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First Steps in Several Complex Variables: Reinhardt Domains

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Preface

There are many excellent books for students introducing them to classical complex analysis of one variable, but only a few that cover several complex variables. Thus we were motivated to write such a book, intended as a textbook for beginning graduate students and as a source book for lectures and seminars. We have developed the main ideas of several complex variables in the context of, but without entering into too many technical details of, a very simple geometry, known as Reinhardt domains. Though many students may know little about this topic, we think it is a good start for beginners in several complex variables. Using this as a base, we add to all topics a selection of remarks and hints relating the discussion to the general theory. Some of the chapters or sections, those marked with a star (*), are more developed than others and can be skipped in a first reading. Moreover, we present some topics that have never appeared in a textbook or are new findings. We hope that these new ideas will motivate the student studying this book to become more deeply involved in the use of several complex variables. Further toward that end, we include in the Bibliography both direct references and a list of monographs and textbooks in complex analysis, thus providing a source for expansion on topics in our book and extensions to new studies.

The book contains many exercises that the reader is asked to work on when encountered, before proceeding with further topics. There are also many points in the proofs that we have marked EXERCISE. By this we mean that the reader should write out the argument in more detail than we have done, to assure mastery of those details in preparation for what is to come. We believe that the study and understanding of mathematics requires continuous interaction between the reader and the text, and this cannot be achieved by passive reading. From time to time we pose open problems (marked by $\boxed{?}\dots\boxed{?}$) that to the best of our knowledge have not yet been solved. We encourage the reader to try to solve them and would be most grateful to hear about such attempts, both successes and interesting failures.

Note that at many places, in order to simplify formulations, some obvious assumptions that guarantee that the considered objects are non-empty are not stated. For example, if we write "Let $D \subset \mathbb{C}^n$ be a Reinhardt domain…", then we always automatically assume that $D \neq \varnothing$. We think that the reader will easily be able to complete the missing assumptions. In the interest of consistency of form and notation, we sometimes send the reader to [Jar-Pfl 1993] or [Jar-Pfl 2000] instead of quoting the original research paper. We nevertheless encourage the reader to seek out those original works in their further studies.

During the process of proofreading we detected some gaps and misprints. Our thanks go especially to Dr. P. Zapałowski who helped us during that process. Nevertheless, according to our experience with former books, we are sure that a number of errors remain about which we would be happy to be informed.

We would be pleased if the reader would send any comments or remarks to one of the following e-mail addresses

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Kraków and Oldenburg, February 2008 Marek Jarnicki
Peter Pflug

Contents

Chapter 1
Reinhardt domains

1.1 Introduction

The notion of a holomorphic function of one complex variable can be based on the notion of a *power series* – a function $f : \Omega \to \mathbb{C}$ (where $\Omega \subset \mathbb{C}$ is open) is holomorphic ($f \in \mathcal{O}(\Omega)$) if for every $a \in \Omega$ there exist a *power series* $\sum_{k=0}^{\infty} c_k (z-a)^k$ centered at a and a neighborhood $U_a \subset \Omega$ of a such that

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k, \quad z \in U_a.$$ [1]

It is well known that the *domain of convergence* of an arbitrary power series

$$\sum_{k=0}^{\infty} b_k (z-a)^k$$

is the disc $K(a,R) := \{z \in \mathbb{C} : |z-a| < R\}$ with the radius (*radius of convergence*)

$$R = \frac{1}{\limsup\limits_{k\to+\infty} \sqrt[k]{|b_k|}} \in [0, +\infty]$$

(where $K(a,0) = \varnothing$, $K(a,+\infty) = \mathbb{C}$). Moreover, if $R > 0$, then the function

$$f(z) := \sum_{k=0}^{\infty} b_k (z-a)^k, \quad z \in K(a,R),$$

is holomorphic.

If $f \in \mathcal{O}(\Omega)$ and $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$, $z \in U_a \subset \Omega$, then the radius of convergence of the series $\sum_{k=0}^{\infty} c_k (z-a)^k$ is not smaller than the Euclidean distance $d_\Omega(a)$ of the point a to $\partial\Omega$ ($d_{\mathbb{C}}(a) :\equiv +\infty$) and $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$, $z \in K(a, d_\Omega(a))$.

The most elementary is the case where $\Omega = K(a,r)$, which, of course, may be reduced to the case $\Omega = \mathbb{D} =$ the *unit disc*. Recall the following well-known

[1] Equivalently: f is differentiable in the complex sense at every point $a \in \Omega$, i.e. the function f has at every $a \in \Omega$ the *complex derivative*

$$f'(a) := \lim_{\mathbb{C}\setminus\{0\}\ni h\to 0} \frac{f(a+h) - f(a)}{h}.$$

issues, whose analogues will be considered in the sequel in a much more general context:

- The structure of the group $\mathrm{Aut}(\mathbb{D})$ of *holomorphic automorphisms* of $\mathbb{D}$. It is well known that
$$\mathrm{Aut}(\mathbb{D}) = \Big\{\mathbb{D} \ni z \mapsto \zeta\frac{z-a}{1-\bar{a}z} \in \mathbb{D} : \zeta \in \mathbb{T},\ a \in \mathbb{D}\Big\},$$
where $\mathbb{T} := \partial\mathbb{D}$. In particular (cf. Exercise 2.1.5 (b)), $\mathrm{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$.
- The *holomorphic geometry* of $\mathbb{D}$. In particular, the theory of *holomorphically invariant* distances, i.e. those distances $d : \mathbb{D} \times \mathbb{D} \to \mathbb{R}_+$, for which
$$d(f(z), f(w)) \le d(z, w), \quad z, w \in \mathbb{D},\ f \in \mathcal{O}(\mathbb{D}, \mathbb{D}), \tag{1.1.1}$$
where $\mathcal{O}(\mathbb{D}, \mathbb{D})$ denotes the set of all holomorphic functions $f : \mathbb{D} \to \mathbb{D}$. The above condition means in particular that any $f \in \mathrm{Aut}(\mathbb{D})$ is an isometry of the metric space $(\mathbb{D}, d)$. Typical examples are:
$$\boldsymbol{m}(z, w) := \left|\frac{z-w}{1-z\bar{w}}\right| \qquad (\textit{Möbius distance}),$$
$$\boldsymbol{p}(z, w) := \frac{1}{2}\log\frac{1+\boldsymbol{m}(z,w)}{1-\boldsymbol{m}(z,w)} \qquad (\textit{Poincaré distance});$$
cf. [Jar-Pfl 1993], Chapter 1.

Exercise 1.1.1. (a) Check (1.1.1) for $d \in \{\boldsymbol{m}, \boldsymbol{p}\}$.
(b) Prove that $\boldsymbol{m}$ and $\boldsymbol{p}$ are distances on $\mathbb{D}$.
(c) Prove that $\boldsymbol{p}(0, b) = \boldsymbol{p}(0, a) + \boldsymbol{p}(a, b)$, $0 < a < b < 1$.

In the next step we substitute power series by *Laurent series*
$$\sum_{k=-\infty}^{\infty} b_k(z-a)^k$$
and, consequently, discs $K(a, r)$ by annuli
$$\mathbb{A}(a, r^-, r^+) := \{z \in \mathbb{C} : r^- < |z-a| < r^+\}, \quad -\infty \le r^- < r^+ \le +\infty,\ r^+ > 0.$$
Note that if $r^- < 0$, then
$$\mathbb{A}(a, r^-, r^+) = K(a, r^+) \quad \text{and} \quad \mathbb{A}(a, 0, r^+) = K(a, r^+) \setminus \{a\} =: K_*(a, r^+).$$
A Laurent series with $b_{-k} = 0$, $k = 1, 2, \dots$, will be always identified with the power series $\sum_{k=0}^{\infty} b_k(z-a)^k$. The *domain of convergence* of a Laurent series is

an annulus $\mathbb{A}(a, R^-, R^+)$ with

$$R^+ := \frac{1}{\limsup\limits_{k\to+\infty} \sqrt[k]{|b_k|}}, \quad R^- := \begin{cases} \limsup\limits_{k\to+\infty} \sqrt[k]{|b_{-k}|} & \text{if } \exists_{k\in\mathbb{N}} : b_{-k} \neq 0, \\ -\infty & \text{if } \forall_{k\in\mathbb{N}} : b_{-k} = 0, \end{cases} \tag{1.1.2}$$

provided that $R^- < R^+$. The function

$$f(z) := \sum_{k=-\infty}^{\infty} b_k (z-a)^k, \quad z \in \mathbb{A}(a, R^-, R^+),$$

is holomorphic. Moreover, for every compact $K \subset \mathbb{A}(a, R^-, R^+)$ there exist $C > 0$, $\theta \in (0,1)$ such that $|b_k(z-a)^k| \leq C\theta^{|k|}$, $z \in K$, $k \in \mathbb{Z}$.

Conversely, every function f holomorphic in an annulus $\mathbb{A}(a, r^-, r^+)$ has a unique representation by a Laurent series. We may always assume that $a = 0$. Notice that for $A := \mathbb{A}(0, 1/R, R)$ $(R > 1)$ we have

$$\operatorname{Aut}(A) = \{A \ni z \mapsto \zeta z \in A : \zeta \in \mathbb{T}\} \cup \{A \ni z \mapsto \zeta/z \in A : \zeta \in \mathbb{T}\}.$$

In particular, the group $\operatorname{Aut}(A)$ does not act transitively; cf. Exercise 2.1.5 (c). The holomorphic geometry of an annulus is much more complicated than the one of $\mathbb{D}$; cf. [Jar-Pfl 1993], Chapter 5.

Notice that for domains (a subset D of a topological space X is said to be a *domain* if D is open and connected) $D \subset \mathbb{C}$ the following three notions coincide:

- D is a domain of convergence of a Laurent series centered at 0;
- D is a domain *invariant under rotations*, i.e. for any $z \in D$ and $\lambda \in \mathbb{T}$ the point λz also belongs to D;
- D is a disc or an annulus centered at 0.

The notion of a power series generalizes in a natural way to the case of several complex variables. By an (*n-fold*) *power series* (centered at $0 \in \mathbb{C}^n$) we mean any series of the form

$$\sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha z^\alpha \quad (z \in \mathbb{C}^n),$$

where $(a_\alpha)_{\alpha\in\mathbb{Z}_+^n} \subset \mathbb{C}$, $\mathbb{Z}_+^n := \{\alpha \in \mathbb{Z}^n : \alpha \geq 0\}$, $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ $(0^0 := 1)$; see § 1.3. The *domain of convergence* $\mathfrak{D}$ of a power series (Definition 1.3.3) has the following important properties:

- For any $a = (a_1, \dots, a_n) \in \mathfrak{D}$, the closed polydisc

  $$\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| \leq |a_j|, \ j = 1, \dots, n\}$$

  is contained in $\mathfrak{D}$, i.e. $\mathfrak{D}$ is a *complete Reinhardt* (*n-circled*) *domain* (Definition 1.3.8).

- The set
$$\log \mathfrak{D} := \{(\log|z_1|, \dots, \log|z_n|) : (z_1, \dots, z_n) \in \mathfrak{D},\ z_1 \cdots z_n \neq 0\}$$
is convex in the geometric sense, i.e. $\mathfrak{D}$ is *logarithmically convex* (Definition 1.5.5, Proposition 1.5.16).
- The series is locally geometrically summable in $\mathfrak{D}$, i.e. for any compact $K \subset \mathfrak{D}$ there exist $C > 0$, $\theta \in (0,1)$ such that $|a_\alpha z^\alpha| \leq C\theta^{|\alpha|}$, $z \in K$, $\alpha \in \mathbb{Z}_+^n$, where $|\alpha| := \alpha_1 + \dots + \alpha_n$ (Remark 1.3.5 (f)).
In the case $n = 1$ the only complete Reinhardt domains are discs $K(r)$ and they are always logarithmically convex. In the case $n \geq 2$ the situation is more complicated. There are infinitely many types of complete Reinhardt domains which are not biholomorphically equivalent (e.g. Euclidean balls $\mathbb{B}(r)$ and polydiscs $\mathbb{P}(r)$; cf. Theorem 2.1.17). Moreover, there are complete Reinhardt domains $D \subset \mathbb{C}^n$ ($n \geq 2$) which are not logarithmically convex, e.g.
$$D := \{(z_1, z_2) \in \mathbb{D}^2 : \min\{|z_1|,\ |z_2|\} < r\} \quad (r \in (0,1)).$$
The function $f(z) := \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha$, $z \in \mathfrak{D}$, is holomorphic. Conversely, every function f holomorphic in a complete Reinhardt domain $D \subset \mathbb{C}^n$ has a "global" expansion into a power series $f(z) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha$, $z \in D$ (cf. Proposition 1.7.15 (c), (d)).

The notion of a Laurent series extends to the notion of an (*n-fold*) *Laurent series* (centered at 0)
$$\sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha;$$
see § 1.6. The *domain of convergence* $\mathfrak{D}$ of a Laurent series (Definition 1.6.1) has the following properties:

- For any $a = (a_1, \dots, a_n) \in \mathfrak{D}$, the torus
$$\{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| = |a_j|,\ j = 1, \dots, n\}$$
is contained in $\mathfrak{D}$, i.e. $\mathfrak{D}$ is a *Reinhardt* (*n-circled*) *domain* (Definition 1.5.2).
- $\mathfrak{D}$ is logarithmically convex (Proposition 1.6.5 (d)).
- For every $j \in \{1, \dots, n\}$, if $\mathfrak{D} \cap V_j \neq \varnothing$,[2] where
$$V_j := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j = 0\},$$
then for every $a = (a_1, \dots, a_n) \in \mathfrak{D}$, the disc
$$\{(a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_n) : |z_j| \leq |a_j|\}$$
is contained in $\mathfrak{D}$ (Proposition 1.6.5 (c)).

[2] Observe that in the case of a power series we have $0 \in \mathfrak{D}$ and, consequently, $\mathfrak{D} \cap V_j \neq \varnothing$ for any j.

- The Laurent series is locally geometrically summable in $\mathfrak{D}$, i.e. for any compact set $K \subset \mathfrak{D}$ there exist $C > 0$, $\theta \in (0,1)$ such that $|a_\alpha z^\alpha| \le C\theta^{|\alpha|}$, $z \in K$, $\alpha \in \mathbb{Z}^n$, where $|\alpha| := |\alpha_1| + \dots + |\alpha_n|$ (Proposition 1.6.5 (a), Lemma 1.6.3).

In the case $n = 1$ the only Reinhardt domains are discs $K(r)$ and annuli $\mathbb{A}(r^-, r^+)$;[3] they are always logarithmically convex.

Every function given by a Laurent series is holomorphic. Conversely, every function f holomorphic in a Reinhardt domain $D \subset \mathbb{C}^n$ has a "global" expansion into a Laurent series $f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha$, $z \in D$ (Proposition 1.7.15 (c)).

