Recent Advances on

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Michel Chipot | Hirokazu Ninomiya

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2004 Swiss-Japanese Seminar

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Zurich, Switzerland • 6–10 December 2004

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World Scientific

NEW JERSEY · LONDON · SINGAPORE · BEIJING · SHANGHAI · HONG KONG · TAIPEI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

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Proceedings of the 2004 Swiss–Japanese Seminar

ISBN 981-256-675-9

Printed in Singapore by World Scientific Printers (S) Pte Ltd

PREFACE

The Swiss-Japanese seminar on elliptic and parabolic issues in applied sciences was held at the University of Zürich (Switzerland) in December 7-9, 2004. This book collects different papers on the research themes that were discussed during this seminar.

We hope that these articles will become a landmark in the field of elliptic and parabolic problems.

We thank JSPS and SNF for having generously supported our reunion. We extend our warm thanks to the University of Zürich for its hospitality and to Ms Zhang and World Scientific for their editing work.

Zürich, October 2005

Michel Chipot
Hirokazu Ninomiya

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CONTENTS

STEADY FREE CONVECTION IN A BOUNDED AND SATURATED POROUS MEDIUM

SAMIR AKESBI†, BERNARD BRIGHI‡ AND JEAN-DAVID HOERNEL♯

In this paper we are interested with a strongly coupled system of partial differential equations that modelizes free convection in a two-dimensional bounded domain filled with a fluid saturated porous medium. This model is inspired by the one of free convection near a semi-infinite impermeable vertical flat plate embedded in a fluid saturated porous medium. We establish the existence and uniqueness of the solution for small data in some unusual spaces.

1. Introduction

In the literature, many papers about free convection in fluid saturated porous media study the case of the semi-infinite vertical flat plate in the framework of boundary layer approximations. This approach allows to introduce similarity variables to reduce the whole system of partial differential equations into one single ordinary differential equation of the third order with appropriate boundary values. This two points boundary value problem can be studied using a shooting method or an auxiliary dynamical system either in the case of prescribed temperature or in the case of prescribed heat flux along the plate.

In this article we first present the derivation of the equations, show how the boundary layer approximation leads to the two points boundary value problem and the similarity solutions, then we rewrite the full problem of free convection in a two-dimensional bounded domain filled with a fluid saturated porous medium. This new model, written in terms of stream function and temperature, consists in two strongly coupled partial differential equations. We establish the existence and uniqueness of its solution for small data.

AMS 2000 Subject Classification: 34B15, 34B40, 35Q35, 76R10, 76S05.
Key words and phrases: Free convection, porous medium, coupled pdes.
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2. The semi-infinite vertical flat plate case

Let us consider a semi-infinite vertical permeable or impermeable flat plate embedded in a fluid saturated porous medium at the ambient temperature T_∞, and a rectangular Cartesian co-ordinates system with the origin fixed at the leading edge of the vertical plate, the x-axis directed upward along the plate and the y-axis normal to it. If we suppose that the porous medium is homogeneous and isotropic, that all the properties of the fluid and the porous medium are constants and that the fluid is incompressible and follows the Darcy-Boussinesq law we obtain the following governing equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u = -\frac{k}{\mu}\left(\frac{\partial p}{\partial x} + \rho g\right),$$

$$v = -\frac{k}{\mu}\frac{\partial p}{\partial y},$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \lambda\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right),$$

$$\rho = \rho_\infty(1 - \beta(T - T_\infty))$$

in which u and v are the Darcy velocities in the x and y directions, ρ, μ and β are the density, viscosity and thermal expansion coefficient of the fluid, k is the permeability of the saturated porous medium, λ is its thermal diffusivity, p is the pressure, T the temperature and g the acceleration of the gravity. The subscript ∞ is used for values taken far from the plate. In our system of co-ordinates there are two main interesting sets of boundary conditions along the plate.

First, the temperature is prescribed on the wall that gives

$$v(x,0) = \omega x^{\frac{m-1}{2}}, \quad T(x,0) = T_w(x) = T_\infty + Ax^m \tag{1}$$

with $m \in \mathbb{R}$ and $A > 0$, see [16], [18], [21], [28] and [32].

Secondly, the heat flux is prescribed along the plate that leads to

$$v(x,0) = \omega x^{\frac{m-1}{3}}, \quad \frac{\partial T}{\partial y}(x,0) = -x^m \tag{2}$$

with $m \in \mathbb{R}$, see [10] and [17].

The parameter $\omega \in \mathbb{R}$ is the mass transfer coefficient. For an impermeable wall we have $\omega = 0$, and for a permeable wall, $\omega < 0$ corresponds to fluid suction and $\omega > 0$ to fluid injection. The boundary conditions far from the plate are the same in both cases (12) and (13)

$$u(x,\infty) = 0, \quad T(x,\infty) = T_\infty.$$

If we introduce the stream function Ψ such that

$$u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}$$

we obtain the system in which it remains only Ψ and T

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = \frac{\rho_\infty \beta g k}{\mu} \frac{\partial T}{\partial y}, \tag{3}$$

$$\lambda \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) = \frac{\partial T}{\partial x} \frac{\partial \Psi}{\partial y} - \frac{\partial T}{\partial y} \frac{\partial \Psi}{\partial x}. \tag{4}$$

Along the wall, the boundary conditions (12) become

$$\frac{\partial \Psi}{\partial x}(x,0) = -\omega x^{\frac{m-1}{2}}, \quad T(x,0) = T_w(x) = T_\infty + A x^m \tag{5}$$

and (13) becomes

$$\frac{\partial \Psi}{\partial x}(x,0) = -\omega x^{\frac{m-1}{3}}, \quad \frac{\partial T}{\partial y}(x,0) = -x^m. \tag{6}$$

The boundary conditions far from the plate become

$$\frac{\partial \Psi}{\partial y}(x,\infty) = 0, \quad T(x,\infty) = T_\infty. \tag{7}$$

We will start from the equations (3)-(4) subjected to the boundary conditions (5) and (7) with $\omega = 0$ to write a new model, settled in a two-dimensional bounded domain, that we will study in the rest of this paper.

Before doing this, let us say a few words about the similarity solutions. Assuming that convection takes place in a thin layer around the plate, we obtain the boundary layer approximation

$$\frac{\partial^2 \Psi}{\partial y^2} = \frac{\rho_\infty \beta g k}{\mu} \frac{\partial T}{\partial y}, \tag{8}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{1}{\lambda} \left(\frac{\partial T}{\partial x} \frac{\partial \Psi}{\partial y} - \frac{\partial T}{\partial y} \frac{\partial \Psi}{\partial x} \right) \tag{9}$$

with the same boundary conditions (5) or (6) and (7) as before.

For the case of prescribed heat, introducing the new dimensionless similarity variables

$$
\begin{aligned}
t &= (Ra_x)^{\frac{1}{2}}\frac{y}{x},\\
\Psi(x,y) &= \lambda(Ra_x)^{\frac{1}{2}}f(t),\\
T(x,y) &= (T_w(x)-T_\infty)\theta(t)+T_\infty
\end{aligned}
$$

with

$$Ra_x = \frac{\rho_\infty \beta g k (T_w(x)-T_\infty)x}{\mu\lambda}$$

the local Rayleigh number, equations (8) and (9) with the boundary conditions (5) and (7) leads to the third order ordinary differential equations

$$f''' + \frac{m+1}{2}ff'' - mf'^2 = 0$$

on $[0,\infty)$ subjected to

$$f(0) = -\gamma, \quad f'(0) = 1 \quad \text{and} \quad f'(\infty) = 0$$

where

$$\gamma = \frac{2\omega}{m+1}\sqrt{\frac{\mu}{\rho_\infty \beta g k A\lambda}}.$$

One can find explicit solutions of this problem for some particular values of γ or m in [5], [6], [9], [20], [26], [28], [30] and [35]. For mathematical results about existence, nonexistence, uniqueness, nonuniqueness and asymptotic behavior, see [2], [5], [6] and [28] for $\gamma = 0$, and [9], [12], [15], [23] and [24] for the general case. Numerical investigations can be found in [2], [7], [16], [18], [28], [30] and [38].

In the case of prescribed heat flux, we introduce the new dimensionless similarity variables

$$
\begin{aligned}
t &= 3^{-\frac{1}{3}}R_a^{\frac{1}{3}}x^{\frac{m-1}{3}}y,\\
\Psi(x,y) &= 3^{\frac{2}{3}}R_a^{\frac{1}{3}}\lambda x^{\frac{m+2}{3}}f(t),\\
T(x,y) &= 3^{\frac{1}{3}}R_a^{-\frac{1}{3}}x^{\frac{2m+1}{3}}\theta(t)+T_\infty
\end{aligned}
$$

and the Rayleigh number

$$R_a = \frac{\rho_\infty \beta g k}{\mu\lambda}.$$

Then, equations (8) and (9) with the boundary conditions (6)-(7) give

$$f''' + (m+2)ff'' - (2m+1)f'^2 = 0$$

and

$$f(0) = -\gamma, \quad f''(0) = -1 \quad \text{and} \quad f'(\infty) = 0$$

where

$$\gamma = \frac{3^{\frac{1}{3}} R_a^{-\frac{1}{3}} \omega}{\lambda(m+2)}.$$

The study of existence, uniqueness and qualitative properties of the solutions of this problem is made in [10]. For a survey of the two cases, see [11]. This equation can also be found in industrial processes such as boundary layer flow adjacent to stretching walls (see [2], [3], [20], [26], [30]) or excitation of liquid metals in a high-frequency magnetic field (see [33]).

One particular case of the two previous equations is the Blasius equation $f''' + ff'' = 0$ introduced in [8] and studied, for example, in [4], [19] and [27].

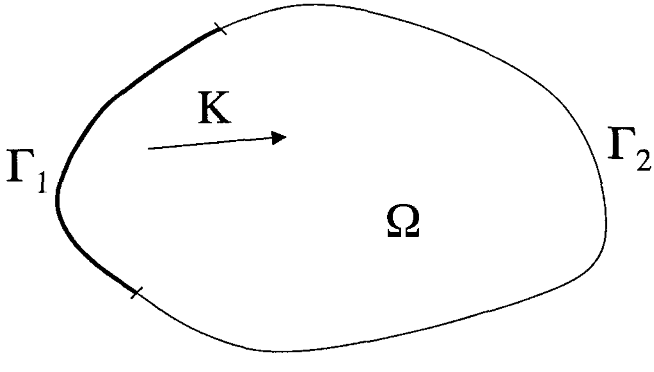
The case of mixed convection $f''' + ff'' + mf'(1-f') = 0$ with $m \in \mathbb{R}$ is interesting too and results about it can be found in [1], [13], [25] and [34]. The Falkner-Skan equation $f''' + ff'' + m(1 - f'^2) = 0$ with $m \in \mathbb{R}$ is in the same family of problems, see [19], [22], [27], [29], [37], [39] and [40] for results about it.

New results about the more general equation $f''' + ff'' + g(f') = 0$ for some given function g can be found in [14], see also [36].

3. A model problem in a bounded domain

Let $\Omega \subset \mathbb{R}^2$ be a simply connected, bounded lipschitz domain whose boundary $\Gamma = \partial\Omega$ is divided in two connected parts Γ_1 and Γ_2 such that

$$\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma \text{ and } \Gamma_1 \cap \Gamma_2 = \emptyset.$$



We start from the previous equations (3)-(4) in terms of the stream function Ψ and the temperature T with $K = \left(0, \frac{\rho_\infty \beta g k}{\mu}\right)$, and assuming that Γ_1 is impermeable and that the temperature $T_w \geq 0$ is known on the whole boundary Γ, we modify the equation (3) by setting $K(x) = (k_1(x), k_2(x)) \in \mathbb{R}^2$ with $0 < \|K\|_\infty < \infty$. Then, we obtain the following new problem in the bounded domain Ω, which consists in finding (Ψ, T)

$$\Psi : \Omega \to \mathbb{R}$$

$$T : \Omega \to \mathbb{R}$$

verifying the equations in Ω

$$\Delta\Psi = K.\nabla T, \tag{10}$$

$$\lambda\Delta T = \nabla T.(\nabla\Psi)^\perp, \tag{11}$$

the boundary conditions on Γ for Ψ

$$\Psi = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial\Psi}{\partial n} = 0 \text{ on } \Gamma_2 \tag{12}$$

and the boundary conditions on Γ for T

$$T = T_w \text{ on } \Gamma \tag{13}$$

where $\lambda \in \mathbb{R}^{+*}$ and for all $x = (u, v) \in \Omega$, let $x^\perp = (v, -u)$.

3.1. *Preliminary results*

Let us assume that $T_w \in H^{\frac{1}{2}}(\Gamma)$ and let Θ be the unique function in $H^1(\Omega)$ verifying

$$\Delta\Theta = 0 \quad \text{in } \Omega, \tag{14}$$

$$\Theta = T_w \quad \text{on } \Gamma. \tag{15}$$

In the following we will need that $\nabla\Theta \in L^\infty(\Omega)$, thus we will suppose that it holds (it is the case if $T_w \in H^{\frac{5}{2}}(\Gamma)$ for example).

If (Ψ, T) is a solution of (10)-(13) and if we set $H = T - \Theta$, then (Ψ, H) is a solution of

$$\Delta\Psi = K.\nabla H + K.\nabla\Theta, \tag{16}$$

$$\lambda\Delta H = \nabla H.(\nabla\Psi)^\perp + \nabla\Theta.(\nabla\Psi)^\perp \tag{17}$$

in the domain Ω with the boundary conditions for Ψ

$$\Psi = 0 \text{ on } \Gamma_1 \quad \text{and} \quad \frac{\partial\Psi}{\partial n} = 0 \text{ on } \Gamma_2 \tag{18}$$

and the boundary conditions for H

$$H = 0 \text{ on } \Gamma. \tag{19}$$

Conversly, it is clear that if (Ψ, H) is a solution of (16)-(19) then $(\Psi, T) := (\Psi, H + \Theta)$ is a solution of (10)-(13).

In the following we set $\|.\|_{L^1(\Omega)} = \|.\|_1$, $\|.\|_{L^2(\Omega)} = \|.\|_2$, $\|.\|_{L^\infty(\Omega)} = \|.\|_\infty$ and

$$(u, v) = \int_\Omega uv dx.$$

Definition 3.1. For $u \in L^\infty(\Omega)$, $v \in H_0^1(\Omega)$ and $w \in H^1(\Omega)$ let

$$a(u, v, w) = (u\nabla v, (\nabla w)^\perp)_{L^2(\Omega), L^2(\Omega)}.$$

Remark 3.1. The trilinear form a is well defined because for $u \in L^\infty(\Omega)$, $v \in H_0^1(\Omega)$ and $w \in H^1(\Omega)$ we have

$$|a(u, v, w)| \leq \|u\|_\infty \|\nabla v\|_2 \|\nabla w\|_2.$$

Proposition 3.1. *For $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $v \in H^1(\Omega)$ we have*

$$a(u, u, v) = 0. \tag{20}$$

Proof. First, let us notice that if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ then $u^2 \in H_0^1(\Omega)$ and $\nabla(u^2) = 2u\nabla u$. Hence

$$\begin{aligned}
a(u, u, v) &= (u\nabla u, (\nabla v)^\perp)_{L^2(\Omega), L^2(\Omega)} \\
&= \frac{1}{2}(\nabla u^2, (\nabla v)^\perp)_{L^2(\Omega), L^2(\Omega)} \\
&= -\frac{1}{2}(\operatorname{div}((\nabla v)^\perp), u^2)_{H^{-1}(\Omega), H_0^1(\Omega)} \\
&= 0
\end{aligned}$$

because $u = 0$ on Γ and $\operatorname{div}((\nabla v)^\perp) = 0$ in $H^{-1}(\Omega)$. □

Remark 3.2. For $u, v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $w \in H^1(\Omega)$ we have

$$a(u, v, w) = -a(v, u, w). \tag{21}$$

3.2. *A priori estimates*

Let

$$W_\Psi = \{u \mid u \in H^1(\Omega) \text{ and } u = 0 \text{ on } \Gamma_1\}$$

and

$$W_H = H_0^1(\Omega) \cap L^\infty(\Omega).$$

The spaces W_Ψ and W_H are equipped with the norms $\|.\|_{W_\Psi}$ and $\|.\|_{W_H}$ defined by

$$\|u\|_{W_\Psi} = \|\nabla u\|_2 \quad \text{and} \quad \|u\|_{W_H}^2 = \|u\|_\infty^2 + \|\nabla u\|_2^2.$$

In the following we will use the notation C for the Poincaré's constant of Ω.

Definition 3.2. We will call $(\Psi, H) \in W_\Psi \times W_H$ a weak solution of the problem (16)-(19) if and only if we have

$$(\nabla\Psi, \nabla u) + (K.\nabla H, u) + (K.\nabla\Theta, u) = 0, \tag{22}$$

$$\lambda(\nabla H, \nabla v) + a(v, H, \Psi) + a(v, \Theta, \Psi) = 0 \tag{23}$$

for all $u \in W_\Psi$ and $v \in W_H$.

Proposition 3.2. *Let $(\Psi, H) \in W_\Psi \times W_H$ be a solution of the problem (22)-(23) and $T = H + \Theta$, then*

$$\inf_\Gamma T_w \le T \le \sup_\Gamma T_w. \tag{24}$$

Proof. Set $l = \sup_\Gamma T_w$ and $T^+ = \sup(T - l, 0)$. As $T^+ \in W_H$, using (23) with $v = T^+$ and noticing that $(\nabla\Theta, \nabla T^+) = 0$ because $\Delta\Theta = 0$, leads to

$$\lambda(\nabla T, \nabla T^+) + a(T^+, T, \Psi) = 0.$$

Using the facts that $\lambda(\nabla T, \nabla T^+) = \lambda(\nabla T^+, \nabla T^+)$ and $a(T^+, T, \Psi) = a(T^+, T^+, \Psi) = 0$ by proposition 3.1 we obtain that

$$\|\nabla T^+\|_2 = 0$$

and as $T^+ \in H_0^1(\Omega)$ we have $T^+ = 0$ on Ω. We proceed in the same way with $l' = \inf_\Gamma T_w$ and $T^- = \inf(T - l', 0)$ for the other inequality. □

Proposition 3.3. *Let $(\Psi, H) \in W_\Psi \times W_H$ be a solution of the problem (22)-(23), then for $\|\nabla\Theta\|_\infty < \frac{\lambda}{2C^2\|K\|_\infty}$ we have*

$$\|\nabla\Psi\|_2 \le 2C\|K\|_\infty\|\nabla\Theta\|_2 \quad \textit{and} \quad \|\nabla H\|_2 \le \|\nabla\Theta\|_2.$$

Proof. Taking $u = \Psi$ in (22) and using Poincaré's inequality we obtain

$$\begin{aligned}\|\nabla\Psi\|_2^2 \leq &|(K.\nabla H, \Psi)| + |(K.\nabla\Theta, \Psi)| \\ \leq &\|K\|_\infty \left(\|\nabla H\|_2 + \|\nabla\Theta\|_2\right) \|\Psi\|_2 \\ \leq &C\|K\|_\infty \left(\|\nabla H\|_2 + \|\nabla\Theta\|_2\right) \|\nabla\Psi\|_2\end{aligned}$$

and

$$\|\nabla\Psi\|_2 \leq C\|K\|_\infty \left(\|\nabla H\|_2 + \|\nabla\Theta\|_2\right). \tag{25}$$

Taking $v = H$ in (23) leads to

$$\lambda(\nabla H, \nabla H) + a(H, H, \Psi) + a(H, \Theta, \Psi) = 0.$$

Then, by proposition 3.1 we have

$$\begin{aligned}\lambda\|\nabla H\|_2^2 &\leq |a(H, \Theta, \Psi)| \\ &\leq \|\nabla\Theta\|_\infty \|H\|_2 \|\nabla\Psi\|_2 \\ &\leq C\|\nabla\Theta\|_\infty \|\nabla H\|_2 \|\nabla\Psi\|_2\end{aligned}$$

using Poincaré's inequality and

$$\|\nabla H\|_2 \leq \frac{C}{\lambda}\|\nabla\Theta\|_\infty \|\nabla\Psi\|_2. \tag{26}$$

Then, combining (25) and (26) leads to

$$\|\nabla\Psi\|_2 \leq C\|K\|_\infty \|\nabla\Theta\|_2 + \frac{C^2\|K\|_\infty}{\lambda}\|\nabla\Theta\|_\infty \|\nabla\Psi\|_2.$$

Thus

$$\left(1 - \frac{C^2\|K\|_\infty}{\lambda}\|\nabla\Theta\|_\infty\right) \|\nabla\Psi\|_2 \leq C\|K\|_\infty \|\nabla\Theta\|_2$$

and as $\frac{C^2\|K\|_\infty}{\lambda}\|\nabla\Theta\|_\infty < 1/2$ we have

$$\|\nabla\Psi\|_2 \leq 2C\|K\|_\infty \|\nabla\Theta\|_2.$$

Using this new inequality in (26), we obtain

$$\|\nabla H\|_2 \leq \|\nabla\Theta\|_2.$$

□

Remark 3.3. As

$$\|\nabla\Theta\|_2 \leq (\text{mes } \Omega)^{\frac{1}{2}} \|\nabla\Theta\|_\infty \quad \text{and} \quad \|\nabla\Theta\|_\infty < \frac{\lambda}{2C^2\|K\|_\infty}$$

we can rewrite the previous result as

$$\|\nabla\Psi\|_2 \leq \frac{\lambda}{C}(\text{mes } \Omega)^{\frac{1}{2}} \quad \text{and} \quad \|\nabla H\|_2 \leq \frac{\lambda}{2C^2\|K\|_\infty}(\text{mes } \Omega)^{\frac{1}{2}}.$$

3.3. *Main results*

Theorem 3.1. *Let* $M = \sup_\Gamma T_w$. *If* $MC\|K\|_\infty < \lambda$, *then the problem* (16)-(19) *admits at most one weak solution* (Ψ, H) *in* $W_\Psi \times W_H$.

Proof. Let (Ψ_1, H_1) and (Ψ_2, H_2) be two solutions of (16)-(19). Setting $\bar{H} = H_1 - H_2$ and $\bar{\Psi} = \Psi_1 - \Psi_2$ we obtain

$$\begin{aligned}(\nabla\bar{\Psi}, \nabla u) + (K.\nabla\bar{H}, u) &= 0,\\ \lambda(\nabla\bar{H}, \nabla v) + a(v, H_1, \Psi_1) - a(v, H_2, \Psi_2) + a(v, \Theta, \bar{\Psi}) &= 0\end{aligned}$$

for $u \in W_\Psi$ and $v \in W_H$. Choosing $u = \bar{\Psi}$ and $v = \bar{H}$ leads to

$$(\nabla\bar{\Psi}, \nabla\bar{\Psi}) + (K.\nabla\bar{H}, \bar{\Psi}) = 0, \tag{27}$$

$$\lambda(\nabla\bar{H}, \nabla\bar{H}) + a(\bar{H}, H_1, \Psi_1) - a(\bar{H}, H_2, \Psi_2) + a(\bar{H}, \Theta, \bar{\Psi}) = 0. \tag{28}$$

From equation (27) we deduce that

$$\|\nabla\bar{\Psi}\|_2 \le C\|K\|_\infty\|\nabla\bar{H}\|_2. \tag{29}$$

Let us compute

$$\begin{aligned}a(\bar{H}, H_1, \Psi_1) - a(\bar{H}, H_2, \Psi_2) &= -a(H_2, H_1, \Psi_1) - a(H_1, H_2, \Psi_2)\\ &= a(H_1, H_2, \Psi_1) - a(H_1, H_2, \Psi_2)\\ &= a(H_1, H_2, \bar{\Psi})\\ &= a(\bar{H}, H_1, \bar{\Psi}).\end{aligned}$$

Thus, using now equation (28) we get

$$\lambda(\nabla\bar{H}, \nabla\bar{H}) + a(\bar{H}, H_1 + \Theta, \bar{\Psi}) = 0$$

and

$$\begin{aligned}\lambda\|\nabla\bar{H}\|_2^2 &\le |a(\bar{H}, H_1 + \Theta, \bar{\Psi})|\\ &\le |a(T_1, \bar{H}, \bar{\Psi})|\\ &\le \|T_1\|_\infty\|\nabla\bar{H}\|_2\|\nabla\bar{\Psi}\|_2\\ &\le M\|\nabla\bar{H}\|_2\|\nabla\bar{\Psi}\|_2\end{aligned}$$

with $M = \sup_\Gamma T_w$. Therefore

$$\|\nabla\bar{H}\|_2 \le \frac{M}{\lambda}\|\nabla\bar{\Psi}\|_2$$

and using (29) we have

$$\|\nabla\bar{H}\|_2 \le \frac{MC\|K\|_\infty}{\lambda}\|\nabla\bar{H}\|_2.$$

Choosing $\frac{MC\|K\|_\infty}{\lambda} < 1$ we obtain $\|\nabla \bar{H}\|_2 = 0$ and $\|\nabla \bar{\Psi}\|_2 = 0$. This complete the proof. $\square$

In the following Theorem, we prove the existence of a strong solution (Ψ, H) of the problem (16)-(19) under some hypothesis on the data. To this aim, let us define the spaces

$$\tilde{W}_\Psi = \left\{ u \mid u \in H^2(\Omega), u = 0 \text{ on } \Gamma_1 \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_2 \right\}$$

and

$$\tilde{W}_H = H_0^1(\Omega) \cap H^2(\Omega).$$

These spaces are equipped with the following norms

$$\begin{aligned} \|u\|^2_{\tilde{W}_\Psi} &= \|\nabla u\|^2_{H^1(\Omega)}, \\ \|v\|^2_{\tilde{W}_H} &= \|\nabla v\|^2_{H^1(\Omega)}, \\ \|(u,v)\|^2_{\tilde{W}_\Psi \times \tilde{W}_H} &= \|u\|^2_{\tilde{W}_\Psi} + \|v\|^2_{\tilde{W}_H} \end{aligned}$$

and

$$\|(u,v)\|_{L^2(\Omega)\times L^2(\Omega)} = \|u\|_2 + \|v\|_2.$$

Theorem 3.2. *Let $M = \sup_\Gamma T_w$. For $\max\{C\|\nabla\Theta\|_\infty, M\} < \frac{\lambda}{C\|K\|_\infty}$ and small values of $\|K.\nabla\Theta\|_2$, there exists a unique solution (Ψ, H) of the problem* (16)-(19) *in the space $\tilde{W}_\Psi \times \tilde{W}_H$.*

Proof. Let us define the operator

$$A \,:\, \tilde{W}_\Psi \times \tilde{W}_H \to L^2(\Omega) \times L^2(\Omega)$$

such that $A(\Psi, H) = (A_1(\Psi, H), A_2(\Psi, H))$ with

$$\begin{aligned} &A_1(\Psi, H) = \Delta\Psi - K.\nabla H, \\ &A_2(\Psi, H) = \lambda\Delta H - \nabla H.(\nabla\Psi)^\perp - \nabla\Theta.(\nabla\Psi)^\perp. \end{aligned}$$

Let us remark that, using the Sobolev embedding theorem, we have $H^1(\Omega) \hookrightarrow L^4(\Omega)$ in such a way that $\nabla H.(\nabla\Psi)^\perp \in L^2(\Omega)$.
In term of the operator A, the equations (16)-(17) can be rewritten as

$$A(\Psi, H) = (K.\nabla\Theta, 0).$$

Notice that $(\Psi, H) = (0,0)$ is a solution of $A(\Psi, H) = (0,0)$ and by the same argument as in Theorem 3.1, it is the only one.

Now we want to show that the solution of $A(\Psi,H)=(K.\nabla\Theta,0)$ also exists for small values of $\|K.\nabla\Theta\|_2$. To this end, let us compute the Fréchet derivative of A. For $\phi\in\tilde{W}_\Psi$ and $G\in\tilde{W}_H$, we have

$$\begin{aligned}A(\phi,G)-(\Delta\phi-K.\nabla G,\lambda\Delta G-\nabla\Theta.(\nabla\phi)^\perp) &:= A(\phi,G)-L(\phi,G)\\ &=(0,-\nabla G.(\nabla\phi)^\perp)\\ &=o(\|(\phi,G)\|_{\tilde{W}_\Psi\times\tilde{W}_H})\end{aligned}$$

because

$$\begin{aligned}\|(0,\nabla G.(\nabla\phi)^\perp)\|_{L^2(\Omega)\times L^2(\Omega)} &= \|\nabla G.(\nabla\phi)^\perp\|_{L^2(\Omega)}\\ &\le \|\nabla G\|_{L^4(\Omega)}\|\nabla\phi\|_{L^4(\Omega)}\\ &\le C_s^2\|\nabla G\|_{H^1(\Omega)}\|\nabla\phi\|_{H^1(\Omega)}\\ &\le C_s^2\|(\phi,G)\|^2_{\tilde{W}_\Psi\times\tilde{W}_H}\end{aligned}$$

where C_s is the Sobolev constant corresponding to the continuity of the embedding $H^1(\Omega)\hookrightarrow L^4(\Omega)$. Thus, L defined by $L(\phi,G)=(\Delta\phi-K.\nabla G,\lambda\Delta G-\nabla\Theta.(\nabla\phi)^\perp)$ is the Fréchet derivative of A at the point $(0,0)$, i.e.

$$A'(0,0).(\phi,G)=(\Delta\phi-K.\nabla G,\lambda\Delta G-\nabla\Theta.(\nabla\phi)^\perp).$$

For f and g in $L^2(\Omega)$ let us now consider the system $A'(0,0).(\phi,G)=(f,g)$ that can be written as

$$-\Delta\phi+K.\nabla G=f, \tag{30}$$
$$-\lambda\Delta G+\nabla\Theta.(\nabla\phi)^\perp=g. \tag{31}$$

To prove the existence of a solution (Ψ,H) of (16)-(19) it remains to show that the linear operator $A'(0,0):\tilde{W}_\Psi\times\tilde{W}_H\to L^2(\Omega)\times L^2(\Omega)$ is invertible. To this end, we must first prove that for every given f and g in $L^2(\Omega)$ the system (30)-(31) admits at least a solution and secondly that for $(f,g)=(0,0)$ only $(\phi,G)=(0,0)$ is a solution of (30)-(31).

- First, we want to prove that for every given f and g in $L^2(\Omega)$ the system (30)-(31) admits at least a solution. To this aim, let us define the operator $T=Q\circ S:G\mapsto G_1$ from $H^1(\Omega)$ into $H^1(\Omega)$ with $S:G\mapsto\phi$ where ϕ is the solution of

$$-\Delta\phi+K.\nabla G=f$$

in Ω with the boundary conditions $\phi=0$ on Γ_1 and $\frac{\partial\phi}{\partial n}=0$ on Γ_2, and $Q:\phi\mapsto G_1$ where G_1 is the solution of

$$-\lambda\Delta G_1+\nabla\Theta.(\nabla\phi)^\perp=g$$

in Ω with the boundary conditions $G_1 = 0$ on Γ.

Suppose now that G and G' are given in $H^1(\Omega)$. Let us consider $\phi = S(G)$, $\phi' = S(G')$ and $G_1 = Q(\phi)$, $G_1' = Q(\phi')$. Setting $\bar{G} = G - G'$, $\bar{\phi} = \phi - \phi'$ and $\bar{G}_1 = G_1 - G_1'$, by (30)-(31) we have the inequalities

$$\begin{aligned}
&\int_\Omega \|\nabla\bar{\phi}\|^2 dx \\
&\quad = -\int_\Omega (K.\nabla\bar{G})\phi dx \\
&\quad \leq C\|K\|_\infty \left(\int_\Omega \|\nabla\bar{\phi}\|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \|\nabla\bar{G}\|^2 dx\right)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\lambda \int_\Omega \|\nabla\bar{G}_1\|^2 dx \\
&\quad = -\int_\Omega \nabla\Theta.(\nabla\bar{\phi})^\perp \bar{G}_1 dx \\
&\quad \leq C\|\nabla\Theta\|_\infty \left(\int_\Omega \|\nabla\bar{\phi}\|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega \|\nabla\bar{G}_1\|^2 dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining these two inequalities, we obtain

$$\|\nabla\bar{G}_1\|_{L^2(\Omega)} \leq \frac{C^2\|K\|_\infty\|\nabla\Theta\|_\infty}{\lambda}\|\nabla\bar{G}\|_{L^2(\Omega)}$$

that shows us that if

$$\frac{C^2\|K\|_\infty\|\nabla\Theta\|_\infty}{\lambda} < 1$$

then T is a contraction from $H^1(\Omega)$ into itself and admits a fixed point $G \in H^2(\Omega)$ that gives us a solution $(\phi, G) \in \tilde{W}_\Psi \times \tilde{W}_H$ of (30)-(31).

- The system (30)-(31) with $(f, g) = (0, 0)$ admits $(0, 0)$ for solution, let us show that this solution is unique. Let us suppose that $(\phi, G) \in \tilde{W}_\Psi \times \tilde{W}_H$ is a solution of (30)-(31), multiplying (30) by ϕ, (31) by G and integrating on Ω leads to

$$\|\nabla G\|_2 \leq \frac{C^2\|K\|_\infty\|\nabla\Theta\|_\infty}{\lambda}\|\nabla G\|_2$$

from which we deduce $G = 0$ and $\phi = 0$ if $C^2\|K\|_\infty\|\nabla\Theta\|_\infty < \lambda$.

This shows that, for small values of $\|K.\nabla\Theta\|_2$, the problem $A(\Psi, H) = (K.\nabla\Theta, 0)$ does have solutions. Thus, for such values of Θ and K and $C^2\|K\|_\infty\|\nabla\Theta\|_\infty < \lambda$, the problem (16)-(19) admits at least one solution (ϕ, G) in $\tilde{W}_\Psi \times \tilde{W}_H$ and, as $\tilde{W}_\Psi \times \tilde{W}_H \subset W_\Psi \times W_H$, by Theorem 3.1 it is unique if, in addition, we have $MC\|K\|_\infty < \lambda$. □

Remark 3.4. Since, in the previous Theorem we have

$$\|K.\nabla\Theta\|_2 \leq \|K\|_\infty\|\nabla\Theta\|_\infty(\text{mes }\Omega)^{\frac{1}{2}}$$

and

$$\|\nabla\Theta\|_\infty < \frac{\lambda}{C^2\|K\|_\infty}$$

the condition $\|K.\nabla\Theta\|_2$ small is realized when $\frac{\lambda}{C^2}(\text{mes }\Omega)^{\frac{1}{2}}$ is small. It is the case, for example, when the domain Ω is large and the parameter λ, that is the thermal diffusivity of the porous medium, is small.

Corollary 3.1. *Let* $T_w \in H^{\frac{5}{2}}(\Gamma)$ *and* $M = \sup_\Gamma T_w$. *If* $\max\{C\|\nabla\Theta\|_\infty, M\} < \frac{\lambda}{C\|K\|_\infty}$ *there exists a unique solution* (Ψ, T) *of the problem* (10)-(13) *in the space* $\tilde{W}_\Psi \times H^2(\Omega)$ *for small values of* $\|K.\nabla\Theta\|_2$.

Proof. It follows immediately from Theorem 3.2 and the fact that problems (10)-(13) and (16)-(19) are equivalent. □

4. Conclusion

In this paper, starting from the model of free convection in a fluid saturated porous medium near a semi-infinite vertical flat plate we have written an extension describing this phenomenon in a two-dimensional bounded domain. This new problem is given by two strongly coupled partial differential equations, that allows us to compute the stream function and the temperature of the fluid in the porous medium.

In a first approach of this complex problem, we have proved existence and uniqueness of a solution for small data when a part of the boundary of the domain is assumed to be impermeable.

Acknowledgement

The authors would like to thank Professor Herbert Amann for suggesting the idea for the existence proof and Professor Michel Chipot for his comments and helpful suggestions for proving Theorem 4.

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QUASILINEAR PARABOLIC FUNCTIONAL EVOLUTION EQUATIONS

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Based on our recent work on quasilinear parabolic evolution equations and maximal regularity we prove a general result for quasilinear evolution equations with memory. It is then applied to the study of quasilinear parabolic differential equations in weak settings. We prove that they generate Lipschitz semiflows on natural history spaces. The new feature is that delays can occur in the highest order nonlinear terms. The general theorems are illustrated by a number of model problems.

Keywords: nonlinear evolution equations with memory, time delays, parabolic functional differential equations, Volterra evolution equations.

Categories: 35R10, 35K90, 45K05: 45D05, 34K99

1. Introduction

In a recent paper [8] we have derived very general existence, uniqueness, and continuity theorems for abstract quasilinear evolution equations of the form

$$\dot{u} + A(u)u = F(u). \tag{1}$$

Here $A(u)$ is for each given u in an appropriate class of functions a bounded measurable function with values in a Banach space of bounded linear operators. Thus $\dot{v} + A(u)v = F(u)$ is for each suitable u a nonautonomous evolution equation on some Banach space. The new feature of our result is that the class of admissible functions, that is, the domain of definition of $A(\cdot)$ and $F(\cdot)$, is the same as the one where a solution of (1) is being sought for. More precisely, given Banach spaces $E_1 \overset{d}{\hookrightarrow} E_0$ and $1 < p < \infty$, we assume that A and F are defined on

$$L_p\big((0,T),E_1\big) \cap H_p^1\big((0,T),E_0\big) \tag{2}$$

and map this space into $L_\infty\big((0,T),\mathcal{L}(E_1,E_0)\big)$ and $L_r\big((0,T),E_0\big)$ for some $r > p$, respectively. Consequently, A and F will be nonlocal operators with respect to the time variable, in general. This distinguishes our work in [8] from all previous studies of nonlinear evolution equations where A and F always have been assumed to be local maps (see [7]).

The fact that we work on the function space (2) allows for great flexibility in applications. In particular, we can use the general results to treat evolution equations depending on the history of their solution (see [4], [6], and [9]).

It is the purpose of this paper to give a rigorous basis for such problems. More precisely, we develop a general existence, uniqueness, and continuity theory for functional evolution equations of the form

$$\dot{u} + A(u_t, u)u = F(u_t, u), \tag{3}$$

where, as usual in the theory of functional differential equations, $u_t(\theta) := u(t+\theta)$ for $t \geq 0$ and $-S \leq \theta \leq 0$. (This notation should not be confused with the partial derivative $\partial_t u$.) In particular, we show that in the autonomous case problems of this type generate semiflows on appropriate history spaces. So far only semilinear equations of the general form (3) have been considered where A is independent of u and u_t. For these problems there is a vast literature for which we refer to [29] and the references therein, for example.

The main results for (3) are given in Section 4. In the section following it we prove a rather general theorem for quasilinear parabolic differential equations with memory. The main new feature is that we can allow memory terms in the top order coefficients and that we derive the continuous dependence of the solution on its history. In the autonomous case this implies that (3) generates a semiflow on the history space, a fact which has, up to now, only been shown in semilinear problems.

Problems of this kind occur in several applications, for instance in climate models (see Section 2) or by regularizing ill posed problems in image processing (see [9]). For simplicity, we restrict ourselves to weak settings. However, it will be clear to the reader that the abstract results can also be applied to parabolic differential equations in strong settings (as in e.g., [6] and [9]).

In Section 2 we illustrate the power of our approach by applying the main result of Section 5 to some model problems. We restrict ourselves to simple cases to give the flavor of the techniques and do not strive for optimal results. In particular, we do not present sophisticated global existence results. Section 3 contains an existence and continuity theorem for parameter dependent quasilinear evolution equations. It is an easy consequence of the results in [7], but is put in a form suitable for the study of (3) in Section 4. In the last section we show how the results for the model cases of Section 2 follow from the basic result of Section 5.

2. Model problems

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, where $n \geq 2$. Assume that Γ_1 is a measurable subset of its boundary, Γ, denote by $\chi : \Gamma \to \{0,1\}$ the characteristic function of Γ_1, and put $\Gamma_0 := \Gamma \backslash \Gamma_1$. The pair (Ω, χ) is said to be (C^2) **regular** if Ω is a C^2 domain and χ is continuous. In this case Γ_0 and Γ_1 are both open (and closed) in Γ. In general, either Γ_0 or Γ_1 can be empty, of course. We write $\vec{\nu}$ for the outer unit normal on Γ (defined a.e. with respect to the $(n-1)$-dimensional Hausdorff measure).

In this section we consider the following evolution system

$$\left.\begin{aligned} \partial_t\big(\boldsymbol{e}(u)\big) + \nabla\cdot\vec{\jmath}(u) &= \boldsymbol{f}(u) && \text{on } \Omega\times(0,\infty), \\ \chi\vec{\nu}\cdot\vec{\jmath}(u) + (1-\chi)\gamma u &= \chi\boldsymbol{g}(u) && \text{on } \Sigma\times(0,\infty), \end{aligned}\right\} \tag{4}$$

γ being the trace operator and $\nabla\cdot$ denoting divergence. We are particularly interested in situations where (4) is history dependent. More precisely, we consider constitutive hypotheses of the following form

$$\left.\begin{array}{l} \bullet \quad \boldsymbol{e}(u) := \mu * u; \\ \bullet \quad \vec{\jmath}(u) := -\nu_0 * \big(a(\cdot,\sigma_0 * u)\nabla u\big) + \nu_1 * \big(b(\cdot,\sigma_1 * u)\nabla u\big); \\ \bullet \quad \boldsymbol{f}(u) := \rho_0 * f(\cdot,\sigma_2 * u); \\ \bullet \quad \boldsymbol{g}(u) := \rho_1 * g(\cdot,\sigma_3 * u), \end{array}\right\} \tag{5}$$

where μ, ν_j, ρ_j, and σ_j are bounded (possibly Banach space valued) Radon measures on $\mathbb{R}$, to be specified more precisely below. Throughout we suppose that

$$\left.\begin{array}{l} \bullet \quad a \in C^{0,1\text{-}}(\overline{\Omega}\times\mathbb{R}^m, \mathbb{R}^{n\times n}) \text{ such that} \\ \quad\;\; a(x,\xi) \text{ is symmetric and positive definit,} \\ \quad\;\; \text{uniformly for } x\in\overline{\Omega} \text{ and } \xi \text{ in bounded intervals;} \\ \bullet \quad b \in C^{0,1\text{-}}(\overline{\Omega}\times\mathbb{R}^m, \mathbb{R}^{n\times n}); \\ \bullet \quad f \in C^{0,1\text{-}}(\overline{\Omega}\times\mathbb{R}^m); \\ \bullet \quad g \in C^{0,1\text{-}}(\Gamma\times\mathbb{R}^m) \end{array}\right\} \tag{6}$$

for some $m \in \mathbb{N}$. (For convenience, we put $[0,\infty] := [0,\infty) = \mathbb{R}^+$ and $[-\infty,0] := (-\infty,0]$.) Here and below, given metric spaces X and Y, an open subset O of $X \times Y$, and a Banach space F, we write $C^{0,1\text{-}}(O,F)$ for the set of all $f \in C(O,F)$ such that for each point (x,y) in O there exists a neighborhood $U \times V$ in O such that $f(\cdot,y) : U \to F$ is Lipschitz continuous, uniformly with respect to $y \in V$. As usual, we omit the symbol F if $F = \mathbb{R}$.

In this section we also suppose that

$$\left.\begin{array}{l} \bullet\quad \text{either } p = 2, \\ \bullet\quad \text{or } n+2 < p < \infty \text{ and } (\Omega,\chi) \text{ is regular.} \end{array}\right\} \tag{7}$$

We set

$$H^1_{p,\chi} := \{\, v \in H^1_p := H^1_p(\Omega)\ ;\ (1-\chi)\gamma v = 0 \,\}, \quad H^{-1}_{p,\chi} := (H^1_{p',\chi})',$$

the dual being determined by means of the L_p duality pairing

$$\langle v, w\rangle := \int_\Omega v \cdot w \, dx, \qquad (v,w) \in L_{p'} \times L_p.$$

Note that

$$H^1_{p,\chi} \overset{d}{\hookrightarrow} L_p \overset{d}{\hookrightarrow} H^{-1}_{p,\chi},$$

where $\overset{d}{\hookrightarrow}$ denotes continuous and dense embedding. Also note that $H^{-1}_{p,\chi} = H^{-1}_p$ if $\chi = 0$, that is, $\Gamma = \Gamma_0$. In this case the second line of (4) reduces to the homogeneous Dirichlet boundary condition $\gamma u = 0$. We also put $H^j_\chi := H^j_{2,\chi}$ for $j = \pm 1$.

Furthermore,

$$W^{1-2/p}_{p,\chi} := \begin{cases} L_2, & \text{if } p = 2, \\ \{\, v \in W^{1-2/p}_p\ ;\ (1-\chi)\gamma v = 0 \,\} & \text{otherwise,} \end{cases}$$

where $W^s_p := W^s_p(\Omega)$ are the usual Sobolev-Slobodeckii spaces of order $s \in [0,1]$. Recall that, except for equivalent norms, $W^s_p = H^s_p$ for $s = 0, 1$.

Let I be an interval with nonempty interior $\mathring{I}$. Then

$$\mathcal{H}^1_{p,\chi}(I) := \mathcal{H}^{1,1}_{p,p,\chi}(\Omega \times I) := L_p(I, H^1_{p,\chi}) \cap H^1_p(\mathring{I}, H^{-1}_{p,\chi})$$

and $\mathcal{H}^1_\chi := \mathcal{H}^1_{2,\chi}$. It will be shown below that

$$\mathcal{H}^1_{p,\chi}(I) \hookrightarrow C_0(\bar{I}, W^{1-2/p}_{p,\chi}), \tag{8}$$

where C_0 denotes the space of continuous functions vanishing at infinity.

For $0 < T \le \infty$ we put $J_T := [0,T)$ and $J_{-T} := (-T,0]$. Furthermore, we usually employ the same symbol for a function and its restriction to any of its subdomains, if no confusion seems likely.

Suppose that $0 < S \le \infty$,

$$v \in C_0\big([-S,0], W_{p,X}^{1-2/p}\big), \tag{9}$$

and $0 < T \le \infty$. By a **solution** (more precisely: an $\mathcal{H}_p^1$ **solution**) of (4) on $(0,T)$ **with history** v we mean a

$$u \in C\big([-S,T), W_{p,X}^{1-2/p}\big)$$

satisfying $u \,|\, J_{-S} = v$ and

$$u \in \mathcal{H}_{p,X}^1(J_\tau), \qquad 0 < \tau < T, \tag{10}$$

as well as, given any $w \in H_{p',X}^1$,

$$\partial_t \langle w, \boldsymbol{e}(u)\rangle + \langle \nabla w, \vec{\jmath}(u)\rangle = \langle w, \boldsymbol{f}(u)\rangle + \langle \gamma w, \boldsymbol{g}(u)\rangle_\Gamma \qquad \text{on } (0,T) \tag{11}$$

in the sense of distributions, where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the $L_p(\Gamma)$ duality pairing (with respect to the Hausdorff volume measure of Γ). In addition, all integrals occurring in (11) have to be well defined. Note that (10) and (11) imply that u is a weak solution in the usual sense if $p = 2$.

A solution u is **maximal** if there does not exist a solution being a proper extension of u. In this case J_T is the maximal existence interval for u.

Before considering some model problems we recall the concept of a semiflow. Let X be a metric space and suppose that $J(x)$ is for each $x \in X$ an open subinterval of $\mathbb{R}^+$ containing 0. Set

$$\mathcal{X} := \bigcup_{x \in X} J(x) \times \{x\}.$$

Then $\varphi : \mathcal{X} \to X$ is said to be a (local) semiflow on X if $\mathcal{X}$ is open in $\mathbb{R}^+ \times X$, $\varphi \in C(\mathcal{X}, X)$, $\varphi(0,x) = x$ for $x \in X$, and, given $(t,x) \in \mathcal{X}$ and $s \in J\big(\varphi(t,x)\big)$, it follows that $s + t \in J(x)$ and $\varphi\big(s, \varphi(t,x)\big) = \varphi(s+t, x)$. It is global if $\mathcal{X} = \mathbb{R}^+ \times X$. Furthermore, φ is a Lipschitz semiflow if, in addition, $\varphi \in C^{0,1\text{-}}(\mathcal{X}, X)$.

Given $T \in (0,\infty]$, a Banach space F, and $u \in C\big([-S,T], F\big)$, we recall that

$$u_t(\theta) := u(t-\theta), \qquad 0 \le \theta \le S, \quad 0 \le t \le T.$$

Note that $u_t \in C\big([-S,0], F\big)$ for $0 \le t \le T$.

Suppose that $\mathcal{V}$ is a Banach space such that $\mathcal{V} \hookrightarrow C\big([-S,0], W_{p,X}^{1-2/p}\big)$. Then we say that (4) is well posed in $\mathcal{H}_p^1$ and generates a semiflow on the history space $\mathcal{V}$ if there exists for each $v \in \mathcal{V}$ a unique maximal $\mathcal{H}_p^1$ solution, $u(v)$, of (4) and the map $(t,v) \mapsto u(v)_t$ is a semiflow on $\mathcal{V}$.

We start with a simple model problem of reaction-diffusion type:

$$\left.\begin{aligned} \partial_t u - \nabla\cdot\vec{\jmath}(u) &= f(u) \quad \text{on } \Omega,\\ u &= 0 \qquad\ \text{on } \Gamma_0,\\ \vec{\nu}\cdot\vec{\jmath}(u) &= g(u) \quad \text{on } \Gamma_1, \end{aligned}\right\} \tag{12}$$

where

$$\vec{\jmath}(u) := -a(\sigma_0 * u)\nabla u,$$

that is, we set $\nu_0 := \delta_0$, where δ_r is the Dirac measure supported in $r \in \mathbb{R}$, and $b := 0$. For notational simplicity, we usually do no longer indicate the x dependence of the nonlinearities.

First we suppose that $m = 1$ and the diffusion matrix depends on suitable space averages of u only. For this we assume that

$$K \in \mathcal{L}\big(L_2, C(\overline{\Omega})\big), \tag{13}$$

where $\mathcal{L}(E,F)$ is the Banach space of all continuous linear maps from the Banach space E into the Banach space F. We also set $\mathcal{L}(E) := \mathcal{L}(E,E)$.

We denote by $\mathcal{M}[0,S]$ the space of all real valued Radon measures of bounded variation on the interval $[0,S]$. We suppose that

$$\alpha \in \mathcal{M}[0,S] \tag{14}$$

and consider the nonlocal time-delayed quasilinear parabolic problem

$$\left.\begin{aligned} \partial_t u - \nabla\cdot\big(a(\alpha * Ku)\nabla u\big) &= f(\alpha * Ku) \quad \text{on } \Omega,\\ u &= 0 \qquad\qquad\ \text{on } \Gamma_0,\\ \vec{\nu}\cdot a(\alpha * Ku)\nabla u &= g(\alpha * Ku) \quad \text{on } \Gamma_1. \end{aligned}\right\} \tag{15}$$

Here and below it is understood that the boundary conditions are taken in the sense of traces. In particular, the right hand side of the third equation of (15) reads more precisely as $g(\gamma\alpha * Ku)$. Observe that

$$(\alpha * Ku)(x,t) = \int_{[0,S]} (Ku)(x,t-\tau)\alpha(d\tau) \in C\big(\overline{\Omega}\times[0,T]\big)$$

for $(x,t) \in \overline{\Omega}\times J_T$ and $T > 0$, provided $u \in C_0\big([-S,T],L_2\big)$.

Theorem 2.1. *Let* (13) *and* (14) *be satisfied. Then* (15) *is well posed in* $\mathcal{H}^1_\chi$ *and generates a Lipschitz semiflow on the history space* $C_0\big([-S,0],L_2\big)$. *It depends Lipschitz continuously on* α *and* K. *If the support of* α *is contained in* $[s,S]$ *for some* $s \in (0,S]$, *then this semiflow is global.*

The proof as well as the proofs of all the following theorems of this section are found in Section 6, where it will be made precise how (15) is a particular instant of (4). What is meant by a semiflow depending Lipschitz continuously on parameters is defined in Section 4.

Corollary 2.1. *If $r > 0$, then the nonlocal retarded problem*

$$\begin{aligned} \partial_t u - \nabla \cdot \big(a(Ku(t-r))\nabla u\big) &= f\big(Ku(t-r)\big) \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \Gamma_0, \\ \vec{\nu} \cdot a\big(Ku(t-r)\big)\nabla u &= g\big(Ku(t-r)\big) \quad \text{on } \Gamma_1 \end{aligned}$$

is well posed in $\mathcal{H}^1_\chi$ and generates a global Lipschitz semiflow on the history space $C\big([-r,0],L_2\big)$. It depends Lipschitz continuously on K.

Proof. It suffices to choose $S := r$ and $\alpha := \delta_r$. □

To treat local reaction terms in a weak setting we replace the hypotheses on f and g in (6) by assuming, for simplicity, that $n \geq 3$, that $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Carathéodory function satisfying $f(\cdot,0) \in L_{2n/(n+2)}$ and

$$|f(\cdot,\xi) - f(\cdot,\eta)| \leq c\big(1 + |\xi|^{2/n} + |\eta|^{2/n}\big)\,|\xi - \eta| \tag{16}$$

for $\xi, \eta \in \mathbb{R}^m$, and that $g_0 \in L_2(\Gamma)$. Observe that (16) is satisfied for the model nonlinearity

$$f(\cdot,\xi) = b\,|\xi|^{2/n}\,\xi + f_0 \tag{17}$$

with $m = 1$, $b \in L_\infty$, and $f_0 \in L_2$. Then we consider quasilinear parabolic problems with nonlocal time-delays in the diffusion matrix only, the reaction term being local, that is,

$$\left.\begin{aligned} \partial_t u - \nabla \cdot \big(a(\alpha * Ku)\nabla u\big) &= f(u) \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \Gamma_0, \\ \vec{\nu} \cdot a(\alpha * Ku)\nabla u &= g_0 \quad \text{on } \Gamma_1. \end{aligned}\right\} \tag{18}$$

Theorem 2.2. *Suppose that assumptions* (13), (14), *and* (16) *hold. Then* (18) *is well posed in $\mathcal{H}^1_\chi$ and generates a semiflow on the history space $C_0\big([-S,0],L_2\big)$ which depends Lipschitz continuously on α and K. It is global if the following additional conditions are satisfied:*

(i) $\operatorname{supp}(\alpha) \subset [s,S]$ *for some* $s \in (0,S]$;
(ii) *there exists a constant κ such that*

$$\big(f(\cdot,\xi) - f(\cdot,0)\big)\xi \leq \kappa(1 + |\xi|^2), \quad \xi \in \mathbb{R}.$$

Corollary 2.2. *If $r > 0$, then the model problem*

$$\begin{aligned} \partial_t u - \nabla \cdot \big(a(Ku(t-r))\nabla u\big) &= b\,|u|^{2/n}\,u + f_0 && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a\big(Ku(t-r)\big)\nabla u &= g_0 && \text{on } \Gamma_1 \end{aligned}$$

is well posed in $\mathcal{H}^1_\chi$ and generates a semiflow on $C_0\big([-r,0],L_2\big)$, depending Lipschitz continuously on K. It is global if $b \le 0$.

Proof. With (17) this follows by choosing $S := r$ and $\alpha := \delta_r$. □

There are many conceivable choices for K. For example, we could set

$$Ku := \langle k, u\rangle, \qquad u \in L_2,$$

for some fixed $k \in L_2$, so that Ku is constant on $\overline{\Omega}$. Nonlocal (non delayed) quasilinear parabolic initial boundary value problems, predominantly with this choice for K, have recently attracted some interest, in particular by M. Chipot and coworkers (cf. [6], [10]–[12], and the references therein).

Another important case is obtained by setting

$$Ku := k \star \widetilde{u}, \qquad u \in L_2,$$

where $k \in L_2(\mathbb{R}^n)$, $\widetilde{u}$ is the extension of u to $\mathbb{R}^n$ by zero in Ω^c, and $\star$ denotes convolution on $\mathbb{R}^n$. In particular, setting $k := \chi_{r\mathbb{B}^n}$, the characteristic function of the ball in $\mathbb{R}^n$ with center at 0 and radius r, it follows that

$$Ku(x) = \int_{\Omega(x,r)} u(y)\,dy, \qquad x \in \Omega,$$

where $\Omega(x,r) := (x + r\mathbb{B}^n) \cap \Omega$. Thus in this case the diffusion matrix (and f and g in the case of Theorem 2.1) depends on a suitably delayed space average of the solution over a neighborhood of x in Ω.

Next we consider model problems where the diffusion matrix depends on u in a local way with respect to the x variable, but not necessarily with respect to t. For this we suppose that $m = 2$ and consider the model problem

$$\left.\begin{aligned} \partial_t u - \nabla \cdot \big(a(u,\alpha * u)\nabla u\big) &= f(u,\alpha * u) && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a(u,\alpha * u)\nabla u &= g(u,\alpha * u) && \text{on } \Gamma_1. \end{aligned}\right\} \tag{19}$$

Then the following analogue to Theorem 2.1 is valid.

Theorem 2.3. *Suppose that (Ω, χ) is regular, $n+2 < p < \infty$, and (14) is satisfied. Then (19) is well posed in $\mathcal{H}^1_{p,\chi}$ and generates a Lipschitz semiflow on the history space $C_0\big([-S,0], W^{1-2/p}_{p,\chi}\big)$ depending Lipschitz continuously on α. If $\operatorname{supp}(\alpha) \subset [s,S]$ for some $s \in (0,S]$ and $(a,f) = (a,f)(\alpha * u)$, then this semiflow is global.*

Remarks 2.1. (a) For simplicity, we have omitted convection terms of the form $\vec{c}(u) \cdot \nabla u$ and $\nabla \cdot \big(\vec{c}(u)u\big)$, where $\vec{c}(u)$ is a suitable nonlocal function of u. It will be clear from the general abstract results how this can be done.

(b) From Theorem 4.2 it will also be clear that we can obtain well posedness results for nonautonomous equations. Of course, in such a case the semiflow property is no longer valid.

(c) We can replace $\overline{\Omega}$ by a smooth submanifold of some Riemannian manifold, provided gradients, divergence, and normals are taken with respect to the corresponding Riemannian metric. □

Problems of the form (19) occur in applications, for example in certain climate models. For instance, in [20] and [21] G. Hetzer studies the quasilinear functional differential equation

$$c(\beta * u)\partial_t u - \nabla \cdot (k \nabla u) = R(t, u, \beta * u) \tag{20}$$

on the Euclidean unit sphere in $\mathbb{R}^3$, assuming that c is a bounded C^2 function being uniformly positive, $\beta \in C^2[0,T]$ for some $T > 0$, and k and R are sufficiently smooth functions with k being uniformly positive. By dividing (20) by $c(\beta * u)$ it is clear, due to Remarks 2.1, that this model fits into the framework of this paper.

In the theory of heat conduction in a rigid body the functions occurring in (4) have the following interpretation: u is the temperature, $\boldsymbol{e}(u)$ the interval energy density, $\vec{\jmath}(u)$ the heat flux, $\boldsymbol{f}(u)$ and $\boldsymbol{g}(u)$, respectively, the density of external heat sources in Ω and on Γ, respectively. Considering bodies with memory one arrives at the following constitutive hypotheses:

$$\boldsymbol{e}(u) = u + h * u$$

and

$$\vec{\jmath}(u) = -a(u, \alpha * u)\nabla u - k * \big(b(u, \alpha * u)\nabla u\big),$$

where we suppose that

$$h \in L_r\big(\mathbb{R}^+, C^1(\overline{\Omega})\big), \quad k \in L_r(\mathbb{R}^+, L_\infty), \quad \alpha \in \mathcal{M}\big(\mathbb{R}^+, C(\overline{\Omega})\big) \tag{21}$$

for some $r \in (1, p']$. Thus one is led to consider the problem

$$\partial_t(u + h * u) - \nabla \cdot \big(a(u, \alpha * u)\nabla u + k * \big(a(u, \alpha * u)\nabla u\big)\big) = f(u, \alpha * u) \quad (22)$$

in Ω, subject to the boundary conditions

$$\left.\begin{aligned} u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot a(u, \alpha * u)\nabla u + k * \big(a(u, \alpha * u)\nabla u\big) &= g(u, \alpha * u) && \text{on } \Gamma_1. \end{aligned}\right\} \quad (23)$$

Observe that, for example,

$$(h * u)(x, t) = \int_0^\infty h(x, \tau)u(x, t - \tau)\, d\tau, \qquad (x, t) \in \Omega \times \mathbb{R}^+,$$

where $u(t) = u(\cdot, t)$ etc.

Theorem 2.4. *Suppose that (Ω, χ) is regular, $n + 2 < p < \infty$, and (21) is satisfied. Then (22), (23) is well posed in $\mathcal{H}_p^1$ and generates a Lipschitz semiflow on the history space $\mathcal{H}_{p,\chi}^1(J_{-S})$, where $S \in (0, \infty]$ is such that*

$$\big(\operatorname{supp}(h) \cup \operatorname{supp}(k)\big) + \operatorname{supp}(\alpha) \subset [0, S].$$

If $h = 0$, then this is true for the history space $C_0\big([-S, 0], W_{p,\chi}^{1-2/p}\big)$.

Setting $h = 0$ and $\alpha = 0$ and assuming that k is real valued we obtain as a particular case the quasilinear Volterra integro differential equation

$$\partial_t u - \nabla \cdot \big(a(u)\nabla u\big) - \int_0^\infty h(\tau)\nabla \cdot \big(a(u(t - \tau))\nabla u(t - \tau)\big)\, d\tau = f(u).$$

Equations of this type, usually with zero Dirichlet boundary conditions, have been studied by many authors, even for more general fully nonlinear equations, by means of maximal regularity results in Hölder and Besov space settings (see [15], [26], and the references therein). Another approach to such equations is based on sophisticated results from the theory of abstract linear Volterra equations (see [27]). Using these techniques it is also possible to obtain existence results in the difficult singular case where the local second order operator $\nabla \cdot \big(a(u)\nabla u\big)$ is not present (cf. [17]–[19], [25], [27], [30], for example, and the references in those papers).

The only results known to the author for problems containing the term $\partial_t(h * a)$ concern linear and semilinear equations (e.g., [13], [14], [22], and [29]).

Another model case is the retarded quasilinear parabolic problem

$$\left.\begin{aligned} \partial_t e(u)(t) - \nabla \cdot \vec{\jmath}(u)(t) &= f\big(u(t), u(t - r)\big) && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \vec{\nu} \cdot \vec{\jmath}(u)(t) &= g\big(u(t), u(t - r)\big) && \text{on } \Gamma_1, \end{aligned}\right\} \quad (24)$$

where $r > 0$,

$$e(u)(t) := u(t) + hu(t-r) \tag{25}$$

and

$$\vec{\jmath}(u)(t) := -a\big(u(t), u(t-r)\big)\nabla u(t) - kb\big(u(t), u(t-r)\big)\nabla u(t-r) \tag{26}$$

with

$$h, k \in C^1(\overline{\Omega}). \tag{27}$$

Theorem 2.5. *Suppose that* (Ω, χ) *is regular and* $n+2 < p < \infty$. *Also suppose that* (25)–(27) *are satisfied for some* $r > 0$. *Then* (24) *is well posed in* $\mathcal{H}_p^1$ *and generates a Lipschitz semiflow on the history space* $\mathcal{H}_{p,\chi}^1(J_{-r})$. *If* $h = 0$, *then this is true for the history space* $C\big([-r, 0], W_{p,\chi}^{1-2/p}\big)$. *These semiflows are global, if* a, b, f, *and* g *depend on* $u(t-r)$ *only and not on* $u(t)$.

A very particular instant of problem (24) is the retarded semilinear parabolic equation

$$\partial_t\big(u(t) + \alpha u(t-r)\big) - a\Delta u(t) - b\Delta u(t-r) = f\big(u(t-r)\big) \tag{28}$$

in $\Omega \times (0, \infty)$, where α, a, and b are constants with $a > 0$, subject to the boundary conditions

$$\left.\begin{aligned} u &= 0 && \text{on } \Gamma_0, \\ a\partial_\nu u(t) + b\partial_\nu u(t-r) &= g\big(u(t-r)\big) && \text{on } \Gamma_1. \end{aligned}\right\} \tag{29}$$

As usual, ∂_ν is the normal derivative on Γ. It follows from Theorem 2.5 that, given any history $v \in \mathcal{H}_{p,\chi}^1(J_{-r})$, problem (28), (29) possesses a unique global $\mathcal{H}_p^1$ solution $u(v)$, and the map

$$\mathbb{R}^+ \times \mathcal{H}_{p,\chi}^1(J_{-r}) \hookrightarrow \mathcal{H}_{p,\chi}^1(J_{-r}), \quad (t, v) \mapsto u(v)_t$$

is a well defined Lipschitz semiflow on $\mathcal{H}_{p,\chi}^1(J_{-r})$, provided (Ω, χ) is regular and $p > n+2$. Furthermore, if $\alpha = 0$, then this remains true if we replace $\mathcal{H}_{p,\chi}^1(J_{-r})$ by the history space $C\big([-r, 0], W_{p,\chi}^{1-2/p}\big)$.

3. Parameter dependent evolution equations

Let E_0 and E_1 be Banach spaces such that $E_1 \overset{d}{\hookrightarrow} E_0$. We fix $p \in (1, \infty)$ and, given a subinterval I of $\mathbb{R}$ with nonempty interior, we put

$$\mathcal{H}_p^1\big(I, (E_1, E_0)\big) := L_p(I, E_1) \cap H_p^1(\mathring{I}, E_1).$$

It follows that

$$\mathcal{H}_p^1(I,(E_1,E_0)) \hookrightarrow C_0(\bar{I},E), \tag{30}$$

where

$$E := (E_0, E_1)_{1/p',p}$$

with $(\cdot,\cdot)_{\theta,p}$ being the real interpolation functor of exponent $\theta \in (0,1)$ and parameter p (cf. Theorem III.4.10.2 of [2]).

Suppose that $a := \inf I > -\infty$ with $a \in I$ and

$$A \in L_\infty\big(I, \mathcal{L}(E_1, E_0)\big).$$

Then A is said to have (the property of) **maximal L_p regularity** (on I with respect to (E_1, E_0)) if the Cauchy problem

$$\dot{u} + Au = f \text{ on } \mathring{I}, \quad u(a) = 0$$

has for each $f \in L_p(I, E_0)$ a unique solution $\mathcal{H}_p^1\big(I,(E_1,E_0)\big)$ and if, in addition, given $\tau, T \in I$ with $\tau < T$, the homogeneous problem

$$\dot{v} + Av = 0 \text{ on } (\tau, T), \quad v(\tau) = 0 \tag{31}$$

possesses in $\mathcal{H}_p^1\big([\tau,T),(E_1,E_0)\big)$ the trivial solution only. The proof of Lemma 4.1 in [5] shows that assumption (31) is equivalent to: A has maximal L_p regularity on every nontrivial bounded subinterval of I being closed on the left. (In Lemma 4.1 of [5] hypothesis (31) is missing.)

We fix a positive number T and set $\mathsf{J} := J_{\mathsf{T}}$. Then we denote by

$$\mathcal{MR}_p(\mathsf{J}) := \mathcal{MR}_p\big(\mathsf{J},(E_1,E_0)\big)$$

the set of all $A \in L_\infty\big(\mathsf{J},\mathcal{L}(E_1,E_0)\big)$ possessing maximal L_p regularity, endowed with the topology induced by $L_\infty\big(\mathsf{J},\mathcal{L}(E_1,E_0)\big)$. We write $\mathcal{MR}_p(E_1,E_0)$ for the subset of $\mathcal{MR}_p(\mathsf{J})$ consisting of all constant maps $t \mapsto A$ therein and assume that

$$\mathcal{MR}_p(E_1,E_0) \neq \emptyset. \tag{32}$$

Let X and Y be nonempty sets and J a subinterval of $\mathbb{R}^+$ containing 0. A function $f : X^J \to Y^J$ is a **Volterra map** if, for each $T \in \mathring{J}$ and each pair $u, v \in X^J$ with $u \,|\, J_T = v \,|\, J_T$, it follows that $f(u) \,|\, J_T = f(v) \,|\, J_T$. Let X and Y be metric spaces. Then $C^{1\text{-}}(X,Y)$ is the space of all maps from X into Y which are bounded on bounded sets and uniformly Lipschitz continuous on such sets. If Y is a Banach space, then $C^{1\text{-}}(X,Y)$ is endowed with the Fréchet topology of uniform convergence on bounded sets of the functions and their first order difference quotients on such sets. Note that

$\mathcal{C}^{1-}(X,Y)$ equals $C^{1-}(X,Y)$, the space of all (locally) Lipschitz continuous maps from X into Y, provided X is finite dimensional.

For abbreviation we put

$$\mathcal{H}_p^1(J_T) := \mathcal{H}_p^1(J_T, (E_1, E_0)), \quad \mathcal{H} := \mathcal{H}_p^1(\mathsf{J})$$

for $0 < T \leq \mathsf{T}$ and assume that

$$\bullet \quad \Xi \text{ is a Banach space and } \alpha \in \mathcal{L}(\Xi, E). \tag{33}$$

We denote by $\gamma_0 \in \mathcal{L}(\mathcal{H}, E)$ the trace map for $t = 0$, that is, $\gamma_0(u) = u(0)$ for $u \in \mathcal{H}$, and set

$$D := \{\, (\xi, u) \in \Xi \times \mathcal{H} \ ; \ \alpha(\xi) = \gamma_0(u) \,\}. \tag{34}$$

Note that D is the kernel of

$$\big((\xi, u) \mapsto \alpha(\xi) - \gamma_0(u)\big) \in \mathcal{L}(\Xi \times \mathcal{H}, E).$$

Thus it is a closed linear subspace of $\Xi \times \mathcal{H}$, hence a Banach space.

For $\xi \in \Xi$ we put

$$\mathcal{H}_{\alpha(\xi)} := \{\, u \in \mathcal{H} \ ; \ \gamma_0(u) = \alpha(\xi) \,\}$$

and assume that

$$\left.\begin{array}{l} \bullet \quad A \in \mathcal{C}^{1-}\big(D, \mathcal{MR}_p(\mathsf{J}, (E_1, E_0))\big); \\ \bullet \quad F \in \mathcal{C}^{1-}\big(D, L_p(\mathsf{J}, E_0)\big) \text{ for some } r \in (p, \infty]; \\ \bullet \quad (A, F)(\xi, \cdot) \text{ is for each } \xi \in \Xi \text{ a Volterra map on } \mathcal{H}_{\alpha(\xi)}. \end{array}\right\} \tag{35}$$

We consider the parameter dependent quasilinear evolution problem

$$\dot{u} + A(\xi, u)u = F(\xi, u) \text{ on } (0, \mathsf{T}), \quad u(0) = \alpha(\xi) \tag{36}$$

for $\xi \in \Xi$.

Theorem 3.1. *Let assumptions* (33) *and* (35) *be satisfied and suppose that* $\xi \in \Xi$. *Then:*

(i) (Existence and Uniqueness) *There exist a maximal open subinterval* $J^* := J(\xi)$ *of* J *containing* 0 *and a unique* $u^* := u(\xi) : J^* \to E_0$ *such that* $u^* \,|\, J_T$ *belongs to* $\mathcal{H}_p^1(J_T)$ *and satisfies*

$$\dot{u}^* + A(\xi, u^*)u^* = F(\xi, u^*) \ \text{on} \ (0, T), \quad u^*(0) = \alpha(\xi)$$

for $0 < T < T^* := \sup J^*$.

(ii) (Global existence) *If* $J^* \neq \mathsf{J}$, *then* $u^* \notin \mathcal{H}_p^1(J^*)$.

(iii) (Continuous dependence on ξ) *If $u^* \in \mathcal{H}$, then put $T_0 := \mathsf{T}$. Otherwise, fix any positive $T_0 < T^*$. Then there exist $r, \kappa > 0$ such that, given any $\xi_j \in \Xi$ satisfying*

$$\|\xi_j - \xi\|_\Xi < r, \qquad j = 1, 2,$$

it follows that $u(\xi_j) \in \mathcal{H}_p^1(J_{T_0})$ and

$$\|u(\xi_1) - u(\xi_2)\|_{\mathcal{H}_p^1(J_{T_0})} \leq \kappa \, \|\xi_1 - \xi_2\|_\Xi.$$

(iv) (Continuous dependence on A and F) *Let T_0 be defined as above and let $\big((A_j, F_j)\big)$ be a sequence such that (A_j, F_j) satisfies (35) for each $j \in \mathbb{N}$ and $(A_j, F_j) \to (A, F)$ in*

$$C^{1\text{-}}\big(D, L_\infty\big(\mathsf{J}, \mathcal{L}(E_1, E_0)\big) \times L_p(\mathsf{J}, E_0)\big).$$

Denote by $u_j(\xi)$ the solution of (36) with (A, F) replaced by (A_j, F_j). Then $u_j(\xi) \to u(\xi)$ in $\mathcal{H}_p^1(J_{T_0})$.

Proof. (i) and (ii) follow from Theorem 2.1 and Remark 4.3 in [8]. Assertions (iii) and (iv) are easily deduced from the proof of Theorem 3.1 therein by modifying appropriately the situation considered in Remark 4.3 of [8]. (In [8] assumption (31) has to be added to the definition of maximal L_p regularity since Lemma 4.1 of [5] is used in the proofs.) □

Remark 3.1. Let Π be a Banach space and suppose that

- $A \in C^{1\text{-}}\big(\Pi \times \mathcal{H}, \mathcal{MR}_p(\mathsf{J}, (E_1, E_0))\big)$;
- $F \in C^{1\text{-}}\big(\Pi \times \mathcal{H}, L_r(\mathsf{J}, E_0)\big)$ for some $r \in (p, \infty]$;
- $(A, F)(\pi, \cdot)$ is for each $\pi \in \Pi$ a Volterra map.

Then the quasilinear parameter dependent initial value problem

$$\dot{u} + A(\pi, u)u = F(\pi, u) \quad \text{on } (0, T), \qquad u_0 = e$$

has for each $e \in E$ a unique maximal solution in the sense specified in (i) of Theorem 3.1. Furthermore, assertions (ii)–(iv) are also valid.

Proof. This follows from the preceding theorem by setting $\Xi := \Pi \times E$ and $\alpha(\xi) := e$ for $\xi = (\pi, e) \in \Xi$. □

4. Functional evolution equations

Now we fix $S \in (0, \infty]$ and suppose that

$$\mathcal{V} \in \{\mathcal{H}_p^1\big((-S,0),(E_1,E_0)\big), C_0\big([-S,0],E\big)\}. \tag{37}$$

We put

$$\mathcal{D} := \{\,(v,w) \in \mathcal{V} \times \mathcal{H}\ ;\ v(0) = w(0)\,\},$$

fix a (parameter) Banach space Π, and suppose that

$$\left.\begin{array}{l} \bullet\quad A \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{D}, \mathcal{MR}_p(\mathsf{J},(E_1,E_0))\big);\\ \bullet\quad F \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{D}, L_r(\mathsf{J},E_0)\big) \text{ for some } r \in (p,\infty];\\ \bullet\quad (A,F)(\pi,v,\cdot) \text{ is for each } (\pi,v) \in \Pi \times \mathcal{V}\\ \quad\ \text{a Volterra map on } \mathcal{H}_{v(0)}. \end{array}\right\} \tag{38}$$

Then, given $\pi \in \Pi$ and $v \in \mathcal{V}$, we consider the following parameter dependent quasilinear functional differential equation

$$\dot u + A(\pi,u_t,u)u = F(\pi,u_t,u) \quad \text{on } (0,\mathsf{T}), \qquad u_0 = v. \tag{39}$$

By an $\mathcal{H}_p^1$ **solution** u of (39) on J_T, where $0 < T \le \mathsf{T}$, we mean a $u: [-S,T) \to E_0$ satisfying $u\,|\,[-S,0) = v$ and $u\,|\,J_\tau \in \mathcal{H}_p^1(J_\tau)$ as well as

$$\dot u(t) + A(\pi,u_t,u)(t)u(t) = F(\pi,u_t,u)(t), \qquad 0 < t < \tau,$$

for $0 < \tau < T$. It is maximal if there does not exist an $\mathcal{H}_p^1$ solution being a proper extension of u. In this case J_T is called maximal existence interval for u.

The following general existence, uniqueness, and continuity theorem is the first main result of this paper.

Theorem 4.1. *Let assumptions* (37) *and* (38) *be satisfied. Then:*

(i) (Existence and Uniqueness) *Problem* (39) *has for each* $(\pi,v) \in \Pi \times \mathcal{V}$ *a unique maximal* $\mathcal{H}_p^1$ *solution* $u(\pi,v)$.

(ii) (Global existence) *Denote by* $J(\pi,v)$ *the maximal existence interval of* $u(\pi,v)$. *If* $J(\pi,v) \ne \mathsf{J}$, *then* $u(\pi,v) \notin \mathcal{H}_p^1\big(J(\pi,v)\big)$.

(iii) (Continuous dependence on π and v) *If* $u(\pi,v) \in \mathcal{H}_p^1(\mathsf{J})$, *then set* $T_0 := \mathsf{T}$. *Otherwise, fix any* $T_0 \in \mathring{J}(\pi,v)$. *Then there exist* $r, \kappa > 0$ *such that, given* $(\pi_j,v_j) \in \Pi \times \mathcal{V}$ *satisfying*

$$\|\pi_j - \pi\|_\Pi + \|v_j - v\|_V < r, \qquad j = 1,2,$$

it follows that $u(\pi_{,}v_j) \in \mathcal{H}_p^1(J_{T_0})$ *and*

$$\|u(\pi_1,v_1) - u(\pi_2,v_2)\|_{\mathcal{H}(J_{T_0})} \le \kappa\big(\|\pi_1 - \pi_2\|_\Pi + \|v_1 - v_2\|_V\big).$$

(iv) (Continuous dependence on A and F) *Let T_0 be defined as in (iii) and let $((A_j, F_j))$ be a sequence such that (A_j, F_j) satisfies* (38) *for each $j \in \mathbb{N}$ and $(A_j, F_j) \to (A, F)$ in*

$$\mathcal{C}^{1\text{-}}(\Pi \times \mathcal{D}, L_\infty(\mathsf{J}, \mathcal{L}(E_1, E_0)) \times L_p(\mathsf{J}, E_0)).$$

Denote by $u_j(\pi, v)$ the maximal solution of (39) *with (A, F) replaced by (A_j, F_j). Then $u_j(\pi, v) \to u(\pi, v)$ in $\mathcal{H}_p^1(J_{T_0})$.*

Proof. For $(v, w) \in \mathcal{D}$ we set

$$v \oplus w(t) := \begin{cases} v(t), & -S < t \leq 0, \\ w(t), & 0 < t < \mathsf{T}. \end{cases}$$

If $\mathcal{V} = C_0([-S, 0], E)$, then it is obvious that

$$\big((v, w) \mapsto (v \oplus w)_t\big) \in \mathcal{C}^{1\text{-}}(\mathcal{D}, \mathcal{V}), \qquad 0 \leq t < \mathsf{T}. \tag{40}$$

In the other case this follows from Lemma 7.1 in [8].

Set $\Xi := \Pi \times \mathcal{V}$ and $\alpha(\xi) := v(0)$ for $\xi = (\pi, v) \in \Xi$. Then (33) is satisfied and $D = \Pi \times \mathcal{D}$.

For $(\xi, u) = (\pi, v, u) \in D$ define $\mathcal{A}(\xi, u)$ and $\mathcal{F}(\xi, u)$ by

$$(\mathcal{A}, \mathcal{F})(\xi, u)(t) := (A, F)\big(\pi, (v \oplus u)_t, u\big)(t), \qquad t \in \mathsf{J}.$$

It follows from (40), $\big((v, u) \to u\big) \in \mathcal{L}(\mathcal{D}, \mathcal{H})$, and (38) that $\mathcal{A}$ and $\mathcal{F}$ satisfy (35). Thus Theorem 3.1 implies the assertions. □

Let X and Y be metric spaces and put $Z := X \times Y$. Suppose that $J(z)$ is for each $z \in Z$ an open subinterval of $\mathbb{R}^+$ containing 0. Set

$$\mathcal{Z} := \bigcup_{z \in Z} J(z) \times \{z\}.$$

Then $\varphi : \mathcal{Z} \to X$ is a parameter dependent Lipschitz semiflow on X, provided

$$\varphi(\cdot, \cdot, y) : \mathcal{Z}_y := \big\{ (t, x) \in \mathbb{R}^+ \times X \ ; \ (t, x, y) \in \mathcal{Z} \big\} \to X$$

is for each $y \in Y$ a Lipschitz semiflow on X. It depends Lipschitz continuously on the parameters $y \in Y$ if $\big((t, z) \mapsto \varphi(t, z)\big) \in \mathcal{D}^{0,1\text{-}}(\mathcal{Z}, X)$.

Suppose that (38) is satisfied for every $\mathsf{T} > 0$. Then the map (A, F) is said to be **autonomous** if, given $s, t \in \mathbb{R}^+$ and $u \in \mathcal{H}_p^1(J_{s+t})$, it follows that, setting $v^s(\tau) := u(s + \tau)$ for $0 \leq \tau \leq t$, that

$$(A, F)(\pi, u_{t+s}, u)(t + s) = (A, F)\big(\pi, (v^s)_t, v^s\big)(t).$$

Note that this is true, in particular, if $(A, F)(\pi, v, \cdot)$ is a local map.

Let (38) be satisfied for every $\mathsf{T} > 0$. Then we consider the quasilinear functional differential equation

$$\dot{u} + A(\pi, u_t, u)u = F(\pi, u_t, u) \text{ on } (0, \infty), \quad u_0 = v. \tag{41}$$

Clearly, u is an $\mathcal{H}_p^1$ solution if it is an $\mathcal{H}_p^1$ solution of (39) for every $\mathsf{T} > 0$.

The following theorem is the second main abstract theorem of this paper.

Theorem 4.2. *Let* (37) *be true and suppose that* (38) *holds for every* $\mathsf{T} > 0$. *Then*

(i) *Problem* (41) *has for each* $(\pi, v) \in \Pi \times \mathcal{V}$ *a unique maximal* $\mathcal{H}_p^1$ *solution,* $u(\pi, v)$.

(ii) *If* $u(\pi, v) \in \mathcal{H}_p^1(J_T \cap J(\pi, v))$ *for every* $T > 0$, *where* $J(\pi, v)$ *is the maximal existence interval for* $u(\pi, v)$, *then* $u(\pi, v)$ *exists globally, that is,* $J(\pi, v) = \mathbb{R}^+$.

(iii) *For each* $T \in \mathring{J}(\pi, v)$ *there are* $r, \kappa > 0$ *such that*

$$\|u(\pi_1, v_1) - u(\pi_2, v_2)\|_{\mathcal{H}_p^1(J_T)} \leq \kappa\big(\|\pi_1 - \pi_2\|_\Pi + \|v_1 - v_2\|_\mathcal{V}\big),$$

whenever $(\pi_j, v_j) \in \Pi \times \mathcal{V}$ *satisfy*

$$\|\pi_j - \pi\|_\Pi + \|v_j - v\|_\mathcal{V} < r, \qquad j = 1, 2.$$

(iv) *If* (A, F) *is autonomous then the map* $(t, v, \pi) \mapsto u(\pi, v)_t$ *defines a Lipschitz semiflow on* $\mathcal{V}$ *depending Lipschitz continuously on* $\pi \in \Pi$.

Proof. (i)–(iii) are obviously implied by (i)–(iii) of Theorem 4.1.

(iv) We fix $\pi \in \Pi$ and omit it from the notation since it does not play a role in the following argument. Given $v \in \mathcal{V}$ and $t, s \in \mathbb{R}^+$ with $t + s \in J(v)$, set $w(t) := u(v)(t + s)$. Then the fact that (A, F) is autonomous implies that

$$\dot{w} + A(\pi, w_t, w)w = F(\pi, w_t, w) \text{ on } \mathring{J}(v) - s$$

and $w_0 = u(v)_s$. Note that $u(v)_s \in \mathcal{V}$ and $w \,|\, J_\tau \in \mathcal{H}_p^1(J_\tau)$ for $\tau \in \mathring{J}(v) - s$. Hence we infer from (i) that $J\big(u(v)_s\big) \supset J(v) - s$ and $w = u\big(u(v)_s\big)$ on $J(v) - s$. On the other hand, set

$$\widetilde{w}(t) := \begin{cases} u(v)(t), & -S \leq t < s, \\ u\big(u(v)_s\big)(t - s), & s \leq t \in J\big(u(v)_s\big) + s. \end{cases}$$

Then using Lemma 7.1 of [8] one verifies that $\widetilde{w}$ is an $\mathcal{H}_p^1$ solution of (41) on $J\big(u(v)_s\big)+s$. Thus, by uniqueness, $J(v)\supset J\big(u(v)_s\big)+s$. This implies that $J(v)=J\big(u(v)_s\big)+s$ and

$$u(v)_{s+t}=u\big(u(v)_s\big)_t$$

for $s\in J(v)$ and $t\in J\big(u(v)_s\big)$. Now the assertion is a consequence of (iii), the strong continuity of the translation group on $C_0(\mathbb{R},E)$ and $\mathcal{H}_p^1(\mathbb{R})$, and, in the case where $\mathcal{V}=C_0\big([-S,0],E\big)$, of (30). □

It is easy to derive from Theorem 4.1(iv) the continuous dependence of $u(\pi,v)$ on A and F. We leave this to the reader.

5. Parabolic boundary value problems

Let I be a nonempty closed interval and F a Banach space. We denote by $\mathcal{M}(I,F)$ the Banach space of all bounded F valued Radon measures on I (see Section 2.2 in [3] for a brief introduction to the theory of vector valued measures and the corresponding integration). We identify $\mathcal{M}(I,F)$ with the closed linear subspace of $\mathcal{M}(\mathbb{R},F)$ consisting of all bounded F valued Radon measures being supported in I. We also identify $L_1(I,F)$ with the closed linear subspace of $\mathcal{M}(I,F)$ consisting of all measures being absolutely continuous with respect to Lebesgue's measure dt. Thus, in particular, we identify $f\in L_1(I,F)$ with its trivial extension (by zero on I^c) in $L_1(\mathbb{R},F)$.

Let F_0, F_1, and F_2 be Banach spaces and $F_1\times F_2\to F_0$ and assume that $(x,y)\mapsto x\bullet y$ is a multiplication, that is, a continuous bilinear form of norm at most 1. In particular, given Banach spaces E and F, we can choose $F_1:=\mathcal{L}(E,F)$, $F_2:=E$, $F_0:=F$, and $A\bullet e:=Ae$ for $A\in\mathcal{L}(E,F)$ and $e\in E$.

We put $\infty-\infty:=\infty$. Given $0\le R\le S\le\infty$ with $S>0$, $0<T\le\infty$, $u\in C_0\big([-S,T],F_1\big)$, and $\mu\in\mathcal{M}\big([0,S-R],F_2\big)$, the convolution integral

$$u*\mu(t):=\int u(t-\tau)\bullet\mu\,(d\tau) \tag{42}$$

is well defined for $-R\le t\le T$. It is not difficult to see that $(u,\mu)\mapsto u*\mu$ defines a multiplication

$$C_0\big([-S,T],F_1\big)\times\mathcal{M}\big([0,S-R],F_2\big)\to BUC\big([-R,T],F_0\big). \tag{43}$$

It also follows from Young's inequality that the map $(v,w)\mapsto v*w$ is a well defined multiplication

$$L_\xi\big((0,S-R),F_1\big)\times L_\eta\big((-S,T),F_2\big)\to L_\zeta\big((-R,T),F_0\big), \tag{44}$$

provided $\xi, \eta, \zeta \in [1, \infty]$ satisfy

$$1/\xi + 1/\eta = 1 + 1/\zeta. \tag{45}$$

For abbreviation, we set

$$X := \begin{cases} L_2, & \text{if } p = 2, \\ C(\overline{\Omega}) & \text{otherwise.} \end{cases}$$

In order to specify the measures appearing in (5) we fix R and S as above and suppose throughout that $1 < s \leq p'$. Then we introduce the following Banach spaces:

$$\begin{aligned} &\mathsf{H}_0 := L_s\big(J_R, \mathcal{L}(H^{-1}_{p,\chi})\big); && \mathsf{H} := L_s\big(J_R, \mathcal{L}(L_p)\big); \\ &\mathsf{P} := \mathcal{M}\big(\overline{J}_R, \mathcal{L}(C(\overline{\Omega}), L_\xi)\big); && \mathsf{P}_\Gamma := \mathcal{M}\big(\overline{J}_R, \mathcal{L}(C(\Gamma), L_\eta(\Gamma))\big); \\ &\Sigma := \mathcal{M}\big(\overline{J}_{S-R}, \mathcal{L}(X, C(\overline{\Omega}, \mathbb{R}^m))\big); && \Sigma_\Gamma := \mathcal{M}\big(\overline{J}_{S-R}, \mathcal{L}(X, C(\Gamma, \mathbb{R}^m))\big), \end{aligned}$$

where $pn/(n+p) \leq \xi \leq \infty$ with $\xi > 1$, and $p(n-1)/n \leq \eta \leq \infty$ with $\eta > 1$, and where we agree to set $L_s(J_R, F) := \{0\}$ if $R = 0$. We put

$$\Pi := \mathsf{H}_0 \times \mathsf{H} \times \mathsf{H} \times \mathsf{P} \times \mathsf{P}_\Gamma \times \Sigma \times \Sigma \times \Sigma \times \Sigma_\Gamma$$

and denote the general point of this Banach space by

$$\pi = (h, h_0, h_1, \rho_0, \rho_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3).$$

Given $\pi \in \Pi$, we set

$$\mu := \delta_0 + h\,dt, \quad \nu_0 := \delta_0 + h_0\,dt, \quad \nu_1 := h_1\,dt. \tag{46}$$

Now we can formulate and prove the third main result of this paper, the following existence, uniqueness, and continuity theorem for problem (4).

Theorem 5.1. *Suppose that assumptions* (6) *and* (7) *are satisfied. Fix* $\pi \in \Pi$ *and define* μ, ν_0, *and* ν_1 *by* (46), *and* (4) *by* (5). *If* $R = \infty$, *then assume in addition that, for* $j = 0, 1$,

$$\left.\begin{array}{l} \textit{either } \rho_j \textit{ is compactly supported} \\ \textit{or it is absolutely continuous with respect to } dt. \end{array}\right\} \tag{47}$$

Finally, suppose that $\mathcal{V}$ *equals either* $\mathcal{H}^1_{p,\chi}(J_{-S})$ *or* $C_0\big(\overline{J}_{-S}, W^{1-2/p}_{p,\chi}\big)$ *with*

$$V = \mathcal{H}^1_{p,\chi}(J_{-S}), \quad \textit{if } h \neq 0. \tag{48}$$

Then:

(i) *Problem* (4) *has for each history* $v \in \mathcal{V}$ *a unique maximal* $\mathcal{H}^1_p$ *solution,* $u(v)$.

(ii) *If* $u(v) \in \mathcal{H}_p^1(J_T \cap J(v))$ *for each* $T > 0$*, where* $J(v)$ *is the maximal existence interval of* $u(v)$*, then* $u(v)$ *exists globally, that is,* $J(v) = \mathbb{R}^+$*.*

(iii) *The map* $(t, v) \mapsto u(v)_t$ *is a Lipschitz semiflow on* $\mathcal{V}$ *depending Lipschitz continuously on* $\pi \in \Pi$ *(subject to condition* (47)*, of course).*

Proof. (1) Set $(E_0, E_1) := (H_{p,\chi}^{-1}, H_{p,\chi}^1)$. Then $E_1 \overset{d}{\hookrightarrow} E_0$, and, except for equivalent norms, $E = W_{p,\chi}^{1-2/p}$. Indeed, if $p = 2$, this follows from Proposition 2.1 and Theorem 15.1 in Chapter I of [24] (also see Section 1.15.10 in [28]). If $p > n + 2$, then it is implied by Theorem 7.2 of [1], for example. (Note that this proves (8).)

(2) Fix $\mathsf{T} > 0$. The Sobolev embedding $W_{p,\chi}^{1-2/p} \hookrightarrow C(\overline{\Omega})$ if $p > n + 2$ and (8) imply, together with the definition of $v \oplus w$, that

$$\big((v, w) \mapsto v \oplus w\big) \in \mathcal{C}^{1\text{-}}\big(\mathcal{D}, C_0([-S, \mathsf{T}], X)\big).$$

Hence we infer from (43) that the map

$$\big(\sigma, (v, w)\big) \mapsto \sigma * (v \oplus w) \tag{49}$$

belongs to $\mathcal{C}^{1\text{-}}\big(\Sigma \times \mathcal{D}, BUC([-R, \mathsf{T}], C(\overline{\Omega}, \mathbb{R}^m))\big)$. Set

$$\big(\widetilde{a}, \widetilde{b}, \widetilde{f}\,\big)(\sigma, v, w) := (a, b, f)\big(\cdot, \sigma * (v \oplus w)\big).$$

Then (6) and the asserted continuity properties of (49) imply that

$$\widetilde{a}, \widetilde{b} \in \mathcal{C}^{1\text{-}}\big(\Sigma \times \mathcal{D}, BUC([-R, \mathsf{T}], C(\overline{\Omega}, \mathbb{R}^{n\times n}))\big) \tag{50}$$

and

$$\widetilde{f} \in \mathcal{C}^{1\text{-}}\big(\Sigma \times \mathcal{D}, BUC([-R, \mathsf{T}], C(\overline{\Omega}))\big). \tag{51}$$

Similarly,

$$\begin{aligned}\widetilde{g} := \big((\sigma, (v, w)) \mapsto g(\cdot, \sigma * (v \oplus w))\big) \\ \in \mathcal{C}^{1\text{-}}\big(\Sigma_\Gamma \times \mathcal{D}, BUC([-R, \mathsf{T}], C(\Gamma))\big).\end{aligned} \tag{52}$$

(3) For $\pi \in \Pi$ and $(v, u) \in \mathcal{D}$ define $A(\pi, v, u)$ by

$$\big\langle \varphi, A(\pi, v, u)w \big\rangle := \big\langle \nabla\varphi, \widetilde{a}(\sigma_0, v, u)\nabla w \big\rangle$$

for $(\varphi, w) \in H_{p',\chi}^1 \times H_{p,\chi}^1$. Then it follows from (50) that

$$A \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{D}, C(\overline{\mathsf{J}}, \mathcal{L}(E_1, E_0))\big). \tag{53}$$

Observe that $\widetilde{a}(\sigma_0, v, u)(x, t)$ is symmetric and uniformly positive semidefinit on $\overline{\Omega} \times \overline{\mathsf{J}}$. Thus, if $p = 2$, well known results on the weak solvability of linear parabolic equations, essentially due to J.-L. Lions [23] (also see Theorem 2 in Chapter XVIII of [16], Chapter 23 in [31], or Theorem 11.7 in [12]), guarantee that

$$A(\pi, v, u) \in \mathcal{MR}_p\big(\mathsf{J}, (E_1, E_0)\big). \tag{54}$$

If $p > n+2$, this will be shown elsewhere. In particular, (32) is satisfied.

