 10.17- the L^p-estimates for the solution inside the domain and near a smooth portion of the boundary, we have

$$(7.2.67) \quad \|z\|_{W^{2,p}(G_{5/8}^{7/8})} \leq c(\nu, \mu, \delta(1)) \left[\int_{G_{1/2}^1} (|\widetilde{F}(x')|^p + |z|^p) dx' \right]^{1/p},$$

$$\forall p > 1, \quad G_{5/8}^{7/8} \subset G_{1/2}^1.$$

Returning to the variable x and the function $v(x)$, from (7.2.64)-(7.2.66) by (7.2.43) and (7.2.4) we obtain

$$(7.2.68) \quad \left(\int_{G_{\rho/2}^\rho} |\nabla v|^p dx \right)^{1/p} \leq c \cdot \rho^{\pi/\omega_0 + 2/p - 1}, \quad \forall p \geq 2$$

and also, considering the smoothness of the boundary function,

$$\sup_{x \in G_{5\rho/8}^{7\rho/8}} |\nabla v(x)| + \rho^{1-2/p} \cdot \sup_{\substack{x,y \in G_{5\rho/8}^{7\rho/8} \\ x \neq y}} \frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^{1-2/p}} \leq$$

$$(7.2.69) \quad \leq c(\nu, \mu, \mu_1, p, \delta(\rho)) \left\{ \rho^{1-2/p} \left(\int_{G_{\rho/2}^\rho} |\nabla v|^{2p} dx \right)^{1/p} + \right.$$

$$+ \left(\int_{G_{\rho/2}^\rho} r^{-p-2} |v|^p dx \right)^{1/p} +$$

$$+ \rho^{1 - \frac{2+\alpha}{p}} \cdot \left\langle \int_{G_{\rho/2}^\rho} \left[r^\alpha \left(f^p + b^{2p} + |\Phi_{xx}|^p \right) + r^{\alpha-p} |\nabla \Phi|^p \right] dx \right\rangle^{1/p} \right\}$$

and

$$(7.2.70) \quad \|v\|_{V_{p,\alpha}^2(G_{5\rho/8}^{7\rho/8})} \leq c(\nu, \mu, \mu_1, \delta(\rho)) \Bigg\{ \bigg[\int\limits_{G_{\rho/2}^\rho} (r^\alpha |\nabla v|^{2p} +$$

$$+ r^{\alpha - 2p} |v|^p) dx \bigg]^{1/p} + \bigg\langle \int\limits_{G_{\rho/2}^\rho} [r^\alpha \cdot (f^p + b^{2p} +$$

$$+ |\Phi_{xx}|^p) + r^{\alpha - p} \cdot |\nabla \Phi|^p] dx \bigg\rangle^{1/p} \Bigg\}.$$

We now note that by the estimate (7.2.5)

$$(7.2.71) \quad \int\limits_{G_0^\rho} r^{-p-2} |v|^p dx \leq c_1^p \cdot \int\limits_{G_0^\rho} r^{(\lambda - 1)p - 2} dx =$$

$$= \omega_0 c_1^p \cdot \frac{\rho^{(\lambda - 1)p}}{(\lambda - 1)p}, \quad \lambda = \pi/\omega_0 > 1$$

and

$$(7.2.72) \quad \int\limits_{G_0^\rho} r^{\alpha - 2p} |v|^p dx \leq c_1^p \cdot \int\limits_{G_0^\rho} r^{\alpha + (\lambda - 2)p} dx = \frac{c_1^p \omega_0}{2 + p(\lambda - 2)} \rho^{2 + \alpha + p(\lambda - 2)},$$

if $\lambda - 2 + \frac{2 + \alpha}{p} > 0$. In addition, we also have

$$(7.2.73) \quad \rho^{1 - 2/p} \bigg(\int\limits_{G_{\rho/2}^\rho} |\nabla v|^{2p} dx \bigg)^{1/p} \leq \varrho M_1^2 \leq M_1^2 \varrho^{\lambda - 1}$$

and

$$(7.2.74) \quad \bigg(\int\limits_{G_{\rho/2}^\rho} r^\alpha |\nabla v|^{2p} dx \bigg)^{1/p} \leq c M_1^2 \varrho^{\frac{\alpha + 2}{p}} \leq c M_1^2 \varrho^{\lambda - 2 + \frac{\alpha + 2}{p}},$$

since $1 < \lambda < 2$.

From (7.2.69), on the basis of (7.2.71), (7.2.73)and (7.2.3), we obtain (7.2.6). This concludes the proof of assertion 2) of Theorem 7.7. The assertion 3) and (7.2.7) follow in exactly the same way from (7.2.70) on the basis of (7.2.72), (7.2.74), and assumption (7.2.3). Finally, suppose the conditions of assertion 4) are satisfied. Returning to (7.2.69), by (7.2.71),

(7.2.73), and (7.2.3) we have

$$|\nabla v(x) - \nabla v(y)| \le c\rho^\varkappa |x - y|^{\pi/\omega_0 - 1 - \varkappa},$$

(7.2.75)

$$\forall x, y \in G^{7\rho/8}_{5\rho/8}, \quad \varkappa = \frac{\pi}{\omega_0} - 2 + \frac{2}{p} \le 0.$$

By the definition of the sets $G^{7\rho/8}_{5\rho/8}$ we have $|x - y|^\varkappa \ge (\frac{7}{4}\rho)^\varkappa$, since $\varkappa \le 0$. Therefore, from (7.2.75) we obtain

$$|\nabla v(x) - \nabla v(y)| \le c \cdot |x - y|^{\pi/\omega_0 - 1}, \quad \forall x, y \in G^{7\rho/8}_{5\rho/8},$$

whence assertion 4) follows. Theorem 7.7 is proved. $\qquad\square$

7.3. Estimates near a conical point

7.3.1. Introduction. In §7.2 we have investigated the behavior of strong solutions to the Dirichlet problem for uniform elliptic quasi-linear second order equation of non-divergent form near an angular point of the boundary of a plane bounded domain. There in particular it is proved that the first order derivatives of the strong solution are Hölder continuous with the exponent $\frac{\pi}{\omega_0} - 1$, if $\frac{\pi}{2} < \omega_0 < \pi$ and this exponent is the best possible. (ω_0 is an angle of intersection of segments of the domain boundary in the angular point.) Two-dimensionality of the domain is stipulated by Nirenberg's method which we have applied to obtain the estimate

(7.3.1) $$|u(x)| \le c_0 |x|^{1+\gamma}$$

with a certain $\gamma \in (0, 1)$ in the neighborhood of an angular point. Other results of §7.2 do not depend upon two-dimensionality of the domain and may be obtained by the methods presented in §7.2 in the multidimensional case. First we build the barrier function and with the aid of the Comparison Principle establish the estimate (7.3.1) with a certain now small $\gamma > 0$. Then, by the layers method, using the results of Ladyzhenskaya-Uraltseva-Lieberman [**217, 219, 224**] and the estimate (7.3.1), we establish the estimate

(7.3.2) $$|\nabla u(x)| \le c_1 |x|^\gamma.$$

On the basis of (7.3.1), (7.3.2) we prove the integral weight estimates for the second generalized derivatives of the solution with the best weight exponent. These estimates allow us to obtain exact estimates of the solution's moduli and its gradient and weight L^q-estimates of the second generalized derivatives of the solution and also to prove the Hölder continuity of the first derivatives of the solution with the best Hölder exponent.

DEFINITION 7.14. *A strong solution of the problem (QL) is the function*

$$u(x) \in W^{2,q}_{loc}(\overline{G} \setminus \mathcal{O}) \cap C^0(\overline{G}), \quad q \ge N$$

satisfying the equation of the problem for almost all $x \in G$ and the boundary condition of the problem for all $x \in \partial G$. The value $M_0 = \max\limits_{x \in \overline{G}} |u(x)|$ is assumed to be known.

We shall further assume throughout that *the below conditions* are satisfied

(S) for $\forall \varepsilon_0 > 0$ there exists $d_0 > 0$ such that

$$G_0^{d_0} = \left\{ x \in G \Big| \arccos\Big(\frac{x_N}{r}\Big) < \frac{\pi}{2} - \varepsilon_0 \right\} \Leftrightarrow G_0^{d_0} \subset \{x_N \geq 0\} \Rightarrow \lambda > 1;$$

(J) $\varphi(x) \in W^{2-\frac{1}{q},q}(\partial G)$, $q \geq N$; $a_{ij}(x, u, z) \in W^{1,q}(\mathfrak{M})$, $q > N$;
 there exist a number $\beta > -1$ and nonnegative number k_1 such that

(7.3.3) $$b(x) + f(x) \leq k_1 |x|^\beta,$$

where functions $b(x), f(x)$ are from the condition (C).

7.3.2. The barrier function.

Let $G_0 = G_0^\infty$ be an infinite cone, where $G_0 \subset \{x_N \geq 0\}$ and Γ_0 is a lateral surface. We consider the second order linear operator

$$L_0 = a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}; \quad a^{ij}(x) = a^{ji}(x), \ x \in G_0,$$

$$\nu \xi^2 \leq a^{ij}(x)\xi_i \xi_j \leq \mu \xi^2 \quad \forall x \in G_0, \forall \xi \in \mathbb{R}^N; \ \nu, \mu = \mathrm{const} > 0.$$

LEMMA 7.15. *(About the existence of the barrier function). There exist a number $h > 0$, determined only by G_0, a number γ_0 and a function $w(x) \in C^1(\overline{G_0}) \cap C^2(G_0)$ depending only on G_0 and ellipticity constants ν, μ of the operator L_0 such that $\forall \gamma \in (0, \gamma_0]$*

(7.3.4) $$L_0 w(x) \leq -\nu h^2 |x|^{\gamma-1}, \quad x \in G_0$$

(7.3.5) $$0 \leq w(x) \leq |x|^{1+\gamma} \quad and \quad |\nabla w(x)| \leq 2(1+h^2)^{1/2}|x|^\gamma, \quad x \in \overline{G_0}.$$

PROOF. We set $x' = (x_1, .. x_{N-2})$, $x = x_{N-1}$, $y = x_N$. In the half-space $y \geq 0$ we consider a cone K with the vertex $\mathcal{O}$ such that $K \supset G_0$ which it is possible since $G_0 \subset \{y \geq 0\}$. Let ∂K be the lateral surface of K and the equation of $\partial K \cap (x\mathcal{O}y)$ is $y = \pm hx$ so that inside K the inequality $y > h|x|$ is true. We consider the function

(7.3.6) $$w(x'; x, y) = (y^2 - h^2 x^2)y^{\gamma-1}, \quad \gamma \in \mathbb{R}.$$

Renaming the operator L_0 coefficients: $a^{N-1,N-1} = a$, $a^{N-1,N} = b$, $a^{N,N} = c$ we get

$$L_0 w = a w_{xx} + 2b w_{xy} + c w_{yy},$$

(7.3.7)

$$\nu \eta^2 \leq a \eta_1^2 + 2b \eta_1 \eta_2 + c \eta_2^2 \leq \mu \eta^2,$$

where $\eta^2 = \eta_1^2 + \eta_2^2$ $\forall \eta_1, \eta_2 \in \mathbb{R}$.

We calculate the operator L_0 on the function (7.3.6)

$$L_0 w = -h^2 y^{\gamma-1} \varphi(\gamma), \quad t = x/y, \quad |t| < 1/h,$$

(7.3.8)

$$\varphi(\gamma) = 2(a - 2bt + ct^2) - (3ct^2 - 4bt + ch^{-2})\gamma - c(h^{-2} - t^2)\gamma^2.$$

Since, by (7.3.7), $\varphi(0) = 2(a - 2bt + ct^2) > 2\nu$ and $\varphi(\gamma)$ is the square function, then it is obvious that there is a number $\gamma_0 > 0$, depending only on ν, μ, h such that $\varphi(\gamma) > \nu$ for $\gamma \in [0, \gamma_0]$. From (7.3.6) and (7.3.8) we now obtain all the statements of our Lemma. □

7.3.3. The weak smoothness of solutions. The above constructed function and the Comparison Principle (see Theorem 4.4) allow us to estimate $u(x)$ in the neighborhood of conical point. Without loss of generality we assume $\varphi(x) \equiv 0$.

THEOREM 7.16. *Let $u(x)$ be a solution of (QL) and satisfy the conditions $(S), (A), (B), (C)$ on the set $\mathfrak{M}^{(u)}$. Then there exist nonnegative numbers $d < d_0, \gamma$ defined only by values $\nu, \mu, N, k_1, \beta, \gamma_0, d_0, M_0$ and the domain G such that in G_0^d the estimate (7.3.1) holds with a constant c_0, independent of $u(x)$ and defined only by the values $\nu, \mu, N, k_1, \beta, \gamma_0, d_0, M_0$ and the domain G.*

PROOF. We consider the linear elliptic operator

$$\widetilde{L} = a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + a^i(x) \frac{\partial}{\partial x_i}, \quad x \in G;$$

$$a^{ij}(x) = a_{ij}(x, u(x), u_x(x)); \text{ and } a^i(x) = b(x)|\nabla u(x)|^{-1} u_{x_i}(x),$$

where we assume $a^i(x) = 0$, $i = 1, \dots, N$ in such points x, for which $|\nabla u(x)| = 0$. Let us introduce the auxiliary function

(7.3.9) $$v(x) = -1 + \exp(\nu^{-1}\mu_1 u(x)).$$

Then we get

$$\widetilde{L}v(x) \equiv \nu^{-1}\mu_1 \left(a^{ij}(x)u_{x_ix_j} + \nu^{-1}\mu_1 a^{ij}(x)u_{x_i}u_{x_j} + b(x)|\nabla u(x)| \right) \times$$

$$\times \exp(\nu^{-1}\mu_1 u(x)) = \nu^{-1}\mu_1 \Big\{ (b(x)|\nabla u(x)| - a(x,u(x),u_x(x)) +$$

$$+ \nu^{-1}\mu_1 a^{ij}(x)u_{x_i}u_{x_j} \Big\} \exp\left(\nu^{-1}\mu_1 u(x) \right) \geq -\nu^{-1}\mu_1 f(x)\exp(\nu^{-1}\mu_1 M_0)$$

in virtue of the assumptions $(B),(C)$. By the condition (7.3.3), now we obtain

$$(7.3.10) \qquad \widetilde{L}v(x) \geq -\nu^{-1}\mu_1 k_1 r^\beta \exp(\nu^{-1}\mu_1 M_0), \quad x \in G_0^d.$$

Let γ_0 be the number defined by the barrier function Lemma and the number γ satisfies the inequality

$$(7.3.11) \qquad\qquad 0 < \gamma \leq \min(\gamma_0, \beta+1).$$

We calculate the operator $\widetilde{L}$ for the barrier function (7.3.6) thusly

$$\widetilde{L}w(x';x,y) = -h^2 y^{\gamma-1}\varphi(\gamma) + |\nabla u|^{-1}b\bigg(\Big(h^2(1-\gamma)x^2 y^{\gamma-2} + (1+\gamma)y^\gamma \Big)\frac{\partial u}{\partial y} -$$

$$- 2h^2 xy^{\gamma-1})\frac{\partial u}{\partial x} \bigg) \leq -\nu h^2 y^{\gamma-1} + 2(1+h)by^\gamma, \quad \forall(x';x,y) \in G_0.$$

Returning to the previous denotations and considering the inequality (7.3.3), we get

$$\widetilde{L}w(x) \leq \left(-\nu h^2 + 2(1+h)k_1 d^{1+\beta} \right)r^{\gamma-1}, \quad x \in G_0^d.$$

Now let the number $d \in (0, d_0)$ satisfy the inequality

$$(7.3.12) \qquad\qquad d \leq \left(\frac{\nu h^2}{4k_1(1+h)} \right)^{1/(1+\beta)} .$$

Then finally we have

$$(7.3.13) \qquad\qquad \widetilde{L}w(x) \leq -\frac{1}{2}\nu h^2 r^{\gamma-1}, \quad x \in G_0^d.$$

Now let us define a number $\mathcal{A}$

$$(7.3.14) \qquad\qquad \mathcal{A} \geq 2k_1\mu_1\nu^{-2}h^{-2}\exp(M_0\mu_1/\nu).$$

Then from (7.3.10) and (7.3.13) with regard to (7.3.11) it follows

$$(7.3.15) \qquad\qquad \widetilde{L}(\mathcal{A}w(x)) \leq \widetilde{L}v(x), \quad x \in G_0^d.$$

In addition, from (7.3.5) and (7.3.9) it follows

$$(7.3.16) \qquad\qquad \mathcal{A}w(x) \geq 0 = v(x), \quad x \in \Gamma_0^d.$$

Now we compare the functions v and w on Ω_d. In virtue of the assumption (S) we have on the set $G \cap \{r = d\} \cap \{x_{N-1} \mathcal{O} x_N\}$ that

$$x_{N-1} = d \sin \vartheta, \quad x_N = d \cos \vartheta, \quad |\vartheta| < \pi/2 - \varepsilon_0, \quad \text{where } d \leq d_0,$$

and there is a cone $K \supset G_0$ such that $0 < h < \tan \varepsilon_0$ (see the proof of the barrier function Lemma). From (7.3.6) it follows

$$(7.3.17) \qquad w\Big|_{r=d} \geq d^{1+\gamma}(\sin \varepsilon_0)^{\gamma-1}(\sin^2 \varepsilon_0 - h^2 \cos^2 \varepsilon_0) > 0.$$

On the other hand, by Theorem 7.3 we have $|u(x)| \leq M_\alpha |x|^\alpha$, where $\alpha \in (0,1)$ is determined by ν^{-1}, μ, N and the domain G, but M_α is determined by the same values and M_0, k_1, β, d_0. Therefore, by the well-known inequality $e^t - 1 \leq 2t$, if $0 < t < 1$, we have

$$(7.3.18) \qquad v(x)\Big|_{r=d} \leq -1 + \exp(\nu^{-1}\mu_1 M_\alpha d^\alpha) \leq 2\nu^{-1}\mu_1 M_\alpha d^\alpha,$$

if d is so small that

$$(7.3.19) \qquad d \leq (2\mu_1 M_\alpha \nu^{-1})^{-1/\alpha}$$

holds. Choosing a number $\mathcal{A}$ so large that the following inequality

$$(7.3.20) \qquad \mathcal{A} \geq 2\nu^{-1}\mu_1 M_\alpha d^{\alpha-1-\gamma}(\sin \varepsilon_0)^{1-\gamma}(\sin^2 \varepsilon_0 - h^2 \cos^2 \varepsilon_0)^{-1}$$

would be satisfied, from (7.3.17) and (7.3.18) it follows that

$$(7.3.21) \qquad \mathcal{A}w(x) \geq v(x), \quad x \in \Omega_d.$$

Thus, if $d \in (0, d_0)$ is chosen according to (7.3.12), (7.3.19), the number γ is chosen according to (7.3.11), and $\mathcal{A}$ is chosen according to (7.3.14), (7.3.20), then from (7.3.15), (7.3.16), (7.3.21) we obtain

$$\widetilde{L}v(x) \geq \widetilde{L}(\mathcal{A}w(x)), \ x \in G_0^d \quad \text{and} \quad v(x) \leq \mathcal{A}w(x), \quad x \in \partial G_0^d.$$

In this case and because of the Comparison Principle (see Theorem 4.4), we have $v(x) \leq \mathcal{A}w(x), \ x \in \overline{G_0^d}$. Returning to the function $u(x)$, from (7.3.9) we obtain

$$u(x) = \nu\mu_1^{-1} \ln(1 + v(x)) \leq \nu\mu_1^{-1} \ln(1 + \mathcal{A}w(x)) \leq \mathcal{A}\nu\mu_1^{-1}w(x), \quad x \in \overline{G_0^d}.$$

In the analogous way the inequality $u(x) \geq -\mathcal{A}\nu\mu_1^{-1}w(x), x \in \overline{G_0^d}$ is proved, if we consider $v(x) = 1 - \exp(-\nu^{-1}\mu_1 u(x))$ as an auxiliary function. By (7.3.5), the proof of our Theorem is complete. □

Basing on the layer method and the assumption (D), we can now prove a gradient bound for solutions near a conical point.

THEOREM 7.17. *Let $u(x)$ be a strong solution of the problem (QL), $q > N$ and the assumptions $(S), (A) - (J)$ on the set $\mathfrak{M}^{(u)}$ are fulfilled. Then in the domain G_0^d, $0 < d \leq \min(d_0, \overline{d})$ the estimate (7.3.2) is true with a constant c_1, depending only on $\nu^{-1}, \mu, \mu_0, \mu_1, \mu_2, u, q, \beta, \gamma, \kappa_1, \kappa_2, M_0$ and the domain G.*

PROOF. Let us consider in the layer $G_{1/2}^1$ the function $v(x') = \rho^{-1-\gamma} u(\rho x')$, taking $u \equiv 0$ outside G. Let us perform the change of variables $x = \rho x'$ in the equation (QL). The function $v(x')$ satisfies the equation

$(QL)'$ $\qquad\qquad a^{ij}(x') v_{x_i' x_j'} = F(x'), \quad x' \in G_{1/2}^1$

where

$$a^{ij}(x') \equiv a_{ij}(\rho x', \rho^{1+\gamma} v(x'), \rho^\gamma v_{x'}(x')),$$
$$F(x') \equiv -\rho^{1-\gamma} a(\rho x', \rho^{1+\gamma} v(x'), \rho^\gamma v_{x'}(x')).$$

By Theorem 7.5 with regard to assumptions $(A) - (D)$

$(7.3.22)$ $\qquad\qquad \operatorname*{vrai\,max}_{G_{1/2}^1} |\nabla' v| \leq M_1',$

where M_1' is determined only by $\nu, \mu, \mu_1, k_1, c_0, M_0, \beta, \gamma, N, q$. Returning to former variables from (7.3.22) we obtain

$$|\nabla u(x)| \leq M_1' \rho^\gamma, \quad x \in G_{\rho/2}^\rho.$$

Taking $|x| = 2\rho/3$, we arrive to the sought estimate (7.3.2). The Theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let us now establish a "weak" solution smoothness of the problem (QL) in the neighborhood of a conical point.

THEOREM 7.18. *Let $u(x)$ be a strong solution of (QL), $q > N$ and the assumptions $(S), (A) - (J)$ are fulfilled. Let γ_0 be the number defined by the barrier function Lemma. Then $u(x) \in G^{1+\gamma}(\overline{G_0^d})$ for some $d \in (0, \min(d_0, \overline{d}))$ and $\forall \gamma \in (0, \gamma^*]$, where $\gamma^* = \min(\gamma_0; \beta + 1; 1 - N/q)$.*

PROOF. Let a number $d \in (0, \min(d_0, \overline{d}))$ be fixed so that the estimates (7.3.1), (7.3.2) are satisfied according to Theorems 7.16, 7.17. Let us consider in the layer $G_{1/2}^1$ the equation of $(QL)'$ for the function $v(x') = \rho^{-1-\gamma} u(\rho x')$. By the Sobolev-Kondrashov imbedding Theorem 1.33

$(7.3.23)$ $\qquad \sup_{\substack{x', y' \in G_{1/2}^1 \\ x' \neq y'}} \dfrac{|\nabla' v(x') - \nabla' v(y')|}{|x' - y'|^{1 - N/q}} \leq c(N, q, G) \|v\|_{2, q; G_{1/2}^1}, \quad q > N.$

Let us verify that for the solution $v(x')$ we can apply Theorem 4.6 about L^q-estimate inside a domain and near a smooth boundary portion. In fact, by the assumption $(A) - (J)$ and the imbedding theorem, the functions $a_{ij}(x, u, z)$ are continuous on the set $\mathfrak{M}^{(u)}$, that is for $\forall \varepsilon > 0$ there exists such $\eta(\varepsilon)$ that

$$|a_{ij}(x, u(x), u_x(x)) - a_{ij}(y, u(y), u_x(y))| < \varepsilon,$$

as soon as

$$|x - y| + |u(x) - u(y)| + |u_x(x) - u_x(y)| < \eta(\varepsilon), \quad \forall x, y \in G^\rho_{\rho/2}, \rho \in (0, \overline{d}).$$

The assumption (D) guarantees the existence of the local *a priori* estimate inside the domain $G^\rho_{\rho/2}$ and near a smooth portion of the boundary $\Gamma^\rho_{\rho/2}$, namely there exist the number $\widetilde{\varkappa} > 0$ and the number $M_1 > 0$ such that

$$|u(x) - u(y)| + |\nabla u(x) - \nabla u(y)| \le M_1 |x - y|^{\widetilde{\varkappa}}, \quad \forall x, y \in G^\rho_{\rho/2}, \ \rho \in (0, d).$$

Then the functions $a^{ij}(x')$ are continuous in $\overline{G^1_{1/2}}$ and consequently are uniform continuous. It means that for any $\varepsilon > 0$ there exists $\delta > 0$ (we choose the number δ such that $\delta d + M_1(\delta d)^{\widetilde{\varkappa}} < \eta$) such that $|a^{ij}(x') - a^{ij}(y')| < \varepsilon$, if only $|x' - y'| < \delta$, $\forall x', y' \in \overline{G^1_{1/2}}$. We see that the assumptions of the theorem about the local L^q *a priori* estimate for the $(QL)'$ are satisfied. By this theorem, we have

$$(7.3.24) \quad \|v\|^q_{2,q:G^1_{1/2}} \le c_4 \int\limits_{G^2_{1/4}} \left(|v|^q + \rho^{q(1-\gamma)} |a(\rho x', \rho^{1+\gamma} v, \rho^\gamma u_{x'})|^q \right) dx'$$

with a constant c_4, independent of v and a, and being determined only by $N, \nu, \mu, \mu_1, \gamma, \beta, k_1, q, M_0, M_1, d_0, \overline{d}$. The estimate (7.3.1) gives rise to

$$(7.3.25) \quad \int\limits_{G^2_{1/4}} |v|^q dx' = \int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{-q(1+\gamma)} |u(x)|^q \varrho^{-N} dx \le c_{q,\gamma} mes\Omega \int\limits_{\rho/4}^{2\rho} \frac{dr}{r} \le$$

$$\le c_{q,\gamma} mes\Omega \ln 8.$$

In the analogous way, by the assumption (C) together with the inequality (7.3.3) and the estimate (7.3.2), we obtain

$$\int_{G^2_{1/4}} \rho^{q(1-\gamma)} |a(\rho x', \rho^{1+\gamma}v, \rho^\gamma v_{x'})|^q dx' \leq \rho^{q(1-\gamma)-N} \int_{G^{2\rho}_{\rho/4}} (\mu_1 |\nabla u|^2 +$$

$$+b(x)|\nabla u| + f(x))^q dx \leq 2^N 3^{q-1} \rho^{q(1-\gamma)} mes\Omega \int_{G_{\rho/4}}^{2\rho} (\mu_1^q c_1^{2q} r^{2q\gamma-1} +$$

$$(7.3.26) \qquad +(k_1 c_1)^q r^{q(\beta+\gamma)-1} + k_1^q r^{q\beta-1}) dr \leq c(N, q, \gamma, \beta, \mu_1, c_1, k_1),$$

since $0 < \gamma \leq 1 + \beta$. From(7.3.24)-(7.3.26) it follows

$$(7.3.27) \qquad \|v\|_{2,q;G^1_{1/2}} \leq c(N, \nu, \mu, \mu_1, \gamma, \beta, k_1, q, M_0, M_1, c_0, c_1).$$

Now from (7.3.23) and (7.3.27) we obtain

$$(7.3.28) \qquad \sup_{\substack{x',y'\in G^1_{1/2} \\ x'\neq y'}} \frac{|\nabla' v(x') - \nabla' v(y')|}{|x'-y'|^{1-N/q}} \leq c_5, \quad q > N,$$

where $c_5 = c(N, \nu, \mu, \mu_1, \gamma, \beta, k_1, q, M_0, M_1, c_0, c_1, G)$.

Returning to the variables x, u, we get

$$(7.3.29) \qquad \sup_{\substack{x,y\in G^\rho_{\rho/2} \\ x\neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{1-N/q}} \leq c_5 \rho^{\gamma-1+N/q}, \quad q > N, \rho \in (0,d).$$

Now let us recall that by the assumptions of our theorem $q \geq N/(1-\gamma)$. Let us put $\tau = \gamma - 1 + N/q \leq 0$, then from (7.3.29) it follows

$$|\nabla u(x) - \nabla u(y)| \leq c_5 \rho^\tau |x-y|^{\gamma-\tau} \quad \forall x, y \in G^\rho_{\rho/2}, \rho \in (0,d).$$

By the definition of the set $G^\rho_{\rho/2}$, $|x-y| \leq 2\rho$ and consequently $|x-y|^\tau \geq (2\rho)^\tau$, since $\tau \leq 0$. Therefore

$$(7.3.30) \qquad \sup_{\substack{x,y\in G^\rho_{\rho/2} \\ x\neq y}} \frac{|\nabla v(x) - \nabla v(y)|}{|x-y|^\gamma} \leq 2^{-\gamma} c_5, \quad \rho \in (0,d).$$

Now, let $x, y \in \overline{G^d_0}$ and $\forall \rho \in (0, d)$. If $x, y \in G^\rho_{\rho/2}$, then (7.3.30) is fulfilled. If $|x - y| > \rho = |x|$, then, by the estimate (7.3.2), we have

$$\frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\gamma} \leq 2\rho^{-\gamma} |\nabla u(x)| \leq 2c_1.$$

From here and (7.3.30) we conclude that

$$\sup_{\substack{x,y \in G_0^d \\ x \neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\gamma} \leq const.$$

This inequality together with the estimates (7.3.1), (7.3.2) means $u(x) \in C^{1+\gamma}(\overline{G_0^d})$. Our theorem is proved. $\qquad\square$

7.3.4. Estimates in weighted spaces. On the basis of the estimates of §7.3.3 let us now derive the weighted integral estimates of the weak second order derivatives of strong solutions and establish the best-possible weighted exponent. For the simplicity, we take $\varphi(x) \equiv 0$.

THEOREM 7.19. *Let $u(x)$ be a strong solution of the problem (QL), $q > N$ and the assumptions $(S), (A) - (J)$ on the set $\mathfrak{M}^{(u)}$ are fulfilled. In addition, suppose*

$$a_{ij}(0,0,0) = \delta_i^j, \quad (i,j = 1..N).$$

Then there exist positive numbers d, c_2, independent of $u(x)$ such that if $b(x), f(x) \in V_{2,\alpha}^0(G)$ and

$$(7.3.31) \qquad\qquad 4 - N - 2\lambda < \alpha \leq 2,$$

then $u(x) \in V_{2,\alpha}^2(G_0^{d/2})$ and the estimate

$$(7.3.32) \quad \int_{G_0^{d/2}} (r^\alpha u_{xx}^2 + r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)dx \leq c_2 \int_{G_0^{2/d}} \Big(u^2 + |\nabla u|^2 + $$

$$+ r^\alpha(b^2(x) + f^2(x)) \Big)dx,$$

is true, where d and c_2 are defined by the values $N, \nu, \mu, \mu_1, \gamma, \beta, k_1, q, d_0, \overline{d}, M_0, M_1, \lambda, \alpha$ and the domain G.

PROOF. **1.** $2 - N \leq a \leq 2$.

In this case, by the estimates (7.3.1), (7.3.2),

$$(7.3.33) \qquad \int_{G_0^d} (r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)dx \leq c(\alpha, N, \gamma)d^{\alpha+N-2+2\gamma}.$$

Now we obtain the weighted estimate of the second order weak derivatives of the solution in the following manner. Let us fix $d \in (0, \min(\overline{d}, d_0)]$ and consider the sets $G^{(k)}, k = 0, 1, 2, \dots$. Let us perform the change of variables

$$x = (2^{-k}d)x', u((2^{-k}d)x') = v(x')$$

in the equation of the problem (QL). As a result the domain $G^{(k)}$ of the space $(x_1, \ldots, x_N)$ transforms at the domain $G^1_{1/2}$ of the space $(x'_1, \ldots, x'_N)$, and the equation takes the form

$$a^{ij}(x')v_{x'_i x'_j} = F(x'), \quad a^{ij}(x') \equiv a_{ij}((2^{-k}d)x', v(x'), d^{-1}2^k v_{x'}),$$
$$F(x') \equiv -(2^{-k}d)^2 a((2^{-k}d)x', v(x'), d^{-1}2^k v_{x'}).$$

To its solution let us apply the L^2-estimate inside the domain and near a smooth boundary portion. (To us the possibility this estimate is substantiated under the proof of Theorem 7.18. See the inequality (7.3.24).)

$$(7.3.34) \qquad \int\limits_{G^1_{1/2}} v^2_{x'x'}dx' \le c_4 \int\limits_{G^2_{1/4}} \left(v^2(x') + F^2(x')\right) dx'$$

where the constant c_4 is independent of v and F and is determined only by the quantities pointed in (7.3.24). In the inequality (7.3.34) we return to former variables and taking into account the definition of the sets $G^{(k)}$ we arrive at

$$(7.3.35) \quad \int\limits_{G^{(k)}} r^\alpha u^2_{xx}dx \le c_4 \int\limits_{G^{(k-1)}\cup G^{(k)}\cup G^{(k+1)}} \left(r^{\alpha-4}u^2 + r^\alpha a^2(x,u,u_x)\right) dx.$$

We sum the inequalities(7.3.35) over $k = 0, 1, \ldots, [\log_2(d/\varepsilon)] \quad \forall \varepsilon \in (0,d)$ and we get

$$(7.3.36) \qquad \int\limits_{G^d_\varepsilon} r^\alpha u^2_{xx}dx \le c_4 \int\limits_{G^{2d}_{\varepsilon/4}} \left(r^{\alpha-4}u^2 + r^\alpha a^2(x,u,u_x)\right) dx.$$

Taking into consideration the finiteness of the integral (7.3.33) and because of the assumption (C) and the estimate (7.3.2) from (7.3.36) it follows that

$$(7.3.37) \quad \int\limits_{G^d_\varepsilon} r^\alpha u^2_{xx}dx \le c_4 c(\gamma, d, c_1) \int\limits_{G^{2d}_0} \Big(r^{\alpha-4}u^2 + r^\alpha f^2(x) + r^\alpha b^2(x) +$$
$$+ r^{\alpha-2}|\nabla u|^2\Big)dx \quad \forall \varepsilon > 0,$$

where c_4 is independent of ε. Therefore, by the Fatou Theorem, in (7.3.37) one can perform the passage to the limit over $\varepsilon \to +0$ and as a result we

get

$$(7.3.38) \quad \int\limits_{G_0^d} r^\alpha u_{xx}^2 dx \le$$

$$\le c_4 \int\limits_{G_0^{2d}} \left(r^{\alpha-4}u^2 + r^\alpha f^2(x) + r^\alpha b^2(x) + r^{\alpha-2}|\nabla u|^2 \right) dx.$$

The inequality (7.3.38) together with (7.3.33) means that $u(x) \in V_{2,\alpha}^2(G_0^d)$.

We are coming now to the derivation of the estimate (7.3.32).

Let $\zeta(r) \in C^2[0,d]$ be the cut-off function on the segment $[0,d]$

$$\begin{cases} \zeta(r) \equiv 1, & \text{if } r \in [0,d/2], \\ 0 \le \zeta(r) \le 1, & \text{if } r \in [d/2,d], \\ \zeta \equiv 0, & \text{if } r \ge d, \\ \zeta(d) = \zeta'(d) = 0. \end{cases}$$

We multiply both parts of the problem (QL) equation by $\zeta^2(r)r^{\alpha-2}u(x)$ and integrate over the domain G_0^d. Taking into account the assumption $a_{ij}(0,0,0) = \delta_i^j$, twice integrating by parts we obtain

$$\int\limits_{G_0^d} \zeta^2(r)r^{\alpha-2}|\nabla u|^2 dx + \frac{2-\alpha}{2}(N+\alpha-4) \int\limits_{G_0^d} \zeta^2(r)r^{\alpha-4}u^2(x)dx =$$

$$(7.3.39) \quad = \int\limits_{G_{d/2}^d} ((N+2\alpha-5)\zeta\zeta'r^{\alpha-3} + \zeta\zeta''r^{\alpha-2} + \zeta'^2 r^{\alpha-2})u^2(x)dx +$$

$$+ \int\limits_{G_0^d} \zeta^2(r)r^{\alpha-2}u(x)\Big(\{a_{ij}(x,u,u_x) - a_{ij}(0,0,0)\}u_{x_i x_j} + a(x,u,u_x) \Big)dx.$$

From the assumptions $(A),(D),(J)$, by the Sobolev Imbedding Theorem, it follows that $a_{ij}(x,u,z)$ are continuous at any point $(x,u,z) \in \mathfrak{M}^{(u)}$ $(i,j = 1,\ldots,N)$ and in particular at the point $(0,0,0)$. This means that for $\forall \delta > 0$ there exists $d_\delta > 0$ such that

$$(7.3.40) \quad |a_{ij}(x,u(x),u_x(x)) - a_{ij}(0,0,0)| < \delta$$

as soon as

$$(7.3.41) \quad |x| + |u(x)| + |\nabla u(x)| < d_\delta.$$

By the estimates (7.3.1), (7.3.2),

$$(7.3.42) \quad |x| + |u(x)| + |\nabla u(x)| \le d + c_0 d^{1+\gamma} + c_1 d^\gamma \quad \forall x \in G_0^d.$$

Let us now choose $d > 0$, maybe more smaller than before such that the inequality

$$(7.3.43) \qquad d + c_0 d^{1+\gamma} + c_1 d^\gamma \leq d_\delta$$

would be fulfilled. Then the inequality (7.3.40) is fulfilled and therefore, by the Cauchy inequality and the (7.3.38), we get

$$(7.3.44) \quad \int_{G_0^d} \zeta^2(r) r^{\alpha-2} \{a_{ij}(x, u, u_x) - a_{ij}(0,0,0)\} u u_{x_i x_j} dx \leq \delta \int_{G_0^d} \zeta^2(r) \times$$

$$\times r^{\alpha-2} |u| |u_{xx}| dx \leq \frac{\delta}{2} \int_{G_0^d} (r^\alpha u_{xx}^2 + r^{\alpha-4} u^2) dx \leq$$

$$\leq \frac{\delta}{2} (1 + c_4) \int_{G_0^{2d}} (r^{\alpha-4} u^2 + r^\alpha f^2(x) + r^\alpha b^2(x) + r^{\alpha-2} |\nabla u|^2) dx \quad \forall \delta > 0.$$

Further, by the assumption (C), the estimates (7.3.1), (7.3.2) and the Cauchy inequality, we have

$$\int_{G_0^d} \zeta^2(r) r^{\alpha-2} u(x) a(x, u, u_x) dx \leq \mu_1 c_0 d^{1+\gamma} \int_{G_0^d} \zeta^2(r) r^{\alpha-2} |\nabla u|^2 dx +$$

$$+ \frac{1}{2} (c_1 d^\gamma + \delta) \int_{G_0^d} \zeta^2(r) r^{\alpha-4} u^2(x) dx + \frac{1}{2} c_1 d^\gamma \int_{G_0^d} \zeta^2(r) r^\alpha b^2(x) dx +$$

$$(7.3.45) \qquad + \frac{1}{2\delta} \int_{G_0^d} \zeta^2(r) r^\alpha f^2(x) dx, \quad \forall \delta > 0.$$

From (7.3.39), (7.3.44), (7.3.45) it follows that

$$\int_{G_0^d} \zeta^2(r) r^{\alpha-2} |\nabla u|^2 dx + \frac{2 - \alpha}{2} (N + \alpha - 4) \int_{G_0^d} \zeta^2(r) r^{\alpha-4} u^2(x) dx \leq$$

$$\leq c_\delta (\delta + d^\gamma) \int_{G_0^{2d}} (r^{\alpha-2} |\nabla u|^2 + r^{\alpha-4} u^2) dx + c_8 \int_{G_0^{2d}} r^\alpha (b^2 + f^2) dx +$$

$$(7.3.46) \qquad + c_7 \int_{G_{d/2}^{2d}} (|\nabla u|^2 + u^2) dx \quad \forall \delta > 0, \quad \text{where}$$

$$c_6 = c(\mu_1, c_0, c_1, c_4), c_7 = c(\mu_1, c_0, c_1, c_4, N, \alpha, \gamma, d), \quad c_8 = c(\delta, \gamma, c_1, c_4, d).$$

If $N + \alpha - 4 \leq 0$, then let us also use the inequality (2.5.3). As a result we have

$$(7.3.47) \quad C(\lambda, N, \alpha) \int\limits_{G_0^{d/2}} r^{\alpha-2}|\nabla u|^2 dx \leq c_9(\delta + d^\gamma) \int\limits_{G_0^{2d}} r^{\alpha-2}|\nabla u|^2 dx +$$

$$+ c_{10} \int\limits_{G_0^{2d}} (|\nabla u|^2 + u^2 + r^\alpha(b^2 + f^2)) dx \quad \forall \delta > 0,$$

where

$$C(\lambda, N, \alpha) = 1 - \frac{2-\alpha}{2}(4 - N - \alpha)H(\lambda, \alpha, N) > 0 \quad \text{(by (7.3.31))},$$

$$c_9 = c(\mu_1, c_0, c_1, c_4, N, \alpha, \lambda), \quad c_{10} = c(\mu_1, c_0, c_1, c_4, N, \alpha, \gamma, d, \delta).$$

Now we choose the numbers δ and d such that

$$(7.3.48) \qquad\qquad \delta = \frac{1}{4}c_9^{-1}C(\lambda, N, \alpha),$$

$$(7.3.49) \qquad\qquad c_9 d^\gamma \leq \frac{1}{4}C(\lambda, N, \alpha)$$

Then from (7.3.47) we finally obtain the inequality

$$(7.3.50) \quad \int\limits_{G_0^{d/2}} r^{\alpha-2}|\nabla u|^2 dx \leq \frac{2c_{10}}{C(\lambda, \alpha, N)} \int\limits_{G_0^{2d}} (|\nabla u|^2 + u^2 + r^\alpha(b^2 + f^2)) dx,$$

being true only for a $d \in (0, \min d_0, \overline{d})$ such that (7.3.49) and (7.3.43) are fulfilled with d_δ being determined by the continuity of $a_{ij}(x, u, z)$ at $(0, 0, 0)$ for a δ from the equality (7.3.48). The inequality (7.3.50) together with (7.3.38) and (2.5.3) leads us to the desired (7.3.32).

2. $4 - N - 2\lambda < \alpha < 2 - N$.

By the assumption (J), we have $b(x), f(x) \in \mathring{W}_{2-N}^0(G_0^d)$, consequently, $u(x) \in \mathring{W}_{2-N}^2(G_0^{d/2})$, that is

$$(7.3.51) \qquad \int\limits_{G_0^{d/2}} (r^{2-N}u_{xx}^2 + r^{-N}|\nabla u|^2 + r^{-N-2}u^2) dx < \infty,$$

which was proved in the case 1).

Now we use the function $r_\varepsilon(x)$ defined in §1.4. We consider again the inequality (7.3.34). We multiply both parts of this inequality by

$(2^{-k}d + \varepsilon)^{\alpha-2}$ with any $\varepsilon > 0$ and take into account that in $G^{(k)}$ we have $2^{-k-1}d + \varepsilon < r + \varepsilon < 2^{-k}d + \varepsilon$. Then returning to former variables we get

$$\int\limits_{G^{(k)}} r^2(r + \varepsilon)^{\alpha-2}u_{xx}^2 dx \le c_4 \int\limits_{G^{(k-1)}\cup G^{(k)}\cup G^{(k+1)}} \Big(r^{-2}(r + \varepsilon)^{\alpha-2}u^2 +$$
$$+ (r + \varepsilon)^{\alpha}a^2(x, u, u_x)\Big)dx.$$

Hence, by the Corollary 1.12, it follows that

$$\int\limits_{G^{(k)}} r^2 r_{\varepsilon}^{\alpha-2}u_{xx}^2 dx \le c_4 \int\limits_{G^{(k-1)}\cup G^{(k)}\cup G^{(k+1)}} (r^{-2}r_{\varepsilon}^{\alpha-2}u^2 + r_{\varepsilon}^{\alpha}a^2(x, u, u_x))dx.$$

Summing this inequalities over all $k = 0, 1, 2...$, we obtain

$$(7.3.52) \qquad \int\limits_{G_0^d} r^2 r_{\varepsilon}^{\alpha-2}u_{xx}^2 dx \le c_4 \int\limits_{G_0^{2d}} (r^{-2}r_{\varepsilon}^{\alpha-2}u^2 + r_{\varepsilon}^{\alpha}a^2(x, u, u_x))dx.$$

Let us now multiply both parts of the problem (QL) equation by $\zeta^2(r)r_{\varepsilon}^{\alpha-2}u(x)$ and integrate over G_0^d; twice having applied the formula of integration by parts. As a result we have

$$\int\limits_{G_0^d} \zeta^2(r)r_{\varepsilon}^{\alpha-2}|\nabla u|^2 dx = \frac{2-\alpha}{2}(4 - N - \alpha)\int\limits_{G_0^d} \zeta^2(r)r_{\varepsilon}^{\alpha-4}u^2(x)dx +$$

$$+ \int\limits_{G_{d/2}^d} u^2(x)(2(\alpha - 2)\zeta\zeta'(x_i - \varepsilon l_i)\frac{x_i}{r}r_{\varepsilon}^{\alpha-4} + N\zeta\zeta'r^{-1}r_{\varepsilon}^{\alpha-2} + \zeta'^2 r_{\varepsilon}^{\alpha-2} +$$

$$+ \zeta\zeta''r_{\varepsilon}^{\alpha-2})dx + \int\limits_{G_0^d} \zeta^2(r)r_{\varepsilon}^{\alpha-2}u(x)\Big((a_{ij}(x, u, u_x) -$$

$$(7.3.53) \qquad - a_{ij}(0, 0, 0))u_{x_i x_j} + a(x, u, u_x)\Big)dx.$$

Let $d \in (0, \min(\bar{d}, d_0)]$ be so small that (7.3.43) is fulfilled, and consequently (7.3.40) is fulfilled too. Then, by the Cauchy inequality,

$$\int_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} (a_{ij}(x, u, u_x) - a_{ij}(0, 0, 0)) u u_{x_i x_j} dx \leq$$

$$(7.3.54) \qquad \leq \delta \int_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} (r |u_{xx}|)(r^{-1}|u|) dx \leq$$

$$\leq \frac{\delta}{2} \int_{G_0^d} \left(\zeta^2(r) r^2 r_\varepsilon^{\alpha-2} u_{xx}^2 + \zeta^2(r) r^{-2} r_\varepsilon^{\alpha-2} u^2 \right) dx, \quad \forall \delta > 0.$$

Similarly, by the assumption (C) in view of the estimates (7.3.1), (7.3.2),

$$(7.3.55) \qquad \int_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} u(x) a(x, u, u_x) dx \leq$$

$$\leq c_{10} \mu_1 d^{1+\gamma} \int_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} |\nabla u|^2 dx + \frac{1}{2}(c_1 d^\gamma + \delta) \int_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-4} u^2 dx +$$

$$+ \frac{1}{2} c_1 d^\gamma \int_{G_0^d} \zeta^2(r) r_\varepsilon^\alpha b^2(x) dx + \frac{1}{2\delta} \int_{G_0^d} \zeta^2(r) r_\varepsilon^\alpha f^2(x) dx, \quad \forall \delta > 0.$$

From (7.3.52) - (7.3.55) with regard to the properties of $r_\varepsilon(x)$ (see §1.4) and $\zeta(r)$ it follows

$$\int_{G_0^{d/2}} r_\varepsilon^{\alpha-2} |\nabla u|^2 dx \leq c_{12}(\mu_1, c_1, c_4, \gamma, d, \delta) \int_{G_0^{2d}} r_\varepsilon^\alpha(x)(b^2 + f^2) dx +$$

$$(7.3.56) \qquad + \frac{1}{2} \left(\delta + c_1 d^\gamma + (2-\alpha)(4-N-\alpha) \right) \int_{G_0^{d/2}} r_\varepsilon^{\alpha-4} u^2 dx +$$

$$+ c_{11}(\mu_1, c_0, c_1, c_4, \gamma) d^{2\gamma} (d+\varepsilon)^2 \int_{G_0^{d/2}} r_\varepsilon^{\alpha-2} |\nabla u|^2 dx +$$

$$+ \frac{\delta}{2}(1 + c_4) \int_{G_0^{2d}} r^{-2} r_\varepsilon^{\alpha-2} u^2 dx.$$

The second integral on the right we estimate with the aid of (2.5.8), but for the bound of the latter integral on the right we use Lemma 2.32. As a

result from (7.3.56) we obtain

$$C(\lambda, N, \alpha) \int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2} |\nabla u|^2 dx \leq \left(\frac{1}{2}(\delta + c_1 d^\gamma) H(\lambda, \alpha, N) + \right.$$

$$(7.3.57) \qquad + c_{11} d^{2\gamma+2} + \frac{\delta(1+c_4)3^{2-\alpha}}{2\lambda(\lambda+N-2)}\right) \int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2} |\nabla u|^2 dx +$$

$$+ c_{14} \int\limits_{G_0^{2d}} (u^2 + |\nabla u|^2 + r^\alpha f^2(x) + r^\alpha b^2(x)) dx, \quad \forall \delta > 0$$

with $C(N, \lambda, \alpha)$ being the same as in (7.3.47). Let us now choose δ and d as the following

$$(7.3.58) \qquad \delta = \frac{1}{2} C(\lambda, \alpha, N) \left(H(\lambda, \alpha, N) + \frac{1+c_4}{\lambda(\lambda+N-2)} \right)^{-1},$$

$$(7.3.59) \qquad \frac{1}{2} c_1 d^\gamma H(\lambda, \alpha, N) + c_{11} d^{2\gamma+2} \leq \frac{1}{4} C(\lambda, \alpha, N).$$

Then from (7.3.57) it follows

$$(7.3.60) \qquad \int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2} |\nabla u|^2 dx \leq \frac{2c_{14}}{C(\lambda, \alpha, N)} \int\limits_{G_0^{2d}} (u^2 + |\nabla u|^2 + r^\alpha f^2(x) +$$

$$+ r^\alpha b^2(x)) dx, \quad \forall \varepsilon > 0.$$

Finally, from (7.3.52) and (2.5.8) with regard to the assumption (C) and the estimates (7.3.1), (7.3.2), because of (7.3.60), we have

$$(7.3.61) \qquad \int\limits_{G_0^{d/2}} (r^2 r_\varepsilon^{\alpha-2} u_{xx}^2 + r_\varepsilon^{\alpha-2} |\nabla u|^2 + r_\varepsilon^{\alpha-4} u^2) dx \leq$$

$$\leq c_2 \int\limits_{G_0^{2d}} (u^2 + |\nabla u|^2 + r^\alpha f^2(x) + r^\alpha b^2(x)) dx$$

for any $\varepsilon > 0$, where $c_2 = c(N, \lambda, \alpha, \nu, \mu, \mu_1, \gamma, \beta, k_1, q, M_0, d_0, \overline{d})$ and independent of ε. The inequality (7.3.61) holds for a $d \in (0, \min(d_0, \overline{d})]$ for which (7.3.59) and (7.3.43) are fulfilled with d_δ, being determined by the continuity of $a_{ij}(x, u, z)$ at $(0, 0, 0)$ for δ, being assigned by (7.3.58). Taking the limit $\varepsilon \to +0$ in the inequality (7.3.61), we get the desired estimate (7.3.32) by the Fatou Theorem. □

REMARK 7.20. By the continuity of the equation leading coefficients at the point $(0,0,0)$ and in virtue of estimates (7.3.1) and (7.3.2), the condition $a_{ij}(0,0,0) = \delta_i^j$, $(i,j = 1, \ldots, N)$ of our theorem is not implied to be restrictive. In fact, there exists the orthogonal transformation of coordinates, which transforms an elliptic equation with leading coefficients which are frozen at the point to canonical form. Main part of this canonical form is Laplacian.

THEOREM 7.21. *Let $u(x)$ be a strong solution of the problem (QL), $q > N$ and the hypotheses of Theorem 7.19 are satisfied. In addition, suppose that $\beta > \lambda - 2$. Then there exist positive numbers d and c_{15} independent of $u(x)$ and being defined only by the quantities from hypotheses $(B) - (J)$ and by G such that $u(x) \in \overset{\circ}{W}{}^2_{4-N}(G_0^{d/2})$ and the inequality*

$$(7.3.62) \qquad |u|_{\overset{\circ}{W}{}^2_{4-N}(G_0^\rho)} \leq c_{15}\rho^\lambda, \quad \rho \in (0, \frac{d}{2})$$

holds.

PROOF. The belonging of $u(x)$ to $W^2_{4-N}(G_0^{d/2})$ follows from Theorem 7.19, therefore it is required to prove only the estimate (7.3.62). We set

$$(7.3.63) \qquad U(\rho) \equiv \int\limits_{G_0^\rho} r^{2-N} |\nabla u|^2 dx.$$

Let us multiply both parts of the (QL) equation by $r^{2-N}u(x)$ and integrate over the domain G_0^ρ, $\rho \in (0, \frac{d}{2})$

$$(7.3.64) \quad U(\rho) = \int\limits_{\Omega} \left(\rho u(\rho,\omega)\frac{\partial u}{\partial r}\Big|_{r=\rho} + \frac{N-2}{2}u^2(\rho,\omega)\right)d\omega +$$

$$+ \int\limits_{G_0^\rho} u(x) r^{2-N}\Big(\{a_{ij}(x,u,u_x) - a_{ij}(0,0,0)\}u_{x_i x_j} + a(x,u,u_x)\Big)dx.$$

Let us use an upper estimate for every integral on the right. The first integral is estimated by Corollary 2.29. By the assumption (C) and the Cauchy inequality with
$\delta = \rho^\varepsilon$, $\forall \varepsilon > 0$ with regard to (7.3.1) and (7.3.2), we have

$$(7.3.65) \quad \int\limits_{G_0^\rho} r^{2-N}u(x)a(x,u,u_x)dx \leq \mu_1 c_0 \rho^{1+\gamma}U(\rho) + \frac{1}{2}c_1\rho^\gamma \int\limits_{G_0^\rho}\Big(r^{-N}u^2 +$$

$$+ r^{4-N}b^2(x)\Big)dx + \frac{1}{2}\int\limits_{G_0^\rho}\Big(\rho^\varepsilon r^{-N}u^2 + \rho^{-\varepsilon}r^{4-N}f^2(x)\Big)dx.$$

Let us also apply the inequality (2.5.3) with $\alpha = 4 - N$ and also (7.3.3). Thusly

$$(7.3.66) \quad \int_{G_0^\rho} r^{2-N} u(x) a(x, u, u_x) dx \leq \left(\mu_1 c_0 \rho^{1+\gamma} + \frac{1}{2} H(\lambda, N, 4 - N) \times \right.$$

$$\times (\rho^\varepsilon + c_1 \rho^\gamma) \Big) U(\rho) + (4\lambda)^{-1} (1 + c_1) k_1^2 \mathrm{meas} \Omega \rho^{2s-\varepsilon}, \quad \forall \varepsilon > 0, \ s = \beta + 2 > \lambda.$$

Further

$$a_{ij}(x, u, z) - a_{ij}(0, 0, 0) = (a_{ij}(0, 0, z) - a_{ij}(0, 0, 0)) +$$

$$+ (a_{ij}(x, u, z) - a_{ij}(0, 0, z)).$$

From the assumption (J), by the Sobolev imbedding Theorem, taking into account the estimates (7.3.1) and (7.3.2), we have

$$\left(\sum_{i,j=1}^N |a_{ij}(x, u(x), u_x(x)) - a_{ij}(0, 0, 0)|^2 \right)^{1/2} \leq \delta(\rho), \ |x| < \rho$$

$$\delta(\rho) = c(N, q, c_0, c_1, \gamma, d) \rho^\gamma, \quad \rho \in (0, \frac{d}{2}),$$

where $0 < \gamma \leq \gamma^* = \min(\gamma_0; 1 + \beta; 1 - \frac{N}{q})$.

Therefore applying the Cauchy inequality, the (7.3.38), the inequality (2.5.3) with $\alpha = 4 - N$, and the condition (7.3.3) we get

$$\int_{G_0^\rho} r^{2-N} u(x) (a_{ij}(x, u, u_x) - a_{ij}(0, 0, 0)) u_{x_i x_j} dx \leq$$

$$\leq \frac{1}{2} \delta(\rho) \int_{G_0^\rho} (r^{4-N} u_{xx}^2 + r^{-N} u^2) dx \leq \frac{1}{2} H(\lambda, N, 4 - N) \delta(\rho) U(\rho) +$$

$$(7.3.67)$$

$$+ \frac{c_4}{2} (H(\lambda, N, 4 - N) + 1) \delta(\rho) U(2\rho) + \frac{k_1^2 c_4}{4\lambda} \mathrm{meas} \Omega \delta(\rho) (2\rho)^{2s}.$$

From (7.3.64) basing upon Corollary 2.29, (7.3.66)-(7.3.67) we conclude that $U(\rho)$ satisfies the inequality for the Cauchy problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} - c\left(\varrho^{\gamma-1} + \varrho^{\varepsilon-1}\right); \quad \mathcal{N}(\varrho) = c\varrho^{\gamma-1};$$

$$\mathcal{Q}(\varrho) = ck_1^2\left(\varrho^{2s+\gamma-1} + \varrho^{2s-\varepsilon-1}\right), \ s > \lambda, \forall \varepsilon \in (0, 2(s-\lambda)) \text{ and}$$

$$V_0 = \int\limits_{G_0^d} r^{4-N}|\nabla u|^2 dx \leq \frac{c_1^2 \text{meas}\Omega}{2(2+\gamma)} d^{2(\gamma+2)}.$$

According to the Theorem 1.57 the estimate (1.10.1) holds, which leads to the estimate $U(\rho) \leq c\rho^{2\lambda}$. (See the proof of Theorem 4.18 in case 1).) This estimate together with (7.3.37) and (2.5.3) gives the desired estimate (7.3.62). $\square$

7.3.5. L^p and pointwise estimates of the solution and its gradient.
Let us make precise the exponent γ (in the estimates (7.3.1) and (7.3.2)) and the Hölder exponent for the first order weak derivatives of the strong solution in the neighborhood of conical point $\mathcal{O}$. We recall that $\varphi(x) \equiv 0$.

THEOREM 7.22. *Let $u(x)$ be a strong solution of the problem (QL), $q > N$ and it is known the value $M_0 = \max\limits_{x \in \overline{G}}|u(x)|$. Let the assumptions (S), $(A) - (J)$ be fulfilled with $\beta > \lambda - 2 > -1$. Then there exist nonnegative numbers $d \leq d^* = \min(d, \overline{d})$ and $\overline{c_0}, \overline{c_1}, \overline{c_2}, \overline{c_3}$, independent of $u(x)$ and being defined only by quantities $N, \lambda, \nu, \mu, \mu_1, \beta, k_1, q, M_0, M_1, d_0, \overline{d}$, and G such that the following assertions hold*

(1) $|u(x)| \leq \overline{c_0}|x|^\lambda$ *and* $|\nabla u(x)| \leq \overline{c_1}|x|^{\lambda-1}$, $x \in G_0^{d/2}$;

(2) $u(x) \in \overset{\circ}{W}_{4-N}^2(G_0^{d/2})$ *and* $\|u\|_{\overset{\circ}{W}_{4-N}^2(G_0^\rho)} \leq \overline{c_2}\rho^\lambda$, $0 < \rho < d/2$;

(3) *if* $\alpha + q(\lambda - 2) + N > 0$, *then* $u(x) \in V_{q,\alpha}^2(G_0^{d/2})$ *and*

$$\|u\|_{V_{q,\alpha}^2(G_0^\rho)} \leq \overline{c_3}\rho^{\lambda-2+\frac{\alpha+N}{q}}, \ 0 < \rho < d/2;$$

(4) *if* $1 < \lambda < 2, q \geq \frac{N}{2-\lambda}$, *then* $u(x) \in C^\lambda(\overline{G_0^{d/2}})$.

PROOF. Assertion 2) is proved in Theorem 7.21. To prove the remaining assertions we consider the sets $G_{\rho/2}^\rho$ and $G_{\rho/4}^{2\rho} \supset G_{\rho/2}^\rho$. Let us perform the transformation of coordinates $x = \rho x'$ in the equation of (QL). The function $v(x') = \rho^{-\lambda}u(\rho x')$ satisfies in $G_{1/4}^2$ the equation $(QL)'$ for $\gamma = \lambda - 1$. For the local boundary L^q-estimate, Theorem 4.6 seems to be applicable to the solution $v(x')$. (For the justification of the possibility of its application

see the proof of Theorem 7.18.)

$$(7.3.68) \quad |v|^q_{2,q;G^1_{1/2}} \le c_4 \int_{G^2_{1/4}} (|v|^q + \rho^{(2-\lambda)q}|a(\rho x', \rho^\lambda v, \rho^{\lambda-1} v_x)|^q) dx',$$

$$\forall q > 1$$

with the constant c_4 independent of v and a.

Let at first $2 \le N < 4$. By the estimate (7.3.62) we have

$$\|v\|^2_{2,2;G^1_{1/2}} \le c(N)\rho^{-2\lambda} \int_{G^\rho_{\rho/2}} (r^{4-N}u^2_{xx} + r^{2-N}|\nabla u|^2 + r^{-N}u^2) dx \le c(N)c^2_{15}$$

Therefore from the Sobolev imbedding theorem it follows

$$\sup_{x' \in G^1_{1/2}} |v(x')| \le c(N,q)\|v\|_{2,2;G^1_{1/2}} \le c(N,q)c_{15} = \overline{c_0} \quad \text{or}$$

$$|u(x)| \le \overline{c_0}\rho^\lambda, x \in G^\rho_{\rho/2}.$$

Putting $|x| = \frac{3}{4}\rho$ hence we obtain the first bound of statement 1) of our theorem. The second bound of that assertion follows from Theorem 7.17 having been considered under $\gamma = \lambda - 1$.

Let now $N \ge 4$. In this case let us apply the local maximum principle (see Theorem 4.5)

$$(7.3.69) \quad \sup_{x' \in G^1_{1/2}} |v(x')| \le c(N, \nu^{-1}, \mu)\Big(\|v\|_{2,G^2_{1/4}} +$$

$$+ \rho^{2-\lambda}\|a(\rho x', \rho^\lambda v, \rho^{\lambda-1} v_{x'})\|_{N,G^2_{1/4}}\Big).$$

Let us estimate from above the summands of the right part of (7.3.69). The first summand is estimated as well as above (see (7.3.62))

$$(7.3.70) \quad \|v\|^2_{2,G^2_{1/4}} \le 2^N \rho^{-2\lambda} \int_{G^{2\rho}_{\rho/4}} r^{-N}u^2 dx \le 2^N c^2_{15}$$

By the assumption (C) in view of (7.3.2), we have

$$\int_{G^2_{1/4}} |a(\rho x', \rho^\lambda v, \rho^{\lambda-1} v_{x'})|^N dx' \le \frac{1}{3}6^N \int_{G^{2\rho}_{\rho/4}} \Big(\mu^N_1 |\nabla u|^{2N} + f^N(x) +$$

$$+b^N(x)|\nabla u|^N\Big)r^{-N}dx \le \frac{1}{3}6^N \int\limits_{G^{2\rho}_{\rho/4}} \Big(\mu_1^N(r^{2-N}|\nabla u|^2)(r^{-2}|\nabla u|^{2N-2})+$$

$$+(r^{2-N}|\nabla u|^2)(k_1^N r^{\beta N-2}|\nabla u|^{N-2}) + k_1^N r^{\beta N-N}\Big)dx \le$$

$$\le \frac{1}{3}6^N\Big(\mu_1^N c_1^{2N-2}\rho^{2\gamma(N-1)-2} + k_1^N c_1^{N-2}\rho^{\gamma(N-2)+\beta N-2}\Big) \int\limits_{G^{2\rho}_{\rho/4}} r^{2-N}|\nabla u|^2 dx+$$

(7.3.71) $\qquad +(3\beta N)^{-1}(6k)^N \text{meas}\Omega(2^{\beta N} - 2^{-\beta N})\rho^{\beta N}, \ \rho \in (0, \frac{d}{2}).$

Hence with regard to (7.3.62) we obtain

(7.3.72) $\quad \rho^{2-\lambda}\|a(\rho x', \rho^\lambda v, \rho^{\lambda-1}v_{x'})\|_{N,G^2_{1/4}} \le c_{16}\rho^{2-\lambda+2\gamma+\frac{2(\lambda-1-\gamma)}{N}} +$

$$+ c_{17}\rho^{2-\lambda+\beta+\gamma+\frac{2(\lambda-1-\gamma)}{N}} + c_{18}\rho^{\beta+2-\lambda}, \ \forall\rho \in (0, \frac{d}{2}).$$

From (7.3.69), (7.3.70), and (7.3.72) with regard to $\beta \ge \lambda - 2$ we get

(7.3.73) $\qquad \sup_{x'\in G^1_{1/2}} |v(x')| \le c_{19} + c_{20}\rho^{2-\lambda+2\gamma+\frac{2(\lambda-1-\gamma)}{N}}.$

Let us recall that $\lambda > 1$ and $\gamma > 0$ which is determined by Theorem 7.16.

Also in the case $2 \le N < 4$ for the validity of assertion 1) of our theorem, it is sufficient to derive the bound

(7.3.74) $\qquad \sup_{x'\in G^1_{1/2}} |v(x')| \le M'_0 = const.$

Let us show that repeating a finite number of times the procedure of deriving of (7.3.73) with different exponents γ, it is possible to deduce the estimate (7.3.74). So let the exponent of ρ in (7.3.73) be negative. Otherwise (7.3.73) means (7.3.74). From (7.3.73) we have

(7.3.75) $\qquad |u(x)| \le c_{21}|x|^{2+2(\lambda-1)/N}$

and from here, by Theorem 7.17 with $\gamma = \gamma_1$

(7.3.76) $\qquad \gamma_1 = 1 + \frac{2}{N}(\lambda - 1)$

we get also the inequality

(7.3.77) $\qquad |\nabla u(x)| \le c_{22}|x|^{\gamma_1}.$

Let us repeat the procedure of the deduction of (7.3.71) and (7.3.72) having applied the inequality (7.3.77) instead of (7.3.2), that is replacing γ on γ_1.

As a result we get

$$(7.3.78) \qquad \sup_{x' \in G^1_{1/2}} |v(x')| \leq c_{19} + c_{20}\rho^{2-\lambda+2(\lambda-1)/N+2\gamma_1(N-1)/N}$$

If the exponent of ρ in this inequality is negative, then putting

$$(7.3.79) \qquad \gamma_2 = 1 + \frac{2}{N}(\lambda-1) + \frac{2(N-1)}{N}\gamma_1,$$

we first obtain by Theorem 7.17 the inequality

$$(7.3.80) \qquad |\nabla u(x)| \leq c_{22}|x|^{\gamma_2}.$$

Next repeating the procedure described above, we get also the bound

$$(7.3.81) \qquad \sup_{x' \in G^1_{1/2}} |v(x')| \leq c_{19} + c_{20}\rho^{2-\lambda+2(\lambda-1)/N+2\gamma_2(N-1)/N}.$$

Let us set

$$(7.3.82) \qquad t = \frac{2(N-1)}{N} \geq \frac{3}{2}, \quad \forall N \geq 4$$

and consider the numerical sequence γ_k

γ_1 is determined by the equality (7.3.76),
$\gamma_2 = (1+t)\gamma_1,$
$\gamma_3 = (1+t+t^2)\gamma_2,$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\gamma_{k+1} = (1+t+\cdots+t^k)\gamma_1 = \frac{t^{k+1}-1}{t-1}, \quad k = 0,1,\cdots.$$

Repeating the expounded procedure k times we get

$$(7.3.83) \qquad \sup_{x' \in G^1_{1/2}} |v(x')| \leq c_{19} + c_{20}\rho^{1-\lambda+\gamma_{k+1}}, \quad \rho \in (0,d/2).$$

Let us show that for $\forall N \geq 4$ we can find such an integer k that

$$(7.3.84) \qquad 1 - \lambda + \gamma_{k+1} \geq 0.$$

In fact, from the definition of the numerical sequence γ_k and (7.3.76) it follows

$$1 - \lambda + \gamma_{k+1} = \frac{t^{k+1}-1}{t-1} + \frac{\lambda-1}{N(t-1)}\left(2t^{k+1} - 2 - Nt + N\right).$$

The first addend on the right is positive. For the second addend from (7.3.82) it follows

$$2t^{k+1} - 2 - Nt + N = 2^{k+2}\left(1 - \frac{1}{N}\right)^{k+1} - N \geq 0,$$

if

$$\left(\frac{2N-2}{N}\right)^{k+1} \geq \frac{N}{2} \quad \text{or} \quad k+1 \geq \frac{\ln\frac{N}{2}}{\ln\frac{2N-2}{N}}.$$

Hence we obtain the validity of (7.3.84) for

$$k = \left[\frac{\ln \frac{N}{2}}{\ln \frac{2N-2}{N}} \right], \quad \forall N \geq 4,$$

where $[a]$ is the integral part of a. Thus statement 1) is proved.

Now let us refer to the proof of statement 3) of our theorem. Multiplying both sides of the inequality (7.3.68) by $\varrho^{\alpha-2q}$ and returning to the variables x, u we rewrite the inequality obtained in such way replacing ρ by $2^{-k}\rho$ and next sum all inequalities over $k = 0, 1, \ldots$. As result we have

$$(7.3.85) \quad \|u\|^q_{V^2_{q,\alpha}(G^\varrho_0)} \leq c_4 \int\limits_{G^{2\varrho}_0} \left(r^\alpha |a(x, u, u_x)|^q + r^{\alpha-2q}|u|^q \right) dx, \quad q > 1.$$

Taking into account the assumption (C) and the bounds from assertion 1) proved above, we obtain

$$|a(x, u, u_x)|^q \leq C(\mu_1, k_1, q, N) \left(|\nabla u|^{2q} + r^{\beta q}|\nabla u|^q + r^{\beta q} \right) \leq$$
$$\leq C \left(r^{2q(\lambda-1)} + r^{q(\beta+\lambda-1)} + r^{\beta q} \right).$$

Hence and from (7.3.85) it follows

$$\|u\|^q_{V^2_{q,\alpha}(G^\varrho_0)} \leq C \operatorname{meas}\Omega \int\limits_0^{2\varrho} \left(r^{\alpha+q(\lambda-2)} + r^{\alpha+2q(\lambda-1)} + r^{\alpha+q(\beta+\lambda-1)} + \right.$$
$$\left. + r^{\alpha+\beta q} \right) r^{N-1} dr.$$

Since $\beta > \lambda - 2$ and therefore $r^{q\beta} < r^{q(\lambda-2)}$, finally we establish

$$(7.3.86) \quad \|u\|^q_{V^2_{q,\alpha}(G^\varrho_0)} \leq C\varrho^{\alpha+N+q(\lambda-2)},$$

provided $\alpha + N + q(\lambda - 2) > 0$. The latter means the required statement 3).

Finally, let us prove statement 4). By the Sobolev-Kondrashov Imbedding Theorem 1.33

$$\sup_{\substack{x', y' \in G^1_{1/2} \\ x' \neq y'}} \frac{|\nabla' v(x') - \nabla' v(y')|}{|x' - y'|^{1-N/q}} \leq c(N, q, G)\|v\|_{W^{2,q}(G^1_{1/2})}, \quad q > N.$$

Returning to the variables x, u, we get

$$\sup_{\substack{x,y\in G^{\rho}_{\rho/2} \\ x\neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^{1-N/q}} \leq C\|u\|_{V^2_{q,0}(G^{\varrho/4}_{2\varrho})} \leq C\rho^{\lambda-2+N/q},$$

$$q > N, \ \rho \in (0, d),$$

in virtue of (7.3.86). Repeating the proof of Theorem 7.18 with $\gamma = \lambda - 1$, provided $N + q(\lambda - 2) \leq 0$, we get the validity of statement 4). $\qquad\square$

7.3.6. Higher regularity results. In this subsection we examine the question of a smoothness rise of the Dirichlet problem solutions for the elliptic second order non-divergence quasi-linear equations near the conical boundary point. Let us consider the strong solution from $W^{2,q}(G) \cap C^{1+\gamma}(\overline{G})$ of (QL). As well as in the linear case the solution smoothness in the quasilinear case depends upon the quantity λ determining value of the cone solid angle in a neighborhood of the point $\mathcal{O}$.

Let us define the set

$$\mathfrak{M}_{M_0,M_1} = \{(x, u, z)|x \in \overline{G}, |u| \leq M_0, |z| \leq M_1\}.$$

As for the equation of the problem (QL) we assume that the following conditions are satisfied on the set $\mathfrak{M}_{M_0,M_1}$

(E) for the uniform ellipticity that is there exist the positive constants ν, μ such that for $\forall (x, y, z) \in \mathfrak{M}_{M_0,M_1}, \forall \xi \in \mathbb{R}^N$

$$\nu\xi^2 \leq a_{ij}(x, u, z)\xi_i\xi_j < \mu\xi^2; \quad a_{ij}(0, 0, 0) = \delta_i^j, \ i, j = 1, \ldots, N;$$

(F) $a_{ij}(x, u, z) \in C^m(\mathfrak{M}_{M_0,M_1})$ $(i, j = 1, ..., N)$ for some integer $m \geq 1$ and the partial derivatives of the functions $a_{ij}(x, u, z)$ over all their arguments up to the order m are bounded on $\mathfrak{M}_{M_0,M_1}$;

(G) there exist generalized partial derivatives of the function $a(x, u, z)$ over all their arguments up to the order $m \geq 1$, nonnegative functions $f_l(x)$ and the numbers $\widetilde{\mu}_l, \widetilde{k}_l$ $(l = 1, ..., m)$ such that the following inequalities

(7.3.87) $\qquad |D_u^{l_1} D_x^{l_2} a(x, u, z)| \leq \mu_1|z|^2 + f_l(x); \ 1 \leq l_1 + l_2 \leq m,$

(7.3.88) $\qquad |D_u^{l_1} D_x^{l_2} D_z^{l_3} a(x, u, z)| \leq \mu_1|z| + f_l(x); \ 0 \leq l_1 + l_2 \leq m - 1,$

(7.3.89) $\qquad |D_u^{l_1} D_x^{l_2} D_z^{l_3} a(x, u, z)| \leq \widetilde{\mu}_l; \ 2 \leq l_1 + l_2 + l_3 \leq m,$

where

(7.3.90) $\qquad f_l(x) \leq \widetilde{k}_l|x|^{\lambda-2-l}, \ f_0 \leq \widetilde{k}_0|x|^{\beta}, \ \beta > \lambda - 2,$

are fulfilled.

THEOREM 7.23. *Let $\lambda > 1, p > N$ be given and let the integer m satisfy the condition*

$$(7.3.91) \qquad\qquad 1 \le m < \lambda - 2 + N/p.$$

Let the assumptions $(A) - (G)$ be satisfied and let the function $u(x) \in V_{p,0}^2(G)$ be a solution of the problem (QL) with

$$M_0 = \max_{x \in \overline{G}} |u(x)| \text{ and } M_1 = \max_{x \in \overline{G}} |\nabla u(x)|.$$

Moreover, let $\varphi(x) \in V_{p,0}^{m+2-1/p}(\partial G) \cap V_{2,4-N}^{3/2}(\partial G)$ and there exist the nonnegative numbers $\widetilde{k}_0', \widetilde{k}_1', ..., \widetilde{k}_m'$ and $s > l$ such that the inequalities

$$(7.3.92) \quad ||\varphi||_{V_{2,4-N}^{3/2}\Gamma_0^\rho} \le \widetilde{k}_0' \rho^s \text{ and } ||\varphi||_{V_{p,0}^{2-1/p+m}\Gamma_{\rho/2}^\rho} \le \widetilde{k}_m' \rho^{\lambda-2-m+N/p},$$

$$\rho \in (0, d)$$

hold. Then $u(x) \in V_{p,0}^{m+2}(G)$, and there exist the numbers $\widetilde{d} \in (0, d)$ and $C_m > 0$ such that

$$(7.3.93) \qquad\qquad ||u||_{V_{p,0}^{m+2}G_0^\rho} \le C_m \rho^{\lambda-2-m+N/p}, \ \rho \in (0, \widetilde{d}),$$

where C_m is determined only by the quantities taking part in the assumptions of the theorem and by G.

PROOF. We apply the usual iteration procedure over m. Let $m = 1$. Let us consider the equation of (QL) in the domain $G_{\rho/2}^\rho, \rho \in (0, d)$. The lateral surface $\Gamma_{\rho/2}^\rho$ of $G_{\rho/2}^\rho$ is unboundedly smooth, because G_0^d is a convex cone. By definition of smooth domains, for every point $x_0 \in \Gamma_{\rho/2}^\rho$ there exists a neighborhood $\Gamma \subset \Gamma_{\rho/2}^\rho$ of this point and a diffeomorphism χ from C^{2+m} rectifying the boundary in Γ. Let $\mathfrak{D} \subset G_{\rho/2}^\rho$ be such that $\Gamma \subset \mathfrak{D}$. Let us perform the transformation $y = \chi(x) = (\chi_1(x), ..., \chi(x))$ and let $\chi(\mathfrak{D}) = \mathfrak{D}', \ \chi(\Gamma) = \Gamma' \subset \partial\mathfrak{D}', (\Gamma'$ is a plane portion of the boundary$\mathfrak{D}'), v(y) = u(\chi^{-1}(y))$. In this case $\chi, \chi^{-1} \in C^{2+m}$ and Jacobian $|\nabla\chi| \neq 0$. Besides, one can suppose the norms in C^{2+m} of transformations χ determining the local representation of the boundary $\Gamma_{\rho/2}^\rho$ to be uniformly bounded with respect to $x_0 \in \Gamma_{\rho/2}^\rho$. In the new variables the equation of (QL) takes the form

$$(QL)' \qquad A_{ij}(y, v, v_y)v_{y_i y_j} + A(y, v, v_y) = 0, \ y \in \mathfrak{D}',$$

where

$$A(y, v, v_y) = a(x, u, u_x) + a_{ij}(x, u, u_x)v_{y_k}\frac{\partial^2 \chi_k}{\partial x_i \partial x_j}$$

(7.3.94)

$$A_{ij}(y, v, v_y) = a_{kl}(x, u, u_x)\frac{\partial \chi_i}{\partial x_k}\frac{\partial \chi_j}{\partial x_l},$$

Let us notice that in $\mathfrak{D}'$ by condition (E)

(7.3.95) $\varkappa_1 \nu \xi^2 \le A_{ij}\xi_i\xi_j \le \varkappa_2 \mu \xi^2,$

where

$$\varkappa_1 = \inf_{x \in \mathfrak{D}'} |\nabla\chi(x)|^2 > 0 \text{ and } \varkappa_2 = \sup_{x \in \mathfrak{D}'} |\nabla\chi(x)|^2 > 0$$

. The coordinate system can be chosen such that the positive axis y_N would be parallel to the normal toward Γ' and the axes $y_1, ..., y_{N-1}$ parallel the rays at plane Γ'. Let $\mathbf{e}_k$ be the fixed coordinate vectors $(k = 1, ..., N - 1)$. For sufficiently small $|h|$ we define the difference quotients

$$v_k(y; h) = \frac{1}{h}\{v(y) - v(y_1, ..., y_{k-1}, y_k - h, y_{k+1}, ..., y_N)\}, \quad k = 1, ..., N - 1.$$

We set

$$y^t = ty + (1 - t)(y - h\mathbf{e}_k); \quad v^t(y) = tv(y) + (1 - t)v(y - h\mathbf{e}_k).$$

Then the function $w(y) \equiv v_k(y, h)$ satisfies the linear equation

(L) $a^{ij}(y)w_{y_iy_j} + a^i(y)w_{y_i} + a(y)w = f(y), \quad y \in \mathfrak{D}',$

where

$$a^{ij}(y) = A_{ij}(y, v(y), v_y(y)),$$

$$a^i(y) = v_{y_py_l}(y - h)\int_0^1 \frac{\partial A_{pl}(y^t, v^t, v_y^t)}{\partial v_{y_i}^t}dt + \int_0^1 \frac{\partial A(y^t, v^t, v_y^t)}{\partial v_{y_i}^t}dt,$$

$$a(y) = v_{y_py_l}(y - h)\int_0^1 \frac{\partial A_{pl}(y^t, v^t, v_y^t)}{\partial v^t}dt + \int_0^1 \frac{\partial A(y^t, v^t, v_y^t)}{\partial v^t}dt,$$

$$-f(y) = v_{y_py_l}(y - h)\int_0^1 \frac{\partial A_{pl}(y^t, v^t, v_y^t)}{\partial y_k^t}dt + \int_0^1 \frac{\partial A(y^t, v^t, v_y^t)}{\partial y_k^t}dt,$$

$k = 1, ..., N - 1$. Since the directories $\mathbf{e}_k(k = 1, ...N - 1)$ are parallel to the tangent plane to Γ, we have $w|_{\Gamma'} = \psi_k(y, h), y \in \Gamma', \psi(y) = \varphi(\chi^{-1}y)$. Let us apply the local L^p - estimate near smooth boundary portion (Theorem 4.6) to the solution $w(y)$. Let us verify the fulfillment of all conditions of the above estimate. (7.3.95) implies the fulfillment of the uniform ellipticity condition for the (L) equation. Since our solution $u(x) \in C^{1+\gamma}(\overline{G})$, the hypothesis (E) guarantees continuity of the coefficients $a^{ij}(y)$ in $\mathfrak{D}'$. Since,

by the assumptions of our theorem $u_{xx} \in L_p(\mathfrak{D}'), p > N$, then by assertions 1) and 3) of Theorem 7.22 in view of the assumptions $(F), (G)$ we have

$$\left\| \left(\sum_{i=1}^{N} |a^i(y)|^2 \right)^{\frac{1}{2}} \right\|_{N,\mathfrak{D}'} \leq C\left(|\chi|_{2,\mathfrak{D}'}\right)\left(\overline{\mu}_1 |u_{xx}|_{N,G_0^{3\rho/2}} + \right.$$

$$+ \left\| \mu + (\overline{\mu}_1 + \widetilde{\mu}_1)|\nabla u| + f_0(x) \right\|_{N,G_{\rho/2}^\rho} \bigg) \leq$$

$$\leq C\left(|\chi|_{2,\mathfrak{D}'}, N, p, \widetilde{k}_0, \mu, \overline{\mu}_1, \widetilde{\mu}_1\right)\left(\rho + \rho^\lambda + \rho^{\lambda-2+N/p} + \rho^{\lambda-1}\right) \leq const,$$

by the inequality (7.3.91). Similarly

$$\|a\|_{p,\mathfrak{D}'} \leq C\left(|\chi|_{2,\mathfrak{D}'}\right)\left(\||\mu_1|\nabla u|^2 + \overline{\mu}_1|\nabla u| + f_0(x)\|_{p,G_{\rho/2}^\rho} + \overline{\mu}_1\|u_{xx}\|_{p,G_0^{3\rho/2}}\right)$$

$$\leq C\left(|\chi|_{2,\mathfrak{D}'}, N, p, \widetilde{k}_0, \mu_1, \overline{\mu}_1\right)\left(\rho^{2(\lambda-1)+N/p} + \rho^{\lambda-2+N/p} + \right.$$

$$+ \rho^{\lambda-1+N/p}\bigg) \leq const;$$

$$\|f\|_{p,\mathfrak{D}'} \leq C\left(|\chi|_{3,\mathfrak{D}'}\right)\left(\||\mu_1|\nabla u|^2 + \overline{\mu}_1|\nabla u| + f_1(x)\|_{p,G_{\rho/2}^\rho} + \overline{\mu}_1\|u_{xx}\|_{p,G_0^{3\rho/2}}\right)$$

$$\leq C\left(|\chi|_{2,\mathfrak{D}'}, N, p, \widetilde{k}_0, \mu_1, \overline{\mu}_1\right)\varrho^{\lambda-3+N/p}.$$

So the local L^p-estimate for the solutions of $(L)'$ gives us the inequality

(7.3.96) $\|w\|_{2,p;\mathfrak{D}''} \leq const(\|w\|_{p,\mathfrak{D}'} + \|f\|_{p;\mathfrak{D}} + \|\varphi_k(y-h)\|_{2-1/p,p;\Gamma'})$,

$$\forall \mathfrak{D}'' \Subset \mathfrak{D}' \cup \Gamma',$$

where *const* is independent of $w, f, , \varphi_k, h$ and depends only on k, p, ν, $\mu, \varkappa_1, \varkappa_2$ and the moduli of continuity of the coefficients $a^{ij}(y)$ on $\mathfrak{D}'$. The latter are estimated in the following way:

$$|a^{ij}(y_1) - a^{ij}(y_2)|_{\mathfrak{D}'} = |A_{ij}(y_1, v(y_1), v_y(y_1)) - A_{ij}(y_2, v(y_2), v_y(y_2))| =$$

$$= \left| a_{kl}(x_1, u(x_1), u_x(x_1)) \frac{\partial \chi_i(x_1)}{\partial x_k} \frac{\partial \chi_j(x_1)}{\partial x_l} - \right.$$

$$- a_{kl}(x_2, u(x_2), u_x(x_2)) \frac{\partial \chi_i(x_2)}{\partial x_k} \frac{\partial \chi_j(x_2)}{\partial x_l} \bigg| \leq$$

$$\leq |a_{kl}(x_1, u(x_1), u_x(x_1)) - a_{kl}(x_2, u(x_2), u_x(x_2))| \cdot |\nabla \chi|^2 +$$

$$+ \mu \left| \frac{\partial \chi_i(x_1)}{\partial x_k} \frac{\partial \chi_j(x_1)}{\partial x_l} - \frac{\partial \chi_i(x_2)}{\partial x_k} \frac{\partial \chi_j(x_2)}{\partial x_l} \right| \leq \mu \varkappa_2^{1/2} |\chi|_{2,\mathfrak{D}} |x_1 - x_2| +$$

$$+ \varkappa_2 \overline{\mu}_1 \left(|x_1 - x_2| + |u(x_1) - u(x_2)| + |\nabla u(x_1) - \nabla u(x_2)| \right) \leq$$

$$\leq 2\rho \left(\varkappa_2 \overline{\mu}_1 + \mu \varkappa_2^{1/2} |\chi|_{2,\mathfrak{D}} + \overline{c}_1 \rho^\gamma \right) + C(2\rho)^\gamma$$

by $u(x) \in C^{1+\gamma}(\overline{G})$. Further, we have by the definition of $w(y)$

(7.3.97) $\|w\|_{p,\mathfrak{D}'} = \left\| \dfrac{v(y) - v(y - h\mathbf{e}_k)}{h} \right\|_{p,\mathfrak{D}'} \leq C(|\chi^{-1}|_1)\|\nabla u(x)\|_{p,G_{\rho/2}^\rho}$

Analogously we obtain

$$(7.3.98) \qquad ||\varphi_k(y,h)||_{2-1/p,p;\Gamma'} \leq C(|\chi^{-1}|_1)||\varphi(x)||_{3-1/p,p;\Gamma^\rho_{\rho/2}},$$

and finally

$$(7.3.99) \quad ||f||_{p,\mathfrak{D}'} \leq C(|\chi|_{2,\mathfrak{D}'})\Big(\big|\big||\mu_1|\nabla u|^2 + \overline{\mu}_1|\nabla u| + f_1(x)\big|\big|_{p,G^\rho_{\rho/2}} + \\ + \overline{\mu}_1||u_{xx}||_{p,G^{3\rho/2}_0}\Big)$$

Now from (7.3.96)-(7.3.99) we obtain the inequality

$$\Big|\Big|\frac{v(y) - v(y - h\mathbf{e}_k)}{h}\Big|\Big|_{2,p;\mathfrak{D}'} \leq const\Big(||\varphi(x)||_{3-1/p,p;\Gamma^\rho_{\rho/2}} + \\ + \big|\big||\nabla u|^2\big|\big|_{p,G^\rho_{\rho/2}} + ||\nabla u(x)||_{p,G^\rho_{\rho/2}} + ||f_1(x)||_{p,G^\rho_{\rho/2}} + ||u_{xx}||_{p,G^{3\rho/2}_0}\Big) \leq \\ \leq const\varrho^{\lambda-3+N/p},$$

where *const* on the right is independent of h. This fact allows us to conclude on the basis of Fatou's theorem that there exists a $v_{y_h} \in W^{2,p}(\mathfrak{D}'')$ and perform passage to the limit $h \to 0$

$$(7.3.100) \quad \Big|\Big|\frac{\partial v}{\partial y_k}\Big|\Big|_{2,p;\mathfrak{D}''} \leq const\Big(||\varphi(x)||_{3-1/p,p;\Gamma^\rho_{\rho/2}} + ||u_{xx}||_{p,G^{3\rho/2}_0} + \\ + \big|\big||\nabla u(x)|^2\big|\big|_{p,G^\rho_{\rho/2}} + ||\nabla u(x)||_{p,G^\rho_{\rho/2}} + ||f_1(x)||_{p,G^\rho_{\rho/2}}\Big) \leq \\ \leq const\varrho^{\lambda-3+N/p}, \quad k = 1,...,N-1.$$

We consider again the equation $(QL)'$ and differentiate it over y_N thusly

$$(7.3.101) \quad A_{NN}(y,v,v_y)v_{y_Ny_Ny_N} = -\Big\{\sum_{k=1}^{N-1} A_{kN}v_{y_ky_Ny_N} + \\ + \sum_{i,j=1}^{N-1} A_{ij}v_{y_iy_iy_N} + \frac{\partial A_{ij}}{\partial v_{y_l}}v_{y_iy_j}v_{y_ly_N} + \frac{\partial A_{ij}}{\partial v}v_{y_iy_j}v_{y_N} + \frac{\partial A_{ij}}{\partial y_N}v_{y_iy_j} + \\ + \frac{\partial A_{ij}}{\partial v_{y_l}}v_{y_ly_N} + \frac{\partial A}{\partial y_N} + \frac{\partial A}{\partial v}v_{y_N}\Big\}, \quad y \in \mathfrak{D}',$$

where

$$A_{NN} = a_{kl}(y,v,v_y)\frac{\partial \chi_N}{\partial x_k}\frac{\partial \chi_N}{\partial x_l} \geq \nu|\nabla\chi_N(x)|^2 \geq \varkappa_1\nu.$$

Since $u(x) \in W^{2,p}(G^\rho_{\rho/2})$, $v_{y_k} \in W^{2,p}(\mathfrak{D}'')$, $1 \leq k \leq N-1$ then from (7.3.101) we obtain $v(y) \in W^{3,p}(\mathfrak{D}'')$, by the assumptions $(F), (G)$. Then by Sobolev's Imbedding theorems 1.32, 1.34 we can derive

1) if $p > 2N$ then $v(y) \in C^2(\mathfrak{D}'')$ and in this case

$$|v|_{2;\mathfrak{D}''} \le c\|v\|_{3,p/2;\mathfrak{D}''};$$

2) if $N < p \le 2N$ then $v(y) \in W^{2,q_1}(\mathfrak{D}'')$ with $q_1 = \frac{Np}{2N-p} > p$ and in particular $v(y) \in W^{2,2p}(\mathfrak{D}'')$ for $p \ge 3N/2$.

By the above statements and equation (7.3.101), we obtain $v(y) \in W^{3,p}(\mathfrak{D}'')$ and therefore $u(x) \in W^{3,p}(G_{5p/8}^{7p/8})$, if $p \ge 3N/2$. Now we need to examine only $p \in (N, 3N/2)$. From above we have $v(y) \in W^{2,q_1}(\mathfrak{D}'')$ and by (7.3.101) $v(y) \in W^{3,q_1/2}(\mathfrak{D}'')$. Let us use again the following imbedding

$$W^{3,q} \subset W^{2,q*}, \quad \text{where } q* = \frac{Nq}{N-q} \text{ and } q < N.$$

As a result, we obtain

$$v(y) \in W^{2,q_2}(\mathfrak{D}''), \quad q_2 = Nq_1/(2N - q_1) = Np/(4N - 3p),$$

if $N < p < 4N/3$ and

$$v(y) \in C^2(\mathfrak{D}''),$$

if $p \ge 4N/3$.

We repeat that procedure s times

$$(7.3.102) \qquad v(y) \in W^{3,q_s/2}(\mathfrak{D}'') \cap W^{2,q_s}(\mathfrak{D}''), \quad q_s = \frac{Np}{N2^s - (2^s - 1)p},$$

if $N < p < N/(1 - 2^{-s})$. We choose an integer number $s \ge 1$ in such way that $q_s \ge 2p$. Solving that inequality we obtain $s = [\log_2((2p-N)/(p-N))]$, where $[a]$ is the integral part of a. Thus from (7.3.102) we find

$$u(x) \in W^{3,p}(G_{5\rho/8}^{7\rho/8}) \cap W^{2,2p}(G_{5\rho/8}^{7\rho/8}), \quad \forall \rho \in (0, d).$$

We proceed to a derivation of the estimate (7.3.93) under $m = 1$. From (7.3.101), by (7.3.100), we have

$$(7.3.103) \quad \left(\int\int_{\mathfrak{D}''} |v_{y_N y_N y_N}|^p dy\right)^{1/p} \le (\nu\chi_1)^{-1}\left\{\mu\chi_2 \sum_{k=1}^{N-1}\left\|\frac{\partial v}{\partial y_k}\right\|_{2,p;\mathfrak{D}''} + \right.$$

$$+ \left(\mu + (\mu_1 + \overline{\mu}_1)|\nabla_y v|_{\mathfrak{D}''} + |f_0(y)|_{\mathfrak{D}''}\right)\|v_{yy}\|_{p,\mathfrak{D}''} +$$

$$+ \left\||\mu_1|\nabla_y v|^3 + (\overline{\mu}_1 + \mu_1)|\nabla_y v|^2 + (\overline{\mu}_1 + f_0(y))|\nabla_y v| + f_1(y)\right\|_{p,\mathfrak{D}''} +$$

$$\left. + \overline{\mu}_1\|v_{yy}\|_{p,\mathfrak{D}''}^2 + \overline{\mu}_1\|v_{yy}\|_{p,\mathfrak{D}''}(1 + |\nabla_y v|_{\mathfrak{D}''})\right\}C(|\chi|_{2,\mathfrak{D}''}).$$

From (7.3.100) and (7.3.103) in the variables $x, u(x)$ taking into account the hypothesis (G), we obtain

$$\|u\|_{3,p;G_{5\rho/8}^{7\rho/8}} \leq C\Big(|\chi|_{3,G_{\rho/2}^{\rho}}, \nu, \mu, N, p, \varkappa_1, \varkappa_2, \mu_1, \overline{\mu}_1\Big)\Big(\|u_{xx}\|_{p,G_0^{3\rho/2}}+$$

$$+\big\||\nabla u|^3 + |\nabla u|^2 + |\nabla u|(1 + f_0(x)) + |f_1(x)| + u_{xx}^2\big\|_{p,G_{\rho/2}^{\rho}}+$$

$$+(1 + |f_0(x) + \nabla u|_{G_{\rho/2}^{\rho}})\|u_{xx}\|_{p,G_{\rho/2}^{\rho}} + \|\varphi\|_{3-1/p,p,\Gamma_{\rho/2}^{\rho}}\Big).$$

From here basing on Theorem 7.22 for $\rho \in (0, d)$

$$(7.3.104) \qquad \|u\|_{3,p;G_{5\rho/8}^{7\rho/8}} \leq C\rho^{\lambda-3+N/p},$$

$$C = C\Big(|\chi|_{3,G_{\rho/2}^{\rho}}, \nu, \mu, p, \varkappa_1, \varkappa_2, \mu_1, \overline{\mu}_1, \lambda, \widetilde{k}_0, \widetilde{k}_1, d, \overline{c}, \overline{c}_3\Big).$$

Replacing in (7.3.104) ρ by $2^{-k}\rho$, summing the inequalities obtained over all $k = 0, 1, 2, ...$, and taking into account (7.3.91) under $m = 1$, we come to the desired estimate (7.3.93) under $m = 1$.