As always, from the point of view of the theory of holomorphic functions, most important are *domains of holomorphy*, i.e. those domains D which are "maximal" in the sense that all holomorphic functions in D cannot be simultaneously extended through a boundary point of D (Definition 1.11.1); let us mention that for $n \ge 2$ there are even pairs of domains $D \subsetneq \widetilde{D} \subset \mathbb{C}^n$ such that *every* function $f \in \mathcal{O}(D)$ extends holomorphically to $\widetilde{D}$. It turns out that in the category of Reinhardt domains the following conditions are equivalent (Theorem 1.11.13):

- D is a domain of holomorphy;
- D is logarithmically convex and *relatively complete*, that is, for every $j \in \{1, \dots, n\}$, if $D \cap \boldsymbol{V}_j \ne \varnothing$, then for every $a = (a_1, \dots, a_n) \in D$, the disc
$$\{(a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_n) : |z_j| \le |a_j|\}$$
is contained in D;
- $D = D^* \setminus M$, where
$$D^* := \operatorname{int} \bigcap_{\substack{(\alpha,c) \in \mathbb{R}^n \times \mathbb{R}:\\ D \subset \boldsymbol{D}_{\alpha,c}}} \boldsymbol{D}_{\alpha,c}, \quad M := \bigcup_{\substack{j \in \{1,\dots,n\}:\\ D \cap \boldsymbol{V}_j = \varnothing}} \boldsymbol{V}_j,$$
$$\boldsymbol{D}_{\alpha,c} := \{(z_1, \dots, z_n) : |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} < e^c\};$$
$\boldsymbol{D}_{\alpha,c}$ is called an *elementary Reinhardt domain*.

In particular, the domain of convergence of a Laurent series is always a domain of holomorphy. Such a simple geometric characterization of domains of holomorphy does not occur in any other category of domains.

The notion of a domain of holomorphy extends in a natural way to an $\mathcal{F}$-domain of holomorphy, when we are only interested in the extendibility of functions from a family $\mathcal{F} \subset \mathcal{O}(D)$ (Definition 1.11.1). If D is not an $\mathcal{F}$-domain of holomorphy, then one can ask whether there exists the maximal domain $\widetilde{D} \subset \mathbb{C}^n$ (the *$\mathcal{F}$-envelope of holomorphy of D*) such that every function $f \in \mathcal{F}$ extends holomorphically to $\widetilde{D}$. The answer is negative in general, even for $\mathcal{F} = \mathcal{O}(D)$ – the $\mathcal{F}$-envelope of holomorphy of D may be non-univalent, i.e. it is a non-univalent Riemann domain

[3] $\mathbb{A}(r^-, r^+) := \mathbb{A}(0, r^-, r^+)$.

spread over $\mathbb{C}^n$. In the category of Reinhardt domains the situation is simpler, namely: For an arbitrary Reinhardt domain $D \subset \mathbb{C}^n$ and an arbitrary rotation-invariant family of functions $\mathcal{F} \subset \mathcal{O}(D)$, the $\mathcal{F}$-envelope of holomorphy of D is again a Reinhardt domain (Theorem 1.12.4).

The above results permit us to reduce many problems concerning Reinhardt domains of holomorphy to the case of elementary Reinhardt domains. We will see that many holomorphic properties of D are encoded in geometric properties of $\log D$. In particular, we will discuss the following problem. Given a Reinhardt domain of holomorphy $D \subset \mathbb{C}^n$ and a family $\mathcal{F} \subsetneq \mathcal{O}(D)$, find geometric conditions under which D is a domain of holomorphy with respect to the family $\mathcal{F}$. For example, we consider as $\mathcal{F}$ the following spaces:

- $\mathcal{H}^\infty(D) =$ the space of bounded holomorphic functions,
- $L_h^p(D) =$ the space of p-integrable holomorphic functions,
- $\mathcal{A}^k(D) =$ the space of all functions $f \in \mathcal{O}(D)$ whose derivatives $D^\alpha f$ extend continuously to $\bar{D}$ for all $|\alpha| \le k$.

Various geometric characterizations of domains of holomorphy with respect to special families of functions will be presented in Chapter 3.

Chapter 2 is devoted to a presentation of different aspects of the problem of biholomorphic equivalence of Reinhardt domains.

Finally, Chapter 4 presents a thorough study of the theory of holomorphically invariant functions and pseudometrics on Reinhardt domains.

1.2 Summable families

The aim of this auxiliary section is to recall some basic notions related to summable families (cf. for instance [Sch 1967] or [Hér 1982]).

Let us fix an arbitrary set $\varnothing \neq Z \subset \mathbb{C}^n$ and let $I \neq \varnothing$ be an arbitrary set of indices. Let $\mathfrak{F}(I)$ be the set of all non-empty finite subsets of I. Consider a family $\boldsymbol{f} = (f_i)_{i \in I}$ of functions $f_i : Z \to \mathbb{C}$.

For example (cf. §§ 1.3, 1.6): $I \subset \mathbb{Z}^n$, $f_\alpha(z) := a_\alpha z^\alpha$, $z \in Z \subset \mathbb{C}^n$, $\alpha \in I$, where $(a_\alpha)_{\alpha \in I} \subset \mathbb{C}$ and the set Z is such that all the powers z^α, $\alpha \in I$, are defined on Z.

In the case where $Z = \{a\}$, instead of a family of functions, we rather should think of a family of complex numbers $(f_i(a))_{i \in I}$.

For $A \in \mathfrak{F}(I)$ put $\boldsymbol{f}_A := \sum_{i \in A} f_i$. Let, moreover, $\boldsymbol{f}_\varnothing := 0$.

Definition 1.2.1. We say that the *family $\boldsymbol{f}$ is uniformly summable on Z* (equivalently: the *series $\sum_{i \in I} f_i$ is uniformly summable on Z*) if there exists a function $\boldsymbol{f}_I : Z \to \mathbb{C}$ such that

$$\forall_{\varepsilon>0}\ \exists_{S(\varepsilon)=S(I,\varepsilon)\in\mathfrak{F}(I)}\ \forall_{A\in\mathfrak{F}(I):\, S(\varepsilon)\subset A}\ \forall_{z\in Z} : |\boldsymbol{f}_A(z) - \boldsymbol{f}_I(z)| \le \varepsilon. \quad (1.2.1)$$

Notice that the case where I is finite is trivial (we take $S(I,\varepsilon) := I$ for any $\varepsilon > 0$).

In the case where $\#Z = 1$ we simply say that the family $\boldsymbol{f}$ (considered as a family of complex numbers) is *summable* or that the series $\sum_{i\in I} f_i$ is *summable*.

It is clear (EXERCISE) that the function $\boldsymbol{f}_I$ is uniquely determined. We write $\boldsymbol{f}_I = \sum_{i\in I} f_i$ and we say that $\boldsymbol{f}_I$ is the *sum* of the family $\boldsymbol{f}$.

Let $\mathcal{S}(I,\mathbb{C}^Z)$ be the set of all families $\boldsymbol{f} = (f_i)_{i\in I}$ that are uniformly summable on Z. More generally, for $T \subset \mathbb{C}$, let $\mathcal{S}(I,T^Z)$ be the set of all uniformly summable families $\boldsymbol{f} = (f_i)_{i\in I}$ with $f_i\colon Z \to T$, $i \in I$.

Exercise 1.2.2. Let $(f_k)_{k\in\mathbb{N}} \in \mathcal{S}(\mathbb{N},\mathbb{C}^Z)$. Prove that the series $\sum_{k=1}^{\infty} f_{\sigma(k)}$ is uniformly convergent in the classical sense for every bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$.

Exercise 1.2.3. Let $I := \mathbb{N}$. Find a convergent (in the classical sense) series $\sum_{k=1}^{\infty} f_k$ of real numbers such that the family $(f_k)_{k\in\mathbb{N}}$ is not summable in the sense of Definition 1.2.1 (cf. Theorem 1.2.12).

Remark 1.2.4. (a) (EXERCISE) If $\boldsymbol{f} = (f_i)_{i\in I}$, $\boldsymbol{g} = (g_i)_{i\in I} \in \mathcal{S}(I,\mathbb{C}^Z)$, $\alpha,\beta \in \mathbb{C}$, then $\alpha\boldsymbol{f} + \beta\boldsymbol{g} := (\alpha f_i + \beta g_i)_{i\in I} \in \mathcal{S}(I,\mathbb{C}^Z)$ and $(\alpha\boldsymbol{f} + \beta\boldsymbol{g})_I = \alpha\boldsymbol{f}_I + \beta\boldsymbol{g}_I$. In particular, $\mathcal{S}(I,\mathbb{C}^Z)$ is a complex vector space and the mapping

$$\mathcal{S}(I,\mathbb{C}^Z) \ni \boldsymbol{f} \mapsto \boldsymbol{f}_I \in \mathbb{C}^Z$$

is $\mathbb{C}$-linear.

(b) (EXERCISE) A family $\boldsymbol{f}$ is uniformly summable iff the families $\operatorname{Re}\boldsymbol{f} := (\operatorname{Re} f_i)_{i\in I}$ and $\operatorname{Im}\boldsymbol{f} := (\operatorname{Im} f_i)_{i\in I}$ are uniformly summable. Moreover, $\operatorname{Re}(\boldsymbol{f}_I) = (\operatorname{Re}\boldsymbol{f})_I$ and $\operatorname{Im}(\boldsymbol{f}_I) = (\operatorname{Im}\boldsymbol{f})_I$.

(c) If $\boldsymbol{f} \in \mathcal{S}(I,\mathbb{C}^Z)$ and all the mappings $f_i\colon Z \to \mathbb{C}$ are bounded, then the family of functions $\{\boldsymbol{f}_A\colon A \in \mathfrak{F}\}$ is uniformly bounded.

Indeed, let $S := S(I,1) \in \mathfrak{F}(I)$ be associated to $\varepsilon = 1$ according to (1.2.1). It suffices to prove that the set $\{\boldsymbol{f}_A : A \in \mathfrak{F}(I\setminus S)\}$ is uniformly bounded. Fix an $A \in \mathfrak{F}(I\setminus S)$. Then we have $|\boldsymbol{f}_A| = |\boldsymbol{f}_{A\cup S} - \boldsymbol{f}_S| \le |\boldsymbol{f}_{A\cup S} - \boldsymbol{f}_I| + |\boldsymbol{f}_S - \boldsymbol{f}_I| \le 2$.

(d) If $\boldsymbol{f} \in \mathcal{S}(I,\mathbb{C}^Z)$, then the set $\{i \in I : f_i \not\equiv 0\}$ is at most countable. Consequently, the most important is the case where I is countable.

Indeed, it suffices to show that for every $\varepsilon > 0$,

$$\{i \in I : \exists_{z\in Z} : |f_i(z)| > 2\varepsilon\} \subset S(\varepsilon),$$

where $S(\varepsilon)$ is chosen according to (1.2.1). Fix $\varepsilon > 0$ and $i \in I\setminus S(\varepsilon)$. Then

$$|f_i| = |\boldsymbol{f}_{\{i\}\cup S(\varepsilon)} - \boldsymbol{f}_{S(\varepsilon)}| \le |\boldsymbol{f}_{\{i\}\cup S(\varepsilon)} - \boldsymbol{f}_I| + |\boldsymbol{f}_{S(\varepsilon)} - \boldsymbol{f}_I| \le 2\varepsilon.$$

Proposition 1.2.5 (Cauchy criterion).

$$\boldsymbol{f} \in \mathcal{S}(I,\mathbb{C}^Z) \iff \forall_{\varepsilon>0}\ \exists_{C(\varepsilon)\in\mathfrak{F}(I)} \forall_{A\in\mathfrak{F}(I\setminus C(\varepsilon))} : |\boldsymbol{f}_A| \le \varepsilon. \tag{1.2.2}$$

Notice that the *Cauchy condition* (1.2.2) permits us to verify the summability of $\boldsymbol{f}$ without determining $\boldsymbol{f}_I$.

Proof. ($\Rightarrow$): Let $\boldsymbol{f} \in \mathcal{S}(I, \mathbb{C}^Z)$. Take an $\varepsilon > 0$ and let $S(I, \varepsilon/2)$ be associated to $\varepsilon/2$ according to (1.2.1). Put $C(\varepsilon) := S(I, \varepsilon/2)$. Then for any $A \in \mathfrak{F}(I \setminus C(\varepsilon))$ we have

$$|\boldsymbol{f}_A| = |\boldsymbol{f}_{A \cup C(\varepsilon)} - \boldsymbol{f}_{C(\varepsilon)}| \le |\boldsymbol{f}_{A \cup C(\varepsilon)} - \boldsymbol{f}_I| + |\boldsymbol{f}_{C(\varepsilon)} - \boldsymbol{f}_I| \le \varepsilon.$$

($\Leftarrow$): Suppose that (1.2.2) is fulfilled. Let $C_\nu := C(1/\nu)$, $F_\nu := \boldsymbol{f}_{C_\nu}$, $\nu \in \mathbb{N}$. Then we have

$$|F_{\nu+k} - F_\nu| = |\boldsymbol{f}_{C_{\nu+k} \setminus C_\nu} - \boldsymbol{f}_{C_\nu \setminus C_{\nu+k}}| \le \frac{1}{\nu} + \frac{1}{\nu + k}, \quad \nu, k \in \mathbb{N}.$$

Consequently, $(F_\nu)_{\nu=1}^\infty$ satisfies the uniform Cauchy condition on Z and, therefore, there exists a function $F_0 \colon Z \to \mathbb{C}$ such that $F_\nu \to F_0$ uniformly on Z. If $k \to +\infty$, the above inequality implies that

$$|F_\nu - F_0| \le \frac{1}{\nu}, \quad \nu \in \mathbb{N}.$$

Now, let $A \in \mathfrak{F}(I)$, $C_n \subset A$. Then we get

$$|\boldsymbol{f}_A - F_0| \le |\boldsymbol{f}_A - \boldsymbol{f}_{C_n}| + |F_n - F_0| \le |\boldsymbol{f}_{A \setminus C_n}| + \frac{1}{n} \le \frac{2}{n},$$

which shows that $\boldsymbol{f}$ is uniformly summable and $\boldsymbol{f}_I = F_0$. ☐

Corollary 1.2.6. *If $(f_i)_{i \in I} \in \mathcal{S}(I, \mathbb{C}^Z)$, then for any non-empty set $J \subset I$ we have $(f_i)_{i \in J} \in \mathcal{S}(J, \mathbb{C}^Z)$. In particular, we may define $\boldsymbol{f}_J := \sum_{i \in J} f_i$, $\varnothing \ne J \subset I$.*

Theorem 1.2.7. *Let $I = \bigcup_{j \in J} I(j)$, $I(j) \ne \varnothing$ and $I(j) \cap I(k) = \varnothing$ for $j \ne k$. If $(f_i)_{i \in I} \in \mathcal{S}(I, \mathbb{C}^Z)$, then $(\boldsymbol{f}_{I(j)})_{j \in J} \in \mathcal{S}(J, \mathbb{C}^Z)$ and*

$$\sum_{j \in J} \boldsymbol{f}_{I(j)} = \boldsymbol{f}_I, \quad \text{i.e.} \quad \sum_{j \in J} \Big(\sum_{i \in I(j)} f_i \Big) = \sum_{i \in I} f_i.$$

Notice that the converse theorem is not true: take for instance $\#Z = 1$, $I = J := \mathbb{N}$, $I(j) := \{2j - 1, 2j\}$, $f_i := (-1)^i$. Then $\boldsymbol{f}_{I(j)} = 0$, $j \in \mathbb{N}$, but the family $(f_i)_{i \in \mathbb{N}}$ is not summable.

Proof. Take an $\varepsilon > 0$, let $S := S(I, \varepsilon/2)$ be taken as in (1.2.1), and let

$$T := \{j \in J : I(j) \cap S \ne \varnothing\}.$$

Observe that $T \in \mathfrak{F}(J)$. We are going to show that $T = S(J, \varepsilon)$ (with respect to the family $(\boldsymbol{f}_{I(j)})_{j \in J}$). Take a $B \in \mathfrak{F}(J)$ with $T \subset B$. Put $N := \#B$. For any $j \in J$ let $S_j := S(I(j), \frac{\varepsilon}{2N})$. We may assume that $S \cap I(j) \subset S_j$, $j \in J$.