(4) For $\pi \in \Pi$ and $(v,u) \in \mathcal{D}$ we define $F_0(\pi, v, u)$ by

$$\begin{aligned} \big\langle w, F_0(\pi, v, u)\big\rangle :=& \big\langle \nabla w, \big(-h_0 * \widetilde{a}(\sigma_0, v, u) + h_1 * \widetilde{b}(\sigma_1, v, u)\big)\nabla u\big\rangle \\ &+ \big\langle w, \rho_0 * \widetilde{f}(\sigma_2, v, u)\big\rangle + \big\langle \gamma w, \rho_1 * \widetilde{g}(\sigma_3, v, u)\big\rangle_\Gamma \end{aligned} \tag{55}$$

for $w \in H^1_{p,\chi}$. Using $\nabla u \in L_p(\mathsf{J}, L_p)$ and (50) we see that

$$\big((\sigma_0, (v,u)) \mapsto \widetilde{a}(\sigma_0, v, u)\nabla u\big) \in \mathcal{C}^{1\text{-}}\big(\Sigma \times \mathcal{D}, L_p(\mathsf{J}, L_p)\big).$$

Thus, setting $1/r := 1/p + 1/s - 1 \in [0, 1/p)$, we infer from (44), (45) that

$$\big((h_0, \sigma_0, (v,u)) \mapsto -h_0 * \widetilde{a}(\sigma_0, v, u)\nabla u\big)$$

belongs to $\mathcal{C}^{1\text{-}}\big(\mathsf{H} \times \Sigma \times \mathcal{D}, L_r(\mathsf{J}, L_p)\big)$. The same is true, if $\widetilde{a}$, h_0, and σ_0 are replaced by $\widetilde{b}$, h_1, and σ_1, respectively. From (43), (47), and (51) we deduce that

$$\big((\rho_0, \sigma_2, (v,u)) \mapsto \rho_0 * \widetilde{f}(\sigma_2, v, u)\big) \in \mathcal{C}^{1\text{-}}\big(\mathsf{P} \times \Sigma \times \mathcal{D}, L_\infty(\mathsf{J}, L_\xi)\big).$$

Note that $H^1_{p',\chi} \hookrightarrow L_{\xi'}$ by Sobolev's embedding theorem. Similarly, (43), (47), and (52) imply

$$\big((\rho_1, \sigma_3, (v,u)) \mapsto \rho_1 * \widetilde{g}(\sigma_3, v, u)\nabla u\big) \in \mathcal{C}^{1\text{-}}\big(\mathsf{P}_\Gamma \times \Sigma_\Gamma \times \mathcal{D}, L_\infty(\mathsf{J}, L_\eta(\Gamma))\big).$$

Furthermore, the trace theorem implies

$$\gamma \in \mathcal{L}\big(H^1_{p',\chi}, L_{\eta'}(\Gamma)\big).$$

From these considerations and the boundedness of J it follows that

$$F_0 \in \mathcal{C}^{1\text{-}}\big(\Pi \times \mathcal{D}, L_r(\mathsf{J}, E_0)\big). \tag{56}$$

(5) Now suppose that $h \neq 0$ so that (48) is satisfied. Since

$$\big((v,u) \mapsto v \oplus u\big) \in \mathcal{C}^{1\text{-}}\big(\mathcal{D}, \mathcal{H}^1_{p,\chi}(J_{-S} \cup \mathsf{J})\big)$$

(see (40)), it follows from

$$\mathcal{H}^1_{p,\chi}(J_{-S} \cup \mathsf{J}, H^{-1}_{p,\chi}) \hookrightarrow L_p(J_{-S} \cup \mathsf{J}) \hookrightarrow L_p(\mathbb{R}, H^{-1}_{p,\chi}) \tag{57}$$

and (44) that

$$\big((h, (v,u)) \mapsto \widetilde{h} * (v \oplus u)^\sim\big) \in \mathcal{C}^{1\text{-}}\big(\mathsf{H}_0 \times \mathcal{D}, L_r(\mathbb{R}, H^{-1}_{p,\chi})\big), \tag{58}$$

where $\sim$ denotes extension by zero, due to $\widetilde{h} \in L_s\big(\mathbb{R}, \mathcal{L}(H^{-1}_{p,\chi})\big)$. Similarly, using Lemma 7.1 of [8] we see that

$$\big((h,(v,u)) \mapsto \widetilde{h} * (\dot{v} \oplus \dot{u})^{\sim}\big) \in C^{1\text{-}}\big(\mathsf{H}_0 \times \mathcal{D}, L_r(\mathbb{R}, H^{-1}_{p,\chi})\big). \tag{59}$$

Since convolution and distributional derivatives commute, it is not difficult to see that

$$-\int_{\mathbb{R}} \dot{\varphi}\langle w, \widetilde{h} * (v \oplus u)^{\sim}\rangle\, dt = \int_{\mathbb{R}} \varphi\langle w, h * (\dot{v} \oplus \dot{u})\rangle\, dt \tag{60}$$

for each smooth φ having compact support in $\mathring{\mathsf{J}}$ and each $w \in H^1_{p',\chi}$ (cf. Lemma 7.1 in [8]).

Given $\big(\pi,(v,w)\big) \in \Pi \times \mathcal{D}$, set

$$F_1(\pi, v, w) := -\partial_t\big(h * (v \oplus w)\big) := -\big[\partial_t\big(\widetilde{h} * (v \oplus w)^{\sim}\big)\big] \,|\, \mathring{\mathsf{J}}.$$

Then we infer from (59) and (60) that

$$F_1 \in C^{1\text{-}}\big(\Pi \times \mathcal{D}, L_r(\mathsf{J}, E_0)\big). \tag{61}$$

(6) Put $F := F_0$ if $h = 0$, and $F := F_0 + F_1$ otherwise. Then assumption (38) is satisfied, due to (53), (54), (56), and (61), since the Volterra property is obvious.

Finally, set

$$(\mathcal{A}, \mathcal{F})(\pi, u_t) := (A, F)(\pi, u \,|\, J_{-S}, u \,|\, \mathsf{J})$$

for $u : J_{-S} \cup \mathsf{J} \to E_0$ with $(u \,|\, J_{-S}, u \,|\, \mathsf{J}) \in \mathcal{D}$. Then, given $(\pi, v) \in \Pi \times \mathcal{V}$, one verifies that u is an $\mathcal{H}^1_p$ solution on J_T of (4) with history $v \in \mathcal{V}$ iff u is an $\mathcal{H}^1_p$ solution on J_T of the parameter dependent functional differential equation

$$\dot{u} + \mathcal{A}(\pi, u_t)u = \mathcal{F}(\pi, u_t), \quad u_0 = v.$$

Hence the assertion follows from Theorem 4.2. □

6. Proofs for the model problems

It is now not difficult to prove the theorems of Section 2 by observing that the corresponding model problems are particular instances of (4), (5).

Proof of Theorem 2.1. (1) Set $m := 1$, $p := 2$, $h := 0$, $h_0 := h_1 := 0$, $\rho_0 := j\delta_0$ with $j : C(\overline{\Omega}) \hookrightarrow L_2$, $\rho_1 := j_\Gamma \delta_0$ with $j_\Gamma : C(\Gamma) \hookrightarrow L_2(\Gamma)$, $\sigma_0 := \sigma_1 := \sigma_2 := K\alpha$, and $\sigma_3 := \gamma K\alpha$. Then

$$\pi := (h, h_0, h_1, \rho_0, \rho_1, \sigma_0, \sigma_1, \sigma_2, \sigma_3) \in \Pi$$

with $\xi := \eta := 2$ and $R := 0$. Hence everything but the last assertion follows from Theorem 5.1.

(2) Suppose that $\operatorname{supp}(\alpha) \subset [s, S]$ for some $s \in (0, S]$. Then

$$\alpha * Ku(t) = \int_{[s,S]} Ku(t-\tau)\,\mu(d\tau) + \int_{[s,S]} Kv(t-\tau)\,\mu(d\tau), \qquad 0 \le t \le s.$$

Thus on the interval $[0, s]$ the diffusion matrix and the nonlinearities are known functions so that (15) reduces on J_s to a linear equation which has a unique $\mathcal{H}_p^1$ solution on J_s with initial value $v(0)$. Next we consider (15) on the interval $[s, 2s]$. Here we are now also faced with a linear problem with initial value $u(s) \in E$ having a unique solution. By iterating this argument we see that (15) is globally solvable since we can 'piece together' the solutions on the intervals $\bigl(k\tau, (k+1)\tau\bigr)$ by means of Lemma 7.1 of [8]. □

It should be observed that the argument of the second part of this proof is the 'method of steps', well known in the theory of retarded differential equations (e.g., [29]).

Problem (18) does not fit completely into the framework of Theorem 5.1 since f is not continuous. However, easy modifications of the proof of the latter theorem give the stated results.

Proof of Theorem 2.2. (1) It follows from (16) and known properties of Nemytskii operators that

$$\bigl(u \mapsto f(\cdot, u)\bigr) \in C^{1\text{-}}\bigl(C([0,T], L_2), C([0,T], L_\xi)\bigr)$$

where $\xi := 2n/(n+2)$. Since $H_\chi^1 \hookrightarrow L_{\xi'}$ we see that $F(u)$ is well defined for $u \in C\bigl([0,T], L_2\bigr)$ by

$$\bigl\langle v, F(u)\bigr\rangle := \bigl\langle v, f(\cdot, u)\bigr\rangle + \langle \gamma v, g_0\rangle_\Gamma, \qquad v \in H_\chi^1,$$

and that

$$F \in C^{1\text{-}}\bigl(C([0,T], L_2), C([0,T], H_\chi^{-1})\bigr).$$

Thus (30) implies

$$F \in C^{1\text{-}}\bigl(\mathcal{H}_\chi^1(J_T), L_\infty(J_T, H_\chi^{-1})\bigr)$$

for every $T > 0$. With this definition of F the proof of Theorem 5.1 remains valid. Thus all but the last assertion follow from that theorem.

(2) If the additional assumptions are satisfied we apply again the method of steps. However, in this case we have to solve at each step a semilinear

equation since the diffusion matrix is known but the right hand side is still a function of u on the corresponding interval.

Using condition (ii) and well known arguments for weak solutions of semilinear parabolic equations we easily deduce that $\|u(\cdot,t)\|_{L_2} \leq c$ for $0 \leq t < \tau$, where u is the maximal solution of the semilinear problem on J_s and $\tau \in (0,s]$ is its maximal existence time. Consequently, $F(u) \in L_\infty(J_\tau, H_\chi^{-1})$. Now maximal regularity implies $u \in \mathcal{H}_\chi^1(J_\tau)$. Hence we infer from Theorem 3.1(ii), for example, applied to the semilinear problem, that u exists on J_s and belongs to $\mathcal{H}_\chi^1(J_s)$. Thus $u(s) \in L_2$ and the method of steps can be carried through. □

Proof of Theorem 2.3. Here we put $m := 2$ and define σ_0 by $\sigma_0 := [\delta_0 \otimes I, \alpha \otimes I]$ with I being the identity in $\mathcal{L}\big(C(\overline{\Omega})\big)$, that is,

$$\langle \sigma_0, \varphi \rangle = \Big(\varphi(0,\cdot), \int_{\mathbb{R}} \varphi(t,\cdot)\,\alpha(dt)\Big), \qquad \varphi \in C_0\big(\mathbb{R}, C(\overline{\Omega})\big).$$

Now the assertions follow by the arguments of the proof of Theorem 2.1.□

It is now clear how Theorem 5.1 can be applied to prove Theorems 2.3 and 2.4. Theorem 2.5 is again obtained by the method of steps. At each step there has to be solved a quasilinear problem to which Theorem 2.1 of [8] can be applied. For this we have to observe that the translation group acts strongly continuously on $H_p^1(\mathbb{R}, H_{p,\chi}^{-1})$ and that it commutes with differentiation. Thus $\partial_t\big(hv(t-\cdot)\big)$ is well defined in $L_p(J_r, H_{p,\chi}^{-1})$. If the solution exists globally on J_r then we can go on to the next step. Otherwise, we have arrived at the maximal solution. This is true for every step which can be carried out. Details are left to the reader.

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A LINEAR PARABOLIC PROBLEM WITH NON-DISSIPATIVE DYNAMICAL BOUNDARY CONDITIONS

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An existence theory for solutions of a parabolic problem $u_t - \operatorname{div}(A(x)\nabla u) + q(x)u = f(x,t)$ for $x \in D$ and $t \in (0,T]$ under a dynamical boundary condition $\sigma(x)u_t + \nabla u^T A(x)n = g(x,t)$ for $x \in \partial D$ and $t \in (0,T]$ is developed and a spectral representation formula is derived. It extends the results of [2] to problems with variable coefficients. We are interested in the case where the dynamical coefficient σ is a sign-changing or negative function. The one-dimensional parabolic problem is well-posed in the space $C([0,T], H^1(D))$. This is not true in higher dimensions. Our approach is based on the spectral theory of an associated elliptic problem with the eigenvalue parameter both in the equation and the boundary condition. By means of the theory of compact operators the spectrum is analyzed. Qualitative properties of the eigenfunctions are derived, e.g. strict positivity of the principal two eigenfunctions follows from a Harnack-type inequality. An interesting phenomenon is the "parameter-resonance", where for a specific value of the mean of the dynamical coefficient $\sigma(x)$, two eigenvalues of the elliptic problem cross.

AMS Subject Classification 2000 35K20, 47A75, 35K60, 35C10.

Key words. Compact operator, variational characterization of eigenvalues, series representation of solutions, Harnack inequality.

1. Introduction

Let $D \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary ∂D and let n be its outer normal defined almost everywhere. For $x \in D$ let $A(x)$ be a uniformly positive definite, symmetric matrix. Assume moreover that $q \in L^\infty(D)$ is a non-negative function and that $\sigma(x)$ is continuous on ∂D

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with $\sigma^- = -\min\{\sigma, 0\} \not\equiv 0$. In this paper we shall discuss parabolic problems of the form

$$u_t - \operatorname{div}(A(x)\nabla u) + q(x)u = f(x,t) \text{ in } D \times (0,T), \tag{1.1}$$

$$\sigma(x)u_t + \nabla u^T A(x)n = g(x,t) \text{ on } \partial D \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \text{ in } D. \tag{1.3}$$

We will study the existence of weak solutions by means of a Hilbert space approach and derive a representation formula for the solutions.

Many authors have studied problems with dynamical boundary conditions and positive dynamical coefficient σ. In this case rather complete existence theories are available, cf. Escher [5] for an approach via semigroups or Bandle, von Below, Reichel [2] for an L^2-theory. In [4] von Below and Pincet Mailly study the blow-up of solutions. Vitillaro [12] considered nonlinear boundary conditions where u_t is replaced by $|u_t|^{m-1}u_t$. Fila and Quittner [8] treated the problem (1.1)-(1.3) with a nonlinear term at the right-hand side of (1.2) of the form $\mu|u|^{q-1}u$. They were mainly interested under what conditions solutions exist globally for all times or when they blow up in finite time. In all of the above papers it is assumed that σ is positive.

The case where σ is a negative constant is less studied, see Bandle, von Below, Reichel [2] and Vazquez, Vittilaro [11]. It turns out that it is much more delicate and gives rise to unexpected phenomena. In fact if $D = (0, L)$ is a one-dimensional interval the parabolic problem with initial conditions in $H^1(D)$ is well-posed in the space $C([0,T], H^1(D))$. In higher space-dimensions the parabolic problem is ill-posed in these spaces. This fact can only be compensated by replacing the space of initial conditions $H^1(D)$ by a subspace $\mathcal{H}^1(D)$, cf. Section 2.3. The case of a sign changing function $\sigma(x)$ has not been treated so far. This will be done in this paper. The goal is to generalize the results of constant coefficients $A(x) = \mathrm{Id}$, $\sigma = \text{const.} < 0$ treated in [2] to the case where $A(x)$ is uniformly elliptic and $\sigma(x)$ sign-changing or negative. It leads to similar results as in the case of constant coefficients, e.g., independently of the matrix $A(x)$ the critical case of parameter-resonance happens when the mean of $\sigma(x)$ is equal to $\sigma_0 = -|D|/|\partial D|$.

For the expansion of the solutions of the heat equation (1.1) with dynamical boundary conditions (1.2) into a Fourier series we are led to the following eigenvalue problem

$$-\operatorname{div}(A(x)\nabla\varphi) + q(x)\varphi = \lambda\varphi \text{ in } D, \quad \nabla\varphi^T A(x)n = \lambda\sigma(x)\varphi \text{ on } \partial D. \tag{1.4}$$

The corresponding Rayleigh quotient reads as

$$R[v] = \frac{\int_D \nabla v^T A(x) \nabla v + q(x) v^2 \, dx}{\int_D v^2 \, dx + \oint_{\partial D} \sigma(x) v^2 \, ds}.$$

Notice that it takes both positive and negative values if $\sigma^- \not\equiv 0$.

The spectral theory for such problems has been treated in [2] if $A = \mathrm{Id}$, $\sigma = \text{const.}$ and by Ercolano and Schechter [7] for formally self-adjoint elliptic operators of second and higher order under lower boundedness assumptions. In our case, where $\sigma(x)$ is negative or sign-changing, we will show that the spectrum has in dimensions $N \geq 2$ two sets of eigenvalues, $\{\lambda_n\}_{n\geq 1}$ with $\lim_{n\to\infty} \lambda_n = \infty$ and $\{\lambda_{-n}\}_{n\geq 1}$ with $\lim_{n\to\infty} \lambda_{-n} = -\infty$. For $N = 1$ there exist at most two negative eigenvalues. The eigenfunctions are complete in $H^1(D)$ except in the *resonance case*

$$q(x) = 0 \quad \text{and} \quad |D| + \oint_{\partial D} \sigma(x) \, ds = 0$$

where they have to be supplemented with an additional element. This is very similar to the case where σ is a negative constant.

The main existence results for the eigenvalue problem and the linear heat equation with dynamical boundary conditions are stated in Sections 2.2 and 2.3. We follow the approach used in [2] which has the advantage of providing an expansion formula for the solutions.

Section 3 deals with qualitative aspects of the eigenvalue problem (1.4). A Harnack inequality is derived for positive solutions of linear problems including the eigenvalue problem. This inequality is used in the discussion of the simplicity of the principal eigenvalues λ_1 and λ_{-1}. Moreover it is shown that the one-dimensional eigenvalue problem on the interval $D = (0, L)$ has at most two negative eigenvalues.

In Section 4 the spectrum of (1.4) will be described completely by means of the theory of compact linear operators. We distinguish the cases $\int_D q(x) \, dx > 0$ and $q(x) \equiv 0$, where in the latter case we need to make further distinctions depending on the mean value

$$\bar{\sigma} := \frac{1}{|\partial D|} \oint_{\partial D} \sigma(x) \, ds$$

of σ. The critical threshold of the mean value is $\sigma_0 := -|D|/|\partial D| < 0$. In the last Section 5 we study the phenomenon of parameter resonance in detail. As the mean value $\bar{\sigma}$ of σ passes through a critical value σ_0 the first positive eigenvalue $\lambda_1(\sigma)$ varies continuously into the first negative eigenvalue $\lambda_{-1}(\sigma)$. At the resonance value σ_0 itself, the system of eigenfunctions is incomplete.

2. Main existence results

2.1. *Notation*

Assume that $D \subset \mathbb{R}^N$ is a bounded Lipschitz domain. For $x \in D$ let $A(x)$ be a symmetric matrix. Assume there exist positive ellipticity constants $\alpha, \beta > 0$ such that $\alpha|\xi|^2 \leq \sum_{i,j=1}^N \xi_i A_{ij}(x)\xi_j \leq \beta|\xi|^2$ for all $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ and almost all $x \in D$. Consider the linear problem

$$u_t - \operatorname{div}(A(x)\nabla u) + q(x)u = f(x,t) \text{ in } D \times \mathbb{R}^+, \tag{2.1}$$

$$\sigma(x)u_t + \nabla u^T A(x)n = g(x,t) \text{ on } \partial D \times \mathbb{R}^+, \tag{2.2}$$

$$u(x,0) = u_0(x) \text{ in } D \tag{2.3}$$

with $q \in L^\infty(D), \sigma \in C(\partial D)$ and $\sigma^- \not\equiv 0$. For $u, v \in H^1(D)$ we set

$$\langle u, v\rangle = \int_D \nabla u^T A(x)\nabla v + q(x)uv\,dx,$$

$$(u,v) = \int_D uv\,dx, \quad (u,v)_0 = \oint_{\partial D} uv\,ds.$$

If $q \geq 0$, $\int_D q\,dx > 0$ then the form $\langle\cdot,\cdot\rangle$ induces a norm which is equivalent to the standard norm of $H^1(D)$.

Let us now define the concept of a weak solution of (2.1)-(2.3). Assume that $f \in L^2((0,T), L^2(D))$, $g \in L^2((0,T), L^2(\partial D))$ and $u_0 \in H^1(D)$.

Definition 2.1. A function $u \in \mathcal{B} := C([0,T], H^1(D))$ is called a *weak solution* of (2.1)-(2.3) if

$$-\int_0^T (u,\phi_t) + (\sigma(x)u,\phi_t)_0\,dt + \int_0^T \langle u,\phi\rangle\,dt$$
$$= \int_0^T (f,\phi)\,dt + \int_0^T (g,\phi)_0\,dt + (u_0,\phi) + (\sigma(x)u_0,\phi)_0$$

for all $\phi \in C^1([0,T], H^1(D))$ with $\phi(\cdot,T) \equiv 0$.

Recall that for domains D with Lipschitz boundary every function $u \in H^1(D)$ has a trace in $H^{1/2}(\partial D)$ and in particular in $L^2(\partial D)$, cf. Alt [1].

2.2. *Results for the eigenvalue problem*

If we are looking for solutions $u(x,t) = e^{-\lambda t}\varphi(x)$ of the homogeneous heat equation (2.1) with $f \equiv 0$, $g \equiv 0$ satisfying the boundary conditions (2.2) then $\varphi(x)$ is a solution of the eigenvalue problem

$$-\operatorname{div}(A(x)\nabla\varphi) + q(x)\varphi = \lambda\varphi \text{ in } D,\ \nabla\varphi^T A(x)n = \sigma(x)\lambda\varphi \text{ on } \partial D. \quad (2.4)$$

In space dimension $N = 1$ and $D = (0, L)$ the above problem reads

$$-(A(x)\varphi')' + q(x)\varphi = \lambda\varphi \text{ in } (0, L)$$

with the boundary condition

$$-A(0)\varphi'(0) = \sigma_1\lambda\varphi(0), \quad A(L)\varphi'(L) = \sigma_2\lambda\varphi(L).$$

We first collect some results on the eigenvalue problem (2.4) and refer to a later Section 4 for the proofs. Define

$$a(u,v) := (u,v) + (\sigma u, v)_0 = \int_D uv\,dx + \oint_{\partial D} \sigma(x)uv\,dx, \quad u,v \in H^1(D).$$

The eigenvalue problem (2.4) can be expressed in the weak form as

$$\langle\varphi, z\rangle = \lambda a(\varphi, z) \quad \forall z \in H^1(D).$$

We shall use in the sequel the following notation

$$\mathbb{N} = \{1, 2 \ldots\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ and } \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}.$$

Let φ_i and λ_i, $i \in I$ denote all the eigenfunctions and eigenvalues of (2.4). We will show that the index set I is countably infinite. A negative (resp. positive) index will stand for a negative (resp. positive) eigenvalue. If zero is an eigenvalue then it will be denoted by λ_0. Our results on the eigenvalue problem are as follows.

Theorem 2.1. *Let $q \geq 0, \int_D q\,dx > 0$. Then there exists a complete set $\{\psi_i\}_{i \in I} \subset H^1(D)$ of eigenfunctions of (2.4) with the property $\langle\psi_i, \psi_j\rangle = \delta_{ij}$. For every $u \in H^1(D)$ we have $u = \sum_{i \in I}\langle u, \psi_i\rangle\psi_i$ in $H^1(D)$.*

(i) If $N \geq 2$ then there are countably many positive and negative eigenvalues, i.e. $I = \mathbb{Z} \setminus \{0\}$.
(ii) Let $N = 1$. If $\sigma_1 \cdot \sigma_2 > 0$ then $I = \{-2, -1\} \cup \mathbb{N}$, i.e., there are exactly two negative eigenvalues. If $\sigma_1 \cdot \sigma_2 \leq 0$ then $I = \{-1\} \cup \mathbb{N}$, i.e, there is exactly one negative eigenvalue.

The case $q \equiv 0$ is more involved. Let $\bar{\sigma} = \frac{1}{|\partial D|} \oint_{\partial D} \sigma(x)\, ds$ and let $\sigma_0 = -|D|/|\partial D|$. Note that $\sigma_0 = -L/2$ if $N = 1$ and $D = (0, L)$. We shall distinguish between two cases: firstly $\bar{\sigma} \neq \sigma_0$ and secondly $\bar{\sigma} = \sigma_0$.

Theorem 2.2. *Assume $q \equiv 0$ and $\bar{\sigma} \neq \sigma_0$. Then there exists a complete set $\{\psi_i\}_{i \in I} \subset H^1(D)$ of eigenfunctions of (2.4) with the property $\langle \psi_i, \psi_j \rangle = \delta_{ij}$ for i and $j \neq 0$. For every $u \in H^1(D)$ we have $u = \sum_{i \in I \setminus \{0\}} \langle u, \psi_i \rangle \psi_i + P(u)$ in $H^1(D)$ where*

$$P(u) := \frac{a(u,1)}{a(1,1)} = \frac{\int_D u\, dx + \oint_{\partial D} \sigma(x) u\, ds}{|D| + \bar{\sigma}|\partial D|}$$

is a projection into the eigenspace corresponding to $\lambda_0 = 0$.

(i) If $N \geq 2$ then $I = \mathbb{Z}$.
(ii) Let $N = 1$, $D = (0, L)$ and $\sigma_1 \cdot \sigma_2 > 0$. If $-L/2 < \bar{\sigma}$ then $I = \{-2, -1\} \cup \mathbb{N}_0$ and if $\bar{\sigma} < -L/2$ then $I = \{-1\} \cup \mathbb{N}_0$.
(iii) Let $N = 1$, $D = (0, L)$ and $\sigma_1 \cdot \sigma_2 \leq 0$. If $-L/2 < \bar{\sigma}$ then $I = \{-1\} \cup \mathbb{N}_0$ and if $\bar{\sigma} < -L/2$ then $I = \mathbb{N}_0$, i.e., in this case there is no negative eigenvalue.

In order to describe the situation in the resonance case $\bar{\sigma} = \sigma_0$ we consider an arbitrary solution w of the boundary value problem

$$-\operatorname{div}(A(x)\nabla w) = 1 \text{ in } D, \quad \nabla w^T A(x) n = \sigma(x) \text{ on } \partial D. \tag{2.5}$$

Note that all eigenfunctions, including the constants, belong to the space

$$\mathcal{V} = \{v \in H^1(D) : a(v, 1) = \int_D v\, dx + \oint_{\partial D} \sigma(x) v\, ds = 0\}.$$

In addition, all eigenfunctions except the constants lie in the subspace

$$\mathcal{V}_w = \{v \in \mathcal{V} : a(v, w) = \int_D vw\, dx + \oint_{\partial D} \sigma(x) vw\, ds = 0\},$$

where w is an arbitrary but fixed solution of (2.5). Hence every element $u \in H^1(D)$ can be split into

$$u = u_w + P(u) + Q(u) w$$

where

$$u_w \in \mathcal{V}_w, \quad P(u) = \frac{a(u,w)}{a(1,w)} - \frac{a(w,w)a(u,1)}{a(1,w)^2}, \quad Q(u) = \frac{a(u,1)}{a(w,1)}.$$

Theorem 2.3. *Assume $q \equiv 0$ and $\bar{\sigma} = \sigma_0$. Then there exists an orthonormal system $\{\psi_i\}_{i \in I} \subset H^1(D)$ of eigenfunctions of (2.4) with $\langle \psi_i, \psi_j \rangle = \delta_{ij}$*

for i and $j \neq 0$. By adding the functions 1 and w the expansion $u = \sum_{i\in I\setminus\{0\}}\langle u,\psi_i\rangle\psi_i + P(u) + Q(u)w$ holds in $H^1(D)$ for every $u \in H^1(D)$.

(i) If $N \geq 2$ then $I = \mathbb{Z}$.
(ii) Let $N = 1$, $D = (0, L)$ and $\sigma_1 \cdot \sigma_2 > 0$. Then $I = \{-1\} \cup \mathbb{N}_0$.
(iii) Let $N = 1$, $D = (0, L)$ and $\sigma_1 \cdot \sigma_2 \leq 0$. Then $I = \mathbb{N}_0$, i.e., in this case there is no negative eigenvalue.

2.3. *Existence results for the linear parabolic problem*

The weak solution of the parabolic problem (2.1)-(2.3) can now be constructed by means of the complete system of eigenfunctions introduced in the previous section. Let ψ_i and w have the same meaning as in Theorems 2.1, 2.2 and 2.3. Let us introduce the following Banach-spaces

$$\mathcal{H}^1(D) = \{u \in H^1(D) : \sum_{i\in I, i<0} \langle u,\psi_i\rangle^2 e^{\lambda_i^2} < \infty\},$$

$$\mathcal{L}^2(D) = \{u \in L^2(D) : \sum_{i\in I, i<0} (u,\psi_i)^2 e^{\lambda_i^2} < \infty\},$$

$$\mathcal{L}^2(\partial D) = \{u \in L^2(\partial D) : \sum_{i\in I, i<0} (u,\psi_i)_0^2 e^{\lambda_i^2} < \infty\}$$

with the norms

$$\|u\|^2_{\mathcal{H}^1(D)} := \sum_{i\in I, i<0} \langle u,\psi_i\rangle^2 e^{\lambda_i^2} + \|u\|^2_{H^2(D)},$$

$$\|u\|^2_{\mathcal{L}^2(D)} := \sum_{i\in I, i<0} (u,\psi_i)^2 e^{\lambda_i^2} + \|u\|^2_{L^2(D)},$$

$$\|u\|^2_{\mathcal{L}^2(D)} := \sum_{i\in I, i<0} (u,\psi_i)_0^2 e^{\lambda_i^2} + \|u\|^2_{L^2(\partial D)}.$$

The following simple result will be helpful for the convergence proof of the formal solution of the parabolic problem.

Lemma 2.1. *There exists a constant $C > 0$ such that for every $f \in L^2(D)$ and $g \in L^2(\partial D)$ one has*

$$\sum_{i\in I}(f,\psi_i)^2 \leq C\|f\|^2_{L^2(D)}, \quad \sum_{i\in I}(g,\psi_i)_0^2 \leq C\|g\|^2_{L^2(\partial D)}.$$

Proof. For $k \in \mathbb{N}$ let $I_k = I \setminus \{0\} \cap \{-k, \dots, k\}$ and let $Z_k = \operatorname{span}[\psi_i, i \in I_k]$. The function $z_k := \sum_{i \in I_k} (f, \psi_i)\psi_i$ satisfies $\langle z_k, \phi \rangle = (f, \phi)$ for all $\phi \in Z_k$. Hence

$$\langle z_k, z_k \rangle = \sum_{i \in I_k} (f, \psi_i)^2 = (f, z_k) \le \|f\|_{L^2(D)} \|z_k\|_{L^2(D)} \le C\|f\|_{L^2(D)} \|z_k\|_{H^1(D)}.$$

Since the constants do not belong to Z_k the bilinear form $\langle \cdot, \cdot \rangle$ produces a norm on Z_k which is equivalent to the $H^1(D)$-norm. Therefore $\langle z_k, z_k \rangle = \sum_{i \in I_k} (f, \psi_i)^2 \le C\|f\|^2_{L^2(D)}$. Letting $k \to \infty$ we get $\sum_{i \in I \setminus \{0\}} (f, \psi_i)^2 \le C\|f\|^2_{L^2(D)}$ and the same holds if $i = 0$ is included. A similar proof works for the second inequality of the lemma. □

Theorem 2.4. *Let $f \in H^1((0,T), \mathcal{L}^2(D))$, $g \in H^1((0,T), \mathcal{L}^2(\partial D))$ and $u_0 \in \mathcal{H}^1(D)$. Then problem (2.1)-(2.3) has a unique solution $u \in C([0,T], H^1(D))$, which is in particular a weak solution in the sense of Definition 2.1. The solution has the following form:*

(i) If $q \ge 0$ and $\int_D q\,dx > 0$ then

$$u(x,t) = \sum_{i \in I} \langle u_0, \psi_i \rangle \psi_i(x) e^{-\lambda_i t} + \sum_{i \in I} h_i(t) \lambda_i \psi_i(x) e^{-\lambda_i t},$$

where $h_i(t) = \int_0^t \left[(f(\cdot, \tau), \psi_i) + (g(\cdot, \tau), \psi_i)_0 \right] e^{\lambda_i \tau}\,d\tau$ for $i \in I$.

(ii) If $q \equiv 0$ and $\bar{\sigma} \ne \sigma_0$ then

$$\begin{aligned} u(x,t) = &\sum_{i \in I \setminus \{0\}} \langle u_0, \psi_i \rangle \psi_i(x) e^{-\lambda_i t} + \frac{a(u_0, 1)}{a(1,1)} \\ &+ \sum_{i \in I \setminus \{0\}} h_i(t) \lambda_i \psi_i(x) e^{-\lambda_i t} + \frac{h_0(t)}{a(1,1)}, \end{aligned}$$

where $h_i(t) = \int_0^t \left[(f(\cdot, \tau), \psi_i) + (g(\cdot, \tau), \psi_i)_0 \right] e^{\lambda_i \tau}\,d\tau$ for $i \in I \setminus \{0\}$ and $h_0(t) = \int_0^t (f(\cdot, \tau), 1) + (g(\cdot, \tau), 1)_0\,d\tau$.

(iii) If $q \equiv 0$ and $\bar{\sigma} = \sigma_0$ then

$$\begin{aligned} u(x,t) = &\sum_{i \in I \setminus \{0\}} \langle u_0, \psi_i \rangle \psi_i(x) e^{-\lambda_i t} \\ &+ \frac{a(u_0, w)}{a(1,w)} - \frac{a(w,w)a(u_0,1)}{a(1,w)^2} + \frac{a(u_0,1)}{a(w,1)}(w(x) - t) \\ &+ \sum_{i \in I \setminus \{0\}} h_i(t) \lambda_i \psi_i(x) e^{-\lambda_i t} + h_0(t) + \tilde{h}(t)(w(x) - t), \end{aligned}$$

where

$$h_i(t) = \int_0^t [(f(\cdot,\tau),\psi_i) + (g(\cdot,\tau),\psi_i)_0]e^{\lambda_i\tau}\,d\tau \ \textit{for}\ i \in I\setminus\{0\},$$

$$\begin{aligned} h_0(t) &= \int_0^t \left(f(\cdot,\tau), \frac{w+\tau}{a(1,w)} - \frac{a(w,w)}{a(1,w)^2}\right) d\tau \\ &\quad + \int_0^t \left(g(\cdot,\tau), \frac{w+\tau}{a(1,w)} - \frac{a(w,w)}{a(1,w)^2}\right)_0 d\tau, \\ \tilde{h}(t) &= \int_0^t \frac{(f(\cdot,\tau),1) + (g(\cdot,\tau),1)_0}{a(w,1)}\,d\tau. \end{aligned}$$

Proof. As an illustration we prove (iii). The proofs of the statements (i) and (ii) are almost the same. In view of Theorem 2.3 we look for a solution of the form

$$u(x,t) = \sum_{j\in I\setminus\{0\}} \alpha_j(t)\psi_i(x) + \alpha_0(t) + \tilde{\alpha}(t)(w(x)-t).$$

First we replace the infinite sum $\sum_{j\in I\setminus\{0\}}$ by a finite sum $\sum_{j\in I_k}$, $I_k = I\setminus\{0\}\cap\{-k,\dots,k\}$ and show that the coefficients $\alpha_j(t)$ have the form given in the theorem. We insert the finite-sum expression u^k into the weak form of (2.1)-(2.3), where u_0 is replaced by the projection of u_0^k into $Z_k = \mathrm{span}[\psi_i : i \in I_k] \oplus \mathrm{span}[1,w]$. For finite sums u^k we can use the concept of classical solution of (2.1)-(2.3). Testing with a test function $\phi \in H^1(D)$ this means

$$a(u_t^k,\phi) + \langle u^k,\phi\rangle = (f,\phi) + (g,\phi)_0.$$

Replacing ϕ successively with ψ_i, 1 and w and keeping in mind that

$$\lambda_i a(\psi_i,\psi_j) = \delta_{ij}, \quad a(\psi_i,1) = a(1,1) = a(\psi_i,w) = 0$$
$$\text{and } \langle\psi_i,\psi_j\rangle = \delta_{ij}, \quad \langle\psi_i,w\rangle = 0,$$

we obtain the following set of equations

$$\begin{aligned} \frac{\dot{\alpha}_i}{\lambda_i} + \alpha_i &= (f,\psi_i) + (g,\psi_i)_0 \text{ if } i \in I\setminus\{0\}, \\ \dot{\tilde{\alpha}}(t)a(w,1) &= (f,1) + (g,1)_0, \\ \dot{\alpha}_0(t)a(w,1) + \dot{\tilde{\alpha}}(t)a(w-t,w) &= (f,w) + (g,w)_0. \end{aligned}$$

The expressions for the coefficients $\alpha_i, \alpha_0, \tilde{\alpha}$ follow by straightforward integration if we impose as initial condition $u^k(0) = u_0^k$. Now we can build the full series defining $u(x,t)$. We will show next that $u \in C([0,T], H^1(D))$. This then establishes that u is a weak solution in the sense of Definition 2.1.

Note that $\langle \cdot, \cdot \rangle$ introduces an equivalent norm on $H^1(D)$ only in the case $\int_D q\,dx > 0$. For $q \equiv 0$ it is an equivalent norm only on the subspaces $\mathcal{V}, \mathcal{V}_w$. But since these subspaces have co-dimension 1 or 2, it is enough to control $\langle u, u \rangle$. Let

$$u_a(x,t) = \sum_{i \in I \setminus \{0\}} \langle u_0, \psi_i \rangle \psi_i(x) e^{-\lambda_i t}, \quad u_b(x,t) = \sum_{i \in I \setminus \{0\}} h_i(t) \lambda_i \psi_i(x) e^{-\lambda_i t},$$

where $h_i(t) = f_i(t) + g_i(t)$ and $f_i(t) = \int_0^t (f, \psi_i) e^{\lambda_i s}\,ds$, $g_i(t) = \int_0^t (g, \psi_i)_0 e^{\lambda_i s}\,ds$. Then for $t \in [0,T]$ one finds

$$\langle u_a, u_a \rangle \leq \sum_{i \in I, i<0} \langle u_0, \psi_i \rangle^2 e^{-2\lambda_i t} + \sum_{i \in I, i>0} \langle u_0, \psi_i \rangle^2 \leq e^{T^2} \|u_0\|^2_{\mathcal{H}^1(D)}$$

by using the trivial inequality $-2\lambda_i t \leq \lambda_i^2 + t^2$. Lebesgue's dominated convergence theorem implies that $u_a(\cdot, t)$ is continuous as a function from $[0,T] \to H^1(D)$. Next we need to show the same for u_b. Note first that

$$\langle u_b, u_b \rangle \leq 2 \sum_{i \in I \setminus \{0\}} (f_i(t)^2 + g_i(t)^2) \lambda_i^2 e^{-2\lambda_i t}. \tag{2.6}$$

For $i \neq 0$ one has

$$\begin{aligned} f_i(t) &= \int_0^t (f, \psi_i) e^{\lambda_i s}\,ds \\ &= \frac{1}{\lambda_i}(f(\cdot,t), \psi_i) e^{\lambda_i t} - \frac{1}{\lambda_i}(f(\cdot,0), \psi_i) - \frac{1}{\lambda_i} \int_0^t (f_t(\cdot,s), \psi_i) e^{\lambda_i s}\,ds \end{aligned}$$

and hence

$$f_i^2(t) \leq \frac{2}{\lambda_i^2}(f(\cdot,t), \psi_i)^2 e^{2\lambda_i t} + \frac{2}{\lambda_i^2}(f(\cdot,0), \psi_i)^2 + \int_0^t (f_t(\cdot,s), \psi_i)^2\,ds \frac{e^{2\lambda_i t} - 1}{\lambda_i^3}.$$

Finally this leads to

$$\begin{aligned} &\sum_{i \in I \setminus \{0\}} f_i(t)^2 \lambda_i^2 e^{-2\lambda_i t} \\ &\leq 2 \sum_{i \in I \setminus \{0\}} (f(\cdot,t), \psi_i)^2 + 2 \sum_{i \in I, i<0} (f(\cdot,0), \psi_i)^2 e^{-2\lambda_i t} + 2 \sum_{i \in I, i>0} (f(\cdot,0), \psi_i)^2 \\ &\quad + \int_0^t \sum_{i \in I, i<0} (f_t(\cdot,s), \psi_i)^2 \frac{e^{-2\lambda_i t}}{|\lambda_i|}\,ds + \int_0^t \sum_{i \in I, i>0} (f_t(\cdot,s), \psi_i)^2 \frac{1}{\lambda_i}\,ds. \end{aligned}$$

Applying Lemma 2.1 we obtain

$$
\begin{aligned}
&\sum_{i\in I\setminus\{0\}} f_i(t)^2\lambda_i^2 e^{-2\lambda_i t} \\
&\le C\Big(\|f(\cdot,t)\|^2_{L^2(D)} + e^{T^2}\|f(\cdot,0)\|^2_{\mathcal{L}^2(D)} + \int_0^t e^{T^2}\|f_t(\cdot,s)\|^2_{\mathcal{L}^2(D)}\,ds\Big) \\
&\le Ce^{T^2}\Big(\max_{t\in[0,T]}\|f(\cdot,t)\|^2_{L^2(D)} + \int_0^T \|f_t(\cdot,s)\|^2_{\mathcal{L}^2(D)}\,ds\Big) \\
&\le Ce^{T^2}\int_0^T \|f(\cdot,s)\|^2_{\mathcal{L}^2(D)} + \|f_t(\cdot,s)\|^2_{\mathcal{L}^2(D)}\,ds.
\end{aligned}
$$

For $\sum_{i\in I\setminus\{0\}} g_i(t)^2\lambda_i^2 e^{-2\lambda_i t}$ a similar estimate by the $H^1((0,T),\mathcal{L}^2(\partial D))$-norm of $g(x,t)$ holds. This show that the series on the right-hand side of (2.6) converges uniformly in t. As before Lebesgue's dominated convergence theorem implies that $u_b(\cdot,t)$ is continuous as a function from $[0,T]\to H^1(D)$. This finishes the proof of the theorem. □

3. Qualitative properties of eigenfunctions

The results of this section concern the simplicity of eigenvalues with eigenfunctions of constant sign and an upper estimate on the number of negative eigenvalues in the case of space dimension $N=1$. For the first purpose we need the following version of the Harnack inequality.

Lemma 3.1. *Suppose $D\subset\mathbb{R}^N$ is a bounded Lipschitz domain. Let $v\in H^1(D)$ be a weak solution of*

$$
-\operatorname{div}(A(x)\nabla v) = a(x)v \text{ in } D, \quad \nabla v^T A(x)n = b(x)v \text{ on } \partial D \tag{3.1}
$$

with $a\in L^\infty(D)$ and $b\in L^\infty(\partial D)$. If $v\ge 0$ in D then there exist constants $C_1,C_2>0$ depending only on A,a,b,D and N such that

$$
\sup_D v(x)\le C_1\|v\|_{L^2(D)},\quad \inf_D v(x)\ge C_2\|v\|_{L^2(D)}.
$$

In particular either $v\equiv 0$ or there exist $K,\delta>0$ such that $K\ge v\ge\delta>0$ a.e. in D and $K\ge \operatorname{trace} v\ge\delta>0$ a.e. on ∂D.

Proof. The proof is based on Moser's iteration method, cf. Gilbarg, Trudinger [9]. We will use the following interpolation inequality: there exists a constant $\tilde C=\tilde C(D)$ such that for every $\epsilon\in(0,1)$ we have

$$
\oint_{\partial D} z^2\,ds \le \frac{\tilde C}{\epsilon}\int_D z^2\,dx + \tilde C\epsilon\int_D |\nabla z|^2\,dx \text{ for every } z\in H^1(D). \tag{3.2}
$$

We begin with the upper estimate stated in the lemma. Let $L > 0$ be fixed and define $\varphi = v\min\{v^{2s}, L^2\}$ with $s > 0$. Then

$$\nabla v^T A(x)\nabla\varphi \geq \alpha|\nabla v|^2(\min\{v^{2s}, L^2\} + 2sv^{2s}\chi_{\{v^s\leq L\}}).$$

Since $\nabla(v\min\{v^s, L\}) = \nabla v\min\{v^s, L\} + sv^s\chi_{\{v^s\leq L\}}\nabla v$ we obtain

$$|\nabla(v\min\{v^s, L\})|^2 \leq \frac{(s+1)}{\alpha}\nabla v^T A(x)\nabla\varphi.$$

Taking φ as above as a test function in (3.1) and setting $\bar{v} = v\min\{v^s, L\}$ we obtain

$$\frac{1}{s+1}\int_D |\nabla\bar{v}|^2\,dx \leq C\Big(\int_D \bar{v}^2\,dx + \oint_D \bar{v}^2\,ds\Big), \tag{3.3}$$

where $C = C(\alpha, \|a\|_\infty, \|b\|_\infty)$. Here and in the following the same symbol C denotes different constants depending only on α, $\|a\|_\infty$ and $\|b\|_\infty$. By choosing $\epsilon = \frac{1}{2C\tilde{C}(s+1)}$ in the interpolation inequality (3.2) we obtain from (3.3)

$$\int_D |\nabla\bar{v}|^2\,dx \leq C(s+1)^2\int_D \bar{v}^2\,dx$$

and by adding the square of the L^2-norm of $\bar{v}$ on both sides and using the Sobolev-inequality we find

$$\|\bar{v}\|_{\frac{2n}{n-2}} \leq C(s+1)\|\bar{v}\|_2. \tag{3.4}$$

Provided $v \in L^{2(s+1)}(D)$ we can let L tend to infinity in (3.4) and obtain $v \in L^{\frac{(s+1)2n}{n-2}}(D)$ and

$$\|v\|_{(s+1)\frac{2n}{n-2}} \leq \big(C(s+1)\big)^{\frac{1}{s+1}}\|v\|_{2(s+1)}. \tag{3.5}$$

Hence, if $s_0 = 0$ and $s_{k+1} + 1 = (s_k + 1)\frac{n}{n-2}$ then

$$\|v\|_{2(s_{k+1}+1)} \leq \big(C(s_k+1)\big)^{\frac{1}{s_k+1}}\|v\|_{2(s_k+1)}.$$

Since $s_k + 1 = (\frac{n}{n-2})^k$, $k \in \mathbb{N}_0$ it follows that

$$\begin{aligned}
\|v\|_\infty = \lim_{k\to\infty}\|u\|_{2(s_{k+1}+1)} &\leq \prod_{k=0}^{\infty}\Big(C(s_k+1)\Big)^{\frac{1}{s_k+1}}\|v\|_2 \\
&= \exp\Big(\sum_{k=0}^{\infty}\frac{\ln C(s_k+1)}{s_k+1}\Big)\|v\|_2 \\
&\leq C\exp\Big(\sum_{k=0}^{\infty}k\big(\frac{n-2}{n}\big)^k\Big)\|v\|_2,
\end{aligned}$$

and since the last sum converges we have obtained the upper estimate of the lemma.

Now we turn to the lower estimate of the lemma. Let $\varphi = \bar{v}^s$ with $s < 0$ where $\bar{v} = v + L$ with $L > 0$. Then

$$\nabla v^T A(x) \nabla \varphi \leq s\alpha \bar{v}^{s-1} |\nabla \bar{v}|^2.$$

Taking φ as a test function in (3.1), we find

$$\begin{aligned} s\alpha \int_D \bar{v}^{s-1} |\nabla \bar{v}|^2 \, dx &\geq \int_D a^-(x) \bar{v}^{s+1} \, dx + \oint_{\partial D} b^-(x) \bar{v}^{s+1} \, ds \\ &\geq -C\Big(\int_D \bar{v}^{s+1} \, dx + \oint_{\partial D} \bar{v}^{s+1} \, ds \Big). \end{aligned} \tag{3.6}$$

If $s \neq -1$ we set $V = \bar{v}^{\frac{s+1}{2}}$ and obtain $|\nabla V|^2 = \big(\frac{s+1}{2}\big)^2 |\nabla \bar{v}|^2 \bar{v}^{s-1}$. If $s = -1$ then we set $V = \log \bar{v}$ and obtain $|\nabla V|^2 = \bar{v}^{-2} |\nabla \bar{v}|^2$. Together with (3.6) this implies

$$\int_D |\nabla V|^2 \, dx \leq \begin{cases} C|s+1| \Big(\int_D V^2 \, dx + \oint_{\partial D} V^2 \, ds \Big) & \text{if } s \neq -1, \\ C & \text{if } s = -1, \end{cases} \tag{3.7}$$

with $C = C(\alpha, \|a\|_\infty, \|b\|_\infty)$. Using the interpolation inequality (3.2) with $\epsilon = \frac{1}{2C\tilde{C}|s+1|}$ this implies

$$\int_D |\nabla V|^2 \, dx \leq C|s+1|^2 \int_D V^2 \, dx,$$

provided $|s+1| \geq |s_0 + 1| > 0$. Adding the square of the L^2-norm of V on both sides and using the Sobolev-inequality we get

$$\|V\|_{\frac{2n}{n-2}} \leq C|s+1| \|V\|_2. \tag{3.8}$$

For $p < 0$ let

$$\Phi(p) = \Big(\int_D \bar{v}^p \, dx \Big)^{1/p}.$$

Then (3.8) implies $\Phi\big((s+1)\frac{n}{n-2}\big)^{\frac{s+1}{2}} \leq C|s+1| \Phi(s+1)^{\frac{s+1}{2}}$, i.e.,

$$\Phi\Big((s+1)\frac{n}{n-2}\Big) \geq (C|s+1|)^{\frac{-2}{|s+1|}} \Phi(s+1). \tag{3.9}$$

This estimate will be iterated. Set $s_{k+1} + 1 = (s_k + 1)\frac{n}{n-2}$ with $s_1 < -1$. Then $s_k + 1 = (s_1 + 1)(\frac{n}{n-2})^{k-1}$ and

$$\Phi(s_{k+1} + 1) \geq (C|s_k + 1|)^{\frac{-2}{|s_k+1|}} \Phi(s_k + 1).$$

Solving this difference inequality it follows that

$$\begin{aligned}\inf_D \bar{v} \geq \lim_{k\to\infty} \Phi(s_{k+1}+1) &\geq \prod_{k=1}^{\infty} (C|s_k+1|)^{\frac{-2}{|s_k+1|}} \Phi(s_1+1) \\ &= \exp\Big(\sum_{k=1}^{\infty} \frac{-2\ln C|s_k+1|}{|s_k+1|}\Big)\Phi(s_1+1) \\ &\geq \frac{C}{\exp\big(\sum_{k=1}^{\infty}(k-1)\big(\frac{n-2}{n}\big)^{k-1}\big)}\Phi(s_1+1),\end{aligned}$$

and since the last sum converges we have obtained that

$$\inf_D \bar{v} \geq C\Phi(s_1+1) \tag{3.10}$$

for some initial number $s_1 < -1$, which we can still choose. It remains to give a lower bound for $\Phi(p)$ for some $p < 0$. For this purpose recall the John-Nirenberg inequality, cf. Gilbarg, Trudinger [9]: *suppose $V \in W^{1,1}(D)$ is such that there exists $C > 0$ with $\int_{B_r} |\nabla V|\,dx \leq Cr^{N-1}$ for every ball $B_r \subset D$. Then there exists a number $p_0 > 0$ such that $\int_D e^{p_0|V-\tilde{V}|}\,dx < C$ where $\tilde{V} = \frac{1}{|D|}\int_D V\,dx$.* We apply this for $V = \log \bar{v}$. Then the second inequality of (3.7) shows that $V \in W^{1,2}(D)$ and hence $\int_{B_r} |\nabla V|\,dx \leq C\|\nabla V\|_2 r^{N/2} \leq C\|\nabla V\|_2 r^{N-1}$ if $N \geq 2$. Thus, the John-Nirenberg inequality applies and together with the trivial estimate $\pm(V-\tilde{V}) \leq |V-\tilde{V}|$ we obtain

$$\int_D e^{p_0 V}\,dx \leq Ce^{p_0\tilde{V}}, \quad \int_D e^{-p_0 V}\,dx \leq Ce^{-p_0\tilde{V}},$$

i.e.,

$$\int_D e^{p_0 V}\,dx \int_D e^{-p_0 V}\,dx \leq C^2.$$

Recalling the definition of $V = \log \bar{v}$ this shows that $\int_D \bar{v}^{p_0}\,dx \int_D \bar{v}^{-p_0}\,dx \leq C^2$ and hence

$$\Big(\int_D \bar{v}^{p_0}\,dx\Big)^{1/p_0} \leq C^{2/p_0}\Big(\int_D \bar{v}^{-p_0}\,dx\Big)^{-1/p_0}.$$

Together with (3.10) this shows that

$$\inf_D \bar{v} \geq C\Phi(-p_0) \geq C'\|\bar{v}\|_{p_0}.$$

Letting $L \to 0$ we obtain the second claim of the lemma. □

In the next theorem we use the existence of the eigenvalues λ_1 and λ_{-1}. This is proved in Section 4. We also use the fact that if $\int_D q(x)\,dx > 0$

then the variational characterization as given in Section 4 guarantees that at least one corresponding eigenfunction has constant sign.

Theorem 3.1. *Assume $q \geq 0$.*

(i) If $\int_D q\,dx > 0$ then the eigenvalues λ_{-1} and λ_1 are simple and their eigenfunctions are of constant sign.
(ii) If $\lambda \in \mathbb{R}$ is an eigenvalue such that one eigenfunction is of constant sign, then $\lambda = \lambda_1$, $\lambda = 0$ or $\lambda = \lambda_{-1}$.

Proof. We begin by deriving inequality (3.11) below. Let ψ be an eigenfunction associated to an arbitrary eigenvalue μ and let φ be a non-negative eigenfunction associated with another eigenvalue $\lambda \neq \mu$. By Lemma 3.1 we find that $0 < \delta \leq \varphi \leq K$ in D. Hence ψ^2/φ is in $H^1(D)$ and can be used as a test function for the φ-equation. This implies

$$\begin{aligned}\int_D \nabla\varphi^T A(x)\frac{2\psi\varphi\nabla\psi - \psi^2\nabla\varphi}{\varphi^2}\,dx &= \int_D (\lambda - q(x))\psi^2\,dx + \oint_{\partial D} \sigma(x)\lambda\psi^2\,dx\\ &= (\lambda-\mu)a(\psi,\psi) + \int_D \nabla\psi^T A(x)\nabla\psi\,dx.\end{aligned}$$

Hence

$$0 \leq \int_D \Big(\frac{\psi}{\varphi}\nabla\varphi - \nabla\psi\Big)^T A(x)\Big(\frac{\psi}{\varphi}\nabla\varphi - \nabla\psi\Big)\,dx = (\mu-\lambda)a(\psi,\psi). \quad (3.11)$$

(i): Let ψ be an eigenfunction associated to λ_{-1}. The variational principle of Lemma 4.3 implies that $\varphi = |\psi|$ is also an eigenfunction to λ_{-1}. By (3.11) we have

$$\Big(\frac{\psi}{\varphi}\nabla\varphi - \nabla\psi\Big)^T A(x)\Big(\frac{\psi}{\varphi}\nabla\varphi - \nabla\psi\Big)\,dx = 0 \qquad \text{a.e. in } D$$

which proves that $\varphi = c\psi$ in D. Hence λ_{-1} is simple and by Lemma 3.1 the associated eigenfunction is bounded away from 0. The same argument works for λ_1.

(ii): If we choose $\mu = \lambda_1$ then it follows from $a(\psi,\psi) > 0$ and (3.11) that $\lambda \leq \lambda_1$. If we take $\mu = \lambda_{-1}$ then $a(\psi,\psi) < 0$ and (3.11) imply $\lambda \geq \lambda_{-1}$. Since there are no eigenvalues in $(\lambda_{-1}, 0)$ and $(0, \lambda_1)$ it follows that $\lambda \in \{\lambda_{-1}, 0, \lambda_1\}$. □

Theorem 3.2. *Assume $q \geq 0$ and $N = 1$. Then the eigenvalue problem (2.4) has at most two negative eigenvalues.*

The proof requires the following simple lemma.

Lemma 3.2. *Let $N = 1$ and $D = (0, L)$. Suppose $\lambda < 0$ is an eigenvalue of* (2.4) *with corresponding eigenfunction φ. Then φ has either no zero on $[0, L]$ or exactly one zero in $(0, L)$.*

Proof. Suppose φ has a first zero at $x_1 \in [0, L]$. Clearly x_1 cannot be at $0, L$, i.e., $x_1 \in (0, L)$. Assume further for contradiction that there is a second zero $x_2 \in (x_1, L)$ such that w.l.o.g $\varphi > 0$ in (x_1, x_2). Since $\lambda < 0$ we have $(A(x)\varphi')' > 0$ in (x_1, x_2), i.e,. $A(x)\varphi'(x) > A(x_1)\varphi'(x_1) > 0$ for all $x \in (x_1, x_2]$. But clearly at the next zero x_2 we must have $\varphi'(x_2) < 0$. This is impossible and thus we have a contradiction to the assumption that φ has two zeroes in $(0, L)$. □

Proof of Theorem 3.2: If $\lambda < 0$ is an eigenvalue then the associated eigenfunction either has no zero or exactly one zero. In the first case Theorem 3.1 shows that λ is unique. Now we show that also in the second case λ is unique. This is done as follows: assume $\lambda, \mu < 0$ are two negative eigenvalues such that the associated eigenfunctions φ, ψ have exactly one zero, i.e., $\varphi(x) = 0 = \psi(y)$ with $0 < x \leq y < L$. Note that[a]

$$\frac{1}{\mu} = \min_{u(y)=0} \frac{\int_0^y u^2\,dt + \sigma_1 u^2(0)}{\int_0^y A(t)u'^2 + q(t)u^2\,dt},$$

where the minimization is done in the $H^1(0, y)$-setting. After extending the function φ by zero on the interval $[x, y]$ it is an admissible function to put into the variational characterization. Thus we find

$$\frac{1}{\mu} \leq \frac{\int_0^y \varphi^2\,dt + \sigma_1\varphi^2(0)}{\int_0^y A(t)\varphi'^2 + q(t)\varphi^2\,dt} = \frac{\int_0^x \varphi^2\,dt + \sigma_1\varphi^2(0)}{\int_0^x A(t)\varphi'^2 + q(t)\varphi^2\,dt} = \frac{1}{\lambda},$$

i.e. $\mu \leq \lambda$. Similarly

$$\frac{1}{\lambda} = \min_{u(x)=0} \frac{\int_x^L u^2\,dt + \sigma_2 u^2(L)}{\int_x^L A(t)u'^2 + q(t)u^2\,dt}, \qquad u \in H^1(x, L).$$

After extending ψ by zero on the interval $[x, y]$ the function ψ is an admissible test function an yields as above $1/\lambda \leq 1/\mu$, i.e, $\lambda \leq \mu$. Hence $\lambda = \mu$, which implies that φ, ψ are linearly dependent and $x = y$. Hence

[a]This variational formulation follows in the same way as Lemma 4.3 (b) in Section 4.1 since due to the Dirichlet condition at y there is no zero-eigenvalue and hence no extra orthogonality condition.

the eigenvalue $\lambda = \mu$ with eigenfunctions having one sign-change is uniquely determined. Together with Lemma 3.2 this finished the proof. $\square$

4. Spectral theory

In this section we shall prove Theorem 2.1, Theorem 2.2 and Theorem 2.3 on the structure of the spectrum of (2.4). We are interested in functions $\sigma \in C(\partial D)$ such that $\sigma^- \not\equiv 0$. However, if $0 \le \sigma \in L^\infty(\partial D)$ then it follows immediately from the arguments given below that the eigenvalue problem (2.4) has countably many positive eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots$ such that $\lambda_k \to \infty$ as $k \to \infty$ and no other eigenvalues except $\lambda_0 = 0$ which only occurs if $q(x) \equiv 0$.

Frequently in this section we use the following well known result, cf. Alt [1]:

Lemma 4.1. *If $\mathcal{V}$ is a closed subspace of $H^1(D)$ not containing the constants, then $(\int_D \nabla u^T A(x) \nabla u\, dx)^{1/2}$ is an equivalent norm on $\mathcal{V}$. In particular, there exist constants $C_1, C_2 > 0$ such that for all $v \in \mathcal{V}$:*

$$\int_D v^2\, dx \le C_1 \int_D \nabla v^T A(x) \nabla v\, dx, \quad \oint_{\partial D} v^2\, ds \le C_2 \int_D \nabla v^T A(x) \nabla v\, dx. \tag{4.1}$$

4.1. *The case* $q(x) \ge 0, \int_D q\, dx > 0$

In this case the form $\langle \cdot, \cdot \rangle$ generates the norm $(\int_D \nabla v^T A(x) \nabla v + q(x) v^2\, dx)^{1/2}$, which is equivalent to the standard norm of $H^1(D)$. To see this note first that by Lemma 4.1

$$\left\| v - \frac{\int_D vq\, dx}{\int_D q\, dx} \right\|^2_{H^1(D)} \le C \int_D \nabla v^T A(x) \nabla v\, dx \tag{4.2}$$

since the space $\{v \in H^1(D) : \int_D vq\, dx = 0\}$ does not contain the constants. It follows from (4.2) that $\|v\|^2_{H^1(D)} \le C(\int_D \nabla v^T A(x) \nabla v + q(x) v^2\, dx)$. Now we can describe the eigenvalues of (2.4) as eigenvalues of a compact operator as follows.

Lemma 4.2. *For $h \in H^1(D)$ there exists a unique $v \in H^1(D)$ such that*

$$-\operatorname{div}(A(x)\nabla v) + q(x) v = h \text{ in } D, \quad \nabla v^T A(x) n = \sigma(x) h \text{ on } \partial D. \tag{4.3}$$

The operator

$$K : \begin{cases} H^1(D) \to H^1(D) \\ h \qquad \mapsto v \end{cases}$$

is compact, invertible and self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$. *Hence it has countably many eigenvalues* $\{\mu_k\}_{k \in I}$ *and the eigenfunctions form a complete system in* $H^1(D)$. *The eigenvalues of (2.4) are the reciprocals* $\lambda_i = \mu_i^{-1}$.

Proof. For $h \in H^1(D)$ the functional $L_h : H^1(D) \to \mathbb{R}$ given by $L_h(\phi) = \int_D h\phi \, dx + \oint_{\partial D} \sigma(x) h\phi \, ds$ is continuous and hence by the Riesz-representation theorem there exists a unique $v \in H^1(D)$ such that $\langle v, \phi \rangle = L_h(\phi)$ for all $\phi \in H^1(D)$. Thus v is the weak solution of (4.3) and the operator K is well defined. Continuity and compactness of K are standard and invertibility and symmetry are immediate. □

Remark. The following is a more general version of Lemma 4.2. Let $\mathcal{W} = \{(f, g) \in L^2(D) \times L^2(\partial D)\}$ be equipped with the norm $\|(f, g)\| = (\|f\|^2_{L^2(D)} + \|g\|^2_{L^2(\partial D)})^{1/2}$. Then for every $(f, g) \in \mathcal{W}$ there exists a unique $v \in H^1(D)$ such that $-\operatorname{div}(A(x)\nabla v) + q(x)v = f$ in D, $\nabla v^T A(x) n = \sigma(x) g$ on ∂D. The corresponding solution operator $T : (f, g) \to v$ from $\mathcal{W}$ to $H^1(D)$ is compact.

As a consequence of Lemma 4.2 eigenvalues of (2.4) can be described variationally as critical values of the functional

$$J(v) := a(v, v) = \int_D v^2 \, dx + \oint_{\partial D} \sigma(x) v^2 \, ds$$

in the set $\{v \in H^1(D) : \int_D \nabla v^T A(x) \nabla v + q(x) v^2 \, dx = 1\}$. The following variational characterization of the eigenvalues of (2.4) is well known, see e.g. De Figueiredo [6]:

Lemma 4.3. *Suppose that*

$$\mu_1 = \sup\{J(v) : \langle v, v \rangle = 1\} > 0$$

and

$$\mu_{-1} = \inf\{J(v) : \langle v, v \rangle = 1\} < 0.$$

Then $\lambda_1 = \mu_1^{-1}$, $\lambda_{-1} = \mu_{-1}^{-1}$ *are the first positive, negative eigenvalues of (2.4). Moreover the following holds:*

(a) *Let $k \in \mathbb{N}$. Suppose $0 < \lambda_1 \leq \ldots \leq \lambda_k$ are the (not necessarily different) first k positive eigenvalues with eigenfunctions $\psi_1, \ldots, \psi_k$. Suppose that*

$$\mu_{k+1} = \sup\Big\{J(v) : \langle v, v\rangle = 1, a(\psi_j, v) = 0, j = 1, \ldots, k\Big\} > 0.$$

Then $\lambda_{k+1} = \mu_{k+1}^{-1}$ is the next positive eigenvalue.

(b) *Let $k \in \mathbb{N}$. Suppose $\lambda_{-k} \leq \ldots \leq \lambda_{-1} < 0$ are the (not necessarily different) first k negative eigenvalues with eigenfunctions $\psi_{-k}, \ldots, \psi_{-1}$. Suppose that*

$$\mu_{-k-1} = \inf\Big\{J(v) : \langle v, v\rangle = 1, a(\psi_j, v) = 0, j = -k, \ldots, -1\Big\} < 0.$$

Then $\lambda_{-k-1} = \mu_{-k-1}^{-1}$ is the next negative eigenvalue.

It is easy to see that the critical values μ_j, μ_{-j} are attained provided they are positive, negative, resp.

Theorem 4.1. *Problem (2.4) has an unbounded sequence of positive eigenvalues.*

(a) *If $N \geq 2$ then (2.4) has an unbounded sequence of negative eigenvalues.*

(b) *Let $N = 1$. If $\sigma_1 \cdot \sigma_2 > 0$ then (2.4) has exactly two negative eigenvalues. If $\sigma_1 \cdot \sigma_2 \leq 0$ then (2.4) has exactly one negative eigenvalue. The multiplicity is always one.*

Proof. For any function $v \in H_0^1(D)$ we find $J(v) > 0$ since the boundary integral vanishes. Thus we see that $\mu_1 > 0$ is attained. Now it suffices to show that for any $k \in \mathbb{N}$ there exists a trial function v such that $a(\psi_j, v) = 0$ for $j = 1, \ldots, k$ and $J(v) > 0$. Such a choice is always possible in any $(k+1)$-dimensional subspace of $H_0^1(D)$.

Part (a): We need to show that $\mu_{-k-1} < 0$ for all $k \in \mathbb{N}_0$. For this purpose we construct a function v such that $a(v, \psi_j) = 0$ for $j = -k, \ldots, -1$. Let $Q \in \partial D$ be a point where $\sigma(Q) < 0$. After rotating and shifting D we may assume that locally near Q the set ∂D is described as the graph $(x', \eta(x'))$ of a Lipschitz function $\eta : B_\epsilon(0) \subset \mathbb{R}^{N-1} \to \mathbb{R}$ where the ball $\overline{B}_\epsilon(0)$ is so small that $\sigma(x', \eta(x')) < 0$ for all $x' = (x_1, \ldots, x_{N-1}) \in \overline{B}_\epsilon(0)$. Moreover, we may assume that $\eta > 0$ on $\overline{B}_\epsilon(0)$ and that the cylindrical piece

$$C = \{(x', t) : x' \in B_\epsilon(0), 0 \leq t < \eta(x')\}$$

lies entirely in D. Next we partition $B_\epsilon(0)$ into $k+1$ pairwise disjoint open sets $D'_1, \dots, D'_{k+1}$ such that $\overline{B}_\epsilon(0) = \bigcup_{i=1}^{k+1} \overline{D}'_i$. Then

$$\overline{C} = \bigcup_{i=1}^{k+1} \overline{D}_i \text{ with } D_i = \{(x', t) : x' \in D'_i, 0 \le t < \eta(x')\}.$$

Next let $0 \not\equiv g_i \in C_0^\infty(D'_i)$, $i = 1, 2 \dots k+1$ and fix $\alpha > 0$. Define $k+1$ functions on the cylindrical piece C by

$$v_i(x) = \begin{cases} x_N^\alpha g_i(x') & \text{if } x = (x', x_N) \in D_i, \\ 0 & \text{else} \end{cases}$$

for $i = 1, \dots, k+1$ and extend theses function by zero to all of D. Let $v = \sum_{i=1}^{k+1} c_i v_i$ and determine the vector $c = (c_1, \dots, c_{k+1})$ from the condition $a(\psi_j, v) = 0$ for $j = -k, \dots, -1$. These k conditions are represented by the linear system

$$\sum_{i=1}^{k+1} c_i a(\psi_j, v_i) = 0 \text{ for } j = -k, \dots, -1,$$

where in case $k = 0$ there are no conditions on the value c. Since the linear system consists of k equations in $k+1$ unknowns $(c_1, \dots, c_{k+1})$ we have at least a one-dimensional space of non-trivial solutions $0 \neq c = (c_1, \dots, c_{k+1}) \in \mathbb{R}^{k+1}$. Setting $e_N = (0, \dots, 0, 1)^T$ we have

$$\begin{aligned} \int_{D_i} v^2\,dx &= c_i^2 \int_{D_i} x_N^{2\alpha} g_i(x')^2\,dx \\ &= \frac{c_i^2}{2\alpha+1} \int_{D_i} \nabla \cdot \left(x_N^{2\alpha+1} g_i(x')^2 e_N \right) dx \\ &= \frac{c_i^2}{2\alpha+1} \oint_{\partial D_i} x_N^{2\alpha+1} g_i(x')^2 e_N \cdot n\,ds \\ &\le \frac{\operatorname{diam} D}{2\alpha+1} \oint_{\partial D_i} v^2\,ds. \end{aligned}$$

Taking into account that $v = 0$ on $\partial D_i \cap D$ we obtain by superposition

$$\int_D v^2\,dx = \int_C v^2\,dx \le \frac{\operatorname{diam} D}{2\alpha+1} \oint_{\partial C \cap \partial D} v^2\,ds = \frac{\operatorname{diam} D}{2\alpha+1} \oint_{\partial D} v^2\,ds.$$

If $\alpha > 0$ is so large that $-\frac{\operatorname{diam} D}{2\alpha+1} > \sigma(x)$ for all $x \in \partial C \cap \partial D$ then $J(v) = \int_D v^2\,dx + \oint_{\partial D} \sigma(x) v^2\,ds < 0$. The remaining degree of freedom is the multiple of c which is chosen such that $\int_D |\nabla v|^2 + q(x) v^2\,dx = 1$. This shows that $\mu_{-k-1} < 0$.

Part (b): Let $D = (0, L)$. Let us first consider the case $\sigma_1, \sigma_2 < 0$. By Theorem 3.2 we know that (2.4) has at most two negative eigenvalues. It remains to show that there are at least two negative eigenvalues. For λ_{-1} it suffices to construct a function v such that $J(v) = \int_0^L v^2\,dx + \sigma_1 v^2(0) + \sigma_2 v^2(L) < 0$. This is achieved by $v(x) = x^\alpha$ with sufficiently large α. For λ_{-2} one needs a function v such that $J(v) < 0$ and $a(v, \psi_{-1}) = \int_0^L v\psi_{-1}\,dx + \sigma_1 v(0)\psi_{-1}(0) + \sigma_2 v(L)\psi_{-1}(L) = 0$. This can be obtained by the function

$$v(x) = \begin{cases} a_1|x - L/2|^\alpha & \text{on } [0, L/2], \\ a_2|x - L/2|^\alpha & \text{on } [L/2, L]. \end{cases}$$

For sufficiently large α the functional is negative independent of the choice of a_1, a_2. By choosing a_1, a_2 appropriately one can achieve $a(v, \psi_{-1}) = 0$.

Now we turn to the case $\sigma_1 < 0$ and $\sigma_2 \geq 0$. By Theorem 3.2 there are at most two negative eigenvalues. Let us show that there is no sign-changing eigenfunction corresponding to a negative eigenvalue. If such an eigenfunction ψ existed then we could assume $\psi(0) > 0$ and $\psi(L) < 0$. Then

$$-A(0)\psi'(0) = \sigma_1\lambda\psi(0) > 0, \quad A(L)\psi'(L) = \sigma_2\lambda\psi(L) \geq 0,$$

i.e. $A(0)\psi'(0) < 0 \leq A(L)\psi'(L)$. But $A(x)\psi'(x)$ is decreasing where ψ is negative, i.e., $A(x)\psi'(x) < 0$ where ψ is negative. This is a contradiction. Thus, there is a most one negative eigenvalue. The existence of one negative eigenvalue is constructed as in the case $\sigma_1, \sigma_2 < 0$. Making use of $\sigma_1 < 0$ one utilizes the test-function $|L - x|^\alpha$ for sufficiently large $\alpha > 0$. This finishes the proof of Part (b). □

4.2. *The case $q(x) \equiv 0$*

Again we want to apply the theory of compact self-adjoint operators in order to describe the eigenvalues of (2.4). Now $\lambda = 0$ is an eigenvalue. Therefore we need the Hilbert-space $\mathcal{V} = \{v \in H^1(D) : a(v, 1) = 0\}$ with $a(u, v) = \int_D uv\,dx + \oint_{\partial D} \sigma(x)uv\,ds$. Recall that $\bar{\sigma} = \frac{1}{|\partial D|}\oint_{\partial D} \sigma(x)\,ds$ and $\sigma_0 = -|D|/|\partial D|$.

In the case $\bar{\sigma} \neq \sigma_0$ the space $\mathcal{V}$ does not contain the constants and hence $(\int_D \nabla v^T A(x) \nabla v\,dx)^{1/2}$ is an equivalent norm on $\mathcal{V}$. Every solution of (2.4) except the constants belongs to $\mathcal{V}$.

However, if $\bar{\sigma} = \sigma_0$ then the constants do belong to $\mathcal{V}$. We must therefore change the setting and define a proper subspace $\mathcal{V}_w$ of $\mathcal{V}$ as follows.

Let $\mathcal{V}_w = \{v \in \mathcal{V} : a(v,w) = 0\}$ where w is a solution of the problem $-\operatorname{div}(A(x)\nabla w) = 1$ in D, $\nabla w^T A(x) n = \sigma(x)$ on ∂D. The constants do not belong to $\mathcal{V}_w$ and $(\int_D \nabla v^T A(x) \nabla v \, dx)^{1/2}$ is an equivalent norm on $\mathcal{V}_w$. The choice of w may seem arbitrary. In Section 5 we show why no other choice for w is possible.

Lemma 4.4.

(i) *Let $\bar\sigma \neq \sigma_0$. For any $h \in \mathcal{V}$ there exists a unique $v \in \mathcal{V}$ such that*

$$-\operatorname{div}(A(x)\nabla v) = h \text{ in } D, \quad \nabla v^T A(x) n = \sigma h \text{ on } \partial D. \tag{4.4}$$

The operator

$$K : \begin{cases} \mathcal{V} \to \mathcal{V} \\ h \mapsto v \end{cases}$$

is compact, invertible and self-adjoint with respect to the inner product $\langle \cdot, \cdot \rangle$. Hence it has countably many eigenvalues $\{\mu_k\}_{k \in I}$ and the eigenfunctions form a complete system in $\mathcal{V}$. The eigenvalues of (2.4) except $\lambda_0 = 0$ are the reciprocals $\lambda_i = \mu_i^{-1}$.

(ii) *The same holds in the case $\bar\sigma = \sigma_0$ if $\mathcal{V}$ is replaced by $\mathcal{V}_w$.*

Proof. We give the proof in the "resonance"-case $\bar\sigma = \sigma_0$. In the "non-resonance"-case $\bar\sigma \neq \sigma_0$ the same proof works by formally setting $w = 0$ in all of the following. For given $h \in \mathcal{V}_w$ the functional $L_h : \mathcal{V}_w \to \mathbb{R}$ given by $L_h(\phi) = \int_D h\phi \, dx + \oint_{\partial D} \sigma(x) h \phi \, ds$ is continuous and hence by the Riesz-representation theorem there exists a unique $v \in \mathcal{V}_w$ such that $\langle v, \phi \rangle = L_h(\phi)$ for all $\phi \in \mathcal{V}_w$. We want to deduce that v is a weak solution of (4.4). Since $H^1(D) = \mathcal{V}_w \oplus \operatorname{span}[1, w]$ this follows once we show that

$$\langle v, \phi \rangle = L_h(\phi) \quad \forall \phi \in \operatorname{span}[1, w]. \tag{4.5}$$

The right-hand side $L_h(\phi) = a(\phi, h)$ in (4.5) vanishes for $\phi \in \{1, w\}$ by the assumption $h \in \mathcal{V}_w$. For $\phi = 1$ also the left-hand side of (4.5) vanishes. It remains to compute $\langle v, w \rangle$. Since w weakly solves the equation $-\operatorname{div}(A(x)\nabla w) = 1$ in D and $\nabla w^T A(x) n = \sigma(x)$ on ∂D we find $\int_D \nabla w^T A(x) \nabla v \, dx = \int_D v \, dx + \oint_{\partial D} \sigma(x) v \, ds = 0$ by definition of $\mathcal{V}_w$. Hence the operator K is well defined. Continuity and compactness of K are again standard, and so are invertibility and symmetry. □

Remark. There is a more general version of Lemma 4.4. If $\bar\sigma \neq \sigma_0$ then let $\mathcal{W} = \{(f, g) \in L^2(D) \times L^2(\partial D) : \int_D f \, dx + \int_D \sigma(x) g \, ds = 0\}$ with the norm

$\|(f,g)\| = (\|f\|^2_{L^2(D)} + \|g\|^2_{L^2(\partial D)})^{1/2}$. For every $(f,g) \in \mathcal{W}$ there exists a unique $v \in \mathcal{V}$ such that $-\operatorname{div}(A(x)\nabla v) = f$ in D, $\nabla v^T A(x) n = \sigma(x) g$ on ∂D. The corresponding solution operator $T : (f,g) \to v$ from $\mathcal{W}$ to $\mathcal{V}$ is compact. If $\bar\sigma = \sigma_0$ then the same result holds if $\mathcal{W}$, $\mathcal{V}$ are replaced by $\mathcal{W}_w$, $\mathcal{V}_w$, where $\mathcal{W}_w = \{(f,g) \in \mathcal{W} : \int_D fw\,dx + \oint_{\partial D} \sigma(x) gw\,ds = 0\}$.

Since $\lambda_0 = 0$ is an eigenvalue with the constants as eigenfunctions, the variational description of the eigenvalues of (2.4) differs slightly from the one given in the case $q(x) \geq 0, \not\equiv 0$. The eigenvalues except 0 are critical values of

$$J(v) := a(v,v) = \int_D v^2\,dx + \sigma \oint_{\partial D} v^2\,ds$$

with respect to the set $\{v \in \mathcal{V} : \int_D \nabla v^T A(x) \nabla v\,dx = 1\}$ or $\{v \in \mathcal{V}_w : \int_D \nabla v^T A(x) \nabla v\,dx = 1\}$. The following variational characterization is standard, see e.g. De Figueiredo [6]:

Lemma 4.5. *Assume $\bar\sigma \neq \sigma_0$. Suppose that*

$$\mu_1 = \sup\{J(v) : v \in \mathcal{V}, \langle v,v\rangle = 1\} > 0$$

and

$$\mu_{-1} = \inf\{J(v) : v \in \mathcal{V}, \langle v,v\rangle = 1\} < 0.$$

Then $\lambda_1 = \mu_1^{-1}$, $\lambda_{-1} = \mu_{-1}^{-1}$ are the first positive, negative eigenvalues of (2.4). Moreover the following holds:

(a) Let $k \in \mathbb{N}$. Suppose $0 = \lambda_0 < \lambda_1 \ldots \leq \lambda_k$ are the (not necessarily different) first $k+1$ non-negative eigenvalues with eigenfunctions $\psi_0, \ldots, \psi_k$. Suppose that

$$\mu_{k+1} = \sup\left\{J(v) : v \in \mathcal{V}, \langle v,v\rangle = 1, a(\psi_j, v) = 0, j = 1,\ldots,k\right\} > 0.$$

Then $\lambda_{k+1} = \mu_{k+1}^{-1}$ is the next positive eigenvalue.

(b) Let $k \in \mathbb{N}$. Suppose $\lambda_{-k} \leq \ldots \leq \lambda_{-1} < \lambda_0 = 0$ are the (not necessarily different) first $k+1$ non-positive eigenvalues with eigenfunctions $\psi_{-k}, \ldots, \psi_0$. Suppose that

$$\mu_{-k-1} = \inf\left\{J(v) : v \in \mathcal{V}, \langle v,v\rangle = 1, a(\psi_j, v) = 0, j = -k,\ldots,-1\right\} < ($$

Then $\lambda_{-k-1} = \mu_{-k-1}^{-1}$ is the next negative eigenvalue.

The same holds in the case $\bar\sigma = \sigma_0$ if $\mathcal{V}$ is replaced by $\mathcal{V}_w$.

Since $\langle v, v\rangle^{1/2}$ is an equivalent norm on $\mathcal{V}$, $\mathcal{V}_w$ any sequence of extremal functions is bounded in the full H^1-norm. Provided $\mu_j > 0$, $\mu_{-j} < 0$ it is easy to see that these values are attained.

Theorem 2.2 and Theorem 2.3 are implied by the following result.

Theorem 4.2. *Problem (2.4) has an unbounded sequence of positive eigenvalues.*

(a) If $N \geq 2$ then (2.4) has an unbounded sequence of negative eigenvalues.

(b) Let $N = 1$, $D = (0, L)$ and $\sigma_1 \cdot \sigma_2 > 0$. If $-L/2 < \bar{\sigma}$ then (2.4) has exactly two negative eigenvalues. If $\bar{\sigma} \leq -L/2$ then (2.4) has exactly one negative eigenvalue.

(c) Let $N = 1$, $D = (0, L)$ and $\sigma_1 \cdot \sigma_2 \leq 0$. If $-L/2 < \bar{\sigma}$ then (2.4) has exactly one negative eigenvalue. If $\bar{\sigma} \leq -L/2$ then there is no negative eigenvalue.

Proof. (a) The proof is based on the variational characterization of Lemma 4.5. It is almost identical with the proof of Theorem 4.1.

(b) By Theorem 3.2 there are at most two negative eigenvalues. Theorem 5.1 of the last section shows that a negative eigenvalue with corresponding eigenfunction of one sign exists only for $-L/2 < \bar{\sigma}$. The assertion of the theorem will follow provided it is possible to establish for all values of $\bar{\sigma}$ the existence of a negative eigenvalue where the corresponding eigenfunction has exactly one sign-change. This eigenvalue is obtained as follows. Fix $y \in (0, L)$ and determine the first negative eigenvalue $\lambda_{-1}(0, y)$ of

$$-(A(x)\varphi')' = \lambda\varphi \text{ in } (0, y) \text{ with } -A(0)\varphi'(0) = \sigma_1\lambda\varphi(0), \quad \varphi(y) = 0.$$

Similarly let $\lambda_{-1}(y, L)$ be the first negative eigenvalue of

$$-(A(x)\varphi')' = \lambda\varphi \text{ in } (y, L) \text{ with } \varphi(y) = 0, \quad A(L)\varphi'(L) = \sigma_2\lambda\varphi(L).$$

It is easy to see that $\lambda_{-1}(0, y)$ and $\lambda_{-1}(y, L)$ are both continuous in y and that $\lim_{y\to 0} \lambda_{-1}(0, y) = \infty$ and $\lim_{y\to L} \lambda_{-1}(y, L) = \infty$. Hence there exists a point $y_0 \in (0, L)$ such that $\lambda_{-1}(0, y_0) = \lambda_{-1}(y_0, L)$. This proves the existence of a negative eigenvalue of (2.4) where the corresponding eigenfunction has exactly one sign-change.

(c) The fact that σ_1 and σ_2 have opposite signs excludes the possibility of negative eigenvalues with sign-changing eigenfunctions. Therefore the question reduces to the existence/non-existence of eigenfunctions of one-sign. This is completely described by Theorem 5.1 of the last section. $\square$

Remark. Note that in contrast to Theorem 3.1 we do not claim that the eigenfunctions associated with λ_{-1}, λ_1 have constant sign. In fact the properties of λ_{-1}, λ_1 depend on the value of $\bar\sigma$ and change near the critical value $\sigma_0 = -|D|/|\partial D|$, see Corollary 5.1 below:

$$\begin{array}{lll} \bar\sigma > \sigma_0 & \Rightarrow & \lambda_{-1} \text{ simple, } \psi_{-1} \text{ has constant sign, } \psi_1 \text{ sign-changing,} \\ \bar\sigma < \sigma_0 & \Rightarrow & \lambda_1 \text{ simple, } \psi_1 \text{ has constant sign, } \psi_{-1} \text{ sign-changing,} \\ \bar\sigma = \sigma_0 & \Rightarrow & \psi_{-1}, \psi_1 \text{ are both sign-changing.} \end{array}$$

5. Eigenvalues in the resonance case

As we have seen the resonance case $q \equiv 0$ and $\bar\sigma = \sigma_0 = -|D|/|\partial D|$ displays special spectral properties, that are discussed in detail in this section.

5.1. *The choice of the space $\mathcal{V}_w$*

Suppose one wants so solve

$$-\operatorname{div}(A(x)\nabla v) = h \text{ in } D, \quad \nabla v^T A(x) n = \sigma(x) h \text{ on } \partial D \tag{5.1}$$

for $h \in H^1(D)$. Then necessarily $h \in \mathcal{V} = \{v \in H^1(D) : a(1,v) = 0\}$, where $a(u,v) = \int_D uv\,dx + \oint_{\partial D} \sigma(x) uv\,ds$. The next lemma explains why in the resonance case $\bar\sigma = \sigma_0$ one has to choose h with the extra condition $a(w,h) = 0$ in order to obtain $v \in \mathcal{V}$. The only possible choice for w is a solution of $-\operatorname{div}(A(x)\nabla w) = 1$ in D and $\nabla w^T A(x) n = \sigma_0$ on ∂D.

Lemma 5.1. *Let $h \in \mathcal{V}$ and let $\sigma \in L^\infty(\partial D)$ be an arbitrary function. Then there exists a one-parameter family $\mathcal{S} = \{v_0 + \gamma\}_{\gamma \in \mathbb{R}} \subset H^1(D)$ of solutions of (5.1).*

Proof. Let $h \in \mathcal{V}$ and define $\bar h = |D|^{-1} \int_D h\,dx$.
Case 1: Assume $\bar\sigma \neq 0$. Let $a, b, c \in H^1(D)$ be solutions of

$$(A) \begin{cases} -\operatorname{div}(A(x)\nabla a) = h - \bar h \text{ in } D, \\ \quad \nabla a^T A(x) n = 0 \quad \text{ on } \partial D, \end{cases}$$

$$(B) \begin{cases} -\operatorname{div}(A(x)\nabla b) = \bar h \quad \text{ in } D, \\ \quad \nabla b^T A(x) n = \dfrac{\sigma_0}{\bar\sigma} \sigma(x) \bar h \text{ on } \partial D, \end{cases}$$

$$(C) \begin{cases} -\operatorname{div}(A(x)\nabla c) = 0 \quad \text{ in } D, \\ \quad \nabla c^T A(x) n = \sigma(x) h - \dfrac{\sigma_0}{\bar\sigma} \sigma(x) \bar h \text{ on } \partial D. \end{cases}$$

Solutions for (A) and (B) exist for every $h \in H^1(D)$ whereas the solution of (C) only exists if additionally $a(1,h) = 0$. Moreover, a, b, c are unique up to additive constants. Finally $v_0 = a + b + c$ solves (5.1).
Case 2: Assume $\bar{\sigma} = 0$. Let $a, b \in H^1(D)$ be solutions of

$$(A)\begin{cases} -\operatorname{div}(A(x)\nabla a) = h - \bar{h} \text{ in } D, \\ \nabla a^T A(x) n = \bar{h}\sigma(x) \text{ on } \partial D, \end{cases}$$

$$(B)\begin{cases} -\operatorname{div}(A(x)\nabla b) = \bar{h} & \text{in } D, \\ \nabla b^T A(x) n = \sigma(x)(h - \bar{h}) & \text{on } \partial D. \end{cases}$$

Solutions for (A) exist for every $h \in H^1(D)$ whereas the solution of (B) only exists if additionally $a(1,h) = 0$. As before a, b are unique up to additive constants and $v_0 = a + b$ solves (5.1). □

Lemma 5.2. *Let $h \in \mathcal{V}$.*

(i) There exists a unique element in $\mathcal{S} \cap \mathcal{V}$ if and only if $\bar{\sigma} \neq \sigma_0$.
(ii) Let $\bar{\sigma} = \sigma_0$. Then $\mathcal{S} \subset \mathcal{V}$ if and only if $h \in \mathcal{V}_w = \{h \in \mathcal{V} : a(w,h) = 0\}$. Furthermore, if $h \in \mathcal{V}_w$ then there exists a unique element in $\mathcal{S} \cap \mathcal{V}_w$.

Proof. (i): A unique solution $v_0 + \gamma \in \mathcal{V}$ can be selected provided $a(1,\gamma) \neq 0$. This is the case if only if $\bar{\sigma} \neq \sigma_0$.

(ii): We use the notation of Case 1 of the previous lemma. If w is the solution of $-\operatorname{div}(A(x)\nabla w) = 1$ in D, $\nabla w^T A(x) n = \sigma(x)$ on ∂D then $b = \bar{h}w$. Testing the equation for w with c and rearranging terms one finds

$$\int_D c\,dx + \oint_{\partial D} \sigma(x) c\,ds = \oint_{\partial D} \sigma(x)(h - \bar{h}) w\,ds,$$

and likewise by testing with a one obtains

$$\int_D a\,dx + \oint_{\partial D} \sigma(x) a\,ds = \int_D (h - \bar{h}) w\,dx.$$

Hence, the condition $v \in \mathcal{V}$ reads

$$0 = \int_D a + b + c\,dx + \oint_{\partial D} \sigma(x)(a + b + c)\,ds = \int_D hw\,dx + \oint_{\partial D} \sigma(x) hw\,ds,$$

i.e., one needs the additional condition $a(w,h) = 0$. Uniqueness of the solution in the space $\mathcal{V}_w$ holds provided $a(1,w) \neq 0$. This is true since $a(1,w) = \int_D \nabla w^T A(x) \nabla w\,dx$. □

5.2. *Behavior of λ_{-1}, λ_1 near $\bar{\sigma} = \sigma_0$*

We consider dynamical coefficients $\sigma \in C(\partial D)$, where $C(\partial D)$ is equipped with the maximum-norm. If σ is close to resonance then the consequences for the principal eigenvalues and eigenfunctions are described as follows.

Theorem 5.1. *There exists $\epsilon > 0$ and a C^1-map $\sigma \mapsto (\lambda(\sigma), v(\sigma))$ for $\sigma \in B_\epsilon(\sigma_0) \subset C(\partial\Omega)$ with values in $\mathbb{R} \times H^1(D)$ such that $(\lambda(\sigma), v(\sigma))$ is an eigenpair for the eigenvalue problem (2.4) with the properties $\int_D v(\sigma)\,dx = |D|$ and*

$$\lambda(\sigma) = \frac{-|\partial D|}{\int_D \nabla w^T A(x) \nabla w\,dx}(\bar{\sigma} - \sigma_0) + O(\|\sigma - \sigma_0\|)^2,$$

$$v(\sigma) = 1 - \frac{|\partial D|(w - \bar{w})}{\int_D \nabla w^T A(x) \nabla w\,dx}(\bar{\sigma} - \sigma_0) + O(\|\sigma - \sigma_0\|)^2.$$

Moreover, if (λ, v) is an eigenpair of (2.4) with $\|\sigma - \sigma_0\| < \epsilon$, $|\lambda| < \epsilon$ and $v > 0$, $\int_D v\,dx = |D|$ then either (λ, v) lies on the curve or $(\lambda, v) = (0, 1)$.

Remark. Note that $\bar{\sigma} > \sigma_0$ implies $\lambda(\sigma) < 0$ and $\bar{\sigma} < \sigma_0$ implies $\lambda(\sigma) > 0$. Hence $\lambda(\sigma)$ parameterizes λ_{-1} if $\bar{\sigma} > \sigma_0$ and λ_1 if $\bar{\sigma} < \sigma_0$. It shows how λ_{-1} passes through 0 and becomes λ_1 as $\bar{\sigma}$ passes through the critical value σ_0, see Figure 1. The positivity of the eigenfunction is also passed on from ψ_{-1} to ψ_1. Note that the min-max principle implies that the eigenvalues are monotone decreasing in σ with respect to the natural ordering in $C(\partial D)$.

Proof. Consider the normalized eigenvalue problem

$$(P) \quad -\operatorname{div}(A\nabla v) = \lambda v \text{ in } D,\ \nabla v^T A(x) n = \sigma(x)\lambda v \text{ on } \partial D,\ \int_D v\,dx = |D|.$$

In the following we describe the solutions of (P) as the zero-set of a nonlinear function $F(\sigma, \lambda, v)$ where $F : C(\partial D) \times \mathbb{R} \times H^1(D) \to \mathbb{R} \times H^1(D)$.