Repeating such procedure by induction we conclude the validity of the assertions of Theorem 7.23. $\qquad\qquad\square$

THEOREM 7.24. *Let all assumptions of Theorem 7.23 excepting of (7.3.91) be fulfilled. If $m \geq 0$ is the integer and*

$$(7.3.105) \qquad m+1 < \lambda \leq m+2 - \frac{N}{p}, \quad p > N,$$

then $u(x) \in C^{\lambda}(\overline{G})$. In addition, there exist constants $\widetilde{c}_k$, $(k = 0, ..., m+1)$ independent of $u(x)$ such that

$$(7.3.106) \qquad |\nabla^k u(x)| \leq \widetilde{C}_k |x|^{\lambda-k}, \quad x \in \overline{G_0^d}, \quad k = 0, ..., m+1.$$

If $\lambda = m+1$, $p \geq N$ then $u \in C^{\lambda-\varepsilon}(\overline{G})$, $\forall \varepsilon > 0$.

PROOF. Let the function $v(x') = \rho^{-\lambda}u(\rho x')$ be a solution in the layer $G_{1/2}^1$ of $(QL)'$. Verbally repeating the proof of Theorem 7.22 and using the results of Theorem 7.23 we obtain all assertions of Theorem 7.24. $\qquad\square$

7.4. Solvability results

Let us include the problem (QL) to a family of one-parametric problems for $t \in [0, 1]$

$$(QL)_t \qquad \begin{cases} a_{ij}(x, u, u_x)u_{x_i, x_j} + ta(x, u, u_x) = 0, & x \in G \\ u(x) = t\varphi(x), & x \in \partial G. \end{cases}$$

With regard to the problem (QL) we assume the hypotheses (S) and $(A) - (J)$ to be satisfied. In addition, suppose

(M) for every solution $u_t(x)$ of the problem $(QL)_t$ the value
$M_0 = \sup\limits_{x \in G} |u_t(x)|$, $\forall t \in [0,1]$ is known,

(K) $\varphi(x) \in \mathring{W}^{\frac{3}{2}}_{4-N}(\partial G) \cap V^{2-\frac{1}{q}}_{q,0}(\partial G)$, $q \geq N$,

there exist nonnegative numbers k_3, k_4, k_5 and $s > \lambda$ such that

$$b(x) + f(x) + |\Phi_{xx}(x)| \leq k_3 d^{\lambda-2}(x), \quad x \in G_\varepsilon, \forall \varepsilon > 0;$$

$$\|\varphi\|_{\mathring{W}^{\frac{3}{2}}_{4-N}(\Gamma^\varrho_0)} \leq k_4 \varrho^s, \quad \|\varphi\|_{V^{2-\frac{1}{q}}_{q,0}(\Gamma^\varrho_{\varrho/2})} \leq k_5 \varrho^{\lambda-2+\frac{N}{q}}, \quad \varrho \in (0,d).$$

THEOREM 7.25. *Let* $\Gamma_d \in W^{2,p}$ *and the assumptions* $(S), (A) - (J)$, $(M), (K)$ *under* $q = p > N$ *be fulfilled. If either* $\lambda \geq 2$ *or* $1 < \lambda < 2$, $N < p < \frac{N}{2-\lambda}$, *then the problem* $(QL)_t$ *has at least one solution* $u_t(x) \in V^2_{p,0}(G)$ *for* $\forall t \in [0,1]$.

THEOREM 7.26. *Let* $\lambda \in (1,2)$, $p \in (N, \frac{N}{2-\lambda})$, $\beta > \lambda - 2$, $q > \frac{N}{2-\lambda}$ *be given numbers, and let* $\Gamma_d \in W^{2,p}$. *Suppose the hypotheses* $(S), (A) - (J)$, $(M), (K)$ *are fulfilled. Then the problem* $(QL)_t$ *has at least one solution*

$$u_t(x) \in W^{2,q}_{loc}(G) \cap V^2_{p,0}(G) \cap C^\lambda(\overline{G})$$

for $\forall t \in [0,1]$.

PROOF. We first shall establish that for some $\gamma \in (0,1)$ and all $t \in [0,1]$ every solution $u_t(x) \in W^{2,q}_{loc}(G) \cap C^0(\overline{G})$ satisfies the inequality

$$(7.4.1) \qquad\qquad |u_t(x)|_{1+\gamma,\overline{G}} \leq K$$

with a constant K being independent of $u_t(x)$ and t. Let us represent $G = G^d_0 \cup G_d$ with some positive sufficiently small d. From Theorem 7.18 we conclude that under given assumptions there exist the positive d and γ_0 such that a $u_t(x) \in C^{1+\gamma}(\overline{G^d_0})$ and the estimate (7.4.1) holds with $\forall \gamma \in (0, \gamma^*]$, where $\gamma^* = \min(\gamma_0; \beta + 1; 1 - N/q)$. The membership $u_t(x) \in C^{1+\gamma}(\overline{G_d})$ and corresponding *a priori* estimate follow from the assumption (D) (local estimates near a smooth boundary portion have been established in [**217, 219, 224**]), but in strictly contained subdomain follows, by the Sobolev-Kondrashov Imbedding theorem 1.33. In such a way the membership $u_t(x) \in C^{1+\gamma}(\overline{G})$ and the *a priori* estimate (7.4.1) are established.

The bound (7.4.1) allows to apply the Leray-Schauder Fixed Point Theorem 1.56. To apply this theorem we fix $\gamma \in (0,1)$ and consider the Banach space $\mathfrak{B} = C^{1+\gamma}(\overline{G})$ for Theorem 7.25 or $\mathfrak{B} = C^{1+\gamma}_0(\overline{G}) = \left\{ v \in C^{1+\gamma}(\overline{G}) \big| v(0) = |\nabla v(0)| = 0 \right\}$ for Theorem 7.26. Let us define the operator $\mathfrak{T}$, by letting $u_t = t\mathfrak{T}v$, as the unique solution from the space $V^2_{p,0}(G)$ (Theorem 7.25) or $W^{2,q}_{loc}(G) \cap V^2_{p,0}(G) \cap C^\lambda(\overline{G})$ (Theorem 7.26) for

any $v \in \mathfrak{B}$ of the linear problem

$$(L)_t \qquad \begin{cases} a^{ij}(x)u_{x_i x_j} = A_t(x), & x \in G, \\ u(x) = t\varphi(x), & x \in \partial G. \end{cases}$$

where $a^{ij}(x) = a_{ij}(x, v(x), v_x(x))$, $A_t(x) = -ta(x, v(x), v_x(x))$. It exists by Theorem 4.48 (Theorem 4.49). In fact, it is not difficult to verify that all hypotheses of these theorems are fulfilled. In particular, by the assumption (A), $a_{ij}(x, v(x), v_x(x)) \in W^{1,p}(\mathfrak{M})$, $p > N$ and therefore by the imbedding theorem $a^{ij}(x) \in C^{1-N/p}(\overline{G})$. In addition, for $u_t(x)$ the bound (4.4.9) holds. In virtue of the assumption (C) it has the form

$$(7.4.2) \quad \|u_t\|_{V^2_{p,0}(G)} \le c\Big(\mu_1|\nabla v|^2 + |\nabla v|\|b(x)\|_{p,G} +$$

$$+ \|f\|_{p,G} + \|\varphi\|_{V^{2-1/p}_{p,0}(\partial G)}\Big), \quad \forall t \in [0,1].$$

It is clear that the solvability of the problem $(QL)_t$ in the corresponding space is equivalent to the solvability of the equation $u_t = t\mathfrak{T}v$ in the Banach space $\mathfrak{B}$. Now we verify that all hypotheses of the Leray-Schauder Fixed Point Theorem 1.56 are fulfilled. This theorem guarantees the existence of a fix point of the map $\mathfrak{T}$.

At first, we verify that $\mathfrak{T}$ is the compact mapping of the space $\mathfrak{B}$ onto itself. From the bound (7.4.2) it follows that the operator $\mathfrak{T}$ maps sets that are bounded in $\mathfrak{B}$ into bounded sets of the space $V^2_{p,0}(G)$, and they are precompact sets in $C^{1+\gamma}(\overline{G})$, if $\gamma < 1 - \frac{N}{p}$. Thus $\mathfrak{T}$ is the compact mapping. Now we verify that $\mathfrak{T}$ is the continuous mapping onto $\mathfrak{B}$. Let the sequence $\{v_k(x) \subset \mathfrak{B}\}$ converge to $v(x) \in \mathfrak{B}$. We set $u_k(x) = \mathfrak{T}v_k(x)$. As stated above, $u_k(x) \subset V^2_{p,0}(G)$. It is well known that in the space $V^2_{p,0}(G)$ every bounded set is weakly compact. We leave the notation $u_k(x)$ for a weak convergent subsequence and denote the weakly limit by $\lim\limits_{k \to \infty} u_k(x) = u(x) \in V^2_{p,0}(G)$. The last statement means

$$\lim_{k \to \infty} \int\limits_G g(x)D^\alpha u_k(x)dx = \int\limits_G g(x)D^\alpha u(x)dx, \quad |\alpha| \le 2,$$

(7.4.3)

$$\forall g(x) \in L^{p'}(G) \text{ with } \frac{1}{p} + \frac{1}{p'} = 1.$$

Since now it is obvious that

$$a_k^{ij}(x)(u_k)_{x_i x_j} - A_k(x) \in L^p(G),$$

where

$$a_k^{ij}(x) = a_{ij}(x, v_k(x), v_{kx}(x)), \quad A_k(x) = -a(x, v_k(x), v_{kx}(x)),$$

then we prove that

$$\lim_{k\to\infty} \int_G g(x) \left(a_k^{ij}(x)(u_k)_{x_ix_j} - A_k(x) \right) dx =$$

(7.4.4)

$$= \int_G g(x) \left(a^{ij}(x)u_{x_ix_j} - A(x) \right) dx, \ \forall g(x) \in L^{p'}(G).$$

In fact, by the continuity of $a_{ij}(x, v(x), v_x(x))$ on $\mathfrak{M}$ and because of $v_k(x) \to v(x)$ in $C^{1+\gamma}(\overline{G})$, we have

(7.4.5) $\quad \lim_{k\to\infty} a_k^{ij}(x) = \lim_{k\to\infty} a_{ij}(x, v_k(x), v_{kx}(x)) =$

$$= a_{ij}\left(x, \lim_{k\to\infty} v_k(x), \lim_{k\to\infty} v_{kx}(x) \right) = a^{ij}(x).$$

Similarly we verify that $\lim_{k\to\infty} A_k(x) = A(x)$. Now for $\forall g(x) \in L^{p'}(G)$ we obtain

$$\int_G g(x) \left(a_k^{ij}(x)(u_k)_{x_ix_j} - a^{ij}(x)u_{x_ix_j} \right) dx \leq$$

(7.4.6)

$$\leq \sup_{x\in G} |a_k^{ij}(x) - a^{ij}(x)| \cdot \|u_{kxx}\|_{p,G} \|g\|_{p',G} +$$

$$+ \int_G \left(u_{kx_ix_j} - u_{x_ix_j} \right) a^{ij}(x)g(x)dx.$$

Since the equation of the problem (QL) is uniformly elliptic then $a^{ij}(x)g(x) \in L^{p'}(G)$ and by (7.4.3) we get that the last summand in (7.4.6) tends to zero as $k \to \infty$. By the proven above $a_k^{ij}(x) \in C^{1-\frac{N}{p}}(G)$ therefore, by the Arzela Theorem, the limit (7.4.5) is uniform and, consequently,

$$\lim_{k\to\infty} \sup_{x\in G} |a_k^{ij}(x) - a^{ij}(x)| = 0.$$

In addition, $\{v_k(x)\}$ is uniformly bounded in $\mathfrak{B}$, hence by the bound (7.4.2) we obtain that $\|u_{kxx}\|_{p,G} \leq const$ for $\forall k$. Hence, the first summand in (7.4.6) tends to zero as $k \to \infty$ too. Thus

$$\lim_{k\to\infty} \int_G g(x) \left(a_k^{ij}(x)(u_k)_{x_ix_j} - a^{ij}(x)u_{x_ix_j} \right) dx = 0.$$

In the same way we verify that

$$\lim_{k\to\infty} \int_G g(x) \left(A_k(x) - A(x) \right) dx = 0.$$

Thus the equality (7.4.4) is proved. Since $u_k = \mathfrak{T}v_k$, then the left side of (7.4.4) is equal to zero and hence

$$\int\limits_G g(x) \left(a^{ij}(x)u_{x_ix_j} - A(x) \right) dx = 0, \; \forall g(x) \in L^{p'}(G).$$

Hence it follows that $a^{ij}(x)u_{x_ix_j} = A(x)$ for almost all $x \in G$. Further, $u_k(x) = \varphi(x)$, $x \in \partial G$ and, by $u_k(x) \subset V^2_{p,0}(G)$ because of the imbedding theorem $u_k(x) \subset C^{1+\gamma}\overline{(G)}$, $0 < \gamma \le 1 - \frac{N}{p}$. Therefore

$$u(x)\Big|_{\partial G} = \lim_{k\to\infty} u_k(x)\Big|_{\partial G} = \varphi(x).$$

Thus we proved the equality $u = \mathfrak{T}v$. But then we have

$$\lim_{k\to\infty} \mathfrak{T}v_k(x) = \lim_{k\to\infty} u_k(x) = u(x) = \mathfrak{T}v(x) = \mathfrak{T}\left(\lim_{k\to\infty} v_k(x) \right),$$

that is $\mathfrak{T}$ is the continuous mapping. All hypotheses of the Leray-Schauder Theorem are verified and Theorem 7.25 is proved.

Theorem 7.26 is proved in the same way. Let us turn our attention to some details only. We consider in the space $\mathfrak{B}$ the bounded set

$$V_K = \left\{ v \in C_0^{1+\gamma}(\overline{G}) \Big| |v(x)|_{1+\gamma,\overline{G}} \le K \right\}.$$

In this case we apply Theorem 4.49 with $\alpha = 0$ for the solvability of the linear problem $(L)_t$. We must verify only the assumption A7) of Theorem 4.49. For this point and by our assumption (J) we get

$$\int\limits_{G_0^\varrho} r^{4-N}A_t^2(x)dx \le \int\limits_{G_0^\varrho} r^{4-N}a^2(x,v,v_x)dx \le c\int\limits_{G_0^\varrho} r^{4-N}\left(\mu_1^2 K^4 + \right.$$

$$\left. + K^2 b^2(x) + f^2(x) \right)dx \le c\int\limits_{G_0^\varrho} \left(r^{4-N} + r^{4-N+2\beta} \right) dx \le$$

$$\le c\,\mathrm{meas}\,\Omega\left(\frac{1}{4}\varrho^4 + \frac{1}{4+2\beta}\varrho^{2\beta+4} \right) \le C\varrho^{2s}, \; s > \lambda; \quad \forall t \in [0,1];$$

$$\int\limits_{G_{\varrho/2}^\varrho} |A_t(x)|^q dx \le c\int\limits_{G_{\varrho/2}^\varrho} \left(\mu_1^q K^{2q} + K^q b^q(x) + f^q(x) \right) dx \le$$

$$\le c\,\mathrm{meas}\,\Omega\left(\varrho^N + \varrho^{q\beta+N} \right) \le C\varrho^{N+(\lambda-2)q}, \quad \forall t \in [0,1].$$

$\square$

7.5. Notes

The condition (D) can be replaced by any other condition which guarantee the existence of the a priori estimate

$$|u|_{1+\gamma;G'} \leq M_1, \quad \gamma \in (0,1)$$

for any smooth subdomain $G' \subset\subset \overline{G} \setminus \mathcal{O}$ (see [85, 224, 329, 129]).

The results of Chapter 7 refer to the problem (QL) with its equation as *non-divergent*. Such problems in *non-smooth* domains have not been studied before. Only the research of I.I. Danilyuk [90] is known here. In this work, using the methods of complex variable function theory and integral equations, the author proved the solvability in the space $W^{2,2+\varepsilon}(G)$, $\varepsilon > 0$ is sufficiently small, $G \subset \mathbb{R}^2$ and contains angular points. However, as will see below (§7.2), the requirements for these problems in this work are too high and the number $\varepsilon > 0$ is not precise. The formulated Theorem 7.7 from §7.2.2 shows

$$0 < \varepsilon < 2 \cdot \frac{\pi/\omega_0 - 1}{2 - \pi/\omega_0}, \text{ if } \frac{\pi}{2} < \omega_0 < \pi.$$

The results of Sections 7.2 -7.4 were first established in [54, 55, 57, 58, 59, 60, 61, 63]. We follow these articles.

N. Fandyushina [122] has investigated the solutions behavior in a neighborhood of the boundary without assumption for its smoothness and convexity on quasilinear elliptic equation with two independent variables.

N. Trudinger [382] has established a necessary and sufficient condition on boundary data for the solvability of the Dirichlet problem for a quasilinear elliptic equation $a_{ij}(u_x)u_{x_i x_j} = 0$.

Solutions to some other quasilinear equations in nonsmooth domains were studied in [10, 102, 295, 296, 333, 334, 335, 336, 410].

The results of this chapter were generalized in [369] on quasilinear elliptic equations whose coefficients may degenerate near a conical boundary point namely, the ellipticity condition on the set $\mathfrak{M}$ has the form

$$\nu|x|^\tau \xi^2 \leq a_{ij}(x,u,z)\xi_i\xi_j \leq \mu|x|^\tau \xi^2, \quad \forall \xi \in \mathbb{R}^N, \quad 0 < \tau \leq 1;$$

$$\lim_{|x|\to +0} |x|^{-\tau} a_{ij}(x,u,z) = \delta_i^j.$$

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CHAPTER 8

Weak solutions of the Dirichlet problem for elliptic quasilinear equations of divergence form

In this chapter we investigate the behavior of weak solutions to the Dirichlet problem for uniformly elliptic quasilinear equations of divergence form in a neighborhood of a boundary conical point. We consider weak solutions $u \in W^{1,m}(G) \cap L^p(G)$, $m > 1$ of the differential equation

$$(DQL) \qquad Q(u, \phi) \equiv \int_G \left\{ a_i(x, u, u_x)\phi_{x_i} + a(x, u, u_x)\phi \right\} dx = 0$$

for all $\phi(x) \in W_0^{1,m}(G) \cap L^p(G)$. We suppose that Q is elliptic in G, namely there are positive constants ν, μ such that

$$(E) \qquad \nu|z|^{m-2}|\xi|^2 \leq \frac{\partial a_i(x, u, z)}{\partial z_j}\xi_i\xi_j \leq \mu|z|^{m-2}|\xi|^2, \quad m > 1$$

$$\text{for all } (x, u, z) \in G \times \mathbb{R} \times \mathbb{R}^N, \quad \forall \xi \in \mathbb{R}^N.$$

8.1. The Dirichlet problem in general domains

THEOREM 8.1. **Maximum principle** (see Theorem 10.9 §10.5 [**129**]). *Let* $u \in W^{1,m}(G)$, $m > 1$ *be a weak solution of* (DQL) *and suppose that* Q *satisfies the structure conditions*

    (i) $a_i(x, u, z)z_i \geq \nu|z|^m - g(x)$,
    (ii) $a(x, u, z)sign\, z \geq -\mu_2|z|^{m-1} - f(x)$,

where $\nu, \mu_2 = const > 0$, *and* $f(x), g(x) \in L^{p/m}(G)$ *are nonnegative measurable functions. Then we have the estimate*

$$\sup_{x \in G} |u(x)| \leq C \left(\|f\|_{p/m, G} + \|g\|_{p/m, G} \right)$$

where $C = C(N, m, \nu, \mu_2, p, meas\, G)$.

THEOREM 8.2. **The weak Harnack inequality** (see Theorem 1.1
[381]). *Let $u \in W^{1,m}(G)$, $m > 1$ be a weak nonnegative solution of (DQL)
and suppose that Q satisfies the structure conditions*

(i) $a_i(x, u, z)z_i \geq \nu|z|^m - \mu_1 u^m$,

(ii) $\sqrt{\sum\limits_{i=1}^{N} (a_i(x, u, z))^2} \leq \mu_2|z|^{m-1} + \mu_3 u^{m-1}$,

(iii) $|a(x, u, z)| \leq \mu_4|z|^{m-1} + \mu_5 u^{m-1}$,

with $\nu, \mu_2 > 0$; $\mu_1, \mu_3, \mu_4, \mu_5 \geq 0$. Then for any ball $B_{3R} \subset G$, there holds

(8.1.1) $$\|u\|_{L^p(B_{2R})} \leq CR^{\frac{N}{p}} \inf_{B_R} u(x),$$

*where C depends only on $m, N, p, \nu, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5$ and $p \in (0, \frac{(m-1)N}{N-m})$ if
$m < N$ or $p \in (0, \infty)$ if $m \geq N$.*

THEOREM 8.3. **Hölder continuity of weak solutions** (see Theorems
2.1 and 2.2 §2, chapter IX [215]).

*Let G be of type (A) (see Definition 7.2). Let $u \in W^{1,m}(G) \cap L^\infty(G)$,
$m > 1$ with vrai $\max\limits_{G} |u| = M_0 < \infty$ being a weak solution of the (DQL)
and suppose that the following assumptions are satisfied*

(a) $a_i(x, u, z)z_i \geq \nu|z|^m - g(x)$;

(b) $\sqrt{\sum\limits_{i=1}^{N} (a_i(x, u, z))^2} \leq \mu_1|z|^{m-1} + \varphi_1(x)$;

(c) $|a(x, u, z)| \leq \mu_1|z|^m + \varphi_2(x)$,

where $1 < m \leq N$, and $\varphi_i(x)$ are nonnegative and

$$\|g(x)\|_{L^{p/m}(G)}, \quad \|\varphi_1(x)\|_{L^{p/(m-1)}(G)}, \quad \|\varphi_2(x)\|_{L^{p/m}(G)} \leq const, \quad p > N.$$

Then $u(x)$ is Hölder continuous in $\overline{G}$.

REMARK 8.4. We observe that the condition (a) follows from the ellipticity condition (E) and the condition (b). In fact, we have

$$a_i(x, u, z)z_i = z_i z_j \int\limits_{0}^{1} \frac{\partial a_i(x, u, \tilde{z})}{\partial \tilde{z}_j}\Big|_{\tilde{z}=tz} dt + z_i a_i(x, u, 0) \geq$$

$$\geq \nu|z|^2 \int\limits_{0}^{1} t^{m-2}|z|^{m-2}dt - z_i a_i(x, u, 0) \geq \frac{\nu}{m-1}|z|^m - \varphi_1(x)|z| \geq$$

$$\geq \left(\frac{\nu}{m-1} - \varepsilon\right)|z|^m - c_\varepsilon \varphi_1^{m'}(x), \quad \forall \varepsilon > 0$$

in virtue of the Young inequality.

THEOREM 8.5. **Existence Theorem** (see Theorem 9.2 §9, chapter IV [**216**]). *Let $1 < m < N$, $1 \le p < \infty$. Let the functions $a_i(x, u, z), a(x, u, z)$ be continuous with respect to u, z and satisfy the conditions*

(i) $Q(u, \phi)$ *is coercive, that is*

$$Q(u, u) \ge h(\|u\|_{W_0^{1,m}(G) \cap L^p(G)}) - c_1 \quad \text{for } \forall u \in W_0^{1,m}(G) \cap L^p(G),$$

where $c_1 > 0$, and $h(t)$ is a continuous positive function such that $\lim\limits_{t \to \infty} h(t) = \infty$,

(ii) $|a_i(x, u, z)| \le \mu|z|^{m-1} + \mu|u|^{\widetilde{p}/m'} + \varphi_1(x), \quad \varphi_1(x) \in L^{m'}(G)$,

(iii) $|a(x, u, z)| \le \mu|z|^{m/\widetilde{p}'} + \mu|u|^{\widetilde{p}-1} + \varphi_2(x), \quad \varphi_2(x) \in L^{\widetilde{p}'}(G)$;
with $\widetilde{p} < \overline{p} = \max\left(\frac{mN}{N-m}, p\right)$, $\widetilde{p}' = \frac{\widetilde{p}}{\widetilde{p}-1}$, $m' = \frac{m}{m-1}$,

(iv) $(a_i(x, u, z) - a_i(x, u, w))(z_i - w_i) \ge \psi(|z - w|) \quad \text{for } x \in \overline{G}$,
$|u| \le M_0, \quad \forall z, w \in \mathbb{R}^N$, *where $\psi(\zeta)$ is a continuous, positive for $\zeta > 0$, nondecreasing function.*

Then the problem (DQL) has at least one weak solution from $W_0^{1,m}(G) \cap L^p(G)$.

REMARK 8.6. If the functions $a_i(x, u, z)$ are differentiable with respect to z, then the condition *(iv)* follows from the ellipticity condition (E). In fact, let the ellipticity condition (E) be satisfied. Then considering the following two cases. For 1) $m \ge 2$ and 2) $1 < m < 2$, we obtain

1) m ≥ 2:

$$(a_i(x, u, z) - a_i(x, u, w))(z_i - w_i) =$$

$$= (z_i - w_i) \int\limits_0^1 \frac{d}{dt} a_i(x, u, w + t(z - w)) \, dt =$$

$$= \int\limits_0^1 \frac{\partial a_i(x, u, w + t(z - w))}{\partial z_j} dt \cdot (z_i - w_i)(z_j - w_j) \ge$$

$$\ge \nu|z - w|^2 \int\limits_0^1 |w + t(z - w)|^{m-2} dt \ge \nu c(m)|z - w|^m$$

in virtue of Lemma 1.7 and $m \ge 2$.

2) $1 < m < 2$: We have again

$$(a_i(x,u,z) - a_i(x,u,w))(z_i - w_i) \geq \nu|z-w|^2 \int_0^1 |w+t(z-w)|^{m-2}dt.$$

But

$$|w+t(z-w)| \leq |w| + t|z-w| \Rightarrow |w+t(z-w)|^{m-2} \geq (|w|+t|z-w|)^{m-2}$$

and therefore

$$\int_0^1 |w+t(z-w)|^{m-2}dt \geq \int_0^1 (|w|+t|z-w|)^{m-2}dt =$$

$$= \frac{1}{|z-w|} \int_{|z|}^{|w|+|z-w|} \tau^{m-2}d\tau = \frac{1}{m-1}\frac{(|w|+|z-w|)^{m-1}-|w|^{m-1}}{|z-w|}.$$

Hence it follows that

$$(a_i(x,u,z) - a_i(x,u,w))(z_i - w_i) \geq$$

$$\geq \frac{\nu|z-w|}{m-1}\left\{(|w|+|z-w|)^{m-1}-|w|^{m-1}\right\}.$$

It is easy to verify that in both cases the function $\psi(\zeta)$ satisfies the conditions of (iv).

THEOREM 8.7. Hölder continuity of the first derivatives of weak solutions (see Theorem 1 [228]).

Let μ, M_0 be positive constants. Let G be a bounded domain in $\mathbb{R}^N$ with $C^{1+\alpha}$, $\alpha \in (0,1]$ boundary. Let $u(x)$ be a bounded weak solution of (DQL) with $|u| \leq M_0$. Suppose (DQL) satisfies the ellipticity condition (E) and the structure conditions

$$\sqrt{\sum_{i=1}^N |a_i(x,u,z) - a_i(y,v,z)|^2} \leq \mu(1+|z|)^{m-1}(|x-y|^\alpha + |u-v|^\alpha),$$

$$|a(x,u,z)| \leq \mu(1+|z|)^m$$

for all $(x,u,z) \in \partial G \times [-M_0, M_0] \times \mathbb{R}^N$ and all $(y,v) \in G \times [-M_0, M_0]$.

Then there is a positive constant $\gamma = \gamma(\alpha, \nu^{-1}\mu, m, N)$ such that $u \in C^{1+\gamma}(\overline{G})$. Moreover we have

$$\|u\|_{C^{1+\gamma}(\overline{G})} \leq C(\alpha, \nu^{-1}\mu, M_0, m, N).$$

8.2. The $m-$Laplace operator with an absorption term

8.2.1. Introduction.
We consider the Dirichlet problem

$$(LPA) \quad \begin{cases} \Delta_m u := -\mathrm{div}\,(|\nabla u|^{m-2}\nabla u) = -a_0(x)u|u|^{q-1} + f(x) & \text{in } G, \\ u(x) = 0 & \text{on } \partial G \setminus \{0\}, \end{cases}$$

where $1 < m < \infty$, $q > 0$ and $a_0(x) \geq 0$, $f(x)$ are measurable functions in G.

DEFINITION 8.8. A function u is called a *generalized* solution of (LPA), if $u \in W^{1,m}(G_\varepsilon) \cap L^{q+1}(G_\varepsilon)$ $\forall \varepsilon > 0$ and it satisfies

$$(II) \qquad \int\limits_G \{|\nabla u|^{m-2}\langle \nabla u, \nabla \eta \rangle + a_0(x)u|u|^{q-1}\eta - f\eta\}\, dx = 0$$

for any $\eta \in W^{1,m}(G) \cap L^{q+1}(G)$ having a compact support in G and $u(x) = 0$ on Γ_ε for all $\varepsilon > 0$ in the sense of traces.

DEFINITION 8.9. A function u is called a *weak* solution of (LPA), if $u \in W_0^{1,m}(G) \cap L^{q+1}(G)$ and satisfies (II) for all $\eta \in W_0^{1,m}(G) \cap L^{q+1}(G)$.

Let us denote

$$(8.2.1) \qquad\qquad a_i(z) := |z|^{m-2}z_i.$$

We verify that the ellipticity condition (E) is satisfied with

$$(8.2.2) \quad \mu = \begin{cases} m - 1 & \text{for } m \geq 2 \\ 1 & \text{for } 1 < m \leq 2 \end{cases}, \text{ and } \nu = \begin{cases} 1 & \text{for } m \geq 2 \\ m - 1 & \text{for } 1 < m \leq 2. \end{cases}$$

THEOREM 8.10. **Weak comparison principle.** *Let* $u, v \in W^{1,m}(G)$ *satisfy* $\Delta_m u \leq \Delta_m v$ *in the weak sense, that is*

$$\int\limits_G (a_i(\nabla u) - a_i(\nabla v))\, \eta_{x_i} dx \leq 0$$

for all nonnegative $\eta \in W_0^{1,m}(G)$ *and let*

$$u \leq v \quad \text{on } \partial G.$$

Then

$$u \leq v \quad \text{in } G.$$

PROOF. Since $u - v \leq 0$ on ∂G, we may set

$$\eta = \max(u - v, 0).$$

By the ellipticity condition (E) and by Remark 8.6, we have

$$\int\limits_G (a_i(\nabla u) - a_i(\nabla v))\, (u_{x_i} - v_{x_i})dx \geq \int\limits_G \psi\,(|\nabla(u - v)|)\, dx > 0$$

because $\psi(\zeta)$ is a continuous, positive for $\zeta > 0$, nondecreasing function. Hence, by standard arguments, we obtain the required assertion.   □

THEOREM 8.11. *Let $u(x)$ be a bounded weak solution of (LPA) with $|u(x)| \le M_0$. Suppose that $a_0(x), f(x) \in L^{p/m}(G)$, $p > N$. If $f(x) \ge 0$ in G, then $u(x) \ge 0$ in G.*

PROOF. Choose $\eta = u^- = \max\{-u(x), 0\}$ as a test function in the integral identity (II). We obtain

$$\int\limits_G \left\langle |\nabla u^-|^m + a_0(x)|u^-|^{q+1} + f(x)u^- \right\rangle dx = 0 \Longrightarrow$$

$$\int\limits_G |\nabla u^-|^m dx + \int\limits_G a_0(x)|u^-|^{q+1} dx = -\int\limits_G f(x)u^- dx \le 0,$$

since $u^- \ge 0$. By Theorem 8.3, $u(x)$ is continuous in $\overline{G}$. Due to $a_0(x) \ge 0$ and $u|_{\partial G} = 0$ we get $u^-(x) = 0$ in G and therefore $u(x) \ge 0$ in G.   □

8.2.2. Singular functions for the m–Laplace operator and the corresponding eigenvalue problem.

The first eigenvalue problem which characterizes the singular behavior of the solutions of (LPA) can be derived by inserting in $\Delta_m v = 0$ the function of the form $v = r^\lambda \phi(\omega)$ which leads to the *nonlinear* eigenvalue problem

$(NEVP)$ $\mathfrak{D}(\lambda, \phi) = 0$ in Ω and $\phi = 0$ on $\partial\Omega$,

where

$$\mathfrak{D}(\lambda, \phi) = -\text{div}_\omega\{(\lambda^2\phi^2 + |\nabla_\omega\phi|^2)^{\frac{m-2}{2}}\nabla_\omega\phi\}-$$

($\mathfrak{D}$) $- \lambda\{\lambda(m-1) + N - m\}(\lambda^2\phi^2 + |\nabla_\omega\phi|^2)^{\frac{m-2}{2}}\phi.$

We formulate the Tolksdorf result as follows

THEOREM 8.12. [374, 375]. *There exists a solution $(\lambda_0, \phi) \in \mathbb{R}_+ \times C^\infty(\overline{\Omega})$ of $(NEVP)$ such that*

$$(8.2.3) \quad \lambda_0 > \max\left\{0, \frac{m-N}{m-1}\right\}, \quad \phi > 0 \ \ in \ \Omega, \quad \phi^2 + |\nabla_\omega\phi|^2 > 0 \ \ in \ \overline{\Omega}.$$

REMARK 8.13. In the case $N = 2$, by direct calculation (see (9.4.14)), we can obtain

$$(8.2.4) \quad \lambda_0 = \begin{cases} \dfrac{m + \varkappa(2-\varkappa)(m-2) + (1-\varkappa)\sqrt{m^2 - \varkappa(2-\varkappa)(m-2)^2}}{2\varkappa(m-1)(2-\varkappa)}, & \text{if} \quad \omega_0 < 2\pi; \\[3mm] \dfrac{m-1}{m}, & \text{if} \quad \omega_0 = 2\pi, \end{cases}$$

where $\varkappa = \frac{\omega_0}{\pi}$.

In order to construct a barrier function which can be used in the weak comparison principle, we prove a solvability property of the operator $\mathfrak{D}$ associated to the eigenvalue problem $(NEVP)$.

THEOREM 8.14. *For $0 \leq \lambda < \lambda_0$ there exists a solution ϕ of the problem*

$$(8.2.5) \qquad \mathfrak{D}(\lambda, \phi) = 1 \quad in \; \Omega \quad and \quad \phi = 0 \quad on \; \partial\Omega,$$

with $\phi > 0$ in Ω.

This theorem will be proved in a sequence of lemmas. In the proofs of these lemmas we frequently use the fact that every solution (λ, ϕ) of (8.2.5) corresponds to a solution of

$$\Delta_m(r^\lambda \phi) = r^{(\lambda-1)(m-1)-1} \quad in \; G_0^d,$$

which, by local regularity of the pseudo-Laplace equation, implies that $\phi \in C^\beta(\overline{\Omega}) \cap W_0^{1+\varepsilon,m}(\Omega)$ for $\beta, \varepsilon > 0$.

LEMMA 8.15. *The problem (8.2.5) is solvable for all $0 \leq \lambda < \lambda_0$.*

PROOF. We prove that Fredholm's alternative holds for (8.2.5) in the sense that if (8.2.5) is not solvable then λ is an eigenvalue of $\mathfrak{D}$. For this purpose, we choose a sufficiently large $\alpha \in \mathbb{R}$ such that the problem

$$\mathfrak{D}(\lambda, \phi) + \alpha|\phi|^{m-2}\phi = g \quad in \; \Omega, \quad \phi = 0 \quad on \; \partial\Omega$$

is uniquely solvable for all $g \in H^{-1,m'}(\Omega)$, $\frac{1}{m} + \frac{1}{m'} = 1$, and denote the solution operator by $\phi = \Phi g$. By the regularity of $\mathfrak{D}$, $\Phi: C^\beta(\overline{\Omega}) \to C^\beta(\overline{\Omega})$ is a compact operator for a $\beta > 0$. Moreover, Φ is homogeneous of degree $\frac{1}{m-1}$. The problem $\mathfrak{D}(\lambda, \phi) = f$ in Ω, $\phi = 0$ on $\partial\Omega$, is then equivalent to

$$(8.2.6) \qquad \phi - \alpha F\phi = \Phi f,$$

where $F\phi = \Phi(|\phi|^{m-2}\phi)$ is compact and homogeneous of degree 1. The operator $Id - \alpha F$ is studied on the unit ball

$$B_1 = \{\phi \in C^\beta(\overline{\Omega}) \mid \|\phi\|_{C^\beta} \leq 1\}$$

If $0 \notin (Id - \alpha F)(\partial B_1)$ then K. Borsuk's theorem states that (8.2.6) is solvable for sufficiently small f. Since (8.2.6) is equivalent to $\mathfrak{D}(\lambda, \phi) = f$ and $\mathfrak{D}(\lambda, \cdot)$ is homogeneous of degree $m - 1$ we can solve $\mathfrak{D}(\lambda, \phi) = f$ for all f. $\qquad \square$

LEMMA 8.16. *Let (λ, ϕ) be a solution of (8.2.5). Then $\phi(\omega) \neq 0$ for all $\omega \in \Omega$.*

PROOF. Let $K = \{(r,\omega) \mid 1 < r < 2, \ \omega \in \Omega\}$. If (λ, ϕ) is a solution of (8.2.5) then $v = r^\lambda \phi(\omega)$ solves

$$(8.2.7) \qquad \Delta_m v = r^{(\lambda-1)(m-1)-1} \text{ in } K, \quad v = 0 \text{ on } (1,2) \times \partial\Omega,$$

$$(8.2.8) \qquad v = c_r \phi \text{ for } r = 1, 2.$$

Assume that $\phi(\omega_0) = 0$ for $\omega_0 \in \Omega$. We apply the weak comparison principle on the domain K using the function v. It follows that every solution of

$$\Delta_m u = f \text{ in } K, \quad u = v \text{ on } \partial K,$$

with $f \in C_0^\infty(K)$, satisfies $u(r, \omega_0) \leq 0$ which is a contradiction.   □

LEMMA 8.17. *For sufficiently small $\lambda \geq 0$, the solution of (8.2.5) is unique and satisfies $\phi > 0$ in Ω.*

PROOF. The operator $\mathfrak{D}(0, \cdot)$ is strictly monotone on $W_0^{1,m}(\Omega)$. Hence, the problem (8.2.5) is uniquely solvable and the comparison principle implies $\phi > 0$ in Ω. Since $\mathfrak{D}(\lambda, \cdot)$ is continuous in λ, the conclusion also holds for sufficiently small $\lambda \geq 0$.   □

LEMMA 8.18. *There exists a constant $c = c(\lambda_1)$ such that $\|\phi\|_{1,m} \leq c$ for all solutions (λ, ϕ) of (8.2.5) satisfying $0 \leq \lambda \leq \lambda_1 < \lambda_0$.*

PROOF. Assuming the converse we obtain a sequence (λ_i, ϕ_i) solving (8.2.5) with

$$\lambda_i \to \lambda, \quad \|\phi_i\|_{1,m} \to \infty.$$

For the normalized functions

$$\tilde{\phi}_i = \frac{\phi_i}{\|\phi_i\|_{1,m}}$$

we obtain that $\mathfrak{D}(\lambda_i, \tilde{\phi}_i) \to 0$ in $W^{-1,m'}(\Omega)$ and, by regularity, $\|\tilde{\phi}_i\|_{1+\epsilon,m} \leq c$. Hence, we can extract a subsequence $\{\tilde{\phi}_{i_k}\}$ such that $\tilde{\phi}_{i_k} \to \phi$ in $W_0^{1,m}(\Omega)$ and $\mathfrak{D}(\lambda, \phi) = 0$ with $\|\phi\|_{1,m} = 1$. This contradicts the fact that there is no eigenvalue of $\mathfrak{D}$ in the interval $[0, \lambda_1)$.   □

PROOF OF THEOREM 8.14. Lemma 8.18 implies a kind of continuity of the solutions (λ, ϕ) in the following sense. If $\lambda_i \to \lambda$ with $0 \leq \lambda_i, \lambda < \lambda_0$, then there exists a subsequence $\{\phi_{i_k}\}$ such that

$$\phi_{i_k} \to \phi \text{ in } C^\alpha(\overline{\Omega}),$$

where (λ, ϕ) is a solution of (8.2.5). Hence, by Lemmas 8.16 and 8.17 there exists a solution (λ, ϕ) with $\phi > 0$ in Ω for all $0 \leq \lambda < \lambda_0$.   □

8.2.3. Eigenvalue problem for m-Laplacian in a bounded domain on the unit sphere.

For technical reasons we consider eigenvalue problem for m-Laplacian in a bounded domain Ω on the unit sphere S^{N-1}.

$$\begin{cases} -\mathrm{div}_\omega(|\nabla_\omega\psi|^{m-2}\nabla_\omega\psi) = \mu|\psi|^{m-2}\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

DEFINITION 8.19. We say that μ is an eigenvalue, if there exists a *continuous* function $\psi \in W_0^{1,m}(\Omega)$, $\psi \not\equiv 0$ such that

$$(\text{II2}) \qquad \int_\Omega \left\{ |\nabla_\omega\psi|^{m-2}\frac{1}{q_i}\frac{\partial\psi}{\partial\omega_i}\frac{\partial\eta}{\partial\omega_i} - \mu|\psi|^{m-2}\psi\eta \right\} d\Omega = 0$$

whenever $\eta(x) \in W_0^{1,m}(\Omega)$. The function ψ is called a *weak* eigenfunction (a weak solution of the eigenvalue problem for m-Laplacian).

We characterize the first eigenvalue $\mu(m)$ of the eigenvalue problem for m-Laplacian by

$$(8.2.9) \qquad \mu(m) = \inf_{\substack{\psi\in W_0^{1,m}(\Omega)\\ \psi\neq 0}} \frac{\int_\Omega |\nabla_\omega\psi|^m d\Omega}{\int_\Omega |\psi|^m d\Omega}.$$

THEOREM 8.20. *There exists a solution (μ, ψ) of the eigenvalue problem for m-Laplacian with $\mu > 0$ and $\psi > 0$ in Ω. Furthermore, the following Wirtinger's inequality holds*

$$(W_m) \qquad \int_\Omega |\psi|^m \, d\omega \le \frac{1}{\mu(m)} \int_\Omega |\nabla_\omega\psi|^m \, d\omega, \quad \forall\psi \in W_0^{1,m}(\Omega)$$

with a sharp constant $\frac{1}{\mu(m)}$.

PROOF. Let us introduce the following functionals on $W^{1,m}(\Omega)$

$$F[u] = \int_\Omega |\nabla_\omega u|^m d\Omega, \quad G[u] = \int_\Omega |u|^m d\Omega,$$

$$H[u] = \int_\Omega \left\langle |\nabla_\omega u|^m - \mu|u|^m \right\rangle d\Omega$$

and the corresponding forms

$$F(u,\eta) = \int_\Omega |\nabla_\omega u|^{m-2}\frac{1}{q_i}\frac{\partial u}{\partial\omega_i}\frac{\partial\eta}{\partial\omega_i} d\Omega, \quad G(u,\eta) = \int_\Omega |u|^{m-2}u\eta d\Omega.$$

Now, we define the set

$$K = \left\{ u \in W_0^{1,m}(\Omega) \,\Big|\, G[u] = 1 \right\}.$$

Since $K \subset W_0^{1,m}(\Omega)$, $F[u]$ is bounded from below for $u \in K$. The greatest lower bound of $F[u]$ for this family we denote by μ

$$\inf_{u \in K} F[u] = \mu.$$

Since $F[v]$ is bounded from below for $v \in K$, there is $\mu = \inf_{v \in K} F[v]$. Consider a sequence $\{v_k\} \subset K$ such that $\lim_{k \to \infty} F[v_k] = \mu$. (Such a sequence exists by the definition of infimum.) From $K \subset W_0^{1,m}(\Omega)$ it follows that v_k is bounded in $W_0^{1,m}(\Omega)$ and therefore compact in $L^m(\Omega)$. Choosing a subsequence we can assume that it is converging in $L^m(\Omega)$. Furthermore,

$$(8.2.10) \qquad \|v_k - v_l\|_{L^m(\Omega)}^m = G[v_k - v_l] < \varepsilon$$

as soon as $k, l > N(\varepsilon)$. Now we use Lemma 1.6

$$\left| \frac{v_k + v_l}{2} \right|^m \geq |v_k|^m + \frac{m}{2} |v_k|^{m-2} v_k (v_l - v_k), \ m > 1.$$

We integrate this inequality over Ω

$$\int_\Omega \left| \frac{v_k + v_l}{2} \right|^m d\Omega \geq \int_\Omega |v_k|^m d\Omega + \frac{m}{2} \int_\Omega |v_k|^{m-2} v_k (v_l - v_k) d\Omega.$$

Further, by the Young inequality (1.2.2) with $p = \frac{m}{m-1}$, $q = m$, we have

$$\left| \frac{m}{2} v_k |v_k|^{m-2} (v_l - v_k) \right| \leq \frac{m}{2} |v_k|^{m-1} |v_l - v_k| \leq \frac{m-1}{2} \delta^{\frac{m}{m-1}} |v_k|^m +$$
$$+ \frac{1}{2\delta^m} |v_l - v_k|^m, \ \forall \delta > 0.$$

This fact yields that

$$\int_\Omega \left| \frac{v_k + v_l}{2} \right|^m d\Omega \geq \left(1 - \frac{m-1}{2} \delta^{\frac{m}{m-1}} \right) \int_\Omega |v_k|^m d\Omega -$$
$$- \frac{1}{2\delta^m} \int_\Omega |v_l - v_k|^m d\Omega, \ \forall \delta > 0.$$

This implies that

$$G\left[\frac{v_k + v_l}{2} \right] \geq \left(1 - \frac{m-1}{2} \delta^{\frac{m}{m-1}} \right) G[v_k] - \frac{1}{2\delta^m} G[v_l - v_k], \ \forall \delta > 0.$$

By using $G[v_k] = G[v_l] = 1$ and $G[v_l - v_k] < \varepsilon_1$ we obtain

$$G\left[\frac{v_k + v_l}{2}\right] > 1 - \frac{m-1}{2}\delta^{\frac{m}{m-1}} - \frac{\varepsilon_1}{2\delta^m}, \quad \forall \delta, \varepsilon_1 > 0$$

for big k, l. Now we choose $\delta^m = \varepsilon_1^{\frac{m-1}{m}}$. By setting $\varepsilon = \frac{m\mu}{2}\varepsilon_1^{\frac{1}{m}}$ we get

(8.2.11) $$G\left[\frac{v_k + v_l}{2}\right] > 1 - \frac{\varepsilon}{\mu}$$

for big k, l. The functionals $F[v]$ and $G[v]$ are homogeneous functionals and therefore their ratio $\frac{F[v]}{G[v]}$ does not change under the passage from v to cv ($c = const \neq 0$) and hence

$$\inf_{v \in W^1(\Omega)} \frac{F[v]}{G[v]} = \inf_{v \in K} F[v] = \mu.$$

Therefore $F[v] \geq \mu G[v]$ for all $v \in W_0^{1,m}(\Omega)$. Since $\frac{v_k + v_l}{2} \in W_0^{1,m}(\Omega)$ together with $v_k, v_l \in K$, then

$$F\left[\frac{v_k + v_l}{2}\right] \geq \mu G\left[\frac{v_k + v_l}{2}\right] > \mu\left(1 - \frac{\varepsilon}{\mu}\right) = \mu - \varepsilon, \quad k, l > N(\varepsilon).$$

Let us take k and l large enough so that $F[v_k] < \mu + \varepsilon$ and $F[v_l] < \mu + \varepsilon$. We apply Clarkson's inequalities (Theorem 1.18)

- 1) $\underline{m \geq 2}$

$$F\left[\frac{v_l - v_k}{2}\right] \leq \frac{1}{2}F[v_l] + \frac{1}{2}F[v_k] - F\left[\frac{v_l + v_k}{2}\right] <$$
$$< \mu + \varepsilon - (\mu - \varepsilon) = 2\varepsilon$$

- 2) $\underline{1 < m \leq 2}$

$$F^{\frac{1}{m-1}}\left[\frac{v_l - v_k}{2}\right] \leq \left(\frac{1}{2}F[v_k] + \frac{1}{2}F[v_l]\right)^{\frac{1}{m-1}} -$$
$$- F^{\frac{1}{m-1}}\left[\frac{v_k + v_l}{2}\right] < (\mu + \varepsilon)^{\frac{1}{m-1}} - (\mu - \varepsilon)^{\frac{1}{m-1}} <$$
$$< \frac{2\varepsilon}{m-1}(\mu + \varepsilon)^{\frac{2-m}{m-1}}$$

by Lemma 1.4. Consequently,

(8.2.12) $$F[v_k - v_l] \to 0, \quad \text{as } k, l \to \infty.$$

From (8.2.10), (8.2.12) it follows that $\|v_k - v_l\|_{W_0^{1,m}(\Omega)} \to 0, \quad$ as $k, l \to \infty$. Hence $\{v_k\}$ converges in $W_0^{1,m}(\Omega)$ and as a result of the completeness of $W_0^{1,m}(\Omega)$ there exists a limit function $u \in W_0^{1,m}(\Omega)$ such that

$$\|v_k - u\|_{W_0^{1,m}(\Omega)} \to 0, \quad \text{as } k \to \infty.$$

In addition, again by Lemma 1.4 and the Hölder inequality

$$|F[v_k] - F[u]| = \left| \int\limits_{\Omega} (|\nabla_\omega v_k|^m - |\nabla_\omega u|^m)\, d\Omega \right| \le$$

$$\le m \int\limits_{\Omega} |\nabla v_k|^{m-1} |\nabla_\omega (v_k - u)|\, d\Omega \le$$

$$\le m \left(\int\limits_{\Omega} |\nabla_\omega (v_k - u)|^m d\Omega \right)^{1/m} \left(\int\limits_{\Omega} |\nabla_\omega v_k|^m d\Omega \right)^{(m-1)/m}$$

$$\to 0, \quad \text{as } k \to \infty,$$

since $v_k \in W_0^{1,m}$. Therefore we get

$$F[u] = \lim_{k \to \infty} F[v_k] = \mu.$$

Analogously one sees that $G[u] = 1$.

Suppose now that η is some function from $W_0^{1,m}(\Omega)$. Consider the ratio $\frac{F[u+\varepsilon\eta]}{G[u+\varepsilon\eta]}$. It is a continuously differentiable function of ε on some interval around the point $\varepsilon = 0$. This ratio has a minimum at $\varepsilon = 0$ equal to μ and by the Fermat Theorem, we have

$$\left(\frac{F[u + \varepsilon\eta]}{G[u + \varepsilon\eta]} \right)'_{\varepsilon=0} = m \frac{F(u,\eta)G[u] - F[u]G(u,\eta)}{G^2[u]} = 0,$$

which by virtue of $F[u] = \mu$, $G[u] = 1$ gives

$$F(u,\eta) - \mu G(u,\eta) = 0, \quad \forall \eta \in W_0^{1,m}(\Omega).$$

Further, if u is an eigenfunction of μ, then it follows from the formula (8.2.9) that $|u|$ is one also. But then, by the weak Harnack inequality, Theorem 8.2, either $|u| > 0$ in the whole domain G or $u \equiv 0$ (the latter case being excluded for eigenfunctions). By continuity, either u or $-u$ is positive in the whole domain G. Indeed, suppose that $u = 0$ at some point $x_0 \in G$. Let $B_{3R}(x_0)$ be a ball with so small R that $B_{3R} \subset G$. Then $\inf\limits_{B_R} u(x) = 0$, so in turn $\|u\|_{L^p(B_{2R})} = 0$ by (8.1.1), that is $u = 0$ in B_{2R}. Chaining then gives the conclusion $u \equiv 0$ in G, thus proving the theorem.

Now we shall prove the inequality (W_m). Consider the described above functionals $F[u], G[u], H[u]$ on $W_0^{1,m}(\Omega)$. We will find the minimum of the functional $F[u]$ on the set K. For this we investigate the minimization of the functional $H[u]$ on all functions $u(\omega)$, for which the integral exists and which satisfy the boundary condition $\psi = 0$ on $\partial\Omega$. We use formally the Lagrange multipliers and get the Euler equation from the condition $\delta H[u] = 0$. By the

calculation (with the help of formulas from Section 1.3) of the first variation δH we have

$$\delta H[u] = \delta \left(\int\limits_{\Omega} \left\{ \left[\sum_{i=1}^{N-1} \frac{1}{q_i} \left(\frac{\partial u}{\partial \omega_i} \right)^2 \right]^{\frac{m}{2}} - \mu (u^2)^{\frac{m}{2}} \right\} d\Omega \right) =$$

$$= -m \int\limits_{\Omega} \delta u \cdot \sum_{i=1}^{N-1} \frac{\partial}{\partial \omega_i} \left(\frac{J(\omega)}{q_i} \cdot |\nabla_\omega u|^{m-2} \frac{\partial u}{\partial \omega_i} \right) d\omega -$$

$$- m\mu \int\limits_{\Omega} \delta u \cdot |u|^{m-2} u \, d\Omega =$$

$$= -m \int\limits_{\Omega} \delta u \cdot \left\{ \frac{1}{J(\omega)} \sum_{i=1}^{N-1} \frac{\partial}{\partial \omega_i} \left(\frac{J(\omega)}{q_i} \cdot |\nabla_\omega u|^{m-2} \frac{\partial u}{\partial \omega_i} \right) + \right.$$

$$\left. + \mu u |u|^{m-2} \right\} d\Omega =$$

$$= -m \int\limits_{\Omega} \delta u \cdot \left\{ \mathrm{div}_\omega (|\nabla_\omega u|^{m-2} \nabla_\omega u) + \mu |u|^{m-2} u \right\} d\Omega \implies$$

the eigenvalue problem for m-Laplacian.

Backwards, let $u(\omega)$ be a solution of the eigenvalue problem for m-Laplacian. We multiply both sides of the equation from this problem by u and integrate over Ω, using the Gauss-Ostrogradskiy formula

$$0 = \int\limits_{\Omega} \left\{ u \cdot \mathrm{div}_\omega \left(|\nabla_\omega u|^{m-2} \nabla_\omega u \right) + \mu |u|^m \right\} d\Omega =$$

$$= \mu \int\limits_{\Omega} |u|^m d\Omega +$$

$$+ \int\limits_{\Omega} u \cdot \sum_{i=1}^{N-1} \frac{\partial}{\partial \omega_i} \left(\frac{J(\omega)}{q_i} \cdot |\nabla_\omega u|^{m-2} \frac{\partial u}{\partial \omega_i} \right) d\omega =$$

$$= \mu \int\limits_{\Omega} |u|^m d\Omega -$$

$$- \sum_{i=1}^{N-1} \frac{J(\omega)}{q_i} |\nabla_\omega u|^{m-2} \left(\frac{\partial u}{\partial \omega_i} \right)^2 d\omega =$$

$$= \int\limits_{\Omega} \left(\mu |u|^m - |\nabla_\omega u|^m \right) d\Omega = \mu G[u] - F[u] =$$

$$\underset{(\text{by } K)}{=} \mu - F[u] \Rightarrow \mu = F[u],$$

consequently, the required minimum is the least eigenvalue of the eigenvalue problem for $m-$ Laplacian.

The existence of a function $u \in K$ such that

$$F[u] \le F[v] \text{ for all } v \in K$$

has been proved above.   □

The one-dimensional Wirtinger inequality.
Now we consider the case $N = 2$ and thus let $\Omega = \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2} \right]$ be an arc on the unit circle. Then our eigenvalue problem is

$$\begin{cases} \left(|\psi'|^{m-2}\psi' \right)' + \mu \psi |\psi|^{m-2} = 0, & \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2} \right); \quad m > 1, \\ \psi \left(\pm \frac{\omega_0}{2} \right) = 0. \end{cases}$$

The Wirtinger inequality in this case take the following form

$$\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\psi|^m d\omega \le \frac{1}{\mu(m)} \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\psi'|^m d\omega, \quad \forall \psi \in W_0^{1,m} \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2} \right).$$

We want to calculate the sharp constant $\mu(m)$. First of all, we note that the solutions of our eigenvalue problem are determined uniquely up to a scalar multiple. We consider the solution normed by the condition $\psi(0) = 1$. In addition, it is easy to see that $\psi(-\omega) = \psi(\omega)$ and therefore $\psi'(0) = 0$. Thus we can suppose

$$0 \le \psi(\omega) \le 1.$$

This we shall take into consideration for the solution of the problem.
Rewriting the equation in the form

$$(m-1)|\psi'|^{m-2}\psi'' + \mu \psi |\psi|^{m-2} = 0$$

and solving it direct by the preset parameter method we obtain

$$|\psi'|^m = \frac{\mu}{m-1} \left(1 - \psi^m \right).$$

By integrating from this equation it follows that

$$\pm \sqrt[m]{\frac{\mu}{m-1}} \cdot \omega = \int\limits_{\psi}^{1} \frac{dt}{\sqrt[m]{1-t^m}}.$$

Taking into account the boundary condition we get

$$\sqrt[m]{\frac{\mu}{m-1}} \cdot \frac{\omega_0}{2} = \int\limits_0^1 \frac{dt}{\sqrt[m]{1-t^m}}.$$

Let $\Gamma(x)$ be a gamma-function and the beta-function $B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$. Then we have (see e.g., formula (16) §1.5.1, Chapter 1 [34])

$$\int\limits_0^1 \frac{dt}{\sqrt[m]{1-t^m}} = \frac{1}{m}B\left(\frac{1}{m}, 1-\frac{1}{m}\right) = \frac{\Gamma\left(\frac{1}{m}\right)\cdot\Gamma\left(1-\frac{1}{m}\right)}{m\Gamma(1)} =$$

$$= \frac{1}{m}\Gamma\left(\frac{1}{m}\right)\cdot\Gamma\left(1-\frac{1}{m}\right) = \frac{\pi}{m\sin\left(\frac{\pi}{m}\right)}.$$

Here we used the formula

$$\Gamma(z)\cdot\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad \mathrm{Re}\, z > 0.$$

Thus we get

$$\boxed{\mu(m) = (m-1)\left(\frac{2}{\omega_0} \cdot \frac{\pi}{m\sin\left(\frac{\pi}{m}\right)}\right)^m, \quad \forall m > 1}$$

Hence, in particular, we have the well-known result

$$\boxed{\mu(2) = \left(\frac{\pi}{\omega_0}\right)^2.}$$

Finally, we calculate $\mu(1) = \lim\limits_{m\to 1+0} \mu(m)$. To this end we rewrite obtained result above in this way

$$\mu^{\frac{1}{m}}(m-1)^{-\frac{1}{m}} = \frac{2}{\omega_0} \cdot \frac{1}{m} \cdot \Gamma\left(\frac{1}{m}\right)\cdot\Gamma\left(1-\frac{1}{m}\right).$$

We multiply this equality by $(m-1)$ and use the formula $z\Gamma(z) = \Gamma(1+z)$ thusly

$$\mu^{\frac{1}{m}}(m-1)^{1-\frac{1}{m}} = \frac{2}{\omega_0} \cdot \Gamma\left(\frac{1}{m}\right)\cdot\left(1-\frac{1}{m}\right)\cdot\Gamma\left(1-\frac{1}{m}\right) =$$

$$= \frac{2}{\omega_0} \cdot \Gamma\left(\frac{1}{m}\right)\cdot\Gamma\left(2-\frac{1}{m}\right) \to \frac{2}{\omega_0} \text{ as } m \to 1+0,$$

since $\Gamma(1) = 1$. On the other hand, by $\lim\limits_{x\to+0} x^x = 1$, we have $\lim\limits_{m\to1+0} \mu^{\frac{1}{m}}(m-1)^{1-\frac{1}{m}} = \mu$. Hence it follows that

$$\boxed{\mu(1) = \frac{2}{\omega_0}.}$$

This equality leads to the Wirtinger inequality for the case $m = 1$

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\psi|\,d\omega \leq \frac{\omega_0}{2} \int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\psi'|\,d\omega, \quad \forall\psi \in W_0^{1,1}\left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right).$$

8.2.4. Integral estimates of solutions. The aim of this section is to present integral estimates for the solutions of (LPA). Moreover, the weak comparison principle is not used in the proof so that it may be applied also to the case of elliptic systems.

THEOREM 8.21. *Let $a_0 \in L_{\frac{m}{m-1-q}}(G)$, if $0 < q < m - 1$ and $0 < a_0 \leq a_0(x) \leq a_1$ (a_0, a_1 − const.), if $q \geq m - 1$. Let $f \in V^0_{\frac{m}{m-1},2}(G)$. Then the weak solution u of the problem (LPA) belongs to $V^1_{m,0}(G)$ the inequality*

$$(8.2.13) \quad \int_G \left(|\nabla u|^m + r^{-m}|u|^m + a_0(x)|u|^{1+q}\right) dx \leq c(N,G) \int_G |rf|^{\frac{m}{m-1}} dx$$

holds.

PROOF. Let us consider the function

$$(8.2.14) \qquad \Theta \in C^\infty(\mathbb{R}), \; \Theta(t) \geq 0, \; \Theta(t) = \begin{cases} 0, & t < 1, \\ 1, & t > 2. \end{cases}$$

Inserting $\eta(x) = u(x)\Theta\left(\frac{|x|}{\varepsilon}\right)$ with $\varepsilon > 0$ into the integral identity (II) we obtain

$$(8.2.15) \quad \int_G \left(|\nabla u|^m + a_0(x)|u|^{1+q}\right) \Theta\left(\frac{|x|}{\varepsilon}\right) dx$$

$$\leq c_1\varepsilon^{-1} \int_{G_\varepsilon^{2\varepsilon}} |u||\nabla u|^{m-1} dx + \int_G |u||f|\Theta\left(\frac{|x|}{\varepsilon}\right) dx.$$

By Young's inequality and (W_m) we get

$$(8.2.16) \quad \varepsilon^{-1} \int\limits_{G_\varepsilon^{2\varepsilon}} |u||\nabla u|^{m-1} dx$$

$$\leq c_2 \varepsilon^{-1} \int\limits_{G_\varepsilon^{2\varepsilon}} \left(r|\nabla u|^m + r^{1-m}|u|^m \right) dx \leq c_3(\mu_0) \int\limits_{G_\varepsilon^{2\varepsilon}} |\nabla u|^m dx.$$

From (8.2.15) and (8.2.16) it follows that

$$(8.2.17) \quad \int\limits_G \left(|\nabla u|^m + a_0(x)|u|^{1+q} \right) \Theta \left(\frac{|x|}{\varepsilon} \right) dx$$

$$\leq c_3 \int\limits_{G_\varepsilon^{2\varepsilon}} |\nabla u|^m dx + \int\limits_G |u||f| \Theta \left(\frac{|x|}{\varepsilon} \right) dx.$$

Passing to the limit as $\varepsilon \to 0$ and applying the Young inequality to the last integral on the right hand side of (8.2.17), we obtain the assertion. $\qquad\square$

COROLLARY 8.22. *Let $m > N$. Under the suppositions of Theorem 8.21, a weak solution $u(x)$ of (LPA) is bounded and Hölder continuous in $\overline{G}$.*

PROOF. This follows from Theorem 8.21 in view of the embedding theorem

$$|u(x)| \leq c_0 |x|^{1-\frac{N}{m}} \left(\int\limits_G |rf|^{\frac{m}{m-1}} \right)^{\frac{1}{m}}, \quad x \in \overline{G}.$$

$$\square$$

We set $\mu_0 = \mu(N)$ and observe that $\mu_0 = \mu_0(\Omega)$ is the smallest positive eigenvalue of the eigenvalue problem for m-Laplacian for $m = N$.

THEOREM 8.23. *Let $m = N$ and let the following condition be satisfied*

$$\int\limits_{G_0^\rho} |rf|^{N/(N-1)} dx \leq c\rho^\kappa.$$

$Let \;\; \chi_0 = \frac{2\sqrt{\mu_0}}{(1+\mu_0)^{(N-2)/N}}.$ *Then for any weak solution of (LPA) the*
bound

$$\int\limits_{G_0^\rho} |\nabla u|^N \, dx \le$$

$$(8.2.18) \quad \le c(N,\mu_0,\Omega) \begin{cases} (\rho/d)^{\chi_0}, & if\; \chi_0 < \kappa; \\ (\rho/d)^{\chi_0} \ln^{\frac{N}{N-1}}(d/\rho), & if\; \chi_0 = \kappa, \quad \rho \in (0,d); \\ (\rho/d)^{\kappa}, & if\; \chi_0 > \kappa \end{cases}$$

is satisfied.

REMARK 8.24. It is well known that if $m = N = 2$, then $\mu_0 = \lambda_0^2 = \frac{\pi^2}{\omega_0^2}$, where ω_0 is the quantity of the angle with the vertex 0. In this case the assertion of the theorem was proved in chapter 5 (see Theorems 5.4, 5.5).

The proof of the theorem will be carried out based on the following lemma.

LEMMA 8.25. *Let* $2 \le m \le N$. *For any function* $u \in W_0^{1,m}(G)$ *with* $\nabla u(\rho, \cdot) \in L^m(\Omega)$ *we have*

$$(8.2.19) \quad \int\limits_{\Omega} \left\{ \rho u u_r + \frac{N-m}{2} u^2 \right\} |\nabla u|^{m-2} \, d\omega \le \frac{\rho^2}{\chi} \int\limits_{\Omega} |\nabla u|^m \, d\omega,$$

where

$$(8.2.20) \quad \chi = \frac{m - N + \sqrt{4\mu + (N-m)^2}}{(1+\mu)^{(m-2)/m}}.$$

PROOF. From the Cauchy inequality we obtain

$$\rho u u_r + \frac{N-m}{2} u^2 \le \frac{\varepsilon + N - m}{2} u^2 + \frac{1}{2\varepsilon} \rho^2 u_r^2, \quad \varepsilon > 0,$$

and hence

$$\int\limits_{\Omega} \left\{ \rho u u_r + \frac{N-m}{2} u^2 \right\} |\nabla u|^{m-2} \, d\omega \le \rho^2 \int\limits_{\Omega} \left\{ \frac{\varepsilon + N - m}{2} \left(\frac{u}{\rho} \right)^2 + \right.$$

$$\left. + \frac{1}{2\varepsilon} u_r^2 Bigr \} |\nabla u|^{m-2} \, d\omega =: A \right.$$

The right hand side is estimated by Young's inequality

$$\left(\frac{u}{\rho} \right)^2 |\nabla u|^{m-2} \le \frac{m-2}{m} \delta^{-2/(m-2)} |\nabla u|^m + \frac{2}{m} \delta \left(\frac{|u|}{\rho} \right)^m,$$

$$u_r^2 |\nabla u|^{m-2} \le \frac{m-2}{m} \delta^{-2/(m-2)} |\nabla u|^m + \frac{2}{m} \delta |u_r|^m, \quad \forall \delta > 0,$$

which implies by Wirtinger's inequality (W_m)

$$
A \leq \rho^2 \int_\Omega \left\{ \frac{m-2}{2m} \delta^{-2/(m-2)} \left(\varepsilon + N - m + \frac{1}{\varepsilon} \right) |\nabla u|^m + \right.
$$

$$
\left. + \frac{2\delta}{m} \left(\frac{\varepsilon + N - m}{2} \left(\frac{|u|}{\rho} \right)^m + \frac{1}{2\varepsilon} |u_r|^m \right) \right\} d\omega \leq
$$

$$
\leq \rho^2 \int_\Omega \left\{ \frac{m-2}{2m} \delta^{-2/(m-2)} \left(\varepsilon + N - m + \frac{1}{\varepsilon} \right) |\nabla u|^m + \right.
$$

$$
\left. + \frac{\delta}{m} \left(\frac{\varepsilon + N - m}{\mu} \left| \frac{\nabla_\omega u}{\rho} \right|^m + \frac{1}{\varepsilon} |u_r|^m \right) \right\} d\omega
$$

We choose $\varepsilon > 0$ such that $\frac{\varepsilon + N - m}{\mu} = \frac{1}{\varepsilon}$, which gives

$$
\varepsilon = \frac{1}{2} \left(m - N + \sqrt{(N-m)^2 + 4\mu} \right),
$$

and hence,

$$
\int_\Omega \left\{ \rho u u_r + \frac{N-m}{2} u^2 \right\} |\nabla u|^{m-2} \, d\omega \leq
$$

$$
\leq \frac{\rho^2}{m\varepsilon} \left(\frac{m-2}{2} \delta^{-2(m-2)} (\mu + 1) + \delta \right) \int_\Omega |\nabla u|^m \, d\omega.
$$

The lemma is proved by choosing $\delta = (1 + \mu)^{(m-2)/m}$. $\square$

REMARK 8.26. For $m = N = 2$ the constant χ is sharp.

Proof of Theorem 8.23.. Let

$$
V(\rho) = \int_{G_0^\rho} |\nabla u|^N \, dx.
$$

From (LPA) it follows that

$$
V(\rho) + \int_{G_0^\rho} a_0(x) |u|^{1+q} \, dx = \rho^{N-2} \int_\Omega \rho u u_r |\nabla u|^{N-2} \, d\omega + \int_{G_0^\rho} u f \, dx.
$$

In view of

$$
V'(\rho) = \rho^{N-1} \int_\Omega |\nabla u|^N \, d\omega,
$$

we obtain from Lemma 8.25

$$
V(\rho) \leq \frac{\rho}{\chi_0} V'(\rho) + \int_{G_0^\rho} |u f| \, dx.
$$

The second term of the right hand side can be estimated by the condition of the theorem and Wirtinger's inequality (W_m),

$$\int\limits_{G_0^\rho} |uf|\, dx \leq \left(\int\limits_{G_0^\rho} r^{-N}|u|^N\, dx\right)^{1/N} \left(\int\limits_{G_0^\rho} |rf|^{N/(N-1)}\, dx\right)^{(N-1)/N} \leq$$

$$\leq c\rho^{\kappa\frac{N-1}{N}} V^{\frac{1}{N}}(\rho).$$

Thus we get the differential inequality for $V(\rho)$

$$V(\rho) \leq \frac{\rho}{\chi_0} V'(\rho) + c\rho^{\kappa\frac{N-1}{N}} V^{\frac{1}{N}}(\rho).$$

In view of Theorem 8.21, as an initial condition for this differential inequality, we can use

$$V(d) \leq \int\limits_G |\nabla u|^N\, dx \leq c \int\limits_G |rf|^{N/(N-1)}\, dx \equiv V_0.$$

By putting $W(\rho) = V^{\frac{N-1}{N}}(\rho)$, we obtain the differential inequality for $W(\rho)$

$$\begin{cases} W(\rho) \leq \frac{N}{N-1}\frac{\rho}{\chi_0} W'(\rho) + c\rho^{\kappa\frac{N-1}{N}}, & 0 < \rho < d \\ W(d) = V_0^{\frac{N-1}{N}}. \end{cases}$$

Solving the Cauchy problem for the corresponding equation, we get

$$W^*(\rho) = \left(\frac{\rho}{d}\right)^{\chi_0\frac{N-1}{N}} \left(V_0^{\frac{N}{N-1}} + \right.$$

$$\left. + \kappa\chi_0 \begin{cases} \frac{N-1}{N}\ln\frac{d}{\rho}, & \text{if } \chi_0 = \kappa, \\ \frac{d^{\frac{N-1}{N}(\kappa-\chi_0)} - \rho^{\frac{N-1}{N}(\kappa-\chi_0)}}{\kappa-\chi_0}, & \text{if } \chi_0 \neq \kappa \end{cases}\right).$$

It is well known that the solution of the differential inequality can be estimated by the solution $W^*(\rho)$ of the corresponding equation, that is $W(\rho) \leq W^*(\rho)$ and hence we obtain finally the required estimate. Theorem 8.23 is proved. $\qquad\square$

LEMMA 8.27. *Let* $q > m - 1$, $a_0(x) \geq a_0 > 0$, $(a_0$ - *const). Let*

$$|f(x)| \leq f_1|x|^\beta, \quad x \in G_0^d, \text{ where } \begin{cases} \beta > -1 & \text{if } m > N, \\ \beta > -m & \text{if } m \leq N. \end{cases}$$

Then for any generalized solution $u(x)$ *of* (LPA) *the inequality*

(8.2.21) $\|u\|_{p;G_{\rho/2}^\rho} \leq c(a_0, m, N, p, q, f_1)\rho^{\frac{N}{p} - \frac{m}{q-m+1}} \quad \forall p > m$

holds.

PROOF. We consider the cut-off function

$$\zeta(r) = \begin{cases} 0, & r \in \left[0, \frac{\rho}{4}\right] \cup [2\rho, \infty), \\ 1, & r \in \left[\frac{\rho}{2}, \rho\right]; \end{cases}$$

$$0 \le \zeta(r) \le 1, \ |\nabla\zeta| \le c\rho^{-1}, \ r \in \left[\frac{\rho}{4}, \frac{\rho}{2}\right] \cup [\rho, 2\rho].$$

By putting in (II)

$$\eta(x) = |u|^t \operatorname{sgn} u \cdot \zeta^s(|x|) \quad \forall t \ge 1, \ s > 0,$$

we obtain

$$(8.2.22) \qquad t \int_{G_{\rho/4}^{2\rho}} \zeta^s(r) \left(|u|^{t-1}|\nabla u|^m + a_0(x)|u|^{t+q}\right) dx \le$$

$$\le s \int_{G_{\rho/4}^{2\rho}} |u|^t |\nabla u|^{m-1} \zeta^{s-1} |\nabla\zeta| dx + \int_{G_{\rho/4}^{2\rho}} |u|^t |f| \zeta^s dx.$$

By the Young inequality

$$s|u|^t |\nabla u|^{m-1} \zeta^{s-1} |\nabla\zeta| \le \frac{m-1}{m} \varepsilon^{\frac{m}{m-1}} |u|^{t-1} |\nabla u|^m \zeta^s +$$

$$+ \frac{1}{m} \varepsilon^{-m} s^m |u|^{t+m-1} |\nabla\zeta|^m \zeta^{s-m}, \qquad \forall \varepsilon > 0,$$

choosing $\varepsilon = \left(\frac{tm}{m-1}\right)^{\frac{m-1}{m}}$ and taking into account that $\nabla\zeta = O(\rho^{-1})$, then from (2.13) we get

$$(8.2.23) \qquad a_0 \int_{G_{\rho/4}^{2\rho}} |u|^{t+q} \zeta^s dx \le c(m) \frac{s^m}{t^{m-1}} \int_{G_{\rho/4}^{2\rho}} r^{-m} |u|^{t+m-1} \zeta^{s-m} dx +$$

$$+ \int_{G_{\rho/4}^{2\rho}} |u|^t |f| \zeta^s dx.$$

Applying the Hölder inequality to integrals with $p = \frac{t+q}{t+m-1} > 1$, $p' = \frac{t+q}{q-m+1}$ we obtain

$$(8.2.24) \qquad \int\limits_{G^{2\rho}_{\rho/4}} r^{-m}(|u|^{t+m-1}\zeta^{s-m})dx \le$$

$$\le \left(\int\limits_{G^{2\rho}_{\rho/4}} r^{-mp'}dx\right)^{1/p'} \left(\int\limits_{G^{2\rho}_{\rho/4}} |u|^{t+q}\zeta^{(s-m)\frac{t+q}{t+m-1}}dx\right)^{\frac{t+m-1}{t+q}} .$$

Let us now choose $s = \frac{m(t+q)}{q-m+1}$. Then from (8.2.23), (8.2.24) it follows that

$$(8.2.25) \qquad a_0 \int\limits_{G^{2\rho}_{\rho/4}} |u|^{t+q}\zeta^s dx \le \int\limits_{G^{2\rho}_{\rho/4}} |u|^t|f|\zeta^s dx +$$

$$+ c_1(m,N,t,q)\frac{s^m}{t^{m-1}}\rho^{\frac{N(q-m+1)}{t+q}-m}\left(\int\limits_{G^{2\rho}_{\rho/4}} |u|^{t+q}\zeta^s dx\right)^{\frac{t+m-1}{t+q}} .$$

We estimate the first right hand side term in (8.2.25) by the Hölder inequality

$$\int\limits_{G^{2\rho}_{\rho/4}} |u|^t|f|\zeta^s dx \le \left(\int\limits_{G^{2\rho}_{\rho/4}} |u|^{t+q}\zeta^s dx\right)^{\frac{t}{t+q}} \left(\int\limits_{G^{2\rho}_{\rho/4}} |f|^{\frac{t+q}{q}}\zeta^s dx\right)^{\frac{q}{t+q}} .$$

Then from (8.2.25) we obtain

$$(8.2.26) \qquad a_0 \left(\int\limits_{G^{2\rho}_{\rho/4}} |u|^{t+q}\zeta^s dx\right)^{\frac{q}{t+q}} \le \left(\int\limits_{G^{2\rho}_{\rho/4}} |f|^{\frac{t+q}{q}}\zeta^s dx\right)^{\frac{q}{t+q}} +$$

$$+ c_1(m,N,t,q)\frac{s^m}{t^{m-1}}\rho^{\frac{N(q-m+1)}{t+q}-m}\left(\int\limits_{G^{2\rho}_{\rho/4}} |u|^{t+q}\zeta^s dx\right)^{\frac{m-1}{t+q}} .$$

Again, by the Young inequality, taking into account that $q > m - 1$, we get

$$(8.2.27) \qquad c_1(m, N, t, q) \frac{s^m}{t^{m-1}} \rho^{\frac{N(q-m+1)}{t+q} - m} \left(\int\limits_{G_{\rho/4}^{2\rho}} |u|^{t+q} \zeta^s dx \right)^{\frac{m-1}{t+q}} \leq$$

$$\leq \frac{a_0}{2} \left(\int\limits_{G_{\rho/4}^{2\rho}} |u|^{t+q} \zeta^s dx \right)^{\frac{q}{t+q}} + c_2(m, N, t, q, a_0) \rho^{\frac{nq}{t+q} - \frac{mq}{q-m+1}}.$$

Now by setting $p = t + q > 1 + q > m$, from (8.2.26) and (8.2.27) we arrive at the inequality (8.2.21) sought for. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

LEMMA 8.28. *Suppose the conditions of the Lemma 8.27 hold. Let $u(x)$ be any generalized solution of (LPA). Then the inequality*

$$(8.2.28) \qquad \int\limits_{G_{\rho/2}^{\rho}} \left(|\nabla u|^m + |u|^{1+q} \right) dx \leq c(a_0, m, N, q, f_1) \rho^{N - \frac{(1+q)m}{1+q-m}}$$

is valid.

PROOF. Let us consider the inequality (8.2.22) with $t = 1$ and $\forall s > 0$

$$(8.2.29) \qquad \int\limits_{G_{\rho/4}^{2\rho}} |\nabla u|^m \zeta^s dx + a_0 \int\limits_{G_{\rho/4}^{2\rho}} |u|^{1+q} \zeta^s dx \leq$$

$$\leq cs \int\limits_{G_{\rho/4}^{2\rho}} r^{-1} |u| |\nabla u|^{m-1} \zeta^{s-1} dx + \int\limits_{G_{\rho/4}^{2\rho}} |u| |f| \zeta^s dx.$$

By estimating the first right side term in (8.2.29) with the help of the Young inequality, we have

$$(8.2.30) \qquad \frac{1}{2} \int\limits_{G_{\rho/4}^{2\rho}} |\nabla u|^m \zeta^s dx + a_0 \int\limits_{G_{\rho/4}^{2\rho}} |u|^{1+q} \zeta^s dx \leq$$

$$\leq c(m) s^m \int\limits_{G_{\rho/4}^{2\rho}} r^{-m} |u|^m \zeta^{s-m} dx + \int\limits_{G_{\rho/4}^{2\rho}} |u| |f| \zeta^s dx.$$

By using the Young inequality once again with $p = \frac{1+q}{m}$, $p' = \frac{1+q}{1+q-m}$ and $\forall \delta > 0$ we have

$$c(m)s^m r^{-m}(|u|^m \zeta^{s-m}) \leq \delta |u|^{1+q} \zeta^{(s-m)\frac{1+q}{m}} +$$

$$+ c(\delta, m, s) r^{-m\frac{1+q}{1+q-m}}.$$

Now we set

(8.2.31) $$s = \frac{(1+q)m}{1+q-m}.$$

As a result from (8.2.30) we get

$$\frac{1}{2} \int_{G^{2\rho}_{\rho/4}} |\nabla u|^m \zeta^s dx + a_0 \int_{G^{2\rho}_{\rho/4}} |u|^{1+q} \zeta^s dx \leq \delta \int_{G^{2\rho}_{\rho/4}} |u|^{1+q} \zeta^s dx +$$

$$+ c(\delta, m, q) \int_{G^{2\rho}_{\rho/4}} r^{-s} dx + \int_{G^{2\rho}_{\rho/4}} |u||f|\zeta^s dx \qquad \forall \delta > 0.$$

Hence, by choosing $\delta = \frac{a_0}{2}$, we obtain

(8.2.32) $$\int_{G^{2\rho}_{\rho/4}} \left(|\nabla u|^m + |u|^{1+q} \right) \zeta^s dx \leq c(a_0, m, q, N)\rho^{N-s} +$$

$$+ \varepsilon \int_{G^{2\rho}_{\rho/4}} r^{-m} |u|^m \zeta^s dx + c_\varepsilon \int_{G^{2\rho}_{\rho/4}} (r|f|)^{\frac{m}{m-1}} \zeta^s(r) dx, \qquad \forall \varepsilon > 0.$$

Taking into account the inequality (W_m) and choosing $\varepsilon > 0$ properly, from (8.2.31) and (8.2.32) we get the inequality (8.2.28) sought for. This fact completes the proof of Lemma 8.28.   □

COROLLARY 8.29. *Let* $q > \frac{mN}{N-m} - 1$, $1 < m < N$ *and the hypothesis of Lemma 8.27 about the functions* $a_0(x)$, $f(x)$ *hold. Then for any generalized solution* $u(x)$ *of* (LPA) *the inequality*

(8.2.33) $$\int_{G^\rho_0} \left(|\nabla u|^m + r^{-m}|u|^m + |u|^{1+q} \right) dx \leq c(a_0, N, m, q, f_1, d),$$

$$\forall \rho \in (0, d)$$

is valid.

PROOF. By replacing ρ with $2^{-k}\rho$ $(k = 0, 1, 2, \dots)$ in (8.2.28) and summing the received inequalities over all k, we obtain (8.2.33).   □

8.2.5. Estimates of solutions for singular right hand sides. We state two results of M. Dobrowolski (Theorems 1, 2 [99]). Let λ_0 be the least positive eigenvalue and $\phi(\omega)$ be the corresponding eigenfunction of $(NEVP)$ (see (8.2.3)).

THEOREM 8.30. *Let $u \in W^{1,m}(G)$ be a weak solution of the problem*

$$(PL)_0 \qquad \begin{cases} \Delta_m u = f(x), & x \in G_0^d, \\ u(x) = g(x), & x \in \Omega_d, \\ u(x) = 0, & x \in \Gamma_0^d. \end{cases}$$

Assume that $g(x) \in C^1(\overline{\Omega_d})$ and

$$(8.2.34) \qquad |f(x)| \le f_1 |x|^\beta \quad with \ f_1 \ge 0, \ \beta > \lambda_0(m-1) - m.$$

Then

$$|u(x)| \le c_0 |x|^{\lambda_0}, \quad |\nabla u(x)| \le c_1 |x|^{\lambda_0 - 1}, \quad for \ x \in G_0^d.$$

THEOREM 8.31. *Assume that $0 \le f(x) \le f_1|x|^\beta$ with $\beta > \lambda_0(m-1) - m$ and $a_0(x) \equiv 0$. Then each nonvanishing weak solution of (LPA) admits the singular expansion*

$$u(r, \omega) = kr^{\lambda_0}\phi(\omega) + v(x)$$

with $k > 0$ and

$$|v(x)| \le c|x|^{\lambda_0 + \delta}, \quad |\nabla v| \le c|x|^{\lambda_0 + \delta - 1}, \quad |v_{xx}| \le c|x|^{\lambda_0 + \delta - 2},$$

where the maximum $\delta > 0$ depends on β and the eigenvalue problem $(NEVP)$.

The proof of these results is based on the weak comparison principle for the pseudo-Laplace operator. Here we shall prove the estimates of the modulus of generalized and weak solutions of (LPA) with $a_0 \ge 0$. Let $d > 0$ be a small fixed number. We also suppose that

$$(8.2.35) \qquad |f(x)| \le f_1|x|^\beta, \quad \beta > -\frac{N}{p}$$

with some $p > \frac{N}{m}$.

Observe that a function $v = r^\alpha \phi(\omega)$ is a weak solution $v \in W_0^{1,m}$, if $\phi(\omega)$ is sufficiently smooth and

$$(8.2.36) \qquad \alpha > \frac{m - N}{m}.$$

Since $\Delta_m v \sim r^{\alpha(m-1)-m}$ and the right-hand side of (LPA)

$$-a_0(x)v|v|^{q-1} + f(x) \sim r^{\alpha q} + r^\beta,$$

hence we obtain that

$$(8.2.37) \qquad r^{\alpha(m-1)-m} \sim r^{\alpha q} + r^{\beta}.$$

These arguments suggest the following theorems to us.

THEOREM 8.32. *Let* $u(x)$ *be a weak solution of* (LPA). *Let* $1 < m < N$, $q > 0$ *be given. Let* $a_0(x) \geq a_0 > 0$ *(a_0 is a constant) and let* $f(x) \in L^p(G)$, $p > \frac{N}{m}$. *Then there exists the constant* $M_0 > 0$, *depending only on* $\|f(x)\|_{L^p(G)}$, $\operatorname{meas} G, N, m, q, p, a_0$, *such that*

$$\|u\|_{L_\infty(G)} \leq M_0.$$

PROOF. Let us introduce the set $A(k) = \{x \in \overline{G}, \quad |u(x)| > k\}$ and let $\chi_{A(k)}$ be a characteristic function of the set $A(k)$. We note that $A(k+d) \subseteq A(k) \quad \forall d > 0$. By setting $\phi(x) = \eta((|u| - k)_+)\chi_{A(k)} \cdot \operatorname{sgn} u$ in *(II)*, where η is defined by Lemma 1.60 and $k \geq k_0$ (without loss of generality we can assume that $k_0 \geq 1$), on the strength of the theorem assumptions, we get the inequality

$$(8.2.38) \qquad \int\limits_{A(k)} |\nabla u|^m \eta'((|u| - k)_+) dx + a_0 \int\limits_{A(k)} |u|^q \eta((|u| - k)_+) dx \leq$$

$$\leq \int\limits_{A(k)} |f(x)| \eta((|u| - k)_+) dx.$$

Now we define the function $w_k(x) := \eta\left(\dfrac{(|u| - k)_+}{m}\right)$. By the definition of $\eta(x)$ (see Lemma 1.60)

$$e^{\kappa(|u| - k)_+} |\nabla u|^m = \left(\frac{m}{\kappa}\right)^m |\nabla w_k|^m, \quad \kappa > 0$$

and by the choice of $\kappa > m$ according to Lemma 1.60, using (1.11.5) - (1.11.7), from (8.2.38), we obtain

$$(8.2.39) \qquad \frac{1}{2}\left(\frac{m}{\kappa}\right)^m \int\limits_{A(k)} |\nabla w_k|^m dx + a_0 k_0^q \int\limits_{A(k)} |w_k|^m dx \leq$$

$$\leq c_7 M \int\limits_{A(k+d)} |f(x)| |w_k|^m dx + c_8 e^{\kappa d} \left\langle \int\limits_{A(k) \setminus A(k+d)} |f(x)| dx \right\rangle.$$

By the assumptions of the theorem we have that $f(x) \in L_p(G)$, $p > \frac{N}{m}$. Then by the Hölder inequality for integrals with the exponents p and p'

$(\frac{1}{p} + \frac{1}{p'} = 1)$ we have

$$(8.2.40) \qquad \int\limits_{A(k+d)} |f||w_k|^m dx \leq \|f(x)\|_{L_p(G)} \left(\int\limits_{A(k)} |w_k|^{mp'} dx \right)^{\frac{1}{p'}}.$$

Letting $m^{\#} = \frac{mN}{N-m}$, from the interpolation inequality (see Lemma 1.16) for L^p–norms, we obtain:

$$\left(\int\limits_{A(k)} |w_k|^{mp'} dx \right)^{\frac{1}{p'}} \leq \left(\int\limits_{A(k)} |w_k|^m dx \right)^{\theta} \cdot \left(\int\limits_{A(k)} |w_k|^{m^{\#}} dx \right)^{\frac{(1-\theta)m}{m^{\#}}}$$

with $\theta \in (0,1)$, which is defined by the equality

$$\frac{1}{p'} = \theta + \frac{(1-\theta)m}{m^{\#}} \implies \theta = 1 - \frac{N}{pm}.$$

Thus from (8.2.40) we get

$$(8.2.41) \qquad \int\limits_{A(k+d)} |f||w_k|^m dx \leq \|f(x)\|_{L_p(G)} \left(\int\limits_{A(k)} |w_k|^m dx \right)^{\theta} \times$$
$$\times \left(\int\limits_{A(k)} |w_k|^{m^{\#}} dx \right)^{\frac{(1-\theta)m}{m^{\#}}}.$$

By using the Young inequality with the exponents $\frac{1}{\theta}$ and $\frac{1}{(1-\theta)}$, from (8.2.41) we obtain

$$(8.2.42) \qquad \int\limits_{A(k+d)} |f||w_k|^m dx \leq \frac{\theta \|f(x)\|_{L_p(G)}^{\frac{1}{\theta}}}{\varepsilon^{1/\theta}} \int\limits_{A(k)} |w_k|^m dx +$$
$$+ \varepsilon^{\frac{1}{(1-\theta)}} (1-\theta) \left(\int\limits_{A(k)} |w_k|^{m^{\#}} dx \right)^{\frac{m}{m^{\#}}}, \quad \forall \varepsilon > 0.$$

It follows from (8.2.39), (8.2.42) that

$$(8.2.43) \quad \frac{1}{2}\left(\frac{m}{\kappa}\right)^m \int\limits_{A(k)} |\nabla w_k|^m dx + a_0 k_0^q \int\limits_{A(k)} |w_k|^m dx \leq$$

$$\leq c_9 \varepsilon^{-1/\theta} \int\limits_{A(k)} |w_k|^m dx + c_{10} \varepsilon^{\frac{1}{(1-\theta)}} \left(\int\limits_{A(k)} |w_k|^{m^\#} dx\right)^{\frac{m}{m^\#}} +$$

$$+ c_{11} \int\limits_{A(k)} |f(x)| dx, \quad \forall \varepsilon > 0,$$

where

$$c_9 = \theta M c_7 \|f(x)\|_{L_p(G)}^{\frac{1}{\theta}},$$

$$c_{10} = (1 - \theta) M c_7,$$

$$c_{11} = c_8 e^{\kappa d}.$$

Now we use the Sobolev imbedding Theorem 1.30. Then from (8.2.43) we get

$$(8.2.44) \quad \frac{1}{2}\left(\frac{m}{c_1\kappa}\right)^m \left(\int\limits_{A(k)} |w_k|^{m^\#} dx\right)^{\frac{m}{m^\#}} + a_0 k_0^q \int\limits_{A(k)} |w_k|^m dx \leq$$

$$\leq c_9 \varepsilon^{-1/\theta} \int\limits_{A(k)} |w_k|^m dx + c_{10} \varepsilon^{\frac{1}{(1-\theta)}} \left(\int\limits_{A(k)} |w_k|^{m^\#} dx\right)^{\frac{m}{m^\#}} +$$

$$+ c_{11} \int\limits_{A(k)} |f(x)| dx, \quad \forall \varepsilon > 0.$$

Now, we can choose ε in order to have

$$(8.2.45) \qquad\qquad c_{10}\varepsilon^{\frac{1}{1-\theta}} = \frac{1}{4}\left(\frac{m}{c_1\kappa}\right)^m$$

and k_0 such that

$$(8.2.46) \qquad\qquad c_9\varepsilon^{-\frac{1}{\theta}} = a_0 k_0^q$$

From (8.2.44) it follows that

$$(8.2.47) \qquad \left(\int\limits_{A(k)} |w_k|^{m^\#} dx \right)^{\frac{m}{m^\#}} \leq c_{12} \int\limits_{A(k)} |f(x)| dx \quad \forall k \geq k_0.$$

At last, by Young's inequality we get

$$\int\limits_{A(k)} |f(x)| dx \leq \|f(x)\|_{L_p(G)} \text{ meas }^{1-\frac{1}{p}} A(k).$$

Therefore from (8.2.47) it follows that

$$(8.2.48) \qquad \left(\int\limits_{A(k)} |w_k|^{m^\#} \right)^{\frac{m}{m^\#}} \leq c_{12} \|f(x)\|_{L_p(G)} \text{ meas }^{1-\frac{1}{p}} A(k).$$

Now let $l > k > k_0$. By (1.11.8) of the preliminaries and the definition of the function $w_k(x)$ we have $|w_k| \geq \frac{1}{m}(|u| - k)_+$, and therefore

$$\int\limits_{A(l)} |w_k|^{m^\#} dx \geq \left(\frac{l-k}{m} \right)^{m^\#} \text{ meas } A(l).$$

From (8.2.48) it now follows that

$$(8.2.49) \qquad \text{meas } A(l) \leq \left(\frac{m}{l-k} \right)^{m^\#} \int\limits_{A(k)} |w_k|^{m^\#} dx \leq$$

$$\leq \left(\frac{m}{l-k} \right)^{m^\#} \left(c_{12} \|f(x)\|_{L_p(G)} \right)^{\frac{m^\#}{m}} \text{ meas }^{\frac{m^\#}{m}\left(1-\frac{1}{p}\right)} A(k),$$

$$\forall l > k \geq k_0.$$

Now we set

$$\psi(k) = \text{meas } A(k).$$

Then from (8.2.49) it follows that

$$(8.2.50) \qquad \psi(l) \leq c_{13} \left(\frac{m}{l-k} \right)^{m^\#} [\psi(k)]^{\frac{m^\#}{m}\left(1-\frac{1}{p}\right)}.$$

From the definition of $m^\#$ and the assumption $p > \frac{N}{m}$ we note that

$$\gamma = \frac{m^\#}{m}\left(1 - \frac{1}{p}\right) > 1.$$

Then from (8.2.50) we get

$$\psi(l) \le \frac{c_{19}}{(l-k)^{m^\#}} \psi^\gamma(k) \quad \forall l > k \ge k_0$$

and therefore we have, according to Lemma 1.59 , that $\psi(k_0 + \delta) = 0$ with δ depending only on the quantities in the formulation of Theorem 8.32. This fact means that $|u(x)| < k_0 + \delta$ for almost all $x \in G$. Theorem 8.32 is proved.   □

COROLLARY 8.33. *Let* $1 < m < N$, $q > \frac{mN}{N-m} - 1$ *and* $\beta > -\frac{N}{s}$ *for some* $s > \frac{N}{m}$ *be given. Let* $a_0(x) \ge a_0 > 0$ *(a_0 - const) and* $|f(x)| \le f_1|x|^\beta$. *Suppose*

$$a_0(x),\ f(x) \in L^{p/m}(G),\ p > N.$$

Then any generalized solution $u(x)$ *of* (LPA) *is Hölder continuous in* $\overline{G}$.