Let $A := \bigcup_{j\in B} S_j \in \mathfrak{F}(I)$. Observe that $S \subset A$. Hence, $|\boldsymbol{f}_A - \boldsymbol{f}_I| \le \varepsilon/2$ and, finally, we get

$$\Big|\boldsymbol{f}_I - \sum_{j\in B} \boldsymbol{f}_{I(j)}\Big| \le \Big|\boldsymbol{f}_I - \sum_{j\in B} \boldsymbol{f}_{S_j}\Big| + \sum_{j\in B} |\boldsymbol{f}_{S_j} - \boldsymbol{f}_{I(j)}| \le |\boldsymbol{f}_I - \boldsymbol{f}_A| + \varepsilon/2 \le \varepsilon.$$

□

Definition 1.2.8. (a) We say that $\boldsymbol{f}$ is *absolutely uniformly summable on Z* if the family $|\boldsymbol{f}| := (|f_i|)_{i\in I}$ is uniformly summable on Z, i.e. $|\boldsymbol{f}| \in \mathcal{S}(I, \mathbb{R}_+^Z)$. In the case where $\#Z = 1$, then we simply say that $\boldsymbol{f}$ is an *absolutely summable family*.

(b) We say that $\boldsymbol{f}$ is *normally summable on Z* if all the functions f_i are bounded on Z and the family of numbers $(\sup_Z |f_i|)_{i\in I}$ is summable.

(c) We say that $\boldsymbol{f}$ is *locally uniformly summable* (resp. *locally normally summable*) on Z if every point $a \in Z$ has an open neighborhood U such that the family $(f_i|_{Z\cap U})_{i\in I}$ is uniformly summable (resp. normally summable) on $Z \cap U$.

In any of the above cases, instead of the family $\boldsymbol{f}$, we can say that the *series $\sum_{i\in I} f_i$ is absolutely uniformly summable, normally summable, etc.*

Remark 1.2.9 (Exercise). (a) Observe that if $|f_i| \le g_i$, $i \in I$, and $(g_i)_{i\in I} \in \mathcal{S}(I, \mathbb{R}_+^Z)$, then, by the Cauchy criterion, $\boldsymbol{f} \in \mathcal{S}(I, \mathbb{C}^Z)$. In particular, if $|\boldsymbol{f}| \in \mathcal{S}(I, \mathbb{R}_+^Z)$, then $\boldsymbol{f} \in \mathcal{S}(I, \mathbb{C}^Z)$. Moreover, $|\boldsymbol{f}_I| \le |\boldsymbol{f}|_I$, i.e. $|\sum_{i\in I} f_i| \le \sum_{i\in I} |f_i|$.

We will see in Theorem 1.2.12 that the converse implication is also true, i.e. if $\boldsymbol{f} \in \mathcal{S}(I, \mathbb{C}^Z)$, then $|\boldsymbol{f}| \in \mathcal{S}(I, \mathbb{R}_+^Z)$.

(b) Using the Cauchy criterion, we conclude that every normally summable family is absolutely uniformly summable. The converse implication is not true as the following standard example shows.

Let $Z := [0,1]$, $I := \mathbb{N}$, $g_k \colon [0,1] \to \mathbb{R}$,

$$g_k(x) := \begin{cases} 1 - \frac{1}{k} & \text{if } 0 \le x \le \frac{1}{k+1}, \\ \frac{1}{2} - \frac{1}{k} + \frac{k+1}{2}x & \text{if } \frac{1}{k+1} \le x \le \frac{1}{k}, \\ 1 - \frac{1}{2k} & \text{if } \frac{1}{k} \le x \le 1, \end{cases}$$

$f_k := g_k - g_{k-1}$, $k \in \mathbb{N}$, with $g_0 := 0$. Then the family $(f_k)_{k\in\mathbb{N}}$ is uniformly summable but is not normally summable (Exercise).

Proposition 1.2.10. *For every family $\boldsymbol{f} = (f_i)_{i\in I} \subset \mathbb{C}$ the following conditions are equivalent:*

(i) *$\boldsymbol{f} \in \mathcal{S}(I, \mathbb{C})$, i.e. $\boldsymbol{f}$ is summable;*

(ii) *the set $\{\boldsymbol{f}_J : J \in \mathfrak{F}(I)\} \subset \mathbb{C}$ is bounded;*

(iii) *$(|f_i|)_{i\in I} \in \mathcal{S}(I, \mathbb{R}_+)$, i.e. $\boldsymbol{f}$ is absolutely summable.*

Proof. The implication (i) ⇒ (ii) follows from Remark 1.2.4 (c).

(ii) $\Rightarrow$ (iii): Taking $\operatorname{Re} \boldsymbol{f}$ and $\operatorname{Im} \boldsymbol{f}$ instead of $\boldsymbol{f}$, we may assume that the numbers f_i are real. In the case where $f_i \geq 0$, $i \in I$, one can easily prove that the number $\boldsymbol{f}_I := \sup\{\boldsymbol{f}_J : J \in \mathfrak{F}(I)\}$ satisfies condition (1.2.1), which implies that $\boldsymbol{f}$ is (absolutely) summable. In the general case put

$$f_i^+ := \begin{cases} f_i & \text{if } f_i \geq 0, \\ 0 & \text{if } f_i < 0, \end{cases} \qquad f_i^- := \begin{cases} 0 & \text{if } f_i \geq 0, \\ -f_i & \text{if } f_i < 0, \end{cases}$$

and observe that $\{\boldsymbol{f}_J^+ : J \in \mathfrak{F}(I)\} \cup \{-\boldsymbol{f}_J^- : J \in \mathfrak{F}(I)\} \subset \{\boldsymbol{f}_J : J \in \mathfrak{F}(I)\}$. Consequently, $(f_i^+)_{i \in I} \in \mathcal{S}(I, \mathbb{R}_+)$ and $(f_i^-)_{i \in I} \in \mathcal{S}(I, \mathbb{R}_+)$. Since $|f_i| = f_i^+ + f_i^-$, $i \in I$, we conclude that the family $(|f_i|)_{i \in I}$ is also summable.

The implication (iii) $\Rightarrow$ (i) is obvious. $\square$

Theorem 1.2.11. *If $(f_i)_{i \in I} \in \mathcal{S}(I, \mathbb{C}^Z)$, $(g_j)_{j \in J} \in \mathcal{S}(J, \mathbb{C}^Z)$ and all the functions $f_i : Z \to \mathbb{C}$, $g_j : Z \to \mathbb{C}$ are bounded, then*

$$(f_i g_j)_{(i,j) \in I \times J} \in \mathcal{S}(I \times J, \mathbb{C}^Z) \quad \textit{and} \quad \sum_{(i,j) \in I \times J} f_i g_j = \boldsymbol{f}_I \boldsymbol{g}_J.$$

Proof. Recall that $(|g_j|)_{j \in J} \in \mathcal{S}(J, \mathbb{R}_+^Z)$ (Theorem 1.2.12) and all the functions $\boldsymbol{f}_A$, $A \in \mathfrak{F}(I)$, $|\boldsymbol{g}|_B$, $B \in \mathfrak{F}(J)$, are uniformly bounded (Remark 1.2.4 (c)). Let $M > 0$ be such that $|\boldsymbol{f}_A| \leq M$, $A \in \mathfrak{F}(I)$, and $|\boldsymbol{g}|_B \leq M$, $B \in \mathfrak{F}(J)$. In particular, $|\boldsymbol{f}_I| \leq M$ and $|\boldsymbol{g}|_J \leq M$.

Fix an $\varepsilon > 0$. Let $S(\varepsilon) = S(I, \varepsilon) \in \mathfrak{F}(I)$ be such that for any $A \in \mathfrak{F}(I)$ with $S(\varepsilon) \subset A$ we have $|\boldsymbol{f}_A - \boldsymbol{f}_I| \leq \varepsilon$. The Cauchy criterion implies that there exists a $C(\varepsilon) = C(J, \varepsilon) \in \mathfrak{F}(J)$ such that for every $B \in \mathfrak{F}(J \setminus C(\varepsilon))$ we have $|\boldsymbol{g}|_B \leq \varepsilon$. Let $K \in \mathfrak{F}(I \times J)$ be such that $S(\varepsilon) \times C(\varepsilon) \subset K$. Define $K(j) := \{i \in I : (i, j) \in K\}$, $j \in J$. Observe that $S(\varepsilon) \subset K(j)$ for $j \in C(\varepsilon)$. We have

$$\Big(\sum_{(i,j) \in K} f_i g_j\Big) - \boldsymbol{f}_I \boldsymbol{g}_J = \sum_{j \in J} (\boldsymbol{f}_{K(j)} - \boldsymbol{f}_I) g_j.$$

Hence

$$\begin{aligned} \Big|\Big(\sum_{(i,j) \in K} f_i g_j\Big) - \boldsymbol{f}_I \boldsymbol{g}_J\Big| &\leq \sum_{j \in C(\varepsilon)} |\boldsymbol{f}_{K(j)} - \boldsymbol{f}_I||g_j| + \sum_{j \notin C(\varepsilon)} |\boldsymbol{f}_{K(j)} - \boldsymbol{f}_I||g_j| \\ &\leq \varepsilon \sum_{j \in J} |g_j| + 2M \sum_{j \notin C(\varepsilon)} |g_j| \leq 3M\varepsilon. \end{aligned}$$

$\square$

Theorem 1.2.12 ([Sie 1910]). *Assume that I is (infinite) countable. For every family $\boldsymbol{f} = (f_i)_{i \in I} \in \mathbb{C}^Z$ the following conditions are equivalent:*

(i) *$\boldsymbol{f} \in \mathcal{S}(I, \mathbb{C}^Z)$;*

(ii) *for every bijection* $\sigma: \mathbb{N} \to I$, *the series* $\sum_{\nu=1}^{\infty} f_{\sigma(\nu)}$ *is uniformly convergent on* Z *(and* $\boldsymbol{f}_I = \sum_{\nu=1}^{\infty} f_{\sigma(\nu)}$*);*

(iii) $|\boldsymbol{f}| \in \mathfrak{S}(I, \mathbb{R}_+^Z)$.[4]

Notice that (ii) may be used as an alternative definition of the uniform summability.

Proof. (i) $\Rightarrow$ (ii): Fix a bijection $\sigma: \mathbb{N} \to I$ and $\varepsilon > 0$. Let $N_0 \in \mathbb{N}$ be such that $S(\varepsilon) \subset \{\sigma(1), \dots, \sigma(N_0)\}$, where $S(\varepsilon) = S(I, \varepsilon)$ is chosen according to (1.2.1). Then, for every $N \geq N_0$, we have $S(\varepsilon) \subset \{\sigma(1), \dots, \sigma(N)\}$, which implies that $|\sum_{\nu=1}^{N} f_{\sigma(\nu)} - \boldsymbol{f}_I| \leq \varepsilon$.

(ii) $\Rightarrow$ (iii): We may assume that $f_i: Z \to \mathbb{R}$, $i \in I$. Suppose that for some $\varepsilon_0 > 0$ the family $(|f_i|)_{i \in I}$ does not satisfy the Cauchy condition. Fix an $i_0 \in I$. The set $C(\varepsilon_0) := \{i_0\}$ does not satisfy (1.2.2). Hence, there exists a set $G(1) \in \mathfrak{F}(I \setminus \{i_0\})$ such that

$$\sup_{z \in Z} \sum_{i \in G(1)} |f_i(z)| > \varepsilon_0.$$

Let $z_1 \in Z$ be such that $\sum_{i \in G(1)} |f_i(z_1)| > \varepsilon_0$. The set $G(1)$ may be divided into two disjoint parts

$$G^+(1) := \{i \in G(1) : f_i(z_1) \geq 0\}, \quad G^-(1) := G(1) \setminus G^+(1).$$

Obviously, $|\boldsymbol{f}_{G^+(1)}(z_1)| > \varepsilon_0/2$ or $|\boldsymbol{f}_{G^-(1)}(z_1)| > \varepsilon_0/2$. Suppose that the first case holds and put $F(1) := G^+(1)$. Then $|\boldsymbol{f}_{F(1)}(z_1)| > \varepsilon_0/2$.

The set $F(1)$ also is not good. Repeating the above argument, we find a set $F(2) \in \mathfrak{F}(I \setminus F(1))$ such that $\sup_{z \in Z} |\boldsymbol{f}_{F(2)}(z)| > \varepsilon_0/2$.

Now, we take $F(1) \cup F(2)$ and we find $F(3) \in \mathfrak{F}(I \setminus (F(1) \cup F(2)))$ such that $\sup_{z \in Z} |\boldsymbol{f}_{F(3)}(z)| > \varepsilon_0/2$.

Finally, we find a sequence $(F(k))_{k=1}^{\infty} \subset \mathfrak{F}(I)$, $F(k) = \{i_{k,1}, \dots, i_{k,n(k)}\}$, of pairwise disjoint sets such that $\sup_{z \in Z} |\boldsymbol{f}_{F(k)}(z)| > \varepsilon_0/2$, $k \in \mathbb{N}$.

Put $F(0) := I \setminus \bigcup_{k=1}^{\infty} F(k)$. If $F(0)$ is finite, $F(0) = \{i_{0,1}, \dots, i_{0,n(0)}\}$ (if $F(0) = \varnothing$, then we put $n(0) := 0$), we define a bijection $\sigma: \mathbb{N} \to I$ via the following sequence:

$$i_{0,1}, \dots, i_{0,n(0)}, i_{1,1}, \dots, i_{1,n(1)}, i_{2,1}, \dots, i_{2,n(2)}, \dots.$$

[4] Let us mention the following general Dvoretzky–Rogers theorem [Dvo-Rog 1950].

Theorem. *Let* $(E, \| \ \|)$ *be a Banach space. Then the following conditions are equivalent:*

(i) *for every sequence* $(f_k)_{k=1}^{\infty} \subset E$ *the following two notions are equivalent:*

- $\sum_{k=1}^{\infty} \|f_k\| < +\infty$ (*i.e. the series* $\sum_{k=1}^{\infty} f_k$ *is* absolutely convergent),
- *for every permutation* $\sigma: \mathbb{N} \to \mathbb{N}$, *the series* $\sum_{k=1}^{\infty} f_{\sigma(k)}$ *is convergent* (*i.e. the series* $\sum_{k=1}^{\infty} f_k$ *is* unconditionally convergent)*;*

(ii) $\dim E < \infty$.

Let $S_\nu := \sum_{k=1}^{\nu} f_{\sigma(k)}$, $\nu \in \mathbb{N}$. We get

$$S_{n(0)+\dots+n(k)} - S_{n(0)+\dots+n(k-1)} = \boldsymbol{f}_{F(k)}, \quad k \in \mathbb{N}.$$

Consequently, the sequence $(S_\nu)_{\nu=1}^{\infty}$ does not satisfy the uniform Cauchy condition, which contradicts (ii).

If $F(0)$ is infinite, $F(0) = \{i_{0,1}, i_{0,2}, \dots\}$, then we define a bijection $\sigma\colon \mathbb{N} \to I$ via the following sequence:

$$i_{0,1}, i_{1,1}, \dots, i_{1,n(1)}, i_{0,2}, i_{2,1}, \dots, i_{2,n(2)}, \dots.$$

In this case we get

$$S_{n(1)+\dots+n(k)+k} - S_{n(1)+\dots+n(k-1)+k} = \boldsymbol{f}_{F(k)}, \quad k \in \mathbb{N},$$

which also contradicts (ii).