Construction of the C^1-map: Define the operator

$$T : \begin{cases} C(\partial D) \times \mathbb{R} \times H^1(D) \to H^1(D), \\ (\sigma, \lambda, v) \qquad\qquad\qquad \mapsto T(\sigma, \lambda, v) := \xi, \end{cases}$$

where ξ is the unique solution of

$$\begin{aligned} &-\operatorname{div}(A(x)\nabla\xi) = \lambda v \text{ in } D, \\ &\quad \nabla\xi^T A(x) n = \sigma(x)\lambda v - \tfrac{\lambda}{|\partial D|}\left(\textstyle\int_D v\,dx + \oint_{\partial D} \sigma(x) v\,ds\right) \text{ on } \partial D, \\ &\quad \textstyle\int_D \xi\,dx = |D|. \end{aligned} \tag{5.2}$$

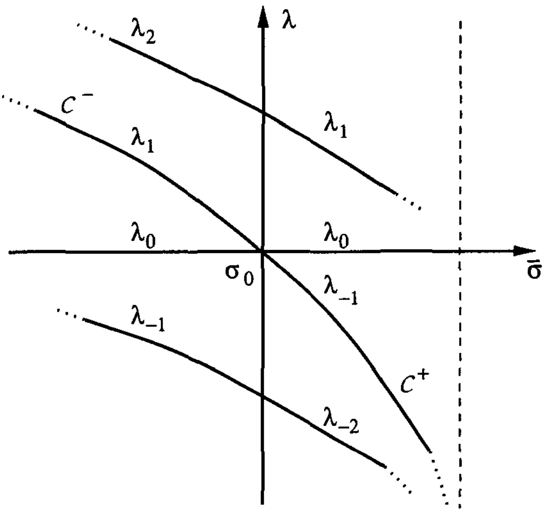

Figure 1. Eigenvalues as functions of $\bar{\sigma}$

Let

$$F : \begin{cases} C(\partial D) \times \mathbb{R} \times H^1(D) \to \mathbb{R} \times H^1(D), \\ (\sigma, \lambda, v) \qquad\qquad \mapsto \Big(\int_D v\, dx + \oint_{\partial D} \sigma(x) v\, ds, T(\sigma, \lambda, v) - v \Big). \end{cases}$$

Note that $F(\sigma_0, 0, 1) = (0, 0)$. Moreover, the following relation between zeroes of F and solutions of (P) holds:

$$\begin{aligned} F(\sigma, \lambda, v) = (0,0) \qquad & \Rightarrow (\sigma, \lambda, v) \text{ solves } (P), \\ \left\{ \begin{array}{l} (\sigma, \lambda, v) \text{ solves } (P) \text{ and} \\ (\lambda, v) \neq (0,1) \text{ or } \bar{\sigma} = \sigma_0 \end{array} \right\} & \Rightarrow F(\sigma, \lambda, v) = (0,0). \end{aligned}$$

Therefore, solving $F(\sigma, \lambda, v) = (0, 0)$ near $\sigma = \sigma_0, \lambda = 0, v = 1$ by the implicit function theorem will give all statements of the theorem since

$$(\lambda(\sigma), v(\sigma)) = (0,1) - \left[\frac{\partial F}{\partial(\lambda, v)} \Big|_{(\sigma_0,0,1)} \right]^{-1} \frac{\partial F}{\partial \sigma} \Big|_{(\sigma_0,0,1)} (\sigma - \sigma_0) + O(\sigma - \sigma_0)^2. \tag{5.3}$$

It remains to show the invertibility of $\frac{\partial F}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}$ and to compute the inverse $(\beta, z) := [\frac{\partial F}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}]^{-1}(\alpha, y)$ for given $(\alpha, y) \in \mathbb{R} \times H^1(D)$. This requires to find the solution (β, z) of

$$\int_D z\, dx + \sigma_0 \oint_{\partial D} z\, ds = \alpha, \quad \frac{\partial T}{\partial(\lambda, v)} \Big|_{(\sigma_0,0,1)} (\beta, z) - z = y. \tag{5.4}$$

Differentiation of (5.2) w.r.t. (λ, v) yields $\frac{\partial T}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}(\beta, z) = \zeta$, where ζ solves

$$-\operatorname{div}(A(x)\nabla\zeta) = \beta \text{ in } D, \quad \nabla\zeta^T A(x)n = \beta\sigma_0 \text{ on } \partial D, \quad \int_D \zeta\, dx = 0,$$

i.e. $\zeta = \beta(w - \bar{w})$. Therefore the solution (β, z) of (5.4) is determined by

$$\int_D z\, dx + \sigma_0 \oint_{\partial D} z = \alpha, \quad \beta(w - \bar{w}) - z = y.$$

The solution (β, z) can now be computed as

$$\beta = \frac{\alpha + \int_D y\, dx + \sigma_0 \oint_{\partial D} y\, ds}{\int_D w\, dx + \sigma_0 \oint_{\partial D} w\, ds}, \tag{5.5}$$

$$z = \frac{\alpha + \int_D y\, dx + \sigma_0 \oint_{\partial D} y\, ds}{\int_D w\, dx + \sigma_0 \oint_{\partial D} w\, ds}(w - \bar{w}) - y. \tag{5.6}$$

The uniqueness of (β, z) shows the invertibility of $\frac{\partial F}{\partial(\lambda,v)}|_{(\sigma_0,0,1)}$. Notice that the denominator in the above formula is $\int_D \nabla w^T A(x)\nabla w\, dx$. Finally it is easy to see that

$$\frac{\partial F}{\partial\sigma}\Big|_{(\sigma_0,0,1)} : \begin{cases} C(\partial D) \to \mathbb{R} \times H^1(D), \\ \quad \Sigma \quad \mapsto (\oint_{\partial D} \Sigma(x)\, ds, 0). \end{cases}$$

Inserting $(\alpha, y) = \frac{\partial F}{\partial\sigma}|_{(\sigma_0,0,1)}(\sigma - \sigma_0) = (|\partial D|(\bar{\sigma} - \sigma_0), 0)$ into (5.5)-(5.6) and (5.3) gives the expansion of $\lambda(\sigma)$ and $v(\sigma)$ as claimed in the theorem□

We know from Theorem 3.1(ii) that eigenfunctions of constant sign can occur only for $\lambda \in \{\lambda_{-1}, 0, \lambda_1\}$. The next lemma sharpens this result.

Lemma 5.3. *Suppose $q(x) \equiv 0$. Let λ be an eigenvalue and let v be a corresponding positive eigenfunction of (2.4). If $\bar{\sigma} < \sigma_0$ then $\lambda \in \{0, \lambda_1\}$ and if $\bar{\sigma} > \sigma_0$ then $\lambda \in \{\lambda_{-1}, 0\}$. For $\bar{\sigma} = \sigma_0$ only $\lambda = 0$ is possible.*

Proof. The following proof idea is attributed to Hess [10]. By the Harnack inequality of Theorem 3.1 there exists $\delta > 0$ such that $v \geq \delta$ in D and trace $v \geq \delta$ on ∂D. Thus, we may write $v = e^y$ with a function $y \in H^1(D)$. For $z \in C^\infty(\overline{D})$ let us use $z^2 e^{-y}$ as a test-function for (2.4). Thus we obtain

$$\int_D -(z\nabla y - \nabla z)^T A(x)(z\nabla y - \nabla z) + \nabla z^T A(x)\nabla z\, dx$$
$$= \lambda\Big(\int_D z^2\, dx + \oint_{\partial D} \sigma(x) z^2\, ds\Big),$$

which implies

$$\int_D \nabla z^T A(x)\nabla z\,dx \geq \lambda\Big(\int_D z^2\,dx + \oint_{\partial D} \sigma(x) z^2\,ds\Big) \quad \forall z \in C^\infty(\overline{D}). \tag{5.7}$$

Now recall that if $\bar\sigma < \sigma_0$ then

$$0 = \lambda_0 = \min\Big\{\int_D \nabla u^T A(x)\nabla u\,dx : \int_D u^2 + \oint_{\partial D} \sigma(x) u^2\,ds = -1\Big\}.$$

Together with (5.7) this implies $\lambda \geq 0$ and hence by Theorem 3.1(ii) we find $\lambda \in \{0, \lambda_1\}$. If $\sigma_0 < \bar\sigma$ then

$$0 = \lambda_0 = \min\Big\{\int_D \nabla u^T A(x)\nabla u\,dx : \int_D u^2 + \oint_{\partial D} \sigma(x) u^2\,ds = 1\Big\},$$

which together with (5.7) and Theorem 3.1(ii) implies $\lambda \in \{\lambda_{-1}, 0\}$.

It remains to treat the case $\bar\sigma = \sigma_0$. In this case the following two characterizations of $\lambda_0 = 0$ hold simultaneously

$$0 = \lambda_0 = \inf\Big\{\int_D \nabla u^T A(x)\nabla u\,dx : \int_D u^2\,dx + \oint_D \sigma(x) u^2\,ds = -1\Big\} \tag{5.8}$$

$$= \inf\Big\{\int_D \nabla u^T A(x)\nabla u\,dx : \int_D u^2\,dx + \oint_D \sigma(x) u^2\,ds = 1\Big\}, \tag{5.9}$$

where neither of the two minimization problems has a minimizer. Together with (5.7) this implies that necessarily $\lambda = 0$. So let us show (5.8) and (5.9). Assume without loss of generality that $\operatorname{diam} D < 1$ and $\bar D \subset \{x \in \mathbb{R}^N : 0 < x_1 < 1\}$. Define $u(x) = 1 \pm x_1^\alpha$ for $\alpha > 1$. Thus, $u(x) \to 1$ in $H^1(D)$ for $\alpha \to \infty$. Furthermore, with $e_1 = (1, 0, \ldots, 0)$ we compute

$$\begin{aligned}\int_D \nabla u^T A(x)\nabla u\,dx &\leq \beta \int_D |\nabla u|^2 = \beta\alpha^2 \int_D x_1^{2\alpha-2}\,dx \\ &= \beta\frac{\alpha^2}{2\alpha-1}\int_D \nabla\cdot(x_1^{2\alpha-1}e_1)\,dx \leq \beta\alpha \oint_{\partial D} x_1^{2\alpha-1}\,ds\end{aligned}$$

and

$$\begin{aligned}a(u,u) &= \int_D (1 \pm x_1^\alpha)^2\,dx + \oint_{\partial D} \sigma(x)(1 \pm x_1^\alpha)^2\,ds \\ &= \int_D \pm 2x_1^\alpha + x_1^{2\alpha}\,dx + \oint_{\partial D} \sigma(x)(\pm 2x_1^\alpha + x_1^{2\alpha})\,ds\end{aligned}$$

since $\bar\sigma = \sigma_0$, i.e., $\int_{\partial D} \sigma(x)\,ds = -|D|$. Continuing the above calculation we find

$$\begin{aligned}a(u,u) &= \int_D \nabla\cdot\Big(\pm\frac{2x_1^{\alpha+1}}{\alpha+1}e_1 + \frac{x_1^{2\alpha+1}}{2\alpha+1}e_1\Big)\,dx + \oint_{\partial D} \sigma(x)(\pm 2x_1^\alpha + x_1^{2\alpha})\,ds \\ &= \oint_{\partial D} \Big(\pm\frac{2x_1^{\alpha+1}}{\alpha+1} + \frac{x_1^{2\alpha+1}}{2\alpha+1}\Big)e_1\cdot n + \sigma(x)(\pm 2x_1^\alpha + x_1^{2\alpha})\,ds.\end{aligned}$$

Note that there exists a sequence $\alpha_k \to \infty$ such that $\oint_{\partial D} 2\sigma(x)x_1^{\alpha_k}\,ds \neq 0$. Thus, for $k \to \infty$ the leading order-term for $a(u_k, u_k)$ is $\pm \oint_{\partial D} 2\sigma(x)x_1^{\alpha_k}\,ds$. Therefore $\int_D \nabla u_k^T A(x) \nabla u_k\,dx / a(u_k, u_k) \to 0$ as $k \to \infty$. If we set $\tilde{u}_k = s_k(1 \pm x_1^{\alpha_k})$ and $\bar{u}_k = t_k(1 \mp x_1^{\alpha_k})$ with appropriate multiples s_k, t_k and the sign chosen appropriately such that $a(\tilde{u}_k, \tilde{u}_k) = 1$, $a(\bar{u}_k, \bar{u}_k) = -1$, then $\tilde{u}_k$, $\bar{u}_k$ are minimizing sequences for (5.8), (5.9), respectively. As a result we get that the values of the minimization problem (5.8) and (5.9) are zero, although they are not attained. This finishes the proof of the claim. □

Corollary 5.1. *If $\bar{\sigma} \in (-\infty, \sigma_0)$ then λ_1 is simple, the eigenfunctions corresponding to λ_1 have constant sign and the eigenfunctions corresponding to λ_{-1} are sign-changing. If $\bar{\sigma} > \sigma_0$ then λ_{-1} is simple, the eigenfunctions corresponding to λ_{-1} have constant sign and the eigenfunctions corresponding to λ_1 are sign-changing.*

Proof. The statements on the sign-change of eigenfunctions follows from Lemma 5.3. It remains to prove the statement on the eigenfunctions with constant sign. Fix a function $\sigma \in C(\partial D)$ with $\bar{\sigma} \neq \sigma_0$ and let us consider the one-parameter family

$$\sigma_t = \begin{cases} t\sigma + (1-t)\sigma_0 & \text{if } \bar{\sigma} > \sigma_0, \\ -t\sigma + (1+t)\sigma_0 & \text{if } \bar{\sigma} < \sigma_0 \end{cases} \tag{5.10}$$

with $t \in \mathbb{R}$. With this choice the mean $\bar{\sigma}_t$ is increasing in t. By Theorem 5.1 there exists a one-parameter curve $t \mapsto (\lambda(t), v(t))$ for $t \in (-\epsilon, \epsilon)$ with values in $\mathbb{R} \times H^1(D)$ such that $F(\sigma_t, \lambda(t), v(t)) = (0,0)$ for all $t \in (-\epsilon, \epsilon)$, $\lambda(0) = 0$, $v(0) = 1$ with

$$\begin{aligned} \lambda(t) &= -t \frac{|\partial D|\,|\bar{\sigma} - \sigma_0|}{\int_D \nabla w^T A(x) \nabla w\,dx} + O(t^2), \\ v(t) &= 1 - t \frac{|\partial D|\,|\bar{\sigma} - \sigma_0|}{\int_D \nabla w^T A(x) \nabla w\,dx}(w - \bar{w}) + O(t^2). \end{aligned}$$

The parameter t now replaces σ. Let $B_\delta(0,1) \subset \mathbb{R} \times H^1(D)$ be the open unit ball or radius δ centered at $(0,1) \in \mathbb{R} \times H^1(D)$. For small $\delta > 0$ we know that

$$\text{degree}(F(\sigma_0, \cdot, \cdot), B_\delta(0,1), (0,0)) \neq 0$$

due to the invertibility of $\frac{\partial F}{\partial(\lambda, v)}|_{(\sigma_0, 0, 1)}$. Therefore, the global continuation theorem, see e.g. [3], applies and shows the existence of two continua $\mathcal{C}^+ \subset [0,\infty) \times \mathbb{R} \times H^1(D)$ and $\mathcal{C}^- \subset (-\infty, 0] \times \mathbb{R} \times H^1(D)$ of solutions

(t, λ, v) of $F(\sigma_t, \lambda, v) = (0,0)$ containing the point $(0,0,1)$. Locally near $(t, \lambda, v) = (0,0,1)$ the two continua $\mathcal{C}^+, \mathcal{C}^-$ are described by the curve $t \to (t, \lambda(t), v(t))$. Note that the condition $\int_D v\,dx = |D|$ shows that $v \neq 0$ for every element $(t, \lambda, v) \in \mathcal{C}^+, \mathcal{C}^-$. Thus, the maximum principle of Lemma 3.1 and a continuity argument show that $v > 0$ for every $(t, \lambda, v) \in \mathcal{C}^+, \mathcal{C}^-$. Similarly, $\lambda > 0$ for every $(t, \lambda, v) \in \mathcal{C}^-$ except for $t = 0$. And likewise $\lambda < 0$ for every $(t, \lambda, v) \in \mathcal{C}^+$ except for $t = 0$. Therefore Theorem 3.1 (ii) shows that $\lambda = \lambda_{\mp 1}$ provided $(t, \lambda, v) \in \mathcal{C}^\pm$ and $t \neq 0$ and that $\mathcal{C}^+, \mathcal{C}^-$ can be parameterized as single-valued continuous curves depending on t. Moreover the λ-part is decreasing in t. Hence the global continuation theorem implies that both $\mathcal{C}^+$ and $\mathcal{C}^-$ are unbounded continua.

Finally let us determine the projection of $\mathcal{C}^\pm$ onto the t-axis. It is clear that $\mathcal{C}^-$ projects onto $(-\infty, 0]$ since $\bar{\sigma}_t$ is increasing in t and $\sigma_0 < 0$, i.e., for every $t \leq 0$ the eigenvalue λ_1 exists. For positive t this is different, since for very large positive t the function σ_t could become entirely positive. In fact we find that $\mathcal{C}^+$ projects onto $[0, \frac{-\sigma_0}{(\sigma_0 - \max \sigma^+)^+})$ in case $\bar{\sigma} < \sigma_0$ and onto $[0, \frac{-\sigma_0}{(-\sigma_0 - \max \sigma^-)^+})$ in case $\bar{\sigma} > \sigma_0$, where the right-end points of the intervals are ∞ if the denominator is 0.

The statement of the corollary it obtained as follows: if $\bar{\sigma} < \sigma_0$ then we set $t = -1$ and obtain from (5.10) the original function σ. This means that $(-1, \lambda(-1), v(-1))$ lies on $\mathcal{C}^-$ and produces for the original function σ the positive eigenvalue λ_1 with a positive eigenfunction. If $\bar{\sigma} > \sigma_0$ then we set $t = 1$ which is an admissible value and also obtain from (5.10) the original function σ. This time it means that $(1, \lambda(1), v(1))$ lies on $\mathcal{C}^+$ and produces for σ a negative eigenvalue λ_{-1} with a positive eigenfunction. $\square$

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REMARKS ON SOME CLASS OF NONLOCAL ELLIPTIC PROBLEMS

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The goal of this paper is to provide a simple introduction to the theory of nonlocal elliptic problems.

1. Local versus nonlocal

Denote by Ω a bounded open set of $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$. Let a be a continuous function satisfying for some positive constants λ, Λ

$$0 < \lambda \leq a(s) \leq \Lambda \qquad \forall\, s \in \mathbb{R}. \tag{1.1}$$

Let Γ_0 be a part of Γ with positive measure. We denote by V or $H_0^1(\Omega;\Gamma_0)$ the subspace of $H^1(\Omega)$ defined by

$$V = H_0^1(\Omega;\Gamma_0) = \{\, v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0 \,\}. \tag{1.2}$$

Then for $f \in V'$ (the dual space of V) we would like to consider the two model problems

$$\begin{cases} -\nabla \cdot (a(u(x))\nabla u) = f \text{ in } \Omega, \\ u \in H_0^1(\Omega;\Gamma_0), \end{cases} \tag{L}$$

and

$$\begin{cases} -\nabla \cdot \left(a\Big(⨍_\Omega u(x)\, dx \Big) \nabla u \right) = f \text{ in } \Omega, \\ u \in H_0^1(\Omega;\Gamma_0). \end{cases} \tag{NL}$$

In the above equations $\nabla\cdot$ denotes the divergence operator, $⨍$ is the average i.e.

$$⨍_\Omega u(x)\, dx = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx \tag{1.3}$$

where $|\cdot|$ denotes the Lebesgue measure of the sets. We refer the reader to [1], [9], [11], for references on Sobolev spaces. The two above problems are meant in a weak sense that is to say for both of them the first equation has to be read

$$\int_\Omega a\nabla u \cdot \nabla v \, dx = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega;\Gamma_0) \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ is the duality bracket between V' and V.

The problem (L) is a "local" problem, that is to say to determine the solution one has available a rate of diffusion varying from a point to another – i.e. determined locally. On the contrary (NL) is a nonlocal problem in the sense that the diffusion coefficient is determined by a global quantity, here the average of the solution. Clearly (L) imposes more constraints on the solution u and one expects this one to be unique. As we will see (NL) leaves more flexibility to the solution – for instance very different functions can have the same average and thus the same value of a in (NL). This will lead to nonuniqueness and even to the possibility of having a continuum of solutions. From a physical point of view, both equations can describe a steady population density u, f is a source term, the diffusion coefficient is supposed to vary in function of $u(x)$ at any point in the case (L) and in function of the total population only in the case of (NL) – see also the next section.

To verify that (L) possesses a unique solution let us introduce

$$A(s) = \int_0^s a(\xi)\, d\xi. \tag{1.5}$$

Due to (1.1) it is clear that u is solution to (L) iff Au is solution in a weak above sense to

$$\begin{cases} -\Delta(Au) = f \text{ in } \Omega, \\ Au \in H_0^1(\Omega;\Gamma_0). \end{cases} \tag{1.6}$$

Now, clearly, (1.6) is a usual linear problem and has a unique solution. Since A is monotone increasing, by (1.1), the solution of (L) is also unique.

Remark 1.1. If $a(u(x))$ is replaced by $a(x, u(x))$ uniqueness can still be preserved. We refer the reader to [2] for such issues.

The rest of the paper will be devoted to problems of the type (NL) or intermediate between (L) and (NL). First in the next section we will show that (NL) can be obtained as a limit of a sequence of local problems. Then we will see that (NL) reduces in fact to solving a nonlinear equation in $\mathbb{R}$.

We will consider in Section 4 the case of nonlocal problems in the calculus of variations. Finally in the last section we will introduce a class of problem interpolating the case of (L) and (NL) and will in particular address the question of uniqueness for this kind of problems.

2. Nonlocal problems as the limit of local ones

Let us first start with the following remark. For $\sigma > 0$ consider the problem of finding v_σ solution to

$$\begin{cases} -\sigma \Delta v_\sigma + v_\sigma = f_\sigma \text{ in } \Omega, \\ \dfrac{\partial v_\sigma}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

By the Lax-Milgram theorem this problem admit a unique weak solution for any $f_\sigma \in L^2(\Omega)$. Then we have

Proposition 2.1. *Suppose that when $\sigma \to +\infty$*

$$f_\sigma \rightharpoonup f_\infty \quad in \quad L^2(\Omega) \tag{2.2}$$

then one has, if Ω is supposed to be connected

$$v_\sigma \to \fint_\Omega f_\infty(x)\,dx \quad in \quad H^1(\Omega). \tag{2.3}$$

Proof. Since f_σ converges in $L^2(\Omega)$ weakly, f_σ is clearly bounded in $L^2(\Omega)$. Considering the weak formulation of (2.1) – i.e.

$$\sigma \int_\Omega \nabla v_\sigma \nabla v\,dx + \int_\Omega v_\sigma v\,dx = \int_\Omega f_\sigma v\,dx \quad \forall\, v \in H^1(\Omega) \tag{2.4}$$

and taking $v = v_\sigma$ in (2.4) we obtain easily by the Cauchy-Schwarz inequality

$$\sigma \big| |\nabla v_\sigma| \big|_2^2 + |v_\sigma|_2^2 \le |f_\sigma|_2 |v_\sigma|_2$$

($|\cdot|_2$ denotes the usual $L_2(\Omega)$-norm, $|\cdot|$ the euclidean norm.) Thus it comes

$$\sigma \big| |\nabla v_\sigma| \big|_2^2 + |v_\sigma|_2^2 \le |f_\sigma|_2^2 \tag{2.5}$$

and we derive that it holds

$$\big| |\nabla v_\sigma| \big|_2^2 + |v_\sigma|_2^2 \le |f_\sigma|_2^2 \quad \forall \sigma \ge 1. \tag{2.6}$$

It follows that v_σ is bounded in $H^1(\Omega)$, when $\sigma \to +\infty$, and up to a subsequence we have for some v_∞

$$v_\sigma \rightharpoonup v_\infty \quad \text{in} \quad H^1(\Omega). \tag{2.7}$$

To determine v_∞, one notices from (2.5) that

$$\big|\big|\nabla v_\sigma\big|\big|_2^2 \le |f_\sigma|_2^2/\sigma \longrightarrow 0 \tag{2.8}$$

when $\sigma \to +\infty$. Thus $\nabla v_\sigma \to 0$ strongly in $L^2(\Omega)$ and by (2.7) we deduce that it holds that

$$\nabla v_\infty = 0 \quad \Longrightarrow \quad v_\infty = \text{cst}\,.$$

To determine this constant taking $v \equiv 1$ in (2.4) leads to

$$\int_\Omega v_\sigma \, dx = \int_\Omega f_\sigma \, dx.$$

Passing to the limit since – up to a subsequence – we can assume that $v_\sigma \to v_\infty$ in $L^2(\Omega)$ we get

$$v_\infty |\Omega| = \int_\Omega f_\infty \, dx$$

and thus

$$v_\infty = \fint_\Omega f_\infty(x)\, dx.$$

Due to the uniqueness of this limit we have that the whole sequence converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ toward u_∞. By (2.8) the strong convergence in $H^1(\Omega)$ follows. This completes the proof of the proposition. $\square$

Remark 2.1. In the case where $f_\sigma = f$, $\forall\, \sigma$ we have of course that

$$v_\infty = \fint_\Omega f \, dx \quad \text{in} \quad H^1(\Omega).$$

Any mixed boundary condition would force u_σ to converge toward 0. We could replace in the above proposition $-\Delta$ by a general elliptic operator depending even nonlinearly on v_σ.

Under the assumptions of Section 1 and for $\sigma > 0$ let us consider the following problem (see [10], [12], and the references there). One is looking for $(u, v) \in H_0^1(\Omega; \Gamma_0) \times H^1(\Omega)$ satisfying in a weak sense

$$\begin{cases} -\sigma \Delta v + v = u & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \Gamma, \\ -\nabla \cdot (a(v) \nabla u) = f & \text{in } \Omega, \\ u = 0 \text{ on } \Gamma_0, \ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma \setminus \Gamma_0. \end{cases} \tag{L_σ}$$

This problem is a local problem. One interpretation could be the following - for instance when $\Gamma_0 = \Gamma$. v is a density of bacteria located in a container Ω, u is a density of nutrient provided at a constant (in time) rate f. The source term of bacteria depends only on the nutrient (many modern societies enjoy this property...) and the diffusion of the nutrient depends on the density of population locally. The term $+v$ in the first equation corresponds to a constant death rate in this population. The boundary conditions are clear. Then we have

Theorem 2.1. *For $\sigma > 0$, $f \in V'$ there exists a weak solution to (L_σ) - i.e. a couple $(u, v) \in H_0^1(\Omega; \Gamma_0) \times H^1(\Omega)$ such that*

$$\begin{cases} \displaystyle\int_\Omega \sigma \nabla v \cdot \nabla \xi + v\xi \, dx = \int_\Omega u\xi \, dx & \forall \xi \in H^1(\Omega), \\ \displaystyle\int_\Omega a(v) \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle & \forall \varphi \in H_0^1(\Omega; \Gamma_0). \end{cases} \tag{2.9}$$

Proof. We suppose that $H_0^1(\Omega; \Gamma_0)$ is equipped with the norm

$$\big|\,|\nabla u|\,\big|_2 = \left\{ \int_\Omega |\nabla u(x)|^2 \, dx \right\}^{1/2}, \tag{2.10}$$

and $H^1(\Omega)$ equipped with

$$\|v\|_{1,2} = \left\{ |v|_2^2 + \big|\,|\nabla v|\,\big|_2^2 \right\}^{1/2}. \tag{2.11}$$

Moreover, we denote by $|\cdot|_*$ the strong dual norm on V' corresponding to (2.10). Taking $\varphi = u$ in the second equation of (2.9) we get from (1.1)

$$\lambda \big|\,|\nabla u|\,\big|_2^2 \le \langle f, u \rangle \le |f|_* \big|\,|\nabla u|\,\big|_2 \tag{2.12}$$

from which it follows

$$\big|\,|\nabla u|\,\big|_2 \le \frac{|f|_*}{\lambda}. \tag{2.13}$$

If C_p denotes the constant in the Poincaré inequality we derive then

$$|u|_2 \le C_p \big|\,|\nabla u|\,\big|_2 \le C_p \frac{|f|_*}{\lambda} := C. \tag{2.14}$$

We set then

$$B_C(0) = \{ w \in L^2(\Omega) \mid |w|_2 \le C \} \tag{2.15}$$

i.e. $B_C(0)$ is the ball of center 0 and radius C in $L^2(\Omega)$. We consider next the following mapping

$$w \mapsto Tw = u \tag{2.16}$$

where u is defined as follows. We let v be the solution to

$$\begin{cases} v \in H^1(\Omega), \\ \sigma \int_\Omega \nabla v \cdot \nabla \xi \, dx + \int_\Omega v \cdot \xi \, dx = \int_\Omega w \cdot \xi \, dx \quad \forall \xi \in H^1(\Omega). \end{cases} \tag{2.17}$$

Since $\sigma > 0$, the existence of v is a simple consequence of the Lax–Milgram theorem. Having found v we define $u = Tw$ as the solution to

$$u \in H_0^1(\Omega; \Gamma_0), \qquad \int_\Omega a(v) \nabla u \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle \quad \forall \varphi \in H_0^1(\Omega; \Gamma_0). \tag{2.18}$$

Since $a(v) = a(v(\cdot)) \in L^\infty(\Omega)$, by (1.1), the existence and uniqueness of u follows also from the Lax–Milgram theorem. Now, clearly u satisfies (2.14) and T maps $B_C(0)$ into itself. Moreover, by (2.13), $T(B_C(0))$ is relatively compact in $B_C(0)$. If T is continuous, by the Schauder fixed point theorem, T will have a fixed point u and (u, v) will be the solution to (2.1). To show the continuity of T consider a sequence w_n such that

$$w_n \in B_C(0), \qquad w_n \longrightarrow w \quad \text{in } L^2(\Omega). \tag{2.19}$$

Denote by v_n the solution to (2.17) corresponding to w_n. It is clear that

$$v_n \longrightarrow v \quad \text{in } H^1(\Omega). \tag{2.20}$$

If $u_n = Tw_n$ one has – due to (2.13)

$$||\nabla u_n||_2 \leq C$$

where C is some constant independent of n and – up to a sequence – we have for some $u_0 \in H_0^1(\Omega)$

$$u_n \rightharpoonup u_0 \text{ in } H_0^1(\Omega; \Gamma_0), \quad u_n \longrightarrow u_0 \text{ in } L^2(\Omega), \quad v_n \longrightarrow v \text{ a.e. in } \Omega. \tag{2.21}$$

Considering the equation

$$\int_\Omega a(v_n(x)) \nabla u_n \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle$$

and noting that

$$\nabla u_n \rightharpoonup \nabla u_0 \text{ in } L^2(\Omega), \qquad a(v_n(x)) \nabla \varphi \longrightarrow a(v(x)) \nabla \varphi \text{ in } L^2(\Omega)$$

we get by passing to the limit

$$\int_\Omega a(v) \nabla u_0 \cdot \nabla \varphi \, dx = \langle f, \varphi \rangle \quad \forall \varphi \in H_0^1(\Omega; \Gamma_0).$$

i.e. $u_0 = u = Tw$. By uniqueness of the limit it is the whole sequence u_n which satisfies

$$u_n = Tw_n \longrightarrow u \quad \text{in} \quad L^2(\Omega).$$

This shows that T is continuous and completes the proof of the theorem. □

Remark 2.2. At this point we do not know if the solution (u, v) of (L_σ) is unique (see also below).

In the theorem below we denote by (u_σ, v_σ) a couple of solutions to (2.9). We would like to study the asymptotic behaviour of (u_σ, v_σ) when $\sigma \to +\infty$. We have

Theorem 2.2. *There exists $(u_\infty, v_\infty) \in H_0^1(\Omega; \Gamma_0) \times H^1(\Omega)$ and a subsequence from σ such hat $(u_\sigma, v_\sigma) \to (u_\infty, v_\infty)$ in $H^1(\Omega)^2$ where u_∞ is solution to*

$$\begin{aligned} u_\infty \in & H_0^1(\Omega; \Gamma_0), \quad \int_\Omega a\Big(\fint_\Omega u_\infty \, dx\Big) \nabla u_\infty \cdot \nabla v \, dx \\ & = \langle f, v \rangle \;\; \forall\, v \in H_0^1(\Omega; \Gamma_0), \end{aligned} \tag{2.22}$$

$$v_\infty = \fint_\Omega u_\infty \, dx. \tag{2.23}$$

Proof. From (2.14) we have

$$|u_\sigma|_2 \le C_p \big|\big|\nabla u_\sigma\big|\big|_2 \le \frac{C_p |f|_*}{\lambda} = C \tag{2.24}$$

where C is independent of σ. From (2.5) we derive

$$\sigma \big|\big|\nabla v_\sigma\big|\big|_2^2 + |v_\sigma|_2^2 \le C^2. \tag{2.25}$$

Since (u_σ, v_σ) is bounded independently of $\sigma > 1$ there exists $(u_\infty, v_\infty) \in H_0^1(\Omega; \Gamma_0) \times H^1(\Omega)$ and a subsequence of $\sigma \to +\infty$ such that

$$u_\sigma \rightharpoonup u_\infty \text{ in } H_0^1(\Omega; \Gamma_0), \quad u_\sigma \to u_\infty \text{ in } L^2(\Omega) \tag{2.26}$$

$$v_\sigma \rightharpoonup v_\infty \text{ in } H^1(\Omega), \quad v_\sigma \to v_\infty \text{ in } L^2(\Omega), \quad v_\sigma \to v_\infty \text{ a.e. in } \Omega. \tag{2.27}$$

By Proposition 2.1 we have clearly (2.23) and a strong convergence of v_σ toward v_∞ in $H^1(\Omega)$. Since without loss of generality we can assume $v_\sigma \to v_\infty$ a.e. in Ω. It is easy to get (2.22) and the strong convergence of u_σ toward u_∞. This completes the proof of the theorem. □

Remark 2.3. If the solution of (2.22) is unique then the whole sequence (u_σ, v_σ) converges toward (u_∞, v_∞). We do not know if the solution (u_σ, v_σ) is unique or is unique for σ large. In general it would be interesting to find conditions on a imposing uniqueness of a solution to (2.9).

3. Simple existence result

In this section we would like to consider problems of the type (NL). We make it slightly more general by setting

$$\ell(u) = \int_\Omega g(x)u(x)\,dx \tag{3.1}$$

where $g \in L^2(\Omega)$ and by considering u solution to

$$\begin{cases} u \in H_0^1(\Omega;\Gamma_0), \\ \displaystyle\int_\Omega a(\ell(u))\nabla u \cdot \nabla v\,dx = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega;\Gamma_0). \end{cases} \tag{3.2}$$

It is clear that (NL) is the particular case of (3.2) when $g = \frac{1}{|\Omega|}$. Let us also introduce φ the solution to

$$\begin{cases} \varphi \in H_0^1(\Omega;\Gamma_0), \\ \displaystyle\int_\Omega \nabla\varphi \cdot \nabla v\,dx = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega;\Gamma_0). \end{cases} \tag{3.3}$$

By the Lax-Milgram theorem, for every $f \in V'$ the dual of $H_0^1(\Omega;\Gamma_0)$, there exists a unique solution to (3.3). Then we have

Theorem 3.1. *The mapping $u \mapsto \ell(u)$ is a one-to-one mapping from the set of solutions of* (3.2) *onto the set of solutions of the equation in* $\mathbb{R}$

$$a(\mu)\mu = \ell(\varphi). \tag{3.4}$$

Proof. First consider u solution to (3.2). By the uniqueness of the solution to (3.3) we have

$$a(\ell(u))u = \varphi. \tag{3.5}$$

Taking ℓ of both sides we obtain

$$a(\ell(u))\ell(u) = \ell(\varphi)$$

and $\ell(u)$ is solution of (3.4), i.e. ℓ goes from the set of solutions to (3.2) into the set of solutions to (3.4).

Let μ be now a solution to (3.4). By the Lax–Milgram theorem there exists a unique $u = u_\mu$ solution to

$$\begin{cases} u_\mu \in H_0^1(\Omega;\Gamma_0), \\ \displaystyle\int_\Omega a(\mu)\nabla u \cdot \nabla v \, dx = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega;\Gamma_0). \end{cases} \tag{3.6}$$

As above we have clearly

$$a(\mu)u = \varphi$$

and thus

$$a(\mu)\ell(u) = \ell(\varphi) = a(\mu)\mu.$$

Since $a > 0$ this implies that $\mu = \ell(u)$ and thus $u = u_\mu$ is solution to (3.2). This shows that the mapping ℓ is onto. To complete the proof it is easy to see that if u_1, u_2 are solutions to (3.2) then $\ell(u_1) = \ell(u_2)$ implies clearly that $u_1 = u_2$. $\square$

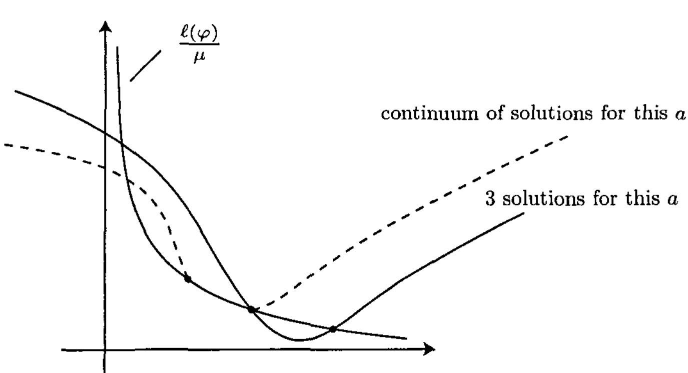
Remark 3.1. What we used here is in fact the homogeneity of ℓ with respect to positive numbers, i.e. the same result would hold true for

$$\ell(u) = \int_\Omega |u(x)|\, dx, \qquad \ell(u) = \int_\Omega g|u(x)|\, dx$$

and could be easily adapted in the case where

$$\ell(\lambda u) = \lambda^\beta u \quad \forall\, \lambda > 0$$

(see [6]). Note also that only the positivity of a was useful here.



From the above theorem we can easily solve (3.2). Suppose for instance the $\ell(\varphi) > 0$. Then the equation (3.4) is equivalent to find $\mu > 0$ such that

$$a(\mu) = \frac{\ell(\varphi)}{\mu}$$

i.e. one has to find the intersection between a branch of hyperbola and the graph of a. Clearly this intersection could give rise to any of the situations described on the above figure (see also [4]- [6] and also [8] for further references and some results on the parabolic case).

4. Nonlocal problems in the calculus of variations

In this section we would like to study certain features of nonlocal problems in the calculus of variations. In particular, as we saw in the previous section for equations, we would like to rely on a problem in $\mathbb{R}$ to solve them. Of course we will consider a simple class of them. We refer to [7] for a more involved analysis.

Set

$$J[u] = \frac{1}{2}a(l(u)) \int_\Omega |\nabla u|^2 \, dx - \int_\Omega fu \, dx \tag{4.1}$$

where $f \in L^2(\Omega)$ and $l(u)$ is a linear form on $L^2(\Omega)$ defined by

$$l(u) = \int_\Omega g(x)u(x)\, dx, \quad g \in L^2(\Omega), \ g \not\equiv 0. \tag{4.2}$$

We want to minimize this functional on $H_0^1(\Omega; \Gamma_0)$. We denote by K_m the closed convex set of $H_0^1(\Omega; \Gamma_0)$ defined by

$$K_m = \{\, v \in H_0^1(\Omega; \Gamma_0) \mid l(v) = m \,\}. \tag{4.3}$$

We set

$$\tilde{J}(m) = \mathrm{Inf}_{K_m} \left\{ \frac{1}{2}a(m) \int_\Omega |\nabla u|^2 dx - \int_\Omega fudx \right\}. \tag{4.4}$$

We suppose that

$$a(m) > 0 \quad \forall\, m \in \mathbb{R}. \tag{4.5}$$

Then we have

Lemma 4.1. *For every m in $\mathbb{R}$, there exists a unique $u_m \in K_m$ such that*

$$\tilde{J}(m) = \frac{1}{2}a(m) \int_\Omega |\nabla u_m|^2 dx - \int_\Omega f u_m dx. \tag{4.6}$$

Moreover, u_m is the unique solution to

$$\begin{cases} u_m \in K_m, \\ \displaystyle\int_\Omega a(m)\nabla u_m \nabla w dx = \int_\Omega f w dx \quad \forall w \in K_0. \end{cases} \tag{4.7}$$

Proof. Since K_m is a nonempty closed convex set of $H_0^1(\Omega;\Gamma_0)$, u_m is the unique solution of the variational inequality:

$$\begin{cases} u_m \in K_m, \\ \displaystyle\int_\Omega a(m)\nabla u_m \nabla (v - u_m) dx \geq \int_\Omega f(v - u_m) dx \quad \forall v \in K_m. \end{cases} \tag{4.8}$$

If u_m is solution of (4.7), for any $v \in K_m$, $v - u_m \in K_0$ and (4.8) holds. Conversely if (4.8) holds, taking $v = \pm w + u_m$ in (4.8) with $w \in K_0$ we deduce easily (4.7). □

Then we can show

Theorem 4.1. *The map*

$$u \mapsto l(u)$$

is a one-to-one mapping from the set of minimizers of J on $H_0^1(\Omega;\Gamma_0)$ onto the set of minimizers of $\tilde{J}$ over $\mathbb{R}$.

Proof. Let u be a minimizer of J on $H_0^1(\Omega;\Gamma_0)$. Let $m_0 = l(u)$. One has

$$\begin{aligned} J[u] &= \mathrm{Inf}_{K_{m_0}}\left\{\frac{1}{2}a(m_0)\int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx\right\} = \tilde{J}(m_0) \\ &\leq J[v] \quad \forall v \in H_0^1(\Omega;\Gamma_0). \end{aligned}$$

Thus, for every $v \in K_m$

$$\begin{aligned} & J[u] = \tilde{J}(m_0) \leq J[v] \quad \forall v \in K_m \quad \forall m \\ \Longrightarrow \quad & J[u] = \tilde{J}(m_0) \leq \tilde{J}(m) \quad \forall m \end{aligned}$$

and $m_0 = l(u)$ is a minimizer of $\tilde{J}$. The mapping $u \mapsto l(u)$ goes from the set of minimizers of J into the set of minimizers of $\tilde{J}$. To show that the mapping is onto, let m_0 be a minimizer of $\tilde{J}$. Let u_{m_0} the solution to (4.7) or (4.8) corresponding to $m = m_0$. It holds that

$$\tilde{J}(m_0) = J[u_{m_0}] \leq J[v] \quad \forall v \in K_{m_0}.$$

Moreover, for $v \in K_m$, $m \neq m_0$ we have

$$\tilde{J}(m_0) \leq \tilde{J}(m) \leq J[v].$$

Thus, we have

$$J[u_{m_0}] \leq J[v] \quad \forall v \in H_0^1(\Omega; \Gamma_0)$$

and the mapping is onto. If u_1, u_2 are two minimizers with $l(u_1) = l(u_2)$ then clearly $u_1 = u_2 = u_{l(u_i)}$ and the injectivity is proved. □

Let us define by $\theta = \theta_g$ the weak solution to

$$\begin{cases} -\Delta\theta_g = g & \text{in } \Omega, \\ \theta_g \in H_0^1(\Omega; \Gamma_0). \end{cases} \tag{4.9}$$

Then we have

Lemma 4.2. *Given $m \in \mathbb{R}$, let u_m be the unique solution to (4.7). Then it holds that u_m is the weak solution to*

$$\begin{cases} -a(m)\Delta u_m = f + c_m g & \text{in} \quad \Omega, \\ u_m \in H_0^1(\Omega; \Gamma_0). \end{cases} \tag{4.10}$$

where c_m is the constant given by

$$c_m = \{a(m)m - (f, \theta)\}/l(\theta), \tag{4.11}$$

and $(f, \theta) = \int_\Omega f\theta \, dx$.

Proof. Since $g \not\equiv 0$, $\theta \neq 0$ and from (4.9) we deduce

$$l(\theta) = \int_\Omega g\theta dx = \int_\Omega |\nabla\theta|^2 dx > 0.$$

Let $v \in H_0^1(\Omega; \Gamma_0)$ and ϱ a function fixed in $\mathcal{D}(\Omega)$ such that

$$l(\varrho) = 1$$

(such choice of ϱ is possible since $g \not\equiv 0$). From (4.7) we have, since $w = v - l(v)\varrho \in K_0$, for every $v \in H_0^1(\Omega; \Gamma_0)$

$$\begin{aligned} \langle -a(m)\Delta u_m - f, v\rangle &= \int_\Omega \{a(m)\nabla u_m \nabla v - fv\} dx \\ &= \int_\Omega \{a(m)\nabla u_m \nabla l(v)\varrho - fl(v)\varrho\} dx \\ &= l(v) \int_\Omega \{a(m)\nabla u_m \nabla\varrho - f\varrho\} dx := c_m l(v). \end{aligned}$$

This proves (4.10) (one can see by (4.7) that c_m is independent of ϱ). To get c_m one multiplies (4.10) by θ -i.e. one uses the weak formulation of (4.10)- to get

$$
\begin{aligned}
& a(m)\int_\Omega \nabla u_m \nabla\theta dx = \int_\Omega f\theta dx + c_m l(\theta) \\
\iff \quad & a(m)l(u_m) = \int_\Omega f\theta dx + c_m l(\theta) \\
\implies \quad & c_m = \{a(m)m - (f,\theta)\}/l(\theta)
\end{aligned}
$$

which completes the proof of the Lemma. □

Theorem 4.2. *To stress the dependence of θ on g we denote it by θ_g for any $g \in L^2(\Omega)$. Then we have*

$$
\tilde{J}(m) = \frac{1}{2(g,\theta_g)}\left\{\frac{(a(m)m-(f,\theta_g))^2-(g,\theta_g)(f,\theta_f)}{a(m)}\right\}. \tag{4.12}
$$

Proof. By the uniqueness of the solution of the problems of the type (1.4) it holds that (see (4.10))

$$
a(m)u_m = \theta_f + c_m\theta_g. \tag{4.13}
$$

From (4.10) we also have by multiplying both sides by u_m and integrating

$$
a(m)\int_\Omega |\nabla u_m|^2 dx = (f,u_m) + c_m m.
$$

Recall that $(\cdot,\cdot)$ denotes the usual scalar product in $L^2(\Omega)$. It follows that we have

$$
\tilde{J}(m) = \frac{1}{2}a(m)\int_\Omega |\nabla u_m|^2 dx - (f,u_m) = \frac{1}{2}\{c_m m - (f,u_m)\}.
$$

Using now (4.13) it comes

$$
\tilde{J}(m) = \frac{1}{2}\left\{c_m m - \frac{1}{a(m)}(f,\theta_f + c_m\theta_g)\right\}.
$$

Using the expression of c_m given by (4.11) we get after easy computations

$$
\begin{aligned}
&\tilde{J}(m) \\
&= \frac{1}{2}\left\{\frac{a(m)^2m^2-(f,\theta_g)a(m)m-(g,\theta_g)(f,\theta_f)-a(m)m(f,\theta_g)+(f,\theta_g)^2}{a(m)(g,\theta_g)}\right\} \\
&\quad = \frac{1}{2(g,\theta_g)}\frac{a(m)^2m^2-2a(m)m(f,\theta_g)+(f,\theta_g)^2-(g,\theta_g)(f,\theta_f)}{a(m)}
\end{aligned}
$$

which completes the proof. □

Remark 4.1. If we set

$$(f,\theta_g)=\alpha \qquad (g,\theta_g)(f,\theta_f)=\beta>0 \tag{4.14}$$

the minimization of $\tilde{J}$ reduces to the minimization of

$$\mathcal{J}(m)=\frac{(a(m)m-\alpha)^2-\beta}{a(m)}. \tag{4.15}$$

Note that $\alpha^2\leq\beta$. It is clear that $\tilde{J}$ and $\mathcal{J}$ are continuous functions of m if a is continuous.

Since $0\in K_0$ one should notice that

$$\tilde{J}(0)\leq 0$$

(this is also clear from $\mathcal{J}(0)=\{\alpha^2-\beta\}/a(0)$). We have shown in Theorem 4.1 that J admits minimizers iff $\mathcal{J}$ admits minimizers on $\mathbb{R}$. This is not always the case. However we have

Theorem 4.3. *Suppose that a is a continuous function satisfying* (4.5). *If for $|m|$ large enough*

$$a(m)\geq\frac{\delta}{|m|} \tag{4.16}$$

where δ is a positive constant such that

$$(\delta-|\alpha|)^2>\beta \tag{4.17}$$

then $J[\cdot]$ and $\mathcal{J}$ admit minimizers. This is sharp in the sense that if for $|m|$ large enough

$$a(m)=\frac{\delta}{|m|}$$

with $(\delta-|\alpha|)^2<\beta$ then $J[\cdot]$ fails to have minimizers.

Proof. In the case where (4.16), (4.17) hold we have for $|m|$ large enough

$$\begin{aligned}\mathcal{J}(m)&=a(m)m^2-2\alpha m+\frac{\alpha^2-\beta}{a(m)}\\&\geq\frac{\delta}{|m|}m^2-2\alpha m+\frac{\alpha^2-\beta}{\delta}|m|\\&=\delta|m|-2\alpha m+\frac{\alpha^2-\beta}{\delta}|m|,\end{aligned}$$

(recall that $\alpha^2 - \beta \leq 0$). This gives

$$\begin{aligned}\mathcal{J}(m) &\geq \delta|m| - 2|\alpha||m| + \frac{\alpha^2 - \beta}{\delta}|m| \\ &= |m|\{\frac{(\delta - |\alpha|)^2 - \beta}{\delta}\} \to +\infty \quad \text{when} \quad |m| \to +\infty.\end{aligned}$$

Thus the minimization of $\mathcal{J}$ reduces to a minimization on a compact set and since $\mathcal{J}$ is a continuous function, a minimizer does exist.

In the case where $a(m) = \frac{\delta}{|m|}$ we have

$$\begin{aligned}\mathcal{J}(m) &= \delta|m| - 2\alpha m + \frac{\alpha^2 - \beta}{\delta}|m| \\ &= |m|\{\frac{(\delta - |\alpha|)^2 - \beta}{\delta}\} \quad \text{for} \quad sign(m) = sign(\alpha).\end{aligned}$$

and $\mathcal{J}$ is not bounded below for $(\delta - |\alpha|)^2 < \beta$. This completes the proof of the theorem. □

Remark 4.2. It is clear that (4.16) holds for instance when

$$a(m) \geq \delta > 0$$

or more generally when

$$a(m) \geq \delta|m|^{-\gamma} \quad \text{for } |m| \text{ large},$$

γ being a constant such that $0 < \gamma < 1$, δ being here an arbitrary positive constant.

Theorem 4.4. *Suppose that*

$$a \geq \delta > 0. \tag{4.18}$$

Then if a is discontinuous $J[\cdot]$ might fail to have a minimizer.

Proof. Indeed let a be a continuous function satisfying (4.18). Then $\mathcal{J}$ admits minimizers. Let m_0 be one of them. One has

$$\mathcal{J}(m_0) = a(m_0)m_0^2 - 2\alpha m_0 + \frac{\alpha^2 - \beta}{a(m_0)}.$$

If m_0 or $(\alpha^2 - \beta) \neq 0$, the function

$$a \to am_0^2 - 2\alpha m_0 + \frac{\alpha^2 - \beta}{a}$$

is clearly increasing and one can change the value of $a(m_0)$ in such a way that m_0 is no longer a minimizer. Thus, this new a – not continuous – will be so that J has no minimizer since $\mathcal{J}$ has none. □

Regarding uniqueness we have

Theorem 4.5. *If $\mathcal{J}$ is strictly convex then J admits a unique minimizer. Otherwise J can have as many minimizers as we wish – even for a smooth a.*

Proof. The first point is clear. Note that

$$\mathcal{J}(m) = a(m)m^2 - 2\alpha m + \frac{\alpha^2 - \beta}{a(m)}$$

and this function is strictly convex, in particular when

$$\mathcal{J}''(m) = a''m^2 + 4a'm + 2a - \frac{(\alpha^2 - \beta)}{a^2}\{a'' - 2\frac{a'^2}{a}\} > 0.$$

This is in particular the case when

$$a'' > 2\frac{a'^2}{a}.$$

Suppose now – this is of course always possible

$$\alpha^2 - \beta < 0.$$

Then consider a function $\mathcal{J}$ having as many minimizers as we wish. It is always possible to find a positive a such that

$$2a\mathcal{J}(m) = (am - \alpha)^2 - \beta \iff a^2m^2 - 2a(\alpha m + \mathcal{J}(m)) + \alpha^2 - \beta = 0.$$

Indeed the discriminant of this equation is

$$\Delta = 4\{(\alpha m + \mathcal{J}(m))^2 - m^2(\alpha^2 - \beta)\} \tag{4.19}$$

and it has its roots in $\mathbb{R}$. Moreover since $\alpha^2 - \beta < 0$ the roots do not have the same signs and one is positive. We call it $a(m)$. It varies of course, continuously with m, and for the corresponding problem of minimizing (4.1) one has as many solutions as $\mathcal{J}$ has of minimizers. □

Remark 4.3. Without any changes we can replace $f, g \in L^2(\Omega)$ by $f, g \in V'$ where V' is the dual of $H_0^1(\Omega; \Gamma_0)$.

5. A class of intermediate problems

In this section we would like to consider a class of problems interpolating between (L) and (NL). More precisely the problems that we will address involve a parameter r. When $r \to 0$ the solution of the problems at hand converges toward the solution of (L) and when r is large enough this is the solution of (NL).

Let us denote by $\Omega(x,r)$ the set

$$\Omega(x,r) = \Omega \cap B(x,r) \tag{5.1}$$

where $B(x,r)$ is the ball of center x and radius $r > 0$. Then we would like to consider the problem of finding u such that

$$\begin{cases} u \in H_0^1(\Omega;\Gamma_0), \\ -\nabla \cdot \left\{ a\left(\fint_{\Omega(x,r)} u(y)\,dy \right) \nabla u \right\} = f \text{ in } \Omega, \end{cases} \tag{5.2}$$

where $f \in V'$ (the notation is as above, $\fint_{\Omega(x,r)} u(y)\,dy = \frac{1}{|\Omega(x,r)|} \cdot \int_{\Omega(x,r)} u(y)\,dy$). Then we have

Theorem 5.1. *Suppose that* (1.1) *holds, then for any* $r > 0$ *there exists a solution to* (5.2).

Proof. We first get a prior estimate for u. Indeed, considering in the weak formulation of (5.2) (see (1.4)) $v = u$, we easily get

$$\lambda \big| |\nabla u| \big|_2^2 \le |f|_* \big| |\nabla u| \big|_2 \tag{5.3}$$

where $|f|_*$ denotes the strong dual norm of f. It follows that it holds

$$|u|_2 \le C_p \big| |\nabla u| \big|_2 \le \frac{C_p |f|_*}{\lambda} = C \tag{5.4}$$

where C_p is the Poincaré constant. Denote then by $B_C(0)$ the ball of center 0 and radius C in $L^2(\Omega)$. For $w \in B_C(0)$ let $u = Tw$ be the solution to

$$\begin{cases} u \in H_0^1(\Omega;\Gamma_0), \\ -\nabla \cdot \left\{ a\left(\fint_{\Omega(x,r)} w(y)\,dy \right) \nabla u \right\} = f \text{ in } \Omega. \end{cases} \tag{5.5}$$

It is clear that such a u exists and is unique. Moreover due to (5.3), (5.4), T maps $B_C(0)$ into itself and $T(B_C(0))$ is relatively compact in $B_C(0)$. To get existence we just need to prove that the mapping T is continuous. For that consider $w_n \in B_C(0)$ such that

$$w_n \to w \quad \text{in} \quad L^2(\Omega).$$

We have

$$\left|\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} w_n(y)\,dy - \frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} w(y)\,dy\right|$$
$$\leq \frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} |w_n - w|\,dy \leq \frac{1}{|\Omega(x,r)|^{1/2}}|w_n - w|_2$$

(by the Cauchy–Schwarz inequality). Thus we deduce that

$$⨍_{\Omega(x,r)} w_n(y)\,dy \longrightarrow ⨍_{\Omega(x,r)} w(y)\,dy \quad \text{a.e. in } \Omega. \tag{5.6}$$

Let us denote by u_n the solution to (5.5) corresponding to w_n – i.e. $u_n = Tw_n$. By (5.4) it holds that

$$\big|\big|\nabla u_n\big|\big|_2 \leq \frac{|f|_*}{\lambda}$$

and u_n is bounded in $H^1(\Omega)$. Thus – up to a subsequence – we can assume that for some $u_\infty \in H_0^1(\Omega;\Gamma_0)$ we have

$$u_n \rightharpoonup u_\infty \quad \text{in} \quad H^1(\Omega), \qquad u_n \to u_\infty \quad \text{in} \quad L^2(\Omega).$$

Passing to the limit in

$$\int_\Omega a\Big(⨍_{\Omega(x,r)} w_n(y)\,dy\Big)\nabla u_n \cdot \nabla v = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega;\Gamma_0)$$

by (5.6) we get that u_∞ satisfies

$$\int_\Omega a\Big(⨍_{\Omega(x,r)} w(y)\,dy\Big)\nabla u_\infty \nabla v = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega;\Gamma_0).$$

Thus we have $u_\infty = Tw$ and by uniqueness of the limit the whole sequence u_n converges toward Tw. This shows that T is continuous and concludes the proof by the Schauder fixed point theorem. □

Let us show that (5.2) somehow is an interpolation between (L) and (NL). It is clear that for r large enough $\Omega(x,r) = \Omega$ for every $x \in \Omega$ and thus in this case (5.2) is nothing but (NL). Moreover we have:

Theorem 5.2. *Let u_r the solution to (5.2). Suppose that*

$$\text{the "sequence" } u_r \text{ is equicontinuous in every } \Omega' \subset\subset \Omega. \tag{5.7}$$

Then we have

$$u_r \to u \quad \text{in} \quad H^1(\Omega), \qquad r \to 0, \tag{5.8}$$

where u is the solution to (L).

Proof. From (5.3), (5.4) we know that u_r is bounded in $H^1(\Omega)$ independently of r. Thus -up to a subsequence- we can assume that

$$u_r \rightharpoonup u_0 \quad \text{in } H^1(\Omega), \quad u_r \to u_0 \quad \text{in } L^2(\Omega), \quad u_r \to u_0 \quad \text{a.e. in } \Omega. \tag{5.9}$$

Let us show that it holds

$$⨍_{\Omega(x,r)} u_r(y)\, dy \to u_0 \quad \text{a.e. in} \quad \Omega. \tag{5.10}$$

Let $\epsilon > 0$ be fixed. Let us select r such that

$$r < \frac{dist(\Omega', \Gamma)}{2} \tag{5.11}$$

Since u_r is an equicontinuous sequence in every subdomain of Ω for every $\epsilon > 0$ there exists r_0 such for $r \leq r_0$ it holds that

$$|u_r(y) - u_r(x)| \leq \frac{\epsilon}{2} \quad \forall x \in \Omega', \quad \forall y \in \Omega(x,r). \tag{5.12}$$

It follows that for r small enough we have

$$\begin{aligned}
|⨍_{\Omega(x,r)} u_r(y)\, dy - u_0(x)| &= |⨍_{\Omega(x,r)} u_r(y)\, dy - u_r(x) + u_r(x) - u_0(x)| \\
&\leq |⨍_{\Omega(x,r)} (u_r(y) - u_r(x))\, dy + u_r(x) - u_0(x)| \\
&\leq ⨍_{\Omega(x,r)} |u_r(y) - u_r(x)|\, dy + |u_r(x) - u_0(x)| \\
&\leq \frac{\epsilon}{2} + |u_r(x) - u_0(x)| \leq \epsilon
\end{aligned}$$

by (5.9). It follows that

$$⨍_{\Omega(x,r)} u_r(y)\, dy \to u_0 \quad \text{a.e. in} \quad \Omega'. \tag{5.13}$$

Since Ω' is arbitrary (5.10) follows. For $v \in H_0^1(\Omega; \Gamma_0)$ from the weak formulation of (5.2) we have

$$\int_\Omega a\Big(⨍_{\Omega(x,r)} u_r(y)\, dy\Big) \nabla u_r \nabla v dx = \langle f, v \rangle. \tag{5.14}$$

By (5.9) and (5.10) we have also

$$\nabla u_r \rightharpoonup \nabla u_0 \quad \text{in } (L^2(\Omega))^n, \tag{5.15}$$

$$a\Big(⨍_{\Omega(x,r)} u_r(y)\, dy\Big) \nabla v \to a(u_0) \nabla v \quad \text{in } (L^2(\Omega))^n. \tag{5.16}$$

Passing to the limit in (5.14) we deduce that it holds

$$\int_\Omega a(u_0)\nabla u_0 \nabla v dx = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega;\Gamma_0) \tag{5.17}$$

i.e. u_0 is the unique solution to (L). Since the limit of the subsequence is unique the whole sequence u_r satisfies (5.9). To show now that u_r converges strongly, taking $v = u_r$ in (5.14) we obtain

$$\int_\Omega a\Big(\fint_{\Omega(x,r)} u_r(y)\,dy\Big)\nabla u_r \nabla u_r dx = \langle f, u_r\rangle. \tag{5.18}$$

Passing to the limit in r we get

$$\lim_{r\to 0}\int_\Omega a\Big(\fint_{\Omega(x,r)} u_r(y)\,dy\Big)\nabla u_r \nabla u_r dx = \langle f, u_0\rangle = \int_\Omega a(u_0)\nabla u_0\nabla u_0 dx. \tag{5.19}$$

It follows that it holds

$$\begin{aligned}\lambda\big|\big|\nabla(u_r-u_0)\big|\big|_2^2 &\le \int_\Omega a\Big(\fint_{\Omega(x,r)} u_r(y)\,dy\Big)\nabla(u_r-u_0)\nabla(u_r-u_0)dx\\ &= \int_\Omega a\Big(\fint_{\Omega(x,r)} u_r(y)\,dy\Big)\nabla u_r\nabla u_r dx - 2\int_\Omega a\Big(\fint_{\Omega(x,r)} u_r(y)\,dy\Big)\nabla u_r\nabla u_0 dx\\ &\qquad + \int_\Omega a(u_0)\nabla u_0\nabla u_0 dx \to 0.\end{aligned} \tag{5.20}$$

when $r \to 0$. This completes the proof of the theorem. □

Remark 5.1. 1) The property (5.7) holds for instance when $n = 1$ due to the estimate

$$|u_r(y) - u_r(x)| = \Big|\int_x^y u_r'(\xi)d\xi\Big| \tag{5.21}$$

$$\le |u_r'|_2|x-y|^{\frac{1}{2}} \le \frac{|f|_*}{\lambda}|x-y|^{\frac{1}{2}}. \tag{5.22}$$

It holds also in any dimension for $f \in L^p(\Omega), p > \frac{n}{2}$ due to the De Giorgi estimates (See [9]).
2) As we have seen, the solution to (L) is unique while (NL) can have infinitely many solutions. It would be interesting to study the variation of the number of solutions to (5.2) when r varies (see also the following theorem).

As a contribution to the uniqueness issue, let us consider the following one dimensional problem where $\Omega = (0,1)$, $\Gamma_0 = \{1\}$. It reads

$$\begin{cases} -\left\{a\left(\fint_{\Omega(x,r)} u(s)\,ds\right)u'\right\}' = f \text{ in } \Omega, \\ u'(1) = 0, \quad u(0) = 0. \end{cases} \tag{5.23}$$

Then we have

Theorem 5.3. *Suppose that $f \in L^2(\Omega)$ and that a satisfies (1.1) together with*

$$|a(s) - a(t)| \le A|s - t| \quad \forall s, t \in \mathbb{R}, \tag{5.24}$$

for some positive constant A. If r is small enough the solution to(5.23) is unique.

Proof. Suppose that v is another solution to the problems (5.23). It is clear by (5.23) that

$$a\left(\fint_{\Omega(x,r)} u(s)\,ds\right)u' \in H^1(\Omega) \tag{5.25}$$

and thus is a continuous function in Ω. From (5.23) we derive also

$$a\left(\fint_{\Omega(x,r)} u(s)\,ds\right)u' = a\left(\fint_{\Omega(x,r)} v(s)\,ds\right)v' + c \tag{5.26}$$

where c is a constant. Since all the functions above are continuous – taking the value at the point 1 we see that $c = 0$. Thus we get

$$a\left(\fint_{\Omega(x,r)} u(s)\,ds\right)(u' - v') = \left\{a\left(\fint_{\Omega(x,r)} v(s)\,ds\right) - a\left(\fint_{\Omega(x,r)} u(s)\,ds\right)\right\}v'$$

which implies

$$u' - v' = \tag{5.27}$$

$$\frac{1}{a(\fint_{\Omega(x,r)} u(s)ds)}\left\{a\left(\fint_{\Omega(x,r)} v(s)ds\right) - a\left(\fint_{\Omega(x,r)} u(s)ds\right)\right\}v'. \tag{5.28}$$

Claim: v' or u' are uniformly bounded. Indeed, from (5.23) integrated between x and 1 we get

$$a\left(\fint_{\Omega(x,r)} u(s)ds\right)u'(x) = \int_x^1 f(s)ds \tag{5.29}$$

which implies

$$|u'(x)| \le \frac{|\int_x^1 f(s)ds|}{\lambda} \le \frac{|f|_2}{\lambda}. \tag{5.30}$$

Integrating (5.27) between 0 and x we get

$$(u-v)(x) = \int_0^x \frac{1}{a(\fint_{\Omega(t,r)} u(s)ds)} \left\{ a\left(\fint_{\Omega(t,r)} v(s)ds\right) - a\left(\fint_{\Omega(t,r)} u(s)ds\right) \right\} v' \, dt.$$

From this follows easily – see (5.30)

$$|(u-v)(x)| \le \frac{A|f|_2}{\lambda^2} \int_0^x \fint_{\Omega(t,r)} |(u-v)(s)| \, dt \, ds. \tag{5.31}$$

Let us set

$$F(x) = |(u-v)(x)|, \qquad C = \frac{A|f|_2}{\lambda^2}.$$

Then (5.31) reads

$$F(x) \le C \int_0^x \fint_{\Omega(t,r)} F(s) \, dt \, ds \qquad \forall\, x \in (0,1). \tag{5.32}$$

We claim that this implies that F vanishes for r small enough. First, it is clear since u', v' are uniformly bounded and $u(0) = v(0) = 0$ that F is uniformly bounded in $(0,1)$. So, suppose that

$$0 \le F(x) \le M \qquad \forall\, x \in (0,1). \tag{5.33}$$

We would like to show that (5.32) implies that

$$F(x) \le \frac{C^k M (x + (k-1)r)^k}{k!} \quad \forall\, k. \tag{5.34}$$

By (5.33) the formula is true for $k = 0$. Suppose it is true for k. Then by (5.32) we get

$$F(x) \le C^{k+1} \frac{M}{k!} \int_0^x \fint_{\Omega(t,r)} (s + (k-1)r)^k \, dt \, ds.$$

Clearly the function $(s+(k-1)r)^k$ is bounded on $\Omega(t,r)$ by $(t+kr)^k$. Thus we get

$$F(x) \le C^{k+1} \frac{M}{k!} \int_0^x (t + kr)^k \, dt.$$

Integrating we obtain

$$\begin{aligned} F(x) &\le C^{k+1}\frac{M}{k+1!}(x+kr)^{k+1}-(kr)^{k+1} \\ &\le C^{k+1}\frac{M}{k+1!}(x+kr)^{k+1}. \end{aligned}$$

which completes the proof of (5.34) by induction. We claim then that for r small, the series of term

$$u_k = \frac{C^k M(x+(k-1)r)^k}{k!}$$

converges. Indeed we have

$$\begin{aligned} u_{k+1}/u_k &= \frac{C}{k+1}\frac{(x+kr)^{k+1}}{(x+(k-1)r)^k} \\ &= \frac{C(x+kr)}{k+1}\cdot\left(\frac{x+kr}{x+(k-1)r}\right)^k \\ &= C\left(\frac{x+kr}{k+1}\right)e^{k\ln\{1+\frac{r}{x+(k-1)r}\}}. \end{aligned}$$

Since for u close to 0 we have

$$\ln(1+u) = u(1+\varepsilon(u))$$

we get

$$u_{k+1}/u_k = C\frac{(x+kr)}{k+1}e^{\frac{kr}{x+(k-1)r}(1+\varepsilon(k))} \quad \text{with } \varepsilon(k)\to 0 \text{ when } k\to+\infty.$$

From that we derive

$$\lim_{k\to+\infty} u_{k+1}/u_k = C\,e\,r < 1 \quad \text{for } r \text{ small.} \tag{5.35}$$

Going back to (5.34) – since the above series converges for $C\,e\,r < 1$ – we get that

$$F = 0.$$

This completes the proof of the theorem. $\square$

Remark 5.2. Since $C = \frac{A|f|_2}{\lambda^2}$, (5.35) gives a precise estimate on the size of r which insures uniqueness.

Acknowledgement: This research has been supported by the Swiss National Science foundation under the contract # 20-105155.

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ON SOME DEFINITIONS AND PROPERTIES OF GENERALIZED CONVEX SETS ARISING IN THE CALCULUS OF VARIATIONS

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We deal with generalized notions of convexity for sets. Namely, the polyconvexity, quasiconvexity, rank one convexity and separate convexity. The question has its origin in the calculus of variations. We try to systematize the results concerning these generalized notions imitating as much as possible the classical approach of convex analysis. Throughout the article, we will discuss the relations between the different convexities, separation and Carathéodory type theorems, the notion of hull of a set and extremal points.

1. Introduction

We discuss here the extension of the notion of convex set to generalized convex sets that are encountered in the vector valued calculus of variations and in partial differential equations. These are: polyconvex, quasiconvex and rank one convex set.

Contrary to classical convex analysis, where the notion of convex set precedes the one of convex function; this is not the case for the generalized ones. This is of course due to historical reasons. Morrey introduced the notions of polyconvex, quasiconvex and rank one convex functions in 1952 (although the terminology is the one of Ball). It was not until the systematic studies of partial differential equations and inclusions by Dacorogna-Marcellini and Müller-Šverák that the equivalent definitions for sets became an important issue. Moreover these notions were essentially seen through the different generalized convex hulls, leading somehow to terminologies that do not exactly covers the same concepts. One of the aims of the present paper is to try to imitate as much as possible the classical approach of convex analysis in the present context. This will, we hope, allow to clarify the situation.

In order to describe the content of our article, we have to get back to classical convex analysis. Here are important facts that we will try to mimic in the generalized context.

1) A set E is convex if and only if its indicator function

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \in E \\ +\infty & \text{if } x \notin E \end{cases}$$

is convex.

2) Important facts concerning convex sets are the separation and Carathéodory theorems.

3) The convex hull of a set E is the smallest convex set, denoted $\operatorname{co} E$, that contains E. As *consequences* of this definition, one finds that if

$$\begin{aligned} \overline{\mathcal{F}}_E &= \{f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} : f|_E \leq 0\} \\ \mathcal{F}_E &= \{f : \mathbb{R}^m \to \mathbb{R} : f|_E \leq 0\} \end{aligned}$$

then

$$\operatorname{co} E = \left\{x \in \mathbb{R}^m : f(x) \leq 0, \text{ for every convex } f \in \overline{\mathcal{F}}_E\right\} \tag{1}$$

$$\overline{\operatorname{co} E} = \{x \in \mathbb{R}^m : f(x) \leq 0, \text{ for every convex } f \in \mathcal{F}_E\} \tag{2}$$

where $\overline{\operatorname{co} E}$ denotes the closure of $\operatorname{co} E$.

4) Minkowski theorem for the convex hull of extreme points of compact sets.

The article is organized as follows.

In Section 3, we define the notions of polyconvex, quasiconvex and rank one convex set. The first and the third one are straightforward and are equivalent, as they should be, to the polyconvexity and rank one convexity of the indicator function. The second one is more delicate. Indeed one would have liked to define it as equivalent to the quasiconvexity of the indicator function; but quasiconvex functions allowed to take the value $+\infty$ are, at the moment, poorly understood. We will give a definition of quasiconvex set which is compatible with many of the desired properties that should have such definition. Notably we will have that

$$E \text{ convex} \Rightarrow E \text{ polyconvex} \Rightarrow E \text{ quasiconvex} \Rightarrow E \text{ rank one convex}$$

and all counterimplications turn out to be false whenever $N, n \geq 2$. This last result is better than the corresponding one for functions, since we have examples of rank one convex functions that are not quasiconvex (cf. Šverák [15]) only when $n \geq 2$ and $N \geq 3$.

Separation and Carathéodory type theorems exist for polyconvex sets and we will discuss these extensions in Section 4.

In Section 5, we consider the definitions of polyconvex, quasiconvex and rank one convex hulls of a given set E denoted respectively $\operatorname{Pco} E, \operatorname{Qco} E, \operatorname{Rco} E$. They are, as they should be, the smallest polyconvex, quasiconvex and rank one convex set, respectively, that contains E. It turns out that for polyconvex sets (and in a similar way for rank one convex sets) we have

$$\operatorname{Pco} E = \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every polyconvex } f \in \overline{\mathcal{F}}_E\right\}$$

as for the convex case. However, the representation of the closure of the hulls analogous to (2) is not true for general sets. We will discuss this question in details introducing three more types of hulls, namely

$$\operatorname{Pco}_f E = \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every polyconvex } f \in \mathcal{F}_E\right\}$$

$$\operatorname{Qco}_f E = \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every quasiconvex } f \in \mathcal{F}_E\right\}$$

$$\operatorname{Rco}_f E = \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every rank one convex } f \in \mathcal{F}_E\right\}.$$

It turns out that, in general,

$$\overline{\operatorname{Pco} E} \underset{\neq}{\subset} \operatorname{Pco}_f E, \ \overline{\operatorname{Qco} E} \underset{\neq}{\subset} \operatorname{Qco}_f E \ \text{ and } \ \overline{\operatorname{Rco} E} \underset{\neq}{\subset} \operatorname{Rco}_f E.$$

However, if E is compact, then

$$\overline{\operatorname{Pco} E} = \operatorname{Pco}_f E.$$

In Section 6 we will introduce the notion of extreme points in these generalized senses and establish Minkowski type theorems.

2. Notations and preliminaries

We recall the notation below (cf. Dacorogna [4]) used in the context of polyconvexity.

Notation 2.1. (i) For $\xi \in \mathbb{R}^{N\times n}$ we let

$$T(\xi) = (\xi, \operatorname{adj}_2 \xi, \ldots, \operatorname{adj}_{N\wedge n} \xi) \in \mathbb{R}^{\tau(N,n)}$$

where $\operatorname{adj}_s\xi$ stands for the matrix of all $s \times s$ subdeterminants of the matrix ξ, $1 \le s \le N \wedge n = \min\{N, n\}$ and where

$$\tau = \tau(N,n) = \sum_{s=1}^{N\wedge n} \binom{N}{s}\binom{n}{s} \text{ and } \binom{N}{s} = \frac{N!}{s!\,(N-s)!}.$$

In particular if $N = n = 2$, then $T(\xi) = (\xi, \det \xi)$.

(ii) For $s \in \mathbb{N}$, let

$$\Lambda_s = \left\{ \lambda = (\lambda_1, ..., \lambda_s) : \ \lambda_i \ge 0, \ \sum_{i=1}^{s} \lambda_i = 1 \right\}.$$

We also introduce a useful notation when defining a quasiconvex set (cf. Definition 3.1).

Notation 2.2. Let Ω be the hypercube $(0,1)^n$ of $\mathbb{R}^n$. For an orthogonal transformation $R \in O(n)$,

– $W^{1,\infty}_{per}(R\,\Omega; \mathbb{R}^N)$ will denote the space of periodic functions in $W^{1,\infty}(R\,\Omega; \mathbb{R}^N)$, i.e. functions u verifying $u(Rx) = u(R(x + e_i))$ for all vectors e_i of the canonical basis of $\mathbb{R}^n$ and all $x \in \Omega$;

– $\mathcal{W}_R$ will denote the space $W^{1,\infty}_{per}(R\,\Omega; \mathbb{R}^N)$ of functions whose gradients take only a finite number of values.

We now recall the different notions of convexity for functions.

Definition 2.1. (i) A function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is said to be *convex* if

$$f(\lambda\xi + (1-\lambda)\eta) \le \lambda f(\xi) + (1-\lambda)\ f(\eta)$$

for every $\lambda \in [0,1]$ and every $\xi, \eta \in \mathbb{R}^m$.

(ii) A function $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ is said to be *polyconvex* if there exists a convex function $g : \mathbb{R}^{\tau(N,n)} \longrightarrow \mathbb{R} \cup \{+\infty\}$ such that

$$f(\xi) = g(T(\xi)).$$

(iii) A Borel measurable function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be *quasiconvex* if

$$f(\xi)\ \operatorname{meas}(U) \le \int_U f(\xi + D\varphi(x))\, dx$$

for every bounded open set $U \subset \mathbb{R}^n$, $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W_0^{1,\infty}\left(U; \mathbb{R}^N\right)$.

(iv) A function $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ is said to be *rank one convex* if

$$f(\lambda\xi + (1-\lambda)\eta) \le \lambda f(\xi) + (1-\lambda)\ f(\eta)$$

for every $\lambda \in [0,1]$ and every $\xi, \eta \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(\xi - \eta) = 1$.

(v) A function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is said to be *separately convex* if

$$f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda)\, f(\eta)$$

for every $\lambda \in [0,1]$ and every $\xi, \eta \in \mathbb{R}^m$ with $\xi - \eta = se_i$, for some $s \in \mathbb{R}$ and $i \in \{1, ..., m\}$ (e_i denoting the i^{th}-vector of the canonical basis of $\mathbb{R}^m$).

(vi) A Borel measurable function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be *quasiaffine* if both f and $-f$ are quasiconvex.

Remark 2.1. A good definition of quasiconvex functions equivalent to the weak lower semicontinuity of the corresponding integral taking the value $+\infty$ is not available at the moment. Moreover, if we allow it in the above definition, then the known implication

$$f \text{ quasiconvex } \Rightarrow f \text{ rank one convex}$$

is no longer true.

Equivalent conditions for polyconvexity and quasiconvexity are given in the next result. For the proofs see, respectively, Dacorogna [4] and Šverák [15].

Theorem 2.1. *(i) A function $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ is polyconvex if and only if*

$$f\left(\sum_{i=1}^{\tau+1} \lambda_i \xi_i\right) \leq \sum_{i=1}^{\tau+1} \lambda_i f(\xi_i)$$

whenever $(\lambda_1, ..., \lambda_{\tau+1}) \in \Lambda_{\tau+1}$ and

$$T\left(\sum_{i=1}^{\tau+1} \lambda_i \xi_i\right) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i).$$

(ii) A Borel measurable function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex if and only if

$$f(\xi) \leq \int_{R\Omega} f(\xi + D\varphi(x))\, dx$$

for $\Omega := (0,1)^n$ and every $R \in O(n)$, $\varphi \in W^{1,\infty}_{per}(R\Omega; \mathbb{R}^N)$ and $\xi \in \mathbb{R}^{N\times n}$.

The different envelopes are then defined as

$$\begin{aligned} Cf &= \sup\{g \le f : g \text{ convex}\}, \\ Pf &= \sup\{g \le f : g \text{ polyconvex}\}, \\ Qf &= \sup\{g \le f : g \text{ quasiconvex}\}, \\ Rf &= \sup\{g \le f : g \text{ rank one convex}\}, \\ Sf &= \sup\{g \le f : g \text{ separately convex}\}. \end{aligned}$$

As well known we have that, provided $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$, the following implications hold

$$\begin{aligned} &f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex} \\ &\Rightarrow f \text{ rank one convex } \Rightarrow f \text{ separately convex} \end{aligned}$$

and thus

$$Cf \le Pf \le Qf \le Rf \le Sf \le f.$$

3. Generalized notions of convexity

We start giving the generalized definitions of convexity for sets.

Definition 3.1. (i) We say that $E \subset \mathbb{R}^m$ is *convex* if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$, then

$$\lambda\xi + (1-\lambda)\eta \in E.$$

(ii) We say that $E \subset \mathbb{R}^{N\times n}$ is *polyconvex* if there exists a convex set $K \subset \mathbb{R}^{\tau(N,n)}$ such that

$$\pi(K \cap T(\mathbb{R}^{N\times n})) = E,$$

where π denotes the orthogonal projection of (the first component of) $\mathbb{R}^{\tau(N,n)}$ in $\mathbb{R}^{N\times n}$. Equivalently, E is polyconvex if there exists a convex set $K \subset \mathbb{R}^{\tau(N,n)}$ such that

$$\{\xi \in \mathbb{R}^{N\times n} : T(\xi) \in K\} = E.$$

(iii) We say that $E \subset \mathbb{R}^{N\times n}$ is *quasiconvex* if we have

$$\left.\begin{array}{l} \xi + D\varphi(x) \in E, \ \text{a.e. } x \in R\Omega, \\ \text{for some } R \in O(n) \text{ and } \varphi \in \mathcal{W}_R \end{array}\right\} \Rightarrow \xi \in E$$

(Ω denoting the hypercube $(0,1)^n$).

(iv) Let $E \subset \mathbb{R}^{N\times n}$. We say that E is *rank one convex* if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$ such that $\operatorname{rank}(\xi - \eta) = 1$, then

$$\lambda\xi + (1-\lambda)\eta \in E.$$

(v) We say that $E \subset \mathbb{R}^m$ is *separately convex* if for every $\lambda \in [0,1]$ and $\xi, \eta \in E$ such that $\xi - \eta = se_i$, for some $s \in \mathbb{R}$ and $i \in \{1, ..., m\}$ (e_i denoting the i^{th}-vector of the canonical basis of $\mathbb{R}^m$), then

$$\lambda\xi + (1-\lambda)\eta \in E.$$

Remark 3.1. (i) The operator π introduced in the above definition is more precisely defined as follows. If

$$X = (X_1, ..., X_{\tau(N,n)}) \text{ then } \pi(X) = (X_1, ..., X_{N\times n}).$$

In particular, if $N = n = 2$ and $X = (\xi, \delta) \in \mathbb{R}^{2\times 2} \times \mathbb{R}$, then $\pi(X) = \xi$.

(ii) The definitions of convex, rank one convex and separately convex sets are standard.

(iii) In what concerns polyconvexity, the more usual way to define it is with the condition in Theorem 3.1 below. However, the two conditions turn out to be equivalent. With our definition we get some coherence with the analogous notion for functions.

We note that one could think, in view of Definition 2.1 (*ii*), that a set E is polyconvex if $T(E)$ is convex. This is however not true. Consider, for example, the polyconvex set $E = \{I, \xi\}$, where I is the identity matrix and $\xi = \operatorname{diag}(2,0)$. Then $T(E) = \{(I,1), (\xi, 0)\}$ which is not convex.

(iv) The best definition for quasiconvex sets is unclear. Several definitions have already been considered (see Dacorogna-Marcellini [5], Müller [11], Zhang [18]). The one we propose here is consistent with known properties for functions and have most properties which are desirable (cf. Theorem 3.2 below).

We first give an equivalent condition for polyconvexity.

Theorem 3.1. *Let $E \subset \mathbb{R}^{N\times n}$. The following conditions are equivalent.*

(i) E is polyconvex.

(ii)

$$\left.\begin{array}{c} \sum_{i=1}^{I} \lambda_i T(\xi_i) = T\left(\sum_{i=1}^{I} \lambda_i \xi_i\right) \\ \xi_i \in E,\ (\lambda_1, ..., \lambda_I) \in \Lambda_I \end{array}\right\} \Rightarrow \sum_{i=1}^{I} \lambda_i \xi_i \in E.$$

Moreover one can take $I = \tau(N, n) + 1$.

(iii) Denoting by $\operatorname{co} T(E)$ *the convex hull of* $T(E)$,

$$E = \pi(\operatorname{co} T(E) \cap T(\mathbb{R}^{N \times n}))$$

or equivalently

$$E = \{\xi \in \mathbb{R}^{N \times n} : \ T(\xi) \in \operatorname{co} T(E)\}.$$

Proof. $(i) \Rightarrow (ii)$. Suppose

$$\sum_{i=1}^{I} \lambda_i T(\xi_i) = T\left(\sum_{i=1}^{I} \lambda_i \xi_i\right), \tag{3}$$

for some $\xi_i \in E$ and $(\lambda_1, ..., \lambda_I) \in \Lambda_I$. By hypothesis, $\xi_i \in \pi(K \cap T(\mathbb{R}^{N \times n}))$ for some convex set $K \subset \mathbb{R}^{\tau(N,n)}$ and so $T(\xi_i) \in K$. Therefore $\sum_{i=1}^{I} \lambda_i T(\xi_i) \in \operatorname{co} K = K$ and, by (3), we conclude that $\sum_{i=1}^{I} \lambda_i \xi_i \in E$.

The fact that we can take $I = \tau(N, n) + 1$ in (ii) is a consequence of Carathéodory theorem (see Dacorogna [4]).

$(ii) \Rightarrow (iii)$. We have to see that $E = \pi(\operatorname{co} T(E) \cap T(\mathbb{R}^{N \times n}))$. Evidently E is contained in the set in the right hand side. For the reverse inclusion, consider $\xi \in \pi(\operatorname{co} T(E) \cap T(\mathbb{R}^{N \times n}))$. So, $T(\xi) \in \operatorname{co} T(E)$ and we can write

$$T(\xi) = \sum_{i=1}^{I} \lambda_i T(\xi_i)$$

for some $\xi_i \in E$ and $(\lambda_1, ..., \lambda_I) \in \Lambda_I$. We then use (ii) to get that $\xi \in E$, as wished.

$(iii) \Rightarrow (i)$ This is immediate. $\square$

The next result shows the relation between the notions of convexity for sets and the corresponding notions for functions (the proof is straightforward).

Proposition 3.1. *Let* $E \subset \mathbb{R}^{N \times n}$ *and* χ_E *denote the indicator function of* E:

$$\chi_E(\xi) = \begin{cases} 0 & \text{if } \xi \in E \\ +\infty & \text{if } \xi \notin E. \end{cases}$$

Then E *is, respectively, convex, polyconvex, rank one convex or separately convex, if and only if* χ_E *is, respectively, convex, polyconvex, rank one convex or separately convex.*

Remark 3.2. One would have liked to have the same result for quasiconvex sets but, as already discussed, quasiconvex functions taking the value $+\infty$ are not considered here.

The convexity conditions are related in the following way.

Theorem 3.2. *Let $E \subset \mathbb{R}^{N\times n}$. We have the following implications*

$$\begin{aligned} &E \text{ convex } \Rightarrow E \text{ polyconvex} \Rightarrow E \text{ quasiconvex} \\ &\Rightarrow E \text{ rank one convex} \Rightarrow E \text{ separately convex.} \end{aligned}$$

All counterimplications are false, as soon as $N, n \geq 2$.

Remark 3.3. We will see (cf. Proposition 5.2) that, as for the convex case: E, respectively, polyconvex, quasiconvex, rank one convex or separately convex implies that $\operatorname{int} E$ is also, respectively, polyconvex, quasiconvex, rank one convex or separately convex. However, this is not anymore true for $\overline{E}$. Indeed we will give (cf. Proposition 5.2) an example of a bounded polyconvex set $E \subset \mathbb{R}^{2\times 2}$ with $\overline{E}$ not even separately convex.

Proof. Part 1. We only prove the implications related to the notion of quasiconvexity since the others are trivial and well known.
(i) We prove that if E is polyconvex then E is quasiconvex. Assume that

$$\xi + D\varphi(x) \in E, \text{ a.e. } x \in R\Omega$$

for some $R \in O(n)$ and $\varphi \in \mathcal{W}_R$. We can write $D\varphi(x) \in \{\eta_1, ..., \eta_k\}$, *a.e.* $x \in R\Omega$ for some η_i such that $\xi + \eta_i \in E$, $i = 1, ..., k$. Defining

$$\lambda_i = \operatorname{meas}\{x \in R\Omega : \ D\varphi(x) = \eta_i\},$$

we have $\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1$. Since φ is periodic and the functions adj_s are quasiaffine ($s = 1, ..., N \wedge n$) we have

$$T(\xi) = \int_{R\Omega} T(\xi + D\varphi(x))\, dx = \sum_{i=1}^{k} \lambda_i T(\xi + \eta_i).$$

Using the polyconvexity of the set E we obtain that $\xi \in E$.

(ii) We now prove that if a set E is quasiconvex then it is rank one convex. Let $\xi, \eta \in E$ be such that $\operatorname{rank}(\xi - \eta) = 1$ and $\lambda \in (0, 1)$. We will prove that $\lambda\xi + (1-\lambda)\eta \in E$. To achieve this, it is enough to find $R \in O(n)$ and $\varphi \in \mathcal{W}_R$ such that

$$\lambda\xi + (1-\lambda)\eta + D\varphi(x) \in \{\xi, \eta\}, \text{ a.e. } x \in R\Omega$$

or equivalently

$$D\varphi(x) \in \{(1-\lambda)(\xi-\eta), -\lambda(\xi-\eta)\}, \ a.e.\ x \in R\Omega.$$

The result will then follows from the quasiconvexity of E. The construction of such φ is standard for relaxation theorems (see, for example, Dacorogna [4]). We just outline the proof. Since $\operatorname{rank}(\xi-\eta)=1$, we can write $\xi-\eta = a\otimes\nu$ with $a\in\mathbb{R}^N$ and ν a unit vector in $\mathbb{R}^n$. Choose $R\in O(n)$ any orthogonal transformation such that $Re_1=\nu$ (e_1 denoting the first vector of the canonical basis) and define the function $h:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$h(s) = \begin{cases} s, & 0\le s\le\lambda \\ \lambda, & \lambda\le s\le 1 \end{cases}$$

and $h(s+1)=h(s)+\lambda$, $\forall\ s\in\mathbb{R}$. Then $\varphi(x) = -\lambda(\xi-\eta)x + a\,h(\langle x;\nu\rangle)$ satisfies the required conditions, which finishes the proof.

Part 2. We will next see that the reverse implications are, in general, not true.

(i) There are polyconvex sets which are not convex. Consider, for example, the set $E=\{\xi,\eta\}\subset\mathbb{R}^{2\times 2}$, where $\xi=\operatorname{diag}(1,0)$ and $\eta=\operatorname{diag}(0,1)$.

(ii) Quasiconvexity does not imply polyconvexity. Consider the matrices (cf. Dacorogna [4])

$$\xi_1=\begin{pmatrix}1&0\\2&0\end{pmatrix},\quad \xi_2=\begin{pmatrix}0&1\\0&1\end{pmatrix},\quad \xi_3=\begin{pmatrix}-1&-1\\0&0\end{pmatrix}$$

and

$$\eta=\begin{pmatrix}0&0\\2/3&1/3\end{pmatrix}.$$

We have

$$T(\eta)=\frac{1}{3}T(\xi_1)+\frac{1}{3}T(\xi_2)+\frac{1}{3}T(\xi_3).$$

The set $E=\{\xi_1,\xi_2,\xi_3\}$ is not a polyconvex set since $\eta\notin E$. However, it is quasiconvex. Suppose $\xi+D\varphi\in E$ for some $\varphi\in\mathcal{W}_R$ where $R\in O(2)$. Since $\operatorname{rank}(\xi_i-\xi_j)=2$ for $i\ne j$, we have that the solution of this three gradient problem is an affine function (cf. Šverák [13], [14], Zhang [20]) that is to say $\xi+D\varphi$ is identically equal to one of the matrices ξ_i. Using then the periodicity of φ it results that $\xi=\xi_i\in E$. We can then conclude that E is quasiconvex.

(iii) Rank one convexity does not imply quasiconvexity. We should again draw the attention to the fact that our result is better for sets than for functions. We prove this assertion in two steps.

Step 1. There are (cf. Kirchheim [7]) $\eta_1, ..., \eta_k \in \mathbb{R}^{2\times 2}$ such that $\operatorname{rank}(\eta_i - \eta_j) = 2$, $\forall\ i \neq j$ and there exists $\xi \notin \{\eta_1, ..., \eta_k\}$ and $u \in u_\xi + W_0^{1,\infty}((0,1)^2; \mathbb{R}^2)$ where $Du_\xi(x) \equiv \xi$ such that $Du(x) \in \{\eta_1, ..., \eta_k\}$, *a.e.* in $(0,1)^2$.

Step 2. Let $E = \{\eta_1, ..., \eta_k\}$. Since there are no rank one connections between the matrices η_i, the set E is rank one convex. We will see that E is not quasiconvex. Let u be the function mentioned in Step 1. Since u is Lipschitz and has affine boundary data, we can write $u = u_\xi + \varphi$ for some $\varphi \in W_0^{1,\infty}((0,1)^2; \mathbb{R}^2)$. Besides $Du(x) = \xi + D\varphi(x) \in E$, *a.e.* in $(0,1)^2$, but $\xi \notin E$, which ensures that E is not quasiconvex.

(iv) Separate convexity does not imply rank one convexity. Indeed, the set $E = \{\xi, \eta\} \subset \mathbb{R}^{2\times 2}$, where

$$\xi = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is separately convex but not rank one convex. □

4. Separation results for polyconvex sets

We next deal with the problem of separating polyconvex sets generalizing in this way known results in the convex context.

Theorem 4.1. *Let E be a polyconvex set of $\mathbb{R}^{N\times n}$.*
(i) If $\eta \notin E$ or $\eta \in \partial E$, then there exists $\beta \in \mathbb{R}^{\tau(N,n)} \setminus \{0\}$ such that

$$\langle \beta; T(\eta) - T(\xi) \rangle \leq 0, \ \forall\ \xi \in E.$$

(ii) If E is compact and $\eta \notin E$, then there exists $\beta \in \mathbb{R}^{\tau(N,n)} \setminus \{0\}$ such that

$$\langle \beta; T(\eta) \rangle < \inf_{\xi \in E} \{\langle \beta; T(\xi) \rangle\}.$$

Proof. (i) Since E is polyconvex, if $\eta \notin E$ then $T(\eta) \notin \operatorname{co} T(E)$; in the case $\eta \in \partial E$ then we get $T(\eta) \in \partial \operatorname{co} T(E)$. In both cases, using the separation theorem for convex sets we obtain the existence of β satisfying

$$\langle \beta; T(\eta) - X \rangle \leq 0, \ \forall\ X \in \operatorname{co} T(E),$$

and, in particular, for $X \in T(E)$ as desired.

(ii) This stronger result can be obtained using the strong separation theorem for the closed convex set $\operatorname{co} T(E)$. □

As a consequence of the previous separation theorem we have the characterization of a polyconvex set given in the following result. This is an extension of the classical version for convex sets which ensures that a closed convex set is the intersection of the closed half-spaces containing the set.

Theorem 4.2. *A compact set $E \subset \mathbb{R}^{N\times n}$ is polyconvex if and only if*

$$E = \{\xi \in \mathbb{R}^{N\times n} : \varphi(\xi) \geq 0, \text{ for every quasiaffine } \varphi \text{ with } \varphi_{|E} \geq 0\}.$$

Proof. Let E be a compact polyconvex set and ξ_0 be such that $\varphi(\xi_0) \geq 0$ for every quasiaffine φ satisfying $\varphi_{|E} \geq 0$. We will see that $\xi_0 \in E$. If this was not the case, then, from Theorem 4.1 *(ii)*,

$$\langle \beta; T(\xi_0)\rangle < c < \inf_{\xi\in E}\{\langle \beta; T(\xi)\rangle\}$$

for some $\beta \in \mathbb{R}^{\tau(N,n)} \setminus \{0\}$ and $c \in \mathbb{R}$. Defining $C = c - \inf_{\xi\in E}\{\langle \beta; T(\xi)\rangle\}$ and the quasiaffine function

$$\psi(\xi) = \langle \beta; T(\xi)\rangle + C - \langle \beta; T(\xi_0)\rangle$$

we get a contradiction since $\psi(\xi_0) = C < 0$ but, since $\psi_{|E} \geq 0$ we should have $\psi(\xi_0) \geq 0$.

The reverse inclusion is evident. □

5. Generalized convex hulls

Having defined the generalized notions of convexity, we are now in position to introduce the concepts of generalized convex hulls. We follow the same procedure as in the classical convex case.

Definition 5.1. The *polyconvex*, *quasiconvex*, *rank one convex* and *separately convex hulls* of a set $E \subset \mathbb{R}^{N\times n}$ are, respectively, the smallest polyconvex, quasiconvex, rank one convex and separately convex sets containing E and are respectively denoted by $\operatorname{Pco} E$, $\operatorname{Qco} E$, $\operatorname{Rco} E$ and $\operatorname{Sco} E$.

From the discussion made in Section 3, the following inclusions hold:

$$E \subset \operatorname{Sco} E \subset \operatorname{Rco} E \subset \operatorname{Qco} E \subset \operatorname{Pco} E \subset \operatorname{co} E.$$

As we note below (cf. Remark 5.2) there are some authors who have adopted other definitions for the rank one convex hull, but this one is more consistent with the convex case. Besides, with the above definitions one has the following result (cf. Dacorogna-Marcellini [5]) whose proof follows in a straightforward manner from Theorem 5.5 below.

Proposition 5.1. *Let E be a subset of $\mathbb{R}^{N\times n}$ and χ_E be its indicator function. Then*

$$\begin{aligned} P\chi_E &= \chi_{\mathrm{Pco}\,E} \\ R\chi_E &= \chi_{\mathrm{Rco}\,E} \\ S\chi_E &= \chi_{\mathrm{Sco}\,E} \end{aligned}$$

where $P\chi_E$, $R\chi_E$ and $S\chi_E$ are, respectively, the polyconvex, rank one convex and separately convex envelopes of χ_E.

In the following we will give some representations of the hulls defined above. We start giving two characterizations of the polyconvex hull of a set. The second one, which has been proved in Dacorogna-Marcellini [5], is a consequence of Carathéodory theorem and is the equivalent to what is obtained in the convex case.