PROOF. This assertion follows from Theorem 8.32 and Theorem 8.3 according to the inequality (8.2.33).   □

THEOREM 8.34. *Let* $1 < m < N$ *and* $q > m - 1$ *be given. Let* $0 < a_0 \le a_0(x) \le a_1$, *($a_0, a_1$ — const) and let (8.2.35) is satisfied with some* $\beta \ge 0$. *Let* $u(x)$ *be any generalized solution of* (LPA). *If, in addition,*

$$(8.2.51) \qquad \lambda_0 < \frac{\beta + m}{m - 1}, \quad and\ q > \frac{mN}{N - m} - 1,$$

then

$$(8.2.52) \qquad |u(x)| \le c_0|x|^{\lambda_0}, \quad x \in G_0^d.$$

PROOF. First we apply Lemma 8.27. From the inequality (8.2.21) under $p \to \infty$ the estimate follows

$$(8.2.53) \qquad |u(x)| \le c|x|^{\frac{m}{m-1-q}}.$$

Hence, in view of (8.2.36) the second inequality (8.2.51) is justified. Now we consider the auxiliary problem

$$(8.2.54) \qquad \begin{cases} \Delta_m v = f_1|x|^\beta, & x \in G_0^d, \\ v(x) = u_+(x), & x \in \Omega_d, \\ v(x) = 0, & x \in \Gamma_0^d \end{cases}$$

with some $d > 0$, $f_1 \ge 0$, where $u_+(x)$ is the positive part of $u(x)$.

Under the assumptions of our theorem, by the existence Theorem 8.5, there is a weak solution of the auxiliary problem (8.2.54). Further, by Theorem 8.7, we have that $u(x) \in C^{1+\gamma}(\overline{G_{d/2}^d})$. Then, in view of Theorem 8.30, we have

$$(8.2.55) \qquad 0 \le v(x) \le c_0|x|^{\lambda_0}, \quad and\ |\nabla v| \le c|x|^{\lambda_0 - 1} \quad for\ x \in G_0^d.$$

We wish to prove that

$$(8.2.56) \qquad u(x) \leq v(x) \quad \text{for } x \in G_0^d,$$

which will prove the theorem. To do this, we apply the proof by contradiction. We suppose that $u(x) > v(x)$ on some set $D \subset G_0^d$ is fulfilled. By Corollary 8.33, the set D is a domain. From (LPA) and (8.2.54) we have

$$\Delta_m u \leq f(x) \leq f_1 |x|^\beta = \Delta_m v, \quad \forall x \in D,$$

that is

$$(8.2.57) \qquad \int\limits_D \left(|\nabla u|^{m-2} u_{x_i} - |\nabla v|^{m-2} v_{x_i} \right) \eta_{x_i} dx \leq 0$$

for $\forall \eta(x) \in W_0^{1,m}(D) \cap L^{q+1}(D)$, $\eta(x) \geq 0$. We put

$$w = u - v, \quad u^t = tu + (1-t)v \quad \forall t \in [0,1],$$

$$a^{ij}(x) = \int\limits_0^1 \frac{\partial a_i(u_x^t)}{\partial u_{x_j}^t} dt,$$

where $a_i(z)$ are defined by (8.2.1). Then from (8.2.57) we obtain

$$(8.2.58) \qquad \int\limits_D a^{ij}(x) w_{x_j} \eta_{x_i} dx \leq 0$$

for $\forall \eta(x) \in W_0^{1,m}(D) \cap L^{q+1}(D)$, $\eta(x) \geq 0$. Recall that the ellipticity condition (E) with (8.2.2) holds. Thus, the function $w(x) > 0$ in D and satisfies the integral inequality (8.2.58). Further, by the conditions of the theorem, the inequality (8.2.33) holds and in particular

$$(8.2.59) \qquad \int\limits_D \left(|\nabla u|^m + r^{-m} |u|^m \right) dx \leq \text{const.}$$

The same inequality is true for the function $v(x)$. In fact (8.2.59) for $v(x)$ follows from (8.2.55), if we take into account (8.2.3) and $m < N$. But now we can state the validity of the inequality

$$(8.2.60) \qquad \int\limits_D \left(|\nabla w|^m + r^{-m} |w|^m \right) dx \leq \text{const.}$$

This circumstance makes it possible to put in (8.2.58) the function $\eta(x) = w(x)\Theta\left(\frac{|x|}{\varepsilon}\right)$ with $\Theta(t)$, defined by (8.2.14). As a result we obtain

$$\int_D \Theta\left(\frac{|x|}{\varepsilon}\right) |\nabla w|^2 \left(\int_0^1 |\nabla u^t|^{m-2} dt\right) dx \leq$$

(8.2.61)
$$\leq c \int_{D \cap G_\varepsilon^{2\varepsilon}} r^{-1} w |\nabla w| \left(\int_0^1 |\nabla u^t|^{m-2} dt\right) dx \leq$$

$$\leq c \int_{D \cap G_\varepsilon^{2\varepsilon}} \left(|\nabla w|^m + r^{-m} w^m + |\nabla v|^m\right) dx,$$

(by the Young inequality). In view of (8.2.55) and (8.2.60) the right hand integral is uniformly bounded over $\varepsilon > 0$. Therefore, it is possible to take the limit as $\varepsilon \to 0$ which produces

(8.2.62)
$$\int_D |\nabla w|^2 \left(\int_0^1 |\nabla u^t|^{m-2} dt\right) dx \leq 0.$$

By the continuity of $w(x)$ and in view of $w(x) = 0$, $x \in \partial D$, from (8.2.64) we get $w(x) \equiv 0 \ \forall x \in D$. The contradiction to our assumption $w(x) > 0$ $\forall x \in D$ is finished. By this fact, (8.2.56) and the assertion of Theorem 8.34 are proved. $\qquad\square$

LEMMA 8.35. *Let $u(x)$ be a weak solution of the problem (LPA). If $f(x) \geq 0$ for a.e. $x \in G$ then $u(x) \geq 0$ a.e. in G.*

PROOF. We define

$$G^- = \{x \in G \mid u(x) < 0\}.$$

Choose $\eta = \max\{-u(x), 0\}$ as a test function in the integral identity (II). We obtain

$$\int_{G^-} \langle |\nabla u|^m + a_0(x)|u|^{q+1}\rangle dx = \int_{G^-} f(x)u(x)dx \leq 0.$$

Hence it follows that $u(x) = 0$, $x \in G^-$. Thus $u(x) \geq 0$ a.e. in G. $\qquad\square$

THEOREM 8.36. *Let $1 < m < N$, $q > 0$ be given. Let $a_0(x) \geq a_0 > 0$ (a_0 is a constant) and let (8.2.35) be satisfied. Let $u(x)$ be a weak bounded*

solution of (LPA) with $\sup\limits_{G} |u(x)| = M_0$. *Suppose, in addition,*

$$f(x) \geq 0; \quad and\, a_0(x) \leq M_0^{-q} f(x) \ a.e. \ in\, G.$$

The following assertion holds. If $\lambda_0 < \frac{\beta+m}{m-1}$, *then*

(8.2.63) $$0 \leq u(x) \leq c_0 |x|^{\lambda_0}, \quad x \in G_0^d.$$

PROOF. From the equation of (LPA) we have

$$\Delta_m u = F(x), \quad where\, F(x) \equiv f(x) - a_0(x)u|u|^{q-1}.$$

By Lemma 8.35, $u \geq 0$. Therefore, in view of our assumptions, we get that $0 \leq F(x) \leq f_1 |x|^\beta$. By the assumption on λ_0, β, the conditions of Theorem 8.31 are satisfied. By this theorem we get (8.2.63). $\qquad\square$

THEOREM 8.37. *Let* $1 < m < N$, $q > 0$ *be given. Let* $a_0(x) \geq a_0 > 0$ *(a_0 is a constant) and let (8.2.35) be satisfied. Let* $u(x)$ *be a weak solution of* (LPA).

The following assertion holds. If $\lambda_0 > \frac{\beta+m}{m-1}$, *then*

(8.2.64) $$|u(x)| \leq c_0 |x|^{\frac{\beta+m}{m-1}}, \quad x \in G_0^d.$$

PROOF. By Theorem 8.32 we verify that $u(x)$ is a bounded function. We set $\lambda = \frac{m+\beta}{m-1}$. By the conditions of our theorem,

$$0 < \lambda < \lambda_0.$$

We take

$$v(x) = A|x|^\lambda \phi(\omega)$$

as the barrier functions, where $\forall A > 0$ and (λ, ϕ) is a solution of (8.2.5). It exists in view of Theorem 8.14. In this connection

$$\begin{cases} \Delta_m v = A^{m-1}|x|^{\lambda(m-1)-m}, & x \in G_0^d; \\ v(x) = Ad^\lambda \phi(\omega) \geq 0, & x \in \Omega_d; \\ v(x) = 0, & x \in \Gamma_0^d. \end{cases}$$

By the function $\phi(\omega)$ properties (see Theorem 8.14 and Lemma 8.18) it is easy to verify that

$$0 \leq v(x) \leq cA|x|^\lambda,$$

and

$$\int\limits_{G} \left(|\nabla v|^m + r^{-m}|v|^m \right) dx \leq const.$$

Wishing to prove that $u(x) \leq v(x)$ for $x \in G_0^d$ (by this the assertion of the theorem will be proved), we suppose by contradiction that on some set $D \subset G_0^d$ the inequality $u(x) > v(x)$ is satisfied. Since $u(x)$ is bounded in G,

then by Theorem 8.3 it is Hölder-continuous. This fact implies that the set D is a domain. Further, we have for $x \in D$

$$\Delta_m u(x) \leq f(x) \leq f_1 |x|^\beta = f_1 |x|^{\lambda(m-1)-m} \leq$$

$$\leq A^{m-1} |x|^{\lambda(m-1)-m} = \Delta_m v(x), \quad \text{if } A \geq f_1^{\frac{1}{m-1}}.$$

Moreover, (8.2.59) is valid by Theorem 8.21. Really, for this fact it obviously suffices to show that $\int\limits_G |rf|^{\frac{m}{m-1}} dx$ is finite. Because of (8.2.35) we have

$$\int\limits_G |rf|^{\frac{m}{m-1}} dx \leq f_1^{\frac{m}{m-1}} \int\limits_0^d r^{\frac{m}{m-1}(\beta+1)+N-1} dr < \infty,$$

if $\frac{m}{m-1}(\beta+1) + N > 0$. But by (8.2.35) and since $N > m$ we obtain

$$\frac{m}{m-1}(\beta+1) + N > \frac{m}{m-1}\left(1 - \frac{N}{p}\right) + N > \frac{m}{m-1}(1-m) + N = N - m > 0.$$

Now we repeat the arguments of the proof of Theorem 8.34 word for word and obtain the required assertion of Theorem 8.37.   □

8.3. Estimates of weak solutions near a conical point

In this section we investigate the behavior of the weak solutions of the (DQL) near a conical point. Let λ_0 be the least positive eigenvalue of the problem (EVD) (see Theorem 8.12). Let us introduce the number

$$q = \frac{(1-t)(m-1)}{t}, \quad \text{for } 0 < t \leq 1.$$

Concerning the equation of the (DQL) we make the following **Assumptions**:

the functions $a_i(x,u,z)$ and $a(x,u,z)$ are continuously differentiable with respect to the x, u, z variables in $\mathfrak{M}_{d,M_0} = \overline{G_0^d} \times [-M_0, M_0] \times \mathbb{R}^N$ and satisfy the following inequalities

E) $\nu |u|^q |z|^{m-2} |\xi|^2 \leq \frac{\partial a_i(x,u,z)}{\partial z_j} \xi_i \xi_j \leq \mu |u|^q |z|^{m-2} |\xi|^2,$

$\forall \xi \in \mathbb{R}^N \setminus \{0\};$

1) $\sqrt{\sum\limits_{i=1}^N \left| \frac{\partial a(x,u,z)}{\partial z_i} \right|^2} \leq \mu |u|^{q-1} |z|^{m-1};$

2) $\frac{\partial a(x,u,z)}{\partial u} \geq \nu |u|^{q-2} |z|^m;$

3) $\left| \frac{\partial a_i(x,u,z)}{\partial z_j} - |u|^q |z|^{m-4} \big(\delta_i^j |z|^2 + (m-2) z_i z_j \big) \right| \leq$

$$\leq c_1(r) r^{\beta + m - \lambda_0(m-1)} |u|^q |z|^{m-2} + c_2(r) r^{\beta + 2 - \lambda_0} |u|^{\frac{1-t}{t}};$$

4) $\left| \frac{\partial a_i(x,u,z)}{\partial x_i} \right| + |a(x,u,z)| \leq c_3(r) r^{\beta - m(\lambda_0 - 1)} |u|^{\frac{m(1-t)}{t}} |z|^m +$

$$+ c_4(r) |u|^{\frac{\beta}{t\lambda_0}} + c_5(r) r^{\beta},$$

where $\nu, \mu > 0, \beta > (m-1)\lambda_0 - m$ *are constants,* $c_i(r)$ *are nonnegative,
continuous at zero functions with* $c_i(0) = 0; \quad i = 1, \ldots, 5.$

At first, we transform our problem (DQL) into such problem in which
the leading coefficients are independent of u explicit.

LEMMA 8.38. *Let us make the change of function*

$$(8.3.1) \qquad u = v|v|^{t-1}; \qquad for \ 0 < t \leq 1.$$

Suppose that

$$(U) \qquad u \frac{\partial a_i(x,u,z)}{\partial u} = \frac{1-t}{t} \cdot \frac{\partial a_i(x,u,z)}{\partial z_j} z_j; \quad i = 1, \ldots, N.$$

Then the problem (DQL) *takes the form*

$$(8.3.2) \qquad Q_t(v, \phi) \equiv \int_G \left\langle \mathcal{A}_i(x, v_x) \phi_{x_i} + \mathcal{A}(x, v, v_x) \phi \right\rangle dx = 0$$

for all $\phi(x) \in W_0^{1,m}(G) \cap L^{\infty}(G)$, *where*

$$\mathcal{A}_i(x, \zeta) \equiv a_i(x, v|v|^{t-1}, t|v|^{t-1}\zeta),$$

$$(8.3.3)$$

$$\mathcal{A}(x, v, \zeta) \equiv a(x, v|v|^{t-1}, t|v|^{t-1}\zeta).$$

PROOF. In fact, by calculating, from (8.3.1)-(8.3.3) it follows that

$$\frac{d\mathcal{A}_i}{dv} = \frac{\partial a_i(x,u,z)}{\partial u} \cdot t|v|^{t-1} + \frac{\partial a_i(x,u,z)}{\partial z_j} \cdot t(t-1)|v|^{t-2}\text{sign}\, v \cdot \zeta_j =$$

$$= \frac{\partial a_i(x,u,z)}{\partial u} \cdot t|v|^{t-1} + \frac{\partial a_i(x,u,z)}{\partial z_j} \cdot (t-1)|v|^{t-2}\text{sign}\, v \cdot |v|^{1-t} z_j =$$

$$= \frac{1}{v} \left(tu \frac{\partial a_i(x,u,z)}{\partial u} + (t-1) \frac{\partial a_i(x,u,z)}{\partial z_j} z_j \right) = 0,$$

which is the required statement. $\square$

REMARK 8.39. It is easy to see that we can take

$$(8.3.4) \qquad \begin{cases} t = 1, & \text{if } \frac{da_i(x,u,z)}{du} = 0; \\ t = 1 - \varepsilon, \ \forall \varepsilon \in (0,1), & \text{if } \frac{da_i(x,u,z)}{du} \neq 0. \end{cases}$$

The change (8.3.1) transforms our assumptions into the following

$(\widetilde{E})$ $\quad \nu|\zeta|^{m-2}|\xi|^2 \leq \frac{\partial A_i(x,\zeta)}{\partial \zeta_j}\xi_i\xi_j \leq \mu|\zeta|^{m-2}|\xi|^2, \ \forall \xi \in \mathbb{R}^N \setminus \{0\};$

$\widetilde{1})$ $\quad \sqrt{\sum\limits_{i=1}^{N}\left|\frac{\partial \mathcal{A}(x,v,\zeta)}{\partial \zeta_i}\right|^2} \leq \mu|v|^{-1}|\zeta|^{m-1};$

$\widetilde{2})$ $\quad \frac{\partial \mathcal{A}(x,v,\zeta)}{\partial v} \geq \nu t^{m+1}|v|^{-2}|\zeta|^m;$

$\widetilde{3})$ $\quad \left|\frac{\partial A_i(x,\zeta)}{\partial \zeta_j} - t^{m-1}|\zeta|^{m-4}\big(\delta_i^j|\zeta|^2 + (m-2)\zeta_i\zeta_j\big)\right| \leq$
$$\leq c_1(r)r^{\beta+m-\lambda_0(m-1)}|\zeta|^{m-2} + c_2(r)r^{\beta+2-\lambda_0};$$

$\widetilde{4})$ $\quad \left|\frac{\partial A_i(x,\zeta)}{\partial x_i}\right| + |\mathcal{A}(x,v,\zeta)| \leq c_3(r)r^{\beta-m(\lambda_0-1)}|\zeta|^m + c_4(r)|v|^{\frac{\beta}{\lambda_0}} +$
$c_5(r)r^\beta.$

The main statement of this section is presented by the following theorems.

THEOREM 8.40. *Let* $u(x) \in W^{1,m}(G) \cap L^\infty(G)$ *for* $1 < m < N$ *be a weak solution of the* (DQL). *Suppose that the assumptions* $E), (U), 1) - 4)$ *are fulfilled. Then there exists a constant* $c_0 > 0$, *depending only on the parameters and norms of functions occuring in the assumptions, such that*

$$(8.3.5) \qquad\qquad |u(x)| \leq c_0|x|^{t\lambda_0}.$$

PROOF. Making the transformation (8.3.1) in the problem (DQL) to the equation $Q_t(v,\phi) = 0$ we shall estimate the function $v(x)$ under the assumptions $(\widetilde{E}), \widetilde{1}) - \widetilde{4})$. At first, for some $d > 0$ we consider the auxiliary problem

$$(8.3.6) \qquad \begin{cases} \Delta_m w = f_1|x|^\beta, & x \in G_0^d; \\ w(x) = v_+(x), & x \in \Omega_d; \\ w(x) = 0, & x \in \Gamma_0^d, \end{cases}$$

where $v_+(x)$ is the positive part of $v(x)$ and the constants

$$f_1 \geq 0, \ \beta > (m-1)\lambda_0 - m.$$

Under the assumptions of our Theorem, by the existence Theorem 8.5, there is a weak solution $w(x)$ of the auxiliary problem (8.3.6). Further, by Theorem 8.7, we have that $v(x) \in C^{1+\gamma}(\overline{G^d_{d/2}})$. Then, in view of Theorem 8.30, we have

$$0 \leq w(x) \leq c_0 |x|^{\lambda_0}, \quad |\nabla w| \leq c_1 |x|^{\lambda_0-1},$$

(8.3.7)

$$|w_{xx}| \leq c_2 |x|^{\lambda_0-2}, \quad x \in G^d_0.$$

Now let $\phi \in L_\infty(G^d_0) \cap W^{1,m}_0(G^d_0)$ be any *nonnegative* function. For the operator Q_t, that is defined by (8.3.2), applying the assumptions $\widetilde{3}) - \widetilde{4})$ and estimates (8.3.7) we obtain

$$Q_t(w,\phi) = \int\limits_{G^d_0} \left\langle \mathcal{A}_i(x,w_x)\phi_{x_i} + \mathcal{A}(x,w,w_x)\phi \right\rangle dx =$$

$$= \int\limits_{G^d_0} \phi(x) \left\langle -\frac{d}{dx_i}\mathcal{A}_i(x,w_x) + \mathcal{A}(x,w,w_x) \right\rangle dx =$$

$$= \int\limits_{G^d_0} \phi(x) \left\langle -\frac{d}{dx_i}\left(\mathcal{A}_i(x,w_x) - t^{m-1}|\nabla w|^{m-2}w_{x_i}\right) + f_1 r^\beta + \right.$$

$$\left. + \mathcal{A}(x,w,w_x) \right\rangle dx = \int\limits_{G^d_0} \phi(x) \left\langle f_1 r^\beta - \frac{\partial \mathcal{A}_i(x,w_x)}{\partial x_i} + \mathcal{A}(x,w,w_x) - \right.$$

$$- \left[\frac{\partial \mathcal{A}_i(x,w_x)}{\partial w_{x_j}} - t^{m-1}|\nabla w|^{m-4}\left(\delta^j_i|\nabla w|^2 + \right.\right.$$

$$\left.\left. + (m-2)w_{x_i}w_{x_j}\right]w_{x_ix_j} \right\rangle dx \geq$$

$$\geq \int\limits_{G^d_0} \phi(x) \left\langle f_1 r^\beta - \left|\frac{\partial \mathcal{A}_i(x,w_x)}{\partial x_i}\right| - |\mathcal{A}(x,w,w_x)| - \right.$$

$$- \left|\frac{\partial \mathcal{A}_i(x,w_x)}{\partial w_{x_j}} - t^{m-1}|\nabla w|^{m-4}\left(\delta^j_i|\nabla w|^2 + \right.\right.$$

$$\left.\left. + (m-2)w_{x_i}w_{x_j}\right| \cdot |w_{xx}| \right\rangle dx \geq$$

$$\geq \int\limits_{G^d_0} \phi(x) \left\langle f_1 r^\beta - c_3(r)r^{\beta-m(\lambda_0-1)}|\nabla w|^m - c_4(r)|w|^{\frac{\beta}{\lambda_0}} - c_5(r)r^\beta - \right.$$

$$- c_1(r)r^{\beta+m-\lambda_0(m-1)}|\nabla w|^{m-2}|w_{xx}| - c_2(r)r^{\beta+2-\lambda_0}|w_{xx}| \Big\rangle dx \geq$$

$$\geq \int\limits_{G_0^d} \phi(x)r^\beta \Big\langle f_1 - \sum_{i=1}^5 c_i(r) \Big\rangle dx.$$

Hence, choosing a $d > 0$ by the continuity of $c_i(r)$, $(i = 1, \ldots, 5)$ so small that $\sum_{i=1}^5 c_i(r) \leq \frac{1}{2}f_1$, we get

$$Q_t(w, \phi) \geq \frac{1}{2}f_1 \int\limits_{G_0^d} \phi(x)r^\beta \geq 0.$$

Thus, from (8.3.2) and (8.3.6) we get

$$\begin{cases} Q(w, \phi) \geq 0 = Q(v, \phi) & \forall \phi \geq 0 \quad \text{in} \quad G_0^d; \\ w(x) \geq v(x), & x \in \partial G_0^d. \end{cases}$$

Besides that, one can readily verify that all the other conditions of the comparison principle (Theorem 9.6) are fulfilled. By this principle we get

$$v(x) \leq w_\varepsilon(x), \quad \forall x \in \overline{G_0^d}.$$

Similarly one can prove that

$$v(x) \geq -w(x), \quad \forall x \in \overline{G_0^d}.$$

Thus, finally, we obtain

$$|v(x)| \leq w(x) \leq c_0|x|^{\lambda_0}, \quad \forall x \in \overline{G_0^d}.$$

Returning to the old variables, in virtue of (8.3.1) we get the required estimate (8.3.5). Our theorem is proved.   □

THEOREM 8.41. *Let $u(x) \in W^{1,m}(G) \cap L^\infty(G)$ for $1 < m < N$ be a weak solution of the (DQL). Suppose that the assumptions $(E), (U), 1) - 4)$ are fulfilled. Suppose, in addition,*

$$\sqrt{\sum_{i=1}^N |a_i(x, u, z) - a_i(y, v, z)|^2} \leq \mu(1 + |z|)^{m-1}(|x - y|^\alpha + |u - v|^\alpha)$$

for all $(x, u, z) \in \partial G \times [-M_0, M_0] \times \mathbb{R}^N$ and all $(y, v) \in G \times [-M_0, M_0]$.

Then there exists a constant $c_1 > 0$, depending only on the the parameters and norms of the functions occurring in the assumptions, such that

(8.3.8) $$|\nabla u(x)| \leq c_1|x|^{t\lambda_0 - 1}.$$

PROOF. Let us consider in the layer $G^1_{1/2}$ the function $v(x') = \varrho^{-t\lambda_0}u(\varrho x')$, taking $u \equiv 0$ outside G. Let us perform in the equation (DQL) the change of variables $x = \varrho x'$. The function $v(x')$ satisfies the equation

$$(DQL)' \qquad \begin{cases} \int\limits_{G^1_{1/2}} \left\{ \widetilde{a}_i(x', v, v_{x'})\phi_{x'_i} + \widetilde{a}(x', v, v_{x'})\phi \right\} dx' = 0, \\[2mm] \qquad \forall \phi(x') \in W^{1,m}_0(G^1_{1/2}) \cap L^\infty(G^1_{1/2}); \\[2mm] \widetilde{a}_i(x', v, v_{x'}) \equiv a_i(\varrho x', \varrho^{t\lambda_0}v, \varrho^{t\lambda_0-1}v_{x'}), \\[2mm] \widetilde{a}(x', v, v_{x'}) \equiv \varrho a(\varrho x', \varrho^{t\lambda_0}v, \varrho^{t\lambda_0-1}v_{x'}). \end{cases}$$

In virtue of the assumptions of our theorem, we can apply the Lieberman Theorem 8.7:

$$\sup_{G^1_{1/2}} |\nabla' v| \le M'_1,$$

where $M'_1 > 0$ is determined only by $t, \lambda_0, \alpha, \nu, \mu, N, G$ and c_0 from (8.3.5). Hence, returning to the function $u(x)$ we get

$$|\nabla u(x)| \le M'_1 \varrho^{t\lambda_0-1}, \quad x \in G^\varrho_{\varrho/2}.$$

Letting $|x| = \frac{2}{3}\varrho$, we obtain the desired inequality (8.3.8). $\square$

COROLLARY 8.42. *From Remark 8.39 it follows that the estimates (8.3.5), (8.3.8) can be rewritten in the following form*

$$(8.3.9) \qquad |u(x)| \le c \begin{cases} |x|^{\lambda_0}, & \text{if } \quad \frac{da_i(x,u,z)}{du} = 0, \\[2mm] |x|^{\lambda_0-\varepsilon}, \ \forall \varepsilon \in (0,1), & \text{if } \quad \frac{da_i(x,u,z)}{du} \ne 0. \end{cases}$$

$$(8.3.10) \qquad |\nabla u(x)| \le c \begin{cases} |x|^{\lambda_0-1}, & \text{if } \quad \frac{da_i(x,u,z)}{du} = 0, \\[2mm] |x|^{\lambda_0-1-\varepsilon}, \ \forall \varepsilon \in (0,1), & \text{if } \quad \frac{da_i(x,u,z)}{du} \ne 0. \end{cases}$$

8.4. Integral estimates of second weak derivatives of solutions

In this section we will derive *a priori* estimates of second derivatives (in terms of the Sobolev weighted norm) of solutions to the (DQL) in a neighborhood of a conical boundary point. We give an example which demonstrates that the estimates obtained are exact.

We define the set $\mathfrak{M} = \overline{G} \times \mathbb{R} \times \mathbb{R}^N$ and we will suppose that the ellipticity condition (E) and the following *assumptions* are fulfilled

there exist a number $\mu > 0$ and nonnegative functions

$$f(x) \in L^2(G) \cap L^{(m+2)/m}(G) \cap L^{p/m}(G),$$

$$g(x) \in L^{2(m+2)/m}(G) \cap L^{(m+2)/(m-1)}(G) \cap L^{p/(m-1)}(G),$$

$$p > N$$

such that

(A) $a_i(x, u, z), a(x, u, z) \in C^1(\mathfrak{M}), \; i = 1, \ldots, N;$

(B) $|a(x, u, z)| + \left| \sum_{i=1}^{N} \frac{\partial a_i(x, u, z)}{\partial x_i} \right| \le \mu |z|^m + f(x) |z|^{\frac{m-2}{2}};$

(C) $\sqrt{\sum_{i=1}^{N} \left| \frac{\partial a_i(x, u, z)}{\partial u} \right|^2} + \sqrt{\sum_{i=1}^{N} \left| \frac{\partial a(x, u, z)}{\partial z_i} \right|^2} \le \mu |z|^{m-1} + g(x) |z|^{\frac{m-2}{2}};$

(D) $|z| \cdot \sqrt{\sum_{i=1}^{N} |a_i(x, u, z) - a_i(y, v, z)|^2} + |a(x, u, z) - a(y, v, z)| \le$

$\le \mu |z|^m \left(|x - y| + |u - v| \right), \quad \forall x, y \in G, \; \forall u, v \in \mathbb{R};$

(F) $\left| \sum_{i=1}^{N} \frac{\partial a_i(x, u, 0)}{\partial x_i} \right| \le f(x); \quad \sqrt{\sum_{i=1}^{N} \left| \frac{\partial a_i(x, u, 0)}{\partial u} \right|^2} \le g(x).$

We make the transformation $x = \varrho x'$. Let $v(x') = u(\varrho x')$ and G' be the image of G under this transformation. Let $d > 0$ be so small that if $\varrho \in (0, d)$, then $G_{1/4}^2 \subset G'$. Further, our problem (DQL) takes the form

$(DQL)'$ $\qquad \begin{cases} \int\limits_{G'} \left\{ \widetilde{a}_i(x', v, v_{x'}) \phi_{x'_i} + \widetilde{a}(x', v, v_{x'}) \phi \right\} dx' = 0, \\ \qquad\qquad \forall \phi(x') \in W_0^{1,m}(G') \cap L^\infty(G'); \\ \widetilde{a}_i(x', v, v_{x'}) \equiv a_i(\varrho x', v, \varrho^{-1} v_{x'}), \\ \widetilde{a}(x', v, v_{x'}) \equiv \varrho a(\varrho x', v, \varrho^{-1} v_{x'}). \end{cases}$

At first we establish the strong interior estimate.

8.4.1. Local interior estimates.

In this subsection we derive local interior integral estimates of weak solutions of the problem (DQL).

THEOREM 8.43. *Let $u(x)$ be a bounded weak solution of the problem (DQL). Let us assume that the hypotheses $(A), (B), (C), (D), (E), (F)$ are fulfilled on the set $\mathfrak{M}$. Let any $\widetilde{G}$ be such $\widetilde{G} \subset\subset G_{\varrho/4}^\varrho \subset G$. Then there exists*

the integral $\int\limits_{\widetilde{G}} \left(|\nabla u|^{m+2} + |\nabla u|^{m-2} u_{xx}^2 \right) dx$ and we have the estimate

$$\int\limits_{\widetilde{G}} \left(|\nabla u|^{m+2} + |\nabla u|^{m-2} u_{xx}^2 \right) dx \leq$$

(8.4.1)

$$\leq C \int\limits_{\widetilde{G}} \left(\rho^{-2} |\nabla u|^m + f^2 + f^{\frac{m+2}{m}} + g^{\frac{m+2}{m-1}} + g^{\frac{2(m+2)}{m}} \right) dx.$$

PROOF. Let the image of $\widetilde{G}$ be $\widetilde{G}' \subset\subset G_{1/4}^2 \subset G'$. For all $x_0' \in \widetilde{G}'$ and all σ such that $0 < \sigma < \text{dist}\,(\widetilde{G}', \partial G_{1/4}^2)$, we take

$$\phi(x') = \triangle_k^{-h} \left(\zeta^2(x') \triangle_k^h v(x') \right)$$

as the test function in the $(DQL)'$, where $\zeta(x') \in C_0^\infty \left(B_{2\sigma}(x_0') \right)$ is a cutoff function such that

$$\zeta(x') = 1 \text{ in } B_\sigma(x_0'), \quad 0 \leq \zeta(x') \leq 1, \ |\nabla'\zeta| \leq c\sigma^{-1} \text{ in } B_{2\sigma}(x_0').$$

Then for sufficiently small $|h| \leq \sigma$, summing formula (1.11.17) by parts, we obtain

$$(8.4.2) \quad \int\limits_{B_{2\sigma}(x_0')} \left\{ \triangle_k^h \widetilde{a}_i(x', v, v_{x'}) \left(\zeta^2 \frac{\partial \triangle_k^h v(x')}{\partial x_i'} + 2\zeta \zeta_{x_i'} \triangle_k^h v(x') \right) + \right.$$

$$\left. + \triangle_k^h v(x') \zeta^2 \triangle_k^h \widetilde{a}(x', v, v_{x'}) \right\} dx' = 0,$$

where

$$\triangle_k^h \widetilde{a}_i(x', v, v_{x'}) = \widetilde{a}^{ij}(x') \frac{\partial \triangle_k^h v(x')}{\partial x_j'} + \widetilde{a}^i(x'),$$

$$\triangle_k^h \widetilde{a}(x', v, v_{x'}) = b^j(x') \frac{\partial \triangle_k^h v(x')}{\partial x_j'} + b(x')$$

with

$$\widetilde{a}^{ij}(x') \equiv \int\limits_0^1 \frac{\partial \widetilde{a}_i(x', v, v_{x'}^t)}{\partial v_{x_j'}^t} dt; \quad b^j(x') \equiv \int\limits_0^1 \frac{\partial \widetilde{a}(x', v, v_{x'}^t)}{\partial v_{x_j'}^t} dt;$$

$$\widetilde{a}^i(x') \equiv \frac{1}{h} \left[\widetilde{a}_i \left(x' + h\mathbf{e}_k, v(x' + h\mathbf{e}_k), v_{x'}(x' + h\mathbf{e}_k) \right) - \right.$$

$$\left. - \widetilde{a}_i \left(x', v(x'), v_{x'}(x' + h\mathbf{e}_k) \right) \right];$$

$$b(x') \equiv \frac{\widetilde{a}\left(x' + h\mathbf{e}_k, v(x' + h\mathbf{e}_k), v_{x'}(x' + h\mathbf{e}_k)\right) - \widetilde{a}\left(x', v(x'), v_{x'}(x' + h\mathbf{e}_k)\right)}{h};$$

$$v^t(x') = (1 - t)v(x') + tv(x' + h\mathbf{e}_k).$$

Thus we get (for brevity we denote $B_{2\sigma} = B_{2\sigma}(x_0')$)

$$\int\limits_{B_{2\sigma}} \widetilde{a}^{ij}(x') \frac{\partial \triangle_k^h v(x')}{\partial x_i'} \frac{\partial \triangle_k^h v(x')}{\partial x_j'} \zeta^2 dx' \leq \int\limits_{B_{2\sigma}} \left\{ \left| \widetilde{a}^{ij}(x') \frac{\partial \triangle_k^h v(x')}{\partial x_j'} 2\zeta \zeta_{x_i'} \triangle_k^h v(x') \right| \right.$$

$$(8.4.3) \qquad + \left| \widetilde{a}^i(x') \frac{\partial \triangle_k^h v(x')}{\partial x_i'} \zeta^2 \right| + \left| \widetilde{a}^i(x') 2\zeta \zeta_{x_i'} \triangle_k^h v(x') \right| +$$

$$+ \left| b^j(x') \frac{\partial \triangle_k^h v(x')}{\partial x_j'} \triangle_k^h v(x') \zeta^2 \right| + \left| b(x') \triangle_k^h v(x') \zeta^2 \right| \right\} dx'.$$

Letting

$$(8.4.4) \qquad P_k(x') \equiv |\nabla' v(x')| + |\nabla' v(x' + h\mathbf{e}_k)|,$$

by assumptions $(C), (D), (E)$, and applying Lemma 1.7, we have

$$\widetilde{a}^{ij}(x') \frac{\partial \triangle_k^h v(x')}{\partial x_i'} \frac{\partial \triangle_k^h v(x')}{\partial x_j'} \geq \nu c(m) \varrho^{1-m} P_k^{m-2}(x') |\nabla' \triangle_k^h v(x')|^2;$$

$$|\widetilde{a}^{ij}(x')| \leq \frac{\mu}{m-1} \varrho^{1-m} P_k^{m-2}(x');$$

$$|\widetilde{a}^i(x')| \leq \mu \varrho^{1-m} P_k^{m-1}(x') \left(1 + |\triangle_k^h v(x')|\right);$$

$$(8.4.5) \qquad |b^j(x')| \leq \varrho^{1-m} \left\langle \mu P_k^{m-1}(x') + \varrho^{\frac{m}{2}} g(\varrho x') P_k^{\frac{m-2}{2}} \right\rangle;$$

$$(i, j = 1, \ldots, N)$$

$$|b(x')| \leq \mu \varrho^{1-m} P_k^m(x') \left(1 + |\triangle_k^h v(x')|\right).$$

Now from (8.4.3)-(8.4.5) it follows that

$$\int_{B_{2\sigma}} P_k^{m-2}(x')|\nabla'\triangle_k^h v(x')|^2\zeta^2 dx' \leq$$

$$\leq c(\nu,\mu,m) \int_{B_{2\sigma}} \Big(P_k^{m-2}(x')|\nabla'\triangle_k^h v(x')||\triangle_k^h v(x')|\zeta|\nabla'\zeta|+$$

$$(8.4.6) \quad P_k^{m-1}(x')|\nabla'\triangle_k^h v(x')|\zeta^2 + P_k^{m-1}(x')|\nabla'\triangle_k^h v(x')||\triangle_k^h v(x'|\zeta^2+$$

$$+P_k^{\frac{m-2}{2}}(x')|\nabla'\triangle_k^h v(x')||\triangle_k^h v(x'|\varrho^{\frac{m}{2}}g(\varrho x')\zeta^2+$$

$$+P_k^{m-1}(x')|\triangle_k^h v(x')|\zeta|\nabla'\zeta| + P_k^{m-1}(x')|\triangle_k^h v(x')|^2\zeta|\nabla'\zeta|+$$

$$+P_k^m(x')|\triangle_k^h v(x'|\zeta^2 + P_k^m(x')|\triangle_k^h v(x'|^2\zeta^2 \Big) dx'.$$

Now we estimate each term on the right using the Cauchy inequality with $\forall \varepsilon > 0$

$$|\nabla'\triangle_k^h v(x')||\triangle_k^h v(x')|\zeta|\nabla'\zeta| \leq \frac{\varepsilon}{2}|\nabla'\triangle_k^h v(x')|^2\zeta^2+$$

$$+\frac{1}{2\varepsilon}|\triangle_k^h v(x')|^2|\nabla'\zeta|^2;$$

$$P_k^{m-1}(x')|\nabla'\triangle_k^h v(x')| \leq \frac{\varepsilon}{2}P_k^{m-2}|\nabla'\triangle_k^h v(x')|^2 + \frac{1}{2\varepsilon}P_k^m;$$

$$P_k^{m-1}(x')|\nabla'\triangle_k^h v(x')||\triangle_k^h v(x'| \leq \frac{\varepsilon}{2}P_k^{m-2}|\nabla'\triangle_k^h v(x')|^2+$$

$$+\frac{1}{2\varepsilon}P_k^m|\triangle_k^h v(x')|^2;$$

$$P_k^{\frac{m-2}{2}}(x')|\nabla'\triangle_k^h v(x')||\triangle_k^h v(x'|\varrho^{\frac{m}{2}}g(\varrho x') \leq \frac{\varepsilon}{2}P_k^{m-2}|\nabla'\triangle_k^h v(x')|^2+$$

$$+\frac{1}{2\varepsilon}|\triangle_k^h v(x')|^2\varrho^m g^2(\varrho x')$$

$$P_k^{m-1}(x')|\triangle_k^h v(x')|\zeta|\nabla'\zeta| \leq \frac{1}{2}P_k^{m-2}(x')|\triangle_k^h v(x')|^2\zeta^2+$$

$$+\frac{1}{2}P_k^m|\nabla'\zeta|^2;$$

$$P_k^{m-1}(x')|\triangle_k^h v(x')|^2\zeta|\nabla'\zeta| \leq \frac{1}{2}P_k^m(x')|\triangle_k^h v(x')|^2\zeta^2+$$

$$+\frac{1}{2}P_k^{m-2}|\triangle_k^h v(x')|^2|\nabla'\zeta|^2;$$

$$P_k^m(x')|\triangle_k^h v(x'|\zeta^2 \leq \frac{1}{2}P_k^m(x')|\triangle_k^h v(x')|^2\zeta^2 + \frac{1}{2}P_k^m\zeta^2.$$

Choosing $\varepsilon > 0$ in an appropriate way we get from (8.4.6)

$$\int\limits_{B_{2\sigma}} P_k^{m-2}(x')|\nabla'\triangle_k^h v(x')|^2\zeta^2 dx' \le c(\nu,\mu,m)\int\limits_{B_{2\sigma}}\Big\{ P_k^m|\triangle_k^h v(x')|^2\zeta^2 +$$

$$+ \left(P_k^{m-2}(x')|\triangle_k^h v(x')|^2 + P_k^m(x')\right)\left(\zeta^2 + |\nabla'\zeta|^2\right) +$$

$$(8.4.7)\qquad\qquad +|\triangle_k^h v(x')|^2\varrho^m g^2(\varrho x')\zeta^2\Big\} dx'.$$

In order to estimate the integral $\int\limits_{B_{2\sigma}} P_k^m|\triangle_k^h v(x')|^2\zeta^2 dx'$ we take

$$\phi(x') = \left(v(x') - v(x'_0)\right)\zeta^2(x')|\triangle_k^h v(x')|^2$$

as the test function in the $(DQL)'$. Then we obtain

$$(8.4.8)\qquad \int\limits_{B_{2\sigma}}\Big\{\widetilde{a}_i(x',v,v_{x'})\left(\left(v(x') - v(x'_0)\right)\zeta^2(x')|\triangle_k^h v(x')|^2\right)_{x'_i} +$$

$$+ \widetilde{a}(x',v,v_{x'})\left(v(x') - v(x'_0)\right)\zeta^2(x')|\triangle_k^h v(x')|^2\Big\}dx' = 0.$$

Now we use the representation

$$\widetilde{a}_i(x',v,z) = \widetilde{a}_{ij}(x',v,z)z_j + \widetilde{a}_i(x',v,0),$$

(8.4.9)

$$\widetilde{a}_{ij}(x',v,z) = \int\limits_0^1 \frac{\partial\widetilde{a}_i(x',v,\tau z)}{\partial(\tau z_j)}d\tau, \quad (i,j = 1,\dots N).$$

Therefore from (8.4.8) it follows that

$$(8.4.10)\qquad \int\limits_{B_{2\sigma}}\widetilde{a}_{ij}(x',v,v_{x'})v_{x'_i}v_{x'_j}|\triangle_k^h v(x')|^2\zeta^2(x')dx' =$$

$$= -\int\limits_{B_{2\sigma}}\Big\{\widetilde{a}(x',v,v_{x'})\left(v(x') - v(x'_0)\right)|\triangle_k^h v(x')|^2\zeta^2(x') +$$

$$+ 2\widetilde{a}_{ij}(x',v,v_{x'})\left(v(x') - v(x'_0)\right)v_{x'_j}\left(|\triangle_k^h v(x')|^2\zeta\zeta_{x'_i} +\right.$$

$$\left.+ \triangle_k^h v(x')\frac{\partial(\triangle_k^h v(x'))}{\partial x'_i}\zeta^2\right) +$$

$$+ \widetilde{a}_i(x',v,0)\left(\left(v(x') - v(x'_0)\right)\zeta^2(x')|\triangle_k^h v(x')|^2\right)_{x'_i}\Big\}dx'.$$

In the last term on the right we integrate by parts and so obtain

$$(8.4.11) \quad \int\limits_{B_{2\sigma}} \widetilde{a}_{ij}(x', v, v_{x'})v_{x'_i}v_{x'_j}|\triangle_k^h v(x')|^2 \zeta^2(x')dx' =$$

$$= - \int\limits_{B_{2\sigma}} (v(x') - v(x'_0)) \left\{ \widetilde{a}(x', v, v_{x'})|\triangle_k^h v(x')|^2 \zeta^2(x') + \right.$$

$$+ 2\widetilde{a}_{ij}(x', v, v_{x'})v_{x'_i} \left(|\triangle_k^h v(x')|^2 \zeta \zeta_{x'_i} + \triangle_k^h v(x') \frac{\partial(\triangle_k^h v(x'))}{\partial x'_i} \zeta^2 \right) -$$

$$- \left(\frac{\partial \widetilde{a}_i(x', v, 0)}{\partial x'_i} + \frac{\partial \widetilde{a}_i(x', v, 0)}{\partial v}v_{x'_i} \right) \zeta^2(x')|\triangle_k^h v(x')|^2 \right\}dx'.$$

After a simple computation, using assumptions $(B), (E), (F)$ and taking into account $0 < \varrho < d < 1$, we obtain from (8.4.11)

$$\int\limits_{B_{2\sigma}} |\nabla' v|^m |\triangle_k^h v|^2 \zeta^2 dx' \le c(\nu, \mu, m) \int\limits_{B_{2\sigma}} |v(x') - v(x'_0)| \left\{ |\nabla' v|^m |\triangle_k^h v|^2 \zeta^2 + \right.$$

$$+ |\nabla' v|^{m-1}|\triangle_k^h v|^2 \zeta|\nabla'\zeta| + |\nabla'(\triangle_k^h v)||\nabla' v|^{m-1}|\triangle_k^h v|\zeta^2 + \varrho^m f(\varrho x')|\triangle_k^h v|^2 \zeta^2$$

$$(8.4.12)$$

$$+ \varrho^{1+\frac{m}{2}} f(\varrho x')|\nabla' v|^{\frac{m-2}{2}}|\triangle_k^h v|^2 \zeta^2 + \varrho^{m-1}g(\varrho x')|\nabla' v||\triangle_k^h v|^2 \zeta^2 \right\}dx'.$$

Taking into consideration Remark 8.4 we observe that all hypotheses of Theorem 8.3 about Hölder continuity of weak solutions are fulfilled and conclude

$$|v(x') - v(x'_0)| \le c\sigma^\alpha, \quad x' \in B_{2\sigma}(x'_0), \quad \alpha \in (0,1).$$

Moreover, we use the Cauchy inequality

$$|\nabla' v|^{m-1}\zeta|\nabla'\zeta| \le \frac{1}{2}|\nabla' v|^m \zeta^2 + \frac{1}{2}|\nabla' v|^{m-2}|\nabla'\zeta|^2,$$

$$|\nabla'(\triangle_k^h v)||\nabla' v|^{m-1}|\triangle_k^h v| \le \frac{1}{2}|\nabla' v|^{m-2}|\nabla'(\triangle_k^h v)|^2 + |\nabla' v|^m |\triangle_k^h v|^2.$$

Hence and from (8.4.12), it follows that

$$
\int\limits_{B_{2\sigma}} |\nabla'v|^m |\triangle_k^h v|^2 \zeta^2 dx' \le c(\nu,\mu,m)\sigma^\alpha \int\limits_{B_{2\sigma}} \Big\{ |\nabla'v|^m |\triangle_k^h v|^2 \zeta^2 +
$$

$$
+ |\nabla'v|^{m-2} |\triangle_k^h v|^2 |\nabla'\zeta|^2 + |\nabla'(\triangle_k^h v)|^2 |\nabla'v|^{m-2} \zeta^2 + \varrho^m f(\varrho x')|\triangle_k^h v|^2 \zeta^2 +
$$

(8.4.13)

$$
+ \varrho^{1+\frac{m}{2}} f(\varrho x')|\nabla'v|^{\frac{m-2}{2}} |\triangle_k^h v|^2 \zeta^2 + \varrho^{m-1} g(\varrho x')|\nabla'v||\triangle_k^h v|^2 \zeta^2 \Big\} dx'.
$$

Now we consider the function $w(x') = v(x' + h\mathbf{e}_k)$. It is easy to observe that this function is the bounded weak solution of the equation

$$
-\frac{d}{dx_i'} \widetilde{a}_i (x' + h\mathbf{e}_k, w(x'), w_{x'}(x')) + \widetilde{a} (x' + h\mathbf{e}_k, w(x'), w_{x'}(x')) = 0.
$$

Then we write the corresponding integral identity with the test function

$$
\phi(x') = (w(x') - w(x_0')) \zeta^2(x')|\triangle_k^h v(x')|^2
$$

and repeat verbatim the deduction of (8.4.13). As a result we get

$$
\int\limits_{B_{2\sigma}} |\nabla'w|^m |\triangle_k^h v|^2 \zeta^2 dx' \le c(\nu,\mu,m)\sigma^\alpha \int\limits_{B_{2\sigma}} \Big\{ |\nabla'w|^m |\triangle_k^h v|^2 \zeta^2 +
$$

$$
+ |\nabla'w|^{m-2} |\triangle_k^h v|^2 |\nabla'\zeta|^2 + |\nabla'(\triangle_k^h v)|^2 |\nabla'w|^{m-2} \zeta^2 +
$$

$$
+ \varrho^m f (\varrho(x' + h\mathbf{e}_k)) |\triangle_k^h v|^2 \zeta^2 + \varrho^{m-1} g (\varrho(x' + h\mathbf{e}_k)) |\nabla'w||\triangle_k^h v|^2 \zeta^2 +
$$

(8.4.14)

$$
+ \varrho^{1+\frac{m}{2}} f (\varrho(x' + h\mathbf{e}_k)) |\nabla'w|^{\frac{m-2}{2}} |\triangle_k^h v|^2 \zeta^2 \Big\} dx'.
$$

Let us sum the estimates (8.4.13) and (8.4.14), applying the inequality (1.2.5) of Lemma 1.5. Then recalling the notation (8.4.4), we have

$$
\int\limits_{B_{2\sigma}} P_k^m(x')|\triangle_k^h v|^2 \zeta^2 dx' \le c(\nu,\mu,m)\sigma^\alpha \int\limits_{B_{2\sigma}} \Big\{ P_k^m(x')|\triangle_k^h v|^2 \zeta^2 +
$$

$$
+ P_k^{m-2}(x')|\triangle_k^h v|^2 |\nabla'\zeta|^2 + |\nabla'(\triangle_k^h v)|^2 P_k^{m-2}(x')\zeta^2 +
$$

$$
+ \varrho^m \langle f(\varrho x') + f (\varrho(x' + h\mathbf{e}_k))\rangle |\triangle_k^h v|^2 \zeta^2 +
$$

$$
+ \varrho^{1+\frac{m}{2}} \langle f(\varrho x') + f (\varrho(x' + h\mathbf{e}_k))\rangle P_k^{\frac{m-2}{2}} (x')|\triangle_k^h v|^2 \zeta^2 +
$$

$$
+ \varrho^{m-1} \langle g(\varrho x') + g (\varrho(x' + h\mathbf{e}_k))\rangle P_k(x')|\triangle_k^h v|^2 \zeta^2 \Big\} dx'.
$$

Choosing, if it is necessary, $\sigma \in (0, \text{dist}(\widetilde{G}, \partial G^2_{1/4}))$ smaller such that $c(\nu, \mu, m)\sigma^\alpha \leq \frac{1}{2}$ hence we obtain

$$\int_{B_{2\sigma}} P_k^m(x') |\triangle_k^h v|^2 \zeta^2 dx' \leq c(\nu, \mu, m)\sigma^\alpha \int_{B_{2\sigma}} \left\{ |\nabla'(\triangle_k^h v)|^2 P_k^{m-2}(x')\zeta^2 + \right.$$

$$+ P_k^{m-2}(x') |\triangle_k^h v|^2 |\nabla'\zeta|^2 + \varrho^m \langle f(\varrho x') + f(\varrho(x' + he_k)) \rangle |\triangle_k^h v|^2 \zeta^2 +$$

$$+ \varrho^{1+\frac{m}{2}} \langle f(\varrho x') + f(\varrho(x' + he_k)) \rangle P_k^{\frac{m-2}{2}}(x') |\triangle_k^h v|^2 \zeta^2 +$$

$$(8.4.15) \qquad + \varrho^{m-1} \langle g(\varrho x') + g(\varrho(x' + he_k)) \rangle P_k(x') |\triangle_k^h v|^2 \zeta^2 \right\} dx'.$$

In the same way from (8.4.7), (8.4.15)

$$\int_{B_{2\sigma}} P_k^{m-2}(x') |\nabla' \triangle_k^h v(x')|^2 \zeta^2 dx' \leq c(\nu, \mu, m) \int_{B_{2\sigma}} \left\{ \left(P_k^{m-2}(x') |\triangle_k^h v|^2 + \right. \right.$$

$$+ P_k^m(x') \right) (\zeta^2 + |\nabla'\zeta|^2) + \varrho^m \langle f(\varrho x') + f(\varrho(x' + he_k)) \rangle |\triangle_k^h v|^2 \zeta^2 +$$

$$+ \varrho^{1+\frac{m}{2}} \langle f(\varrho x') + f(\varrho(x' + he_k)) \rangle P_k^{\frac{m-2}{2}}(x') |\triangle_k^h v|^2 \zeta^2 +$$

$$+ \varrho^{m-1} \langle g(\varrho x') + g(\varrho(x' + he_k)) \rangle P_k(x') |\triangle_k^h v|^2 \zeta^2 +$$

$$(8.4.16) \qquad + \varrho^m g^2(\varrho x') |\triangle_k^h v(x')|^2 \zeta^2 \right\} dx'.$$

From (8.4.15) and (8.4.16) the estimate follows as

$$\int_{B_{2\sigma}} \left(P_k^m(x') |\triangle_k^h v|^2 + P_k^{m-2}(x') |\nabla' \triangle_k^h v(x')|^2 \right) \zeta^2 dx' \leq$$

$$\leq c(\nu, \mu, m) \int_{B_{2\sigma}} \left\{ \left(P_k^{m-2}(x') |\triangle_k^h v|^2 + P_k^m(x') \right) (\zeta^2 + |\nabla'\zeta|^2) + \right.$$

$$+ \varrho^m \langle f(\varrho x') + f(\varrho(x' + he_k)) \rangle |\triangle_k^h v|^2 \zeta^2 + \varrho^m g^2(\varrho x') |\triangle_k^h v(x')|^2 \zeta^2 +$$

$$+ \varrho^{1+\frac{m}{2}} \langle f(\varrho x') + f(\varrho(x' + he_k)) \rangle P_k^{\frac{m-2}{2}}(x') |\triangle_k^h v|^2 \zeta^2 +$$

$$(8.4.17) \qquad + \varrho^{m-1} \langle g(\varrho x') + g(\varrho(x' + he_k)) \rangle P_k(x') |\triangle_k^h v|^2 \zeta^2 \right\} dx'.$$

Further, by the Young inequality, we have for $\forall \varepsilon > 0$

$$\varrho^{m-1} g(\varrho x') P_k(x') |\triangle_k^h v|^2 \leq \frac{\varepsilon}{m} P_k^m(x') |\triangle_k^h v|^2 +$$

$$+ \frac{m-1}{m} \varepsilon^{\frac{1}{1-m}} \varrho^m g^{\frac{m}{m-1}}(\varrho x') |\triangle_k^h v|^2.$$

Then choosing $\varepsilon > 0$ from the equality $\frac{2\varepsilon}{m}c(\nu,\mu,m) = \frac{1}{2}$, we can rewrite (8.4.17) in the following way

$$\int\limits_{B_{2\sigma}} \left(P_k^m(x')|\triangle_k^h v|^2 + P_k^{m-2}(x')|\nabla'\triangle_k^h v(x')|^2\right)\zeta^2 dx' \le$$

$$\le c(\nu,\mu,m)\int\limits_{B_{2\sigma}}\left\{\left(P_k^{m-2}(x')|\triangle_k^h v|^2 + P_k^m(x')\right)\left(\zeta^2 + |\nabla'\zeta|^2\right) +\right.$$

$$+\varrho^{1+\frac{m}{2}}\langle f(\varrho x') + f\left(\varrho(x' + h\mathbf{e}_k))\right) P_k^{\frac{m-2}{2}}(x')|\triangle_k^h v|^2\zeta^2 +$$

$$+\varrho^m\Big\langle f(\varrho x') + f\left(\varrho(x' + h\mathbf{e}_k)\right) + g^2(\varrho x') +$$

$$\text{(8.4.18)}\qquad +g^{\frac{m}{m-1}}(\varrho x') + g^{\frac{m}{m-1}}\left(\varrho(x' + h\mathbf{e}_k)\right)\Big\rangle|\triangle_k^h v|^2\zeta^2\Big\}dx'.$$

Again, by the Young inequality, we have for $\forall\delta > 0$

$$\varrho^m f(\varrho x')|\triangle_k^h v|^2 \le \frac{2\delta}{m+2}|\triangle_k^h v|^{m+2} +$$

$$+\frac{m}{m+2}\delta^{-\frac{2}{m}}\varrho^{m+2}f^{\frac{m+2}{m}}(\varrho x'),$$

$$\varrho^{1+\frac{m}{2}}f(\varrho x')P_k^{\frac{m-2}{2}}(x')|\triangle_k^h v|^2 \le \frac{\delta}{2}P_k^{m-2}|\triangle_k^h v|^4 + \frac{1}{2\delta}\varrho^{m+2}f^2(\varrho x'),$$

$$\varrho^m g^2(\varrho x')|\triangle_k^h v|^2 \le \frac{2\delta}{m+2}|\triangle_k^h v|^{m+2} +$$

$$+\frac{m}{m+2}\delta^{-\frac{2}{m}}\varrho^{m+2}g^{\frac{2(m+2)}{m}}(\varrho x'),$$

$$\varrho^m g^{\frac{m}{m-1}}(\varrho x')|\triangle_k^h v|^2 \le \frac{2\delta}{m+2}|\triangle_k^h v|^{m+2} +$$

$$+\frac{m}{m+2}\delta^{-\frac{2}{m}}\varrho^{m+2}g^{\frac{m+2}{m-1}}(\varrho x')$$

and therefore from (8.4.18), it follows that

$$\int\limits_{B_{2\sigma}} \left(P_k^m(x')|\triangle_k^h v|^2 + P_k^{m-2}(x')|\nabla'\triangle_k^h v(x')|^2\right)\zeta^2 dx' \le$$

$$\le c(\nu,\mu,m)\int\limits_{B_{2\sigma}}\left\{\delta\left(|\triangle_k^h v|^{m+2} + P_k^{m-2}(x')|\triangle_k^h v|^4\right) +\right.$$

$$+\left(P_k^{m-2}(x')|\triangle_k^h v|^2 + P_k^m(x')\right)\left(\zeta^2 + |\nabla'\zeta|^2\right) +$$

$$+\delta^{-1}\varrho^{m+2}\Big\langle f^2(\varrho x') + f^2\left(\varrho(x'+h\mathbf{e}_k)\right)\Big\rangle\zeta^2 +$$

$$+\delta^{-\frac{2}{m}}\varrho^{m+2}\Big\langle f^{\frac{m+2}{m}}(\varrho x') + f^{\frac{m+2}{m}}\left(\varrho(x'+h\mathbf{e}_k)\right) +$$

$$(8.4.19)\qquad +g^{\frac{m+2}{m-1}}(\varrho x') + g^{\frac{m+2}{m-1}}(\varrho x') + g^{\frac{2(m+2)}{m}}\left(\varrho(x')\right)\zeta^2\Big\rangle dx'.$$

If $\delta > 0$ is sufficiently small, we get

$$\int\limits_{B_{2\sigma}}\Big\{P_k^{m-2}(x')|\nabla'\triangle_k^h v(x')|^2 + \Big\langle P_k^m|\triangle_k^h v|^2 -$$

$$-c(\nu,\mu,m)\delta\left(|\triangle_k^h v|^{m+2} + P_k^{m-2}(x')|\triangle_k^h v|^4\right)\Big\rangle\Big\}\zeta^2 dx' \le$$

$$\le c(\nu,\mu,m)\int\limits_{B_{2\sigma}}\Big\{\left(P_k^{m-2}(x')|\triangle_k^h v|^2 + P_k^m(x')\right)\left(\zeta^2 + |\nabla'\zeta|^2\right) +$$

$$+\delta^{-1}\varrho^{m+2}\Big\langle f^2(\varrho x') + f^2\left(\varrho(x'+h\mathbf{e}_k)\right)\Big\rangle\zeta^2 +$$

$$+\delta^{-\frac{2}{m}}\varrho^{m+2}\Big\langle f^{\frac{m+2}{m}}(\varrho x') + f^{\frac{m+2}{m}}\left(\varrho(x'+h\mathbf{e}_k)\right) +$$

$$(8.4.20)\qquad +g^{\frac{m+2}{m-1}}(\varrho x') + g^{\frac{m+2}{m-1}}(\varrho x') + g^{\frac{2(m+2)}{m}}\left(\varrho(x')\right)\zeta^2\Big\rangle dx'.$$

Now we verify that

$$\lim_{h\to 0}\int\limits_{B_{2\sigma}}\left(P_k^{m-2}(x')|\triangle_k^h v|^2 + P_k^m(x')\right)\left(\zeta^2 + |\nabla'\zeta|^2\right) dx' =$$

$$= \left(2^{m-2} + 2^m\right)\int\limits_{B_{2\sigma}}|\nabla'v|^m\left(\zeta^2 + |\nabla'\zeta|^2\right) dx'.$$

In fact, by virtue of Lemma 1.66, $\triangle_k^h v$ converges to $D_k u$ in the norm L^m almost everywhere and, by the Egorov Theorem almost uniformly. Analogously, the almost uniform convergence of $f\left(\varrho(x'+h\mathbf{e}_k)\right)$, $g\left(\varrho(x'+h\mathbf{e}_k)\right)$ to $f(\varrho x')$, $g(\varrho x')$ respectively is verified.

Thus, we can apply the Fatou Theorem and take the limit as $h \to 0$ in (8.4.20). We obtain as a result

$$
\int_{B_{2\sigma}} \left(\left[2^m - c(\nu,\mu,m) \cdot \delta(1 + 2^{m-2}) \right] |\nabla' v|^m \left| \frac{\partial v}{\partial x'_k} \right|^2 + |\nabla' v|^{m-2} \left| \frac{\partial |\nabla' v|}{\partial x'_k} \right|^2 \right)
$$

$$
\times \zeta^2(x') dx' \leq c_1 \int_{B_{2\sigma}} |\nabla' v|^m \left(\zeta^2 + |\nabla' \zeta|^2 \right) dx' +
$$

$$
+ c_2 \varrho^{m+2} \int_{B_{2\sigma}} \left\langle f^2(\varrho x') + f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{2(m+2)}{m}}(\varrho x') \left(\varrho(x') + g^{\frac{m+2}{m-1}}(\varrho x') \right) \right\rangle dx',
$$

$$
k = 1, \dots, N.
$$

Let us now choose $\delta > 0$ from the equality $c(\nu,\mu,m) \cdot \delta = \frac{2^{m-1}}{1 + 2^{m-2}}$. Then we get

(8.4.21)
$$
\int_{B_{2\sigma}} \left(|\nabla' v|^m \left| \frac{\partial v}{\partial x'_k} \right|^2 + |\nabla' v|^{m-2} \left| \frac{\partial |\nabla' v|}{\partial x'_k} \right|^2 \right) \zeta^2(x') dx' \leq
$$

$$
\leq c_1 \int_{B_{2\sigma}} |\nabla' v|^m \left(\zeta^2 + |\nabla' \zeta|^2 \right) dx' +
$$

$$
+ c_2 \varrho^{m+2} \int_{B_{2\sigma}} \left\langle f^2(\varrho x') + f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{2(m+2)}{m}}(\varrho x') \left(\varrho(x') + g^{\frac{m+2}{m-1}}(\varrho x') \right) \right\rangle dx',
$$

$$
k = 1, \dots, N.
$$

After summing up over all $k = 1, \dots, N$, by the properties of the function $\zeta(x')$, we establish

$$
\int_{B_\sigma} \left(|\nabla' v|^{m+2} + |\nabla' v|^{m-2} |v_{x'x'}|^2 \right) dx' \leq c_1 \int_{B_{2\sigma}} |\nabla' v|^m dx' +
$$

(8.4.22)
$$
+ c_2 \varrho^{m+2} \int_{B_{2\sigma}} \left\langle f^2(\varrho x') + f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{m+2}{m-1}}(\varrho x') + g^{\frac{2(m+2)}{m}}(\varrho x') \right\rangle dx'.
$$

By the covering argument we obtain

$$\int\limits_{\widetilde{G}'} \left(|\nabla'v|^{m+2} + |\nabla'v|^{m-2}|v_{x'x'}|^2\right) dx' \leq c_1 \int\limits_{\widetilde{G}'} |\nabla'v|^m dx' +$$

(8.4.23)

$$+ c_2 \varrho^{m+2} \int\limits_{\widetilde{G}'} \left\langle f^2(\varrho x') + f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{m+2}{m-1}}(\varrho x') + g^{\frac{2(m+2)}{m}}(\varrho(x'))\right\rangle dx'.$$

Returning to previous variables x, u we get the desired (8.4.1). $\qquad\square$

8.4.2. Local estimates near a boundary smooth portion. In this subsection we derive local integral estimates near a boundary smooth portion of weak solutions of the problem (DQL). Let $x_0' \in \widetilde{\Gamma}' \subset \Gamma_{1/4}^2$ and let $U'(x_0') \subset \overline{G_{1/4}^2}$ be a neighborhood of x_0'. Since our assumption on the boundary of G is such that $\partial G \setminus \mathcal{O}$ is smooth, then there exists a diffeomorphism $U'(x_0') \longrightarrow B_{2\sigma}^+(x_0')$, which flattens the boundary that is maps $\widetilde{\Gamma}'$ onto $\Sigma_{2\sigma} \subset \{x_N' = 0\}$ being a plane part of $\partial B_{2\sigma}^+(x_0')$.

So we may suppose that $G' = B_{2\sigma}^+(x_0')$ in the $(DQL)'$ that takes the form

$$(DQL)_0' \quad \begin{cases} \int\limits_{B_{2\sigma}^+(x_0')} \left\{\widetilde{a}_i(x', v, v_{x'})\phi_{x_i'} + \widetilde{a}(x', v, v_{x'})\phi\right\} dx' = 0, \\[2mm] \qquad\qquad \forall\phi(x') \in W_0^{1,m}(B_{2\sigma}^+(x_0')) \cap L^\infty(B_{2\sigma}^+(x_0')); \\[4mm] \widetilde{a}_i(x', v, v_{x'}) \equiv a_i(\varrho x', v, \varrho^{-1}v_{x'}), \\[2mm] \widetilde{a}(x', v, v_{x'}) \equiv \varrho a(\varrho x', v, \varrho^{-1}v_{x'}). \end{cases}$$

We denote $U(x_0)$ as the preimage of $U'(x_0')$ under the transformation $x = \varrho x'$. It is obvious that $U(x_0) \subset \overline{G_{\varrho/4}^{2\varrho}}$.

THEOREM 8.44. *Let $u(x)$ be a weak bounded solution of the problem (DQL). Let us assume that the hypotheses $(A), (B), (C), (D), (E), (F)$ are fulfilled on the set $\mathfrak{M}$. Let $\forall \widetilde{G} \subset \overline{G_{\varrho/4}^{2\varrho}} \subset \overline{G}$. Then we have the estimate*

$$\int\limits_{\widetilde{G}} \left(|\nabla u|^{m+2} + |\nabla u|^{m-2}u_{xx}^2\right) dx \leq$$

(8.4.24)

$$\leq C \int\limits_{\widetilde{G}} \left(\rho^{-2}|\nabla u|^m + f^2 + f^{\frac{m+2}{m}} + g^{\frac{m+2}{m-1}} + g^{\frac{2(m+2)}{m}}\right) dx.$$

PROOF. Repeating verbatim the procedure of the deduction of the estimate (8.4.21), we establish

$$
\int\limits_{B_{2\sigma}^+} \left(|\nabla' v|^m \left| \frac{\partial v}{\partial x_k'} \right|^2 + |\nabla' v|^{m-2} \left| \frac{\partial |\nabla' v|}{\partial x_k'} \right|^2 \right) \zeta^2(x') dx' \le
$$

$$
(8.4.25) \qquad \le c_1 \int\limits_{B_{2\sigma}^+} |\nabla' v|^m \left(\zeta^2 + |\nabla' \zeta|^2 \right) dx' +
$$

$$
+ c_2 \varrho^{m+2} \int\limits_{B_{2\sigma}^+} \left\langle f^2(\varrho x') + f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{2(m+2)}{m}}(\varrho(x') + g^{\frac{m+2}{m-1}}(\varrho x')) \right\rangle dx',
$$

$$
k = 1, \ldots, N-1.
$$

It remains only to consider the case $k = N$. For this case, by Theorem 8.43, using the covering argument we can easily establish that

$$
\phi(x') a_i(x', v, v_x') \in W_0^{1,1}(\widetilde{G}'), \ i = 1, \ldots N,
$$

$$
\forall \phi(x') \in W_0^{1,m}(\widetilde{G}') \cap L^\infty(\widetilde{G}), \ \forall \widetilde{G}' \subset\subset G'.
$$

Therefore we have from $(DQL)'$

$$
-\frac{d}{dx_i'} a_i(x', v, v_x') + a(x', v, v_x') = 0 \text{ a.e. } x' \in G'.
$$

Then we obtain

$$
\frac{\partial \widetilde{a}_N(x', v, v_{x'})}{\partial v_{x_N'}} v_{x_N' x_N'} = \widetilde{a}(x', v, v_{x'}) - \sum_{i=1}^{N-1} \frac{\partial \widetilde{a}_i(x', v, v_{x'})}{\partial v_{x_j'}} v_{x_i' x_j'} -
$$

$$
- \frac{\partial \widetilde{a}_i(x', v, v_{x'})}{\partial v} v_{x_i'} - \frac{\partial \widetilde{a}_i(x', v, v_{x'})}{\partial x_i'}.
$$

Hence, in virtue of assumptions $(B), (C), (E)$, the next inequality follows

$$
|\nabla' v|^{m-2} \left| v_{x_N' x_N'} \right| \le c(\nu, \mu, m) \Bigg\{ |\nabla' v|^{m-2} \sum_{i=1}^{N-1} \left| v_{x_i' x_j'} \right| + |\nabla' v|^m +
$$

$$
+ \varrho^{1+\frac{m}{2}} f(\varrho x') |\nabla' v|^{\frac{m-2}{2}} + \varrho^{\frac{m}{2}} g(\varrho x') |\nabla' v|^{\frac{m}{2}} \Bigg\}.
$$

Further using the Young inequality, it is easy to obtain the inequality

$$(8.4.26) \quad |\nabla'v|^{m-2}\left|v_{x'_N x'_N}\right|^2 \le c(\nu,\mu,m)\Bigg\{|\nabla'v|^{m-2}\sum_{i=1}^{N-1}\left|v_{x'_i x'_j}\right|^2 +$$

$$+ |\nabla'v|^{m+2} + \varrho^{m+2}f^2(\varrho x') + \varrho^{m+2}g^{\frac{2(m+2)}{m}}(\varrho x')\Bigg\}.$$

Let $\zeta(x') \in C_0^\infty\left(B_{2\sigma}^+(x'_0)\right)$ be a cutoff function such that

$$\zeta(x') = 1 \text{ in } B_\sigma^+(x'_0), \quad 0 \le \zeta(x') \le 1, \ |\nabla'\zeta| \le c\sigma^{-1} \text{ in } B_{2\sigma}^+(x'_0).$$

Let us now multiply both sides of this inequality by $\zeta^2(x')$ and integrate over $B_{2\sigma}^+(x'_0)$. As a result we deduce

$$(8.4.27) \quad \int_{B_{2\sigma}^+} |\nabla'v|^{m-2}\left|v_{x'_N x'_N}\right|^2 \zeta^2(x')dx' \le$$

$$\le c(\nu,\mu,m)\int_{B_{2\sigma}^+}\Bigg\{|\nabla'v|^{m+2}\zeta^2(x') + |\nabla'v|^{m-2}\sum_{i=1}^{N-1}\left|v_{x'_i x'_j}\right|^2 \zeta^2(x')+$$

$$+ \varrho^{m+2}f^2(\varrho x') + \varrho^{m+2}g^{\frac{2(m+2)}{m}}(\varrho x')\Bigg\}dx'.$$

We estimate the first term on the right in (8.4.27) by means of (8.4.25)

$$\int_{B_{2\sigma}^+} |\nabla'v|^{m-2}\left|v_{x'_N x'_N}\right|^2 \zeta^2(x')dx' \le c_1\int_{B_{2\sigma}^+}\Bigg\langle|\nabla'v|^{m+2}\zeta^2(x')+$$

$$+|\nabla'v|^m\left(\zeta^2 + |\nabla'\zeta|^2\right)\Bigg\rangle dx' + c_2\varrho^{m+2}\int_{B_{2\sigma}^+}\Bigg\langle f^2(\varrho x')+$$

$$(8.4.28) \quad +f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{2(m+2)}{m}}(\varrho x') + g^{\frac{m+2}{m-1}}(\varrho x')\Bigg\rangle dx'.$$

Summing (8.4.25) and (8.4.28) we obtain

$$\int_{B_{2\sigma}^+} |\nabla'v|^{m-2}\left|v_{x'x'}\right|^2 \zeta^2(x')dx' \le c_1\int_{B_{2\sigma}^+}\Bigg\langle|\nabla'v|^{m+2}\zeta^2(x')+$$

$$+|\nabla'v|^m\left(\zeta^2 + |\nabla'\zeta|^2\right)\Bigg\rangle dx' + c_2\varrho^{m+2}\int_{B_{2\sigma}^+}\Bigg\langle f^2(\varrho x')+$$

$$(8.4.29) \quad +f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{2(m+2)}{m}}(\varrho x') + g^{\frac{m+2}{m-1}}(\varrho x')\Bigg\rangle dx'.$$

We embark on the estimating of $\int\limits_{B_{2\sigma}^+} |\nabla' v|^{m+2}\zeta^2(x')dx'$. Because of

(8.4.25), it is sufficient to estimate the integral $\int\limits_{B_{2\sigma}^+} |\nabla' v|^m v_{x'_N}^2 \zeta^2(x')dx'$. For

this estimating we turn again to the $(DQL)'_0$ and we take

$$\phi(x') = (v(x') - v(x'_0)) v_{x'_N}^2 \zeta^2(x')$$

as a test function. Then, in virtue of the representation (8.4.9), we have

$$\int\limits_{B_{2\sigma}^+} \widetilde{a}_{ij}(x', v, v_{x'}) v_{x'_i} v_{x'_j} v_{x'_N}^2 \zeta^2(x')dx' =$$

$$= -\int\limits_{B_{2\sigma}^+} (v(x') - v(x'_0)) \left\langle \widetilde{a}_{ij}(x', v, v_{x'}) v_{x'_j} \left(2 v_{x'_N} v_{x'_i x'_N} \zeta^2(x') + \right. \right.$$

$$\left. + 2\zeta(x')\zeta_{x'_i} v_{x'_N}^2 \right) + \widetilde{a}(x', v, v_{x'}) v_{x'_N}^2 \zeta^2(x') \right\rangle dx' -$$

$$- \int\limits_{B_{2\sigma}^+} \widetilde{a}_i(x', v, 0) \left\langle (v(x') - v(x'_0)) v_{x'_N}^2 \zeta^2(x') \right\rangle_{x'_i} dx'.$$

Hence integrating by parts in the last term on the right and applying the assumption (E), we obtain

$$\frac{\nu}{m-1}\varrho^{1-m} \int\limits_{B_{2\sigma}^+} |\nabla' v|^m v_{x'_N}^2 \zeta^2(x')dx' \leq$$

$$\leq \int\limits_{B_{2\sigma}^+} |v(x') - v(x'_0)| \left\langle 2|\widetilde{a}_{ij}(x', v, v_{x'})| |\nabla' v| |v_{x'_N}| |v_{x'x'}| \zeta^2(x') + \right.$$

$$+ 2|\widetilde{a}_{ij}(x', v, v_{x'})| |\nabla' v| \zeta(x') |\nabla' \zeta| v_{x'_N}^2 + |\widetilde{a}(x', v, v_{x'})| v_{x'_N}^2 \zeta^2(x') +$$

$$+ \left| \frac{\partial \widetilde{a}_i(x', v, 0)}{\partial x'_i} + \frac{\partial \widetilde{a}_i(x', v, 0)}{\partial v} v_{x'_i} \right| v_{x'_N}^2 \zeta^2(x') \right\rangle dx'.$$

Now we observe again that all hypotheses of Theorem 8.3 about Hölder continuity of weak solutions are fulfilled and conclude

$$|v(x') - v(x'_0)| \leq c\sigma^\alpha, \quad x' \in B_{2\sigma}^+(x'_0), \quad \alpha \in (0,1).$$

Therefore, by assumptions $(B), (E), (F)$, we get

$$
\int\limits_{B_{2\sigma}^+} |\nabla' v|^m v_{x'_N}^2 \zeta^2(x') dx' \le c(\nu, \mu, m) \sigma^\alpha \int\limits_{B_{2\sigma}^+} \Big\langle |\nabla' v|^{m-1} |v_{x'_N}| |v_{x'x'}| \zeta^2(x') +
$$

$$
+ |\nabla' v|^{m-1} \zeta(x') |\nabla' \zeta| v_{x'_N}^2 + |\nabla' v|^m v_{x'_N}^2 \zeta^2(x') + \varrho^m f(\varrho x') v_{x'_N}^2 \zeta^2(x') +
$$

(8.4.30)

$$
+ \varrho^{1 + \frac{m}{2}} f(\varrho x') |\nabla' v|^{\frac{m-2}{2}} v_{x'_N}^2 \zeta^2(x') + \varrho^{m-1} g(\varrho x') |\nabla' v| v_{x'_N}^2 \zeta^2(x') \Big\rangle dx'.
$$

Let us apply again the Cauchy-Young inequalities

$$
|\nabla' v|^{m-1} |v_{x'_N}| |v_{x'x'}| \le \frac{1}{2} |\nabla' v|^{m-2} |v_{x'x'}|^2 + \frac{1}{2} |\nabla' v|^m v_{x'_N}^2;
$$

$$
|\nabla' v|^{m-1} \zeta(x') |\nabla' \zeta| v_{x'_N}^2 \le \frac{1}{2} |\nabla' v|^m v_{x'_N}^2 \zeta^2 + \frac{1}{2} |\nabla' v|^m |\nabla' \zeta|^2;
$$

$$
\varrho^m f(\varrho x') v_{x'_N}^2 \le \frac{2}{m+2} |v_{x'_N}|^{m+2} + \frac{m}{m+2} \varrho^{m+2} f^{\frac{m+2}{m}}(\varrho x') \le
$$

$$
\le \frac{2}{m+2} |\nabla' v|^m v_{x'_N}^2 + \frac{m}{m+2} \varrho^{m+2} f^{\frac{m+2}{m}}(\varrho x');
$$

$$
\varrho^{1 + \frac{m}{2}} f(\varrho x') |\nabla' v|^{\frac{m-2}{2}} v_{x'_N}^2 \le \frac{1}{2} |\nabla' v|^m v_{x'_N}^2 + \frac{1}{2} \varrho^{m+2} f^2(\varrho x');
$$

$$
\varrho^{m-1} g(\varrho x') |\nabla' v| v_{x'_N}^2 \le \frac{1}{m} |\nabla' v|^m v_{x'_N}^2 + \frac{m-1}{m} \varrho^m g^{\frac{m}{m-1}}(\varrho x') v_{x'_N}^2 \le
$$

$$
\le \frac{3}{m+2} |\nabla' v|^m v_{x'_N}^2 + \frac{m-1}{m+2} \varrho^{m+2} g^{\frac{m+2)}{m}}(\varrho x').
$$

Hence and from (8.4.30) we finally obtain

$$
\int\limits_{B_{2\sigma}^+} |\nabla' v|^m v_{x'_N}^2 \zeta^2(x') dx' \le c_1 \sigma^\alpha \int\limits_{B_{2\sigma}^+} \Big\langle |\nabla' v|^{m-2} v_{x'x'}^2 \zeta^2(x') +
$$

$$
+ |\nabla' v|^m v_{x'_N}^2 \zeta^2(x') + |\nabla' v|^m |\nabla' \zeta|^2 \Big\rangle dx' + c_2 \varrho^{m+2} \int\limits_{B_{2\sigma}^+} \Big\langle f^2(\varrho x') +
$$

(8.4.31)

$$
+ f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{m+2}{m}}(\varrho x') \Big\rangle \zeta^2(x') dx'.
$$

Combining (8.4.25), (8.4.29), (8.4.31), choosing σ sufficiently small and using the properties of $\zeta(x')$, we get

$$\int\limits_{B_\sigma^+} \left(|\nabla' v|^{m+2} + |\nabla' v|^{m-2} |v_{x'x'}|^2 \right) dx' \leq c_1 \int\limits_{B_{2\sigma}^+} |\nabla' v|^m dx' +$$

(8.4.32)

$$+ c_2 \varrho^{m+2} \int\limits_{B_{2\sigma}^+} \left\langle f^2(\varrho x') + f^{\frac{m+2}{m}}(\varrho x') + g^{\frac{m+2}{m-1}}(\varrho x') + g^{\frac{2(m+2)}{m}}(\varrho(x') \right\rangle dx'.$$

By the covering argument and returning to the previous variables x, u, we get the desired (8.4.24).   □

From Theorems 8.43 and 8.44 the following theorem follows immediately

THEOREM 8.45. *Let $u(x)$ be a weak bounded solution of the problem (DQL). Let us assume that the hypotheses $(A), (B), (C), (D), (E)$ and (F) are fulfilled on the set $\mathfrak{M}$. Then we have the estimate*

$$\int\limits_{G_\varrho^{2\varrho}} \left(|\nabla u|^{m+2} + |\nabla u|^{m-2} u_{xx}^2 \right) dx \leq$$

(8.4.33)

$$\leq C \int\limits_{G_{\varrho/2}^{4\varrho}} \left(r^{-2} |\nabla u|^m + f^2 + f^{\frac{m+2}{m}} + g^{\frac{m+2}{m-1}} + g^{\frac{2(m+2)}{m}} \right) dx, \quad \forall \rho \in (0, d).$$

8.4.3. The local estimate near a conical point.

THEOREM 8.46. *Let $u(x)$ be a weak bounded solution of the problem (DQL). Let λ_0 be the least eigenvalue of the problem $(NEVP)$ (it is determined by Theorem 8.12) and $t \in (0, 1]$ be the number that is determined by (8.3.4). Let us assume that the hypotheses $(U), 1) - 4)$ from Section 8.3 and $(A), (B), (C), (D), (E), (F)$ are fulfilled. In addition, suppose*

$$(8.4.34) \quad \int\limits_{G_{\varrho/2}^{4\varrho}} \left(f^2 + f^{\frac{m+2}{m}} + g^{\frac{m+2}{m-1}} + g^{\frac{2(m+2)}{m}} \right) dx \leq K \varrho^{N-2+m(t\lambda_0-1)}.$$

If $\gamma > 2 - N - m(t\lambda_0 - 1)$, then we have the estimate

$$(8.4.35) \quad \int\limits_{G_0^\varrho} \left(r^\gamma |\nabla u|^{m-2} u_{xx}^2 + r^{\gamma-2} |\nabla u|^m + r^{\gamma-2-m} |u|^m \right) dx \leq$$

$$\leq C \varrho^{\gamma+N-2+m(t\lambda_0-1)}, \quad \forall \varrho \in (0, d).$$

PROOF. By Theorem 8.45 together with (8.4.34) we have
(8.4.36)
$$\int\limits_{G_\varrho^{2\varrho}} r^\gamma |\nabla u|^{m-2} u_{xx}^2 \, dx \le C \int\limits_{G_{\varrho/2}^{4\varrho}} r^{\gamma-2} |\nabla u|^m dx + K\varrho^{\gamma+N-2+m(t\lambda_0-1)}.$$

Let us now apply Theorems 8.40 and 8.41. According to the estimates
(8.3.5), (8.3.8) and (8.4.36) we obtain

$$(8.4.37) \quad \int\limits_{G_\varrho^{2\varrho}} \left(r^\gamma |\nabla u|^{m-2} u_{xx}^2 + r^{\gamma-2}|\nabla u|^m + r^{\gamma-2-m}|u|^m \right) dx \le$$

$$\le C\varrho^{\gamma+N-2+m(t\lambda_0-1)}, \quad \forall \varrho \in (0,d).$$

Let us define the sequence $\varrho_k = 2^{1-k}\varrho$. We rewrite the inequality (8.4.37)
replacing ϱ by ϱ_k. Then we get

$$\int\limits_{G_{\varrho_k}^{2\varrho_k}} \left(r^\gamma |\nabla u|^{m-2} u_{xx}^2 + r^{\gamma-2}|\nabla u|^m + r^{\gamma-2-m}|u|^m \right) dx \le$$

$$(8.4.38) \qquad \le C2^{(1-k)\varkappa}\varrho^\varkappa, \quad \forall \varrho \in (0,d),$$

$$\varkappa = \gamma + N - 2 + m(t\lambda_0 - 1) > 0.$$

Summing the inequalities (8.4.38) over all $k = 1, 2, \ldots$ we have

$$\int\limits_{G_0^{2\varrho}} \left(r^\gamma |\nabla u|^{m-2} u_{xx}^2 + r^{\gamma-2}|\nabla u|^m + r^{\gamma-2-m}|u|^m \right) dx \le$$

$$\le C\varrho^\varkappa \sum_{k=1}^\infty 2^{(1-k)\varkappa} = \frac{C}{1-2^{-\varkappa}}\varrho^\varkappa,$$

since $\varkappa > 0$. $\square$

Example.
Let us look at the problem

$$\begin{cases} \Delta_m u := -\text{div}\left(|\nabla u|^{m-2}\nabla u\right) = 0 & \text{in } G_0, \\ u\Big|_{\omega=\pm\frac{1}{2}\omega_0} = 0, \end{cases}$$

where $m > 1$ and

$$G_0 = \left\{ x = (r,\omega) \Big| 0 < r < \infty, |\omega| \le \frac{\omega_0}{2} \right\}, \quad \omega_0 \in (0, 2\pi)$$

is the plane angle. We use the results of Chapter 9. In Subsection 9.4 of Chapter 9 we constructed the solution of our problem in the form

$$w(x) = r^\lambda \Phi(\omega), \quad \omega \in [-\frac{\omega_0}{2}, \frac{\omega_0}{2}], \quad \lambda > 0$$

with $\Phi(\omega) \geq 0$ and $\lambda = \lambda_0$, determined by (8.2.4). By the properties of $\Phi(\omega)$, established in Subsection 9.4, it is not difficult to deduce the following estimates

(8.4.39) $0 < u(x) \leq r^{\lambda_0}, \quad |\nabla u| \leq c_1 r^{\lambda_0 - 1}, \quad |u_{xx}| \leq c_2 r^{\lambda_0 - 2}.$

Now we can establish the condition of the finiteness of the integral

$$\int\limits_{G_0^\varrho} \left(r^\gamma |\nabla u|^{m-2} u_{xx}^2 + r^{\gamma-2} |\nabla u|^m + r^{\gamma-2-m} |u|^m \right) dx.$$

From the estimates (8.4.39) it follows that the integral above is finite, if the integral $\int\limits_0^\varrho r^{\gamma-1+m(\lambda_0-1)} dr$ is convergent. This fact holds under the condition $\gamma > (1 - \lambda_0)m$ and it shows that the statement of Theorem 8.46 is precise.

8.5. Notes

The properties of weak solutions of the (LPA) in the neighborhood of isolated singularities have been studied by many authors (see e.g. [157, 393] and the literature cited therein). We point out the great cycle of the L. Veron works [384] - [397].

The behavior of solutions near a conical boundary points is treated only in special cases in [375, 99, 59] for $a_0(x) \equiv 0$, in [52] for bounded solutions and for $m = 2$. In this chapter we extend these results to the more general quasilinear case $m \neq 2$.

The problem $(NEVP)$ was studied by P.Tolksdorf [374, 375, 376, 378] and a more detailed analysis is carried out by Aronsson [11], Krol [204, 205] and §9.5.2, Chapter 9.

The solvability property of the operator $\mathfrak{D}$ associated with the eigenvalue problem $(NEVP)$, Theorem 8.14, as proved here is due to M. Dobrowolski [99, 68].

There is a number of works relating to the estimation of the first eigenvalue of the $m-$Laplacian in a Riemannian manifold (see, e.g., [223, 411, 371, 155]). Apropos to the one-dimensional Wirtinger inequality, see also Theorems 256, 257 [142].

The other $L^\infty-$ estimates of weak solutions of the problem (DQL) can be found in §10.5, Chapter 10 [129] and in §7, Chapter IV [216].

Integral estimates of second weak derivatives of the (DQL) weak solutions in smooth domains were established in [**215, 216, 217, 401**]. In Section 8.4 we make these estimates more precise in the case of smooth domains as well as establish new estimates for nonsmooth domains and we follow [**57, 70**].

G. Savaré [**354**] obtained recently the certain new regularity results for solutions of the Dirichlet and Neumann problems to some linear and quasilinear elliptic equation of the variational structure in the Lipschitz domains. M. Fuchs & Li Gongbao [**125**] established L^∞−bound for weak solutions of the Dirichlet problem for the quasilinear elliptic equation on Orlicz- Sobolev spaces. S. Knobloch [**158**] considered the Neumann problem for (DQL) in a plane domain with corners.

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CHAPTER 9

The boundary value problems for elliptic quasilinear equations with triple degeneration in a domain with boundary edge

9.1. Introduction. Assumptions.

This chapter is devoted to the estimate of weak solutions to the boundary value problems for elliptic quasilinear *degenerate* second order equations. We investigate the behavior of weak solutions of the first and mixed boundary value problems for quasilinear elliptic equation of the second order with triple degeneracy and singularity in the coefficients in a neighborhood of singular boundary point.

Let G be a domain in $\mathbb{R}^N$, $N \geq 3$, bounded by $(N-1)-$ dimensional manifold ∂G and let Γ_1, Γ_2 be open nonempty submanifolds of ∂G, possessing the following properties: $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\partial G = \overline{\Gamma_1} \cup \overline{\Gamma_2}$, where $\overline{\Gamma_1} \cap \overline{\Gamma_2}$ is smooth $(N-2)-$ dimensional submanifold that contains an edge $\Gamma_0 \subseteq \overline{\Gamma_1} \cap \overline{\Gamma_2}$. We also fix a partition of $\{0, 1, 2\}$ into two subsets $\mathcal{N}$ and $\mathcal{D}$. The union of the Γ_j with $j \in \mathcal{D}$ is going to be the part of the boundary where we consider a Dirichlet boundary condition, but with $j \in \mathcal{N}$ is going to be the part of the boundary where we consider first order boundary conditions either Neumann or the third BVP. In what follows we suppose $\{0, 1\} \in \mathcal{D}$. If $2 \in \mathcal{D}$, then our problem is the Dirichlet problem, if $2 \in \mathcal{N}$, then our problem is the mixed BVP.

We derive *an almost exact estimate* of the weak solution in a neighborhood of an edge of the boundary for the problem

$$(BVP) \quad \begin{cases} -\dfrac{d}{dx_i} a_i(x, u, u_x) + a_0 a(x, u, u_x) + b(x, u, u_x) = f(x), \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in G, \text{ where } a_0 \geq 0; \\ u(x) = 0, \quad x \in \partial G, \text{ if } 2 \in \mathcal{D} \quad \text{and } x \in \partial G \setminus \Gamma_2, \text{ if } 2 \in \mathcal{N}; \\ a_i(x, u, u_x) n_i(x) + \sigma(x, u) = g(x), \quad x \in \Gamma_2, \text{ if } 2 \in \mathcal{N}. \end{cases}$$

(summation over repeated indices from 1 to N is understood.) Here: $n_i(x)$, $i = 1, \ldots, N$ are components of the unit outward normal to Γ_2.

For $x = (x_1, \ldots, x_N)$ let us define the cylindrical coordinates $(\overline{x}, r, \omega)$ as

$$\overline{x} = (x_1, \ldots, x_{N-2}), \quad r = \sqrt{x_{N-1}^2 + x_N^2}, \text{ and } \omega = \arctan \frac{x_{N-1}}{x_N}.$$

For the sufficiently small number $d > 0$ we also define the following sets as

$$G_0^d = G \cap \{(\overline{x}, r, \omega) \mid \overline{x} \in \mathbb{R}^{N-2}, \ 0 < r < d, \ \omega \in (-\omega_0/2, \omega_0/2)\};$$

$$\Gamma_j^d = \Gamma_j \cap \overline{G_0^d} \subset \partial G_0^d, \quad j = 0, 1, 2;$$

$$\Omega_d = G \cap \{(\overline{x}, r, \omega) \mid \overline{x} \in \mathbb{R}^{N-2}, \ r = d, \ \omega \in [-\omega_0/2, \omega_0/2]\} \subset \partial G_0^d;$$

$$\omega_0 \in (0, 2\pi).$$

We shall assume the following

- $\partial G \setminus \Gamma_0$ is smooth submanifold in $\mathbb{R}^N$;
- there exists a number $d > 0$ such that

$$\Gamma_0^d = \{(\overline{x}, 0, 0) \mid |\overline{x}| < d\} \subset \Gamma_0$$

  is the straight edge with the center in the origin;
- G_0^d is locally diffeomorphic to the dihedral cone

$$\mathbb{D}_d = \{(r, \omega) \mid 0 < r < d, \ \omega \in (-\omega_0/2, \omega_0/2)\} \times \mathbb{R}^{N-2}; \quad 0 < \omega_0 < 2\pi;$$

  thus we assume that $G_0^d \subset G$ and, consequently, the domain G is a *"wedge"* in some vicinity of the edge.
- $\omega \mid_{\Gamma_1} = -\omega_0/2$; and $\omega \mid_{\Gamma_2} = \omega_0/2$.

Let $C^0(\overline{G})$ be the set of continuous functions on $\overline{G}$ and let $L_m(G)$ and $W^{k,m}(G), m > 1$ be the usual Lebesgue and Sobolev spaces respectively. By $\mathfrak{N}_{m,q}^1(\nu, \nu_0, G)$ we shall denote a set of functions $u(x) \in L_\infty(G)$ having first weak derivatives with the finite integral

$$(9.1.1) \qquad \int\limits_G \left(\nu(x)|u|^q |\nabla u|^m + \nu_0(x)|u|^{q+m} \right) dx < \infty, \quad q \geq 0, \ m > 1,$$

where $\nu_0(x)$ and $\nu(x)$ are two nonnegative measurable in G functions such that

$$\nu_0^{-1}(x) \in L_t(G), \ \nu^{-1}(x) \in L_t(G); \ \nu_0(x) \in L_s(G), \ \frac{1}{s} + \frac{1}{t} < \frac{m}{N};$$

$$(9.1.2)$$

$$1 + \frac{1}{t} < m < N \left(1 + \frac{1}{t} \right), \quad t > \max(N, \frac{N}{m-1}), \quad N > m > 1.$$

If $X(G)$ is one of the above spaces, then by $X(G, \Gamma)$ with any $\Gamma \subseteq \partial G$ we denote a subset of functions $u(x) \in X(G)$ vanishing on Γ in the sense of traces. Now we define *the space V*

$$V := \begin{cases} \mathfrak{N}_{m,q}^1(\nu, \nu_0, G, \partial G), & \text{if BVP is the Dirichlet problem,} \\ \mathfrak{N}_{m,q}^1(\nu, \nu_0, G, \partial G \setminus \Gamma_2), & \text{if BVP is the mixed problem.} \end{cases}$$

We set also V_0 is V for $q = 0$. Let us define for $\forall \varepsilon \geq 0$ the number

$$\theta_\varepsilon := \begin{cases} \frac{1}{2}(\omega_0 + \varepsilon), & \text{if BVP is the Dirichlet problem,} \\ \omega_0 + \varepsilon, & \text{if BVP is the mixed problem,} \end{cases}$$

and let λ be the *least positive number* satisfying

$$\int\limits_0^{+\infty} \frac{[(m-1)y^2 + \lambda^2](y^2 + \lambda^2)^{\frac{m-4}{2}} dy}{(m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m+\tau)(y^2+\lambda^2)^{\frac{m-2}{2}} - a_0} =$$

(9.1.3) $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \theta_0,$

(9.1.4) $\quad \lambda^m(q + m - 1 + \mu) + \lambda^{m-1}(2 - m + \tau) > a_0.$

We shall use the following notation $(|u| - k)_+ := \max(|u| - k; 0)$.