The implication (iii) $\Rightarrow$ (i) is obvious. □

Corollary 1.2.13. *Assume that I is countable and let $\boldsymbol{f} = (f_i)_{i\in I} \in \mathcal{S}(I, \mathbb{C}^Z)$.*

(a) *Let $z^0 \in Z$ be fixed. If each f_i is continuous at z^0, then $\boldsymbol{f}_I$ is continuous at z^0.*

(b) *If Z is Lebesgue measurable, $\Lambda_{2n}(Z) < +\infty$,*[5] *and each f_i is Lebesgue integrable on Z, then $(\int_Z f_i\, d\Lambda_{2n})_{i\in I} \in \mathcal{S}(I, \mathbb{C})$, $\boldsymbol{f}_I$ is Lebesgue integrable on Z, and*

$$\int_Z \boldsymbol{f}_I \, d\Lambda_{2n} = \sum_{i\in I} \int_Z f_i \, d\Lambda_{2n}.$$

Proof. Exercise. □

Using induction and Theorem 1.2.11 one gets the following corollary (Exercise).

Corollary 1.2.14. (a) *The geometric series*

$$\sum_{\alpha\in\mathbb{Z}_+^n} \frac{z^\alpha}{r^\alpha} = \sum_{\alpha\in\mathbb{Z}_+^n} \Big(\frac{z_1}{r_1}\Big)^{\alpha_1} \cdots \Big(\frac{z_n}{r_n}\Big)^{\alpha_n} = \prod_{j=1}^{n} \sum_{k=0}^{\infty} \Big(\frac{z_j}{r_j}\Big)^k,$$

where $r = (r_1, \dots, r_n) \in \mathbb{R}_{>0}^n$,[6] *is locally normally summable in $\mathbb{P}(r)$*[7] *to the function*

$$\mathbb{P}(r) \ni (z_1, \dots, z_n) \mapsto \prod_{j=1}^{n} \frac{1}{1 - z_j/r_j}.$$

---

[5] Λ_k denotes the Lebesgue measure in $\mathbb{R}^k$.

[6] $A_{>0} := \{a \in A : a > 0\}$. To simplify notation we write $\mathbb{R}_{>0}^n$ instead of $(\mathbb{R}_{>0})^n$.

[7] $\mathbb{P}(a,r) := K(a_1,r_1)\times\dots\times K(a_n,r_n)$, $K(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$, $\mathbb{P}(r) := \mathbb{P}(0,r)$, $K(r) := K(0,r)$.

(b) *The series*

$$\sum_{\alpha\in\mathbb{Z}^n}\Big(\frac{z_1}{r_1}\Big)^{|\alpha_1|}\cdots\Big(\frac{z_n}{r_n}\Big)^{|\alpha_n|}=\prod_{j=1}^{n}\sum_{k=-\infty}^{\infty}\Big(\frac{z_j}{r_j}\Big)^{|k|},$$

where $r=(r_1,\dots,r_n)\in\mathbb{R}^n_{>0}$, *is locally normally summable in* $\mathbb{P}(r)$ *to the function*

$$\mathbb{P}(r)\ni(z_1,\dots,z_n)\mapsto\prod_{j=1}^{n}\frac{1+z_j/r_j}{1-z_j/r_j}.$$

1.3 Domains of convergence of power series

Definition 1.3.1. Any series of the form

$$\sum_{\alpha\in\mathbb{Z}^n_+}a_\alpha(z-a)^\alpha\quad(z\in\mathbb{C}^n),$$

where $(a_\alpha)_{\alpha\in\mathbb{Z}^n_+}\subset\mathbb{C}$, $a\in\mathbb{C}^n$ ($w^\alpha:=w_1^{\alpha_1}\cdots w_n^{\alpha_n}$), is called an ($n$-*fold*) *power series with center at* a.

Fix a power series (centered at 0):

$$S=\sum_{\alpha\in\mathbb{Z}^n_+}a_\alpha z^\alpha.$$

Remark 1.3.2 (Abel's lemma). Assume that

$$|a_\alpha|r^\alpha\le C,\quad\alpha\in\mathbb{Z}^n_+,$$

where $r\in\mathbb{R}^n_{>0}$. Then for every $0<\theta<1$ we have

$$|a_\alpha z^\alpha|\le C\theta^{|\alpha|},\quad z\in\mathbb{P}(\theta r),\ \alpha\in\mathbb{Z}^n_+\quad(\text{Exercise}).$$

In particular, the series $\sum_{\alpha\in\mathbb{Z}^n_+}a_\alpha z^\alpha$ is locally normally summable in $\mathbb{P}(r)$.

Definition 1.3.3. Given a power series S, put

$$\mathfrak{B}=\mathfrak{B}_S:=\Big\{z\in\mathbb{C}^n:\ \sup_{\alpha\in\mathbb{Z}^n_+}|a_\alpha z^\alpha|<+\infty\Big\},$$

$$\mathfrak{C}=\mathfrak{C}_S:=\Big\{z\in\mathbb{C}^n:\ \text{the series}\sum_{\alpha\in\mathbb{Z}^n_+}a_\alpha z^\alpha\text{ is summable}\Big\},$$

$$\mathfrak{D}=\mathfrak{D}_S:=\operatorname{int}\mathfrak{C}.$$

Clearly $\mathfrak{D}\subset\mathfrak{C}\subset\mathfrak{B}$. The set $\mathfrak{D}$ is traditionally called the *domain of convergence* of the power series S.

Exercise 1.3.4. Determine $\mathfrak{B}_S$, $\mathfrak{C}_S$, and $\mathfrak{D}_S$ for the following power series:

(a) $\sum_{\mu\in\mathbb{Z}_+} \mu! z_1^{\mu} z_2$;
(b) $\sum_{\mu\in\mathbb{Z}_+} z_1^{\mu} z_2$;
(c) $\sum_{\mu,\nu\in\mathbb{Z}_+} z_1^{\mu} z_2^{\nu}$;
(d) $\sum_{\mu,\nu\in\mathbb{Z}_+} \mu! z_1^{\mu} z_2^{\nu}$;
(e) $\sum_{\mu\in\mathbb{Z}_+} (z_1 z_2)^{\mu}$;
(f) $\sum_{\mu,\nu\in\mathbb{N}} \frac{\mu}{\nu!} z_1^{\mu} z_2^{\nu}$;
(g) $\sum_{\mu,\nu\in\mathbb{N}} \frac{(\mu+\nu)!}{\mu!\nu!} z_1^{\mu} z_2^{\nu}$.

Remark 1.3.5. (a) If $n = 1$ and $\varnothing \neq \mathfrak{D} \neq \mathbb{C}$, then $\bar{\mathfrak{B}} = \bar{\mathfrak{C}} = \bar{\mathfrak{D}} = \bar{K}(R)$, where R is the radius of convergence of S.

(b) If $S := \sum_{\mu\in\mathbb{Z}_+} \mu! z_1^{\mu} z_2$, then $\mathfrak{C} = (\mathbb{C}\times\{0\})\cup(\{0\}\times\mathbb{C}) = V_0$ and $\mathfrak{D} = \varnothing$. In particular, for $n \geq 2$ we may have $\bar{\mathfrak{C}} \not\subset \bar{\mathfrak{D}}$.

(c) For every point $a = (a_1, \dots, a_n) \in \mathfrak{B}$ (resp. $\mathfrak{C}$) the closed polydisc

$$\bar{\mathbb{P}}((|a_1|, \dots, |a_n|)) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| \leq |a_j|,\ j = 1, \dots, n\}$$

is contained in $\mathfrak{B}$ (resp. $\mathfrak{C}$).

(d) $\mathfrak{D} = \operatorname{int} \mathfrak{B} = \operatorname{int} \bar{\mathfrak{B}}$. In particular, $\mathfrak{D}$ is fat. (An open set $\Omega \subset \mathbb{R}^k$ is said to be *fat* if $\Omega = \operatorname{int} \bar{\Omega}$.)

Indeed, fix an $a = (a_1, \dots, a_n) \in \operatorname{int} \bar{\mathfrak{B}}$. Observe that for small $\varepsilon > 0$ the point $b = (b_1, \dots, b_n)$, with

$$b_j := \begin{cases} a_j(1+\varepsilon) & \text{if } a_j \neq 0, \\ \varepsilon & \text{if } a_j = 0, \end{cases} \qquad j = 1, \dots, n,$$

also belongs to $\operatorname{int} \bar{\mathfrak{B}}$. Let $c = (c_1, \dots, c_n) \in \mathfrak{B}$ be such that

$$|c_j - b_j| < \begin{cases} |a_j|\varepsilon & \text{if } a_j \neq 0, \\ \varepsilon & \text{if } a_j = 0, \end{cases} \qquad j = 1, \dots, n.$$

Consequently,

$$r_j := |c_j| \geq |b_j| - |c_j - b_j| > \begin{Bmatrix} |a_j|(1+\varepsilon) - |a_j|\varepsilon & \text{if } a_j \neq 0 \\ \varepsilon - \varepsilon & \text{if } a_j = 0 \end{Bmatrix} = |a_j|, \qquad j = 1, \dots, n.$$

Thus $a \in \mathbb{P}(r)$. Now, by Abel's lemma, we conclude that $a \in \mathbb{P}(r) \subset \mathfrak{D}$.

(e) In view of (d), $\bar{\mathbb{P}}((|a_1|, \dots, |a_n|)) \subset \mathfrak{D}$ for every $a = (a_1, \dots, a_n) \in \mathfrak{D}$. Observe that any closed polydisc $\bar{\mathbb{P}}((|a_1|, \dots, |a_n|))$ is obviously connected. In particular, $\mathfrak{D}$ is connected and so, $\mathfrak{D}$ is really a *domain*.

(f) For every compact $K \subset \mathfrak{D}$ there exist $C > 0$ and $0 < \theta < 1$ such that

$$|a_\alpha z^\alpha| \le C\theta^{|\alpha|}, \quad z \in K,\ \alpha \in \mathbb{Z}_+^n.$$

Consequently, the series S is locally normally summable in $\mathfrak{D}$. In particular, the function $f(z) := \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha z^\alpha$, $z \in \mathfrak{D}$, is continuous (Corollary 1.2.13 (a)).

Indeed, take a point $a \in \mathfrak{D}$ and let $r \in \mathbb{R}_{>0}^n \cap \mathfrak{B}$ and $0 < \theta < 1$ be such that $a \in \mathbb{P}(\theta r)$. Next use Abel's lemma (Exercise).

Exercise 1.3.6. Let

$$f(z) := \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha z^\alpha, \quad z \in \mathfrak{D}_S.$$

Prove that for every $\mathbb{P}(a,r) \subset \mathfrak{D}_S$ there exists a power series $\sum_{\gamma\in\mathbb{Z}_+^n} b_\gamma (z-a)^\gamma$ centered at a such that

$$f(z) = \sum_{\gamma\in\mathbb{Z}_+^n} b_\gamma (z-a)^\gamma, \quad z \in \mathbb{P}(a,r)$$

(cf. Step 3 of the proof of Proposition 1.3.12).

Exercise 1.3.7. Check whether there exists a power series S such that

$$\mathfrak{D}_S = \{(z_1,z_2) \in \mathbb{D}^2 : \text{ either } |z_1| < r_1 \text{ or } |z_2| < r_2\}$$

with $0 < r_1, r_2 < 1$ (cf. Fig. 1.5.2).

We are led to the very important notion of a complete Reinhardt set.

Definition 1.3.8. We say that a set $A \subset \mathbb{C}^n$ is *complete Reinhardt* (*n-circled*) if for every point $a = (a_1,\dots,a_n) \in A$ and for every $\lambda = (\lambda_1,\dots,\lambda_n) \in \overline{\mathbb{D}}^n$, the point $\lambda \cdot a = (\lambda_1 a_1,\dots,\lambda_n a_n)$ belongs to A; equivalently,

$$A = \bigcup_{a=(a_1,\dots,a_n)\in A} \overline{\mathbb{P}}((|a_1|,\dots,|a_n|)).$$

Exercise 1.3.9. (a) The domain of convergence of a power series is a complete Reinhardt domain.

(b) If $A \subset \mathbb{C}^n$ is complete Reinhardt, then A is arcwise connected.

(c) If $A \subset \mathbb{C}^n$ is complete Reinhardt, then $\overline{A}$ and $\operatorname{int} A$ are complete Reinhardt.

Exercise 1.3.10. Let $S = \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha z^\alpha$, $T = \sum_{\beta\in\mathbb{Z}_+^n} b_\beta z^\beta$ be arbitrary power series. Using Theorem 1.2.7, prove that

$$\sum_{\alpha,\,\beta\in\mathbb{Z}_+^n} a_\alpha b_\beta z^{\alpha+\beta} = \sum_{\gamma\in\mathbb{Z}_+^n} c_\gamma z^\gamma, \quad z \in \mathfrak{D}_S \cap \mathfrak{D}_T,$$

where

$$c_\gamma := \sum_{\alpha\in\mathbb{Z}_+^n:\,\alpha\le\gamma} a_\alpha b_{\gamma-\alpha}, \quad \gamma\in\mathbb{Z}_+^n.$$

The power series on the right-hand side is called the *Cauchy product* of the series S and T.

We are going to study the function $f(z) := \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha z^\alpha$, $z\in\mathfrak{D}_S$, defined by the series S. First we need some notation.

Let $\Omega\subset\mathbb{C}^n$ be open. We say that a function $g\colon\Omega\to\mathbb{C}$ is *Fréchet differentiable in the complex* (*resp. real*) *sense* at a point $a\in\Omega$ if one of the following two equivalent conditions is satisfied (details are left to the reader as an Exercise):

(i) there exists a $\mathbb{C}$-linear (resp. $\mathbb{R}$-linear) mapping $L\colon\mathbb{C}^n\to\mathbb{C}$ such that

$$g(a+h) = g(a) + L(h) + o(\|h\|) \quad \text{when } h\to 0;$$

(ii) there exist a $\mathbb{C}$-linear (resp. $\mathbb{R}$-linear) mapping $L\colon\mathbb{C}^n\to\mathbb{C}$ and functions $g_1,\dots,g_n\colon\Omega-a\to\mathbb{C}$, continuous at 0, with $g_1(0)=\dots=g_n(0)=0$, such that

$$g(a+h) = g(a) + L(h) + \sum_{j=1}^n g_j(h)h_j, \quad h=(h_1,\dots,h_n)\in\Omega-a.$$ [8]

Obviously, the above operator L is uniquely determined; we write $g'(a) = g'_{\mathbb{C}}(a) := L$ (resp. $g'_{\mathbb{R}}(a) := L$) and we say that $g'_{\mathbb{C}}(a)$ (resp. $g'_{\mathbb{R}}(a)$) is the *complex* (resp. *real*) *Fréchet differential* of g at a.

Exercise 1.3.11. Find a function $g\colon\mathbb{C}\to\mathbb{C}$ such that $g'_{\mathbb{R}}(0)$ exists, but $g'_{\mathbb{C}}(0)$ does not exist.