Theorem 5.1. *Let $E \subset \mathbb{R}^{N\times n}$. Then*
(i) $\mathrm{Pco}\,E = \pi(\mathrm{co}\,T(E) \cap T(\mathbb{R}^{N\times n}))$,
(ii) $\mathrm{Pco}\,E =$

$$\left\{\xi \in \mathbb{R}^{N\times n} :\ T(\xi) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i),\ \xi_i \in E,\ (\lambda_1, ..., \lambda_{\tau+1}) \in \Lambda_{\tau+1}\right\}.$$

In particular, if E is compact, then $\mathrm{Pco}\,E$ *is also compact and if E is open, then* $\mathrm{Pco}\,E$ *is also open.*

Proof. (i) We prove the first representation of $\mathrm{Pco}\,E$. It is clear that $\mathrm{Pco}\,E \subset \pi(\mathrm{co}\,T(E) \cap T(\mathbb{R}^{N\times n}))$. For the other inclusion we start noting that, since $\mathrm{Pco}\,E$ is polyconvex, by definition,

$$\mathrm{Pco}\,E = \pi(K \cap T(\mathbb{R}^{N\times n}))$$

for some convex set $K \subset \mathbb{R}^{\tau(N,n)}$. Since $E \subset \mathrm{Pco}\,E$, K must contain $T(E)$ and, consequently, must contain $\mathrm{co}\,T(E)$, from that the desired inclusion follows.

(ii) For this second representation of $\mathrm{Pco}\,E$, denoting by Y the set on the right hand side, it immediately follows, from the definition of polyconvex set, that $Y \subset \mathrm{Pco}\,E$. Moreover, one easily verifies that Y is a polyconvex set containing E which implies that $\mathrm{Pco}\,E \subset Y$.

For the assertion concerning compact sets, it is trivial that $\mathrm{Pco}\,E$ is bounded if E is compact. Let then $\xi_\nu \in \mathrm{Pco}\,E$ with $\xi_\nu \to \xi$. By the first representation of $\mathrm{Pco}\,E$, $T(\xi_\nu) \in \mathrm{co}\,T(E)$, which is a compact set since

$T(E)$ is compact. Then $T(\xi) = \lim T(\xi_\nu) \in \operatorname{co} T(E)$ and thus $\xi \in \operatorname{Pco} E$ as wished.

Finally, it can be seen, using an inductive argument, that, if

$$T(\xi) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i),$$

for some $\xi, \xi_i \in \mathbb{R}^{N\times n}$ and $(\lambda_1, ..., \lambda_{\tau+1}) \in \Lambda_{\tau+1}$, then

$$T(\xi + \eta) = \sum_{i=1}^{\tau+1} \lambda_i T(\xi_i + \eta), \ \forall \ \eta \in \mathbb{R}^{N\times n}.$$

From this and (ii), it easily follows that $\operatorname{Pco} E$ is open if E is open. □

We now give a different representation of the polyconvex hull, using the separation results of the previous section.

Theorem 5.2. *Let $E \subset \mathbb{R}^{N\times n}$ be such that* $\operatorname{Pco} E$ *is compact. Then*

$$\operatorname{Pco} E = \{\xi \in \mathbb{R}^{N\times n} : \ \varphi(\xi) \geq 0, \textit{ for every quasiaffine } \varphi \textit{ with } \varphi_{|E} \geq 0\}.$$

Proof. The set in the right hand side is polyconvex and contains E, then it contains $\operatorname{Pco} E$. On the other hand, since $\operatorname{Pco} E$ is polyconvex and compact then, by Theorem 4.2 we have

$$\operatorname{Pco} E = \{\xi \in \mathbb{R}^{N\times n} : \ \varphi(\xi) \geq 0, \text{ for every quasiaffine } \varphi \text{ with } \varphi_{|\operatorname{Pco} E} \geq 0\}.$$

Since any quasiaffine function φ with $\varphi_{|\operatorname{Pco} E} \geq 0$ verifies also $\varphi_{|E} \geq 0$, one gets

$$\{\xi \in \mathbb{R}^{N\times n} : \ \varphi(\xi) \geq 0, \text{ for every quasiaffine } \varphi \text{ with } \varphi_{|E} \geq 0\} \subset \operatorname{Pco} E,$$

which finishes the proof. □

We next give a representation for the quasiconvex hull, similar to (ii) of Theorem 5.1. This representation is however weaker than the one obtained in the polyconvex case since we cannot obtain the representation formula in a prescribed finite number of steps.

Theorem 5.3. *Let $E \subset \mathbb{R}^{N\times n}$. Let $\mathrm{Q}_0\mathrm{co}E = E$ and define by induction the sets*

$$\mathrm{Q}_{i+1}\mathrm{co}E = \left\{ \xi \in \mathbb{R}^{N\times n} : \begin{array}{c} \exists \ R \in O(n), \ \varphi \in \mathcal{W}_R \textit{ such that} \\ \xi + D\varphi(x) \in \mathrm{Q}_i\mathrm{co}E, \ \textit{a.e. } x \in R\Omega \end{array} \right\}, \ i \geq 0.$$

Then $\operatorname{Qco} E = \cup_{i\in\mathbb{N}} \mathrm{Q}_i\mathrm{co}E$.

In particular, if E is open, then $\operatorname{Qco} E$ *is also open.*

Proof. By definition of quasiconvex set and by induction, we have $\mathrm{Q}_i\mathrm{co}E \subset \mathrm{Qco}\, E$, for every i and thus $\cup_{i\in\mathbb{N}}\mathrm{Q}_i\mathrm{co}E \subset \mathrm{Qco}\, E$. The reverse inclusion follows from the fact that $\cup_{i\in\mathbb{N}}\mathrm{Q}_i\mathrm{co}E$ is, as we will see, a quasiconvex set.

Let $R \in O(n)$, $\varphi \in \mathcal{W}_R$ and $\xi + D\varphi(x) \in \cup_{i\in\mathbb{N}}\mathrm{Q}_i\mathrm{co}E$, a.e. $x \in R\Omega$. One has

$$D\varphi(x) \in \{\eta_1, ..., \eta_k\} \text{ a.e. } x \in R\Omega, \text{ with}$$

$$\mathrm{meas}\{x \in R\Omega : \ D\varphi(x) = \eta_i\} > 0, \ i = 1, ..., k.$$

Moreover, $\xi + \eta_i \in \mathrm{Q}_{\alpha(i)}\mathrm{co}E$ for some $\alpha(i) \in \mathbb{N}$. Let $s = \max\{\alpha(1), ..., \alpha(k)\}$. Since $\mathrm{Q}_i\mathrm{co}E \subset \mathrm{Q}_{i+1}\mathrm{co}E$, we have, for all $i = 1, ..., k$, $\xi + \eta_i \in \mathrm{Q}_s\mathrm{co}E$. Thus $\xi + D\varphi(x) \in \mathrm{Q}_s\mathrm{co}E$ and, by definition, we get $\xi \in \mathrm{Q}_{s+1}\mathrm{co}E \subset \cup_{i\in\mathbb{N}}\mathrm{Q}_i\mathrm{co}E$; the quasiconvexity of this last set follows.

Under the hypothesis of E being an open set, one easily gets, using induction arguments, that each $\mathrm{Q}_i\mathrm{co}E$ is open. By the preceding representation of $\mathrm{Qco}\, E$ it follows that this set is also open. □

The analogous representation for the rank one convex hull of a set is given in the result below (for the proof, see Dacorogna-Marcellini [5]).

Theorem 5.4. *Let $E \subset \mathbb{R}^{N\times n}$. Let $\mathrm{R}_0\mathrm{co}E = E$ and define by induction the sets*

$$\mathrm{R}_{i+1}\mathrm{co}E = \left\{ \xi \in \mathbb{R}^{N\times n} : \begin{array}{c} \xi = \lambda A + (1-\lambda)B, \ \lambda \in [0,1], \\ A, B \in \mathrm{R}_i\mathrm{co}E, \ \mathrm{rank}(A-B) = 1 \end{array} \right\}, \ i \geq 0.$$

Then $\mathrm{Rco}\, E = \cup_{i\in\mathbb{N}}\mathrm{R}_i\mathrm{co}E$.

In particular, if E is open, then $\mathrm{Rco}\, E$ *is also open.*

Remark 5.1. (i) Similar construction and results can be obtained for $\mathrm{Sco}\, E$.

(ii) The last assertion of the theorem follows, as in the quasiconvex case, from the fact that each $\mathrm{R}_i\mathrm{co}E$ is open if E itself is open.

(iii) In general it is not true that rank one convex hulls or separately convex hulls of compact sets are compact (see Aumann-Hart [1] and Kolář [9]).

We will now consider representations of the convex hulls through functions as we can get in the convex case.

Notation 5.1. Given a set $E \subset \mathbb{R}^{N\times n}$, we consider the following sets of functions

$$\begin{aligned}\overline{\mathcal{F}}_E &= \left\{f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\} : f|_E \leq 0\right\} \\ \mathcal{F}_E &= \left\{f : \mathbb{R}^{N\times n} \to \mathbb{R} : f|_E \leq 0\right\}.\end{aligned}$$

With the above notation, one has, for $E \subset \mathbb{R}^{N\times n}$,

$$\operatorname{co} E = \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every convex } f \in \overline{\mathcal{F}}_E\right\} \tag{4}$$

$$\overline{\operatorname{co} E} = \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every convex } f \in \mathcal{F}_E\right\} \tag{5}$$

where $\overline{\operatorname{co} E}$ denotes the closure of the convex hull of E.

Analogous representations to (4) can be obtained in the polyconvex, rank one convex and separately convex cases. However, (5) can only be generalized to the polyconvex case if the sets are compact (see Theorem 5.6). When dealing with the other notions of convexity, (5) is not true, even if compact sets are considered.

Theorem 5.5. *Let $E \subset \mathbb{R}^{N\times n}$, then*

$$\begin{aligned}\operatorname{Pco} E &= \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \textit{ for every polyconvex } f \in \overline{\mathcal{F}}_E\right\} \\ \operatorname{Rco} E &= \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \textit{ for every rank one convex } f \in \overline{\mathcal{F}}_E\right\} \\ \operatorname{Sco} E &= \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \textit{ for every separately convex } f \in \overline{\mathcal{F}}_E\right\}.\end{aligned}$$

Proof. We prove the first identity, the others being analogous. Let us call X the set in the right hand side. Evidently X is a polyconvex set containing E and thus $\operatorname{Pco} E \subset X$. Consider now $\xi \in X$. Since $\chi_{\operatorname{Pco} E}$ is a polyconvex function of $\overline{\mathcal{F}}_E$, one has $\chi_{\operatorname{Pco} E}(\xi) \leq 0$ and consequently $\xi \in \operatorname{Pco} E$ obtaining the other inclusion. □

We next introduce some new sets which will allow a better understanding of the closure of the different hulls.

Definition 5.2. For a set E of $\mathbb{R}^{N\times n}$, let

$$\begin{aligned}\operatorname{co}_f E &= \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every convex } f \in \mathcal{F}_E\right\} \\ \operatorname{Pco}_f E &= \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every polyconvex } f \in \mathcal{F}_E\right\} \\ \operatorname{Qco}_f E &= \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every quasiconvex } f \in \mathcal{F}_E\right\} \\ \operatorname{Rco}_f E &= \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every rank one convex } f \in \mathcal{F}_E\right\} \\ \operatorname{Sco}_f E &= \left\{\xi \in \mathbb{R}^{N\times n} : f(\xi) \leq 0, \text{ for every separately convex } f \in \mathcal{F}_E\right\}.\end{aligned}$$

Remark 5.2. (i) As well known,

$$\mathrm{co}_f\, E = \overline{\mathrm{co}\, E}.$$

(ii) The above sets are all closed because any separately convex function taking only finite values is continuous. Besides, they are, respectively, (according to our definitions) convex, polyconvex, quasiconvex, rank one convex and separately convex.

(iii) Some authors (see, for example, Müller-Šverák [12], Šverák [16], Zhang [19]), when dealing with quasiconvexity and rank one convexity, have adopted the above definitions for the hull of a set (in the generalized senses). They call laminate convex hull what we have called $\mathrm{Rco}\, E$.

(iv) As in Theorem 5.1, it can easily be shown that

$$\mathrm{Pco}_f\, E = \pi(\mathrm{co}_f\, T(E) \cap T(\mathbb{R}^{N\times n})).$$

We next see the relations between the closures of the convex hulls and the sets introduced in the above definition.

Theorem 5.6. *Given any set $E \subset \mathbb{R}^{N\times n}$ and denoting by $\overline{\mathrm{Pco}\, E}$, $\overline{\mathrm{Qco}\, E}$, $\overline{\mathrm{Rco}\, E}$ and $\overline{\mathrm{Sco}\, E}$ the closure of, respectively, the polyconvex, quasiconvex, rank one convex and separately convex hulls of E, we have*

$$\overline{\mathrm{Pco}\, E} \subset \mathrm{Pco}_f\, E$$
$$\overline{\mathrm{Qco}\, E} \subset \mathrm{Qco}_f\, E$$
$$\overline{\mathrm{Rco}\, E} \subset \mathrm{Rco}_f\, E$$
$$\overline{\mathrm{Sco}\, E} \subset \mathrm{Sco}_f\, E.$$

In general, the four inclusions are strict. However if E is compact, then

$$\mathrm{Pco}\, E = \overline{\mathrm{Pco}\, E} = \mathrm{Pco}_f\, E.$$

Remark 5.3. We call the attention to the fact that, contrary to what was stated in Dacorogna-Marcellini [5] , in general, $\overline{\mathrm{Pco}\, E} \neq \mathrm{Pco}_f\, E$, unless E is compact. We should also draw the attention (cf. Proposition 5.2) that in general the sets $\overline{\mathrm{Pco}\, E}$, $\overline{\mathrm{Qco}\, E}$, $\overline{\mathrm{Rco}\, E}$, $\overline{\mathrm{Sco}\, E}$ are not even separately convex.

Proof. Since $\mathrm{Pco}_f\, E$ is a closed polyconvex set containing E then $\overline{\mathrm{Pco}\, E} \subset \mathrm{Pco}_f\, E$. In the same way we get the inclusions for the quasiconvex, rank one convex and separately convex cases.

We now deal with the fact that the inclusions are strict. The first one follows (cf. Proposition 5.2 below) from the fact that there are polyconvex sets whose closure is not polyconvex though $\mathrm{Pco}_f\, E$ is always a polyconvex set. If we assume E *to be compact then we have, as we will see,*

$$\mathrm{Pco}\, E = \overline{\mathrm{Pco}\, E} = \mathrm{Pco}_f\, E.$$

By Theorem 5.1, in this case, $\mathrm{Pco}\, E$ is compact and then $\mathrm{Pco}\, E = \overline{\mathrm{Pco}\, E}$. We will prove that $\mathrm{Pco}_f\, E \subset \mathrm{Pco}\, E$. We start noting that, since E is compact, $T(E)$ is compact and thus $\mathrm{co}\, T(E)$ is also compact. Considering $\xi \in \mathrm{Pco}_f\, E$ then, since the function $\eta \mapsto \mathrm{dist}(T(\eta), \mathrm{co}\, T(E))$ is a polyconvex function, $\mathrm{dist}(T(\xi), \mathrm{co}\, T(E)) = 0$. Since $\mathrm{co}\, T(E)$ is closed, we can deduce that $T(\xi) \in \mathrm{co}\, T(E)$ and thus, $\xi \in \mathrm{Pco}\, E$.

Next we use an example due to Casadio [2] (or equivalent examples by Aumann-Hart [1] and Tartar [17]) which will give at once $\overline{\mathrm{Qco}\, E} \underset{\neq}{\subset} \mathrm{Qco}_f\, E$, $\overline{\mathrm{Rco}\, E} \underset{\neq}{\subset} \mathrm{Rco}_f\, E$ and $\overline{\mathrm{Sco}\, E} \underset{\neq}{\subset} \mathrm{Sco}_f\, E$. The second non inclusion was already observed in Dacorogna-Marcellini [5] . Consider the following four diagonal matrices of $\mathbb{R}^{2\times 2}$

$$\xi_1 = \mathrm{diag}(-1, 0),\ \xi_2 = \mathrm{diag}(1, -1),\ \xi_3 = \mathrm{diag}(2, 1),\ \xi_4 = \mathrm{diag}(0, 2).$$

Since $\mathrm{rank}(\xi_i - \xi_j) = 2$ for $i \neq j$, the set $E = \{\xi_1, \xi_2, \xi_3, \xi_4\}$ is rank one convex. It is also quasiconvex, the argument is the same as in the proof of Theorem 3.2, assertion *(ii)* of Part 2, here using the non existence of non-affine Lipschitz functions whose gradient takes four possible values with no rank one connections (cf. Chlebík-Kirchheim [3]). However, any separately convex function $f \in \mathcal{F}_E$ and consequently any rank one convex or quasiconvex function in $\mathcal{F}_E$, has $f(0) \leq 0$ (see [5]). Thus $0 \in \mathrm{Sco}_f\, E$, but $0 \notin \overline{\mathrm{Qco}\, E}$. □

We can write

$$\overline{\mathrm{Sco}\, E} \subset \overline{\mathrm{Rco}\, E} \subset \overline{\mathrm{Qco}\, E} \subset \overline{\mathrm{Pco}\, E} \subset \overline{\mathrm{co}\, E} = \mathrm{co}_f\, E$$

and also

$$\mathrm{Sco}_f\, E \subset \mathrm{Rco}_f\, E \subset \mathrm{Qco}_f\, E \subset \mathrm{Pco}_f\, E \subset \overline{\mathrm{co}\, E} = \mathrm{co}_f\, E.$$

Moreover, the same example and arguments used in the proof of Theorem 5.6 (see also Proposition 5.2) shows that, in general,

$$\mathrm{Sco}_f\, E \not\subset \overline{\mathrm{Rco}\, E}, \qquad \mathrm{Rco}_f\, E \not\subset \overline{\mathrm{Qco}\, E} \qquad \text{and} \qquad \mathrm{Qco}_f\, E \not\subset \overline{\mathrm{Pco}\, E}.$$

However, if E is compact one has $\mathrm{Qco}_f\, E \subset \overline{\mathrm{Pco}\, E}$.

We draw the attention to the fact that several characterizations of the sets in Definition 5.2 have been used in the literature according to the specific needs of each situation. These sets can be written in terms of measures (cf. Kirchheim [8], Müller [11]) or using the distance function (cf. Zhang [18]): if $E \subset \mathbb{R}^{N\times n}$ is compact, then

$$\mathrm{Qco}_f\, E = \left\{\xi \in \mathbb{R}^{N\times n} :\ Q\mathrm{dist}(\xi, E) = 0\right\},$$

where $Q\mathrm{dist}(\cdot, E)$ is the quasiconvex envelope of the function $\mathrm{dist}(\cdot, E)$.

We next prove, as already mentioned in Remark 3.3, that the interior of generalized convex sets keeps the convexity (in the generalized sense), but that, contrary to the classical convex case, this is not true for the closure.

Proposition 5.2. *(i) Let $E \subset \mathbb{R}^{N\times n}$ be, respectively, a polyconvex, quasiconvex, rank one convex or separately convex set. Then* $\mathrm{int}E$ *is also, respectively, polyconvex, quasiconvex, rank one convex or separately convex.*

(ii) There is $E \subset \mathbb{R}^{2\times 2}$ a polyconvex and bounded set such that $\overline{E}$ is not separately convex.

Proof. (i) We present the proof in the context of polyconvexity. For the other convexities the proof is analogous. It is sufficient to prove that $\mathrm{Pco}(\mathrm{int}E) = \mathrm{int}E$. The non trivial inclusion is $\mathrm{Pco}(\mathrm{int}E) \subset \mathrm{int}E$. Since E is polyconvex, evidently

$$\mathrm{Pco}(\mathrm{int}E) \subset \mathrm{Pco}\, E = E. \tag{6}$$

On the other hand, $\mathrm{int}E$ is open and thus (cf. Theorem 5.1) $\mathrm{Pco}(\mathrm{int}E)$ is also open. From (6), it follows then the desired inclusion.

(ii) We define

$$E = \left\{\pm \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} :\ 0 < x < 1\right\}.$$

It is a bounded set and $\overline{E}$ is not separately convex. In fact, let $\xi_1 = \mathrm{diag}(1,0)$ and $\xi_2 = \mathrm{diag}(-1,0)$, one has $\xi_1, \xi_2 \in \overline{E}$, but $\lambda\xi_1 + (1-\lambda)\xi_2 \notin \overline{E}$ for any $0 < \lambda < 1$.

We now show that E is polyconvex. Let $\xi_1, ..., \xi_6 \in E$ and suppose

$$T(\xi) = \sum_{i=1}^{6} \lambda_i T(\xi_i), \text{ for some } (\lambda_1, ..., \lambda_6) \in \Lambda_6. \tag{7}$$

We have to see that $\xi \in E$. We can write $\{1, ..., 6\} = I_+ \cup I_-$ for some I_+ and I_- such that

$$\xi_i = \begin{pmatrix} 1 & 0 \\ 0 & x_i \end{pmatrix} \text{ if } i \in I_+ \text{ and } \xi_i = \begin{pmatrix} -1 & 0 \\ 0 & -x_i \end{pmatrix} \text{ if } i \in I_-,$$

where $0 < x_i < 1,\ i = 1, ..., 6$. In any case $\det \xi_i = x_i$.

If $I_+ = \emptyset$ or $I_- = \emptyset$ then it is clear that $\xi \in E$. We will see that the other case: $I_+ \neq \emptyset$ and $I_- \neq \emptyset$, is not an admissible one. In fact, from (7), we can write

$$\xi = \begin{pmatrix} \sum\limits_{i \in I_+} \lambda_i - \sum\limits_{i \in I_-} \lambda_i & 0 \\ 0 & \sum\limits_{i \in I_+} \lambda_i x_i - \sum\limits_{i \in I_-} \lambda_i x_i \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

and $\det \xi = \alpha\beta = \sum\limits_{i=1}^{6} \lambda_i x_i$.

Then $|\alpha| < \sum_{i=1}^{6} \lambda_i = 1$, $|\beta| < \sum_{i=1}^{6} \lambda_i x_i$ and thus $|\alpha\beta| < \sum_{i=1}^{6} \lambda_i x_i$, which is a contradiction. □

6. Extreme points

An important tool in convex analysis is the notion of extreme point. In a straightforward manner we can define it for generalized convex sets as follows (cf. Dacorogna-Marcellini [5]).

Definition 6.1. (i) If $E \subset \mathbb{R}^m$ is convex, $\xi \in E$ is said to be an extreme point of E in the convex sense if

$$\left.\begin{array}{l} \xi = \lambda\xi_1 + (1-\lambda)\xi_2 \\ \lambda \in (0,1),\ \xi_1, \xi_2 \in E \end{array}\right\} \Rightarrow \xi_1 = \xi_2 = \xi.$$

For an arbitrary set $E \subset \mathbb{R}^m$, the set of extreme points of $\operatorname{co} E$ will be denoted E^c_{ext}.

(ii) If $E \subset \mathbb{R}^{N\times n}$ is polyconvex, $\xi \in E$ is said to be an extreme point of E in the polyconvex sense if

$$\left.\begin{array}{l} T(\xi) = \sum\limits_{i=1}^{\tau+1} \lambda_i T(\xi_i), \\ (\lambda_1, ..., \lambda_{\tau+1}) \in \Lambda_{\tau+1},\ \lambda_i > 0,\ \xi_i \in E \end{array}\right\} \Rightarrow \xi_i = \xi,\ i = 1, ..., \tau+1.$$

For an arbitrary set $E \subset \mathbb{R}^{N\times n}$, the set of extreme points of $\operatorname{Pco} E$ will be denoted E^p_{ext}.

(iii) If $E \subset \mathbb{R}^{N\times n}$ is quasiconvex, $\xi \in E$ is said to be an extreme point of E in the quasiconvex sense if

$$\left.\begin{array}{l} \xi + D\varphi(x) \in E,\ a.e.\ x \in R\Omega, \\ \Omega = (0,1)^n,\ R \in O(n),\ \varphi \in \mathcal{W}_R \end{array}\right\} \Rightarrow D\varphi \equiv 0.$$

For an arbitrary set $E \subset \mathbb{R}^{N \times n}$, the set of extreme points of $\operatorname{Qco} E$ will be denoted E^q_{ext}.

(iv) If $E \subset \mathbb{R}^{N \times n}$ is rank one convex, $\xi \in E$ is said to be an extreme point of E in the rank one convex sense if

$$\left.\begin{array}{c} \xi = \lambda \xi_1 + (1-\lambda)\xi_2 \\ \lambda \in (0,1),\ \xi_1, \xi_2 \in E,\ \operatorname{rank}(\xi_1 - \xi_2) \le 1 \end{array}\right\} \Rightarrow \xi_1 = \xi_2 = \xi.$$

For an arbitrary set $E \subset \mathbb{R}^{N \times n}$, the set of extreme points of $\operatorname{Rco} E$ will be denoted E^r_{ext}.

(v) If $E \subset \mathbb{R}^m$ is separately convex, $\xi \in E$ is said to be an extreme point of E in the separately convex sense if

$$\left.\begin{array}{c} \xi = \lambda \xi_1 + (1-\lambda)\xi_2 \\ \lambda \in (0,1),\ \xi_1, \xi_2 \in E,\ \xi_1 - \xi_2 = s\, e_i, \\ \text{with } s \in \mathbb{R} \text{ and } e_i \text{ a vector of the canonical basis of } \mathbb{R}^m \end{array}\right\} \Rightarrow \xi_1 = \xi_2 = \xi.$$

For an arbitrary set $E \subset \mathbb{R}^m$, the set of extreme points of $\operatorname{Sco} E$ will be denoted E^s_{ext}.

We next see the relations between the sets of extreme points for the different notions of convexity.

Proposition 6.1. *Let $E \subset \mathbb{R}^{N \times n}$. Then*

$$E^c_{ext} \subset E^p_{ext} \subset E^q_{ext} \subset E^r_{ext} \subset E^s_{ext} \subset E.$$

Proof. The non trivial inclusions are those related to E^q_{ext}, the set of extreme points of $\operatorname{Qco} E$, but it can be obtained with the same arguments used in the proof of Theorem 3.2, Part 1, and we opt not to repeat them. □

Minkowski theorem (often better known as Krein-Milman theorem which is its infinite dimensional version) assures that the convex hull of a compact set coincides with the convex hull of its extreme points. We next deal with the generalization of this result to the other convexities. We start with the polyconvex case (see also Dacorogna-Tanteri [6]).

Theorem 6.1. *Let $E \subset \mathbb{R}^{N \times n}$ be a compact set. Then*

$$\operatorname{Pco} E = \operatorname{Pco} E^p_{ext}.$$

Proof. One inclusion is trivial: $\operatorname{Pco} E^p_{ext} \subset \operatorname{Pco} E$, since $E^p_{ext} \subset \operatorname{Pco} E$. We will next show the reverse inclusion. We start remarking that

$$\operatorname{Pco} E = \pi(\operatorname{co} T(E) \cap T(\mathbb{R}^{N\times n}))$$
$$\operatorname{Pco} E^p_{ext} = \pi(\operatorname{co} T(E^p_{ext}) \cap T(\mathbb{R}^{N\times n})).$$

Let $\xi \in \operatorname{Pco} E$. We will see that $\xi \in \operatorname{Pco} E^p_{ext}$. By the above characterization of $\operatorname{Pco} E$ we have $T(\xi) \in \operatorname{co} T(E)$. Moreover, by Minkowski theorem, and using the fact that $T(E)$ is compact, we have

$$\operatorname{co} T(E) = \operatorname{co}(T(E)^c_{ext}),$$

where $T(E)^c_{ext}$ is the set of extreme points of $\operatorname{co} T(E)$ (in the convex sense).

We will next prove that

$$T(E)^c_{ext} \subset T(E^p_{ext}),$$

which will finish the proof.

Let then $X \in T(E)^c_{ext}$. In particular, $X \in T(E)$ and we can write $X = T(\eta)$ with $\eta \in E$. It suffices then to see that $\eta \in E^p_{ext}$. Suppose that

$$T(\eta) = \sum_{i=1}^{\tau+1} \lambda_i T(\eta_i)$$

for some $(\lambda_1, ..., \lambda_{\tau+1}) \in \Lambda_{\tau+1}$, $\lambda_i > 0$, $\eta_i \in \operatorname{Pco} E$. Noting that, since $\eta_i \in \operatorname{Pco} E$ then $T(\eta_i) \in \operatorname{co} T(E)$, it immediately follows, from the fact that $T(\eta)$ is an extreme point of $\operatorname{co} T(E)$, that $\eta_i = \eta$ for every i, that is to say η is an extreme point of $\operatorname{Pco} E$. The proof is finished. □

As remarked in Kirchheim [8], the result above is not true for quasiconvex, rank one convex or separately convex hulls (see Example 6.1 below). Even though, for these cases, a weaker result can be proved (cf. Theorem 6.2). We reproduce the proof of Matoušek-Plecháč [10], which is also seen to apply to the quasiconvex case. See also Zhang [18] for the quasiconvex case.

Theorem 6.2. *Let $E \subset \mathbb{R}^{N\times n}$ be a bounded set and E^{qf}_{ext}, E^{rf}_{ext}, E^{sf}_{ext} denote, respectively, the set of extreme points of $\operatorname{Qco}_f E$ (in the quasiconvex sense), the set of extreme points of $\operatorname{Rco}_f E$ (in the rank one convex sense) and the set of extreme points of $\operatorname{Sco}_f E$ (in the separately convex sense). Then*

$$\operatorname{Qco}_f E = \operatorname{Qco}_f E^{qf}_{ext} \quad \operatorname{Rco}_f E = \operatorname{Rco}_f E^{rf}_{ext} \quad \text{and} \quad \operatorname{Sco}_f E = \operatorname{Sco}_f E^{sf}_{ext}.$$

Proof. We divide the proof in two steps. The first is common to the three convexities and we present it in the context of quasiconvexity. In the second step we consider separately the quasiconvex and the rank one convex cases (this last being analogous to the separately convex case). In all what follows we will denote by $\overline{E}^{qf}_{ext}$ the closure of E^{qf}_{ext}

Step 1. We remark that, for any set $K \subset \mathbb{R}^{N\times n}$, since Qco_f is automatically closed, $\mathrm{Qco}_f K = \mathrm{Qco}_f \overline{K}$. Thus, it is enough to prove that $\mathrm{Qco}_f E = \mathrm{Qco}_f \overline{E}^{qf}_{ext}$. The inclusion $\mathrm{Qco}_f \overline{E}^{qf}_{ext} \subset \mathrm{Qco}_f E$ is trivial. It remains to verify the reverse inclusion. We use a contradiction argument.

Suppose there is some $\eta \in \mathrm{Qco}_f E \backslash \mathrm{Qco}_f \overline{E}^{qf}_{ext}$, then, by definition, there exists a quasiconvex function $f : \mathbb{R}^{N\times n} \longrightarrow \mathbb{R}$ with $f \in \mathcal{F}_{\overline{E}^{qf}_{ext}}$, such that $f(\eta) > 0$.

Now let

$$M = \max_{\mathrm{Qco}_f E} f \qquad \text{and} \qquad \mathcal{A} = \{\xi \in \mathrm{Qco}_f E : \ f(\xi) = M\}.$$

This set is nonempty and compact (since $\mathrm{Qco}_f E$ is compact and f is a continuous function). Thus, considering $\mathbb{R}^{N\times n}$ with the lexicographic order (the elements of $\mathbb{R}^{N\times n}$ being seen as vectors) one can consider the maximum element of $\mathcal{A}$, say ξ_0. We have $\xi_0 \notin E^{qf}_{ext}$, which follows from

$$0 < f(\eta) \leq \max_{\mathrm{Qco}_f E} f = M = f(\xi_0).$$

As we will see in Step 2 this will lead to the existence of an element in $\mathcal{A}$ greater than ξ_0 for the lexicographic order, which is absurd.

Step 2. *Quasiconvex case.* Since $\xi_0 \in \mathrm{Qco}_f E \backslash E^{qf}_{ext}$, there are $R \in O(n)$ and $\varphi \in \mathcal{W}_R$ such that

$$\xi_0 + D\varphi(x) \in \mathrm{Qco}_f E, \ a.e. \ x \in R\Omega, \quad \text{with } D\varphi \not\equiv 0.$$

We can write

$$D\varphi(x) \in \{\xi_1, ..., \xi_k\} \text{ and } \lambda_i = \mathrm{meas}\{x \in R\Omega : \ D\varphi(x) = \xi_i\} > 0.$$

Since $\xi_0 + \xi_i \in \mathrm{Qco}_f E$, we have $f(\xi_0 + \xi_i) \leq M$. Consequently, by the quasiconvexity of f we get

$$M = f(\xi_0) \leq \int_{R\Omega} f(\xi_0 + D\varphi(x))\, dx = \sum_{i=1}^{k} \lambda_i f(\xi_0 + \xi_i) \leq M$$

implying $f(\xi_0 + \xi_i) = M$, $i = 1, ..., k$ that is $\xi_0 + \xi_i \in \mathcal{A}$. Finally, from the fact that $D\varphi \not\equiv 0$ and $0 = \int_{R\Omega} D\varphi(x)\, dx = \sum_{i=1}^{k} \lambda_i \xi_i$ we conclude that among the elements $\xi_0 + \xi_i$ there must be at least one which is greater than ξ_0 (in the lexicographic order) which contradicts the fact that ξ_0 is the maximum element of $\mathcal{A}$.

Rank one convex case. We recall that in this case the function f is a rank one convex function. Since $\xi_0 \in \operatorname{Rco}_f E \setminus E_{ext}^{rf}$, there are $\eta_1, \eta_2 \in \operatorname{Rco}_f E$, with $\operatorname{rank}(\eta_1 - \eta_2) \leq 1$ such that $\xi_0 = \lambda\eta_1 + (1-\lambda)\eta_2$ and $\xi_0 \neq \eta_1$, $\xi_0 \neq \eta_2$. As in the quasiconvex case we get $f(\eta_1) = f(\eta_2) = M$ and from $\xi_0 = \lambda\eta_1 + (1-\lambda)\eta_2$ it follows that η_1 or η_2 must be greater than ξ_0, which is a contradiction. □

As observed by Kirchheim [8], the example of Casadio [2] (or those of Aumann-Hart [1] and Tartar [17]) considered in the proof of Theorem 5.6 shows that, in general,

$$\operatorname{Qco} E_{ext}^q \neq \operatorname{Qco} E, \quad \operatorname{Rco} E_{ext}^r \neq \operatorname{Rco} E \quad \text{and} \quad \operatorname{Sco} E_{ext}^s \neq \operatorname{Sco} E.$$

Example 6.1. We consider a set of diagonal matrices which we identify with elements of $\mathbb{R}^2$. In particular, rank one convexity and separate convexity coincide.

Let

$$E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5,$$

where

$$E_1 = \{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1\},$$
$$E_2 = \{(x,1) \in \mathbb{R}^2 : \ 1 \leq x \leq 2\}, \quad E_3 = \{(0,y) \in \mathbb{R}^2 : \ 1 \leq y \leq 2\},$$
$$E_4 = \{(x,0) \in \mathbb{R}^2 : \ -1 \leq x \leq 0\}, E_5 = \{(1,y) \in \mathbb{R}^2 : \ -1 \leq y \leq 0\}.$$

Note that E is a compact rank one convex set and

$$E_{ext}^q \subset E_{ext}^r = \{\xi_1, \xi_2, \xi_3, \xi_4\},$$

where

$$\xi_1 = (-1,0), \ \xi_2 = (1,-1), \ \xi_3 = (2,1), \ \xi_4 = (0,2).$$

Thus, since there are no rank one connections between the elements ξ_i, $\operatorname{Qco} E_{ext}^q = E_{ext}^q$ and $\operatorname{Rco} E_{ext}^r = E_{ext}^r$. However, $E_{ext}^q \subset E_{ext}^r \subsetneqq E = \operatorname{Rco} E \subset \operatorname{Qco} E$.

In Dacorogna-Tanteri [6], it was also proved the existence of the Choquet function for the polyconvex case. The result is the following.

Theorem 6.3. *Let $E \subset \mathbb{R}^{N\times n}$ be a nonempty compact polyconvex set. Then there exists a polyconvex function $\varphi : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ such that*

$$E^p_{ext} = \{x \in E : \varphi(x) = 0\} \qquad \text{and} \qquad \varphi(x) \le 0 \Leftrightarrow x \in E.$$

Acknowledgements: We would like to thank Gisella Croce for interesting discussions on the subject of the article. The research of A. M. Ribeiro was partially supported by portuguese Fundação para a Ciência e Tecnologia (BD/10042/02).

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NOTE ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO AN ANISOTROPIC CRYSTALLINE CURVATURE FLOW

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The motion of closed polygonal curves in the plane moving under anisotropic crystalline curvature flows is investigated. It is known that the flow develops a singularity in finite time. We will discuss various kinds of situations of singularities, especially for convex curves. For nonconvex curves, a simple but suggestive example of singularity will be showed.

1. Introduction

In this paper we study the motion of shrinking polygonal curves in the plane $\mathbb{R}^2$, which is governed by an anisotropic crystalline curvature flow. Crystalline curvature flows were introduced by J.E. Taylor [20] and S. Angenent and M.E. Gurtin[4] (precise history is found in e.g., [1]). They established a new flamework of moving curves in the case where an interfacial energy density, defined on the curves, is not smooth and its Wulff shape is a convex polygon.

Crystalline curvature flows are formulated as follows: Let $\mathcal{P}$ be a simple closed curve in $\mathbb{R}^2$, and let $f(\boldsymbol{n})$ be an interfacial energy defined on the unit circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, i.e., f is a function of inward unit normal vector $\boldsymbol{n}$ of curve $\mathcal{P}$. The gradient flow of total interfacial energy on $\mathcal{P}$, $\int_{\mathcal{P}} f(\boldsymbol{n})\, ds$,

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yields a weighted curvature flow $v = (\sigma + \sigma'')K$. Here ds is an arc-length parameter, v is a velocity in the $\boldsymbol{n}$ direction, K is a curvature in the $\boldsymbol{n}$ direction ($K \equiv 1$ if $\mathcal{P}$ is a unit circle), and $\sigma(\theta) = f(\boldsymbol{n})$ (θ is a normal angle which satisfies $\boldsymbol{n} = -(\cos\theta, \sin\theta)$). We denoted $\sigma'' = d^2\sigma/d\theta^2$.

If the Wulff shape of σ, defined by $\mathcal{W}_\sigma = \bigcap_{\theta\in S^1}\{(x,y) \in \mathbb{R}^2 \mid x\cos\theta + y\sin\theta \le \sigma(\theta)\}$, is a convex polygon, then σ is not differentiable and the weighted curvature flow $v = (\sigma + \sigma'')K$ is not well-defined in the usual sense. In this case, $f = \sigma$ is called *crystalline energy.* When $\mathcal{W}_\sigma$ is an N_σ-sided polygon ($N_\sigma \ge 3$), its normal angle's set is defined as

$$\Phi_{N_\sigma} = \{\varphi_0, \varphi_1, \cdots, \varphi_{N_\sigma - 1}\},$$

where $\varphi_n \in S^1$ is a normal angle of the n-th edge satisfying $\varphi_0 < \varphi_1 < \cdots < \varphi_{N_\sigma-1} < \varphi_0 + 2\pi$ with $\varphi_{n+1} - \varphi_n < \pi$ for all n ($\varphi_{N_\sigma} = \varphi_0 \bmod 2\pi$). Then the Wulff polygon $\mathcal{W}_\sigma$ can be restated as follows:

$$\mathcal{W}_\sigma = \bigcap_{\varphi\in\Phi_{N_\sigma}} \{(x,y) \in \mathbb{R}^2 \mid x\cos\varphi + y\sin\varphi \le \sigma(\varphi)\}.$$

When σ is a crystalline, we restrict curves to piecewise linear curves in a specific class in the following way: A curve $\mathcal{P}$ has N vertices (x_j, y_j) ($j = 0, 1, \ldots, N-1$), which are labeled in an anticlockwise order with $(x_N, y_N) = (x_0, y_0)$. Let $\mathcal{S}_j = \{(1-t)(x_j, y_j) + t(x_{j+1}, y_{j+1}) \mid 0 \le t \le 1\}$ be the j-th edge of $\mathcal{P}$. Then we may express $\mathcal{P}$ as $\mathcal{P} = \bigcup_{j=0}^{N-1} \mathcal{S}_j$. Let θ_j be a normal angle of $\mathcal{S}_j$. We say that $\mathcal{P}$ is an *N-admissible curve* if the all normal angles belong to Φ_{N_σ} and the angles of adjacent edges in $\mathcal{P}$ are adjacent in Φ_{N_σ} ($\theta_N = \theta_0 \bmod 2\pi$).

For an admissible curve $\mathcal{P}$, the total interfacial crystalline energy is given by $\sum_{j=0}^{N-1} \sigma(\theta_j)d_j$ (d_j is the length of $\mathcal{S}_j$), and then the gradient flow (in the family of admissible curves) of this yields $v_j = \chi_j l_\sigma(\theta_j)/d_j$, where v_j is a normal velocity at $\mathcal{S}_j$ in the inward normal direction $\boldsymbol{n}_j = -(\cos\theta_j, \sin\theta_j)$, $l_\sigma(\theta_j)$ is a length of the n-th edge of $\mathcal{W}_\sigma$ satisfying $\varphi_n = \theta_j$, and χ_j is a transition number, which takes $+1$ (resp. -1) if $\mathcal{P}$ is convex (resp. concave) at $\mathcal{S}_j$ in the $\boldsymbol{n}_j$ direction; otherwise we set $\chi_j = 0$. The quantity

$$H_j = \chi_j \frac{l_\sigma(\theta_j)}{d_j}$$

is called *crystalline curvature*, and then $v_j = H_j$ is called *crystalline curvature flow.* Note that $\chi_j \equiv +1$ for all j if $\mathcal{P}$ is a convex polygon, and that the every crystalline curvature of $\mathcal{W}_\sigma$ equals $+1$ on each edge.

In this paper we consider the following generalized crystalline curvature flow

$$v_j = a(\theta_j)\chi_j |H_j|^\alpha \quad \text{on } \mathcal{S}_j, \tag{1}$$

for $j = 0, 1, \ldots, N-1$, where α is a positive parameter and $a(\theta)$ is a positive function which describes anisotropy of mobility.

Under the generalized crystalline curvature flow, each edge $\mathcal{S}_j$ keeps the same normal angle but moves in the $\boldsymbol{n}_j$ direction with the velocity v_j. Then we have the following system of ordinary differential equations

$$\dot{d}_j(t) = (\cot(\theta_{j+1}-\theta_j)+\cot(\theta_j-\theta_{j-1}))v_j - \frac{v_{j-1}}{\sin(\theta_j-\theta_{j-1})} - \frac{v_{j+1}}{\sin(\theta_{j+1}-\theta_j)},$$

for $j = 0, 1, \ldots, N-1$. Here and hereafter we denote $\dot{(\cdot)} = d(\cdot)/dt$. See, e.g., M.E. Gurtin [11]. The local existence and uniqueness of solutions of this problem follow from a general theory of system of ODEs. Therefore, if the initial curve $\mathcal{P}(0)$ is admissible, then the admissibility of a solution curve $\mathcal{P}(t)$ is preserved as long as all edges of the solution curve exist.

In this paragraph let us "decompose" the quantity H_j as follows. The following operator on $\mathbb{R}^{N_\sigma}$ is useful for this or other purposes: When $\varphi_n = \theta_j$, we define $\gamma_\sigma(\varphi_n)$ by

$$\gamma_\sigma(\varphi_n) = \frac{1-\cos(\varphi_n-\varphi_{n-1})}{\sin(\varphi_n-\varphi_{n-1})} + \frac{1-\cos(\varphi_{n+1}-\varphi_n)}{\sin(\varphi_{n+1}-\varphi_n)} > 0,$$

and define the operator Δ_σ by

$$(\Delta_\sigma\, u)_n = \frac{(\mathrm{D}_+\, u)_n - (\mathrm{D}_+\, u)_{n-1}}{\gamma_\sigma(\varphi_n)}, \quad (\mathrm{D}_+\, u)_n = \frac{u_{n+1}-u_n}{\sin(\varphi_{n+1}-\varphi_n)}.$$

Note that $\{\varphi_{n-1}, \varphi_{n+1}\} = \{\theta_{j-1}, \theta_{j+1}\}$. Thus we have $l_\sigma(\theta_j) = \gamma_\sigma(\theta_j)(\sigma + \Delta_\sigma\,\sigma)_j$ with $\sigma_j = \sigma(\theta_j)$. Therefore we can decompose curvatures into $H_j = (\sigma+\Delta_\sigma\,\sigma)_j K_j$ and $K_j = \chi_j\gamma_\sigma(\theta_j)/d_j$. K_j is sometimes called *discrete curvature*. The correspondence $K_j \sim K$ and $(\sigma+\Delta_\sigma\,\sigma)_j K_j \sim (\sigma+\sigma'')K$ are not only discrete analogue but also approximation when $N_\sigma \to \infty$, respectively. This is the reason why crystalline curvature flows are used as approximation scheme of curvature flows of smooth curves (see, e.g., [6], [8], [9], [21] and [22]).

The organization of this paper is as follows. In Section 2, several results will be shown in the case where the initial curve is a convex polygon. The theoretical results will be mentioned in the former half. We will consider the sublinear case $\alpha \in (0,1)$ mainly and show a blow-up rate in degenerate pinching (defined below) case. In the latter half, we will discuss some open problems. In Section 3, we will show an example of singularity in the case where the initial curve is a nonconvex. In Appendix, we will mention a numerical algorithm which is based on the numerical estimating method.

2. Convex case

A convex polygon is admissible if and only if $N = N_\sigma$ holds and $\theta_j = \varphi_j$ holds for all $j \in \mathcal{I} = \{0, 1, \dots, N-1\}$. Then evolution equations are given by

$$\dot{d}_j(t) = -\gamma_\sigma(\theta_j)(\Delta_\sigma v + v)_j, \quad j \in \mathcal{I},$$

where $v_j = a(\theta_j)H_j^\alpha$ is the j-th normal velocity with a positive crystalline curvature $H_j = l_\sigma(\theta_j)/d_j$. We note again that the normal angle of each edge does not change under the crystalline curvature flow. Therefore, if the initial convex polygon $\mathcal{P}(0)$ is admissible, then the admissibility and convexity of a solution polygon are preserved as long as the solution polygon exists.

The above evolution equations can be restated as the following system of ordinary differential equations with respect to $\{v_j\}$:

$$\text{(P)} \qquad \begin{cases} \dot{v}_j(t) = \alpha b_j^{-1/\alpha} {v_j}^{(\alpha+1)/\alpha}(\Delta_\sigma v + v)_j, & j \in \mathcal{I},\ t > 0, \\ v_N(t) = v_0(t),\ v_{-1}(t) = v_{N-1}(t), & t \geq 0, \\ v_j(0) = a(\theta_j)H_j(0)^\alpha, & j \in \mathcal{I}. \end{cases}$$

Here b_j's are positive constants given by $b_j = a(\theta_j)(\sigma + \Delta_\sigma \sigma)_j^\alpha$ (note that $v_j = b_j K_j^\alpha$ and $K_j = \gamma_\sigma(\theta_j)/d_j$), and $H_j(0)$ is the crystalline curvature of the j-th edge $\mathcal{S}_j$ of the initial polygon $\mathcal{P}(0)$.

It is known that for any initial admissible polygon, the maximum of solution $\{v_j\}$ blows up to infinity in a finite time, say T. This means that for any initial admissible polygon, the maximal time of preserving the admissibility is finite and this flow develops a singularity at $t = T$.

M.-H. Giga and Y. Giga [7] showed the detailed information on limiting shape at the final time T: if $\alpha \geq 1$ or there are no parallel edges, the solution polygon $\mathcal{P}(t)$ shrinks to a single point, i.e., $d_j(t) \to 0$ as $t \to T$ for all $j \in \mathcal{I}$; and if $\alpha \in (0, 1)$ and there exists at least one pair of parallel edges, the solution polygon $\mathcal{P}(t)$ shrinks to a single point or collapses to a line segment with a positive length. The latter phenomenon is called *degenerate pinching*. B. Andrews [2] gave a sufficient condition of degenerate pinching. In any case, the enclosed area of a solution polygon becomes zero at the final time T. See Figure 1 for numerical examples of single point extinction and degenerate pinching.

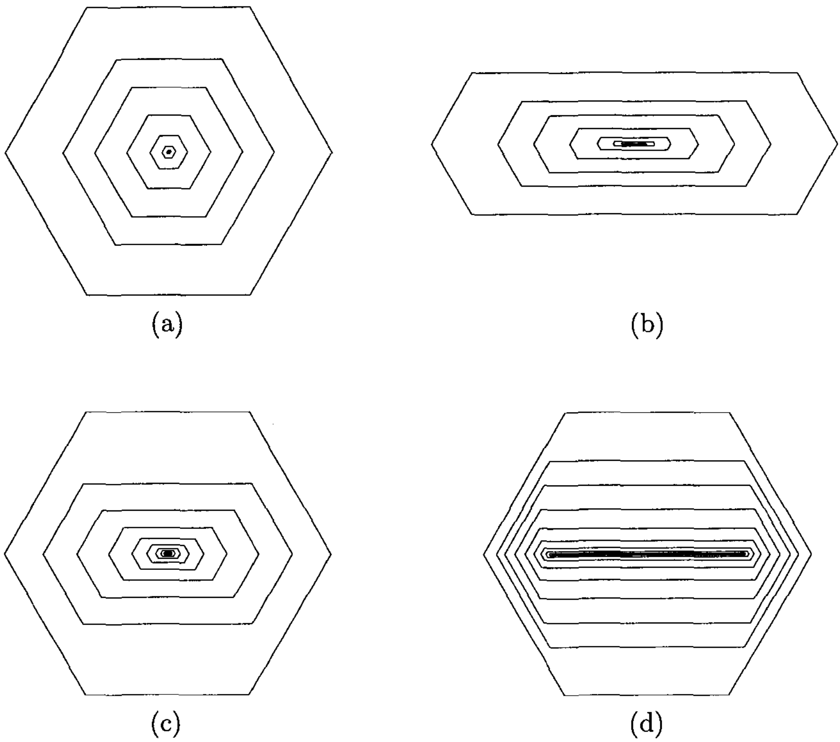

Figure 1. Single point extinction (a)(c) and degenerate pinching (b)(d). In each figure, the initial polygon $\mathcal{P}(0)$ is the outmost hexagon ($N = 6$), and from outside to inside, time evolution of $\mathcal{P}(t)$ is plotted. The initial polygons in (a), (c) and (d) are all the same regular hexagons (i.e., $d_j(0) \equiv const.$ for all j) and the initial polygon in (b) is not a regular hexagon with $d_0 = d_2 = d_3 = d_5$ and $d_1 = d_4 = 4d_0$. The parameters are the following: in (a) and (b), $\alpha = 0.5$ and $b_j \equiv const.$; in (c), $\alpha = 1.5$ and $b_0 = b_2 = b_3 = b_5$ and $b_1 = b_4 = 5b_0$; and in (d), $\alpha = 0.5$ and b_j's are the same as in (c).

Blow-up rate

We will characterize blow-up rate of solutions. As mentioned above, in single point extinction case $v_j(t)$ blows up for all j since $d_j(t)$ vanishes at $t = T$ simultaneously for all j (case Figure 1 (a)(c)), and in degenerate pinching case $v_{\min}(t)$ is bounded since $\liminf_{t\to T} d_{\max}(t) > 0$ holds (case Figure 1 (b)(d)); while $\lim_{t\to T} v_{\max}(t) = +\infty$ always holds (this is proved from $\liminf_{t\to T} d_{\min}(t) = 0$). In general, $v_{\min}(t)$ and $v_{\max}(t)$ are estimated from above and below by a specific blow-up rate, respectively, as follows (see [19] in case $\alpha = 1$, and its generalization [18]):

$$\begin{cases} v_{\min}(t) \le b_{\max}^{1/(\alpha+1)}((\alpha+1)(T-t))^{-\alpha/(\alpha+1)}, \\ v_{\max}(t) \ge b_{\min}^{1/(\alpha+1)}((\alpha+1)(T-t))^{-\alpha/(\alpha+1)}. \end{cases}$$

Here and hereafter, we use the notation $u_{\max}$ and $u_{\min}$ for $\max_{j\in\mathcal{I}} u_j$ and $\min_{j\in\mathcal{I}} u_j$, respectively.

This result implies that the generic lower bound of blow-up rate is $(T-t)^{-\alpha/(\alpha+1)}$. Moreover, if $\alpha > 1$, then there exists a positive constant C such that $v_{\max}(t) \le C(T-t)^{-\alpha/(\alpha+1)}$, that is, the blow-up rate in the case $\alpha > 1$ is exactly $(T-t)^{-\alpha/(\alpha+1)}$ (see [2]). The order $(T-t)^{-\alpha/(\alpha+1)}$ specifies blow-up rate in the following sense: If $b_j \equiv 1$ for all $j \in \mathcal{I}$, then $v_j(t) \equiv ((\alpha+1)(T-t))^{-\alpha/(\alpha+1)}$ is a special solution of (P) and a corresponding solution polygon shrinks to a single point homothetically. This is called *self-similar* solution.

Using this order, blow-up solutions can be classified as follows: We say that the solution undergoes a "type I blow-up" if the blow-up rate of the maximum of solution $\{v_j\}$ is at most the self-similar rate, that is,

$$\sup_{0<t<T} v_{\max}(t)(T-t)^{\alpha/(\alpha+1)} < \infty, \tag{2}$$

and that the solution undergoes a "type II blow-up" if (2) does not hold. A type II blow-up is sometimes called *fast blow-up*, because a solution $\{v_j\}$ undergoes a type II blow-up if and only if

$$\limsup_{t\to T} v_{\max}(t)(T-t)^{\alpha/(\alpha+1)} = \infty.$$

If $\alpha > 1$, the problem (P) has no fast blow-up solutions, that is, a type I blow-up occurs only, and if $\alpha = 1$, a type I and a type II blow-up are intermixed [2]; while if $\alpha \ge 1$, a solution polygon shrinks to a single point [7], even if a type II blow-up occurs. In general, the next lemma holds:

Lemma 2.1. *Let $\alpha > 0$. If a solution $\{v_j\}$ of (P) undergoes a type I blow-up, then the solution polygon shrinks to a single point.*

From this lemma, it holds that if degenerate pinching occurs, then a type II blow-up occurs. If $\alpha \in (0,1)$, and if the initial admissible polygon $\mathcal{P}(0)$ has at least one pair of parallel edges and a distance between them is sufficiently small compared with length of the edges, then degenerate pinching occurs [2]. Hence, by this lemma, a type II blow-up solution exists if $\alpha \in (0,1)$. The next proposition is the lower bound of a type II blow-up rate when degenerate pinching occurs.

Proposition 2.1. *Let $\alpha \in (0,1)$. Suppose that there exists a pair of parallel edges S_{j_0} and S_{j_1}, and that they do not disappear at the final time T, then for all $j \neq j_0, j_1$ the solutions v_j blow up to infinity at least the following rate:*

$$v_j(t) \geq C(T-t)^{-\alpha}, \quad j \neq j_0, j_1, \quad t \in [0,T),$$

for a generic constant $C > 0$.

Under an additional condition on monotonicity of a solution, we obtain the exact type II blow-up rate in degenerate pinching case.

Proposition 2.2. *Assume*

$$\text{(A)} \qquad (\Delta_\sigma v(0) + v(0))_j \geq 0, \quad j \in \mathcal{I}.$$

Let $\alpha \in (0,1)$. Suppose that there exists a pair of parallel edges S_{j_0} and S_{j_1}, and that they do not disappear at the final time T, then it holds that

$$v_j(t) \sim (T-t)^{-\alpha}, \quad j \neq j_0, j_1, \quad t \in [0,T).$$

Here and hereafter, $u \sim w$ means $c_1 u \leq w \leq c_2 u$ for generic constants $c_1 > 0$ and $c_2 > 0$.

Lemma 2.1 and Proposition 2.1 and Proposition 2.2 were proved in [17] (with the Wulff shape being regular) and in [18] (in general case), respectively.

In our knowledge, the above results (in case $\alpha \in (0,1)$) and several results (in case $\alpha = 1$) in [2] are only known about type II blow-up, and then its rate has not been completely classified. In addition, existence of type II blow-up solutions in single point extinction case, especially for $\alpha \in (0,1)$, is still open problem.

Discussion

We will discuss "possibilities" of behavior of type II blow-up solutions. Note that the condition (A) leads the monotonicity of the solutions by maximun principle, that is, $\dot{v}_j(t) \geq 0$ for all $j \in \mathcal{I}$ and $t \geq 0$. By the same argument, we can easily obtain that if the following condition

$$\text{(EM)} \qquad (\Delta_\sigma v(t_0) + v(t_0))_j \geq 0, \quad j \in \mathcal{I},$$

holds for some time $t = t_0 \geq 0$, then $\dot{v}_j(t) \geq 0$ for any $j \in \mathcal{I}$ and $t \geq t_0$. For curve-shortening problem $v = K$ (in case $\sigma(\theta) \equiv 1$ for $\theta \in S^1$ on smooth curves), it is shown that the solutions eventually become monotone increasing in time (so-called "eventual monotonicity") under certain conditions on initial data ([3]). What is the initial condition that has such eventual monotonicity hold? This is an open problem. However, the following lemma holds.

Lemma 2.2. *Suppose that the Wulff shape has no parallel edges. Then type II blow-up solutions do not satisfy (EM) for any* $t_0 \geq 0$.

Sketch of Proof. Assume that there exists a type II blow-up solution which satisfies (EM) for some $t_0 > 0$. We can prove the same assertion as in Theorem 4 and Corollary 3 in [18] under the condition (EM), from which it follows that there exists at least one pair of parallel edges. This contradicts the assumption on the Wulff shape. ∎

When the Wulff shape has no parallel edges, single point extinction only occurs, that is, all $v_j(t)$ blow up. However, this lemma says that for any $t \geq 0$ there exists $j_t \in \mathcal{I}$ such that $\dot{v}_{j_t}(t) < 0$. Namely, this lemma indicates that, in this case, all type II blow-up solutions "oscillate" infinite many times near the blow-up time. Is there such a blow-up solution? This is an open problem. And the blow-up rate of type II solutions in single point extinction case, especially for $\alpha \in (0,1)$, is also still open.

Next, let us consider the case where the Wulff shape has at least one pair of parallel edges $\mathcal{S}_{j_0}$ and $\mathcal{S}_{j_1}$ and, in addition, degenerate pinching singularity occurs. Assume that $\mathcal{S}_{j_0}$ and $\mathcal{S}_{j_1}$ do not disappear at $t = T$, and $\lim_{t \to T} d_j(t) = 0$ holds for $j \neq j_0, j_1$. The distance function $w(t)$ between $\mathcal{S}_{j_0}$ and $\mathcal{S}_{j_1}$ in their normal direction is given by $w(t) = \sum_{j_0 < j < j_1} d_j |\sin\theta_j|$. Then we have

$$w(t) \leq C(\min_{j \neq j_0, j_1} v_j)^{-1/\alpha}.$$

On the other hand, since $-\dot{w}(t) = v_{j_0} + v_{j_1}$ holds, and v_{j_0} and v_{j_1} do not blow up, we have $w(t) \sim T - t$. Thus we have

$$\min_{j \neq j_0, j_1} v_j \leq C(T-t)^{-\alpha}.$$

By virtue of Proposition 2.1, we get the following lemma.

Lemma 2.3. *Let* $\alpha \in (0,1)$. *Suppose that there exists a pair of parallel edges* $\mathcal{S}_{j_0}$ *and* $\mathcal{S}_{j_1}$, *and that they do not disappear at the final time* T, *then*

$$\min_{j \neq j_0, j_1} v_j \sim C(T-t)^{-\alpha}.$$

This lemma says that the slowest blow-up rate is exactly $(T-t)^{-\alpha}$. If the solutions satisfy (EM) at $t = t_0 \geq 0$, we can get the upper estimate $v_j(t) \leq$

$C(T-t)^{-\alpha}$ by the same argument as in the proof of Proposition 2.2. If (EM) does not hold, then $v_{\max}(t)$ could blow up faster than $(T-t)^{-\alpha}$. Therefore, blow-up set may decompose several parts by blow-up rates. By numerical conjectures in [15], even if there exist such type II blow-up solutions, the upper bound of their blow-up rate is at most $(T-t)^{-\alpha-\varepsilon}$ for any $\varepsilon > 0$. The numerical method for blow-up solutions is based on [13, 14]. For reader's convenience, we will mention the algorithm of this numerical scheme in Appendix.

3. An example of degenerate pinching singularity in nonconvex case

When the initial admissible curve is nonconvex, the asymptotic behavior of the solution curve, especially on limiting shape, may have variety, which is a contrast to the convex case: Convex solution curve shrinks to a single point or a line segment. We will show an example of variety later, in which a solution curve shrinks to the specific shape other than a point and a line segment. This example will indicate difficulty of convexity criterion. In general, if $\alpha \geq 1$ and $\sigma(\theta+\pi) = \sigma(\theta)$, $a(\theta+\pi) = a(\theta)$, then a solution curve with an N-admissible initial curve converges to a single point or an N'-admissible curve with $N' < N$ as t tends to $t' < \infty$, and eventually the solution curve shrinks to a single point in a finite time $T \geq t'$ ([7]). Although it seems that a solution curve becomes convex before it shrinks to a point at the final time T ([7]), it is known that some examples of point-extinction solutions preserving non-convexity ([16] and Remark 3.2 stated below). A gap is recognized in between these results, and so convexity criterion is ambiguous at this stage. For a smooth interfacial energy density σ, the Grayson-type convexity theorem is known ([10] and [5]).

Now we will show an example of a "capital L"-shaped (L-shaped in short) degenerate pinching singularity. Let the Wulff shape be a square ($N_\sigma = 4$) with $\varphi_n = \pi n/2$ and $l_\sigma(\varphi_n) \equiv 1$ for $n = 0, 1, 2, 3$, and let the initial curve $\mathcal{P}(0)$ be a 6-admissible elbow-like curve with $\theta_j = \varphi_j$ ($j = 0, 1, 2, 3$) and $\theta_j = \varphi_{j-4}$ ($j = 4, 5$). We assume symmetry of $\mathcal{P}(0)$ such as $d_0(0) = d_5(0)$, $d_1(0) = d_4(0)$ and $d_2(0) = d_3(0) = d_0(0) + d_1(0)$. See Figure 2.

Suppose $\alpha \in (0, 1)$, and $a(\varphi_n) = \mu > 0$ for $n = 0, 1$ and $a(\varphi_n) = 1$ for $n = 2, 3$. Then, from the symmetry of $\mathcal{P}(0)$, we have the following evolution equations:

$$\dot{d}_1 = -d_2^{-\alpha}, \quad \dot{d}_2 = -\mu d_1^{-\alpha} - d_2^{-\alpha},$$

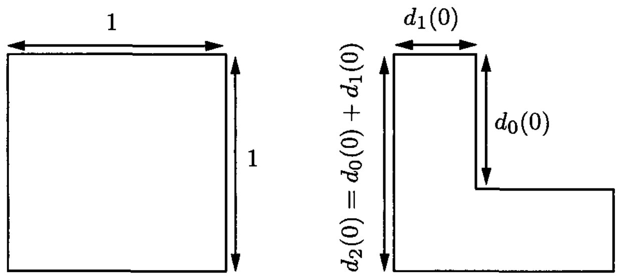

Figure 2. The Wulff square $\mathcal{W}_\sigma$ (left) and the *initial symmetric 6-admissible* elbow-like curve $\mathcal{P}(0)$ (right).

since $v_j = a(\theta_j)\chi_j d_j^{-\alpha}$, and $d_2(t) = d_0(t) + d_1(t)$ for $t \geq 0$.

Let us analyze the evolution equations. Put $d_2(t) = r(t)d_1(t)$. Note that $r(0) > 1$. Then we have

$$\dot{r} = d_1^{-(1+\alpha)} f(r), \quad f(r) = r^{1-\alpha} - r^{-\alpha} - \mu.$$

Since $f(1) = -\mu < 0$, $f'(r) > 0$, $f''(r) < 0$ and $\lim_{r\to\infty} = \infty$, there exists $r_* > 1$ such that $f(r) > 0$ if $r > r_*$. Therefore, if $r(0) > r_*$, then $\dot{r}(t) > 0$ and so $r(t) > r_*$ for any $t \geq 0$. Moreover, if $r(0) > r_*$, then we get $\dot{d}_2 \geq -(\mu + r(0)^{-\alpha})d_1^{-\alpha}$. Inequality $\mu + r(0)^{-\alpha} \leq 1$ holds if and only if $\mu \in (0,1)$ and $r(0) \geq (1-\mu)^{-1/\alpha}$. Hence we have $\dot{d}_2 \geq -d_1^{-\alpha}$ if $r(0) \geq \max\{r_* + \varepsilon, (1-\mu)^{-1/\alpha}\}$ for any $\varepsilon > 0$ and $\mu \in (0,1)$. From $\dot{d}_1 = -d_2^{-\alpha}$, we have

$$d_2^{-\alpha}\dot{d}_2 \geq d_1^{-\alpha}\dot{d}_1.$$

This yields

$$d_2(t)^{1-\alpha} \geq d_2(0)^{1-\alpha} + d_1(t)^{1-\alpha} - d_1(0)^{1-\alpha} \geq d_2(0)^{1-\alpha} - d_1(0)^{1-\alpha} > 0,$$

from which it follows that

$$d_2(t) \geq (d_2(0)^{1-\alpha} - d_1(0)^{1-\alpha})^{1/(1-\alpha)} \equiv C_0 > 0.$$

On the other hand, we have

$$\dot{d}_1 = -d_2^{-\alpha} \leq -d_2(0)^{-\alpha}.$$

Hence it holds that

$$d_1(t) \leq d_1(0) - t/d_2(0)^{\alpha}.$$

Now we have the following:

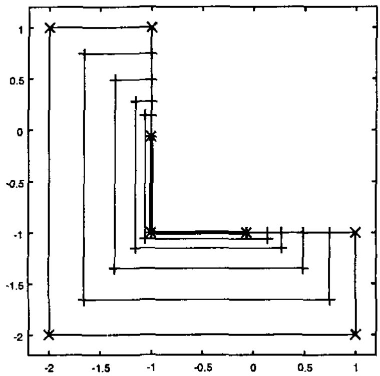

Figure 3. An L-shaped degenerate pinching singularity under the same assumption as in Lemma 3.1 with $\alpha = 1/2$, $\mu = 1/\sqrt{6}$ and $d_0(0) = d_5(0) = 2$, $d_1(0) = d_4(0) = 1$, $d_2(0) = d_3(0) = 3$. The outmost curve is the initial elbow-like curve. The solution curve evolves from outside to inside and finally it converges to an L-shaped curve.

Lemma 3.1. *Assume that* $\alpha \in (0,1)$, $\mu \in (0,1)$ *and that* $d_2(0)/d_1(0) \geq \max\{r_* + \varepsilon, (1-\mu)^{-1/\alpha}\}$ *for any fixed* $\varepsilon > 0$, *where* $r_* > 1$ *satisfies* $f(r_*) = 0$. *Then there exists* $T > 0$ *such that* $\lim_{t\to T} d_1(t) = 0$ *and* $\inf_{0<t<T} d_2(t) \geq C_0$ *hold.*

Remark 3.1. One can estimate r_* such as $r_* < (1+\mu)^{1/(1-\alpha)}$, and so if $d_2(0)/d_1(0) \geq \max\{(1-\mu)^{-1/\alpha}, (1+\mu)^{1/(1-\alpha)}\}$ held, then it is enough.

This lemma shows that the limiting shape of $\mathcal{P}(t)$ is an L-shaped curve. See Figure 3. Furthermore, one can easily obtain the extinction rate or the blow-up rate of solutions as follows: From $C_0 \leq d_2(t) \leq d_2(0)$, we have $\dot{d}_1 \sim -1$. Then we get $d_1(t) \sim T - t$ and $v_1(t) \sim (T-t)^{-\alpha}$.

Remark 3.2. (Nonconvex self-similar solutions) Put $\beta = d_2(0)/d_1(0)$. If $\mu = \mu(\beta) = (\beta - 1)/\beta^\alpha$, then there is the self-similar solution $\mathcal{P}(t) = R(t)\mathcal{P}(0)$, $R(t) = ((T-t)/T)^{1/\alpha}$ with $T = d_1(0)d_2(0)^\alpha/(1+\alpha)$. We have the following three cases: (i) Case $\alpha \in (0,1)$. For any $m > 0$ there exists a unique $\beta > 1$ such that $m = \mu(\beta)$ and the solution is self-similar. (ii) Case $\alpha = 1$. For any $m \in (0,1)$ there exists a unique $\beta > 1$ such that $m = \mu(\beta)$ and the solution is self-similar. (iii) Case $\alpha > 1$. Let $\beta_* = \alpha/(\alpha-1) > 1$ and $m_* = \mu(\beta_*)$. For $\mu = m_*$ the solution is self-similar if and only if $\beta = \beta_*$. Moreover, for any $m \in (0, m_*)$ there exists two constants $\beta_1 \in (1, \beta_*)$ and $\beta_2 > \beta_*$ such that $m = \mu(\beta_1) = \mu(\beta_2)$ and the solutions are self-similar,

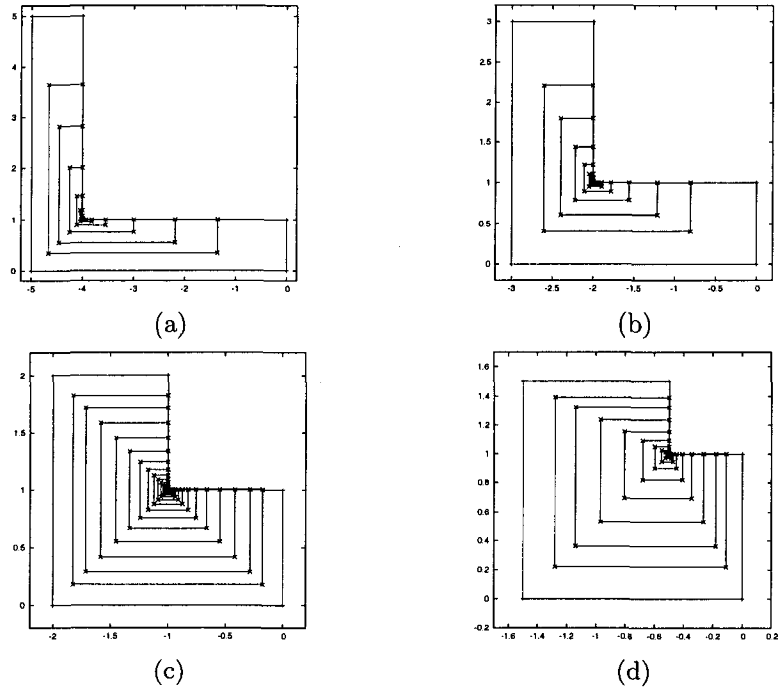

Figure 4. Numerical examples of nonconvex self-similar solutions starting from several initial elbow-like curves (see Remark 3.2): (a) $\alpha = 1/2 \in (0,1)$, $\beta = 5$, (b) $\alpha = 1/2 \in (0,1)$, $\beta = 3$, (c) $\alpha = 2 > 1$, $\beta = 2$ and (d) $\alpha = 1$, $\beta = 3/2$. The outmost curve is the initial elbow-like curve. The solution curve evolves from outside to inside homothetically and finally it shrinks to a single point. Note that the initial curve in (b) is the same as in Figure 3.

respectively. See Figure 4 for numerical examples.

These results coincide with the three cases in [16] exactly, after the following transformation has been done: $a = d_0$, $b = d_1$, $z_0 = (\mu-1)/(\mu+1)$, $c = d_1(0)/d_0(0)$, and the time rescale $t \mapsto \sqrt{2}^{1-\alpha} t/(1 - z_0)$.

Acknowledgments

The research was partially supported by Grant-in-Aid for Encouragement of Young Scientists (Ishiwata: No. 15740056; Yazaki: No. 15740073).

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Appendix: Numerical method of estimating blow-up time and rate

In [13, 14], Hirota and Ozawa developed a new numerical estimating method of blow-up time and $(T-t)^{-p}$ type blow-up rate of solutions to a system of ordinary differential equations.

We will estimate some blow-up rates numerically by using their method. Roughly speaking, their method is based on the following three parts:

(1) Arc-length transformation technique:
Let us consider the initial value problem for the following system of ordinary differential equations

$$\frac{d}{dt}y_j(t) = f_j(t, y_0, \dots, y_{N-1}), \quad j \in \mathcal{I}.$$

The next transformation is called *arc-length transformation*:

$$\frac{d}{ds}\begin{pmatrix} t(s) \\ y_0(s) \\ \vdots \\ y_{N-1}(s) \end{pmatrix} = \frac{1}{\sqrt{1+\sum_{k=0}^{N-1} f_k^2}}\begin{pmatrix} 1 \\ f_0 \\ \vdots \\ f_{N-1} \end{pmatrix}, \quad t(0) = 0.$$

From this transformation, a solution of a new problem never blows up in a finite time even if the solution of the original problem blows up in a finite time.

(2) Generate a linearly convergent sequence to T:
Assume that there is only $(T-t)^{-p}$ type singularity. Here $p > 0$, and T is a blow-up time of the original problem. We note that blow-up time is given by

$$T = \int_0^{\infty} \frac{ds}{\sqrt{1+\sum_{k=0}^{N-1} f_k^2}}.$$

Let $\{s_n\}$ be the geometric sequence given by

$$s_n = s_0 r^n \quad (s_0 > 0,\ r > 1,\ n = 0, 1, 2, \ldots),$$

and let $\{t_n\}$ be the time sequence given by

$$t_n = \int_0^{s_n} \frac{ds}{\sqrt{1+\sum_{k=0}^{N-1} f_k^2}}.$$

Then $\{t_n\}$ converges to T linearly, that is, $\lim_{n\to\infty} |e_n/e_{n-1}| = r^{-1/p}$, where $e_n = T - t_n$.

(3) Acceleration by the Aitken Δ^2 method:
The Aitken Δ^2 method can be applied to linearly convergent sequence in order to accelerate the convergence. Thus, we obtain an *approximation of the blow-up time*, say $\tilde{T}$. Using $\tilde{T}$ instead of T, we can calculate an approximate value of p by $p \simeq p_n = -\log r / \log |\tilde{e}_n/\tilde{e}_{n-1}|$, where $\tilde{e}_n = \tilde{T} - t_n$.

For a numerical integrator of ODEs from $s = s_{n-1}$ to $s = s_n$, we use the `DOPRI5` code (see [12]) with parameters `ITOL=0` and `RTOL=ATOL=1.d-15`. Computations are performed by using the double precision IEEE arithmetic. In Figure 4, we set $s_n = 1 \cdot 2^n$ ($s_0 = 1$ and $r = 2$) and apply the Aitken Δ^2 method three times.

A REACTION-DIFFUSION APPROXIMATION TO A CROSS-DIFFUSION SYSTEM

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In this paper it is discussed whether reaction and linear diffusion bring about a effect of nonlinear diffusion or not. It is proved that a cross-diffusion system for two competitive species is realized in a singular limit of a reaction–diffusion system with a small parameter under some assumptions.

1. Introduction

In this paper the following type of parabolic equations is called a *reaction–diffusion system*:

$$\boldsymbol{u}_t = \boldsymbol{D}\triangle \boldsymbol{u} + \boldsymbol{f}(\boldsymbol{u}), \tag{1}$$

where

$$\boldsymbol{u} = \boldsymbol{u}(x,t) = {}^t(u_1(x,t), \cdots, u_M(x,t)), \quad \boldsymbol{f}(\boldsymbol{u}) = {}^t(f_1(\boldsymbol{u}), \cdots, f_M(\boldsymbol{u})),$$

and $\boldsymbol{D}$ is a diagonal matrix whose elements are positive (or non-negative). In other words, a reaction–diffusion system consists of two parts: one is a kinetic term $\boldsymbol{f}(\boldsymbol{u})$; the other is a diffusion one $\boldsymbol{D}\triangle\boldsymbol{u}$. Many manuscripts reveals various dynamics of reaction–diffusion systems. Thus we meet the questions: "What sort of behavior can be exhibited by solutions to the reaction–diffusion system ?", or "How rich are the dynamics of the reaction–diffusion system ?" One of the ways to answer these questions is to "realize" in reaction–diffusion systems the dynamics of parabolic systems which do not belong to reaction–diffusion systems (cf. [3, 14]). This is also important for modelling. From the morphological point of view, Mimura et al. [12] showed that reaction–diffusion systems can "realize" density–dependent diffusion models. They considered the colonies of some species of bacteria which exhibit the various spatial patterns. Though several density–dependent diffusion models for such spatial patterns had already been proposed, they obtained the similar spatial patterns even from a reaction–diffusion system by introducing "inactive state" of a bacterium explicitly. Their concept of modelling motivated us to "reaction–diffusion approximation".

Our aim is to find a reaction–diffusion system which approximates a cross–diffusion system

$$\begin{cases} u_t^* = \Delta\alpha(u^*) + f(u^*, v^*), & x \in \Omega,\ t > 0, \\ v_t^* = \Delta\beta(u^*, v^*) + g(u^*, v^*), & x \in \Omega,\ t > 0 \end{cases} \tag{2}$$

under the Neumann boundary condition

$$\frac{\partial u^*}{\partial n} = 0, \ \frac{\partial v^*}{\partial n} = 0, \qquad x \in \partial\Omega,\ t > 0 \tag{3}$$

and the initial condition

$$u^*(x, 0) = u_0(x),\ v^*(x, 0) = v_0(x), \quad x \in \Omega. \tag{4}$$

For a typical example,

$$\begin{aligned} &\alpha(u) = (\alpha_0 + \alpha_1 u)u, \quad \beta(u, v) = (\beta_0 + \beta_1 u + \beta_2 v)v, \\ &f(u, v) = (f_0 - f_1 u - f_2 v)u, \quad g(u, v) = (g_0 - g_1 u - g_2 v)v, \end{aligned}$$

where $\alpha_0, \beta_0, f_j, g_j$ are positive constants and $\alpha_1, \beta_1, \beta_2$ are nonnegative ones. This example is one of the ecological models which Shigesada et al. [15] proposed in order to introduce the population pressure by interference between individuals into a Lotka–Volterra competition system. In this case u^* and v^* stand for population densities for two competing species. The species for v^* has a tendency to move towards where u^* is less distributed (also see [13]). Namely this system includes the "negative chemotactic effect". This effect induces the complex dynamics including the Hopf bifurcations and the segregation of a convex habitat between two similar species (see [5, 7, 9, 10, 11]). It is well-known in [6] that if Ω is convex there are no stable inhomogeneous equilibria in the competition–diffusion system, i.e., $\alpha_1 = \beta_1 = \beta_2 = 0$. It is shown in [9, 10] that the stable spatial segregation takes place under some assumptions with $\beta_1 > 0$, which is called *cross–diffusion induced instability*. In this paper we will show that the cross–diffusion system (2) is actually a singular limit of a reaction–diffusion system with a small parameter. Though reaction–diffusion systems do not seem to bring about the negative chemotactic effect, this fact might imply that reaction–diffusion systems include such an effect. This viewpoint also leads us to the relationship between Turing's instability and the cross–diffusion induced instability, which is shown in [4].

Hereafter we assume that Ω is a bounded domain in $\mathbf{R}^N$ with a smooth boundary $\partial\Omega$, and α, β, f, g are smooth functions satisfying

$$\alpha \in C^4(\mathbf{R}),\ \beta \in C^4(\mathbf{R}^2),\ f \in C^2(\mathbf{R}^2),\ g \in C^2(\mathbf{R}^2), \tag{5}$$

$$\inf_{u>0} \alpha'(u) > 0, \qquad \inf_{u>0, v>0} \beta_v(u,v) > 0, \tag{6}$$

and

$$\begin{aligned} &u_0 \in C^4(\overline{\Omega}), \quad v_0 \in C^4(\overline{\Omega}), \\ &u_0(x) \geq 0, \quad v_0(x) \geq 0 \qquad \text{in } \overline{\Omega}. \end{aligned}$$

We can take constants d_1, d_2, d_3 and d_4 satisfying

$$\begin{cases} 0 < d_1 < \inf_{u>0} \alpha'(u), & 0 < d_2 < \inf_{u>0, v>0} \beta_v(u,v), \\ d_3 > 0, \ \ d_3 \neq d_1, & d_4 > 0, \ \ d_4 \neq d_2. \end{cases}$$

Set

$$a(u) := \alpha(u) - d_1 u, \quad b(u,v) := \beta(u,v) - d_2 v.$$

For a small positive parameter ϵ, we consider an auxiliary semilinear parabolic system with fast reactions in w and z:

$$\begin{cases} u_t = d_1 \Delta u + \Delta w + f(u,v), & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v + \Delta z + g(u,v), & x \in \Omega,\ t > 0, \\ w_t = d_3 \Delta w + \dfrac{1}{\epsilon}(a(u) - w), & x \in \Omega,\ t > 0, \\ z_t = d_4 \Delta z + \dfrac{1}{\epsilon}(b(u,v) - z), & x \in \Omega,\ t > 0 \end{cases} \tag{7}$$

under the boundary condition

$$\frac{\partial u}{\partial n} = 0,\ \frac{\partial v}{\partial n} = 0,\ \frac{\partial w}{\partial n} = 0,\ \frac{\partial z}{\partial n} = 0, \qquad x \in \partial\Omega,\ t > 0 \tag{8}$$

and the initial condition

$$\begin{cases} u(x,0) = u_0(x),\ v(x,0) = v_0(x), \\ w(x,0) = a(u_0(x)),\ z(x,0) = b(u_0(x), v_0(x)), \quad x \in \Omega. \end{cases} \tag{9}$$

Since we can rewrite (2) as

$$\begin{cases} u_t^* = d_1 \Delta u^* + \Delta a(u^*) + f(u^*, v^*), & x \in \Omega,\ t > 0, \\ v_t^* = d_2 \Delta v^* + \Delta b(u^*, v^*) + g(u^*, v^*), & x \in \Omega,\ t > 0, \end{cases} \tag{10}$$

we may expect that (w, z) approximates to $(a(u), b(u, v))$ in (7) and that (u, v) converges to the solution of (10) as ϵ tends to 0. Actually we will show later that the dynamics of (7) under (8) and (9) is close to that of (10) under (3) and (4) as $\epsilon \to +0$, if they are restricted to any bounded region. Notice that the system (7) is almost a reaction–diffusion system. Indeed, applying the linear transformation

$$\tilde{u} = u - \frac{w}{d_3 - d_1}, \quad \tilde{v} = v - \frac{z}{d_4 - d_2}, \quad \tilde{w} = w, \quad \tilde{z} = z,$$

i.e.,

$$u = \tilde{u} + \frac{\tilde{w}}{d_3 - d_1}, \quad v = \tilde{v} + \frac{\tilde{z}}{d_4 - d_2}, \quad w = \tilde{w}, \quad z = \tilde{z},$$

we obtain the following reaction–diffusion system

$$\begin{cases} \tilde{u}_t = d_1 \Delta \tilde{u} + f(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}, \tilde{v} + \dfrac{\tilde{z}}{d_4 - d_2}) \\ \qquad - \dfrac{1}{(d_3 - d_1)\epsilon} \Big(a(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}) - \tilde{w} \Big), \\ \tilde{v}_t = d_2 \Delta \tilde{v} + g(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}, \tilde{v} + \dfrac{\tilde{z}}{d_4 - d_2}) \\ \qquad - \dfrac{1}{(d_4 - d_2)\epsilon} \Big(b(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}, \tilde{v} + \dfrac{\tilde{z}}{d_4 - d_2}) - \tilde{z} \Big), \\ \tilde{w}_t = d_3 \Delta \tilde{w} + \dfrac{1}{\epsilon} \Big(a(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}) - \tilde{w} \Big), \\ \tilde{z}_t = d_4 \Delta \tilde{z} + \dfrac{1}{\epsilon} \Big(b(\tilde{u} + \dfrac{\tilde{w}}{d_3 - d_1}, \tilde{v} + \dfrac{\tilde{z}}{d_4 - d_2}) - \tilde{z} \Big) \end{cases} \tag{11}$$

for $x \in \Omega,\ t > 0$. It is not clear whether $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{z}$ in (11) can stand for some biological quantities. However, using the same idea, Iida et al. [4] shows that (2) can be approximated by another reaction–diffusion system under additional assumptions on α and β and that the solutions of the later reaction–diffusion system like (11) stand for the population densities of some parts of the competing species which are described by the model of Shigesada et al. [15]; besides, the later approximation gives us a better understanding of cross–diffusion in biological models.

We remark that the existence of local solutions of (7) follows from that of (11).

Theorem 1.1. *Assume* (5) *and* (6). *Fix positive numbers* d_1, d_2, d_3, d_4 *and functions* $a(r), b(r, s)$ *as above. For positive constants* R_1 *and* R_2, *there* *exist functions* $\tilde{a}(r), \tilde{b}(r,s), \tilde{f}(r,s)$ *and* $\tilde{g}(r,s)$ *such that*

$$\tilde{a}(r) = a(r), \quad \tilde{b}(r,s) = b(r,s), \quad \tilde{f}(r,s) = f(r,s), \quad \tilde{g}(r,s) = g(r,s) \\ \text{for any } (r,s) \in [0, R_1] \times [0, R_2] \qquad (12)$$

and that the solution $(u,v,w,z) = (u(x,t), v(x,t), w(x,t), z(x,t))$ *of* (7)–(9) *with* a, b, f, g *replaced by* $\tilde{a}$, $\tilde{b}$, $\tilde{f}$, $\tilde{g}$ *respectively exists globally in time.*

If the solution $(u^*, v^*) = (u^*(x,t), v^*(x,t))$ *of* (2)–(4) *belongs to* $C^4(\overline{\Omega} \times [0,T]) \times C^4(\overline{\Omega} \times [0,T])$ *and*

$$0 \le u^*(x,t) \le R_1, \quad 0 \le v^*(x,t) \le R_2 \qquad in\ \overline{\Omega} \times [0,T]$$

for some positive constant T, *then the following inequalities hold*

$$\begin{cases} \|u - u^*\|_{C^0([0,T];L^2(\Omega))} & \le c_1 \epsilon, \\ \|v - v^*\|_{C^0([0,T];L^2(\Omega))} & \le c_1 \epsilon, \\ \|w - a(u^*)\|_{C^0([0,T];L^2(\Omega))} & \le c_1 \epsilon, \\ \|z - b(u^*, v^*)\|_{C^0([0,T];L^2(\Omega))} & \le c_1 \epsilon \end{cases} \qquad (13)$$

for $\epsilon > 0$ *where* c_1 *is a positive constant independent of* ϵ *and* (u,v,w,z). *Moreover, if* $N \le 4$, *then the following inequalities also hold:*

$$\begin{cases} \|\nabla(u - u^*)\|_{C^0([0,T];L^2(\Omega))} & \le c_2 \epsilon^{3/4}, \\ \|\nabla(v - v^*)\|_{C^0([0,T];L^2(\Omega))} & \le c_2 \epsilon^{3/4}, \\ \|\nabla(w - a(u^*))\|_{C^0([0,T];L^2(\Omega))} & \le c_2 \epsilon^{3/4}, \\ \|\nabla(z - b(u^*, v^*))\|_{C^0([0,T];L^2(\Omega))} & \le c_2 \epsilon^{3/4}, \\ \|\triangle(u - u^*)\|_{C^0([0,T];L^2(\Omega))} & \le c_3 \epsilon^{1/4}, \\ \|\triangle(v - v^*)\|_{C^0([0,T];L^2(\Omega))} & \le c_3 \epsilon^{1/4}, \\ \|\triangle(w - a(u^*))\|_{C^0([0,T];L^2(\Omega))} & \le c_3 \epsilon^{1/4}, \\ \|\triangle(z - b(u^*, v^*))\|_{C^0([0,T];L^2(\Omega))} & \le c_3 \epsilon^{1/4} \end{cases} \qquad (14)$$

for $0 < \epsilon \le \epsilon_0$ *where* c_2, c_3 *and* ϵ_0 *are positive constants independent of* ϵ *and* (u,v,w,z).

As long as $(u(x,t), v(x,t))$ belongs to the region $[0,R_1] \times [0,R_2]$, (u,v,w,z) in Theorem 1.1 is the solution of (7)–(9) without the replacement of a, b, f and g. Thus this theorem implies that solutions to the cross–diffusion system (2) can be approximated by the linear combinations of solutions to the reaction–diffusion system (11) in any bounded region in the phase space.

The proof of this theorem will be given in §2. We will construct $\tilde{a}$, $\tilde{b}$, $\tilde{f}$ and $\tilde{g}$ by suitably truncating a, b, f and g respectively around the bounded region $[0,R_1] \times [0,R_2]$. The constants c_1, c_2 and c_3 depend on R_1, R_2, u^*, v^*, and thus on T.

See [1] for the existence, uniqueness, and regularity of a local solution of (2)-(4), where $(u_0, v_0) \in W^1_p(\Omega)^2$ for $p > N$. In particular, if α, β, f and g are sufficiently smooth, then the local solution instantly becomes sufficiently smooth up to the boundary. See also [2] and the references therein. Thus the assumptions for u^* and v^* are not so restricted. As for its global existence, see, e.g., [8, 16, 17].

2. Modification of equations

To prove Theorem 1.1, we will construct the functions $\tilde{a}$, $\tilde{b}$, $\tilde{f}$, and $\tilde{g}$ in this section. First we introduce the following stronger assumption for a, b, f and g than (5) and (6):

(A) There exist positive constants k_i $(i = 1, 2, 3, 4)$ satisfying

$$\begin{cases} a'(r) \geq k_1, \\ b_v(r,s) \geq k_2, \\ \displaystyle\sum_{j=1}^{3} \left| \frac{d^j}{du^j} a(r) \right| + \sum_{1 \leq j+l \leq 3,\ j,l \geq 0} \left| \frac{\partial^{j+l}}{\partial u^j \partial v^l} b(r,s) \right| \\ \qquad + \left| \displaystyle\int_0^s b_{uu}(r,\sigma) d\sigma \right| + \left| \displaystyle\int_0^s b_{uuu}(r,\sigma) d\sigma \right| \leq k_3, \\ |f_u(r,s)| + |f_v(r,s)| + |f_{uu}(r,s)| + |f_{uv}(r,s)| + |f_{vv}(r,s)| \leq k_4, \\ |g_u(r,s)| + |g_v(r,s)| + |g_{uu}(r,s)| + |g_{uv}(r,s)| + |g_{vv}(r,s)| \leq k_5 \end{cases} \tag{15}$$

for any $r, s \in \mathbf{R}$.

Theorem 2.1. *Let* $(u^*, v^*) = (u^*(x,t), v^*(x,t))$ *(resp.* $(u, v, w, z) = (u(x,t), v(x,t),\ w(x,t), z(x,t))$*) be the solution of* (2) - (4) (resp. (7) - (9)) *in* $t \in [0, T]$. *Assume* (A) *and*

$$\begin{aligned} &\|u_t^*\|_{L^\infty(\Omega)} + \|v_t^*\|_{L^\infty(\Omega)} + \|\nabla u^*\|_{L^\infty(\Omega)} + \|\nabla v^*\|_{L^\infty(\Omega)} \\ &+ \|d_3 \Delta a(u^*) - a'(u^*) u_t^*\|_{H^1(\Omega)} \\ &+ \|d_4 \Delta b(u^*, v^*) - b_u(u^*, v^*) u_t^* - b_v(u^*, v^*) v_t^*\|_{H^1(\Omega)} \leq M_1 \end{aligned} \tag{16}$$

for $0 \leq t \leq T$. *Then* (13) *holds.*

The proof will be given in the next section.

For positive numbers δ and R we can easily choose a C^∞-function $\chi(x; \delta, R)$ as follows:

$$\chi(x; \delta, R) = \begin{cases} 1 & \text{for } x \in [0, R], \\ 0 & \text{for } x \in (-\infty, -2\delta] \cup\ [R + 2\delta, \infty), \end{cases}$$

and

$$0 \le \chi(x;\delta,R) \le 1, \qquad \sup_{-\infty<x<\infty} |\chi'(x;\delta,R)| \le \frac{1}{\delta}.$$

Lemma 2.1. *Assume* (5) *and* (6)*. Let* R_1 *and* R_2 *be positive numbers, and set*

$$m_1 := \min_{u\in[0,R_1]} a'(u), \qquad m_2 := \min_{(u,v)\in[0,R_1]\times[0,R_2]} b_v(u,v).$$

If δ_1 *and* δ_2 *are positive but so small, then* (A) *holds true for the following functions* $\tilde{a}$, $\tilde{b}$, $\tilde{f}$, $\tilde{g}$ *and some positive constants* $k_1, \cdots, k_5$*:*

$$\begin{aligned}
\tilde{a}(u) &:= m_1 u + a(0) + \int_0^u \chi_1(s)\,(a'(s) - m_1)\,ds,\\
\tilde{b}(u,v) &:= m_2 v + \chi_2(v)\left(\chi_1(u)b(u,0) + \int_0^v \chi_3(u,s)\,(b_v(u,s) - m_2)\,ds\right),\\
\tilde{f}(u,v) &:= \chi_3(u,v)f(u,v),\\
\tilde{g}(u,v) &:= \chi_3(u,v)g(u,v),
\end{aligned}$$

where

$$\begin{aligned}
\chi_1(u) &:= \chi(u;\delta_1,R_1),\\
\chi_2(v) &:= \chi(v;\frac{1}{\delta_2},R_2),\\
\chi_3(u,v) &:= \chi(u;\delta_1,R_1)\chi(v;\delta_1,R_2).
\end{aligned}$$

We can easily check (12). Since the support of

$$\int_0^v \chi_3(u,s)\,(b_v(u,s) - m_2)\,ds$$

is not compact, we cannot obtain the boundedness of

$$\int_0^v \int_0^\sigma \chi_3(u,s)\,(b_v(u,s) - m_2)\,dsd\sigma$$

and its derivatives. Therefore it is necessary to multiply

$$\int_0^v \chi_3(u,s)\,(b_v(u,s) - m_2)\,ds$$

by $\chi_2(v)$ in the definition of $\tilde{b}$.

Proof. We show (15) only for $\tilde{b}$. If δ_1 is so small, then

$$\chi_3(u,v)\,(b_v(u,v) - m_2) \ge -\frac{m_2}{4} \quad \text{for } (u,v) \in \mathbf{R}^2.$$

We can choose δ_2 so small that

$$\chi_2'(v)\left(\chi_1(u)b(u,0)+\int_0^v \chi_3(u,s)\left(b_v(u,s)-m_2\right)ds\right)\geq -\frac{m_2}{4}$$

for $(u,v)\in \mathbf{R}^2$. Differentiating $\tilde{b}$ in v, we have

$$\begin{aligned}\tilde{b}_v(u,v) &= m_2+\chi_2'(v)\left(\chi_1(u)b(u,0)+\int_0^v \chi_3(u,s)\left(b_v(u,s)-m_2\right)ds\right)\\ &\quad +\chi_2(v)\chi_3(u,v)\left(b_v(u,v)-m_2\right)\\ &\geq m_2-\frac{m_2}{4}-\frac{m_2}{4}\\ &=\frac{m_2}{2}.\end{aligned}$$

The other conditions of (15) can be checked. □

Proof of Theorem 1.1. The inequalities (13) are a direct consequence of Lemma 2.1 and Theorem 2.1. Notice that the global existence of (u,v,w,z) is guaranteed by the fact: the grow-up rates of the nonlinear terms in (11) are less than or equal to some affine functions of $(\tilde{u},\tilde{v},\tilde{w},\tilde{z})$ after the replacement of (a,b,f,g) with $(\tilde{a},\tilde{b},\tilde{f},\tilde{g})$. Since the fourth derivatives of $\tilde{a}$ and $\tilde{b}$ in Lemma 2.1 are bounded, the latter part of Theorem 1.1 is deduced from Lemma 2.1 and the following theorem which will be proved in the next section.