Concerning the equation of (BVP) we make the following **assumptions.**

Let $1 < m < N$, $l > N$, $q \geq 0$ and $0 \leq \mu < 1$ be given numbers and let $\alpha(x), \alpha_0(x), b_0(x)$ be nonnegative functions.

   1) $f(x), \alpha(x), \alpha_0(x), b_0(x)$ and $g(x)$ are measurable functions such that

$$\nu_0^{-1}(x)(\alpha_0(x) + b_0(x) + f(x)) \in L_p(G); \quad \alpha(x) \in L_{m'}(G); \quad g(x) \in L_\alpha(\Gamma_2);$$

$$\frac{1}{p} < \frac{m}{N} - \frac{1}{t} - \frac{1}{s}, \quad \alpha > \frac{N-1}{m-1-\frac{N}{t}}, \quad \frac{1}{m} + \frac{1}{m'} = 1;$$

$a_i(x, u, \xi)$, $i = 1, \ldots, N$; $a(x, u, \xi), b(x, u, \xi)$ and $\sigma(x, u)$ are *Caratheodory functions $G \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, possessing the properties*

   2) $a_i(x, u, \xi)\xi_i \geq \nu(x)|u|^q|\xi|^m - \alpha_0(x)$; $\quad a(x, u, \xi)u \geq \nu_0(x)|u|^{q+m}$; $\sigma(x, u) \cdot \text{sign } u \geq 0$;

   3) $|b(x, u, \xi)| \leq \mu\nu(x)|u|^{q-1}|\xi|^m + b_0(x)$;

4) $\sqrt{\sum_{i=1}^{n} a_i^2(x,u,\xi)} \leq \nu(x)|u|^q|\xi|^{m-1} + \nu^{\frac{1}{m}}(x)\nu_0^{1/m'}(x)|u|^{q+m-1} +$
$+ \alpha(x)\nu^{\frac{1}{m}}(x);$

5) $|a(x,u,\xi)| \leq \nu^{\frac{1}{m'}}(x)\nu_0^{1/m}(x)|u|^q|\xi|^{m-1} + \nu_0(x)|u|^{q+m-1} +$
$+ \alpha(x)\nu_0^{1/m}(x)|u|^q;$

6) $\int_{\Gamma_2} |\sigma(x,u)|ds < \infty \quad \forall u \in L_\infty(G \cup \Gamma_2).$

In addition, suppose that *the functions* $a_i(x,u,\xi), a(x,u,\xi), b(x,u,\xi)$ *and* $\sigma(x,u)$ *are continuously differentiable with respect to the* x, u, ξ *variables in* $\mathfrak{M}_{d,M_0} = \overline{G_0^d} \times [-M_0, M_0] \times \mathbb{R}^N$ *and satisfy in* $\mathfrak{M}_{d,M_0}$

7) $(m-1)u\frac{\partial a_i(x,u,\xi)}{\partial u} = q\frac{\partial a_i(x,u,\xi)}{\partial \xi_j}\xi_j; \quad i = 1,\dots,N;$

8) $\frac{\partial a_i(x,u,\xi)}{\partial \xi_j}p_ip_j \geq \gamma_{m,q}\nu(x)|u|^q|\xi|^{m-2}p^2, \quad \forall p \in \mathbb{R}^N \setminus \{0\};$

9) $\sqrt{\sum_{i=1}^{N}\left|\frac{\partial b(x,u,\xi)}{\partial \xi_i}\right|^2} \leq \nu(x)|u|^{q-1}|\xi|^{m-1};$

10) $\frac{\partial b(x,u,\xi)}{\partial u} \geq \nu(x)|u|^{q-2}|\xi|^m; \quad \frac{\partial a(x,u,\xi)}{\partial u} \geq \gamma_{m,q}\nu_0(x)|u|^{q+m-2};$
$\frac{\partial \sigma(x,u)}{\partial u} \geq 0;$

11) $\sqrt{\sum_{i=1}^{N}\left|a_i(x,u,\xi) - r^\tau|u|^q|\xi|^{m-2}\xi_i\right|^2} \leq c_1(r)r^\tau|u|^q|\xi|^{m-1} + \psi_1(r);$

12) $\left|\frac{\partial a_i(x,u,\xi)}{\partial \xi_j} - r^\tau|u|^q|\xi|^{m-4}\left(\delta_i^j|\xi|^2 + (m-2)\xi_i\xi_j\right)\right| \leq$
$\leq c_2(r)r^\tau|u|^q|\xi|^{m-2} + c_2(r)\psi_2(r)|u|^{\frac{q}{m-1}};$

13) $\left|\frac{\partial a_i(x,u,\xi)}{\partial x_i} - \tau r^{\tau-2}|u|^q|\xi|^{m-2}x_i\xi_i\right| \leq c_3(r)r^{\tau-1}|u|^q|\xi|^{m-1} + \psi_3(r);$

14) $\left|a(x,u,\xi) - r^{\tau-m}u|u|^{q+m-2}\right| + \left|b(x,u,\xi) + \mu r^\tau u|u|^{q-2}|\xi|^m\right| \leq$
$\leq c_4(r)r^\varkappa|u|^q|\xi|^{m-1} + |u|^q\psi_4(r),$

where $\gamma_{m,q} > 0$, $c_i(r)$ *are nonnegative functions that are continuous at zero with* $c_i(0) = 0$. *In addition, let there exist numbers* $k_i \geq 0$, *such that* $\psi_i(r) \leq k_i r^{\beta_i}$, $i = 1, ..., 4$, *where*

$$\beta_1 = \frac{l(N-1) - N(m-1)}{l(N-m)}\tau - \frac{2}{l}(m-1) + \lambda(q+m-1);$$

$$\beta_2 = \tau - m + 2 + \lambda(q+m-1)\frac{m-2}{m-1};$$

(9.1.5) $$\beta_3 = \tau - m + \lambda(q+m-1);$$

$$\beta_4 = \frac{(l-m)N}{l(N-m)}\tau - \frac{2}{l}m + \lambda(m-1),$$

$$\varkappa = \frac{l(N-m+1) - N}{l(N-m)}\tau - \frac{2}{l} + \varepsilon, \ \forall \varepsilon > 0.$$

REMARK 9.1. Our assumptions *11)-14)* essentially mean that the coefficients of the (BVP) near the edge Γ_0 are close to coefficients of model equation

$$-\frac{d}{dx_i}\left(r^\tau |u|^q |\nabla u|^{m-2} u_{x_i}\right) + a_0 r^{\tau - m} u|u|^{q+m-2} -$$

(ME) $$-\mu r^\tau |u|^{q-1} |\nabla u|^m \text{sign } u = f(x),$$

$$0 \leq \mu < 1, \quad q \geq 0, \quad m > 1, \quad a_0 \geq 0, \quad \tau \geq m - 2.$$

DEFINITION 9.2. Function $u(x)$ is called a *weak* solution of (BVP) provided that $u(x) \in V$ and satisfies the integral identity

(II) $$\int_G \{a_i(x, u, u_x)\phi_{x_i} + a_0 a(x, u, u_x)\phi + b(x, u, u_x)\phi\} \, dx =$$

$$= \int_G f(x)\phi dx + \int_{\Gamma_2} \{g(x) - \sigma(x, u)\} \, \phi ds$$

for all $\phi(x) \in V$.

One can easily verify that assumptions *1)-6)* together with (9.1.2) guarantee the correctness of such a definition.

We need the following auxiliary statements

LEMMA 9.3. *Let* $m^\#$ *denote the number associated to* m *by the relation*

(9.1.6) $$\frac{1}{m^\#} = \frac{1}{m}\left(1 + \frac{1}{t}\right) - \frac{1}{N}$$

and suppose that assumption (9.1.2) holds. Then there exist constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$ depending only on meas $G, \omega_0, N, m, t, ||\nu_0^{-1}||_{L_t(G)}$, $||\nu^{-1}||_{L_t(G)}$ such that

$$(9.1.7) \qquad \int\limits_G \nu_0(x)|v|^m dx \leq c_1 \int\limits_G \nu(x)|\nabla v|^m dx$$

and

$$(9.1.8) \qquad \left(\int\limits_G |v|^{m^\#} dx \right)^{\frac{m}{m^\#}} \leq c_2 \int\limits_G \left(\nu_0(x)|v|^m + \nu(x)|\nabla v|^m \right) dx$$

for any $v(x) \in V_0$ and also

$$(9.1.9) \qquad \int\limits_G \nu_0(x)|u|^{q+m} dx \leq c_3 \int\limits_G \nu(x)|u|^q |\nabla u|^m dx,$$

for any $u(x) \in V$.

PROOF. The proof for (9.1.7) had been given either in §1.5 [100] or in the statements 3.2 - 3.5 [316]. The inequality (9.1.9) is obtained from (9.1.7) by performing in the latter the following substitution

$$u = v|v|^{\sigma-1}, \qquad \sigma = \frac{m}{q+m}.$$

Now we prove the inequality (9.1.8) following the Theorem 3.1 [316]. We shall deduce the inequality (9.1.8) from the corresponding ones for the imbedding Sobolev Theorem 1.31, namely if $1 < m < N$ then

$$(9.1.10) \qquad ||v||_{L^{\frac{mN}{N-m}}(G)} \leq C||v||_{W^{1,m}(G)}, \quad \forall v \in W^{1,m}(G).$$

If we put $\frac{1}{\varkappa} = 1 + \frac{1}{t}$ then we have from (9.1.2)

$$1 < m\varkappa < N \quad \text{and} \quad \varkappa + \frac{\varkappa}{t} = 1.$$

Now, by using the Hölder integral inequality with $p = \frac{1}{\varkappa}$, $p' = \frac{1}{1-\varkappa}$, we obtain

$$||v||_{L^{m\varkappa}(G)} = \left(\int\limits_G |v|^{m\varkappa} \nu_0^\varkappa(x) \nu_0^{-\varkappa}(x) dx \right)^{\frac{1}{m\varkappa}} \leq$$

$$(9.1.11)$$

$$\leq ||\nu_0^{-1}(x)||_{L^t(G)}^{\frac{1}{m}} \cdot \left(\int\limits_G \nu_0(x)|v|^m dx \right)^{\frac{1}{m}}.$$

Similarly,

$$
\|\nabla v\|_{L^{m\varkappa}(G)} = \left(\int_G |\nabla v|^{m\varkappa} \nu^{\varkappa}(x) \nu^{-\varkappa}(x) dx \right)^{\frac{1}{m\varkappa}} \le
$$

(9.1.12)

$$
\le \|\nu^{-1}(x)\|_{L^{t}(G)}^{\frac{1}{m}} \cdot \left(\int_G \nu(x) |\nabla v|^m dx \right)^{\frac{1}{m}}.
$$

We consider now the inequality (9.1.10) replacing m by $m\varkappa$. (In this connection we verify that $\frac{Nm\varkappa}{N-m\varkappa} = m^{\#}$). Then we obtain

$$
\|v\|_{L^{m^{\#}}(G)} \le C \left(\|v\|_{L^{m\varkappa}(G)} + \|\nabla v\|_{L^{m\varkappa}(G)} \right).
$$

Hence and from (9.1.11), (9.1.12) it follows the required inequality (9.1.8). $\square$

LEMMA 9.4. *There exists a constant $c_4 > 0$ depending on N, m, t, G, Γ_2 such that for any $v(x) \in \mathfrak{N}_{m,0}^1(\nu, \nu_0, G, \partial G \setminus \Gamma_2)$*

(9.1.13)
$$
\left(\int_{\Gamma_2} |v|^{\alpha^*} ds \right)^{\frac{1}{\alpha^*}} \le c_4 \Big\{ \int_G \left(\nu_0(x)|v|^m + \nu(x)|\nabla v|^m \right) dx \Big\}^{\frac{1}{m}},
$$

where

(9.1.14)
$$
\alpha^* = \frac{m(N-1)}{N - m + \frac{N}{t}}.
$$

PROOF. By the theorem of trace for Sobolev spaces (Theorem 1.35), we have

$$
\|v\|_{L^{\alpha^*}(\Gamma_2)} \le c \|v\|_{W^{1,m\varkappa}(G)}
$$

with α^* from (9.1.14). Hence and from the inequalities (9.1.11), (9.1.12) it follows the desired inequality (9.1.13). $\square$

COROLLARY 9.5. (From Lemmas 9.3, 9.4).

(9.1.15)
$$
\left(\int_G |v|^{m^{\#}} dx \right)^{\frac{m}{m^{\#}}} + \left(\int_{\Gamma_2} |v|^{\alpha^*} ds \right)^{\frac{m}{\alpha^*}} \le
$$

$$
\le c_5 \int_G \left(\nu_0(x)|v|^m + \nu(x)|\nabla v|^m \right) dx
$$

for any $v(x) \in \mathfrak{N}_{m,0}^1(\nu, \nu_0, G, \partial G \setminus \Gamma_2)$, where the constant $c_5 > 0$ depends on $N, m, t, G, \Gamma_2, \|\nu_0^{-1}\|_{L_t(G)}, \|\nu^{-1}\|_{L_t(G)}$.

The main statement of this chapter is in the following theorem.

Main Theorem. *Let $u(x)$ be a weak solution to (BVP) and let λ be least positive solution of (9.1.3) and (9.1.4). Suppose that the assumptions (9.1.2) and 1) - 14) with $m \geq 2$ are fulfilled. Let there be nonnegative constants f_1, g_1 such that*

$$|f(x)| \leq f_1 r^{\tau - m + \lambda(q + m - 1)}, \ x \in G_0^d \text{ and}$$

(9.1.16)

$$|g(x)| \leq g_1 r^{\tau - m + 1 + \lambda(q + m - 1)}, \ x \in \Gamma_2^d.$$

Then for every $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$, depending only on the parameters and norms of functions occuring in the assumptions such that

(9.1.17) $$|u(x)| \leq c_\varepsilon r^{\lambda - \varepsilon}.$$

9.2. A weak comparison principle. The E. Hopf strong maximum principle

Now we shall prove a weak comparison principles for the quasilinear equation which extend the corresponding results in chapter 10, Theorem 10.7 [129] and in chapter 3, Lemma 3.1 [375] (see also [298]).

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with lipschitzian boundary $\partial\Omega = \overline{\partial_1\Omega} \cup \overline{\partial_2\Omega}$. We consider the second order quasilinear degenerate operator Q of the form

$$Q(v, \phi) \equiv \int_\Omega \Big\langle \mathcal{A}_i(x, v_x)\phi_{x_i} + \mathcal{A}(x, v)\phi + \mathcal{B}(x, v, v_x)\phi -$$

(9.2.1)

$$- f(x)\phi \Big\rangle dx + \int_{\partial_2\Omega} \Big\langle \Sigma(x, v) - g(x) \Big\rangle \phi ds$$

for $v \in \mathfrak{N}^1_{m,0}(\nu, \nu_0, \Omega, \partial\Omega \setminus \partial_2\Omega)$ and for all nonnegative ϕ belonging to the set $\mathfrak{N}^1_{m,0}(\nu, \nu_0, \Omega, \partial\Omega \setminus \partial_2\Omega)$ under following assumptions

the functions $f(x), g(x)$ are summable on Ω and $\partial_2\Omega$ respectively; the functions $\mathcal{A}_i(x, \eta), \mathcal{A}(x, v), \mathcal{B}(x, v, \eta), \pm(x, v)$ are Caratheodory, continuously differentiable with respect to the v, η variables in $\mathfrak{M} = \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N$ and satisfy in $\mathfrak{M}$ the following inequalities:

(i) $\frac{\partial \mathcal{A}_i(x,\eta)}{\partial \eta_j} p_i p_j \geq \gamma_m \nu(x) |\eta|^{m-2} p^2, \ \forall p \in \mathbb{R}^N \setminus \{0\};$

(ii) $\sqrt{\sum_{i=1}^N \left| \frac{\partial \mathcal{B}(x,v,\eta)}{\partial \eta_i} \right|^2} \leq \nu(x) |v|^{-1} |\eta|^{m-1};$

(iii) $\frac{\partial B(x,v,\eta)}{\partial v} \geq \nu(x)|v|^{-2}|\eta|^m$; $\frac{\partial A(x,v)}{\partial v} \geq \gamma_m \nu_0(x)|v|^{m-2}$; $\frac{\partial \Sigma(x,v)}{\partial v} \geq 0$.

Here: $m > 1$, $\gamma_m > 0$ and $\nu_0(x)$ and $\nu(x)$ are the functions defined by (9.1.2).

THEOREM 9.6. *Let operator Q satisfy assumptions (i) - (iii). Let the functions $v, w \in \mathfrak{N}_{m,0}^1(\nu, \nu_0, \Omega, \partial\Omega \setminus \partial_2\Omega)$ satisfy the inequality*

(9.2.2) $Q(v, \phi) \leq Q(w, \phi)$

for all non-negative $\phi \in \mathfrak{N}_{m,0}^1(\nu, \nu_0, \Omega, \partial\Omega \setminus \partial_2\Omega)$ and also the inequality

(9.2.3) $v(x) \leq w(x)$, *on* $\partial\Omega \setminus \partial_2\Omega$

holds in the weak sense. Then

(9.2.4) $v(x) \leq w(x)$, *a.e. in* Ω.

PROOF. Let us define

$$z = v - w; \text{ and } v^t = tv + (1-t)w, \ t \in [0,1].$$

Then we have $0 \geq Q(v, \phi) - Q(w, \phi) =$

$$= \int_\Omega \left\langle \phi_{x_i} z_{x_j} \int_0^1 \frac{\partial A_i(x, v_x^t)}{\partial v_{x_j}^t} dt + z\phi \int_0^1 \frac{\partial A(x, v^t)}{\partial v^t} dt + \right.$$

(9.2.5) $$\left. + \phi z_{x_i} \int_0^1 \frac{\partial B(x, v^t, v_x^t)}{\partial v_{x_i}^t} dt + \phi z \int_0^1 \frac{\partial B(x, v^t, v_x^t)}{\partial v^t} dt \right\rangle dx +$$

$$+ \int_{\partial_2\Omega} \phi z \int_0^1 \frac{\partial \Sigma(x, v^t)}{\partial v^t} dt ds$$

for all non-negative $\phi \in \mathfrak{N}_{m,0}^1(\nu, \nu_0, \Omega, \partial\Omega \setminus \partial_2\Omega)$.

Now let $k \geq 1$ be any odd number. We define the set

$$\Omega_+ := \{x \in \overline{\Omega} \mid v(x) > w(x)\}.$$

As the test function in the integral inequality (9.2.2), we choose

$$\phi = \max\{(v - w)^k, 0\}.$$

By assumptions *(i)-(iii)* we then obtain

$$k\gamma_m \int_{\Omega_+} \nu(x)z^{k-1}\Big(\int_0^1 |\nabla v^t|^{m-2}dt\Big)|\nabla z|^2 dx+$$

$$\gamma_m \int_{\Omega_+} \nu_0(x)z^{k+1}\Big(\int_0^1 |v^t|^{m-2}dt\Big)dx + \int_{\Omega_+} \nu(x)z^{k+1}\Big(\int_0^1 |v^t|^{-2}|\nabla v^t|^m dt\Big)dx$$

$$(9.2.6) \qquad \le \int_{\Omega_+} \nu(x)z^k\Big(\int_0^1 |v^t|^{-1}|\nabla v^t|^{m-1}dt\Big)|\nabla z|dx.$$

Now we use the Cauchy inequality

$$z^k|\nabla z||v^t|^{-1}|\nabla v^t|^{m-1} = \Big(|v^t|^{-1}z^{\frac{k+1}{2}}|\nabla v^t|^{m/2}\Big)\cdot\Big(z^{\frac{k-1}{2}}|\nabla z||\nabla v^t|^{m/2-1}\Big)$$

$$\le \frac{\varepsilon}{2}|v^t|^{-2}z^{k+1}|\nabla v^t|^m + \frac{1}{2\varepsilon}z^{k-1}|\nabla z|^2|\nabla v^t|^{m-2}, \ \forall\varepsilon > 0.$$

Hence, taking $\varepsilon = 2$, we obtain from (9.2.6) the inequality

$$(9.2.7) \qquad \Big(k\gamma_m - \frac{1}{4}\Big)\int_{\Omega_+} \nu(x)z^{k-1}|\nabla z|^2\Big(\int_0^1 |\nabla v^t|^{m-2}dt\Big)dx \le 0.$$

Now choosing the odd number $k \ge \max\Big(1; \frac{1}{2\gamma_m}\Big)$ in view of $z(x) \equiv 0$ almost everywhere on $\partial\Omega_+$, we get from (9.2.7) $z(x) \equiv 0$ almost everywhere in Ω_+. We have finished with the contradiction to our definition of the set Ω_+. By this fact the (9.2.4) is proved. $\qquad\square$

REMARK 9.7. The operator Q, generated by the model equation *(ME)* with $q = 0$, satisfies all assumption *(i)-(iii)*. In fact, we have for this case

$$\nu(x) = r^\tau, \ \nu_0(x) = a_0 r^{\tau-m}, \ \mathcal{A}_i(x,\eta) = \nu(x)|\eta|^{m-2}\eta_i,$$

$$\mathcal{A}(x,v) = \nu_0(x)v|v|^{m-2}, \ \mathcal{B}(x,v,\eta) = -\mu\nu(x)v^{-1}|\eta|^m.$$

Therefore

$$\nu^{-1}(x)\frac{\partial\mathcal{A}_i(x,\eta)}{\partial\eta_j} = \delta_i^j|\eta|^{m-2} + (m-2)|\eta|^{m-4}\eta_i\eta_j$$

and hence

$$\nu^{-1}(x)\frac{\partial\mathcal{A}_i(x,\eta)}{\partial\eta_j}p_ip_j = |\eta|^{m-2}|p|^2 + (m-2)|\eta|^{m-4}(p_i\eta_i)^2 \ge \gamma_m|\eta|^{m-2}|p|^2,$$

where

$$\gamma_m = \begin{cases} 1, & \text{if } m \geq 2; \\ m - 1, & \text{if } 1 < m \leq 2, \end{cases}$$

that is *(i)* holds.

Furthermore,

$$\frac{\partial \mathcal{B}(x, v, \eta)}{\partial \eta_i} = -\mu m \nu(x) v^{-1} |\eta|^{m-2} \eta_i$$

and hence *(ii)* holds. At last

$$\frac{\partial \mathcal{A}(x, v)}{\partial v} = (m - 1)\nu_0(x)|v|^{m-2}, \quad \frac{\partial \mathcal{B}(x, v, \eta)}{\partial v} = \mu \nu(x)|v|^{-2}|\eta|^m,$$

and therefore *(iii)* holds as well.

Now we want to prove the strong Hopf maximum principle (cf. §3.2 [375].) In addition to *(i)-(iii)* we shall suppose

$$(v) \quad |\eta| \cdot \left| \sum_{i=1}^{N} \frac{\partial \mathcal{A}_i(x, \eta)}{\partial \eta_i} \right| + \left| \sum_{i=1}^{N} \frac{\partial \mathcal{A}_i(x, \eta)}{\partial x_i} \right| + \left| \mathcal{B}(x, v, \eta) + \mu \nu(x) v^{-1} |\eta|^m \right| \leq$$

$$\leq \tilde{\gamma}_m \nu(x) |\eta|^{m-1}$$

for some non-negative constants $\tilde{\gamma}_m, \mu$.

LEMMA 9.8. *Let $B_d(y)$ be an open ball of radius $d > 0$ centered at y, contained in $\Omega \subset \mathbb{R}^N$ and $v(x) \in \mathfrak{N}^1_{m,0}(\nu, \nu_0, B_d(y)) \cap C^1\overline{(B_d(y))}$ be a solution of*

$$(9.2.8) \qquad \mathcal{Q}_0(v, \phi) \equiv \int\limits_{B_d(y)} \left\langle \mathcal{A}_i(x, v_x)\phi_{x_i} + \mathcal{B}(x, v, v_x)\phi \right\rangle dx = 0$$

for all nonnegative $\phi \in L_\infty(B_d(y)) \cap W^{1,m}(B_d(y), \partial B_d(y))$. Suppose that assumptions $(i) - (v)$ are fulfilled. Assume that

$$(9.2.9) \qquad v(x) > 0, \ x \in B_d(y) \ and \ v(x_0) = 0 \ for \ some \ x_0 \in \partial B_d(y).$$

Then

$$(9.2.10) \qquad |\nabla v(x_0)| \neq 0.$$

PROOF. We consider the annular region

$$\mathcal{R} = B_d(y) \setminus B_{d/2}(y) = \left\{ x \mid \frac{d}{2} < |x - y| < d \right\}$$

and the function

$$w(x) = e^{-\sigma|x-y|^2} - e^{-\sigma d^2}, \quad x \in \mathcal{R}, \ \sigma > 0.$$

Direct calculation gives

$$(9.2.11) \qquad 0 \le w(x) \le e^{-\sigma|x-y|^2};$$

$$(9.2.12) \quad w_{x_i} = -2\sigma(x_i - y_i)e^{-\sigma|x-y|^2}; \quad |\nabla w| = 2\sigma|x - y|e^{-\sigma|x-y|^2};$$

$$(9.2.13) \qquad w_{x_i x_j} = \left\langle 4\sigma^2(x_i - y_i)(x_j - y_j) - 2\sigma\delta_i^j \right\rangle e^{-\sigma|x-y|^2};$$

$$\mathcal{L}(\varepsilon w) \equiv -\frac{d\mathcal{A}_i(x, \varepsilon w_x)}{dx_i} + \mathcal{B}(x, \varepsilon w, \varepsilon w_x) =$$

$$= -\varepsilon \frac{\partial \mathcal{A}_i(x, \varepsilon w_x)}{\partial(\varepsilon w_{x_j})} w_{x_i x_j} - \frac{\partial \mathcal{A}_i(x, \varepsilon w_x)}{\partial x_i} + \mathcal{B}(x, \varepsilon w, \varepsilon w_x) =$$

$$= -4\varepsilon\sigma^2 e^{-\sigma|x-y|^2} \frac{\partial \mathcal{A}_i(x, \varepsilon w_x)}{\partial(\varepsilon w_{x_j})}(x_i - y_i)(x_j - y_j) +$$

$$+ 2\varepsilon\sigma e^{-\sigma|x-y|^2} \frac{\partial \mathcal{A}_i(x, \varepsilon w_x)}{\partial(\varepsilon w_{x_i})} - \frac{\partial \mathcal{A}_i(x, \varepsilon w_x)}{\partial x_i} + \mathcal{B}(x, \varepsilon w, \varepsilon w_x), \quad \forall \varepsilon > 0.$$

By assumptions *(i)* and *(v)* it follows that

$$\mathcal{L}(\varepsilon w) \le -\varepsilon^{m-1}\nu(x)|\nabla w|^{m-2}e^{-\sigma|x-y|^2} \cdot \left\langle 4\gamma_m|x - y|^2\sigma^2 - \right.$$

$$(9.2.14)$$

$$\left. -2\tilde{\gamma}_m\sigma - 4|x - y|\tilde{\gamma}_m\sigma \right\rangle - \mu\varepsilon^{m-1}\nu(x)w^{-1}|\nabla w|^m, \ \forall \varepsilon > 0.$$

Now we observe by (9.2.11), (9.2.12) that

$$(9.2.15) \qquad \frac{|\nabla w|}{w} > 2\sigma|x - y|$$

and therefore we have from (9.2.14) in the region $\mathcal{R}$

$$\mathcal{L}(\varepsilon w) \le -\varepsilon^{m-1}\nu(x)|\nabla w|^{m-2}e^{-\sigma|x-y|^2} \cdot \left\langle (\gamma_m + \mu)d^2\sigma^2 - 2(1 + 2d)\tilde{\gamma}_m\sigma \right\rangle,$$

$$\varepsilon > 0.$$

If we choose $\sigma \ge \frac{2(1+2d)\tilde{\gamma}_m}{(\gamma_m+\mu)d^2}$, then we obtain

$$(9.2.16) \qquad \mathcal{L}(\varepsilon w) \le 0 \quad \text{in } \mathcal{R}, \ \forall \varepsilon > 0.$$

Since $v > 0$ on $\partial B_{d/2}(y)$ there is a constant $\varepsilon > 0$ for which $v - \varepsilon w \ge 0$ on $\partial B_{d/2}(y)$. This inequality is also satisfied on $\partial B_d(y)$ where $w = 0$. By virtue of (9.2.16) we have

$$\mathcal{Q}_0(\varepsilon w, \phi) = \int_{B_d(y)} \phi \mathcal{L}(\varepsilon w) dx \le 0 = \mathcal{Q}_0(v, \phi).$$

Thus we obtain

$$(9.2.17) \qquad \begin{cases} \mathcal{Q}_0(v, \phi) \geq \mathcal{Q}_0(\varepsilon w, \phi) \text{ in } \mathcal{R}; \\ v \geq \varepsilon w \text{ on } \partial \mathcal{R}. \end{cases}$$

By the weak comparison principle (Theorem 9.6) from (9.2.17) it follows that

$$(9.2.18) \qquad v \geq \varepsilon w \quad \text{throughout } \mathcal{R}.$$

Since $x_0 \in \partial B_d(y)$ and $w(x_0) = 0$ we now have

$$\frac{v(x) - v(x_0)}{|x - x_0|} \geq \varepsilon \frac{w(x) - w(x_0)}{|x - x_0|}$$

and therefore

$$|\nabla v(x_0)| \geq \varepsilon |\nabla w(x_0)| = 2\varepsilon \sigma d e^{-\sigma d^2} > 0, \quad \text{Q.E.D.}$$

$\square$

THEOREM 9.9. **(Strong maximum principle of E.Hopf).** *Assume that Ω is connected and $v(x) \in \mathfrak{N}^1_{m,0}(\nu, \nu_0, \Omega) \cap C^1(\Omega)$ is a non-negative weak solution of*

$$\int_\Omega \Big\langle \mathcal{A}_i(x, v_x)\phi_{x_i} + \mathcal{B}(x, v, v_x)\phi \Big\rangle dx = 0$$

for all nonnegative $\phi \in L_\infty(\Omega) \cap W^{1,m}(\Omega, \partial\Omega)$. Assume that $v(x) \not\equiv 0$. Suppose that assumptions $(i) - (v)$ are fulfilled. Then

$$(9.2.19) \qquad v(x) > 0, \ x \in \Omega$$

PROOF. Assume that $v(x_0) = 0$ for some $x_0 \in \Omega$. Then, we can find a ball $B_d(y) \subset \Omega$, satisfying the hypotheses of Lemma 9.8, that is $x_0 \in \partial B_d(y)$. By this Lemma we have $|\nabla v(x_0)| \neq 0$. But $0 = v(x_0) = \inf_{x \in \Omega} v(x)$ and therefore $|\nabla v(x_0)| = 0$. This, however, is a contradiction. Therefore, the conclusion of the theorem must be true. $\square$

LEMMA 9.10. *Let $u(x)$ be a weak solution of (BVP) and let the assumptions 2), 3) with $\alpha_0(x) \equiv 0$, $b_0(x) \equiv 0$ be fulfilled. If in addition*

$$f(x) \geq 0, \ g(x) \geq 0 \text{ for a.e. } x \in G$$

then $u(x) \geq 0$ a.e. in G.

PROOF. Choose $\phi = u^- = \max\{-u(x), 0\}$ as a test function in the integral identity (II). We obtain

$$\int\limits_{G} \Big\langle a_i(x, -u^-, -u_x^-)(-u_{x_i}^-) + a_0 \cdot a(x, -u^-, -u_x^-)(-u^-) +$$

$$+ b(x, -u^-, -u_x^-)(-u^-) + f(x)u^- \Big\rangle dx =$$

$$= -\int\limits_{\Gamma_2} \big\langle (-u^-)\sigma(x, -u^-) + g(x)u^- \big\rangle ds.$$

By virtue of assumptions $2)$ and $3)$

$$(1-\mu)\int\limits_{G} \nu(x)|u^-|^q |\nabla u^-|^m dx + a_0 \int\limits_{G} \nu_0(x)|u^-|^{q+m} dx \leq$$

$$\leq -\int\limits_{G} fu^- dx - \int\limits_{\Gamma_2} \big\langle (-u^-)\sigma(x, -u^-) + g(x)u^- \big\rangle ds \leq 0,$$

since $u^- \geq 0$. Due to $\mu < 1, a_0 \geq 0$ and $u\big|_{\partial G \setminus \Gamma_2} = 0$ we get $u^-(x) = 0$ a.e. in G, i.e. $u(x) \geq 0$ a.e. in G. $\square$

9.3. The boundedness of weak solutions

The goal of this section is to derive $L_\infty(G)-a$ *priori* estimate of the weak solution to problem (BVP). The main statement of this section is the following theorem.

THEOREM 9.11. *Let $u(x)$ be a weak solution of (BVP) and assumptions (9.1.2), 1) - 3) hold. Then there exists the constant $M_0 > 0$, depending only on $\|g\|_{L_\alpha(\Gamma_2)}, \|\nu^{-1}(x), \nu_0^{-1}(x)\|_{L_t(G)}$, $measG, \omega_0, N, m, \mu, q, p, t, s, a_0$, $\|\nu_0^{-1}(x)(\alpha_0(x) + b_0(x) + |f(x)|)\|_{L_p(G)}$, such that*

$$\|u\|_{L_\infty(G)} \leq M_0.$$

PROOF. Let us introduce the set $A(k) = \{x \in \overline{G}, \quad |u(x)| > k\}$ and let $\chi_{A(k)}$ be a characteristic function of the set $A(k)$. We note that $A(k+d) \subseteq A(k) \quad \forall d > 0$. By setting $\phi(x) = \eta((|u| - k)_+)\chi_{A(k)} \cdot \text{sign} u$ in (II), where η is defined by Lemma 1.60 and $k \geq k_0$ (without loss of generality we can assume $k_0 \geq 1$), on the strength of the assumptions 2) and 3) we get the

inequality

$$\int\limits_{A(k)} \nu(x)|u|^q |\nabla u|^m \eta'((|u| - k)_+)dx +$$

$$+ a_0 \int\limits_{A(k)} \nu_0(x)|u|^{q+m-1}\eta((|u| - k)_+)dx +$$

$$+ \int\limits_{\Gamma_2 \cap A(k)} \sigma(x, u)(\text{sign } u)\eta((|u| - k)_+)ds \leq$$

(9.3.1)
$$\leq \mu \int\limits_{A(k)} \nu(x)|\nabla u|^m |u|^{q-1}\eta((|u| - k)_+)dx +$$

$$+ \int\limits_{A(k)} \big(b_0(x) + |f(x)|\big)\eta((|u| - k)_+)dx +$$

$$+ \int\limits_{A(k)} \alpha_0(x)\eta'((|u| - k)_+)dx + \int\limits_{\Gamma_2 \cap A(k)} |g(x)|\eta((|u| - k)_+)ds.$$

Now we define the function $w_k(x) := \eta\left(\dfrac{(|u| - k)_+}{m}\right)$. By (1.11.7) from Lemma 1.60 we have

(9.3.2)
$$\int\limits_{\Gamma_2 \cap A(k)} |g(x)|\eta((|u| - k)_+)ds \leq M \cdot \int\limits_{\Gamma_2 \cap A(k+d)} |g(x)||w_k|^m ds +$$

$$+ e^{\varkappa d} \cdot \int\limits_{\Gamma_2 \cap \{A(k+d) \setminus A(k)\}} |g(x)|ds$$

Now we apply Lemma 9.4. In virtue of Hölder's inequality and (9.1.13), (9.1.14) we get

$$\int\limits_{\Gamma_2 \cap A(k+d)} |g(x)||w_k|^m ds \leq \left(\int\limits_{\Gamma_2 \cap A(k+d)} |w_k|^{\alpha^*}ds\right)^{\frac{m}{\alpha^*}} \cdot \|g\|_{L_{\frac{N-1}{m-1-\frac{N}{t}}}(\Gamma_2)} \leq$$

$$\leq c_4 \|g\|_{L_{\frac{N-1}{m-1-\frac{N}{t}}}(\Gamma_2)} \cdot \int\limits_{A(k)} \big(\nu(x)|\nabla w_k|^m + \nu_0(x)|w_k|^m\big)dx.$$

Then by assumptions *2)* from (9.3.1) and (9.3.2) it follows that

$$\int\limits_{A(k)} \nu(x)|u|^q|\nabla u|^m \Big\langle \eta'((|u|-k)_+) - \mu\eta((|u|-k)_+) \Big\rangle dx \le$$

$$\le Mc_4\|g\|_{L_{\frac{N-1}{m-1-\frac{N}{t}}}(\Gamma_2)} \cdot \int\limits_{A(k)} \Big(\nu(x)|\nabla w_k|^m + \nu_0(x)|w_k|^m\Big)dx +$$

$$(9.3.3) \qquad + \int\limits_{A(k)} \Big\langle \alpha_0(x)\eta'((|u|-k)_+) + (b_0 + |f|)\eta((|u|-k)_+) \Big\rangle dx +$$

$$+ e^{\varkappa d} \cdot \int\limits_{\Gamma_2 \cap A(k)} |g(x)|ds.$$

By the definition of $\eta(x)$ (see Lemma 1.60) and $w_k(x)$

$$e^{\varkappa(|u|-k)_+}|\nabla u|^m = \left(\frac{m}{\varkappa}\right)^m |\nabla w_k|^m, \quad \varkappa > 0$$

and by the choice of $\varkappa > m + 2\mu$ according to Lemma 1.60, using (1.11.5)-(1.11.7), from (9.3.3), we obtain

$$(9.3.4) \quad \frac{1}{2}\left(\frac{m}{\varkappa}\right)^m k_0^q \int\limits_{A(k)} \nu(x)|\nabla w_k|^m dx \le c_7 M \int\limits_{A(k+d)} h(x)|w_k|^m dx +$$

$$+ Mc_4\|g\|_{L_{\frac{N-1}{m-1-\frac{N}{t}}}(\Gamma_2)} \cdot \int\limits_{A(k)} \Big(\nu(x)|\nabla w_k|^m + \nu_0(x)|w_k|^m\Big)dx +$$

$$+ c_8 e^{\varkappa d}\Big\langle \int\limits_{A(k)\setminus A(k+d)} h(x)dx + \int\limits_{\Gamma_2 \cap A(k)} |g(x)|ds \Big\rangle,$$

where

$$(9.3.5) \qquad h(x) = \alpha_0(x) + b_0(x) + |f(x)|.$$

Now, by (9.1.7) from (9.3.4) it follows that

$$(9.3.6) \quad (k_0^q - c_9)\int\limits_{A(k)} \nu(x)|\nabla w_k|^m dx \le c_{10} \int\limits_{A(k+d)} h(x)|w_k|^m dx +$$

$$+ c_{11} e^{\varkappa d}\Big\langle \int\limits_{A(k)\setminus A(k+d)} h(x)dx + \int\limits_{\Gamma_2 \cap A(k)} |g(x)|ds \Big\rangle,$$

where

$$(9.3.7) \quad c_9 = 2\left(\frac{\varkappa}{m}\right)^m (1 + c_1) M c_4 \|g\|_{L_{\frac{N-1}{m-1-\frac{N}{t}}}(\Gamma_2)}; \quad c_{10} = 2\left(\frac{\varkappa}{m}\right)^m M c_7;$$

$$c_{11} = 2\left(\frac{\varkappa}{m}\right)^m c_8.$$

By assumptions *1)* we get that $\nu_0^{-1}(x)h(x) \in L_p(G)$, where p is such that $\frac{1}{p} < \frac{m}{N} - \frac{1}{t} - \frac{1}{s}$. By Hölder's inequality with exponents p and p' ($\frac{1}{p} + \frac{1}{p'} = 1$)

$$(9.3.8) \quad \int\limits_{A(k+d)} h|w_k|^m dx \leq \left\|\nu_0^{-1}(x)h(x)\right\|_{L_p(G)} \left(\int\limits_{A(k)} \nu_0^{p'}(x)|w_k|^{mp'} dx\right)^{\frac{1}{p'}}.$$

From the inequality $\frac{1}{p} < \frac{m}{N} - \frac{1}{t} - \frac{1}{s}$ it follows that $mp' < m^{\#}$, where $m^{\#}$ is defined in (9.1.6). Let j be a real number such that $mp' < j < m^{\#}$. From the interpolation inequality

$$\left(\int\limits_{A(k)} \nu_0^{p'}(x)|w_k|^{mp'} dx\right)^{\frac{1}{p'}} \leq$$

$$\leq \left(\int\limits_{A(k)} \nu_0(x)|w_k|^m dx\right)^{\theta} \cdot \left(\int\limits_{A(k)} \nu_0^{j/m}(x)|w_k|^j dx\right)^{\frac{(1-\theta)m}{j}}$$

with $\theta \in (0,1)$, which is defined by the equality $\frac{1}{mp'} = \frac{\theta}{m} + \frac{1-\theta}{j}$, on the strength of Hölder's inequality with exponents $\frac{m^{\#}}{j}$ and $\frac{m^{\#}}{m^{\#}-j}$ from (9.3.8), we get

$$(9.3.9) \quad \begin{cases} \displaystyle\int\limits_{A(k+d)} h|w_k|^m dx \leq c_{12} \left(\int\limits_{A(k)} \nu_0(x)|w_k|^m dx\right)^{\theta} \times \\ \qquad\qquad\qquad \times \left(\int\limits_{A(k)} |w_k|^{m^{\#}} dx\right)^{\frac{(1-\theta)m}{m^{\#}}}, \\ c_{12} = \left\|\nu_0^{-1}(x)h(x)\right\|_{L_p(G)} \|\nu_0(x)\|_{L_s(G)}^{1-\theta} \end{cases}$$

provided we choose $j = \frac{smm^{\#}}{sm+m^{\#}} \in (mp', m^{\#})$ in virtue of (9.1.6) and (9.1.2).
By using the Young inequality with exponents $\frac{1}{\theta}$ and $\frac{1}{(1-\theta)}$ from (9.2.9) we obtain

(9.3.10)
$$
\begin{cases}
\displaystyle\int\limits_{A(k+d)} h|w_k|^m dx \leq \frac{c_{13}}{\varepsilon^{1/\theta}} \int\limits_{A(k)} \nu_0(x)|w_k|^m dx + \\[4mm]
\qquad\qquad +\varepsilon^{\frac{1}{(1-\theta)}}(1-\theta) \left(\displaystyle\int\limits_{A(k)} |w_k|^{m^{\#}} dx \right)^{\frac{m}{m^{\#}}}, \\[6mm]
c_{13} = \theta \left\| \nu_0^{-1}(x)h(x) \right\|_{L_p(G)}^{\frac{1}{\theta}} \left\| \nu_0(x) \right\|_{L_s(G)}^{\frac{1-\theta}{\theta}}, \quad \forall \varepsilon > 0.
\end{cases}
$$

It follows from (9.3.6), (9.3.10) that

(9.3.11)
$$
(k_0^q - c_9) \int\limits_{A(k)} \nu(x)|\nabla w_k|^m dx \leq c_{14}\varepsilon^{-1/\theta} \int\limits_{A(k)} \nu_0(x)|w_k|^m dx +
$$
$$
+c_{16}\left\langle \int\limits_{A(k)} h(x)dx + \int\limits_{\Gamma_2 \cap A(k)} |g(x)|ds \right\rangle +
$$
$$
+c_{15}\varepsilon^{\frac{1}{(1-\theta)}} \left(\int\limits_{A(k)} |w_k|^{m^{\#}} dx \right)^{\frac{m}{m^{\#}}},
$$

where $\forall \varepsilon > 0,\ c_{14} = c_{13}c_{10},\ c_{15} = (1-\theta)c_{10},\ c_{16} = c_{11}e^{\varkappa d}.$
  Further, from (9.3.11), by (9.1.7), we get

$$
(k_0^q - c_9 - c_1 c_{14}\varepsilon^{-\frac{1}{\theta}}) \int\limits_{A(k)} \nu(x)|\nabla w_k|^m dx \leq c_{15}\varepsilon^{\frac{1}{(1-\theta)}} \left(\int\limits_{A(k)} |w_k|^{m^{\#}} dx \right)^{\frac{m}{m^{\#}}}
$$

(9.3.12) $+c_{16}\left\langle \displaystyle\int\limits_{A(k)} h(x)dx + \int\limits_{\Gamma_2 \cap A(k)} |g(x)|ds \right\rangle, \quad \forall \varepsilon > 0,\ \forall k \geq k_0.$

Let us choose

(9.3.13) $\begin{cases} c_1 c_{14}\varepsilon^{-\frac{1}{\theta}} = \frac{1}{2}k_0^q \quad \Rightarrow \varepsilon = (2c_1 c_{14})^{\theta} k_0^{-q\theta}; \\ k_0^q \geq 4c_9. \end{cases}$

By virtue of (9.1.15) we obtain

$$\left(\frac{1}{4c_5}k_0^q - c_{15}\varepsilon^{\frac{1}{(1-\theta)}}\right)\left\{\left(\int_{A(k)}|w_k|^{m^\#}dx\right)^{\frac{m}{m^\#}} + \left(\int_{\Gamma_2\cap A(k)}|w_k|^{\alpha^*}ds\right)^{\frac{m}{\alpha^*}}\right\} \le$$

$$(9.3.14) \qquad \le c_{16}\left\langle \int_{A(k)}h(x)dx + \int_{\Gamma_2\cap A(k)}|g(x)|ds\right\rangle,$$

if we choose

$$(9.3.15) \qquad \frac{1}{8c_5}k_0^q \ge c_{15}\varepsilon^{\frac{1}{(1-\theta)}}.$$

So by (9.3.13), (9.3.15) we choose

$$(9.3.16) \qquad k_0 \ge \max\left\{1, (8c_5c_{15})^{\frac{1-\theta}{q}}(2c_1c_{14})^{\frac{\theta}{q}}, (4c_9)^{\frac{1}{q}}\right\}.$$

Therefore from (9.3.15), it follows that

$$(9.3.17) \qquad \left(\int_{A(k)}|w_k|^{m^\#}dx\right)^{\frac{m}{m^\#}} + \left(\int_{\Gamma_2\cap A(k)}|w_k|^{\alpha^*}ds\right)^{\frac{m}{\alpha^*}} \le$$

$$\le c_{17}\left\langle \int_{A(k)}h(x)dx + \int_{\Gamma_2\cap A(k)}|g(x)|ds\right\rangle \quad \forall k \ge k_0,$$

where

$$c_{17} = \max\{4^\theta c_1^{-\theta}c_5^\theta c_{14}^{-\theta}c_{15}^{\theta-1}c_{16}, \; 2c_5c_9^{-1}c_{16}, \; 8c_5c_{16}\}.$$

At last, by Young's inequality with exponents $p, s, \frac{1}{1-\frac{1}{p}-\frac{1}{s}}$, we get

$$\int_{A(k)}h(x)dx \le \left\|\nu_0^{-1}(x)h(x)\right\|_{L_p(G)}\left\|\nu_0(x)\right\|_{L_s(G)}\text{meas}^{\,1-\frac{1}{p}-\frac{1}{s}}A(k).$$

In just the same way

$$\int_{\Gamma_2\cap A(k)}|g(x)|ds \le \|g\|_{L_\alpha(\Gamma_2)}\cdot[\text{meas}(\Gamma_2\cap A(k))]^{\frac{1}{\alpha'}}, \quad \frac{1}{\alpha}+\frac{1}{\alpha'}=1.$$

Therefore from (9.3.17) it follows that

$$\left(\int_{A(k)} |w_k|^{m^\#}\right)^{\frac{m}{m^\#}} + \left(\int_{\Gamma_2 \cap A(k)} |w_k|^{\alpha^*} ds\right)^{\frac{m}{\alpha^*}} \le$$

(9.3.18) $\qquad \le 17\left\langle \left\|\nu_0^{-1}(x)h(x)\right\|_{L_p(G)} \|\nu_0(x)\|_{L_s(G)} \text{meas}^{1-\frac{1}{p}-\frac{1}{s}} A(k)+\right.$

$$\left. +\|g\|_{L_\alpha(\Gamma_2)} \cdot \left[\text{meas}(\Gamma_2 \cap A(k))\right]^{\frac{1}{\alpha'}}\right\rangle,$$

where $1 - \frac{1}{p} - \frac{1}{s} > 0$ in virtue of (9.1.2) and assumptions *1)*.

Let now $l > k > k_0$. By (1.11.8) and the definition of the function $w_k(x)$ we have $|w_k| \ge \frac{1}{m}(|u| - k)_+$ and therefore

$$\int_{A(l)} |w_k|^{m^\#} dx \ge \left(\frac{l-k}{m}\right)^{m^\#} \cdot \text{meas } A(l)$$

and

$$\int_{\Gamma_2 \cap A(l)} |w_k|^{\alpha^*} ds \ge \left(\frac{l-k}{m}\right)^{\alpha^*} \cdot \text{meas } (\Gamma_2 \cap A(l)).$$

From (9.3.18) it now follows that

(9.3.19) $\qquad \text{meas } A(l) + \left[\text{meas}(\Gamma_2 \cap A(l))\right]^{\frac{m^\#}{\alpha^*}} \le$

$$\le \left(\frac{m}{l-k}\right)^{m^\#} \cdot \left\{\int_{A(k)} |w_k|^{m^\#} dx + \left(\int_{\Gamma_2 \cap A(k)} |w_k|^{\alpha^*} ds\right)^{\frac{m^\#}{\alpha^*}}\right\} \le$$

$$\le \frac{1}{2}\left(\frac{m}{l-k}\right)^{m^\#} \cdot (2c_{17})^{\frac{m^\#}{m}} \left(\left\|\nu_0^{-1}(x)h(x)\right\|_{L_p(G)} \|\nu_0(x)\|_{L_s(G)} +\right.$$

$$\left. +\|g\|_{L_\alpha(\Gamma_2)}\right)^{\frac{m^\#}{m}} \times \left\{\text{meas}^{\frac{m^\#}{m}\left(1-\frac{1}{p}-\frac{1}{s}\right)} A(k) + \left[\text{meas } (\Gamma_2 \cap A(k))\right]^{\frac{m^\#}{m\alpha'}}\right\},$$

$$\forall l > k \ge k_0.$$

Now we set

$$\psi(k) = \text{meas } A(k) + \left[\text{meas}(\Gamma_2 \cap A(k))\right]^{\frac{m^\#}{\alpha^*}}.$$

Then from (9.3.19) it follows that

(9.3.20) $\qquad \psi(l) \le c_{18}\left(\frac{m}{l-k}\right)^{m^\#} \cdot \left\langle [\psi(k)]^{\frac{m^\#}{m}\left(1-\frac{1}{p}-\frac{1}{s}\right)} + [\psi(k)]^{\frac{\alpha^*}{m\alpha'}}\right\rangle.$

Relying on (9.1.2), (9.1.6), (9.1.14) and assumptions *1)* we note that

$$\gamma = \min\left\{\frac{m^{\#}}{m}\left(1 - \frac{1}{p} - \frac{1}{s}\right), \frac{\alpha^*}{ma'}\right\} > 1.$$

From (9.3.20) we then get

$$\psi(l) \le \frac{c_{19}}{(l-k)^{m^{\#}}}\psi^{\gamma}(k) \quad \forall l > k \ge k_0$$

and therefore we have, because of Lemma 1.59, that $\psi(k_0 + \delta) = 0$ with δ depending only on quantities in the formulation of Theorem 9.11. This fact means that $|u(x)| < k_0 + \delta$ for almost all $x \in G$. Theorem 9.11 is proved. $\quad\square$

To complete this section let us derive some *a priori* integral estimates of these solutions.

THEOREM 9.12. *Let $u(x)$ be a weak solution of (BVP) and assumptions (9.1.2), 1) - 3) hold. Let us suppose in addition that*

$$\int_G \nu_0^{\frac{1}{1-q-m}}(x)\big(b_0(x) + |f(x)|\big)^{\frac{q+m}{q+m-1}} < \infty, \quad g(x) \in L_{\frac{m+q}{q}}(\Gamma_2).$$

Then the inequality

$$(9.3.21) \quad \int_G \Big(\nu(x)|u|^q|\nabla u|^m + \nu_0(x)|u|^{q+m}\Big)dx \le$$

$$\le C\Bigg\{\int_G \nu_0^{\frac{1}{1-q-m}}(x)\big(b_0(x) + |f(x)|\big)^{\frac{q+m}{q+m-1}}dx + \big\|\nu_0^{-1}(x)\alpha_0(x)\big\|_p\|\nu_0(x)\|_s +$$

$$+ \big\|\nu^{-1}(x)\big\|_{L_t(G)}^{\frac{1}{m-1}} + \big\|\nu_0^{-1}(x)\big\|_{L_t(G)}^{\frac{1}{m-1}} + \int_{\Gamma_2} |g(x)|^{\frac{m+q}{q}}ds\Bigg\}$$

holds, where $C > 0$ is a constant depending only on N, m, q, μ, a_0, $measG$.

PROOF. By setting in *(II)* $\phi = u$ we get, in virtue of assumptions *2)* and *3)*

$$(9.3.22) \quad (1-\mu)\int_G \nu(x)|u|^q|\nabla u|^m dx + a_0 \int_G \nu_0(x)|u|^{q+m}dx \le$$

$$\le \int_G \alpha_0(x)dx + \int_G \big(b_0(x) + |f(x)|\big)|u(x)|dx + \int_{\Gamma_2} |g(x)||u(x)|ds.$$

By the Young inequality with $p = q + m$ and $p' = \frac{q+m}{q+m-1}$

$$\left(b_0(x) + |f(x)|\right)|u(x)| = \left(\nu_0^{\frac{1}{q+m}}(x)|u(x)|\right)\left(\nu_0^{\frac{-1}{q+m}}\left(b_0(x) + |f(x)|\right)\right) \le$$

$$(9.3.23) \quad \le \varepsilon\nu_0(x)|u|^{q+m} + c_\varepsilon \nu_0^{\frac{1}{1-q-m}}(x)\left(b_0(x) + |f(x)|\right)^{\frac{q+m}{q+m-1}} \ textand$$

$$|g(x)||u(x)| \le \varepsilon|u(x)|^{\frac{m+q}{m}} + c_\varepsilon|g(x)|^{\frac{m+q}{q}} \text{ for } \forall\varepsilon > 0.$$

Further, by Lemma 1.29 and by Young's inequality we have

$$\int\limits_{\Gamma_2} |u(x)|^{\frac{m+q}{m}} ds \le c_6 \int\limits_G \left(|u(x)|^{\frac{m+q}{m}} + \frac{m+q}{m}|u(x)|^{\frac{q}{m}}|\nabla u|\right)dx \le$$

$$\le \frac{m+q}{m}c_6 \int\limits_G \left\langle \left(\nu^{\frac{1}{m}}(x)|u|^{\frac{q}{m}}|\nabla u|\right)\nu^{-\frac{1}{m}}(x) + \left(\nu_0^{\frac{1}{m}}(x)|u|^{\frac{q+m}{m}}\right)\nu_0^{-\frac{1}{m}}(x)\right\rangle dx \le$$

$$(9.3.24) \qquad \le \frac{m+q}{m}c_6 \int\limits_G \left\langle \frac{1}{m}\left(\nu(x)|u|^q|\nabla u|^m + \nu_0(x)|u|^{q+m}\right)+\right.$$

$$\left. +\frac{1}{m'}\left(\nu^{-\frac{m'}{m}}(x) + \nu_0^{-\frac{m'}{m}}(x)\right)\right\rangle dx.$$

In addition, we have

$$\int\limits_G \alpha_0(x)dx \le \left\|\nu_0^{-1}(x)\alpha_0(x)\right\|_p \left\|\nu_0(x)\right\|_s \left\|1\right\|_{1-\frac{1}{p}-\frac{1}{s}} \text{ and}$$

$$(9.3.25)$$

$$\int\limits_G \nu^{-\frac{m'}{m}}(x)dx = \int\limits_G \left(\nu^{-1}(x)\right)^{\frac{1}{m-1}}(x)dx \le \left\|\nu^{-1}(x)\right\|_{L_t(G)}^{\frac{1}{m-1}} \cdot (\text{meas}G)^{1-\frac{1}{t(m-1)}},$$

where $t(m-1) > 1$ by (9.1.2).

From (9.3.22)-(9.3.25) it follows that

$$(1 - \mu) \int\limits_G \nu(x)|u|^q |\nabla u|^m dx + a_0 \int\limits_G \nu_0(x)|u|^{q+m} dx \le$$

$$(9.3.26) \qquad \le \varepsilon_1 \int\limits_G \nu(x)|u|^q |\nabla u|^m dx + \varepsilon_2 \int\limits_G \nu_0(x)|u|^{q+m} dx +$$

$$+ c(\varepsilon_1, \varepsilon_2, m, q, N, t, \text{meas} G) \Bigg\{ \int\limits_{\Gamma_2} |g(x)|^{\frac{m+q}{q}} ds + \left\| \nu^{-1}(x) \right\|_{L_t(G)}^{\frac{1}{m-1}} +$$

$$+ \left\| \nu_0^{-1}(x) \right\|_{L_t(G)}^{\frac{1}{m-1}} + \int\limits_G \nu_0^{\frac{1}{1-q-m}}(x) \big(b_0(x) + |f(x)| \big)^{\frac{q+m}{q+m-1}} dx +$$

$$+ \left\| \nu_0^{-1}(x)\alpha_0(x) \right\|_p \left\| \nu_0(x) \right\|_s \Bigg\}.$$

Now, if $a_0 > 0$, then we choose $\varepsilon_1 = \frac{1-\mu}{2}$, $\varepsilon_2 = \frac{a_0}{2}$. And if $a_0 = 0$, we then take advantage of (9.1.9) and choose $\varepsilon_1 = \varepsilon_2 c_3 = \frac{1-\mu}{4}$. For both cases, from (9.3.26) we obtain the required (9.3.21). Theorem 3.2 is proved. $\square$

9.4. The construction of the barrier function

Let us set

$$\nu(x) = r^\tau, \ \nu_0(x) = r^{\tau-m}, \ \tau \ge m - 2 \text{ for } m \ge 2.$$

In this section, for $N-$ dimensional *infinite* dihedral cone

$$G_0 = \left\{ x = (\overline{x}, r, \omega) \big| \ \overline{x} \in \mathbb{R}^{N-2}, \ 0 < r < \infty, \ -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2}, \ \omega_0 \in (0, 2\pi) \right\}$$

with the edge $\Gamma_0 = \left\{ (\overline{x}, 0, 0) \big| \ \overline{x} \in \mathfrak{R}^{N-2} \right\}$, that contains the origin, and lateral faces

$$\Gamma_1 = \left\{ (\overline{x}, r, -\frac{\omega_0}{2}) \big| \ \overline{x} \in \mathbb{R}^{N-2}, \ 0 < r < \infty \right\};$$

$$\Gamma_2 = \left\{ (\overline{x}, r, +\frac{\omega_0}{2}) \big| \ \overline{x} \in \mathbb{R}^{N-2}, \ 0 < r < \infty \right\}$$

we shall consider only the homogeneous boundary value problem

$$
\begin{cases}
\mathcal{L}w \equiv -\dfrac{d}{dx_i}\left(r^\tau |w|^q |\nabla w|^{m-2} w_{x_i}\right) + a_0 r^{\tau-m} w|w|^{q+m-2} - \\
\qquad\qquad\qquad -\mu r^\tau w|w|^{q-2}|\nabla w|^m = 0, \quad x \in G_0, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (BVP)_0 \\
w(x) = 0, \ x \in \Gamma_0 \cup \Gamma_1 \cup \Gamma_2; \qquad \text{for the Dirichlet problem;} \\
w(x) = 0, \ x \in \Gamma_0 \cup \Gamma_1; \quad \frac{\partial w}{\partial n} = 0, \ x \in \Gamma_2 \text{ for the mixed problem;} \\
\qquad a_0 \geq 0, \quad 0 \leq \mu < 1, \quad q \geq 0, \quad m \geq 2, \quad \tau \geq m-2
\end{cases}
$$

and construct the function that will be *the barrier* for the non-homogeneous problem. We shall seek a solution of the problem $(BVP)_0$ of the form

$$(9.4.1) \qquad w(x) = r^\lambda \Phi(\omega), \quad \omega \in [-\frac{\omega_0}{2}, \frac{\omega_0}{2}], \quad \lambda > 0$$

with $\Phi(\omega) \geq 0$ and λ satisfying (9.1.3)-(9.1.4). By substituting the function (9.4.1) in $(BVP)_0$ and calculating in the cylindrical coordinates for the function $\Phi(\omega)$, we get the following Sturm-Liouville boundary problem

$$
\begin{cases}
\dfrac{d}{d\omega}\left[\left(\lambda^2 \Phi^2 + \Phi'^2\right)^{\frac{m-2}{2}} |\Phi|^q \Phi'\right] + \\
\quad +\lambda[\lambda(q+m-1) - m + 2 + \tau]\Phi|\Phi|^q \left(\lambda^2 \Phi^2 + \Phi'^2\right)^{\frac{m-2}{2}} = \\
\quad = a_0 \Phi|\Phi|^{q+m-2} - \mu\Phi|\Phi|^{q-2}\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m}{2}}, \ \omega \in (-\omega_0/2, \omega_0/2), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (StL) \\
\Phi(-\omega_0/2) = \Phi(\omega_0/2) = 0 \qquad \text{for the Dirichlet problem;} \\
\Phi(-\omega_0/2) = \Phi'(\omega_0/2) = 0 \qquad \text{for the mixed problem.}
\end{cases}
$$

By setting $\Phi'/\Phi = y$, we arrive at the Cauchy problem for $y(\omega)$

$$
\begin{cases}
\left[(m-1)y^2 + \lambda^2\right](y^2 + \lambda^2)^{\frac{m-4}{2}} y' + (m-1+q+\mu)(y^2+\lambda^2)^{\frac{m}{2}} + \\
\qquad +\lambda(2-m+\tau)(y^2+\lambda^2)^{\frac{m-2}{2}} = a_0, \quad \omega \in (-\omega_0/2, \omega_0/2), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (CPE) \\
y(0) = 0 \qquad \text{for the Dirichlet problem;} \\
y(\omega_0/2) = 0 \qquad \text{for the mixed problem.}
\end{cases}
$$

From the equation of (CPE) we get:

$$-\left[(m-1)y^2 + \lambda^2\right](y^2 + \lambda^2)^{\frac{m-4}{2}} y' =$$

$$= (m - 1 + q + \mu)(y^2 + \lambda^2)^{\frac{m}{2}} + \lambda(2 - m + \tau)(y^2 + \lambda^2)^{\frac{m-2}{2}} - a_0 =$$

(9.4.2)

$$= (y^2 + \lambda^2)^{\frac{m-2}{2}} \left[(m - 1 + q + \mu)(y^2 + \lambda^2) + \lambda(2 - m + \tau)\right] - a_0 \geq$$

$$\geq (y^2 + \lambda^2)^{\frac{m-2}{2}}[\lambda^2(m - 1 + q + \mu) + \lambda(2 - m + \tau)] - a_0 \geq$$

$$\geq \lambda^m(m - 1 + q + \mu) + \lambda^{m-1}(2 - m + \tau) - a_0 > 0$$

by virtue of (9.1.4). Thus, it is proved that $y'(\omega) < 0$, $\omega \in (-\omega_0/2, \omega_0/2)$. Therefore $y(\omega)$ *decreases* on the interval $(-\omega_0/2, \omega_0/2)$.

9.4.1. Properties of the function $\Phi(\omega)$. We turn in detail our attention to the properties of the function $\Phi(\omega)$. The case of Dirichlet problem see as well [72]. First of all, we note that the solutions of (StL) are determined uniquely up to a scalar multiple. We consider the solution normed by the condition

$$(9.4.3) \qquad 1 = \begin{cases} \Phi(0) & \text{for the Dirichlet problem;} \\ \Phi(\frac{\omega_0}{2}) & \text{for the mixed problem.} \end{cases}$$

We rewrite the equation of (StL) in the following form

$$-\Phi\left[(m-1)\Phi'^2 + \lambda^2\Phi^2\right]\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-4}{2}}\Phi'' = (q + \mu)\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m}{2}} +$$

$$+\Phi^2\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-4}{2}}\left\{\lambda[\lambda(m - 1) - m + 2 + \tau]\left(\lambda^2\Phi^2 + \Phi'^2\right) + \right.$$

(9.4.4) $$\left. +(m - 2)\lambda^2\Phi'^2\right\} - a_0\Phi^m.$$

Now, since $m \geq 2$ from (9.4.4) it follows that

$$(9.4.5) \quad -\Phi\left[(m-1)\Phi'^2 + \lambda^2\Phi^2\right]\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-4}{2}}\Phi'' \geq -a_0\Phi^m +$$

$$+\left(\lambda^2\Phi^2 + \Phi'^2\right)^{\frac{m-2}{2}}\left\{(q+\mu)\left(\lambda^2\Phi^2 + \Phi'^2\right) + \lambda[\lambda(m-1) - m + 2 + \tau]\Phi^2\right\} \geq$$

$$\geq \Phi^m\{(q + \mu + m - 1)\lambda^m + (2 - m + \tau)\lambda^{m-1} - a_0\} > 0.$$

(Here we take into account that $(q + \mu + m - 1)\lambda^2 + (2 - m + \tau)\lambda > 0$ by (9.1.4).)

Summarizing the above we obtain the following properties of function $\Phi(\omega)$

$$(9.4.6) \qquad \Phi(\omega) \geq 0, \quad \Phi''(\omega) < 0 \quad \forall \omega \in (-\omega_0/2, \omega_0/2).$$

COROLLARY 9.13.

$$(9.4.7) \qquad \max_{[-\omega_0/2,\omega_0/2]} \Phi(\omega) = 1 \ \Rightarrow\ 0 \le \Phi(\omega) \le 1 \quad \forall \omega \in [-\omega_0/2, \omega_0/2].$$

Let us proceed with the problem of (CPE) solvability. Rewriting the equation of (CPE) in the form resolved with respect to the derivative $y' = g(y)$, we observe that by (9.4.2) $g(y) \ne 0 \ \forall y \in \mathbb{R}$. Moreover, $g(y)$ and $g'(y)$ being rational functions with non-zero denominators are *continuous* functions. By the theory of ordinary differential equations the Chauchy problem (CPE) is uniquely solvable in the interval $\left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right]$. By integrating (StL) - (CPE) we obtain

$$(9.4.8) \qquad \Phi(\omega) = \exp \begin{cases} \int\limits_0^\omega y(\xi) d\xi & \text{for the Dirichlet problem;} \\[2mm] \int\limits_{\omega_0/2}^\omega y(\xi) d\xi & \text{for the mixed problem.} \end{cases}$$

$$\int_0^y \frac{[(m-1)z^2 + \lambda^2](z^2 + \lambda^2)^{\frac{m-4}{2}} dz}{(m-1+q+\mu)(z^2+\lambda^2)^{\frac{m}{2}} + \lambda(2-m+\tau)(z^2+\lambda^2)^{\frac{m-2}{2}} - a_0} =$$

$$(9.4.9) \qquad = \begin{cases} -\omega & \text{for the Dirichlet problem;} \\[2mm] \frac{\omega_0}{2} - \omega & \text{for the mixed problem.} \end{cases}$$

Hence, in particular, we get from (9.1.3) that $\lim\limits_{\omega \to -\frac{\omega_0}{2}+0} y(\omega) = +\infty$. The last allows us to prove the solvability of the eigenvalue problem (StL). The expression (9.1.3) yields the equation for the sharp finding of the exponent λ in (9.4.1).

9.4.2. About solutions of (9.1.3) and (9.1.4). We may calculate the exponent λ explicitly for $m = 2$ or $a_0 = 0$. In fact, integrating the (9.1.3) we obtain

$$m = 2.$$

$$(9.4.10) \qquad \lambda = \frac{\sqrt{\tau^2 + (\pi/\theta_0)^2 + 4a_0(1+q+\mu)} - \tau}{2(1+q+\mu)}.$$

$$a_0 = 0, \ m \ne 2.$$

We denote the value λ in this case by λ_0 as follows

$$(9.4.11) \quad \frac{\lambda_0(m-2)(m-1+q+\mu) + (m-1)(2-m+\tau)}{\sqrt{(m-1+q+\mu)^2\lambda_0^2 + (m-1+q+\mu)(2-m+\tau)\lambda_0}} =$$

$$= (m-2)(1-\varkappa) + \varkappa\tau,$$

where $\varkappa = \frac{2\theta_0}{\pi}$. Hence we get the quadratic equation and it followsthat

$$\lambda_0 =$$

$$(9.4.12) \quad = \begin{cases} \frac{m(m-2)+[(1-\varkappa)(m-2)+\varkappa\tau]\sqrt{m^2+\varkappa(2-m+\tau)[(m-2)(2-\varkappa)+\varkappa\tau]}}{2\varkappa(m-1+q+\mu)[(m-2)(2-\varkappa)+\varkappa\tau]} + \\ \qquad\qquad + \frac{m-2-\tau}{2(m-1+q+\mu)}, \text{ if } \theta_0 < \pi, \\ \\ \frac{m(m-2)-4\tau(\tau+2-m)+(2\tau+2-m)\sqrt{m^2+4\tau(\tau+2-m)}}{8\tau(m-1+q+\mu)}, \quad \text{if } \theta_0 = \pi. \end{cases}$$

It is easily to see that $\lambda_0 > 0$. From (9.4.12) we have for $\theta_0 = \frac{\pi}{2}$ also

$$(9.4.13) \quad \lambda_0 = \frac{m-2-\tau}{2(m-1+q+\mu)} + \frac{m(m-2)+\tau\sqrt{\tau^2+4(m-1)}}{2\varkappa(m-1+q+\mu)(m-2+\tau)}.$$

Now from (9.4.12) - (9.4.13) we deduce following special cases of value λ_0.

$$\boxed{\tau = 0}$$

$$(9.4.14) \quad \lambda_0 = \begin{cases} \frac{m+\varkappa(2-\varkappa)(m-2)+(1-\varkappa)\sqrt{m^2-\varkappa(2-\varkappa)(m-2)^2}}{2\varkappa(m-1+q+\mu)(2-\varkappa)}, & \text{if } \theta_0 < \pi, \\ \\ \frac{(m-1)^2}{m(m-1+q+\mu)}, & \text{if } \theta_0 = \pi. \end{cases}$$

PROOF. We prove the second equality of (9.4.14). Applying the Taylor formula $\sqrt{1 \pm t} = 1 \pm \frac{1}{2}t + o(t)$ for $t \to 0$, from (9.4.12) we obtain

$$\lambda_0\Big|_{\varkappa=2} = \frac{m(m-2)-4\tau^2+4(m-2)\tau+m(2\tau+2-m)\sqrt{1+\frac{4\tau(\tau+2-m)}{m^2}}}{8\tau(m-1+q+\mu)}$$

$$= \frac{m(m-2)-4\tau^2+4(m-2)\tau+m(2\tau+2-m)\left[1+\frac{2\tau(\tau+2-m)}{m^2}+o(\tau)\right]}{8\tau(m-1+q+\mu)}$$

$$= \frac{m(-2\tau+3m-4)+(\tau+2-m)(2\tau+2-m)}{4m(m-1+q+\mu)} + \frac{o(\tau)}{\tau}.$$

Hence it follows that

$$\lambda_0\Big|_{\substack{\varkappa=2\\\tau=0}}= \lim_{\tau\to0}\lambda_0\Big|_{\varkappa=2} = \frac{(m-1)^2}{m(m-1+q+\mu)}, \quad \text{Q.E.D.}$$

Similarly, on the other hand from the first equal of (9.4.14) we have

$$\lambda_0\Big|_{\tau=0} =$$

$$= \frac{1}{2\varkappa(m-1+q+\mu)}\left\{\varkappa(m-2)+m\frac{1+(1-\varkappa)\sqrt{1-\frac{\varkappa(2-\varkappa)(m-2)^2}{m^2}}}{2-\varkappa}\right\} =$$

$$= \frac{1}{2\varkappa(m-1+q+\mu)}\left\{\varkappa(m-2)+m-\frac{\varkappa(1-\varkappa)(m-2)^2}{2m}+\frac{o(2-\varkappa)}{2-\varkappa}\right\}.$$

Hence it follows that

$$\lambda_0\Big|_{\substack{\varkappa=2\\\tau=0}}= \lim_{\varkappa\to2-0}\lambda_0\Big|_{\tau=0} = \frac{(m-1)^2}{m(m-1+q+\mu)},$$

Q.E.D. □

$$\boxed{\tau = m - 2}$$

From (9.4.12) it immediately follows

$$\lambda_0 = \frac{m\pi}{4\theta_0(m-1+q+\mu)}, \quad \theta_0 \le \pi.$$

$$\boxed{\omega_0 \to 0}$$

We want to investigate the behavior of λ_0 for $\varkappa \to 0$. For this fact we rewrite (9.4.12) in the following way

$$\lambda_0 = \frac{m-2-\tau}{2(m-1+q+\mu)}+$$

$$+\frac{m(m-2)+[m-2+\varkappa(\tau-m+2)]m\sqrt{1+\frac{\varkappa^2(2-m+\tau)+2(m-2)(2-m+\tau)\varkappa}{m^2}}}{2\varkappa^2(m-1+q+\mu)(\tau-m+2)+4\varkappa(m-2)(m-1+q+\mu)}$$

$$=\frac{m(m-2)+[m-2+\varkappa(\tau-m+2)]\left\{m+\frac{\varkappa}{2m}(2-m+\tau)(\varkappa+2m-4)\right\}}{2\varkappa^2(m-1+q+\mu)(\tau-m+2)+4\varkappa(m-2)(m-1+q+\mu)}$$

$$+\frac{m-2-\tau}{2(m-1+q+\mu)}+\frac{o(\varkappa)}{\varkappa}=\frac{m\left(1-\frac{\varkappa(\tau-m+2)}{2(m-2)}+o(\varkappa)\right)}{2\varkappa(m-1+q+\mu)}+$$

$$+\frac{m-2-\tau}{2(m-1+q+\mu)}+\frac{o(\varkappa)}{\varkappa}+$$

$$+\frac{m(\tau-m+2)+\frac{1}{2m}(2-m+\tau)(\varkappa+2m-4)[\varkappa(\tau-m+2)+m-2]}{2\varkappa(m-1+q+\mu)(\tau-m+2)+4(m-2)(m-1+q+\mu)}=$$

$$=\frac{m}{2(m-1+q+\mu)}\cdot\frac{1}{\varkappa}+O(1).$$

Hence we finally get

$$\lambda_0=\frac{m\pi}{4(m-1+q+\mu)}\cdot\frac{1}{\theta_0}+O(1)\quad\text{for }\theta_0\to 0.$$

This fact coincides with the Krol result for the pseudo-Laplacian ($q=\mu=\tau=a_0=0$), see p. 145 [205].

$$\boxed{m\to+\infty}$$

We want to investigate the behavior of λ_0 for $m\to+\infty$. For this fact we rewrite (9.4.12) in the following way

1) if $\varkappa<2$,

$$\lambda_0=\frac{m-2-\tau}{2(m-1+q+\mu)}+\frac{m^2-2m}{2\varkappa(2-\varkappa)m^2+O(m)}+$$

$$+\frac{[(1-\varkappa)m+2\varkappa-2+\varkappa\tau]\sqrt{(1-\varkappa)^2m^2+O(m)}}{2\varkappa(2-\varkappa)m^2+O(m)}=$$

$$=\frac{m-2-\tau}{2(m-1+q+\mu)}+\frac{[1+(1-\varkappa)|1-\varkappa|]m^2+O(m^{3/2})}{2\varkappa(2-\varkappa)m^2}.$$

Hence it follows that

$$\lim_{m\to+\infty}\lambda_0=\frac{1}{2}+\frac{1+(1-\varkappa)|1-\varkappa|}{2\varkappa(2-\varkappa)}=\begin{cases}\frac{1}{\varkappa(2-\varkappa)}, & \text{if }\varkappa\le 1,\\ 1, & \text{if }1\le\varkappa<2;\end{cases}$$

2) if $\varkappa = 2$,

$$\lambda_0 = \frac{m^2 - 2m + 4m\tau - 4\tau^2 - 8\tau}{8\tau(m - 1 + q + \mu)} +$$

$$+ \frac{(2\tau + 2 - m)m\sqrt{1 + \frac{4\tau(\tau + 2 - m)}{m^2}}}{8\tau(m - 1 + q + \mu)} =$$

$$= \frac{m^2 - 2m(1 - 2\tau) - 4\tau(\tau + 2)}{8\tau(m - 1 + q + \mu)} -$$

$$- \frac{[m^2 - 2m(\tau + 1)]\left(1 + \frac{2\tau(\tau + 2 - m)}{m^2} + o(\frac{1}{m^2})\right)}{8\tau(m - 1 + q + \mu)} =$$

$$= \frac{8m\tau + O(\frac{1}{m})}{8m\tau + O(1)} \implies \lim_{\substack{\varkappa \to 2 - 0 \\ m \to +\infty}} \lambda_0 = 1.$$

Thus we finally have

$$\lim_{m \to +\infty} \lambda_0 = \begin{cases} \frac{\pi^2}{4\theta_0(\pi - \theta_0)}, & \text{if} \quad 0 < \theta_0 \le \frac{\pi}{2}, \\ 1, & \text{if} \quad \frac{\pi}{2} \le \theta_0 \le \pi. \end{cases}$$

This fact coincides with the Aronsson result for the pseudo-Laplacian $(q = \mu = \tau = a_0 = 0)$, see [11].

9.4.3. About the solvability of (9.1.3) and (9.1.4) with $a_0 > 0$.
We set

$$\mathcal{F}(\lambda, a_0, \omega_0) = -\theta_0 +$$

(9.4.15)

$$+ \int_0^{+\infty} \frac{[(m-1)y^2 + \lambda^2](y^2 + \lambda^2)^{\frac{m-4}{2}} \, dy}{(m - 1 + q + \mu)(y^2 + \lambda^2)^{\frac{m}{2}} + \lambda(2 - m + \tau)(y^2 + \lambda^2)^{\frac{m-2}{2}} - a_0}.$$

By making the substitution: $y = t\lambda$, $t \in (0, +\infty)$ we obtain

$$\mathcal{F}(\lambda, a_0, \omega_0) = -\theta_0 + \int_0^{+\infty} \Lambda(\lambda, a_0, t) dt,$$

where

$$\Lambda(\lambda, a_0, t) \equiv$$

(9.4.16)

$$\equiv \frac{[(m-1)t^2 + 1](t^2 + 1)^{\frac{m-4}{2}}}{\lambda(m - 1 + q + \mu)(t^2 + 1)^{\frac{m}{2}} + (2 - m + \tau)(t^2 + 1)^{\frac{m-2}{2}} - a_0\lambda^{1-m}}.$$

Then the equation (9.1.3) takes the form

(9.4.17) $$\mathcal{F}(\lambda, a_0, \omega_0) = 0.$$

According to the above, we have

(9.4.18) $$\mathcal{F}(\lambda_0, 0, \omega_0) = 0.$$

The direct calculations yield

$$\frac{\partial \Lambda}{\partial \lambda} = -[(m-1)t^2 + 1](t^2 + 1)^{\frac{m-4}{2}} \times$$

(9.4.19)

$$\times \frac{(m - 1 + q + \mu)(t^2 + 1)^{\frac{m}{2}} + a_0(m-1)/\lambda^m}{\left[\lambda(m - 1 + q + \mu)(t^2 + 1)^{\frac{m}{2}} + (2 - m + \tau)(t^2 + 1)^{\frac{m-2}{2}} - a_0\lambda^{1-m}\right]^2} < 0$$

$$\forall t, \ \lambda, \ a_0;$$

$$\frac{\partial \Lambda}{\partial a_0} = \frac{\lambda^{1-m}[(m-1)t^2 + 1](t^2 + 1)^{\frac{m-4}{2}}}{\left[\lambda(m - 1 + q + \mu)(t^2 + 1)^{\frac{m}{2}} + (2 - m + \tau)(t^2 + 1)^{\frac{m-2}{2}} - a_0\lambda^{1-m}\right]^2}$$

(9.4.20)

$$> 0 \quad \forall t, \ \lambda, \ a_0.$$

Therefore, we can apply the theorem about implicit functions. In a certain neighborhood of the point $(\lambda_0, 0)$ the equation (9.4.17) (and so the equation (9.1.3) as well) determines $\lambda = \lambda(a_0, \omega_0)$ as a single-valued *continuous* function of a_0, depending continuously on the parameter ω_0 and having continuous partial derivatives $\frac{\partial \lambda}{\partial a_0}, \frac{\partial \lambda}{\partial \omega_0}$. Now, we analyze the properties of λ as the function $\lambda(a_0, \omega_0)$. First from (9.4.17) we get:

$$\frac{\partial \mathcal{F}}{\partial \lambda} \frac{\partial \lambda}{\partial a_0} + \frac{\partial \mathcal{F}}{\partial a_0} = 0 \text{ and } \frac{\partial \mathcal{F}}{\partial \lambda} \frac{\partial \lambda}{\partial \omega_0} + \frac{\partial \mathcal{F}}{\partial \omega_0} = 0.$$

Hence it follows that

(9.4.21) $$\frac{\partial \lambda}{\partial a_0} = -\frac{\left(\frac{\partial \mathcal{F}}{\partial a_0}\right)}{\left(\frac{\partial \mathcal{F}}{\partial \lambda}\right)} \text{ and } \frac{\partial \lambda}{\partial \omega_0} = -\frac{\left(\frac{\partial \mathcal{F}}{\partial \omega_0}\right)}{\left(\frac{\partial \mathcal{F}}{\partial \lambda}\right)}.$$

But, on the strength of (9.4.19), (9.4.20) we have

$$\frac{\partial \mathcal{F}}{\partial a_0} = \int\limits_0^{+\infty} \frac{\partial \Lambda}{\partial a_0} dt > 0, \quad \frac{\partial \mathcal{F}}{\partial \lambda} = \int\limits_0^{+\infty} \frac{\partial \Lambda}{\partial \lambda} dt < 0, \quad \frac{\partial \mathcal{F}}{\partial \theta_0} = -1 \quad \forall (\lambda, a_0)$$

(9.4.22)

and $\quad \dfrac{\partial \mathcal{F}}{\partial \omega_0} = \dfrac{\partial \mathcal{F}}{\partial \theta_0} \cdot \dfrac{d\theta_0}{d\omega_0} = \begin{cases} -\frac{1}{2}, & \text{if BVP is the Dirichlet problem,} \\ -1, & \text{if BVP is the mixed problem.} \end{cases}$

From (9.4.21) and (9.4.22) we get:

(9.4.23) $\qquad \dfrac{\partial \lambda}{\partial a_0} > 0 \text{ and } \dfrac{\partial \lambda}{\partial \omega_0} < 0 \quad \text{for any } a_0 \geq 0.$

So, *the function* $\lambda(a_0, \omega_0)$ *increases with respect to* a_0 *and decreases with respect to* ω_0. Applying the analytic continuation method, we obtain the solvability of the equation (9.1.3) $\forall a_0$.

COROLLARY 9.14.

$$\lambda = \lambda(a_0, \omega_0) \geq \lambda_0 > 0 \quad \text{for any } a_0 \geq 0.$$

Further, multiplying the equation of (StL) by $\Phi(\omega)$ and integrating over the interval $(-\frac{\omega_0}{2}, \frac{\omega_0}{2})$, we get

(9.4.24) $\quad (1 - \mu) \displaystyle\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^q (\lambda^2 \Phi^2 + \Phi'^2)^{\frac{m-2}{2}} \Phi'^2 d\omega = -a_0 \displaystyle\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+m} d\omega +$

$+ [\lambda^2 (m - 1 + q + \mu) + \lambda(2 - m + \tau)] \displaystyle\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+2} (\lambda^2 \Phi^2 + \Phi'^2)^{\frac{m-2}{2}} d\omega \geq$

$\geq \left\langle \lambda^m (m - 1 + q + \mu) + \lambda^{m-1}(2 - m + \tau) - a_0 \right\rangle \displaystyle\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+m} d\omega > 0,$

by virtue of (9.1.4).

LEMMA 9.15. *We have the inequality*

(9.4.25) $\qquad \displaystyle\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^q |\Phi'|^m d\omega \leq c(q, \mu, m, \tau, \lambda) \displaystyle\int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+m} d\omega.$

PROOF. From (9.4.24), by Young's inequality with $p = \frac{m}{m-2}$, $p' = \frac{m}{2}$, it follows that

$$(1 - \mu) \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^q |\Phi'|^m d\omega \leq$$

$$\leq [\lambda^2(m - 1 + q + \mu) + \lambda(2 - m + \tau)] \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^q \Phi^2 (\lambda^2 \Phi^2 + \Phi'^2)^{\frac{m-2}{2}} d\omega \leq$$

$$\leq \epsilon \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^q (\lambda^2 \Phi^2 + \Phi'^2)^{\frac{m}{2}} d\omega + c_\epsilon \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+m} d\omega \leq$$

$$\leq \epsilon \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^q |\Phi'|^m d\omega + c_\epsilon \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^{q+m} d\omega \quad \forall \epsilon > 0,$$

since $m \geq 2$. Choosing $\epsilon = \frac{1-\mu}{2}$, we obtain the required (9.4.25).   □

LEMMA 9.16. *Let the inequality (9.1.4) hold and, in addition,*

$$(9.4.26) \qquad\qquad q + \mu < 1.$$

Then

$$(9.4.27) \qquad\qquad \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi'|^m d\omega \leq c(q, \mu, m, \tau, \lambda, \omega_0).$$

PROOF. For dividing the equation of (StL) by $\Phi |\Phi|^{q-2}$, we get

$$\Phi \frac{d}{d\omega} \left[\left(\lambda^2 \Phi^2 + \Phi'^2 \right)^{\frac{m-2}{2}} \Phi' \right] + q\Phi'^2 \left(\lambda^2 \Phi^2 + \Phi'^2 \right)^{\frac{m-2}{2}} +$$

$$+ \lambda[\lambda(q + m - 1) - m + 2 + \tau]\Phi^2 \left(\lambda^2 \Phi^2 + \Phi'^2 \right)^{\frac{m-2}{2}} =$$

$$= a_0 |\Phi|^m - \mu \left(\lambda^2 \Phi^2 + \Phi'^2 \right)^{\frac{m}{2}}.$$

On integrating the obtained equality we have

$$(9.4.28) \quad (1 - q - \mu) \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} (\lambda^2 \Phi^2 + \Phi'^2)^{\frac{m}{2}} d\omega + a_0 \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega =$$

$$= \lambda(\lambda m + 2 - m + \tau) \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi^2 (\lambda^2 \Phi^2 + \Phi'^2)^{\frac{m-2}{2}} d\omega.$$

Since $q + \mu < 1$, $\lambda(\lambda m + 2 - m + \tau) > 0$ and $m \geq 2$ we shall get the required (9.4.27), if we apply the Young inequality with $p = \frac{m}{m-2}$, $p' = \frac{m}{2}$, $\forall \varepsilon > 0$. Finally, if there were $q + \mu \geq 1$, then from (9.4.28) we would get

$$(q - 1 + \mu)\lambda^m \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega + \lambda^{m-1}(\lambda m + 2 - m + \tau) \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega \leq$$

$$\leq a_0 \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi|^m d\omega,$$

which would contradict (9.1.4), by virtue of $\Phi \not\equiv 0$. The lemma is proved. $\square$

(9.4.1), (9.1.3) give us the function $w = r^\lambda \Phi(\omega)$, which will be *a barrier* for our boundary problem (BVP).

LEMMA 9.17. *Let* $\zeta(r) \in C_0^\infty[0, d]$. *Then*

$$\zeta(r)w(x) \in \mathfrak{N}_{m,q}^1(r^\tau, r^{\tau-m}, G_0^d, G_0^d \setminus \Gamma_2^d).$$

If (9.1.4) and (9.4.26) hold, then

$$\zeta(r)w(x) \in \mathfrak{N}_{m,0}^1(r^\tau, r^{\tau-m}, G_0^d, G_0^d \setminus \Gamma_2^d).$$

PROOF. At first, we observe that $w \in L_\infty(G_0^d)$ since $\lambda > 0$. Now we shall prove that

$$(9.4.29) \qquad I_q[w] \equiv \int\limits_{G_0^d} \left(r^{\tau-m}|w|^{q+m} + r^\tau |w|^q |\nabla w|^m \right) dx < \infty.$$

The direct calculations give

$$(9.4.30) \qquad |\nabla w|^m = r^{m(\lambda-1)}(\lambda^2 \Phi^2 + \Phi'^2)^{\frac{m}{2}}.$$

Therefore

$$I_q[w] = \int\limits_{G_0^d} \left\{ r^{\tau+m(\lambda-1)+q\lambda} \Phi^{m+q}(\omega) + r^{\tau+m(\lambda-1)+q\lambda} |\Phi|^q (\lambda^2\Phi^2 + \Phi'^2)^{\frac{m}{2}} \right\} dx$$

$$\leq c(\lambda,m) \int\limits_0^d r^{\tau+(m+q)\lambda-m+1} dr \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \left(|\Phi|^q |\Phi'|^m + |\Phi|^{q+m} \right) d\omega.$$

It is clear that, by virtue of Lemma 9.15, $I_q[w]$ is finite. To prove the second assertion of the lemma we have to demonstrate that

(9.4.31) $$I[w] \equiv \int\limits_{G_0^d} \left(r^{\tau-m}|w|^m + r^\tau |\nabla w|^m \right) dx < \infty.$$

We again have

$$I[w] = \int\limits_{G_0^d} \left\{ r^{\tau+m(\lambda-1)} (\lambda^2\Phi^2 + \Phi'^2)^{\frac{m}{2}} + r^{\tau+m(\lambda-1)} \Phi^m(\omega) \right\} dx \leq$$

$$\leq c(\lambda,m) \int\limits_0^d r^{\tau+m\lambda+1-m} dr \int\limits_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \left\{ (\lambda^2\Phi^2 + \Phi'^2)^{\frac{m}{2}} + \Phi^m \right\} d\omega.$$

$I[w]$ is finite by Lemma 9.16. Thus,

$$I[w] \leq c(m, \lambda, N, q, \mu, \omega_0, d).$$

Lemma 9.17 is proved.   □

EXAMPLE 9.18. Let $m = 2$ and we shall consider the boundary value problem $(BVP)_0$ for the equation

$$\frac{d}{dx_i} \left(r^\tau |w|^q w_{x_i} \right) = a_0 r^{\tau-2} w|w|^q - \mu r^\tau w|w|^{q-2}|\nabla w|^2, \quad x \in G_0,$$

(9.4.32)

$$a_0 \geq 0, \; 0 \leq \mu < 1, \; q \geq 0, \; \tau \geq 0.$$

From (9.4.8), (9.1.3) it follows that the solution of our problem is the function

$$w(r,\omega) = r^\lambda \times \begin{cases} \cos^{\frac{1}{1+q+\mu}} \left(\dfrac{\pi\omega}{\omega_0} \right) & \text{for the Dirichlet problem,} \\[3mm] \cos^{\frac{1}{1+q+\mu}} \left(\dfrac{\pi\omega}{2\omega_0} - \dfrac{\pi}{4} \right) & \text{for the mixed problem.} \end{cases}$$

where

$$\lambda = \frac{\sqrt{\tau^2 + (\pi/\theta_0)^2 + 4a_0(1 + q + \mu)} - \tau}{2(1 + q + \mu)}$$

(see (9.4.10)). It is easy to check that for such a λ the inequality (9.1.4) is satisfied.

By calculating $\Phi'(\omega)$ one can readily see that all the properties of the function $\Phi(\omega)$ hold. Moreover, we have:

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} \Phi'^2(\omega)d\omega = \frac{\pi}{(1 + q + \mu)^2 \omega_0} \cdot \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1-q-\mu}{2(1+q+\mu)})}{\Gamma(\frac{2+q+\mu}{1+q+\mu})} \times$$

(9.4.33) $\times \begin{cases} 1, & \text{if BVP is the Dirichlet problem,} \\ \frac{1}{4}, & \text{if BVP is the mixed problem} \end{cases}$

provided $q + \mu < 1$. This integral is *nonconvergent*, if $q + \mu \geq 1$. At the same time $\forall q > 0$ we have

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi(\omega)|^q \Phi'^2(\omega)d\omega = \frac{\pi}{(1 + q + \mu)^2 \omega_0} \cdot \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1-\mu}{2(1+q+\mu)})}{\Gamma(\frac{2+\frac{3}{2}q+\mu}{1+q+\mu})} \times$$

(9.4.34) $\times \begin{cases} 1, & \text{if BVP is the Dirichlet problem,} \\ \frac{1}{4}, & \text{if BVP is the mixed problem} \end{cases}$

since $\mu < 1$. This fact completely agrees with Lemmas 9.15-9.17 since

$$\int_{-\frac{\omega_0}{2}}^{\frac{\omega_0}{2}} |\Phi(\omega)|^{q+2}d\omega = \frac{\omega_0}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{q+\frac{3+\mu}{2}}{(1+q+\mu)})}{\Gamma(\frac{2+\frac{3}{2}q+\mu}{1+q+\mu})}.$$

This fact demonstrates that $w(x) \in \mathfrak{N}_{2,0}^1(r^\tau, r^{\tau-m}, G_0^d)$, if $q + \mu < 1$, and $w(x) \notin \mathfrak{N}_{2,0}^1(r^\tau, r^{\tau-m}, G_0^d)$, if $q + \mu \geq 1$. For the latter case we have $w(x) \in \mathfrak{N}_{2,q}^1(r^\tau, r^{\tau-m}, G_0^d)$.

9.5. The estimate of weak solutions in a neighborhood of a boundary edge

In this section we derive an almost exact estimate of the weak solution of (BVP) in a neighborhood of a boundary edge. For our purpose we are going to apply the comparison principle (see Theorem 9.6) and use the barrier function constructed in §9.4. It is easy to verify by assumptions *8)-10)* that all assumptions *(i)-(iii)* of the comparison principle (see §9.2) are fulfilled.