It is clear that if $g'_{\mathbb{C}}(a)$ exists, then $g'_{\mathbb{R}}(a)$ exists and $g'_{\mathbb{R}}(a) = g'_{\mathbb{C}}(a)$. If $g'_{\mathbb{R}}(a)$ exists, then g is continuous at a. If $g'_{\mathbb{C}}(a)$ (resp. $g'_{\mathbb{R}}(a)$) exists, then g has at a all *complex* (*resp. real*) *partial derivatives*

$$\frac{\partial g}{\partial z_j}(a) := \lim_{\mathbb{C}_*\ni h\to 0}\frac{g(a+he_j)-g(a)}{h},$$ [9]

$$\text{(resp. } \frac{\partial g}{\partial x_j}(a) := \lim_{\mathbb{R}_*\ni h\to 0}\frac{g(a+he_j)-g(a)}{h},$$

$$\frac{\partial g}{\partial y_j}(a) := \lim_{\mathbb{R}_*\ni h\to 0}\frac{g(a+ihe_j)-g(a)}{h}), \quad j=1,\dots,n,$$

[8] The implication (ii) $\Rightarrow$ (i) is elementary. If (i) is satisfied, then we put

$$g_j(h) := \frac{\bar h_j}{\|h\|^2}(g(a+h) - g(a) - L(h)).$$

[9] If $A\subset\mathbb{C}^k$, then $A_* := A\setminus\{0\}$.

where $(e_1, \dots, e_n)$ is the canonical basis of $\mathbb{C}^n$. Moreover,

$$g'_{\mathbb{C}}(a)(h) = \sum_{j=1}^{n} \frac{\partial g}{\partial z_j}(a) h_j$$

$$\Big(\text{resp. } g'_{\mathbb{R}}(a)(h) = \sum_{j=1}^{n} \frac{\partial g}{\partial z_j}(a) h_j + \sum_{j=1}^{n} \frac{\partial g}{\partial \bar{z}_j}(a) \bar{h}_j\Big), \quad h = (h_1, \dots, h_n) \in \mathbb{C}^n,$$

where

$$\frac{\partial g}{\partial z_j}(a) := \frac{1}{2}\Big(\frac{\partial g}{\partial x_j}(a) - i\frac{\partial g}{\partial y_j}(a)\Big), \quad \frac{\partial g}{\partial \bar{z}_j}(a) := \frac{1}{2}\Big(\frac{\partial g}{\partial x_j}(a) + i\frac{\partial g}{\partial y_j}(a)\Big)$$

denote the *formal partial derivatives of g at a*.[10] If $g'_{\mathbb{R}}(a)$ exists, then the following conditions are equivalent (Exercise):

(i) $g'_{\mathbb{C}}(a)$ exists;

(ii) $g'_{\mathbb{R}}(a)$ is $\mathbb{C}$-linear;

(iii) $\frac{\partial g}{\partial \bar{z}_j}(a) = 0$, $j = 1, \dots, n$;

(iv) the complex partial derivatives $\frac{\partial g}{\partial z_j}(a)$, $j = 1, \dots, n$, exist.

The above result frequently permits us to transport theorems from real analysis to the complex case.

The notion of the Fréchet differentiability extends in a standard way (componentwise) to mappings $g\colon \Omega \to \mathbb{C}^m$. Then the complex Fréchet differential of g at a is a $\mathbb{C}$-linear mapping $g'(a)\colon \mathbb{C}^n \to \mathbb{C}^m$, which may be identified with an $m \times n$-dimensional matrix. In view of the above identification, one can define *k-th complex Fréchet differentials $g^{(k)}(a)$* and *k-th order complex partial derivatives*

$$\frac{\partial^k g}{\partial z_{j_k} \dots \partial z_{j_1}}(a) := \frac{\partial}{\partial z_{j_k}}\Big(\frac{\partial^{k-1} g}{\partial z_{j_{k-1}} \dots \partial z_{j_1}}\Big)(a), \qquad \begin{array}{l} 1 \le j_1, \dots, j_k \le n, \\ k = 2, 3, \dots. \end{array}$$

One can prove that if $g^{(k)}(a)$ exists, then g has at a all complex partial derivatives of order k, the derivatives are independent of the order of differentiation, and

$$g^{(k)}(a)(h) = \sum_{\alpha \in \mathbb{Z}_+^n \colon |\alpha| = k} \frac{k!}{\alpha!} D^{\alpha} g(a) h^{\alpha}, \quad h \in \mathbb{C}^n,$$ [11]

where

$$D^{\alpha} g(a) = \Big(\frac{\partial}{\partial z_1}\Big)^{\alpha_1} \circ \dots \circ \Big(\frac{\partial}{\partial z_n}\Big)^{\alpha_n} g(a).$$

[10] The reader should always decipher from the context whether $\frac{\partial g}{\partial z_j}(a)$ denotes the complex or formal partial derivative!

[11] $\alpha! := \alpha_1! \cdots \alpha_n!$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$; notice that formally $g^{(k)}(a)$ is a k-linear symmetric mapping $(\mathbb{C}^n)^k \to \mathbb{C}$, which is, as always, identified with the homogeneous polynomial $\mathbb{C}^n \to \mathbb{C}$ of degree k.

If $g^{(k)}(a)$ exists for every $k \in \mathbb{N}$, then we define the *Taylor series* of g at a as the power series

$$T_a g(z) := \sum_{\alpha \in \mathbb{Z}_+^n} \frac{1}{\alpha!} D^\alpha g(a)(z-a)^\alpha.$$

The number

$$d(T_a g) := \sup\{r \geq 0 : T_a g \text{ is uniformly summable in } \overline{\mathbb{P}}(a,r)\} \in [0, +\infty]$$

is called the *radius of convergence of* $T_a g$. Observe that

$$T_a g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} g^{(k)}(a)(z-a).$$

Proposition 1.3.12. *Assume that* $\mathfrak{D}_S \neq \varnothing$ *and let*

$$f(z) := \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha z^\alpha, \quad z \in \mathfrak{D}_S.$$

For $\beta \in \mathbb{Z}_+^n$ *let* $D^\beta S$ *denote the power series*

$$\sum_{\alpha \in \mathbb{Z}_+^n : \, \alpha \geq \beta} \binom{\alpha}{\beta} \beta! \, a_\alpha \, z^{\alpha - \beta}.$$[12]

Then f *has all complex Fréchet differentials in* $\mathfrak{D}_S$,[13] $\mathfrak{D}_S \subset \mathfrak{D}_{D^\beta S}$, *and*

$$D^\beta f(z) = \sum_{\alpha \in \mathbb{Z}_+^n : \, \alpha \geq \beta} \binom{\alpha}{\beta} \beta! \, a_\alpha \, z^{\alpha - \beta}, \quad z \in \mathfrak{D}_S, \ \beta \in \mathbb{Z}_+^n. \tag{1.3.1}$$

In particular, $f(z) = T_0 f(z)$, $z \in \mathfrak{D}_S$.

Notice the following difference between one and several variables. For $n = 1$ the radius of convergence of S is equal to the radius of convergence of the series of derivatives. This is no longer true for $n \geq 2$, for instance if S is the power series

$$\sum_{\nu=0}^{\infty} z_1^\nu + \sum_{\nu=0}^{\infty} z_2^\nu,$$

then $\mathfrak{D}_S = \mathbb{D} \times \mathbb{D}$, but $\mathfrak{D}_{\frac{\partial S}{\partial z_1}} = \mathbb{D} \times \mathbb{C}$.[14]

[12] $\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, $\beta \leq \alpha$.

[13] In fact, f is holomorphic – cf. Theorem 1.7.19.

[14] $\frac{\partial S}{\partial z_1} = \sum_{\nu=1}^{\infty} \nu z_1^{\nu-1}$.

Proof. Step 1. First observe that, for every $j \in \{1, \dots, n\}$, the series

$$\frac{\partial S}{\partial z_j} = \sum_{\alpha \in \mathbb{Z}_+^n :\, \alpha \geq e_j} \alpha_j a_\alpha \, z^{\alpha - e_j}$$

is locally normally summable in $\mathfrak{D}_S$. It is sufficient to prove that if $R \in \mathbb{R}_{>0}^n \cap \mathfrak{B}_S$, then the series $\frac{\partial S}{\partial z_j}$ is locally normally summable in $\mathbb{P}(R)$. Let $C > 0$ be such that $|a_\alpha| R^\alpha \leq C$, $\alpha \in \mathbb{Z}_+^n$. Then for any $0 < \theta < 1$ we have

$$\sum_{\alpha \in \mathbb{Z}_+^n :\, \alpha \geq e_j} \sup_{\mathbb{P}(\theta R)} |\alpha_j a_\alpha \, z^{\alpha - e_j}| \leq \frac{C}{\theta R_j} \sum_{\alpha \in \mathbb{Z}_+^n :\, \alpha \geq e_j} \alpha_j \theta^{|\alpha|},$$

which gives the normal summability in $\mathbb{P}(\theta R)$.

In particular, the function F_j defined by the series $\frac{\partial S}{\partial z_j}$ is continuous on $\mathfrak{D}_S$, $j = 1, \dots, n$ (Corollary 1.2.13 (a)).

Step 2. We have

$$f(h) = f(0) + \sum_{j=1}^n a_{e_j} h_j + \sum_{j=1}^n f_j(h) h_j, \quad h = (h_1, \dots, h_n) \in \mathfrak{D}_S,$$

where

$$f_1(h) := \sum_{|\alpha| \geq 2,\, \alpha \geq e_1} a_\alpha h^{\alpha - e_1}, \quad f_2(h) := \sum_{\substack{|\alpha| \geq 2,\, \alpha \geq e_2 \\ \alpha_1 = 0}} a_\alpha h^{\alpha - e_2}, \dots,$$

$$f_{n-1}(h) := \sum_{\substack{|\alpha| \geq 2,\, \alpha \geq e_{n-1} \\ \alpha_1 = \dots = \alpha_{n-2} = 0}} a_\alpha h^{\alpha - e_{n-1}}, \quad f_n(h) := \sum_{\substack{|\alpha| \geq 2,\, \alpha \geq e_n \\ \alpha_1 = \dots = \alpha_{n-1} = 0}} a_\alpha h^{\alpha - e_n}.$$

Observe that all the above series are normally summable in a neighborhood U of 0 (EXERCISE). In particular, the functions $f_1, \dots, f_n$ are continuous in U. Note that $f_1(0) = \dots = f_n(0) = 0$. Thus $f'(0)$ exists and $\frac{\partial f}{\partial z_j}(0) = a_{e_j} = F_j(0)$, $j = 1, \dots, n$.

Step 3. If $\mathbb{P}(a, r) \Subset \mathfrak{D}_S$, then the series

$$\sum_{\substack{\alpha, \gamma \in \mathbb{Z}_+^n \\ \alpha \geq \gamma}} a_\alpha \binom{\alpha}{\gamma} (z - a)^\gamma a^{\alpha - \gamma}$$

is normally summable in $\mathbb{P}(a, r)$.

Indeed, let $R \in \mathfrak{B}_S \cap \mathbb{R}_{>0}^n$ and $\theta \in (0, 1)$ be such that $|a_j| + r_j \leq \theta R_j$,

$j = 1, \dots, n$, and let $|a_\alpha| R^\alpha \le C$, $\alpha \in \mathbb{Z}_+^n$. Then

$$\sum_{\substack{\alpha,\gamma\in\mathbb{Z}_+^n\\ \alpha\ge\gamma}} |a_\alpha| \binom{\alpha}{\gamma} \sup_{z\in\mathbb{P}(a,r)} |(z-a)^\gamma a^{\alpha-\gamma}| \le \sum_{\alpha\in\mathbb{Z}_+^n} |a_\alpha| \sup_{z\in\mathbb{P}(a,r)} \prod_{j=1}^n (|z_j - a_j| + |a_j|)^{\alpha_j}$$
$$\le \sum_{\alpha\in\mathbb{Z}_+^n} |a_\alpha| (\theta R)^\alpha \le C \sum_{\alpha\in\mathbb{Z}_+^n} \theta^{|\alpha|} < +\infty.$$

Step 4. Fix $\mathbb{P}(a,r) \Subset \mathfrak{D}_S$. By Step 3 and Theorem 1.2.7, we have

$$\begin{aligned} f(z) &= \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha (z + a - a)^\alpha = \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha \sum_{\gamma\le\alpha} \tbinom{\alpha}{\gamma} (z-a)^\gamma a^{\alpha-\gamma} \\ &= \sum_{\gamma\in\mathbb{Z}_+^n} \Big(\sum_{\alpha\ge\gamma} a_\alpha \tbinom{\alpha}{\gamma} a^{\alpha-\gamma} \Big) (z-a)^\gamma =: \sum_{\gamma\in\mathbb{Z}_+^n} b_\gamma (z-a)^\gamma, \quad z \in \mathbb{P}(a,r). \end{aligned}$$

Hence, by Step 2, the function $\mathbb{P}(r) \ni z \overset{g}{\mapsto} f(a+z)$ is Fréchet differentiable at 0 and $\frac{\partial g}{\partial z_j}(0) = b_{e_j}$, $j = 1, \dots, n$. Consequently, f is differentiable at a and $\frac{\partial f}{\partial z_j}(a) = \frac{\partial g}{\partial z_j}(0) = b_{e_j} = F_j(a)$, $j = 1, \dots, n$.

Step 5. Iterating the above procedure shows that f has all complex Fréchet differentials and (1.3.1) holds for arbitrary β (EXERCISE). □

Exercise* 1.3.13. Assume that $\mathfrak{D}_S \ne \varnothing$,

$$f(z) := \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha z^\alpha, \quad z \in \mathfrak{D}_S,$$

and $f(0) = a_0 \ne 0$. Find a power series $\sum_{\beta\in\mathbb{Z}_+^n} b_\beta z^\beta$ such that

$$\frac{1}{f(z)} = \sum_{\beta\in\mathbb{Z}_+^n} b_\beta z^\beta$$

for z in a neighborhood of 0.

1.4 Maximal affine subspace of a convex set I

As we have already mentioned in the Introduction, the logarithmic image $X := \log D \subset \mathbb{R}^n$ of a Reinhardt domain $D \subset \mathbb{C}^n$ will play an important role in various characterizations of the structure of holomorphic functions on D. In all essential cases the domain X will be convex. For the convenience of the reader we collect below some basic properties of convex domains in $\mathbb{R}^n$ which will be used in the sequel.

Recall that a set $X \subset \mathbb{R}^n$ is said to be *convex* if for every $a, b \in X$, the *segment* $[a, b] := \{(1-t)a + tb : t \in [0, 1]\}$ is contained in X.

Remark 1.4.1 (Properties of convex sets; the reader is asked to complete details).

(a) For any family $(X_i)_{i \in I} \subset \mathbb{R}^n$ of convex sets, the set $\bigcap_{i \in I} X_i$ is convex. In particular, for any set $A \subset \mathbb{R}^n$, there exists the smallest convex set $\operatorname{conv} A$ containing A.

(b) If $A, B \subset \mathbb{R}^n$ are convex, then

$$\operatorname{conv}(A \cup B) = \{(1-t)a + tb : a \in A,\ b \in B,\ t \in [0, 1]\} =: X.$$

(c) If $X \subset \mathbb{R}^n$ is convex, then $\overline{X}$ is convex.

(d) If $X \subset \mathbb{R}^n$ is convex, then $\operatorname{int} X$ is convex. In particular, for any family $(X_i)_{i \in I} \subset \mathbb{R}^n$ of convex sets, the set $\operatorname{int} \bigcap_{i \in I} X_i$ is a convex domain.

(e) For every $\alpha \in (\mathbb{R}^n)_*$, $c \in \mathbb{R}$, the *open halfspace*

$$\boldsymbol{H}_{\alpha,c} := \{x \in \mathbb{R}^n : \langle x, \alpha \rangle < c\},$$

where $\langle x, y \rangle := \sum_{j=1}^{n} x_j y_j$ is the standard scalar product in $\mathbb{R}^n$, is convex. Moreover, we put $\boldsymbol{H}_{0,c} := \begin{cases} \mathbb{R}^n & \text{if } c > 0 \\ \varnothing & \text{if } c \le 0 \end{cases}$. Notice that $\boldsymbol{H}_{\alpha,c}$ is fat.