Theorem 2.2. *Assume* (A), (16), $N\leq 4$ *and*

$$\begin{aligned}&\|\Delta u^*\|_{L^\infty(\Omega)}+\|\Delta v^*\|_{L^\infty(\Omega)}+\left\|\frac{\partial}{\partial n}\Delta a(u^*)\right\|_{L^2(\partial\Omega)}+\left\|\frac{\partial}{\partial n}\Delta b(u^*,v^*)\right\|_{L^2(\partial\Omega)}\\ &\quad+\|d_3\Delta a(u^*)-a'(u^*)u_t^*\|_{H^2(\Omega)}\\ &\quad+\|d_4\Delta b(u^*,v^*)-b_u(u^*,v^*)u_t^*-b_v(u^*,v^*)v_t^*\|_{H^2(\Omega)}\leq M_2 \qquad (17)\end{aligned}$$

for $0\leq t\leq T$. *Then* (14) *holds.*

3. Proof

Let $\|\cdot\|$ be a L^2-norm and $(\cdot,\cdot)$ an inner product in $L^2(\Omega)$. Let $(u^*,v^*)=(u^*(x,t),v^*(x,t))$ and $(u,v,w,z)=(u(x,t),v(x,t),w(x,t),w(x,t))$ be as in Theorem 2.1. Hereafter for the simplicity of notation, the positive constants independent of ϵ and (u,v,w,z) (namely, depending only on $d_1,\cdots,d_4,k_1,\cdots,k_5,M_1,M_2,T,\Omega,N,$ and ϵ_0) is denoted by c_i $(i=1,2,\cdots)$.

Set

$$U := u - u^*, \quad V := v - v^*, \quad W := w - a(u^*), \quad Z := z - b(u^*, v^*),$$
$$w^* := a(u^*), \quad z^* := b(u^*, v^*),$$

which satisfy

$$\begin{cases} U_t = d_1 \triangle U + \triangle W + f(u^* + U, v^* + V) - f(u^*, v^*), \\ V_t = d_2 \triangle V + \triangle Z + g(u^* + U, v^* + V) - g(u^*, v^*), \\ W_t = d_3 \triangle W + \dfrac{1}{\epsilon}(a(u^* + U) - a(u^*) - W) + d_3 \triangle w^* - w_t^*, \\ Z_t = d_4 \triangle Z + \dfrac{1}{\epsilon}(b(u^* + U, v^* + V) - b(u^*, v^*) - Z) + d_4 \triangle z^* - z_t^*. \end{cases} \tag{18}$$

Define

$$\begin{aligned} A(u) &:= \int_0^u a(s) ds, \\ B(u, v) &:= \int_0^v b(u, s) ds, \\ E_1(t) &:= \int_\Omega (A(u^* + U) - A(u^*) - A'(u^*)U)\, dx, \\ E_2(t) &:= \int_\Omega \Big(B(u^* + U, v^* + V) - B(u^*, v^*) \\ &\qquad -B_u(u^*, v^*)U - B_v(u^*, v^*)V \Big) dx. \end{aligned}$$

Proof of Theorem 2.1. Differentiating E_1 in t, we have

$$\begin{aligned} &\frac{dE_1}{dt} = \frac{d}{dt} \int_\Omega (A(u^* + U) - A(u^*) - A'(u^*)U)\, dx \\ &= \int_\Omega \{A'(u^* + U)(u_t^* + U_t) - A'(u^*)u_t^* - A''(u^*)u_t^* U - A'(u^*)U_t\}\, dx \\ &= \int_\Omega \Big\{(a(u^* + U) - a(u^*))U_t + \Big(a(u^* + U) - a(u^*) - a'(u^*)U\Big)u_t^*\Big\}\, dx. \end{aligned}$$

Substituting (18) into the above equality, we can calculate dE_1/dt as

$$\begin{aligned} &\int_\Omega \Big\{ \Big(a(u^* + U) - a(u^*)\Big)\Big(d_1 \triangle U + \triangle W + f(u^* + U, v^* + V) - f(u^*, v^*)\Big) \\ &+ \int_0^1 \int_0^1 a''(u^* + \theta_1\theta_2 U)\theta_1 U^2 u_t^* d\theta_1 d\theta_2 \Big\} dx. \end{aligned} \tag{19}$$

The first term of the right hand side of (19) is estimated as follows:

$$\begin{aligned}
&\int_\Omega (a(u^*+U)-a(u^*))(d_1\Delta U+\Delta W+f(u^*+U,v^*+V)-f(u^*,v^*))dx \\
&\quad\le -\int_\Omega \Big(a'(u^*+U)(\nabla u^*+\nabla U)-a'(u^*)\nabla u^*\Big)\cdot d_1\nabla U dx \\
&\qquad+\int_\Omega (a(u^*+U)-a(u^*))\Delta W dx+k_3k_4\|U\|(\|U\|+\|V\|) \\
&\quad\le \int_\Omega (a(u^*+U)-a(u^*))\Delta W dx-\int_\Omega d_1a'(u^*+U)|\nabla U|^2dx \\
&\qquad+d_1k_3M_1\|U\|\ \|\nabla U\|+k_3k_4\|U\|(\|U\|+\|V\|) \\
&\quad\le (a(u^*+U)-a(u^*),\Delta W)-\frac{d_1k_1}{2}\|\nabla U\|^2+c_4(\|U\|^2+\|V\|^2), \quad (20)
\end{aligned}$$

where $c_4 := 3k_3k_4/2+d_1k_3^2M_1^2/(2k_1)$. Substituting (20) into (19), we get

$$\frac{dE_1}{dt}\le (a(u^*+U)-a(u^*),\Delta W)-\frac{d_1k_1}{2}\|\nabla U\|^2+c_5(\|U\|^2+\|V\|^2), \quad (21)$$

where $c_5 := c_4+k_3M_1/2$. Taking an inner product between the third equation of (18) and $-\Delta W$ yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla W\|^2 &= -d_3\|\Delta W\|^2-\frac{1}{\epsilon}(a(u^*+U)-a(u^*),\Delta W) \\
&\quad+(\nabla(d_3\Delta w^*-w_t^*),\nabla W)-\frac{1}{\epsilon}\|\nabla W\|^2 \\
&\le -d_3\|\Delta W\|^2-\frac{1}{\epsilon}(a(u^*+U)-a(u^*),\Delta W) \\
&\quad-\frac{1}{2\epsilon}\|\nabla W\|^2+c_6\epsilon, \qquad (22)
\end{aligned}$$

where $c_6 := M_1^2/2$. The above two inequalities (21) and (22) immediately imply

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|\nabla W\|^2+\frac{1}{\epsilon}E_1\right) &\le -d_3\|\Delta W\|^2-\frac{1}{2\epsilon}\|\nabla W\|^2-\frac{d_1k_1}{2\epsilon}\|\nabla U\|^2 \\
&\quad+\frac{c_5}{\epsilon}(\|U\|^2+\|V\|^2)+c_6\epsilon. \qquad (23)
\end{aligned}$$

Next we consider the derivative of E_2:

$$\begin{aligned}
\frac{dE_2}{dt} &= \frac{d}{dt}\int_\Omega \Big(B(u^*+U,v^*+V)-B(u^*,v^*)-B_u(u^*,v^*)U-B_v(u^*,v^*)V\Big)dx \\
&= \int_\Omega \Big\{B_u(u^*+U,v^*+V)(u_t^*+U_t)+B_v(u^*+U,v^*+V)(v_t^*+V_t) \\
&\quad -B_u(u^*,v^*)u_t^*-B_v(u^*,v^*)v_t^*-B_{uu}(u^*,v^*)u_t^*U-B_{uv}(u^*,v^*)v_t^*U \\
&\quad -B_u(u^*,v^*)U_t-B_{uv}(u^*,v^*)u_t^*V-B_{vv}(u^*,v^*)v_t^*V-B_v(u^*,v^*)V_t\Big\}dx \\
&= \int_\Omega \Big\{\Big(B_u(u^*+U,v^*+V)-B_u(u^*,v^*)-B_{uu}(u^*,v^*)U-B_{uv}(u^*,v^*)V\Big)u_t^* \\
&\quad +\Big(B_v(u^*+U,v^*+V)-B_v(u^*,v^*)-B_{uv}(u^*,v^*)U-B_{vv}(u^*,v^*)V\Big)v_t^* \\
&\quad +\Big(B_u(u^*+U,v^*+V)-B_u(u^*,v^*)\Big)U_t \\
&\quad +(b(u^*+U,v^*+V)-b(u^*,v^*))V_t\Big\}dx \\
&\le \frac{k_3M_1}{2}(\|U\|^2+\|V\|^2)+(b(u^*+U,v^*+V)-b(u^*,v^*),\triangle Z)-d_2k_2\|\nabla V\|^2 \\
&\quad +k_3\{M_1(\|U\|+\|V\|)+\|\nabla U\|+\|\nabla V\|\}(d_1\|\nabla U\|+\|\nabla W\|) \\
&\quad +d_2k_3\{M_1(\|U\|+\|V\|)+\|\nabla U\|\}\|\nabla V\|+2k_3(k_4+k_5)(\|U\|^2+\|V\|^2) \\
&\le (b(u^*+U,v^*+V)-b(u^*,v^*),\triangle Z)-\frac{d_2k_2}{2}\|\nabla V\|^2 \\
&\quad +c_7(\|U\|^2+\|V\|^2+\|\nabla U\|^2+\|\nabla W\|^2) \qquad (24)
\end{aligned}$$

with some positive constant c_7. Similarly we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla Z\|^2 &= -d_4\|\triangle Z\|^2-\frac{1}{\epsilon}(b(u^*+U,v^*+V)-b(u^*,v^*),\triangle Z) \\
&\quad -\frac{1}{\epsilon}\|\nabla Z\|^2+(\nabla(d_4\triangle z^*-z_t^*),\nabla Z) \\
&\le -d_4\|\triangle Z\|^2-\frac{1}{\epsilon}(b(u^*+U,v^*+V)-b(u^*,v^*),\triangle Z) \\
&\quad -\frac{1}{2\epsilon}\|\nabla Z\|^2+c_8\epsilon, \qquad (25)
\end{aligned}$$

where $c_8 := M_1^2/2$. Combining two inequalities (24) and (25), we obtain

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\|\nabla Z\|^2+\frac{1}{\epsilon}E_2\right) &\le -d_4\|\triangle Z\|^2-\frac{1}{2\epsilon}\|\nabla Z\|^2-\frac{d_2k_2}{2\epsilon}\|\nabla V\|^2 \\
&\quad +\frac{c_7}{\epsilon}(\|U\|^2+\|V\|^2+\|\nabla U\|^2+\|\nabla W\|^2)+c_8\epsilon. \qquad (26)
\end{aligned}$$

Combine (23) and (26). If $\gamma \geq \max\{4c_7, 4c_7/(d_1k_1)\}$, then we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|\nabla Z\|^2 + \frac{1}{\epsilon}E_2 + \frac{\gamma}{2}\|\nabla W\|^2 + \frac{\gamma}{\epsilon}E_1\right) \\
&\leq -\gamma d_3\|\triangle W\|^2 - d_4\|\triangle Z\|^2 - \frac{\gamma}{4\epsilon}\|\nabla W\|^2 - \frac{1}{2\epsilon}\|\nabla Z\|^2 - \frac{\gamma d_1k_1}{4\epsilon}\|\nabla U\|^2 \\
&\quad -\frac{d_2k_2}{2\epsilon}\|\nabla V\|^2 + \frac{\gamma c_5 + c_7}{\epsilon}(\|U\|^2 + \|V\|^2) + (\gamma c_6 + c_8)\epsilon. \qquad (27)
\end{aligned}$$

The assumption (A) implies

$$\begin{aligned}
&E_1 \geq \frac{k_1}{2}\|U\|^2, \\
&E_2 \geq -\frac{k_3}{2}\|U\|^2 - k_3\|U\|\|V\| + \frac{k_2}{2}\|V\|^2 \geq -\left(\frac{k_3}{2} + \frac{k_3^2}{k_2}\right)\|U\|^2 + \frac{k_2}{4}\|V\|^2.
\end{aligned}$$

Taking γ so large as

$$\gamma \geq \max\left\{\frac{4k_3}{k_1}\left(\frac{1}{2} + \frac{k_3}{k_2}\right), 4c_7, \frac{4c_7}{d_1k_1}\right\},$$

we have

$$\gamma E_1 + E_2 \geq c_9(\|U\|^2 + \|V\|^2),$$

where

$$c_9 := \min\left\{\frac{k_2}{4}, \frac{k_1\gamma}{4}\right\}.$$

Thus, (27) and the above inequality mean

$$\begin{aligned}
&\frac{d}{dt}\left\{\left(\frac{1}{2}\|\nabla Z\|^2 + \frac{1}{\epsilon}E_2 + \frac{\gamma}{2}\|\nabla W\|^2 + \frac{\gamma}{\epsilon}E_1\right)e^{-c_{10}t}\right\} \\
&\leq -\left(\gamma d_3\|\triangle W\|^2 + d_4\|\triangle Z\|^2 + \frac{\gamma}{4\epsilon}\|\nabla W\|^2 + \frac{1}{2\epsilon}\|\nabla Z\|^2 + \frac{\gamma d_1k_1}{4\epsilon}\|\nabla U\|^2\right. \\
&\quad \left. + \frac{d_2k_2}{2\epsilon}\|\nabla V\|^2\right)e^{-c_{10}t} + (\gamma c_6 + c_8)\epsilon e^{-c_{10}t}
\end{aligned}$$

for $c_{10} := (\gamma c_5 + c_7)/c_9$. Finally, we obtain the first and second inequalities of (13).

Lemma 3.1. *Let λ and ξ be constants. If a C^1-function $X(t)$ satisfies*

$$X'(t) \leq \lambda(\xi - X)$$

for $0 < t \leq T$ and $X(0) \leq \xi$, then $X(t) \leq \xi$ for $0 \leq t \leq T$.

This lemma can be easily checked. So, the proof is omitted.

We will show the inequality for $Z = z - b(u^*, v^*)$ in (13). By (18), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|Z\|^2 &\leq -d_4\|\nabla Z\|^2 + \frac{k_3}{\epsilon}(\|U\| + \|V\|)\|Z\| - \frac{1}{\epsilon}\|Z\|^2 + M_1\|Z\| \\
&\leq -d_4\|\nabla Z\|^2 + \frac{k_3^2}{\epsilon}(\|U\|^2 + \|V\|^2) + M_1^2\epsilon - \frac{1}{4\epsilon}\|Z\|^2 \\
&\leq (2k_3^2c_1^2 + M_1^2)\epsilon - \frac{1}{4\epsilon}\|Z\|^2.
\end{aligned}$$

By Lemma 3.1,

$$\|Z\|^2 \leq 4(2k_3^2c_1^2 + M_1^2)\epsilon^2.$$

The inequality for $W = w - a(u^*)$ in (13) can be proved similarly. This completes the proof of Theorem 2.1.

Remark 3.1. It is difficult to estimate the terms $\triangle W$ and $\triangle Z$ in the first and second equations of (18). To overcome this difficulty, we have introduced the functionals $E_1(t)$ and $E_2(t)$ instead of $\|U\|^2$ and $\|V\|^2$. For example, we have chosen $E_1(t)$ in order that $(a(u^* + U) - a(u^*), \triangle W)$ in (21) cancels out that of (22).

We prepare the following lemma for the proof of Theorem 2.2.

Lemma 3.2. *Let $\lambda(t;\epsilon), \rho(t;\epsilon)$ be non-negative continuous functions in t and satisfy*

$$\int_0^T \lambda(t;\epsilon)dt \leq \overline{\lambda}, \quad \int_0^T \rho(t;\epsilon)dt \leq \overline{\rho}(\epsilon)$$

where $\overline{\lambda}$ is independent of ϵ. Assume a non-negative C^1-function $X(t;\epsilon)$ and a non-negative continuous function $Y(t;\epsilon)$ satisfy

$$X_t \leq -Y + \lambda(t;\epsilon)X + \rho(t;\epsilon)$$

for $0 < t \leq T$. Then,

$$X(t;\epsilon) \leq \{X(0;\epsilon) + \overline{\rho}(\epsilon)\}e^{\overline{\lambda}}, \quad \int_0^t Y(s;\epsilon)ds \leq \{X(0;\epsilon) + \overline{\rho}(\epsilon)\}e^{\overline{\lambda}} \tag{28}$$

for $0 \leq t \leq T$.

Proof. Since

$$\frac{d}{dt}\left(Xe^{-\int_0^t \lambda(\tau;\epsilon)d\tau}\right) \leq (-Y+\rho)e^{-\int_0^t \lambda(\tau;\epsilon)d\tau},$$

we have

$$X(t;\epsilon)+\int_0^t Y(s;\epsilon)e^{\int_s^t \lambda(\tau;\epsilon)d\tau}ds \leq X(0;\epsilon)e^{\int_0^t \lambda(\tau;\epsilon)d\tau}+\int_0^t \rho(s;\epsilon)e^{\int_s^t \lambda(\tau;\epsilon)d\tau}ds.$$

We can easily check (28). □

Proof of Theorem 2.2. Owing to Lemma 3.2, it follows from the first and second inequalities of (13) and (27) that

$$\int_0^T (\|\triangle W\|^2+\|\triangle Z\|^2)dt \leq c_{11}\epsilon.$$

By (18), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla U\|^2 &\leq -d_1\|\triangle U\|^2+\|\triangle W\|\,\|\triangle U\|+k_4(\|U\|+\|V\|)\|\triangle U\| \\ &\leq -\frac{d_1}{2}\|\triangle U\|^2+\frac{1}{d_1}\left\{\|\triangle W\|^2+k_4^2(\|U\|+\|V\|)^2\right\}.\end{aligned}$$

Lemma 3.2 shows us that these inequalities and (13) imply

$$\int_0^T \|\triangle U\|^2 dt \leq c_{12}\epsilon. \tag{29}$$

It is similarly seen that

$$\int_0^T \|\triangle V\|^2 dt \leq c_{12}\epsilon. \tag{30}$$

Multiplying the first equation of (18) by $-a'(u^*)\triangle U$ and integrating over Ω yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla U|^2 a'(u^*)dx-\frac{1}{2}\int_\Omega |\nabla U|^2 a''(u^*)u_t^* dx+\int_\Omega U_t a''(u^*)\nabla U\cdot\nabla u^* dx \\ &\quad\leq -d_1\int_\Omega |\triangle U|^2 a'(u^*)dx-(\triangle W, a'(u^*)\triangle U)+k_3k_4(\|U\|+\|V\|)\|\triangle U\|.\end{aligned}$$

Since

$$\|U_t\| \leq d_1\|\triangle U\|+\|\triangle W\|+k_4(\|U\|+\|V\|),$$

the above inequality implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla U|^2 a'(u^*)dx \leq -\frac{d_1k_1}{2}\|\triangle U\|^2-(\triangle W, a'(u^*)\triangle U)+\frac{1}{4}\|\triangle W\|^2 \\ +c_{13}(\|U\|^2+\|V\|^2+\|\nabla U\|^2).\end{aligned} \tag{31}$$

Similarly, operating $\triangle$ to the third equation of (18), multiplying it by $\triangle W$, and integrating over Ω, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle W\|^2 &= -d_3\|\nabla\triangle W\|^2 + d_3\int_{\partial\Omega}\triangle W\frac{\partial}{\partial n}\triangle W dS \\ &+(\triangle(d_3\triangle w^* - w_t^*), \triangle W) \\ &+\frac{1}{\epsilon}(\triangle(a(u^*+U) - a(u^*)), \triangle W) - \frac{1}{\epsilon}\|\triangle W\|^2. \quad (32)\end{aligned}$$

Lemma 3.3. *Assume* (17). *Then, there exists a positive constant* c_{14} *independent of* ϵ *and* (u, v, w, z) *such that*

$$d_3\left|\int_{\partial\Omega}\triangle W\frac{\partial}{\partial n}\triangle W dS\right| \le \frac{d_3}{4}\|\nabla\triangle W\|^2 + \frac{1}{4\epsilon}\|\triangle W\|^2 + c_{14}(\epsilon^{1/2}+\epsilon), \quad (33)$$

$$d_4\left|\int_{\partial\Omega}\triangle Z\frac{\partial}{\partial n}\triangle Z dS\right| \le \frac{d_4}{4}\|\nabla\triangle Z\|^2 + \frac{1}{4\epsilon}\|\triangle Z\|^2 + c_{14}(\epsilon^{1/2}+\epsilon). \quad (34)$$

Proof. The equations (7) and (8) imply

$$\frac{\partial}{\partial n}\triangle w = 0 \quad \text{on } \partial\Omega,$$

and hence

$$\frac{\partial}{\partial n}\triangle W = -\frac{\partial}{\partial n}\triangle a(u^*) \quad \text{on } \partial\Omega.$$

Then,

$$\begin{aligned}\left|\int_{\partial\Omega}\triangle W\frac{\partial}{\partial n}\triangle W dS\right| &\le \left\|\frac{\partial}{\partial n}\triangle W\right\|_{L^2(\partial\Omega)}\|\triangle W\|_{L^2(\partial\Omega)} \\ &\le M_2\|\triangle W\|_{L^2(\partial\Omega)} \\ &\le c_{15}\|\triangle W\|_{H^{1/2}(\Omega)} \\ &\le c_{16}\left(\|\nabla\triangle W\|^{1/2}\|\triangle W\|^{1/2} + \|\triangle W\|\right),\end{aligned}$$

which is reduced to (33). Similarly, (34) can be checked. □

Set

$$I_1 := \triangle(a(u^*+U) - a(u^*)) - a'(u^*)\triangle U.$$

It follows from the chain rule that

$$\begin{aligned}I_1 &= a''(u^*+U)(|\nabla u^* + \nabla U|^2 - |\nabla u^*|^2) + (a''(u^*+U) - a''(u^*))|\nabla u^*|^2 \\ &+(a'(u^*+U) - a'(u^*))\triangle u^* + (a'(u^*+U) - a'(u^*))\triangle U.\end{aligned}$$

Then, we have

$$
\begin{aligned}
|(I_1, \triangle W)| \leq c_{17}(\|U\| + \|\nabla U\|)\|\triangle W\| + c_{17}\|\nabla U\|^2_{L^4(\Omega)}\|\triangle W\| \\
+c_{17}\|U\|_{L^4(\Omega)}\|\triangle U\| \; \|\triangle W\|_{L^4(\Omega)}.
\end{aligned} \tag{35}
$$

The assumption $N \leq 4$ ensures the inclusion $H^1(\Omega) \subset L^4(\Omega)$ and the existence of a positive constant c_{18} such that

$$
\|U\|_{L^4(\Omega)} \leq c_{18}(\|\nabla U\| + \|U\|), \quad \|\nabla U\|_{L^4(\Omega)} \leq c_{18}(\|\triangle U\| + \|U\|).
$$

Here we also used an elliptic estimate for U under the boundary conditions (3) and (8). There exists a positive constant c_{19} such that

$$
\begin{aligned}
& c_{17}\|\nabla U\|^2_{L^4(\Omega)}\|\triangle W\| \\
& \quad \leq 2c_{17}c_{18}^2(\|\triangle U\|^2 + \|U\|^2)\|\triangle W\| \\
& \quad \leq \frac{d_1 k_1}{4}\|\triangle U\|^2 + \frac{1}{8}\|\triangle W\|^2 + c_{19}\|U\|^4 + c_{19}\|\triangle U\|^2\|\triangle W\|^2, \\
& c_{17}\|U\|_{L^4(\Omega)}\|\triangle U\| \; \|\triangle W\|_{L^4(\Omega)} \\
& \quad \leq c_{17}c_{18}^2(\|\nabla U\| + \|U\|)\|\triangle U\|(\|\nabla \triangle W\| + \|\triangle W\|) \\
& \quad \leq \frac{d_3 \epsilon}{4}\|\nabla \triangle W\|^2 + \frac{c_{19}}{\epsilon}(\|U\|^2 + \|\nabla U\|^2)\|\triangle U\|^2 \\
& \quad \quad +c_{19}(\|U\|^2 + \|\nabla U\|^2) + c_{19}\|\triangle U\|^2\|\triangle W\|^2.
\end{aligned}
$$

Using the above inequalities, we can estimate $(I_1, \triangle W)$ as follows:

$$
\begin{aligned}
|(I_1, \triangle W)| \leq & \frac{d_3 \epsilon}{4}\|\nabla \triangle W\|^2 + \frac{1}{4}\|\triangle W\|^2 + \frac{d_1 k_1}{4}\|\triangle U\|^2 \\
& +(4c_{17}^2 + c_{19})(\|U\|^2 + \|\nabla U\|^2) \\
& +\frac{c_{19}}{\epsilon}\|\triangle U\|^2 \left(\|U\|^2 + \|\nabla U\|^2 + 2\epsilon\|\triangle W\|^2\right) + c_{19}\|U\|^4.
\end{aligned} \tag{36}
$$

Combining (32), (33), and (36) yields

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\triangle W\|^2 \\
& \leq -\frac{d_3}{2}\|\nabla \triangle W\|^2 - \frac{1}{2\epsilon}\|\triangle W\|^2 + c_{14}(\epsilon^{1/2} + \epsilon) + M_2\|\triangle W\| \\
& \quad +\frac{d_1 k_1}{4\epsilon}\|\triangle U\|^2 + \frac{1}{\epsilon}(a'(u^*)\triangle U, \triangle W) + \frac{4c_{17}^2 + c_{19}}{\epsilon}(\|U\|^2 + \|\nabla U\|^2) \\
& \quad +\frac{c_{19}}{\epsilon^2}\|\triangle U\|^2 \left(\|U\|^2 + \|\nabla U\|^2 + 2\epsilon\|\triangle W\|^2\right) + \frac{c_{19}}{\epsilon}\|U\|^4.
\end{aligned} \tag{37}
$$

The inequalities (31) and (37) imply that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\epsilon}\int_\Omega |\nabla U|^2 a'(u^*)dx + \|\triangle W\|^2\right)\\
&\leq -\frac{d_1k_1}{4\epsilon}\|\triangle U\|^2 - \frac{d_3}{2}\|\nabla\triangle W\|^2 - \frac{1}{8\epsilon}\|\triangle W\|^2 + c_{14}(\epsilon^{1/2}+\epsilon) + 2M_2^2\epsilon\\
&\quad + \frac{c_{13}+4c_{17}^2+c_{19}}{\epsilon}(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2)\\
&\quad + \frac{c_{19}}{\epsilon^2}\|\triangle U\|^2\left(\|U\|^2 + \|\nabla U\|^2 + 2\epsilon\|\triangle W\|^2\right) + \frac{c_{19}}{\epsilon}\|U\|^4. \qquad (38)
\end{aligned}$$

Recall that

$$\frac{1}{\epsilon}\int_\Omega |\nabla U|^2 a'(u^*)dx + \|\triangle W\|^2 \geq \frac{k_1}{\epsilon}\|\nabla U\|^2 + \|\triangle W\|^2.$$

Fix an arbitrary positive number ϵ_0. Taking (13) into account, we can derive from (38)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\frac{1}{\epsilon}\int_\Omega |\nabla U|^2 a'(u^*)dx + \|\triangle W\|^2\right)\\
&\leq -\frac{d_1k_1}{4\epsilon}\|\triangle U\|^2 - \frac{d_3}{2}\|\nabla\triangle W\|^2 - \frac{1}{8\epsilon}\|\triangle W\|^2 + c_{20}\epsilon^{1/2} + c_{20}\|\triangle U\|^2\\
&\quad + c_{20}\left(1+\frac{1}{\epsilon}\|\triangle U\|^2\right)\left(\frac{k_1}{\epsilon}\|\nabla U\|^2 + \|\triangle W\|^2\right), \qquad (39)
\end{aligned}$$

if $0<\epsilon\leq\epsilon_0$. Since (29) guarantees that the assumptions of Lemma 3.2 hold true with

$$\rho(t;\epsilon) = 2c_{20}(\epsilon^{1/2} + \|\triangle U\|^2), \quad \lambda(t;\epsilon) = 2c_{20}\left(1+\frac{1}{\epsilon}\|\triangle U\|^2\right),$$

we can apply Lemma 3.2 to (39) and obtain

$$\|\nabla U\|^2 \leq c_{21}\epsilon^{3/2}, \quad \int_0^T \|\triangle U\|^2 dt \leq c_{21}\epsilon^{3/2}, \quad \int_0^T \|\nabla\triangle W\|^2 dt \leq c_{21}\epsilon^{1/2} \qquad (40)$$

for $0<\epsilon\leq\epsilon_0$.

Next we will show the inequality for $\|\nabla V\|^2$. Multiplying the second equation of (18) by $-b_v(u^*,v^*)\triangle V$ and integrating over Ω yield

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla V|^2 b_v(u^*,v^*)dx &\leq -\frac{d_2k_2}{2}\|\triangle V\|^2 - (\triangle Z, b_v(u^*,v^*)\triangle V) + \frac{1}{4}\|\triangle Z\|^2\\
&\quad + c_{22}(\|U\|^2 + \|V\|^2 + \|\nabla V\|^2). \qquad (41)
\end{aligned}$$

Let us operate $\triangle$ to the last equation of (18), multiply it by $\triangle Z$, and integrate over Ω:

$$\frac{1}{2}\frac{d}{dt}\|\triangle Z\|^2 = -d_4\|\nabla\triangle Z\|^2 + d_4\int_{\partial\Omega}\triangle Z\frac{\partial}{\partial n}\triangle Z dS + (\triangle(d_4\triangle z^* - z_t^*), \triangle Z)$$
$$+\frac{1}{\epsilon}(\triangle(b(u^*+U, v^*+V) - b(u^*, v^*)), \triangle Z) - \frac{1}{\epsilon}\|\triangle Z\|^2. \quad (42)$$

Setting

$$I_2 := \triangle(b(u^*+U, v^*+V) - b(u^*, v^*)) - b_v(u^*, v^*)\triangle V,$$

we can see, in the similar manner to the argument to obtain (35), that

$$\begin{aligned}|(I_2, \triangle Z)| \le\ & c_{23}(\|U\| + \|\nabla U\| + \|V\| + \|\nabla V\|)\|\triangle Z\| \\ & + c_{23}\left(\|\nabla U\|^2_{L^4(\Omega)} + \|\nabla V\|^2_{L^4(\Omega)}\right)\|\triangle Z\| + c_{23}\|\triangle U\|\ \|\triangle Z\| \\ & + c_{23}\left(\|U\|_{L^4(\Omega)} + \|V\|_{L^4(\Omega)}\right)\|\triangle V\|\ \|\triangle Z\|_{L^4(\Omega)}.\end{aligned}$$

As in deriving (36), the above inequality is reduced to

$$\begin{aligned}|(I_2, \triangle Z)| \le\ & \frac{d_4\epsilon}{4}\|\nabla\triangle Z\|^2 + \frac{1}{4}\|\triangle Z\|^2 + \frac{d_2k_2}{4}\|\triangle V\|^2 + c_{24}\|\triangle U\|^2 \\ & + c_{24}\left(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2 + \|\nabla V\|^2\right) \\ & + \frac{c_{24}}{\epsilon}\|\triangle V\|^2\left(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2 + \|\nabla V\|^2\right) \\ & + c_{24}\left(\|\triangle U\|^2 + \|\triangle V\|^2\right)\|\triangle Z\|^2 + c_{24}(\|U\|^4 + \|V\|^4). \quad (43)\end{aligned}$$

Combining (41), (42), (43) and (34), we have

$$\begin{aligned}& \frac{1}{2}\frac{d}{dt}\left(\frac{1}{\epsilon}\int_\Omega |\nabla V|^2 b_v(u^*, v^*)dx + \|\triangle Z\|^2\right) \\ & \le -\frac{d_2k_2}{4\epsilon}\|\triangle V\|^2 - \frac{d_4}{2}\|\nabla\triangle Z\|^2 - \frac{1}{8\epsilon}\|\triangle Z\|^2 + c_{14}(\epsilon^{1/2} + \epsilon) + 2M_2^2\epsilon \\ & \quad + \frac{c_{24}}{\epsilon}\|\triangle U\|^2 + \frac{c_{22}+c_{24}}{\epsilon}\left(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2 + \|\nabla V\|^2\right) \\ & \quad + \frac{c_{24}}{\epsilon^2}\|\triangle V\|^2\left(\|U\|^2 + \|\nabla U\|^2 + \|V\|^2 + \|\nabla V\|^2\right) \\ & \quad + \frac{c_{24}}{\epsilon}\left(\|\triangle U\|^2 + \|\triangle V\|^2\right)\|\triangle Z\|^2 + \frac{c_{24}}{\epsilon}(\|U\|^4 + \|V\|^4). \quad (44)\end{aligned}$$

Hence, we can obtain

$$\|\nabla V\|^2 \le c_{25}\epsilon^{3/2}, \quad \int_0^T \|\triangle V\|^2 dt \le c_{25}\epsilon^{3/2}, \quad \int_0^T \|\nabla\triangle Z\|^2 dt \le c_{25}\epsilon^{1/2} \quad (45)$$

for $0 < \epsilon \le \epsilon_0$, using (13), (30), (40) and Lemma 3.2.

Due to (22),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla W\|^2 &\le \frac{k_3}{\epsilon}\left(M_1\|U\| + \|\nabla U\|\right)\|\nabla W\| - \frac{1}{2\epsilon}\|\nabla W\|^2 + c_6\epsilon \\ &\le \frac{1}{4\epsilon}\left(c_{26}\epsilon^{3/2} - \|\nabla W\|^2\right),\end{aligned}$$

where $0 < \epsilon \le \epsilon_0$. Lemma 3.1 and the above inequality imply the third inequality of (14). The fourth inequality of (14) can be also seen similarly.

Hereafter we will prove the remaining four inequalities of (14) for $0 < \epsilon \le \epsilon_0$. Since

$$\frac{\partial}{\partial n}(d_1\triangle U + \triangle W) = 0 \quad \text{on } \partial\Omega$$

by (18), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle U\|^2 &\le -d_1\|\nabla\triangle U\|^2 + \|\nabla\triangle W\|\,\|\nabla\triangle U\| \\ &\quad + \|\nabla(f(u^*+U, v^*+V) - f(u^*, v^*))\|\,\|\nabla\triangle U\| \\ &\le -\frac{d_1}{2}\|\nabla\triangle U\|^2 + \frac{1}{d_1}\|\nabla\triangle W\|^2 + c_{27}\epsilon^{3/2}.\end{aligned}$$

Integrating the above in $t \in [0, T]$ and using (40) yield the fifth inequality of (14). Similarly we can show the sixth inequality of (14).

Consider the estimates of $\|\triangle W\|$ and $\|\triangle Z\|$. By (37), (42), (34) and (43), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\triangle W\|^2 &\le -\frac{d_3}{2}\|\nabla\triangle W\|^2 + c_{28}\epsilon^{1/2} + \frac{c_{28}}{\epsilon}\|\triangle U\|^2 + \frac{c_{19}}{\epsilon}\|\triangle U\|^2\|\triangle W\|^2, \\ \frac{1}{2}\frac{d}{dt}\|\triangle Z\|^2 &\le -\frac{d_4}{2}\|\nabla\triangle Z\|^2 + c_{29}\epsilon^{1/2} + \frac{c_{29}}{\epsilon}(\|\triangle U\|^2 + \|\triangle V\|^2) \\ &\quad + \frac{c_{29}}{\epsilon}(\|\triangle U\|^2 + \|\triangle V\|^2)\|\triangle Z\|^2.\end{aligned}$$

The last two inequalities of (14) follow from the above inequalities, (40), (45) and Lemma 3.2.

Acknowledgments

The authors would like to express their gratitude to Professor Masayasu Mimura and Professor Wei-Ming Ni for fruitful discussions with them.

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BIFURCATION DIAGRAMS TO AN ELLIPTIC EQUATION INVOLVING THE CRITICAL SOBOLEV EXPONENT WITH THE ROBIN CONDITION *

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The uniqueness and the multiplicity of radial solutions to the Brezis-Nirenberg equation on the unit ball with the Robin condition are discussed. The scalar-field equation is also treated and the unified approach is presented. Moreover, depending on the Robin condition parameter, the difference between the structure of solutions in the three dimension and that in a higher dimension is shown.

1. Introduction

The investigation of the global structure of solutions is one of the main topics in the study of elliptic partial differential equations. Especially, the homogeneous Dirichlet problem is well investigated. For example, Korman, Li and Ouyang [11, 12] and Ouyang and Shi [15, 16] considered various nonlinearity under which the exact multiplicity of solutions is verified. Moreover, since the pioneering work by Brezis and Nirenberg [3], the elliptic equations with the critical Sobolev exponent has been intensively studied. Here we study the problem

$$\begin{cases} \Delta u + \lambda u + u^{(n+2)/(n-2)} = 0 \quad \text{in } B = \{x \in \mathbf{R}^n : \ |x| < 1\}, \\ u > 0 \quad \text{in } B, \\ \kappa \dfrac{\partial u}{\partial \nu} + u = 0 \quad \text{on } \partial B, \end{cases} \tag{1.1}$$

with dimension $n \geq 3$, where ν is the outward unit normal vector to ∂B,

*Supported in part by the Grant-in-Aid for Scientific Research (C) (2)(No. 15540211), Japan Society for the Promotion of Science.

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$\kappa \geq 0$ and $\lambda < \lambda_1(n;\kappa)$ (the first eigenvalue of $-\Delta$ subject to $\kappa\partial u/\partial\nu + u = 0$).

We vary λ from positive to negative for each fixed $\kappa \geq 0$. For $\kappa = 0$ (Dirichlet Problem), the existence and uniqueness is obtained by Brezis and Nirenberg together with Kwong and Li [13] or Zhang [22].

For Neumann problem ($\kappa = \infty$), no solution exists for $\lambda > 0$ region and the constant solution $u \equiv (-\lambda)^{4/(n-2)}$ and non-trivial solutions bifurcate from the constant solution. Moreover, Budd, Knaap and Peletier [4] showed that there exists $\lambda_* > 0$ such that (1.1) with the Neumann condition has only the trivial solution for $\lambda \in (-\lambda_*, 0)$ if $n = 3$. However, for the higher dimensional case ($n = 4,\ 5,\ 6$), Adimurthi and Yadava [1] have proved that there exists a nontrivial positive solution to (1.1) under the Neumann condition. The result shows the difference between the three dimensional case and the higher dimensional ones. However, strangely, for $n \geq 7$, it is proved that there exists $\lambda(n) > 0$ such that the problem (1.1) has only a constant solution for $\lambda \in [-\lambda(n), 0]$ with the Neumann condition by Adimurthi and Yadava [2].

In between, what will be expected for the solution structure in the case of the Robin condition? We will answer the question. We emphasize here that the three dimensional case is the exceptional case and we state our results on the three dimensional case first.

Theorem 1.1. **(Theorem 1.2 of** [9]**)** *Let $n = 3$. For $0 \leq \kappa \leq 1$,*

(a) *if $\mu_1^2 < \lambda < \mu_0^2 = \lambda_1(3;\kappa)$, then (1.1) has a unique radial solution.*
(b) *if $-\zeta^2 < \lambda \leq \mu_1^2$, then (1.1) has no solution,*

where μ_0, μ_1 and ζ are defined by $1 - \mu_0 \cot \mu_0 = 1/\kappa$, $\mu_1 \tan \mu_1 = 1/\kappa - 1$ for $0 < \kappa \leq 1$, $\mu_1 = \pi/2$ for $\kappa = 0$, and $\zeta \coth \zeta = 1/\kappa$ for $0 < \kappa < 1$, $\zeta = \infty$ for $\kappa = 0$, and $\zeta = 0$ for $\kappa = 1$, respectively.

Theorem 1.2. **(Theorem 1.3 of** [9]**)** *Let $n = 3$. For $\kappa > 1$, (1.1) has a unique radial solution if $-\mu_2^2 \leq \lambda \leq \mu_0^2$, where μ_2 is defined by $\mu_2 \tanh \mu_2 = (\kappa - 1)/\kappa$.*

What will happen for $\lambda < -\mu_2^2$? Does the value μ_2 appear due to the technical reason? We answer that the value μ_2 is essential.

Theorem 1.3. ([7]) *Let $n = 3$. Suppose that $\kappa > 1$. There exists $\varepsilon_0 > 0$ such that (1.1) has at least two solutions $u_1(r,\lambda)$, $u_2(r,\lambda)$ for $\lambda \in [-\mu_2^2 -$*

Figure 1. The bifurcation diagram of the equation (1.1) with $n = 3$ and $\kappa = 0.5$. The horizontal axis is λ and the vertical axis is $u(0)$.

$\varepsilon_0, -\mu_2^2)$. *They are monotone decreasing and* $u_1(0, \lambda)$ *is uniformly bounded while* $u_2(0, \lambda) \to \infty$ *as* $\lambda \uparrow -\mu_2^2$.

For higher dimension, we first note the result on the Dirichlet problem.

Theorem A.1. ([3]) *For* $\kappa = 0$, *if* $n \geq 4$, *then* (1.1) *has a radial solution if and only if* $0 < \lambda < \lambda_1(n; 0)$.

Thus, for a higher dimensional case with $\lambda > 0$ and $\kappa > 0$, the structure of solutions must be different from the three dimensional case.

For a generic dimension, we need detailed informations of the Bessel, the Neumann, and the modified Bessel functions (for formulae on Bessel functions, see e.g., Watson [19]). Even for the three dimensional case, the knowledge of the special functions are required, but only in this case, they can be expressed by simple combinations of sin, cos, sinh and cosh. The following are due to [5] and [6].

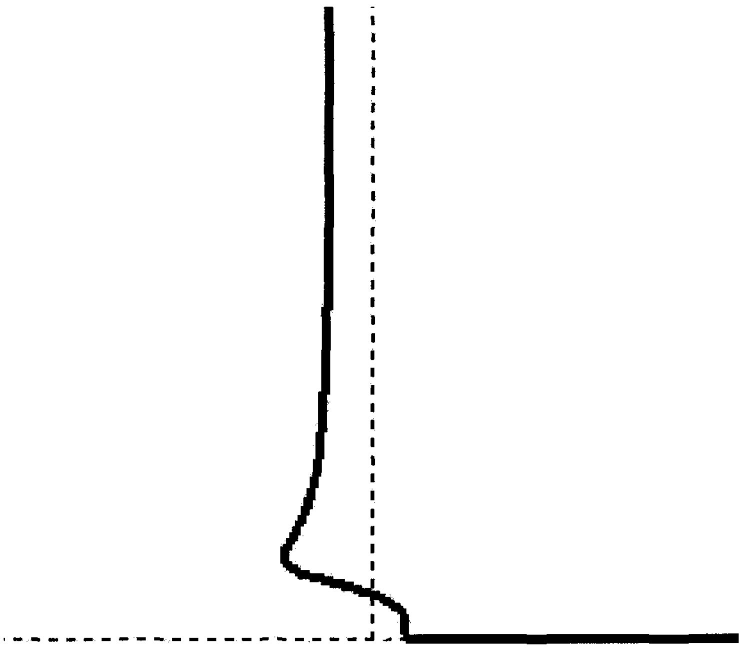

Figure 2. The bifurcation diagram of the equation (1.1) with $n = 3$ and $\kappa = 4$. The horizontal axis is λ and the vertical axis is $u(0)$.

Theorem 1.4. *Let $n \geq 4$ and $0 \leq \kappa < 1/(n-2)$.*

(a) *There exists $\varepsilon_0 > 0$ such that (1.1) has a unique radial solution u_λ for $0 < \lambda < \varepsilon_0$ and $u_\lambda(0) \to \infty$ as $\lambda \to +0$.*
(b) *There exists $\varepsilon_1 > 0$ such that (1.1) has no solution for $-\varepsilon_1 < \lambda < 0$.*

We can prove the existence of a positive solution to (1.1) for any $0 < \lambda < \lambda_1(n;\kappa)$. Theorem 1.4 emphasizes the uniqueness of the solutions for small $\lambda > 0$. The value $\kappa = 1/(n-2)$ is a critical value in the sense that the structure of solutions changes. However, for the Robin condition, there is no difference between $n = 4, 5, 6$ and $n \geq 7$ unlike the results by Adimurthi and Yadava [1, 2].

Theorem 1.5. *Let $n \geq 4$ and $\kappa > 1/(n-2)$.*

(a) *There exists $\varepsilon_2 > 0$ such that (1.1) has a unique radial solution u_λ for $0 \leq \lambda < \varepsilon_2$.*
(b) *There exists $\varepsilon_3 > 0$ such that (1.1) has at least two solutions $u_{1,\lambda}$*

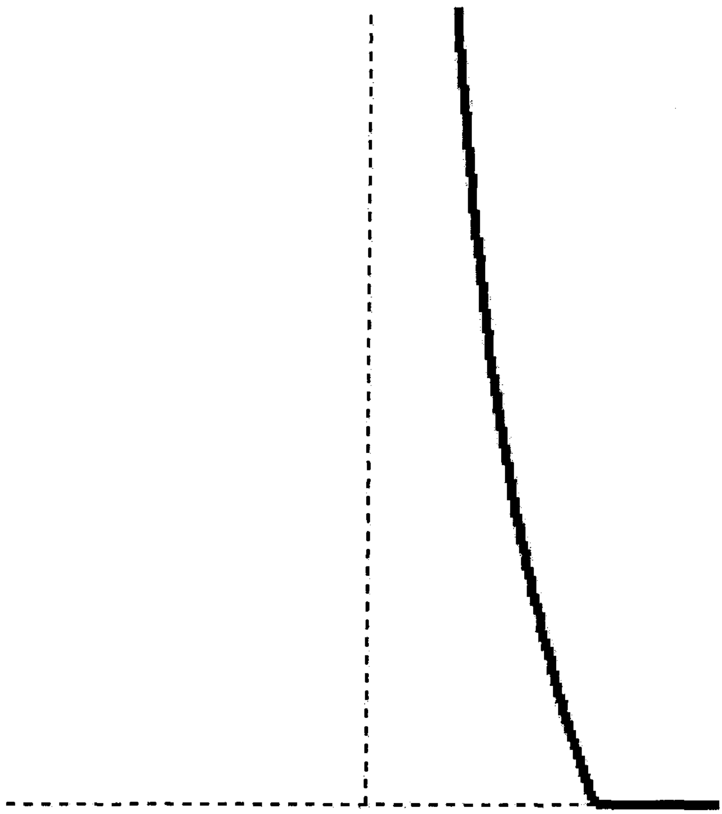

Figure 3. The bifurcation diagram of the equation (1.1) with $n = 6$ and $\kappa = 0.01$. The horizontal axis is λ and the vertical axis is $u(0)$.

and $u_{2,\lambda}$, which are both monotone decreasing for $-\varepsilon_3 < \lambda < 0$. Moreover, $u_{1,\lambda}$ blows up as $\lambda \to -0$ while $u_{2,\lambda}$ is uniformly bounded.

From the bifurcation-theoretic point of view, there exists a solution branch bifurcated from $(\lambda_1(\kappa), 0)$ which has at least one bending point in $\lambda < 0$ region and the branch goes to infinity in $\mathbf{R} \times C([0,1])$ space as $\lambda \uparrow 0$. On the other hand, if $n = 3$, the bending of the solution branch occurs for $\lambda < -\mu_2^2$. For the higher dimension, unlike the three dimensional case, $\lambda = 0$ is always a blowup point for any $\kappa > 1/(n-2)$.

This paper is organized as follows. In Section 2, we transform (1.1) to an exterior Neumann problem as in [9]. A basic structure theorem is presented

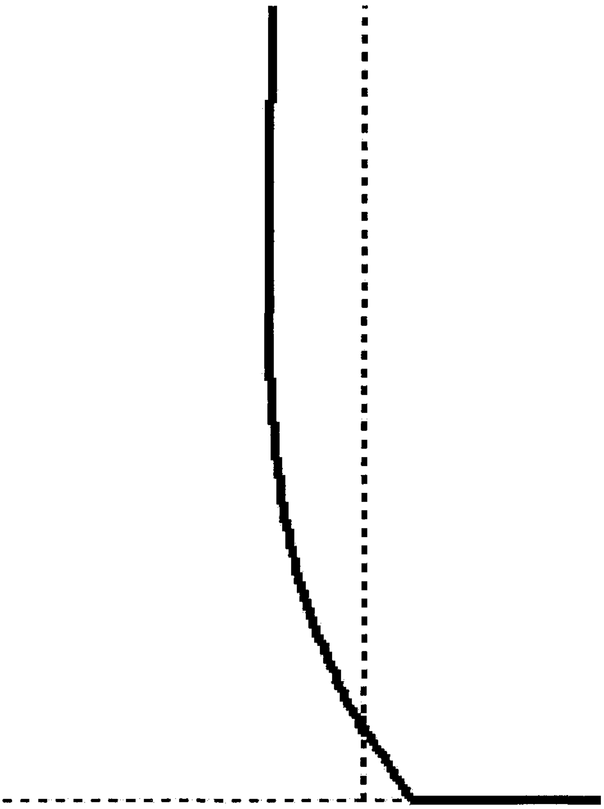

Figure 4. The bifurcation diagram of the equation (1.1) with $n = 6$ and $\kappa = 1$. The horizontal axis is λ and the vertical axis is $u(0)$.

in Section 3. Lemmas peculiar to the critical exponent are discussed in Section 4. In Section 5, the three dimensional case treated. Section 6 is divoted to higher dimensional case. Related topics, especially the imperfect bifurcations are discussed in Section 7.

2. Transformation to the exterior problem

In this section, mainly following Section 2 of [9], we transform (1.1) to the exterior Neumann problem

$$\begin{cases} \dfrac{1}{\tau^{n-1}}(\tau^{n-1}w_\tau)_\tau + Q(\tau)w^{(n+2)/(n-2)} = 0, \ \tau > \rho^{1/(n-2)} \\ w_\tau(\rho^{1/(n-2)}) = 0, \end{cases} \tag{2.1}$$

with some $\rho > 0$. Let $\varphi(r)$ be defined as

$$\varphi(r) := \begin{cases} 2^{(n-2)/2}\Gamma(\frac{n}{2})\dfrac{J_{(n-2)/2}(\sqrt{\lambda}r)}{(\sqrt{\lambda}r)^{(n-2)/2}}, & \lambda > 0, \\ 1, \ \lambda = 0, & \\ 2^{(n-2)/2}\Gamma(\frac{n}{2})\dfrac{I_{(n-2)/2}(\sqrt{-\lambda}r)}{(\sqrt{-\lambda}r)^{(n-2)/2}}, & \lambda < 0, \end{cases} \tag{2.2}$$

where Γ is the gamma function, J_ν and I_ν are the Bessel function and the modified Bessel of the first kind, respectively, of order ν. When $n = 3$, φ is expressed as

$$\varphi(r) := \begin{cases} \dfrac{\sin\sqrt{\lambda}r}{\sqrt{\lambda}r}, & \lambda > 0, \\ 1, \ \lambda = 0, & \\ \dfrac{\sinh\sqrt{-\lambda}r}{\sqrt{-\lambda}r}, & \lambda < 0, \end{cases} \tag{2.3}$$

We define

$$g(r) := r^{n-1}(\varphi(r))^2.$$

For the transformation, we do not need to restrict ourselves to the critical power, we consider the following generic problem

$$\begin{cases} u_{rr} + \dfrac{n-1}{r}u_r + \lambda u + u^p = 0, \quad r \in (0,1), \\ u > 0, \quad r \in (0,1), \\ u(0) < \infty, \ u(1) + \kappa u_r(1) = 0. \end{cases} \tag{2.4}$$

The first step is to transform (2.4) to a special form, whose proof is expressed in Lemma 2.1 of [9].

Lemma 2.1. *For $\lambda < \lambda_1(n;\kappa)$, set $v = u/\varphi$. Then (2.4) is equivalent to*

$$\begin{cases} \dfrac{1}{g}(gv_r)_r + r^{-(n-1)(p-1)/2}g^{(p-1)/2}v^p = 0, \quad r \in (0,1), \\ v > 0, \quad r \in (0,1), \\ v(0) < \infty, \ v(1) + \rho g(1)v_r(1) = 0, \end{cases} \tag{2.5}$$

where $\rho = \rho(\kappa, \lambda)$ is given by

$$\rho = \frac{\kappa}{\varphi(1)\{\varphi(1) + \kappa\varphi_r(1)\}} \tag{2.6}$$

Next, we transform (2.5) to an n-dimensional exterior Neumann problem.

Note that for $\lambda = \lambda_1(n;\kappa)$, φ satisfies $\varphi(r) > 0$ for $r \in [0,1]$ and $\varphi(1) + \kappa\varphi_r(1) = 0$ (φ is an eigenfunction). Hence, if $\lambda < \lambda_1(n;\kappa)$, it follows from Prüfer's comparison theorem that $\varphi(r) > 0$ for $r \in [0,1]$ and $\varphi(1) + \kappa\varphi_r(1) > 0$. Thus, for $\lambda < \lambda_1(\kappa)$, we have $\rho > 0$ if $\kappa > 0$ and $\rho = 0$ if $\kappa = 0$.

Now, we transform (2.5) to an exterior Neumann problem.

Lemma 2.2. *For $\lambda < \lambda_1(n;\kappa)$, set*

$$w(\tau) = \frac{v(r)}{\tau^{n-2}}, \quad \tau^{n-2} = \frac{h(r)}{g(r)},$$

with

$$h(r) = g(r)\Big(\int_r^1 \frac{ds}{g(s)} + \rho\Big). \tag{2.7}$$

Then (2.5) is equivalent to

$$\begin{cases} \dfrac{1}{\tau^{n-1}}(\tau^{n-1}w_\tau)_\tau + Q(\tau)w^p = 0, \quad \tau \in (\rho^{1/(n-2)}, \infty), \\ w > 0, \quad \tau \in (\rho^{1/(n-2)}, \infty), \\ w_\tau(\rho^{1/(n-2)}) = 0, \ \lim\limits_{\tau\to\infty} \tau^{n-2}w(\tau) < \infty, \end{cases} \tag{2.8}$$

where

$$Q(\tau) := (n-2)^2\tau^{(n-2)p+n-4}g^{(p+3)/2}r^{-(n-1)(p-1)/(n-2)}. \tag{2.9}$$

Remark 2.1. Since $\tau^{n-2} = \int_r^1 g(s)^{-1}\,ds + \rho$ and $g(r) = r^{n-1}\varphi^2$, we see that $r \sim 1/\tau$ near $r = 0$.

Proof. By definition, we have

$$g\frac{d}{dr} = -\frac{1}{(n-2)\tau^{n-3}}\frac{d}{d\tau}.$$

Hence we obtain

$$gv_r = -\frac{1}{(n-2)\tau^{n-3}}(\tau^{n-2}w)_\tau) = -\frac{1}{n-2}(\tau w_\tau + (n-2)w)$$

and

$$\begin{aligned} (gv_r)_r &= -\frac{1}{n-2}\frac{d}{d\tau}(\tau w_\tau + (n-2)w)\frac{d\tau}{dr} \\ &= \frac{1}{(n-2)^2\tau^{n-3}g}(\tau w_{\tau\tau} + (n-1)w_\tau) \\ &= \frac{1}{(n-2)^2\tau^{2n-5}g}(\tau^{n-1}w_\tau)_\tau. \end{aligned}$$

Since $\varphi(r) = g(r)^{1/2} r^{-(n-1)/2}$, we see from (2.5) that w satisfies

$$\frac{1}{\tau^{n-1}}(\tau^{n-1} w_\tau)_\tau + Q(\tau) w^p = 0 \tag{2.10}$$

with (2.9). Here, τ varies on $(\rho^{1/(n-2)}, \infty)$ as r varies on $(0, 1)$ in view of (2.7).

As for the boundary condition, note that $r = 1$ corresponds to $\tau = \rho^{1/(n-2)}$. Hence, we have

$$g(1)v_r(1) = -\frac{1}{n-2}\Big\{\rho^{1/(n-2)} w_\tau(\rho^{1/(n-2)}) + (n-2) w(\rho^{1/(n-2)})\Big\}$$

and $v(1) = \tau^{n-2}|_{r=1} w = \rho w(\rho^{1/(n-2)})$. Thus $v(1) + \rho g(1) v_r(1) = 0$ implies $w_\tau(\rho^{1/(n-2)}) = 0$. Finally, we see that $v(0) < \infty$ implies $\lim_{\tau\to\infty} \tau^{n-2} w(\tau) < \infty$ by the definition of $w(\tau)$. $\square$

In the critical Sobolev case $p = (n+2)/(n-2)$, we have

$$\begin{aligned} Q(\tau) &= (n-2)^2 \tau^{2(n-1)} g^{2(n-1)/(n-2)} r^{-2(n-1)/(n-2)} \\ &= (n-2)^2 \big(\frac{h(r)}{r}\big)^{2(n-1)/(n-2)}. \end{aligned} \tag{2.11}$$

Then, as in [9], the investigation of the behavior of Q_τ, i.e., $rh_r - h$ is crucial. Useful Lemmas are given in Section 4.

We enumerate concrete expressions of ρ and h. First, we consider the case where $\lambda > 0$. Let $\lambda = \mu^2$. From the definition of φ, we have

$$\varphi(1) + \kappa\varphi_r(1) = 2^{(n-2)/2}\Gamma(\frac{n}{2})\mu^{-(n-2)/2}\Big\{J_{(n-2)/2}(\mu) - \kappa\mu J_{n/2}(\mu)\Big\}.$$

and

$$\rho = \frac{\mu^{n-2}\kappa}{2^{n-2}\Gamma^2(n/2) J_{(n-2)/2}(\mu)\big(J_{(n-2)/2}(\mu) - \kappa\mu J_{n/2}(\mu)\big)}.$$

We can calculate $h(r)$ explicitly as

$$\begin{aligned} h(r) &= g(r)\Big(\int_r^1 \frac{ds}{g(s)} + \rho\Big) \\ &= (\mu r) J^2_{(n-2)/2}(\mu r) \int_r^1 \frac{ds}{\mu s J^2_{(n-2)/2}(\mu s)} \\ &\qquad + \frac{\kappa r J^2_{(n-2)/2}(\mu r)}{J_{(n-2)/2}(\mu)(J_{(n-2)/2}(\mu) - \kappa\mu J_{n/2}(\mu))}. \end{aligned}$$

Note that

$$\int_r^1 \frac{ds}{\mu s J^2_{(n-2)/2}(\mu s)} = \frac{1}{\mu}\int_{\mu r}^{\mu} \frac{dt}{tJ^2_{(n-2)/2}(t)}$$
$$= \begin{cases} -\dfrac{\pi}{2\mu \sin\frac{n-2}{2}\pi}\Big[\dfrac{J_{-(n-2)/2}(t)}{J_{(n-2)/2}(t)}\Big]_{\mu r}^{\mu}, & n: \text{ odd },\\[2ex] \dfrac{\pi}{2\mu}\Big[\dfrac{N_{(n-2)/2}(t)}{J_{(n-2)/2}(t)}\Big]_{\mu r}^{\mu}, & n: \text{ even.}\end{cases}$$

Let $n = 2k-1$ $(k \geq 3)$. Then we have

$$h(r) = rJ^2_{k-3/2}(\mu r)\Big\{\frac{(-1)^k \pi J_{-(k-3/2)}(\mu r)}{J_{(k-3/2)}(\mu r)} + \frac{(-1)^{k+1}\pi J_{-(k-3/2)}(\mu)}{2J_{k-3/2}(\mu)}$$
$$+\frac{\kappa}{J_{k-3/2}(\mu)\big(J_{k-3/2}(\mu) - \kappa\mu J_{k-1/2}(\mu)\big)}\Big\}.$$

For an even dimensional case $n = 2k$ $(k \geq 2)$, we have

$$h(r)$$
$$= rJ^2_{k-1}(\mu r)\Big\{-\frac{\pi N_{k-1}(\mu r)}{2J_{k-1}(\mu r)} + \frac{\pi N_{k-1}(\mu)}{2J_{k-1}(\mu)}$$
$$+\frac{\kappa}{J_{k-1}(\mu)\big(J_{k-1}(\mu) - \kappa\mu J_k(\mu)\big)}\Big\}.$$

Next, we consider the case where $\lambda < 0$. Let $\lambda = -\xi^2$. Similar to the case where $\lambda > 0$, we have

$$\varphi(1) + \kappa\varphi_r(1) = 2^{(n-2)/2}\Gamma(\frac{n}{2})\xi^{-(n-2)/2}\Big\{I_{(n-2)/2}(\xi) + \kappa\xi I_{n/2}(\xi)\Big\}.$$

and

$$\rho = \frac{\xi^{n-2}\kappa}{2^{n-2}\Gamma^2(n/2)I_{(n-2)/2}(\xi)\big\{I_{(n-2)/2}(\xi) + \kappa\xi I_{n/2}(\xi)\big\}}.$$

We can calculate $h(r)$ explicitly as

$$h(r) = g(r)\Big(\int_r^1 \frac{ds}{g(s)} + \rho\Big)$$
$$= (\xi r)I^2_{(n-2)/2}(\xi r)\int_r^1 \frac{ds}{\xi s I^2_{(n-2)/2}(\xi s)}$$
$$+\frac{\kappa r I^2_{(n-2)/2}(\xi r)}{I_{(n-2)/2}(\xi)\{I_{(n-2)/2}(\xi) + \kappa\xi I_{n/2}(\xi)\}}.$$

Since

$$\int_r^1 \frac{ds}{\xi s I^2_{(n-2)/2}(\xi s)} = \frac{1}{\xi}\int_{\xi r}^{\xi} \frac{dt}{t I^2_{(n-2)/2}(t)}$$
$$= \frac{1}{\xi}\Big[- \frac{K_{(n-2)/2}(t)}{I_{(n-2)/2}(t)} \Big]_{\xi r}^{\xi},$$

we obtain

$$h(r)$$
$$= r I^2_{(n-2)/2}(\xi r)\Bigg\{ \Big(\frac{K_{(n-2)/2}(\xi r)}{I_{(n-2)/2}(\xi r)} - \frac{K_{(n-2)/2}(\xi)}{I_{(n-2)/2}(\xi)} \Big)$$
$$+ \frac{\kappa}{I_{(n-2)/2}(\xi)\{I_{(n-2)/2}(\xi) + \kappa \xi I_{n/2}(\xi)\}} \Bigg\}.$$

The exact form of $h(r)$ is necessary for the investigation of the structure of solutions by using the structure theorem in Section 3.

3. Structure of solutions to the exterior problem

Following the argument in Section 3 of [9], we show the structure theorem for the exterior Neumann problem. Proofs of Lemmas and Theorem in this section are found in Yanagida and Yotsutani [20, 21], Kawano, Yanagida and Yotsutani [10], Ni and Yotsutani [14], and we omit their proofs.

In this section, we consider the auxiliary initial value problem

$$\begin{cases} \dfrac{1}{\tau^{n-1}}(\tau^{n-1} w_\tau)_\tau + Q(\tau) w_+^p = 0, \quad \tau \in (\rho, \infty), \\ w(\rho) = \beta > 0, \quad \lim\limits_{\tau \downarrow \rho} \tau^{n-1} w_\tau(\tau) = 0, \end{cases} \tag{3.1}$$

where $w_+ = \max\{w, 0\}$ and $\rho \geq 0$.

When $\rho = 0$, the last condition in (3.1) is automatically satisfied provided that $\lim_{\tau \downarrow 0} w(\tau)$ exists and is positive, and moreover $w_\tau(0) = 0$ if and only if

$$\lim_{\tau \downarrow 0} \tau^{-(n-1)} \int_0^\tau s^{n-1} Q(s) ds = 0$$

(see, e.g., [14]). When $\rho > 0$, the Neumann boundary condition $w_\tau(\rho) = 0$ is imposed. In this section, we denote $\rho^{1/(n-2)}$ by $\tilde{\rho}$. We also impose the following general conditions on $Q(\tau)$:

$$(\mathrm{Q}) \quad \begin{cases} Q(\tau) \in C^1((\tilde{\rho}, \infty)), \\ Q(\tau) \geq 0, \ \not\equiv 0 \quad \text{on } (\tilde{\rho}, \infty), \\ \tau Q(\tau) \in L^1((\tilde{\rho}, \rho')), \\ \tau^{1-(n-2)p} Q(\tau) \in L^1((\rho', \infty)), \end{cases}$$

where $\rho' \in (\tilde{\rho}, \infty)$ is an arbitrarily fixed constant. It is easy to verify that $Q(\tau)$ given by (2.9) satisfies (Q). We denote the solution of (3.1) by $w(\tau; \beta)$ or simply by $w(\tau)$.

We can show as in the proof of Lemma 7.2 of [14] that if a solution w of (3.1) is positive on $(\tilde{\rho}, \infty)$, then $(\tau^{n-2} w)_\tau > 0$ on $(\tilde{\rho}, \infty)$. Therefore, according to the behavior as $\tau \to \infty$, we can classify solutions of (3.1) into one of the following three types.

Definition.

(i) w is said to be a *rapidly decaying solution* if $w > 0$ on $[\tilde{\rho}, \infty)$ and the limit $\lim_{\tau \to \infty} \tau^{n-2} w(\tau)$ exists and is positive.

(ii) w is said to be a *slowly decaying solution* if $w > 0$ on $[\tilde{\rho}, \infty)$ and the limit $\lim_{\tau \to \infty} \tau^{n-2} w(\tau) = \infty$.

(iii) w is said to be a *crossing solution* if w has a zero in $(\tilde{\rho}, \infty)$.

We remark that a regular solution of (1.1) corresponds to a rapidly decaying solution of (2.8), and a singular solution of (1.1) corresponds to a slowly decaying solution of (2.8).

We introduce the Pohozaev identity which is effective to study the exterior problem. Define

$$P(\tau; w) := \frac{1}{2} \tau^{n-1} w_\tau \{\tau w_\tau + (n-2) w\} + \frac{\tau^n}{p+1} Q(\tau) w_+^{p+1},$$

$$G(\tau) := \frac{1}{p+1} \Big\{ \tau^n Q(\tau) - \frac{(n-2)(p+1)}{2} \int_{\tilde{\rho}}^{\tau} s^{n-1} Q(s)\, ds \Big\},$$

and

$$H(\tau) := \frac{1}{p+1} \Big\{ \tau^{2-(n-2)p} Q(\tau) - \frac{(n-2)(p+1)}{2} \int_{\tau}^{\infty} s^{1-(n-2)p} Q(s)\, ds \Big\}.$$

By (Q), the function $H(\tau)$ is well-defined. The following is a fundamental property of the Pohozaev identity.

Lemma 3.1. *Any solution w of* (3.1) *satisfies the identity*

$$\frac{d}{d\tau}P(\tau;w) = G_\tau(\tau)w_+^{p+1} \tag{3.2}$$

and its integral form

$$P(\tau,w) = G(\tau)w_+^{p+1} - (p+1)\int_{\tilde\rho}^{\tau} G(s)w_+^p w_\tau\, ds. \tag{3.3}$$

Note that

$$G_\tau(\tau) \equiv \tau^{(n-2)(p+1)}H_\tau(\tau) = \frac{\tau^{(n-2)(p+1)/2}}{p+1}(\tau^{-\theta}Q)_\tau \tag{3.4}$$

with

$$\theta = \frac{(n-2)p-(n+2)}{2}.$$

Let us put

$$\tau_G := \inf\{\tau \in [\tilde\rho,\infty) \,|\, G(\tau) < 0\},$$
$$\tau_H := \sup\{\tau \in [\tilde\rho,\infty) \,|\, H(\tau) < 0\}.$$

Here we define $\tau_G = \infty$ if $G(\tau) \geq 0$ on $(\tilde\rho,\infty)$ and $\tau_H = \tilde\rho$ if $H(\tau) \geq 0$ on $(\tilde\rho,\infty)$.

The purpose of this section is to introduce the following theorem, which is essential to the proofs of the uniqueness part of Theorems 1.4 and 1.5.

Theorem 3.1. (Structure Theorem) *The structure of solutions to* (3.1) *is as follows under the assumption (Q).*

(i) *If $\tau_G = \infty$, then the structure is of type C: $w(\tau;\beta)$ is a crossing solution for any $\beta > 0$.*
(ii) *If $\tau_H = 0$, then the structure is of type S: $w(\tau;\beta)$ is a slowly decaying solution for any $\beta > 0$.*
(iii) *If $0 < \tau_H \leq \tau_G < \infty$, then the structure is of type M: there exists $\beta^* > 0$ such that $w(\tau;\beta)$ is a slowly decaying solution for any $\beta \in (0,\beta^*)$, $w(\tau;\beta^*)$ is a rapidly decaying solution and $w(\tau;\beta)$ is a crossing solution for $\beta \in (\beta^*,\infty)$.*

The relation between the structure for (3.1) and that in (1.1) is as follows:

Theorem 3.2.

(i) *If the structure for* (3.1) *is of type C of type S, then there exists no solution for* (1.1).

(ii) *If the structure for* (3.1) *is of type M, then the structure for* (1.1) *is also of type M. That is, there exists a unique solution to* (1.1).

Theorem 3.1 is a slight extension of Theorem 1 of [20] in which only the case $\tilde{\rho} = 0$ is treated. See Theorem 3.3 of [9]. As is sated in the following, the exterior problem has a peculiar property, which is not necessarily possessed by the entire space problem.

Lemma 3.2. *Let $\tilde{\rho} > 0$. If $\beta > 0$ is sufficiently large, the unique solution $w(\tau;\beta)$ of* (3.1) *is a crossing solution.*

For a small initial value, the behavior of $w(\tau;\beta)$ and $P(\tau;w)$ has a specific limiting behavior.

Lemma 3.3. *For a solution w to* (3.1) *satisfies*

$$\lim_{\beta\downarrow 0}\frac{w(\tau)}{\beta} = 1, \quad \lim_{\beta\downarrow 0}\frac{1}{\beta^{p+1}}P(\tau;w) = G(\tau)$$

uniformly on $[\tilde{\rho}, T]$ with any $T > \tilde{\rho}$.

By the above Lemma, we can find an open interval of small initial values for which a solution $w(\tau;\beta)$ is a crossing solution under a specific condition of G. This lemma helps us to prove the multiplicity of solutions.

Lemma 3.4. *If there exists $T > \tilde{\rho}$ such that $G(\tau) \geq G(T) > 0$ on $[T,\infty)$, then $w(\tau;\beta)$ to* (3.1) *is a crossing solution for any sufficiently small $\beta > 0$.*

4. Critical exponent case

In this section, Lemmas necessary to prove Theorems 1.4 and 1.5 are given. We state peculiar properties of Q in (2.11) here. That is, we are concentrated on the problem

$$\begin{cases} \dfrac{1}{\tau^{n-1}}(\tau^{n-1}w_\tau)_\tau + (n-2)^2(\dfrac{h(r)}{r})^{2(n-1)/(n-2)}w_+^{(n+2)/(n-2)} = 0, & \tau > \rho, \\ w(\rho) = \beta > 0, \quad \lim\limits_{\tau\downarrow\rho}\tau^{n-1}w_\tau(\tau) = 0, & \end{cases} \tag{4.1}$$

Since we are concerned with $p = (n+2)/(n-2)$, we have $\theta = 0$ and $G_\tau(\tau) = \{(n-2)/2n\}\tau^n Q_\tau(\tau)$ by (3.4). Thus, for our problem, since

$$Q(\tau) = (n-2)^2(\frac{h(r)}{r})^{2(n-1)/(n-2)},$$

we obtain

$$G_\tau(\tau) = \frac{(n-1)(n-2)^3}{2n}\left(\frac{h(r)}{r}\right)^{n/(n-2)}\left(\frac{h(r)-rh_r(r)}{r^2}\right)g(r)\tau^{2n-3}, \quad (4.2)$$

in view of $d\tau/dr = -(n-2)g(r)\tau^{n-3}$. Thus the investigation of $h - rh_r$ is crucial for our purpose and its concrete calculation is necessary.

First, we give a sufficient condition for (iii) of Theorem 3.1.

Lemma 4.1. *Suppose that there exists $r_0 \in (0,1]$ such that $h - rh_r < 0$ on $[0, r_0)$ and $h - rh_r > 0$ on $(r_0, 1]$. If there exists $c_0 > 0$ such that $|h - rh_r| \geq c_0 r^\ell$ near $r = 0$ with some $\ell \leq n+1$, then the structure of solutions to* (4.1) *is of type M.*

Proof. By assumption, (4.2) implies that there exists $T_0 \geq \rho$ such $G_\tau > 0$ on $[\rho, T_0)$ and $G_\tau < 0$ on (T_0, ∞). Hence we see $\tau_G \geq \tau_H$ in view of (3.4).

Next, we prove that $\tau_G < \infty$. By (4.2), since $r \sim 1/\tau$, since $h(r) \sim r$ and since $g \sim r^{n-1}$, we have

$$|G_\tau(\tau)| \geq c_1 r^{n+\ell-3}\tau^{2n-3} \sim \tau^{n-\ell}$$

with some $c_1 > 0$ near $\tau = \infty$. Since $\ell \leq n+1$, $\tau^{n-\ell} \notin L^1([\rho, \infty))$. Thus, we see $\int_\rho^\infty G_\tau \, d\tau = -\infty$ and $\tau_G < \infty$. From (iii) of Theorem 3.1, the structure is of type M. □

Unfortunately, the opposite sign case of $h - rh_r$ in Lemma 4.1 does happen in our problem.

Instead, we give a sufficient condition for (i) of Theorem 3.1.

Lemma 4.2. *If $h - rh_r > 0$ on $(0,1]$, then the structure of solutions to* (4.1) *is of type C.*

Proof. Since $h - rh_r > 0$ on $(0,1]$, $G_\tau > 0$ on (ρ, ∞) by (4.2). This implies that $G > 0$ on $[\rho, \infty)$ in view of $G(\rho) = \{(n-2)/(2n)\}\rho^n Q(\rho) > 0$. Thus, by (i) of Theorem 3.1, the conclusion holds. □

Using the explicit form of h, we write down $h - rh_r$.

For $\lambda = \mu^2 > 0$ and $n = 2k-1 (k \geq 3)$, we have

$$\begin{aligned} h - rh_r &= \frac{(-1)^{k+1}\pi\mu r^2}{2}\left\{J'_{k-3/2}(\mu r)J_{-(k-3/2)}(\mu r) + J_{k-3/2}(\mu r)J'_{-(k-3/2)}(\mu r)\right\} \\ &\qquad -2C(2k-1,\kappa,\mu)\mu r^2 J_{k-3/2}(\mu r)J'_{k-3/2}(\mu r) \end{aligned} \quad (4.3)$$

with

$$C(2k-1,\kappa,\mu)=\frac{2\kappa+(-1)^{k+1}\pi J_{-(k-3/2)}(\mu)\{J_{k-3/2}(\mu)-\kappa\mu J_{k-1/2}(\mu)\}}{2J_{k-3/2}(\mu)(J_{k-3/2}(\mu)-\kappa\mu J_{k-1/2}(\mu))}.$$

For $\lambda=\mu^2>0$ and $n=2k(k\geq 2)$,

$$\begin{aligned} h-rh_r=\frac{\pi\mu r^2}{2}&\Big\{J'_{k-1}(\mu r)N_{k-1}(\mu r)+J_{k-1}(\mu r)N'_{k-1}(\mu r)\Big\}\\ &-2C(2k,\kappa,\mu)\mu r^2J_{k-1}(\mu r)J'_{k-1}(\mu r)\end{aligned}\tag{4.4}$$

with

$$C(2k,\kappa,\mu)=\frac{2\kappa+\pi N_{k-1}(\mu)\{J_{k-1}(\mu)-\kappa\mu J_k(\mu)\}}{2J_{k-1}(\mu)(J_{k-1}(\mu)-\kappa\mu J_k(\mu))}$$

and for $\lambda=-\xi^2<0$,

$$\begin{aligned} h-rh_r=-\xi r^2&\Big\{I'_{(n-2)/2}(\xi r)K_{(n-2)/2}(\xi r)+I_{(n-2)/2}(\xi r)K'_{(n-2)/2}(\xi r)\Big\}\\ &-2C(n,\kappa,\xi)\xi r^2I_{(n-2)/2}(\xi r)I'_{(n-2)/2}(\xi r)\end{aligned}\tag{4.5}$$

with

$$C(n,\kappa,\xi)=\frac{\kappa}{I_{(n-2)/2}(\xi)\{I_{(n-2)/2}(\xi)+\kappa\xi I_{n/2}(\xi)\}}-\frac{K_{(n-2)/2}(\xi)}{I_{(n-2)/2}(\xi)}.$$

If $n=3$, then the expressions are much simpler and expressed as follows:

$$h(r)=\frac{1}{2\mu}\Big\{\sin 2\mu r+\frac{\kappa(\cot\mu+\mu)-\cot\mu}{\kappa(\mu\cot\mu-1)+1}(1-\cos 2\mu r)\Big\}$$

for $\lambda=\mu^2$, and

$$h(r)=\frac{1}{2\xi}\Big\{\sinh 2\xi r+\frac{\kappa(\xi-\coth\xi)+\coth\xi}{\kappa(\xi\coth\xi-1)+1}(1-\cosh 2\xi r)\Big\}$$

for $\lambda=-\xi^2$.

In the following two Sections, we investigate $h-rh_r$. We will show that Lemmas 4.1 and 4.2 are applicable for some range of λ and for some other range, Lemma 3.4 is applicable.

5. Three dimensional case

In this section, we concentrate on the three dimensional case. Since proofs of Theorem 1.1 and 1.2 are shown in [9], we give sketchy proofs.

For $n = 3$, we have

$$h - rh_r = \begin{cases} -\nu\mu r^2 + O(r^3) & \text{for } \lambda = \mu^2 > 0, \\ (1-\kappa)r^2 & \text{for } \lambda = 0, \\ \eta\mu r^2 + O(r^3) & \text{for } -\mu_2^2 < \lambda = -\xi^2 < 0, \end{cases} \tag{5.1}$$

at $r = 0$, and if $\lambda = -\mu_2^2$, then

$$h - rh_r = -\frac{4}{3}\mu_2^2 r^3 + O(r^4)$$

at $r = 0$.

Moreover, we see that

$$(h - rh_r)_r = \begin{cases} 2\mu r(\sin 2\mu r - \nu\cos 2\mu r) & \text{for } \lambda = \mu^2 > 0, \\ 2(1-\kappa)r^2 & \text{for } \lambda = 0, \\ 2\xi r(-\sinh 2\xi r + \eta\cosh 2\xi r) & \text{for } \lambda = -\xi^2 < 0. \end{cases} \tag{5.2}$$

Taking these behaviors into account, we can check the number of zeroes of $h - rh_r$.

Proof of Theorem 1.1 . Let $\kappa \in [0,1]$. For $\lambda \in (\mu_1^2, \mu_0^2)$, we can prove that the conditions in Lemma 4.1 are fulfilled. Then by (ii) of Theorem 3.2, we see the existence of a unique solution to (1.1).

For $\lambda \in (-\zeta^2, \mu_1^2]$, the conditions in Lemma 4.2 are satisfied. By (i) of Theorem 3.2, we see the nonexistence of a solution. $\square$

Proof of Theorem 1.2. Let $\kappa > 1$. For $\lambda \in [-\mu_2^2, \mu_0^2)$, the conditions in Lemma 4.1 are satisfied. By (ii) of Theorem 3.2, we see the existence of a unique solution to (1.1). $\square$

Unfortunately, for $\lambda \leq -\mu_2^2$, we cannot expect Lemma 4.1. What we have is Lemma 3.4. However, by using the global bifurcation Theorem due to Rabinowitz [17] and the result by [4], we can prove Theorem 1.3.

Proof of Theorem 1.3. By Rabinowitz [17], the solution-curve $\{(\lambda, u) \in \mathbf{R} \times C([0,1])\}$ which bifurcates from $(\mu_0^2, 0)$ goes to ∞ in $\mathbf{R} \times C([0,1])$. For $\lambda \in [-\mu_2^2, \mu_0^2)$, by Theorem 1.2, the cannot bend back to this region. Moreover, due to the result by [4] on the Neumann problem, the solution curve cannot intersect with the solution curve under the Neumann condition unless $u \equiv 0$. Since the non-trivial solution-branch blows up at $\lambda = -\mu_*^2$ with μ_* satisfying $\mu_* \tanh \mu_* = 1$, our solution-branch blows up between $[-\mu_*^2, -\mu_2^2]$. By using Lemma 3.4, we can see that the blow-up point is

$\lambda = -\mu_2^2$ and obtain the multiplicity of solutions and the blow-up behavior.

□

6. Higher dimensional case

In this Section, in order to prove Theorems 1.4 and 1.5, we investigate the behavior of $h - rh_r$. First, we consider the case where $\lambda = \mu^2 > 0$. We obtain the expansion of $h - rh_r$.

Lemma 6.1. *Suppose that $\mu > 0$ is sufficiently small. On $[0, 1]$, there holds for $n = 2k - 1$ with $k \geq 3$,*

$$\begin{aligned} &h - rh_r \\ &= -\frac{4\mu^2 r^3}{(2k-1)(2k-3)(2k-5)} - \{(2k-3)\kappa - 1 + O(\mu^2)\} r^{2k-2} \\ &\qquad + O(\mu^2 r^{2k} + \mu^4 r^5), \end{aligned} \tag{6.1}$$

for $n = 2k$ with $k \geq 3$,

$$\begin{aligned} &h - rh_r \\ &= -\frac{\mu^2 r^3}{2k(k-1)(k-2)} - \{(2k-2)\kappa - 1 + O(\mu^2)\} r^{2k-1} \\ &\qquad + O(\mu^2 r^{2k+1} + \mu^4 r^5), \end{aligned} \tag{6.2}$$

and for $n = 4$,

$$\begin{aligned} &h - rh_r \\ &= \frac{\mu^2 r^3}{2} \log\frac{\mu r}{2} - \{(2\kappa - 1) + \frac{\mu^2}{2}\log\frac{\mu}{2} + O(\mu^2)\} r^3 + \frac{1}{4}(2\gamma + \frac{1}{2})\mu^2 r^3 \\ &\qquad + O(\mu^2 r^5 + \mu^4 r^5 \log(\mu r)), \end{aligned} \tag{6.3}$$

where γ is the Euler number.

We are now ready to prove (a) of Theorem 1.5.

Proof of (a) of Theorem 1.5. Suppose that $\mu > 0$ is sufficiently small. If $\kappa > 1/(2k-3)$ in (6.1), or if $\kappa > 1/(2k-2)$ in (6.2), or if $\kappa > 1/2$ in (6.3) (in either case, these conditions are expressed as $\kappa > 1/(n-2)$), then $h - rh_r < 0$ on $(0, 1]$. Moreover, since $|h - rh_r| \geq c_0 r^3$ holds near $r = 0$, with $c_0 > 0$, from Lemma 4.1, the structure of (4.1) is of type M, i.e., there exists a unique radial solution to (1.1) by (ii) of Theorem 3.2. □

In case of $0 \le \kappa < 1/(n-2)$, $h - rh_r$ changes its sign on $(0,1]$. We can prove that the zero of $h - rh_r$ is unique and thus we can prove a part of (a) of Theorem 1.4.

Proof of the uniqueness part of (a) of Theorem 1.4. Let $r_0 := r_0(k,\mu)$ be a zero of $h - rh_r$. In view of the expansion in Lemma 6.1, we have, for $n = 2k-1$,

$$r_0 = \left[\frac{4}{\{1-(2k-3)\kappa\}(2k-1)(2k-3)(2k-5)}\right]^{\frac{1}{2k-5}} \mu^{\frac{2}{2k-5}} + O(\mu^{1+\frac{6}{2k-5}}), \tag{6.4}$$

for $n = 2k$,

$$r_0 = \left[\frac{1}{\{1-2(k-1)\kappa\}2k(k-1)(k-2)}\right]^{\frac{1}{2k-4}} \mu^{\frac{1}{k-2}} + O(\mu^{1+\frac{3}{k-2}}), \tag{6.5}$$

and for $n = 4$, we have

$$r_0 = M \exp\{-\frac{2(1-2\kappa)}{\mu^2}\}\left[1 + O(\exp\{-\frac{2(1-2\kappa)}{\mu^2}\})\right] \tag{6.6}$$

with some constant $M > 0$. In either case, at $r = r_0$, we can prove that $h - rh_r$ is non-degenerate and the uniqueness of r_0 follows. Since we have already seen $|h - rh_r| \ge c(n)\mu^2 r^3$ near $r = 0$ with $c(n) > 0$ as in the proof of (a) of Theorem 1.5, the fact that the structure of (4.1) is of type M follows by Lemma 4.1. Hence the uniqueness of solutions to (1.1) are ensured by (ii) of Theorem 3.2. $\square$

As for the blowup property described in Theorem 1.4, we need information of structure at $\lambda = 0$.

Lemma 6.2. *Let* $\lambda = 0$.

(i) *If* $0 \le \kappa \le 1/(n-2)$, *then the structure of solutions to* (4.1) *is of type C.*
(ii) *If* $\kappa > 1/(n-2)$, *then the structure of solutions to* (4.1) *is of type M.*

For $\kappa > 1/(n-2)$with $\lambda = 0$, we have a unique radial solution to (1.1) of the form

$$U(r) = \left[\left\{\frac{n(n-2)}{(n-2)\kappa - 1}\right\}^{-1/2} + \frac{1}{n(n-2)}\left\{\frac{n(n-2)}{(n-2)\kappa - 1}\right\}^{1/2} r^2\right]^{-(n-2)/2} \tag{6.7}$$

and note that

$$U(0) = \alpha_0 := \left\{\frac{n(n-2)}{(n-2)\kappa - 1}\right\}^{(n-2)/4}.$$

Let

$$U(r;\alpha) := \left[\alpha^{-2/(n-2)} + \frac{1}{n(n-2)}\alpha^{2/(n-2)}r^2\right]^{-(n-2)/2} \tag{6.8}$$

with $\alpha > 0$. We can prove that $V(r) := \partial U(r)/\partial\alpha\big|_{\alpha=\alpha_0}$ does not satisfy the boundary condition $\kappa V_r(1) + V(1) = 0$. This implies that $U(r)$ can be uniquely continued to the region where $\lambda < 0$. Thus, we have the following.

Lemma 6.3. *The unique solution* (6.7) *is nondegenerate at* $\lambda = 0$.

We now prove the blow-up behavior.

Proof of (a) of Theorem 1.4 completed. At $\lambda = 0$, the structure of solutions to (4.1) is of type C. Since the zero of the solution $w(\tau;\lambda;\beta)$ to (4.1) is continuously dependent on λ and β, the set of (λ,β) so that $w(\tau;\lambda;\beta)$ has a zero is open. Hence, for any fixed $\beta > 0$, we can take $\lambda(\beta) > 0$ such that $w(\tau;\lambda;\beta)$ has a zero for any $0 < \lambda < \lambda(\beta)$. Thus, in view of the type M structure of (4.1) for small $\lambda > 0$, $\beta > 0$ such that $w(\tau;\lambda;\beta)$ has a zero satisfies $\beta > \beta_*$, where β_* is the initial value for the unique solution $w(\tau;\lambda;\beta_*)$. This implies that $\beta_* \to 0$ as $\lambda \downarrow 0$. Since a solution $u(r)$ to the equation of (1.1) with $u(0) = \alpha > 0$ sufficiently small never satisfies the boundary condition (see, Lemma 5.2 of [9]), the initial value corresponding to $w(\tau;\lambda;\beta_*)$ tends to infinity as $\lambda \downarrow 0$. Thus $u(0) \to \infty$ as $\lambda \downarrow 0$. □

Finally, we consider the case where $\lambda = -\xi^2 < 0$. Similar to Lemma 6.1, we have the expansion of $h - rh_r$.

Lemma 6.4. *Suppose that* $\xi > 0$ *is sufficiently small. Then, on* $[0,1]$, *there hold for* $n = 2k-1$,

$$\begin{aligned} h - rh_r &= \frac{4\xi^2 r^3}{(2k-1)(2k-3)(2k-5)} - \{(2k-3)\kappa - 1 + O(\xi^2)\}r^{2k-2} \\ &\quad + O(\xi^2 r^{2k} + \xi^4 r^5), \end{aligned} \tag{6.9}$$

for $n = 2k$ with $k \geq 3$,

$$h - rh_r = \frac{\xi^2 r^3}{2k(k-1)(k-2)} - \{(2k-2)\kappa - 1 + O(\xi^2)\}r^{2k-1} + O(\xi^2 r^{2k+1} + \xi^4 r^5), \tag{6.10}$$

and for $n = 4$,

$$h - rh_r = -\frac{\xi^2 r^3}{2}\log\frac{\xi r}{2} - \Big[(2\kappa - 1) + \frac{\xi^2}{2}\log\frac{\xi}{2} + \{\kappa(\kappa-1) + \frac{3}{4} + \gamma + O(\xi)\}\xi^2\}\Big]r^3 - \frac{1}{4}(2\gamma + \frac{1}{2})\xi^2 r^3 + O(\xi^2 r^5 + \xi^4 r^5 \log \xi r). \tag{6.11}$$

Proof of (b) of Theorem 1.4. Suppose that $\xi > 0$ is sufficiently small. If $\kappa < 1/(2k-3)$ in (6.9), or if $\kappa < 1/(2k-2)$ in (6.10), or if $\kappa < 1/2$ in (6.11) (in either case, these conditions are expressed as $\kappa < 1/(n-2)$), then $h - rh_r > 0$ on $(0, 1]$. Hence, from Lemma 4.2, the structure of (4.1) is of type C. By (i) of Theorem 3.2, the original problem does not have a solution. Thus, there exists $\varepsilon_1 > 0$ such that (1.1) does not have a solution for $-\varepsilon_1 < \lambda < 0$. □

At $r = 1$, in view of the expansion (6.9), (6.10), or (6.11), if $\xi > 0$ is sufficiently small, we have

$$h - rh_r\Big|_{r=1} < -\frac{(n-2)\kappa - 1}{2} < 0.$$

This implies that the structure of (4.1) is neither of type C or of type M. We will prove that there exists a rapidly decaying solution with small initial value. (See Theorem 2 of Yanagida and Yotsutani [20]). Indeed, we have the following lemma.

Lemma 6.5. *Let $\kappa > 1/(n-2)$. Then, if $\xi > 0$ is sufficiently small, there exists a unique $r_1(k, \xi) > 0$ such that $h - rh_r > 0$ on $(0, r_1(k, \xi))$ and $h - rh_r < 0$ on $(r_1(k, \xi), 1)$.*

Proof. From Lemma 6.4, we see that $h - rh_r > 0$ near $r = 0$ if $\xi > 0$ is sufficiently small. We prove the uniqueness of the zeros of $h - rh_r$. Since the coefficients in $h - rh_r$ with $\xi > 0$ is exactly the same as that with

$\mu > 0$ for $n \geq 5$, we can follow the proof of (a) of Theorem 1.4 to get the conclusion. Indeed, Let $r_1 := r_1(k, \mu)$ be a zero of $h - rh_r$. In view of the expansion in Lemma 6.4, we have

$$r_1 = \left[\frac{4}{\{(2k-3)\kappa - 1\}(2k-1)(2k-3)(2k-5)}\right]^{\frac{1}{2k-5}} \xi^{\frac{2}{2k-5}} + O(\xi^{1+\frac{6}{2k-5}}) \tag{6.12}$$

for $n = 2k - 1$,

$$r_1 = \left[\frac{1}{\{2(k-1)\kappa - 1\}2k(k-1)(k-2)}\right]^{\frac{1}{2k-4}} \xi^{\frac{1}{k-2}} + O(\xi^{1+\frac{3}{k-2}}) \tag{6.13}$$

for $n = 2k$,

$$r_1 = M_1 \xi^{-2} \exp\{-\frac{2(2\kappa-1)}{\xi^2}\}\Big\{1 + O(\exp\{-\frac{2(2\kappa-1)}{\xi^2}\})\Big\}, \tag{6.14}$$

with

$$M_1 = 4[\exp\{-2\gamma - 1 - \kappa(\kappa-1)\}](1 + o(1)).$$

for $n = 4$. In either case, at $r = r_1$, we can prove that $h - rh_r$ is non-degenerate and the uniqueness of r_1 follows. □

From Lemma 6.5, the following holds immediately.

Lemma 6.6. *If $\kappa > 1/(n-2)$ and $\xi > 0$ is sufficiently small. Then there exists $\beta_0 = \beta(\kappa, \xi)$ such that a solution $w(\tau; \xi; \beta)$ to (4.1) has a finite zero for any $\beta \in (0, \beta_0)$.*

Moreover, we can prove the existence of a solution to (1.1) with a large initial value. Intuitively, this fact is explained as follows. Take $\beta > 0$ and $\xi_0 > 0$ sufficiently small so that $w(\tau; \xi_0; \beta)$ has a finite zero. When $\xi = 0$, $w(\tau; 0; \beta)$ is a positive slowly decaying solution. Thus, in-between, we can find a suitable $\xi_1 > 0$ such that $w(\tau; \xi_1; \beta)$ is a rapidly-decaying solution.

Lemma 6.7. *For $\kappa > 1/(n-2)$, if $\beta > 0$ is sufficiently small, then there exists sufficiently small $\xi_* = \xi_*(\beta, \kappa) > 0$ ($\xi_*(\beta; \kappa) \to 0$ as $\beta \to 0$) such that $w(\tau; \xi_*; \beta)$ is a rapidly decaying solution.*

We translate this Lemma to the original problem. Note that small $w(1; \xi_1; \beta)$ corresponds to a solution for (1.1) with large initial value. A large solution is somewhat close to the exact solution at $\lambda = 0$ as expressed in (6.8) and this smallness and largeness correspondence follows.

Proof of (b) of Theorem 1.5. By the non-degeneracy of $U(r;\alpha_0)$, the solution obtained by the continuation of $U(r;\alpha_0)$ to $\lambda < 0$ region is not a solution derived by $w(\tau;\xi_*;\beta)$ in Lemma 6.7 by the transformation in Section 2. Thus, we obtain at least two different solutions.

As for the blowup property, we argue as in the proof of Theorem 1.4. As $\xi_* \downarrow 0$, the initial value of the solution $u(r;\xi)$ of (1.1) corresponding to $w(\tau;\xi_*;\beta)$ goes to infinity. Hence the blowup property is proved. □

7. Related topics

Our obtained solution branch is a part of the imperfect bifurcation branches. Rigorous and mathematical analysis is done in Kabeya, Morishita and Ninomiya [8]. See the Figure 5 below. On each connected branch, the number of the critical points of the solution on the branch is constant (solutions have the same mode). Thus in Figure 5, the left branch is that of the solutions having one critical point and the right branch is that of solution without any critical points (monotone decreasing solutions). From the perturbation point of view, how the branches vary from the connected bifurcation diagrams on the homogeneous Neumann problem is stated in [8].

Similar bifurcation diagrams are numerically obtained by Stingelin [18], who is a former graduate student of Professor C. Bandle.

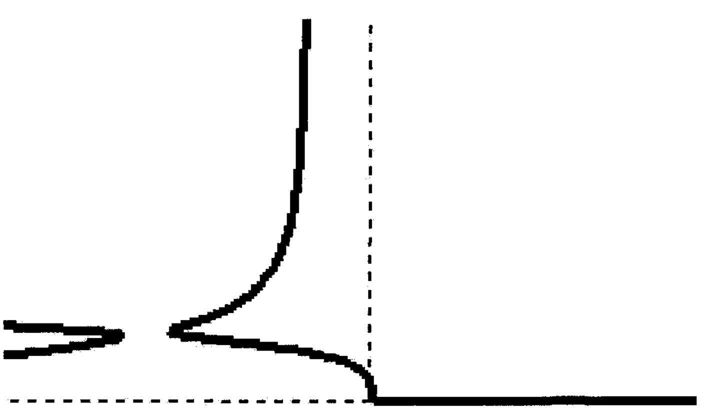

Figure 5. The bifurcation diagram of the problem (1.1) with $n = 3$ and $\kappa = 2000$. The horizontal axis is λ and the vertical axis is $u(0)$.