Let us make some transformations. At first we introduce the change of
function

(9.5.1) $$u = v|v|^{t-1} \text{ with } t = \frac{m-1}{q+m-1}.$$

By virtue of the assumption 7), the problem (BVP) takes the form

$(\overline{II})$ $$Q(v, \phi) \equiv \int\limits_{G} \Big\langle \mathcal{A}_i(x, v_x)\phi_{x_i} + a_0\mathcal{A}(x, v)\phi + \mathcal{B}(x, v, v_x)\phi -$$

$$-f(x)\phi \Big\rangle dx + \int\limits_{\Gamma_2} \Big\langle \pm(x, v) - g(x) \Big\rangle \phi ds = 0$$

for $v(x) \in V_0$ and any $\phi(x) \in V_0$, where

$$\mathcal{A}_i(x, \eta) \equiv a_i(x, v|v|^{t-1}, t|v|^{t-1}\eta), \quad \mathcal{A}(x, v) \equiv a(x, v|v|^{t-1}, t|v|^{t-1}\eta),$$

$$\mathcal{B}(x, v, \eta) \equiv b(x, v|v|^{t-1}, t|v|^{t-1}\eta), \quad \pm(x, v) \equiv \sigma(x, v|v|^{t-1}).$$

And by assumptions 11)-14) we have

$\overline{11)}$ $$\sqrt{\sum_{i=1}^{N}\left|\mathcal{A}_i(x, \eta) - t^{m-1}r^\tau|\eta|^{m-2}\eta_i\right|^2} \le c_1(r)r^\tau|\eta|^{m-1} + \psi_1(r);$$

$\overline{12)}$ $$\left|\frac{\partial\mathcal{A}_i(x,\eta)}{\partial\eta_j} - t^{m-1}r^\tau|\eta|^{m-4}\big(\delta_i^j|\eta|^2 + (m-2)\eta_i\eta_j\big)\right| \le$$

$$\le c_2(r)r^\tau|\eta|^{m-2} + c_2(r)\psi_2(r);$$

$\overline{13)}$ $$\left|\frac{\partial\mathcal{A}_i(x,\eta)}{\partial x_i} - \tau t^{m-1}r^{\tau-2}|\eta|^{m-2}x_i\eta_i\right| \le c_3(r)r^{\tau-1}|\eta|^{m-1} + \psi_3(r);$$

$\overline{14)}$ $$\left|\mathcal{A}(x, v) - r^{\tau-m}v|v|^{m-2}\right| + \left|\mathcal{B}(x, v, \eta) + \mu t^m r^\tau v^{-1}|\eta|^m\right| \le$$

$$\le c_4(r)r^\varkappa|\eta|^{m-1} + |v|^{tq}\psi_4(r).$$

REMARK 9.19. Our assumptions 11)-14) essentially mean that the oper-
ator of the problem (BVP) is approximated near the edge Γ_0 by the operator
of the problem for the (ME). Furthermore, by the assumption 7) coefficients
$a_i(x, u, u_x)$ $i = 1, ..., N$ after the substitution (9.5.1) do not depend on v ex-
plicit. For instance, the model equation (ME) satisfies these assumptions.
In fact, after the substitution (9.5.1) the (ME) takes the form

$$\mathcal{L}_0 v(x) \equiv -t^{m-1}\frac{d}{dx_i}\left(r^\tau|\nabla v|^{m-2}v_{x_i}\right) + a_0 r^{\tau-m}v|v|^{m-2} - \mu t^m r^\tau v^{-1}|\nabla v|^m$$

$$= f(x), \quad x \in G.$$

We shall make additional studies. Let us set

$$(9.5.2) \qquad \overline{a}_0 = t^{1-m}a_0, \quad \overline{\mu} = t\mu, \quad \overline{\lambda} = \frac{1}{t}\lambda \text{ and } \overline{\Phi}(\omega) = \Phi^{\frac{1}{t}}(\omega),$$

where $t = \frac{m-1}{q+m-1}$.

Now the function

$$(9.5.3) \qquad\qquad\qquad \overline{w} = r^{\overline{\lambda}}\overline{\Phi}(\omega)$$

will play the role of the barrier. By (StL)-(CPE), (9.1.3) one can easily check that $(\overline{\lambda}, \overline{\Phi}(\omega))$ is a solution to the problem

$$
\begin{cases}
\dfrac{d}{d\omega}\left[\left(\overline{\lambda}^2\overline{\Phi}^2 + \overline{\Phi}'^2\right)^{\frac{m-2}{2}}\overline{\Phi}'\right] + \overline{\mu}\dfrac{1}{\overline{\Phi}}\left(\overline{\lambda}^2\overline{\Phi}^2 + \overline{\Phi}'^2\right)^{\frac{m}{2}} = \\
\qquad = \overline{a}_0\overline{\Phi}|\overline{\Phi}|^{m-2} - \overline{\lambda}[\overline{\lambda}(m-1) - m + 2 + \tau]\overline{\Phi}\left(\overline{\lambda}^2\overline{\Phi}^2 + \overline{\Phi}'^2\right)^{\frac{m-2}{2}}, \\
\qquad\qquad\qquad\qquad \omega \in (-\omega_0/2, \omega_0/2), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (NEVP) \\
\overline{\Phi}(-\omega_0/2) = \overline{\Phi}(\omega_0/2) = 0 \qquad \text{for the Dirichlet problem,} \\
\overline{\Phi}(-\omega_0/2) = \overline{\Phi}'(\omega_0/2) = 0 \qquad \text{for the mixed problem.} \\
\text{and} \\
\displaystyle\int_0^{+\infty} \dfrac{\left[(m-1)y^2 + \overline{\lambda}^2\right](y^2 + \overline{\lambda}^2)^{\frac{m-4}{2}}\,dy}{(m-1+\overline{\mu})(y^2 + \overline{\lambda}^2)^{\frac{m}{2}} + \overline{\lambda}(2-m+\tau)(y^2 + \overline{\lambda}^2)^{\frac{m-2}{2}} - \overline{a}_0} = \theta_0.
\end{cases}
$$

It is evident that the properties of (λ, Φ) established in §9.4.1 - 9.4.2 also remain valid for the $(\overline{\lambda}, \overline{\Phi}(\omega))$. In particular, (9.1.4) takes the form

$$(9.5.4) \qquad P_m(\overline{\lambda}) \equiv (m-1+\overline{\mu})\overline{\lambda}^m + (2-m+\tau)\overline{\lambda}^{m-1} - \overline{a}_0 > 0.$$

We consider the perturbation of the problem (NEVP). Namely, for $\forall \varepsilon \in (0, 2\pi - \omega_0)$ on the segment $[-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2}]$, we define the problem for $(\lambda_\varepsilon, \Phi_\varepsilon)$

$$
\begin{cases}
\dfrac{d}{d\omega}\left[\left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + \Phi_\varepsilon'^2\right)^{\frac{m-2}{2}} \Phi_\varepsilon'\right] + \overline{\mu}\dfrac{1}{\Phi_\varepsilon}\left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + \Phi_\varepsilon'^2\right)^{\frac{m}{2}} = \\[2mm]
\quad = (\overline{a}_0 - \varepsilon)\Phi_\varepsilon |\Phi_\varepsilon|^{m-2} - \lambda_\varepsilon[\lambda_\varepsilon(m-1) - m + 2 + \tau]\Phi_\varepsilon\left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + \Phi_\varepsilon'^2\right)^{\frac{m-2}{2}}, \\[2mm]
\qquad\qquad\qquad\qquad \omega \in \left(-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2}\right), \\[2mm]
\hfill (NEVP)_\varepsilon \\[2mm]
\Phi_\varepsilon\left(-\frac{\omega_0 + \varepsilon}{2}\right) = \Phi_\varepsilon\left(\frac{\omega_0 + \varepsilon}{2}\right) = 0 \qquad \text{for the Dirichlet problem}, \\[2mm]
\Phi_\varepsilon\left(-\frac{\omega_0 + \varepsilon}{2}\right) = \Phi_\varepsilon'\left(\frac{\omega_0 + \varepsilon}{2}\right) = 0 \qquad \text{for the mixed problem} \\[2mm]
\text{and} \\[2mm]
\displaystyle\int_0^{+\infty} \dfrac{\left[(m-1)y^2 + \lambda_\varepsilon^2\right](y^2 + \lambda_\varepsilon^2)^{\frac{m-4}{2}}\, dy}{(m - 1 + \overline{\mu})(y^2 + \lambda_\varepsilon^2)^{\frac{m}{2}} + \lambda_\varepsilon(2 - m + \tau)(y^2 + \lambda_\varepsilon^2)^{\frac{m-2}{2}} + \varepsilon - \overline{a}_0} = \theta_\varepsilon \\[2mm]
\text{as well as} \\[2mm]
P_m(\lambda_\varepsilon) + \varepsilon > 0.
\end{cases}
$$

The problem $(NEVP)_\varepsilon$ is obtained from the problem (NEVP) by replacing in the latter ω_0 by $\omega_0 + \varepsilon$ and $\overline{a}_0$ by $\overline{a}_0 - \varepsilon$. In virtue of the monotonicity of the function $\overline{\lambda}(\omega_0, \overline{a}_0)$, established in §9.4.2 (see. (9.4.23)), we get

$$
(9.5.5) \qquad\qquad 0 < \lambda_\varepsilon < \overline{\lambda}, \quad \lim_{\varepsilon \to +0} \lambda_\varepsilon = \overline{\lambda}.
$$

We denote by $\overline{\lambda}_0$ the value of $\overline{\lambda}$ for $\overline{a}_0 = 0$. It clearly follows from (9.4.12) that $\overline{\lambda}_0 = \lambda_0\Big|_{q=0; \mu=\overline{\mu}}$. In just the same way as in §9.4.2 we calculate that $\overline{\lambda} > \overline{\lambda}_0$. From (9.5.5) it follows that

$$
(9.5.6) \qquad\qquad 0 < \frac{1}{2}\overline{\lambda}_0 < \lambda_\varepsilon < \overline{\lambda}
$$

for a sufficiently small $\varepsilon > 0$.

Next we shall consider separately the case of the Dirichlet problem and the case of the mixed boundary value problem.

Dirichlet problem.

LEMMA 9.20. *There exists an $\varepsilon^* > 0$ such that*

$$
(9.5.7) \qquad\qquad \Phi_\varepsilon\left(\frac{\omega_0}{2}\right) \geq \frac{\varepsilon}{\omega_0 + \varepsilon}, \qquad \forall \varepsilon \in (0, \varepsilon^*).
$$

PROOF. We turn to (9.5.4), that is $P_m(\overline{\lambda}) > 0$. Since $P_m(\overline{\lambda})$ is a polynomial, by continuity, there exists a δ^*- neighborhood of $\overline{\lambda}$, in which (9.5.4)

is satisfied as before, that is there exists $\delta^* > 0$ such that $P_m(\lambda) > 0$ for $\forall \lambda \mid |\lambda - \overline{\lambda}| < \delta^*$. We choose the number $\delta^* > 0$ in the same way. In particular, the inequality

$$P_m(\overline{\lambda} - \delta) > 0 \quad \forall \delta \in (0, \delta^*)$$

holds. We recall that $\overline{\lambda}$ solves $(NEVP)$. By virtue of (9.5.5), now for every $\delta \in (0, \delta^*)$ we can put

$$\lambda_\varepsilon = \overline{\lambda} - \delta$$

and solve $(NEVP)_\varepsilon$ together with this λ_ε with respect to ε. Let $\varepsilon(\delta) > 0$ be the obtained solution. Since (9.5.5) is true,

$$\lim_{\delta \to +0} \varepsilon(\delta) = +0.$$

Thus we have the sequence of problems $(NEVP)_\varepsilon$ with respect to

(9.5.8) $(\lambda_\varepsilon, \Phi_\varepsilon(\omega)) \quad \forall \varepsilon \mid 0 < \varepsilon < \min(\varepsilon(\delta); \pi - \omega_0) = \varepsilon^*(\delta), \ \forall \delta \in (0, \delta^*).$

We consider $\Phi_\varepsilon(\omega)$ with $\forall \varepsilon$ from (9.5.8). In the same way as (9.4.5) we verify that

$$\Phi_\varepsilon''(\omega) < 0, \quad \forall \omega \in \left[-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2} \right].$$

But this inequality means that the function $\Phi_\varepsilon(\omega)$ is convex upon $\left[-\frac{\omega_0+\varepsilon}{2}, \frac{\omega_0+\varepsilon}{2} \right]$, that is

$$\Phi_\varepsilon(\alpha_1 \omega_1 + \alpha_2 \omega_2) \geq \alpha_1 \Phi_\varepsilon(\omega_1) + \alpha_2 \Phi_\varepsilon(\omega_2), \quad \forall \omega_1, \omega_2 \in \left[-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2} \right]$$

$$\text{for } \alpha_1 \geq 0, \ \alpha_2 \geq 0 \mid \alpha_1 + \alpha_2 = 1.$$

We put

$$\alpha_1 = \frac{\omega_0}{\varepsilon + \omega_0}, \quad \alpha_2 = \frac{\varepsilon}{\varepsilon + \omega_0}; \quad \omega_1 = \frac{\varepsilon + \omega_0}{2}, \quad \omega_2 = 0.$$

By $(NEVP)_\varepsilon$ we get

$$\Phi_\varepsilon \left(\frac{\omega_0}{2} \right) \geq \frac{\varepsilon}{\omega_0 + \varepsilon} \Phi_\varepsilon(0) = \frac{\varepsilon}{\omega_0 + \varepsilon},$$

q.e.d. the lemma is proved.   □

COROLLARY 9.21.

(9.5.9) $\dfrac{\varepsilon}{\omega_0 + \varepsilon} \leq \Phi_\varepsilon(\omega) \leq 1,$

for any $\omega \in [-\omega_0/2, \omega_0/2]$ *and for any* $\varepsilon \in (0, \varepsilon^*)$.

Mixed problem.

LEMMA 9.22. *There exists an $\varepsilon^* > 0$ such that*

$$(9.5.10) \qquad \Phi_\varepsilon\left(-\frac{\omega_0}{2}\right) \geq \frac{\varepsilon}{2(\omega_0 + \varepsilon)} \text{ and for any } \varepsilon \in (0, \varepsilon^*).$$

PROOF. We turn back to (9.5.4), that is $\quad P_m(\overline{\lambda}) > 0$. Since $P_m(\overline{\lambda})$ is a polynomial, then, by the continuity, there exists a δ^*-neighborhood of the point $\overline{\lambda}$, in which (9.5.4) remains valid, that is there exists $\delta^* > 0$ such that

$$P_m(\lambda) > 0 \,\forall\lambda \quad \big| \, |\lambda - \overline{\lambda}| < \delta^*.$$

We choose the number $\delta^* > 0$ to guarantee this. Particulary the inequality

$$(9.5.11) \qquad P_m(\overline{\lambda} - \delta) > 0 \quad \forall\delta \in (0, \delta^*)$$

holds. Let us recall that $\overline{\lambda}$ is a solution of (NEVP). By (9.5.5) we can now put for every $\delta \in (0, \delta^*)$

$$(9.5.12) \qquad\qquad \lambda_\varepsilon = \overline{\lambda} - \delta$$

and solve (NEVP)$_\varepsilon$ together with λ_ε with respect to ε. Let $\varepsilon(\delta) > 0$ be the obtained solution. Since (9.5.5) holds, then

$$\lim_{\delta \to +0} \varepsilon(\delta) = +0.$$

Thus, we get the sequence of the problems (NEVP)$_\varepsilon$ with respect to

$$(9.5.13) \quad (\lambda_\varepsilon, \Phi_\varepsilon(\omega)) \quad \forall\varepsilon \text{ such that } 0 < \varepsilon < \min\{\varepsilon(\delta), 2\pi - \omega_0\} = \varepsilon^*(\delta),$$
$$\forall\delta \in (0, \delta^*).$$

We consider $\Phi_\varepsilon(\omega)$ from (9.5.13). In the same way as in (9.4.5) we verify that

$$\Phi_\varepsilon''(\omega) < 0 \quad \forall\omega \in \left(-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2}\right).$$

This inequality means that the function $\Phi_\varepsilon(\omega)$ is convex on the segment $[-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2}]$, that is

$$\Phi_\varepsilon(\alpha_1\omega_1 + \alpha_2\omega_2) \geq \alpha_1\Phi_\varepsilon(\omega_1) + \alpha_2\Phi_\varepsilon(\omega_2) \quad \forall\omega_1, \omega_2 \in \left[-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2}\right]$$

$$\text{for } \alpha_1 \geq 0, \ \alpha_2 \geq 0 \,|\alpha_1 + \alpha_2 = 1.$$

We put

$$\alpha_1 = \frac{\omega_0 + \frac{1}{2}\varepsilon}{\varepsilon + \omega_0}, \quad \alpha_2 = \frac{\varepsilon}{2(\varepsilon + \omega_0)} \text{ and } \omega_1 = -\frac{\varepsilon + \omega_0}{2}, \quad \omega_2 = \frac{\varepsilon + \omega_0}{2}.$$

By (NEVP)$_\varepsilon$ we get

$$\Phi_\varepsilon\left(-\frac{\omega_0}{2}\right) \geq \frac{\varepsilon}{2(\omega_0 + \varepsilon)}\Phi_\varepsilon\left(\frac{\omega_0 + \varepsilon}{2}\right) = \frac{\varepsilon}{2(\omega_0 + \varepsilon)}.$$

The Lemma is proved.   □

COROLLARY 9.23.

$$(9.5.14) \qquad \frac{\varepsilon}{2(\omega_0 + \varepsilon)} \leq \Phi_\varepsilon(\omega) \leq 1 \quad \forall \omega \in [-\omega_0/2, \omega_0/2], \quad \forall \varepsilon \in (0, \varepsilon^*).$$

LEMMA 9.24. *For any* $\varepsilon > 0$ *the inequalities*

$$(9.5.15) \qquad 0 < \Phi'_\varepsilon(\omega) \leq 2\varepsilon^{-1}, \quad \omega \in \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right]$$

and

$$(9.5.16) \qquad -C(q, \overline{\mu}, \tau, m, \lambda, \omega_0)\varepsilon^{-3} \leq \Phi''_\varepsilon < 0, \quad \omega \in \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right]$$

hold, where $C(q, \overline{\mu}, \tau, m, \lambda, \omega_0) > 0$.

PROOF. From $(\text{NEVP})_\varepsilon$, (9.5.10) and (9.5.14) we have

$$(9.5.17) \qquad \frac{\varepsilon}{2(\varepsilon + \omega_0)} \leq \Phi_\varepsilon(\omega) \leq 1, \ \Phi'_\varepsilon(\omega) \geq 0, \ \Phi''_\varepsilon(\omega) < 0,$$

$$\forall \omega \in \left(-\frac{\omega_0 + \varepsilon}{2}, \frac{\omega_0 + \varepsilon}{2}\right) \text{ and } \Phi_\varepsilon\left(-\frac{\omega_0 + \varepsilon}{2}\right) = 0, \quad \forall \varepsilon > 0.$$

Hence it follows that $\Phi'_\varepsilon(\omega)$ decreases on $\left(-\dfrac{\omega_0 + \varepsilon}{2}, \dfrac{\omega_0 + \varepsilon}{2}\right)$. By the Lagrange mean value theorem, we have

$$\Phi_\varepsilon\left(-\frac{\omega_0 + \varepsilon}{2}\right) - \Phi_\varepsilon\left(-\frac{\omega_0}{2}\right) = -\frac{\varepsilon}{2}\Phi'_\varepsilon(\overline{\omega}) \ \Rightarrow \ \varepsilon\Phi'_\varepsilon(\overline{\omega}) \leq 2$$

with some $\overline{\omega} \in \left(-\dfrac{\omega_0 + \varepsilon}{2}, -\dfrac{\omega_0}{2}\right)$. Hence, by decreasing of $\Phi'_\varepsilon(\omega)$ we get (9.5.15). From the equation of $(\text{NEVP})_\varepsilon$ for Φ_ε it follows that

$$-\Phi''_\varepsilon = \frac{1}{\Phi_\varepsilon\left[(m-1)\Phi'^2_\varepsilon + \lambda^2_\varepsilon\Phi^2_\varepsilon\right]\left(\lambda^2_\varepsilon\Phi^2_\varepsilon + \Phi'^2_\varepsilon\right)^{\frac{m-4}{2}}}\left\{\overline{\mu}\left(\lambda^2_\varepsilon\Phi^2_\varepsilon + \Phi'^2_\varepsilon\right)^{\frac{m}{2}} + \right.$$

$$+ \Phi^2_\varepsilon\left(\lambda^2_\varepsilon\Phi^2_\varepsilon + \Phi'^2_\varepsilon\right)^{\frac{m-4}{2}}\left\{\lambda_\varepsilon[\lambda_\varepsilon(m-1) - m + 2 + \tau]\left(\lambda^2_\varepsilon\Phi^2_\varepsilon + \Phi'^2_\varepsilon\right) + \right.$$

$$(9.5.18) \qquad \left. + (m-2)\lambda^2_\varepsilon\Phi'^2_\varepsilon\right\} + (\varepsilon - \overline{a}_0)\Phi^m_\varepsilon\right\}$$

and therefore by virtue of (9.5.6), (9.5.15) and (9.5.17)

$$-\Phi''_\varepsilon(\omega) \leq \left[\lambda^2_\varepsilon(2m - 3 + \overline{\mu}) + (2 - m + \tau)\lambda_\varepsilon + \varepsilon\lambda^{2-m}_\varepsilon\right]\Phi_\varepsilon + \overline{\mu}\Phi^{-1}_\varepsilon\Phi'^2_\varepsilon \leq$$

$$\leq C(q, \overline{\mu}, \tau, m, \lambda, \omega_0)\varepsilon^{-3}.$$

                                                                □

LEMMA 9.25. *There exists a positive constant* $c_0 = c_0(m, q, \mu, \tau, \omega_0)$ *such that*

(9.5.19) $$\Phi'_\varepsilon\left(\frac{\omega_0}{2}\right) \geq c_0 \varepsilon^{m+3}, \quad 0 < \varepsilon \ll 1.$$

PROOF. By the Lagrange mean value theorem in virtue of (NEVP)$_\varepsilon$ we have

(9.5.20) $$\Phi'_\varepsilon\left(\frac{\omega_0}{2}\right) = \Phi'_\varepsilon\left(\frac{\omega_0}{2}\right) - \Phi'_\varepsilon\left(\frac{\omega_0 + \varepsilon}{2}\right) = -\frac{\varepsilon}{2}\Phi''_\varepsilon(\omega)$$

with some $\omega \in \left(\dfrac{\omega_0}{2}, \dfrac{\omega_0 + \varepsilon}{2}\right)$. From the equation (9.5.18) it follows that

$$-\Phi''_\varepsilon \Phi_\varepsilon \left[(m-1)\Phi'^2_\varepsilon + \lambda^2_\varepsilon \Phi^2_\varepsilon\right]\left(\lambda^2_\varepsilon \Phi^2_\varepsilon + \Phi'^2_\varepsilon\right)^{\frac{m-4}{2}} \geq$$

(9.5.21) $$\geq \Phi^m_\varepsilon \left\langle \left[(\overline{\mu} + m - 1)\lambda^m_\varepsilon + (2 + \tau - m)\lambda^{m-1}_\varepsilon - \overline{a}_0\right] + \varepsilon \right\rangle > \varepsilon\Phi^m_\varepsilon,$$

by (9.5.11), (9.5.12) and (9.5.4).

Since $\Phi'_\varepsilon(\omega)$ is a decreasing continuous function and $\Phi'_\varepsilon\left(\dfrac{\omega_0 + \varepsilon}{2}\right) = 0$, then for sufficiently small $\varepsilon > 0$, we can assert that $0 < \Phi'_\varepsilon(\omega) < 1$, $\omega \in \left(\dfrac{\omega_0}{2}, \dfrac{\omega_0 + \varepsilon}{2}\right)$. Therefore we obtain the following statements.

1) If $m \geq 4$, then

$$\left[(m-1)\Phi'^2_\varepsilon + \lambda^2_\varepsilon \Phi^2_\varepsilon\right]\left(\lambda^2_\varepsilon \Phi^2_\varepsilon + \Phi'^2_\varepsilon\right)^{\frac{m-4}{2}} \leq \left[(m-1)\Phi'^2_\varepsilon + \lambda^2_\varepsilon \Phi^2_\varepsilon\right]^{\frac{m-2}{2}} \leq$$

$$\leq \left(m - 1 + \lambda^2_\varepsilon\right)^{\frac{m-2}{2}} \leq \left(m - 1 + \overline{\lambda}^2\right)^{\frac{m-2}{2}}, \quad \omega \in \left(\frac{\omega_0}{2}, \frac{\omega_0 + \varepsilon}{2}\right),$$

by (9.5.5). Hence from (9.5.14) and (9.5.21) it follows that

$$-\Phi''_\varepsilon(\omega) > \left(m - 1 + \overline{\lambda}^2\right)^{\frac{2-m}{2}} \varepsilon\Phi^{m-1}_\varepsilon \geq \left(m - 1 + \overline{\lambda}^2\right)^{\frac{2-m}{2}} \frac{1}{[2(1 + \omega_0)]^{m-1}}\varepsilon^m,$$

and, in virtue of (9.5.20), the required (9.5.19) is proved.

2) If $2 \leq m < 4$, then from (9.5.21), by (9.5.6)

$$-\Phi''_\varepsilon(\omega) > \varepsilon\Phi^{m-1}_\varepsilon \frac{\left(\lambda^2_\varepsilon \Phi^2_\varepsilon + \Phi'^2_\varepsilon\right)^{\frac{4-m}{2}}}{(m-1)\Phi'^2_\varepsilon + \lambda^2_\varepsilon \Phi^2_\varepsilon} > \varepsilon\lambda^{4-m}_\varepsilon \Phi^3_\varepsilon \frac{1}{\overline{\lambda}^2 + m - 1} >$$

$$> \varepsilon\left(\frac{\overline{\lambda}_0}{2}\right)^{4-m} \frac{1}{8(\overline{\lambda}^2 + m - 1)(1 + \omega_0)^3}\varepsilon^3, \quad \omega \in \left(\frac{\omega_0}{2}, \frac{\omega_0 + \varepsilon}{2}\right)$$

and by virtue of (9.5.20), we again obtain (9.5.19).   □

9.6. Proof of the main theorem

PROOF. Let $(\lambda_\varepsilon, \Phi_\varepsilon(\omega))$ be a solution of the problem $(NEVP)_\varepsilon$ with fixed $\varepsilon \in (0, \varepsilon^*)$, where ε^* is determined by (9.5.13). We define the function which we shall use as barrier function. Namely, let us consider the function

$$w_\varepsilon(\overline{x}, r, \omega) = A r^{\lambda_\varepsilon} \Phi_\varepsilon(\omega), \quad \forall \overline{x} \in \mathbb{R}^{N-2}, \ r \in [0, d], \ \omega \in [-\omega_0/2, \omega_0/2],$$

where $A > 0$ is a number to be chosen below. Let us apply the comparison principle (Theorem 9.6) to the problem $(\overline{II})$, comparing its solution $v(x)$ with barrier function $w_\varepsilon(x)$ in the domain G_0^d. The direct calculations demonstrate that

$$\mathcal{L}_0 w_\varepsilon(\overline{x}, r, \omega) = (At)^{m-1} r^{(m-1)\lambda_\varepsilon - m + \tau} \left\{ -\frac{d}{d\omega} \left[\left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + \Phi_\varepsilon'^2 \right)^{\frac{m-2}{2}} \Phi_\varepsilon' \right] - \right.$$

$$- \lambda_\varepsilon [\lambda_\varepsilon(m-1) - m + 2 + \tau] \Phi_\varepsilon \left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + \Phi_\varepsilon'^2 \right)^{\frac{m-2}{2}} + \overline{a}_0 \Phi_\varepsilon^{m-1}$$

$$\left. - \overline{\mu} \frac{1}{\Phi_\varepsilon} \left(\lambda_\varepsilon^2 \Phi_\varepsilon^2 + \Phi_\varepsilon'^2 \right)^{\frac{m}{2}} \right\} = \varepsilon r^{(m-1)\lambda_\varepsilon - m + \tau} \left(tA\Phi_\varepsilon(\omega) \right)^{m-1}.$$

By virtue of (9.5.1), $(NEVP)_\varepsilon$ and by (9.5.14), we obtain

$$(9.6.1) \qquad \mathcal{L}_0 w_\varepsilon(\overline{x}, r, \omega) \geq \varepsilon^m \left[\frac{A(m-1)}{2\theta_\varepsilon(q + m - 1)} \right]^{m-1} r^{(m-1)\lambda_\varepsilon - m + \tau}.$$

Further, in virtue of (9.5.5) and (9.5.14) we have

$$(9.6.2) \qquad\qquad\qquad w_\varepsilon(x) \big|_{\Omega_d} \geq \frac{A\varepsilon}{2\theta_\varepsilon} d^{\overline{\lambda}}.$$

In addition,

$$(9.6.3) \qquad\qquad w_\varepsilon(x) \geq 0 = v(x), \quad x \in \partial G_0^d \setminus \left(\Omega_d \cup \Gamma_2^d \right).$$

Finally it is not difficult to calculate
(9.6.4)
$$c_0 A \varepsilon^{m+1} r^{\lambda_\varepsilon - 1} \leq |\nabla w_\varepsilon| \leq c_1 A \varepsilon^{-1} r^{\lambda_\varepsilon - 1} \text{ and } |\nabla^2 w_\varepsilon| \leq c_2 A \varepsilon^{-3} r^{\lambda_\varepsilon - 2}$$

if we take into account (9.5.15) and (9.5.16) from Lemma 9.24 and (9.5.19) from Lemma 9.25.

Now let $\phi \in L_\infty(G_0^d) \cap W^{1,m}(G_0^d, \partial G_0^d \setminus \Gamma_2^d)$ be any *nonnegative* function. For the operator Q that is defined by $(\overline{II})$ we obtain

$$Q(w_\varepsilon, \phi) = \int\limits_{G_0^d} \phi(x) \Big\langle -\frac{d}{dx_i} \mathcal{A}_i(x, \nabla w_\varepsilon) + a_0 \mathcal{A}(x, w_\varepsilon) + \mathcal{B}(x, w_\varepsilon, \nabla w_\varepsilon) -$$

$$- f(x) \Big\rangle dx + \int\limits_{\Gamma_2^d} \phi(x) \Big\langle \mathcal{A}_i(x, \nabla w_\varepsilon) n_i(x) + \pm(x, w_\varepsilon) - g(x) \Big\rangle ds.$$

And hence, by the definition of the operator $\mathcal{L}_0$ we have

$$Q(w_\varepsilon, \phi) = \int\limits_{G_0^d} \phi(x) \Big\langle -\frac{d}{dx_i} \Big(\mathcal{A}_i(x, \nabla w_\varepsilon) - t^{m-1} r^\tau |\nabla w_\varepsilon|^{m-2} w_{\varepsilon x_i} \Big)$$

$$+ \mathcal{L}_0 w_\varepsilon(x) + a_0 \Big(\mathcal{A}(x, w_\varepsilon) - r^{\tau-m} |w_\varepsilon|^{m-2} w_\varepsilon \Big) +$$

(9.6.5)
$$+ \Big(\mathcal{B}(x, w_\varepsilon, \nabla w_\varepsilon) + \mu t^m r^\tau |\nabla w_\varepsilon|^m w_\varepsilon^{-1} \Big) - f(x) \Big\rangle dx +$$

$$+ \int\limits_{\Gamma_2^d} \phi(x) \Big\langle \pm(x, w_\varepsilon) - g(x) + t^{m-1} r^\tau |\nabla w_\varepsilon|^{m-2} \frac{\partial w_\varepsilon}{\partial n} +$$

$$+ \Big(\mathcal{A}_i(x, \nabla w_\varepsilon) - t^{m-1} r^\tau |\nabla w_\varepsilon|^{m-2} w_{\varepsilon x_i} \Big) n_i(x) \Big\rangle ds.$$

By the assumption *2)*,

(9.6.6)
$$\pm(x, w_\varepsilon) = \sigma(x, w_\varepsilon^t) \geq 0, \quad \text{since} \quad w_\varepsilon(x) \geq 0.$$

Further,

(9.6.7)
$$\frac{\partial w_\varepsilon}{\partial n}\Big|_{\Gamma_2^d} = \frac{1}{r} \frac{\partial w_\varepsilon}{\partial \omega}\Big|_{\omega = \frac{\omega_0}{2}} = A r^{\lambda_\varepsilon - 1} \Phi_\varepsilon'\Big(\frac{\omega_0}{2}\Big) \geq c_0 A \varepsilon^{m+3} r^{\lambda_\varepsilon - 1},$$

by Lemma 9.25. Therefore, in virtue of (9.6.1), (9.6.6) and(9.6.7) from (9.6.5) it follows that

$$
\begin{aligned}
Q(w_\varepsilon, \phi) \geq \int\limits_{G_0^d} \phi(x) \Bigg\langle & \varepsilon^m \left[\frac{A(m-1)}{2(\omega_0 + \varepsilon)(q + m - 1)} \right]^{m-1} r^{(m-1)\lambda_\varepsilon - m + \tau} \\
& - \sum_{i,j=1}^{N} |w_{\varepsilon x_i x_j}| \cdot \left| \frac{\partial \mathcal{A}_i(x, \nabla w_\varepsilon)}{\partial w_{\varepsilon x_j}} - t^{m-1} r^\tau |\nabla w_\varepsilon|^{m-4} \times \right. \\
& \times \left. \left(\delta_i^j |\nabla w_\varepsilon p|^2 + (m-2) w_{\varepsilon x_i} w_{\varepsilon x_j} \right) \right| - \\
& - \left| \frac{\partial \mathcal{A}_i(x, \nabla w_\varepsilon)}{\partial x_i} - \tau t^{m-1} r^{\tau-2} |\nabla w_\varepsilon|^{m-2} x_i w_{\varepsilon x_i} \right| - \\
& - a_0 \left| \mathcal{A}(x, w_\varepsilon) - r^{\tau-m} |w_\varepsilon|^{m-2} w_\varepsilon \right| - \\
& \left| \mathcal{B}(x, w_\varepsilon, \nabla w_\varepsilon) + \mu t^m r^\tau |\nabla w_\varepsilon|^m w_\varepsilon^{-1} \right| - |f(x)| \Bigg\rangle dx + \\
& + \int\limits_{\Gamma_2^d} \phi(x) \Bigg\langle c_0 A t^{m-1} \varepsilon^{m+3} r^{\tau + \lambda_\varepsilon - 1} |\nabla w_\varepsilon|^{m-2} - \\
& - \sqrt{\sum_{i=1}^{N} \left| \mathcal{A}_i(x, \nabla w_\varepsilon) - t^{m-1} r^\tau |\nabla w_\varepsilon|^{m-2} w_{\varepsilon x_i} \right|^2} - \\
& - |g(x)| \Bigg\rangle ds.
\end{aligned}
$$

(9.6.8)

Now, taking into account the assumptions (9.1.16), $\overline{11)} - \overline{14)}$ (9.1.5) and the inequalities (9.6.4), from (9.6.8) we get

$$
\begin{aligned}
Q(w_\varepsilon, \phi) \geq & \, C_1(m, q, \varepsilon, A, \omega_0) \int\limits_{G_0^d} \phi(x) r^{(m-1)\lambda_\varepsilon - m + \tau} \Big\langle 1 - c_2(r) r^{(m-2)(\overline{\lambda} - \lambda_\varepsilon)} - \\
& - c(r) - (f_1 + k_3) r^{(m-1)(\overline{\lambda} - \lambda_\varepsilon)} - k_4 r^{t(m-1)(\overline{\lambda} - \lambda_\varepsilon)} \Big\rangle dx + \\
& + C_2(m, q, \mu, \tau, \varepsilon, A, \omega_0) \int\limits_{\Gamma_2^d} \phi(x) r^{\tau + (m-1)(\lambda_\varepsilon - 1)} \Big\langle 1 - c_1(r) -
\end{aligned}
$$

$$-(k_1+g_1)r^{(m-1)(\bar\lambda-\lambda_\varepsilon)}\Big)ds,$$

where $c(r)=c_2(r)+c_3(r)+c_4(r)$. Fixing $A>0$ and $\varepsilon>0$, we can choose $d>0$ so small (because of the continuity of the functions $c_1(r),c_2(r),c_3(r),$ $c_4(r)$ at zero) that

$$(9.6.9) \qquad\qquad Q(w_\varepsilon,\phi)\geq 0, \quad \forall\phi\geq 0.$$

Further, by Theorem 9.11

$$v(x)\Big|_{\Omega_d}\leq M_0^{1/t},$$

therefore, by (9.6.2)

$$(9.6.10)\qquad\qquad w_\varepsilon\Big|_{\Omega_d}\geq\frac{A\varepsilon}{2\theta_\varepsilon}d^{\bar\lambda}\geq M_0^{1/t}\geq v(x)\Big|_{\Omega_d}$$

provided that $A>0$ is chosen sufficiently large

$$(9.6.11)\qquad\qquad A\geq\frac{2\theta_\varepsilon}{\varepsilon}\Big(\frac{M_0}{d^\lambda}\Big)^{\frac{q+m-1}{m-1}}.$$

Thus, from (9.6.9), (9.6.3), (9.6.10) and $(\overline{II})$ we get

$$\begin{cases} Q(w_\varepsilon,\phi)\geq 0=Q(v,\phi) & \forall\phi\geq 0 \quad\text{in}\quad G_0^d,\\ w_\varepsilon(x)\geq v(x), & x\in\partial G_0^d\setminus\Gamma_2^d. \end{cases}$$

Besides that, one can readily verify that all the other conditions of the comparison principle (Theorem 9.6) are fulfilled. By this principle we get

$$v(x)\leq w_\varepsilon(x), \quad \forall x\in\overline{G_0^d}.$$

Similarly one can prove that

$$v(x)\geq -w_\varepsilon(x), \quad \forall x\in\overline{G_0^d}.$$

Thus, finally, we obtain

$$|v(x)|\leq w_\varepsilon(x)\leq Ar^{\lambda_\varepsilon}, \quad \forall x\in\overline{G_0^d}.$$

On returning to the old variables in virtue of (9.5.1) we get the required estimate (9.1.17). The main theorem is proved. $\qquad\qquad\square$

9.7. Notes

The presentation of this chapter follows [67]. Boundary value problems
in smooth domains for quasilinear degenerate elliptic second order equa-
tions have been intensely studied recently (see [6, 26, 39, 49, 75, 78,
88, 100, 136, 144, 147, 221, 222, 352] etc., and the vast bibliography
therein). Less studied are the problems of this kind in the domains with
a non-smooth boundary. In the paper [212], the existence results for the
quasilinear degenerate elliptic boundary value problems are considered. The
paper [241] examines the well posedness and regularity of the solution of
degenerate quasilinear elliptic equations arising from bimaterial problems
in elastic-plastic mechanics in lipschitzian domain. The papers [58], [59],
[68], [99], [375] are devoted to the study of the weak solutions behavior
for the special cases of the (BVP) equation in the neighborhood of a con-
ical boundary point. In [72] the Dirichlet problem is studied in a domain
with an edge on the boundary for the model equation (ME). In [133] the
properties of the (BVP) solutions for the Laplace operator have been in-
vestigated in a plain domain with a polygonal boundary (see there Chapter
4). Such studies are important for numerical solving of the boundary value
problems (see, for example, [98]). The Hölder continuity of weak solutions
of the Dirichlet problem for the degenerate elliptic linear and quasilinear
divergence equations was proved in Section 3 [118] (linear equation) and in
§2 [26] (quasilinear equation with $m = 2$.)

Recently, C. Ebmeyer and J. Frehse [104, 105] have considered the
mixed boundary value problems for the quasilinear elliptic equations and
systems of the divergent form in a polyhedron. They have proved $W^{s,2}$, $s <$
$\frac{3}{2}$ -regularity and L^p- properties of the first and the second derivatives of
a solution.

CHAPTER 10

Sharp estimates of solutions to the Robin boundary value problem for elliptic non divergence second order equations in a neighborhood of the conical point

The present chapter is devoted to investigating the behavior of strong solutions to the the Robin boundary value problem for the second order elliptic equations (linear and quasilinear) in the neighborhood of a conical boundary point. Such a problem arises, for example, in heat conduction problems as well as in physical geodesy (see e.g., [143]).

Let $G \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with the boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near the point $\mathcal{O}$ it is a *convex* conical surface with its vertex at $\mathcal{O}$. We consider the following elliptic value problems

(LRP)
$$\begin{cases} \mathcal{L}[u] \equiv a^{ij}(x)u_{x_i x_j} + a^i(x)u_{x_i} + a(x)u = f(x), \\ \qquad\qquad\qquad\qquad\qquad a^{ij} = a^{ji}, \ x \in G, \\ \mathcal{B}[u] \equiv \frac{\partial u}{\partial \vec{n}} + \frac{1}{|x|}\gamma(x)u = g(x), \ x \in \partial G \setminus \mathcal{O}. \end{cases}$$

and

$(QLRP)$
$$\begin{cases} a_{ij}(x, u, u_x)u_{x_i x_j} + a(x, u, u_x) = 0, \ a^{ij} = a^{ji}, \ x \in G, \\ \frac{\partial u}{\partial \vec{n}} + \frac{1}{|x|}\gamma(x)u = g(x), \ x \in \partial G \setminus \mathcal{O}. \end{cases}$$

The summation over repeated indices from 1 to N is understood. $\vec{n}$ denotes the unite outward normal to $\partial G \setminus \mathcal{O}$. We obtain *the best possible estimates* of the strong solutions of these problems near a conical boundary point.

A principal new feature here is the consideration of equations with coefficients whose smoothness is *the minimal possible!* Our examples demonstrate this fact. The exact solution estimates near singularities on the boundary are obtained under the condition that leading coefficients of the equation satisfy the Dini condition and the lowest coefficients can increase. The rate of the solution decrease in the neighborhood of a conical point is

characterized by the smallest eigenvalue ϑ_0 of the Laplace-Beltrami opera-
tor in a domain Ω on the unit sphere (see $(EVR)_0$ §2.4.2).

Let us refer to the problem $(QLRP)$. We obtain *the best possible esti-
mates* of the strong solutions of the problem $(QLRP)$ near a conical bound-
ary point. Our theorems also show that the quasilinear problem solutions
have the same regularity (near a conical point) as the linear problem solu-
tions.

10.1. The linear problem

10.1.1. Formulation of the main result.

DEFINITION 10.1. *A strong solution of the problem* (LRP) *is a function*
$u(x) \in W_{loc}^{2,N}(G) \cap W^2(G_\varepsilon) \cap C^0(\overline{G})$ *that for each* $\varepsilon > 0$ *satisfies the equation
for almost all* $x \in G_\varepsilon$ *and the boundary condition in the sense of traces on*
Γ_ε.

We assume the existence $d > 0$ such that G_0^d is *the convex rotational
cone* with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (\frac{\pi}{2}, \pi)$ (see (1.3.13)).
Regarding the equation we assume that the following *conditions* are satisfied

(a) *the condition of the uniform ellipticity*

$$\nu \xi^2 \le a^{ij}(x)\xi_i\xi_j \le \mu\xi^2, \quad \forall x \in \overline{G}, \quad \forall \xi \in \mathbb{R}^N;$$

$\nu, \mu = const > 0,$ *and* $a^{ij}(0) = \delta_i^j$ *(the Kronecker symbol)*,

(b) $a^{ij} \in C^0(\overline{G}),$ $a^i \in L^p(G),$ $p > N,$ $a, f \in L^N(G)$ *and for these the
inequalities*

$$\left(\sum_{i,j=1}^N |a^{ij}(x) - a^{ij}(y)|^2\right)^{\frac{1}{2}} \le \mathcal{A}(|x-y|)$$

and

$$|x|\left(\sum_{i=1}^N |a^i(x)|^2\right)^{\frac{1}{2}} + |x|^2|a(x)| \le \mathcal{A}(|x|)$$

hold for $x, y \in \overline{G},$ *where* $\mathcal{A}(r)$ *is a monotonically increasing, non-
negative function, continuous at* $0,$ $\mathcal{A}(0) = 0,$

(c) *there exist numbers* $f_1 \ge 0,$ $g_1 \ge 0,$ $s > 1,$ $\beta \ge s - 2,$ $\gamma_0 > \tan \frac{\omega_0}{2}$
such that

$$|f(x)| \le f_1|x|^\beta, \quad |g(x)| \le g_1|x|^{s-1}, \quad \gamma(x) \ge \gamma_0$$

and

$$\gamma(x) \in L^\infty(\partial G) \cap C^1(\partial G \setminus \mathcal{O}),$$

(d) $a(x) \le 0$ *in* G.

We denote $M_0 = \max\limits_{x \in \overline{G}} |u(x)|$ (see Proposition 10.11). Our main results are the following theorems. Let λ be the number that is defined by (2.5.11) or (2.5.19) from Section 2.5.

THEOREM 10.2. *Let* u *be a strong solution of the problem* (LRP) *and assumptions* (a) - (d) *are satisfied with* $\mathcal{A}(r)$ *Dini continuous at zero. Suppose, in addition, that*

$$g(x) \in \overset{\circ}{W}{}^{\frac{1}{2}}_{4-N}(\partial G),$$

as well as $a(x) \in \overset{\circ}{W}{}^{0}_{4-N}(G), \gamma(x) \in \overset{\circ}{W}{}^{\frac{1}{2}}_{2-N}(\partial G),$ *if* $u(0) \ne 0$

and there exist numbers

$$k_s =: \sup_{\varrho > 0} \varrho^{-s} \Big(\|f\|_{\overset{\circ}{W}{}^{0}_{4-N}(G^\varrho_0)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma^\varrho_0)} + $$

$$+ |u(0)| \big(\|a\|_{\overset{\circ}{W}{}^{0}_{4-N}(G^\varrho_0)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-N}(\Gamma^\varrho_0)} \big) \Big),$$

(10.1.1)

$$\varkappa_s =: \sup_{\varrho > 0} \varrho^{1-s} \Big(\|f\|_{N, G^\varrho_{\varrho/2}} + |u(0)| \|a\|_{N, G^\varrho_{\varrho/2}} \Big).$$

Then there are $d \in (0, 1)$ *and a constant* $C > 0$ *depending only on* $\nu, \mu, d, s,$ $N, \lambda, \gamma_0, \|\gamma\|_{C^1(\partial G \setminus \mathcal{O})},$ *meas* G *and on the quantity* $\int\limits_0^d \frac{\mathcal{A}(r)}{r} dr$ *such that* $\forall x \in G^d_0$

$$|u(x) - u(0)| \le C \Big(|u|_{0, G} + \|f\|_{\overset{\circ}{W}{}^{0}_{4-N}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-N}(\partial G)} + g_1 + $$

$$+ k_s + \varkappa_s + |u(0)| \big(1 + \|a\|_{\overset{\circ}{W}{}^{0}_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-N}(\partial G)} \big) \Big) \times$$

(10.1.2)
$$\times \begin{cases} |x|^\lambda, & \text{if } s > \lambda, \\ |x|^\lambda \ln^{3/2}\big(\frac{1}{|x|}\big), & \text{if } s = \lambda, \\ |x|^s, & \text{if } s < \lambda. \end{cases}$$

If, in addition, there is a number

$$\tau_s =: \sup_{\varrho > 0} \varrho^{-s} \Big(\|f + u(0)a\|_{V^0_{p, 2p-N}(G^\varrho_{\varrho/2})} + \|g\|_{V^{1-\frac{1}{p}}_{p, 2p-N}(\Gamma^\varrho_{\varrho/2})} + $$

(10.1.3)
$$|u(0)| \|\gamma\|_{V^{1-\frac{1}{p}}_{p, p-N}(\Gamma^\varrho_{\varrho/2})} \Big),$$

then

$$|\nabla u(x)| \le C\Big(|u|_{0,G} + \tau_s + \|f\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-N}(\partial G)} + g_1 +$$

$$+ k_s + \varkappa_s + |u(0)|\Big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-N}(\partial G)}\Big)\Big) \times$$

(10.1.4)

$$\times \begin{cases} |x|^{\lambda-1}, & \text{if } s > \lambda, \\ |x|^{\lambda-1}\ln^{3/2}\big(\frac{1}{|x|}\big), & \text{if } s = \lambda, \\ |x|^{s-1}, & \text{if } s < \lambda. \end{cases}$$

THEOREM 10.3. *Let* u *be a strong solution of the problem* (LRP) *and the assumptions of Theorem 10.2 are satisfied with* $\mathcal{A}(r)$ *that is a function continuous at zero but not Dini continuous at zero. Then there are* $d \in (0,1)$ *and for each* $\varepsilon > 0$ *a constant* $C_\varepsilon > 0$ *depending only on* $\varepsilon, \nu, \mu, d, s, N, \lambda, \gamma_0,$ $\|\gamma\|_{C^1(\partial G \backslash \mathcal{O})},$ *meas* G *and on* $\mathcal{A}(diamG)$ *such that* $\forall x \in G_0^d$

$$|u(x) - u(0)| \le C_\varepsilon\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-N}(\partial G)} + g_1 +$$

(10.1.5)

$$+ |u(0)|\Big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-N}(\partial G)}\Big) + k_s + \varkappa_s\Big) \times$$

$$\times \begin{cases} |x|^{\lambda-\varepsilon}, & \text{if } s > \lambda, \\ |x|^{s-\varepsilon}, & \text{if } s \le \lambda \end{cases}$$

and

$$|\nabla u(x)| \le C_\varepsilon\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-N}(\partial G)} + g_1 +$$

(10.1.6)

$$+ |u(0)|\Big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-N}(\partial G)}\Big) + k_s + \varkappa_s + \tau_s\Big) \times$$

$$\times \begin{cases} |x|^{\lambda-1-\varepsilon}, & \text{if } s > \lambda, \\ |x|^{s-1-\varepsilon}, & \text{if } s \le \lambda. \end{cases}$$

THEOREM 10.4. *Let* u *be a strong solution of the problem* (LRP) *and the assumptions of Theorem 10.2 are satisfied with* $s \ge \lambda,$ $\mathcal{A}(r)\ln\frac{1}{r} \le const,$ $r > 0$ *and* $\mathcal{A}(0) = 0.$ *Then there are* $d \in (0,1)$ *and the constants* $C > 0, c > 0$ *depending only on* $\nu, \mu, d, N, \lambda, \gamma_0, \|\gamma\|_{C^1(\partial G \backslash \mathcal{O})},$ *meas* G *and on* $\mathcal{A}(diamG)$ *such that* $\forall x \in G_0^d$

$$(10.1.7) \quad |u(x) - u(0)| \le C\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-N}(\partial G)} + g_1 +$$

$$+ |u(0)|\Big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-N}(\partial G)}\Big) + k_s + \varkappa_s\Big)|x|^\lambda \ln^{c+1}\frac{1}{|x|}$$

and

$$(10.1.8) \quad |\nabla u(x)| \leq C\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}{}^{1/2}_{4-N}(\partial G)} + g_1 +$$

$$+ |u(0)|\big(1 + \|a\|_{\overset{\circ}{W}{}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{2-N}(\partial G)}\big) + k_s + \varkappa_s + \tau_s\Big)|x|^{\lambda-1}\ln^{c+1}\frac{1}{|x|}.$$

10.1.2. The Lieberman global and local maximum principle. The comparison principle.

DEFINITION 10.5. Let the domain G be at least Lipschitz. A vector $\overrightarrow{\beta}$ is said to *point into* $\overline{G}$ at $x_0 \in \partial G$ if there is a positive constant t_0 such that $x_0 + t\overrightarrow{\beta} \in \overline{G}$ for $0 < t < t_0$. A vector field $\overrightarrow{\beta}$, defined on some subset T of ∂G, points into $\overline{G}$ if $\overrightarrow{\beta}(x_0)$ points into $\overline{G}$ at x_0 for all $x_0 \in T$.

In this section we consider the linear elliptic oblique derivative problem

$$(OP) \quad \begin{cases} \mathcal{L}[u] \equiv a^{ij}(x)u_{x_ix_j} + a^i(x)u_{x_i} + a(x)u = f(x), \\ \qquad\qquad\qquad\qquad\qquad a^{ij} = a^{ji}, \ x \in G, \\ \mathcal{B}_0[u] \equiv \beta^i(x)\frac{\partial u}{\partial x_i} + \gamma(x)u = g(x), \ x \in \partial G. \end{cases}$$

DEFINITION 10.6. It is said that the operator $\mathcal{B}_0$ (or the vector field $\overrightarrow{\beta}$) is *oblique* at a point $x_0 \in \partial G$ if there is a coordinate system $(x_1, x') = (x_1, \ldots, x_N)$ centered at x_0 such that $\overrightarrow{\beta}(x_0)$ is parallel to the positive x_1−axis and if there is a Lipschitz function χ defined on some $(N-1)-$dimensional ball $B_d(x_0)$ such that

$$G \cap B_d(x_0) = \{x \in \mathbb{R}^N | x_1 > \chi(x'), \ |x| < d\}.$$

DEFINITION 10.7. It is said that a vector field $\overrightarrow{\beta} = (\beta^1, \beta')$ defined in a neighborhood of some $x_0 \in \partial G$ has *modulus of obliqueness* δ near x_0 if, for any $\varepsilon > 0$, there is a coordinate system such that

$$G \cap B_d(x_0) = \{x \in \mathbb{R}^N | x_1 > \chi(x'), \ |x| < d\}$$

with a Lipschitz function χ such that

$$\sup |\nabla\chi| \sup \frac{|\beta'|}{|\beta^1|} \leq \delta + \varepsilon.$$

Here $\beta' = (\beta^2, \ldots, \beta^N)$.

DEFINITION 10.8. Let $x = (x_1, x')$ be a point in $\mathbb{R}^N$ and $\overrightarrow{\beta}$ be a vector field such that

$$< \overrightarrow{\beta}, \overrightarrow{n} >= \sum_{i=1}^N \beta^i \cos(\overrightarrow{n}, x_i) < 0.$$

We assume that there are positive constants d, h, m_1 and $\varepsilon < 1$ such that

$$G_0^d = \{x \in \mathbb{R}^N \big| x_1 > h|x'|, \ |x| < d\} \subset G \quad \text{and}$$

$$|\beta'| \leq m_1 \beta^1 \text{ on } \Gamma_0^d, \quad \text{where} \quad hm_1 \leq 1 - \varepsilon.$$

By Definition 10.7, this inequality means that *the modulus of obliqueness at* $x_0 \in \Gamma_0^d$ *is less than 1.*

REMARK 10.9. In the definitions above the vector field $\overrightarrow{\beta}$ can have discontinuities and ∂G is allowed to be piecewise. In this connection see also [227].

REMARK 10.10. For the *convex* rotational cone G_0^d with the vertex at $\mathcal{O}$, the aperture $\omega_0 \in (0, \pi)$ and the vector $\overrightarrow{\beta} = -\overrightarrow{n}$ on Γ_0^d we have (see Lemma 1.10), by (1.3.13) - (1.3.14)

$$h = \cot \frac{\omega_0}{2}, \quad \beta^1 = \sin \frac{\omega_0}{2} \quad \text{and}$$

$$\beta'^2 = \sum_{i=2}^{N} (\beta^i)^2 = \frac{\cot^4 \frac{\omega_0}{2} \sin^2 \frac{\omega_0}{2}}{x_1^2} \sum_{i=2}^{n} x_i^2 = \cos^2 \frac{\omega_0}{2} \Rightarrow$$

$$|\beta'| = \cos \frac{\omega_0}{2} \leq m_1 \sin \frac{\omega_0}{2} \Rightarrow h \leq m_1.$$

Hence it follows that the modulus of obliqueness at $x_0 \in \Gamma_0^d$ is less than 1, if

$$h = \cot \frac{\omega_0}{2} < 1 \quad \Rightarrow \quad \omega_0 > \frac{\pi}{2}.$$

PROPOSITION 10.11. **The global maximum principle** (see Lemma 1.1 [225], Proposition 2.1 [234]; see as well [233]).

Let G be a bounded domain in $\mathbb{R}^N$ with the C^1-boundary $\partial G \backslash \Gamma_0^d$ and G_0^d be a convex rotational cone with vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (\frac{\pi}{2}, \pi)$. Let $u(x)$ be a strong solution of the problem (LRP). Suppose the operator $\mathcal{L}$ is uniformly elliptic with the ellipticity constants $0 < \nu \leq \mu$, $a^i(x), f(x) \in L^N(G), g(x) \in L^\infty(\partial G)$, $a(x) \leq 0$ in G, $\gamma(x) \geq \gamma_0 > 0$ on ∂G. Then

$$\max_{x \in \overline{G}} |u(x)| \leq C \left(\|g\|_{L^\infty(\partial G)} + \|f\|_{L^N(G)} \right),$$

where $C = C(\nu, \gamma_0, N, diam G, \|a^i\|_{L^N(G)})$.

REMARK 10.12. We observe that the vector $-\overrightarrow{n}$ points into $\overline{G}$ if G is a bounded domain in $\mathbb{R}^N$ with the C^1-boundary $\partial G \backslash \Gamma_0^d$ and G_0^d be a convex rotational cone G_0^d with vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (\frac{\pi}{2}, \pi)$.

PROPOSITION 10.13. **The strong maximum principle** (see Corollary 3.2 [234]).

Let G be a bounded domain in $\mathbb{R}^N$ with the C^1-boundary $\partial G \backslash \Gamma_0^d$ and G_0^d be a convex rotational cone with vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (\frac{\pi}{2}, \pi)$. Suppose $u(x) \in C^0(\overline{G})$ has nonnegative maximum at some $x_0 \in \Gamma_0^d$, and suppose there is a positive constant d such that $u \in W_{loc}^{2,N}(G_0^d)$. Suppose the operator $\mathcal{L}$ is uniformly elliptic with the ellipticity constants $0 < \nu \leq \mu$, $a^i(x), a(x) \in L^N(G_0^d)$, $a(x) \leq 0$ in G_0^d, as well $\gamma(x) \in L^\infty(\Gamma_0^d)$, $\gamma(x) \geq \gamma_0 > 0$ on Γ_0^d. If

$$(10.1.9) \qquad \mathcal{L}[u] \geq 0 \ in \ G_0^d, \quad \mathcal{B}[u] \leq 0 \ on \ \Gamma_0^d,$$

then u is constant in G_0^d.

PROPOSITION 10.14. **The local maximum principle** (see Theorem 3.3 [225], Theorem 4.3 [234]; see as well [233]).

Let the hypotheses of Proposition 10.11 hold. In addition, suppose $a^i(x) \in L^p(G), p > N$ and $a(x) \in L^N(G)$. Then for any $q > 0$ and $\sigma \in (0,1)$, we have

$$\sup_{G_0^{\sigma R}} |u(x)| \leq C \left\{ \left(\frac{\int_{G_0^R} |u|^q dx}{meas G_0^R} \right)^{1/q} + R \left(\|f\|_{L^N(G^R)} + \|g\|_{L^\infty(\partial G)} \right) \right\},$$

where $C = C(\nu, \mu, \gamma_0, N, p, R, G, \|a^i\|_{L^p(G)}, \|a\|_{L^N(G)})$.

PROPOSITION 10.15. **The maximum principle.**

Let G be a bounded domain in $\mathbb{R}^N$ with the C^1-boundary $\partial G \backslash \Gamma_0^d$ and G_0^d be a convex rotational cone with vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (\frac{\pi}{2}, \pi)$. Let $u(x)$ be a strong solution of the problem

$$\begin{cases} \mathcal{L}[u] = f(x) & in \quad G_0^d, \\ \mathcal{B}[u] = g(x) & on \quad \Gamma_0^d, \\ \quad u = h(x) & on \quad \Omega_d \cup \mathcal{O} \end{cases}$$

and suppose the operator $\mathcal{L}$ is uniformly elliptic with the ellipticity constants $0 < \nu \leq \mu$, $a^i(x), a(x) \in L_{loc}^\infty(G_0^d)$, $a(x) \leq 0$ in G_0^d, as well $\gamma(x) \in L^\infty(\Gamma_0^d), g(x) \in L^\infty(\Gamma_0^d), h(x) \in L^\infty(\Omega_d \cup \mathcal{O})$ $\gamma(x) \geq \gamma_0 > 0$ on Γ_0^d. In addition, suppose that the functions $w_1(x), w_2(x)$ can be found which satisfy the following inequalities:

$$\begin{cases} \mathcal{L}[w_1] \leq f(x) & in \quad G_0^d, \\ \mathcal{B}[w_1] \geq g(x) & on \quad \Gamma_0^d, \\ \quad w_1 \geq h(x) & on \quad \Omega_d \cup \mathcal{O} \end{cases}$$

and

$$\begin{cases} \mathcal{L}[w_2] \geq f(x) & in \quad G_0^d, \\ \mathcal{B}[w_2] \leq g(x) & on \quad \Gamma_0^d, \\ \quad w_2 \leq h(x) & on \quad \Omega_d \cup \mathcal{O} \end{cases}$$

respectively. Then the solution u satisfies the inequalities

$$w_2(x) \leq u(x) \leq w_1(x) \ in \ \overline{G_0^d}.$$

PROOF. Under such circumstance, the function $v = u - w_1$ satisfies these three inequalities

$$\begin{cases} \mathcal{L}[v] \geq 0 & in \ G_0^d, \\ \mathcal{B}[v] \leq 0 & on \ \Gamma_0^d, \\ \quad v \leq 0 & on \ \Omega_d \cup \mathcal{O}. \end{cases}$$

According to the E. Hopf strong maximum principle, Theorem 4.3 , if v is not identically constant, it can only have a nonnegative maximum at a point on the boundary. By Proposition 10.13, v cannot have a nonnegative maximum on Γ_0^d unless it is a constant. Thus v can only have a nonnegative maximum on $\Omega_d \cup \mathcal{O}$ and therefore we conclude that $v \leq 0$ in G_0^d. To obtain a lower bound we consider the function $v = w_2 - u$ manner reasoning in the same as we did for w_1. $\qquad\square$

PROPOSITION 10.16. **The comparison principle.**
  Let G_0^d be a convex rotational cone with vertex at $\mathcal{O}$ and the aperture $\omega_0 \in (\frac{\pi}{2}, \pi)$. Let $\mathcal{L}$ be uniformly elliptic in G_0^d with the ellipticity constants $0 < \nu \leq \mu$, $a^i(x), a(x) \in L^\infty_{loc}(G_0^d)$, $a(x) \leq 0$ in G_0^d. Let $\gamma(x) \in L^\infty(\Gamma_0^d)$, $\gamma(x) \geq \gamma_0 > 0$ on Γ_0^d. Suppose that v and w are functions in $W^{2,N}_{loc}(G_0^d) \cap C^0(\overline{G_0^d})$ satisfying

$$(10.1.10) \qquad \begin{cases} \mathcal{L}[w(x)] \leq \mathcal{L}[v(x)], & x \in G_0^d; \\ \mathcal{B}[w(x)] \geq \mathcal{B}[v(x)], & x \in \Gamma_0^d; \\ \quad w(x) \geq v(x), & x \in \Omega_d \cup \mathcal{O}. \end{cases}$$

Then $v(x) \leq w(x)$ in $\overline{G_0^d}$.

PROOF. This proposition is the direct consequence of Proposition 10.15. $\qquad\square$

THEOREM 10.17. L_p—**estimate of solutions of the elliptic oblique problem in the smooth domain** (see Theorem 15.3 of [4]).
  Let G be a domain in $\mathbb{R}^N$ with a C^2 boundary portion $T \subset \partial G$. Let $\mathcal{L}$ be uniformly elliptic in G with the ellipticity constants $0 < \nu \leq \mu$ and

$u \in W^{2,p}(G)$, $p > 1$ be a strong solution of the problem

$$\begin{cases} \mathcal{L}[u] = f & in \ G, \\ \mathcal{B}[u] = g & on \ T \end{cases}$$

in the weak sense, where

(i) $a^{ij}(x), a^i(x), a(x) \in C^0(G)$; $\gamma(x) \in C^1(T)$ and

(ii) $f(x) \in L^p(G)$, $g(x) \in W^{1-\frac{1}{p},p}(T)$.

Then, for any domain $G' \subset\subset G \cup T$ we have

$$\|u\|_{W^{2,p}(G')} \leq C \left(\|u\|_{L^p(G)} + \|f\|_{L^p(G)} + \|g\|_{W^{1-\frac{1}{p},p}(T)}, \right)$$

where the constant C is independent of u and depends only on $N, p, \nu, \mu,$ $T, G', G, \|a^i(x)\|_{C^0(G)}, \|a(x)\|_{C^0(G)}, \|\gamma(x)\|_{C^1(T)}$ and the moduli of continuity of the coefficients $a^{ij}(x)$ on G'.

10.1.3. The barrier function. The preliminary estimate of the solution modulus. Let G_0^d be a convex rotational cone with a solid angle $\omega_0 \in (0, \pi)$ and the lateral surface Γ_0^d such that $G_0^d \subset \{x_1 \geq 0\}$. Let us define the following linear elliptic operator

$$\mathcal{L}_0 \equiv a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}; \ a^{ij}(x) = a^{ji}(x), \ x \in G_0^d,$$

where

$$\nu \xi^2 \leq a^{ij}(x)\xi_i \xi_j \leq \mu \xi^2, \ \forall x \in G_0^d, \ \forall \xi \in \mathbb{R}^N \ and \ \nu, \mu = const > 0$$

and the boundary operator

$$\mathcal{B} \equiv \frac{\partial}{\partial \overrightarrow{n}} + \frac{1}{|x|}\gamma(x), \quad \gamma(x) \geq \gamma_0 > 0, \ x \in \Gamma_0^d.$$

LEMMA 10.18. *(Existence of the barrier function).*

Fix the numbers $\gamma_0 > \tan\frac{\omega_0}{2}, \delta > 0, g_1 \geq 0, d \in (0,1)$. *There exist* $h > 0$ *depending only on* ω_0, *the number* $\varkappa_0 \in (0, \gamma_0 \cot\frac{\omega_0}{2} - 1)$, *a number* $B > 0$ *and a function* $w(x) \in C^1(\overline{G_0}) \cap C^2(G_0)$ *that depend only on* ω_0, *the ellipticity constants* ν, μ *of the operator* $\mathcal{L}_0$ *and the quantities* γ_0, δ, g_1, *such that for any* $\varkappa \in (0; \min(\delta, \varkappa_0))$ *the following inequalities hold:*

(10.1.11) $\mathcal{L}_0[w(x)] \leq -\nu h^2 |x|^{\varkappa-1}$; $x \in G_0^d$;

(10.1.12) $\mathcal{B}[w(x)] \geq g_1 |x|^\delta$; $x \in \Gamma_0^d \setminus \mathcal{O}$;

(10.1.13) $0 \leq w(x) \leq C_0(\varkappa_0, B, \omega_0)|x|^{\varkappa+1}$; $x \in \overline{G_0^d}$;

(10.1.14) $|\nabla w(x)| \leq C_1(\varkappa_0, B, \omega_0)|x|^\varkappa$; $x \in \overline{G_0^d}$.

PROOF. Let $(x, y, x') \in \mathbb{R}^N$, where $x = x_1, y = x_2, x' = (x_3, ..., x_N)$. In $\{x_1 \geq 0\}$ we consider the cone K with the vertex in $\mathcal{O}$, such that $K \supset G_0^d$. (We recall that $G_0^d \subset \{x_1 \geq 0\}$.) Let ∂K be the lateral surface of K and let $\partial K \cap y\mathcal{O}x = \Gamma_\pm$ be $x = \pm hy$, where $h = \cot \frac{\omega_0}{2}, 0 < \omega_0 < \pi$, such that in the interior of K the inequality $x > h|y|$ holds. We shall consider the following function

$$w(x; y, x') \equiv x^{\varkappa - 1}(x^2 - h^2 y^2) + Bx^{\varkappa + 1},$$

(10.1.15)

with some $\varkappa \in (0; 1)$, $B > 0$.

Let the coefficients of the operator $\mathcal{L}_0$ be $a^{2,2} = a$, $a^{1,2} = b$, $a^{1,1} = c$. Then we have

(10.1.16)
$$\mathcal{L}_0 w = aw_{yy} + 2bw_{xy} + cw_{xx}$$

where

$$\nu \eta^2 \leq a\eta_1^2 + 2b\eta_1\eta_2 + c\eta_2^2 \leq \mu\eta^2 \quad \text{and}$$

(10.1.17)
$$\eta^2 = \eta_1^2 + \eta_2^2; \; \forall \eta_1, \eta_2 \in \mathbb{R}.$$

Let us calculate the operator $\mathcal{L}_0$ on the function (10.1.15). For $t = \frac{y}{x}, |t| < \frac{1}{h}$ we obtain

$$\mathcal{L}_0 w = -h^2 x^{\varkappa - 1} \phi(\varkappa),$$

where

$$\phi(\varkappa) = 2a - 4bt + 4bt\varkappa - ch^{-1}(1 + B)(\varkappa^2 + \varkappa) + ct^2\varkappa^2 - 3ct^2\varkappa + 2ct^2 =$$
$$= c(t^2 - h^{-2}(1 + B))\varkappa^2 + (4bt - ch^{-2}(1 + B) - 3ct^2)\varkappa + 2(ct^2 - 2bt + a)$$

and

$$c(t^2 - h^{-2}(1 + B)) = c\left(\frac{y^2}{x^2} - \frac{1 + B}{h^2}\right) \leq -c\frac{B}{h^2} < 0.$$

Because of (10.1.17), we have $\phi(0) = 2(ct^2 - 2bt + a) \geq 2\nu$ and since $\phi(\varkappa)$ is a square function there exists the number $\varkappa_0 > 0$ depending only on ν, μ, h such that $\phi(\varkappa) \geq \nu$ for $\varkappa \in [0; \varkappa_0]$. Therefore we obtain (10.1.11).

Now, let us notice that

(10.1.18) $\Gamma_\pm : x = \pm hy, \; h = \cot \frac{\omega_0}{2}, \; 0 < \omega_0 < \pi.$

Then we have

$$\text{on } \Gamma_+ : \begin{cases} x = r\cos\frac{\omega_0}{2}, \\ y = r\sin\frac{\omega_0}{2} \end{cases} \begin{cases} \angle(\vec{n}, x) = \frac{\pi}{2} + \frac{\omega_0}{2}, \\ \angle(\vec{n}, y) = \frac{\omega_0}{2} \end{cases}$$

$$\text{on } \Gamma_- : \begin{cases} x = r\cos\frac{\omega_0}{2}, \\ y = -r\sin\frac{\omega_0}{2} \end{cases} \begin{cases} \angle(\vec{n}, x) = \frac{\pi}{2} + \frac{\omega_0}{2}, \\ \angle(\vec{n}, y) = \pi + \frac{\omega_0}{2} \end{cases}$$

$$\sin\frac{\omega_0}{2} = \frac{1}{\sqrt{1+h^2}}, \quad \cos\frac{\omega_0}{2} = \frac{h}{\sqrt{1+h^2}}.$$

Therefore we obtain

$$w_x = (1 + \varkappa)x^\varkappa(1 + B) - (\varkappa - 1)h^2 y^2 x^{\varkappa - 2} \Rightarrow w_x\big|_{\Gamma_\pm} = [2 + B(1 + \varkappa)]x^\varkappa,$$

(10.1.19)

$$w_y = -2h^2 y x^{\varkappa - 1} \Rightarrow w_y\big|_{\Gamma_\pm} = \mp 2h x^\varkappa.$$

Because of

$$\frac{\partial w}{\partial \vec{n}}\bigg|_{\Gamma_\pm} = w_x \cos\angle(\vec{n}, x)\bigg|_{\Gamma_\pm} + w_y \cos\angle(\vec{n}, y)\bigg|_{\Gamma_\pm}$$

and (10.1.19), we get

$$\frac{\partial w}{\partial \vec{n}}\bigg|_{\Gamma_\pm} = -r^\varkappa \frac{h^\varkappa}{(1+h^2)^{\frac{\varkappa+1}{2}}}[2(1+h^2) + B(1+\varkappa)].$$

Hence it follows that

$$\mathcal{B}[w]\big|_{\Gamma_\pm} \geq \frac{h^\varkappa}{(1+h^2)^{\frac{\varkappa+1}{2}}} r^\varkappa \big[Bh\gamma_0 - B(1+\varkappa) - 2(1+h^2)\big].$$

Since $h > \frac{1}{\gamma_0}$ for $\varkappa \leq \varkappa_0$ we obtain

$$\mathcal{B}[w]\big|_{\Gamma_\pm^d} \geq \frac{h^{\varkappa_0} r^\varkappa}{(1+h^2)^{\frac{\varkappa_0+1}{2}}} \big[B(h\gamma_0 - 1 - \varkappa_0) - 2(1+h^2)\big] \geq g_1 r^\delta, \ 0 < r < d < 1$$

if we choose

$$\varkappa \leq \delta \Rightarrow r^\varkappa \geq r^\delta \quad \text{and}$$

(10.1.20)

$$B \geq \left\{ \frac{g_1(1+h^2)^{\frac{\varkappa_0+1}{2}}}{h^{\varkappa_0}} + 2(1+h^2) \right\} \cdot \frac{1}{h\gamma_0 - 1 - \varkappa_0}.$$

(It should be pointed out that we can choose, if necessary, $\varkappa_0$ so small that $\varkappa_0 < h\gamma_0 - 1$).

Now will show (10.1.13). Let us rewrite the function (10.1.15) in spherical coordinates. Recalling that $h = \cot \frac{\omega_0}{2}$ we obtain

$$w(x; y, x') = (1 + B)(r \cos \omega)^{1+\varkappa} - h^2 r^2 \sin^2 \omega (r \cos \omega)^{\varkappa - 1} =$$

$$= r^{1+\varkappa} \cos^{\varkappa - 1} \omega \left(B \cos^2 \omega + \frac{\chi(\omega)}{\sin^2 \frac{\omega_0}{2}} \right), \quad \forall \omega \in \left[-\frac{\omega_0}{2}, \frac{\omega_0}{2} \right],$$

where

$$\chi(\omega) = \sin \left(\frac{\omega_0}{2} - \omega \right) \cdot \sin \left(\frac{\omega_0}{2} + \omega \right).$$

We find $\chi'(\omega) = -\sin 2\omega$ and $\chi'(\omega) = 0$ for $\omega = 0$. Now we see that $\chi''(0) = -2 \cos 0 = -2 < 0$. In this way we have

$$\max_{\omega \in [-\omega_0/2, \omega_0/2]} \chi(\omega) = \chi(0) = \sin^2 \frac{\omega_0}{2}$$

and therefore

$$w(x; y, x') \leq r^{1+\varkappa} \cos^{\varkappa - 1} \omega (B \cos^2 \omega + 1) \leq r^{1+\varkappa} \cos^{\varkappa + 1} \omega \left(B + \frac{1}{\cos^2 \omega} \right)$$

$$\leq r^{1+\varkappa} \left(B + \frac{1}{\cos^2 \omega} \right).$$

Hence (10.1.13) follows. Finally, (10.1.14) follows in virtue of (10.1.19). $\square$

Now we can estimate $|u(x)|$ for (LRP) in the neighborhood of a conical point.

THEOREM 10.19. *Let $u(x)$ be a strong solution of the problem (LRP) and satisfy assumptions (a)-(d). Then there exist numbers $d \in (0, 1)$ and $\varkappa > 0$ depending only on $\nu, \mu, N, \varkappa_0, \omega_0, f_1, \beta, \gamma_0, s, g_1, M_0$ and the domain G such that*

$$(10.1.21) \qquad |u(x) - u(0)| \leq C_0 |x|^{\varkappa + 1}, \quad x \in G_0^d,$$

where the positive constant C_0 depends only on $\nu, \mu, N, f_1, g_1, \beta, s, \gamma_0, M_0$ and the domain G, and does not depend on $u(x)$.

PROOF. Without loss of generality we may suppose that $u(0) \geq 0$. Let us take the barrier function $w(x)$ defined by (10.1.15) with $\varkappa \in (0, \varkappa_0)$ and the function $v(x) = u(x) - u(0)$. For them we shall show

$$(10.1.22) \qquad \begin{cases} \mathcal{L}(Aw(x)) \leq \mathcal{L}v(x), \ x \in G_0^d, \\ \mathcal{B}[Aw(x)] \geq \mathcal{B}[v(x)], \ x \in \Gamma_0^d, \\ \quad Aw(x) \geq v(x), \qquad x \in \Omega_d \cup \mathcal{O}. \end{cases}$$

Let us calculate the operator $\mathcal{L}$ on these functions. Because of Lemma 10.18 and the assumptions $(b), (d)$, we obtain

$$\mathcal{L}v(x) = \mathcal{L}u(x) - a(x)u(0) = f(x) - a(x)u(0) \geq f(x) \geq -f_1 r^\beta$$

and

$$\mathcal{L}w(x) \le \mathcal{L}_0 w + a^i(x) w_{x_i} \le -\nu h^2 r^{\varkappa - 1} + \frac{\mathcal{A}(r)}{r} C_1 r^\varkappa \le -\frac{1}{2} \nu h^2 r^{\varkappa_0 - 1}.$$

By the continuity of $\mathcal{A}(r)$, $d > 0$ has been chosen so small that

$$(10.1.23) \qquad C_1 \mathcal{A}(r) \le C_1 \mathcal{A}(d) \le \frac{1}{2} \nu h^2 \text{ for } r \le d.$$

Since $0 < \varkappa < \varkappa_0$, hence it follows that

$$\mathcal{L}[Aw(x)] \le -\frac{1}{2} A \nu h^2 r^{\varkappa_0 - 1} \le \mathcal{L}v(x), \quad x \in G_0^d,$$

if numbers $\varkappa_0, A$ are chosen such that

$$(10.1.24) \qquad \varkappa_0 \le \beta + 1, \ A \ge \frac{2f_1}{\nu h^2}.$$

From (10.1.12) we get

$$(10.1.25) \qquad \mathcal{B}[Aw]\Big|_{\Gamma_{\pm}^d} \ge A g_1 r^\delta.$$

Let us calculate $\mathcal{B}[v]$ on $\Gamma_{\pm}^d$. If $A \ge 1$ and $0 < \delta \le s - 1$ then

$$(10.1.26) \qquad \begin{aligned} \mathcal{B}[v(x)] &= \frac{\partial u}{\partial \bar{n}} + \frac{1}{|x|} \gamma(x) \left(u(x) - u(0) \right) = g(x) - \frac{1}{|x|} \gamma(x) u(0) \\ &\le g(x) \le g_1 r^{s-1} \le g_1 r^\delta \le \mathcal{B}[Aw], \quad x \in \Gamma_{\pm}^d \end{aligned}$$

by (10.1.25).

Now we compare $v(x)$ and $w(x)$ on Ω_d. Since $x^2 \ge h^2 y^2$ in $\overline{K}$, from (10.1.15) we have

$$(10.1.27) \qquad w(x)\Big|_{r=d} \ge B|x|^{1+\varkappa}\Big|_{r=d} = B d^{1+\varkappa} \cos^{\varkappa+1} \frac{\omega_0}{2}.$$

On the other hand

$$(10.1.28) \qquad v(x)\Big|_{\Omega_d} = \left(u(x) - u(0) \right)\Big|_{\Omega_d} \le M_0$$

and therefore from (10.1.27) and (10.1.28), in virtue of (10.1.20), we obtain

$$Aw(x)\Big|_{\Omega_d} \ge A B d^{1+\varkappa} \cos^{\varkappa+1} \frac{\omega_0}{2} \ge A \left\{ \frac{g_1 \left(1 + h^2\right)^{\frac{\varkappa_0 + 1}{2}}}{h^{\varkappa_0}} + 2(1 + h^2) \right\} \times$$

$$\times \frac{1}{h\gamma_0 - 1 - \varkappa_0} d^{1+\varkappa_0} h^{1+\varkappa_0} (1 + h^2)^{-\frac{1+\varkappa_0}{2}} \ge$$

$$\ge M_0 \ge v\Big|_{\Omega_d},$$

where A is made great enough to satisfy

$$(10.1.29) \qquad A \geq \frac{M_0(h\gamma_0 - 1 - \varkappa_0)}{hd^{1+\varkappa_0}\left[g_1 + 2h^{\varkappa_0}\left(1 + h^2\right)^{\frac{1-\varkappa_0}{2}}\right]}.$$

Thus, if we choose the small number $d > 0$ according to (10.1.23) and large numbers $B > 0, A \geq 1$ according to (10.1.20), (10.1.24), (10.1.29), we provide the validity of (10.1.22).

Therefore the functions $v(x), Aw(x)$ satisfy the comparison principle, Proposition 10.16, and we have

$$(10.1.30) \qquad u(x) - u(0) \leq Aw(x), \ x \in \overline{G_0^d}.$$

Similarly, we derive the estimate

$$u(x) - u(0) \geq -Aw(x),$$

if we consider an auxiliary function $v(x) = u(0) - u(x)$. The theorem is proved, in virtue of (10.1.13). □

10.1.4. Global integral weighted estimate.

THEOREM 10.20. *Let $u(x)$ be a strong solution of the problem (LRP). Let assumptions (a) - (c) be satisfied. Suppose, in addition, that $g(x) \in \overset{\circ}{W}_\alpha^{\frac{1}{2}}(\partial G)$, where*

$$(10.1.31) \qquad 4 - N < \alpha < 2.$$

Then $u(x) \in \overset{\circ}{W}_\alpha^2(G)$ and

$$(10.1.32) \qquad \|u\|_{\overset{\circ}{W}_\alpha^2(G)} \leq C\big(\|u\|_{2,G} + \|f\|_{\overset{\circ}{W}_\alpha^0(G)} + \|g\|_{\overset{\circ}{W}_\alpha^{\frac{1}{2}}(\partial G)}\big),$$

where the constant $C > 0$ depends only on $\nu, \mu, \alpha, N, \|a^i\|_{p,G}, \ i = 1, \dots, N$ and $\|a\|_{N,G}, \gamma_0, \|\gamma\|_{C^1(\partial G \backslash \mathcal{O})}$, the moduli of continuity of the coefficients a^{ij} and the domain G.

PROOF. Since $a^{ij}(0) = \delta_i^j$, we have

$$(10.1.33) \quad \Delta u(x) = f(x) - \left(a^{ij}(x) - a^{ij}(0)\right)D_{ij}u(x) - a^i(x)D_i u(x) - \\ - a(x)u(x) \quad in \ G.$$

Integrating by parts, using the Gauss-Ostrogradskiy formula, we show that

$$
\begin{aligned}
\int_{G_\varepsilon} r^{\alpha-2} u \Delta u\, dx &= -\varepsilon^{\alpha-2} \int_{\Omega_\varepsilon} u \frac{\partial u}{\partial r} d\Omega_\varepsilon + \int_{\Gamma_\varepsilon} r^{\alpha-2} u \frac{\partial u}{\partial \overrightarrow{n}} ds - \\
&\quad - \int_{G_\varepsilon} \left\langle \nabla u, \nabla \left(r^{\alpha-2} u\right) \right\rangle dx = -\varepsilon^{\alpha-2} \int_{\Omega_\varepsilon} u \frac{\partial u}{\partial r} d\Omega_\varepsilon + \\
&\quad + \int_{\Gamma_\varepsilon} r^{\alpha-2} u \frac{\partial u}{\partial \overrightarrow{n}} ds - \int_{G_\varepsilon} r^{\alpha-2} |\nabla u|^2 dx + \\
&\quad + (2-\alpha) \int_{G_\varepsilon} r^{\alpha-4} u \langle x, \nabla u \rangle dx.
\end{aligned}
$$

Integrating again by parts we obtain

$$
\begin{aligned}
\int_{G_\varepsilon} r^{\alpha-4} u \langle x, \nabla u \rangle dx &= \frac{1}{2} \int_{G_\varepsilon} \langle r^{\alpha-4} x, \nabla u^2 \rangle dx - \\
&\quad - \frac{1}{2}\varepsilon^{\alpha-3} \int_{\Omega_\varepsilon} u^2 d\Omega_\varepsilon + \frac{1}{2} \int_{\Gamma_\varepsilon} r^{\alpha-4} u^2 x_i \cos(\overrightarrow{n}, x_i) ds - \\
&\quad - \frac{1}{2} \int_{G_\varepsilon} u^2 \sum_{i=1}^{N} D_i(r^{\alpha-4} x_i) dx = -\frac{1}{2}\varepsilon^{\alpha-3} \int_{\Omega_\varepsilon} u^2 d\Omega_\varepsilon + \\
&\quad + \int_{\Gamma_d} r^{\alpha-3} u^2 \frac{\partial r}{\partial \overrightarrow{n}} ds - \frac{N+\alpha-4}{2} \int_{G_\varepsilon} r^{\alpha-4} u^2 dx,
\end{aligned}
$$

because of

$$
\sum_{i=1}^{N} D_i(r^{\alpha-4} x_i) = N r^{\alpha-4} + (\alpha-4) r^{\alpha-5} \sum_{i=1}^{N} \frac{x_i^2}{r} = (N+\alpha-4) r^{\alpha-4}
$$

and (1.3.14) of Lemma 1.10

Thus, multiplying both sides of (10.1.33) by $r^{\alpha-2} u(x)$ and integrating over G_ε and because of the boundary condition of the (LRP), we obtain

$$
(10.1.34) \quad \int_{G_\varepsilon} r^{\alpha-2} |\nabla u|^2 dx + \frac{2-\alpha}{2} \varepsilon^{\alpha-3} \int_{\Omega_\varepsilon} u^2 d\Omega_\varepsilon + \int_{\Gamma_\varepsilon} r^{\alpha-3} \gamma(x) u^2 ds +
$$

$$+ \frac{2-\alpha}{2}(N+\alpha-4) \int\limits_{G_\varepsilon} r^{\alpha-4}u^2 dx = -\varepsilon^{\alpha-2} \int\limits_{\Omega_\varepsilon} u \frac{\partial u}{\partial r} d\Omega_\varepsilon + \int\limits_{\Gamma_\varepsilon} r^{\alpha-2} g(x) u ds +$$

$$+ \frac{2-\alpha}{2} \int\limits_{\Gamma_d} r^{\alpha-3}u^2 \frac{\partial r}{\partial \overrightarrow{n}} ds + \int\limits_{G_\varepsilon} r^{\alpha-2} u \Big(-f(x) + \big(a^{ij}(x) - a^{ij}(0) \big) D_{ij}u(x) +$$

$$+ a^i(x) D_i u(x) + a(x) u(x) \Big) dx.$$

Let us estimate the integral over Ω_ε in the above equality. To this end we consider the function

$$M(\varepsilon) = \max_{x \in \Omega_\varepsilon} |u(x)|.$$

LEMMA 10.21.

(10.1.35) $$\lim_{\varepsilon \to +0} \varepsilon^{\alpha-2} \int\limits_{\Omega_\varepsilon} u \frac{\partial u}{\partial r} d\Omega_\varepsilon = 0, \ \forall \alpha \in (4-N, 2].$$

PROOF. We consider the set $G_\varepsilon^{2\varepsilon}$. We have $\Omega_\varepsilon \subset \partial G_\varepsilon^{2\varepsilon}$. Now we use the inequality (1.6.1)

$$\int\limits_{\Omega_\varepsilon} |w| d\Omega_\varepsilon \le c \int\limits_{G_\varepsilon^{2\varepsilon}} (|w| + |\nabla w|) dx.$$

Setting $w = u \frac{\partial u}{\partial r}$ we find $|w| + |\nabla w| \le c(r^2 u_{xx}^2 + |\nabla u|^2 + r^{-2}u^2)$. Therefore we get

(10.1.36) $$\int\limits_{\Omega_\varepsilon} \left| u \frac{\partial u}{\partial r} \right| d\Omega_\varepsilon \le c \int\limits_{G_\varepsilon^{2\varepsilon}} (r^2 u_{xx}^2 + |\nabla u|^2 + r^{-2}u^2) dx.$$

Let us now consider the sets $G_{\varepsilon/2}^{5\varepsilon/2}$ and $G_\varepsilon^{2\varepsilon} \subset G_{\varepsilon/2}^{5\varepsilon/2}$ and new variables x' defined by $x = \varepsilon x'$. Then the function $w(x') = u(\varepsilon x')$ satisfies in $G_{1/2}^{5/2}$ the problem

$(LPR)'$
$$\begin{cases} a^{ij}(\varepsilon x') \frac{\partial^2 w}{\partial x_i' \partial x_j'} + \varepsilon a^i(\varepsilon x') \frac{\partial w}{\partial x_i'} + \varepsilon^2 a(\varepsilon x') w = \varepsilon^2 f(\varepsilon x'), \\ \hspace{5cm} x' \in G_{1/2}^{5/2}, \\ \frac{\partial w}{\partial \overrightarrow{n'}} + \frac{1}{|x'|} \gamma(\varepsilon x') w = \varepsilon g(\varepsilon x'), \quad x' \in \Gamma_{1/2}^{5/2}. \end{cases}$$

Because of the interior and near a smooth portion of the boundary L^2—estimate, Theorem 10.17, for the equation $(LPR)'$ solution we have:

$$\int_{G_1^2} \left(w_{x'x'}^2 + |\nabla'w|^2\right) dx' \leq C_1 \int_{G_{1/2}^{5/2}} \left(\varepsilon^4 f^2 + w^2\right) dx' +$$

$$+ C_2\varepsilon^2 \inf \int_{G_{1/2}^{5/2}} (|\nabla'\mathcal{G}|^2 + |\mathcal{G}|^2)dx',$$

where infimum is taken over all $\mathcal{G}$ such that $\mathcal{G}|_{\Gamma_{1/2}^{5/2}} = g$ and the constants $C_1, C_2 > 0$ depend only on $\nu, \mu, \max_{x',y' \in G_{1/2}^{5/2}} \mathcal{A}(|x' - y'|)$, $\|\gamma\|_{C^1(\Gamma_{1/2}^{5/2})}$ and the domain G.

Returning to the variable x, we obtain

$$(10.1.37) \quad \int_{G_\varepsilon^{2\varepsilon}} \left(r^2|D^2u|^2 + |\nabla u|^2 + r^{-2}u^2\right) dx \leq$$

$$\leq c \int_{G_{\varepsilon/2}^{5\varepsilon/2}} \left(r^2f^2 + r^{-2}u^2\right) dx + C_2 \inf \int_{G_{\varepsilon/2}^{5\varepsilon/2}} (r^2|\nabla\mathcal{G}|^2 + |\mathcal{G}|^2)dx.$$

By the Mean Value Theorem 1.58 with regard to $u \in C^0(\overline{G})$, we have

$$(10.1.38) \quad \int_{G_{\varepsilon/2}^{5\varepsilon/2}} r^{-2}u^2dx = \int_{\varepsilon/2}^{5\varepsilon/2} r^{N-3} \int_\Omega u^2(r,\omega)d\Omega dr$$

$$\leq 2\varepsilon(\theta_1\varepsilon)^{N-3} \int_\Omega u^2(\theta_1\varepsilon, \omega)d\Omega$$

$$\leq 2\varepsilon^{N-2}\theta_1^{N-3}M^2(\theta_1\varepsilon) \text{ meas } \Omega$$

for some $\frac{1}{2} < \theta_1 < \frac{5}{2}$. From (10.1.36), (10.1.37), and (10.1.38) we obtain

$$(10.1.39) \quad \int_{\Omega_\varepsilon} \left|u\frac{\partial u}{\partial r}\right| d\Omega_\varepsilon \leq c_1\varepsilon^{N-2}M^2(\varepsilon) + c_2 \int_{G_{\varepsilon/2}^{5\varepsilon/2}} r^2f^2dx +$$

$$+ C_2 \inf \int_{G_{\varepsilon/2}^{5\varepsilon/2}} (r^2|\nabla \mathcal{G}|^2 + |\mathcal{G}|^2)dx \leq c_1 \varepsilon^{N-2} M^2(\varepsilon) +$$

$$+ c_3 \varepsilon^{2-\alpha} \int_{G_{\varepsilon/2}^{5\varepsilon/2}} \left\{ r^\alpha f^2 + r^\alpha |\nabla \mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2 \right\} dx, \quad \forall \alpha \leq 2.$$

Also we have

$$(10.1.40) \quad \varepsilon^{\alpha-2} \int_{\Omega_\varepsilon} \left| u \frac{\partial u}{\partial r} \right| d\Omega_\varepsilon \leq c_1 \varepsilon^{\alpha+N-4} M^2(\varepsilon) +$$

$$+ c_3 \int_{G_{\varepsilon/2}^{5\varepsilon/2}} \left\{ r^\alpha f^2 + r^\alpha |\nabla \mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2 \right\} dx, \quad \forall \alpha \leq 2.$$

By the hypotheses of our Theorem, we have $f \in \mathring{W}_\alpha^0(G)$, $g(x) \in \mathring{W}_\alpha^{\frac{1}{2}}(\partial G)$, hence

$$(10.1.41) \qquad \lim_{\varepsilon \to +0} \int_{G_{\varepsilon/2}^{5\varepsilon/2}} \left\{ r^\alpha f^2 + r^\alpha |\nabla \mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2 \right\} dx = 0.$$

Because $u \in C^0(\overline{G})$ and $4 - N < \alpha \leq 2$, from (10.1.40) and (10.1.41), we deduce the validity of the statement (10.1.35) of our lemma. $\qquad \square$

Now we estimate each integral from the right hand side of (10.1.34).

1)

$$\int_{\Gamma_d} r^{\alpha-3} u^2 \frac{\partial r}{\partial \overrightarrow{n}} ds \leq d^{\alpha-3} \int_{\Gamma_d} u^2 ds \quad \text{since } r \geq d, \ \alpha \leq 2;$$

hence, applying (1.6.2), we get

$$(10.1.42) \quad \int_{\Gamma_d} r^{\alpha-3} u^2 \frac{\partial r}{\partial \overrightarrow{n}} ds \leq \delta d^{\alpha-3} \int_{G_d} |\nabla u|^2 dx + c_\delta \int_{G_d} |u|^2 dx; \ \forall \delta > 0.$$

2) Using the Cauchy inequality we obtain

$$(10.1.43) \quad \begin{aligned} \int_{\Gamma_\varepsilon} r^{\alpha-2}|u||g|ds &= \int_{\Gamma_\varepsilon} \left(r^{\frac{\alpha-1}{2}} \frac{1}{\sqrt{\gamma(x)}} |g| \right) \left(r^{\frac{\alpha-3}{2}} \sqrt{\gamma(x)} |u| \right) ds \leq \\ &\leq \frac{\delta}{2} \int_{\Gamma_\varepsilon} r^{\alpha-3} \gamma(x) u^2 ds + \frac{1}{2\delta\gamma_0} \int_{\partial G} r^{\alpha-1} g^2 ds; \ \forall \delta > 0. \end{aligned}$$

3) We get, by the Cauchy inequality,

(10.1.44)
$$\int\limits_{G_\varepsilon} r^{\alpha-2}u(x)f(x)dx = \int\limits_{G_\varepsilon} (r^{\alpha/2-2}u(x))(r^{\alpha/2}f(x))dx$$
$$\leq \frac{\delta}{2}\int\limits_{G_\varepsilon} r^{\alpha-4}u^2 dx + \frac{1}{2\delta}\int\limits_{G_\varepsilon} r^\alpha f^2(x)dx, \ \forall \delta > 0.$$

4) Applying the assumption b) together with the Cauchy inequality we obtain

(10.1.45) $\quad r^{\alpha-2}u\left((a^{ij}(x) - a^{ij}(0))D_{ij}u(x) + a^i(x)D_i u(x) + a(x)u(x)\right)$
$$\leq \mathcal{A}(r)\left((r^{\frac{\alpha}{2}}|D^2 u|)(r^{\frac{\alpha}{2}-2}u) + r^{\alpha-2}|\nabla u|(r^{-1}u) + r^{\alpha-4}u^2\right)$$
$$\leq \mathcal{A}(r)\left(r^\alpha|D^2 u|^2 + r^{\alpha-2}|\nabla u|^2 + 2r^{\alpha-4}u^2\right).$$

Finally, by (10.1.42) and (10.1.45) from (10.1.34), we obtain

(10.1.46) $\quad \displaystyle\int\limits_{G_\varepsilon} r^{\alpha-2}|\nabla u|^2 dx + \frac{2-\alpha}{2}(N+\alpha-4)\int\limits_{G_\varepsilon} r^{\alpha-4}u^2 dx +$
$$+ \frac{1}{2}\int\limits_{\Gamma_\varepsilon} r^{\alpha-3}\gamma(x)u^2 ds \leq \varepsilon^{\alpha-2}\int\limits_{\Omega_\varepsilon} u\frac{\partial u}{\partial r}d\Omega_\varepsilon + \delta\int\limits_{G_\varepsilon} r^{\alpha-4}|u|^2 dx +$$
$$+ \frac{2-\alpha}{2}c\int\limits_{G_d}(|\nabla u|^2 + |u|^2)dx + c_\delta\int\limits_G r^\alpha f^2(x)dx + \frac{1}{2\gamma_0}\int\limits_{\partial G} r^{\alpha-1}g^2 ds +$$
$$+ \int\limits_{G_\varepsilon} \mathcal{A}(|x|)\left(r^\alpha|D^2 u|^2 + r^{\alpha-2}|\nabla u|^2 + 2r^{\alpha-4}u^2\right)dx$$

for $\forall \delta > 0$.