(f) If

$$\varnothing \neq X := \operatorname{int} \bigcap_{(\alpha,c) \in A} \boldsymbol{H}_{\alpha,c}, \quad A \subset (\mathbb{R}^n)_* \times \mathbb{R},$$

then we may always assume that the *representation of X is minimal* in the following sense: for every $(\alpha, c) \in A$ we have $\partial X \cap \partial \boldsymbol{H}_{\alpha,c} \neq \varnothing$, i.e. $\boldsymbol{H}_{\alpha,c} = H_\alpha^a$ for some $a \in \partial X$, where

$$H_\alpha^a := \{x \in \mathbb{R}^n : \langle x - a, \alpha \rangle < 0\}.$$

Indeed, we proceed in two steps:

– Define

$$B := \operatorname{pr}_{\mathbb{R}^n}(A),\ [15] \quad \boldsymbol{c}(\alpha) := \sup\{\langle x, \alpha \rangle : x \in X\},\ \alpha \in B.$$

We have got a function $\boldsymbol{c} \colon B \to \mathbb{R}$. Observe that $\boldsymbol{c}(\alpha) \le \inf\{c : (\alpha, c) \in A\}$. Then $X = \operatorname{int} \overline{\bigcap_{\alpha \in B} \boldsymbol{H}_{\alpha,\boldsymbol{c}(\alpha)}}$.

– Let $B_0 := \{\alpha \in B : \partial X \cap \partial \boldsymbol{H}_{\alpha,\boldsymbol{c}(\alpha)} \neq \varnothing\}$. Then $X = \operatorname{int} \bigcap_{\alpha \in B_0} \boldsymbol{H}_{\alpha,\boldsymbol{c}(\alpha)}$.

We only need to show that if $\alpha_0 \in B \setminus B_0$, then

$$X = \operatorname{int} \bigcap_{\alpha \in B \setminus \{\alpha_0\}} \boldsymbol{H}_{\alpha,\boldsymbol{c}(\alpha)} =: X_0.$$

[15] $\operatorname{pr}_X \colon X \times Y \to X$, $\operatorname{pr}_X(x, y) := x$. Notice that the same notation will be also used in the case where a vector space V is a direct sum of subspaces X and Y, $V = X + Y$, and then $\operatorname{pr}_X \colon V \to X$, $\operatorname{pr}_X(x + y) := x$. In the sequel, the context will indicate which of the above cases occurs.

Suppose that $x_0 \in X_0 \setminus X$, i.e. $x_0 \in X_0 \setminus \boldsymbol{H}_{\alpha_0,c(\alpha_0)}$. Take a $y_0 \in X$ and let $z_0 \in [x_0, y_0] \cap \partial \boldsymbol{H}_{\alpha_0,c(\alpha_0)}$. Let $U \subset X_0$ be a convex neighborhood of z_0. Then $U \cap \boldsymbol{H}_{\alpha_0,c(\alpha_0)} \subset X$ and, therefore, $z_0 \in \partial X \cap \partial \boldsymbol{H}_{\alpha_0,c(\alpha_0)}$; a contradiction.

(g) If $X \subsetneq \mathbb{R}^n$ is a convex domain, then for every $a \in \partial X$ there exists an $\alpha \in (\mathbb{R}^n)_*$ such that $X \subset H_\alpha^a$. In particular, there exists a mapping $\Theta\colon \partial X \to (\mathbb{R}^n)_*$ such that $X = \operatorname{int} \bigcap_{a \in \partial X} \boldsymbol{H}_{\Theta(a)}^a$.

(h) If $X = \operatorname{int} \bigcap_{i \in I} X_i$, where each X_i is a fat domain, then X is fat. In particular, any convex domain is fat.

(i) If X is a closed convex set, $\operatorname{int} X \neq \varnothing$, then for any $a \in \operatorname{int} X$ and $b \in X$ we have $[a, b) := \{(1-t)a + tb : t \in [0, 1)\} \subset \operatorname{int} X$. In particular, $X = \overline{\operatorname{int} X}$.

For any set $A \subset \mathbb{R}^n$, we define its *orthogonal complement* $A^\perp$ by the formula

$$A^\perp := \{x \in \mathbb{R}^n : \forall_{a \in A} : \langle x, a \rangle = 0\}.$$

For any vector subspace F of $\mathbb{R}^n$ let $\operatorname{pr}_F : \mathbb{R}^n \to F$ denote the orthogonal projection onto F. For $A \subset \mathbb{R}^n$, let $[A]$ or span A denote the vector subspace of $\mathbb{R}^n$ spanned by A.

The rest of this section is based on [Jar-Pfl 1985] and [Jar-Pfl 1987].

Remark 1.4.2. Let $X \subsetneq \mathbb{R}^n$ be a convex domain and let $F \subset \mathbb{R}^n$ be a vector space. Then the following conditions are equivalent:

(i) $X + F = X$;

(ii) there exists a point $x^0 \in \overline{X}$ such that $x^0 + F \subset \overline{X}$;

(iii) $\overline{X} + F = \overline{X}$;

(iv) $(\partial X) + F = \partial X$;

(v) $X = F + Y$, where Y is a convex domain in $F^\perp$ (observe that $Y = \operatorname{pr}_{F^\perp}(X)$).

In fact, it is trivial that (i) $\Rightarrow$ (ii). To prove that (ii) $\Rightarrow$ (iii), observe that

$$(1 - \tfrac{1}{k})x + \tfrac{1}{k}(x^0 + ky) \xrightarrow[k \to +\infty]{} x + y, \quad x \in \overline{X},\ y \in F.$$

To prove that (iii) $\Rightarrow$ (i), observe that by Remark 1.4.1 (i), for every $y \in F$ we get

$$X + y = \operatorname{int}(X + y) \subset \operatorname{int} \overline{X} = X.$$

Now it is clear that (i) + (iii) $\Rightarrow$ (iv). Obviously (iv) $\Rightarrow$ (ii). The implication (v) $\Rightarrow$ (i) is obvious. To prove the converse implication, define $Y := \operatorname{pr}_{F^\perp}(X)$. Obviously, $X \subset F + Y$. Take $y \in F$ and $x'' = \operatorname{pr}_{F^\perp}(x)$ with $x \in X$. Let $x' := \operatorname{pr}_F(x)$. Then $y + x'' = (y - x') + x \in F + X = X$. Thus $F + Y \subset X$.

Definition 1.4.3. A vector subspace F of $\mathbb{R}^n$ is of *rational type*, if F is generated by $F \cap \mathbb{Q}^n$, i.e. $F = [F \cap \mathbb{Q}^n]$. Otherwise, we say that F is of *irrational type*.

Remark 1.4.4. Let $F \subset \mathbb{R}^n$ be a vector space, $d := \dim F$.

(a) F is of rational type iff $F = [F \cap \mathbb{Z}^n]$.

(b) Let $L \in \mathbb{GL}(n, \mathbb{Q})$.[16] Then F is of rational type iff $L(F)$ is of rational type.

(c) The following conditions are equivalent:

(i) F is of rational type;

(ii) $F^\perp$ is of rational type;

(iii) there exist $\alpha^1, \dots, \alpha^{n-d} \in \mathbb{Z}^n$ such that $F = \{\alpha^1, \dots, \alpha^{n-d}\}^\perp$;

(iv) there exists a family $B \subset \mathbb{Q}^n$ such that $F = B^\perp$;

(v) $\dim(F^\perp \cap \mathbb{Q}^n)^\perp = \dim F$;

(vi) there exists a non-singular matrix $L \in \mathbb{M}(n \times n, \mathbb{Z})$ such that $F = L(\mathbb{R}^d \times \{0\}^{n-d})$ and $F^\perp = L(\{0\}^d \times \mathbb{R}^{n-d})$.

Indeed, to see that (i) $\Leftrightarrow$ (ii), let $\alpha^1, \dots, \alpha^d \in \mathbb{Q}^n$ be a basis of F. Then $F^\perp = \{\alpha^1, \dots, \alpha^d\}^\perp$. To simplify notation, suppose that

$$\Delta := \det[\alpha_k^j]_{j,k=1,\dots,d} \neq 0.$$

Then, using Cramer's formulas, we conclude that the space $F^\perp$ is spanned by the rational vectors

$$v^k := (\Delta_{1,k}/\Delta, \dots, \Delta_{d,k}/\Delta, 0, \dots, 0, \underset{k\text{-th place}}{-1}, 0, \dots, 0), \quad k = d+1, \dots, n, \tag{1.4.1}$$

where

$$\Delta_{j,k} := \det \begin{bmatrix} \alpha_1^1 & \dots & \alpha_{j-1}^1 & \alpha_k^1 & \alpha_{j+1}^1 & \dots & \alpha_d^1 \\ \vdots & & & & & & \vdots \\ \alpha_1^d & \dots & \alpha_{j-1}^d & \alpha_k^d & \alpha_{j+1}^d & \dots & \alpha_d^d \end{bmatrix}, \quad j = 1, \dots, d,\ k = d+1, \dots, n. \tag{1.4.2}$$

The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are obvious.

(ii) $\Leftrightarrow$ (v): Observe that always we have $(F^\perp \cap \mathbb{Q}^n)^\perp \supset F$. Hence it holds that $\dim(F^\perp \cap \mathbb{Q}^n)^\perp = \dim F \Leftrightarrow (F^\perp \cap \mathbb{Q}^n)^\perp = F \Leftrightarrow [F^\perp \cap \mathbb{Q}^n] = F^\perp$.

(i) $\Rightarrow$ (vi): We only need to take

$$L = \begin{bmatrix} \alpha_1^1 & \dots & \alpha_1^n \\ \alpha_2^1 & \dots & \alpha_2^n \\ \vdots & & \vdots \\ \alpha_n^1 & \dots & \alpha_n^n \end{bmatrix},$$

[16] $\mathbb{GL}(n, \mathbb{F}) := \{L \in \mathbb{M}(n \times n; \mathbb{F}) : \det L \neq 0\}$, $\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

where $\alpha^1, \dots, \alpha^d \in \mathbb{Z}^n$ is a basis of F and $\alpha^{d+1}, \dots, \alpha^n \in \mathbb{Z}^n$ is a basis of $F^\perp$.

The implication (vi) $\Rightarrow$ (i) is obvious.

(d) Let $F = \{\alpha^1, \dots, \alpha^d\}^\perp$, where $\alpha^1, \dots, \alpha^d \in (\mathbb{R}^n)_*$ and

$$d = \operatorname{rank}[\alpha^1, \dots, \alpha^d] \quad (1 \le d \le n-1).$$

Assume that $\Delta := \det[\alpha^i_j]_{i,j=1,\dots,d} \neq 0$. Then F is of rational type iff $\Delta_{j,k}/\Delta \in \mathbb{Q}$ (where $\Delta_{j,k}$ is as in (1.4.2)), $j = 1, \dots, d$, $k = d+1, \dots, n$.

Indeed, we already know that by Cramer's formulas, the vectors v^{d+1}, …, v^n (as in (1.4.1)) form a basis of F. Thus, if all the numbers $\Delta_{j,k}/\Delta$ are rational, then v^{d+1}, …, v^n is a basis of $F \cap \mathbb{Q}^n$. Conversely, if F is of rational type, then there exists a non-singular matrix $L = [L_{i,j}] \in \mathbb{M}((n-d) \times (n-d), \mathbb{R})$ such that the vectors $L_{i,1}v^{d+1} + \dots + L_{i,n-d}v^n$, $i = 1, \dots, n-d$, give a basis of $F \cap \mathbb{Q}^n$. In particular,

$$-L_{i,j-d} = L_{i,1}v^{d+1}_j + \dots + L_{i,n-d}v^n_j \in \mathbb{Q}, \quad i = 1, \dots, n-d, \; j = d+1, \dots, n.$$

Hence $L \in \mathbb{M}((n-d) \times (n-d), \mathbb{Q})$ and, consequently, $v^{d+1}, \dots, v^n \in \mathbb{Q}^n$.

(e) If $F = \bigcap_{i \in I} F_i$, where F_i is of rational type, then F is of rational type. In particular, for every subspace $F \subset \mathbb{R}^n$ there exists the smallest subspace of rational type $\boldsymbol{K}(F)$ with $F \subset \boldsymbol{K}(F)$.

Indeed, we only need to use (c)(iv).

Definition 1.4.5. Let $\varnothing \neq X \subset \mathbb{R}^n$ be a convex domain. We denote by $\boldsymbol{E}(X)$ a vector subspace of $\mathbb{R}^n$ such that:

(a) $X + \boldsymbol{E}(X) = X$,

(b) for any vector subspace $F \subset \mathbb{R}^n$ with $X + F = X$ we have $\dim F \le \dim \boldsymbol{E}(X)$.[17]

The definition extends in an obvious way to the case where X is a convex domain of a vector subspace $H \subset \mathbb{R}^n$ and we are interested in the maximal vector space $F \subset H$ such that $X + F = X$ – in this case we write $\boldsymbol{E}_H(X)$.

Exercise 1.4.6. Prove that $\boldsymbol{E}(X) = \{0\} \Leftrightarrow X$ does not contain an affine line.

Remark 1.4.7. Let $X \subset \mathbb{R}^n$ be a convex domain.

(a) If $F_1, F_2 \subset \mathbb{R}^n$ are vector subspaces such that $X + F_1 = X + F_2 = X$, then $X + (F_1 + F_2) = X$. In particular,

- the space $\boldsymbol{E}(X)$ is uniquely determined,
- if F is a vector subspace of $\mathbb{R}^n$ such that $X + F = X$, then $F \subset \boldsymbol{E}(X)$.

(b) If $X \subset Y$ (Y is another convex domain), then $\boldsymbol{E}(X) \subset \boldsymbol{E}(Y)$. For any $y^0 \in \mathbb{R}^n$ we have $\boldsymbol{E}(X + y^0) = \boldsymbol{E}(X)$. If $L\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism, then $\boldsymbol{E}(L(X)) = L(\boldsymbol{E}(X))$.

[17] Below, in Remark 1.4.7 (a), we will see that $\boldsymbol{E}(X)$ is uniquely determined.

(c) $\boldsymbol{E}(\boldsymbol{H}_{\alpha,c}) = \alpha^{\perp}$.

(d) $\dim \boldsymbol{E}(X) = n$ iff $X = \mathbb{R}^n$.

(e) If $X = \operatorname{int} \bigcap_{i \in I} X_i$, where each X_i is a convex domain, then $\boldsymbol{E}(X) = \bigcap_{i \in I} \boldsymbol{E}(X_i)$. In particular, if $X = \operatorname{int} \bigcap_{(\alpha,c) \in A} \boldsymbol{H}_{\alpha,c}$, where $A \subset \mathbb{R}^n \times \mathbb{R}$, then $\boldsymbol{E}(X) = B^{\perp}$, where $B := \operatorname{pr}_{\mathbb{R}^n}(A)$.

Indeed, the inclusion $\boldsymbol{E}(X) \subset \bigcap_{i \in I} \boldsymbol{E}(X_i) =: F$ is obvious. We have

$$X + F \subset \Big(\bigcap_{i \in I} X_i\Big) + F \subset \bigcap_{i \in I} (X_i + F) = \bigcap_{i \in I} X_i.$$

Since the set $X + F$ is open, we get $X + F \subset \operatorname{int} \bigcap_{i \in I} X_i = X$, which proves that $F \subset \boldsymbol{E}(X)$.