Acknowledgments

Numerically computed graphs are due to Dr. Hiroshi Morishita of Hyogo University, Japan and the author is grateful for him.

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GINZBURG-LANDAU FUNCTIONAL IN A THIN LOOP AND LOCAL MINIMIZERS

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We consider the Ginzburg-Landau functional in a 3-dimensional loop which has thin cross section. It is formally shown that this functional is approximated by a reduced functional in a 1-dimensional ring. In this article we rigorously prove that if the reduced functional has a nondegenerate local minimizer and if the cross section is sufficiently thin, there exists a local minimizer of the 3-dimensional one.

1. Introduction

In the Ginzburg-Landau theory of superconductivity a macroscopic superconducting state in a superconductor is represented by a complex order parameter Ψ. Taking account of the magnetic effect into a model yields the celebrated Ginzburg-Landau energy functional in the superconductivity [3]. The Ginzburg-Landau equations are the Euler-Lagrange equations for this functional and a macroscopic physical state is realized by a solution of the Ginzburg-Landau equations.

In this paper we are dealing with a mathematical problem concerned with a superconducting phenomenon in a thin superconducting sample with an applied magnetic field. More precisely we consider a 3-dimensional tubular loop of the sample and consider local minimizers of the Ginzburg-Landau functional provided that the loop is very thin. We set up such a domain. Let

$$L := \{p(s) \in \mathbb{R}^3 : s \in \mathbb{R}\}$$

be a 1-dimensional loop, where $p = p(s)$ is an $\mathbb{R}^3$-valued function of class C^∞ with period ℓ (say $C^\infty(\mathbb{R}/\ell\mathbb{Z};\mathbb{R}^3)$) and s is the arc length parameter of L. We denote by $\{[n_1(s), n_2(s), n_3(s)] : s \in \mathbb{R}\}$ a family of orthonormal basis such that

$$n_1(s) = \frac{dp}{ds}(s) \quad (\forall s \in \mathbb{R}), \qquad n_j \in C^\infty(\mathbb{R}/\ell\mathbb{Z}; \mathbb{R}^3) \quad (j = 1, 2, 3)$$

and the linear map $y \to \sum_{j=1}^{3} y_j n_j(s)$ on $\mathbb{R}^3$ is orientation-preserving for each s. Define a vector-valued function P_ε on $\mathbb{R}^3$

$$P_\varepsilon(y) := p(y_1) + \varepsilon\, y_2\, \rho_2(y_1)\, n_2(y_1) + \varepsilon\, y_3\, \rho_3(y_1)\, n_3(y_1), \tag{1}$$

where ε is the positive parameter and $\rho_2 = \rho_2(s)$ and $\rho_3 = \rho_3(s)$ are smooth positive functions with period ℓ. With the aid of $P_\varepsilon(y)$ we define a tubular loop $\Omega(\varepsilon)$ by

$$\Omega(\varepsilon) := \left\{ x = P_\varepsilon(y) : 0 \leq y_1 \leq \ell,\ |y'| = \left| \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \right| < 1 \right\}. \tag{2}$$

We take $\varepsilon_\rho > 0$ small so that for each $\varepsilon \in (0, \varepsilon_\rho)$ the mapping $P_\varepsilon : \mathbb{R}/\ell\mathbb{Z} \times \{|y'| < 1\} \to \Omega(\varepsilon)$ is a bijection.

Now we consider the Ginzburg-Landau energy in the domain $\Omega(\varepsilon)$ with an applied magnetic field

$$\begin{aligned} \mathcal{G}_\varepsilon(\Psi, A) = \frac{1}{2} \int_{\Omega(\varepsilon)} & \left\{ |(\nabla - iA)\Psi|^2 + \frac{\alpha}{2}(1 - |\Psi|^2)^2 \right\} dx \\ & + \frac{\beta}{2} \int_{\mathbb{R}^3} |\mathrm{rot}\, A - H|^2\, dx, \end{aligned} \tag{3}$$

where A is a magnetic vector potential, α and β are positive constants, and H is the applied magnetic field. We let A_H be a vector potential of H, namely it satisfies

$$H = \mathrm{rot}\, A_H, \quad \mathrm{div}\, A_H = 0, \quad A_H \in H^1(\mathbb{R}^3\,;\, \mathbb{R}^3) \cap C^1(\mathbb{R}^3\,;\, \mathbb{R}^3). \tag{4}$$

For the study of the functional (3) with small ε, it is natural to consider a limiting behavior as $\varepsilon \to 0$. Scaling $\mathcal{G}_\varepsilon/\varepsilon^2$, we can formally compute the limiting behavior as $\mathcal{G}_\varepsilon/\varepsilon^2 \to G$,

$$G(\psi) := \frac{1}{2} \int_0^\ell \left\{ \left| \frac{d\psi}{ds} - i a_1(s) \psi \right|^2 + \frac{\alpha}{2}(1 - |\psi|^2)^2 \right\} \pi m(s)\, ds, \tag{5}$$

where $\psi(s)$ is a $\mathbb{C}$-valued function with period ℓ, $a_1(s)$ and $m(s)$ are defined as

$$a_1(s) := \langle n_1(s), A_H(p(s)) \rangle_{\mathbb{R}^3}, \quad m(s) := \rho_2(s)\, \rho_3(s) \tag{6}$$

(refer to [1] and [12]). We here used the Euclidean inner product $\langle \xi, \eta \rangle_{\mathbb{R}^3} = \sum_{j=1}^{3} \xi_j \eta_j$ for $\xi, \eta \in \mathbb{R}^3$. Since the limiting functional (5) is much simpler than (3), mathematical justification of this reduction is an important problem.

In this article we establish that if there exists a *nondegenerate* local minimizer ψ_0 of (5), (3) allows a local minimizer $(\Psi_\varepsilon, A_\varepsilon)$ near $\{(\Psi, A) = (e^{i\gamma}\psi_0, A_H) : \gamma \in \mathbb{R}\}$ with respect to a norm, where *nondegenerate* implies that the normal direction to the continuum of the solutions $\{e^{i\gamma}\psi_0 : \gamma \in \mathbb{R}\}$ is hyperbolic, (see (A2) in Section 2). Thus (5) certainly works as a simplified Ginzburg-Landau model of (3) under a certain situation.

We remark that there are some mathematical results for the reduction of the Ginzburg-Landau energy in a thin domain to the one in a lower dimensional domain. In fact the present paper is closely related to the works [4] and [10]. The readers also refer to [1], [11], [12], and [13]. As for some studies for the reduction energy see [7], [8], and [9].

The rest of this article is organized as follows: In Section 2 we state the main result and present some examples of the domain. In Section 3 we show a key lemma which gives an estimate of the energy. Finally we prove the main theorem in Section 4.

2. Main result

Hereafter we regard the space $\mathbb{C}$ as a vector space $\mathbb{R}^2$ with scalar $\mathbb{R}$. We thus identify a complex valued function $\Psi(x) = \operatorname{Re}\Psi(x) + i \operatorname{Im}\Psi(x)$ with a vector valued one $(\operatorname{Re}\Psi(x), \operatorname{Im}\Psi(x))^{\mathrm{T}}$.

Let us define some function spaces which are used in this article. We denote by $L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z}; \mathbb{C})$ a Hilbert space of $\mathbb{C}$-valued functions with period ℓ in $L^2_{\mathrm{loc}}(\mathbb{R})$ equipped with the inner product

$$(\psi, \phi)_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} := \operatorname{Re} \int_0^\ell \psi(s)\, \phi(s)^* \, \pi m(s)\, ds,$$

where $\phi(s)^*$ stands for the complex conjugate of $\phi(s)$. We also define a Hilbert space $H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z}; \mathbb{C})$ as a subspace of $L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z}; \mathbb{C})$ which consists of all functions whose weak derivative is in $L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z}; \mathbb{C})$. The bilinear form

$$(\psi, \phi)_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} = (\psi, \phi)_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} + \left(\frac{d\psi}{ds}, \frac{d\phi}{ds} \right)_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}$$

is an inner product of $H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z}; \mathbb{C})$. Let $G^{(k)}(\psi, \phi)$ $(k = 1, 2)$ be derivatives of G defined by

$$G^{(k)}(\psi,\phi)=\left(\frac{d^k}{dt^k}G(\psi+t\,\phi)\right)\bigg|_{t=0}\qquad(\psi,\phi\in H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})).\tag{7}$$

In this paper we assume

(A1) ψ_0 belongs to $H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})$ and satisfies

$$\begin{cases}\|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}\neq 0,\\ G^{(1)}(\psi_0,\phi)=0 \qquad\qquad (\forall\phi\in H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z}\,;\mathbb{C})),\end{cases}$$

(A2) there exists $\mu_0>0$ such that

$$G^{(2)}(\psi_0,\phi)\geq\mu_0\,\|\phi\|^2_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z}\,;\mathbb{C})},\quad\forall\phi\in\langle i\psi_0\rangle^{\perp L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}$$

where

$$\langle i\psi_0\rangle^{\perp L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}=\{\phi\in H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C}):(\phi,i\psi_0)_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}=0\}.$$

The invariance of rotation tells that a continuum $\{e^{i\gamma}\psi_0\,:\,\gamma\in\mathbb{R}\}$ is a set of local minimizers of the energy $G(\psi)$. We shall show that a local minimizer of $\mathcal{G}_\varepsilon(\Psi,A)$ exists in a neighborhood of the continuum $\{(\Psi,A)=(e^{i\gamma}\psi_0,A_H)\,:\,\gamma\in\mathbb{R}\}$ if ε is sufficiently small. To deal with the energy $\mathcal{G}_\varepsilon(\Psi,A)$ in the neighborhood, we set up function spaces. Define

$$Y:=\{B=(B_1,B_2,B_3)^{\mathrm{T}}\in L^6(\mathbb{R}^3;\mathbb{R}^3):\nabla B_j\in L^2(\mathbb{R}^3;\mathbb{R}^3)\}.\tag{8}$$

Then Y is a Banach space with the norm

$$\|B\|_Y:=\|\nabla B\|_{L^2(\mathbb{R}^3)}=\left\{\int_{\mathbb{R}^3}\sum_{j=1}^{3}|\nabla B_j|^2dx\right\}^{1/2}$$

by virtue of the Sobolev inequality. Taking the gauge invariance into account, we fix the gauge so that a subspace Z of Y is given by

$$Z:=\{B\in Y\,:\,\operatorname{div}B=0\}.$$

Let

$$\tilde{\Omega}:=\{y\in\mathbb{R}^3:0<y_1<\ell,\quad|y'|<1\}$$

be a stretched domain of $\Omega(\varepsilon)$. Given function $\Psi=\Psi(x)$ $(x\in\Omega(\varepsilon))$, we denote a transformed function $\tilde{\Psi}=\tilde{\Psi}(y)$ by

$$\tilde{\Psi}(y):=\Psi(x),\quad x=P_\varepsilon(y)\quad(y\in\tilde{\Omega})$$

and define inner products as

$$(\tilde{\Phi},\tilde{\Psi})_{L^2_m(\tilde{\Omega};\mathbb{C})} := \mathrm{Re}\int_{\tilde{\Omega}} \tilde{\Phi}(y)\,\tilde{\Psi}(y)^*\,m(y_1)\,dy \qquad (\Phi,\Psi\in L^2(\Omega(\varepsilon);\mathbb{C}))\,,$$

$$(\tilde{\Phi},\tilde{\Psi})_{H^1_m(\tilde{\Omega};\mathbb{C})} := \mathrm{Re}\int_{\tilde{\Omega}} \langle\nabla_y\tilde{\Phi}(y),\nabla_y\tilde{\Psi}(y)\rangle_{\mathbb{C}^3}\,m(y_1)\,dy + (\tilde{\Phi},\tilde{\Psi})_{L^2_m(\tilde{\Omega};\mathbb{C})}$$

$$(\Phi,\Psi\in H^1(\Omega(\varepsilon);\mathbb{C}))$$

where $m(y_1)$ is defined in (6) and $\langle\cdot,\cdot\rangle_{\mathbb{C}^3}$ is the Hermitian inner product $\langle\xi,\eta\rangle_{\mathbb{C}^3} = \xi^{\mathrm{T}}\,\eta^* = \sum_{j=1}^3 \xi_j\,\eta_j^* \in \mathbb{C}$ $(\xi,\eta\in\mathbb{C}^3)$. Put

$$a_j = a_j(s) := \langle n_j(s), A_H(p(s))\rangle_{\mathbb{R}^3}, \quad (j=2,3),$$
$$a' = a'(s) := \begin{pmatrix} a_2(s)\\ a_3(s)\end{pmatrix}, \quad R = R(s) := \begin{pmatrix}\rho_2(s) & 0\\ 0 & \rho_3(s)\end{pmatrix}$$

and

$$e_\varepsilon(y) := \exp\left(i\,\varepsilon\,\langle R(y_1)\,a'(y_1), y'\rangle_{\mathbb{R}^2}\right). \tag{9}$$

We note that

$$\langle P_\varepsilon(y) - p(y_1), A_H(p(y_1))\rangle_{\mathbb{R}^3} = \varepsilon\langle R(y_1)\,a'(y_1), y'\rangle_{\mathbb{R}^2}.$$

Using ψ_0 and (9), we define an approximate solution as

$$\Psi_{0,\varepsilon}(x) = \tilde{\Psi}_{0,\varepsilon}(y) := \psi_0(y_1)\,e_\varepsilon(y), \quad y = P_\varepsilon^{-1}(x) \quad (x\in\Omega(\varepsilon)) \tag{10}$$

and a δ-neighborhood of the continuum $\{e^{i\gamma}\Psi_{0,\varepsilon} \,:\, \gamma\in\mathbb{R}\}$ as

$$\Sigma_\varepsilon(\delta) := \{\Psi\in H^1(\Omega(\varepsilon);\mathbb{C}) \,:\, \inf_{\gamma\in[0,2\pi]} \|\tilde{\Psi} - e^{i\gamma}\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} < \delta\}. \tag{11}$$

Now we state the main result.

Theorem 2.1. *Assume* (A1) *and* (A2). *Then there exist a positive constant* $\delta_0 > 0$ *and a positive function* $\varepsilon_0 = \varepsilon_0(\delta) > 0$ $(\forall\delta\in(0,\delta_0))$ *such that for each* $\delta\in(0,\delta_0)$ *and* $\varepsilon\in(0,\varepsilon_0(\delta))$, *the Ginzburg-Landau energy* $\mathcal{G}_\varepsilon(\Psi,A)$ *has a local minimizer* $(\Psi,A) = (\Psi_\varepsilon,A_\varepsilon)\in\Sigma_\varepsilon(\delta)\times Z$.

We here remark that the Ginzburg-Landau equation corresponding to (3) is written as

$$\begin{cases} (\nabla - iA)^2\,\Psi + \alpha\,(1-|\Psi|^2)\Psi = 0, & x\in\Omega(\varepsilon),\\ \langle(\nabla - iA)\,\Psi,\nu\rangle_{\mathbb{C}^3} = 0, & x\in\partial\Omega(\varepsilon),\\ \beta\,\mathrm{rot}\,(\mathrm{rot}\,A - H) - \mathrm{Im}\,(\Psi^*(\nabla - iA)\Psi)\,\chi_{\Omega(\varepsilon)} = 0, & x\in\mathbb{R}^3, \end{cases}$$

where ν is the outward normal vector on $\partial\Omega(\varepsilon)$ and $\chi_{\Omega(\varepsilon)}$ is the characteristic function. On the other hand the simplified Ginzburg-Landau equation corresponding to (5) is

$$\begin{cases} \dfrac{1}{m(s)^2}\left\{m(s)\left(\dfrac{d}{ds}-ia_1(s)\right)\right\}^2\psi+\alpha(1-|\psi|^2)\psi=0, & s\in\mathbb{R}, \\ \psi(s+\ell)=\psi(s), & s\in\mathbb{R}. \end{cases}$$

As an application of Theorem 2.1 we consider a simple situation where the tubular domain $\Omega(\varepsilon)$ has a uniform thinness, that is, $\rho_2(s)=\rho_3(s)\equiv 1$. Then the simplified functional (5) has an Euler-Lagrange equation

$$\left(\frac{d}{ds}-ia_1(s)\right)^2\psi+\alpha(1-|\psi|^2)\psi=0, \quad \psi(s+\ell)=\psi(s). \tag{12}$$

Put

$$\mu=\frac{1}{\ell}\int_0^\ell a_1(\tau)d\tau.$$

Given $k\in\mathbb{Z}$, the equation (12) has a solution

$$\psi_k(s)=\sqrt{1-(2k\pi/\ell-\mu)^2/\alpha}\exp\left(i(2k\pi/\ell-\mu)s+i\int_0^s a_1(\tau)d\tau\right) \tag{13}$$

if $\alpha>(2k\pi/\ell-\mu)^2$. It is known that this solution satisfies the assumptions of Theorem 2.1 for $\alpha>3(2k\pi/\ell-\mu)^2-2\pi^2/\ell^2$ (see [9] and [15]). Hence the theorem tells that there exists a local minimizer of (3) near the solution (13).

Before concluding this section we will compute $a_1(s)$ for specific loops. We let the applied magnetic field H be constant and perpendicular to the x_1x_2-plane, namely $H=h_a(0,0,1)^{\mathrm{T}}$ for a constant h_a. Then we obtain a vector potential

$$A_H=(h_a/2)(-x_2,x_1,0)^{\mathrm{T}}$$

and we can compute $a_1(s)$ by (6) with $n_1=dp/ds$.

First consider

$$p=p_\varphi(s)=(\cos s,\cos\varphi\sin s,\sin\varphi\sin s)^{\mathrm{T}} \quad (s\in\mathbb{R}), \tag{14}$$

for a fixed $\varphi\in[0,2\pi)$. The loop of (14) is obtained by rotating the unit circle $p_0(s)=(\cos s,\sin s,0)^{\mathrm{T}}$ around the x_1-axis with the angle φ. From

$$\frac{dp_\varphi}{ds}=(-\sin s,\cos\varphi\cos s,\sin\varphi\cos s)^{\mathrm{T}},$$

we easily see $|dp_\varphi/ds| = 1$ and

$$a_1(s) = \langle dp_\varphi/ds, A_H(p(s))\rangle_{\mathbb{R}^3} = (h_a/2)\cos\varphi.$$

This implies that the effect by the applied magnetic field vanishes when $\varphi = \pi/2, 3\pi/2$, and it is maximized if the angle φ takes 0 or π.

Next consider

$$p = p_\delta(s) = (\sin\tau\cos(\cos\tau), \sin\tau\sin(\cos\tau), \delta\cos\tau)^{\mathrm{T}}, \quad \tau = \tau(s), \qquad (15)$$

where δ is a nonzero constant and $\tau(s)$ is taken so that

$$|dp_\delta/ds| = (\cos^2\tau + \sin^4\tau + \delta^2\sin^2\tau)^{1/2}|d\tau/ds| = 1$$

holds. Note that if $\delta = 0$, the curve of (15) is lying in the x_1x_2-plane and it intersects itself at the origin (See Fig. 1).

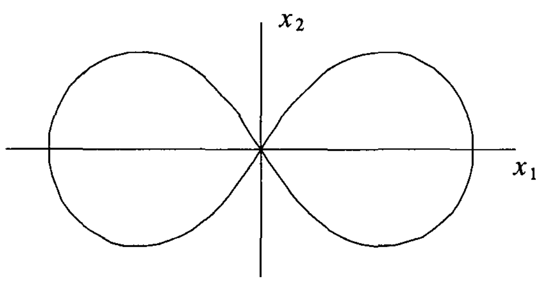

Fig. 1: A curve given by putting $\delta = 0$ in (15)

For $p_\delta(s)$ we easily compute

$$a_1(s) = -\frac{h_a\sin^3\tau}{2}\frac{d\tau}{ds}.$$

In this case it holds that

$$\int_0^\ell a_1(s)ds = -\frac{h_a}{2}\int_0^{2\pi}\sin^3\tau\, d\tau = 0,$$

from which $\mu = 0$ in (13) follows.

3. Lower estimate of the functional

Define

$$E_\varepsilon(\Psi, B) := \frac{1}{\varepsilon^2}\mathcal{G}_\varepsilon(\Psi, A_H + B).$$

In this section we estimate $E_\varepsilon(\Psi, B) - E_\varepsilon(\Psi_{0,\varepsilon}, 0)$ (see Lemma 3.9 below). As shown later, a lower bound of this energy difference plays an important role to prove the theorem.

3.1. *Notation and preliminary*

We here introduce some notation to estimate the energy. For each $\gamma \in \mathbb{R}$ and $\Psi \in H^1(\Omega(\varepsilon);\mathbb{C})$, we put

$$\Psi_{\gamma,\varepsilon} := e^{i\gamma}\Psi_{0,\varepsilon} \quad (\text{i.e. } \tilde{\Psi}_{\gamma,\varepsilon}(y) = e^{i\gamma}e_\varepsilon(y)\,\psi_0(y_1)),$$

and

$$\begin{aligned}
Q_1 &:= \frac{1}{2\varepsilon^2}\int_{\Omega(\varepsilon)} \{|D_{A_H+B}\Psi|^2 - |D_{A_H}\Psi|^2\}\,dx + \frac{\beta}{2\varepsilon^2}\int_{\mathbb{R}^3} |\operatorname{rot} B|^2\,dx,\\
Q_2 &:= \frac{1}{\varepsilon^2}\operatorname{Re}\int_{\Omega(\varepsilon)} \{\langle D_{A_H}\Psi_{\gamma,\varepsilon}, D_{A_H}\Phi\rangle_{\mathbb{C}^3} - \alpha(1-|\Psi_{\gamma,\varepsilon}|^2)\Psi_{\gamma,\varepsilon}\Phi^*\}\,dx,\\
Q_3 &:= \frac{1}{2\varepsilon^2}\int_{\Omega(\varepsilon)} \{|D_{A_H}\Phi|^2 + 2\alpha(\operatorname{Re}(\Psi_{\gamma,\varepsilon}\Phi^*))^2 - \alpha(1-|\Psi_{\gamma,\varepsilon}|^2)|\Phi|^2\}\,dx,\\
Q_4 &:= \frac{1}{2\varepsilon^2}\int_{\Omega(\varepsilon)} \left\{2\alpha\operatorname{Re}(\Psi_{\gamma,\varepsilon}\Phi^*)|\Phi|^2 + \frac{\alpha}{2}|\Phi|^4\right\}dx,
\end{aligned}$$

where $D_A = \nabla_x - iA$ and $\Phi = \Psi - \Psi_{\gamma,\varepsilon}$. Then a direct calculation implies

$$\begin{aligned}
E_\varepsilon(\Psi,B) - E_\varepsilon(\Psi_{0,\varepsilon},0) &= E_\varepsilon(\Psi,B) - E_\varepsilon(\Psi_{\gamma,\varepsilon},0)\\
&= Q_1 + Q_2 + Q_3 + Q_4.
\end{aligned}$$

We shall estimate each Q_j in the next subsection. To carry out it, we need to use the change of variables $x = P_\varepsilon(y)$ frequently. For convenience of computations we introduce the notation below. Let Dx/Dy be the Jacobian matrix

$$\frac{Dx}{Dy} := \left(\frac{\partial}{\partial y_1}P_\varepsilon, \frac{\partial}{\partial y_2}P_\varepsilon, \frac{\partial}{\partial y_3}P_\varepsilon\right). \tag{16}$$

Then it is clear that for $x = P_\varepsilon(y)$,

$$\nabla_y\tilde{\Psi}(y) = \frac{Dx}{Dy}^{\mathrm{T}}\nabla_x\Psi(x), \quad \nabla_x\Psi(x) = \left(\frac{Dx}{Dy}^{-1}\right)^{\mathrm{T}}\nabla_y\tilde{\Psi}(y)$$

if Dx/Dy^{-1} exists. Put

$$\begin{aligned}
\tau_1 = \tau_1(y) &:= y_2\,\rho_2(y_1)\left\langle \frac{dn_2}{ds}(y_1), n_1(y_1)\right\rangle_{\mathbb{R}^3} \\
&\quad + y_3\,\rho_3(y_1)\left\langle \frac{dn_3}{ds}(y_1), n_1(y_1)\right\rangle_{\mathbb{R}^3}, \\
\tau_2 = \tau_2(y) &:= \frac{y_2}{\rho_2(y_1)}\frac{d\rho_2}{ds}(y_1) + \frac{y_3\,\rho_3(y_1)}{\rho_2(y_1)}\left\langle \frac{dn_3}{ds}(y_1), n_2(y_1)\right\rangle_{\mathbb{R}^3}, \\
\tau_3 = \tau_3(y) &:= \frac{y_2\,\rho_2(y_1)}{\rho_3(y_1)}\left\langle \frac{dn_2}{ds}(y_1), n_3(y_1)\right\rangle_{\mathbb{R}^3} + \frac{y_3}{\rho_3(y_1)}\frac{d\rho_3}{ds}(y_1), \\
\tau' &= \begin{pmatrix}\tau_2\\ \tau_3\end{pmatrix}.
\end{aligned}$$

Then a simple calculation implies

$$\frac{Dx}{Dy} = \left((1+\varepsilon\,\tau_1)\,n_1 + \varepsilon\,\rho_2\,\tau_2\,n_2 + \varepsilon\,\rho_3\,\tau_3\,n_3,\ \varepsilon\,\rho_2\,n_2,\ \varepsilon\,\rho_3\,n_3\right), \tag{17}$$

$$\frac{Dy}{Dx} := \frac{Dx}{Dy}^{-1} = \begin{pmatrix} (1+\varepsilon\tau_1)^{-1}n_1^{\mathrm{T}} \\ -\tau_2(1+\varepsilon\tau_1)^{-1}n_1^{\mathrm{T}} + (\varepsilon\rho_2)^{-1}n_2^{\mathrm{T}} \\ -\tau_3(1+\varepsilon\tau_1)^{-1}n_1^{\mathrm{T}} + (\varepsilon\rho_3)^{-1}n_3^{\mathrm{T}}, \end{pmatrix}, \tag{18}$$

$$\frac{Dy}{Dx}\frac{Dy}{Dx}^{\mathrm{T}} = \frac{1}{(1+\varepsilon\tau_1)^2}\begin{pmatrix}1\\ -\tau'\end{pmatrix}\left(1, -(\tau')^{\mathrm{T}}\right) + \frac{1}{\varepsilon^2}\begin{pmatrix}0 & 0\\ 0 & R(y_1)^{-2}\end{pmatrix}, \tag{19}$$

$$\det\frac{Dx}{Dy} = \varepsilon^2\,(1+\varepsilon\tau_1(y))\,m(y_1). \tag{20}$$

3.2. *Estimate of Q_j*

We first estimate Q_1. Put

$$\varepsilon_1 := \min\{\varepsilon_\rho, 1/2\|\tau_1\|_{L^\infty(\tilde\Omega)}\}.$$

Then we have the following lemma:

Lemma 3.1. *If $\varepsilon \in (0,\varepsilon_1]$, there exists a positive constant $C_1 = C_1(A_H,\tilde\Omega,\tau_j,\rho_j)$ such that*

$$Q_1 \geq -\frac{C_1}{\beta}\|\tilde\Psi\|^2_{H^1_m(\tilde\Omega;\mathbb{C})}\left(\varepsilon^{4/3}\|\tilde\Psi\|^2_{H^1_m(\tilde\Omega;\mathbb{C})} + \varepsilon^{-2/3}\|R^{-1}\nabla_{y'}\tilde\Psi\|^2_{L^2_m(\tilde\Omega;\mathbb{C})}\right).$$

Proof. A direct calculation yields

$$
\begin{aligned}
Q_1 = &\frac{1}{2\varepsilon^2}\int_{\Omega(\varepsilon)} |\Psi(x)B|^2 dx + \frac{1}{\varepsilon^2}\int_{\Omega(\varepsilon)} \langle A_H, B\rangle_{\mathbb{R}^3} |\Psi(x)|^2\, dx \\
&- \frac{1}{\varepsilon^2}\int_{\Omega(\varepsilon)} \mathrm{Im}\langle \nabla_x \Psi, \Psi\, B\rangle_{\mathbb{C}^3} dx + \frac{\beta}{2\varepsilon^2}\int_{\mathbb{R}^3} |\nabla B|^2 dx, \qquad (21)
\end{aligned}
$$

where we used $\|\operatorname{rot} B\|_{L^2(\mathbb{R}^3)} = \|\nabla B\|_{L^2(\mathbb{R}^3)}$ for $B \in Z$. By the Hölder inequality we have

$$
\begin{aligned}
&\left|\frac{1}{\varepsilon^2}\int_{\Omega(\varepsilon)} \langle A_H, B\rangle_{\mathbb{R}^3} |\Psi(x)|^2\, dx\right| \\
&\leq \varepsilon^{-2} \|A_H\|_{L^\infty(\Omega(\varepsilon))} \|B\|_{L^6(\mathbb{R}^3)} \|\Psi\|^2_{L^{12/5}(\Omega(\varepsilon);\mathbb{C})}.
\end{aligned}
$$

With the aid of the Sobolev inequality and changing valuables,

$$
\begin{aligned}
\|B\|_{L^6(\mathbb{R}^3)} &\leq (\text{const.})\, \|\nabla B\|_{L^2(\mathbb{R}^3)}, \\
\|\Psi\|_{L^{12/5}(\Omega(\varepsilon);\mathbb{C})} &= \left(\int_{\tilde{\Omega}} |\tilde{\Psi}(y)|^{12/5} \varepsilon^2 \left(1 + \varepsilon\tau_1(y)\right) m(y_1)\, dy\right)^{5/12} \\
&\leq \left(1 + \varepsilon\|\tau_1\|_{L^\infty(\tilde{\Omega})}\right)^{5/12} \varepsilon^{5/6}\, \|\tilde{\Psi}\|_{L^{12/5}_m(\tilde{\Omega};\mathbb{C})} \\
&\leq (\text{const.})\, \varepsilon^{5/6} \|\tilde{\Psi}\|_{H^1_m(\tilde{\Omega};\mathbb{C})}.
\end{aligned}
$$

Thus there exist a positive constant C_{11} such that

$$
\begin{aligned}
&\left|\frac{1}{\varepsilon^2}\int_{\Omega(\varepsilon)} \langle A_H, B\rangle_{\mathbb{R}^3} |\Psi(x)|^2\, dx\right| \\
&\leq C_{11}\, \varepsilon^{-1/3} \|A_H\|_{L^\infty(\Omega(\varepsilon))} \|\tilde{\Psi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})} \|\nabla B\|_{L^2(\mathbb{R}^3)} \\
&\leq \frac{C_{11}^2\, \varepsilon^{4/3}}{\beta} \|A_H\|^2_{L^\infty(\Omega(\varepsilon))} \|\tilde{\Psi}\|^4_{H^1_m(\tilde{\Omega};\mathbb{C})} + \frac{\beta}{4\varepsilon^2} \|\nabla B\|^2_{L^2(\mathbb{R}^3)}. \qquad (22)
\end{aligned}
$$

Similarly, using the Hölder inequality, the Sobolev inequality, and changing valuables, we obtain

$$
\begin{aligned}
&\left|\frac{1}{\varepsilon^2}\int_{\Omega(\varepsilon)} \mathrm{Im}\langle \nabla_x \Psi, \Psi B\rangle_{\mathbb{C}^3} dx\right| \\
&\leq \varepsilon^{-2} \|\nabla\Psi\|_{L^2(\Omega(\varepsilon);\mathbb{C})} \|\Psi\|_{L^3(\Omega(\varepsilon);\mathbb{C})} \|B\|_{L^6(\mathbb{R}^3)} \\
&\leq C_{12}\, \varepsilon^{-4/3} \|\nabla\Psi\|_{L^2(\Omega(\varepsilon);\mathbb{C})} \|\tilde{\Psi}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} \|\nabla B\|_{L^2(\mathbb{R}^3)} \\
&\leq \frac{C_{12}^2}{\beta\, \varepsilon^{2/3}} \|\nabla\Psi\|^2_{L^2(\Omega(\varepsilon);\mathbb{C})} \|\tilde{\Psi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})} + \frac{\beta}{4\varepsilon^2} \|\nabla B\|^2_{L^2(\mathbb{R}^3)}, \qquad (23)
\end{aligned}
$$

where C_{12} is a positive constant. Thus applying (22) and (23) to (21) yields

$$Q_1 \geq -\frac{C_{11}^2 \varepsilon^{4/3}}{\beta} \|A_H\|^2_{L^\infty(\Omega(\varepsilon))} \|\tilde{\Psi}\|^4_{H^1_m(\tilde{\Omega};\mathbb{C})} - \frac{C_{12}^2}{\beta \varepsilon^{2/3}} \|\nabla \Psi\|^2_{L^2(\Omega(\varepsilon);\mathbb{C})} \|\tilde{\Psi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})}. \tag{24}$$

It follows from (19) that

$$\begin{aligned}
|\nabla_x \Psi(x)|^2 &= \left\langle \frac{Dy}{Dx}^{\mathrm{T}} \nabla_y \tilde{\Psi}, \frac{Dy}{Dx}^{\mathrm{T}} \nabla_y \tilde{\Psi} \right\rangle_{\mathbb{C}^3} \\
&= \left\langle \nabla_y \tilde{\Psi}, \frac{Dy}{Dx} \frac{Dy}{Dx}^{\mathrm{T}} \nabla_y \tilde{\Psi} \right\rangle_{\mathbb{C}^3} \\
&= \frac{|\partial_{y_1} \tilde{\Psi} - \langle \nabla_y \tilde{\Psi}, \tau' \rangle_{\mathbb{C}^2}|^2}{(1+\varepsilon\tau_1)^2} + \frac{|R(y_1)^{-1} \nabla_{y'} \tilde{\Psi}|^2}{\varepsilon^2}
\end{aligned}$$

and hence

$$\|\nabla \Psi\|^2_{L^2(\Omega(\varepsilon);\mathbb{C})} = \varepsilon^2 \int_{\tilde{\Omega}} \frac{|\partial_{y_1} \tilde{\Psi} - \langle \nabla_y \tilde{\Psi}, \tau' \rangle_{\mathbb{C}^2}|^2}{1+\varepsilon\tau_1} m(y_1)\, dy + \int_{\tilde{\Omega}} |R^{-1} \nabla_{y'} \tilde{\Psi}|^2 (1+\varepsilon\tau_1)\, m(y_1)\, dy.$$

If $\varepsilon \in (0, \varepsilon_1]$, we obtain

$$\|\nabla \Psi\|^2_{L^2(\Omega(\varepsilon);\mathbb{C})} \leq 2\varepsilon^2 \left(1 + \|\tau'\|^2_{L^\infty(\tilde{\Omega})}\right) \|\nabla_y \tilde{\Psi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} + \frac{3}{2} \|R^{-1} \nabla_{y'} \tilde{\Psi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}.$$

Combining (24) with this inequality leads us to the desired inequality. □

Next we estimate Q_2.

Lemma 3.2. *For any $\gamma \in \mathbb{R}$, if $\varepsilon \in (0, \varepsilon_1]$, there exists a positive constant $C_2 = C_2(\psi_0, \alpha, A_H, \tilde{\Omega}, \tau_j, \rho_j)$ such that*

$$Q_2 \geq -\frac{1}{8\varepsilon} \|R^{-1} \nabla_{y'} \tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} - C_2 \varepsilon \left(1 + \|\tilde{\Phi}\|_{H^1_m(\tilde{\Omega};\mathbb{C})}\right)$$

where $\Phi = \Psi - \Psi_{\gamma,\varepsilon}$.

Proof. Put

$$\begin{cases} \tilde{A}_j(y) := \langle n_j(y_1), A_H(P_\varepsilon(y)) \rangle_{\mathbb{R}^3} \quad (j = 1, 2, 3), \\ \tilde{A}'(y) := \begin{pmatrix} \tilde{A}_2(y) \\ \tilde{A}_3(y) \end{pmatrix}. \end{cases} \tag{25}$$

By changing valuables and (19), we have

$$
\begin{aligned}
&\langle D_{A_H}\Psi_{\gamma,\varepsilon}, D_{A_H}\Phi\rangle_{\mathbb{C}^3} \\
&= \langle \nabla_x\Psi_{\gamma,\varepsilon} - i\Psi_{\gamma,\varepsilon}A_H, \nabla_x\Phi - i\Phi A_H\rangle_{\mathbb{C}^3} \\
&= \left\langle \frac{Dy}{Dx}^{\mathrm{T}}\left(\nabla_y\tilde{\Psi}_{\gamma,\varepsilon} - i\tilde{\Psi}_{\gamma,\varepsilon}\frac{Dx}{Dy}^{\mathrm{T}}\tilde{A}_H\right), \frac{Dy}{Dx}^{\mathrm{T}}\left(\nabla_y\tilde{\Phi} - i\tilde{\Phi}\frac{Dx}{Dy}^{\mathrm{T}}\tilde{A}_H\right)\right\rangle_{\mathbb{C}^3} \\
&= \left\langle \nabla_y\tilde{\Psi}_{\gamma,\varepsilon} - i\tilde{\Psi}_{\gamma,\varepsilon}\frac{Dx}{Dy}^{\mathrm{T}}\tilde{A}_H, \frac{Dy}{Dx}\frac{Dy}{Dx}^{\mathrm{T}}\left(\nabla_y\tilde{\Phi} - i\tilde{\Phi}\frac{Dx}{Dy}^{\mathrm{T}}\tilde{A}_H\right)\right\rangle_{\mathbb{C}^3} \\
&= \left(\frac{\partial_{y_1}\tilde{\Psi}_{\gamma,\varepsilon} - \langle\nabla_{y'}\tilde{\Psi}_{\gamma,\varepsilon},\tau'\rangle_{\mathbb{C}^2}}{1+\varepsilon\tau_1} - i\tilde{A}_1\tilde{\Psi}_{\gamma,\varepsilon}\right)\left(\frac{\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi},\tau'\rangle_{\mathbb{C}^2}}{1+\varepsilon\tau_1} - i\tilde{A}_1\tilde{\Phi}\right)^* \\
&\quad + \frac{1}{\varepsilon^2}\left\langle R^{-1}\nabla_{y'}\tilde{\Psi}_{\gamma,\varepsilon} - i\varepsilon\tilde{\Psi}_{\gamma,\varepsilon}\tilde{A}', R^{-1}\nabla_{y'}\tilde{\Phi} - i\varepsilon\tilde{\Phi}\tilde{A}'\right\rangle_{\mathbb{C}^2}.
\end{aligned}
$$

Let

$$\psi_\gamma(s) := e^{i\gamma}\psi_0(s).$$

Since $\tilde{\Psi}_{\gamma,\varepsilon} = \psi_\gamma(y_1)e_\varepsilon(y)$,

$$\partial_{y_1}\tilde{\Psi}_{\gamma,\varepsilon} = \frac{d\psi_\gamma}{ds}e_\varepsilon(y) + \frac{\partial e_\varepsilon}{\partial y_1}\psi_\gamma,$$

and

$$\nabla_{y'}\tilde{\Psi}_{\gamma,\varepsilon} = i\,\varepsilon\,\tilde{\Psi}_{\gamma,\varepsilon}(y)\,R(y_1)\,a'(y_1), \tag{26}$$

we have

$$
\begin{aligned}
&\frac{\partial_{y_1}\tilde{\Psi}_{\gamma,\varepsilon} - \langle\nabla_{y'}\tilde{\Psi}_{\gamma,\varepsilon},\tau'\rangle_{\mathbb{C}^2}}{1+\varepsilon\tau_1} - i\,\tilde{A}_1\,\tilde{\Psi}_{\gamma,\varepsilon} \\
&= e_\varepsilon\left(\frac{d\psi_\gamma}{ds}(y_1) - ia_1\psi_\gamma(y_1) + W_1(\varepsilon)\right)
\end{aligned}
$$

where

$$
\begin{aligned}
W_1(\varepsilon) = &-\frac{\varepsilon\,\tau_1}{1+\varepsilon\,\tau_1}\frac{d\psi_\gamma}{ds} + \frac{i\,\varepsilon\,\psi_\gamma}{1+\varepsilon\,\tau_1}\left(\frac{\partial}{\partial y_1}\langle R\,a', y'\rangle_{\mathbb{R}^2} - \langle R\,a',\tau'\rangle_{\mathbb{R}^2}\right) \\
&+ i\,(a_1 - \tilde{A}_1)\,\psi_\gamma.
\end{aligned}
$$

By a simple calculation, it holds that

$$
\begin{aligned}
&\left(\frac{\partial_{y_1}\tilde{\Phi} - \langle\nabla_y\tilde{\Phi},\tau'\rangle_{\mathbb{C}^2}}{(1+\varepsilon\,\tau_1)} - i\,\tilde{A}_1\,\tilde{\Phi}\right)(1+\varepsilon\tau_1) \\
&= \frac{1}{e_\varepsilon^*}\left(\frac{\partial}{\partial y_1}(\tilde{\Phi}e_\varepsilon^*) - ia_1(\tilde{\Phi}e_\varepsilon^*)\right) - \langle\nabla_{y'}\tilde{\Phi},\tau'\rangle_{\mathbb{C}^2} + W_2(\varepsilon)\,\tilde{\Phi}
\end{aligned}
$$

where

$$W_2(\varepsilon) = i\varepsilon \left(\frac{a_1 - \tilde{A}_1}{\varepsilon} - \tau_1 \tilde{A}_1 + \frac{\partial}{\partial y_1} \langle R\, a', y' \rangle_{\mathbb{R}^2} \right).$$

By using (26) again, we have

$$\begin{aligned}
&\frac{1}{\varepsilon^2} \left\langle R^{-1} \nabla_{y'} \tilde{\Psi}_{\gamma,\varepsilon} - i\,\varepsilon\, \tilde{\Psi}_{\gamma,\varepsilon} \tilde{A}', R^{-1} \nabla_{y'} \tilde{\Phi} - i\,\varepsilon\, \tilde{\Phi}\, \tilde{A}' \right\rangle_{\mathbb{C}^2} \\
&= \frac{i\, \tilde{\Psi}_{\gamma,\varepsilon}}{\varepsilon} \left\langle a' - \tilde{A}', R^{-1} \nabla_{y'} \tilde{\Phi} \right\rangle_{\mathbb{C}^2} + \tilde{\Psi}_{\gamma,\varepsilon} \left\langle \tilde{A}' - a', \tilde{\Phi}\, \tilde{A}' \right\rangle_{\mathbb{C}^2}.
\end{aligned}$$

Taking $dx = \varepsilon^2 (1 + \varepsilon\, \tau_1)\, m(y_1)\, dy$ into account, we have

$$\begin{aligned}
Q_2 = \operatorname{Re} \int_{\tilde{\Omega}} &\left(\frac{d\psi_\gamma}{ds}(y_1) - i a_1 \psi_\gamma(y_1) + R_1(\varepsilon) \right) \\
&\left(\frac{\partial}{\partial y_1} (\tilde{\Phi} e_\varepsilon^*) - i a_1 (\tilde{\Phi} e_\varepsilon^*) - e_\varepsilon^* \langle \nabla_{y'} \tilde{\Phi}, \tau' \rangle_{\mathbb{C}^2} + W_2(\varepsilon)\, \tilde{\Phi} e_\varepsilon^* \right)^* m\, dy \\
&+ \operatorname{Re} \int_{\tilde{\Omega}} \frac{i\, \tilde{\Psi}_{\gamma,\varepsilon}}{\varepsilon} \left\langle a' - \tilde{A}', R^{-1} \nabla_{y'} \tilde{\Phi} \right\rangle_{\mathbb{C}^2} (1 + \varepsilon \tau_1)\, m\, dy \\
&+ \operatorname{Re} \int_{\tilde{\Omega}} \tilde{\Psi}_{\gamma,\varepsilon} \left\langle \tilde{A}' - a', \tilde{\Phi}\, \tilde{A}' \right\rangle_{\mathbb{C}^2} (1 + \varepsilon \tau_1)\, m\, dy \\
&- \operatorname{Re} \int_{\tilde{\Omega}} \alpha\, (1 - |\psi_0|^2)\, \tilde{\Psi}_{\gamma,\varepsilon}\, \tilde{\Phi}^* \, (1 + \varepsilon\, \tau_1)\, m\, dy \\
= \operatorname{Re} &\frac{1}{\pi} \int_{|y'|<1} G^{(1)}(\psi_\gamma, (e_\varepsilon^*\, \tilde{\Phi})(\cdot, y'))\, dy' \\
&- \operatorname{Re} \int_{\tilde{\Omega}} e_\varepsilon \left\langle \left(\frac{d\psi_\gamma}{ds} - i\, a_1\, \psi_\gamma \right) R\, \tau' - \frac{i\, \psi_\gamma}{\varepsilon} (a' - \tilde{A}'), R^{-1} \nabla_{y'} \tilde{\Phi} \right\rangle_{\mathbb{C}^2} m\, dy \\
&+ \operatorname{Re} \int_{\tilde{\Omega}} W_3(\varepsilon)\, m(y_1)\, dy
\end{aligned}$$

where $W_3(\varepsilon)$ is a term of $\mathcal{O}(\varepsilon)$ given by

$$\begin{aligned}
W_3(\varepsilon) = W_1(\varepsilon) &\left(\frac{\partial}{\partial y_1} (\tilde{\Phi} e_\varepsilon^*) - i a_1 (\tilde{\Phi} e_\varepsilon^*) - e_\varepsilon^* \, \langle \nabla_{y'} \tilde{\Phi}, \tau' \rangle_{\mathbb{C}^2} + W_2(\varepsilon)\, \tilde{\Phi}\, e_\varepsilon^* \right)^* \\
&+ W_2(\varepsilon)^* \left(\frac{d\psi_\gamma}{ds} - i a_1 \psi_\gamma \right) e_\varepsilon\, \tilde{\Phi}^* + i\, \tau_1\, \psi_\gamma\, e_\varepsilon \left\langle a' - \tilde{A}', R^{-1} \nabla_{y'} \tilde{\Phi} \right\rangle_{\mathbb{C}^2} \\
&+ (1 + \varepsilon \tau_1)\, \psi_\gamma\, e_\varepsilon \left\langle \tilde{A}' - a', \tilde{\Phi}\, \tilde{A}' \right\rangle_{\mathbb{C}^2} - \varepsilon\, \tau_1\, \alpha\, (1 - |\psi_0|^2)\, \psi_\gamma\, e_\varepsilon\, \tilde{\Phi}^*.
\end{aligned}$$

Thus there exist constants $C_{21} \geq 0$ and $C_{22} \geq 0$ such that

$$Q_2 \geq -C_{21} \| R^{-1} \nabla_{y'} \tilde{\Phi} \|_{L^2_m(\tilde{\Omega}; \mathbb{C})} - C_{22}\, \varepsilon\, \| \tilde{\Phi} \|_{H^1_m(\tilde{\Omega}; \mathbb{C})}.$$

From

$$C_{21}\|R^{-1}\nabla_{y'}\tilde{\Phi}\|_{L^2_m(\tilde{\Omega};\mathbb{C})} \le \frac{1}{8\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} + 2\,\varepsilon\, C_{21}^2,$$

the lemma follows. □

Next we estimate Q_3. Let

$$\rho' = \rho'(s) = \begin{pmatrix}\rho_2(s)\\ \rho_3(s)\end{pmatrix}, \quad \varepsilon_2 := \min\{\varepsilon_1, 1/3, 1/2\|\rho'\|^2_{L^\infty(0,\ell)}\}.$$

For $\psi_\gamma = e^{i\gamma}\psi_0$ and $\Phi \in H^1(\Omega(\varepsilon);\mathbb{C})$, we define

$$\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) := \int_{\tilde{\Omega}} \Big\{ |\,(\partial/\partial y_1 - i a_1)\,\tilde{\Phi}|^2 + |\nabla_{y'}\tilde{\Phi}|^2 + 2\alpha(\mathrm{Re}(\psi_\gamma\tilde{\Phi}^*))^2 \\ -\alpha(1-|\psi_\gamma|^2)|\tilde{\Phi}|^2\Big\}\, m(y_1)\, dy. \tag{27}$$

Then we have the following estimate.

Lemma 3.3. *For any $\gamma \in \mathbb{R}$, if $\varepsilon \in (0, \varepsilon_2]$, there exists a positive constant $C_3 = C_3(\psi_0, \alpha, A_H, \tilde{\Omega}, \tau_j, \rho_j)$ such that*

$$Q_3 \ge \frac{1}{2}\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) + \frac{1}{8\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} + \frac{1}{16\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Psi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} \\ - C_3\,\varepsilon\,\left(\|\tilde{\Phi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})} + 1\right)$$

where $\Phi = \Psi - \Psi_{\gamma,\varepsilon}$.

Proof. Changing valuables and (19), we have

$$|D_{A_H}\Phi|^2 = \frac{1}{(1+\varepsilon\,\tau_1)^2}\left|\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} - i\,(1+\varepsilon\,\tau_1)\,\tilde{A}_1\,\tilde{\Phi}\right|^2 \\ + \left|\frac{1}{\varepsilon}R^{-1}\nabla_{y'}\tilde{\Phi} - i\,\tilde{\Phi}\,\tilde{A}'\right|^2$$

where $\tilde{A}_1$ and $\tilde{A}'$ are defined in (25). In terms of the inequality

$$|\xi-\eta|^2 \ge (1-C)|\xi|^2 + (1-1/C)|\eta|^2 \quad (C>0), \tag{28}$$

we estimate

$$\left|\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} - i\,(1+\varepsilon\,\tau_1)\,\tilde{A}_1\,\tilde{\Phi}\right|^2 \\ = \left|\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} - i\,a_1\tilde{\Phi} - i\,(\tilde{A}_1 - a_1 + \varepsilon\,\tau_1\,\tilde{A}_1)\tilde{\Phi}\right|^2 \\ \ge \frac{1}{1+\varepsilon}\left|\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} - i\,a_1\tilde{\Phi}\right|^2 - \varepsilon\left|\left(\frac{\tilde{A}_1 - a_1}{\varepsilon} + \tau_1\,\tilde{A}_1\right)\tilde{\Phi}\right|^2,$$

where $C = \varepsilon/(1+\varepsilon)$ in (28). With $C = 1 - 3\varepsilon$ in (28) we obtain

$$\left|\frac{1}{\varepsilon}R^{-1}\nabla_{y'}\tilde{\Phi} - i\tilde{\Phi}\tilde{A}'\right|^2 \geq \frac{3}{\varepsilon}|R^{-1}\nabla_{y'}\tilde{\Phi}|^2 - \frac{\varepsilon}{1/3-\varepsilon}|\tilde{\Phi}\tilde{A}'|^2.$$

By $\langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} = \langle R^{-1}\nabla_{y'}\tilde{\Phi}, R\tau'\rangle_{\mathbb{C}^2}$ and $C = 2|R\tau'|^2\varepsilon/(1+2|R\tau'|^2\varepsilon)$ in (28), it also holds that

$$\left|\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} - i\,a_1\,\tilde{\Phi}\right|^2 \geq \frac{1}{1+2|R\,\tau'|^2\varepsilon}\left|\frac{\partial\tilde{\Phi}}{\partial y_1} - ia_1\tilde{\Phi}\right|^2 - \frac{|R^{-1}\nabla_{y'}\tilde{\Phi}|^2}{2\varepsilon}.$$

Since $dx = (1+\varepsilon\tau_1)\,\varepsilon^2\,m(y_1)\,dy$ and $(1+\varepsilon\tau_1) \geq 1/2$ for $\varepsilon \in (0, \varepsilon_2]$,

$$\begin{aligned}
Q_3 &= \frac{1}{2}\int_{\tilde{\Omega}} \frac{1}{1+\varepsilon\tau_1}\left|\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} - i(1+\varepsilon\tau_1)\tilde{A}_1\tilde{\Phi}\right|^2 m(y_1)\,dy \\
&\quad + \frac{1}{2}\int_{\tilde{\Omega}}(1+\varepsilon\tau_1)\left|\frac{1}{\varepsilon}R^{-1}\nabla_{y'}\tilde{\Phi} - i\tilde{\Phi}\tilde{A}'\right|^2 m(y_1)\,dy \\
&\quad + \frac{1}{2}\int_{\tilde{\Omega}}(1+\varepsilon\tau_1)\left\{2\alpha(\mathrm{Re}(\tilde{\Psi}_{\gamma,\varepsilon}\tilde{\Phi}^*))^2 - \alpha(1-|\psi_\gamma|^2)|\tilde{\Phi}|^2\right\} m(y_1)\,dy \\
&\geq \frac{\mathcal{K}_{\psi_\gamma}(\tilde{\Phi})}{2} + \frac{1}{2}\int_{\tilde{\Omega}}\left(\frac{|R^{-1}\nabla_{y'}\tilde{\Phi}|^2}{\varepsilon} - |\nabla_{y'}\tilde{\Phi}|^2\right) m(y_1)\,dy \\
&\quad + \frac{1}{2}\int_{\tilde{\Omega}} W_4(\varepsilon)\,m(y_1)\,dy
\end{aligned}$$

where we put $W_4(\varepsilon)$ as

$$\begin{aligned}
W_4(\varepsilon) &= -\frac{\varepsilon\tau_1}{1+\varepsilon\tau_1}\left|\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} - i(1+\varepsilon\tau_1)\tilde{A}_1\tilde{\Phi}\right|^2 \\
&\quad - \frac{\varepsilon}{1+\varepsilon}\left|\partial_{y_1}\tilde{\Phi} - \langle\nabla_{y'}\tilde{\Phi}, \tau'\rangle_{\mathbb{C}^2} - i\,a_1\,\tilde{\Phi}\right|^2 - \varepsilon\left|\left(\frac{\tilde{A}_1 - a_1}{\varepsilon} + \tau_1\,\tilde{A}_1\right)\tilde{\Phi}\right|^2 \\
&\quad - \frac{2|R\,\tau'|^2\varepsilon}{1+2|R\,\tau'|^2\varepsilon}\left|\frac{\partial\tilde{\Phi}}{\partial y_1} - i\,a_1\,\tilde{\Phi}\right|^2 - \frac{3\varepsilon}{2-6\varepsilon}|\tilde{\Phi}\tilde{A}'|^2 \\
&\quad + 2\alpha\left\{(1+\varepsilon\tau_1)(\mathrm{Re}(e_\varepsilon\psi_\gamma\tilde{\Phi}^*))^2 - (\mathrm{Re}(\psi_\gamma\tilde{\Phi}^*))^2\right\} \\
&\quad - \varepsilon\,\tau_1\,\alpha\,(1-|\psi_\gamma|^2)\,|\tilde{\Phi}|^2.
\end{aligned}$$

We easily see $W_4(\varepsilon)$ is $\mathcal{O}(\varepsilon)$. From the inequalities $|\nabla_{y'}\tilde{\Phi}| \leq |R^{-1}\nabla_{y'}\tilde{\Phi}||\rho'|$ and $\|\rho'\|^2_{L^\infty(\mathbb{R}/\ell\mathbb{Z})} \leq 1/2\varepsilon$ for $\varepsilon \in (0, \varepsilon_2]$, it follows that

$$\frac{1}{2}\int_{\tilde{\Omega}}\left(\frac{|R^{-1}\nabla_{y'}\tilde{\Phi}|^2}{\varepsilon} - |\nabla_{y'}\tilde{\Phi}|^2\right) m(y_1)\,dy \geq \frac{1}{4\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}.$$

Since $\Phi = \Psi - \Psi_{\gamma,\varepsilon}$ and (26), we have

$$\begin{aligned}|R^{-1}\nabla_{y'}\tilde{\Phi}|^2 &= |R^{-1}\nabla_{y'}\tilde{\Psi} - R^{-1}\nabla_{y'}\tilde{\Psi}_{\gamma,\varepsilon}|^2\\ &= |R^{-1}\nabla_{y'}\tilde{\Psi} - i\,\varepsilon\, e^{i\gamma}\, e_\varepsilon \psi_0\, a'|^2\\ &\geq \frac{1}{2}|R^{-1}\nabla_{y'}\tilde{\Psi}|^2 - \varepsilon^2|\psi_0\, a'|^2\end{aligned}$$

and then

$$\begin{aligned}\frac{1}{4\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega})} \geq \frac{1}{8\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega})} + \frac{1}{16\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Psi}\|^2_{L^2_m(\tilde{\Omega})}\\ - \frac{\varepsilon}{8}\|\psi_0\, a'\|^2_{L^2_m(\tilde{\Omega})}.\end{aligned}$$

We thereby obtain the desired estimate. □

We finally give a lemma for an estimate of Q_4.

Lemma 3.4. *For any $\gamma \in \mathbb{R}$, if $\varepsilon \in (0, \varepsilon_\rho]$, there exists a positive constant $C_4 = C_4(\psi_0, \alpha, \tilde{\Omega}, \tau_1)$ such that $Q_4 \geq -C_4\|\tilde{\Phi}\|^3_{H^1_m(\tilde{\Omega})}$, where $\Phi = \Psi - \Psi_{\gamma,\varepsilon}$.*

Since the assertion of the lemma immediately follows from the Sobolev inequality, we omit the proof.

3.3. *Estimate of $\mathcal{K}_{\psi_\gamma}(\tilde{\Phi})$*

In this section we estimate $\mathcal{K}_{\psi_\gamma}(\tilde{\Phi})$, which is defined in (27).

Lemma 3.5. *Assume* (A1) *and* (A2). *Then there exists a constant $\mu_1 > 0$, which is independent of $\gamma \in \mathbb{R}$, such that*

$$\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) \geq \mu_1\|\tilde{\Phi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})}, \quad \forall \Phi \in \langle i\psi_\gamma\rangle^{\perp L^2_m(\tilde{\Omega};\mathbb{C})}$$

where $\langle i\psi_\gamma\rangle^{\perp L^2_m(\tilde{\Omega};\mathbb{C})} = \{\Phi \in H^1(\Omega(\varepsilon);\mathbb{C}) : (\tilde{\Phi}, i\psi_\gamma)_{L^2_m(\tilde{\Omega};\mathbb{C})} = 0\}$.

Proof. Without loss of generality, we may assume

$$\mu_0 = \inf\left\{G^{(2)}(\psi_\gamma, \phi)/\|\phi\|^2_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} : \phi \in \langle i\psi_\gamma\rangle^{\perp L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}\right\}.$$

We here remark that μ_0 is independent of γ. Let

$$\mu_0' := \inf\left\{\mathcal{K}_{\psi_\gamma}(\tilde{\Phi})/\|\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} : \Phi \in \langle i\psi_\gamma\rangle^{\perp L^2_m(\tilde{\Omega};\mathbb{C})}\right\},$$

which is also independent of γ, and

$$\lambda_2 := \inf\left\{\|\nabla_{y'}u\|^2_{L^2(\Omega')}/\|u\|^2_{L^2(\Omega')} : u \in H^1(\Omega';\mathbb{R}), \int_{\Omega'} u(y')\,dy' = 0\right\}$$

where $\Omega' = \{|y'| < 1\} \subset \mathbb{R}^2$. This λ_2 is the second eigenvalue of the operator $-\Delta_{y'}$ with the Neumann boundary condition. Take u_2 as an eigenfunction with respect to λ_2, that is, $\|u_2\|_{L^2(\Omega')} \neq 0$ and

$$-\Delta_{y'} u_2 = \lambda_2 u_2 \quad \text{in } \Omega', \qquad \frac{\partial u_2}{\partial \nu} = 0 \quad \text{on } \partial\Omega', \qquad \int_{\Omega'} u_2(y')\, dy' = 0.$$

By taking $\tilde{\Phi}(y) = i\,\psi_\gamma(y_1)\,u_2(y')$, we have $\Phi \in \langle i\psi_\gamma\rangle^{\perp L^2_m(\tilde{\Omega};\mathbb{C})}$ and

$$\mathcal{K}_{\psi_\gamma}(\tilde{\Phi})/\|\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} = G^{(2)}(\psi_\gamma, i\psi_\gamma)/\|\psi_\gamma\|^2_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} + \lambda_2.$$

It thus follows from $G^{(2)}(\psi_\gamma, i\psi_\gamma) = 0$ that $\mu_0' \leq \lambda_2$. On the other hand, if $\phi \in \langle i\psi_\gamma\rangle^{\perp L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}$,

$$\mathcal{K}_{\psi_\gamma}(\phi)/\|\phi\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} = G^{(2)}(\psi_\gamma, \phi)/\|\phi\|^2_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}.$$

Thus $\mu_0' \leq \mu_0$ holds and hence $\mu_0' \leq \min\{\mu_0, \lambda_2\}$.

Let $\Phi_{\mu_0'} \in \langle i\psi_\gamma\rangle^{\perp L^2_m(\tilde{\Omega};\mathbb{C})}$ attain the minimum, that is,

$$\mu_0' = \mathcal{K}_{\psi_\gamma}(\tilde{\Phi}_{\mu_0'})/\|\tilde{\Phi}_{\mu_0'}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}, \quad (\tilde{\Phi}_{\mu_0'}, i\psi_\gamma)_{L^2_m(\tilde{\Omega};\mathbb{C})} = 0.$$

Then $\Phi_{\mu_0'}$ satisfies

$$\left(\frac{d}{dt}\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}_{\mu_0'} + t\tilde{\Phi})\right)\Bigg|_{t=0} = 2\,\mu_0'\,(\tilde{\Phi}_{\mu_0'}, \tilde{\Phi})_{L^2_m(\tilde{\Omega};\mathbb{C})} \tag{29}$$

for all $\Phi \in \langle i\psi_\gamma\rangle^{\perp L^2_m(\tilde{\Omega};\mathbb{C})}$. Let

$$\phi_{\mu_0'}(s) := \frac{1}{\pi}\int_{\Omega'} \tilde{\Phi}_{\mu_0'}(s, y')\, dy'.$$

Then $\phi_{\mu_0'} \in \langle i\psi_\gamma\rangle^{\perp L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}$ and

$$2\,G^{(2)}(\psi_\gamma, \phi_{\mu_0'}) = 2\,\mu_0'\,\|\phi_{\mu_0'}\|^2_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}$$

by taking $\tilde{\Phi}(y) = \phi_{\mu_0'}(y_1)$ in (29). Thus $\mu_0' \geq \mu_0$ if $\|\phi_{\mu_0'}\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} \neq 0$.

Consider the case $\|\phi_{\mu_0'}\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} = 0$, that is,

$$\phi_{\mu_0'}(y_1) = \frac{1}{\pi}\int_{\Omega'} \tilde{\Phi}_{\mu_0'}(y_1, y')\, dy' \equiv 0 \quad (\forall y_1 \in \mathbb{R}).$$

By the definition of λ_2,

$$\int_{\Omega'} |\nabla_{y'}\tilde{\Phi}_{\mu_0'}(y_1, y')|^2\, dy' \geq \lambda_2 \int_{\Omega'} |\tilde{\Phi}_{\mu_0'}(y_1, y')|^2\, dy' \quad (\forall y_1 \in \mathbb{R}).$$

It thus follows from $G^{(2)}(\psi_\gamma, \phi) \geq 0$ for $\forall \phi \in H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z}; \mathbb{C})$ that

$$\begin{aligned}\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}_{\mu_0'}) &= \frac{1}{\pi}\int_{\Omega'} G^{(2)}(\psi_\gamma, \tilde{\Phi}_{\mu_0'}(\cdot, y'))\, dy' + \|\nabla_{y'}\tilde{\Phi}_{\mu_0'}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} \\ &\geq \lambda_2 \|\tilde{\Phi}_{\mu_0'}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}.\end{aligned}$$

Consequently we obtain $\mu_0' \geq \min\{\mu_0, \lambda_2\}$ and hence

$$\mu_0' = \min\{\mu_0, \lambda_2\} > 0.$$

By the definition of μ_0', we have the inequality

$$\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) \geq \mu_0' \|\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}, \quad \forall \Phi \in \langle i\psi_\gamma \rangle^{\perp L^2_m(\tilde{\Omega};\mathbb{C})},$$

while by the definition of $\mathcal{K}_{\psi_\gamma}(\tilde{\Phi})$, there exists a constant $M = M(\alpha, \psi_0)$ such that

$$\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) \geq \int_{\tilde{\Omega}} \left\{ |\,(\partial/\partial y_1 - ia_1)\,\tilde{\Phi}|^2 + |\nabla_{y'}\tilde{\Phi}|^2 \right\} m(y_1)\, dy - M\|\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}.$$

Since

$$|\,(\partial/\partial y_1 - ia_1)\,\tilde{\Phi}|^2 \geq (1 - K_1)|\partial\tilde{\Phi}/\partial y_1|^2 + (1 - 1/K_1)|a_1\tilde{\Phi}|^2, \quad (K_1 > 0),$$

we have

$$\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) \geq (1 - K_1)\|\nabla\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} - \left((1/K_1 - 1)\|a_1\|^2_{L^\infty(\tilde{\Omega})} + M\right) \|\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})},$$

if $K_1 \in (0,1)$. For $K_2 \in (0,1)$ it is clear that

$$\begin{aligned}\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) &= (1 - K_2)\,\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) + K_2\,\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) \\ &\geq (1 - K_2)\,\mu_0'\,\|\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} + K_2\,(1 - K_1)\,\|\nabla\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} \\ &\quad - K_2\left((1/K_1 - 1)\|a_1\|^2_{L^\infty(\tilde{\Omega})} + M\right) \|\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} \\ &= \mu_{11}\,\|\nabla\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} + \mu_{12}\,\|\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}\end{aligned}$$

holds where

$$\begin{aligned}\mu_{11} &= K_2\,(1 - K_1), \\ \mu_{12} &= (1 - K_2)\mu_0' - K_2\left((1/K_1 - 1)\|a_1\|^2_{L^\infty(\tilde{\Omega})} + M\right).\end{aligned}$$

Compute (K_1, K_2) which maximizes μ_{11} and μ_{12} under the condition $\mu_{11} = \mu_{12}$. Then

$$\begin{aligned}K_1 &:= \frac{\|a_1\|_{L^\infty(\tilde{\Omega})}}{\|a_1\|_{L^\infty(\tilde{\Omega})} + \sqrt{\mu_0' + M}}, \\ K_2 &:= \frac{(\|a_1\|_{L^\infty(\tilde{\Omega})} + \sqrt{\mu_0' + M})\,\mu_0'}{\{1 + (\|a_1\|_{L^\infty(\tilde{\Omega})} + \sqrt{\mu_0' + M})^2\}\sqrt{\mu_0' + M}}.\end{aligned}$$

Letting $\mu_1 := \mu_{11} = \mu_{12}$, we obtain the lemma. $\square$

3.4. *Adjustment of γ in $\Psi - e^{i\gamma}\Psi_{0,\varepsilon}$*

In the first part of this subsection we define $\theta_\varepsilon(\Psi)$ for each Ψ so that $\Psi - e^{i\gamma}\Psi_{0,\varepsilon}$ and $ie^{i\gamma}\psi_0$ are orthogonal at $\gamma = \theta_\varepsilon(\Psi)$.

Definition 3.1. For each $\Psi \in L^2(\Omega(\varepsilon);\mathbb{C})$, let $\theta_\varepsilon(\Psi)$ be a solution $\gamma = \theta_\varepsilon(\Psi) \in \mathbb{R}/2\pi\mathbb{Z}$ to the equation

$$\begin{cases} (\tilde{\Psi} - e^{i\gamma}\tilde{\Psi}_{0,\varepsilon}, ie^{i\gamma}\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} = 0, \\ \left|\operatorname{Arg}\left(e^{-i\gamma}\int_{\tilde{\Omega}} \tilde{\Psi}(y)\,\psi_0(y_1)^*\,m(y_1)\,dy\right)\right| < \dfrac{\pi}{2}, \end{cases}$$

where $\tilde{\Psi}_{0,\varepsilon} = e_\varepsilon(y)\psi_0(y_1)$.

It is not clear whether $\theta_\varepsilon(\Psi)$ exists or not. The following lemma presents an existence condition of $\theta_\varepsilon(\Psi)$.

Lemma 3.6. *For each $\Psi \in L^2(\Omega(\varepsilon);\mathbb{C})$ such that*

$$\int_{\tilde{\Omega}} \tilde{\Psi}(y)\,\psi_0(y_1)^*\,m(y_1)\,dy \neq 0,$$

the solution $\theta_\varepsilon(\Psi)$ exists if $\varepsilon > 0$ is sufficiently small. Moreover $\exp(i\theta_\varepsilon(\Psi))$ is unique and the function $\exp(i\theta_\varepsilon(\cdot)) : L^2(\Omega(\varepsilon);\mathbb{C}) \to \mathbb{C}$ is continuous.

Proof. It is clear that

$$(\tilde{\Psi} - e^{i\gamma}\tilde{\Psi}_{0,\varepsilon}, ie^{i\gamma}\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} = (\tilde{\Psi}, ie^{i\gamma}\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} - (\tilde{\Psi}_{0,\varepsilon}, i\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})}.$$

From (9) it follows

$$(\tilde{\Psi}_{0,\varepsilon}, i\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} = \int_{\tilde{\Omega}} \sin(\varepsilon\langle Ra', y'\rangle_{\mathbb{R}^2})\,|\psi_0(y_1)|^2\,m(y_1)\,dy.$$

On the other hand, a simple calculation implies

$$(\tilde{\Psi}, ie^{i\gamma}\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} = \operatorname{Im}\left(e^{-i\gamma}\xi\right) = |\xi|\sin\left(\operatorname{Arg}(e^{-i\gamma}\xi)\right)$$

where

$$\xi = \int_{\tilde{\Omega}} \tilde{\Psi}(y)\,\psi_0(y_1)^*\,m(y_1)\,dy.$$

Thus the equation $(\tilde{\Psi} - e^{i\gamma}\tilde{\Psi}_{0,\varepsilon}, ie^{i\gamma}\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} = 0$ is equivalent to

$$|\xi|\sin\left(\operatorname{Arg}(e^{-i\gamma}\xi)\right) = \int_{\tilde{\Omega}} \sin(\varepsilon\langle Ra', y'\rangle_{\mathbb{R}^2})\,|\psi_0(y_1)|^2 m(y_1)\,dy.$$

Therefore this equation has a unique solution satisfying $|\operatorname{Arg}(e^{-i\gamma}\xi)| < \pi/2$ if $\xi \neq 0$ and ε is small, and hence the lemma was proved. $\square$

Next we prove that $\theta_\varepsilon(\Psi)$ exists for all $\Psi \in \Sigma_\varepsilon(\delta)$ if $\varepsilon > 0$ and $\delta > 0$ are sufficiently small. Put

$$\varepsilon_3 := \sup\left\{\bar{\varepsilon} \in (0, \varepsilon_\rho) \;:\; \|1 - e_\varepsilon\|_{L^\infty(\tilde{\Omega};\mathbb{C})} < 1/2, \quad \forall\varepsilon \in (0, \bar{\varepsilon})\right\},$$
$$\delta_1 := \|\psi_0\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}/2.$$

Then we have the following:

Lemma 3.7. *If $\varepsilon \in (0, \varepsilon_3]$ and $\delta \in (0, \delta_1]$, the solution $\theta_\varepsilon(\Psi)$ exists for all $\Psi \in \Sigma_\varepsilon(\delta)$.*

Proof. For each $\Psi \in \Sigma_\varepsilon(\delta)$, there exist $c \in [0, 2\pi)$ and $\Psi_\delta \in H^1(\Omega(\varepsilon);\mathbb{C})$ such that

$$\Psi = e^{ic}\Psi_{0,\varepsilon} + \Psi_\delta, \quad \|\tilde{\Psi}_\delta\|_{H^1_m(\tilde{\Omega};\mathbb{C})} < \delta.$$

It thus holds that

$$\begin{aligned}
&\left|\int_{\tilde{\Omega}} \tilde{\Psi}(y)\, \psi_0(y_1)^*\, m(y_1)\, dy\right| \\
&= \left|e^{ic}\int_{\tilde{\Omega}} |\psi_0|^2\, m\, dy + e^{ic}\int_{\tilde{\Omega}} (e_\varepsilon - 1)\, |\psi_0|^2\, m\, dy + \int_{\tilde{\Omega}} \tilde{\Psi}_\delta\, \psi_0^*\, m\, dy\right| \\
&\geq (1 - \|1 - e_\varepsilon\|_{L^\infty(\tilde{\Omega};\mathbb{C})})\|\psi_0\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} - \|\tilde{\Psi}_\delta\|_{L^2_m(\tilde{\Omega};\mathbb{C})}\, \|\psi_0\|_{L^2_m(\tilde{\Omega};\mathbb{C})} \\
&> \left(\|\psi_0\|_{L^2_m(\tilde{\Omega};\mathbb{C})}/2 - \delta\right) \|\psi_0\|_{L^2_m(\tilde{\Omega};\mathbb{C})} \geq 0.
\end{aligned}$$

Therefore the lemma follows from Lemma 3.6. $\square$

Next we define a modified δ-neighborhood of the continuum $\{e^{i\gamma}\Psi_{0,\varepsilon} : \gamma \in \mathbb{R}\}$, which is used to prove the theorem.

Definition 3.2. For each $\varepsilon \in (0, \varepsilon_3]$ and $\delta \in (0, \delta_1]$, define a subset $F_\varepsilon(\delta) \subset H^1(\Omega(\varepsilon);\mathbb{C})$ as

$$F_\varepsilon(\delta) := \{\Psi \in H^1(\Omega(\varepsilon);\mathbb{C}) : \|\tilde{\Psi} - e^{i\theta_\varepsilon(\Psi)}\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} < \delta\}.$$

This neighborhood is close to $\Sigma_\varepsilon(\delta)$ in the following sense.

Lemma 3.8. *There exist constants $\varepsilon_4 \in (0, \varepsilon_3)$, $\delta_2 \in (0, \delta_1)$, and $\sigma > 1$ such that*

$$\Sigma_\varepsilon(\delta) \subset F_\varepsilon(\sigma\delta) \quad (\forall\varepsilon \in (0, \varepsilon_4], \quad \forall\delta \in (0, \delta_2]), \tag{30}$$
$$F_\varepsilon(\delta) \subset \Sigma_\varepsilon(\delta) \quad (\forall\varepsilon \in (0, \varepsilon_3], \quad \forall\delta \in (0, \delta_1]). \tag{31}$$

Proof. It is clear that (31) follows from the definitions of $F_\varepsilon(\delta)$ and $\Sigma_\varepsilon(\delta)$. We thus prove (30). Let $\Psi \in \Sigma_\varepsilon(\delta)$. Then there exist $c \in [0, 2\pi)$ and $\Psi_\delta \in H^1(\Omega(\varepsilon); \mathbb{C})$ such that

$$\Psi = e^{ic}\Psi_{0,\varepsilon} + \Psi_\delta, \quad \|\tilde{\Psi}_\delta\|_{H^1_m(\tilde{\Omega};\mathbb{C})} < \delta.$$

For $\theta = \theta_\varepsilon(\Psi)$,

$$\begin{aligned} \|\tilde{\Psi} - e^{i\theta}\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} &= \|\tilde{\Psi}_\delta + (e^{ic} - e^{i\theta})\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} \\ &\le \|\tilde{\Psi}_\delta\|_{H^1_m(\tilde{\Omega};\mathbb{C})} + |e^{ic} - e^{i\theta}|\, \|\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} \\ &< \delta + \sqrt{2(1 - \cos(c - \theta))}\, \|\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})}. \end{aligned} \tag{32}$$

A simple calculation implies

$$\begin{aligned} \nabla_y \tilde{\Psi}_{0,\varepsilon} &= e_\varepsilon \nabla_y \psi_0 + \psi_0 \nabla_y e_\varepsilon \\ &= e_\varepsilon \begin{pmatrix} d\psi_0/ds \\ 0 \\ 0 \end{pmatrix} + i\,\varepsilon\, \psi_0\, e_\varepsilon \nabla_y \langle R(y_1)a'(y_1), y' \rangle_{\mathbb{R}^2} \end{aligned}$$

and hence

$$\|\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} \le C_5 \|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} \tag{33}$$

where

$$C_5 = 1 + \varepsilon_3 \|\nabla_y \langle Ra', y' \rangle_{\mathbb{R}^2}\|_{L^\infty(\tilde{\Omega})}.$$

We show $\sqrt{2(1 - \cos(c - \theta))} = \mathcal{O}(\delta)$. Since $\theta = \theta_\varepsilon(\Psi)$ satisfies $(\tilde{\Psi} - e^{i\theta}\tilde{\Psi}_{0,\varepsilon}, ie^{i\theta}\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} = 0$, we have

$$\begin{aligned} ((e^{i(c-\theta)} - 1)\, \psi_0, i\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} &= -((e^{i(c-\theta)} - 1)\, (e_\varepsilon - 1)\, \psi_0, i\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} \\ &\quad - (\tilde{\Psi}_\delta, ie^{i\theta}\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})}. \end{aligned} \tag{34}$$

It is obvious

$$((e^{i(c-\theta)} - 1)\, \psi_0, i\psi_0)_{L^2_m(\tilde{\Omega};\mathbb{C})} = \sin(c - \theta)\, \|\psi_0\|^2_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}.$$

Thus substituting this equality into (34) yields

$$|\sin(c-\theta)| \le \sqrt{2(1 - \cos(c - \theta))}\, \|e_\varepsilon - 1\|_{L^\infty(\tilde{\Omega};\mathbb{C})} + \delta / \|\psi_0\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}. \tag{35}$$

Put

$$\begin{aligned} \varepsilon_4 &:= \sup \left\{ \bar{\varepsilon} \in (0, \varepsilon_3) : \|e_\varepsilon - 1\|_{L^\infty(\tilde{\Omega};\mathbb{C})} < 1/4\sqrt{2},\ \forall \varepsilon \in (0, \bar{\varepsilon}) \right\}, \\ \delta_{21} &:= \|\psi_0\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}/2\sqrt{2}. \end{aligned}$$

Then it follows from (35) that

$$|\sin(c-\theta)| \le 1/\sqrt{2}, \quad (\forall \varepsilon \in (0,\varepsilon_4], \quad \forall \delta \in (0,\delta_{21}]). \tag{36}$$

Let

$$\begin{aligned} \xi_1 &= e^{i(c-\theta)} \int_{\tilde{\Omega}} |\psi_0|^2 \, m \, dy, \\ \xi_2 &= e^{i(c-\theta)} \int_{\tilde{\Omega}} (e_\varepsilon - 1) \, |\psi_0|^2 \, m \, dy + e^{-i\theta} \int_{\tilde{\Omega}} \tilde{\Psi}_\delta \, \psi_0^* \, m \, dy. \end{aligned}$$

Then we have

$$\begin{aligned} &|\xi_2|/|\xi_1| \le 3/4\sqrt{2} < 1/\sqrt{2}, \quad (\forall \varepsilon \in (0,\varepsilon_4], \quad \forall \delta \in (0,\delta_{21}]), \\ &|\operatorname{Arg}(\xi_1+\xi_2)| = \left| \operatorname{Arg}\left(e^{-i\theta} \int_{\tilde{\Omega}} \tilde{\Psi} \, \psi_0^* \, m \, dy \right) \right| < \pi/2, \end{aligned}$$

by the definition of $\theta = \theta_\varepsilon(\Psi)$. It thus follows from

$$|\operatorname{Arg}\{\xi_1/(\xi_1+\xi_2)\}| = |\operatorname{Arg}\{(\xi_1+\xi_2)/\xi_1\}| \le \sin^{-1}(|\xi_2|/|\xi_1|) < \pi/4$$

that

$$|\operatorname{Arg} e^{i(c-\theta)}| = |\operatorname{Arg} \xi_1| = |\operatorname{Arg}\{(\xi_1+\xi_2)\,[\xi_1/(\xi_1+\xi_2)]\}| < 3\pi/4. \tag{37}$$

Since (36) and (37), it holds that $\sqrt{1-\cos(c-\theta)} \le |\sin(c-\theta)|$. Using this inequality and (35), we have

$$\sqrt{1-\cos(c-\theta)} \le 4\delta/3\|\psi_0\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}, \quad \forall \varepsilon \in (0,\varepsilon_4], \forall \delta \in (0,\delta_{21}]. \tag{38}$$

Put

$$\sigma := 1 + \frac{4\sqrt{2}\, C_5 \, \|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}}{3\|\psi_0\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}}. \tag{39}$$

From (32), (33), (38), and (39), we can estimate

$$\begin{aligned} \|\tilde{\Psi} - e^{i\theta}\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} &< \delta + \sqrt{2(1-\cos(c-\theta))}\, \|\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} \\ &\le \delta + \frac{4\sqrt{2}\,\delta\, C_5 \|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}}{3\|\psi_0\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})}} \\ &\le \sigma\delta \qquad (\forall \varepsilon \in (0,\varepsilon_4], \ \forall \delta \in (0,\delta_{21}]). \end{aligned}$$

Taking $\delta_2 := \min\{\delta_{21}, \delta_1/\sigma\}$, we obtain the lemma. $\square$

3.5. *Lower estimate of* $E_\varepsilon(\Psi, B) - E_\varepsilon(\Psi_{0,\varepsilon}, 0)$

In the last part of Section 3, we show a lower estimate of $E_\varepsilon(\Psi,B) - E_\varepsilon(\Psi_{0,\varepsilon},0)$ which is used in the proof of Theorem 2.1. Combining Lemmas 3.1 to 3.8, we obtain the following lemma.

Lemma 3.9. *Assume* (A1) *and* (A2). *Then there exist* $\varepsilon_5 \in (0,\varepsilon_4]$, $\delta_3 \in (0,\delta_2]$ *and* $C_0 = C_0(\psi_0,\alpha,\tilde{\Omega},A_H,\tau_j,\rho_j) > 0$ *such that for each* $\varepsilon \in (0,\varepsilon_5]$ *and* $\delta \in (0,\delta_3]$ *it holds that*

$$E_\varepsilon(\Psi,B) - E_\varepsilon(\Psi_{0,\varepsilon},0) \geq \frac{\mu_1}{4}\|\tilde{\Psi} - e^{i\theta_\varepsilon(\Psi)}\tilde{\Psi}_{0,\varepsilon}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})} - C_0\varepsilon$$

for all $(\Psi,B) \in \Sigma_\varepsilon(\delta) \times Z$ *where* $\theta_\varepsilon(\Psi)$ *is defined in* Definition 3.1.

Proof. It follows from (33) that for each $\varepsilon \in (0,\varepsilon_3]$ and $\delta \in (0,\delta_1]$

$$\|\tilde{\Psi}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} < \delta + C_5\|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} \quad (\forall\Psi \in \Sigma_\varepsilon(\delta)). \tag{40}$$

Combining Lemmas 3.1 to 3.4 with

$$\Psi \in \Sigma_\varepsilon(\delta), \quad B \in Z, \quad \Phi = \Psi - e^{i\gamma}\Psi_{0,\varepsilon}, \quad \psi_\gamma = e^{i\gamma}\psi_0, \quad \gamma \in \mathbb{R},$$

if $\varepsilon \in (0,\varepsilon_2)$ we have

$$\begin{aligned}
&E_\varepsilon(\Psi,B) - E_\varepsilon(\Psi_{0,\varepsilon},0)\\
&\geq -\frac{C_1\varepsilon^{4/3}}{\beta}\|\tilde{\Psi}\|^4_{H^1_m(\tilde{\Omega};\mathbb{C})} - \frac{C_1}{\beta\varepsilon^{2/3}}\|\tilde{\Psi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})}\|R^{-1}\nabla_{y'}\tilde{\Psi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}\\
&\quad -\frac{1}{8\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} - C_2\varepsilon(1 + \|\tilde{\Phi}\|_{H^1_m(\tilde{\Omega};\mathbb{C})})\\
&\quad +\frac{1}{2}\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) + \frac{1}{8\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Phi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})} + \frac{1}{16\varepsilon}\|R^{-1}\nabla_{y'}\tilde{\Psi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}\\
&\quad - C_3\varepsilon(1 + \|\tilde{\Phi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})}) - C_4\|\tilde{\Phi}\|^3_{H^1_m(\tilde{\Omega};\mathbb{C})}\\
&\geq \frac{1}{2}\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) - C_4\|\tilde{\Phi}\|^3_{H^1_m(\tilde{\Omega};\mathbb{C})}\\
&\quad + \left(\frac{1}{16\varepsilon} - \frac{C_1}{\beta\varepsilon^{2/3}}\|\tilde{\Psi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})}\right)\|R^{-1}\nabla_{y'}\tilde{\Psi}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}\\
&\quad - (C_1 + C_2 + C_3)\varepsilon\left(1 + \|\tilde{\Phi}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} + \|\tilde{\Phi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})} + \frac{\varepsilon^{1/3}}{\beta}\|\tilde{\Psi}\|^4_{H^1_m(\tilde{\Omega};\mathbb{C})}\right).
\end{aligned}$$

Put $\gamma = \theta_\varepsilon(\Psi)$. Then $(\tilde{\Phi}, i\psi_\gamma)_{L^2_m(\tilde{\Omega};\mathbb{C})} = 0$. Applying Lemma 3.5, we have

$$\mathcal{K}_{\psi_\gamma}(\tilde{\Phi}) \geq \mu_1\|\tilde{\Phi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})}.$$

Since $\Sigma_\varepsilon(\delta) \subset F_\varepsilon(\sigma\delta)$ for $\forall\varepsilon \in (0,\varepsilon_4]$ and $\forall\delta \in (0,\delta_2]$, it holds that

$$\|\tilde{\Phi}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} = \|\tilde{\Psi} - e^{i\theta_\varepsilon(\Psi)}\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})} < \sigma\delta \quad (\forall\Psi \in \Sigma_\varepsilon(\delta)).$$

Thus,

$$\begin{aligned}
&E_\varepsilon(\Psi,B) - E_\varepsilon(\Psi_{0,\varepsilon},0)\\
&\geq \left(\frac{\mu_1}{2} - C_4\,\sigma\,\delta\right)\|\tilde{\Phi}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})}\\
&\quad + \left(\frac{1}{16\varepsilon} - \frac{C_1(\delta + C_5\|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})})^2}{\beta\varepsilon^{2/3}}\right)\|\nabla_{y'}\tilde{\Psi}\,R^{-1}\|^2_{L^2_m(\tilde{\Omega};\mathbb{C})}\\
&\quad - (C_1+C_2+C_3)\varepsilon\left\{1 + \sigma\delta + \sigma^2\delta^2 + \frac{\varepsilon^{1/3}}{\beta}(\delta + C_5\|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})})^4\right\}\\
&\qquad\qquad (\forall\varepsilon \in (0,\min\{\varepsilon_2,\varepsilon_4\}], \quad \forall\delta \in (0,\delta_2])\,,
\end{aligned}$$

where we used (40). Define

$$\begin{aligned}
\delta_3 &:= \min\{\delta_2, \mu_1/4C_4\sigma\},\\
\varepsilon_5 &:= \min\left\{\varepsilon_2, \varepsilon_4, \frac{\beta^3}{16^3 C_1^3(\delta_3 + C_5\|\psi_0\|_{L^2_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})})^6}\right\},
\end{aligned}$$

and

$$C_0 := (C_1+C_2+C_3)\left\{1 + \sigma\delta_3 + \sigma^2\delta_3^2 + \frac{\varepsilon_5^{1/3}}{\beta}(\delta_3 + C_5\|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})})^4\right\}.$$

Then the desired estimate follows. □

4. Proof of the theorem

First we prove the following lemma:

Lemma 4.1. *$E_\varepsilon(\Psi,B)$ is weakly lower semi-continuous on $H^1(\Omega(\varepsilon);\mathbb{C}) \times Z$.*

Proof. By the definition of $E_\varepsilon(\Psi,B)$, it is written as

$$\begin{aligned}
E_\varepsilon(\Psi,B) = \frac{1}{2\varepsilon^2}\int_{\Omega(\varepsilon)} &\Big\{|\nabla_x\Psi|^2 - 2\,\mathrm{Im}\langle\nabla_x\Psi, \Psi\,(A_H+B)\rangle_{\mathbb{C}^3} + |A_H+B|^2\,|\Psi|^2\\
&+ \alpha\left(1 - 2|\Psi|^2 + |\Psi|^4\right)/2\Big\}dx + \frac{\beta}{2\varepsilon^2}\int_{\mathbb{R}^3}|\nabla B|^2\,dx
\end{aligned}$$

for $(\Psi, B) \in H^1(\Omega(\varepsilon);\mathbb{C}) \times Z$. If $\Psi_k \rightharpoonup \Psi_\infty$ weakly in $H^1(\Omega(\varepsilon);\mathbb{C})$ and $B_k \rightharpoonup B_\infty$ weakly in Y as $k \to \infty$, then

$$\Psi_k \to \Psi_\infty \text{ strongly in } L^q(\Omega(\varepsilon);\mathbb{C}),$$
$$(A_H + B_k)|_{\Omega(\varepsilon)} \to (A_H + B_\infty)|_{\Omega(\varepsilon)} \text{ strongly in } L^q(\Omega(\varepsilon);\mathbb{R}^3)$$

as $k \to \infty$ for $q \in [2, 6)$ by Sobolev embedding theorem. By using the inequalities

$$\|\nabla\Psi_\infty\|_{L^2(\Omega(\varepsilon);\mathbb{C})} \le \liminf_{k\to\infty} \|\nabla\Psi_k\|_{L^2(\Omega(\varepsilon);\mathbb{C})},$$
$$\|\nabla B_\infty\|_{L^2(\mathbb{R}^3)} \le \liminf_{k\to\infty} \|\nabla B_k\|_{L^2(\mathbb{R}^3)}$$

we obtain the lemma. □

Now we are in a position to prove the theorem.

Proof of Theorem 2.1. Take a minimizing sequence $(\Psi_k, B_k) \in \Sigma_\varepsilon(\delta) \times Z$ such that

$$E_\varepsilon(\Psi_k, B_k) \searrow \inf_{(\Psi,B)\in\Sigma_\varepsilon(\delta)\times Z} E_\varepsilon(\Psi, B) \quad (k \to \infty).$$

Since $(\Psi_{0,\varepsilon}, 0) \in \Sigma_\varepsilon(\delta) \times Z$, we may assume

$$E_\varepsilon(\Psi_k, B_k) \le E_\varepsilon(\Psi_{0,\varepsilon}, 0) \quad (\forall k \in \mathbb{N}). \tag{41}$$

Thus by the equality $\|\operatorname{rot} B\|_{L^2(\mathbb{R}^3)} = \|\nabla B\|_{L^2(\mathbb{R}^3)}$ for $B \in Z$,

$$\|B_k\|_Y^2 \le 2\varepsilon^2 E_\varepsilon(\Psi_k, B_k)/\beta \le 2\varepsilon^2 E_\varepsilon(\Psi_{0,\varepsilon}, 0)/\beta \quad (\forall k \in \mathbb{N}).$$

From (40), it follows that

$$\|\tilde{\Psi}_k\|_{H^1_m(\tilde{\Omega};\mathbb{C})} \le \delta + C_5\|\psi_0\|_{H^1_{\pi m}(\mathbb{R}/\ell\mathbb{Z};\mathbb{C})} \quad (\forall k \in \mathbb{N}).$$

As shown in Lemma 4.1, the functional $E_\varepsilon(\Psi, B)$ is weakly lower semicontinuous. Applying the direct method of the variational theory, there exist $(\Psi_\varepsilon, B_\varepsilon) \in \overline{\Sigma_\varepsilon(\delta)} \times Z$ and a subsequence $\{k'\} \subset \mathbb{N}$ such that

$$\Psi_{k'} \rightharpoonup \Psi_\varepsilon \text{ weakly in } H^1(\Omega(\varepsilon);\mathbb{C}) \qquad (k' \to \infty),$$
$$\Psi_{k'} \to \Psi_\varepsilon \text{ strongly in } L^q(\Omega(\varepsilon);\mathbb{C}) \quad (2 \le q < 6) \qquad (k' \to \infty),$$
$$B_{k'} \rightharpoonup B_\varepsilon \text{ weakly in } Y \qquad (k' \to \infty),$$
$$E_\varepsilon(\Psi_\varepsilon, B_\varepsilon) = \inf_{(\Psi,B)\in\Sigma_\varepsilon(\delta)\times Z} E_\varepsilon(\Psi, B).$$

Thus it suffices for verifying the theorem to prove $\Psi_\varepsilon \in \Sigma_\varepsilon(\delta)$, that is, $\Psi_\varepsilon \notin \partial\Sigma_\varepsilon(\delta)$. Applying Lemma 3.9 if $\varepsilon \in (0, \varepsilon_5]$ and $\delta \in (0, \delta_3]$ we obtain

$$4C_0\varepsilon/\mu_1 \ge \|\tilde{\Psi}_k - e^{i\theta_\varepsilon(\Psi_k)}\tilde{\Psi}_{0,\varepsilon}\|^2_{H^1_m(\tilde{\Omega};\mathbb{C})} \qquad (\forall k \in \mathbb{N})$$

and hence

$$2\sqrt{C_0\varepsilon/\mu_1} \geq \liminf_{k'\to\infty} \|\tilde{\Psi}_{k'} - e^{i\theta_\varepsilon(\Psi_{k'})}\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})}$$
$$\geq \|\tilde{\Psi}_\varepsilon - e^{i\theta_\varepsilon(\Psi_\varepsilon)}\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})}.$$

Put

$$\delta_0 := \min\left\{\delta_3, 4\sqrt{C_0\varepsilon_5/\mu_1}\right\}, \quad \varepsilon_0 = \varepsilon_0(\delta) := \mu_1\delta^2/16C_0.$$

Then if $\delta \in (0, \delta_0]$ and $\varepsilon \in (0, \varepsilon_0(\delta)]$,

$$\delta > \delta/2 \geq 2\sqrt{C_0\varepsilon/\mu_1} \geq \|\tilde{\Psi}_\varepsilon - e^{i\theta_\varepsilon(\Psi_\varepsilon)}\tilde{\Psi}_{0,\varepsilon}\|_{H^1_m(\tilde{\Omega};\mathbb{C})},$$

that is, $\Psi_\varepsilon \in F_\varepsilon(\delta)$ holds, where $F_\varepsilon(\delta)$ is defined in Definition 3.2. By (31) in Lemma 3.8, the proof of the theorem was completed. □

Acknowledgments.

The authors would like to express their thanks to Professor Ninomiya Hirokazu for the useful discussion. The first author was supported in part by the Grant-in-Aid for JSPS Fellows, No. 9335. The second author was supported in part by the Grant-in-Aid for Scientific Research, (B) No. 15340037, Japan Society for the Promotion of Science.

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SINGULAR LIMIT FOR SOME REACTION DIFFUSION SYSTEM*

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1. Introduction

Habitat segregation phenomena in mathematical ecology supply us with various problems which are interesting from the aspect of interfacial dynamics. We mathematically discuss regional partition by competitive two species and their competition for their own habitats. When the competition between two species is bitter, they cannot coexist at the same point. In such cases we can expect that the two species with a suitable initial state segregate their habitats and compete on the interface between both the habitats. Then it is significant to understand the dynamics of the segregation patterns.