Let us now estimate the last integral in (10.1.46). Due to assumption b), we have

(10.1.47) $\qquad \forall \delta > 0 \quad \exists d > 0$ such that $\mathcal{A}(r) < \delta$ for all $0 < r < d$.

Let $2\varepsilon < d$. From (10.1.37), (10.1.38) it follows that

(10.1.48) $\quad \displaystyle\int\limits_{G_\varepsilon^{2\varepsilon}} r^\alpha|D^2 u|^2 dx \leq c\varepsilon^{\alpha-2}\int\limits_{G_\varepsilon^{2\varepsilon}} r^2|D^2 u|^2 dx \leq c\varepsilon^{\alpha+N-4}M^2(\varepsilon) +$
$$+ c\int\limits_{G_{\varepsilon/2}^{5\varepsilon/2}}\left(r^\alpha f^2 + r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2\right)dx,$$

and consequently

$$\int\limits_{G_\varepsilon} \mathcal{A}(r)r^\alpha|D^2u|^2dx = \int\limits_{G_\varepsilon^{2\varepsilon}} \mathcal{A}(r)r^\alpha|D^2u|^2dx + \int\limits_{G_{2\varepsilon}^d} \mathcal{A}(r)r^\alpha|D^2u|^2dx +$$

$$+ \int\limits_{G_d} \mathcal{A}(r)r^\alpha|D^2u|^2dx \le c\mathcal{A}(2\varepsilon) \int\limits_{G_{\varepsilon/2}^{5\varepsilon/2}} \Big(r^\alpha f^2 + r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2\Big)dx +$$

$$+ \delta \int\limits_{G_\varepsilon^{2d}} \Big(r^\alpha f^2(x) + r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2 + r^{\alpha-4}u^2\Big)dx +$$

(10.1.49) $$\qquad + c\mathcal{A}(2\varepsilon)\varepsilon^{\alpha+N-4} + c \max_{r\in[d,\text{diam }G]} \mathcal{A}(r) \int\limits_{G_d} |D^2u|^2dx$$

for $\forall\delta > 0$ and $0 < \varepsilon < d/2$. Here c does not depend on ε.

Applying all these estimates to the inequality (10.1.46), we obtain

(10.1.50) $$\int\limits_{G_\varepsilon} r^{\alpha-2}|\nabla u|^2dx + \frac{2-\alpha}{2}(N+\alpha-4)\int\limits_{G_\varepsilon} r^{\alpha-4}u^2dx \le$$

$$\le c\mathcal{A}(2\varepsilon)\Bigg(\varepsilon^{\alpha+N-4} + \int\limits_{G_{\varepsilon/2}^{5\varepsilon/2}} \Big(r^\alpha f^2 + r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2\Big)dx\Bigg) +$$

$$+ \delta\int\limits_{G_\varepsilon} (r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)dx + c\int\limits_{G_d} \Big(|D^2u|^2 + |\nabla u|^2 + u^2\Big)dx +$$

$$+ c\int\limits_{G} \Big(r^\alpha f^2 + r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2\Big)dx + c\int\limits_{\partial G} r^{\alpha-1}g^2ds + \varepsilon^{\alpha-2}\int\limits_{\Omega_\varepsilon} u\frac{\partial u}{\partial r}d\Omega_\varepsilon$$

for $\forall\delta > 0$ and $0 < \varepsilon < d/2$.

Finally, we apply L^2−estimate, Theorem 10.17, to the solution u of the (LRP) in G_d

(10.1.51) $$\int\limits_{G_d} \Big(|D^2u|^2 + |\nabla u|^2\Big)dx \le c\int\limits_{G_{d/2}} (u^2 + f^2)dx + c\|g\|^2_{W^{1/2,2}(\Gamma_{d/2})}.$$

Now we use the inequality

(10.1.52) $$\int\limits_{\Gamma_0^d} r^{\alpha-1}g^2(x)ds \le C\|g\|^2_{\overset{\circ}{W}^{1/2}_\alpha(\Gamma_0^d)}$$

(see Lemma 1.40). Then from (10.1.50), (10.1.51), and (10.1.52) we obtain

$$(10.1.53) \quad \int_{G_\varepsilon} r^{\alpha-2}|\nabla u|^2 dx + \frac{2-\alpha}{2}(N+\alpha-4)\int_{G_\varepsilon} r^{\alpha-4}u^2 dx \le$$

$$\le \varepsilon^{\alpha-2}\int_{\Omega_\varepsilon} u\frac{\partial u}{\partial r}d\Omega_\varepsilon + c\mathcal{A}(2\varepsilon)\int_{G_{\varepsilon/2}^{5\varepsilon/2}}\left(r^\alpha f^2 + r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2\right)dx +$$

$$+ \delta\int_{G_\varepsilon}(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)dx + c\big(\|u\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}_\alpha^0(G)}^2 + \|g\|_{\overset{\circ}{W}_\alpha^{\frac{1}{2}}(\partial G)}^2\big) +$$

$$+ c\mathcal{A}(2\varepsilon)\varepsilon^{\alpha+N-4}$$

for $\forall\delta > 0$ and $0 < \varepsilon < d/2$.

Now, since $4 - N < \alpha < 2$, we can choose $\delta = \min\left(\frac{1}{2}; \frac{(2-\alpha)(N+\alpha-4)}{4}\right)$. Then

$$(10.1.54) \quad c_{\alpha,N}\int_{G_\varepsilon}(r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)dx \le \varepsilon^{\alpha-2}\int_{\Omega_\varepsilon}u\frac{\partial u}{\partial r}d\Omega_\varepsilon +$$

$$+ c\mathcal{A}(2\varepsilon)\left(\int_{G_{\varepsilon/2}^{5\varepsilon/2}}\left(r^\alpha f^2 + r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2\right)dx + \varepsilon^{\alpha+N-4}\right) +$$

$$+ c\big(\|u\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}_\alpha^0(G)}^2 + \|g\|_{\overset{\circ}{W}_\alpha^{\frac{1}{2}}(\partial G)}^2\big).$$

We observe that the constant c in (10.1.54) does not depend on ε. Therefore we can perform the passage to the limit as $\varepsilon \to +0$ by the Fatou theorem. Indeed, we apply Lemma 10.21, (10.1.41) and use the continuity of $\mathcal{A}(r)$ and $\mathcal{A}(0) = 0$. Thus, we get

$$(10.1.55) \quad \int_G (r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)dx \le c\big(\|u\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}_\alpha^0(G)}^2 +$$

$$+ \|g\|_{\overset{\circ}{W}_\alpha^{\frac{1}{2}}(\partial G)}^2\big).$$

Now from (10.1.37) we obtain

$$
\text{(10.1.56)} \quad \int\limits_{G_\varepsilon^{2\varepsilon}} r^\alpha |D^2 u|^2 dx \le c \int\limits_{G_{\varepsilon/2}^{5\varepsilon/2}} \left(r^\alpha f^2 + r^{\alpha-4} u^2 \right) dx +
$$

$$
+ C_2 \inf \int\limits_{G_{\varepsilon/2}^{5\varepsilon/2}} (r^\alpha |\nabla \mathcal{G}|^2 + r^{\alpha-2} |\mathcal{G}|^2) dx.
$$

Let $\varepsilon = 2^{-k} d$, $(k = 0, 1, 2, ...)$ and let us sum the obtained inequalities over all k. Then we have

$$
\text{(10.1.57)} \quad \int\limits_{G_0^d} \left(r^\alpha u_{xx}^2 + r^{\alpha-2} |\nabla u|^2 \right) dx \le C_3 \int\limits_{G_0^{2d}} r^{\alpha-4} u^2 dx +
$$

$$
+ C_4 \|g\|^2_{\overset{\circ}{W}{}_\alpha^{1/2}(\Gamma_0^{2d})} + C_5 \|f\|^2_{\overset{\circ}{W}{}_\alpha^0(G_0^{2d})}.
$$

From (10.1.55), (10.1.57), and (10.1.51) we deduce the validity of our theorem. $\qquad\square$

THEOREM 10.22. *Let $u(x)$ be a strong solution of the problem (LRP). Let $N \ge 3$ and assumptions (a)-(c) be satisfied. Suppose, in addition, that $g(x) \in \overset{\circ}{W}{}_2^{\frac{1}{2}}(\partial G)$. Then $u(x) \in \overset{\circ}{W}{}_2^2(G)$ and*

$$
\text{(10.1.58)} \quad \|u\|_{\overset{\circ}{W}{}_2^2(G)} \le C \big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}{}_2^0(G)} + \|g\|_{\overset{\circ}{W}{}_2^{\frac{1}{2}}(\partial G)} \big),
$$

where the constant $C > 0$ depends only on $\nu, \mu, N, \|a^i\|_{p,G}$, $i = 1, \dots, N$ and $\|a\|_{N,G}, \gamma_0, \|\gamma\|_{C^1(\partial G \setminus \mathcal{O})}$, the moduli of continuity of the coefficients a^{ij} and the domain G.

PROOF. We repeat verbatim the proof of Theorem 10.20 with $\alpha = 2$. Then from (10.1.53) and (10.1.40) we have

$$
\int\limits_{G_\varepsilon} |\nabla u|^2 dx \le c_4 \int\limits_{G_{\varepsilon/2}^{5\varepsilon/2}} \left\{ r^2 f^2 + r^2 |\nabla \mathcal{G}|^2 + |\mathcal{G}|^2 \right\} dx +
$$

$$
+ c\mathcal{A}(2\varepsilon)\varepsilon^{N-2} + \delta_1 \int\limits_{G_\varepsilon} |\nabla u|^2 dx +
$$

$$
\text{(10.1.59)}
$$

$$
+ \delta_2 \int\limits_{G_\varepsilon} r^{-2} u^2 dx + c_1 \varepsilon^{N-2} M^2(\varepsilon) +
$$

$$
+ c \big(\|u\|_{2,G}^2 + \|f\|^2_{\overset{\circ}{W}{}_2^0(G)} + \|g\|^2_{\overset{\circ}{W}{}_2^{\frac{1}{2}}(\partial G)} \big)
$$

for any $\delta_1 > 0$, $\delta_2 > 0$ and $0 < \varepsilon < d/2$. Now, since $N \geq 3$ we can estimate

$$\int\limits_{G_0^d} r^{-2}u^2 dx \leq |u|_{0,G}^2 \mathrm{meas}\Omega \int_0^d r^{N-3}dr < \frac{d^{N-2}}{N-2}\mathrm{meas}\Omega|u|_{0,G}^2.$$

Therefore, for $\delta_1 = \frac{1}{2}$ it follows from (10.1.59) that

$$
\begin{aligned}
\frac{1}{2}\int\limits_{G_\varepsilon} |\nabla u|^2 dx \leq c_4 \int\limits_{G_{\varepsilon/2}^{5\varepsilon/2}} &\left\{ r^2 f^2 + r^2|\nabla\mathcal{G}|^2 + |\mathcal{G}|^2 \right\}dx+\\
&+ c_1\varepsilon^{N-2}M^2(\varepsilon) + c\mathcal{A}(2\varepsilon)\varepsilon^{N-2}+\\
&+ c\big(|u|_{0,G}^2 + \|f\|^2_{\overset{\circ}{W}_2^0(G)} + \|g\|^2_{\overset{\circ}{W}_2^{\frac{1}{2}}(\partial G)}\big)
\end{aligned}
$$

(10.1.60)

for any $\varepsilon \in (0, d/2)$. Performing the passage to the limit as $\varepsilon \to +0$ by the Fatou theorem, we deduce the validity of our theorem. $\qquad\square$

Now we consider $\alpha = 4 - N$, $N \geq 2$. In order to do this, we turn to Theorem 10.19, based on Lemma 10.18 about the existence of the barrier function.

THEOREM 10.23. *Let u be a strong solution of the problem (LRP). Let assumptions (a)-(d) be satisfied. Suppose, in addition, that $g(x) \in \overset{\circ}{W}_{4-N}^{\frac{1}{2}}(\partial G)$ and $a(x) \in \overset{\circ}{W}_{4-N}^0(G), \gamma(x) \in \overset{\circ}{W}_{2-N}^{\frac{1}{2}}(\partial G)$, if $u(0) \neq 0$.*

Then $\big(u(x) - u(0)\big) \in \overset{\circ}{W}_{4-N}^2(G)$ and

$$
(10.1.61) \quad \left(\int\limits_{\partial G} r^{1-N}\gamma(x)|u(x) - u(0)|^2 ds\right)^{\frac{1}{2}} + \|u(x) - u(0)\|_{\overset{\circ}{W}_{4-N}^2(G)} \leq
$$

$$
\leq C\Bigg(|u|_{0,G} + \|f\|_{\overset{\circ}{W}_{4-N}^0(G)} + |u(0)| \cdot \Big(1 + \|\gamma\|_{\overset{\circ}{W}_{2-N}^{1/2}(\partial G)}+
$$

$$
+ \|a\|_{\overset{\circ}{W}_{4-N}^0(G)}\Big) + \|g\|_{\overset{\circ}{W}_{4-N}^{1/2}(\partial G)}\Bigg),
$$

where the constant $C > 0$ depends only on $\nu, \mu, N, \|a^i\|_{p,G}$, $i = 1, \ldots, N$ and $\|a\|_{N,G}, \gamma_0, \|\gamma\|_{C^1(\partial G\setminus\mathcal{O})}$, the moduli of continuity of the coefficients a^{ij} and the domain G.

PROOF. Setting $v(x) = u(x) - u(0)$ we have $v \in C^0(\overline{G})$, $v(0) = 0$ and v is a strong solution of the problem

$$(LRP)_0 \quad \begin{cases} a^{ij}(x)v_{x_i x_j} + a^i(x)v_{x_i} + a(x)v = f(x) - a(x)u(0) \equiv \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \equiv f_0(x), \ x \in G, \\ \frac{\partial v}{\partial \vec{n}} + \frac{1}{|x|}\gamma(x)v = g(x) - \frac{1}{|x|}\gamma(x)u(0) \equiv g_0(x), \ x \in \partial G \setminus \mathcal{O}. \end{cases}$$

We repeat verbatim the arguments of the proof of Theorem 10.20 with $\alpha = 4 - N$. Then from (10.1.34) with regard to $a(x) \leq 0$ we have

$$(10.1.62) \quad \int\limits_{G_\varepsilon} r^{2-N}|\nabla v|^2 dx + \int\limits_{\Gamma_\varepsilon} r^{1-N}\gamma(x)v^2 ds \leq \varepsilon^{2-N} \left| \int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r}d\Omega_\varepsilon \right| +$$

$$+ \int\limits_{\Gamma_\varepsilon} r^{2-N}g(x)vds + |u(0)| \int\limits_{\Gamma_\varepsilon} r^{1-N}\gamma(x)|v|ds + \frac{N-2}{2}\int\limits_{\Gamma_d} r^{1-N}v^2\frac{\partial r}{\partial\vec{n}}ds$$

$$+ \int\limits_{G_\varepsilon} r^{2-N}v\left(-f(x)+u(0)a(x)+\left(a^{ij}(x) - a^{ij}(0)\right)D_{ij}v(x)+a^i(x)D_iv(x)\right)dx.$$

We estimate each term of (10.1.62). First (10.1.40) has the form

$$(10.1.63) \quad \varepsilon^{2-N}\int\limits_{\Omega_\varepsilon}\left|v\frac{\partial v}{\partial r}\right|d\Omega_\varepsilon \leq c_1 \max_{x \in \Omega_\varepsilon}|u(x) - u(0)|^2 +$$

$$+ c_3\int\limits_{G^{5\varepsilon/2}_{\varepsilon/2}}\left\{r^{4-N}f^2 + r^{4-N}|\nabla\mathcal{G}|^2 + r^{2-N}|\mathcal{G}|^2\right\}dx +$$

$$+ c_4 u^2(0)\int\limits_{G^{5\varepsilon/2}_{\varepsilon/2}}\left\{r^{4-N}a^2(x) + r^{2-N}|\nabla\gamma|^2 + r^{-N}\gamma^2(x)\right\}dx.$$

By the hypotheses of our theorem, we get

$$(10.1.64) \qquad \lim_{\varepsilon\to+0}\varepsilon^{2-N}\int\limits_{\Omega_\varepsilon} v\frac{\partial v}{\partial r}d\Omega_\varepsilon = 0.$$

Using the Cauchy inequality we get

$$(10.1.65) \quad |u(0)|\int\limits_{\Gamma_\varepsilon} r^{1-N}\gamma(x)|v|ds \leq \frac{\delta}{2}\int\limits_{\Gamma_\varepsilon} r^{1-N}\gamma(x)|v|^2 ds +$$

$$+ \frac{1}{2\delta}|u(0)|^2\int\limits_{\Gamma_\varepsilon} r^{1-N}\gamma(x)ds, \ \forall\delta > 0.$$

Since $\gamma(x) \geq \gamma_0 > 0$ and because of (10.1.52),

$$(10.1.66) \qquad \int_{\Gamma_\varepsilon} r^{1-N} \gamma(x) ds \leq \frac{1}{\gamma_0} \int_{\Gamma_\varepsilon} r^{1-N} \gamma^2(x) ds \leq \frac{c}{\gamma_0} \|\gamma\|^2_{\overset{\circ}{W}_{2-N}^{1/2}(\partial G)}.$$

From (10.1.62) and (10.1.65) (with $\delta = 1$) and (10.1.66) it follows that

$$(10.1.67) \qquad \int_{G_\varepsilon} r^{2-N} |\nabla v|^2 dx + \frac{1}{2} \int_{\Gamma_\varepsilon} r^{1-N} \gamma(x) v^2 ds \leq \varepsilon^{2-N} \left| \int_{\Omega_\varepsilon} v \frac{\partial v}{\partial r} d\Omega_\varepsilon \right| +$$

$$+ \int_{\Gamma_\varepsilon} r^{2-N} g(x) v ds + |u(0)|^2 \frac{c}{2\gamma_0} \|\gamma\|^2_{\overset{\circ}{W}_{2-N}^{1/2}(\partial G)} + \frac{N-2}{2} \int_{\Gamma_d} r^{1-N} v^2 \frac{\partial r}{\partial \overrightarrow{n}} ds$$

$$+ \int_{G_\varepsilon} r^{2-N} v \Big(-f(x) + u(0) a(x) + \big(a^{ij}(x) - a^{ij}(0)\big) D_{ij} v(x) + a^i(x) D_i v(x) \Big) dx.$$

Taking into account the estimates (10.1.42), (10.1.43) (with $\delta = 1$), (10.1.44), (10.1.45), (10.1.49), (10.1.51), (10.1.52) we obtain

$$(10.1.68) \qquad \frac{1}{2} \int_{\Gamma_\varepsilon} r^{1-N} \gamma(x) |v|^2 ds + \int_{G_\varepsilon} r^{2-N} |\nabla v|^2 dx \leq \varepsilon^{2-N} \left| \int_{\Omega_\varepsilon} v \frac{\partial v}{\partial r} d\Omega_\varepsilon \right| +$$

$$+ c |u(0)|^2 \Big(\|\gamma\|^2_{\overset{\circ}{W}_{2-N}^{1/2}(\partial G)} + \|a\|^2_{\overset{\circ}{W}_{4-N}^0(G)} \Big) +$$

$$+ c \mathcal{A}(2\varepsilon) \Big(1 + \int_{G_{\varepsilon/2}^{5\varepsilon/2}} \big(r^{4-N} f^2 +$$

$$+ r^{4-N} |\nabla \mathcal{G}|^2 + r^{2-N} |\mathcal{G}|^2 \big) dx \Big) +$$

$$+ \delta_1 \int_{G_\varepsilon} r^{2-N} |\nabla v|^2 dx + \delta_2 \int_{G_\varepsilon} r^{-N} v^2 dx +$$

$$+ c \Big(\|v\|^2_{2,G} + \|f\|^2_{\overset{\circ}{W}_{4-N}^0(G)} + \|g\|^2_{\overset{\circ}{W}_{4-N}^{\frac{1}{2}}(\partial G)} \Big)$$

for any $\delta_1, \delta_2 > 0$ and $0 < \varepsilon < d/2$.

Finally, we apply Theorem 10.19. The assumptions of our theorem guarantee the fulfilment of all suppositions of this theorem. Therefore we

can estimate

$$\int\limits_{G_0^d} r^{-N}v^2 dx \le C_0^2 \cdot \text{meas}\Omega \int_0^d r^{2\varkappa+1}dr \le cd^{2\varkappa+2}, \ \varkappa > 0; \Longrightarrow$$

(10.1.69)
$$\int\limits_G r^{-N}v^2 dx < \infty.$$

Now choosing $\delta_1 = \frac{1}{2}$, because of (10.1.69), we may perform the passage to the limit as $\varepsilon \to +0$ by the Fatou Theorem in (10.1.68). By (10.1.64), we get

(10.1.70)
$$\int\limits_G r^{2-N}|\nabla v|^2 dx + \int\limits_{\partial G} r^{1-N}\gamma(x)v^2 ds \le$$

$$\le \delta \int\limits_G r^{-N}u^2 dx + c_1\left(\|u\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}{}_{4-N}^0(G)}^2 + \right.$$

$$\left. + \|g\|_{\overset{\circ}{W}{}_{4-N}^{\frac{1}{2}}(\partial G)}^2\right) + c_2 u^2(0)\left(\|a\|_{\overset{\circ}{W}{}_{4-N}^0(G)}^2 + \|\gamma\|_{\overset{\circ}{W}{}_{2-N}^{1/2}(\partial G)}^2\right)$$

for any $\delta > 0$.

From (10.1.69) and (10.1.70) it follows that $v \in \overset{\circ}{W}{}_{2-N}^1(G)$; moreover, $v(0) = 0$. This makes possible to apply the Hardy-Friedrichs-Wirtinger inequality (2.5.12). Therefore choosing appropriatly small $\delta > 0$ we deduce from (10.1.70) the inequality

$$\int\limits_G \left(r^{2-N}|\nabla v|^2 + r^{-N}v^2\right) dx + + \int\limits_{\partial G} r^{1-N}\gamma(x)v^2 ds \le c_1\left(\|v\|_{2,G}^2 + \right.$$

(10.1.71)

$$\left. + \|f\|_{\overset{\circ}{W}{}_{4-N}^0(G)}^2 + \|g\|_{\overset{\circ}{W}{}_{4-N}^{\frac{1}{2}}(\partial G)}^2\right) + c_2 u^2(0)\left(\|a\|_{\overset{\circ}{W}{}_{4-N}^0(G)}^2 + \|\gamma\|_{\overset{\circ}{W}{}_{2-N}^{1/2}(\partial G)}^2\right).$$

Finally, putting in (10.1.57) $\alpha = 4 - N$ and replacing f by f_0 and g by g_0 from the $(LRP)_0$ we obtain

$$\int\limits_{G_0^d} r^{4-N}v_{xx}^2 dx \le C_3 \int\limits_{G_0^{2d}} r^{-N}v^2 dx + C_4\|g\|_{\overset{\circ}{W}{}_\alpha^{1/2}(\Gamma_0^{2d})}^2 +$$

(10.1.72)
$$+ C_5\|f\|_{\overset{\circ}{W}{}_\alpha^0(G_0^{2d})}^2 + C_6 u^2(0)\left(\|a\|_{\overset{\circ}{W}{}_{4-N}^0(G)}^2 + \right.$$

$$\left. + \|\gamma\|_{\overset{\circ}{W}{}_{2-N}^{1/2}(\partial G)}^2\right).$$

From (10.1.71) and (10.1.72) follows the desired estimate (10.1.61). □

THEOREM 10.24. *Let u be a strong solution of the problem (LRP) and λ be as above (see (2.5.11) or (2.5.19))* . *Let assumptions (a)-(d) with $\beta > \lambda - 2$ be satisfied. Suppose, in addition, that $g(x) \in \overset{\circ}{W}^{\frac{1}{2}}_\alpha(\partial G)$, where*

$$4 - N - 2\lambda < \alpha < 4 - N$$

and $a(x) \in \overset{\circ}{W}^0_\alpha(G), \gamma(x) \in \overset{\circ}{W}^{\frac{1}{2}}_{\alpha-2}(\partial G)$, if $u(0) \neq 0$.

Then $\left(u(x) - u(0)\right) \in \overset{\circ}{W}^2_\alpha(G)$ and

$$(10.1.73) \quad \left(\int\limits_{\partial G} r^{\alpha-3}\gamma(x)\left(u(x) - u(0)\right)^2 ds\right)^{\frac{1}{2}} + \|u(x) - u(0)\|_{\overset{\circ}{W}^2_\alpha(G)} \le$$

$$\le C\left(|u|_{0,G} + \|f\|_{\overset{\circ}{W}^0_\alpha(G)} + \|g\|_{\overset{\circ}{W}^{1/2}_\alpha(\partial G)} + \right.$$

$$\left. + |u(0)|\left(1 + \|a\|_{\overset{\circ}{W}^0_\alpha(G)} + \|\gamma\|_{\overset{\circ}{W}^{1/2}_{\alpha-2}(G)}\right)\right),$$

where the constant $C > 0$ depends only on $\nu, \mu, \alpha, N, \|a^i\|_{p,G}$, $i = 1, \ldots, N$; $\|a\|_{N,G}, \lambda, \gamma_0, \|\gamma\|_{C^1(\partial G \setminus \mathcal{O})}$, the moduli of continuity of the coefficients a^{ij} and the domain G.

PROOF. We consider the function $v(x) = u(x) - u(0)$ which satisfies the problem $(LRP)_0$ and multiply both sides of the equation of the $(LRP)_0$ by $r_\varepsilon^{\alpha-2}v(x)$ and integrate over G. We obtain:

$$(10.1.74) \quad \int\limits_G r_\varepsilon^{\alpha-2}v\triangle v dx = \int\limits_G r_\varepsilon^{\alpha-2}v\{f(x) - a(x)u(0) - $$

$$-\langle(a^{ij}(x) - a^{ij}(0))v_{x_ix_j} + a^i(x)v_{x_i} + a(x)v\rangle\}, \forall \varepsilon > 0.$$

We transform the integral from the left in (10.1.74) by the Gauss-Ostrogradskiy formula

$$(10.1.75) \quad \int\limits_G r_\varepsilon^{\alpha-2}v\triangle v dx = \int\limits_{\partial G} r_\varepsilon^{\alpha-2}v\frac{\partial v}{\partial \vec{n}}ds - \int\limits_G r_\varepsilon^{\alpha-2}|\nabla v|^2 dx + $$

$$+ \frac{2-\alpha}{2}\int\limits_G r_\varepsilon^{\alpha-3}\frac{\partial v^2}{\partial x_i}\frac{\partial r_\varepsilon}{\partial x_i}dx.$$

Because of the boundary condition of the $(LRP)_0$, we obtain

$$(10.1.76) \quad \int\limits_G r_\varepsilon^{\alpha-2} v \Delta v dx = \frac{2-\alpha}{2} \int\limits_G r_\varepsilon^{\alpha-3} \frac{\partial v^2}{\partial x_i} \frac{\partial r_\varepsilon}{\partial x_i} dx - $$

$$- \int\limits_G r_\varepsilon^{\alpha-2} |\nabla v|^2 dx + \int\limits_{\partial G} r_\varepsilon^{\alpha-2} v \{ g(x) - \frac{1}{r} \gamma(x) u(0) - \frac{1}{r} \gamma(x) v \} ds, \quad \forall \varepsilon > 0.$$

Now we transform the second integral from the right in (10.1.76). For this we use the Gauss-Ostrogradskiy formula once more

$$(10.1.77) \quad \int\limits_G r_\varepsilon^{\alpha-3} \frac{\partial r_\varepsilon}{\partial x_i} \frac{\partial v^2}{\partial x_i} dx = \int\limits_{\partial G} r_\varepsilon^{\alpha-3} v^2 \frac{\partial r_\varepsilon}{\partial x_i} \cos(\vec{n}, x_i) ds - $$

$$- \int\limits_G v^2 \frac{\partial}{\partial x_i} \left(r_\varepsilon^{\alpha-3} \frac{\partial r_\varepsilon}{\partial x_i} \right) dx.$$

Because of $\frac{\partial r_\varepsilon}{\partial x_1} = \frac{x_1 + \varepsilon}{r_\varepsilon}$, $\frac{\partial r_\varepsilon}{\partial x_i} = \frac{x_i}{r_\varepsilon}$ $(i \geq 2)$, $\partial G = \Gamma_0^d \cup \Gamma_d$ and by (1.3.14), we obtain

$$(10.1.78) \quad \int\limits_{\partial G} r_\varepsilon^{\alpha-3} v^2 \frac{\partial r_\varepsilon}{\partial x_i} \cos(\vec{n}, x_i) ds = -\varepsilon \sin \frac{\omega_0}{2} \int\limits_{\Gamma_0^d} r_\varepsilon^{\alpha-4} v^2 ds + $$

$$+ \int\limits_{\Gamma_d} r_\varepsilon^{\alpha-3} v^2 \frac{\partial r_\varepsilon}{\partial \vec{n}} ds.$$

However, by the fourth property of r_ε, we have

$$(10.1.79) \quad - \int\limits_G v^2 \frac{\partial}{\partial x_i} \left(r_\varepsilon^{\alpha-3} \frac{\partial r_\varepsilon}{\partial x_i} \right) dx = (4 - N - \alpha) \int\limits_G r_\varepsilon^{\alpha-4} v^2 dx.$$

From (10.1.77)-(10.1.79) it follows that

$$(10.1.80) \quad \frac{2-\alpha}{2} \int\limits_G r_\varepsilon^{\alpha-3} \frac{\partial v^2}{\partial x_i} \frac{\partial r_\varepsilon}{\partial x_i} dx = \frac{2-\alpha}{2} \int\limits_{\Gamma_d} r_\varepsilon^{\alpha-3} v^2 \frac{\partial r_\varepsilon}{\partial \vec{n}} ds - $$

$$- \frac{2-\alpha}{2} \varepsilon \sin \frac{\omega_0}{2} \int\limits_{\Gamma_0^d} r_\varepsilon^{\alpha-4} v^2 ds + \frac{(2-\alpha)(4-N-\alpha)}{2} \int\limits_G r_\varepsilon^{\alpha-4} v^2 dx, \quad \forall \varepsilon > 0.$$

From (10.1.74), (10.1.75) and (10.1.80) with regard to $a(x) \leq 0$ we obtain the following equality

$$(10.1.81) \quad \int\limits_G r_\varepsilon^{\alpha-2} |\nabla v|^2 dx + \varepsilon \frac{2-\alpha}{2} \sin\frac{\omega_0}{2} \int\limits_{\Gamma_0^d} r_\varepsilon^{\alpha-4} v^2 ds +$$

$$+ \int\limits_{\partial G} r_\varepsilon^{\alpha-2} \frac{1}{r} \gamma(x) v^2 ds = \frac{(2-\alpha)(4-\alpha-N)}{2} \int\limits_G r_\varepsilon^{\alpha-4} v^2 dx +$$

$$+ \int\limits_G r_\varepsilon^{\alpha-2} v \left\{ \left(a^{ij}(x) - a^{ij}(0)\right) v_{x_i x_j} + a^i(x) v_{x_i} + a(x) u(0) - f(x) \right\} dx +$$

$$+ \frac{2-\alpha}{2} \int\limits_{\Gamma_d} r_\varepsilon^{\alpha-3} v^2 \frac{\partial r_\varepsilon}{\partial \vec{n}} ds + \int\limits_{\partial G} r_\varepsilon^{\alpha-2} v g(x) ds - u(0) \int\limits_{\partial G} r_\varepsilon^{\alpha-2} \frac{1}{r} \gamma(x) v ds, \; \forall \varepsilon > 0.$$

Now we estimate the integral over Γ_d. Since we have of on Γ_d that $r_\varepsilon \geq hr \geq hd \Rightarrow (\alpha-3)\ln r_\varepsilon \leq (\alpha-3)\ln(hd)$, and since $\alpha < 2$, we have $r_\varepsilon^{\alpha-3}|_{\Gamma_d} \leq (hd)^{\alpha-3}$ and therefore

$$(10.1.82) \quad \frac{2-\alpha}{2} \int\limits_{\Gamma_d} r_\varepsilon^{\alpha-3} v^2 \frac{\partial r_\varepsilon}{\partial \vec{n}} ds \leq \frac{2-\alpha}{2} (hd)^{\alpha-3} \int\limits_{\Gamma_d} v^2 ds.$$

By (1.6.2), we obtain

$$(10.1.83) \quad \int\limits_{\Gamma_d} v^2 ds \leq C_\delta \int\limits_{G_d} v^2 dx + \delta \int\limits_{G_d} |\nabla v|^2 dx, \; \forall \delta > 0.$$

By the Cauchy inequality,

$$vg = \left(r^{\frac{1}{2}} \frac{1}{\sqrt{\gamma(x)}} |g|\right) \left(r^{-\frac{1}{2}} \sqrt{\gamma(x)} |v|\right) \leq \frac{\delta}{2} r^{-1} \gamma(x) v^2 + \frac{1}{2\delta\gamma_0} r g^2, \; \forall \delta > 0.$$

Taking into account property 1) of r_ε, we obtain

$$(10.1.84) \quad \int\limits_{\partial G} r_\varepsilon^{\alpha-2} |v| |g| d\sigma \leq \frac{\delta}{2} \int\limits_{\partial G} r_\varepsilon^{\alpha-2} \frac{1}{r} \gamma(x) v^2 ds +$$

$$+ \frac{1}{2\delta\gamma_0} h^{\alpha-2} \int\limits_{\partial G} r^{\alpha-1} g^2 ds, \; \forall \delta > 0.$$

From assumptions $(a) - (b)$ we have

$$\max_{\Gamma_0^d} |a^{ij}(x) - a^{ij}(0)| \leq \mathcal{A}(d), \quad \text{and} \quad \max_{\Gamma_d} |a^{ij}(x) - a^{ij}(0)| \leq 1 + \mu.$$

Hence by the Cauchy inequality and assumption (b), we obtain

$$(10.1.85) \quad \int\limits_{G_0^d} r_\varepsilon^{\alpha-2} v\big\{ \left(a^{ij}(x) - a^{ij}(0)\right) v_{x_i x_j} + |x| a^i(x) r^{-1} v_{x_i} \big\} dx \le$$

$$\le \mathcal{A}(d) C_1(N) \int\limits_{G_0^d} \left(r^2 r_\varepsilon^{\alpha-2} v_{xx}^2 + r_\varepsilon^{\alpha-2} |\nabla v|^2 + r^{-2} r_\varepsilon^{\alpha-2} v^2 \right) dx.$$

Similarly, we have

$$(10.1.86) \quad \int\limits_{G_d} r_\varepsilon^{\alpha-2} v\big\{ (a^{ij}(x) - a^{ij}(0)) v_{x_i x_j} + a^i(x) v_{x_i} \big\} dx \le$$

$$\le C_2(N, diam G)(hd)^{\alpha-2} \int\limits_{G_d} \left(v_{xx}^2 + |\nabla v|^2 \right) dx.$$

Further, from the Cauchy inequality we obtain

$$(10.1.87) \quad \int\limits_G r_\varepsilon^{\alpha-2} v f(x) dx \le \frac{\delta}{2} \int\limits_G r^{-2} r_\varepsilon^{\alpha-2} v^2 dx +$$

$$+ \frac{1}{2\delta} \int\limits_G r^2 r_\varepsilon^{\alpha-2} f^2(x) dx, \ \forall \delta > 0;$$

$$(10.1.88) \quad |u(0)| \int\limits_{\partial G} r_\varepsilon^{\alpha-2} r^{-1} \gamma(x) |v| ds \le \frac{\delta}{2} \int\limits_{\partial G} r_\varepsilon^{\alpha-2} r^{-1} \gamma(x) |v|^2 ds +$$

$$+ \frac{1}{2\delta} |u(0)|^2 \int\limits_{\partial G} r_\varepsilon^{\alpha-2} r^{-1} \gamma(x) ds, \ \forall \delta > 0;$$

$$(10.1.89) \quad \int\limits_G r_\varepsilon^{\alpha-2} v(x) u(0) a(x) dx \le \frac{\delta}{2} \int\limits_G r^{-2} r_\varepsilon^{\alpha-2} v^2 dx +$$

$$+ \frac{1}{2\delta} |u(0)|^2 \int\limits_G r^2 r_\varepsilon^{\alpha-2} a^2(x) dx, \ \forall \delta > 0.$$

As a result from (10.1.81)-(10.1.89) we obtain with $\forall \delta > 0$:

$$(10.1.90) \quad \int\limits_G r_\varepsilon^{\alpha-2} |\nabla v|^2 dx + \int\limits_{\partial G} r^{-1} r_\varepsilon^{\alpha-2} \gamma(x) v^2 ds \le$$

$$\leq \frac{(2-\alpha)(4-\alpha-N)}{2}\int\limits_{G} r_{\varepsilon}^{\alpha-4}v^2 dx + \delta\int\limits_{\partial G} r^{-1}r_{\varepsilon}^{\alpha-2}\gamma(x)v^2 ds +$$

$$+\frac{1}{2\gamma_0}h^{\alpha-2}\int\limits_{\partial G} r^{\alpha-1}g^2 ds + \mathcal{A}(d)C_3(\delta,\lambda,N)\int\limits_{G_0^d}\Big(r^2 r_{\varepsilon}^{\alpha-2}v_{xx}^2 + r_{\varepsilon}^{\alpha-2}|\nabla v|^2 +$$

$$+r^{-2}r_{\varepsilon}^{\alpha-2}v^2\Big)dx + C_\delta\int\limits_{G} r^2 r_{\varepsilon}^{\alpha-2}f^2(x)dx + \delta\int\limits_{G}\Big(r_{\varepsilon}^{\alpha-2}|\nabla v|^2 + r^{-2}r_{\varepsilon}^{\alpha-2}v^2\Big)dx +$$

$$+C_4(\alpha,h,d,\operatorname{diam}G)\int\limits_{G_d}\Big(v_{xx}^2 + |\nabla v|^2\Big)dx + C_\delta|u(0)|^2\Big(\int\limits_{\partial G} r_{\varepsilon}^{\alpha-2}r^{-1}\gamma(x)ds +$$

$$+\int\limits_{G} r^2 r_{\varepsilon}^{\alpha-2}a^2(x)dx\Big).$$

Now we consider two sets $G_{\rho/4}^{2\rho}$ and $G_{\rho/2}^{\rho} \subset G_{\rho/4}^{2\rho}$, $\rho > 0$. We make the coordinate transformation $x = \rho x'$. The function $z(x') = v(\rho x')$ in $G_{1/4}^2$ satisfies the equation

$$(LRP)'' \qquad \begin{cases} a^{ij}(\rho x')z_{x_i' x_j'} + \rho a^i(\rho x')z_{x_i'} + \rho^2 a(\rho x')z = \rho^2 f(\rho x'), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x' \in G_{1/4}^2 \\ \frac{\partial z}{\partial \vec{n}'} + \frac{1}{|x'|}\gamma(\rho x')z = \rho g(\rho x'), \quad x' \in \Gamma_{1/4}^2. \end{cases}$$

Because of the interior and near a smooth portion of the boundary L^2−estimates, Theorem 10.17, for the equation of the $(LRP)''$ solution, we have:

$$\int\limits_{G_{1/2}^1}\Big(z_{x'x'}^2 + |\nabla' z|^2\Big)dx' \leq C_5\int\limits_{G_{1/4}^2}\Big(\rho^4 f^2 + z^2\Big)dx' +$$

$$+ C_6\varrho^2\inf\int\limits_{G_{1/4}^2}\Big(|\nabla'\mathcal{G}|^2 + |\mathcal{G}|^2\Big)dx',$$

where infimum is taken over all $\mathcal{G}$ such that $\mathcal{G}\big|_{\Gamma_{1/4}^2} = g$ and the constants $C_5, C_6 > 0$ depend only on ν, μ, $\max\limits_{x' \in G_{1/4}^2}\mathcal{A}(|x'|)$, $\|\gamma\|_{C^1(\Gamma_{1/4}^2)}$ and the domain G. Multiplying both sides of this inequality by $(\varrho + \varepsilon)^{\alpha-2}$ and returning to

the variable x, we obtain

$$\int\limits_{G^\rho_{\rho/2}} \Big\langle \varrho^2(\varrho+\varepsilon)^{\alpha-2} v_{xx}^2 + (\varrho+\varepsilon)^{\alpha-2}|\nabla v|^2 \Big\rangle dx \leq C_5 \int\limits_{G^{2\rho}_{\rho/4}} \Big\langle (\varrho^2(\varrho+\varepsilon)^{\alpha-2} f^2 +$$

$$+ \varrho^{-2}(\varrho+\varepsilon)^{\alpha-2} v^2 \Big\rangle dx + C_6(\varrho+\varepsilon)^{\alpha-2} \inf \int\limits_{G^{2\varrho}_{\varrho/4}} (\varrho^2|\nabla\mathcal{G}|^2 + |\mathcal{G}|^2) dx, \; \forall \varepsilon > 0.$$

Now, in the domain $G^\rho_{\rho/2}$, we have

$$\frac{\varrho}{2} < r < \varrho \Rightarrow r < \varrho < 2r \Rightarrow \varrho + \varepsilon < 2r + \varepsilon \leq \frac{3}{h} r_\varepsilon \text{ by the property 1) of } r_\varepsilon$$

$$\Rightarrow \quad (\varrho+\varepsilon)^{\alpha-2} \geq (3h^{-1})^{\alpha-2} r_\varepsilon^{\alpha-2}, \text{ since } \alpha < 2.$$

Similarly in the domain $G^{2\rho}_{\rho/4}$ we have

$$\frac{\varrho}{4} < r < 2\varrho \Rightarrow \frac{1}{2}r < \varrho < 4r \Rightarrow \varrho + \varepsilon \geq \frac{1}{2}r + \varepsilon > \frac{1}{2}(r+\varepsilon) \geq \frac{1}{2}r_\varepsilon \Rightarrow$$

$$(\varrho+\varepsilon)^{\alpha-2} \leq 2^{2-\alpha} r_\varepsilon^{\alpha-2}, \text{ since } \alpha < 2.$$

Thus we obtain

$$\int\limits_{G^\rho_{\rho/2}} (r^2 r_\varepsilon^{\alpha-2} v_{xx}^2 + r_\varepsilon^{\alpha-2}|\nabla v|^2) dx \leq$$

$$\leq C_7(h,\alpha) \Big\{ C_5 \int\limits_{G^{2\rho}_{\rho/4}} (r^\alpha f^2 + r^{-2} r_\varepsilon^{\alpha-2} v^2) dx +$$

$$+ C_6 \inf \int\limits_{G^{2\varrho}_{\varrho/4}} (r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}|\mathcal{G}|^2) dx \Big\}, \; \forall \varepsilon > 0.$$

Let $\rho = 2^{-k}d$, $(k = 0,1,2,...)$ and let us sum the obtained inequalities over all k. Then we have

$$(10.1.91) \quad \int\limits_{G^d_0} \left(r^2 r_\varepsilon^{\alpha-2} v_{xx}^2 + r_\varepsilon^{\alpha-2}|\nabla v|^2 \right) dx \leq C_8 \int\limits_{G^{2d}_0} r^{-2} r_\varepsilon^{\alpha-2} v^2 dx +$$

$$+ C_9 \|g\|^2_{W_\alpha^{1/2}(\Gamma^{2d}_0)} + C_{10}\|f\|^2_{\overset{\circ}{W}_\alpha^0(G^{2d}_0)}, \; \forall \varepsilon > 0.$$

Finally, we use once more the interior and near a smooth portion of the boundary L^2- estimate for the equation (L) solution. We obtain analogously

$$(10.1.92) \quad \int_{G_d} \left(v_{xx}^2 + |\nabla v|^2 \right) dx \le C_{11} \int_{G_{d/2}} \left(f^2 + v^2 \right) dx +$$

$$+ C_{12}\|g\|^2_{\overset{\circ}{W}{}^{1/2}(\Gamma_{d/2})} \le C(d, \operatorname{diam} G)\left(\|f\|^2_{\overset{\circ}{W}{}^{0}_{\alpha}(G_{d/2})} + \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{\alpha}(\Gamma_{d/2})} \right) +$$

$$+ C_{11} \int_{G_{d/2}} v^2 \, dx.$$

Since $\alpha < 2$ and by the property 1) of r_ε we have $r_\varepsilon^{\alpha-2} \le r^{\alpha-2}$ and therefore with regard to (10.1.52) we get

$$(10.1.93) \quad \int_{\partial G} r_\varepsilon^{\alpha-2} r^{-1}\gamma(x)ds \le \int_{\partial G} r^{\alpha-3}\gamma(x)ds \le \frac{1}{\gamma_0} \int_{\partial G} r^{\alpha-3}\gamma^2(x)ds \le$$

$$\le \frac{c}{\gamma_0} \|\gamma\|^2_{\overset{\circ}{W}{}^{1/2}_{\alpha-2}(G)}.$$

From (10.1.90)-(10.1.93) we obtain

$$(10.1.94) \quad \int_{\partial G} r^{-1} r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds + \int_G (r^2 r_\varepsilon^{\alpha-2}v_{xx}^2 + r_\varepsilon^{\alpha-2}|\nabla v|^2)dx \le$$

$$\le \frac{(2-\alpha)(4-\alpha-N)}{2} \int_G r_\varepsilon^{\alpha-4}v^2 dx + \delta \int_{\partial G} r^{-1} r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds +$$

$$+ \left(A(d) + \delta \right) C_{13}\left(d, \lambda, N \right) \int_G (r_\varepsilon^{\alpha-2}|\nabla v|^2 + r^{-2} r_\varepsilon^{\alpha-2}v^2)dx +$$

$$+ C_{14}(\alpha, d, h, \delta, \gamma_0 \operatorname{diam} G)\left(\|v\|^2_{2,G} + \|f\|^2_{\overset{\circ}{W}{}^{0}_{\alpha}(G)} + \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{\alpha}(\partial G)} + \right.$$

$$\left. + |u(0)|^2 \left(\|a\|^2_{\overset{\circ}{W}{}^{0}_{\alpha}(G)} + \|\gamma\|^2_{\overset{\circ}{W}{}^{1/2}_{\alpha-2}(G)} \right) \right), \quad \forall \varepsilon > 0.$$

By our assumptions of the theorem, in virtue of the obvious embedding

$$\overset{\circ}{W}{}^{0}_{\alpha}(G) \hookrightarrow \overset{\circ}{W}{}^{0}_{\beta}(G), \overset{\circ}{W}{}^{1/2}_{\alpha}(\partial G) \hookrightarrow \overset{\circ}{W}{}^{1/2}_{\beta}(\partial G), \quad \forall \beta \ge \alpha,$$

we obtain

$$g(x) \in \overset{\circ}{W}{}^{\frac{1}{2}}_{4-N}(\partial G), \ a(x) \in \overset{\circ}{W}{}^{0}_{4-N}(G), \gamma(x) \in \overset{\circ}{W}{}^{\frac{1}{2}}_{2-N}(\partial G),$$

and therefore by Theorem 10.23 $v(x) \in \overset{\circ}{W}{}^2_{4-N}(G)$. But then we can apply Theorem 2.18 and according to (2.5.13) we have

$$\int_\Omega v^2(r,\omega)d\Omega \le \frac{1}{\lambda(\lambda + N - 2)}\Big\{\int_\Omega |\nabla_\omega v(r,\omega)|^2 d\Omega + \int_{\partial\Omega} \gamma(r,\omega)v^2 ds\Big\},$$

$$\text{for a.e. } r \in (0,d).$$

Multiplying both sides of this inequality by $(\varrho + \varepsilon)^{\alpha-2}r^{N-3}$ and integrating over $r \in (\frac{\varrho}{2}, \varrho)$ we obtain

$$\int_{G^\varrho_{\varrho/2}} (\varrho + \varepsilon)^{\alpha-2}r^{-2}v^2 dx \le \frac{1}{\lambda(\lambda + N - 2)}\Big\{\int_{G^\varrho_{\varrho/2}} (\varrho + \varepsilon)^{\alpha-2}|\nabla v|^2 dx +$$

$$+ \int_{\Gamma^\varrho_{\varrho/2}} r^{-1}(\varrho + \varepsilon)^{\alpha-2}\gamma(x)v^2 ds\Big\}, \quad \forall \varepsilon > 0$$

or since $\varrho + \varepsilon \sim r_\varepsilon$

$$\int_{G^\varrho_{\varrho/2}} r_\varepsilon^{\alpha-2}r^{-2}v^2 dx \le \frac{1}{\lambda(\lambda + N - 2)}\Big\{\int_{G^\varrho_{\varrho/2}} r_\varepsilon^{\alpha-2}|\nabla v|^2 dx +$$

$$+ \int_{\Gamma^\varrho_{\varrho/2}} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds\Big\}, \quad \forall \varepsilon > 0.$$

Letting $\rho = 2^{-k}d$, $(k = 0, 1, 2, ...)$ and summing the obtained inequalities over all k we get

(10.1.95) $$\int_{G^d_0} r_\varepsilon^{\alpha-2}r^{-2}v^2 dx \le \frac{1}{\lambda(\lambda + N - 2)}\Big\{\int_{G^d_0} r_\varepsilon^{\alpha-2}|\nabla v|^2 dx +$$

$$+ \int_{\Gamma^d_0} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds\Big\}, \quad \forall \varepsilon > 0.$$

Therefore from (10.1.94), (10.1.95) it follows that

(10.1.96) $$\int_{\partial G} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds + \int_G (r^2 r_\varepsilon^{\alpha-2}v_{xx}^2 + r_\varepsilon^{\alpha-2}|\nabla v|^2)dx \le$$

$$\leq \frac{(2-\alpha)\,(4-\alpha-N)}{2}\int\limits_{G} r_\varepsilon^{\alpha-4}v^2 dx +$$

$$+\,(\mathcal{A}(d)+\delta)C_{15}\,(d,\lambda,N)\left(\int\limits_{G} r_\varepsilon^{\alpha-2}|\nabla v|^2 dx + \int\limits_{\partial G} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds\right) +$$

$$+\,C_{14}(\alpha,d,h,\delta,\gamma_0,\operatorname{diam} G)\left(\|v\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}{}_\alpha^0(G)}^2 + \|g\|_{\overset{\circ}{W}{}_\alpha^{1/2}(\partial G)}^2 +\right.$$

$$\left.+\,|u(0)|^2\left(\|a\|_{\overset{\circ}{W}{}_\alpha^0(G)}^2 + \|\gamma\|_{\overset{\circ}{W}{}_{\alpha-2}^{1/2}(G)}^2\right)\right).$$

Finally, we use Lemma 2.37 and take into account that $r_\varepsilon \geq r$, because of the convexity of G_0^d. Then from (10.1.96) we get

$$\int\limits_{\partial G} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds + \int\limits_{G}(r^2 r_\varepsilon^{\alpha-2}v_{xx}^2 + r_\varepsilon^{\alpha-2}|\nabla v|^2)dx \leq$$

$$\leq \frac{2\,(2-\alpha)\,(4-\alpha-N)}{(4-N-\alpha)^2+4\lambda(\lambda+N-2)}\left\{\int\limits_{G} r_\varepsilon^{\alpha-2}|\nabla v|^2 dx +\right.$$

$$\left.+\int\limits_{\partial G} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds\right\} + C_{14}|u(0)|^2\left(\|a\|_{\overset{\circ}{W}{}_\alpha^0(G)}^2 + \|\gamma\|_{\overset{\circ}{W}{}_{\alpha-2}^{1/2}(G)}^2\right) +$$

$$+C_{15}\big(\mathcal{A}(d)+\delta\big)\left(\int\limits_{G} r_\varepsilon^{\alpha-2}|\nabla v|^2 dx + \int\limits_{\partial G} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)v^2 ds\right) +$$

$$(10.1.97)\qquad +C_{14}\left(\|v\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}{}_\alpha^0(G)}^2 + \|g\|_{\overset{\circ}{W}{}_\alpha^{1/2}(\partial G)}^2\right),\ \forall\varepsilon>0.$$

In our case, by $4-N-2\lambda < \alpha < 4-N$, we have

$$\frac{2\,(2-\alpha)\,(4-\alpha-N)}{(4-N-\alpha)^2+4\lambda(\lambda+N-2)} < 1$$

and therefore we can rewrite (10.1.97) in the form

$$\left(1-\frac{2\,(2-\alpha)\,(4-\alpha-N)}{(4-N-\alpha)^2+4\lambda(\lambda+N-2)}\right)\left\{\int\limits_{G} r_\varepsilon^{\alpha-2}|\nabla v|^2)dx +\right.$$

$$+ \int\limits_{\partial G} r^{-1} r_\varepsilon^{\alpha-2} \gamma(x) v^2 ds \Big\} + \int\limits_{G} r^2 r_\varepsilon^{\alpha-2} v_{xx}^2 dx \le$$

$$\le C_{15}\big(\mathcal{A}(d) + \delta\big) \left\{ \int\limits_{G} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx + \int\limits_{\partial G} r^{-1} r_\varepsilon^{\alpha-2} \gamma(x) v^2 ds \right\} +$$

$$+ C_{14}(\alpha, d, h, \delta, \gamma_0, \operatorname{diam} G) \bigg(\|v\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}{}^0_\alpha(G)}^2 + \|g\|_{\overset{\circ}{W}{}^{1/2}_\alpha(\partial G)}^2 +$$

$$+ |u(0)|^2 \Big(\|a\|_{\overset{\circ}{W}{}^0_\alpha(G)}^2 + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{\alpha-2}(G)}^2 \Big) \bigg).$$

In this case we choose

$$\delta = \frac{1}{4C_{15}} \left(1 - \frac{2(2-\alpha)(4-\alpha-N)}{(4-N-\alpha)^2 + 4\lambda(\lambda+N-2)} \right)$$

and next $d > 0$ such that, by the continuity of $\mathcal{A}(r)$ at zero,

$$C_{15}\mathcal{A}(d) \le \frac{1}{4} \left(1 - \frac{2(2-\alpha)(4-\alpha-N)}{(4-N-\alpha)^2 + 4\lambda(\lambda+N-2)} \right).$$

Thus we have

$$\int\limits_{G} r^2 r_\varepsilon^{\alpha-2} v_{xx}^2 dx + \int\limits_{G} r_\varepsilon^{\alpha-2} |\nabla v|^2 dx + \int\limits_{\partial G} r^{-1} r_\varepsilon^{\alpha-2} \gamma(x) v^2 ds \le$$

$$\le C_{16}(\alpha, d, h, \delta, \gamma_0, \operatorname{diam} G) \bigg(\|v\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}{}^0_\alpha(G)}^2 + \|g\|_{\overset{\circ}{W}{}^{1/2}_\alpha(\partial G)}^2 +$$

$$(10.1.98) \qquad + |u(0)|^2 \Big(\|a\|_{\overset{\circ}{W}{}^0_\alpha(G)}^2 + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{\alpha-2}(G)}^2 \Big) \bigg), \quad \forall \varepsilon > 0.$$

We observe that the right hand side of (10.1.98) does not depend on ε. Therefore we can perform the passage to the limit as $\varepsilon \to +0$ by the Fatou Theorem. Hence it follows that

$$(10.1.99) \qquad \int\limits_{\partial G} r^{\alpha-3} \gamma(x) v^2 ds + \int\limits_{G} (r^\alpha v_{xx}^2 + r^{\alpha-2} |\nabla v|^2) dx \le$$

$$\le C_{16}(\alpha, d, h, \delta, \gamma_0, \operatorname{diam} G) \bigg(\|v\|_{2,G}^2 + \|f\|_{\overset{\circ}{W}{}^0_\alpha(G)}^2 + \|g\|_{\overset{\circ}{W}{}^{1/2}_\alpha(\partial G)}^2 +$$

$$+ |u(0)|^2 \Big(\|a\|_{\overset{\circ}{W}{}^0_\alpha(G)}^2 + \|\gamma\|_{\overset{\circ}{W}{}^{1/2}_{\alpha-2}(G)}^2 \Big) \bigg).$$

Now, by the Hardy-Friedrichs-Wirtinger inequality (2.5.12), from (10.1.99) we get the desired estimate (10.1.73).   □

10.1.5. Local integral weighted estimates.

THEOREM 10.25. *Let* $u(x)$ *be a strong solution of the problem (LRP) and assumptions (a)-(d) be satisfied for* $\mathcal{A}(r)$ *being Dini continuous at zero. Suppose, in addition, that*

$$g(x) \in \mathring{W}^{1/2}_{4-N}(\partial G) \quad and$$

$$a(x) \in \mathring{W}^{0}_{4-N}(G), \ \gamma(x) \in \mathring{W}^{1/2}_{2-N}(\partial G), \ if\, u(0) \neq 0,$$

and there is k_s *from (10.1.1).*

Then $(u(x) - u(0)) \in \mathring{W}^{2}_{4-N}(G)$ *and there are* $d \in (0,1)$ *and a constant* $C > 0$ *depending only on* $\nu, \mu, d, \mathcal{A}(d), N, s, \lambda, \gamma_0, g_1, \|\gamma\|_{C^1(\partial G \setminus \mathcal{O})}, measG$ *and on the quantity* $\int_0^d \frac{\mathcal{A}(r)}{r} dr$, *such that* $\forall \varrho \in (0, d)$

$$\|u(x) - u(0)\|_{\mathring{W}^{2}_{4-N}(G_0^\varrho)} \leq C \Big(|u|_{0,G} + \|f\|_{\mathring{W}^{0}_{4-N}(G)} + \|g\|_{\mathring{W}^{1/2}_{4-N}(\partial G)} +$$

(10.1.100)

$$+|u(0)| \big(1 + \|a\|_{\mathring{W}^{0}_{4-N}(G)} + \|\gamma\|_{\mathring{W}^{1/2}_{2-N}(\partial G)}\big) + k_s \Big) \begin{cases} \varrho^\lambda, & if\, s > \lambda, \\ \varrho^\lambda \ln^{3/2}\big(\frac{1}{\varrho}\big), & if\, s = \lambda, \\ \varrho^s, & if\, s < \lambda. \end{cases}$$

PROOF. From Theorem 10.23 it follows that $v(x) = u(x) - u(0)$ belongs to $\mathring{W}^{2}_{4-N}(G)$, so it is enough to prove the estimate (10.1.100). We set

$$(10.1.101) \qquad V(\rho) \equiv \int\limits_{G_0^\rho} r^{2-N} |\nabla v|^2 dx + \int\limits_{\Gamma_0^\varrho} r^{1-N} \gamma(x) v^2 ds$$

and multiply both sides of the $(L)_0$ equation by $r^{2-N} v(x)$ and integrate over the domain $G_0^\rho, 0 < \rho < d$. As the result we obtain

$$(10.1.102) \quad V(\varrho) = \int\limits_\Omega \Big(\varrho v \frac{\partial v}{\partial r} + \frac{N-2}{2} v^2 \Big) d\Omega + \int\limits_{\Gamma_0^\varrho} r^{2-N} v g ds -$$

$$- u(0) \int\limits_{\Gamma_0^\varrho} r^{1-N} v \gamma(x) ds + \int\limits_{G_0^\varrho} r^{2-N} v \Big\{ \big(a^{ij}(x) - a^{ij}(0)\big) v_{x_i x_j} +$$

$$+ a^i(x) v_{x_i} + a(x) v - f(x) + u(0) a(x) \Big\} dx.$$

We shall obtain an upper bound for each integral on the right. According to Lemma 2.35, we estimate the first integral.

LEMMA 10.26.

$$\int\limits_{G_0^\varrho} r^{4-N} v_{xx}^2 dx \le C_1 \left(\nu, \mu, N, d, \mathcal{A}(d), g_1, \|\gamma\|_{C^1(\partial G \backslash \mathcal{O})}\right) \left(V(2\varrho)+\right.$$

$$(10.1.103) \qquad +\|f\|^2_{\overset{0}{W}_{4-N}(G_0^{2\varrho})} + \|g\|^2_{\overset{0}{W}^{1/2}_{4-N}(\Gamma_0^{2\varrho})}+$$

$$\left. +u^2(0)\left(\|a\|^2_{\overset{0}{W}_{4-N}(G_0^{2\varrho})} + \|\gamma\|^2_{\overset{0}{W}^{1/2}_{2-N}(\Gamma_0^{2\varrho})}\right)\right).$$

PROOF. The proof is analogous to reasoning deriving (10.1.57). □

Now we estimate the second integral in (10.1.102). By the Cauchy inequality and Lemma 1.40

$$(10.1.104) \quad \int\limits_{\Gamma_0^\varrho} r^{2-N} |v||g| ds \le \frac{\delta}{2} \int\limits_{\Gamma_0^\varrho} r^{1-N} \gamma(x) v^2 ds+$$

$$+ \frac{C_2}{\delta\gamma_0} \|g\|^2_{\overset{0}{W}^{1/2}_{4-N}(\Gamma_0^\varrho)}, \quad \forall \delta > 0.$$

By (10.1.65) and (10.1.66), we obtain

$$(10.1.105) \quad |u(0)| \int\limits_{\Gamma_0^\varrho} r^{1-N} \gamma(x) |v| ds \le \frac{\delta}{2} \int\limits_{\Gamma_0^\varrho} r^{1-N} \gamma(x) |v|^2 ds+$$

$$+ |u(0)|^2 \frac{C_2}{\delta\gamma_0} \|\gamma\|^2_{\overset{0}{W}^{1/2}_{2-N}(\Gamma_0^\varrho)}, \quad \forall \delta > 0.$$

To estimate the last integral in (10.1.102) we use the Cauchy inequality, (2.5.12) with $\alpha = 4 - N$ and with the assumption (b) regarding the equation coefficients. We get

$$(10.1.106) \quad \int\limits_{G_0^\varrho} r^{2-N} v\left\{\left(a^{ij}(x) - a^{ij}(0)\right) v_{x_i x_j} + a^i(x) v_{x_i} + a(x) v\right\} dx \le$$

$$\le \mathcal{A}(\varrho) \int\limits_{G_0^\varrho} r^{4-N} v_{xx}^2 dx + \mathcal{A}(\varrho) C_2 \left(\lambda, N\right) V(\varrho)$$

and

$$(10.1.107) \quad \int_{G_0^\varrho} r^{2-N}|v(x)||-f(x)+u(0)a(x)|dx \le \frac{\delta}{2}\int_{G_0^\varrho} r^{-N}v^2(x)dx+$$

$$+\frac{1}{\delta}\int_{G_0^\varrho} r^{4-N}\left(f^2(x)+|u(0)|^2a^2(x)\right)dx \le \frac{\delta}{2}H(\lambda,N,4-N)V(\varrho)+$$

$$+\frac{1}{\delta}k_s^2\varrho^{2s}, \ \forall\delta>0,$$

because of the supposition (10.1.1). By Lemma 2.35 and (10.1.103)-(10.1.107), we get from (10.1.102) the differential inequality

$$(10.1.108) \quad V(\varrho) \le \frac{\varrho}{2\lambda}V'(\varrho)+C_1\mathcal{A}(\varrho)V(2\varrho)+C_4\left(\delta+\mathcal{A}(\varrho)\right)V(\varrho)+$$

$$+C_5\delta^{-1}k_s^2\varrho^{2s}, \ \forall\delta>0, \quad 0<\varrho<d.$$

We adjoin the initial condition $V(d) \le V_0$ to it. By Theorem 10.23 for $\alpha=4-N$ we have

$$(10.1.109) \quad V(d) = \int_{G_0^d} r^{2-N}|\nabla u|^2dx + \int_{\Gamma_0^d} r^{1-N}\gamma(x)v^2ds \le$$

$$\le C\left(|u|_{0,G}^2 + \|f\|_{\overset{\circ}{W}{}_{4-N}^0(G)}^2 + \|g\|_{\overset{\circ}{W}{}_{4-N}^{1/2}(\partial G)}^2 + \right.$$

$$\left. + |u(0)|^2 \cdot \left(\|\gamma\|_{\overset{\circ}{W}{}_{2-N}^{1/2}(\partial G)}^2 + \|a\|_{\overset{\circ}{W}{}_{4-N}^0(G)}^2\right)\right) \equiv V_0.$$

1) $s > \lambda$

Setting $\delta = \varrho^\varepsilon$ we obtain, from (10.1.108), the problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} - C_7\left(\frac{\mathcal{A}(\varrho)}{\varrho}+\varrho^{\varepsilon-1}\right); \quad \mathcal{N}(\varrho) = 2\lambda C_1\frac{\mathcal{A}(\varrho)}{\varrho};$$

$$\mathcal{Q}(\varrho) = k_s^2 C_6\varrho^{2s-1-\varepsilon}, \ \forall\varepsilon>0.$$

Now we have, by (1.10.2),

$$\int_\varrho^d \mathcal{P}(\tau)d\tau = 2\lambda\ln\frac{d}{\varrho} - C_7\left(\int_\varrho^d \frac{\mathcal{A}(\tau)}{\tau}d\tau + \frac{d^\varepsilon-\varrho^\varepsilon}{\varepsilon}\right) \Rightarrow$$

$$\exp\left(\int_\varrho^{2\varrho}\mathcal{P}(\tau)d\tau\right) \le 2^{2\lambda}; \quad \int_\varrho^d \mathcal{B}(\tau)d\tau \le 2^{2\lambda+1}\lambda C_1 \int_0^d \frac{\mathcal{A}(\tau)}{\tau}d\tau$$

and

$$\exp\left(-\int_{\varrho}^{d}\mathcal{P}(\tau)d\tau\right) \leq \left(\frac{\varrho}{d}\right)^{2\lambda}\exp\left(C_7\int_{0}^{d}\frac{\mathcal{A}(\tau)}{\tau}d\tau\right)\exp(C_7\varepsilon^{-1}d^{\varepsilon}) =$$

$$= C_8\left(\frac{\varrho}{d}\right)^{2\lambda}.$$

In this case we also have

$$\int_{\varrho}^{d}\mathcal{Q}(\tau)\exp\left(-\int_{\varrho}^{\tau}\mathcal{P}(\sigma)d\sigma\right)d\tau \leq k_s^2 C_9\varrho^{2\lambda}\int_{\varrho}^{d}\tau^{2s-2\lambda-\varepsilon-1}d\tau \leq k_s^2 C_{10}\varrho^{2\lambda},$$

since $s > \lambda$.

Now we apply Theorem 1.57, and then from (1.10.1), by virtue of the deduced inequalities and with regard to (10.1.103), we obtain the first statement of (10.1.100).

2) $s = \lambda$

Taking in (10.1.108) any function $\delta(\varrho) > 0$ instead of $\delta > 0$, we obtain the problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda(1-\delta(\varrho))}{\varrho} - C_7\frac{\mathcal{A}(\varrho)}{\varrho}; \quad \mathcal{N}(\varrho) = 2\lambda C_1\frac{\mathcal{A}(\varrho)}{\varrho};$$

$$\mathcal{Q}(\varrho) = k_s^2 C_6\delta^{-1}(\varrho)\varrho^{2\lambda-1}.$$

We choose

$$\delta(\varrho) = \frac{1}{2\lambda\ln\left(\frac{ed}{\varrho}\right)}, \ 0 < \varrho < d,$$

where e is the Euler number. Then we obtain

$$\exp\left(\int_{\varrho}^{2\varrho}\mathcal{P}(\tau)d\tau\right) \leq 2^{2\lambda}; \quad \int_{\varrho}^{d}\mathcal{B}(\tau)d\tau \leq 2^{2\lambda+1}\lambda C_1\int_{0}^{d}\frac{\mathcal{A}(\tau)}{\tau}d\tau$$

and

$$-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)d\tau \le \ln\Big(\frac{\varrho}{d}\Big)^{2\lambda} + \int\limits_{\varrho}^{d} \frac{d\tau}{\tau\ln\big(\frac{ed}{\tau}\big)} + C_7\int\limits_{0}^{d} \mathcal{A}(\tau)d\tau =$$

$$= \ln\Big(\frac{\varrho}{d}\Big)^{2\lambda} + \ln\ln\Big(\frac{ed}{\varrho}\Big) + C_7\int\limits_{0}^{d} \mathcal{A}(\tau)d\tau \Rightarrow$$

$$\exp\Big(-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)d\tau\Big) \le \Big(\frac{\varrho}{d}\Big)^{2\lambda}\ln\Big(\frac{ed}{\varrho}\Big)\exp\Big(C_7\int\limits_{0}^{d}\frac{\mathcal{A}(\tau)}{\tau}d\tau\Big),$$

because of (1.10.2). In this case we also have

$$\int\limits_{\varrho}^{d} \mathcal{Q}(\tau)\exp\Big(-\int\limits_{\varrho}^{\tau}\mathcal{P}(\sigma)d\sigma\Big)d\tau \le k_s^2 C_{11}\varrho^{2\lambda}\int\limits_{\varrho}^{d}\delta^{-1}(\tau)\tau^{-1}\ln\Big(\frac{ed}{\tau}\Big)d\tau \le$$

$$\le k_s^2 C_{12}\varrho^{2\lambda}\ln^3\Big(\frac{ed}{\varrho}\Big).$$

Now we apply Theorem 1.57, and from (1.10.1), by virtue of the deduced inequalities, we obtain

$$V(\varrho) \le C_{17}(V_0 + k_s^2)\varrho^{2\lambda}\ln^3\frac{1}{\varrho}, \quad 0 < \varrho < d < \frac{1}{e}.$$

Taking into account (10.1.103), we obtain the second statement of (10.1.100).

3) $0 < s < \lambda$

From (10.1.108) we obtain the problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda(1-\delta)}{\varrho} - C_7\frac{\mathcal{A}(\varrho)}{\varrho}, \quad \mathcal{N}(\varrho) = 2\lambda C_1\frac{\mathcal{A}(\varrho)}{\varrho}$$

and

$$\mathcal{Q}(\varrho) = k_s^2 C_6\delta^{-1}\varrho^{2s-1}, \; \forall\delta > 0.$$

Now similar to case 1) we have

$$\exp\Big(\int\limits_{\varrho}^{2\varrho}\mathcal{P}(\tau)d\tau\Big) \le 2^{2\lambda(1-\delta)}, \quad \int\limits_{\varrho}^{d}\mathcal{B}(\tau)d\tau \le 2^{2\lambda+1}\lambda C_1\int\limits_{0}^{d}\frac{\mathcal{A}(\tau)}{\tau}d\tau$$

and

$$\exp\left(-\int_{\varrho}^{d} \mathcal{P}(\tau)d\tau\right) \le \left(\frac{\varrho}{d}\right)^{2\lambda(1-\delta)} \exp\left(C_7 \int_{0}^{d} \frac{\mathcal{A}(\tau)}{\tau}d\tau\right) = C_{13}\left(\frac{\varrho}{d}\right)^{2\lambda(1-\delta)},$$

because of (1.10.2).

In this case we also have

$$\int_{\varrho}^{d} \mathcal{Q}(\tau)\exp\left(-\int_{\varrho}^{\tau} \mathcal{P}(\sigma)d\sigma\right)d\tau \le k_s^2 C_{16}\delta^{-1}\varrho^{2\lambda(1-\delta)}\int_{\varrho}^{d} \tau^{2s-2\lambda(1-\delta)-1}d\tau \le$$

$$\le k_s^2 C_{14}\varrho^{2s},$$

if we choose $\delta \in (0, \frac{\lambda-s}{\lambda})$.

Now we apply Theorem 1.57, and then from (1.10.1), by virtue of the deduced inequalities, we obtain

$$V(\varrho) \le C_{15}\left(V_0\varrho^{2\lambda(1-\delta)} + k_s^2\varrho^{2s}\right) \le C_{16}(V_0 + k_s^2)\varrho^{2s},$$

because of chosen δ.

Taking into account (10.1.103), we deduce the third statement of (10.1.100).   □

Theorems 10.27 and 10.28 together with examples from Subsection 10.2.7 show that the assumptions about the smoothness of the coefficients of (L) in Theorem 10.25 (i.e. *Dini continuity* of the function $\mathcal{A}(r)$ at zero from the hypothesis (b)) are essential for their validity.

THEOREM 10.27. *Let $u(x)$ be a strong solution of the problem (LRP) and the assumptions of Theorem 10.25 be satisfied with $\mathcal{A}(r)$, that is continuous at zero but not Dini continuous at zero. Then there are $d \in (0,1)$ and for each $\varepsilon > 0$ a constant $C_\varepsilon > 0$ depending only on $\varepsilon, \nu, \mu, d, s, N, \lambda, \gamma_0$, $\|\gamma\|_{C^1(\partial G\setminus \mathcal{O})}, g_1, \text{meas}G$, such that $\forall \varrho \in (0, d)$*

$$\|u(x) - u(0)\|_{\overset{\circ}{W}^2_{4-N}(G_0^\varrho)} \le C\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}^{1/2}_{4-N}(\partial G)} +$$

(10.1.110)

$$+|u(0)|\big(1 + \|a\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}^{1/2}_{2-N}(\partial G)}\big) + k_s\Big)\begin{cases}\varrho^{\lambda-\varepsilon}, & \text{if } s > \lambda, \\ \varrho^{s-\varepsilon}, & \text{if } s \le \lambda.\end{cases}$$

PROOF. As above in Theorem 10.25, we get the problem (CP), that is (10.1.108) and (10.1.109), with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho}\Big(1 - \frac{\delta}{2} - C_7\mathcal{A}(\varrho)\Big), \quad \forall \delta > 0, \quad \mathcal{N}(\varrho) = 2\lambda C_1\frac{\mathcal{A}(\varrho)}{\varrho}$$

and

$$\mathcal{Q}(\varrho) = k_s^2 C_{17} \varrho^{2s-1}.$$

Therefore we have

$$-\int_\varrho^d \mathcal{P}(\tau)d\tau = 2\lambda(1 - \frac{\delta}{2})\ln\frac{\varrho}{d} + 2\lambda C_7 \int_\varrho^d \frac{\mathcal{A}(\tau)}{\tau}d\tau.$$

Now we apply the mean value theorem for integrals

$$\int_\varrho^d \frac{\mathcal{A}(\tau)}{\tau}d\tau \le \mathcal{A}(d)\ln\frac{d}{\varrho}$$

and choose $d > 0$ by continuity of $\mathcal{A}(r)$ so that $2C_7\mathcal{A}(d) < \delta$. Thus we obtain

$$\exp\left(-\int_\varrho^d \mathcal{P}(\tau)d\tau\right) \le \left(\frac{\varrho}{d}\right)^{2\lambda(1-\delta)}, \ \forall \delta > 0.$$

Similarly we have

$$\exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)d\sigma\right) \le \left(\frac{\varrho}{\tau}\right)^{2\lambda(1-\delta)}, \ \forall \delta > 0.$$

Further it is obvious that

$$\int_\varrho^{2\varrho} \mathcal{P}(\tau)d\tau \le 2\lambda\ln 2$$

and with regard to (1.10.2)

$$\int_\varrho^d \mathcal{B}(\tau)d\tau \le 2\lambda 2^{2\lambda}C_7 \int_\varrho^d \frac{\mathcal{A}(\tau)}{\tau}d\tau \le 2\lambda 2^{2\lambda}C_7\mathcal{A}(d)\ln\frac{d}{\varrho} \le \delta\lambda 2^{2\lambda}\ln\frac{d}{\varrho} \Rightarrow$$

$$\exp\left(\int_\varrho^d \mathcal{B}(\tau)d\tau\right) \le \left(\frac{\varrho}{d}\right)^{-\delta\lambda 2^{2\lambda}}, \ \forall \delta > 0.$$

Hence, by (1.10.1) of Theorem 1.57, we have

$$(10.1.111) \quad V(\varrho) \le \left(\frac{\varrho}{d}\right)^{-\delta\lambda 2^{2\lambda}}\left\{V_0\left(\frac{\varrho}{d}\right)^{2\lambda(1-\delta)} + \right.$$

$$\left. + \int\limits_{\varrho}^{d} \mathcal{Q}(\tau)\exp\left(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)d\sigma\right)d\tau\right\}, \quad \forall\delta > 0.$$

Now we estimate the last integral

$$\int\limits_{\varrho}^{d} \mathcal{Q}(\tau)\exp\left(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)d\sigma\right)d\tau \le k_s^2 C_{17}\varrho^{2\lambda(1-\delta)}\int\limits_{\varrho}^{d} \tau^{2s-2\lambda(1-\delta)-1}d\tau =$$

(10.1.112)

$$= k_s^2 C_{17}\varrho^{2\lambda(1-\delta)}\frac{d^{2s-2\lambda(1-\delta)} - \varrho^{2s-2\lambda(1-\delta)}}{2s - 2\lambda(1-\delta)} \le k_s^2 C_{18}\begin{cases}\varrho^{2\lambda(1-\delta)}, & \text{if } s \ge \lambda, \\ \varrho^{2s}, & \text{if } 0 < s < \lambda.\end{cases}$$

(We choose $\delta > 0$ so that $\delta \ne \frac{\lambda-s}{\lambda}$.)

From (10.1.111) - (10.1.112) and because of (10.1.103) Lemma 10.26 follows the desired estimate (10.1.110).   □

We can improve Theorem 10.27 in the case $s \ge \lambda$, if $\mathcal{A}(r)\ln\frac{1}{r} \le const.$

THEOREM 10.28. *Let $u(x)$ be a strong solution of the problem (LRP) and the assumptions of Theorem 10.25 be satisfied with $s \ge \lambda$ and $\mathcal{A}(r)\ln\frac{1}{r} \le const, \mathcal{A}(0) = 0$. Then there are $d \in (0,1)$ and the constants $C > 0, c > 0$ depending only on $\nu, \mu, d, N, \lambda, \gamma_0, g_1, \|\gamma\|_{C^1(\partial G\setminus\mathcal{O})}, measG,$ such that*

$$\|u(x) - u(0)\|_{\overset{\circ}{W}^2_{4-N}(G_0^\varrho)} \le C\left(|u|_{0,G} + \|f\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}^{1/2}_{4-N}(\partial G)} + \right.$$

(10.1.113)

$$\left. + |u(0)|\left(1 + \|a\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}^{1/2}_{2-N}(\partial G)}\right) + k_\lambda\right)\varrho^\lambda\ln^{c+1}\frac{1}{\varrho}, \quad 0 < \varrho < d.$$

PROOF. As above in Theorem 10.25, we get the problem (CP), that is (10.1.108) and (10.1.109). Taking in (10.1.108) any function $\delta(\varrho) > 0$ instead of $\delta > 0$ we obtain the problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda(1 - \delta(\varrho))}{\varrho} - C_7\frac{\mathcal{A}(\varrho)}{\varrho}, \quad \mathcal{N}(\varrho) = 2\lambda C_1\frac{\mathcal{A}(\varrho)}{\varrho}$$

and

$$\mathcal{Q}(\varrho) = k_s^2 C_6\delta^{-1}(\varrho)\varrho^{2s-1}.$$

We choose

$$\delta(\varrho) = \frac{1}{2\lambda \ln\left(\frac{ed}{\varrho}\right)}, \quad 0 < \varrho < d,$$

where e is the Euler number. Because of (1.10.2), since $\mathcal{A}(\varrho) \leq C\delta(\varrho)$, we have

$$\exp\left(\int\limits_{\varrho}^{2\varrho} \mathcal{P}(\tau)d\tau\right) \leq 2^{2\lambda}; \quad \exp\left(\int\limits_{\varrho}^{d} \mathcal{B}(\tau)d\tau\right) \leq \ln^c\left(\frac{ed}{\varrho}\right), \quad c > 0;$$

$$-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)d\tau \leq \ln\left(\frac{\varrho}{d}\right)^{2\lambda} + c\int\limits_{\varrho}^{d} \frac{d\tau}{\tau \ln\left(\frac{ed}{\tau}\right)} = \ln\left(\frac{\varrho}{d}\right)^{2\lambda} + c\ln\ln\left(\frac{ed}{\varrho}\right) \Rightarrow$$

$$\exp\left(-\int\limits_{\varrho}^{d} \mathcal{P}(\tau)d\tau\right) \leq \left(\frac{\varrho}{d}\right)^{2\lambda} \ln^c\left(\frac{ed}{\varrho}\right)$$

for suitable small $d > 0$.. In this case we also have

$$\int\limits_{\varrho}^{d} \mathcal{Q}(\tau)\exp\left(-\int\limits_{\varrho}^{\tau} \mathcal{P}(\sigma)d\sigma\right)d\tau \leq k_s^2 C_{19}\varrho^{2\lambda} \int\limits_{\varrho}^{d} \delta^{-1}(\tau)\tau^{2(s-\lambda)} \ln^c\left(\frac{e\tau}{\varrho}\right)\frac{d\tau}{\tau}$$

$$\leq k_s^2 C_{20}\varrho^{2\lambda} \ln^{c+2}\left(\frac{ed}{\varrho}\right),$$

because $s \geq \lambda$.

Now we apply Theorem 1.57 and then from (1.10.1), by virtue of the deduced inequalities, we obtain

(10.1.114) $\qquad V(\varrho) \leq C_{21}(V_0 + k_s^2)\varrho^{2\lambda} \ln^{2c+2}\frac{1}{\varrho}, \quad 0 < \varrho < d < \frac{1}{e}.$

From (10.1.114) and because of (10.1.103) the desired estimate (10.1.113) follows. $\qquad\qquad\square$

10.1.6. The power modulus of continuity at the conical point for strong solutions. In this section we prove Theorems 10.2, 10.3, 10.4.

Proof of Theorem 10.2.

We define the functions $v(x) = u(x) - u(0)$ and

$$
(10.1.115) \qquad \psi(\varrho) = \begin{cases} \varrho^\lambda, & \text{if } s > \lambda, \\ \varrho^\lambda \ln^{3/2}\left(\frac{1}{\varrho}\right), & \text{if } s = \lambda, \\ \varrho^s, & \text{if } s < \lambda \end{cases}
$$

for $0 < \varrho < d$ and consider two sets $G^{2\varrho}_{\varrho/4}$ and $G^\varrho_{\varrho/2} \subset G^{2\varrho}_{\varrho/4}$, $\varrho > 0$. We perform the change of variables $x = \varrho x'$ and $v(\varrho x') = \psi(\varrho) z(x')$. Because of $(LRP)_0$, the function $z(x')$ satisfies the problem

$$
(LRP)'_0 \qquad \begin{cases} a^{ij}(\varrho x')z_{x'_i x'_j} + \varrho a^i(\varrho x')z_{x'_i} + \varrho^2 a(\varrho x')z = \\ \qquad = \frac{\varrho^2}{\psi(\varrho)}\Big(f(\varrho x') - u(0)a(\varrho x')\Big), \quad x' \in G^2_{1/4} \\ \frac{\partial z}{\partial \vec{n}'} + \frac{1}{|x'|}\gamma(\varrho x')z = \frac{\varrho}{\psi(\varrho)}\Big(g(\varrho x') - u(0)\frac{\gamma(\varrho x')}{\varrho|x'|}\Big)] \leq \\ \qquad\qquad\qquad \leq \frac{\varrho}{\psi(\varrho)}g(\varrho x'), \quad x' \in \Gamma^2_{1/4}, \end{cases}
$$

since without loss of generality we can suppose that $u(0) \geq 0$. We apply now Proposition 10.14. Because of the estimates proved there, we have

$$
\begin{aligned}
(10.1.116) \qquad \sup_{G^1_{1/2}} |z(x')| \leq C\Bigg\{ &\left(\int\limits_{G^2_{1/4}} z^2 dx' \right)^{\frac{1}{2}} + \frac{\varrho}{\psi(\varrho)} \sup_{G^2_{1/4}} |g(\varrho x')| + \\
&+ \frac{\rho^2}{\psi(\varrho)}\left(\int\limits_{G^2_{1/4}} |f(\varrho x') - u(0)a(\varrho x')|^N dx' \right)^{\frac{1}{N}} \Bigg\},
\end{aligned}
$$

where the constant $C > 0$ depends only on $\left\| \left(\sum\limits_1^N |a^i|^2\right)^{1/2} \right\|_{L_q(G^2_{1/4})}$,

$\|a\|_{L_N(G^2_{1/4})}$, $M_0, g_1, \gamma_0, \|\gamma\|_{L^\infty(\partial G)}, N, \nu, \operatorname{diam} G, \omega_0,$ $\sup\limits_{\frac{1}{4}<\varrho<2} \frac{\rho}{\psi(\varrho)}$ and

$\int\limits_0^d \frac{A(r)}{r} dr$. Returning to the variable x and the function $u(x)$, by Theorem

10.25 with (10.1.115), we obtain

$$(10.1.117) \quad \int_{G^2_{1/4}} z^2 dx' = \frac{1}{\psi^2(\varrho)} \int_{G^{2\varrho}_{\varrho/4}} \varrho^{-N} |u(x) - u(0)|^2 dx \le C\Big((|u|_{0,G} +$$

$$+ \|f\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}^{1/2}_{4-N}(\partial G)} + |u(0)| \big(1 + \|a\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}^{1/2}_{2-N}(\partial G)}\big) + k_s \Big)^2$$

and

$$\left(\int_{G^2_{1/4}} |f(\varrho x') - u(0)a(\varrho x')|^n dx' \right)^{\frac{1}{N}} = \varrho^{-1} \left(\int_{G^{2\varrho}_{\varrho/4}} |f(x) - u(0)a(x)|^N dx \right)^{\frac{1}{N}}$$

$$\Rightarrow \frac{\varrho^2}{\psi(\varrho)} \left(\int_{G^2_{1/4}} |f(x) - u(0)a(x)|^N dx' \right)^{\frac{1}{N}} \le \|f + u(0)a\|_{N, G^{2\varrho}_{\varrho/4}} \frac{\varrho}{\psi(\varrho)} \le$$

$$(10.1.118) \qquad\qquad \le \varkappa_s \frac{\varrho^s}{\psi(\varrho)} \le const(N, s, \lambda, d) \cdot \varkappa_s$$

by our assumptions. From (10.1.116), (10.1.117), and (10.1.118) we get:

$$\sup_{G^\varrho_{\rho/2}} |u(x) - u(0)| \le C\Big(|u|_{0,G} + \|f\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|g\|_{\overset{\circ}{W}^{1/2}_{4-N}(\partial G)} +$$

(10.1.119)

$$+ g_1 + |u(0)|\big(1 + \|a\|_{\overset{\circ}{W}^0_{4-N}(G)} + \|\gamma\|_{\overset{\circ}{W}^{1/2}_{2-N}(\partial G)}\big) + k_s + \varkappa_s \Big) \psi(\varrho).$$

Putting now $|x| = \frac{2}{3}\varrho$ we finally obtain the desired estimate (10.1.2).

By the Sobolev Imbedding Theorems we have

$$(10.1.120) \qquad\qquad \sup_{x' \in G^1_{1/2}} |\nabla' z(x')| \le c\|z\|_{W^{2,p}(G^1_{1/2})}, \quad p > N.$$

By the local L^p a priori estimate, Theorem 10.17, for the solution of the equation of the $(LRP)'_0$ inside the domain and near a smooth portion of the boundary we have

$$(10.1.121) \quad \|z\|_{W^{2,p}(G^1_{1/2})} \le c(N, \nu, \mu, A(2)) \left\{ \frac{\varrho^2}{\psi(\varrho)} \|f + u(0)a\|_{L^p(G^2_{1/4})} + \right.$$

$$+ \frac{\varrho}{\psi(\varrho)} \left\| g + u(0) \frac{\gamma(\varrho x')}{\varrho |x'|} \right\|_{W^{1-1/p,p}(\Gamma^2_{1/4})} + \left. \|z\|_{L^p(G^2_{1/4})} \right\}.$$

Returning back to the variables x, from (10.1.120) and (10.1.121), it follows that

$$\sup_{G_{\varrho/2}^{\varrho}} |\nabla v| \leq c\varrho^{-1}\big\{\varrho^{-N/p}\|v\|_{L^p(G_{\varrho/4}^{2\varrho})} + \varrho^{2-N/p}\|f + u(0)a\|_{p,G_{\varrho/4}^{2\varrho}} +$$

$$+ \varrho^{2-N/p}\|g\|_{V_{p,0}^{1-1/p}(\Gamma_{\varrho/4}^{2\varrho})} + \varrho^{1-N/p}|u(0)|\|\gamma\|_{V_{p,0}^{1-1/p}(\Gamma_{\varrho/4}^{2\varrho})}\big\}$$

or

$$(10.1.122) \quad \sup_{G_{\varrho/2}^{\varrho}} |\nabla v| \leq c\varrho^{-1}\big\{|v|_{0,G_{\varrho/4}^{2\varrho}} + |u(0)|\|a\|_{V_{p,2p-N}^0(G_{\varrho/4}^{2\varrho})} +$$

$$+ \|f\|_{V_{p,2p-N}^0(G_{\varrho/4}^{2\varrho})} + \|g\|_{V_{p,2p-N}^{1-1/p}(\Gamma_{\varrho/4}^{2\varrho})} + |u(0)|\|\gamma\|_{V_{p,p-N}^{1-1/p}(\Gamma_{\varrho/4}^{2\varrho})}\big\}.$$

Because of (10.1.119), (10.1.2) and by the assumption (10.1.3), we get from (10.1.122) the required result (10.1.4).

Proof of Theorem 10.3

We repeat verbatim the proof of Theorem 10.2 taking

$$\psi(\varrho) = \begin{cases} \varrho^{\lambda-\varepsilon}, & \text{if } s > \lambda, \\ \varrho^{s-\varepsilon}, & \text{if } s \leq \lambda \end{cases}$$

and applying Theorem 10.27.

Proof of Theorem 10.4

We repeat verbatim the proof of Theorem 10.2 taking

$$\psi(\varrho) = \varrho^{\lambda} \ln^{c+1} \frac{1}{\varrho}$$

and applying Theorem 10.28.

10.1.7. Examples. We present the examples that show that the conditions of Theorems 10.2 - 10.4 (in particular the Dini condition for the function $\mathcal{A}(r)$ in condition (b) at the point $\mathcal{O}$ in Theorem 10.2) are essential for their validity. We recall also Remark 2.38. Suppose $N = 2$, the domain G lies inside the corner

$$G_0 = \{(r,\omega)\,|r > 0;\; -\frac{\omega_0}{2} < \omega < \frac{\omega_0}{2}\}, \quad \omega_0 \in]0,\pi[,$$

$\mathcal{O} \in \partial G$ and in some neighborhood of $\mathcal{O}$ the boundary ∂G coincides with the sides of the corner $\omega = -\frac{\omega_0}{2}$ and $\omega = \frac{\omega_0}{2}$. We denote

$$\Gamma_{\pm} = \{(r,\omega)\,|\,r > 0;\; \omega = \pm\frac{\omega_0}{2}\}$$

and we put

$$\gamma(x)\Big|_{\omega=\pm\frac{\omega_0}{2}} = \gamma_\pm = const > 0.$$

I.

$$\begin{cases} \triangle u = 0, & x \in G_0; \\ \left(\frac{\partial u}{\partial n} + \frac{1}{r}\gamma_\pm u\right)\Big|_{\Gamma_\pm} = 0. \end{cases}$$

We verify that the function $u(r,\omega) = r^\lambda \psi(\omega)$ is a solution of our problem, if λ^2 is the least positive eigenvalue of the problem

$$\begin{cases} \psi'' + \lambda^2\psi = 0, & \omega \in \left(-\frac{\omega_0}{2}, \frac{\omega_0}{2}\right) \\ (\pm\psi' + \gamma_\pm\psi)\Big|_{\omega=\pm\frac{\omega_0}{2}} = 0 \end{cases}$$

and $\psi(\omega)$ is a regular eigenfunction associated with λ^2. Precisely λ is defined from the transcedence equation

(10.1.123) $$\tan(\lambda\omega_0) = \frac{\lambda(\gamma_+ + \gamma_-)}{\lambda^2 - \gamma_+\gamma_-}.$$

Then we find the eigenfunction

(10.1.124) $$\psi(\omega) = \lambda\cos\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right] - \gamma_+\sin\left[\lambda\left(\omega - \frac{\omega_0}{2}\right)\right].$$

The existence of the positive solution of (10.1.123) may be deduced by the graphic method (see Figure 2). This example shows that the exponent λ in (10.1.2) cannot be increased.