(f) $X = \boldsymbol{E}(X) + Y$, where $Y := \operatorname{pr}_{\boldsymbol{E}(X)^{\perp}}(X)$ is a convex domain in $\boldsymbol{E}(X)^{\perp}$ with $\boldsymbol{E}_{\boldsymbol{E}(X)^{\perp}}(Y) = \{0\}$.

In particular, there exists an $L \in \mathbb{GL}(n, \mathbb{R})$ such that

$$L(\boldsymbol{E}(X)) = \mathbb{R}^d \times \{0\}^{n-d}, \quad L(\boldsymbol{E}(X)^{\perp}) = \{0\}^d \times \mathbb{R}^{n-d}, \quad L(X) = \mathbb{R}^d \times Y,$$

where $d := \dim \boldsymbol{E}(X)$ and $Y \subset \mathbb{R}^{n-d}$ is a convex domain with $\boldsymbol{E}(Y) = \{0\}$.

Definition 1.4.8. A convex domain $X \subset \mathbb{R}^n$ is of *rational* (resp. *irrational*) *type* if $\boldsymbol{E}(X)$ is of rational (resp. irrational) type.

Exercise 1.4.9. Let

$$X := \{(x_1, x_2) \in \mathbb{R}^2 : c + \mu x_1 < x_2 < d + \mu x_1\} \quad (c, d, \mu \in \mathbb{R}).$$

Decide when X is of rational type.

Remark 1.4.10. If $X = \operatorname{int} \bigcap_{i \in I} X_i$, where each X_i is a convex domain of rational type, then X is of rational type. In particular, for every convex domain $X \subset \mathbb{R}^n$ there exists the smallest convex domain of rational type $\boldsymbol{K}(X)$ with $X \subset \boldsymbol{K}(X)$.

Lemma 1.4.11. *Assume that $X \subsetneq \mathbb{R}^n$, $n \geq 2$, is a convex domain. Then the following conditions are equivalent:*

(i) *$\boldsymbol{E}(X)$ is of rational type;*

(ii) *there exists a non-singular matrix $L \in \mathbb{M}(n \times n, \mathbb{Z})$ such that $X = L(\mathbb{R}^d \times Y)$, where $d := \dim \boldsymbol{E}(X)$ and $Y \subset \mathbb{R}^{n-d}$ is a convex domain with $\boldsymbol{E}(Y) = \{0\}$;*

(iii) *for every $x^0 \notin \overline{X}$ there exists an open set $U \subset \boldsymbol{E}(X)^{\perp}$ such that $X \subset \bigcap_{\beta \in U} H_{\beta}^{x^0}$; in particular, there exists a basis $\alpha^1, \dots, \alpha^{n-d} \in \boldsymbol{E}(X)^{\perp} \cap \mathbb{Z}^n$ of $\boldsymbol{E}(X)^{\perp}$ such that $X \subset \bigcap_{j=1}^{n-d} \boldsymbol{H}_{\alpha^j}^{x^0}$;*

(iv) *$X = \operatorname{int} \bigcap_{(\alpha,c) \in A} \boldsymbol{H}_{\alpha,c}$, where $A \subset \mathbb{Z}^n \times \mathbb{R}$.*

Proof. The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) are elementary. To prove that (ii) $\Rightarrow$ (iii) we may assume that $\boldsymbol{E}(X) = \{0\}$ (EXERCISE). Fix an $x^0 \notin \overline{X}$. We may assume that $x^0 = 0$. Let C denote the open convex cone (with vertex at $0 \in \mathbb{R}^n$) generated by X ($C := \{tx : t > 0,\ x \in X\}$). Observe that $\boldsymbol{E}(C) = \{0\}$.

Indeed, suppose that $C + L \subset C$, where $L \subset \mathbb{R}^n$ is a real line. Consider any two-dimensional real space $P \subset \mathbb{R}^n$ with $L \subset P$ and $X' := X \cap P \neq \varnothing$. We have $\boldsymbol{E}_P(X') = \{0\}$. Let C' be the open cone in P generated by X'. Obviously, $C' = C \cap P$. Hence $C' + L \subset C'$ and the proof is reduced to the case $n = 2$. In the case $n = 2$ we only need to observe that if $\boldsymbol{E}(X) = \{0\}$, then there exist two different half-planes $\boldsymbol{H}_{\alpha^1,0}, \boldsymbol{H}_{\alpha^2,0}$ with $X \subset \boldsymbol{H}_{\alpha^1,0} \cap \boldsymbol{H}_{\alpha^2,0}$; a contradiction.

Consequently, there exists a $\beta^0 \in (\mathbb{R}^n)_*$ such that

$$\overline{C} \cap \{x \in \mathbb{R}^n : \langle x, \beta^0 \rangle = 0\} = \{0\}.$$

Indeed, we use induction on n. The case $n = 2$ is obvious. In the general case, take any $\alpha \in \mathbb{R}^n$, $\|\alpha\| = 1$, with $\overline{C} \subset \{x \in \mathbb{R}^n : \langle x, \alpha \rangle \le 0\}$. Put

$$P := \{x \in \mathbb{R}^n : \langle x, \alpha \rangle = 0\}$$

and define $X' := X \cap P$, $C' := C \cap P = \{tx' : t > 0,\ x' \in X'\}$. Note that $\boldsymbol{E}_P(X') = \{0\}$. Hence, by the inductive assumption, there exists an $(n-2)$-dimensional vector subspace V of P such that $\overline{C}' \cap V = \{0\}$. Fix a $u \in P \cap V^\perp$ with $\langle u, \alpha \rangle \le 0$ and $\|u\| = 1$. We are going to prove that $\overline{C} \cap \{x \in \mathbb{R}^n : \langle x, \alpha - \varepsilon u \rangle = 0\} = \{0\}$ for sufficiently small $\varepsilon > 0$. Suppose that for each $\varepsilon > 0$ there exists an $x^\varepsilon \in \overline{C}$, $\|x^\varepsilon\| = 1$, with $\langle x^\varepsilon, \alpha + \varepsilon u \rangle = 0$. Write $x^\varepsilon = v^\varepsilon + t_\varepsilon u + \tau_\varepsilon \alpha$. We have $0 = \langle x^\varepsilon, \alpha - \varepsilon u \rangle = \tau_\varepsilon - \varepsilon t_\varepsilon$. Hence $\tau_\varepsilon = \varepsilon t_\varepsilon$. Moreover, $t_\varepsilon = \langle x^\varepsilon, \alpha \rangle \le 0$ and $1 = \|x^\varepsilon\|^2 = \|v^\varepsilon\|^2 + t_\varepsilon^2(1 + \varepsilon^2)$. Take $\varepsilon_k \to 0$. We may assume that $v^{\varepsilon_k} \to v^0$ and $t_{\varepsilon_k} \to t_0 \le 0$. We have $x^{\varepsilon_k} \to v^0 + t_0 u \in \overline{C}'$ and $t_0 = -\sqrt{1 - \|v^0\|^2}$. Since $v^0 = (v^0 + t_0 u) + (-t_0)u \in V \cap \overline{C}'$, we conclude that $t_0 = 0$ and $v^0 = 0$ – contradiction.

It follows that $\overline{C} \cap \{x \in \mathbb{R}^n : \langle x, \beta \rangle = 0\} = \{0\}$ for β from an open neighborhood of β^0, which directly implies (iii). Indeed, suppose that $\beta^\nu \to \beta^0$ is such that $\langle y^\nu, \beta^\nu \rangle = 0$ for some $y^\nu \in \overline{C}$, $y^\nu \neq 0$, $\nu = 1, 2, \dots$. Since $\overline{C}$ is a cone, we may assume that $\|y^\nu\| = 1$, $\nu = 1, 2, \dots$, and next that $y^\nu \to y^0 \in \overline{C}$, $y^0 \neq 0$. Then $\langle y^0, \beta^0 \rangle = 0$ – contradiction. $\square$

Lemma 1.4.12. *Let $X \subset \mathbb{R}^n$ be a convex domain. Then the following conditions are equivalent:*

(i) $\boldsymbol{E}(X) = \{0\}$*;*

(ii) *there exist a non-singular matrix* $A := \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix} \in \mathbb{M}(n \times n, \mathbb{Z})$ *and a vector* $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ *such that* $X \subset \boldsymbol{H}_{\alpha^1,c_1} \cap \dots \cap \boldsymbol{H}_{\alpha^n,c_n}$*;*

(iii) *there exist a matrix* $A \in \mathbb{GL}(n,\mathbb{Z}) := \{A \in \mathbb{M}(n \times n;\mathbb{Z}) : |\det A| = 1\}$ *and a vector* $c \in \mathbb{R}^n$ *such that* $X \subset \boldsymbol{H}_{\alpha^1,c_1} \cap \dots \cap \boldsymbol{H}_{\alpha^n,c_n}$.

Proof. By Lemma 1.4.11, we only need to prove that (ii) $\Rightarrow$ (iii). Let A and c be as in (ii),

$$X \subset \boldsymbol{H}_{\alpha^1,c_1} \cap \dots \cap \boldsymbol{H}_{\alpha^n,c_n} =: H(A,c).$$

Suppose that $|\det A| > 1$. Put

$$S(A,c) := \{\beta \in \mathbb{Z}^n : \exists_{d=d_\beta \in \mathbb{R}} : H(A,c) \subset \boldsymbol{H}_{\beta,d}\}.$$

Then $S(A,c) = \mathbb{Z}^n \cap (\mathbb{Q}_+\alpha^1 + \dots + \mathbb{Q}_+\alpha^n)$.

Indeed, obviously the set on the right-hand side is contained on the left-hand one. Now take a $\beta \in S(A,c)$. Then there exists a $d \in \mathbb{R}$ such that $H(A,c) \subset \boldsymbol{H}_{\beta,d}$. Write $\beta = \sum_{j=1}^n t_j\alpha^j = ((tA)_1,\dots,(tA)_n)$, where $t := (t_1,\dots,t_n)$. Then $t = \beta A^{-1}$, i.e. all the t_j's are rational numbers. It remains to show that all of them are non-negative. Observe that the linear map

$$L\colon \mathbb{R}^n \to \mathbb{R}^n, \quad L(x) := (\langle x,\alpha^1\rangle,\dots,\langle x,\alpha^n\rangle),$$

gives an isomorphism satisfying

$$\begin{aligned}\{y \in \mathbb{R}^n : y_j < c_j,\ j = 1,\dots,n\} &= L(H(A,c))\\ &\subset L(\boldsymbol{H}_{\beta,d}) = \{y \in \mathbb{R}^n : \langle t,y\rangle < d\}.\end{aligned}$$

Hence, $t \in \mathbb{R}^n_+$ (EXERCISE). Note that the set

$$Q(A,c) := \mathbb{Z}^n \cap \big(([0,1)\cap\mathbb{Q})\alpha^1 + \dots + ([0,1)\cap\mathbb{Q})\alpha^n\big) \cup \{\alpha^1,\dots,\alpha^n\}$$

is finite. Therefore,

$$Q(A,c) = \Big\{\sum_{j=1}^n \frac{p_{k,j}}{q_{k,j}}\alpha^j : k = 1,\dots,N\Big\},$$

where $p_{j,k} \in \mathbb{Z}_+$, $q_{j,k} \in \mathbb{N}$ and the pairs $p_{j,k}, q_{j,k}$ are relatively prime. Then we denote by $s = s(A,c)$ the least common multiple of all denominators $q_{j,k}$.

Let $x \in \mathbb{Q}^n$ with $xA \in \mathbb{Z}^n$. Write $x_j = u_j + v_j$, where $u_j := x_j - v_j \in [0,1)\cap\mathbb{Q}$ and $v_j := \lfloor x_j \rfloor \in \mathbb{Z}$. Here $\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \le x\}$ = the integer part of $x \in \mathbb{R}$. Then $vA \in \mathbb{Z}^n$ and $(x-v)A = \sum_{j=1}^n (x_j - v_j)\alpha^j \in Q(A,c)$. Thus, $s(x-v) \in \mathbb{Z}^n$. Hence, $sx \in \mathbb{Z}^n$.

Let $r = r(A,c)$ be the smallest number in $\mathbb{N}$ such that if $xA \in \mathbb{Z}^n$ for an $x \in \mathbb{Q}^n$, then $rx \in \mathbb{Z}^n$. Comparing with the former paragraph it follows that $r \le s$.

Let $\tilde\alpha^j$ denote the j-th row of the inverse matrix A^{-1} of A. Note that $\tilde\alpha^j \in \mathbb{Q}$ and $\tilde\alpha^j A \in \mathbb{Z}^n$. Therefore, $r\tilde\alpha^j \in \mathbb{Z}^n$ and so $rA^{-1} \in \mathbb{M}(n \times n;\mathbb{Z})$. Consequently, $r^n = \det(rA^{-1})\det A$, i.e. $|\det A|$ divides r.

Observe that $1 < |\det A| \le r \le s$. Therefore there exists a vector $\hat{\alpha} \in Q(A,c) \setminus \{\alpha^1, \dots, \alpha^n\}$; in particular, $\hat{\alpha}^1 \in \mathbb{Z}^n$. So we may assume that there exists a $\tau \in \mathbb{R}^n_+$, $\tau_1 \in (0,1)$, such that $\hat{\alpha} = \sum_{j=1}^n \tau_j \alpha^j \in S(A,c)$. Moreover, if $\hat{A}$ denotes the matrix with rows $\hat{\alpha}^1, \alpha^2, \dots, \alpha^n$, then $|\det \hat{A}| = \tau_1^n |\det A| < |\det A|$.

If $|\det \hat{A}| = 1$, then we are done. If not, repeating the above procedure the proof will be finished after a finite number of steps. $\square$

Lemma 1.4.13. *Let $X \subset \mathbb{R}^n$ be a convex domain. Then the following conditions are equivalent:*

(i) *there exists a sequence $(x_k)_{k=1}^\infty \subset X$ such that the sequences $(x_{k,j})_{k=1}^\infty$, $j = 1, \dots, n-1$, are bounded and $x_{k,n} \to -\infty$;*

(ii) $X + \mathbb{R}_- \cdot e_n = X$.

Proof. The implication (ii) $\Rightarrow$ (i) is trivial. Conversely, take an arbitrary $x_0 \in X$ and $t < 0$. Put $\varepsilon_k := t/x_{k,n}$, $k \gg 1$. We may assume that $0 < \varepsilon_k < 1$. Obviously, $\varepsilon_k \to 0$. Since X is convex, we get $y_k := (1-\varepsilon_k)x_0 + \varepsilon_k x_k \in [x_0, x_k] \subset X$. Moreover, $y_k \to x_0 + te_n$. Hence $x_0 + \mathbb{R}_- \cdot e_n \subset \overline{X}$. Consequently, $X + \mathbb{R}_- \cdot e_n \subset \operatorname{int} \overline{X} = X$. $\square$

Definition 1.4.14. Let $X \subset \mathbb{R}^n$ be a domain which is *starlike* with respect to 0, i.e. $[0,x] \subset X$ for every $x \in X$. Then the function $h_X \colon \mathbb{R}^n \to \mathbb{R}_+$ defined by the formula

$$h_X(x) := \inf\{t > 0 : x/t \in X\}, \quad x \in \mathbb{R}^n,$$

is called the *Minkowski function of X*.

Remark 1.4.15. Before we continue, let us recall the following important notion of semicontinuity.