In this article we treat a competition-diffusion system for two species in competition of the Lotka-Volterra type:

$$\begin{cases} u_t = d_1 \Delta u + (a_1 - b_1 u - c_1 v)u, & \text{in } \Omega \times (0, \infty), \\ v_t = d_2 \Delta v + (a_2 - b_2 v - c_2 u)v, & \text{in } \Omega \times (0, \infty), \end{cases}$$

with Neumann zero boundary condition on $\partial\Omega$. Here a_k, b_k, c_k and d_k ($k = 1, 2$) are positive constants; $u = u(t,x)$ and $v = v(t,x)$ are the population densities of competitive two species. Our concern is the situation where the interspecific competition is exceedingly bitter: in particular, the situation close to the singular limit as $c_1, c_2 \to \infty$ with c_1/c_2 fixed. Thus we simply

[a]This is a joint work with Georgia Karali (University of Toronto), Masato Iida (Iwate university), Masayasu Mimura (Meiji university), Eiji Yanagida (Tohoku university), and Tohru Wakasa (Waseda university) ([7], [9]).

rewrite the above system as

$$\begin{cases} u_t = \Delta u + (a-u)u - bMuv, & \text{in } \Omega \times (0,\infty), \\ \\ v_t = D\Delta v + (d-v)v - cMuv, & \text{in } \Omega \times (0,\infty), \end{cases} \tag{1}$$

where a, b, c, d, D are fixed positive constants and M is a huge parameter. As seen in the following section, the spatial supports of u and v satisfying (3) become separated from each other by an interface in a short time-period. Then after that the *segregated* (u, v) behaves like a solution of a two phase free boundary problem for the Fisher equation. We will establish a rigorous mathematical theory both for the formation of interfaces at the initial stage and for the motion of those interfaces in the later stage. More precisely, we will show that, given virtually arbitrary smooth initial data, the solution develops interfaces within the time scale of $O(\epsilon^2)$. We will then prove that the motion of the interfaces converges to the following free boundary problem as $\epsilon \to 0$.

$$\begin{cases} u_t^* = \Delta u^* + (a-u^*)u^*, \quad v^* \equiv 0 \text{ in } R(t), \\ \\ v_t^* = D\Delta v^* + (d-v^*)v^*, \, u^* \equiv 0 \text{ in } \Omega \backslash R(t), \\ c\dfrac{\partial u^*}{\partial \nu} + bD\dfrac{\partial v^*}{\partial \nu} = 0 \quad \text{on } \Gamma(t), \end{cases} \tag{2}$$

where

$$\Gamma(t) = \partial R(t),$$

and ν an inner normal to $\Gamma(t)$.

There are several related works on singular limits of some reaction-diffusion systems as the effect of interaction tends to infinity: [1], [3], [4], [5] and [11] investigate the *fast reaction limit* of chemical reaction systems (see also the references therein). As for competition-diffusion systems, [2] investigates singular limits of the stationary problems as the interspecific competition rate tends to infinity. The most related work is [6], which we will mention after giving the formal derivation of the singular limit.

2. Formal derivation of the singular limit

We rewrite (1) as

$$\begin{cases} u_t = \Delta u + (a-u)u - \dfrac{b}{\epsilon^3}uv, & \text{in } \Omega \times (0,\infty), \\ \\ v_t = D\Delta v + (d-v)v - \dfrac{c}{\epsilon^3}uv, & \text{in } \Omega \times (0,\infty), \end{cases} \tag{3}$$

with the boundary condition and the initial condition such that

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \qquad \text{on } \partial\Omega \times (0, \infty),$$
$$u(x,0) = u_0(x),\ v(x,0) = v_0(x), \qquad \text{in } \Omega.$$

Here $u_0(x) > 0$, $v_0(x) > 0$, in $\bar{\Omega}$, n is an outer normal to $\partial\Omega$, and ϵ is a small parameter, especially. In this section we present a formal derivation of the singular limit of (3).

Set

$$R(0) = \{\, x \in \Omega \mid cu_0(x) > bv_0(x)\,\}, \qquad \Omega \backslash \overline{R(0)} = \{\, x \in \Omega \mid cu_0(x) < bv_0(x)\,\},$$

and assume that both of $R(0)$ and $\Omega \backslash \overline{R(0)}$ possess interior points.

Let us consider the first stage of a short time period from $t = 0$ until $t = \epsilon^2$. Since the initial data is smooth, it is heuristically seen that the behavior of the solution of (3) is formally approximated by that of $(\tilde{u}, \tilde{v})$ below during the very early stage, where the diffusion terms, $u(a-u)$ and $v(d-v)$ are relatively small compared with the competition terms.

$$\begin{cases} \tilde{u}_t = -\dfrac{b\tilde{u}\tilde{v}}{\epsilon^3} \\[2ex] \tilde{v}_t = -\dfrac{c\tilde{u}\tilde{v}}{\epsilon^3} \\[2ex] \tilde{u}(x,0) = u_0(x), \quad \tilde{v}(x,0) = v_0(x). \end{cases} \tag{4}$$

The solution of (4) is given by

$$\tilde{u}(x,t) = \phi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big), \quad \tilde{v}(x,t) = \psi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big), \tag{5}$$

where $(\phi(\tau;\xi,\eta), \psi(\tau;\xi,\eta))$ is a solution of

$$\begin{cases} \dot{\phi} = -b\phi\psi, & \phi(0) = \xi > 0, \\ \dot{\psi} = -c\phi\psi, & \psi(0) = \eta > 0. \end{cases} \tag{6}$$

Set $A(\xi,\eta) = c\xi - b\eta$, then we can easily observe that $A(\phi(\tau), \psi(\tau))$ is preserved for any $\tau > 0$; so that

$$\dot{\phi} = (A(\xi,\eta) - c\phi)\phi, \qquad \phi(0) = \xi. \tag{7}$$

Solving (7) explicitly, we have

$$\phi(\tau;\xi,\eta) = \frac{\xi A e^{A\tau}}{A + c\xi(e^{A\tau} - 1)}, \quad \psi(\tau;\xi,\eta) = \frac{\eta A e^{-A\tau}}{A + b\eta(1 - e^{-A\tau})}, \tag{8}$$

therefore,

$$\lim_{\tau\to+\infty}\phi(\tau;\xi,\eta)=\max\left\{\frac{A(\xi,\eta)}{c},0\right\},\quad \lim_{\tau\to+\infty}\psi(\tau;\xi,\eta)=\max\left\{0,-\frac{A(\xi,\eta)}{b}\right\}. \tag{9}$$

Then it follows that the solution becomes close to the continuous function

$$(u_1(x),v_1(x))=\begin{cases}(\omega(x)/c,\ 0) & \text{in } R(0),\\ (0,\ -\omega(x)/b) & \text{in } \Omega\backslash\overline{R(0)},\end{cases} \tag{10}$$

after a short period of time scale t. The non-degeneracy of $\nabla\omega$ on $\partial R(0)=\{x|\ \omega(x)=0\}$ causes the gap of $(\nabla u_1,\nabla v_1)$ across the surface $\partial R(0)$. Thus sharp transition of $(\nabla u,\ \nabla v)$ appears near $\partial R(0)$. Namely the *corner layer* of $(u(t,\cdot),\ v(t,\cdot))$ is generated along the surface $\partial R(0)$ in a short time-period.

The second stage of the dynamics of (3) describes the propagation of the corner layer. The stretching (u,v) with a suitable scale makes the analysis of the corner layer easier. To rescale the system in the best possible way, we need to estimate the length scale $\epsilon=\epsilon(M)$ of the width of the corner layer. We note that u_1, v_1 are continuous functions with bounded gradients and that the mean curvature of the surface $\partial R(0)$ is bounded. It is natural to assume in the second stage that $u=O(\epsilon)$, $v=O(\epsilon)$, $u_t=O(1)$ and $\Delta u=O(\epsilon^{-1})$ on the corner layer for huge M and that the effects of Δu and Muv in (3) are well-balanced. Then we have $\epsilon=O(M^{-1/3})$.

Taking account of (10) and the argument for the first stage, we can expect that $u(t,x;\epsilon)$ almost vanishes in some region in Ω, namely $\mathbb{R}^N\backslash R^\epsilon(t)$, on the other hand $v(t,x;\epsilon)$ vanishes in $R^\epsilon(t)$. Further the corner layer of $(u(t,\cdot;\epsilon),\ v(t,\cdot;\epsilon))$ remains along the interface $\partial R^\epsilon(t)$. Around each point $y\in\partial R^\epsilon(t)$ we introduce a local orthogonal coordinate system (ξ,σ) such that $\sigma=(\sigma_1,\ldots,\sigma_{N-1})$ is a local coordinate along $\partial R^\epsilon(t)$ whereas $\xi=\xi(x,\ \partial R^\epsilon(t))$ is the signed distance from x to $\partial R^\epsilon(t)$ locally defined near y so that $\xi>0$ in $R^\epsilon(t)$. Around the corner layer we stretch the solution and see it using a moving coordinate system (t,ρ,σ), where $\rho=\xi/\epsilon$ is a rescaled coordinate in the normal direction to $\partial R^\epsilon(t)$. Suppose that $(u(t,x;\epsilon),\ v(t,x;\epsilon))$ is asymptotically written as

$$(u,v)=\begin{cases}(u^*,v^*)+O(\epsilon) & \text{away from the layer (outer expansion),}\\[2ex] \epsilon(U_1,V_1)+\epsilon^2(U_2,V_2)+O(\epsilon^3) & \text{around the layer (inner expansion),}\end{cases}$$

where (u^*,v^*) is a bounded continuous function of the fixed coordinate (t,x) and (U_1,V_1) and (U_2,V_2) are smooth functions of the moving coordinate (t,ρ,σ) with a bounded gradient; all of them are independent of ϵ. By a formal argument based on the *matched asymptotic expansion* method, we can formally conclude that (u^*, v^*) satisfy (2) and (U_1, V_1) satisfy

$$\begin{cases} U_{1\rho\rho} = cU_1V_1, \quad -\infty < \rho < +\infty, \\ dV_{1\rho\rho} = bU_1V_1, \ -\infty < \rho < +\infty, \\ (U_1(t,\rho,\sigma),\, V_1(t,\rho,\sigma)) = \left(0,\, -\rho\dfrac{\partial v^*}{\partial \nu^o}(t,y)\right) \text{ as } \rho \to -\infty, \\ (U_1(t,\rho,\sigma),\, V_1(t,\rho,\sigma)) = \left(\rho\dfrac{\partial u^*}{\partial \nu^i}(t,y),\, 0\right) \quad \text{as } \rho \to +\infty, \end{cases} \tag{11}$$

and (U_2, V_2) satisfies (30) which is given later.

Here $R(t)$ is the formal limit of $R^\epsilon(t)$ as $\epsilon \to +0$, $\nu^i(\nu^o)$ inner (outer) normal to $\partial R(t)$, and y a point on $\partial R(t)$ corresponding to the coordinate $(0, \sigma)$. In (11) the boundary conditions at $\rho = \pm\infty$ reflect the request that (u^*, v^*) and $\epsilon(U_1, V_1)$ should be matched. The boundary condition on $\partial R(t)$ in (2) is requested for (u^*, v^*) in order that the elliptic boundary value problem (11) possesses a solution. Consequently, in the second stage the supports of $u(t, \cdot; \epsilon)$ and $v(t, \cdot; \epsilon)$ are almost separated by the corner layer which remains in a narrow range of $O(\epsilon)$ along the propagating interface $\partial R(t)$. The dynamics of the segregation pattern is essentially determined by the free boundary problem (2). We see from the elliptic equations in (11) that the population on the interface supplied by the diffusion from both the habitats instantly disappears by the strong competition between two species.

3. Main result

The formal derivation of the free boundary problem (2) from (3) as $\epsilon \to +0$ is justified by [6] on a bounded domain in $\mathbb{R}^N$ under the no-flux boundary condition in the framework of weak topology of H^1. It also gives a result on the uniqueness and existence of a Hölder-continuous weak solution to (2). However we need to justify the derivation of (2) at least in the framework of C^0-topology in order to investigate the dynamics of the segregating interface. To accomplish this end we impose the existence of a classical solution to (2) as follows.

Before stating the results, we will make some assumptions.

Assumption 3.1. *(nondegeneracy condition) Suppose that*

$$\inf_{\Gamma(0)} |c\nabla u_0 - b\nabla v_0| > 0.$$

Here $\Gamma(0) = \partial R(0)$

Remark 3.1. Assumption 3.1 assures that Γ_0 is an $N-1$ dimensional hypersurface with bounded mean curvature.

Let $(u^*(x,t), v^*(x,t), \Gamma(t))$ be a solution to the free boundary problem (2) with an initial data

$$u^*(x,0) = \frac{\omega(x)}{c}, \quad \text{in } R(0), \qquad v^*(x,0) = -\frac{\omega(x)}{b} \quad \text{in } \Omega\backslash\overline{R(0)}. \qquad (12)$$

Assumption 3.2. $(u^*(x,t), v^*(x,t), \Gamma(t))$ *satisfies* (2) *with the initial data* (12) *in a classical sense for* $(x,t) \in \Omega \times [0,T]$. $\Gamma(t)$ *is a closed hypersurface in* Ω *and is in* C^2 *for each* t *and in* C^1 *with respect to* t.

Assumption 3.3. u^* *and* v^* *be nonnegative continuous functions,* $|u^*|, |\nabla u^*|, |\Delta u^*|$ *are bounded in* $R(t)$ *uniformly with respect to* t, *and* $|v^*|, |\nabla v^*|, |\Delta v^*|$ *are bounded in* $\Omega\backslash\overline{R(t)}$ *uniformly with respect to* t;

Assumption 3.4.

$$\inf_{y\in\Gamma(t)} \lim_{\substack{x \to y \\ x \in R(t)}} |\nabla u^*(x)| > 0, \qquad \inf_{y\in\Gamma(t)} \lim_{\substack{x \to y \\ x \in \Omega\backslash\overline{R(t)}}} |\nabla v^*(x)| > 0.$$

Remark 3.2. If the free boundary condition in (2) is replaced by

$$\mu\frac{d}{dt}\Gamma(t) = c\frac{\partial u^*}{\partial \nu^i} - bD\frac{\partial v^*}{\partial \nu^o} \quad \text{on } \Gamma(t),$$

where μ is a positive constant and $\frac{d}{dt}\Gamma(t)$ denotes the propagation speed of $\Gamma(t)$ in the outer normal direction, then the regularity of $\Gamma(t)$ will be assured by the parabolicity as treated in [8] and [10]. However, in our case which corresponds to the case $\mu = 0$, it is not easy to deduce the regularity of $\Gamma(t)$ in (2), because the parabolicity is partially broken on $\Gamma(t)$. Nevertheless, a recent result in [11] suggests that the partial regularity of $\Gamma(t)$ in the classical sense can hold also for (2). Thus we believe the above assumptions natural.

Now we will give our main theorem.

Theorem 3.1. *There exist a positive constant* $C > 0$ *such that for sufficiently small* $\epsilon > 0$, *the following hold:*

$$|u_\epsilon(x,t) - u^*(t,x)| < C\epsilon|\log\epsilon|,$$

$$|v_\epsilon(x,t) - v^*(t,x)| < C\epsilon|\log\epsilon| \quad \textit{for } (t,x) \in [\epsilon^2, T] \times \Omega,$$

where $(u_\epsilon(x,t),\, v_\epsilon(x,t))$ *is a nonnegative solution of (3). More precisely, there exists* $C', C'', C''' > 0$ *such that for sufficiently small* ϵ*, the following holds:*

$$|u_\epsilon(x,t)| < C' \exp(-\frac{C'''|\tilde{d}(x,t)|}{\epsilon}),$$

$$for\{x \in \Omega\backslash\overline{R(t)}\ ;\ |\tilde{d}(x,t)| > C''\epsilon|\log\epsilon|\},$$

$$|v_\epsilon(x,t)| < C' \exp(-\frac{C'''|\tilde{d}(x,t)|}{\epsilon}),$$

$$for\{x \in R(t)\ ;\ |\tilde{d}(x,t)| > C''\epsilon|\log\epsilon|\}.$$

Theorem 1 shows that , for virtually arbitrary smooth initial data, the solution develops interfaces in time $t = \epsilon^2$ and the motion of the interface is approximated by the free boundary problem (2) for $t \in [\epsilon^2, T]$.

Our main tool for deriving the above results is the method of upper and lower solutions. We will use two different pairs of upper and lower solutions, namely $(U^\pm, V^\pm)$ and $(u^\pm, v^\pm)$. The first one $(U^\pm, V^\pm)$ is used to analyze the generation of the interface that takes place in a very fast time scale. The second one $(u^\pm, v^\pm)$ is used to study the motion of the interface in a relatively slow time scale. The transition from the initial stage to the second stage occurs within a time scale of ϵ^2. Since the behaviors of solutions are so different between the two stages, it is important to construct suitable upper and lower solutions for each stage and to know the right timing to switch from $(U^\pm, V^\pm)$ to $(u^\pm, v^\pm)$.

In the following Section 4, we deal with the generation of the interface, and in Section 6, the motion of the interface. Section 4 is depend on [9], and Section 6 is on [7].

4. Generation of interface

In this section we study the generation of interface that takes place in the initial stage. We will construct an upper and lower solution for this stage.

As we have mentioned in Section 2, we can expect that the solution $(u(x,t), v(x,t))$ would be approximated by

$$(\phi(\frac{t}{\epsilon^3}; u_0(x), v_0(x)),\ \psi(\frac{t}{\epsilon^3}; u_0(x), v_0(x))) \tag{13}$$

by a formal argument. Let $d^* > 0$ be sufficiently small constant such that $dist(x, \Gamma(t))$ be signed distance function defined in $\{x \in \Omega \mid dist(x, \Gamma(t)) \leq$

$3d^*$ } We will introduce cut-off functions. We define $\tilde{d}$ as a modification of $dist(x,\Gamma(t))$ such that $\tilde{d}\, d \geq 0$.

$$\tilde{d}(x,t) = \begin{cases} dist(x,\Gamma(t)) & \text{if} \quad |dist(x,\Gamma(t))| \leq d^*, \\ d^* \leq |dist(x,\Gamma(t))| \leq 2d^* & \text{if} \quad d^* \leq |dist(x,\Gamma(t))| \leq 2d^*, \\ |dist(x,\Gamma(t))| = 2d^* & \text{for } \Omega \setminus \{x \in \Omega \mid dist(x,\Gamma(t)) \leq 2d^* \}. \end{cases} \tag{14}$$

Set

$$\Gamma_0 = \Gamma(0).$$

It is easily seen that there exists $0 < C_0 < C_1$ such that

$$C_0|\tilde{d}(x,0)| < |\omega(x)| < C_1|\tilde{d}(x,0)|.$$

Therefore, we obtain the following theorem:

Theorem 4.1. (*Nakashima-Wakasa* [9]) *Then there exist C_1, $C_2 > 0$ such that for sufficiently small $\epsilon > 0$, the solution (u_ϵ, v_ϵ) of (3) satisfies the following estimate:*

$$\left|u_\epsilon(x,t) - \phi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\right| < C_1\epsilon, \ (x,t) \in \Omega \times (0,\epsilon^2),$$

$$\left|v_\epsilon(x,t) - \psi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\right| < C_1\epsilon \ \ (x,t) \in \Omega \times (0,\epsilon^2),$$

and

$$\left|u_\epsilon(x,\epsilon^2) - \max\Big\{\frac{\omega(x)}{c}, 0\Big\}\right| < C_2\epsilon, \ \ x \in \Omega$$

$$\left|v_\epsilon(x,\epsilon^2) - \max\Big\{0, -\frac{\omega(x)}{b}\Big\}\right| < C_2\epsilon \ x \in \Omega.$$

Moreover, under Assumption 3.1, there exist $C_3, C_4, C_5 > 0$ such that for sufficiently small $\epsilon > 0$,

$$|u_\epsilon(x,\epsilon^2)| < C_5 \exp(-\frac{C_3|\ \tilde{d}(x,0)|}{\epsilon}), \ in \ \{x \in \Omega \backslash \overline{R(0)}\ ;\ |\tilde{d}(x,0)| > C_4\epsilon|\log\epsilon|\},$$

$$|v_\epsilon(x,\epsilon^2)| < C_5 \exp(-\frac{C_3|\ \tilde{d}(x,0)|}{\epsilon}), \ in \ \{x \in R(0)\ ;\ |\tilde{d}(x,0)| > C_4\epsilon|\log\epsilon|\}. \tag{15}$$

Theorem 2 shows that, for virtually arbitrary initial data, the solution forms interfaces in time $t = \epsilon^2$. More precisely, at time $t = \epsilon^2$, $(u^{\pm}, v^{\pm})$ stays between another pair of upper and a lower solution which are given in the next section, Motion of interface. This makes it possible to combine two different pairs of upper and lower solutions.

5. Proof of Theorem 4.1

In this section, we will prove Theorem 4.1 The proof of this theorem is due to constructing upper and lower solutions, which are modifications of the approxìmate solutions $(\tilde{u}, \tilde{v})$ in (4).

5.1. *Some estimates for solutions to O.D.E. system*

Let us consider (ϕ, ψ), solutions to (6). We will give some estimates for several quantities of ϕ, ψ, and their derivatives with respect to ξ and η. From (8) and (9), we can see that sign of $A(\xi, \eta)$ plays an essential role to determine asymptotic behavior of (ϕ, ψ) as $\tau \to +\infty$. In order to show Theorem 4.1, we need two kind of different estimates. One of these are given uniformly in A, that is, these estimates are independent of $A(\xi, \eta)$. We also need estimates for ϕ (resp. ψ), if $(\xi, \eta) \in \{A(\xi, \eta) < 0\}$ (resp. if $(\xi, \eta) \in \{A(\xi, \eta) > 0\}$).

Lemma 1.

(i) For all $\tau, \xi, \eta > 0$,

$$0 < \phi(\tau; \xi, \eta) - \max\left\{\frac{A(\xi, \eta)}{c}, 0\right\} \leq \frac{\xi}{(1 + c\xi\tau)},$$

and

$$0 < \psi(\tau; \xi, \eta) - \max\left\{0, -\frac{A(\xi, \eta)}{b}\right\} \leq \frac{\eta}{(1 + b\eta\tau)}.$$

(ii) If $\xi, \eta > 0$ *satisfies* $A(\xi, \eta) < 0$ *(resp.* $A(\xi, \eta) > 0$*), then*

$$0 < \phi(\tau; \xi, \eta) \leq \xi e^{A(\xi,\eta)\tau} \quad (\textit{resp. } 0 < \psi(\tau; \xi, \eta) \leq \eta e^{-A(\xi,\eta)\tau})$$

for any $\tau > 0$.

Lemma 2. *For all* $\tau > 0, \xi > 0, \eta > 0$,

$$0 < \phi_\xi(\tau; \xi, \eta) < 1, \quad -\frac{b}{c} < \phi_\eta(\tau; \xi, \eta) < 0,$$

and

$$-\frac{b}{c} < \psi_\xi(\tau;\xi,\eta) < 0, \quad 0 < \psi_\eta(\tau;\xi,\eta) < 1.$$

Lemma 3. *If $\xi,\eta > 0$ satisfies $A(\xi,\eta) < 0$ (resp. $A(\xi,\eta) > 0$), then there exists $M_0 > 0$, which is independent of ξ and η, such that*

$$|\phi_\xi(\tau;\xi,\eta)| \le M_0(1+\xi\tau)e^{A(\xi,\eta)\tau}, \quad |\phi_\eta(\tau;\xi,\eta)| \le M_0\xi\tau e^{A(\xi,\eta)\tau}$$

$$(\textit{resp. } |\psi_\xi(\tau;\xi,\eta)| \le M_0\eta\tau e^{-A(\xi,\eta)\tau}, \quad |\psi_\eta(\tau;\xi,\eta)| \le M_0(1+\eta\tau)e^{-A(\xi,\eta)\tau})$$

for any $\tau > 0$.

Lemma 4. *(i) For all $\tau > 0$, $\xi > 0$, $\eta > 0$, it holds that*

$$\left|\frac{\phi(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| < 2\xi, \qquad \left|\frac{\psi(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| < 2\eta.$$

(ii) There exist $M_1, M_2 > 0$, which are independent of ξ and η, such that

$$\left|\frac{\phi_{\xi\xi}(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| + \left|\frac{\phi_{\xi\eta}(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| + \left|\frac{\phi_{\eta\eta}(\tau;\xi,\eta)}{\phi_\xi(\tau;\xi,\eta)}\right| \le \frac{M_1}{\xi} + M_2\tau,$$
$$\left|\frac{\psi_{\xi\xi}(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| + \left|\frac{\psi_{\xi\eta}(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| + \left|\frac{\psi_{\eta\eta}(\tau;\xi,\eta)}{\psi_\eta(\tau;\xi,\eta)}\right| \le \frac{M_1}{\eta} + M_2\tau,$$

for all $\tau > 0$, $\xi > 0$, $\eta > 0$.

Proofs of above lemmas are ommited.

5.2. *Definition of upper and lower solutions*

Let $(u(x,t), v(x,t))$ be a smooth function defined on $\bar{\Omega} \times [t_0, t_1]$. We say (u,v) is an *upper solution* for equation (3) (in the time interval $t_0 \le t \le t_1$) if it satisfies

$$\begin{cases} u_t - \Delta u - (a-u)u + \dfrac{buv}{\epsilon^3} \ge 0, \\ \\ v_t - d\Delta v - (d-v)v + \dfrac{cuv}{\epsilon^3} \le 0 \end{cases} \tag{16}$$

for $x \in \Omega, t_0 \le t \le t_1$ along with the boundary condition

$$\frac{\partial u}{\partial n} \ge 0, \frac{\partial v}{\partial n} \le 0, \ (x \in \partial\Omega, \ t_0 \le t \le t_1).$$

We say (u, v) is a *lower solution* for equation (3) if it satisfies

$$\begin{cases} u_t - \Delta u - u(a-u) + \dfrac{buv}{\epsilon^3} \leq 0, \\ \\ v_t - d\Delta v - v(d-v) + \dfrac{cuv}{\epsilon^3} \geq 0 \end{cases} \tag{17}$$

for $x \in \Omega, t_0 \leq t \leq t_1$ along with the boundary condition

$$\frac{\partial u}{\partial n} \leq 0, \frac{\partial v}{\partial n} \geq 0, \ (x \in \partial\Omega, \ t_0 \leq t \leq t_1).$$

The following is a consequence of the maximum principle.

Proposition 1. *Let (u^+, v^+) be an upper solution and (u^-, v^-) be a lower solution of (3) for $t_0 \leq t \leq t_1$. Suppose that a solution (u, v) of (3) satisfies $u^-(x, t_0) \leq u(x, t_0) \leq u^+(x, t_0)$, $v^-(x, t_0) \geq v(x, t_0) \geq v^+(x, t_0)$ for $x \in \bar{\Omega}$. Then the solution (u, v) satisfies $u^-(x, t) \leq u(x, t) \leq u^+(x, t)$ and $v^-(x, t) \geq v(x, t) \geq v^+(x, t)$ for $t \in [t_0, t_1]$ and $x \in \bar{\Omega}$.*

The following is also an immediate consequence of the above proposition:

Cororrary 1. **Comparison principle**. If (u, v) and $(\tilde{u}, \tilde{v})$ are two solutions of (3) and if $u \leq \tilde{u}$ and $v \geq \tilde{v}$ for $t = t_0$, then $u \leq \tilde{u}$ and $v \geq \tilde{v}$ for $t \geq t_0$.

Remark. This comparison principle reduces to Proposition 1 in the case of the ODE system (6). More precisely

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \succeq \begin{pmatrix} \tilde{\xi} \\ \tilde{\eta} \end{pmatrix} \quad \text{implies} \quad \begin{pmatrix} \phi(\tau; \xi, \eta) \\ \psi(\tau; \xi, \eta) \end{pmatrix} \succeq \begin{pmatrix} \phi(\tau; \tilde{\xi}, \tilde{\eta}) \\ \psi(\tau; \tilde{\xi}, \tilde{\eta}) \end{pmatrix} \text{ for } \tau \geq 0. \tag{18}$$

5.3. *Construction of an upper and a lower solution*

The upper and lower solutions for the early stage are constructed by modifying the solution of the following problem:

Define

$$\begin{aligned} U^+(x, t) &= \phi(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1 \epsilon \exp(\frac{t}{\epsilon^2}), \ v_0(x) - \gamma_2 \epsilon \exp(\frac{t}{\epsilon^2})), \\ V^+(x, t) &= \psi(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1 \epsilon \exp(\frac{t}{\epsilon^2}), v_0(x) - \gamma_2 \epsilon \exp(\frac{t}{\epsilon^2})), \\ U^-(x, t) &= \phi(\frac{t}{\epsilon^3}; u_0(x) - \gamma_1 \epsilon \exp(\frac{t}{\epsilon^2}), \ v_0(x) + \gamma_2 \epsilon \exp(\frac{t}{\epsilon^2})), \\ V^-(x, t) &= \psi(\frac{t}{\epsilon^3}; u_0(x) - \gamma_1 \epsilon \exp(\frac{t}{\epsilon^2}), \ v_0(x) + \gamma_2 \epsilon \exp(\frac{t}{\epsilon^2})), \end{aligned} \tag{19}$$

where $\gamma_1, \gamma_2 > 0$ is constants determined later.

Lemma 5. *There exists $\gamma_1 > 0$, $\gamma_2 > 0$ such that for sufficiently small $\epsilon > 0$, the functions $U^\pm, V^\pm$ are pair of upper and lower solutions of* (3) *for $0 \le t \le \epsilon^2$.*

Proof. We first consider the case where

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0.$$

on $\partial\Omega$. Consequently, we have

$$\frac{\partial U^+}{\partial n} = \frac{\partial U^-}{\partial n} = \frac{\partial V^+}{\partial n} = \frac{\partial V^-}{\partial n} = 0$$

on $\partial\Omega$. The general case will be considered in Remark below. We will show that (U^+, V^+), (U^-, V^-) satisfy inequalities (16) and (17) respectively. We set

$$\begin{cases} \mathcal{L}_1(u,v) = u_t - \Delta u - (a-u)u + \dfrac{buv}{\epsilon^3}, \\ \mathcal{L}_2(u,v) \;= v_t - d\Delta v - (d-v)v + \dfrac{cuv}{\epsilon^3}. \end{cases} \tag{20}$$

Our goal is to show that

$$\mathcal{L}_1(U^+, V^+) \ge 0, \quad \mathcal{L}_1(U^-, V^-) \le 0, \quad \mathcal{L}_2(U^+, V^+) \le 0, \quad \mathcal{L}_2(U^-, V^-) \ge 0.$$

We will only prove $\mathcal{L}_1(U^+, V^+) \ge 0$ and $\mathcal{L}_2(U^+, V^+) \le 0$, since the other inequalities can be proved similarly. $\mathcal{L}_1(U^+, V^+)$ and $\mathcal{L}_2(U^+, V^+)$ are given by

$$\begin{aligned} \mathcal{L}_1(U^+, V^+) =& \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2})(\gamma_1\phi_\xi - \gamma_2\phi_\eta) \\ & - \phi_{\xi\xi}|\nabla u_0(x)|^2 - 2\phi_{\xi\eta}\nabla u_0(x)\nabla v_0(x) - \phi_{\eta\eta}|\nabla v_0(x)|^2 \\ & - \phi_\xi\Delta u_0(x) - \phi_\eta\Delta v_0(x) - (a-\phi)\phi \\ =& \phi_\xi(\gamma_1 \cdot \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2}) - R_1) + (-\phi_\eta)(\gamma_2 \cdot \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2}) - R_2), \end{aligned} \tag{21}$$

and

$$\begin{aligned}\mathcal{L}_2(U^+,V^+) =& \frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2})(\gamma_1\psi_\xi - \gamma_2\psi_\eta)\\ &- D\psi_{\xi\xi}|\nabla u_0(x)|^2 - 2D\psi_{\xi\eta}\nabla u_0(x)\nabla v_0(x) - D\psi_{\eta\eta}|\nabla v_0(x)|^2\\ &- D\psi_\xi\Delta u_0(x) - D\psi_\eta\Delta v_0(x) - (d-\psi)\psi\\ =& D\psi_\xi(\frac{\gamma_1}{D}\cdot\frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2}) - R_3) + (-D\psi_\eta)(\frac{\gamma_2}{D}\cdot\frac{1}{\epsilon}\exp(\frac{t}{\epsilon^2}) - R_4).\end{aligned} \tag{22}$$

Here R_i $(i = 1,\cdots,4)$ are

$$\begin{aligned} R_1 &= \tfrac{\phi}{\phi_\xi}(a-\phi) + \Delta u_0 + \tfrac{\phi_{\xi\xi}}{\phi_\xi}|\nabla u_0|^2 + 2\tfrac{\phi_{\xi\eta}}{\phi_\xi}\nabla u_0\nabla v_0 + \tfrac{\phi_{\eta\eta}}{\phi_\xi}|\nabla v_0|^2,\\ R_2 &= -\Delta v_0, \quad R_3 = \Delta u_0,\\ R_4 &= -\tfrac{\psi}{\psi_\eta}(d-\psi) - \Delta v_0 - \tfrac{\psi_{\xi\xi}}{\psi_\eta}|\nabla u_0|^2 - 2\tfrac{\psi_{\xi\eta}}{\psi_\eta}\nabla u_0\nabla v_0 - \tfrac{\psi_{\eta\eta}}{\psi_\eta}|\nabla v_0|^2.\end{aligned}$$

In above expressions, we also use several notations as follows:

$$\begin{aligned}\phi &= \phi(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}), v_0(x) - \gamma_1\epsilon\exp(\frac{t}{\epsilon^2})),\\ \phi_\xi &= \frac{\partial\phi}{\partial\xi}(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}), v_0(x) - \gamma_1\epsilon\exp(\frac{t}{\epsilon^2})),\\ \phi_{\xi\xi} &= \frac{\partial^2\phi}{\partial\xi^2}(\frac{t}{\epsilon^3}; u_0(x) + \gamma_1\epsilon\exp(\frac{t}{\epsilon^2}), v_0(x) - \gamma_1\epsilon\exp(\frac{t}{\epsilon^2})), etc.\end{aligned}$$

By Lemma 2, we can see followings: ϕ_ξ, $-\phi_\eta$ are positive and ψ_ξ, $-\psi_\eta$ are negative. Additionally, from Lemma 4, we can observe that if ϵ is sufficiently small, then

$$\begin{aligned}\max\{|R_1|,|R_3|\} &\le M_1\Big(\frac{1}{\inf_{x\in\Omega}u_0(x)} + \frac{t}{\epsilon^3}\Big),\\ \max\{|R_2|,|R_4|\} &\le M_2\Big(\frac{1}{\inf_{x\in\Omega}v_0(x)} + \frac{t}{\epsilon^3}\Big),\end{aligned} \tag{23}$$

for $x \in \Omega$, and $t > 0$. Here M_1, M_2 are positive constants and, depend on bounds of $|u_0|$, $|\nabla u_0|$, $|\Delta u_0|$, $|v_0|$, $|\nabla v_0|$, $|\Delta v_0|$ in Ω. Therefore applying (23) to (21) and (22), we obtain $\mathcal{L}_1(U^+,V^+) \ge 0$ for sufficiently large γ_1, $\gamma_2 > 0$ independently of $\epsilon > 0$. The proof is complete. □

Lemma 6. *There exist $C_1, C_2 > 0$ such that for sufficiently small $\epsilon > 0$, the solution (u_ϵ, v_ϵ) of (3) satisfies the following estimate:*

$$\Big|U^\pm(x,t) - \phi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\Big| < C_1\epsilon,\ (x,t)\in\Omega\times(0,\epsilon^2),$$

$$\Big|V^\pm(x,t) - \psi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\Big| < C_1\epsilon\ \ (x,t)\in\Omega\times(0,\epsilon^2),$$

and

$$\left|U^{\pm}(x,\epsilon^2) - \max\left\{\frac{\omega(x)}{c}, 0\right\}\right| < C_2\epsilon, \quad x \in \Omega$$

$$\left|V^{\pm}(x,\epsilon^2) - \max\left\{0, -\frac{\omega(x)}{b}\right\}\right| < C_2\epsilon \; x \in \Omega.$$

Moreover, for any $\beta > 0$, there exist $C_3 > 0$ such that for sufficiently small $\epsilon > 0$,

$$|U^{\pm}(x,\epsilon^2)| < C_3\epsilon^{\beta}, \; in \; \{x \in \Omega\backslash\overline{R(0)} \; ; \; \omega(x) < -\beta\epsilon|\log\epsilon|\},$$

$$|V^{\pm}(x,\epsilon^2)| < C_3\epsilon^{\beta}, \; in \; \{x \in R(0) \; ; \; \omega(x) > \beta\epsilon|\log\epsilon|\}.$$

Proof of Theorem 4.1. If we apply Proposition 1 to Lemma 5, then for $(x,t) \in \Omega \times [0,\epsilon^2]$, $U^-(x,t) \le u_\epsilon(x,t) \le U^+(x,t)$, $V^-(x,t) \ge v_\epsilon(x,t) \ge V^+(x,t)$ and especially,

$$\left|u_\epsilon(x,t) - \phi\Big(\frac{t}{\epsilon^3}; u_0(x), v_0(x)\Big)\right| \le \max_{U^+,U^-} \left|U^{\pm}(x,t) - \phi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\right|,$$

$$\left|v_\epsilon(x,t) - \phi\Big(\frac{t}{\epsilon^3}; u_0(x), v_0(x)\Big)\right| \le \max_{V^+,V^-} \left|V^{\pm}(x,t) - \psi\Big(\frac{t}{\epsilon^3}, u_0(x), v_0(x)\Big)\right|.$$

and

$$\left|u_\epsilon(x,\epsilon^2) - \max\left\{\frac{\omega(x)}{c}, 0\right\}\right| \le \max_{U^+,U^-} \left|U^{\pm}(x,\epsilon^2) - \max\left\{\frac{\omega(x)}{c}, 0\right\}\right|,$$

$$\left|v_\epsilon(x,\epsilon^2) - \max\left\{0, -\frac{\omega(x)}{b}\right\}\right| \le \max_{V^+,V^-} \left|V^{\pm}(x,t) - \max\left\{0, -\frac{\omega(x)}{b}\right\}\right|.$$

Therefore from Lemma 6, we obtain the proof. □

Proof of Lemma 6. We only show the inequalities for U^+, since the other inequalities are shown in the same way. Using mean value theorem, we have

$$\begin{aligned}
&\left|U^+(x,t) - \phi\Big(\frac{t}{\epsilon^3} \; ; \; u_0(x), v_0(x)\Big)\right| \\
\le&\epsilon\gamma_1 \exp\frac{t}{\epsilon^2}\phi_\xi\Big(\frac{t}{\epsilon^3} \; ; \; u_0(x) + \epsilon\theta_1\gamma_1\exp\frac{t}{\epsilon^2}, v_0(x) - \epsilon\theta_2\gamma_2\exp\frac{t}{\epsilon^2}\Big) \qquad (24)\\
&- \epsilon\gamma_2 \exp\frac{t}{\epsilon^2}\phi_\eta\Big(\frac{t}{\epsilon^3} \; ; \; u_0(x) + \epsilon\theta_3\gamma_1\exp\frac{t}{\epsilon^2}, v_0(x) - \epsilon\theta_4\gamma_2\exp\frac{t}{\epsilon^2}\Big),
\end{aligned}$$

for some $0 \le \theta_i \le 1$ $(i = 1, 2, 3, 4)$. It follows from Lemma 2 below and (24) that there exists $C_1' > 0$ such that

$$\left| U^+(x,t) - \phi\Big(\frac{t}{\epsilon^3}\ ;\ u_0(x), v_0(x)\Big) \right| \le C_1'\epsilon, \quad \text{for } (x,t) \in \Omega \times [0, \epsilon^2]. \tag{25}$$

Hereafter, C_i' $(i \in \mathbb{N})$ denotes a positive constant independent of $\epsilon > 0$. Set $t = \epsilon^2$. It holds that

$$\begin{aligned} \left| U^+(x,\epsilon^2) - \max\Big\{\frac{\omega(x)}{c}, 0\Big\} \right| \le &\left| U^+(x,\epsilon^2) - \phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x), v_0(x)\Big) \right| \\ &+ \left| \phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x), v_0(x)\Big) - \max\Big\{\frac{\omega(x)}{c}, 0\Big\} \right| \end{aligned} \tag{26}$$

By (i) of Lemma 1,

$$\left| \phi\Big(\frac{1}{\epsilon}; u_0(x), v_0(x)\Big) - \max\Big\{\frac{\omega(x)}{c}, 0\Big\} \right| < c\epsilon$$

in Ω. Hence combining this inequality, (25), and (26) we obtain the second inequality for U^+ in Lemma 6. Now let us consider the third inequality for U^+ in Lemma 6. Fix $\beta > 0$, and recall (24) and (26). Using of (ii) of Lemma 1 and Lemma 3 at $t = \epsilon^2$ $(\tau = \epsilon^{-1})$, if $\omega(x) < -\beta\epsilon|\log\epsilon|$, then for sufficiently small $\epsilon > 0$,

$$\begin{aligned} |U^+(x,\epsilon^2)| \le &\left| U^+(x,\epsilon^2) - \phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x), v_0(x)\Big) \right| + \left| \phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x), v_0(x)\Big) \right| \\ \le &\epsilon\gamma_1 e \phi_\xi\Big(\frac{1}{\epsilon}\ ;\ u_0(x) + \epsilon\theta_1\gamma_1 e, v_0(x) - \epsilon\theta_2\gamma_2 e\Big) \\ &- \epsilon\gamma_2 e \phi_\eta\Big(\frac{1}{\epsilon}\ ;\ u_0(x) + \epsilon\theta_3\gamma_1 e, v_0(x) - \epsilon\theta_4\gamma_2 e\Big) \\ &+ \left| \phi\Big(\frac{1}{\epsilon}\ ;\ u_0(x), v_0(x)\Big) \right| \\ \le &\epsilon(\gamma_1 + \gamma_2)e \cdot M_0' \frac{1}{\epsilon} \sup_{x\in\Omega} u_0(x) \exp\frac{\omega(x) + O(\epsilon)}{\epsilon} \\ &+ \sup_{x\in\Omega} u_0(x) \exp\frac{\omega(x)}{\epsilon} \\ \le &C_3 \exp(-\beta|\log\epsilon|) \le C_3\epsilon^\beta. \end{aligned} \tag{27}$$

Therefore we obtain the inequality for U^+. The proof is complete. □

Remark We finally consider the case where $u_0(x)$ and $v_0(x)$ do not satisfy the Neumann zero boundary conditions. Since ϵ is small, $d(x, \partial\Omega)$ is smooth in $\{x \in \Omega \mid dist(x, \partial\Omega) \leq 3\epsilon\}$. Set

$$\theta(x) = \begin{cases} 0 & \text{if} \quad dist(x, \partial\Omega) \geq 2\epsilon, \\ \\ 1 & \text{if} \quad dist(x, \partial\Omega) \leq \epsilon \end{cases} \tag{28}$$

Choose $\eta_1, \eta_2, \eta_3, \eta_4$ such that

$$-\eta_1 < \min_{x\in\partial\Omega}\left\{\frac{\partial u_0(x)}{\partial n}, 0\right\} \leq 0,\ \eta_2 > \max_{x\in\partial\Omega}\left\{\frac{\partial v_0(x)}{\partial n}, 0\right\} \geq 0,$$

$$\eta_3 > \max_{x\in\partial\Omega}\left\{\frac{\partial u_0(x)}{\partial n}, 0\right\} \geq 0, \quad -\eta_4 < \min_{x\in\partial\Omega}\left\{\frac{\partial v_0(x)}{\partial n}, 0\right\} \leq 0,$$

we define

$$u_0^+(x) = u_0(x) + \eta_1 dist(x, \partial\Omega)\theta(x),$$

$$v_0^+(x) = v_0(x) - \eta_2 dist(x, \partial\Omega)\theta(x),$$

$$u_0^-(x) = u_0(x) - \eta_3 dist(x, \partial\Omega)\theta(x),$$

$$v_0^-(x) = v_0(x) + \eta_4 dist(x, \partial\Omega)\theta(x),$$

If we replace $u_0(x)$ and $v_0(x)$ in (U^+, V^+) in (19) by u_0^+ and v_0^+, respectively, we obtain the upper solutions. On the other hand, if we replace $u_0(x)$ and $v_0(x)$ in (U^-, V^-) in (19) by u_0^- and v_0^-, respectively, we obtain the lower solutions. The proof is completed by repeating the proof of Lemmas 5 and 6. Almost all the arguments are the same as the case of Neumann zero except for two differences. One is that $\Delta u_0^\pm, \Delta v_0^\pm$ are $O(1/\epsilon)$ in Lemma 5. Therefore, we obtain

$$\begin{aligned} \max\{|R_1|, |R_3|\} &\leq M_3\Big(\frac{1}{\epsilon} + \frac{1}{\inf_{x\in\Omega} u_0(x)} + \frac{t}{\epsilon^3}\Big), \\ \max\{|R_2|, |R_4|\} &\leq M_4\Big(\frac{1}{\epsilon} + \frac{1}{\inf_{x\in\Omega} v_0(x)} + \frac{t}{\epsilon^3}\Big), \end{aligned} \tag{29}$$

insted of (23). The other difference is that we need to check the Neumann zero boundary condition of u_0^+, v_0^+, u_0^-, and v_0^-. To show this we remark that

$$\theta(x) = 1, \ \frac{\partial\theta}{\partial n} \geq 0, \ \text{ on } \ \partial\Omega.$$

Then it follows that

$$\frac{\partial}{\partial n}(u_0^+(x)) = \frac{\partial u_0(x)}{\partial n} + \eta_1\theta(x) + \eta_1 dist(x, \partial\Omega)\frac{\partial\theta}{\partial n} = 0 \ \text{ on } \ \partial\Omega.$$

The conditions for v_0^+, u_0^-, v_0^- can be obtained in the same way.

6. Motion of interface

In this section we construct another pair of upper and lower solutions for the second stage, motion of interface. This upper and lower solutions $(u^\pm, v^\pm)$ has interface near $\Gamma(t)$, the solution of the free boundary problem (2).

We first construct upper and lower solutions $(U_{in}^\pm, V_{in}^\pm)$ in a tubular neighborhood of $\Gamma(t)$ by modifying the first two terms of the inner expansion. After that we construct an upper and a lower solution $(U_{out}^\pm, V_{out}^\pm)$ outside the tubular neighborhood using the first term of outer expansion. Then we match $(U_{in}^\pm, V_{in}^\pm)$ and $(U_{out}^\pm, V_{out}^\pm)$, then obtain $(u^\pm, v^\pm)$. Once $(u^\pm, v^\pm)$ are obtained, they will later be combined with another set of upper and lower solutions $(U^\pm, V^\pm)$ that take care of the generation of interface at the initial stage.

6.1. *An upper and a lower solution near the interface*

We define an upper and a lower solutions in the following form:

$$
\begin{aligned}
U_{in}^+(x,t) &= \epsilon U_1\left(\frac{\tilde{d}(x,t)}{\epsilon} - \eta(t), \sigma\right) + \epsilon^2 U_2\left(\frac{\tilde{d}(x,t)}{\epsilon} - \eta(t), \sigma, t\right) + \epsilon^3 q(t),\\
V_{in}^+(x,t) &= \epsilon V_1\left(\frac{\tilde{d}(x,t)}{\epsilon} - \eta(t), \sigma\right) + \epsilon^2 V_2\left(\frac{\tilde{d}(x,t)}{\epsilon} - \eta(t), \sigma, t\right) - \epsilon^3 \hat{q}(t),\\
U_{in}^-(x,t) &= \epsilon U_1\left(\frac{\tilde{d}(x,t)}{\epsilon} + \eta(t), \sigma\right) + \epsilon^2 U_2\left(\frac{\tilde{d}(x,t)}{\epsilon} + \eta(t), \sigma, t\right) - \epsilon^3 q(t),\\
V_{in}^-(x,t) &= \epsilon V_1\left(\frac{\tilde{d}(x,t)}{\epsilon} + \eta(t), \sigma\right) + \epsilon^2 V_2\left(\frac{\tilde{d}(x,t)}{\epsilon} + \eta(t), \sigma, t\right) + \epsilon^3 \hat{q}(t).
\end{aligned}
$$

Here $\tilde{d}(x,t)$ is defined in (14),

$$
\eta(t) = \left(\log\frac{1}{\epsilon}\right)\gamma \exp(Mt), \quad q(t) = \sigma \exp(Mt), \quad \hat{q}(t) = \hat{\sigma}\exp(Mt),
$$

where $\gamma, \sigma, \hat{\sigma}$ and M are positive constants to be determined appropriately, and (U_1, V_1) satisfies (11) and (U_2, V_2) satisfies

$$\begin{cases} -U_{2\xi\xi} + c(U_1V_2 + U_2V_1) = -U_{1\xi}(d_t - \Delta d) & -\infty < \rho < +\infty, \\ -DV_{2\xi\xi} + b(U_1V_2 + U_2V_1) = -V_{1\xi}(d_t - D\Delta d) & -\infty < \rho < +\infty, \\ (U_2(t,\rho,\sigma),\ V_2(t,\rho,\sigma)) = (0,\ 0) & \text{as } \rho \to -\infty, \\ (U_2(t,\rho,\sigma),\ V_2(t,\rho,\sigma)) = (0,\ 0) & \text{as } \rho \to +\infty. \end{cases} \tag{30}$$

(30) is obtained by the formal argument based on the *matched asymptotic expansion.* The following lemma assures the existence of the first and second term of upper and lower solutions, whose proofs are omitted.

Lemma 1. *(i) There exists a unique positive solution of* (11).
(ii) There exists a solution of (30).

Since the first two terms of $(U_{in}^{\pm}, V_{in}^{\pm})$ are determined, we choose appropriate q and $\hat{q}$ so that $(U_{in}^{\pm}, V_{in}^{\pm})$ are an upper and lower solutions.

6.2. *Upper and lower solutions away from the interface*

In this subsection we will construct upper and lower solutions away from the interface modifying the first term of outer expansion.

Let g be a smooth function satisfying

$$g(s) = 0 \ \text{ if } \ s < 0, \quad g(s) = 1 \ \text{ if } \ s > 1$$

$$g'(0) = g'(1) = 0, \quad g'(s) \geq 0 \ \text{ for } \ 0 \leq s \leq 1$$

and set

$$\lambda_1(s) = g\left(\frac{s}{\epsilon} + \tilde{R}|\log\epsilon|\right), \quad \lambda_2(s) = g\left(-\frac{s}{\epsilon} - \tilde{R}|\log\epsilon|\right).$$

Moreover let δ satisfy $0 < \delta \ll d^*$ and define

$$H(s) = \begin{cases} -\beta\epsilon|\log\epsilon|(s+\delta)^2 + \beta\delta\tilde{R}\epsilon^2|\log\epsilon|^2 + \dfrac{\beta\delta^2}{\tilde{R}}\epsilon|\log\epsilon|, \\ \qquad\qquad (-\delta - \tilde{R}\epsilon|\log\epsilon| \leq s \leq -\tilde{R}\epsilon|\log\epsilon|), \\ \beta\delta\tilde{R}\epsilon^2|\log\epsilon|^2 + \dfrac{\beta\delta^2}{\tilde{R}}\epsilon|\log\epsilon|, \quad (s \leq -\delta - \tilde{R}\epsilon|\log\epsilon|). \end{cases}$$

Now we will define upper and lower solutions in the following form:

$$U^{+}_{out}(x,t)=\begin{cases} u^*(x,t)+\epsilon|\log\epsilon|\alpha\exp(Lt)-H(\tilde{d}(x,t)), & d(x,t)\le-\tilde{R}\epsilon|\log\epsilon| \\ (1-\lambda_1(\tilde{d}(x,t)))U^{+}_{\epsilon}+\lambda_1(\tilde{d}(x,t))\epsilon^4, & d(x,t)>\tilde{R}\epsilon|\log\epsilon| \end{cases}$$

$$V^{+}_{out}(x,t)=\begin{cases} 0, & d(x,t)\le-\tilde{R}\epsilon|\log\epsilon|, \\ v^*(x,t)-\epsilon|\log\epsilon|\alpha\exp(Lt)+H(-\tilde{d}(x,t)), & d(x,t)>\tilde{R}\epsilon|\log\epsilon| \end{cases}$$

$$U^{-}_{out}(x,t)=\begin{cases} u^*(x,t)-\epsilon|\log\epsilon|\alpha\exp(Lt)+H(\tilde{d}(x,t)), & d(x,t)\le-\tilde{R}\epsilon|\log\epsilon| \\ 0, & d(x,t)>\tilde{R}\epsilon|\log\epsilon| \end{cases}$$

$$V^{-}_{out}(x,t)=\begin{cases} (1-\lambda_2(\tilde{d}(x,t)))W^{-}_{\epsilon}+\lambda_2(\tilde{d}(x,t))\epsilon^4, & d(x,t)\le-\tilde{R}\epsilon|\log\epsilon| \\ v^*(x,t)+\epsilon|\log\epsilon|\alpha\exp(Lt)-H(-\tilde{d}(x,t)), & d(x,t)>\tilde{R}\epsilon|\log\epsilon|. \end{cases}$$

Here $\alpha,\beta,\tilde{R}$ are positive constants to be specified appropriately.

$(U^{\pm}_{out},V^{\pm}_{out})$ are chosen so as to satisfy the following condition.

- $(U^{\pm}_{out},V^{\pm}_{out})$ is an upper and a lower solution for $|d(x,t)|>\tilde{R}\epsilon|\log\epsilon|$.
- The entire upper and lower solution given by (31) below is not smooth for $|d(x,t)|=\tilde{R}\epsilon|\log\epsilon|$. (We need to care about the derivative of $(U^{\pm}_{in},V^{\pm}_{in})$ and $(U^{\pm}_{out},V^{\pm}_{out})$ at $|d(x,t)|=\tilde{R}\epsilon|\log\epsilon|$.) $(U^{\pm}_{out},V^{\pm}_{out})$ are determined so that $(u^{\pm},v^{\pm})$ given below become an upper and a lower solutions.
- $(U^{\pm}_{out},V^{\pm}_{out})$ has the following estimate.
$$(U^{\pm}_{out},V^{\pm}_{out})=(u^*,v^*)+O(\epsilon|log\epsilon|).$$

6.3. *Entire solution for the motion of interface*

The entire solution is given by

$$(u^{\pm},v^{\pm})=\begin{cases} (U^{\pm}_{in},V^{\pm}_{in}) & |d(x,t)|\le\tilde{R}\epsilon|\log\epsilon|, \\ (U^{\pm}_{out},V^{\pm}_{out}) & |d(x,t)|>\tilde{R}\epsilon|\log\epsilon|. \end{cases} \tag{31}$$

Let $(u^*_m.v^*_m,R_m)$ be a solution to (2) with an arbitrary initial data $(u^*_m(x,0).v^*_m(x,0))$. Assume

Assumption 6.1. *Assumption 3.2 and Assumption 3.3 hold with replacing* (u^*, v^*, R) *by* (u_m^*, v_m^*, R_m).

Set $\Gamma_m = \partial R_m$, and let $\tilde{d}_m$ be difined by (15) with Γ replaced by Γ_m. They give the following result:

Theorem 6.1. (*Iida-Karali-Mimura-Nakashima-Yanagida* [7]) *For any sufficiently large* $\gamma > 0$ *and any sufficiently small* $\sigma > 0$, *there exist* $C_5, C_6, C_7 > 0$ *such that for sufficiently small* $\epsilon > 0$ *and the initial data satisfying*

$$|u_\epsilon(x,0) - u_m^*(x,0)| < C_5\epsilon|\log\epsilon|, \qquad |v_\epsilon(x,0) - v_m^*(x,0)| < C_5\epsilon|\log\epsilon|. \tag{32}$$

$$|u_\epsilon(x,0)| < C_5\exp(-\frac{\sigma|\tilde{d}_m(x,0)|}{\epsilon}), \qquad \textit{for}\ \{x \in \Omega\backslash\overline{R_m(0)}\ ;\ |\tilde{d}_m(x,0)| > \gamma\epsilon|\log\epsilon|\},$$

$$|v_\epsilon(x,0)| < C_5\exp(-\frac{\sigma|\tilde{d}_m(x,0)|}{\epsilon}), \qquad \textit{for}\ \{x \in R_m(0)\ ;\ |\tilde{d}_m(x,0)| > \gamma\epsilon|\log\epsilon|\}, \tag{33}$$

it holds that

$$|u_\epsilon(x,t) - u_m^*(x,t)| < C_6\epsilon|\log\epsilon|, \qquad |v_\epsilon(x,t) - v_m^*(x,t)| < C_6\epsilon|\log\epsilon|.$$

$$|u_\epsilon(x,t)| < C_6\exp(-\frac{\sigma|\tilde{d}_m(x,t)|}{\epsilon}), \qquad \textit{for}\ \{x \in \Omega\backslash\overline{R_m(t)}\ ;\ |\tilde{d}_m(x,t)| > C_7\epsilon|\log\epsilon|\},$$

$$|v_\epsilon(x,t)| < C_6\exp(-\frac{\sigma|\tilde{d}_m(x,t)|}{\epsilon})\ , \qquad \textit{for}\ \{x \in R_m(t)\ ;\ |\tilde{d}_m(x,t)| > C_7\epsilon|\log\epsilon|\}.$$

7. Proof of Theorem 1

Combining the estimate in Theorem 2 and expressions of $(u^\pm, v^\pm)$, we have

$$u^-(x,\epsilon^2) \le U^-(x,\epsilon^2) \le U^+(x,\epsilon^2) \le u^+(x,\epsilon^2),$$

$$v^-(x,\epsilon^2) \ge V^-(x,\epsilon^2) \ge V^+(x,\epsilon^2) \ge v^+(x,\epsilon^2).$$

This and Theorems 2 and 3 implies that for arbitrarily chosen initial data satisfying Assumption 1, the solution of (3) stays between (U^-, V^-) and (U^+, V^+) for $t \in (0, \epsilon^2]$, and stays between (u^-, v^-) and (u^+, v^+) for $t \in [\epsilon^2, T]$. Using the estimate in Theorem 3, the proof is completed.

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RAYLEIGH-BÉNARD CONVECTION IN A RECTANGULAR DOMAIN

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1. Introduction

There has been a lot of studies on a convection patterns in the Rayleigh-Bénard problem. Consider the Bussinesq approximation (Oberbeck-Boussinesq model) with up-down symmetric boundary conditions. It is a well-known fact that a hexagonal and a roll patterns appear when the Rayleigh number exceeds its critical value and only the roll pattern is stable. See for example [14]. In [9] they obtained general bifurcational structure under the up-down symmetric boundary conditions including the Boussinesq approximation. However, both of them have not obtained the eigenvalues about the mixed mode solutions such as the hexagonal pattern. On the other hand, it is rather easy to obtain a hexagonal pattern under the same up-down symmetric situation by a 3D numerical simulation from small random initial data. Therefore we shall exactly calculate the coefficients of the cubic normal form about the critical point where both the roll and hexagonal patterns appear and study the dynamics of them. By the cubic normal form we can study the invariant torus which includes the fixed points corresponding to the hexagonal patterns inside. To determine the motion on the torus we need to calculate the normal form up to higher order. But here, we only discuss the stability of the invariant torus and calculate the eigenvalues for the transversal direction to the torus. We can show that it is true that the invariant torus of the hexagonal pattern has positive eigenvalues but they are small compared to the absolute value of the negative ones. The invariant torus is a saddle for its transversal directions and it will take quite a long time to observe unstable dynamics. It is consistent to the classical theoretical results and also the fact that we frequently observe "unstable" hexagonal patterns in numerical simulations. Notice that a hexagonal pattern can be stable in the case when the two boundary conditions are different each other so that they break the up-down symmetry. In fact the normal form has quadratic terms which correspond to the hexagonal resonance([4]).

In the study of hexagonal patterns it is convenient to consider a rectangular fluid container with particular size so that there exist 3 roll solutions which have the same critical wave length and cross each other by the angle of 120 degrees. We can consider more general situations by changing the size of the container. It is still difficult to study the reduced dynamics for all the variations of the system size. However, we found there are stable mixed mode solutions (patchwork quilt type) by taking the size of the container and the Prandtl number appropriately. We shall also consider the 2-dimensional problem, where the flow is assumed to depend only on (x, z)-directions. We show that the pure mode solution (roll) is unstable while a mixed mode solution is stable when the Prandtl number is small. Part of these results are already anounced in [13]. Here, we shall take a different algorithm to calculate the normal form and mention more detailed results.

We are specially interested in small rectangular container for the bifurcation analysis. Since otherwise if we consider the large system size the bifurcation structure for the stationary solutions might become close to that of the Ginzburg-Landau equation. Notice that the Ginzburg-Landau equation is considered to be the reduced simplified model for the Oberbeck-Boussinesq model. And only the pure roll solutions are stable for the Ginzburg-Landau equations, since their nonlinear terms are cubic. See [4] in detail.

2. The Rayleigh Bénard problem

We consider the Boussinesq approximation for the Rayleigh-Bénard convection. Variation equations about the conductive state can be written in the following non-dimensional form.

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\nabla p + R\theta \mathbf{e}_z + \Delta \boldsymbol{u} \\ \theta_t + (\boldsymbol{u}\cdot\nabla)\theta = (w + \Delta\theta)/P \\ \nabla\cdot\boldsymbol{u} = 0 \end{cases} \tag{1}$$

Here, $\boldsymbol{u} = (u, v, w)$ is a velocity vector and $\mathbf{e}_z = (0, 0, 1)$. Equations (1) are considered in the region $\mathbf{R}^2 \times (0, 1)$. Two constants R and P are called the Rayleigh and the Prandtl numbers, respectively.

Let the boundary conditions be free-slip for both the top and the bottom:

$$u_z = v_z = w = \theta = 0 \quad (z = 0, 1). \tag{2}$$

These boundary conditions simplify the normal form calculation. In fact, we can naturally extend the functions $\boldsymbol{u}, \theta, p$ on $z \in [0, 1]$ to the periodic functions for z-direction whose periods are 2 from the following reasons. First, extend u, v, p as even functions on $[-1, 1]$ and w, θ as odd functions on $[-1, 1]$. Then from the boundary conditions we know that these functions are C^1-continuous at $z = 0$ and they turned out to be C^2 from the equations. Second, we further extend those functions to $z \in \mathbf{R}$ so that they become 2-periodic, i.e., $u(z) = u(z + 2)$. Similar argument shows that the extended functions are also C^2 at $z = 1$. Therefore solutions to equations (1) with the boundary conditions (2) are included in the space of 2-periodic functions for z-direction. Conversely, if a 2-periodic functions for z-direction satisfying (1) and moreover their u, v, p components are even and w, θ components are odd then they are solutions of equations (1) and (2) . This is called hidden symmetry relating to the Neumann boundary conditions (see [8]).

Now, let us assume the periodicity for both x and y directions with the periods $(2\pi/\alpha, 2\pi/\beta)$. We can finally represent each unknown variables by the Fourier expansion.

$$u = \sum_{(m,n,l)\in\mathbf{Z}^3} u_{m,n,l}\; e^{i(m\alpha x + n\beta y + l\pi z)}, \; v = \sum_{(m,n,l)\in\mathbf{Z}^3} v_{m,n,l}\; e^{i(m\alpha x + n\beta y + l\pi z)},$$
$$w = \sum_{(m,n,l)\in\mathbf{Z}^3} w_{m,n,l}\; e^{i(m\alpha x + n\beta y + l\pi z)}, \; \theta = \sum_{(m,n,l)\in\mathbf{Z}^3} \theta_{m,n,l}\; e^{i(m\alpha x + n\beta y + l\pi z)},$$
$$p = \sum_{(m,n,l)\in\mathbf{Z}^3} p_{m,n,l}\; e^{i(m\alpha x + n\beta y + l\pi z)}.$$

We use an abbreviated notation $\mathbf{m} = (m, n, l)$ for the mode vector and $u_{\mathbf{m}} = u_{m,n,l}$.

Since all the unknown variables are real valued and their Fourier coefficients have the Hermitian symmetry: $u_{\mathbf{m}} = \overline{u_{-\mathbf{m}}}$. They also satisfy the following properties which correspond to the even and odd symmetry of the corresponding functions.

$$\begin{aligned} u_{m,n,l} &= u_{m,n,-l}, \\ v_{m,n,l} &= v_{m,n,-l}, \\ w_{m,n,l} &= -w_{m,n,-l}, \\ \theta_{m,n,l} &= -\theta_{m,n,-l}, \\ p_{m,n,l} &= p_{m,n,-l} \end{aligned} \tag{3}$$

Now we rewrite the equations (1) by using the Fourier coefficients as follows.

$$\begin{pmatrix} \dot{u_{\mathbf{m}}} \\ \dot{v_{\mathbf{m}}} \\ \dot{w_{\mathbf{m}}} \\ \dot{\theta_{\mathbf{m}}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega_{\mathbf{m}}^2 & 0 & 0 & 0 & -im\alpha \\ 0 & -\omega_{\mathbf{m}}^2 & 0 & 0 & -in\beta \\ 0 & 0 & -\omega_{\mathbf{m}}^2 & R & -il\pi \\ 0 & 0 & 1/P & -\omega_{\mathbf{m}}^2/P & 0 \\ im\alpha & in\beta & il\pi & 0 & 0 \end{pmatrix} \begin{pmatrix} u_{\mathbf{m}} \\ v_{\mathbf{m}} \\ w_{\mathbf{m}} \\ \theta_{\mathbf{m}} \\ p_{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} \{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)v\}_{m} \\ \{(\boldsymbol{u}\cdot\nabla)w\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)\theta\}_{\mathbf{m}} \\ 0 \end{pmatrix} \tag{4}$$

Here, $\omega_{\mathbf{m}}^2 = m^2\alpha^2 + n^2\beta^2 + l^2\pi^2$.

Note that the problem (1) and (2) admit another type of symmetry: up-down symmetry which means that they are invariant under the mapping:

$$(u, v, w, \theta, p)(t, x, y, z) \mapsto (u, v, -w, -\theta, p)(t, x, y, 1 - z).$$

Equation (4) inherits up-down symmetry from the original problem. More precisely, equation (4) is invariant under the transformation:

$$\begin{aligned} u_{m,n,l} &\to (-1)^l\, u_{m,n,l}, \\ v_{m,n,l} &\to (-1)^l\, v_{m,n,l}, \\ w_{m,n,l} &\to (-1)^l\, w_{m,n,l}, \\ \theta_{m,n,l} &\to (-1)^l\, \theta_{m,n,l}, \\ p_{m,n,l} &\to (-1)^l\, p_{m,n,l}. \end{aligned} \tag{5}$$

Equation (4) with the even-odd symmetry (3) is equivalent to (1) with the boundary condition (2). Moreover the Fourier coefficients for the pressure p and w can be eliminated by the fifth equation of (4) and finally we obtain the following system of ordinary differential equations for $\mathbf{m} \neq \mathbf{0}$.

$$\begin{pmatrix} \dot{u_{\mathbf{m}}} \\ \dot{v_{\mathbf{m}}} \\ \dot{\theta_{\mathbf{m}}} \end{pmatrix} = M_{\mathbf{m}} \begin{pmatrix} u_{\mathbf{m}} \\ v_{\mathbf{m}} \\ \theta_{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} \{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)v\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)\theta\}_{\mathbf{m}} \end{pmatrix} + k_{\mathbf{m}} \begin{pmatrix} m\alpha \\ n\beta \\ 0 \end{pmatrix} \quad (l \neq 0) \tag{6}$$

$$\begin{aligned} \begin{pmatrix} \dot{u_{\mathbf{m}}} \\ \dot{v_{\mathbf{m}}} \end{pmatrix} &= M_{(m,n,0)} \begin{pmatrix} u_{\mathbf{m}} \\ v_{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} \{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\boldsymbol{u}\cdot\nabla)v\}_{\mathbf{m}} \end{pmatrix} \\ &\quad + k_{(m,n,0)} \begin{pmatrix} m\alpha \\ n\beta \end{pmatrix} \quad (l = 0 \text{ and } \mathbf{m} \neq \mathbf{0}) \end{aligned} \tag{7}$$

Here,

$$k_{\mathbf{m}} = \frac{1}{\omega^2}(m\alpha\{(\boldsymbol{u}\cdot\nabla)u\}_{\mathbf{m}} + n\beta\{(\boldsymbol{u}\cdot\nabla)v\}_{\mathbf{m}} + l\pi\{(\boldsymbol{u}\cdot\nabla)w\}_{\mathbf{m}}),$$

$$M_{\mathbf{m}} = \begin{pmatrix} -\omega^2 & 0 & -ml\pi\alpha R/\omega^2 \\ 0 & -\omega^2 & -nl\pi\beta R/\omega^2 \\ -m\alpha/l\pi P & -n\beta/l\pi P & -\omega^2/P \end{pmatrix}, \quad (l \neq 0),$$

$$M_{\mathbf{m}} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix}, \quad (l = 0 \text{ and } \mathbf{m} \neq \mathbf{0})$$

Notice that the mean flow should be zero, that is $u_{\mathbf{0}} = v_{\mathbf{0}} = p_{\mathbf{0}} = 0$ and it holds that $w_{\mathbf{0}} = \theta_{\mathbf{0}} = 0$ by the symmetry (3). It is easy to see that the linearized matrix $M_{\mathbf{m}}$ has 0-eigenvalue if and only if $l \neq 0$ and $R = R(k) = (k^2 + l^2\pi^2)^3/k^2$ where k is the wave number with $k^2 = m^2\alpha^2 + n^2\beta^2$. $R(k)$ takes its minimum value $R_c = 27\pi^4/4$ at the critical wavelength $k_c = \sqrt{2}\pi/2$. R_c is called the critical Rayleigh number. We are interested in the case when the first instability takes place as we increase the Rayleigh number. Therefore, we have only to consider the case $l = 1$ and $R = R(k) = (k^2 + \pi^2)^3/k^2$.

3. 2-D problem and stability of mixed mode solutions

Let us study the 2 dimensional case where the problem depends only on (x, z)-direction, since it might be easier to explain our analysis in the 2-D problem first, and we basically take similar strategy for the 3-D problem. We refer [12] and [11] where they have obtained the same results in this section. Now the unknown variables are u, w, θ, p and their time evolution can be described as follows. Notice that we use the notations $\mathbf{m} = (m, l) \in \mathbf{Z^2}$ for the mode vector and $\omega_{\mathbf{m}}^2 = m^2\alpha^2 + l^2\pi^2$.

$$\begin{pmatrix} \dot{u_{\mathbf{m}}} \\ \dot{\theta_{\mathbf{m}}} \end{pmatrix} = M_{\mathbf{m}} \begin{pmatrix} u_{\mathbf{m}} \\ \theta_{\mathbf{m}} \end{pmatrix} - \begin{pmatrix} \{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\mathbf{u}\cdot\nabla)\theta\}_{\mathbf{m}} \end{pmatrix} + k_{\mathbf{m}} \begin{pmatrix} m\alpha \\ 0 \end{pmatrix}, (l \neq 0) \quad (8)$$

$$\dot{u_{\mathbf{m}}} = -\omega_{\mathbf{m}}^2 u_{\mathbf{m}}, (l = 0, \mathbf{m} \neq (0,0)). \quad (9)$$

Here, $M_{\mathbf{m}}$ and $k_{\mathbf{m}}$ are defined as follows. We use the same notation as the

3 dimensional case as long as it is clear.

$$k_{\mathbf{m}} = (m\alpha\{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}} + l\pi\{(\mathbf{u}\cdot\nabla)w\}_{\mathbf{m}})/\omega_{\mathbf{m}}, \tag{10}$$

$$M_{\mathbf{m}} = \begin{pmatrix} -\omega_{\mathbf{m}}^2 & -ml\pi\alpha R/\omega_{\mathbf{m}}^2 \\ -m\alpha/l\pi P & -\omega_{\mathbf{m}}^2/P \end{pmatrix}. \tag{11}$$

$$\tag{12}$$

Now let us caluculate the convolution terms.

$$\{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1\neq 0}} \frac{i\alpha(m_2l_1 - m_1l_2)}{l_1} u_{\mathbf{m}_1}u_{\mathbf{m}_2} \tag{13}$$

$$\{(\mathbf{u}\cdot\nabla)\theta\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1\neq 0}} \frac{i\alpha(m_2l_1 - m_1l_2)}{l_1} u_{\mathbf{m}_1}\theta_{\mathbf{m}_2} \tag{14}$$

$$\{(\mathbf{u}\cdot\nabla)w\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1l_2\neq 0}} \frac{-im_2\alpha^2(m_2l_1 - m_1l_2)}{l_1l_2\pi} u_{\mathbf{m}_1}u_{\mathbf{m}_2} \tag{15}$$

Here we denote $\mathbf{m}_i = (m_i, l_i)$.

These coupled systems of countably many ordinary differential equations have a trivial zero solution. We study the local bifurcation about the trivial solution. It is necessary to calculate the normal form on the center manifolds which is locally spanned by critical eigenvectors of $M_{\mathbf{m}}$ for each set of parameter values of (R, α, β).

When $R = R(\alpha, m, l) := (m^2\alpha^2 + l^2\pi^2)^3/m^2\alpha^2$ holds, (m, l)-mode becomes critical in the linearized problem about zero to (8)-(9). Therefore at most two critical modes become critical at the same time. More precisely, for a given α there exists a number R^* such that all the eigenvalues about zero is negative for $R < R^*$, and moreover one of the following folds. (See also Figure 1.)

- **Simple critical case:** There exists a natural number κ such that $R^* = R(\alpha; \pm\kappa, 1)$ and if $|m| \neq \kappa$ then $R^* < R(\alpha; m, 1)$. We call the pair of parameter values (α, R^*) a simple critical point.
- **Multiple critical case:** There exists a nutural number $\kappa \geq 2$ such that $R^* = R(\alpha; \pm\kappa, 1) = R(\alpha; \pm(\kappa - 1), 1)$ and if $|m| \neq \kappa, \kappa - 1$ then $R^* < R(\alpha; m, 1)$. We call the pair of parameter values (α, R^*) a multiple critical point.

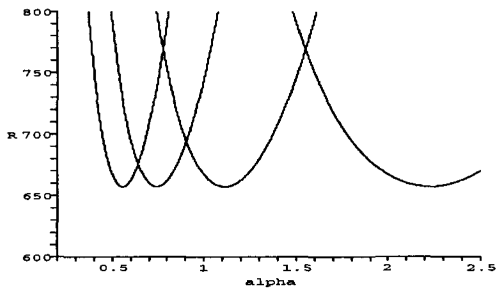


Figure 1. Neutrral stability curves drawn in (α, R)-plane. They correspond the critical curves for $R(\alpha; m) = R(\alpha; 1, 1)$, $\cdots$, $R(\alpha; 4, 1)$ respectively from left.

3.1. *Simple critical case*

It is easy to see that a roll solution bifurcates at a simple critical point as a super-critical pitchfork bifurcation. In fact, $M_{\mathbf{m}}$ has simple 0-eigevalues if and only if $\mathbf{m} \in S := \{(\pm\kappa, \pm 1), (\pm\kappa, \mp 1)\}$. The critical eigenvectors are not $u_{\mathbf{m}}$ but $\tilde{u}_{\mathbf{m}}, \mathbf{m} \in S$ The linear transformation:

$$\begin{pmatrix} \tilde{u}_{\mathbf{m}} \\ \tilde{\theta}_{\mathbf{m}} \end{pmatrix} = T_{\mathbf{m}} \begin{pmatrix} u_{\mathbf{m}} \\ \theta_{\mathbf{m}} \end{pmatrix}, T_{\mathbf{m}} = \frac{1}{(1+P)m\alpha l\pi\omega_{\mathbf{m}}^2} \begin{pmatrix} m\alpha & -lP\pi\omega_{\mathbf{m}}^2 \\ m\alpha & l\pi\omega_{\mathbf{m}}^2 \end{pmatrix} \tag{16}$$

makes the linear part of the equation for $\mathbf{m} \in S$ diagonal as

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{u}}_{\mathbf{m}} \\ \dot{\tilde{\theta}}_{\mathbf{m}} \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1+P}{P}\omega_{\mathbf{m}}^2 \end{pmatrix} \begin{pmatrix} \tilde{u}_{\mathbf{m}} \\ \tilde{\theta}_{\mathbf{m}} \end{pmatrix} \\ &- T_{\mathbf{m}} \begin{pmatrix} \{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\mathbf{u}\cdot\nabla)\theta\}_{\mathbf{m}} \end{pmatrix} + k_{\mathbf{m}} T_{\mathbf{m}} \begin{pmatrix} m\alpha \\ 0 \end{pmatrix}, (l \neq 0). \end{aligned} \tag{17}$$

Now the center manifolds about the simple critical point can be described by $\tilde{u}_{\mathbf{m}}(\mathbf{m} \in S)$. The other modes: $\tilde{\theta}_{\mathbf{m}}(\mathbf{m} \in S)$ and $u_{\mathbf{m}}, \theta_{\mathbf{m}}(\mathbf{m} \notin S)$ are the slave modes. Moreover, it holds that $\tilde{u}_{\kappa,1} = \tilde{u}_{\kappa,-1}$ by even-odd symmetry(3). Therefore $\tilde{u}_{\kappa,1}, \overline{\tilde{u}}_{\kappa,1}$ gives the local coordinate on the center manifolds. We are interested in the small solutions $|\tilde{u}_{\kappa,1}| < \delta$ near the critical point. Then all the slave modes are $O(\delta^2)$ by the center manifold theorem. To obtain the effective normal form on the center manifolds we pick up the nonlinear term from the equation of the critical modes in (17) up to $O(\delta^3)$. The nonlinear terms for the equation of $\tilde{u}_{\kappa,1}$ in (17) consist of $u_{\mathbf{m}_1}, u_{\mathbf{m}_2}$ and $u_{\mathbf{m}_1}, \theta_{\mathbf{m}_2}$ with $\mathbf{m}_1 + \mathbf{m}_2 = (\kappa, 1)$. The combinations of $(\mathbf{m}_1, \mathbf{m}_2)$ which give nonlinear terms up to $O(\delta^3)$ are $\{((\kappa,1),(0,0)), ((-\kappa,1),(2\kappa,0)), ((\kappa,-1),(0,2))((-\kappa,-1),(2\kappa,2))\}$. Since

$\theta_{m,0} = 0$ holds by the up-down symmetry and $u_{m,0} = 0$ holds in the 2-D setting, the nonlinear terms come from the first two combinations of above are zero. It is also zero for$(\mathbf{m}_1, \mathbf{m}_2) = ((-\kappa, -1), (2\kappa, 2))$ by (13),(14) and (15). Therefore the slave modes up to $O(\delta^3)$ which relate to the equation for $A := \tilde{u}_{\kappa,1}$ are only $B := u_{0,2}$ and $C := \theta_{0,2}$. To obtain the normal form on the center manifold we need to calculate the approximation of the center manifolds by the coordinate A . The quadratic approximation of the center manifolds $u_{0,2} = h^u_{0,2} := h^u_{0,2}(\tilde{u}_{\kappa,1}, \tilde{u}_{-\kappa,-1}, \tilde{u}_{-\kappa,1}, \tilde{u}_{\kappa,-1})$ and $\theta_{0,2} = h^\theta_{0,2} := h^\theta_{0,2}(\tilde{u}_{\kappa,1}, \tilde{u}_{-\kappa,-1}, \tilde{u}_{-\kappa,1}, \tilde{u}_{\kappa,-1})$ are as follows.

$$
\begin{aligned}
h^u_{0,2} &= O(\delta^3),\\
h^\theta_{0,2} &= \frac{i\omega^2_{\kappa,1}P}{\pi}|A|^2 + O(\delta^3).
\end{aligned}
$$

From (17),

$$
\begin{aligned}
\dot{A} &= \mu A + \frac{P}{(1+P)\kappa\alpha}\{(\mathbf{u}\cdot\nabla)\theta\}_{\kappa,1} \quad ,\\
\{(\mathbf{u}\cdot\nabla)\theta\}_{\kappa,1} &= 2\pi i\alpha\kappa\omega^2_{\kappa,1}Ah^\theta_{0,2} \quad .
\end{aligned}
$$

Here we assume $\mu = R - R^* = O(\delta^2)$.Finally we obtain

$$
\dot{A} = \mu A - 2\frac{P^2\omega^4_{\kappa,1}}{1+P}|A|^2A + O(\delta^4). \tag{18}
$$

It shows that a super-critical pitchfork bifurcation to a roll solution occurs at the critical point.

3.2. *Multiple critical case*

In this subsection, we consider the multiple critical case in the 2D problem. M has simple 0-eigenvalue if and only if

$$
\mathbf{m} \in S := \{(\pm\kappa', \pm1), (\pm\kappa', \mp1), (\pm\kappa, \pm1), (\pm\kappa, \mp1), \}
$$

where $\kappa' = \kappa - 1$. After taking the similar linear transformation which diagonalize the matrix $M_{\mathbf{m}}$ the 4 critical modes are represented by

$$
A := \tilde{u}_{\kappa,1},\ \overline{A} = \tilde{u}_{-\kappa,-1},\ B := \tilde{u}_{\kappa',1},\ \overline{B} := \tilde{u}_{-\kappa',-1}.
$$

Remember that we are interested in small solutions $|A|, |B| < \delta$ near the critical point. Moreover, the slave modes coming into the equation for A are

$C := u_{0,2}$, $D := \theta_{0,2}$, $E := u_{\kappa-\kappa',2}$, $F := \theta_{\kappa-\kappa',2}$, $G := u_{\kappa+\kappa',2}$, $H := \theta_{\kappa+\kappa',2}$

up to $O(\delta^3)$. We obtain the following normal form when $\kappa \geq 2$ by calculating quadratic approximation of center manifolds.(The calculation is similar as simple critical case.)

Since, equation (17) inherited up-down symmetry from (5), any quadratic resonance does not occur. Notice that it has quadratic resonance term when asymmetry case and we need a different approach to analyze the normal form as one can see in [2] .(see also [3].)

$$\begin{cases} \dot{A} = (\mu_1 + a|A|^2 + b|B|^2)A + O(\delta^4) \\ \dot{B} = (\mu_2 + c|A|^2 + d|B|^2)B + O(\delta^4) \end{cases} \tag{19}$$

It can be separated into the equation for the modulus(amplitude) and the argument(angle) by the polar coordinate.And the equations for the amplitudes are

$$\begin{cases} \dot{r} = (\mu_1 + ar^2 + bs^2)r + O(\delta^4) \\ \dot{s} = (\mu_2 + cr^2 + ds^2)s + O(\delta^4) \end{cases} \tag{20}$$

Here, we denote $r = |A|, s = |B|$ and $\mu_1 = R(\alpha; \kappa, 1) - R^*, \mu_2 = R(\alpha; \kappa', 1) - R^*$. To study the above equations (20) we consider the cut-off equations:

$$\begin{cases} \dot{r} = (\mu_1 + ar^2 + bs^2)r, \\ \dot{s} = (\mu_2 + cr^2 + ds^2)s. \end{cases} \tag{21}$$

Lemma 3.1.

Assume $a, b, d < 0, ad - bc > 0$. Moreover, if μ_1, μ_2 satisfies the inequality: $c\mu_1/a < \mu_2 < d\mu_1/b$ then the equilibrium of (21)$F(r,s) = (r_*, s_*), r_*, s_* \neq 0$ is asymptotically stable.

proof

From $\dot{r} = \dot{s} = 0$, we can obtain r_*, s_* as follows.

$$r = r_* := \sqrt{(b\mu_2 - d\mu_1)/(ad - bc)}$$
$$s = s_* := \sqrt{(c\mu_1 - a\mu_2)/(ad - bc)}$$

Since we assumed $ad - bc > 0$ and $a, b, d < 0$, if $c\mu_1/a < \mu_2 < d\mu_1/b$ then,$(r, s) = (r_*, s_*)$ is an equilibrium of (20). The linearized matrix for (20) about (r_*, s_*) is as follows.

$$\begin{pmatrix} \mu_1 + 3ar_*^2 + bs_*^2 & 2br_*s_* \\ 2cr_*s_* & \mu_2 + cr_*^2 + 3ds_*^2 \end{pmatrix} = \begin{pmatrix} 2ar_*^2 & 2br_*s_* \\ 2cr_*s_* & 2ds_*^2 \end{pmatrix}. \tag{22}$$

So, the eigenvalues about (r_*, s_*) are given as the solutions of quadratic equation:

$$\Lambda^2 - 2(ar_*^2 + ds_*^2)\Lambda + 4(ad - bc)r_*^2 s_*^2 = 0. \tag{23}$$

This completes the proof. □

Proposition 3.1.

Let $\kappa \in \{2,3,4\}$. Then, there exist $P^* > 0$ such that if $P < P^*$ then $ad - bc > 0$. Moreover, if $0 < P < P^*$ then the invariant torus which corresponds to the mixed mode solution is asymptotically stable.

proof

Notice that the coefficients of normal form(20) a, d are as follows.

$$a = -2\frac{P^2\omega_{\kappa,1}^4}{1+P} < 0 \ , \ d = -2\frac{P^2\omega_{\kappa-1,1}^4}{1+P} < 0.$$

By the lemma 3.1 if $ad - bc > 0$ then the mixed mode solution is asymptotic stable. The $ad - bc$ and $\frac{d}{dP}(ad - bc)$ are written by P as follows.

$$ad - bc = \frac{-d_4P^4 - d_3P^3 - d_2P^2 - d_1P + d_0}{(1+P)^2} \tag{24}$$

$$\frac{d}{dP}(ad - bc) = \frac{-D_4P^4 - D_3P^3 - D_2P^2 - D_1P - D_0}{(1+P)^3} \tag{25}$$

Here d_i, D_i are written by α, μ_i, κ, and the table 1 shows that approximate values of d_i for each $\kappa \in \{2,3,4\}$. Moreover $\frac{d}{dP}(ad - bc) < 0$, for all P, and if $P = 0$ then $ad - bc > 0$.(See also Figure 2.) This completes the proof. □

It should be mentioned that we can show the same statement as the above proposition for a given $\kappa > 4$, say $\kappa = 100$. However, we don't rigorously know that it holds for an arbitrary natural number κ. Let $\Delta_\kappa = \lim_{P\to 0} ad - bc$ then we can show that $\lim_{\kappa\to\infty} \Delta_\kappa = 0$. However, it is sufficient in the sense that we are interested in the small size container. Notice that the critical Prandtl numbers for the stability of mixed mode solutions in figure 2 are the same as those obtained by [12].

Table 1.

Approximate values of d_i for each κ.

κ	d_4	d_3	d_2	d_1	d_0
2	7.83×10^5	4.65×10^4	1.80×10^4	-7.26×10^3	-6.69×10^3
3	6.40×10^5	1.30×10^4	5.05×10^3	-1.74×10^3	-1.64×10^3
4	6.08×10^5	6.20×10^3	2.41×10^3	7.99×10^2	7.57×10^2

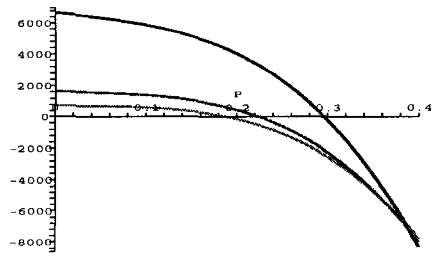


Figure 2. The stability of $(\kappa-1,1)-(\kappa,1)$ mixed mode for $\kappa=2$ (black), $\kappa=3$ (dark gray) and $\kappa=4$ (light gray) . Values of $ad-bc$ with respect to values of P are drawn and if $ad-bc>0$ then the mixed mode is stable.

4. 3-D problem

In this section we consider 3 dimensional case. We diagonalize the linear parts of the equations(6). Take the linear transformation:

$$\begin{pmatrix} u_{\mathbf{m}} \\ v_{\mathbf{m}} \\ \theta_{\mathbf{m}} \end{pmatrix} = \Phi_{\mathbf{m}} \begin{pmatrix} \tilde{u}_{\mathbf{m}} \\ \tilde{v}_{\mathbf{m}} \\ \tilde{\theta}_{\mathbf{m}} \end{pmatrix}$$

$$\Phi_{\mathbf{m}} = \begin{pmatrix} -\omega_{\mathbf{m}}^2 ml\pi\alpha/k^2 & n\beta & m\alpha \\ -\omega_{\mathbf{m}}^2 nl\pi\beta/k^2 & -m\alpha & n\beta \\ 1 & 0 & k^2/\omega_{\mathbf{m}}^2 l\pi P \end{pmatrix} \tag{26}$$

so that the linear parts of the equations(6) become diagonal as

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{u}}_{\mathbf{m}} \\ \dot{\tilde{v}}_{\mathbf{m}} \\ \dot{\tilde{\theta}}_{\mathbf{m}} \end{pmatrix} &= \begin{pmatrix} \mu_{\mathbf{m}}^+ & 0 & 0 \\ 0 & -\omega_{\mathbf{m}}^2 & 0 \\ 0 & 0 & \mu_{\mathbf{m}}^- \end{pmatrix} \begin{pmatrix} \tilde{u}_{\mathbf{m}} \\ \tilde{v}_{\mathbf{m}} \\ \tilde{\theta}_{\mathbf{m}} \end{pmatrix} \\ &- \Phi_{\mathbf{m}}^{-1} \begin{pmatrix} \{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\mathbf{u}\cdot\nabla)v\}_{\mathbf{m}} \\ \{(\mathbf{u}\cdot\nabla)\theta\}_{\mathbf{m}} \end{pmatrix} + \Phi_{\mathbf{m}}^{-1} k_{\mathbf{m}} \begin{pmatrix} m\alpha \\ n\beta \\ 0 \end{pmatrix} . \end{aligned} \tag{27}$$

Here

$$\mu_{\mathbf{m}}^{\pm} = \frac{-(P+1)}{2P}\omega_{\mathbf{m}}^2 \pm \frac{1}{P}\sqrt{\{(P+1)/2\}^2\omega_{\mathbf{m}}^4 + P\{R(m^2\alpha^2+n^2\beta^2) - \omega_{\mathbf{m}}^6\}}$$

are eigenvalues of $M_{\mathbf{m}}$ and if $R = R(k) = (k^2+l^2)^3/k^2$ then, $\mu_{\mathbf{m}}^+ = 0$, $\mu_{\mathbf{m}}^- = -(1+P)\omega_{\mathbf{m}}^2/P$. For the sake of convenience we define the set

$$T_{m,n,l} = \{(\pm m, \pm n, \pm l), (\pm m, \pm n, \mp l), (\pm m. \mp n, \pm l), (\pm m, \mp n, \mp l)\},$$

and

$$S := \{\mathbf{m} = (m,n,l) | R = (m^2\alpha^2 + n^2\beta^2 + l^2\pi^2)^3/(m^2\alpha^2+n^2\beta^2)\}.$$

This means that if $\mathbf{m} \in S$ *then* $M_{\mathbf{m}}$ *has simple 0-eigenvalue.*

4.1. *Neutral stability surface for 3-D problem*

If we consider the special case when $(\alpha,\beta) = k_c(1/2, \sqrt{3}/2)$, where we usually study hexagonal mode interaction, the 3 critical roll modes:$(2,0,1), (1,\pm1,1)$ become unstable exactly at the critical Rayleigh number R_c. On the other hand, the first instability occurs at $R > R_c$ in general. It is convenient to define the neutral stability surface for each mode $(m,n,1)$ (or simply we denote (m,n)) as follows.

$$G_{m,n} = \Big\{(\alpha,\beta,R)\ ;\ R = R_{m,n}(\alpha,\beta) := \frac{(m^2\alpha^2+n^2\beta^2+\pi^2)^3}{(m^2\alpha^2+n^2\beta^2)},\ \alpha,\beta \in (0,\infty)\Big\} \tag{28}$$

The (m,n)-mode instability occurs on the surface $G_{m,n}$. Remember that we have set $l = 1$ since we are interested in the first instability. Therefore, for a given (α,β), the first instability occurs as (m_*, n_*)-mode where $R_{m_*,n_*}(\alpha,\beta) \le R_{m,n}(\alpha,\beta)$ for any $(m,n) \in \mathbf{Z}^2$. There can be multiple critical mode. In fact when $(\alpha,\beta) = k_c(1/2,\sqrt{3}/2)$,both $(2,0)$ and $(1,1)$ are critical at $R = R_c$. More precisely, a set of critical modes S is $S := T_{2,0,1} \cup T_{1,1,1}$ in this case. By using the Hermitian and even-odd symmetries we have essentially 3 critical modes:

$$\tilde{u}_{2,0,1},\ \tilde{u}_{-1,1,1},\ \tilde{u}_{-1,-1,1}.$$

$R_{m,n}(\alpha,\beta)$ attains its minimum R_c on the curve

$$C_{m,n} = \Big\{(\alpha,\beta); m^2\alpha^2 + n^2\beta^2 = k_c^2, \alpha,\beta \in (0,\infty)\Big\}.$$

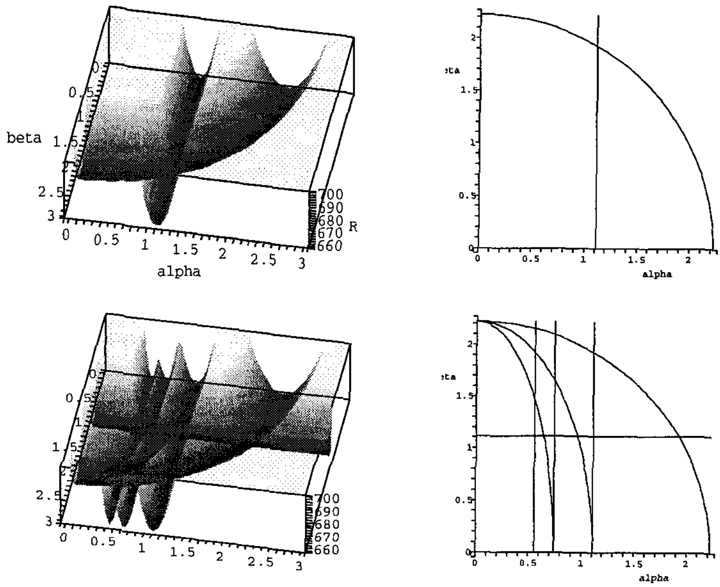


Figure 3. [Above left:Neutral stability surface for $(2,0)$ and $(1,1)$], [Above right:$C_{2,0}$ and $C_{1,1}$], [Below left:Neutral stability surface for $(m,n) = (1,1),(2,0),(0,2),(3,0),(2,1),(4,0)$ and $(3,1)$], [Below right:$C_{m,n}$ for $(m,n) = (1,1),(2,0),(0,2),(3,0),(2,1),(4,0)$ and $(3,1)$].

Next, if we proportionally increase or decrease the ratio of system size β/α to some extent then we have the same set of critical modes but the value of R is larger than R_c. There are so many possibilities of multiple critical points. (See Figure3,4.) In this article we are interested in the critical points which has the critical modes:$S := T_{\kappa,0,1} \cup T_{\tau,\tau',1}$. where $\kappa, \tau, \tau' \in \mathbf{N}$. We call the point (α, β, R) which satisfies this property the pseudo hexagonal critical point since it has essentially 3 critical roll modes $\tilde{u}_{\kappa,0,1}, \tilde{u}_{\tau,\tau',1}, \tilde{u}_{\tau,-\tau',1}$. In other words the set of pseudo hexagonal critical points are on the curve $G_{\kappa,0} \cap G_{\tau,\tau'}$.

4.2. *Approximation of center manifolds*

In this subsection we calculate quadratic approximation of center manifolds to obtain the normal form.

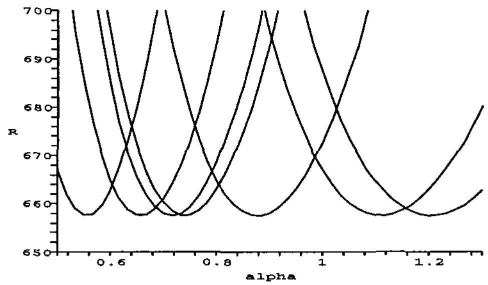


Figure 4. Vertical section of neutral critical surface on the line $\beta = 1.55\alpha$ .Each curves in the figure in the section of $G_{1,1}$, $G_{2,0}$, $G_{2,1}$, $G_{3,0}$, $G_{0,2}$, $G_{3,1}$ and $G_{4,0}$ from the right.