REMARK 10.29. In order to have $\lambda > 1$ we show that the condition $\gamma(x) \geq \gamma_0 > \tan\frac{\omega_0}{2}$ from the assumption (c) of our Theorems is justified. In fact, we rewrite the equation (10.1.123) in the equivalent form

(10.1.125) $$\lambda = \frac{1}{\omega_0}\left(\arctan\frac{\gamma_+}{\lambda} + \arctan\frac{\gamma_-}{\lambda}\right).$$

Hence it follows that

$$1 < \lambda < \frac{1}{\omega_0}\left(\arctan\gamma_+ + \arctan\gamma_-\right) \quad \Rightarrow$$

(10.1.126) $$\omega_0 < \arctan\frac{\gamma_+ + \gamma_-}{1 - \gamma_+\gamma_-}, \text{ provided } \gamma_+\gamma_- < 1$$

has to be fulfilled. But our condition from the assumption (c) means that $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2}$. Hence we obtain

$$\frac{\gamma_+ + \gamma_-}{1 - \gamma_+\gamma_-} \geq \frac{2\gamma_0}{1 - \gamma_0^2} > \frac{2\tan\frac{\omega_0}{2}}{1 - \tan^2\frac{\omega_0}{2}} = \tan\omega_0, \quad \omega_0 < \frac{\pi}{2}.$$

Thus we get (10.1.126). In the case $\gamma_\pm \geq \gamma_0 > \tan\frac{\omega_0}{2} \geq 1$ for $\omega_0 \in [\frac{\pi}{2}, \pi)$ the inequality $\lambda > 1$ is fulfilled a fortiori, because of the property of the monotonic increase of the eigenvalues together with the increase of $\gamma(x)$ (see for example Theorem 6 §2, chapter VI [87]).
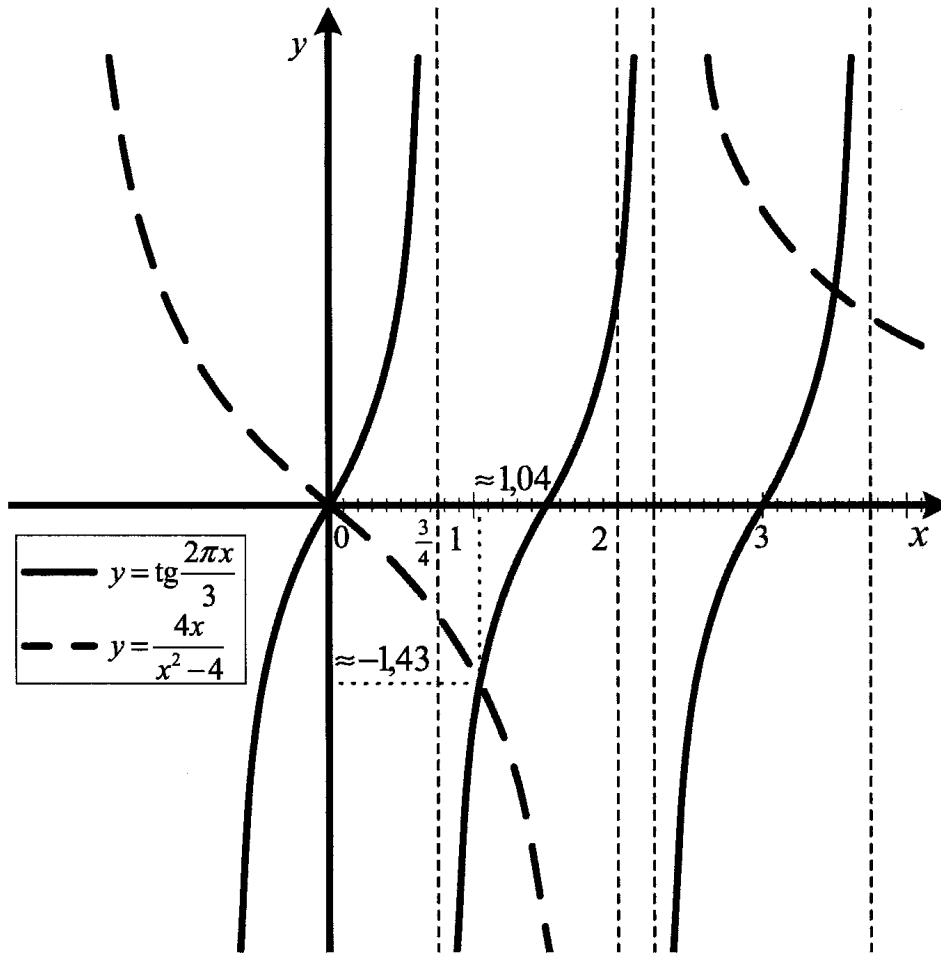
Figure 2

II. The function

$$u(r,\omega) = r^\lambda \left(\ln\frac{1}{r}\right)^{\frac{\lambda-1}{\lambda+1}} \psi(\omega)$$

with λ and $\psi(\omega)$ defined by (10.1.123) - (10.1.124) is a solution of the problem

$$\begin{cases} \sum_{i,j=1}^{N} a^{ij}(x)u_{x_i x_j} = 0, \ x \in G_0, \\ \left(\frac{\partial u}{\partial n} + \frac{1}{r}\gamma_\pm u\right)\Big|_{\Gamma_\pm} = 0, \ \gamma_\pm > 0 \end{cases}$$

in the corner G_0, where

$$a^{11}(x) = 1 - \frac{2}{\lambda + 1} \cdot \frac{x_2^2}{r^2 \ln 1/r}, \quad r > 0;$$

$$a^{12}(x) = a^{21}(x) = \frac{2}{\lambda + 1} \cdot \frac{x_1 x_2}{r^2 \ln 1/r}, \quad r > 0;$$

$$a^{22}(x) = 1 - \frac{2}{\lambda + 1} \cdot \frac{x_1^2}{r^2 \ln 1/r}, \quad r > 0;$$

$$a^{ij}(0) = \delta_i^j, \ (i,j = 1,2).$$

In the domain G_0^d, $d < e^{-2}$ the equation is uniformly elliptic with ellipticity constants $\mu = 1$ and $\nu = 1 - \frac{2}{\ln(1/d)}$. Further, $\mathcal{A}(r) = \frac{2}{\lambda+1}\ln^{-1}\left(\frac{1}{r}\right)$, i.e., the function $\mathcal{A}(r)$ does not satisfy the Dini condition at zero. Moreover, $a^{ij}(x)$ are continuous at the point $\mathcal{O}$. This example shows that the condition of Theorem 10.2 about Dini-continuity of the leading coefficients of the (LRP) are essential, and it illustrates the precision of the assumptions of Theorem 10.4 as well.

III. The function

$$u(r,\omega) = r^\lambda \ln\frac{1}{r}\psi(\omega)$$

λ and $\psi(\omega)$ defined by (10.1.123) - (10.1.124) is a solution of the problem

$$\begin{cases} \triangle u + \frac{2\lambda}{r^2 \ln \frac{1}{r}}u = 0, \ x \in G_0, \\ \left(\frac{\partial u}{\partial n} + \frac{1}{r}\gamma_\pm u\right)\Big|_{\Gamma_\pm} = 0, \ \gamma_\pm > 0 \end{cases}$$

in the corner G_0. This example shows that the assumptions of Theorem 10.4 on the lowest coefficients of the (LRP) are precise and essential.

IV. The function

$$u(r,\omega) = r^\lambda \ln\frac{1}{r}\psi(\omega)$$

with λ and $\psi(\omega)$ defined by (10.1.123) - (10.1.124) is a solution of the problem

$$\begin{cases} \triangle u = -2\lambda r^{\lambda-2}\psi(\omega), \ x \in G_0, \\ \left(\frac{\partial u}{\partial n} + \frac{1}{r}\gamma_{\pm}u\right)\Big|_{\Gamma_{\pm}} = 0, \ \gamma_{\pm} > 0 \end{cases}$$

in the corner G_0. All assumptions of Theorem 10.3 are fulfilled with $s = \lambda$. This example shows the precision of the assumptions for the right hand side of the (LRP) in Theorem 10.2.

10.2. The quasilinear problem

10.2.1. Introduction. In this Section we consider the elliptic value problem $(QLRP)$. We obtain *the best possible estimates* of the problem $(QLRP)$ strong solutions near a conical boundary point. The analogous results were established in Chapter 7 for the Dirichlet problem.

DEFINITION 10.30. *A strong solution of the problem (QLRP) is a function* $u(x) \in C^0(\overline{G}) \cap W^1(G) \cap W^{2,q}_{loc}(\overline{G} \setminus \mathcal{O})$, $q \geq N$ *that satisfies the equation for almost all* $x \in G$, *and the boundary condition in the sense of traces on* $\partial G \setminus \mathcal{O}$.

We assume that $M_0 = \max\limits_{x \in \overline{G}} |u(x)|$ is known.

Let us recall some known facts about $W^{2,p}_{loc}(G)$–solutions $(p > N)$ of the quasilinear oblique derivative problem in *smooth* domains.

THEOREM 10.31. **Local gradient bound estimate** (see Theorems 13.13 and 13.14 [**237**]).

Let $G' \subset\subset \overline{G} \setminus \mathcal{O}$ *be any subdomain with a* C^2 *boundary portion* $T = (\partial G' \cap \partial G) \subset \partial G \setminus \mathcal{O}$. *Let* $u \in W^{2,p}(G') \cap C^1(T)$, $p > N$ *be a strong solution of the problem*

$$\begin{cases} a_{ij}(x,u,u_x)u_{x_i,x_j} + a(x,u,u_x) = 0, & x \in G', \\ \frac{\partial u}{\partial \vec{n}} + \frac{1}{|x|}\gamma(x)u = g(x), & x \in T \end{cases}$$

with $|u| \leq M_0$. *Suppose that*

$$a_{ij}(x,u,z), \ a(x,u,z) \in C^1(\overline{G'} \times [-M_0, M_0] \times \mathbb{R}^N),$$

$$\gamma(x), g(x) \in C^1(T)$$

and there are positive constants ν, μ, μ_1, K *such that* $a_{ij}(x, u, z)$, $a(x, u, z)$ *satisfy*

$$\nu\xi^2 \le a_{ij}(x, u, z)\xi_i\xi_j \le \mu\xi^2, \quad \forall\xi \in \mathbb{R}^N;$$

$$|z|^2\left|\frac{\partial a_{ij}}{\partial z_k}\right| + |z|\left|\frac{\partial a_{ij}}{\partial u}\right| + \left|\frac{\partial a_{ij}}{\partial x_k}\right| \le \mu_1|z|;$$

$$|a(x, u, z)| \le \mu_1\left(1 + |z|^2\right);$$

$$|z|^2\left|\frac{\partial a}{\partial u}\right| + |z|^2\left|\frac{\partial a}{\partial z_k}\right| + \left|\frac{\partial a}{\partial x_k}\right| \le \mu_1|z|^3$$

for $|z| \ge K$. *Then for any subdomain* $G'' \subset\subset G' \cup T$ *there is a constant* $M_1 > 0$ *depending only on* $N, \nu, \mu, \mu_1, \|\gamma\|_{C^1(T)}, \|g\|_{C^1(T)}, M_0, K$ *and* G', G'', T *such that*

$$\sup_{G''} |\nabla u| \le M_1.$$

THEOREM 10.32. **Local Hölder gradient estimate** (see Lemma 2.3 [236]). *Let* $G' \subset\subset \overline{G} \setminus \mathcal{O}$ *be any subdomain with a* C^2 *boundary portion* $T = (\partial G' \cap \partial G) \subset \partial G \setminus \mathcal{O}$. *Let* $u \in W^{2,p}(G') \cap C^1(T)$, $p > N$ *be a strong solution of the problem*

$$\begin{cases} a_{ij}(x, u, u_x)u_{x_i, x_j} + a(x, u, u_x) = 0, & x \in G', \\ \frac{\partial u}{\partial \vec{n}} + \frac{1}{|x|}\gamma(x)u = g(x), & x \in T \end{cases}$$

with $|u| \le M_0$, $|\nabla u| \le M_1$. *Suppose that*

$$a_{ij}(x, u, z), a(x, u, z) \in C^1(\overline{G'} \times [-M_0, M_0] \times [-M_1, M_1]),$$

$$\gamma(x), g(x) \in C^1(T)$$

and there are positive constants ν, μ, μ_1 *such that* $a_{ij}(x, u, z)$, $a(x, u, z)$ *satisfy*

$$\nu\xi^2 \le a_{ij}(x, u, z)\xi_i\xi_j \le \mu\xi^2, \quad \forall\xi \in \mathbb{R}^N;$$

$$\left|\frac{\partial a_{ij}}{\partial z_k}\right| + \left|\frac{\partial a_{ij}}{\partial u}\right| + \left|\frac{\partial a_{ij}}{\partial x_k}\right| + |a(x, u, z)| \le \mu_1$$

for $|u| \le M_0$, $|\nabla u| \le M_1$. *Then for any subdomain* $G'' \subset\subset G' \cup T$ *there are constants* $C > 0, \widetilde{\varkappa} \in (0, 1)$ *depending only on* $N, \nu, \mu, \mu_1, \|\gamma\|_{C^1(T)}$, $\|g\|_{C^1(T)}, M_0, M_1$ *and* G', G'', T *such that*

$$|u|_{1+\widetilde{\varkappa}, G''} \le C.$$

We assume the existence $d > 0$ such that G_0^d is *the convex rotational cone with the vertex at* $\mathcal{O}$ *and the aperture* $\omega_0 \in (\frac{\pi}{2}, \pi)$ (see (1.3.13)). Let $\mathfrak{M} = \{(x, u, z) | x \in \overline{G}, u \in \mathbb{R}, z \in \mathbb{R}^N\}$. Regarding the equation we assume that the following *conditions* are satisfied on $\mathfrak{M}$

(A) $a_{ij}(x, u, z) \in W^{1,q}(\mathfrak{M})$, $q > N$; $\gamma(x) \in L^\infty(\partial G) \cap C^1(\partial G \setminus \mathcal{O})$;
the condition of Caratheodory: functions

$$a(x, u, z), \frac{\partial a(x, u, z)}{\partial u} \in CAR,$$

that is

(i) *they are measurable on G as functions of variable x for $\forall u, z$,*

(ii) *they are continuous with respect to u, z for almost all $x \in G$,*

(B) *the condition of uniform ellipticity*

$$\nu \xi^2 \le a_{ij}(x, u, z) \xi_i \xi_j \le \mu \xi^2,$$

$$\forall \xi \in \mathbb{R}^N, x \in G, u \in \mathbb{R}, z \in \mathbb{R}^N \text{ and } \nu, \mu = const > 0,$$

(C) $\frac{\partial a(x, u, z)}{\partial u} \le 0$,

(D) *there exist numbers $\beta > -1, \gamma_0 > \tan \frac{\omega_0}{2}$, $\gamma_1 \ge \gamma_0$, nonnegative
constants δ, μ_1, k_1, g_0 and functions $b(x), f(x) \in L^q_{loc}(\overline{G} \setminus \mathcal{O}), q \ge N$
such that on $\mathfrak{M}$ the inequalities*

$$|a(x, u, z)| + \left| \frac{\partial a(x, u, z)}{\partial u} \right| \le \mu_1 |z|^2 + b(x)|z| + f(x),$$

$$b(x) + f(x) \le k_1 |x|^\beta, \quad |g(x)| \le g_0 |x|^\delta,$$

$$\gamma_0 \le \gamma(x) \le \gamma_1$$

hold,

(E) *the problem $(QLRP)$ coefficients satisfy such conditions that guar-
antee $u \in C^{1+\tilde{\varkappa}}(G')$ and the existence of the local a priori estimate*

$$|u|_{1+\tilde{\varkappa}, G'} \le M_1, \quad \tilde{\varkappa} \in (0, 1)$$

for any smooth $G' \subset\subset \overline{G} \setminus \{\mathcal{O}\}$ (see Theorems 10.31 and 10.32).

PROPOSITION 10.33. **The local maximum principle** (see Theorem
3.3 **[225]**, Theorem 4.3 **[234]**; see as well **[233]**).

*Let G be a bounded domain in $\mathbb{R}^N$ with the C^1-boundary $\partial G \setminus \Gamma^d_0$ and
G^d_0 be a convex rotational cone with the vertex at $\mathcal{O}$ and the aperture $\omega_0 \in
(\frac{\pi}{2}, \pi)$. Let $u(x)$ be a strong solution of the problem $(QLRP)$ with $|u| \le M_0$.
Suppose the conditions $(A), (B), (C)$ are satisfied. In addition, suppose that
there are nonnegative number μ_1 and nonnegative functions $b(x) \in L^s(G)$,
$s > N$, $f(x) \in L^N(G)$, such that*

$$|a(x, u, z)| \le \mu_1 |z|^2 + b(x)|z| + f(x).$$

Suppose finally that $g \in L^\infty(\partial G)$.

Then for any $q > 0$ and $\sigma \in (0,1)$, there is a constant $C = C(\nu, \mu, \mu_1, M_0, \gamma_0, \omega_0, N, p, R, G, \|b\|_{L^s(G)}, \|f\|_{L^N(G)})$ such that

$$\sup_{G_0^{\sigma R}} |u(x)| \leq C \left\{ \left(\frac{1}{\operatorname{meas} G_0^R} \int_{G_0^R} |u|^q dx \right)^{1/q} + \right.$$

$$\left. + R \left(\|f\|_{L^N(G_0^R)} + \|g\|_{L^\infty(\partial G)} \right) \right\}.$$

10.2.2. Weak smoothness of the strong solution. First we estimate $|u(x)|$ for the (QLRP) in the neighborhood of a conical point. To this end we use the barrier function, constructed in Lemma 10.18, Subsection 10.1.2.

THEOREM 10.34. *Let $u(x)$ be a strong solution of the problem (QLRP) and assumptions (A)-(D) be satisfied. Then there exist the numbers $d > 0$ and $\varkappa > 0$ depending only on $\nu, \mu, \mu_1, N, \varkappa_0, \omega_0, k_1, \beta, \delta, \gamma_0, g_0, M_0$ and the domain G such that*

$$(10.2.1) \qquad |u(x) - u(0)| \leq C_0 |x|^{\varkappa+1}, \quad x \in G_0^d,$$

where the positive constant C_0 does not depend on u but depends only on $\nu, \mu, \mu_1, g_0, N, k_1, \beta, \gamma_0, M_0$ and the domain G.

PROOF. Let us take the linear elliptic operator

$$\widetilde{\mathcal{L}} \equiv a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + a^i(x) \frac{\partial}{\partial x_i}, \quad x \in G,$$

where

$$(10.2.2) \qquad a^{ij}(x) = a_{ij}(x, u(x), u_x(x)); \quad a^i(x) = b(x) |\nabla u(x)|^{-1} u_{x_i}(x).$$

Here we suppose that $a^i(x) = 0$, $i = 1, ..., N$ in such points x, where $|\nabla u(x)| = 0$. Let us take the barrier function (10.1.15) and define the auxiliary function $v(x)$ as follows

$$(10.2.3) \qquad v(x) = -1 + \exp(\nu^{-1}\mu_1(u(x) - u(0))).$$

For those functions we shall show that

$$(10.2.4) \qquad \begin{cases} \widetilde{\mathcal{L}}(Aw(x)) \leq \widetilde{\mathcal{L}}v(x), & x \in G_0^d; \\ \mathcal{B}[Aw(x)] \geq \mathcal{B}[v(x)], & x \in \Gamma_0^d \setminus \mathcal{O}; \\ Aw(x) \geq v(x), & x \in \Omega_d \cup \mathcal{O}. \end{cases}$$

Let us calculate the operator $\widetilde{\mathcal{L}}$ on the function (10.2.3). We obtain

$$\widetilde{\mathcal{L}}v(x) \equiv \nu^{-1}\mu_1(a^{ij}(x)u_{x_ix_j} + \nu^{-1}\mu_1 a^{ij}(x)u_{x_i}u_{x_j} + b(x)|\nabla u(x)|) \times$$

$$\times \exp(\nu^{-1}\mu_1(u(x) - u(0))) = \nu^{-1}\mu_1\big[b(x)|\nabla u(x)| - a(x,u(x),u_x(x))$$

$$+ a^{ij}(x)u_{x_i}u_{x_j}\big]\exp(\nu^{-1}\mu_1(u(x) - u(0))) \geq$$

$$\geq -\nu^{-1}\mu_1 f(x)\exp(2\nu^{-1}\mu_1 M_0)$$

in virtue of assumptions (C) and (D). Since $f(x) \leq k_1 r^\beta$, we obtain

(10.2.5) $\widetilde{\mathcal{L}}v(x) \geq -\nu^{-1}\mu_1 k_1 r^\beta \exp(2\nu^{-1}\mu_1 M_0), \quad x \in G_0^d.$

Let us calculate the operator $\widetilde{\mathcal{L}}$ on the barrier function (10.1.15). Let the number $\varkappa_0$ be such that Lemma 10.18 holds and suppose $\varkappa$ satisfies the inequality

$$0 < \varkappa \leq \min(\delta, \varkappa_0, \beta + 1).$$

By (10.1.11) and (10.2.2), we obtain

$$\widetilde{\mathcal{L}}w = \mathcal{L}_0 w + b(x)\frac{u_{x_i}}{|\nabla u(x)|}w_{x_i} \leq -\nu h^2|x|^{\varkappa-1} + b(x)|\nabla w| \leq$$

$$\leq -\nu h^2|x|^{\varkappa-1} + b(x)|x|^\varkappa\sqrt{2 + 4h^2 + B(1 + \varkappa_0)^2}.$$

Because of $b(x) \leq k_1 r^\beta$, we get in G_0^d

$$\widetilde{\mathcal{L}}w \leq r^{\varkappa-1}\Big(-\nu h^2 + d^{\beta+1}k_1\sqrt{2 + 4h^2 + B(1 + \varkappa_0)^2}\Big) \leq -\frac{1}{2}\nu h^2 r^{\varkappa-1},$$

if

$$k_1\sqrt{2 + 4h^2 + B(1 + \varkappa_0)^2}d^{\beta+1} \leq \frac{1}{2}\nu h^2 \Rightarrow$$

(10.2.6) $d \leq \Big(\dfrac{\nu h^2}{2k_1\sqrt{2 + 4h^2 + B(1 + \varkappa_0)^2}}\Big)^{\frac{1}{1+\beta}}.$

Hence, in virtue of (10.2.5), it follows that

$$\widetilde{\mathcal{L}}[Aw(x)] \leq \widetilde{\mathcal{L}}v(x), \quad x \in G_0^d,$$

if we define the number A such that

(10.2.7) $A \geq 2\mu_1 k_1 \nu^{-2}h^{-2}d^{1-\varkappa_0+\beta}\exp(2\nu^{-1}\mu_1 M_0).$

From (10.1.12) we get

(10.2.8) $\mathcal{B}[Aw]\Big|_{\Gamma_\pm^d} \geq Ag_0 r^\delta$

Let us calculate the operator $\mathcal{B}$ on the function $v(x)$ that is defined by (10.2.3)

$$\mathcal{B}[v(x)] \equiv \frac{\partial v}{\partial\vec{n}} + \frac{1}{|x|}\gamma(x)v(x), \quad x \in \Gamma_\pm^d \setminus \mathcal{O}.$$

By the $(QLRP)$ boundary condition, we have

$$
(10.2.9) \quad
\begin{aligned}
\left.\frac{\partial v}{\partial \vec{n}}\right|_{\Gamma_\pm^d \setminus \mathcal{O}} &= \nu^{-1}\mu_1 \exp(\nu^{-1}\mu_1(u(x) - u(0)))\left.\frac{\partial u}{\partial \vec{n}}\right|_{\Gamma_\pm^d \setminus \mathcal{O}} = \\
&= \nu^{-1}\mu_1 \exp(\nu^{-1}\mu_1(u(x) - u(0)))\langle g(x) - \frac{\gamma(x)}{|x|}u(x)\rangle.
\end{aligned}
$$

Using (10.2.9) and our assumptions, we calculate

$$
(10.2.10) \quad
\begin{aligned}
\left.\mathcal{B}[v]\right|_{\Gamma_\pm^d \setminus \mathcal{O}} &\le \nu^{-1}\mu_1 \exp(2\nu^{-1}\mu_1 M_0)\langle g_0 r^\delta - \frac{\gamma(x)}{|x|}u(x)\rangle + \frac{\gamma(x)}{|x|}v(x) \le \\
&\le \exp(2\nu^{-1}\mu_1 M_0)g_0 r^\delta + \frac{\gamma(x)}{|x|}\left[v - (1+v)\nu^{-1}\mu_1 u(x)\right], \quad v > -1.
\end{aligned}
$$

Because of (10.2.3), we have

$$
\begin{aligned}
\exp(\nu^{-1}\mu_1(u(x) - u(0))) &= v + 1 \Rightarrow \nu^{-1}\mu_1(u(x) - u(0)) = \ln(1 + v) \Rightarrow \\
&\Rightarrow (1+v)\nu^{-1}\mu_1 u(x) = (1+v)\ln(1+v) + (1+v)\nu^{-1}\mu_1 u(0),
\end{aligned}
$$

and, therefore, from (10.2.10) we obtain

$$
(10.2.11) \quad
\begin{aligned}
\left.\mathcal{B}[v]\right|_{\Gamma_\pm^d \setminus \mathcal{O}} &\le \exp(2\nu^{-1}\mu_1 M_0)g_0 r^\delta + \frac{\gamma(x)}{|x|}\left[v - (1+v)\ln(1+v)\right] \le \\
&\le \exp(2\nu^{-1}\mu_1 M_0)g_0 r^\delta, \quad v > -1,
\end{aligned}
$$

if only $u(0) \ge 0$. Indeed, if we denote $f(v) = v - (1+v)\ln(1+v)$, $v > -1$ we get $f'(v) =$
$= -\ln(1+v)$ and $f''(v) = -\frac{1}{1+v}$. We see that $f'(v) = 0 \Leftrightarrow v = 0$ and $f''(0) =$
$= -1 < 0$. Then we obtain

$$
\max_{v>-1} f(v) = f(0) = 0 \implies (10.2.11).
$$

Taking into account (10.2.8) and (10.2.11), we choose

$$
(10.2.12) \quad A \ge \nu^{-1}\mu_1 g_0^{-1} \exp(2\nu^{-1}\mu_1 M_0)
$$

and we obtain

$$
\mathcal{B}[Aw] \ge \mathcal{B}[v] \quad \text{on } \Gamma_\pm^d \setminus \mathcal{O}.
$$

If $u(0) < 0$, instead of the function $v(x)$, defined by (10.2.3), we have to take the function

$$
(10.2.13) \quad z(x) := 1 - \exp(-\nu^{-1}\mu_1(u(x) - u(0))).
$$

Now we compare $v(x)$ and $w(x)$ on Ω_d. Since $x^2 \geq h^2 y^2$ in $\overline{K}$, from (10.1.15) we have

$$(10.2.14) \qquad w(x)\Big|_{r=d} \geq B|x|^{1+\varkappa}\Big|_{r=d} = Bd^{1+\varkappa}\cos^{\varkappa+1}\frac{\omega_0}{2}.$$

On the other hand

$$(10.2.15) \qquad \begin{aligned} v(x)\Big|_{\Omega_d} &= -1 + \exp(\nu^{-1}\mu_1(u(x) - u(0)))\Big|_{\Omega_d} \leq \\ &\leq -1 + \exp(2\nu^{-1}\mu_1 M_0) \end{aligned}$$

and, therefore, from (10.2.14) and(10.2.15), in virtue of (10.1.20), we obtain:

$$\begin{aligned} Aw(x)\Big|_{\Omega_d} &\geq ABd^{1+\varkappa}\cos^{\varkappa+1}\frac{\omega_0}{2} \geq A\left\{\frac{g_0\left(1+h^2\right)^{\frac{\varkappa_0+1}{2}}}{h^{\varkappa_0}} + 2(1+h^2)\right\} \times \\ &\times \frac{1}{h\gamma_0 - 1 - \varkappa_0}d^{1+\varkappa_0}h^{1+\varkappa_0}(1+h^2)^{-\frac{1+\varkappa_0}{2}} \geq \\ &\geq \exp(2\nu^{-1}\mu_1 M_0) - 1 \geq v\Big|_{\Omega_d}, \end{aligned}$$

if we choose A enough great

$$(10.2.16) \qquad A \geq \frac{[\exp(2\nu^{-1}\mu_1 M_0) - 1](h\gamma_0 - 1 - \varkappa_0)}{hd^{1+\varkappa_0}\left[g_0 + 2h^{\varkappa_0}\left(1+h^2\right)^{\frac{1-\varkappa_0}{2}}\right]}.$$

Thus, if we choose a small number $d > 0$ according to (10.2.6) and large numbers A, B according to (10.1.20), (10.2.7), (10.2.12) and (10.2.16), we provide the validity of (10.2.4).

Therefore the functions (10.1.15) and (10.2.3) satisfy the comparison principle, Proposition 10.16, and, by it, we have

$$(10.2.17) \qquad v(x) \leq Aw(x), \ x \in \overline{G_0^d}.$$

Returning to the function $u(x)$ from (10.2.3), on the basis of (10.2.17), we have

$$u(x) - u(0) = \nu\mu_1^{-1}\ln(v(x) + 1) \leq \nu\mu_1^{-1}\ln(Aw(x) + 1) \leq \nu\mu_1^{-1}Aw(x).$$

Similarly, we derive the estimate

$$u(x) - u(0) \geq -\nu\mu_1^{-1}Aw(x),$$

if we consider an auxiliary function (10.2.13). In virtue of (10.1.13), the theorem is proved. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we will estimate the gradient modulus of the problem $(QLRP)$ solution near a conical point.

THEOREM 10.35. *Let $u(x)$ be a strong solution of the problem $(QLRP)$, $q > N$ and suppose that assumptions (A)-(E) are satisfied. Let $\varkappa > 0$ be a number defined by Theorem 10.34. Then there exists the number $d > 0$ such that*

$$(10.2.18) \qquad |\nabla u(x)| < C_1 |x|^{\varkappa}, \quad x \in G_0^d,$$

where the constant C_1 does not depend on u, but depends only on $\nu, \mu, N, k_1, \beta, g_0, \gamma_0, M_1$ and the domain G.

PROOF. Let us consider the set $G_{\rho/2}^{\rho} \subset G$, $0 < \rho < d$. We make the transformation $x = \rho x'$; $v(x') = \rho^{-1-\varkappa} u(\rho x')$. The function $v(x')$ satisfies the problem

$$(QLRP)' \qquad \begin{cases} a^{ij}(x') v_{x_i' x_j'} = F(x'), \quad x' \in G_{1/2}^1, \\ \frac{\partial v}{\partial \vec{n}'} + \frac{1}{|x'|} \gamma(\rho x') v(x') = \rho^{-\varkappa} g(\rho x'), \quad x \in \Gamma_{1/2}^1, \end{cases}$$

where

$$a^{ij}(x') \equiv a_{ij}(\rho x', \rho^{1+\varkappa} v(x'), \rho^{\varkappa} v_{x'}(x'))$$

and

$$F(x') \equiv -\rho^{1-\varkappa} a(\rho x', \rho^{1+\varkappa} v(x'), \rho^{\varkappa} v_{x'}(x')).$$

Now we apply the assumption (E)

$$(10.2.19) \qquad \max_{x' \in G_{1/2}^1} |\nabla' v(x')| \le M_1'.$$

Returning to the variable x and the function $u(x)$ we obtain from (10.2.19)

$$|\nabla u(x)| \le M_1 \rho^{\varkappa}, \quad x \in G_{\rho/2}^{\rho}, \quad 0 < \rho < d.$$

Putting now $|x| = \frac{2}{3}\rho$ we obtain the desired estimate (10.2.18). $\qquad \square$

COROLLARY 10.36. *Let $u(x)$ be a strong solution of the problem $(QLRP)$, $q > N$ and suppose that assumptions (A)-(E) are satisfied. Then $u(0) = 0$ and therefore the inequality (10.2.1) take a form*

$$(10.2.20) \qquad |u(x)| \le C_0 |x|^{\varkappa+1}, \quad x \in G_0^d.$$

PROOF. From the problem boundary condition it follows that

$$\gamma(x) u(x) = |x| g(x) - |x| \frac{\partial u}{\partial n}, \quad x \in \partial G \setminus \mathcal{O}.$$

By the assumption (D) and the estimate (10.2.18), we obtain

$$\gamma_0 |u(x)| \le \gamma(x) |u(x)| \le |x| |g(x)| + |x| |\nabla u| \le g_0 |x|^{\delta+1} + C_1 |x|^{\varkappa+1}.$$

By letting $|x|$ tend to 0 we get, because of the continuity of $u(x)$, that $\gamma_0 |u(0)| = 0$ and taking into account $\gamma_0 > 0$, we find $u(0) = 0$. $\qquad \square$

THEOREM 10.37. *Let $u(x)$ be a strong solution of the problem $(QLRP)$, $q > N$ and suppose that assumptions (A)-(E) are satisfied. Let $\varkappa_0$ be the number from the lemma about the barrier function. In addition, let $g(x) \in V_{q,q}^{1-\frac{1}{q}}(\partial G)$ and*

$$(10.2.21) \qquad \|g\|_{V_{q,q}^{1-\frac{1}{q}}(\Gamma_{\rho/4}^{2\rho})} \leq C\rho^{\varkappa+\frac{N}{q}}.$$

Then $u(x) \in C^{1+\varkappa}(\overline{G_0^d})$, $0 < \varkappa \leq \min(\widetilde{\varkappa}, \varkappa_0, \beta+1, 1-\frac{N}{q})$ for some $d \in (0,1)$.

PROOF. Let $d > 0$ be a number such that estimates (10.1.21) and (10.2.18) are satisfied. Let us consider the set $G_{\rho/2}^{\rho} \subset G$; $0 < \rho < d$. We make the transformation $x = \rho x'$; $v(x') = \rho^{-1-\varkappa}u(\rho x')$, where $\varkappa > 0$ is defined by Theorem 10.34. The function $v(x')$ satisfies the problem $(QLRP)'$. By the Sobolev Embedding Theorem, we have

$$(10.2.22) \qquad \sup_{\substack{x',y' \in G_{1/2}^1 \\ x' \neq y'}} \frac{|\nabla'v(x') - \nabla'v(y')|}{|x'-y'|^{1-\frac{N}{q}}} \leq C(N,q,G)\|v\|_{W^{2,q}(G_{1/2}^1)},$$

where $q > N$.

We shall verify that the local interior and near a smooth boundary portion L^q *a-priori* estimate (Theorem 4.8) for the solution of the $(QLRP)'$ equation holds. On the basis of assumption (E) we have that the functions $a_{ij}(x, u, z)$ are continuous on $\mathfrak{M}$, that is for $\forall \varepsilon > 0$ there exists such η that

$$|a_{ij}(x, u(x), u_x(x)) - a_{ij}(y, u(y), u_x(y))| < \varepsilon,$$

if only

$$|x-y| + |u(x) - u(y)| + |u_x(x) - u_x(y)| < \eta, \quad \forall x, y \in G_{\rho/2}^{\rho}, \ \rho \in (0, d).$$

Assumption (E) guarantees the existence of the local interior and near a smooth boundary portion *a priori* $C^{1+\widetilde{\varkappa}}$-estimate. There exist a number $\widetilde{\varkappa} > 0$ and a number $M_1 > 0$ such that

$$|u(x) - u(y)| + |\nabla u(x) - \nabla u(y)| \leq M_1|x-y|^{\widetilde{\varkappa}}, \quad \forall x, y \in G_{\rho/2}^{\rho}, \ \rho \in (0, d).$$

Then functions $a^{ij}(x')$ are uniformly continuous in $\overline{G_{1/2}^1}$. It means that for $\forall \varepsilon > 0$ exists $\delta > 0$ (we choose the number δ such that $\delta d + M_1(\delta d)^{\widetilde{\varkappa}} < \eta$) such that $|a^{ij}(x') - a^{ij}(y')| < \varepsilon$, if only $|x' - y'| < \delta$, $\forall x', y' \in \overline{G_{1/2}^1}$. We see that the assumptions of Theorem 10.17 about the local $L^q - a$ *priori*

estimate for the $(QLRP)'$ are satisfied. By this theorem, we have

$$\|v\|^q_{W^{2,q}(G^1_{1/2})} \leq C_3 \int\limits_{G^2_{1/4}} \langle |v|^q + \rho^{q(1-\varkappa)}|a(\rho x', \rho^{1+\varkappa}v(x'), \rho^\varkappa v_{x'}(x))|^q \rangle dx' +$$

$$(10.2.23) \qquad + C_4 \inf \int\limits_{G^2_{1/4}} (|\nabla' \mathcal{G}|^q + |\mathcal{G}|^q) dx' \cdot \rho^{-q\varkappa},$$

where the constants C_3, C_4 do not depend on v. Returning to the variable x and using the estimate (10.1.21) we obtain

$$\int\limits_{G^2_{1/4}} |v|^q dx' = \int\limits_{G^{2\rho}_{\frac{\rho}{4}}} \rho^{-q(1+\varkappa)}|u(x)|^q \rho^{-N} dx \leq$$

$$(10.2.24)$$

$$\leq C^q_0 \operatorname{meas} \Omega C(q, \varkappa) \int\limits_{\frac{\rho}{4}}^{2\rho} \frac{dr}{r} = C^q_0 mes\Omega C(q, \varkappa) \ln 8.$$

Similarly, by the assumption (D), the estimate (10.2.18) and the inequality

$$\left(\sum_{i=1}^N c_i\right)^t \leq N^{t-1} \sum_{i=1}^N c^t_i \text{ for any} c_i > 0 \text{ and } t \geq 1,$$

we have

$$\int\limits_{G^2_{1/4}} \rho^{q(1-\varkappa)}|a(\rho x', \rho^{1+\varkappa}v(x'), \rho^\varkappa v_{x'}(x))|^q dx' \leq$$

$$\leq \rho^{q(1-\varkappa)-N} \int\limits_{G^{2\rho}_{\frac{\rho}{4}}} \left(\mu_1|\nabla u|^2 + b(x)|\nabla u| + f(x)\right)^q dx \leq$$

$$(10.2.25) \quad \leq 2^N 3^{q-1} \rho^{q(1-\varkappa)} mes\Omega \int\limits_{\frac{\rho}{4}}^{2\rho} \left(\mu^q_1 C^{2q}_1 r^{2q\varkappa-1} + (k_1 C_1)^q r^{q(\beta+\varkappa)-1} + \right.$$

$$+ k^q_1 r^{q\beta-1}\big) dr \leq C(N, q, \varkappa, \beta, \mu_1, C_1, k_1),$$

if only $0 < \varkappa \le 1 + \beta$. Because of the assumption (10.2.21) of our theorem, we have

(10.2.26)
$$\int_{G^2_{1/4}} (|\nabla' \mathcal{G}|^q + |\mathcal{G}|^q) dx' \cdot \rho^{-q\varkappa} = \int_{G^{2\rho}_{\frac{\rho}{4}}} (\rho^q |\nabla \mathcal{G}|^q + |\mathcal{G}|^q) \rho^{-q\varkappa - N} dx \le$$

$$\le C \int_{G^{2\rho}_{\frac{\rho}{4}}} \left(r^q |\nabla \mathcal{G}|^q + |\mathcal{G}|^q \right) dx \cdot \rho^{-\varkappa q - N} \le \text{const.}$$

In virtue of (10.2.23), we obtain from (10.2.24)-(10.2.26)

(10.2.27)
$$\|v\|^q_{W^{2,q}(G^1_{1/2})} \le C.$$

From (10.2.22) and (10.2.27) we have

(10.2.28)
$$\sup_{\substack{x', y' \in G^1_{1/2} \\ x' \ne y'}} \frac{|\nabla' v(x') - \nabla' v(y')|}{|x' - y'|^{1 - \frac{N}{q}}} \le C_5, \quad q > N$$

Returning to the variable x and the function u we have
(10.2.29)
$$\sup_{\substack{x, y \in G^\rho_{\rho/2} \\ x \ne y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^{1 - \frac{N}{q}}} \le C_5 \rho^{\varkappa - 1 + \frac{N}{q}}, \quad q > N, \ \rho \in (0, d).$$

Let us recall that from the assumptions of our theorem, we have $\varkappa \le \varkappa^* \le 1 - \frac{N}{q}$. From this we obtain $q \ge \frac{N}{1 - \varkappa}$. We take $\tau = \varkappa - 1 + \frac{N}{q} \le 0$. Then from (10.2.29) it follows that

(10.2.30) $|\nabla u(x) - \nabla u(y)| \le C_5 \rho^\tau |x - y|^{\varkappa - \tau}, \quad \forall x, y \in G^\rho_{\rho/2}, \ \rho \in (0, d)$

Because $x, y \in G^\rho_{\rho/2}$, then $|x - y| \le 2\rho$ and because $\tau \le 0$, $|x - y|^\tau \ge (2\rho)^\tau$. That is the way we obtain

$$|\nabla u(x) - \nabla u(y)| \le C_5 2^{-\tau} |x - y|^\varkappa, \quad \forall x, y \in G^\rho_{\rho/2}, \ \rho \in (0, d) \Rightarrow$$

(10.2.31)
$$\sup_{\substack{x, y \in G^\rho_{\rho/2} \\ x \ne y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\varkappa} \le C_5 2^{-\tau}, \quad \rho \in (0, d).$$

Let now $x, y \in \overline{G_0^d}$. If $x, y \in G_{\rho/2}^\rho, \forall \rho \in (0, d)$ we have (10.2.31). If $|x - y| > \rho = |x|$ then, because of (10.2.18), we have

$$\frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\varkappa} \leq 2\rho^{-\varkappa}|\nabla u(x)| \leq 2C_1.$$

From this inequality and (10.2.31) it follows that

$$(10.2.32) \qquad \sup_{\substack{x, y \in G_0^d \\ x \neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\varkappa} \leq const.$$

Because of (10.2.32), (10.2.1) and (10.2.18), we get that $u \in C^{1+\varkappa}(\overline{G_0^d})$.   □

10.2.3. Integral weighted estimates. On the basis of the obtained in Subsection 10.2.2 estimates, we deduce integral weighted estimates of second order generalized derivatives of a strong solution and establish the best possible exponent of the weight. Let λ be the number that is defined by (2.5.11) or (2.5.19) from Section 2.5.

THEOREM 10.38. *Let $u(x)$ be a solution of the problem (QLRP), $q > N$. Suppose that assumptions (A)-(E) are satisfied. In addition, suppose that*

(AA) $a_{ij}(0, u(0), 0) = \delta_i^j$ $(i, j = 1, ..., N)$ - the Kronecker symbol.

Then there exist the numbers $d, C > 0$, which do not depend on u, such that if $b(x), f(x) \in \overset{\circ}{W}_\alpha^0(G), g(x) \in \overset{\circ}{W}_\alpha^{1/2}(\partial G)$ and $\gamma(x) \in \overset{\circ}{W}_{\alpha-2}^{1/2}(\partial G)$ for

$$(10.2.33) \qquad 4 - N - 2\lambda < \alpha \leq 2,$$

then $u(x) \in \overset{\circ}{W}_\alpha^2(G_0^{d/2})$ and

$$(10.2.34) \quad \int_{G_0^{d/2}} \left(r^\alpha u_{xx}^2 + r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}|u(x)|^2\right)dx \leq C\Bigg\{|u|_0^2 +$$

$$+ \|g\|_{\overset{\circ}{W}_\alpha^{1/2}(\Gamma_0^{2d})}^2 + \int_{G_0^{2d}} \left(|\nabla u|^2 + r^\alpha(b^2(x) + f^2(x))\right)dx + 1\Bigg\}.$$

PROOF. We break the proof into three steps.

Step 1. $4 - N < \alpha \leq 2$, $N \geq 3$.
By (10.2.18), we obtain

$$(10.2.35) \quad \int_{G_0^d} (r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2)dx \leq C meas\Omega \int_0^d r^{\alpha-2+2\varkappa} \cdot r^{N-1}dr +$$

$$+ M_0^2 meas\Omega \int_0^d r^{\alpha-4+N-1}dr \leq C(\alpha, N, M_0, meas\Omega, \varkappa)\Big(d^{\alpha+N-2+2\varkappa} +$$

$$+ d^{\alpha+N-4}\Big) \leq const.$$

By assumption (A), we have $a_{ij}(x, u, z) \in W^{1,q}(\mathfrak{M}), q > N$ and, by the embedding theorem, $a_{ij}(x, u, z)$, $i, j = 1, ..., N$ are uniformly continuous on $\mathfrak{M}$. Therefore for $\forall \delta > 0$ there exists $d_\delta > 0$ such that

$$(10.2.36) \quad |a_{ij}(x, u(x), u_x(x)) - a_{ij}(y, u(y), u_x(y))| < \delta$$

if only

$$(10.2.37) \quad |x - y| + |u(x) - u(y)| + |u_x(x) - u_x(y)| < d_\delta.$$

By (10.2.1), (10.2.18), and (10.2.32) we get:

$$(10.2.38) \quad |x - y| + |u(x) - u(y)| + |u_x(x) - u_x(y)| < d + C_0 d^{1+\varkappa} + C_1 d^\varkappa,$$

$$\forall x \in G_0^d.$$

Now we choose $d > 0$ such that the inequality

$$(10.2.39) \quad d + C_0 d^{1+\varkappa} + C_1 d^\varkappa \leq d_\delta$$

holds. For such d we may guarantee (10.2.36) in G_0^d.

Now we shall estimate the second derivatives of the problem $(QLRP)$ solution. We make the transformation $x = \varrho x'$, $u(\varrho x') = v(x')$. Then $(x_1, ..., x_N) \in G_{\varrho/2}^\varrho \to G_{1/2}^1 \ni (x_1', ..., x_N')$ and the function $v(x')$ satisfies the problem

$$(QLRP)'' \quad \begin{cases} a^{ij}(x')v_{x_i'x_j'} = F(x'), & x' \in G_{1/2}^1, \\ \frac{\partial v}{\partial \vec{n}'} + \frac{1}{|x'|}\gamma(\varrho x')v(x') = \varrho g((\varrho x'), & x \in \Gamma_{1/2}^1, \end{cases}$$

where

$$(10.2.40) \quad \begin{aligned} a^{ij}(x') &\equiv a_{ij}(\varrho x', v(x'), \varrho^{-1}v_{x'}(x')), \\ F(x') &\equiv -\varrho^2 a(\varrho x', v(x'), \varrho^{-1}v_{x'}(x')). \end{aligned}$$

Because of (10.2.36), we can apply Theorem 10.17 about interior and near a smooth portion of the boundary L^2–estimate to the solution of the $(QLRP)''$ equation:

$$(10.2.41) \quad \int\limits_{G^1_{1/2}} (v^2_{x'x'} + |\nabla'v|^2)dx' \le C_4 \int\limits_{G^2_{1/4}} \Big(v^2(x') + F^2(x') +$$

$$+ \varrho^2(|\nabla'\mathcal{G}|^2 + \mathcal{G}^2)dx',$$

where the constant C_4 does not depend on v, F, g and it is defined by ν, μ, $\|\gamma(x)\|_{C^1(\Gamma^2_{1/4})}$, the continuity moduli of $a^{ij}(x')$ and $G^2_{1/4}$. Returning to the variable x and the function $u(x)$ in (10.2.41) we obtain

$$(10.2.42) \quad \int\limits_{G^\varrho_{\varrho/2}} r^\alpha u^2_{xx}dx \le C_4 \int\limits_{G^{2\varrho}_{\varrho/4}} \Big(r^{\alpha-4}u^2 + r^\alpha a^2(x,u,u_x) + r^\alpha|\nabla\mathcal{G}|^2 +$$

$$+ r^{\alpha-2}\mathcal{G}^2 \Big)dx.$$

Putting in (10.2.42) $\varrho = 2^{-k}d$ and summing up over $k = 0, 1, ..., \log_2(d/\varepsilon)$ $\forall \varepsilon \in (0, d)$ we get

$$(10.2.43) \quad \int\limits_{G^d_\varepsilon} r^\alpha u^2_{xx}dx \le C_4 \int\limits_{G^{2d}_{\varepsilon/4}} \Big(r^{\alpha-4}u^2 + r^\alpha a^2(x,u,u_x) +$$

$$+ r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}\mathcal{G}^2 \Big)dx.$$

By the assumption (D) and (10.2.18) with regard to (10.2.35), we have

$$(10.2.44) \quad \int\limits_{G^d_\varepsilon} r^\alpha u^2_{xx}dx \le C_4 \int\limits_{G^{2d}_0} \Big(r^{\alpha-4}u^2 + r^\alpha f^2(x) + r^\alpha b^2(x) +$$

$$+ r^{\alpha-2}|\nabla u|^2 + r^\alpha|\nabla\mathcal{G}|^2 + r^{\alpha-2}\mathcal{G}^2 \Big)dx, \quad \forall \varepsilon > 0,$$

where the constant C_4 does not depend on ε. Therefore we can perform the passage to the limit as $\varepsilon \to +0$, by the Fatou theorem, and we get

$$(10.2.45) \quad \int\limits_{G^d_0} r^\alpha u^2_{xx}dx \le C_4 \Big\{ \int\limits_{G^{2d}_0} \Big(r^{\alpha-4}u^2 + r^\alpha f^2(x) + r^\alpha b^2(x) +$$

$$+ r^{\alpha-2}|\nabla u|^2 \Big)dx + \|g\|^2_{\overset{o}{W}{}^{1/2}_\alpha(\Gamma^{2d}_0)} \Big\}.$$

On the basis of the inequalities (10.2.35) and (10.2.45), we have $u(x) \in \overset{o}{W}{}^2_\alpha(G^d_0)$. Now we shall prove (10.2.34). Let $\zeta(r) \in C^2[0, d]$ be a cut off

function such that

$$\zeta(r) \equiv 1 \ for \ r \in [0, d/2], \quad 0 \le \zeta(r) \le 1 \ for \ r \in [d/2, d],$$

$$\zeta(r) \equiv 0 \ for \ r > d, \quad \zeta(d) = \zeta'(d) = 0,$$

$$|\zeta'(r)| \le \frac{C}{r} \ for \ r \in [d/2, d].$$

We multiply both sides of the $(QLRP)$ equation by $\zeta^2(r)r^{\alpha-2}u(x)$ and integrate over G_0^d. We get

$$(10.2.46) \quad \int_{G_0^d} \zeta^2(r)r^{\alpha-2}u \bigtriangleup u \, dx = -\int_{G_0^d} \zeta^2(r)r^{\alpha-2}u\{(a_{ij}(x, u, u_x)-$$

$$- a_{ij}(0, u(0), 0))u_{x_i x_j} + a(x, u, u_x)\}dx.$$

We apply the Gauss-Ostrogradskiy formula:

$$\int_{G_0^d} \zeta^2(r)r^{\alpha-2}u \bigtriangleup u \, dx = \int_{\partial G_0^d} \zeta^2(r)r^{\alpha-2}u \frac{\partial u}{\partial \vec{n}}ds-$$

$$- \int_{G_0^d} \zeta^2(r)r^{\alpha-2}|\nabla u|^2 dx-$$

$$(10.2.47)$$

$$- \int_{G_0^d} \zeta(r)\zeta'(r)x_i r^{\alpha-3}\frac{\partial u^2}{\partial x_i}dx+$$

$$+ \frac{2-\alpha}{2} \int_{G_0^d} \zeta^2(r)x_i r^{\alpha-4}\frac{\partial u^2}{\partial x_i}dx.$$

Because of the $(QLRP)$ boundary condition and by the properties of $\zeta(r)$, we obtain

$$\int_{G_0^d} \zeta^2(r)r^{\alpha-2}u \bigtriangleup u \, dx = -\int_{G_0^d} \zeta^2(r)r^{\alpha-2}|\nabla u|^2 dx-$$

$$- \int_{G_0^d} \zeta(r)\zeta'(r)x_i r^{\alpha-3}\frac{\partial u^2}{\partial x_i}dx+$$

$$(10.2.48)$$

$$+ \frac{2-\alpha}{2} \int_{G_0^d} \zeta^2(r)x_i r^{\alpha-4}\frac{\partial u^2}{\partial x_i}dx+$$

$$+ \int_{\Gamma_0^d} \zeta^2(r)r^{\alpha-2}u\{g(x) - \frac{1}{r}\gamma(x)u\}ds.$$

Now we calculate the second and the third integral from the right in (10.2.48). For this we use the Gauss-Ostrogradskiy formula once more

$$\int\limits_{G_0^d} \zeta(r)\zeta'(r)x_i r^{\alpha-3}\frac{\partial u^2}{\partial x_i}dx = \int\limits_{\partial G_0^d} \zeta(r)\zeta'(r)r^{\alpha-3}u^2 x_i \cos(\vec{n},x_i)ds-$$

$$-\int\limits_{G_0^d} u^2 \frac{\partial}{\partial x_i}\big(\zeta(r)\zeta'(r)x_i r^{\alpha-3}\big)dx = \int\limits_{\Omega_d\cup\Gamma_0^d} \zeta(r)\zeta'(r)r^{\alpha-3}u^2 x_i \cos(\vec{n},x_i)ds-$$

(10.2.49)

$$-\int\limits_{G_0^d} u^2 \Big\langle \zeta'^2(r)r^{\alpha-2} + \zeta(r)\zeta''(r)r^{\alpha-2} + (\alpha+N-3)\zeta(r)\zeta'(r)r^{\alpha-3}\Big\rangle dx$$

and

$$\int\limits_{G_0^d} \zeta^2(r)x_i r^{\alpha-4}\frac{\partial u^2}{\partial x_i}dx = \int\limits_{\partial G_0^d} \zeta^2(r)r^{\alpha-4}u^2 x_i \cos(\vec{n},x_i)ds-$$

$$-\int\limits_{G_0^d} u^2 \frac{\partial}{\partial x_i}\langle\zeta^2(r)x_i r^{\alpha-4}\rangle dx = \int\limits_{\Omega_d\cup\Gamma_0^d} \zeta^2(r)r^{\alpha-4}u^2 x_i \cos(\vec{n},x_i)ds-$$

(10.2.50) $$\qquad -\int\limits_{G_0^d} u^2 \Big\langle 2\zeta(r)\zeta'(r)r^{\alpha-3} + (\alpha+N-4)\zeta^2(r)r^{\alpha-4}\Big\rangle dx.$$

Since $\zeta(r)\big|_{\Omega_d} = 0$, $\zeta'(r)\big|_{\Gamma_0^{d/2}} = 0$ and $x_i\cos(\vec{n},x_i)\big|_{\Gamma_0^d} = 0$, from (10.2.46)-(10.2.50) it follows that

$$\int\limits_{G_0^d} \zeta^2(r)r^{\alpha-2}|\nabla u|^2 dx + \frac{(2-\alpha)(\alpha+N-4)}{2}\int\limits_{G_0^d} \zeta^2(r)r^{\alpha-4}u^2 dx+$$

(10.2.51) $$\qquad +\int\limits_{\Gamma_0^d} \zeta^2(r)r^{\alpha-3}\gamma(x)u^2 ds = \int\limits_{\Gamma_0^d} \zeta^2(r)r^{\alpha-2}ug(x)ds+$$

$$+\int\limits_{G_{d/2}^d} u^2 \Big\langle (2\alpha+N-5)\zeta(r)\zeta'(r)r^{\alpha-3} + \zeta(r)\zeta''(r)r^{\alpha-2} + (\zeta'(r))^2 r^{\alpha-2}\Big\rangle dx+$$

$$+\int\limits_{G_0^d} \zeta^2(r)r^{\alpha-2}u\{(a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0))u_{x_i x_j} + a(x,u,u_x)\}dx.$$

Now, by the Cauchy inequality, we estimate the first integral from the right

$$\int_{\Gamma_0^d} \zeta^2(r)r^{\alpha-2}|u||g|ds = \int_{\Gamma_0^d} \zeta^2(r)\Big(r^{\frac{\alpha-1}{2}}\frac{1}{\sqrt{\gamma(x)}}|g|\Big)\Big(r^{\frac{\alpha-3}{2}}\sqrt{\gamma(x)}|u|\Big)ds \le$$

$$(10.2.52) \qquad \le \frac{1}{2}\int_{\Gamma_0^d} \zeta^2(r)r^{\alpha-3}\gamma(x)u^2 ds + \frac{1}{2\gamma_0}\int_{\partial G} r^{\alpha-1}g^2 ds.$$

Choosing an adequate δ in (10.2.52) we obtain from (10.2.51) the estimate

$$\int_{G_0^d} \zeta^2(r)r^{\alpha-2}|\nabla u|^2 dx + \frac{(2-\alpha)(\alpha+N-4)}{2}\int_{G_0^d} \zeta^2(r)r^{\alpha-4}u^2 dx +$$

$$+\frac{1}{2}\int_{\Gamma_0^d} \zeta^2(r)r^{\alpha-3}\gamma(x)u^2 ds \le \int_{G_{d/2}^d} u^2 \langle (2\alpha+N-5)\zeta(r)\zeta'(r)r^{\alpha-3} +$$

$$+\zeta(r)\zeta''(r)r^{\alpha-2} + (\zeta'(r))^2 r^{\alpha-2}\rangle dx + \int_{G_0^d} \zeta^2(r)r^{\alpha-2}ua(x,u,u_x)dx +$$

$$+\int_{G_0^d} \zeta^2(r)r^{\alpha-2}uu_{x_ix_j}(a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0))dx +$$

$$(10.2.53) \qquad +\int_{G_0^d} \zeta^2(r)r^{\alpha-2}ua(x,u,u_x)dx + \frac{1}{2\gamma_0}\int_{\partial G} r^{\alpha-1}g^2 ds.$$

Using the Cauchy inequality, (10.2.36) and (10.2.45) we obtain

$$(10.2.54) \quad \int_{G_0^d} \zeta^2(r)r^{\alpha-2}uu_{x_ix_j}(a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0))dx \le$$

$$\le \delta \int_{G_0^d} r^{\alpha-2}|u_{xx}||u|dx \le \frac{\delta}{2}\int_{G_0^d} \langle r^\alpha |u_{xx}|^2 + r^{\alpha-4}u^2\rangle dx \le$$

$$\le \delta C_5 \int_{G_0^{2d}} \langle r^{\alpha-4}u^2 + r^\alpha f^2(x) + r^\alpha b^2(x) + r^{\alpha-2}|\nabla u|^2\rangle dx +$$

$$+\frac{1}{2\gamma_0}\int_{\partial G} r^{\alpha-1}g^2 ds, \ \forall \delta > 0.$$

From the assumption (D) and by (10.2.1) and (10.2.18) we get

$$r^{\alpha-2}ua(x,u,u_x) \le r^{\alpha-2}|u|\langle \mu_1|\nabla u|^2 + b(x)|\nabla u| + f(x)\rangle \le$$

$$\le |\nabla u|\langle \mu_1 r^{\alpha-2}|u||\nabla u| + r^{\alpha-2}b(x)|u|\rangle + r^{\alpha-2}|u||f| \le$$

$$\le C_1 d^\varkappa \langle r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2 + r^\alpha b^2(x)\rangle + \frac{\delta}{2}r^{\alpha-4}u^2 +$$

$$+ \frac{1}{2\delta}r^\alpha f^2, \; \forall \delta > 0$$

and therefore

$$(10.2.55) \quad \int\limits_{G_0^d} \zeta^2(r)r^{\alpha-2}ua(x,u,u_x)dx \le Cd^\varkappa \int\limits_{G_0^d} \zeta^2(r)r^{\alpha-2}|\nabla u|^2 dx +$$

$$+ \left(Cd^\varkappa + \frac{\delta}{2}\right) \int\limits_{G_0^d} \zeta^2(r)r^{\alpha-4}u^2(x)dx + Cd^\varkappa \int\limits_{G_0^d} \zeta^2(r)r^\alpha b^2(x)dx +$$

$$+ \frac{1}{2\delta}\int\limits_{G_0^d} \zeta^2(r)r^\alpha f^2(x)dx, \; \forall\delta > 0.$$

From (10.2.53) and (10.2.54), (10.2.55) it follows that

$$\int\limits_{G_0^{d/2}} r^{\alpha-2}|\nabla u|^2 dx + \frac{(2-\alpha)(\alpha+N-4)}{2} \int\limits_{G_0^{d/2}} r^{\alpha-4}u^2 dx \le$$

$$\le C_5(\delta + d^\varkappa) \int\limits_{G_0^{d/2}} \langle r^{\alpha-2}|\nabla u|^2 + r^{\alpha-4}u^2\rangle dx +$$

$$(10.2.56)$$

$$+ C_6 \int\limits_{G_0^{2d}} r^\alpha(b^2(x) + f^2(x))dx +$$

$$+ C_7 \int\limits_{G_{d/2}^{2d}} (|\nabla u|^2 + u^2)dx + \frac{1}{2\gamma_0}\int\limits_{\partial G} r^{\alpha-1}g^2 ds, \; \forall\delta > 0.$$

In our case $N + \alpha - 4 > 0$. If $\alpha < 2$ then we choose d, δ appropriately positive small and obtain

$$(10.2.57) \quad \int\limits_{G_0^{d/2}} r^{\alpha-2}|\nabla u|^2 dx \le C\Big\{|u|_0^2 + \frac{1}{2\gamma_0}\int\limits_{\partial G} r^{\alpha-1}g^2 ds +$$

$$+ \int\limits_{G_0^{2d}} \{|\nabla u|^2 + r^\alpha(b^2(x) + f^2(x))\} dx\Big\}.$$

If $\alpha = 2$ then again for appropriate positive small d, δ and, because of (10.2.35), we get the validity of (10.2.57). Now we use Lemma 1.40. From (10.2.45) and (10.2.57) with regard to (10.2.35), (10.2.34) follows.

Step 2. $\alpha = 4 - N, \ N \geq 2$.

Because of (10.2.1) and (10.2.18), we obtain

$$(10.2.58) \quad \int\limits_{G_0^d} \left(r^{2-N} |\nabla u|^2 + r^{-N} |u(x)|^2 \right) dx \leq$$

$$\leq C \operatorname{meas} \Omega \int\limits_0^d r^{2-N+2\varkappa} \cdot r^{N-1} dr \leq C d^{2+2\varkappa} \leq \operatorname{const.}$$

Hence it follows that $u \in \overset{\circ}{W}{}^1_{2-N}(G)$. We repeat verbatim the arguments of the deduction of (10.2.45) and (10.2.56) for $\alpha = 4 - N$. We obtain

$$\int\limits_{G_0^d} r^{4-N} u_{xx}^2 dx \leq C_4 \Big\{ \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma_0^{2d})} + \int\limits_{G_0^{2d}} \big(r^{-N} u^2 + r^{4-N} f^2(x) +$$

$$(10.2.59) \qquad\qquad + r^{4-N} b^2(x) + r^{2-N} |\nabla u|^2 \big) dx \Big\}$$

and

$$(10.2.60) \quad \frac{1}{2} \int\limits_{\Gamma_0^{d/2}} r^{1-N} \gamma(x) u^2(x) ds + \int\limits_{G_0^{d/2}} r^{2-N} |\nabla u|^2 dx \leq$$

$$\leq C_5(\delta + d^\varkappa) \int\limits_{G_0^{d/2}} \langle r^{2-N} |\nabla u|^2 + r^{-N} u^2 \rangle dx + C_6 \int\limits_{G_0^{2d}} r^{4-N} (b^2(x) + f^2(x)) dx +$$

$$+ C_7 \int\limits_{G_{d/2}^{2d}} (|\nabla u|^2 + u^2) dx + C_8 \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma_0^{2d})}, \quad \forall \delta > 0.$$

Since $u \in \overset{\circ}{W}{}^1_{2-N}(G)$ we can apply the Hardy-Friedrichs-Wirtinger inequality (2.5.12) for $\alpha = 4 - N$. Then from (10.2.59) and (10.2.60) we obtain again the validity of (10.2.34).

Step 3. $4 - N - 2\lambda < \alpha < 4 - N.$

From the assumption (D) it follows that $b(x), f(x) \in \overset{\circ}{W}{}^0_{4-N}(G^d_0)$. In the second step we proved that $u(x) \in \overset{\circ}{W}{}^2_{4-N}(G^{d/2}_0)$. It means

$$(10.2.61) \qquad \int_{G^{d/2}_0} \left(r^{4-N}u^2_{xx} + r^{2-N}|\nabla u|^2 + r^{-N}|u(x)|^2\right)dx < \infty.$$

In this step we use the quasi-distance $r_\varepsilon(x) = \sqrt{(x_1 + \varepsilon)^2 + \sum_{i=2}^{N} x_i^2}$ (see §1.4).

Similar to (10.2.41) we obtain

$$\int_{G^1_{1/2}} (u^2_{x'x'} + |\nabla'u|^2)dx' \leq C_4 \int_{G^2_{1/4}} \left(u^2(x') + \varrho^4 a^2(\varrho x', u, \varrho^{-1}u_{x'}) + \right.$$

$$(10.2.62) \qquad\qquad \left. + \varrho^2(|\nabla'\mathcal{G}|^2 + \mathcal{G}^2)\right)dx'.$$

We put $\varrho = 2^{-k}d$ and notice that

$$2^{-k-1}d + \varepsilon < r + \varepsilon < 2^{-k}d + \varepsilon$$

in $G^{(k)}$ and

$$2^{-k-2}d + \varepsilon < r + \varepsilon < 2^{-k+1}d + \varepsilon$$

in $G^{(k-1)} \cup G^{(k)} \cup G^{(k+1)}$. We multiply the inequality (10.2.62) by $(2^{-k}d + \varepsilon)^{\alpha-2}$. Returning to the variable x, we obtain

$$(10.2.63) \quad \int_{G^{(k)}} \left(r^2(r + \varepsilon)^{\alpha-2}u^2_{xx} + (r + \varepsilon)^{\alpha-2}|\nabla u|^2\right)dx \leq$$

$$\leq C \int_{G^{(k-1)}\cup G^{(k)}\cup G^{(k+1)}} \left(r^{-2}(r + \varepsilon)^{\alpha-2}u^2 + r^2(r + \varepsilon)^{\alpha-2}a^2(x, u, u_x) + \right.$$

$$\left. + r^2(r + \varepsilon)^{\alpha-2}|\nabla\mathcal{G}|^2 + (r + \varepsilon)^{\alpha-2}\mathcal{G}^2\right)dx, \quad \forall\varepsilon > 0.$$

And, in virtue of property 2) for $r_\varepsilon(x)$, we have

$$(10.2.64) \quad \int\limits_{G^{(k)}} \left(r^2 r_\varepsilon^{\alpha-2} u_{xx}^2 + r_\varepsilon^{\alpha-2} |\nabla u|^2 \right) dx \le$$

$$\le C \int\limits_{G^{(k-1)} \cup G^{(k)} \cup G^{(k+1)}} \left(r^{-2} r_\varepsilon^{\alpha-2} u^2 + r^2 r_\varepsilon^{\alpha-2} a^2(x, u, u_x) + \right.$$

$$\left. + r^2 r_\varepsilon^{\alpha-2} |\nabla \mathcal{G}|^2 + r_\varepsilon^{\alpha-2} \mathcal{G}^2 \right) dx, \ \forall \varepsilon > 0.$$

Hence, by summing up over all $k = 0, 1, 2, \ldots$, we get

$$(10.2.65) \quad \int\limits_{G_0^d} \left(r^2 r_\varepsilon^{\alpha-2} u_{xx}^2 + r_\varepsilon^{\alpha-2} |\nabla u|^2 \right) dx \le \int\limits_{G_0^{2d}} \left(r^{-2} r_\varepsilon^{\alpha-2} u^2 + \right.$$

$$\left. + r^2 r_\varepsilon^{\alpha-2} a^2(x, u, u_x) + r^2 r_\varepsilon^{\alpha-2} |\nabla \mathcal{G}|^2 + r_\varepsilon^{\alpha-2} \mathcal{G}^2 \right) dx, \ \forall \varepsilon > 0.$$

Now we multiply both sides of the $(QLRP)_0$ equation by $\zeta^2(r) r_\varepsilon^{\alpha-2} u(x)$ and integrate over G_0^d. We obtain

$$(10.2.66) \quad \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} u \, \triangle \, u dx = - \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} u \Big\{ a(x, u, u_x) + $$

$$+ (a_{ij}(x, u, u_x) - a_{ij}(0, u(0), 0)) u_{x_i x_j} \Big\} dx, \ \forall \varepsilon > 0.$$

Using the Gauss-Ostrogradskiy formula in the integral from the left and the $(QLRP)_0$ boundary condition we obtain

$$\int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} u \, \triangle \, u dx = - \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} |\nabla u|^2 dx - $$

$$- \int\limits_{G_0^d} \zeta(r) \zeta'(r) r_\varepsilon^{\alpha-2} \frac{x_i}{r} \frac{\partial u^2}{\partial x_i} dx + $$

$$(10.2.67) \qquad\qquad + \frac{2-\alpha}{2} \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-3} \frac{\partial r_\varepsilon}{\partial x_i} \frac{\partial u^2}{\partial x_i} dx + $$

$$+ \int\limits_{\Gamma_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} u \Big\{ g(x) - \frac{1}{r} \gamma(x) u \Big\} ds.$$

From (10.2.66), (10.2.67) it follows that

$$(10.2.68) \quad \int\limits_{G_0^d} \zeta^2(r)r_\varepsilon^{\alpha-2}|\nabla u|^2 dx + \int\limits_{\Gamma_0^d} \zeta^2(r)r_\varepsilon^{\alpha-2}u^2\frac{1}{r}\gamma(x)ds =$$

$$= \int\limits_{G_0^d} \frac{\partial u^2}{\partial x_i}\Big\{\frac{2-\alpha}{2}\zeta^2(r)r_\varepsilon^{\alpha-3}\frac{\partial r_\varepsilon}{\partial x_i} - \zeta(r)\zeta'(r)r_\varepsilon^{\alpha-2}\frac{x_i}{r}\Big\}dx+$$

$$+ \int\limits_{G_0^d} \zeta^2(r)r_\varepsilon^{\alpha-2}u\left\{(a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0))u_{x_i x_j} + a(x,u,u_x)\right\}dx+$$

$$+ \int\limits_{\Gamma_0^d} \zeta^2(r)r_\varepsilon^{\alpha-2}ug(x)ds.$$

We shall estimate the first integral from the right. To this end we use the Gauss-Ostrogradskiy formula once more and take into account property 5) of $r_\varepsilon(x)$ and Lemma 1.10

$$x_i\cos(\vec{n},x_i)\Big|_{\Gamma_0^d} = 0, \quad \cos(\vec{n},x_1)\Big|_{\Gamma_0^d} = -\sin\frac{\omega_0}{2}, \quad \zeta(d) = 0.$$

As a result we obtain

$$(10.2.69) \quad \int\limits_{G_0^d} \frac{\partial u^2}{\partial x_i}\Big\{\frac{2-\alpha}{2}\zeta^2(r)r_\varepsilon^{\alpha-3}\frac{\partial r_\varepsilon}{\partial x_i} - \zeta\zeta'r_\varepsilon^{\alpha-2}\frac{x_i}{r}\Big\} =$$

$$= -\frac{2-\alpha}{2}\varepsilon\sin\frac{\omega_0}{2}\cdot\int\limits_{\Gamma_0^d} \zeta^2(r)r_\varepsilon^{\alpha-4}u^2 ds + \int\limits_{G_0^d} u^2\frac{\partial}{\partial x_i}\left(\zeta\zeta'r_\varepsilon^{\alpha-2}\frac{x_i}{r}\right)dx+$$

$$+\frac{\alpha-2}{2}\int\limits_{G_0^d} u^2\frac{\partial}{\partial x_i}\left(\zeta^2(r)r_\varepsilon^{\alpha-3}\frac{\partial r_\varepsilon}{\partial x_i}\right)dx = \frac{\alpha-2}{2}\varepsilon\sin\frac{\omega_0}{2}\cdot\int\limits_{\Gamma_0^d}\zeta^2(r)r_\varepsilon^{\alpha-4}u^2 ds+$$

$$+\frac{(\alpha-2)(\alpha+N-4)}{2}\int\limits_{G_0^d}\zeta^2(r)r_\varepsilon^{\alpha-4}u^2 dx+$$

$$+2(\alpha-2)\int\limits_{G_0^d}\zeta\zeta'r_\varepsilon^{\alpha-4}\left(r+\varepsilon\frac{x_1}{r}\right)u^2 dx + \int\limits_{G_0^d}u^2 r_\varepsilon^{\alpha-2}\Big(\zeta'^2 + \zeta\zeta''+$$

$$+\frac{N-1}{r}\zeta\zeta'\Big)dx.$$

Finally, from (10.2.68) and (10.2.69) we get

$$
\int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} |\nabla u|^2 dx + \frac{2-\alpha}{2}\varepsilon \sin\frac{\omega_0}{2} \cdot \int\limits_{\Gamma_0^d} \zeta^2(r) r_\varepsilon^{\alpha-4} u^2 ds +
$$

$$
+ \int\limits_{\Gamma_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} u^2 \frac{1}{r}\gamma(x)ds = \frac{(2-\alpha)(4-\alpha-N)}{2} \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-4} u^2 dx +
$$

$$
+ 2(\alpha-2)\int\limits_{G_0^d} \zeta\zeta' r_\varepsilon^{\alpha-4}\left(r + \varepsilon\frac{x_1}{r}\right) u^2 dx + \int\limits_{G_0^d} u^2 r_\varepsilon^{\alpha-2}\Big(\zeta'^2 + \zeta\zeta'' +
$$

$$
+ \frac{N-1}{r}\zeta\zeta'\Big)dx + \int\limits_{\Gamma_0^d}\zeta^2(r) r_\varepsilon^{\alpha-2} u g(x)ds + \int\limits_{G_0^d}\zeta^2(r) r_\varepsilon^{\alpha-2} u\Big\{a(x,u,u_x) +
$$

$$
+ (a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0))u_{x_i x_j}\Big\}dx,\ \forall\varepsilon > 0.
$$

Using the properties of the function $\zeta(r)$ hence follows

$$
(10.2.70)\qquad \int\limits_{G_0^d}\zeta^2(r) r_\varepsilon^{\alpha-2}|\nabla u|^2 dx + \int\limits_{\Gamma_0^d}\zeta^2(r) r_\varepsilon^{\alpha-2} u^2\frac{1}{r}\gamma(x)ds \le
$$

$$
\le \frac{(2-\alpha)(4-\alpha-N)}{2}\int\limits_{G_0^d}\zeta^2(r) r_\varepsilon^{\alpha-4} u^2 dx + c\int\limits_{G_{d/2}^d} u^2 dx +
$$

$$
+ \int\limits_{\Gamma_0^d}\zeta^2(r) r_\varepsilon^{\alpha-2}|u||g(x)|ds + \int\limits_{G_0^d}\zeta^2(r) r_\varepsilon^{\alpha-2} u\Big\{a(x,u,u_x) +
$$

$$
+ (a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0))u_{x_i x_j}\Big\}dx,\ \forall\varepsilon > 0.
$$

Let $d > 0$ be such that (10.2.39) and (10.2.36) hold. Using the Cauchy inequality we obtain

$$
(10.2.71)\quad \int\limits_{G_0^d}\zeta^2(r) r_\varepsilon^{\alpha-2}(a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0))vv_{x_i x_j}dx \le
$$

$$
\le \delta\int\limits_{G_0^d}\zeta^2(r) r_\varepsilon^{\alpha-2}(r|u_{xx}|)(r^{-1}|u|)dx \le
$$

$$
\le \frac{\delta}{2}\int\limits_{G_0^d}\Big\langle\zeta^2(r) r^2 r_\varepsilon^{\alpha-2} u_{xx}^2 + \zeta^2(r) r^{-2} r_\varepsilon^{\alpha-2} u^2\Big\rangle dx,\ \forall\delta > 0
$$

In addition, by the assumption (D) and the estimates $(10.2.1)$ and $(10.2.18)$, we get

$$(10.2.72) \quad \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} |u| |a(x, u, u_x)| dx \le$$

$$\le \frac{1}{2}(C_1 d^\varkappa + \delta) \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} r^{-2} u^2 dx + \frac{1}{2\delta} \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} r^2 f^2 dx +$$

$$+ \mu_1 C_0 d^{1+\varkappa} \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} |\nabla u|^2 dx + \frac{1}{2} C_1 d^\varkappa \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} r^2 b^2 dx, \quad \forall \delta > 0.$$

Because of the property of $r_\varepsilon(x)$, we have $r_\varepsilon \ge r$. From $\alpha \le 2$ it follows that $r_\varepsilon^{\alpha-2} \le r^{\alpha-2}$. We know also that $b(x), f(x) \in \overset{\circ}{W}{}_\alpha^0(G)$ and therefore

$$\frac{1}{2\delta} \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} r^2 f^2 dx \le \frac{1}{2\delta} \int\limits_G r^\alpha f^2 dx \text{ and}$$

$$(10.2.73)$$

$$\frac{1}{2} C_1 d^\varkappa \int\limits_{G_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} r^2 b^2 dx \le \frac{1}{2} C_1 d^\varkappa \int\limits_G r^\alpha b^2 dx.$$

By the Cauchy inequality with regard to $\gamma(x) \ge \gamma_0 > 0$,

$$|u||g| = \left(r^{1/2} \frac{1}{\sqrt{\gamma(x)}} |g| \right) \left(r^{-1/2} \sqrt{\gamma(x)} |u| \right) \le \frac{\delta}{2} r^{-1} \gamma(x) u^2 + \frac{1}{2\delta\gamma_0} r g^2,$$

$$\forall \delta > 0.$$

Taking into account the first property of r_ε we obtain

$$(10.2.74) \quad \int\limits_{\Gamma_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} u g(x) ds \le \frac{\delta}{2} \int\limits_{\Gamma_0^d} \zeta^2(r) r_\varepsilon^{\alpha-2} \frac{1}{r} \gamma(x) u^2 ds +$$

$$+ \frac{1}{2\delta\gamma_0} \int\limits_{\Gamma_0^d} r^{\alpha-1} g^2 ds, \quad \forall \delta > 0.$$

From (10.2.70) - (10.2.74) it follows that

$$\int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2}|\nabla u|^2 dx + (1-\delta)\int\limits_{\Gamma_0^{d/2}} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)ds \le$$

$$\le \frac{\delta}{2}\int\limits_{G_0^d} r^2 r_\varepsilon^{\alpha-2}u_{xx}^2 dx + \frac{(2-\alpha)(4-N-\alpha)}{2}\int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-4}u^2 dx +$$

$$(10.2.75)\quad +\frac{1}{2}(C_1 d^\varkappa + 2\delta)\int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2}r^{-2}u^2 dx + Cd^{1+\varkappa}\int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2}|\nabla u|^2 dx +$$

$$+C\int\limits_{G_0^{2d}} (u^2+|\nabla u|^2)dx + C\int\limits_{G_0^{2d}} r^\alpha(b^2+f^2)dx +$$

$$+\frac{1}{2\delta\gamma_0}\int\limits_{\Gamma_0^d} r^{\alpha-1}g^2 ds, \quad \forall \delta > 0.$$

Taking into account (10.2.65) and (10.2.75) and choosing $\delta > 0$ sufficiently small, we get

$$\int\limits_{G_0^{d/2}} (r^2 r_\varepsilon^{\alpha-2}u_{xx}^2 + r_\varepsilon^{\alpha-2}|\nabla u|^2)dx + \int\limits_{\Gamma_0^{d/2}} r^{-1}r_\varepsilon^{\alpha-2}\gamma(x)ds \le$$

$$\le \frac{(2-\alpha)(4-N-\alpha)}{2}\int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-4}u^2 dx + C(d^\varkappa+\delta)\int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2}r^{-2}u^2 dx +$$

$$(10.2.76)\quad +C\int\limits_{G_0^{2d}} \Big(u^2+|\nabla u|^2+r^\alpha(b^2+f^2)+r^\alpha|\nabla \mathcal{G}|^2+r^{\alpha-2}\mathcal{G}^2\Big)dx +$$

$$+\frac{1}{2\delta\gamma_0}\int\limits_{\Gamma_0^d} r^{\alpha-1}g^2 ds + Cd^{2\varkappa}\int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2}|\nabla u|^2 dx, \quad \forall \delta > 0, \ \forall \varepsilon > 0.$$

Since by (10.2.61) $u(x) \in \overset{\circ}{W}{}^2_{4-N}(G_0^{d/2})$, we can apply Theorem 2.20 and then we have (see the inequality (2.5.13))

$$\int\limits_\Omega u^2(r,\omega)d\Omega \le \frac{1}{\lambda(\lambda+N-2)}\Big\{\int\limits_\Omega |\nabla_\omega u(r,\omega)|^2 d\Omega + \int\limits_{\partial\Omega} \gamma(x)u^2(x)d\sigma\Big\},$$

for a.e. $r \in (0, d)$. Multiplying both sides of this inequality by $(\varrho + \varepsilon)^{\alpha - 2} r^{N-3}$ and integrating over $r \in (\frac{\varrho}{2}, \varrho)$ we obtain

$$\int\limits_{G^{\varrho}_{\varrho/2}} (\varrho + \varepsilon)^{\alpha - 2} r^{-2} u^2 dx \leq \frac{1}{\lambda(\lambda + N - 2)} \int\limits_{G^{\varrho}_{\varrho/2}} (\varrho + \varepsilon)^{\alpha - 2} |\nabla u|^2 dx +$$

$$+ \frac{1}{\lambda(\lambda + N - 2)} \int\limits_{\Gamma^{\varrho}_{\varrho/2}} r^{-1} (\varrho + \varepsilon)^{\alpha - 2} \gamma(x) u^2 ds, \quad \forall \varepsilon > 0$$

or since $\varrho + \varepsilon \sim r_\varepsilon$

$$\int\limits_{G^{\varrho}_{\varrho/2}} r_\varepsilon^{\alpha - 2} r^{-2} u^2 dx \leq \frac{1}{\lambda(\lambda + N - 2)} \Big\{ \int\limits_{G^{\varrho}_{\varrho/2}} r_\varepsilon^{\alpha - 2} |\nabla u|^2 dx +$$

$$+ \int\limits_{\Gamma^{\varrho}_{\varrho/2}} r^{-1} r_\varepsilon^{\alpha - 2} \gamma(x) u^2 ds \Big\}, \quad \forall \varepsilon > 0.$$

Letting $\rho = 2^{-k} d$, $(k = 0, 1, 2, ...)$ and summing up the obtained inequalities over all k, we get

$$(10.2.77) \quad \int\limits_{G^d_0} r_\varepsilon^{\alpha - 2} r^{-2} u^2 dx \leq \frac{1}{\lambda(\lambda + N - 2)} \Big\{ \int\limits_{G^d_0} r_\varepsilon^{\alpha - 2} |\nabla u|^2 dx +$$

$$+ \int\limits_{\Gamma^d_0} r^{-1} r_\varepsilon^{\alpha - 2} \gamma(x) u^2 ds \Big\}, \quad \forall \varepsilon > 0.$$

In addition, Corollary 10.36 and Lemma 2.37 hold. Therefore from (10.2.76) in virtue of (10.2.77) and (2.5.18) we obtain

$$K(\lambda, N, \alpha) \left\{ \int\limits_{G^{d/2}_0} r_\varepsilon^{\alpha - 2} |\nabla u|^2 dx + \int\limits_{\Gamma^{d/2}_0} r^{-1} r_\varepsilon^{\alpha - 2} \gamma(x) u^2(x) ds \right\} +$$

$$+ \int\limits_{G_0^{d/2}} r^2 r_\varepsilon^{\alpha-2} u_{xx}^2 dx \le \frac{1}{2\delta\gamma_0} \int\limits_{\Gamma_0^d} r^{\alpha-1} g^2 ds + C \int\limits_{G_0^{2d}} \Big(u^2 + |\nabla u|^2 +$$

$$(10.2.78) \qquad + r^\alpha (b^2 + f^2) + r^\alpha |\nabla \mathcal{G}|^2 + r^{\alpha-2}\mathcal{G}^2 \Big) dx +$$

$$+ C(d^\varkappa + \delta) \left\{ \int\limits_{G_0^{d/2}} r_\varepsilon^{\alpha-2} |\nabla u|^2 dx + \int\limits_{\Gamma_0^{d/2}} r^{-1} r_\varepsilon^{\alpha-2} \gamma(x) u^2(x) ds \right\}$$

for any $\delta > 0$ and $\varepsilon > 0$, where

$$K(\lambda, N, \alpha) = 1 - \frac{(2-\alpha)(4-N-\alpha)}{2} H(\lambda, N, \alpha) =$$

$$= 1 - \frac{2(2-\alpha)(4-N-\alpha)}{(4-N-\alpha)^2 + 4\lambda(\lambda+N-2)} > 0,$$

because of $4 - N - 2\lambda < \alpha < 4 - N$. We choose $\delta = \frac{K(\lambda,N,\alpha)}{4C}$ and $d > 0$ such that $d^\varkappa \le \frac{K(\lambda,N,\alpha)}{4C}$. As a result we get

$$\int\limits_{G_0^{d/2}} \left(r^2 r_\varepsilon^{\alpha-2} u_{xx}^2 + r_\varepsilon^{\alpha-2} |\nabla u|^2 \right) dx + \int\limits_{\Gamma_0^{d/2}} r^{-1} r_\varepsilon^{\alpha-2} \gamma(x) u^2(x) ds \le$$

$$\le C \int\limits_{G_0^{2d}} \left(u^2 + |\nabla u|^2 + r^\alpha (b^2 + f^2) + r^\alpha |\nabla \mathcal{G}|^2 + r^{\alpha-2}\mathcal{G}^2 \right) dx +$$

$$(10.2.79) \qquad + C \int\limits_{\Gamma_0^d} r^{\alpha-1} g^2 ds, \ \forall \varepsilon > 0.$$

We observe that the right side does not depend on ε. Therefore we can perform the passage to the limit as $\varepsilon \to +0$, by the Fatou Theorem. Hence we get

$$(10.2.80) \quad \int\limits_{G_0^{d/2}} \left(r^\alpha u_{xx}^2 + r^{\alpha-2} |\nabla u|^2 \right) dx \le C \int\limits_{G_0^{2d}} \Big(u^2 + |\nabla u|^2 +$$

$$+ r^\alpha (b^2 + f^2) + r^\alpha |\nabla \mathcal{G}|^2 + r^{\alpha-2}\mathcal{G}^2 \Big) dx + C \int\limits_{\Gamma_0^d} r^{\alpha-1} g^2 ds.$$

Finally, using Lemma1.40 we obtain the desired estimate (10.2.34).   □

THEOREM 10.39. *Let $u(x)$ be a solution of the problem (QLRP), $q > N$. Suppose that assumptions (A) - (E) are satisfied for $\beta > \lambda - 2$. Suppose, in*

addition, that $g(x) \in \overset{\circ}{W}{}^{1/2}_{4-N}(\partial G)$ and there is

(10.2.81) $\displaystyle\sup_{\rho>0}\rho^{-s}\|g\|_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma_0^\rho)} := k_2, \ s > \lambda.$

Then there exist numbers $d, C > 0$ not depending on u such that
$u(x) \in \overset{\circ}{W}{}^2_{4-N}(G_0^{d/2})$ *and*

(10.2.82) $\|u(x)\|_{\overset{\circ}{W}{}^2_{4-N}(G_0^\rho)} \leq C\Big(\|u\|_{W^1(G)} + k_1 + k_2\Big)\rho^\lambda, \ \rho \in (0, \frac{d}{2}).$

PROOF. The belonging $u(x) \in \overset{\circ}{W}{}^2_{4-N}(G_0^{d/2})$ follows from Theorem 10.38. So it is enough to derive the estimate (10.2.82). We set

(10.2.83) $\displaystyle V(\rho) \equiv \int\limits_{G_0^\rho} r^{2-N}|\nabla u|^2 dx + \int\limits_{\Gamma_0^\rho} r^{1-N}\gamma(x)u^2 ds$

and multiply both sides of the $(QLRP)_0$ equation by $r^{2-N}u(x)$ and integrate over G_0^ρ, $\rho \in (0, \frac{d}{2})$. As a result we obtain

$$V(\rho) = \int\limits_\Omega \Big(\rho u\frac{\partial u}{\partial r} + \frac{N-2}{2}u^2\Big)d\omega + \int\limits_{\Gamma_0^\rho} r^{2-N}ugds+$$

(10.2.84) $\displaystyle + \int\limits_{G_0^\rho} u(x)r^{2-N}\Big\{(a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0))u_{x_ix_j}+$

$$+ a(x,u,u_x)\Big\}dx, \ \rho \in (0, \frac{d}{2}).$$

We shall obtain an upper bound for each integral on the right. First of all, we use Lemma 2.35

(10.2.85) $\displaystyle \int\limits_\Omega \Big(\varrho u\frac{\partial u}{\partial r} + \frac{N-2}{2}u^2\Big)d\Omega \leq \frac{\varrho}{2\lambda}V'(\varrho).$

We estimate the second integral in (10.2.84). By the Cauchy inequality with regard to Lemma1.40, we get

(10.2.86) $\displaystyle \int\limits_{\Gamma_0^\rho} r^{2-N}ugds = \int\limits_{\Gamma_0^\rho} \big(r^{\frac{1-N}{2}}\gamma^{1/2}(x)u(x)\big)\big(r^{\frac{3-N}{2}}\gamma^{-1/2}(x)g(x)\big)ds \leq$

$$\leq \frac{\delta}{2}\int\limits_{\Gamma_0^\rho} r^{1-N}\gamma(x)u^2(x)ds + \frac{1}{2\delta\gamma_0}\int\limits_{\Gamma_0^\rho} r^{3-N}g^2(x)ds \leq$$

$$\leq \frac{\delta}{2}\int\limits_{\Gamma_0^\rho} r^{1-N}\gamma(x)u^2(x)ds + C\|g(x)\|^2_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma_0^\rho)}, \ \forall\delta > 0.$$

To estimate the third integral in (10.2.84) we use the assumption (A). Then we have

$$a_{ij}(x,u,z) \in W^{1,q}(\mathfrak{M}), q > N \Rightarrow a_{ij}(x,u,z) \in C^{\delta}(\mathfrak{M}), \ 0 < \delta < 1 - \frac{N}{q},$$

$$(i,j = 1,...,N),$$

by the embedding theorem. The last together with (10.2.38) means that

$$|a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0)| \le \delta(\varrho), \ |x| \le \varrho,$$

where

(10.2.87) $\delta(\varrho) \sim \varrho^{\delta \varkappa}, \ \delta \in \left(0, 1 - \frac{N}{q}\right).$

Therefore, by the Cauchy and the Hardy-Friedrichs-Wirtinger (2.5.12) inequalities, we obtain

(10.2.88) $\displaystyle\int_{G_0^\rho} r^{2-N}|u(x)||u_{x_i x_j}| \, |a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0)| \, dx \le$

$$\le \delta(\rho) \int_{G_0^\rho} r^{4-N}|u_{xx}|^2 dx + C\delta(\rho)V(\rho).$$

We apply the inequality (10.2.59) and once more the Hardy-Friedrichs-Wirtinger inequality (2.5.12). Then from (10.2.88) we get

(10.2.89) $\displaystyle\int_{G_0^\rho} r^{2-N}|u(x)||u_{xx}| \, |a_{ij}(x,u,u_x) - a_{ij}(0,u(0),0)| \, dx \le$

$$\le C\delta(\rho)\Big\{ V(\rho) + V(2\rho) + \|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G_0^{2\rho})} + \|b\|^2_{\overset{\circ}{W}{}^0_{4-N}(G_0^{2\rho})} +$$

$$+ \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma_0^{2\rho})} \Big\}.$$

Similar to (10.2.55), considering the Hardy-Friedrichs-Wirtinger inequality (2.5.12), we obtain

(10.2.90) $\displaystyle\int_{G_0^\rho} r^{2-N}u(x)a(x,u,u_x)dx \le C\Big\{ (\rho^\varkappa + \delta)V(\rho) +$

$$+ \rho^\varkappa \int_{G_0^\rho} r^{4-N}b^2(x))dx + \frac{1}{2\delta}\int_{G_0^\rho} r^{4-N}f^2(x)dx \Big\}, \quad \forall \delta > 0.$$

From (10.2.84), in virtue of (10.2.86) - (10.2.90) for $\delta = \varrho^\varepsilon$, $\forall \varepsilon > 0$, it follows that

(10.2.91)
$$V(\varrho) \le \frac{\varrho}{2\lambda} V'(\varrho) + C\delta(\varrho)V(2\varrho) + C\left(\delta(\varrho) + \varrho^\varkappa + \varrho^\varepsilon\right)V(\varrho) +$$
$$+ C\left\{\|b\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\varrho}_0)} + \varrho^{-\varepsilon}\|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\varrho}_0)} + \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma^{2\rho}_0)}\right\}.$$

Hence the Cauchy problem for the differential inequality follows

$$(CP) \quad \begin{cases} V'(\rho) - \mathcal{P}(\varrho)V(\varrho) + \mathcal{N}(\rho)V(2\rho) + \mathcal{Q}(\rho) \ge 0, & 0 < \rho < d, \\ V(d) \le V_0, \end{cases}$$

where

(10.2.92)
$$\mathcal{P}(\rho) = \frac{2\lambda}{\rho} - C\left(\frac{\delta(\rho)}{\rho} + \rho^{\varkappa - 1} + \rho^{\varepsilon - 1}\right);$$

(10.2.93)
$$\mathcal{N}(\rho) = C\frac{\delta(\rho)}{\rho};$$

and

(10.2.94)
$$\mathcal{Q}(\rho) = C\left\{\|b\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\varrho}_0)} + \varrho^{-\varepsilon}\|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\varrho}_0)} +\right.$$
$$\left. + \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma^{2\rho}_0)}\right\}\rho^{-1}.$$

We adjoin it to the initial condition $V(d) \le V_0$. By Theorem 10.38 for $\alpha = 4 - N$,

(10.2.95)
$$V(d) = \int\limits_{G^d_0} r^{2-N}|\nabla u|^2 dx + \int\limits_{\Gamma^d_0} r^{1-N}\gamma(x)u^2 ds \le C\left\{|u|^2_0 +\right.$$
$$+ \int\limits_{G^{2d}_0} \left(|\nabla u|^2 + r^{4-N}(b^2(x) + f^2(x))\right)dx +$$
$$\left. + \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma^{2d}_0)} + 1\right\} \equiv V_0.$$

By Theorem 1.57,

(10.2.96)
$$V(\varrho) \le \exp\left(\int\limits_\varrho^d \mathcal{B}(\tau)d\tau\right)\left\{V_0 \exp\left(-\int\limits_\varrho^d \mathcal{P}(\tau)d\tau\right) +\right.$$
$$\left. + \int\limits_\varrho^d \mathcal{Q}(\tau)\exp\left(-\int\limits_\varrho^\tau \mathcal{P}(\sigma)d\sigma\right)d\tau\right\}$$

with

$$\mathcal{B}(\varrho) = \mathcal{N}(\varrho) \exp\left(\int\limits_{\varrho}^{2\varrho} \mathcal{P}(\sigma) d\sigma\right).$$

Now, by means of simple calculations, from (10.2.92) and (10.2.93) with regard to (10.2.87) we have

$$(10.2.97) \quad \exp\left(-\int\limits_{\rho}^{d} \mathcal{P}(\tau) d\tau\right) \le C\left(\frac{\rho}{d}\right)^{2\lambda} \quad \text{and} \quad \int\limits_{\rho}^{d} \mathcal{B}(\tau) d\tau \le C = const.$$

In addition,

$$(10.2.98) \quad \mathcal{Q}(\tau) \exp\left(-\int\limits_{\rho}^{\tau} \mathcal{P}(\sigma) d\sigma\right) \le C\varrho^{2\lambda} \tau^{-2\lambda-1} \Big\{ \|b\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\tau}_0)} +$$

$$+ \tau^{-\varepsilon} \|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\tau}_0)} + \|g\|^2_{\overset{\circ}{W}{}^{1/2}_{4-N}(\Gamma^{2\tau}_0)} \Big\}, \ \forall \varepsilon > 0.$$

Let us recall the assumption (D)

$$\|b\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\tau}_0)} \le ck^2_1 \tau^{2\beta+4} \quad \text{and} \quad \tau^{-\varepsilon} \|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\tau}_0)} \le ck^2_1 \tau^{2\beta+4-\varepsilon}.$$

Since $\beta > \lambda - 2$, we can put $\beta = \lambda - 2 + \varepsilon$, $\forall \varepsilon > 0$. Therefore we get

$$(10.2.99) \quad \|b\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\tau}_0)} + \tau^{-\varepsilon} \|f\|^2_{\overset{\circ}{W}{}^0_{4-N}(G^{2\tau}_0)} \le ck^2_1 \tau^{2\lambda+\varepsilon}, \ \forall \varepsilon > 0.$$

From (10.2.98) and (10.2.99) with regard to (10.2.81) we obtain

$$(10.2.100) \quad \int\limits_{\varrho}^{d} \mathcal{Q}(\tau) \exp\left(-\int\limits_{\rho}^{\tau} \mathcal{P}(\sigma) d\sigma\right) d\tau \le C(k^2_1 + k^2_2)\varrho^{2\lambda}.$$

Finally, from (1.10.1), by (10.2.95), (10.2.97) and (10.2.100), it follows that

$$(10.2.101) \quad V(\rho) \le C(N, \lambda, d, \varkappa)\Big\langle \|u\|^2_{W^{1,2}(G)} + k^2_1 + k^2_2 \Big\rangle \varrho^{2\lambda}.$$

At last, from (10.2.59) and (10.2.101) we deduce the validity of (10.2.82). $\square$

10.2.4. The power modulus of continuity at the conical point for strong solutions. Now we shall make the exponent $\varkappa$ in the estimates (10.2.1) and (10.2.18) more precise and prove the Hölder continuity of the first derivatives of the strong solutions in the neighborhood of a conical point. Let λ be the number that is defined by (2.5.11) or (2.5.19) from Section 2.5.

THEOREM 10.40. *Let $\lambda > 1$ and let $u(x)$ be the problem (QLRP) strong solution, $q > N$. Suppose the assumptions (A), (AA), (C) - (E) are satisfied for $\beta > \lambda - 2 > -1$, $\delta \geq \lambda - 1 > 0$. Suppose that the function $g(x)$ satisfies the conditions of Theorem 10.39. Then there exist numbers $d > 0$, $\overline{C}_0, \overline{C}_1$ not depending on $u(x)$, but depending only on $N, \lambda, \nu, \mu, \mu_1, \beta, k_1, k_2, q, g_0, M_0, M_1$ and the domain G, such that*

1) $|u(x)| \leq \overline{C}_0 |x|^{\lambda}$ *and* $|\nabla u(x)| \leq \overline{C}_1 |x|^{\lambda-1}$, $x \in G_0^{d/2}$.

In addition, if $g(x) \in V_{q,\alpha}^{1-1/q}(\partial G)$ and

$$(10.2.102) \qquad \|g(x)\|_{V_{q,\alpha}^{1-1/q}(\Gamma_0^\varrho)} \leq C\varrho^{\lambda-2+\frac{\alpha+N}{q}}, \quad 0 < \rho < d/2$$

then there exist numbers $d > 0$, $\overline{C}_2$, not depending on $u(x)$ but only on $N, \lambda, \nu, \mu, \mu_1, \beta, k_1, k_2, q, g_0, M_0, M_1$ and the domain G, such that

2) *if $\alpha + q(\lambda - 2) + N > 0$ then $u(x) \in V_{q,\alpha}^2(G)$ and*

$$\|u(x)\|_{V_{q,\alpha}^2(G_0^\rho)} \leq \overline{C}_2 \rho^{\lambda-2+\frac{N+\alpha}{q}}, \quad 0 < \rho < d/2$$

and

3) *if $1 < \lambda < 2$, $q > \frac{N}{2-\lambda}$ then $u(x) \in C^\lambda(\overline{G_0^{d/2}})$.*

PROOF. Let us consider the sets $G_{\rho/2}^\rho$ and $G_{\rho/4}^{2\rho} \supset G_{\rho/2}^\rho, 0 < \rho < d/2$. We make the transformation $x = \rho x'$; $w(x') = \rho^{-\lambda} u(\rho x')$. The function $w(x')$ satisfies the problem

$(QLRP)_0'$ $\qquad \begin{cases} a^{ij}(x')w_{x_i'x_j'} = F(x'), \ x' \in G_{1/2}^1, \\ \frac{\partial w}{\partial \vec{n}'} + \frac{1}{|x'|}\gamma(\rho x')w(x') = \varrho^{1-\lambda}g(\rho x'), \ x \in \Gamma_{1/2}^1, \end{cases}$

where

$$a^{ij}(x') \equiv a_{ij}(\rho x', u, \rho^{\lambda-1}w_{x'}(x'))$$

and

$$F(x') \equiv -\rho^{2-\lambda}a(\rho x', \rho^\lambda w(x'), \rho^{\lambda-1}w_{x'}(x')).$$

The L^q- estimate (10.2.23) is satisfied for the function $w(x')$ (see the proof of Theorem 10.37), that is

$$(10.2.103) \quad \|w\|_{W^{2,q}(G_{1/2}^1)}^q \leq C_3 \int_{G_{1/4}^2} \langle |w|^q + \rho^{q\lambda}|\nabla'w|^{2q} + \varrho^q|b|^q|\nabla'w|^q +$$

$$+ \rho^{q(2-\lambda)}|f|^q \rangle dx' + C_4 \rho^{q(1-\lambda)} \int_{G_{1/4}^2} (|\nabla'g|^q + |g|^q)dx',$$

where the constants C_3, C_4 do not depend on w.

At first we consider the case $\underline{2 \leq N < 4}$. By the Sobolev Imbedding Theorem, we have

$$(10.2.104) \qquad \sup_{x' \in G^1_{1/2}} |w(x')| \leq C \|w\|_{2,2;G^1_{1/2}}.$$

Returning to the variable x and because of the estimate (10.2.82) of Theorem 10.39, we get

$$\|w\|^2_{2,2;G^1_{1/2}} = \int\limits_{G^1_{1/2}} \left(|w_{x'x'}|^2 + |\nabla' w|^2 + w^2 \right) dx' \leq$$

$$(10.2.105) \qquad \leq C(N) \rho^{-2\lambda} \int\limits_{G^1_{1/2}} \left(r^{4-N} |u_{xx}|^2 + r^{2-N} |\nabla u|^2 + \right.$$

$$\left. + r^{-N} u^2 \right) dx \leq C.$$

From (10.2.104) and (10.2.105) it follows that

$$\sup_{x' \in G^1_{1/2}} |w(x')| \leq \overline{C_0},$$

and returning to the variable x we get

$$|u(x)| \leq \overline{C_0} \rho^\lambda, \quad x \in G^\rho_{\rho/2}.$$

Putting now $|x| = \frac{2}{3}\rho$ we obtain the first estimate of 1) of our theorem.

Let us now $N \geq 4$. We apply the Lieberman local maximum principle, Proposition 10.33. Then, by the condition (D), we have

$$(10.2.106) \quad \sup_{x' \in G^1_{1/2}} w(x') \leq C \left\{ \left(\int\limits_{G^2_{1/4}} w^2 dx' \right)^{\frac{1}{2}} + \rho^{1-\lambda+\delta} g_0 + \right.$$

$$\left. + \rho^{2-\lambda} \left(\int\limits_{G^2_{1/4}} |a(\rho x', \rho^\lambda w(x'), \rho^{\lambda-1} w_{x'})|^N dx' \right)^{\frac{1}{N}} \right\}.$$

We shall estimate each integral from the right hand side of (10.2.106). We estimate the first integral by (10.2.82) of Theorem 10.39 as

$$(10.2.107) \qquad \int\limits_{G^2_{1/4}} w^2 dx' \leq \rho^{-2\lambda} \int\limits_{G^{2\rho}_{\rho/4}} r^{-N} u^2 dx \leq C.$$

Because of the assumption (D) and the estimate (10.2.18), we have

$$(10.2.108) \quad \int\limits_{G_{1/4}^2} |a(\rho x', \rho^\lambda w(x'), \rho^{\lambda-1} w_{x'})|^N dx' \le c(N) \int\limits_{G_{\rho/4}^{2\rho}} \Big(\mu_1^N |\nabla u|^{2N} +$$

$$+ b^N(x) |\nabla u|^N + f^N(x) \Big) r^{-N} dx \le c(N) \int\limits_{G_{\rho/4}^{2\rho}} \Big(\mu_1^N (r^{2-N} |\nabla u|^2) \times$$

$$\times (r^{-2} |\nabla u|^{2N-2}) + (r^{2-N} |\nabla u|^2) \cdot (k_1^N r^{\beta N-2} |\nabla u|^{N-2}) + k_1^N r^{\beta N-N} \Big) dx \le$$

$$\le c(N) \big(\mu_1^N C_1^{2N-2} \rho^{2\varkappa(N-1)-2} + k_1^N C_1^{N-2} \rho^{\varkappa(N-2)+\beta N-2} \big) \int\limits_{G_{\rho/4}^{2\rho}} r^{2-N} |\nabla u|^2 dx$$

$$+ c(N)(\beta)^{-1}(k_1)^N \text{ meas}\Omega \ \big(2^{\beta N} - 2^{-2\beta N} \big) \rho^{\beta N}, \ 0 < \rho < d/2.$$

Because of (10.2.82), hence we obtain

$$(10.2.109) \quad \rho^{2-\lambda} \left(\int\limits_{G_{1/4}^2} |a(\rho x', \rho^\lambda w(x'), \rho^{\lambda-1} w_{x'})|^N dx' \right)^{\frac{1}{N}} \le$$

$$\le C \Big(\rho^{2-\lambda+\frac{2(\lambda-1)}{N}+\frac{2\varkappa(N-1)}{N}} + \rho^{2-\lambda+\beta+\frac{2(\lambda-1)}{N}+\frac{\varkappa(N-2)}{N}} + \rho^{\beta+2-\lambda} \Big), \ \forall \rho \in (0, \frac{d}{2}).$$

We recall that $\beta > \lambda - 2$, $\delta \ge \lambda - 1$. Hence and from (10.2.106),(10.2.107) and (10.2.109) it follows that

$$(10.2.110) \quad \sup_{x' \in G_{1/2}^1} w(x') \le C_1 + C_2 \rho^{2-\lambda+\frac{2(\lambda-1)}{N}+\frac{2\varkappa(N-1)}{N}}.$$

We recall as well as that $\lambda > 1$ and $\varkappa > 0$. To prove the validity of 1) (as in the first case) it is enough to obtain the following estimate

$$(10.2.111) \quad \sup_{x' \in G_{1/2}^1} w(x') \le \text{const.}$$

We shall show that the repetition by the finite time of the procedure of the (10.2.110) receiving for various $\varkappa$ can lead to the estimate (10.2.111).

Let the exponent of ρ in (10.2.110) be negative (otherwise the (10.2.110) means the (10.2.111)). Returning to the function $u(x)$ in (10.2.110) and putting $|x| = \frac{2}{3}\rho$ we obtain

$$(10.2.112) \quad u(x) \le C|x|^{2+\frac{2(\lambda-1)}{N}},$$

and hence, by Theorem 10.35 for $\varkappa = \varkappa_1$,

$$(10.2.113) \quad \varkappa_1 = 1 + \frac{2(\lambda-1)}{N},$$

we get

(10.2.114) $|\nabla u(x)| \leq C|x|^{\varkappa_1}.$

Let us repeat the procedure of receiving the inequalities (10.2.109) and (10.2.110), applying the estimate (10.2.114) instead of the (10.2.18) (i.e. changing $\varkappa$ for $\varkappa_1$). As a result we obtain

(10.2.115) $\displaystyle\sup_{x' \in G^1_{1/2}} w(x') \leq C_1 + C_2 \rho^{2-\lambda+\frac{2(\lambda-1)}{N}+\frac{2\varkappa_1(N-1)}{N}}.$

If the exponent of ρ in (10.2.115) is negative, then letting

(10.2.116) $\varkappa_2 = 1 + \dfrac{2(\lambda-1)}{N} + \dfrac{2(N-1)}{N}\varkappa_1,$

by Theorem 10.35 for $\varkappa = \varkappa_2$, we get

(10.2.117) $|\nabla u(x)| \leq C_{19}|x|^{\varkappa_2},$

and repeating the above procedure we get the estimate

(10.2.118) $\displaystyle\sup_{x' \in G^1_{1/2}} w(x') \leq C_1 + C_2 \rho^{2-\lambda+\frac{2(\lambda-1)}{N}+\frac{2\varkappa_2(N-1)}{N}}.$

Letting

(10.2.119) $t = \dfrac{2(N-1)}{N} \geq \dfrac{3}{2} \quad \forall N \geq 4,$

we consider the following number sequence $\{\varkappa_k\}$

$$\varkappa_1 \text{ defined by (10.2.113)},$$

$$\varkappa_2 = \varkappa_1(1+t),$$

$$\varkappa_3 = \varkappa_2(1+t+t^2),$$

$$\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb\dotsb$$

(10.2.120) $\varkappa_{k+1} = \varkappa_1(1+t+\dots+t^k) = \varkappa_1\dfrac{t^{k+1}-1}{t-1}, \qquad k = 0, 1, \dots.$

Repeating the stated process k times we obtain the estimates

(10.2.121) $\displaystyle\sup_{x' \in G^1_{1/2}} w(x') \leq C_1 + C_2 \rho^{1-\lambda+\varkappa_{k+1}}, \quad 0 < \rho < d/2;$

$$k = 0, 1, \dots.$$

Now we shall show that for $\forall N \geq 4$ exists integer k such that

(10.2.122) $1 - \lambda + \varkappa_{k+1} \geq 0.$

From (10.2.113) and (10.2.120) we have

$$1 - \lambda + \varkappa_{k+1} = \dfrac{t^{k+1}-1}{t-1} + \dfrac{\lambda-1}{N(t-1)}(2t^{k+1} - 2 - Nt + N).$$

The first term on the right is positive, by (10.2.119). For the second term from (10.2.119) it follows that

$$2t^{k+1} - 2 - Nt + N = 2^{k+2}(1 - 1/N)^{k+1} - N \geq 0$$

if $\{(2N - 2)/N\}^{k+1} \geq N/2$. Hence we get that (10.2.122) holds if

$$k + 1 \geq \frac{\ln \frac{N}{2}}{\ln \frac{2N-2}{N}}.$$

Choosing $k = \left[\frac{\ln \frac{N}{2}}{\ln \frac{2N-2}{N}} \right]$, where $[a]$ is an integer part of a, we guarantee (10.2.122) $\forall N \geq 4$. By this, the validity of 1) of our theorem is proved.

The validity of the second estimate we get from Theorem 10.35 for $\varkappa = \lambda - 1$.

Now we shall prove the validity of 2). Returning to the variable x and the function $u(x)$ in (10.2.103) we have

$$\int\limits_{G^{\varrho}_{\varrho/2}} \left(|u_{xx}|^q + \varrho^{-q}|\nabla u|^q + \varrho^{-2q}|u|^q \right) dx \leq C_4 \int\limits_{G^{2\varrho}_{\varrho}} \left(\varrho^{-2q}|u|^q + |\nabla u|^{2q} + \right.$$

$$\left. + |b|^q|\nabla u|^q + |f|^q + |\nabla g|^q + \varrho^{-q}|g|^q \right) dx.$$

Multiplying this inequality by ϱ^α, replacing ϱ by $2^{-k}\varrho$ and summing up over all $k = 0, 1, \ldots$ we obtain

$$\|u\|^q_{V^2_{q,\alpha}(G^{\varrho}_0)} \leq C_4 \int\limits_{G^{2\varrho}_0} \left(r^{\alpha-2q}|u|^q + r^\alpha|\nabla u|^{2q} + r^\alpha|b|^q|\nabla u|^q + \right.$$

$$\left. + r^\alpha|f|^q + r^\alpha|\nabla g|^q + r^{\alpha-q}|g|^q \right) dx, \quad \forall q > 1.$$

Using the estimates from 1), by the assumption (D) and the assumption (10.2.102) of our theorem, taking into consideration $\beta > \lambda - 2 > -1$ we get

$$(10.2.123) \qquad\qquad \|u\|_{V^2_{q,0}(G^{\varrho}_0)} \leq C\varrho^{\lambda-2+\frac{\alpha+N}{q}}$$

if only $\alpha + N + (\lambda - 2)q > 0$. From (10.2.123) we obtain the validity of 2) of our theorem.

Finally, repeating verbatim the proof of Theorem 10.37 for $\varkappa = \lambda - 1$, we obtain the validity of 3) of our theorem. $\square$

10.3. Notes

Many mathematicians have considered the third boundary value problem. The oblique derivative problem for elliptic equations in non-smooth domains was investigated by M.Faierman [121], M.Garroni, V.A.Solonnikov and M.Vivaldi [127],P. Grisvard [133], G. Liberman [225, 226, 232, 234, 227], H.Reisman [350] and others.

P. Grisvard has investigated (Chapter 4 [133]) the properties of the second weak derivatives of the weak solutions of the oblique problem for the Laplace operator in a plane domain with a polygonal boundary. He has established $W^{2,p}$—a priori estimates for such solutions and conditions, when such estimates hold.

M. Dauge and S. Nicaise [94] have investigated oblique derivative and interface problems associated to the Laplace operator on a polygon. They have obtained index formulae, a calculus of the dimension of the kernel, an expansion of the weak solutions into regular and singular parts, and formulae for the coefficients of the singularities in such expansions

M. Faierman [121] has extended the P. Grisvard results to the elliptic operator of the form

$$L = - \sum_{i=1}^{N} a_{ii}(x) D_i^2 + \sum_{i=1}^{N} a^i(x) D_i + a(x),$$

in a $N-$ dimensional rectangle.

H.Reisman [350] considered elliptic boundary value problems for the equation from (L) with infinitely differentiable coefficients in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ with non smooth boundary that has dihedral edges. He considered the boundary conditions that are an oblique derivative on one side of the edge and an oblique derivative or a Dirichlet condition on the other side of the edge. The main results in his work are uniqueness, existence, and regularity theorems for such problems in weighted Sobolev spaces.

M.G.Garroni, V.A.Solonnikov and M.A.Vivaldi [127] have considered the following elliptic boundary value problem for the Poisson equation on the infinite angle

$$\begin{cases} -\Delta u + su = f(x), \ x \in d_\vartheta, \\ \left(\frac{\partial u}{\partial \overrightarrow{n}} + h_i \frac{\partial u}{\partial r} \right)\Big|_{\gamma_i} = \varphi_i(r), \ i = 0, 1, \end{cases}$$

where $d_\vartheta \subset \mathbb{R}^2$ is the infinite angle of the opening $\vartheta \in (0, 2\pi]$ with the sides γ_0 and γ_1 given by

$$\gamma_0 = \{0 \leq x_1 < \infty, \ x_2 = 0\},$$

$$\gamma_1 = \{x_1 = r\cos\vartheta, x_2 = r\sin\vartheta, 0 \le r = \sqrt{x_1^2 + x_2^2} < \infty\}$$

in a Cartesian coordinate system $\{x_1, x_2\}$. Here $\vec{n}$ is the exterior normal to γ_i, h_0 and h_1 are given real constants, s is a complex parameter with $Re\, s \equiv a^2 \ge 0$. The authors have obtained the estimates of the above problem solution, which are uniform with respect to s in weighted Sobolev spaces introduced by V.A.Kondrat'ev for the investigation of elliptic boundary value problems in domains with angular and conical points at the boundary. In these spaces the distance $|x|$ from the origin, with an appropriate exponent, is the weight. The spaces, in which the solution exists, depend on the sign of $h_0 + h_1$.

At last, the oblique derivative problem in Lipschitz domains has been investigated by G. Lieberman [**225, 226, 232, 234, 227**]. He has studied the problem of the existence and the regularity of solutions in Lipschitz domains for elliptic equations with Hölder continuous coefficients. He has proved [**225, 234**] the local and global maximum principle (see Propositions 10.11, 10.14) for the oblique derivative problem for general second order linear and quasilinear elliptic equations in arbitrary Lipschitz domains. Without making any continuity assumptions on the known functions, he has derived the Harnack and Hölder estimates for strong solutions near the boundary of the domain. He as well has bounded the maximum of the solution modulus in terms of an appropriate norms and the known functions.

An important element in the study of elliptic equations is the modulus of continuity estimate for the gradient of the solutions. Usually this modulus of continuity estimate is in fact a Hölder estimate, so it is often referred to as a Hölder gradient estimate. For elliptic nonlinear oblique boundary value problem in a smooth domain, the Hölder gradient estimate first has been proved by G. Lieberman [**236, 237**] and by Lieberman-Trudinger [**238**].

M. Dauge and S. Nicaise [**94**] have investigated the oblique derivative and interface problems on polygonal domains.

L. Lanzani and Zh. Shen [**220**] have obtained existence and uniqueness results for harmonic functions satisfying the Robin boundary condition with boundary data in $L_p(\partial G)$, $1 < p < 2 + \varepsilon$ and G being a bounded Lipschitz domain.

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Bibliography

[1] R.A. Adams, *Sobolev Spaces*, New York: Academic Press (1975).

[2] V. Adolfsson, L^2- *integrability of the second order derivatives for Poisson's equation in nonsmooth domains*, // Math. Scand., **70** (1992), 146-160.

[3] V. Adolfsson and D. Jerison, L^p- *integrability of the second order derivatives for the Neumann problem in convex domains*, // Indiana Univ. Math. J., **43** (1994), 1123-1138.

[4] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, Comm. Pure Appl. Math., **12** (1959), 623-727.

[5] M. Agranovich and M. Vishik, *Elliptic problems with parameter and parabolic problems of general type*, Uspekhi Mat. Nauk, **20**, 5 (1965), 3-120 (in Russian).

[6] Y.Akdim, E. Azroul and A. Benkirane, *Existence of solutions for quasilinear degenerate elliptic equations*, // EJDE, **71**, (2001), 1-19.

[7] F. Ali Mehmeti, M. Bochniak, S. Nicaise and A.-M. Sändig, *Quasilinear elliptic systems of second order in domains with corners and edges*, Preprint, Mathematisches Institut A, Universität Stuttgart, 2000.

[8] Ju.A. Alkhutov and V.A. Kondrat'ev, *The solvability of the Dirichlet problem for elliptic second order equations in a convex domain*, Differ. Uravn. **28**, 5 (1992), 806-818, (in Russian).

[9] T. Apel and S. Nicaise, *Elliptic problems in domains with edges: anisotropic regularity and anisotropic finite element meshes*, Progress in nonlinear differential equations and their applications. **22** (1996), 18-34 (in "Partial differential equations and functional analysis: in memory of Pierre Grisvard"). Birkhäuser.

[10] D. Apushkinskaya and A. Nazarov, *The Dirichlet problem for quasilinear elliptic equations in domains with smooth closed edges*, Problems of Math. Analysis. St. Petersburg. **21** (2000), 3-29

[11] G. Aronsson, *Construction of singular solutions to the p-harmonic equations and its limit equation for $p = \infty$*, Manus. Math. **56** (1986), 135-158.

[12] J.-P. Aubin, *Approximation of elliptic boundary value problems*, New York: Wiley-Interscience (1971).

[13] A. Avantagiatti and M. Troisi, *Spazi di Sobolev con peso e problemi ellitici in un angelo I - III,* // Annali Mat. Pura Appl., **95** (1973), 361-408; **97** (1973), pp. 207-252; **99** (1974), 1-51.

[14] A. Avantagiatti and M. Troisi, *Ulteriori contributi allo studio dei problemi ellitici in un angelo,*// Annali Mat. Pura Appl. **100** (1974), 153-168.

[15] F.K. Aziz and R.B. Kellog, *On homeomorphisms for an elliptic equation in domains with corners,* // Diff. Integral Equat., **8** 2 (1995), 333-352.

[16] A. Azzam, *Behaviour of solutions of Dirichlet problem for elliptic equations at a corner,* Indian J. Pure Appl. Math. **10** (1979), 1453-1459.

[17] A. Azzam, *On the first boundary value problem for elliptic equations in regions with corners,* Arabian J. Sci. Engrg. **4** (1979), 129-135.

[18] A. Azzam, *Smoothness properties of bounded solutions of Dirichlet's problem for elliptic equations in regions with corners on the boundary,* Canad. Math. Bull. **23** (1980), 213-226.

[19] A. Azzam, *On Dirichlet's problem for elliptic equations in sectionally smooth n–dimensional domains,* SIAM J. Math. Anal. **11** (1980), 248-253.

[20] A. Azzam, *Smoothness properties of solutions of mixed boundary value problems for elliptic equations in sectionally smooth n– dimensional domains,* // Annales Polonici Mathematici. **40** 1 (1981), 81-93.

[21] A. Azzam and V. Kondrat'ev, *Schauder–type estimates of solutions of second order elliptic systems in divergence form in non–regular domains,* Comm. PDE's **16** (1991), 1857-1878.

[22] A. Azzam and E. Kreyszig, *Über das gemischte Randwertproblem für elliptische Gleichungen in n–dimensionalen Gebieten mit Kanten,* Ann. Acad. Sci. Fenn. Ser. AI Math. **5** (1980), 341-346.

[23] A. Azzam and E. Kreyszig, *On solutions of elliptic equations satisfying mixed boundary conditions,* // SIAM J. Math. Anal.,**13** 2 (1982), 254-262.

[24] L. Bagirov and V. Kondratiev, *On the asymptotics of solutions of differential equations in a Hilbert space,* Math. USSR Sb. **72**, 2 (1992), 485-501.

[25] L. Bagirov and V. Kondratiev, *On the properties of generalized solutions of elliptic equations,* // Russian journal of mathematical physics. **1** 2 (1993), 139-164.

[26] A. Baldes, *Degenerate elliptic operators diagonal systems and variational integrals,* Manus. Math. **55** (1986), 467-486.

[27] J. Banasiak, *On L_2- solvability of mixed boundary value problems for elliptic equations in plane nonsmooth domains*, // Journal of Diff. Equat., **97** 1 (1992), 99-111.

[28] J. Banasiak, *On asymptotics of solutions of elliptic mixed boundary value problems of second order in domains with vanishing edges*, // SIAM J. Math. Anal., **23** 5 (1992), 1117-1124.

[29] J. Banasiak, *Domains of fractional powers of operators arising in mixed boundary value problems in nonsmooth domains and applications*, // Applicable Analysis, **55** (1994), 79-89.

[30] J. Banasiak, *Remark on a trace theorem for transmission problems*, // Math. Meth. in the Appl. Sci., **18** (1995), 413-421.

[31] J. Banasiak and G.F. Roach , *On mixed boundary value problem of Dirichlet oblique-derivative type in plane domains with piecewise differentiable boundary*, Journal of differential equations. **79**, 1 (1989), 111-131.

[32] J. Banasiak and G.F. Roach , *On corner singularities of solutions to mixed boundary-value problems for second-order elliptic and parabolic equations*, // Proc. R. Soc. Lond. A. **433** (1991), 209-217.

[33] C. Bandle and M. Essen, *On positive solutions of Emden equations in cone-like domains*, // Arch. Ration. Mech. Anal. **112** (1990), 319-338.

[34] G. Bateman and A. Erdélyi, *Higher transcendental functions*, v. 1. MC Graw-Hill book company, INC., New York, Toronto, London (1953).

[35] H. Beirão da Veiga, *Sur la régularité des solutions de l'équation $div\,A(x, u, \nabla u) = B(x, u, \nabla u)$ avec des conditions aux limites unilatérales et mêlées*, Ann. mat. pura appl., **93** (1972), 173-230.

[36] T. Bartsch and Z.Q. Wang, *On the existence of sign changing solutions for semilinear Dirichlet problems*, // Topological methods in nonlinear analysis, **7** 1 (1996), 115-132.

[37] E. Beckenbach and R. Bellman, *Inequalities*, Springer-Verlag (1961).

[38] R. Bellman, *Stability theory of differential equations*, New York, Toronto, London (1953).

[39] J. Below and H. Kaul, *Nonlinear degenerate elliptic equations and axially symmetric problems*, Calc. Var., **7** (1998), 41-51.

[40] A. Bensoussan and J. Frehse, *Regularity results for nonlinear elliptic systems and applications.* Applied mathematical sciences **151**, Springer-Verlag (2002).

[41] Ch. Bernardi, M. Dauge and Y. Maday, *Interpolation of nullspaces for polynomial approximation of divergence-free functions in a cube*, //Lecture notes in pure and applied mathematics. **167** (1995), 27-46.

[42] Ch. Bernardi, M. Dauge and Y. Maday, *Spectral methods for axisymmetric domains*, Guuthier-Villars, Paris (1999).

[43] L. Bers, F. John and M. Schechter, *Partial differential equations*, Interscience Publishers: New York - London -Sydney (1964).

[44] M.F. Bidaut-Veron, *Behavior near zero and near infinity of solutions to elliptic equalities and inequalities*, // Electronic Journal of Differential Equations. Conf. **06** (2001), 29-44.

[45] M.F. Bidaut-Veron and L. Veron, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, // Inventiones Math. **106** (1991), 489-539.

[46] M.F. Bidaut-Veron, V. Galaktionov, Ph. Grillot and L. Veron, *Singularities in for a semilinear elliptic equation with a non-Lipschitz nonlinearity*, // J. Diff. Equ. **154** (1999), 318-338.

[47] M.Sh. Birman and G.E. Skvortsov, *On the quadratic integrability of the highest derivatives of the Dirichlet problem in a domain with piecewise smooth boundary*, // Izv. Vyssh. Uchebn. Zaved., Mat., **5** (1962), 12-21 (in Russian).

[48] H. Blum and R. Rannacher, *On the boundary value problem of the biharmonic operator on domains with angular corners*, Math. Meth. Appl. Sci. **2** (1980), 556-581.

[49] L. Boccardo, F. Murat and J. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl. (IV). **152** (1988), 183-196.

[50] M. Bochniak and M.Borsuk, *Dirichlet problem for linear elliptic equations degenerating at a conical boundary point*, Analysis. Munchen, Germany **23**, 3 (2003), 225-248.

[51] M. Bochniak and A.-M. Sändig, *Local solvability and regularity results for a class of semilinear elliptic problems in nonsmooth domains*, Math. Bohem. **124** (1999), 245-254.

[52] M.V. Borsuk, *Behavior of generalized solutions of the Dirichlet problem for second-order quasilinear elliptic equations of divergence type near a conical point*, Sibirsk. Mat. Zh. **31**, 6 (1990), 25-38.

[53] M.V. Borsuk, *Best possible estimates for solutions of the Dirichlet problem for linear elliptic nondivergence equations of second order in a neighborhood of a conical point of the boundary*, Mat. Sb. **182**, 10 (1991), 1446-1462 = Math. USSR Sbornik **74**, 1, (1993), 185-201.

[54] M.V. Borsuk, *Estimates of solutions of the Dirichlet problem for a quasilinear non divergence elliptic equation of a second order near a corner boundary point*, St. Petersburg Math. J. **3**, 3 (1992), 1281-1302.

[55] M.V. Borsuk, *Behavior of solutions of the Dirichlet problem for a second-order quasilinear elliptic equation of general form near a corner point*, Ukrainan Mat. Zh. **44**, 2 (1992), 167-173.

[56] M.V. Borsuk, *On the solvability of the Dirichlet problem for second-order linear elliptic equations in a domain with a conical points*, Uspekhi mat. nauk. Moscow. **48**, 4 (1993), 176-177.

[57] M.V. Borsuk, *The first boundary value problem for second order elliptic equations in domains with corner or conical points on the boundary*, Lvov Franko University. 1993, 248 p. Doctoral habilitation thesis.

[58] M.V. Borsuk, *Estimates of solutions of Dirichlet problem for elliptic nondivergence second-order equations in a neighbourhood of a conical boundary point*, Differ. Uravn. **30**, 1 (1994), 104-108.

[59] M.V. Borsuk, *Estimates of generalized solutions of the Dirichlet problem for quasilinear second-order elliptic equations in a domain with a conical boundary point*, Differ. Uravn. **31**, 6 (1995), 1001-1007.

[60] M.V. Borsuk, *On the solvability of the Dirichlet problem for second-order quasilinear nondivergence elliptic equation in a domain with conical points*, Ukrainian Math. Dokl. 1 (1995), 14-15.

[61] M.V. Borsuk, *On the solvability of the Dirichlet problem for second-order nondivergence elliptic equations in a domain with a conical point*, Ukrainan Mat. Zh. **48**, 1 (1996), 13-24 (in Ukrainian).

[62] M.V. Borsuk, *Behavior of solutions of the Dirichlet problem for weakly nonlinear elliptic nondivergence equations in a neighborhood of a conical point of the boundary*, Differ. Uravn. **33**, 8 (1997), 1085-1094.

[63] M.V. Borsuk, *On the solvability of the first boundary value problem for second-order elliptic equations in a domain with a conical point on the boundary*, Mat. Fiz. Anal. Geom. **4**, 4 (1997), 428-452 (in Russian).

[64] M.V. Borsuk, *Dini-continuity of first derivatives of solutions of the Dirichlet problem for second-order linear elliptic equations in a non-smooth domain*, Sibirsk. Mat. Zh. **39**, 2 (1998), 261-280; Ann. Polon. Math. **69** (1998), 129-154.

[65] M.V. Borsuk, *On the behavior of solutions of the Dirichlet problem for elliptic nondivergence second order equations near the conical boundary point*, Matemat. Studii **10**, 2 (1998), 163-172.

[66] M. Borsuk, *Elliptic boundary value problems in nonsmooth domains*, // Proc. of the 5th environmental mathematical conference Rzeszow-Lublin-Lesko. 1998. Lublin (1999), 33-43 (in Polish).

[67] M.V. Borsuk, *The behavior of solutions of elliptic quasilinear degenerate equations near a boundary edge*, Nonlinear boundary value problems. Donetsk. **12** (2002), 32-43, Nonlinear Analysis: Theory, Methods and Applications **56**, 3 (2004), 347-384.

[68] M.V. Borsuk and M. Dobrowolski, *On the behavior of solutions of the Dirichlet problem for a class of degenerate elliptic equations in the neighborhood of conical boundary points*, Nonlinear boundary value problems **9** (1999), 29-34.

[69] M.V. Borsuk and V.A. Kondrat'ev, *On the behavior of solutions of Dirichlet problem for semilinear second order elliptic equations in a neighborhood of conical boundary point*, // Nonlinear boundary value problems. Kiev. **7** (1997), 47-56.

[70] M.V. Borsuk and M. Plesha, *Estimates for the generalized solutions of the Dirichlet problem for second-order quasilinear elliptic equations in a domain with a conical boundary point*, Ukrainian Mat. Zh. **50**, 10 (1998), 1299-1309.

[71] M.V. Borsuk and D. Portnyagin, *Barriers on cones for degenerate quasilinear elliptic operators*, Electr. J. Diff. Equat. **11** (1998), 1-8.

[72] M.V. Borsuk and D. Portnyagin, *On the Dirichlet problem for a quasilinear elliptic second order equation with triple degeneracy and singularity in a domain with edge on the boundary*, GAKUTO Intern. Ser. Math. Sci. Appl. **14**, II (2000), 47-60.

[73] M.V. Borsuk and A.Zawadzka, *Best possible estimates of solutions to the Robin boundary value problem for linear elliptic non divergence second order equations in a neighborhood of the conical point*, // Journal of Diff. Equations, **207**, 2 (2004), 303-359.

[74] C. Burch, *The Dini condition and regularity of weak solutions of elliptic equations*,// J. Diff. Equations, **30** (1978), 308-323.

[75] P. Caldiroli and R. Musina, *On a variational degenerate elliptic problem*, // NoDEA, **7** (2000), 187-199.

[76] A.P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math.**88**(1952), 85-139.

[77] T. Carleman, *Über das Neumann - Poincarésche Problem für ein Gebiet mit Ecken*. Thesis, Uppsala. 1916.

[78] G.R. Cirmi and M.M. Porzio, $L^\infty-$ *solutions for some nonlinear degenerate elliptic and parabolic equations*, Ann. mat. pura ed appl. (IV) **169** (1995), 67-86.

[79] S.A. Chaplygin, *The new method of the approximate integration of the differential equations*, Moscow - Leningrad, 1932 (in Russian).

[80] Ya. Chen and L. Wu, *Second order elliptic equations and elliptic systems*, AMS. Translations of mathematical monographs, **174** (1991).

[81] V. Chernetskiy, *Best possible estimates of solutions of mixed problem for second order linear elliptic equations near corner point*, Ukrain. Mat. Zhurn. **49**, 11 (1997), 1529-1542 (in Ukrainian).

[82] W. Chiskouche, D. Mercier and S. Nicaise, *Regularity of the solution of some unilateral boundary value problems in polygonal and polyhedral domains*, Communications in partial differential equations, **29**, 1&2 (2004), 43-70.

[83] H. Choe, *Degenerate elliptic and parabolic equations and variational inequalities*, Seoul National University, Lecture notes series. **16** (1993).

[84] Ph. Clement and P. Grisvard, *Sommes dóperateurs et régularite L^p dans les problèmes aux limites*, C.r. Acad. Sci. Ser. 1. **314**, 11 (1992), 821-824.

[85] H.O. Cordes, *Über die erste Randwertaufgabe bei quasilinearen Differentialgleichungen zweiter Ordnung in mehr als zwei Variablen*, Mathematische Annalen. **131** (1956), 278-312.

[86] M. Costabel and M. Dauge, *General edge asymptotics of solutions of second-order elliptic boundary value problems I,II*, Proc. Royal Soc. Edinburgh **123A** (1993), 109-155, 157-184.

[87] R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, Bd. 1, Springer-Verlag, Berlin, 1931.

[88] M. Cuesta and P. Takáč, *A strong comparison principle for positive solutions of degenerate elliptic equations,* // Differential and integral equations, **13** 4 - 6 (2000), 721-746.

[89] S. Dahlke and R.A. DeVore, *Besov regularity for elliptic boundary value problems,* // Comm. PDE, **22** 1 - 2 (1997), 1-16.

[90] I.I. Danilyuk, *The Dirichlet problem for two dimensional quasilinear differential equations of elliptic type,* Dokl. Akad. Nauk Ukrain, SSSR, Ser. A **41** (1987), 3-7.

[91] M. Dauge, *Elliptic boundary value problems on corner domains*, Lecture notes in Mathem. **1341** (1988), 1-257.

[92] M. Dauge, *Neumann and mixed problems on curvilinear polyhedra*, Integral equations operator theory, **1341** (1992), 227-261.

[93] M. Dauge, *Strongly elliptic problems near cuspidal points and edges*. In Progress in nonlinear differential equations and their applications. **22** (1996), 93-110. PDE & Functional Analysis. In Memory of Pierre Grisvard. Birkhäuser.

[94] M. Dauge and S. Nicaise, *Oblique derivative and interface problems on polygonal domains and networks,* // Communications in partial differential equations, **14** 8-9 (1989), 1147-1192.

[95] M. Dauge and S. Nicaise, *Coefficients of the singularities on domains with conical points,* // PDE-I. Banach center publications, **27** 1 (1992), 91-99.

[96] M. Dauge, S. Nicaise, M. Bourlard and J.M. Lubuma, *Coefficients des singularités pour des problèmes aux limites elliptiques sur un domaine à*

points coniques. I: Résultats généraux pour le problème de Dirichlet, //
Mathematical modelling and numerical analysis, **24** 1 (1990), 27-52.

[97] J.I. Diaz and L. Veron, *Local vanishing properties of solutions of elliptic and parabolic quasilinear equations, //* Trans. Amer. Math. Soc. **290** (1985), 787-814.

[98] M. Dobrowolski, *Numerical approximation of elliptic interface and corner problems,* Habschrift, Universität Bonn, 1981 (see also: Z.A.A.M. 1984. V.**64**, 270-271.)

[99] M. Dobrowolski, *On quasilinear elliptic equations in domains with conical boundary points,* J. reine und angew. Math. **394** (1989), 186-195.

[100] P. Drábek, A. Kufner and F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities,* Walter de Gruyter, Berlin, 1997.

[101] N. Dunford and J.T. Schwartz, *Linear operators, Part I: General Theory.* Interscince Publishers, New York, London, 1958.

[102] R. Dzhafarov, *The nonlinear Dirichlet problem in domains with angular and conical points,* Dissertation. Donetsk. 1999, 120 p.

[103] G. Dziuk, *Das Verhalten von Lösungen semilinearer elliptischer Systeme an Ecken eines Gebietes,* Math. Z. **159** (1978), 89-100.

[104] C. Ebmeyer, *Mixed boundary value problems for nonlinear elliptic systems in n–dimensional Lipschitzian domains,* ZAA **18** (1999), 539-556.

[105] C. Ebmeyer and J. Frehse, *Mixed boundary value problems for nonlinear elliptic equations in multidimensional nonsmooth domains,* Mathematische Nachrichten. **203** (1999), 47-74.

[106] D.E. Edmunds and B. Opic, *Weighted Poincaré and Fridrichs inequalities,* J. London Math. Soc. (2). **47** (1993), 79-96.

[107] H. Egnell, *Positive solutions of semilinear equations in cones,* Transactions of the AMS. **330**, 1 (1992), 191-201.

[108] Yu. Egorov and V. Kondratiev, *On spectral theory of elliptic operators,* Birkhäuser Verlag. 1996 (Operator theory, **89**).

[109] Yu. Egorov and V. Kondratiev, *On the asymptotic behavior of solutions to a nonlinear elliptic boundary problem, //* Nonlinear boundary value problems, Donetsk. **10** (2000), 61-74.

[110] Yu. Egorov and V. Kondratiev, *On the asymptotic behavior of solutions of a semilinear elliptic boundary problem in an unbounded cone,* C.R. Acad. Sci. Paris, **332**, Ser. I (2001), 705-710.

[111] Yu. Egorov, V. Kondratiev and O. Oleynik, *Asymptotic behavior of solutions of a nonlinear elliptic and parabolic systems in cylindrical domains, //* Matem. sb., **189** 3 (1998), 45-68.

[112] Yu. Egorov, V. Kondratiev and B.-W. Schulze, *On completeness of eigenfunctions of an elliptic operator on a manifold with conical points.* Universitat Potsdam - Institut für Mathematik. Preprint 2001/04, 11 p.

[113] Yu. Egorov and B.-W. Schulze, *Pseudo-differential operators, singularities, applications* Birkhäuser Verlag. 1997 (Operator theory, **93**).

[114] S. Eidelman and M. Matiichuk, *The Cauchy problem for parabolic systems with coefficients having low smoothness,* Ukrain. Mat. Zh. **22**, 1 (1970), 22-36 (in Russian).

[115] G.I. Eskin, *General boundary values problems for equations of principal type in a plane domain with angular points,* // Uspekhi Mat. Nauk, **18**, 3 (1963), 241-242 (in Russian).

[116] G.I. Eskin, *Boundary values problems for second order elliptic equations in domains with corners,* Proc. Symp. Pure Math. **43**, 2 (1985), 105-131.

[117] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions,* CRC Press, Boca Raton, 1992.

[118] E. Fabes, K. Kenig and R. Serapioni, *The local regularity of solutions of degenerate elliptic equations,* Comm. PDE's., **7** (1982), 77-116.

[119] J. Fabbri and L. Veron, *Équations elliptiques non-linéaires singulieres au bord dans des domaines non reguliers,* // C. R. Acad. Sci. Paris, **320** (1995), 941-946.

[120] J. Fabbri and L. Veron, *Singular boundary value problems for nonlinear elliptic equations in nonsmooth domains,* // Adv. Diff. Equat., **1** 6 (1996), 1075-1098.

[121] M. Faierman, *Regularity of solutions of an elliptic boundary value problem in a rectangle,* Comm. in PDE., **12** (1987), 285-305.

[122] N. Fandyushina, *On quasilinear elliptic equations with two independent variables,* // Vestnik of Moscow University, 2 (1972), 30-38.

[123] V.I. Feygin, *Boundary value problems for quasielliptic equations in noncylindrical regions,* // Soviet Math. Dokl., **12** (1971).

[124] A. Friedman and L.Veron, *Singular solutions of some quasilinear elliptic equations,* // Arch. Rat. Mech.Anal. **96** (1986), 359-387.

[125] M. Fuchs and Li Gongbao, $L^\infty-$*bounds for elliptic equations on Orlicz - Sobolev spaces,* // Arch. Matrh., **72** (1999), 293-297.

[126] V. Fufajev, *On the Dirichlet problem for domains with angles ,* Dokl. Akad. Nauk SSSR. **131**, 1 (1960), 37-39 (in Russian).

[127] M.G. Garroni, V.A. Solonnikov and M.A. Vivaldi, *On the oblique derivative problem in an infinite angle.* Topological methods in nonlinear analysis, **7**, 2 (1996), 299-325.

[128] M. Giaquinta, *Introduction to regularity theory for nonlinear elliptic systems,* Birkhäuser Verlag, 1993.

[129] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed. Revised third printing. Springer, 1998.

[130] A. Gmira and L. Veron, *Boundary singularities of solutions of nonlinear elliptic equations*, // Duke J. Math. **64** (1991), 271-324.

[131] N. Grachev and V. G. Maz'ya, *Estimates for fundamental solutions of the Neumann problem in a polyhedron*. Preprint, LiTH-MAT-R-91-28, Linkoping University, 1991.

[132] N. Grachev and V. G. Maz'ya, *Solvability of boundary integral equations in a polyhedron*. Preprint, LiTH-MAT-R-91-50, Linkoping University. 1991.

[133] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman Advanced Publishing Program, Boston - London - Melbourne, 1985.

[134] P. Grisvard, *Edge behaviour of the solution of an elliptic problem*, Math. Nachr. **132** (1987), 281-299.

[135] P. Grisvard, *Singular behaviour of elliptic problems in non hilbertian Sobolev spaces*, Journal de meth. pures at appliquées. **74**, 1 (1995), 3-33.

[136] K. Gröger, *A $W^{1,p}$—estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, // Math. Ann., **283** (1989), 679-687.

[137] T. Gronwall, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. Math., **20**, 2 (1919), 292-296.

[138] M. Guedda and L. Veron, *Local and global properties of solutions of quasilinear elliptic equations*, // J. Diff. Equ. **76** (1988), 159-189.

[139] M. Guedda and L. Veron, *Quasilinear elliptic equation involving critical Sobolev exponents*, // Nonlinear Anal. T.M.& A. **13** (1989), 879-902.

[140] N.M. Günther, *La théorie du potentiel*, Paris, 1934.

[141] M.S. Hanna and K.T. Smith, *Some remarks on the Dirichlet problem in piecewise smooth domains*, // Comm. Pure Appl. Math., **20** (1967), 575-593.

[142] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, // 1934; University Press, Cambridge, 1952.

[143] W. Heiskanen and H. Moritz, *Physical Geodesy*, San Francisco / Londres: W.H. Freeman and Co., 1967.

[144] Hong Xie, $L^{2,\mu}(\Omega)$—*estimate to the mixed boundary value problem for second order elliptic equations and its application in the thermistor problem*, // Nonlinear Analysis, **24** 1 (1995), 9-27.

[145] E. Hebey, *Sobolev spaces on riemannian manifolds*, Lecture Notes in Mathematics, **1635** (1996), Springer - Verlag.

[146] L. Hörmander, *On the division of distributions by polynomials*, Ark. Mat., **3**, **53** (1958), 555.

[147] Y.X. Huang, *Existence of positive solutions for a class of the p—Laplace equations,* // J. Austral. Math. Soc., Ser.B **36** (1994), 249-264.

[148] J. Huentutripay, M. Jazar and L. Veron, *A dynamical system approach to the construction of singular solutions of some degenerate elliptic equations,* // J. Diff. Equat., **195** (2003), 175-193.

[149] E.M. Il'in, *Singularities of weak solutions of elliptic equations with discontinuous higher coefficients. II. Corner points of the lines of discontinuity,* // Zap. Nuchn. Semin. LOMI, **47** (1974), 166-169.

[150] D. Jerison and C.E. Kenig, *Boundary value problems on Lipschitz domains,* in "Studies in partial differential equations" (W. Littman, Ed.), MAA Studies in Math., **23** (1982), 1-68.

[151] D. Jerison and C.E. Kenig, *The inhomogeneous Dirichlet problem in Lipschitz domains,* J. Funct. Anal. **130** (1995), 161-219.

[152] S. Kamin and L. Veron, *Existence and uniqueness of the very singular solution of the porous media equation with absorption,* // J. d' Analyse Math. **51** (1988), 245-258.

[153] S. Kamin and L. Veron, *Dead core properties associated to the p—Laplace operator.* Preprint: Laboratoire de Math. et applications. Université François Rabelais - Tours. **31** (1991), 1-10.

[154] D. Kapanadze and B.-W. Schulze, *Crack theory and edge singularities. I.* Universitat Potsdam - Institut für Mathematik. Preprint 2001/05, 1-351.

[155] Sh. Kawai and N. Nakauchi, *The first eigenvalue of the p—Laplacian on a compact Riemannian manifold,* Nonlinear Analysis **55** (2003), 33-46.

[156] B. Kawohl, *On nonlinear mixed boundary value problem for second order elliptic differential equations on domains with corners,* Proceedigs of the Royal of Edinburgh. **87 A** (1980), 33-51.

[157] S. Kichenassamy and L. Veron, *Singular solution of the p-Laplace equation,* // Math. Annalen, **275** (1986), 599-615.

[158] S. Knobloch, *Zum Neumannschen Randwertproblem quasilinearer elliptischer Differentialgleichungen zweiter Ordnung in Gebiete mit Ecken,* Mathematische Nachrichten. **216** (2000) 119-144.

[159] A. Komech and A. Merzon, *General boundary value problems in regions with corners,* Oper. Theory: Adv. Appl. **57** (1992), 171-183.

[160] V.A. Kondrat'ev, *Boundary value problems for elliptic equations in conical regions,* // Soviet Math. Dokl., 4 (1963).

[161] V.A. Kondrat'ev, *Boundary problems for elliptic equations in domains with conical or angular points,* Trans. Moscow Math. Soc. **16** (1967), 209-292.

[162] V.A. Kondrat'ev, *Asymptotic of solution of the Navier - Stokes equation near the angular point of the boundary*, // Prikl. Mat. & Mech., **31** (1967), 119-123 (in Russian).

[163] V.A. Kondrat'ev, *On the smoothness of the solution of the Dirichlet problem for elliptic equations of second order in domain with piecewise-smooth boundary*, Differ. Uravn. **6**, 10 (1970), 1831-1843 (in Russian).

[164] V.A. Kondrat'ev, *Singularities of solutions of the Dirichlet problem for an elliptic equation of second order in the neighbourhood of an edge* (Russian), Differ. Uravn. **13**, 11 (1977), 2026-2032.

[165] V.A. Kondrat'ev, *On solutions of weakly nonlinear elliptic equations in the neighborhood of a conic point at the boundary*, Differ. Uravn. **29**, 2 (1993), 298-305.

[166] V.A. Kondrat'ev, *On certain nonlinear boundary value problems in tube domains*, Trudy Sem. Petrovsk. **19** (1996), 235-261 (in Russian).

[167] V.A. Kondrat'ev and M.V. Borsuk, *The behavior of the solution of the Dirichlet problem for a second-order quasilinear elliptic equation near a corner point*, Differ. Uravn. **24**, 10 (1988), 1778-1784 (in Russian).

[168] V.A. Kondrat'ev and S.D. Eidelman, *About conditions on boundary surface in the theory on elliptic boundary value problems*, // Soviet Math. Dokl., **246** 4 (1979), 812-815 (in Russian).

[169] V.A. Kondrat'ev and Yu.V. Egorov, *On global solutions to a nonlinear elliptic boundary problem in an unbounded domain*, // Nonlinear boundary value problems, Donetsk. **10** (2000), 99-108.

[170] V.A. Kondrat'ev, I. Kopachek and O.A. Oleinik, *On the behaviour of generalized solutions of second order elliptic equations and systems of elasticity theory in a neighbourhood of a boundary point*, Trudy Sem. Petrovsk. **8** (1982), 135-152 (in Russian); English transl.: J. Soviet Math. **32** (1986).

[171] V.A. Kondrat'ev, I. Kopachek and O.A. Oleinik, *On the best Hölder exponents for generalized solutions of the Dirichlet problem for a second-order elliptic equation*, Mat. Sb. **131**, 1 (1986), 113-125 (in Russian); English transl.: Math. USSR Sb. **59** (1988), pp. 113-127.

[172] V.A. Kondrat'ev and E.M. Landis, *On qualitative properties of solutions of a nonlinear equation of second order*, Mat. Sb. **135**, 3 (1988), 346-360 (in Russian); English transl.: Math. USSR Sb. **63**, 2 (1989), 337-351.

[173] V.A. Kondrat'ev and E.M. Landis, *Semilinear second order equations with nonnegative characteristic form*, Matem. Zametki **44**, 4 (1988), 457-468 (in Russian).

[174] V.A. Kondrat'ev and V.A. Nikishkin, *On isolated singularities of solutions of the Emden - Fowler type equations*, // Differ. uravn., **29**, 6 (1993), 1025-1038 (in Russian).

[175] V.A. Kondrat'ev and V.A. Nikishkin, *On positive solutions of singular value problems for the equation* $\Delta u = u^k$, Russian J. Math. Phys., **1** (1993), 123-138 (in Russian).

[176] V.A. Kondrat'ev and V.A. Nikishkin, *On positive solutions of a semilinear equation*, Trudy Sem. Petrovsk. **18** (1995), 157-169 (in Russian).

[177] V.A. Kondrat'ev and O.A. Oleinik, *Boundary value problems for partial differential equations in nonsmooth domains*, Russian Math. Surveys, **38** 2 (1983), 1-86.

[178] V.A. Kondrat'ev and O.A. Oleinik, *Boundary value problems for nonlinear elliptic equations in cylindrical domains*, Journal of the partial differential equations, **6** 1 (1993), 10-16.

[179] V.A. Kondrat'ev and O.A. Oleinik, *On asymptotic of solutions of nonlinear second order elliptic equations in cylindrical domains*. Progress in nonlinear diff. Eqns and their applications. **22** (1996). PDE & Func. Anal. In memory of Pierre Grisvard, 160-173.

[180] V.A. Kondrat'ev and O.A. Oleinik, *On behavior of solutions of a class of nonlinear elliptic second order equations in a neighborhood of a conic point of the boundary*, Lecture notes in pure and applied mathematics. **167** (1995), 151-159.

[181] V.A. Kondrat'ev and L. Veron, *Asymptotic behaviour of solutions of some nonlinear parabolic or elliptic equations*, // Asymptotic Analysis, **14** (1997), 117-156 .

[182] A.A. Kosmodem'yanski, *The properties of solutions to elliptic equations in convex domains*, // Nonlinear boundary value problems, Donetsk. **11** (2001), 98-99.

[183] A.I. Koshelev, *Regularity of solutions of elliptic equations and systems.* "Nauka", Moscow, 1986 (in Russian).

[184] A.I. Koshelev, *Regularity problem for quasilinear elliptic and parabolic systems.* Springer Verlag, 1995.

[185] A.I. Koshelev and S.I. Chelkak *Regularity of solutions of some boundary value problems for quasilinear elliptic and parabolic systems.* St. Petersburg University Verlag, 2000 (in Russian).

[186] V.A. Kozlov, *Singularities of solutions to the Dirichlet problem for elliptic equations in the neighborhood of angular points*, // Algebra & Analiz, **4** (1989), pp. 161 - 177 (in Russian); Engl. transl.: Leningrad Math. J., **1** (1990), 967-982.

[187] V.A. Kozlov, *On the Dirichlet problem for elliptic equations on domains with conical points*, // Diff. Equat., **26** (1990), 739-747.

[188] V.A. Kozlov and V.G. Maz'ya, L^p-estimates and asymptotics of solutions of elliptic boundary value problems in a cone I, Seminar Analysis 1985/86, Berlin, 1986.

[189] V.A. Kozlov and V.G. Maz'ya, Estimates of L_p-means and asymptotics of solutions of elliptic boundary value problems in a cone II. Operators with variable coefficients, Math. Nachr. **137** (1988), 113-139.

[190] V.A. Kozlov and V.G. Maz'ya, Singularities in solutions to mathematical physics problems in non-smooth domains, // Partial differential equations and functional analysis. Progr. Nonlinear Differential Equations Appl., Birkhauser, Boston, MA. **22** (1996), 174-206.

[191] V.A. Kozlov and V.G. Maz'ya, On "power-logarithmic" solutions of the Dirichlet problem for elliptic systems in $K_d \times \mathbb{R}^{N-d}$, where K_d is a d-dimensional cone, // Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **7** 1 (1996), 17-30.

[192] V.A. Kozlov and V.G. Maz'ya, Angle singularities of solutions to the Neumann problem for the two-dimensional Riccati equation, // Asymptot. Anal. **19** 1 (1999), 57-79.

[193] V.A. Kozlov and V.G. Maz'ya, Boundary singularities of solutions to quasilinear elliptic equations, // Journees Equations aux erivees partilelles, 31 mai-4 juin 1999 GDR 1151 (CNRS), VII-1-9, 131-148.

[194] V.A. Kozlov and V.G. Maz'ya, Boundary behavior of solutions to linear and nonlinear elliptic equations in plane convex domains, // Mathematical Research Letters , **8** (2001), 1-5.

[195] V.A. Kozlov and V.G. Maz'ya, Asymptotic formula for solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients near the boundary, // Ann. Scuola. Norm. Sup. Pisa , **2** 3 (2003), 551-600.

[196] V.A. Kozlov and V.G. Maz'ya, Asymptotic formula for solutions to elliptic equations near the Lipschitz boundary, // Annali di Matematica, (2004), 29 p.

[197] V.A. Kozlov and V.G. Maz'ya, Sharp conditions for the classical asymptotic behaviour near a point for solutions to quasilinear elliptic systems, // Asymptotic Analysis , **38** 2 (2004), 143-165.

[198] V.A. Kozlov, V.G. Maz'ya and J. Rossmann, Elliptic Boundary Value Problems in Domains with Point Singularities, AMS. Mathematical surveys and monographs, **52** (1997).

[199] V.A. Kozlov, V.G. Maz'ya and J. Rossman , Spectral problems associated with corner singularities of solutions to elliptic equations, AMS. Mathematical surveys and monographs, **85** (2001).

[200] V.A. Kozlov, V.G. Maz'ya and C. Schwab, *On singularities of solutions of the displacement problem of linear elasticity near the vertex of a cone,* // Arch. Rational Mech. Anal. **119** 3 (1992), 197-227.

[201] V.A. Kozlov, V.G. Maz'ya and C. Schwab, *On singularities of solutions to the Dirichlet problem of hydrodynamics near the vertex of a cone,* // J. Reine Angew. Math., **456** (1994), 65-97.

[202] V.A. Kozlov and J. Rossmann *Singularities of solutions of elliptic boundary value problems near conical points,* // Math. Nachr., **170** (1994), 161-181.

[203] M.A. Krasnoselskiĭ, *Positive solutions of operator equations,* Noordhoff Publ., Groningen (1972).

[204] I.N. Krol, *On solutions of the equation* $\operatorname{div}\{|\nabla u|^{p-2} \cdot \nabla u\} = 0$ *with a particularity at a boundary point,* Tr. Mat. Inst. Steklova **125** (1973), 127-139 (in Russian).

[205] I.N. Krol, *On the behaviour of the solutions of a quasilinear equation near a peak of the boundary,* Tr. Mat. Inst. Steklova **125** (1973), 140-146 (in Russian).

[206] I.N. Krol and V. G. Maz'ya, *The lack of continuity and Hölder continuity of solutions of a certain quasilinear equation,* // Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **14** (1969), 89-91 (Russian).

[207] I.N. Krol and V.G. Maz'ya, *On Hölder continuity and discontinuity of solutions to quasilinear elliptic equations near nonregular boundary,* // Trudy Moskov. Mat. Obsc. **26** (1972), pp. 75–94; // American Math. Soc. (1974), 73-93.

[208] N.V. Krylov, *Nonlinear elliptic and parabolic second order equations,* Nauka, Moscow, 1985 (in Russian).

[209] A. Kufner, *Weighted Sobolev spaces,* Teubner Texte zur Mathematik. **31** (1980), 1-115.

[210] A. Kufner, O. John and S. Fucik, *Function Spaces,* Noordhoff International Publishing, Leyden, 1977.

[211] A. Kufner and S. Fucik, *Nonlinear differential equations,* Elsevier Sc. Publ. Company, Amsterdam - Oxford - New York, 1980.

[212] A. Kufner and S. Leonardi, *Solvability of degenerate elliptic boundary value problems: another approach,* // Mathematica Bohemica, **119** 3 (1994), 255-274.

[213] A. Kufner and A.-M. Sándig, *Some applications of weighted Sobolev spaces,* Teubner Texte zur Mathematik. **100** (1987), 1-268.

[214] O.A. Ladyzhenskaya, *Boundary value problems of mathematical physics,* Nauka, Moscow, 1973 (in Russian).

[215] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[216] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, Nauka, Moscow, 1973 (in Russian).

[217] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Survey of results on the solvability of boundary value problems for uniformly elliptic and parabolic second order equations, having unbounded singularities* Uspekhi Mat. Nauk. **41**, 5 (1986), 59-83.

[218] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Lipschitz estimate at boundary points for solutions of quasilinear divergence equations*, Sib. Mat. Journal. **28**, 4 (1987), 145-153.

[219] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Estimates at boundary of domain of first derivatives of functions, satisfying elliptic or parabolic inequality*, Proc. of Steklov Math. Inst., **179** (1988), 102-125.

[220] L. Lanzani and Zh. Shen, *On the Robin boundary condition for Laplace's equation in Lipschitz domains*, Comm. in partial eqs., **29** 1-2 (2004), 91-109.

[221] Lee Junjie, *Existence of C^1−solutions to certain non-uniformly degenerate elliptic equations*, J. Partial Diff. Eqs., **11** 1 (1998), 9-24.

[222] S. Leonardi, *Solvability of degenerate quasilinear elliptic equations*, Nonlinear Analyse, **26** 6 (1996), 1053-1060.

[223] P. Li and S.-T. Yau, *Estimates of eigenvalues of a compact Riemannian manifold*, Proc. Symp. Pure Math. Inst. **36** (1980), 205-239.

[224] Gary M. Lieberman, *The Dirichlet problem for quasilinear elliptic equations with continuously differentiable boundary data*, Comm. Partial Differential Equations **11** (1986), 167-229.

[225] G.M. Lieberman, *Local estimates for subsolutions and supersolutions of oblique derivative problems for general second order elliptic equations*. Trans. of AMS., **304**(1987), 343-353.

[226] G.M. Lieberman, *Oblique derivative problems in Lipschitz domains. I. Continuous boundary data*. Bull. Un. Mat. Ital.(7), 1-B (1987), 1185-1210.

[227] G.M. Lieberman, *Oblique derivative problems in Lipschitz domains. II. Discontinous boundary data*. Journal für reine und angewandte Mathematik. **389** (1988), 1-21.

[228] Gary M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlineare Analysis. **12**, 11 (1988), 1203-1219.

[229] Gary M. Lieberman, *Hölder continuity of the gradient at a corner for the capillary problem and related results*, Pacific Journal of Mathematics. **133**, 1 (1988), 115-135.

[230] Gary M. Lieberman, *Optimal Hölder regularity for mixed boundary value problems*, // J. Math. Anal. & Appl., **143**, (1989), 572-586.

[231] G.M. Lieberman, *The conormal derivative problem for equations of variational type in nonsmooth domains.* Trans. of AMS, **330** (1992), 41-67.

[232] G.M. Lieberman, *The Perron process applied to oblique derivative problems.* Advances in Math., **55** (1985), 161-172.

[233] G.M. Lieberman, *The maximum principle for equations with composite coefficients,* // EJDE, **38** (2000), 1-17.

[234] G.M. Lieberman, *Pointwise estimates for oblique derivative problems in nonsmooth domains,* J. Diff. Equat. **173** (2001), 1, 178-211.

[235] G.M. Lieberman, *Regularized distance and its applications,* Pacific Journal of Mathematics **117** (1985), 2, 329-352.

[236] G.M. Lieberman, *The nonlinear oblique derivative problem for quasilinear elliptic equations.*// Nonlinear Analysis. Theory, Methods and Applications, **8** (1984), 49-65.

[237] Gary M. Lieberman, *Second order parabolic differential equations,* World Scientific, Singapore - New Jersey - London - Hong Kong, 1996, 439 p.

[238] G.M. Lieberman and N.S. Trudinger, *Nonlinear oblique boundary value problems for nonlinear elliptic equations.*// Trans. of AMS, **295** (1986), 509-546.

[239] Ch. Lin, *Interpolation inequalities with weights,* Comm. Partial Differential Equations **11**, 4 (1986), 1515-1538.

[240] P. Lindqvist, *On the equation* $div(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0,$ // Proc. AMS, **109** 1 (1990), 157-164.

[241] W.B Liu, *Degenerate quasilinear elliptic equations arising from bimaterial problems in elastic-plastic mechanics,* // Nonlinear Analysis, **35** (1999), 517-529.

[242] Ya.B. Lopatinskiy, *On one type of singular integral equtions,* // Teoret. i Prikl. Mat. (Lvov) **2** (1963), 53-57 (in Russian).

[243] J. Lubuma and S. Nicaise *Regularity of the solutions of Dirichlet problem in polihedral domains,*// Lecture notes in pure and applied mathematics, **167** (1995), 171-184.

[244] F.I. Mamedov, *Regularity of the solutions of linear and quasilinear elliptic equations in divergent form,* // Matem. zametki, **53** 1 (1993), 68-82.

[245] M. Marcus and L. Veron, *Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations,* // Ann. I.H.P. Anal. Nonlineaire, **14** 2 (1997), 237-274.

[246] M. Marcus and L. Veron, *Existence and uniqueness results for large solutions of general nonlinear equations,* // Journal of evolution equations, **3** (2004), 637-652.

[247] M. Marcus and L. Veron, *Singular nonlinear boundary problems for semilinear elliptic equations in piecewise regular domains,* (to appear).

[248] E. Marušić-Paloka, *On the mixed boundary value problem for Laplace equation in thin domain,* // Bollettino U.M.I. (7) **9-B** (1995), 1-18.

[249] V. Maslennikova and M. Bogovskiy, *Elliptic boundary value problems in unbounded domains with noncompact and nonsmooth boundaries,* Rendiconti del Seminario Matematico e Fisico di Milano **106** (1986), 125-138.

[250] V.G. Maz'ya, *The solvability in of the Dirichlet problem for a region with a smooth irregular boundary,* // Vestnik Leningrad. Univ. **19**, 7 (1964), 163-165 (Russian).

[251] V.G. Maz'ya, *On the modulus of continuity of a solution of the Dirichlet problem near an irregular boundary.* Problems Math. Anal. Boundary Value Problems Integr. Equations (Russian) Izdat. Leningrad. Univ., Leningrad. 1966, 45-58 (Russian).

[252] V.G. Maz'ya, *Solvability in $\mathring{W}_2^2$ of the Dirichlet problem in a region with a smooth irregular boundary,* // Vestnik Leningrad. Univ. **22**, 7 (1967), 87-95 (Russian).

[253] V.G. Maz'ya, *The behavior near the boundary of the solution of the Dirichlet problem for an elliptic equation of the second order in divergence form,* // Mat. Zametki **2** (1967), 209-220 (Russian).

[254] V.G. Maz'ya, *Examples of nonregular solutions of quasilinear elliptic equations with analytic coefficients,* // Funkcional. Anal. i Prilozen. **2**, 3 (1968), 53-57 (Russian).

[255] V.G. Maz'ya, *The Neumann problem in regions with nonregular boundaries,* // Sibirsk. Mat. Z. **9** (1968), 1322-1350 (Russian).

[256] V.G. Maz'ya, *The continuity at a boundary point of the solutions of quasi-linear elliptic equations,* // Vestnik Leningrad. Univ. **25**, 13 (1970), pp. 42-55; erratum: Vestnik Leningrad. Univ. **27**, 1, p. 160 (Russian). English translation: Vestnik Leningrad. Univ. Math. **3** (1976), 225-242.

[257] V.G. Maz'ya, *The Neumann problem for elliptic operators of arbitrary order in domains with nonregular boundaries,* // Vestnik Leningrad. Univ. 1, (1972), 26-33 (Russian).

[258] V.G. Maz'ya, *The removable singularities of bounded solutions of quasilinear elliptic equations of arbitrary order,* // Boundary value problems of mathematical physics and related questions in the theory of functions. **6.** Zap. Naucn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **27** (1972), 116-130 (Russian).

[259] V.G. Maz'ya, *The coercivity of the Dirichlet problem in a domain with irregular boundary*, // Izv. Vyss. Ucebn. Zaved. Matematika, 4 (1973), 64-76 (Russian).

[260] V.G. Maz'ya, *The oblique derivative problem in a domain with edges of various dimensions*, // Vestnik Leningrad. Univ. 7 Mat. Meh. Astronom. 2 (1973), 34-39 (Russian).

[261] V.G. Maz'ya, *Sobolev spaces*, Springer Verlag, Springer Series in Soviet Mathematics, 1985.

[262] V. Maz'ya, J. Elschner, J. Rehberg and G. Schmidt, *Solutions for quasilinear, nonsmooth evolution systems in L^p*, // Archive for Rat. Mech. Analysis , **171** 2 (2004), 219-262.

[263] V.G. Maz'ya, N. F. Morozov, B. A. Plamenevskii and L. Stupialis, *Elliptic Boundary Value Problems*, // Amer. Math. Soc. Transl. **123**, Ser. 2 (1984), 268 p.

[264] V.G. Maz'ya and S. A. Nazarov, *Singularities of solutions of the Neumann problem at a conic point*, // Sibirsk. Mat. Zh. 30:3, 52-63, 218 (Russian); translation in Siberian Math. J. **30** 3 (1989), 387-396.

[265] V.G. Maz'ya, S. A. Nazarov and B. A. Plamenevskiy, *Asymptotic behavior of solutions of elliptic boundary value problems under singular perturbations of the domain*. Tbilis. Gos. Univ., Inst. Prikl. Mat., Tbilisi, 1981, 207 p. (Russian) with Georgian and English summaries.

[266] V.G. Mazja, S. A. Nazarov and B. A. Plamenevskiy, *Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten*, I. Störungen isolierter Randsingularitäten. Mathematische Lehrbucher und Monographien, II. Abteilung: Mathematische Monographien, **82**, Akademie-Verlag, Berlin, 1991, 432 p.; II. Nichtlokale Störungen. Mathematische Lehrbucher und Monographien, II. Abteilung: Mathematische Monographien, **83**, Akademie-Verlag, Berlin, 1991, 319 p.

[267] V.G. Mazja, S. A. Nazarov and B. A. Plamenevskiy, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains*, vol. 1-2, Operator Theory. Edvances and Applications, **111**, XXIII+435, **112**, XXIII+323, Birkhauser, 2000.

[268] V.G. Maz'ya and B.A. Plamenevskiy, *The oblique derivative problem in a domain with a piecewise smooth boundary*, // Funkcional. Anal. i Prilozen. **5**, 3 (1971), 102-103 (Russian).

[269] V.G. Maz'ya and B.A. Plamenevskiy, *Elliptic boundary value problems in a domain with a piecewise smooth boundary*, // (Russian) Proceedings of the Symposium on Continuum Mechanics and Related Problems of Analysis (Tbilisi, 1971), **1** (1973), 171-181. Izdat. "Mecniereba", Tbilisi.

[270] V.G. Maz'ya and B.A. Plamenevskiy, *The behaviour of the solutions of quasilinear elliptic boundary–value problems in the neighbourhood of a*

conical point, // Zap. Nauchn. Sem. LOMI **38** (1973), 94-97 = J. Soviet Math **8** (1977), 423-426.

[271] V.G. Maz'ya and B.A. Plamenevskiy, *The fundamental solutions of elliptic boundary value problems, and the Miranda-Agmon maximum principle in domains with conical points*, // Sakharth. SSSR Mecn. Akad. Moambe, **73** (1974), 277-280 (Russian).

[272] V.G. Maz'ya and B.A. Plamenevskiy, *Coefficients in the asymptotics of the solutions of elliptic boundary value problems in a cone*, // Dokl. Akad. Nauk SSSR, **219** (1974), 286-289; // Zap. Nauchn. Sem. LOMI **52** (1975), 110-127 = J. Soviet. Math **9** (1978), 750-764.

[273] V.G. Maz'ya and B.A. Plamenevskiy, *On the coefficients in the asymptotics of solutions of elliptic boundary value problems in domains with conical points*, Math. Nachr. **76** (1977), 29-60.

[274] V.G. Maz'ya and B.A. Plamenevskiy, *Elliptic boundary value problems on manifolds with singularities*, Probl. Mat. Anal. **6** (1977), 85-142.

[275] V.G. Maz'ya and B.A. Plamenevskiy, *Estimates in L_p and in Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary*, Math. Nachr. **81** (1978), 25-82.

[276] V.G. Maz'ya and B.A. Plamenevskiy, *Schauder Estimates of Solutions of Elliptic Boundary Value Problems in Domain with Edges on the Boundary*, Proc. Sem. Sobolev **2** (1978) = Amer. Math. Soc. Transl. **123** (1984), 141-169.

[277] V.G. Maz'ya and B.A. Plamenevskiy, L_p- *estimates of solutions of elliptic boundary value problems in domains with edges*, Trudy Moscov. Mat. Obva **37** (1978) = Trans. Moscow Math. Soc. **1** (1980), 49-97.

[278] V.G. Maz'ya and B.A. Plamenevskiy, *Estimates of Green's functions and Schauder estimates for solutions of elliptic boundary problems in a dihedral angle*, Sib. Math. J. **19** (1978), 752-764.

[279] V.G. Maz'ya and B.A. Plamenevskiy, *Weighted spaces with nonhomogeneous norms and boundary value problems in domains with conical points*, Elliptische Gleichungen (Meeting, Rostock, 1977), Wilhelm–Piek–Univ., Rostock, 1978, 161-189 = Amer Math. Soc. Transl. **123** (1984), 89-107.

[280] V.G. Maz'ya and B.A. Plamenevskiy, *Asymptotic behaviour of the fundamental solutions of elliptic boundary value problems in domains with conical points*, Probl. Mat. Anal. **7** (1979), 100-145 = Sel. Math. Sov. **4** (1985), 363-397.

[281] V.G. Maz'ya and B.A. Plamenevskiy, *On solution of a nonlinear Dirichlet problem with a strong singularity near a corner point*, // Uspekhi Mat. Nauk, **37** 4 (1982), 101.

[282] V.G. Maz'ya and B.A. Plamenevskiy, *Weighted spaces with nonhomogeneous norms and boundary value problems in domains with conical points*, Amer Math. Soc. Transl. **123** (1984), 89-107.

[283] V.G. Maz'ya and J. Rossmann, *Über die Lösbarkeit und die Asymptotik der Lösungen elliptischer Randwertaufgaben in Gebieten mit Kanten I*, Preprint P–MATH–07/84, Berlin, 1984.

[284] V.G. Maz'ya and J. Rossmann, *Über die Lösbarkeit und die Asymptotik der Lösungen elliptischer Randwertaufgaben in Gebieten mit Kanten II*, Preprint P–MATH–30/84, Berlin, 1984.

[285] V.G. Maz'ya and J. Rossmann, *Über die Lösbarkeit und die Asymptotik der Lösungen elliptischer Randwertaufgaben in Gebieten mit Kanten III*, Preprint P–MATH–31/84, Berlin, 1984.

[286] V.G. Maz'ya and J. Rossmann, *Über die Asymptotik der Lösungen elliptischer Randwertaufgaben in der Umgebung der Kanten*, Math. Nachr. **138** (1988), 27-53.

[287] V.G. Maz'ya and J. Rossmann, *On the Agmon-Miranda maximum principle for solutions of elliptic equations in polyhedral and polygonal domains*, // Ann. Global Anal. Geom. **9** 3 (1991), 253-303.

[288] V.G. Maz'ya and J. Rossmann, *On the Agmon-Miranda maximum principle for solutions of strongly elliptic equations in domains of with conical points*, // Ann. Global Anal. Geom. **10** 2 (1992), 125-150.

[289] V.G. Maz'ya and J. Rossmann, *Stable asymptotics of the solution to the Dirichlet problem for elliptic equations of second order in domains with angular points or edges*, // Operator calculus and spectral theory (Lambrecht, 1991), 215-224, Oper. Theory Adv. Appl., **57**(1992), Birkhauser, Basel.

[290] V.G. Maz'ya and J. Rossmann, *On the behaviour of solutions to the Dirichlet problem for second order elliptic equations near edges and polyhedral vertices with critical angles*, // Z. Anal. Anwendungen, **13** 1 (1994), 19-47.

[291] V.G. Maz'ya and J. Rossmann, *Point estimates for Green's matrix to boundary value problems for second order elliptic systems in a polyhedral cone*, // Z. Angew. Math. Mech. , **82** 5 (2002), 291-316 .

[292] V.G. Maz'ya and J. Rossmann, *Weighted estimates of solutions to boundary value problems for second order elliptic systems in polyhedral domains*, // Z. Angew. Math. Mech. , **83** 7 (2003), 435-467.

[293] V.G. Maz'ya and J. Rossmann, *Schauder estimates for solutions to boundary value problems for second order elliptic systems in polyhedral domains*, // Appl. Analysis , **83** 3 (2004), 271-308.

[294] V.G. Maz'ya and V. D. Shapoznikova, *Solution of the Dirichlet and Neumann problems for irregular domains by potential-theoretic methods,* // Dokl. Akad. Nauk SSSR. **159** (1964), 1221-1223 (Russian).

[295] V.G. Maz'ya and A.S. Slutskii, *An asymptotic solution of a nonlinear Dirichlet problem with strong singularity at the corner point, I.* // Preprint, LiTH-MAT-R-92-15, Linkoping University. 1992

[296] V.G. Maz'ya and A.S. Slutskii, *Asymptotic solution to the Dirichlet problem for a two-dimensional Riccati's type equation near a corner point,* // Asymptotic Analysis (to appear).

[297] A. McIntosh, *Second order properly elliptic boundary value problems on irregular plane domains,* J. Differential Equations, **34** (1979), 361-392.

[298] A. McNabb, *Strong comparison theorems for elliptic equations of second order,* J. Math. Mech., **10** (1961), 431-440.

[299] R.C. McOwen, *The behavior of the Laplacian on weighted Sobolev spaces,* Comm. Pure Appl. Math., **32** (1979), 783-795.

[300] F.A. Mehmeti, M. Bochniak , S. Nicaise and A.-M. Saendig, *Quasilinear elliptic systems of second order in domains with corners and edges: Nemyckij operator, local existence and asymptotic behaviour,* // Z. für Analysis und ihre Anwendungen, **21** 1 (2002), 57-90.

[301] E. Miersemann, *Zur Regularität verallgemeinerter Lösungen von quasilinearen elliptischen Differentialgleichungen zweiter Ordnung in Gebieten mit Ecken,* ZAA **1** (1982), 59-71.

[302] E. Miersemann, *Quasilineare elliptische Differentialgleichungen zweiter Ordnung in mehrdimensionalen Gebieten mit Kegelspitzen,* ZAA **2** (1983), 361-365.

[303] E. Miersemann, *Zur Gleichung der Fläche mit gegebener mittlerer Krümmung in zweidimensionalen eckigen Gebieten,* Math. Nachr. **110** (1983), 231-241.

[304] E. Miersemann, *Zur gemischten Randaufgabe für die Minimalflächengleichung,* Math. Nachr. **115** (1984), 125-136.

[305] E. Miersemann, *On capillary free surfaces without gravity,* ZAA **4** (1985), pp. 429-436.

[306] E. Miersemann, *Asymptotic expansion at a corner for the capillary problem,* Pacific J. Math. **134** (1988), 299-311.

[307] E. Miersemann, *Asymptotic expansion of solutions of the Dirichlet problem for quasilinear elliptic equations of second order near a conical point,* Math. Nachr. **135** (1988), 239-274.

[308] E. Miersemann, *On the behaviour of capillaries at a corner,* Pacific J. Math **140** (1989), 149-153.

[309] E. Miersemann, *Asymptotic expansion at a corner for the capillary problem: the singular case*, Pacific J. Math. **157** (1993), 95-107.

[310] S.G Mikhlin, *Linear partial differential equations* , Vyssh. Shkola, Moscow. 1977, 431 p.

[311] K. Miller, *Barriers on cones for uniformly elliptic operators*, // Ann. Mat. Pura appl., **76** 4 (1967), 93-105.

[312] K. Miller, *Extremal barriers on cones with Phragmen - Lindelöf theorems and other applications*, // Annali di Matematica pura appl.(4), **90** (1971), 297-329.

[313] C. Miranda, *Partial differential equations of elliptic type*, Springer Verlag New York, Heidelberg, Berlin (1970).

[314] C.B. Morrey, *Functions of several variables and absolute continuity*, Duke Math. J. **6** (1940), 187-215.

[315] C.B. Morrey, *Multiple integrals in the calculus of variations*, Springer. 1966, 506 p.

[316] M.K.V. Murthy and G. Stampacchia, *Boundary value problem for some degenerate elliptic operators* Ann. Math. Pura Appl. Ser.IV, **80** (1968), 1-122.

[317] M.K.V. Murthy and G. Stampacchia, A variational inequality with mixed boundary conditions , // Israel J. of Math., **13** (1972), 188-224.

[318] S.A. Nazarov, *Asymptotic behavior near an angular boundary point of the solution to a nonlinear equation*, // Matem. Zametki, **31** 3 - 4 (1982), 211-216.

[319] S.A. Nazarov, *Estimates of solutions of the Neumann problem for elliptic systems near edges*, Vest. LGU (1988), 37-42.

[320] S.A. Nazarov and K.I. Piletskas, *Asymptotics of the solution of the nonlinear Dirichlet problem having a strong singularity near a corner point*, Math. USSR Izvestiya **25** (1985), 531-550.

[321] S.A. Nazarov and B.A. Plamenevsky, *The Neumann problem for self-adjoint elliptic systems in a domain with piecewise-smooth boundary*, Amer. Math. Soc. Transl. (2) **155** (1993), 169-206.

[322] S.A. Nazarov and B.A. Plamenevsky, *Elliptic Problems in Domains with Piecewise Smooth Boundaries*, Walter de Gruyter, Berlin, 1994.

[323] S.A. Nazarov and O.R. Polyakova, *The asymptotic form of the stress-strain state near a spatial singularity of the boundary of the beak tip type*, // J. Appl. Math. Mech., **57** 5 (1993), 887-902.

[324] S.A. Nazarov and O.R. Polyakova, *Rupture criteria, asymptotic conditions at crack tips, and selfadjoint extensions of the Lamé operator*, Trans. Moscow Math. Soc., **57** (1996), 13-66.

[325] S. Nicaise, *Differential equations in Hilbert spaces and applications to boundary value problems in nonsmooth domains*, // J. Funct. Anal., **96** 2 (1991), 195-218.

[326] V.A. Nikishkin, *Singularities of solution of elliptic Dirichlet problem for a second order differential equation in the neighbourhhod of an edge*, Vest. Mosc. Univ. (1979), 51-62 (in Russian).

[327] S.M. Nikol'skiy, *The Dirichlet problem in domains with corners*, Dokl. Akad. Nauk SSSR **109**, 1 (1956), 33-35 (in Russian).

[328] S.M. Nikol'skiy, *Approximation of functions of several variables and embedding theorems, 1-st ed.*, "Nauka", Moscow, 1969 (Russian); English transl. Springer Verlag, 1975.

[329] L. Nirenberg, *On nonlinear elliptic partial differential equations and Hölder continuity*, Comm. Pure Appl. Math. **6** (1953), 103-157.

[330] D. Pahk and J. Seo, *Regularity for solutions of a certain elliptic equation near a nonsmooth interface*, // Nonlinear Analysis, Theory, Methods & Appl., **23** 7 (1994), 949-952.

[331] G. Peano, *Sull' integrabilità delle equazione differenziali di primo ordine*, Atti R. Accad. Torino, **21** (1885/1886), 667-685.

[332] T. Petersdorff and E. Stephan, *Decompositions in edge and corner singularities for the solution of the Dirichlet problem of the Laplacian in a polyhadron*, Mathematische Nachrichten. **149** (1990), 71-104.

[333] M. Plesha, *The behavior of solutions to the Dirichlet problem for quasilinear second order elliptic equations in a neighborhood of an edge*, // Herald of Lviv university, Mathematics and Mechanics, **53** (1999), 67-76 (in Ukrainian).

[334] M. Plesha, *Barriers on domains with edges for quasilinear second order elliptic equations*, // Herald of national university "Lviv polytechnics", Applied mathematics, **407** (2000), 90-93 (in Ukrainian).

[335] M. Plesha, *Boundary value problems for second order elliptic equations in domains with edges at boundary*, Dissertation. Lvov State University. (2001), 166 p. (in Ukrainian).

[336] M. Plesha, *Best possible integral estimate of solutions of the Dirichlet problem for second order quasilinear elliptic equations near the edge*, Ukrain. Mat. Zhurn. **53**, 6 (2001), 860-865 (in Ukrainian).

[337] S.I. Pohozaev, *On the solutions smoothness of some superlinear elliptic equations*, // Doklady Akademii nauk USSR., **327** 3 (1992), 299-302.

[338] S.I. Pohozaev, *On sharp apriori estimates for some superlinear elliptic equations*, // Doklady Akademii nauk USSR., **327** 4 - 6 (1992), 433-437.

[339] S.I. Pohozaev, *On sharp apriori estimates for some superlinear elliptic equations*, // Doklady Akademii nauk USSR., **327** 4 - 6 (1992), 433-437.

[340] S.I. Pohozaev, *Sharp apriori estimates for one quasilinear degenerate elliptic problem,* // Matem. sbornik, **184** 8 (1993), 3-16.

[341] S.I. Pohozaev and L. Veron, *Multiple positive solutions of some quasilinear Neumann problems,* // Applicable Analysis, **74** (2000), 363-390.

[342] S.I. Pohozaev and L. Veron, *Nonexistence results of solutions of semilinear differential inequalities on the Heisenberg group,* // Manuscripta Mathematica, **102** (2000), 85-99.

[343] A. Porretta and L. Veron, *Symmetry properties of solutions of semilinear elliptic equations in the plane,* // Manuscripta Math., (2004), 20 p.

[344] M.H. Protter and H.F. Weinberger, *Maximum principles in differential equations,*
Englewood Cliffs, N.J.:Prentice - Hall, 1967.

[345] P. Pucci and J. Serrin, *The strong maximum principle revisited,* // J. Diff. Equat., **196** (2004), 1-66.

[346] V. Rabinovich, B.-W. Schulze and N. Tarkhanov, *A calculus of BVP in domains with non-Lipschitz singular points.* Universitat Potsdam - Institut für Mathematik. Preprint 1997/05, 58 p.

[347] V. Rabinovich, B.-W. Schulze and N. Tarkhanov, *Boundary value problems in domains with corners,* Preprint. Univ. Potsdam, **19** (1999), 32 p.

[348] V. Rabinovich, B.-W. Schulze and N. Tarkhanov, *Local algebra of a non-symmetric corner,* Birkhäuser Verlag. 2001 (Operator theory, **126**), 275-280.

[349] A. Rathsfeld, *Piecewise polynomial collocation for the double layer potential equation over polyhedral boundaries,* // Lecture notes in pure and applied mathematics, **167** (1995), 219-253.

[350] H. Reisman, *Second order elliptic boundary value problem in a domain with edges.* Comm. in PDE, **6** (1995), 1023-1042.

[351] Y. Saito, *The limiting equation for Neumann Laplacians on shrinking domains,* // EJDE, **31** (2000), 1-25.

[352] S. Sakaguchi, *Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems,* Ann. Sci. Norm. Super., Pisa, Cl. Sci. **14** (1987), 403-420.

[353] J. Sanchez Hubert and E. Sanchez Palencia, *Vibration and Coupling of Continuous Systems. Asymptotic Methods,* Springer–Verlag, Berlin, 1989.

[354] G. Savare, *Regularity results for elliptic equations in lipschitz domains,* // Journal of functional analysis, **152** (1998), 176-201.

[355] B. Schmutzler,*Branching asymptotics for elliptic BVP in a wedge,* // Lecture notes in pure and applied mathematics, **167** (1995), 255-267.

[356] B.-W. Schulze, *Boundary value problems and singular pseudo-differential operators*, J.Wiley, Chichester. 1998.

[357] J. Serrin, *Local behavior of solutions of quasilinear elliptic equations.* Acta Math., **111** (1964), 247-302.

[358] P. Shi and S. Wright, *Higher integrability of the gradient in linear elasticity*, // Math. Ann., **299** (1994), 435-448.

[359] L. Simon, *Regularity of capillary surfaces over domains with corners*, Pacific J. Math. **88** (1980), 363-377.

[360] I. Skrypnik, *Methods of investigation of nonlinear elliptic boundary value problems.* "Nauka", Moscow, 1990 (in Russian).

[361] S.L. Sobolev, *Introduction to the theory of cubature formulae*, "Nauka", Moscow, 1974 (in Russian).

[362] S.L. Sobolev, *Selected questions of the theory of functional spaces and generalized functions* "Nauka", Moscow, 1989 (in Russian).

[363] S.L. Sobolev, *Some applications of functional analysis in mathematical physics*, Trans. of Math. Monographs, **90** (1991). AMS, Providence, Rhode Island.

[364] V.A. Solonnikov, *Estimates of solutions to the Neumann problem for elliptic second order equations in domains with edges on the boundary*, // Preprint P-4-83, LOMI, Leningrad (in Russian).

[365] E. Sperner, *Schauder's existence theorem for α-Dini continuous data*, Ark. Mat. **19**, 2 (1981), 193-216.

[366] G. Stampacchia, *Some limit cases of L^p-estimates for solutions of second order elliptic equations*, Comm. Pure Appl. Math. **16** (1963), 505-510.

[367] M. Struwe, *Variational methods. Applications to nonlinear partial differential equations and hamiltonian systems.* 3d Ed., Springer Verlag, 2000, 274 p.

[368] L. Stupelis, *On the Neumann problem*, // Lithuánian Math. Sb., **30** 2 (1990), 377-381 (in Russian).

[369] V.V. Sushkov, *Estimates of solutions to the Dirichlet problem for second order elliptic equations with the degeneration in a neighborhood of the conical point.* Degree work. Lvov University, 1997, 49 p.

[370] G. Sweers, *Semilinear elliptic problems on domains with corners*, Communications in partial differential equations. **14**, 8-9 (1989), 1229-1247, // 92-110.

[371] H. Takeuchi, *On the first eigenvalue of the p-Laplacian in a Riemannian manifold*, Tokio J. Math. **21** (1998), 137-140.

[372] F. Thelin, *Quelques résultats d'existence et de non-existence pour une E.D.P. elliptique non linéaire*, // C.R. Acad. Sc. Paris, Ser. I, **299** 18 (1984), 911-914.

[373] P. Tolksdorf, *On quasilinear boundary value problems in domains with corners*, Nonlinear Analysis **5** (1981), 721-735.

[374] P. Tolksdorf, *Zum Dirichletproblem für quasilineare Gleichungen in Gebieten mit konischen Randpunkten*, Dissertation, Universität Bonn, 1982.

[375] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Part. Diff. Equat. **8** (1983), 773-817.

[376] P. Tolksdorf, *On the behaviour near the boundary of solutions of quasilinear equations*, Analysis **3** (1983), 55-78.

[377] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, Journal of Diff. Equat., **51**, 1 (1984), 126-150.

[378] P. Tolksdorf, *Invariance properties and special structures near conical boundary points*, Lecture notes in mathematics, **1121** (1985), 308-318.

[379] G. Trombetti, *Problemi ellitici in un angelo*, Ann. Mat. Pura Appl., **107** (1975), 95-129.

[380] G. Trombetti, *Problemi ellitici in un cono*, Ricerche Mat., **26** (1977), 103-134.

[381] N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math **20** (1967), 721-747.

[382] Neil S. Trudinger, *Lipschitz continuous solutions of elliptic equations of the form $\mathcal{A}(Du)D^2u = 0$*, Math. Z., **109** (1969), 211-216.

[383] N.S. Trudinger, *Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations*, Invent. Math. **61** (1980), 67-79.

[384] L. Veron, *Asymptotic behaviour of the solutions of some nonlinear elliptic equations*, // Nonlinear Problems of Analysis in Geometry and Mechanics, Pitman Ed., (1980), 198-208.

[385] L. Veron, *Singular solutions of some nonlinear elliptic equations*, // Nonlinear Anal. T. M. & A , **5** (1981), 225-242.

[386] L. Veron, *Singularites isotropes des solutions d' equations elliptiques non lineaires*, Portugaliae Math., **41** (1982), 88-102.

[387] L. Veron, *Behavior of solutions of nonlinear elliptic equations near a singularity of codimension 2*, // Contribution to Nonlinear Partial Differential Equations, Pitman Ed., (1984), 274-284.

[388] L. Veron, *Global behaviour and symmetry properties of singular solutions of nonlinear elliptic equations*, // Ann. Fac. Sci. Toulouse, **6** (1984), 1-31.

[389] L. Veron, *Weak and strong singularities of nonlinear elliptic equations*, // Proc. Symp. Pure Math., **45** (1986), 477-495.

[390] L. Veron, *Singularities of some degenerate elliptic equations,* // Contributions to Nonlinear Partial Differential Equations II, Pitman Ed., (1987), 288-308 .

[391] L. Veron, *Singularities of some quasilinear equations,* // Nonlinear Diffusion Equations and their Equilibrium States, Math. Sci. Res. Inst. Series 13, Springer-Verlag Ed., (1987), 335-365 .

[392] L. Veron, *Limit behaviour of singular solutions of some semilinear elliptic equations,* // Banach Center Publ. **19** (1987), 311-350.

[393] L. Veron, *Singularities of some quasilinear equations,* Math. Sci. Res. Inst. **13** (1988), 333–365.

[394] L. Veron,*Semilinear elliptic equations with uniform blow-up on the boundary,* // J. d' Analyse Math., **59** (1992), 231-250.

[395] L. Veron,*Some existence and uniqueness results for solutions of some quasilinear elliptic equations on compact Riemannian manifolds,* // Colloquia Math. Soc. Janos Bolyai, **62** (1994), 317-352.

[396] L. Veron,*Singularities of solutions of second order quasilinear equations,* Pitman Research Notes in Mathematics Series, Addisson-Wesley-Longman. **353** (1996), 1-388.

[397] L. Veron,*Generalized boundary values problems for nonlinear elliptic equations,* Elec. J. Diff. Equ. Conf., **06** (2001), 313-342.

[398] G.M. Verzhbinsky and V.G. Maz'ya, *Asymptotic behaviour of the solutions of second order elliptic equations near the boundary I,* Siberian Math. J. **12** (1971), 874-899.

[399] G.M. Verzhbinsky and V.G. Maz'ya, *Asymptotic behaviour of the solutions of second order elliptic equations near the boundary II,* Siberian Math. J. **13** (1972), 858-885.

[400] G.M. Verzhbinsky and V.G. Maz'ya, *The closure in L_p of the operator of the Dirichlet problem in a domain with conical points,* Izv. Vysch. Uchebn. Zaved. Mat. (1974), 8-19.

[401] J.N. Voinov, *The boundedness of gradients of generalized solutions of boundary value problems for quasilinear elliptic equations near the buondary,* Vestnik Leningrad. Univ., Mat., Mech., Astron. **7** (1974), 5-13 (in Russian).

[402] J. Voldřich, *Application of Sobolev weight spaces to the solution of elliptic boundary value problems.* Thesis. Math. Inst. Acad. Sci., Prague. 1986.

[403] E.A. Volkov, *On the differential properties of boundary value problems for the Laplace and Poisson equtions on a rectangle,* Proc. Steklov Inst. Math. **77** (1965), 89-112 (in Russian).

[404] E.T. Whittaker and G.N. Watson, *A cours of modern Analysis,* II, 4th Ed. Cambridge: University Press, 1927.

[405] K.-O. Widman, *On the boundary behavior of solutions to a class of elliptic partial differential equations,* Arkiv för matematik. **6**, 26 (1966), 485-533.

[406] K.-O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations,* Mathematica scandinavica. **21**, 1 (1967), 17-37.

[407] N. Wigley, *Mixed boundary value problem in plane domains with corners,* Mathematische Zeitschrift. **115** (1970), 33-52.

[408] N. Wigley, *Schauder estimates in domains with corners,* Arch. Ration. Mech. and Anal. **104**, 3 (1988), 271-276.

[409] W. Zajączkowski and V.A. Solonnikov, *On the Neumann problem for second order elliptic equations in domains with edges at the boundary,* Zap. Nauchn. Sem. LOMI **127**, 15 (1983), 7-48 = J. Soviet Math. **27** (1984), 2561-2586.

[410] J.Zhiren and L. Kirk, *Behavior of solutions for some Dirichlet problems near reentrant corners,* // Indiana University Mathematics Journal. **46** 3 (1997), 827-862.

[411] J.Q. Zhong and H.C. Yang, *On the estimates of the first eigenvalue of a compact Riemannian manifold,* Sci. Sinica, Ser. A **27** (12) (1984), 1265-1273.

[412] W.P. Ziemer, *Weakly Differentiable Functions,* Springer–Verlag, New York, 1989.

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