Proposition 4.1. *Let* $\mathbf{m}_i \in S, (i = 1, 2, ...12)$,$\mathbf{m}_s \notin S$,$(\mathbf{m}_s = (m_s, n_s, l_s))$. *The quadratic approximation of the center manifold is given by the graph of the functions* $u_{\mathbf{m}_s} = h^u_{\mathbf{m}_s}$,$v_{\mathbf{m}_s} = h^v_{\mathbf{m}_s}$, $\theta = h^\theta_{\mathbf{m}_s}$*:*

$$\begin{pmatrix} h^u_{\mathbf{m}_s} \\ h^v_{\mathbf{m}_s} \\ h^\theta_{\mathbf{m}_s} \end{pmatrix} = M^{-1}_{\mathbf{m}_s} \begin{pmatrix} \{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}_s} \\ \{(\mathbf{u}\cdot\nabla)v\}_{\mathbf{m}_s} \\ \{(\mathbf{u}\cdot\nabla)\theta\}_{\mathbf{m}_s} \end{pmatrix} - M^{-1}_{\mathbf{m}_s} k_{\mathbf{m}_s} \begin{pmatrix} m\alpha \\ n\beta \\ 0 \end{pmatrix}, \ (l_s \neq 0) \ . \tag{29}$$

$$\begin{pmatrix} h^u_{\mathbf{m}_s} \\ h^v_{\mathbf{m}_s} \end{pmatrix} = M^{-1}_{\mathbf{m}_s} \begin{pmatrix} \{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}_s} \\ \{(\mathbf{u}\cdot\nabla)v\}_{\mathbf{m}_s} \end{pmatrix} - M^{-1}_{\mathbf{m}_s} k_{\mathbf{m}_s} \begin{pmatrix} m\alpha \\ n\beta \end{pmatrix}, \ (l_s = 0) \ . \tag{30}$$

Proof

Rewrite equation(27) as follows

$$
\begin{aligned}
&\dot{\mu}^+_{\mathbf{m}} = 0 \cdot \mu^+_{\mathbf{m}} \\
&\begin{pmatrix} \dot{\tilde{u}}_{\mathbf{m}} \\ \dot{\tilde{v}}_{\mathbf{m}} \\ \dot{\tilde{\theta}}_{\mathbf{m}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\omega^2_{\mathbf{m}} & 0 \\ 0 & 0 & \mu^-_{\mathbf{m}} \end{pmatrix} \begin{pmatrix} \tilde{u}_{\mathbf{m}} \\ \tilde{v}_{\mathbf{m}} \\ \tilde{\theta}_{\mathbf{m}} \end{pmatrix} \\
&\qquad - \Phi^{-1}_{\mathbf{m}} \begin{pmatrix} \{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}} \\ \{(\mathbf{u}\cdot\nabla)v\}_{\mathbf{m}} \\ \{(\mathbf{u}\cdot\nabla)\theta\}_{\mathbf{m}} \end{pmatrix} + \Phi^{-1}_{\mathbf{m}} k_{\mathbf{m}} \begin{pmatrix} m\alpha \\ n\beta \\ 0 \end{pmatrix} + \begin{pmatrix} \mu^+_{\mathbf{m}} \tilde{u}_{\mathbf{m}} \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}
$$

By the center manifold theory ([6] Section2) we obtain

$$
\begin{aligned}
&\begin{pmatrix} \frac{\partial h^u_{\mathbf{m}_s}}{\partial \tilde{u}_{\mathbf{m}_1}} & \cdots & \frac{\partial h^u_{\mathbf{m}_s}}{\partial \tilde{u}_{\mathbf{m}_{12}}} \\ \frac{\partial h^v_{\mathbf{m}_s}}{\partial \tilde{u}_{\mathbf{m}_1}} & \cdots & \frac{\partial h^v_{\mathbf{m}_s}}{\partial \tilde{u}_{\mathbf{m}_{12}}} \\ \frac{\partial h^\theta_{\mathbf{m}_s}}{\partial \tilde{u}_{\mathbf{m}_1}} & \cdots & \frac{\partial h^\theta_{\mathbf{m}_s}}{\partial \tilde{u}_{\mathbf{m}_{12}}} \end{pmatrix} \begin{pmatrix} \dot{\tilde{u}}_{\mathbf{m}_1} \\ \cdot \\ \cdot \\ \cdot \\ \dot{\tilde{u}}_{\mathbf{m}_{12}} \end{pmatrix} \\
&= M_{\mathbf{m}_s} \begin{pmatrix} h^u_{\mathbf{m}_s} \\ h^v_{\mathbf{m}_s} \\ h^\theta_{\mathbf{m}_s} \end{pmatrix} - \begin{pmatrix} \{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}_s} \\ \{(\mathbf{u}\cdot\nabla)v\}_{\mathbf{m}_s} \\ \{(\mathbf{u}\cdot\nabla)\theta\}_{\mathbf{m}_s} \end{pmatrix} + k_{\mathbf{m}_s} \begin{pmatrix} m\alpha \\ n\beta \\ 0 \end{pmatrix}. \qquad (31)
\end{aligned}
$$

If $|\tilde{u}_{\mathbf{m}_i}| < \delta$ and $|\mu^+_{\mathbf{m}_i}| < \delta^2$ then left hand side of (31) is $O(\delta^3)$. Since $M_{\mathbf{m}_s}$ is a regular matrix for each $\mathbf{m}$, we obtain (29). Similarly we obtain (30) for the case $l_s = 0$. This completes the proof. □

Now let us calculate the convolution terms.

$$
\begin{aligned}
\{(\mathbf{u}\cdot\nabla)u\}_{\mathbf{m}} = &\sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1\neq 0}} \Big(\frac{i\alpha(m_2 l_1 - m_1 l_2)}{l_1} u_{\mathbf{m}_1} u_{\mathbf{m}_2} \\
&\qquad + \frac{i\beta(n_2 l_1 - n_1 l_2)}{l_1} v_{\mathbf{m}_1} u_{\mathbf{m}_2} \Big) \\
&+ \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1=0, n_1\neq 0}} \frac{i\alpha(m_2 n_1 - m_1 n_2)}{n_1} u_{\mathbf{m}_1} u_{\mathbf{m}_2} \\
&+ \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m} \\ l_1=0, n_1=0}} i n_2 \beta v_{\mathbf{m}_1} u_{\mathbf{m}_2}
\end{aligned}
$$

$$\{(\mathbf{u}\cdot\nabla)v\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_1\neq 0}} \Big(\frac{i\alpha(m_2l_1-m_1l_2)}{l_1}u_{\mathbf{m}_1}v_{\mathbf{m}_2} + \frac{i\beta(n_2l_1-n_1l_2)}{l_1}v_{\mathbf{m}_1}v_{\mathbf{m}_2}\Big) + \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_1=0,n_1\neq 0}} \frac{i\alpha(m_2n_1-m_1n_2)}{n_1}u_{\mathbf{m}_1}v_{\mathbf{m}_2} + \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_1=0,n_1=0}} in_2\beta v_{\mathbf{m}_1}v_{\mathbf{m}_2}$$

$$\{(\mathbf{u}\cdot\nabla)\theta\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_1\neq 0}} \Big(\frac{i\alpha(m_2l_1-m_1l_2)}{l_1}u_{\mathbf{m}_1}\theta_{\mathbf{m}_2} + \frac{i\beta(n_2l_1-n_1l_2)}{l_1}v_{\mathbf{m}_1}\theta_{\mathbf{m}_2}\Big) + \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_1=0,n_1\neq 0}} \frac{i\alpha(m_2n_1-m_1n_2)}{n_1}u_{\mathbf{m}_1}\theta_{\mathbf{m}_2} + \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_1=0,n_1=0}} in_2\beta v_{\mathbf{m}_1}\theta_{\mathbf{m}_2}$$

$$\begin{aligned}\{(\mathbf{u}\cdot\nabla)w\}_{\mathbf{m}} = \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_1l_2\neq 0}} \Big(&\frac{-i\alpha^2m_2(m_2l_1-m_1l_2)}{l_1l_2\pi}u_{\mathbf{m}_1}u_{\mathbf{m}_2} + \frac{-i\beta^2n_2(n_2l_1-n_1l_2)}{l_1l_2\pi}v_{\mathbf{m}_1}v_{\mathbf{m}_2} \\ &+ \frac{-i\alpha\beta n_2(m_2l_1-m_1l_2)}{l_1l_2\pi}u_{\mathbf{m}_1}v_{\mathbf{m}_2} + \frac{-i\alpha\beta m_2(n_2l_1-n_1l_2)}{l_1l_2\pi}v_{\mathbf{m}_1}u_{\mathbf{m}_2}\Big) \\ + \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_2\neq 0,n_1\neq 0,l_1=0}} \Big(&\frac{-im_2\alpha^2(m_2n_1-n_2m_1)}{n_1l_2\pi}u_{\mathbf{m}_1}u_{\mathbf{m}_2} - \frac{in_2\alpha\beta(m_2n_1-n_2m_1)}{n_1l_2\pi}u_{\mathbf{m}_1}v_{\mathbf{m}_2}\Big) \\ + \sum_{\substack{\mathbf{m}_1+\mathbf{m}_2=\mathbf{m}\\ l_1=n_1=0,n_2\neq 0}} \Big(&\frac{-in_2m_2\alpha\beta}{l_2\pi}v_{\mathbf{m}_1}u_{\mathbf{m}_2} - \frac{in_2^2\beta^2}{l_2\pi}v_{\mathbf{m}_1}v_{\mathbf{m}_2}\Big)\end{aligned}$$

Here we denote $\mathbf{m}_j = (m_j, n_j, l_j)$.

Finally, by using the similar argument to the previous section(2-D case) we can obtain the normal form on the center manifold as follows.

$$\begin{cases} \dot{A}_1 = (\mu_1 + a|A_1|^2 + b|A_2|^2 + b|A_3|^2)A_1 \\ \dot{A}_2 = (\mu_2 + c|A_1|^2 + d|A_2|^2 + e|A_3|^2)A_2 \\ \dot{A}_3 = (\mu_2 + c|A_1|^2 + e|A_2|^2 + d|A_3|^2)A_3 \end{cases} \tag{32}$$

Here we denote

$$A_1 := \tilde{u}_{\kappa,0,1}\ , A_2 := \tilde{u}_{-\tau,\tau',1},\ A_3 := \tilde{u}_{-\tau,-\tau',1}\ ,$$
$$\mu_1 := \mu^+_{\kappa,0,1},\ \mu_2 := \mu^+_{-\tau,\tau',1}\ (\equiv \mu^+_{-\tau,-\tau',1})\ .$$

In the case of hexagonal critical point($(\alpha,\beta,R) = (k_c/2, \sqrt{3}K_c/2, R_c)$ it holds that $\mu_1 = \mu_2, a = d, b = c = e, for\ \ all\ \ P > 0$.

We extract the equations for the amplitudes from (32) by taking the polar coordinates $A_j = r_j e^{i\phi_j}$:

$$\begin{cases} \dot{r_1} = (\mu_1 + ar_1{}^2 + br_2{}^2 + br_3{}^2)r_1 \\ \dot{r_2} = (\mu_2 + cr_1{}^2 + dr_2{}^2 + er_3{}^2)r_2 \\ \dot{r_3} = (\mu_2 + cr_1{}^2 + er_2{}^2 + dr_3{}^2)r_3 \end{cases} \tag{33}$$

Notice that the equation(33) can have the following equilibriums:

$$\begin{aligned} &\cdot(O): \quad (0,0,0) \\ &\cdot(R): \quad (r^*,0,0)\ ,\ (0,s^*,0)\ ,\ (0,0,s^*) \\ &\cdot(PQ): \quad (r_*,s_*,0)\ ,\ (0,s_*,s_*)\ ,\ (r_*,0,s_*) \\ &\cdot(H): \quad (r_{**},s_{**},s_{**}) \end{aligned}$$

Here, each of $r^*, s^*, r_*, s_*, r_{**}, s_{**}$ is non-zero. We call (O):zero, (R):roll, (PQ):patchwork quilt and (H):pseudo hexagonal solution, respectively.

In the later sections, we discuss about the stability of the patterns in the sense of (33). That means we can show that existence of the invariant torus which includes the fixed point corresponding to the each pattern. When the hexagonal case, for example, the equilibrium:(H) might include the three patterns:regular-triangle, up-hexagon and down-hexagon. To determine the dynamics on the invariant torus, we need to calculate the normal form up to higher order term, but we only discuss the stability of the invariant torus and caluculate the eigenvalues for the transversal direction to the torus.

4.3. *Hexagonal case*

In this subsection we study the Hexagonal case, i.e., $(\kappa, 0) = (2, 0)$ and $(\tau, \tau') = (1, 1)$. Remember that hexagonal critical point is $(\alpha, \beta, R) = (k_c/2, \sqrt{3}k_c/2, R_c)$.

The number of positive eigenvalues about each solution in the sense of (33) is $(O) : 3$, $(R) : 0$, $(PQ) : 1$, $(H) : 2$, respectively. The coefficients of the normal form (33) are as follows.

$$a = \frac{-9\pi^4 P^2}{2(1+P)},\ b = \frac{-9\pi^4(12737P^2 + 1113P + 1728)}{16250(1+P)}.$$

Especially, the eigenvalues about (H) are $a - b > 0$, $a - b > 0$, $a + 2b < 0$ and we can show the ratio $|(a-b)/(a+2b)|$ between the absolute values of positive and negative eigenvalues is small(see Figure5). More precisely we can calculate the eigenvalues and the ratio of them as follows.

$$a - b = \frac{9\pi^4(4612P^2 + 1113P + 1728)}{16250(1+P)}$$

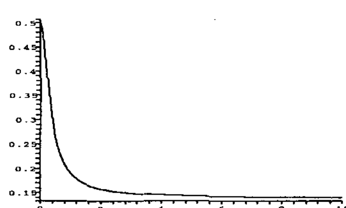
$$a + 2b = \frac{-9\pi^4(33599P^2 + 2226P + 3456)}{16250(1+P)}$$

$$\left|\frac{a-b}{a+2b}\right| = \frac{4612P^2 + 1113P + 1728}{33599P^2 + 2226P + 3456}$$

Furthermore, it is easy to see that

$$\frac{d}{dP}\left(\left|\frac{a-b}{a+2b}\right|\right) < 0.$$

This implies that $|(a-b)/(a+2b)|$ is monotone decreasing with respect to P.



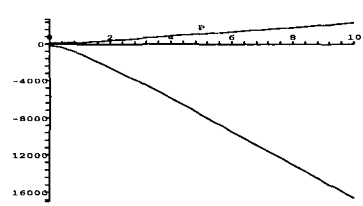


Figure 5. The stability of hexagonal pattern. Left figure correspond to ratio of positive eigenvalues and negative one.($|(a-b)/(a+2b)|$). Right figure correspond to eigenvalues $a - b$ and $a + 2b$.

4.4. *Existence of the stable Patchwork-Quilt pattern*

In this subsection we study the stability of the invariant torus which corresponds to the patchwork quilt pattern. We will show that when $(\tau, \tau') = (\kappa - 1, 1), \kappa = 2, 3, 4, 5, 6, 7$, the equilibriums of (33) which correspond to patchwork quilt patterns can be stable by taking α, β, R and P suitably.

We will begin by considering the property of the curve $G_{m1,n1} \cap G_{m_2,n_2}$. If

$$\mathbf{m}_1 = (m_1, n_1, 1),\ \mathbf{m}_2 = (m_2, n_2, 1),\quad m_1 > m_2 > 0,\ n_2 > n_1 \geq 0.$$

are critical modes and (α, β) satisfies

$$\beta/\alpha = \sqrt{(m_1^2 - m_2^2)/(n_2^2 - n_1^2)}.$$

then, $G_{m_1,n_1} = G_{m_2,n_2}$. Moreover, if both $\mathbf{m}_1$ and $\mathbf{m}_2$ becomes unstable at $R = R_c$ then, α, β satisfies

$$\alpha = \alpha_c := k_c\sqrt{(n_1^2 - n_2^2)/(n_1^2 m_2^2 - m_1^2 n_2^2)}\ ,$$

$$\beta = \beta_c := k_c\sqrt{(m_2^2 - m_1^2)/(n_1^2 m_2^2 - m_1^2 n_2^2)}\ .$$

We define $\eta_c, V_{\mathbf{m}_1,\mathbf{m}_2}$ and $H_{\mathbf{m}_1,\mathbf{m}_2}$ as follows.

$$\eta_c := \sqrt{(m_1^2 - m_2^2)/(n_2^2 - n_1^2)}\ ,$$

$$V_{\mathbf{m}_1,\mathbf{m}_2} := \left\{(\alpha, \beta, R) \in G_{m_1,n_1}; \beta/\alpha = \eta_c, (\alpha, \beta, R) \in \mathbf{R}^3\right\} \setminus \{(\alpha_c, \beta_c, R_c)\},$$

$$H_{\mathbf{m}_1,\mathbf{m}_2} := (G_{m1,n1} \cap G_{m2,n2}) \setminus V_{\mathbf{m}_1,\mathbf{m}_2}\quad .$$

Notice that $G_{m1,n_1} \cap G_{m_2,n_2} = H_{\mathbf{m}_1,\mathbf{m}_2} \cup V_{\mathbf{m}_1,\mathbf{m}_2}$. We are interested in the pseudo hexagonal points:$(m_1, n_1) = (\kappa, 0)$ and $(m_2, n_2) = (\kappa - 1, 1)$. However, when $(\alpha, \beta, R) \in V_{(\kappa,0,1),(\kappa-1,1,1)}$ and both $(\kappa, 0, 1)$ and $(\kappa - 1, 1, 1)$ are first instability, it is expected that we can not obtain the nontrivial stable mixed mode. In fact, for each $\kappa = 2, 3, 4, 5, 6, 7$, we can see the invariant torus corresponding to the mixed mode solutions(patchwork quilt or pseudo hexagonal pattern) can not be stable(unstable) by the normal form analysis. So we are interested in the case when $(\alpha, \beta, R) \in H_{(\kappa,0,1),(\kappa-1,1,1)}$.

To analyze about the pseudo hexagonal critical point, we introduce a new parameter $\eta := \beta/\alpha$. We show that the pseudo hexagonal point:$(\alpha, \beta, R) \in H_{\mathbf{m}_1,\mathbf{m}_2}$ can be parameterized by η. This implies that we can control the ratio of the system size (α, β) by the only one parameter. Remark that if $\mathbf{m}_1$ and $\mathbf{m}_2$ are pseudo hexagonal modes then $n_1 = 0$, but we can prove the next lemma is correct for general n_1.

Lemma 4.1. *Let*

$$\mathbf{m}_1 = (m_1, n_1, 1), \ \mathbf{m}_2 = (m_2, n_2, 1)$$

are critical modes. Then the multiple critical point

$$(\alpha, \beta, R) = (\alpha_*, \beta_*, R_*) \in H_{\mathbf{m}_1, \mathbf{m}_2}$$

is parameterized by $\eta := \beta/\alpha$ as follows:

$$\alpha_*(\eta) = \pi\sqrt{\frac{1}{(\varphi_1\varphi_2)^{1/3}(\varphi_1^{1/3} + \varphi_2^{1/3})}} \tag{34}$$

$$\beta_*(\eta) = \eta\alpha_*(\eta) \ , \tag{35}$$

$$R_*(\eta) = R_{m_1,n_1}(\alpha_*(\eta), \beta_*(\eta)) \tag{36}$$

where $\varphi_j = m_j^2 + n_j^2\eta^2$.

proof

If $\mathbf{m}_1, \mathbf{m}_2$ are critical modes. Then

$$R_{m_1,n_1}(\alpha, \eta\alpha) = R_{m_2,n_2}(\alpha, \eta\alpha) \tag{37}$$

holds. More precisely it hold that

$$\frac{(m_1^2\alpha^2 + n_1^2\eta^2\alpha^2 + \pi^2)^3}{m_1^2\alpha^2 + n_1^2\eta^2\alpha^2} = \frac{(m_2^2\alpha^2 + n_2^2\eta^2\alpha^2 + \pi^2)^3}{m_2^2\alpha^2 + n_2^2\eta^2\alpha^2} \ . \tag{38}$$

If $(\alpha, \beta, R) \in V_{\mathbf{m}_1,\mathbf{m}_2}$ (38) hold for all α, β. Thus the critical point:(α, β, R) on $V_{\mathbf{m}_1,\mathbf{m}_2}$ can not be parameterized by η. Denote that $m_j^2 + n_j^2\eta^2 = \varphi_j$ then (38) becomes

$$(\varphi_1\alpha^2 + \pi^2)^3 - \left\{\left(\frac{\varphi_1}{\varphi_2}\right)^{1/3}(\varphi_2\alpha^2 + \pi^2)\right\}^3 = 0. \tag{39}$$

We have

$$(\varphi_1\alpha^2 + \pi^2) - \left(\frac{\varphi_1}{\varphi_2}\right)^{1/3}(\varphi_2\alpha^2 + \pi^2) = 0 \quad , \tag{40}$$

or

$$(\varphi_1\alpha^2 + \pi^2)^2 + (\varphi_1\alpha^2 + \pi^2)\left\{\left(\frac{\varphi_1}{\varphi_2}\right)^{1/3}(\varphi_2\alpha^2 + \pi^2)\right\}$$
$$+\left\{\left(\frac{\varphi_1}{\varphi_2}\right)^{1/3}(\varphi_2\alpha^2 + \pi^2)\right\}^2 = 0. \tag{41}$$

Since, equation (41) does not hold, we obtain (34),(35), (36)from (40) . □

Next, we study the stability of the invariant torus in (33).

Lemma 4.2. *Assume*

(1) : $a, b < 0, e < d < 0, ad-bc > 0$, *or* (2) : $a, c < 0, e < d < 0, ad-bc > 0$. *in (33). Then (33) have two asymptotically stable equilibriums:* $(r_1, r_2, r_3) = (r_*, 0, s_*), (r_*, s_*, 0)$ *by a suitable choice of* (μ_1, mu_2).

proof

From $\dot{r_1} = \dot{r_2} = \dot{r_3} = 0$, we can obtain r_*, s_* as follows.

$$r_* = \sqrt{\frac{b\mu_2 - d\mu_1}{ad - bc}} \ , \quad s_* = \sqrt{\frac{c\mu_1 - a\mu_2}{ad - bc}} \quad . \tag{42}$$

By assumption(1), if we take μ_1, μ_2 which satisfies $\frac{c}{a}\mu_1 < \mu_2 < \frac{d}{b}\mu_1$ then $r_*, s_* \in \mathbf{R}$. On the other hand, if assumption(2) and $\frac{b}{d}\mu_2 < \mu_1 < \frac{a}{c}\mu_2$ hold, then $r_*, s_* \in \mathbf{R}$. Thus $(r_1, r_2, r_3) = (r_*, 0, s_*), (r_*, s_*, 0)$ are equilibriums of (33). Moreover, one of the linearized eigenvalue about $(r_*, 0, s_*)$(or $(r_*, s_*, 0)$) is

$$(d - e)(a\mu_2 - c\mu_1)/(ad - bc) \ . \tag{43}$$

And another two eigenvalues are given as the solutions of the following quadratic equation:

$$\lambda^2 - 2(ar_*^2 + ds_*^2)\lambda + 4(ad - bc)r_*^2 s_*^2 \ . \tag{44}$$

This completes the proof. □

Finally, by Lemmas4.1,4.2, we can show that stability of the invariant torus which corresponds to the patchwork quilt pattern for each ratio of the system size. We shall concentrate on the case when there are three critical modes which are given by $(\kappa, 0, 1)$, $(\kappa - 1, 1, 1)$, $(\kappa - 1, -1, 1)$. Here $\kappa \in \{2, 3, 4, 5, 6, 7\}$ and take the limit $P \to 0$. (Remark that if $P \neq 0$ then $a, d < 0$ and $a, d \to 0$ *as* $P \to 0$, here a, d are coefficients of (33)) Then there are three cases as follows.

- **case1**:
  Let $\kappa \in \{2, 3\}$, then there exists $\eta_*^{(\kappa)}$ for each κ such that if $\frac{\kappa}{2} < \eta < \eta_*^{(\kappa)}$ then $ad - bc > 0$. Moreover, if $\frac{\kappa}{2} < \eta < \eta_*^{(\kappa)}$ then the patchwork quilt pattern can be stable in the sense of (33).
- **case2**:
  Let $\kappa \in \{4, 5, 6\}$, then there exist $\eta_1^{(\kappa)}, \eta_2^{(\kappa)}$ with $\eta_1^{(\kappa)} < \eta_2^{(\kappa)}$ for each κ such that if $\frac{\kappa}{2} < \eta < \eta_1^{(\kappa)}$ or $\eta_2^{(\kappa)} < \eta < 2\sqrt{\kappa - 1}$ then $ad - bc > 0$. Moreover, if $\frac{\kappa}{2} < \eta < \eta_1^{(\kappa)}$ or $\eta_2^{(\kappa)} < \eta < 2\sqrt{\kappa - 1}$ then the patchwork quilt pattern can be stable in the sense of (33).

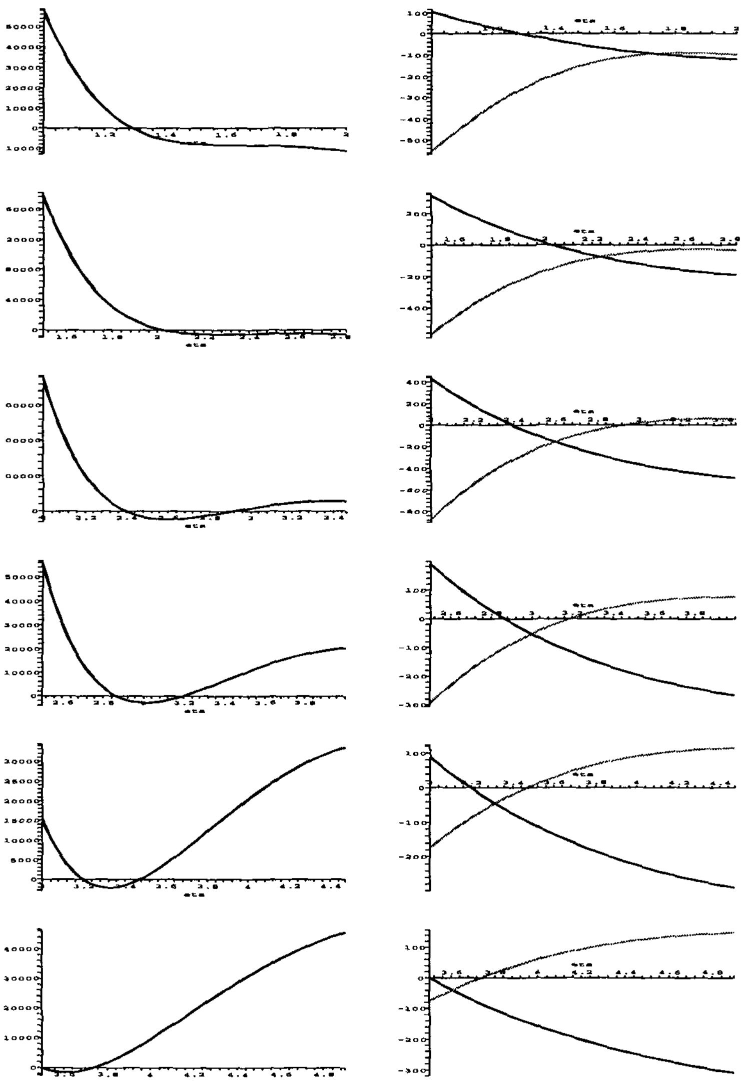


Figure 6. Behaviors of normal form coefficients at each pseudo hexagonal critical point on $H_{(\kappa,0,1)(\kappa-1,1,1)}$. Left side figures: Graphs of $ad-bc$ vs. $\eta \in [\kappa/2, 2\sqrt{\kappa-1}]$. Right side figures: Graphs of b(black) and c(gray) vs. $\eta \in [\kappa/2, 2\sqrt{\kappa-1}]$. Each pair of figures in the same column corresponds to $\kappa = 2,3,4,5,6,7$ from the top.

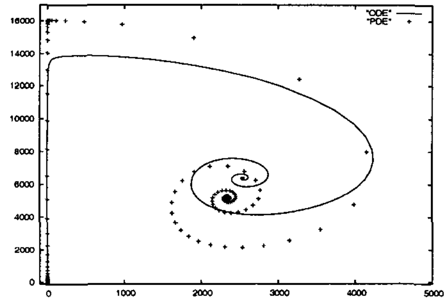


Figure 7. Qualitative comparison between the PDE numerical simulation(greek cross) and the dynamics on the center manifold by the normal form(line). In both figures horizontal and vertical axis represent the amplitudes of $(2,0,1)$-mode and $(1,1,1)$-mode, respectively. The PDE simulation was obtained at the parameter values: $P = 0.2, (\alpha, \beta, R) = (1.2, 1.44, 673)$. The same values of P, R were used for the normal form.

- **case3**:
  Let $\kappa = 7$, then there exists η^* such that if $\eta^* < \eta < 2\sqrt{6}$ then $ad - bc > 0$. Moreover, if $\eta^* < \eta < 2\sqrt{6}$ then the patchwork quilt pattern can be stable in the sense of (33).

Remark

We can calculate the coefficients of the normal form for $\eta \in [0, \infty)$. In fact, for $\kappa \in \{2,3,4,5,6,7\}$, if $\eta = \kappa/2$ or $\eta = 2\sqrt{\kappa - 1}$ then, another modes become unstable at the same time. More precisely, if $\eta = \kappa/2$ then $G_{\kappa,0} = G_{0,2}$, that means the $(0,2,1)$ mode becomes unstable as well as $(\kappa, 0, 1)$ and $(\kappa - 1, 1, 1)$. On the other hand if $\eta = 2\sqrt{\kappa - 1}$ then $G_{\kappa,0} = G_{\kappa-2,1}$,that means the $(\kappa-2, 1, 1)$ mode becomes unstable as well as $(\kappa, 0, 1)$ and $(\kappa - 1, 1, 1)$. On the contrast, both $(\kappa, 0, 1)$ and $(\kappa - 1, \pm 1, 1)$ modes are first instability, when η satisfies $\kappa/2 < \eta < 2\sqrt{\kappa - 1}$ for each $\kappa \in \{2,3,4,5,6,7\}$.

Figure 6 shows that the patch work quilt patterns can be stable. Notice that $e < d = 0$ holds and $b = c$ at $\eta = \eta_c$ for each κ .

This implies that if we take P sufficiently small then $a < 0$ and $e < d < 0$ hold. Moreover, for $\kappa = 2,3,4,5,6$ there exists η which satisfies $b < 0, ad - bc > 0$. On the other hand, for $\kappa = 4,5,6,7$, there exist η which satisfies $c < 0, ad - bc > 0$ for each $\kappa = 4,5,6,7$. Thus, we can show, in any neighborhood of $(\alpha_*(\eta), \beta_*(\eta), R_*(\eta))$ there is parameter value (α, β, R) where the patchwork quilt pattern is stable. We can actually observe the corresponding stable patchwork quilt pattern in the PDE numerical simulation (see Figure 7).

Acknowledgements

In the PDE numerical simulations we use the code written by Tsutomu Ikeda (Ryukoku university) and Tadanobu Kimura (Sharp Corporation). We greatly appreciate their hospitality.

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SOME CONVERGENCE RESULTS FOR ELLIPTIC PROBLEMS WITH PERIODIC DATA

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We address some convergence issues regarding the solution of elliptic problems when the data are periodic and the size of the domain becomes unbounded.

1. Introduction

In [3], [5] we have considered elliptic problems of the type:

$$\begin{cases} u_n \in V_n, \\ -\partial_{x_i}(a_{ij}\partial_{x_j}u_n) + a(x)u_n = f(x) \quad \text{in } \Omega_n, \end{cases} \tag{1.1}$$

where $a_{ij}(x)$ are bounded functions in $\mathbb{R}^k$ satisfying a uniform ellipticity condition, $a(x)$ is a nonnegative function, Ω_n is a bounded domain whose size is becoming infinite when n goes to infinity, V_n is a closed linear space such that

$$H_0^1(\Omega_n) \subset V_n \subset H^1(\Omega_n).$$

We assumed in (1.1) that all the coefficients and f were periodic with respect to x. We have then shown in [3] [5] that the periodic data force the solution u_n to (1.1) to be periodic when the domain Ω_n becomes larger and larger. To be precise, we derived H^1-convergence of u_n towards a periodic function when $n \to \infty$ in the nondegenerate case ($a(x) \not\equiv 0$) and L^∞-weak convergence of u_n in the degenerate case ($a(x) \equiv 0$).

In this note, we will discuss the issue when the domain where the problems is posed is arbitrary, which means that we will show that such convergence is independent of the shape of the domain. We want to study the asymptotic behavior of u_n when $n \to \infty$, we will in particular also address the question of the rate of convergence of u_n and obtain some exponential convergence. Moreover, we will show that the convergence of u_n to a periodic function extends to a large class of nonlinear operators. For parabolic problems, we refer the readers to [4].

2. Convergence in the linear case

Let T be a positive constant and Ω_n be a bounded domain in $\mathbb{R}^k$ whose size is going to become infinite when n approaches to infinity. For simplicity, we will always assume that

$$(-nT, nT)^k \subset \Omega_n.$$

Suppose $a_{ij}(x), (i, j = 1, \cdots, k), a(x)$ to be bounded functions in $\mathbb{R}^k$ such that

$$\begin{gathered} \exists \lambda > 0 \text{ such that } a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^k, \text{a.e. } x \in \mathbb{R}^k \\ a(x) \geq 0, \qquad \text{a.e. } x \in \mathbb{R}^k, a(x) \not\equiv 0. \end{gathered}$$

Denote by $f_i(x), i = 0, \cdots, k$ functions in $L^2_{\text{loc}}(\mathbb{R}^k)$. Moreover, assume that $a_{ij}(x)$, $a(x)$ and $f_i(x)$ are T-periodic, i.e.

$$a_{ij}(x)(resp.\ a(x), f_i(x)) = a_{ij}(x + Te_\ell)(resp.\ a(x + Te_\ell), f_i(x + Te_\ell)), \qquad \text{a.e. } x \in \mathbb{R}^k,$$

for every ℓ, where (e_ℓ) is the canonical basis of $\mathbb{R}^k$.

If u_n is the unique solution to

$$\begin{cases} u_n \in H^1_0(\Omega_n) \\ \displaystyle\int_{\Omega_n} a_{ij}(x)\partial_{x_j} u_n \partial_{x_i} v + a(x) u_n v \, \mathrm{d}x \\ \qquad = \displaystyle\int_{\Omega_n} f_0 v + f_i \partial_{x_i} v \, \mathrm{d}x \qquad \forall v \in H^1_0(\Omega_n), \end{cases} \tag{2.1}$$

and u_∞ is the unique solution to

$$\begin{cases} u_\infty \in H^1_{\text{per}}(Q) \\ \displaystyle\int_Q a_{ij}(x)\partial_{x_j} u_\infty \partial_{x_i} v + a(x) u_\infty v \, \mathrm{d}x \\ \qquad = \displaystyle\int_Q f_0 v + f_i \partial_{x_i} v \, \mathrm{d}x \qquad \forall v \in H^1_{\text{per}}(Q), \end{cases} \tag{2.2}$$

(here we use Einstein convention of repeated indices) where $Q = (0, T)^k$ and

$$H^1_{\text{per}}(Q) = \{v \in H^1(Q) | v(x + Te_j) = v(x) \quad \text{a.e. } x \in \partial Q \cap \{x_j = 0\}\},$$

we have:

Theorem 2.1. *Suppose that u_n and u_∞ are solutions to* (2.1) *and* (2.2) *respectively. If*
1) the functions f_i on the right-hand side of (2.1), (2.2) *satisfy*

$$f_0 \in L^{\frac{s}{2}}_{loc}(\mathbb{R}^k), f_i \in L^s_{loc}(\mathbb{R}^k), i = 1, \cdots, k, s > k, \tag{2.3}$$

or
2) it exists $\beta > 1$, $M > 0$ and n' related to n, such that

$$\begin{gathered}(-nT, nT)^k \subset \Omega_n \subset (-n'T, n'T)^k, \\ n' < Mn^\beta,\end{gathered} \tag{2.4}$$

when n is large enough (see the figure below), then we have that for any $n_0 > 0$,

$$u_n \longrightarrow u_\infty \qquad \text{in } H^1(Q_{n_0}),$$

where $Q_{n_0} = (-n_0T, n_0T)^k$.

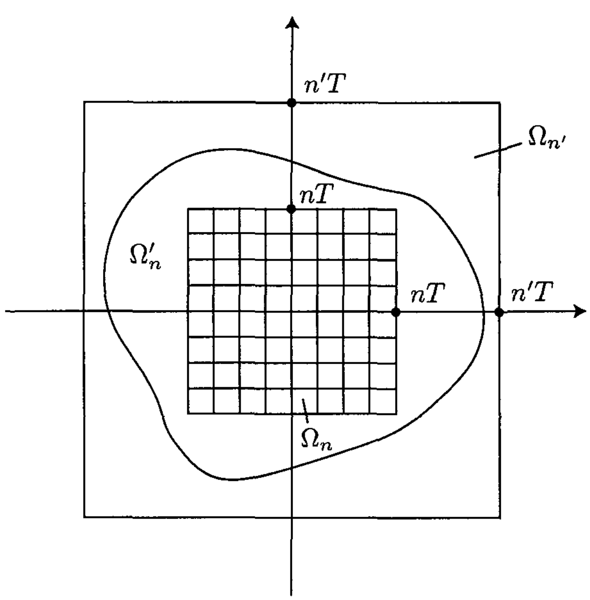

Figure 2.1.

The proof is based on several lemmas:

Lemma 2.1. *(see [3] [6] for a proof) Under the periodicity assumptions on the coefficients and the f_i, if u_∞ is extended by periodicity to $\mathbb{R}^k$, it holds that*

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j}u_\infty) + a(x)u_\infty = f_0 - \partial_{x_i} f_i \quad \text{in } \mathcal{D}'(\mathbb{R}^k). \tag{2.5}$$

In particular, for any bounded domain Ω, we have

$$\int_\Omega a_{ij}(x)\partial_{x_j}u_\infty \partial_{x_i} v + a(x)u_\infty v \,\mathrm{d}x = \int_\Omega f_0 v + f_i \partial_{x_i} v \,\mathrm{d}x \quad \forall v \in H_0^1(\Omega). \tag{2.6}$$

Next, let us quote a maximum principle from [8].

Lemma 2.2. *Suppose (2.3) holds. Then, if u is a $W^{1,2}(\Omega)$ solution to*

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j}u) + a(x)u = f_0 - \partial_{x_i} f_i \quad \text{in } \Omega,$$

we have, for all ball $B_{2R}(y) \subset \Omega$ and $p > 1$,

$$sup_{B_R(y)}|u(x)| \le c(R^{-\frac{k}{p}} \parallel u \parallel_{L^p(B_{2R}(y))} + c_1(R)),$$

where c and $c_1(R)$ are constants depending on k, λ, $max\{a_{ij}\}$, R, s and p.

Furthermore we will need the following lemma (see [3]):

Lemma 2.3. *(Poincaré Inequality) For any $n_0 > 0$, $Q_{n_0} = (-n_0 T, n_0 T)^k$, if $a(x)$ is nonnegative and not identically equal to 0, $a(x)$ is T-periodic in all directions, it holds that*

$$\int_{Q_{n_0}} |\nabla v|^2 + v^2 \,\mathrm{d}x \le C \int_{Q_{n_0}} |\nabla v|^2 + a(x)v^2 \,\mathrm{d}x \qquad \forall\, v \in H^1(Q_{n_0}), \tag{2.7}$$

where C is a constant, independent of n_0, depending only on $Q = (0,T)^k$ and $a(x)$.

We can then complete the proof of Theorem 2.1:

Proof. Set

$$Q = (0,T)^k, \quad Q_n = (-nT, nT)^k.$$

Let ρ be a smooth nonnegative function such that

$$\rho \equiv 1 \text{ on } (-\frac{1}{2}, \frac{1}{2}), \quad \rho = 0 \text{ outside } (-1,1), \quad |\nabla\rho| \text{ is bounded.}$$

For any $n_1 \leq n$ (we also assume that n_1 is an even number),

$$(u_n - u_\infty)\Pi_{i=1}^k \rho^2(\frac{x_i}{n_1 T}) := (u_n - u_\infty)\Pi^2 \tag{2.8}$$

is a test function in $H_0^1(Q_{n_1})$. It follows that from (2.1) and (2.6) that

$$\begin{aligned}\int_{Q_{n_1}} & a_{ij}(x)\partial_{x_j} u_n \partial_{x_i}\{(u_n - u_\infty)\Pi^2\} + a(x)u_n(u_n - u_\infty)\Pi^2 \,\mathrm{d}x \\ &= \int_{Q_{n_1}} a_{ij}(x)\partial_{x_j} u_\infty \partial_{x_i}\{(u_n - u_\infty)\Pi^2\} + a(x)u_\infty(u_n - u_\infty)\Pi^2 \,\mathrm{d}x,\end{aligned}$$

i.e.

$$\int_{Q_{n_1}} a_{ij}(x)\partial_{x_j}(u_n - u_\infty)\partial_{x_i}\{(u_n - u_\infty)\Pi^2\} + a(x)(u_n - u_\infty)^2\Pi^2 \,\mathrm{d}x = 0.$$

The above formula leads to

$$\begin{aligned}\int_{Q_{n_1}} & \{a_{ij}(x)\partial_{x_j}(u_n - u_\infty)\partial_{x_i}(u_n - u_\infty) + a(x)(u_n - u_\infty)^2\}\Pi^2 \,\mathrm{d}x \\ &= -\frac{2}{n_1 T}\int_{Q_{n_1}} a_{ij}(x)\partial_{x_j}(u_n - u_\infty)\partial_{x_i}\Pi(u_n - u_\infty)\Pi \,\mathrm{d}x.\end{aligned}$$

Using the ellipticity condition on the left-hand side and applying Cauchy-Schwarz inequality on the right-hand side we obtain:

$$\begin{aligned}&\int_{Q_{n_1}} \{\lambda|\nabla(u_n - u_\infty)|^2 + a(x)(u_n - u_\infty)^2\}\Pi^2 \,\mathrm{d}x \\ &\leq \frac{c}{n1}\int_{Q_{n_1}} |\nabla(u_n - u_\infty)||u_n - u_\infty|\Pi \,\mathrm{d}x \\ &\leq \frac{c}{n1}\left\{\int_{Q_{n_1}} |\nabla(u_n - u_\infty)|^2\Pi^2 \,\mathrm{d}x\right\}^{\frac{1}{2}}\left\{\int_{Q_{n_1}} (u_n - u_\infty)^2 \,\mathrm{d}x\right\}^{\frac{1}{2}}.\end{aligned}$$

This leads to:

$$\int_{Q_{n_1}} \{|\nabla(u_n - u_\infty)|^2 + a(x)(u_n - u_\infty)^2\}\Pi^2 \,\mathrm{d}x \leq \frac{c}{n_1^2}\int_{Q_{n_1}} (u_n - u_\infty)^2 \,\mathrm{d}x, \tag{2.9}$$

where c is a constant independent of n.

Since $\Pi \equiv 1$ on $Q_{\frac{n_1}{2}}$, by Lemma 2.3, (2.9) implies that

$$\begin{aligned}
&\int_{Q_{\frac{n_1}{2}}} |\nabla(u_n - u_\infty)|^2 + (u_n - u_\infty)^2 \,\mathrm{d}x \\
&\le c \int_{Q_{\frac{n_1}{2}}} \{|\nabla(u_n - u_\infty)|^2 + a(x)(u_n - u_\infty)^2\} \,\mathrm{d}x \\
&\le c \int_{Q_{n_1}} \{|\nabla(u_n - u_\infty)|^2 + a(x)(u_n - u_\infty)^2\}\Pi^2 \,\mathrm{d}x \\
&\le \frac{c}{n_1^2} \int_{Q_{n_1}} (u_n - u_\infty)^2 \,\mathrm{d}x \\
&\le \frac{c}{n_1^2} \int_{Q_{n_1}} |\nabla(u_n - u_\infty)|^2 + (u_n - u_\infty)^2 \,\mathrm{d}x \qquad (2.10)
\end{aligned}$$

for some constant c independent of n_1.

Choosing $n_1 = \frac{n}{2^{\ell-1}}$ (ℓ is a positive integer, large enough) in (2.10) and iterating the above inequality $(\ell - 1)$ times, we get

$$\begin{aligned}
&\int_{Q_{\frac{n}{2^\ell}}} |\nabla(u_n - u_\infty)|^2 + (u_n - u_\infty)^2 \,\mathrm{d}x \\
&\le \frac{c}{n^{2(\ell-1)}} \int_{Q_{\frac{n}{2}}} |\nabla(u_n - u_\infty)|^2 + (u_n - u_\infty)^2 \,\mathrm{d}x \\
&\le \frac{c}{n^{2\ell}} \int_{Q_n} (u_n - u_\infty)^2 \,\mathrm{d}x \\
&\le \frac{2c}{n^{2\ell}} \left\{ \int_{Q_n} u_n^2 \,\mathrm{d}x + \int_{Q_n} u_\infty^2 \,\mathrm{d}x \right\} \\
&\le \frac{2c}{n^{2\ell}} \left\{ \int_{Q_n} u_n^2 \,\mathrm{d}x + (2n)^k \int_{Q} u_\infty^2 \,\mathrm{d}x \right\}, \qquad (2.11)
\end{aligned}$$

since u_∞ is periodic.

Now we need to estimate $\displaystyle\int_{Q_n} u_n^2 \,\mathrm{d}x$.

• In the case 1), we recall from Lemma 2.1 that:

$$-\partial_{x_i}(a_{ij}(x)\partial_{x_j}u_\infty) + a(x)u_\infty = f_0 - \partial_{x_i} f_i \quad \text{in } \mathcal{D}'(\mathbb{R}^k).$$

Since $f_0 \in L^{\frac{s}{2}}_{\mathrm{loc}}(\mathbb{R}^k)$, $f_i \in L^s_{\mathrm{loc}}(\mathbb{R}^k)$ $(s > k)$, by Lemma 2.2 we have

$$sup_{B_{\sqrt{k}T}(0)}|u_\infty| \le c(\sqrt{k}T^{-\frac{k}{2}} \,\|\, u_\infty \,\|_{L^2(B_{2\sqrt{k}}(0))} + c_1(\sqrt{k}T)).$$

Due to the facts that

$$(-T,T)^k \subset B_{\sqrt{k}T}(0), \quad u_\infty \text{ is periodic},$$

this shows that u_∞ is bounded by a positive constant. One has also

$$\begin{cases} -\partial_{x_i}(a_{ij}(x)\partial_{x_j}(u_n - u_\infty)) + a(x)(u_n - u_\infty) = 0 & \text{in } \Omega_n, \\ u_n - u_\infty = -u_\infty \quad \text{on } \partial\Omega_n. \end{cases}$$

Therefore, by the maximum principle, we obtain that

$$max_{\Omega_n}|u_n - u_\infty| \leq max|u_\infty| \leq M, \tag{2.12}$$

where M is a constant depending on the coefficients of the operator, k, s and independent of n. In other words,

$$|u_n| \leq 2M.$$

Going back to (2.11), this leads to

$$\begin{aligned} \int_{Q_{\frac{n}{2^\ell}}} |\nabla(u_n - u_\infty)|^2 + (u_n - u_\infty)^2 \, dx &\leq \frac{c}{n^{2\ell}} \left\{ 4M^2|Q_n| + (2n)^k \int_Q u_\infty^2 \, dx \right\} \\ &\leq \frac{c}{n^{2\ell - k}}. \end{aligned}$$

Choosing $\frac{n}{2^\ell} \geq n_0$, $2\ell - k > 0$, we derive that u_n converges towards u_∞.

• In the case 2), one takes $v = u_n \in H_0^1(\Omega_n)$ in (2.1). It comes:

$$\begin{aligned} &\int_{\Omega_n} a_{ij}(x)\partial_{x_j} u_n \partial_{x_i} u_n + a(x) u_n^2 \, dx \\ &= \int_{\Omega_n} f_0 u_n + f_i \partial_{x_i} u_n \, dx \\ &\leq \left\{ \sum_{i=0}^{k} \| f_i \|_{L^2(\Omega_n)}^2 \right\}^{\frac{1}{2}} \left\{ \int_{\Omega_n} |\nabla u_n|^2 + u_n^2 \, dx \right\}^{\frac{1}{2}}. \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} \int_{\Omega_n} |\nabla u_n|^2 + u_n^2 \, dx &\leq c \sum_{i=0}^{k} \int_{\Omega_n} f_i^2 \, dx \\ &\leq c \sum_{i=0}^{k} \int_{Q_{n'}} f_i^2 \, dx \\ &\leq c \sum_{i=0}^{k} (2n')^k \int_Q f_i^2 \, dx \\ &\leq c n^{k\beta} \end{aligned}$$

Therefore, one derives from the inequality (2.11) that

$$\begin{aligned}
&\int_{\Omega_{\frac{n}{2^\ell}}} |\nabla(u_n - u_\infty)|^2 + (u_n - u_\infty)^2 \,\mathrm{d}x \\
&\le \frac{c}{n^{2\ell}} \left\{ \int_{Q_n} u_n^2 \,\mathrm{d}x + (2n)^k \int_Q u_\infty^2 \,\mathrm{d}x \right\} \\
&\le \frac{c}{n^{2\ell}} \left\{ \int_{\Omega_n} u_n^2 \,\mathrm{d}x + (2n)^k \int_Q u_\infty^2 \,\mathrm{d}x \right\} \\
&\le \frac{c}{n^{2\ell - k\beta}}.
\end{aligned}$$

The proof is complete by letting $\frac{n}{2^\ell} \ge n_0$ and $2\ell - k\beta > 0$. □

From Theorem 2.1, we deduce that the convergence rate of u_n towards u_∞ is any power of n. However, if we consider that $a(x)$ is bounded away from 0, i.e. if

$$\exists \lambda_0 \text{ such that } a(x) \ge \lambda_0 > 0,$$

a higher convergence can yet be achieved.

We first consider the case of the Laplace operator. Set u_n and u_∞ the solutions to

$$\begin{cases} u_n \in H_0^1(Q_n), \\ -\Delta u_n + a(x) u_n = f_0 - \partial_{x_i} f_i \quad \text{in } Q_n, \end{cases} \tag{2.13}$$

and

$$\begin{cases} u_\infty \in H_{\mathrm{per}}^1(Q), \\ -\Delta u_\infty + a(x) u_\infty = f_0 - \partial_{x_i} f_i \quad \text{in } Q. \end{cases} \tag{2.14}$$

Furthermore, assume that u_∞, solution to (2.14), is bounded, we can then show that:

Theorem 2.2. *For u_n and u_∞(bounded) solutions to* (2.13) *and* (2.14) *respectively, for any domain $Q_{n_0} = (-n_0 T, n_0 T)^k$, there exist constants $c_1 = c_1(n_0, a)$, $c_2 = c_2(a) > 0$ such that*

$$|u_n - u_\infty|_\infty \le c_1 e^{-c_2 n T}, \tag{2.15}$$

and

$$|u_n - u_\infty|_{H^1(Q_{n_0})} \le c_1 e^{-c_2 n T}, \tag{2.16}$$

where $|\cdot|_\infty$ denotes the L^∞-norm in Q_{n_0}.

Proof. If $ch(x) = \frac{1}{2}\{e^x + e^{-x}\}$, define

$$\Pi_n(x) = \sum_{i=1}^{k} \frac{ch(c_2 x_i)}{ch(c_2 nT)},$$

where c_2 is a positive constant satisfying

$$c_2^2 \le \lambda_0 \le a(x).$$

Then one has that

$$-\Delta\Pi_n(x) = -c_2^2 \sum_{i=1}^{k} \frac{ch(c_2 x_i)}{ch(c_2 nT)} = -c_2^2 \Pi_n(x),$$

i.e. it holds that

$$\begin{cases} -\Delta\Pi_n(x) + c_2^2\Pi_n(x) = 0 & \text{in } Q_n, \\ \Pi_n(x) \ge 1 & \text{on } \partial Q_n. \end{cases}$$

Since u_n and u_∞ satisfy (2.13), (2.14), we have:

$$\begin{cases} -\Delta(u_n - u_\infty) + a(x)(u_n - u_\infty) = 0 & \text{in } Q_n, \\ u_n - u_\infty = -u_\infty & \text{on } \partial Q_n. \end{cases} \tag{2.17}$$

If we set $w = |u_\infty|_\infty \Pi_n - (u_n - u_\infty)$, we obtain that

$$\begin{aligned} -\Delta w &= -|u_\infty|_\infty \Delta\Pi_n + \Delta(u_n - u_\infty) \\ &= -|u_\infty|_\infty c_2^2 \Pi_n + a(x)(u_n - u_\infty) \\ &= -a(x)|u_\infty|_\infty \Pi_n + \{a(x) - c_2^2\}|u_\infty|_\infty \Pi_n + a(x)(u_n - u_\infty) \\ &= -a(x)w + \{a(x) - c_2^2\}|u_\infty|_\infty \Pi_n, \end{aligned}$$

$\Longrightarrow$ w satisfies

$$\begin{cases} -\Delta w + a(x)w = \{a(x) - c_2^2\}|u_\infty|_\infty \Pi_n \ge 0 & \text{in } Q_n, \\ w \ge |u_\infty|_\infty + u_\infty \ge 0 & \text{on } \partial Q_n. \end{cases}$$

By the maximum principle, one has that

$$w \ge 0 \quad \text{in } Q_n,$$

i.e.

$$|u_\infty|_\infty \Pi_n \ge u_n - u_\infty \quad \text{in } Q_n.$$

Arguing the same way with $w = -|u_\infty|_\infty \Pi_n - (u_n - u_\infty)$, one derives that

$$-|u_\infty|_\infty \Pi_n \le u_n - u_\infty \quad \text{in } Q_n,$$

which is equivalent to

$$|u_n - u_\infty| \le |u_\infty|_\infty \Pi_n \quad \text{in } Q_n. \tag{2.18}$$

For any fixed domain K with $K \subset B(0, d_K)$ we derive

$$\begin{aligned} |u_n - u_\infty|_{\infty,K} &\le |u_\infty|_\infty \Big(k \frac{ch(c_2 d_K)}{ch(nc_2T)}\Big) \\ &\le c_1 e^{-c_2 T n}. \end{aligned} \tag{2.19}$$

This completes the proof of (2.15).

Now, we introduce a smooth nonnegative function $\rho(x)$ such that

$$\rho \equiv 1 \text{ in } Q_{n_0}, \rho \equiv 0 \text{ outside } Q_{n_0+1}, |\nabla \rho| \text{ is bounded,}$$

$(n_0+1 \le n)$. Plugging $v = (u_n - u_\infty)\rho^2$ into the weak formulation of (2.17) we get:

$$\int_{Q_{n_0+1}} \nabla(u_n - u_\infty)\nabla\{(u_n - u_\infty)\rho^2\} + a(x)(u_n - u_\infty)^2 \rho^2 \,\mathrm{d}x = 0,$$

i.e.

$$\begin{aligned} &\int_{Q_{n_0+1}} \{|\nabla(u_n - u_\infty)|^2 + a(x)(u_n - u_\infty)^2\}\rho^2 \,\mathrm{d}x \\ &= -2 \int_{Q_{n_0+1}\setminus Q_{n_0}} \nabla(u_n - u_\infty)\nabla\rho(u_n - u_\infty)\rho \,\mathrm{d}x \\ &\le c \int_{Q_{n_0+1}\setminus Q_{n_0}} |\nabla(u_n - u_\infty)||u_n - u_\infty|\rho \,\mathrm{d}x \\ &\le c \left\{ \int_{Q_{n_0+1}} |\nabla(u_n - u_\infty)|^2 \rho^2 \,\mathrm{d}x \right\}^{\frac{1}{2}} \left\{ \int_{Q_{n_0+1}\setminus Q_{n_0}} (u_n - u_\infty)^2 \,\mathrm{d}x \right\}^{\frac{1}{2}}. \end{aligned}$$

By Lemma 2.3 we obtain that

$$\begin{aligned} &\int_{Q_{n_0}} |\nabla(u_n - u_\infty)|^2 + (u_n - u_\infty)^2 \,\mathrm{d}x \\ &\le c \int_{Q_{n_0+1}\setminus Q_{n_0}} (u_n - u_\infty)^2 \,\mathrm{d}x \\ &\le c|u_n - u_\infty|^2_{\infty, Q_{n_0+1}} |Q_{n_0+1}\setminus Q_{n_0}| \\ &\le c_1 e^{-2c_2 nT}. \end{aligned}$$

This completes the proof of (2.16). □

Remark 2.1. Instead of the Laplace operator, if we are considering the general elliptic operator $L = -\partial_{x_i}(a_{ij}\partial_{x_j}) + a(x)$ where the coefficients $a_{ij}(x)$ are bounded, T-periodic and such that

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{a.e. } x \in \mathbb{R}^k, \forall \xi \in \mathbb{R}^k,$$

we can derive the same exponential convergence rate if we assume that

$$\exists D > 0 \quad \text{such that } |\partial_{x_i} a_{ij}(x)| \le D,$$

and

$$a(x) \ge \lambda_0 > 0.$$

Indeed setting $w = \sum_{i=1}^k \frac{ch(\alpha x_i)}{ch(\alpha nT)}$ (α will be determined later), one sees that

$$\partial_{x_j} w = \alpha \frac{sh(\alpha x_j)}{ch(\alpha nT)} \qquad (sh(x) = \frac{1}{2}\{e^x - e^{-x}\}),$$

and

$$\begin{aligned}
&-\partial_{x_i}(a_{ij}(x)\partial_{x_j} w) + a(x)w \\
&= -\partial_{x_i}\left(a_{ij}(x)\alpha\frac{sh(\alpha x_j)}{ch(\alpha nT)}\right) + a(x)w \\
&= -\partial_{x_i}(a_{ij}(x))\alpha\frac{sh(\alpha x_j)}{ch(\alpha nT)} - a_{ij}(x)\partial_{x_i}\left(\alpha\frac{sh(\alpha x_j)}{ch(\alpha nT)}\right) + a(x)w \\
&= -\partial_{x_i}(a_{ij}(x))\alpha\frac{sh(\alpha x_j)}{ch(\alpha nT)} - a_{ij}(x)\alpha^2\frac{ch(\alpha x_j)}{ch(\alpha nT)}\delta_{ij} + a(x)w \\
&\ge -D\alpha k^2 w - \Lambda\alpha^2 k w + a(x)w \\
&\ge \{a(x) - D\alpha k^2 - \Lambda\alpha^2 k\}w.
\end{aligned}$$

Choosing $\alpha > 0$ small enough we obtain that

$$\begin{cases} -\partial_{x_i}(a_{ij}(x)\partial_{x_j} w) + a(x)w \ge 0 & \text{in } Q_n, \\ w \ge 1 \quad \text{on } \partial Q_n, \end{cases}$$

i.e. w is a super solution associated with the operator L.

Hence we obtain that by setting $u = |u_\infty|_\infty w - (u_n - u_\infty)$

$$\begin{cases} -\partial_{x_i}(a_{ij}(x)\partial_{x_j} u) + a(x)u \ge 0 & \text{in } Q_n, \\ u \ge 0 \quad \text{on } \partial Q_n. \end{cases}$$

Mimicking the above proof, one can obtain an exponential rate of convergence of u_n towards u_∞.

3. Convergence in some nonlinear cases

In this section, we will discuss some quasilinear cases. We recall that we denote that Q_n and Q the bounded domains in $\mathbb{R}^k$ defined by

$$Q_n = (-nT, nT)^k, \quad Q = (0,T)^k,$$

T is a positive constant, the period of the data which will still be periodic.

More precisely, suppose that $a_{ij}(x,p)$ $(i,j = 1,\cdots,k)$ are bounded, Carathédory functions in $\mathbb{R}^k$,

$$p \to a_{ij}(x,p) \text{ is continuous a.e. } x \in \mathbb{R}^k,$$
$$x \to a_{ij}(x,p) \text{ is measurable for } \forall p \in \mathbb{R}^k.$$

We also assume that the $a_{ij}(x,p)$ satisfy the elliptic condition:

$$\exists\lambda > 0 \text{ such that } \lambda|\xi|^2 \le a_{ij}(x,p)\xi_i\xi_j \quad \text{a.e. } x \in \mathbb{R}^k, \forall p \in \mathbb{R}, \forall \xi \in \mathbb{R}^k. \tag{3.1}$$

Moreover, we suppose that there exists a nondecreasing positive function $\omega(x)$ such that

$$|a_{ij}(x,u) - a_{ij}(x,v)| \le \omega(|u-v|) \quad \forall i,j, \tag{3.2}$$

where

$$\int_{0+} \frac{\mathrm{d}s}{\omega(s)^2} = +\infty,$$

(such a condition holds for instance when $a_{ij}(x)$ is Hölder continuous of order less or equal to $\frac{1}{2}$).

Without loss of generality, for any positive constant ϵ, we can assume that it holds

$$\int_\epsilon^{+\infty} \frac{\mathrm{d}s}{\omega(s)^2} < \infty.$$

Assume also that $a(x)$ is positive and bounded, $f(x)$ is bounded, $a_{ij}(x)$, $a(x)$ and $f(x)$ are T-periodic. To be precise for every ℓ:

$$a_{ij}(x,p)(resp.\ a(x), f(x)) = a_{ij}(x+Te_\ell, p)(resp.\ a(x+Te_\ell), f(x+Te_\ell)) \quad \text{a.e. } x \in \mathbb{R}^k.$$

Define u_n and u_∞ to be the solutions to

$$\begin{cases} u_n \in H_0^1(Q_n), \\ -\partial_{x_i}(a_{ij}(x,u_n)\partial_{x_j}u_n) + a(x)u_n = f(x) \quad \text{in } Q_n, \end{cases} \tag{3.3}$$

and

$$\begin{cases} u_\infty \in H^1_{\text{per}}(Q), \\ -\partial_{x_i}(a_{ij}(x, u_\infty)\partial_{x_j} u_\infty) + a(x)u_\infty = f(x) \quad \text{in } Q, \end{cases} \tag{3.4}$$

where $H^1_{\text{per}}(Q) = \{v \in H^1(Q) \mid v(x) = v(x + Te_j) \quad \text{a.e. } x \in \partial Q \cap \{x_j = 0\}\}$. We know that (3.3) and (3.4) admit a weak solution and especially (3.3) has a unique one (see [1], [2] for reference). For the reader's convenience, we would like to give a proof of this last assertion in the case of (3.4). We have:

Proposition 3.1. (3.4) *possesses a unique solution.*

Proof. Let u_1 and u_2 be two solutions to (3.4). It is obvious that

$$u_1 - u_2 \in H^1_{\text{per}}(Q).$$

Define

$$F_\epsilon(x) = \begin{cases} \frac{1}{I_\epsilon} \displaystyle\int_\epsilon^x \frac{\mathrm{d}s}{\omega(s)^2} & x > \epsilon > 0, \\ 0 \quad x \le \epsilon, \end{cases} \tag{3.5}$$

with

$$I_\epsilon = \int_\epsilon^{+\infty} \frac{\mathrm{d}s}{\omega(s)^2}.$$

It is clear that, when $x > 0$ and $\epsilon \to 0$, we have

$$F_\epsilon \to 1.$$

From the definition of F_ϵ, we have also:

$$F_\epsilon(0) = 0, \quad F'_\epsilon = \frac{1}{I_\epsilon \omega^2(x)} < \infty \quad \text{for } x > \epsilon.$$

Thus, (see [1]), we have

$$F_\epsilon(u_1 - u_2) \in H^1_{\text{per}}(Q).$$

Using this function into formula (3.4), we obtain

$$\int_Q a_{ij}(x, u_1)\partial_{x_j} u_1 \partial_{x_i} F_\epsilon(u_1 - u_2) + a(x)u_1 F_\epsilon(u_1 - u_2)\,\mathrm{d}x$$
$$= \int_Q a_{ij}(x, u_2)\partial_{x_j} u_2 \partial_{x_i} F_\epsilon(u_1 - u_2) + a(x)u_2 F_\epsilon(u_1 - u_2)\,\mathrm{d}x,$$

i.e.

$$\int_Q a_{ij}(x,u_1)\partial_{x_j}(u_1-u_2)\partial_{x_i}F_\epsilon(u_1-u_2)+a(x)(u_1-u_2)F_\epsilon(u_1-u_2)\,\mathrm{d}x$$
$$=-\int_Q\{a_{ij}(x,u_1)-a_{ij}(x,u_2)\}\partial_{x_j}u_2\partial_{x_i}F_\epsilon(u_1-u_2)\,\mathrm{d}x,$$

$\Longrightarrow$

$$\int_{Q_\epsilon} a_{ij}(x,u_1)\partial_{x_j}(u_1-u_2)\partial_{x_i}(u_1-u_2)\frac{1}{I_\epsilon\omega^2}+a(x)(u_1-u_2)F_\epsilon(u_1-u_2)\,\mathrm{d}x$$
$$\leq k\int_{Q_\epsilon}\omega(u_1-u_2)|\nabla u_2||\nabla(u_1-u_2)|\frac{1}{I_\epsilon\omega^2}\mathrm{d}x,$$

where $Q_\epsilon=\{x\in Q\mid u_1-u_2>\epsilon\}$. By the ellipticity condition and using Cauchy-Schwarz inequality on the right-hand side we get:

$$\int_{Q_\epsilon}\lambda\frac{|\nabla(u_1-u_2)|^2}{I_\epsilon\omega^2}+a(x)(u_1-u_2)F_\epsilon(u_1-u_2)\,\mathrm{d}x$$
$$\leq k\left\{\int_{Q_\epsilon}\frac{|\nabla(u_1-u_2)|^2}{I_\epsilon\omega^2}\,\mathrm{d}x\right\}^{\frac{1}{2}}\left\{\int_{Q_\epsilon}\frac{1}{I_\epsilon}|\nabla u_2|^2\,\mathrm{d}x\right\}^{\frac{1}{2}},$$

$\Longrightarrow$

$$\int_Q a(x)(u_1-u_2)F_\epsilon(u_1-u_2)\,\mathrm{d}x\leq\frac{k^2}{I_\epsilon}\int_Q|\nabla u_2|^2\,\mathrm{d}x.$$

Letting $\epsilon\to 0$, one arrives to

$$\int_Q a(x)(u_1-u_2)^+\,\mathrm{d}x\leq 0.$$

Since $a(x)>0$, we obtain that

$$(u_1-u_2)^+\equiv 0.$$

Similarly

$$(u_1-u_2)^-=(u_2-u_1)^+=0.$$

Hence we proved the uniqueness for (3.4). □

We remark that in the nonlinear case, if u_∞ is the periodic solution to (3.4) extended by periodicity in $\mathbb{R}^k$, it holds that

Lemma 3.1. *u_∞, solution to (3.4), satisfies:*

$$-\partial_{x_i}(a_{ij}(x,u_\infty)\partial_{x_j}u_\infty)+a(x)u_\infty=f\quad\mathit{in}\ \mathcal{D}'(\mathbb{R}^k).$$

(This is nothing but Lemma 2.1 where $a_{ij}(x,u_\infty)$ is considered as $a_{ij}(x)$).
We have then:

Theorem 3.1. *Assuming that*

$$a(x) \geq \lambda_0 > 0,$$

and that there exists a positive constant D, such that:

$$|\partial_{x_j} a_{ij}(x,p)| \leq D \quad a.e.\ x \in \mathbb{R}^k, \forall p \in \mathbb{R}, \forall i,j = 1,\cdots,k,$$

let u_n and u_∞ be the solutions to (3.3) and (3.4) respectively. For every $\gamma > 0$ it holds that

$$|u_n - u_\infty|_{L^1(Q_{n_0})} \leq \frac{c}{n^\gamma},$$

where c is independent of $n > 0$, $Q_{n_0} = (-n_0T, n_0T)^k$.

Proof. Consider $F_\epsilon(x)$ and $\Pi(x)$ defined as in (3.5) and (2.8). Take $v = F_\epsilon(u_n - u_\infty)\Pi^2 \in H_0^1(Q_{n_1})$ into the weak formulation of (3.3) and (3.4), we obtain

$$\begin{aligned}
&\int_{Q_n} a_{ij}(x,u_n)\partial_{x_j}u_n\partial_{x_i}\{F_\epsilon(u_n-u_\infty)\Pi^2\} + a(x)u_nF_\epsilon(u_n-u_\infty)\Pi^2\,\mathrm{d}x \\
&= \int_{Q_n} a_{ij}(x,u_\infty)\partial_{x_j}u_\infty\partial_{x_i}\{F_\epsilon(u_n-u_\infty)\Pi^2\} + a(x)u_\infty F_\epsilon(u_n-u_\infty)\Pi^2\,\mathrm{d}x.
\end{aligned}$$

Therefore one derives that:

$$\begin{aligned}
&\int_{Q_n} a(x)(u_n-u_\infty)F_\epsilon\Pi^2\,\mathrm{d}x \\
&\quad + \int_{Q_n} \{a_{ij}(x,u_n)\partial_{x_j}u_n - a_{ij}(x,u_\infty)\partial_{x_j}u_\infty\}\partial_{x_i}\Pi^2 F_\epsilon\,\mathrm{d}x \\
&= -\int_{Q_n} \{a_{ij}(x,u_n)\partial_{x_j}u_n - a_{ij}(x,u_\infty)\partial_{x_j}u_\infty\}\partial_{x_i}\{F_\epsilon(u_n-u_\infty)\}\Pi^2\,\mathrm{d}x \\
&= -\int_{Q_n} a_{ij}(x,u_n)\partial_{x_j}(u_n-u_\infty)\partial_{x_i}\{F_\epsilon(u_n-u_\infty)\}\Pi^2\,\mathrm{d}x \\
&\quad - \int_{Q_n} \{a_{ij}(x,u_n) - a_{ij}(x,u_\infty)\}\partial_{x_j}u_\infty\partial_{x_i}\{F_\epsilon(u_n-u_\infty)\}\Pi^2\,\mathrm{d}x.
\end{aligned}$$

If we denote the left-hand side of this equality by I, it holds that:

$$
\begin{aligned}
I = &-\int_{Q_n^\epsilon} a_{ij}(x,u_n)\partial_{x_j}(u_n-u_\infty)\partial_{x_i}(u_n-u_\infty)\frac{1}{I_\epsilon\omega^2}\Pi^2\,\mathrm{d}x \\
&-\int_{Q_n^\epsilon} \{a_{ij}(x,u_n)-a_{ij}(x,u_\infty)\}\partial_{x_j}u_\infty\partial_{x_i}(u_n-u_\infty)\frac{1}{I_\epsilon\omega^2}\Pi^2\,\mathrm{d}x \\
\leq &-\lambda\int_{Q_n^\epsilon}\frac{|\nabla(u_n-u_\infty)|^2}{I_\epsilon\omega^2}\Pi^2\,\mathrm{d}x + k\int_{Q_n^\epsilon}\frac{|\nabla(u_n-u_\infty)||\nabla u_\infty|}{I_\epsilon\omega}\Pi^2\,\mathrm{d}x \\
\leq &-\lambda\int_{Q_n^\epsilon}\frac{|\nabla(u_n-u_\infty)|^2}{I_\epsilon\omega^2}\Pi^2\,\mathrm{d}x \\
&+\lambda\int_{Q_n^\epsilon}\frac{|\nabla(u_n-u_\infty)|^2}{I_\epsilon\omega^2}\Pi^2\,\mathrm{d}x + \frac{c(\lambda)}{I_\epsilon}\int_{Q_n}|\nabla u_\infty|^2\Pi^2\,\mathrm{d}x \\
= &\frac{c(\lambda)}{I_\epsilon}\int_{Q_n}|\nabla u_\infty|^2\Pi^2\,\mathrm{d}x,
\end{aligned}
$$

where $Q_n^\epsilon = \{x\in Q_n \mid u_n-u_\infty>\epsilon\}$. Hence we obtain that

$$
\begin{aligned}
&\int_{Q_n} a(x)(u_n-u_\infty)F_\epsilon\Pi^2\,\mathrm{d}x \\
&\qquad+\int_{Q_n}\{a_{ij}(x,u_n)\partial_{x_j}u_n - a_{ij}(x,u_\infty)\partial_{x_j}u_\infty\}\partial_{x_i}\Pi^2F_\epsilon\,\mathrm{d}x \\
&\qquad\qquad\leq \frac{c(\lambda)}{I_\epsilon}\int_{Q_n}|\nabla u_\infty|^2\Pi^2\,\mathrm{d}x.
\end{aligned}
$$

Letting $\epsilon\to 0$ and recalling that $a(x)\geq\lambda_0>0$, one has for a constant c

$$
\begin{aligned}
&\int_{Q_{n_1}}(u_n-u_\infty)^+\Pi^2\,\mathrm{d}x \\
&+c\int_{Q_{n_1}}\{a_{ij}(x,u_n)\partial_{x_j}u_n - a_{ij}(x,u_\infty)\partial_{x_j}u_\infty\}\partial_{x_i}\Pi^2\chi\{u_n-u_\infty>0\}\,\mathrm{d}x\leq 0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\int_{Q_{n_1}}(u_n-u_\infty)^+\Pi^2\,\mathrm{d}x \\
&\leq -c\int_{Q_{n_1}}\{a_{ij}(x,u_n)\partial_{x_j}u_n - a_{ij}(x,u_\infty)\partial_{x_j}u_\infty\}\partial_{x_i}\Pi^2\chi\{u_n-u_\infty>0\}\,\mathrm{d}x.
\end{aligned}
\tag{3.6}
$$

Set

$$
A_{ij}(x,p) = \int_0^p a_{ij}(x,s)\,\mathrm{d}s,
$$

then

$$a_{ij}(x,u_n)\partial_{x_j}u_n - a_{ij}(x,u_\infty)\partial_{x_j}u_\infty = \partial_{x_j}\{A_{ij}(x,u_n) - A_{ij}(x,u_\infty)\} - \int_{u_\infty}^{u_n} \partial_{x_j}a_{ij}(x,s)\,\mathrm{d}s.$$

Applying the above formula into (3.6), we obtain that

$$\begin{aligned}
&\int_{Q_{n_1}} (u_n - u_\infty)^+\Pi^2\,\mathrm{d}x \\
&\leq -\int_{Q_{n_1}} \{\partial_{x_j}\{A_{ij}(x,u_n) - A_{ij}(x,u_\infty)\} \\
&\quad - \int_{u_\infty}^{u_n} \partial_{x_j}a_{ij}(x,s)\,\mathrm{d}s\}\partial_{x_i}\Pi^2\chi\{u_n - u_\infty > 0\}\,\mathrm{d}x \\
&= \int_{Q_{n_1}} \{A_{ij}(x,u_n) - A_{ij}(x,u_\infty)\}\partial_{x_ix_j}\Pi^2\chi\{u_n - u_\infty > 0\}\,\mathrm{d}x \\
&\quad - \int_{Q_{n_1}}\int_{u_\infty}^{u_n} \partial_{x_j}a_{ij}(x,s)\,\mathrm{d}s\partial_{x_i}\Pi^2\chi\{u_n - u_\infty > 0\}\,\mathrm{d}x \\
&\leq \int_{Q_{n_1}} \Lambda(u_n - u_\infty)^+|\partial_{x_ix_j}\Pi^2|\,\mathrm{d}x + \int_{Q_{n_1}} D(u_n - u_\infty)^+|\partial_{x_i}\Pi^2|\,\mathrm{d}x \\
&\leq \frac{c}{n_1}\int_{Q_{n_1}} (u_n - u_\infty)^+\,\mathrm{d}x,
\end{aligned}$$

(we assume that $|a_{ij}| \leq \Lambda$). The fact that the above argument also holds for $(u_n - u_\infty)^-$ leads to:

$$\int_{Q_{\frac{n_1}{2}}} |u_n - u_\infty|\,\mathrm{d}x \leq \int_{Q_{n_1}} |u_n - u_\infty|\Pi^2\,\mathrm{d}x \leq \frac{c}{n_1}\int_{Q_{n_1}} |u_n - u_\infty|\,\mathrm{d}x.$$

If we iterate this inequality after setting $n_1 = \frac{n}{2^{\ell-1}}$, we get:

$$\begin{aligned}
\int_{Q_{\frac{n}{2^\ell}}} |u_n - u_\infty|\,\mathrm{d}x &\leq \frac{c}{n^\ell}\int_{Q_n} |u_n - u_\infty|\,\mathrm{d}x \\
&\leq \frac{c}{n^\ell}\left\{\int_{Q_n} |u_n|\,\mathrm{d}x + \int_{Q_n} |u_\infty|\,\mathrm{d}x\right\} \\
&\leq \frac{c}{n^\ell}\left\{|u_n|_{H^1(Q_n)}|Q_n| + (2n)^k|u_\infty|_{L^1(Q)}\right\}.
\end{aligned}$$

To estimate $|u_n|_{H^1(Q_n)}$, the usual H^1-norm of u_n, we use (3.3) with $v = u_n$ to obtain $|u_n|_{H^1(Q_n)} \leq cn^k$ (see [3], [4] or the proof of theorem 2.1, case 2).

Therefore we have for some constant c independent of n,

$$\int_{Q_{\frac{n}{2^\ell}}} |u_n - u_\infty|\, \mathrm{d}x \leq \frac{c}{n^{\ell-2k}}.$$

The result holds when $\ell - 2k > \gamma$. □

Remark 3.1. We can also consider the problem in a more general domain, for instance the one in Theorem 2.1. We can prove by the same argument the convergence of u_n towards u_∞.

Acknowledgement: This research has been supported by the Swiss National Science foundation under the contract #20-103300/1.

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ON GLOBAL UNBOUNDED SOLUTIONS FOR A SEMILINEAR PARABOLIC EQUATION

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1. Introduction

We consider the Cauchy problem

$$\text{(E)} \qquad \begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \mathbf{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases}$$

where $u = u(x,t)$, Δ is the Laplace operator with respect to x, $p > 1$, and u_0 is a continuous function on $\mathbf{R}^N$. We note that the local existence of a solution can be shown by a standard method, and the solution can be continued as long as it is bounded.

Since the pioneering work of Fujita [3], many of papers have been published about the blow-up of solutions. On the other hand, it has not been known until recently whether or not global unbounded solutions exist. Indeed, the existence of a global unbounded solution is not an easy question to answer. If we consider a spatially homogeneous solution $u = u(t)$ with the initial data $u_0 \equiv \alpha$, then we can easily obtain the solution explicitly as

$$u(t) = \left\{ \frac{1}{\alpha^{p-1}} - (p-1)t \right\}^{-1/(p-1)}.$$

This solution tends to $+\infty$ (or blows up) as $t \uparrow (p-1)^{-1}\alpha^{1-p}$. This implies that in order to discuss the existence of global unbounded solutions, we must inevitably deal with spatially inhomogeneous time-dependent solutions.

The aim of this article is to survey recent development concerning global unbounded solutions for (E). In the following sections, we describe the existence and non-existence, grow-up rate, and grow-up sets.

2. Definitions

Before discussing global unbounded solutions, we introduce definitions of some critical exponents and important numbers.

It is well known that there are various critical exponents for (E) at which the structure of solutions drastically changes. Fujita [3] showed that the so-called Fujita exponent

$$p_F := \begin{cases} \dfrac{N+2}{N} & \text{for } N > 2, \\ \infty & \text{for } N \le 2, \end{cases}$$

is critical for the existence of positive global solutions. That is, if $1 < p \le (N+2)/N$, then any positive solution of (E) blows up in finite, while if $p > 1 + 2/N$, then the solution of (E) exists globally provided that initial data u_0 are sufficiently small.

Concerning the existence of positive steady states, the Sobolev exponent

$$p_S := \begin{cases} \dfrac{N+2}{N-2} & \text{for } N > 2, \\ \infty & \text{for } N \le 2, \end{cases}$$

is critical. Namely, (E) has a one-parameter family of positive radial steady states, i.e., solutions of

$$\Delta\varphi + \varphi^p = 0 \quad \text{on } \mathbf{R}^N,$$

if and only if $p \ge p_S$. We denote the solution by $\varphi = \varphi_\alpha(|x|)$, where $\alpha = \varphi_\alpha(0) > 0$. For each $\alpha > 0$, the solution φ_α is strictly decreasing in $r = |x|$ and satisfies $\varphi(r) \to 0$ as $r \to \infty$. We can extend the family to all $\alpha \in \mathbf{R}$ by setting

$$\varphi_\alpha = -\varphi_{-\alpha} \ \text{ for } \alpha < 0 \quad \text{and} \quad \varphi_0 \equiv 0.$$

We note that ϕ_α is obtained as a solution of the following initial value problem:

$$\begin{cases} \varphi_{rr} + \dfrac{N-1}{r}\varphi_r + |\varphi|^{p-1}\varphi = 0, & r > 0, \\ \varphi(0) = \alpha. \end{cases}$$

In this article, the following critical value of p plays a crucial role:

$$p_c := \begin{cases} \dfrac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N > 10, \\ \infty & \text{if } N \le 10. \end{cases}$$

Gui, Ni and Wang [5, 6] found that $p = p_c$ is a critical exponent where a change in stability properties of the nontrivial steady states occurs. More precisely, for $p < p_c$ any nontrivial steady state $u = \varphi_\alpha$ is unstable in any reasonable sense (in fact, for any $u_0 > \varphi_\alpha$, $\alpha > 0$, the solution of (E) blows up in finite time), whereas for $p \geq p_c$, $u = \phi_\alpha$ is stable under perturbations in some weighted L^∞ space. These stability properties essentially come from the fact that for $p_S \leq p < p_c$ any two steady states intersect each other, but for $p \geq p_c$, φ_α is strictly increasing in α for each x. Moreover, for $p \geq p_c$, φ_α satisfies

$$\lim_{\alpha \to 0} \varphi_\alpha(r) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} \varphi_\alpha(r) = \varphi_\infty(r), \qquad r > 0,$$

where φ_∞ is a singular steady state given by

$$u = \varphi_\infty(|x|) = L|x|^{-m}, \qquad |x| > 0$$

with

$$m := \frac{2}{p-1} \quad \text{and} \quad L := \{m\,(N-2-m)\}^{m/2}.$$

It is also shown in [5] that each positive regular steady has the asymptotic behavior

$$\varphi_\alpha(r) = \begin{cases} Lr^{-m} - ar^{-m-\lambda_1} + h.o.t. & \text{if } p > p_c \\ Lr^{-m} - ar^{-m-\lambda_1}\log r + h.o.t. & \text{if } p = p_c \end{cases}$$

as $r \to \infty$. Here λ_1 is a positive constant given by

$$\lambda_1 = \lambda_1(N,p) := \frac{N-2-2m-\sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

and $a = a(\alpha, N, p)$ is a positive number that is monotone decreasing in α. We note that the quadratic equation

$$\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0$$

has two positive roots if and only if $p > p_c$; the smaller root is λ_1 and the larger root is given by

$$\lambda_2 = \lambda_2(N,p) := \frac{N-2-2m+\sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2}.$$

These roots will play an important role in this paper.

3. Existence of global unbounded solutions

We first summarize known results about the non-existence of global unbounded solutions.

For $p \le p_F$, there are no positive global (bounded or unbounded) solutions at all by the result of Fujita [3]. For $p < p_S$, it seems that not only all positive global solutions must be bounded, they have to decay to 0 as $t \to \infty$. So far this result has been proved under extra conditions on p or u_0. For example, it is known to be true for global solutions whose initial data u_0 have fast decay at spatial infinity or at least are square integrable (see [7, 12]). For $p_S \le p$, the same result is of course not valid in general (steady states are bounded solutions that do not converge to zero), but Mizoguchi [8] proved that it does hold provided the initial data are radially symmetric, have compact support and other technical conditions are satisfied. Without any additional requirements on u_0, the questions whether global solutions may be unbounded has been open for $p_S \le p < p_c$.

The existence of global unbounded solutions for (E) was first proved by Poláčik and Yanagida in the case of $p \ge p_c$.

Theorem 3.1. (Poláčik-Yanagida [10]) *Let $p \ge p_c$. Suppose that u_0 satisfies*

$$\text{(I)} \qquad -\varphi_\infty(|x|) \le u_0(x) \le \varphi_\infty(|x|), \qquad |x| > 0,$$

and

$$\lim_{|x|\to\infty} |x|^{m+\lambda_1}\{\varphi_\infty(|x|) - u_0(x)\} = 0 \qquad \text{if } p > p_c,$$

$$\lim_{|x|\to\infty} |x|^{m+\lambda_1}(\log|x|)^{-1}\{\varphi_\infty(|x|) - u_0(x)\} = 0 \quad \text{if } p = p_c.$$

Then the solution of (E) *exists globally in time and satisfies $\|u(\cdot,t)\|_{L^\infty} \to \infty$ as $t \to \infty$.*

The proof of this result in [10] is based on global attractivity properties of the steady states. Let us consider initial data satisfying (I). Then by using the comparison technique, we can show that the solution of (E) exists globally in time and is bounded by $+\varphi_\infty$ and $-\varphi_\infty$. Moreover, if u_0 is sufficiently close to a regular steady state $\varphi_\alpha(|x|)$ near $x = \infty$, then the solution converges to $\varphi_\alpha(|x|)$ uniformly. Such global stability implies that if u_0 is larger than φ_α near $x = \infty$, then by comparison the solution of (E) eventually becomes larger than $\varphi_\alpha(|x|)$ in finite time. Therefore, if u_0 is closer to φ_∞ than any other steady state as $|x| \to \infty$, then the solution becomes larger than any other steady state. Since φ_α approaches the singular steady state as $\alpha \to \infty$, the solution must approach the singular steady state from below as $t \to \infty$. We call such phenomena as grow-up.

By using Theorem 3.1 and the continuity of solutions with respect to initial data, we can show the existence of global unbounded solutions that behave in a rather complicated way. Indeed, it was shown in [10] that if the initial data oscillate between 0 and φ_∞ as $r \to \infty$, the solution of (E) may oscillate between the trivial steady state and the singular steady state as t increases, that is,

$$\liminf_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = 0,$$

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} = \infty.$$

4. Grow-up rate

Once we know the existence of grow-up solutions, the next step is to determine the grow-up rate. It turns out that the grow-up rate depends on how close the initial data are to the singular steady state near $|x| = \infty$.

The following upper bound of the grow-up rate is given in Proposition 3.3 of [1].

Theorem 4.1. (Fila-Winkler-Yanagida [1]) *Let $p \geq p_c$. Suppose that u_0 satisfies*

$$0 \leq u_0(x) \leq L|x|^{-m} - b|x|^{-l}, \qquad |x| > R$$

with some constants $l > m + \lambda_1$ and $b, R > 0$. Then there exist positive constants C and T such that the solution of (E) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \leq Ct^{\frac{m(l-m-\lambda_1)}{2\lambda_1}}$$

for all $t > T$.

Concerning the lower bound, only a partial result was obtained in [1] in the case of $l \in (m + \lambda_1, m + \lambda_2]$.

Theorem 4.2. (Fila-Winkler-Yanagida [1]) *Let $p > p_c$. Suppose that u_0 satisfies* (I) *and*

$$L|x|^{-m} - b|x|^{-l} \leq u_0(x) \leq L|x|^{-m}, \qquad |x| > 0$$

with some constants $l \in (m + \lambda_1, m + \lambda_2]$ and $b > 0$. Then there exists a positive constant C such that the solution of (E) *satisfies*

$$u(0,t) \geq Ct^{\frac{m(l-m-\lambda_1)}{2\lambda_1}}$$

for all $t > 0$.

In order to prove Theorems 4.1 and 4.2, we first carry out formal analysis to investigate the asymptotic behavior of solutions. Then it turns out that the grow-up solution behaves in a different way for small r and large r, and we can determine the expected grow-up rate by matching the inner expansion for small r and the outer expansion for large r at some intermediate r. Based on the formal asymptotic analysis, we can obtain a rigorous proof by constructing appropriate comparison functions.

Theorem 4.2 implies that the upper bound obtained in Theorem 4.1 is optimal for $l \in (m+\lambda_1, m+\lambda_2]$. However, the upper bound in Theorem 4.1 is not optimal for large l. In fact, it is shown in [1] that there is a universal upper bound independent of initial data. A sharp universal upper bound was found by Mizoguchi.

Theorem 4.3. (Mizoguchi [9]) *Let $p > p_c$. Suppose that u_0 satisfies* (I). *Then there exist positive constants C_1 and T such that the solution of* (E) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \leq C_1 t^{\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}}$$

for all $t > T$. Moreover, there exists u_0 satisfying

$$L|x|^{-m} - be^{-|x|^2/4} \leq u_0(x) \leq L|x|^{-m}, \qquad |x| > R$$

with some $b, R > 0$ such that the solution of (E) *satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \geq C_2 t^{\frac{m(\lambda_2-\lambda_1+2)}{2\lambda_1}}$$

with some $C_2 > 0$ for all $t > 0$.

It is clear from the universal upper bound in Theorem 4.3 that Theorem 4.2 cannot be extended as it is at least for $l > m+\lambda_2+2$. An optimal lower bound of the grow-up rate in the case of $l > m+\lambda_2$ was obtained recently by Fila, King, Winkler and Yanagida,

Theorem 4.4. (Fila-King-Winkler-Yanagida [2]) *Let $p > p_c$. Suppose that u_0 satisfies* (I) *and*

$$L|x|^{-m} - b|x|^{-l} \leq u_0(x) \leq L|x|^{-m}, \quad |x| > 0$$

with some $l \in (m+\lambda_1, m+\lambda_2+2)$ and $b > 0$. Then there exists a positive constant C such that the solution of (E) *satisfies*

$$u(0,t) \geq C t^{\frac{m(l-m-\lambda_1)}{2\lambda_1}}$$

for all $t > 0$.

In the case of $l \geq m + \lambda_2 + 2$, the next result is obtained immediately from Theorem 4.4.

Theorem 4.5. (Fila-King-Winkler-Yanagida [2]) *Let $p > p_c$. Suppose that u_0 satisfies* (I) *and*

$$L|x|^{-m} - b|x|^{-l} \leq u_0(x) \leq L|x|^{-m}, \qquad |x| > 0$$

with some $l \geq m + \lambda_2 + 2$ and $b > 0$. Then for any small $\varepsilon > 0$, there exists a positive constant C such that the solution of (E) *satisfies*

$$u(0,t) \geq C t^{\frac{m(\lambda_2 - \lambda_1 + 2)}{2\lambda_1} - \varepsilon}$$

for all $t > 0$.

A Key idea in [2] is to consider the outer expansion more precisely. In [2]), we used only the asymptotic behavior of the outer solution as $r \to \infty$, while in [2]) we used more global information about the outer expansion.

When the condition (I) is dropped, the solution of (E) may grow up faster. The following result is due to Mizoguchi.

Theorem 4.6. (Mizoguchi [9]) *Let $p > p_c$. Then for each nonnegative even integer n, there exists a radially symmetric global solution of* (E) *with n intersections with $\varphi_\infty(r)$ such that*

$$\|u(\cdot,t)\|_{L^\infty} = t^{b_n} + h.o.t. \qquad as\ t \to \infty,$$

where

$$b_n = \frac{m(\lambda_2 - \lambda_1 + 2 + 2n)}{2\lambda_1} > 0.$$

We note that the intersection points of u and φ_∞ do not vanish for all $t > 0$, and must move toward $r = \infty$.

Here we mention the work by Galaktionov and King [4]. They considered the Dirichlet problem

$$\text{(D)} \qquad \begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in B, \quad t > 0, \\ u(x,t) = LR^{-m}, & x \in \partial B, t > 0, \\ u(x,0) = u_0(x), & x \in B, \end{cases}$$

where B is a ball with radius R. Notice that this problem has a singular steady state $u = L|x|^{-m}$. It was shown in [4] that if $0 \leq u_0 \leq L|x|^{-m}$, then the solution of (D) satisfies $\|u(\cdot,t)\|_{L^\infty} \to \infty$ exponentially as $t \to \infty$, and the grow-up rate does not depend on the initial data.

5. Grow-up set

For a global unbounded solution of (E), we say that $\xi \in \mathbf{R}^N$ is a grow-up point if there exists a sequence $\{(\xi_i, t_i)\}$ with $\xi_i \to \xi$ and $t_i \to \infty$ as $i \to \infty$ such that

$$|u(\xi_i, t_i)| \to \infty \quad \text{as } i \to \infty.$$

The set of all grow-up points is called a grow-up set. We note that if the initial data satisfy (I), then the solution also satisfies (I). Therefore, the grow-up set is $\{0\}$ in this case. Now the question is what if the condition (I) is dropped.

Theorem 5.1. (Poláčik-Yanagida [11]) *Let $p \geq p_c$. Given any closed subset G of $\mathbf{R}^N$, there exist positive initial data u_0 such that the solution of* (E) *exists globally in time and the grow-up set is exactly equal to G.*

If we consider sign-changing solutions, we can show the following result.

Theorem 5.2. (Poláčik-Yanagida [11]) *Let $p \geq p_c$. Given any closed subsets G^+ and G^- of $\mathbf{R}^N$, there exist initial data u_0 such that the solution of* (E) *exists globally in time and satisfies the following properties:*

(i) $\liminf_{t\to\infty} u(x,t) = \infty$ *for any* $x \in G^+$,

(ii) $\limsup_{t\to\infty} u(x,t) \leq L\,\{\mathrm{dist}(x, G^+)\}^{-2/(p-1)}$ *for any* $x \notin G^+$,

(iii) $\liminf_{t\to\infty} u(x,t) = -\infty$ *for any* $x \in G^-$,

(iv) $\limsup_{t\to\infty} u(x,t) \geq -L\,\{\mathrm{dist}(x, G^-)\}^{-2/(p-1)}$ *for any* $x \notin G^-$,

where

$$\mathrm{dist}(x, G^\pm) := \min_{y \in G^\pm} |y - x|.$$

Proofs of these theorems are based on the following observation. So far, we have considered solutions that are localized near the origin. However, by careful construction of initial data that are not bounded by $+\varphi_\infty(|x|)$ and $-\varphi_\infty(|x|)$, we can find a solution which goes through the birth-and-death process of a localized peak.

Lemma 5.1. (Poláčik -Yanagida [11]) *Let $p \geq p_c$. For any sequence $\{(\alpha_i, \xi_i, \varepsilon_i)\}$ with $\alpha_i \in \mathbf{R}$, $\xi_i \in \mathbf{R}^N$ and $\varepsilon_i > 0$, there exist initial data u_0 such that the solution of* (E) *exists globally in time and satisfies the following properties:*

(i) *There exists a sequence of positive numbers* $\{t_i\}$ *such that*

$$\|u(\cdot,t_i) - \varphi_{\alpha_i}(|\cdot - \xi_i|)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i.$$

(ii) *There exists a sequence of positive numbers* $\{\hat{t}_i\}$ *with* $\hat{t}_i \in (t_i, t_{i+1})$ *such that*

$$\|u(\cdot,\hat{t}_i)\|_{L^\infty(\mathbf{R}^N)} < \varepsilon_i.$$

By virtue of this lemma, we can prove Theorems 5.1 and 5.2 by choosing the sequence $(\xi, \alpha_i, \varepsilon_i)\}$ suitably.

Theorems 5.1 and 5.2 imply that for any prescribed grow-up set, we can always find a global unbounded solution. Perhaps, the definition of grow-up set introduced as above would be too weak. It is an interesting question to ask what if we adopt a stronger definition of a grow-up point. For example, we may define a grow-up point as a point $\xi \in \mathbf{R}^N$ such that $|u(\xi,t)| \to \infty$ as $t \to \infty$.

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