Let X be a topological space. We say that a function $u \colon X \to \overline{\mathbb{R}}$ is *upper semicontinuous* ($u \in \mathcal{C}^\uparrow(X)$) if for every $t \in \mathbb{R}$ the set $\{x \in X : u(x) < t\}$ is open. We say that u is *lower semicontinuous* ($u \in \mathcal{C}^\downarrow(X)$) if $-u \in \mathcal{C}^\uparrow(X)$.

Directly from the definition we get the following properties (Exercise):

- $u \in \mathcal{C}^\downarrow(X)$ iff for every $t \in \mathbb{R}$ the set $\{x \in X : u(x) > t\}$ is open.
- $\mathcal{C}(X, \overline{\mathbb{R}}) = \mathcal{C}^\uparrow(X) \cap \mathcal{C}^\downarrow(X)$.
- $u \in \mathcal{C}^\uparrow(X)$, $f \in \mathcal{C}(Y,X) \Rightarrow u \circ f \in \mathcal{C}^\uparrow(Y)$.
- $\mathbb{R}_{>0} \cdot \mathcal{C}^\uparrow(X) = \mathcal{C}^\uparrow(X)$.
- If $u, v \in \mathcal{C}^\uparrow(X)$ and $u(x) + v(x)$ is well defined for every $x \in X$, then $u + v \in \mathcal{C}^\uparrow(X)$.
- $u, v \in \mathcal{C}^\uparrow(X) \Rightarrow \max\{u, v\} \in \mathcal{C}^\uparrow(X)$.
- $(u_\alpha)_{\alpha \in A} \subset \mathcal{C}^\uparrow(X) \Rightarrow \inf\{u_\alpha : \alpha \in A\} \in \mathcal{C}^\uparrow(X)$. In particular, if $\mathcal{C}^\uparrow(X) \ni u_\nu \searrow u$ pointwise on X, then $u \in \mathcal{C}^\uparrow(X)$.
- If $\mathcal{C}^\uparrow(X, \mathbb{R}) \ni u_\nu \to u$ locally uniformly in X, then $u \in \mathcal{C}^\uparrow(X)$.

- If (X, ρ) is a metric space, then $u \in \mathcal{C}^{\uparrow}(X) \Leftrightarrow \forall_{a \in X} : \ \limsup_{x \to a} u(x) = u(a)$.
- (Weierstrass theorem) If (X, ρ) is a compact space and $u \in \mathcal{C}^{\uparrow}(X, \mathbb{R}_{-\infty})$, then there exists a point $x_0 \in X$ such that $u(x_0) = \sup u(X)$.
- (Baire theorem; cf. [Łoj 1988]) If (X, ρ) is a metric space, then for every $u \in \mathcal{C}^{\uparrow}(X)$, there exists a sequence $(u_\nu)_{\nu=1}^{\infty} \subset \mathcal{C}(X, \overline{\mathbb{R}})$ such that $u_\nu \searrow u$ pointwise on X. Moreover, if $u \in \mathcal{C}^{\uparrow}(X, \mathbb{R}_{-\infty})$, then the sequence $(u_\nu)_{\nu=1}^{\infty}$ may be chosen in $\mathcal{C}(X, \mathbb{R})$.

Exercise 1.4.16. Let $X \subset \mathbb{R}^n$ be a domain which is *starlike* with respect to 0. Prove the following properties of the Minkowski function:

(a) $h_X(tx) = t h_X(x)$, $t \geq 0$, $x \in \mathbb{R}^n$.

(b) $X = \{x \in X : h_X(x) < 1\}$.

(c) h_X is uniquely determined by (a) and (b).

(d) h_X is upper semicontinuous.

(e) X is convex iff h_X satisfies the *triangle inequality*:

$$h_X(x + y) \leq h_X(x) + h_X(y), \quad x, y \in \mathbb{R}^n.$$

(f) If X is convex, then h_X is continuous.

(g) If X is convex and symmetric with respect to 0, then h_X is a seminorm, i.e. h_X is absolutely homogeneous ($h_X(tx) = |t| h_X(x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^n$) and satisfies the triangle inequality.

Lemma 1.4.17. *Let $X \subset \mathbb{R}^n$ be an unbounded convex domain which is contained in $\bigtimes_{j=1}^{n} (-\infty, R)$ for a certain number R. Then, for any point $a \in X$, there exist a vector $v \in \mathbb{R}^n_- \setminus \{0\}$ and a neighborhood $V = V(a) \subset X$ such that $V + \mathbb{R}_+ v \subset X$.*

Proof. We may assume that $a = 0$. Then the continuity of the Minkowski function h of X (cf. Exercise 1.4.16) and the unboundedness of X lead to a vector v on the unit sphere with $h(v) = 0$. Obviously, $v \in \mathbb{R}^n_- \setminus \{0\}$ and $\mathbb{R}_+ v \subset X$. Finally, using the convexity of X, we see that for any open ball $V \subset X$ with center 0 the following inclusion holds: $V + \mathbb{R}_+ v \subset X$. $\square$

1.5 Reinhardt domains

We collect here some basic definitions related to the class of Reinhardt domains which is a natural generalization of the class of complete Reinhardt domains from Definition 1.3.8.

For any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{T}^n$, let $\boldsymbol{T}_\lambda : \mathbb{C}^n \to \mathbb{C}^n$ be the *n-rotation* given by the formula $\boldsymbol{T}_\lambda(z) = \boldsymbol{T}_\lambda(z_1, \dots, z_n) := \lambda \cdot z = (\lambda_1 z_1, \dots, \lambda_n z_n)$.

Remark 1.5.1. $\boldsymbol{T}_{\lambda \cdot \mu} = \boldsymbol{T}_\lambda \circ \boldsymbol{T}_\mu = \boldsymbol{T}_\mu \circ \boldsymbol{T}_\lambda = \boldsymbol{T}_{\mu \cdot \lambda}$, $\quad \boldsymbol{T}_{\mathbf{1}} = \mathrm{id}_{\mathbb{C}^n}$, $\quad (\boldsymbol{T}_\lambda)^{-1} = \boldsymbol{T}_{\lambda^{-1}}$, where $\mathbf{1} := (1, \dots, 1) \in \mathbb{T}^n$ and $\lambda^{-1} = (\lambda_1^{-1}, \dots, \lambda_n^{-1})$.

Definition 1.5.2. A set $A \subset \mathbb{C}^n$ is called *Reinhardt* (*n-circled*) if $\boldsymbol{T}_\lambda(A) = A$ for every $\lambda \in \mathbb{T}^n$.

Let $\boldsymbol{R}\colon \mathbb{C}^n \to \mathbb{R}^n_+$, $\boldsymbol{R}(z_1, \dots, z_n) := (|z_1|, \dots, |z_n|)$.

Remark 1.5.3. (a) A set $A \subset \mathbb{C}^n$ is Reinhardt iff $A = \boldsymbol{R}^{-1}(\boldsymbol{R}(A))$. Consequently, any Reinhardt set $A \subset \mathbb{C}^n$ is completely determined by its *absolute image* $\boldsymbol{R}(A) = A \cap \mathbb{R}^n_+$.

(b) The mapping $\boldsymbol{R}\colon \mathbb{C}^n \to \mathbb{R}^n_+$ is open (Exercise). Consequently, if $\Omega \subset \mathbb{C}^n$ is Reinhardt, then Ω is open in $\mathbb{C}^n$ iff $\boldsymbol{R}(\Omega)$ is open in $\mathbb{R}^n_+$ (in the induced topology).

(c) If a set $B \subset \mathbb{R}^n_+$ is connected, then so is $\boldsymbol{R}^{-1}(B)$.

Indeed, to see that $A := \boldsymbol{R}^{-1}(B)$ is connected for connected $B \subset \mathbb{R}^n_+$ we may argue as follows. Suppose that $A = U \cup V$, where U, V are open in A, disjoint, and non-empty. Since $\mathbb{T}^n$ is connected, we conclude that U, V must be Reinhardt (Exercise). Consequently, if we put $U' := \boldsymbol{R}(U)$ and $V' := \boldsymbol{R}(V)$, then $B = U' \cup V'$, U', V' are open in B, disjoint, and non-empty; a contradiction.

(d) Let $D \subset \mathbb{C}^n$ be Reinhardt. Then D is a domain in $\mathbb{C}^n$ iff $\boldsymbol{R}(D)$ is a domain in $\mathbb{R}^n_+$ (in the induced topology). Observe that a relatively open set $U \subset \mathbb{R}^n_+$ is connected iff it is arcwise connected.

(e) If $A \subset \mathbb{C}^n$ is Reinhardt, then $\overline{A}$ and $\operatorname{int} A$ are Reinhardt.

For any Reinhardt set $A \subset \mathbb{C}^n$, let

$$\begin{aligned}\hat{A}^{(j)} &:= \{\lambda \cdot z : \lambda \in \{1\}^{j-1} \times \overline{\mathbb{D}} \times \{1\}^{n-j},\ z \in A\}, \quad j = 1, \dots, n,\\ \hat{A} &:= \{\lambda \cdot z : \lambda \in \overline{\mathbb{D}}^n,\ z \in A\} = (\dots(\hat{A}^{(1)})\widehat{\ }^{(2)} \dots)\widehat{\ }^{(n)}.\end{aligned}$$

Obviously, $A \subset \hat{A}^{(j)} \subset \hat{A}$.

Remark 1.5.4. (a) Let $A \subset \mathbb{C}^n$ be Reinhardt. Then A is complete Reinhardt iff $A = \hat{A}$.

(b) If D is a Reinhardt domain, then so are $\widehat{D}^{(1)}$, …, $\widehat{D}^{(n)}$, and $\hat{D}$.

For every Reinhardt set $A \subset \mathbb{C}^n$ put

$$\log A := \{x \in \mathbb{R}^n : e^x \in A\} = \{(\log|z_1|, \dots, \log|z_n|) : (z_1, \dots, z_n) \in A \cap \mathbb{C}^n_*\},$$

where $e^x := (e^{x_1}, \dots, e^{x_n})$. The set $\log A$ is called the *logarithmic image* of A.

For any set $B \subset \mathbb{R}^n$ let $\exp B$ be the unique Reinhardt subset of $\mathbb{C}^n_*$ such that $\log(\exp B) = B$, i.e.

$$\exp B = \{(z_1, \dots, z_n) \in \mathbb{C}^n_* : (\log|z_1|, \dots, \log|z_n|) \in B\}.$$

Observe that $\boldsymbol{R}(\exp B) = \{e^x : x \in B\}$. Moreover, for every Reinhardt set $A \subset \mathbb{C}^n$ we have $\exp(\log A) = A \cap \mathbb{C}^n_*$.

Definition 1.5.5. We say that a Reinhardt set A is *logarithmically convex* (*log-convex*) if the set $\log A$ is convex.

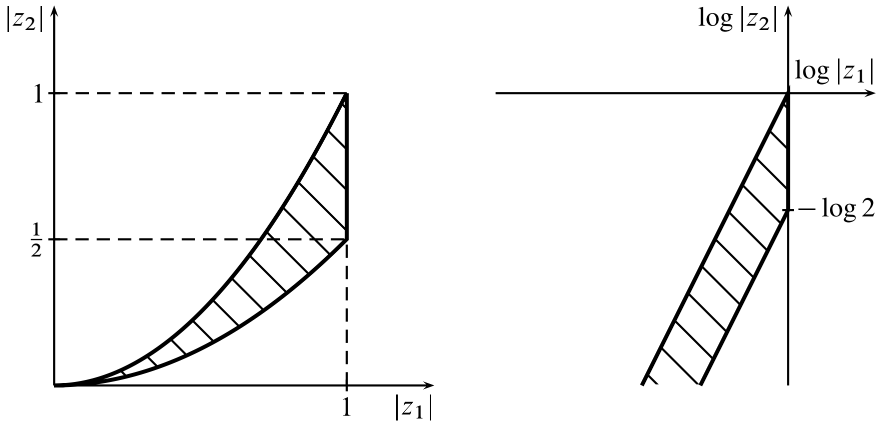

Figure 1.5.1. An example of a log-convex non-complete Reinhardt domain: $D := \{(z_1, z_2) \in \mathbb{D}^2 : \frac{1}{2}|z_1|^2 < |z_2| < |z_1|^2\}$.

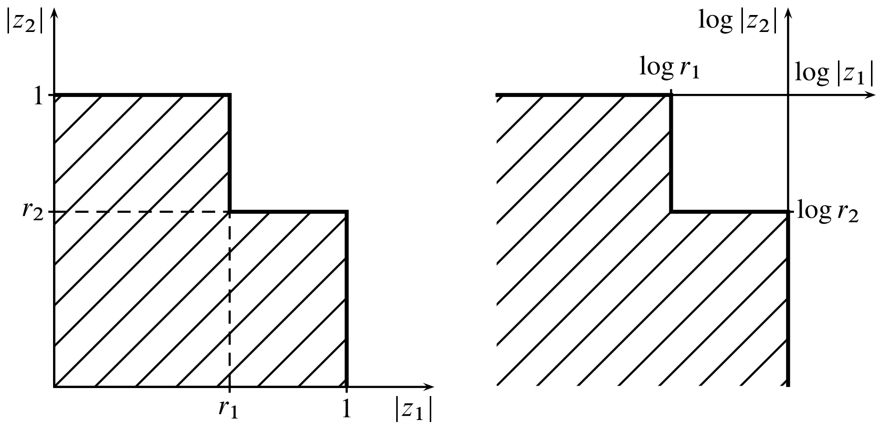

Figure 1.5.2. An example of a complete Reinhardt domain that is not log-convex.

Define

$$V_j = V_j^n := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j = 0\} = \mathbb{C}^{j-1} \times \{0\} \times \mathbb{C}^{n-j}$$

for $j = 1, \dots, n$, and

$$V_0 = V_0^n := V_1 \cup \dots \cup V_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0\}.$$

Remark 1.5.6. (All details are left as EXERCISE.) (a) Let A be a Reinhardt set. Then

$$\operatorname{int}(\log A) = \log(\operatorname{int} A), \quad \log \bar{A} = \overline{\log A}.$$

Consequently, for any set $B \subset \mathbb{R}^n$, we have

$$\log(\operatorname{int} \overline{\exp B}) = \operatorname{int}(\log \overline{\exp B}) = \operatorname{int} \overline{\log(\exp B)} = \operatorname{int} \bar{B}.$$

In particular, if $X \subset \mathbb{R}^n$ is a fat domain (e.g. if X is a convex domain), then $D := \operatorname{int} \overline{\exp X}$ is a fat Reinhardt domain with $\log D = X$. Conversely, if G is a Reinhardt domain with $\log G = X$, then $\operatorname{int} \bar{G} = D$. In fact, if $\log G = X$, then $G \setminus V_0 = \exp X$. Hence, $\bar{G} = \overline{\exp X}$, and finally, $\operatorname{int} \bar{G} = D$.

(b) A Reinhardt set $A \subset \mathbb{C}^n$ is logarithmically convex iff

$$(x_1^{1-t} y_1^t, \dots, x_n^{1-t} y_n^t) \in A, \quad (x_1, \dots, x_n),\ (y_1, \dots, y_n) \in A \cap \mathbb{R}^n_{>0},\ t \in [0, 1].$$

(c) If $D \subset \mathbb{C}^n$ is a Reinhardt domain, then $D \setminus V_0$ is a domain.

Indeed, it suffices to show that for any domain $D \subset \mathbb{C}^n$, the set $D \setminus V_j$ is connected, $j = 1, \dots, n$. Assume that $j = n$. We only need to observe that, for every $a = (a_1, \dots, a_n) \in D \cap V_n$, if $\mathbb{P}(a, r) \subset D$, then

$$\mathbb{P}(a, r) \setminus V_n = K(a_1, r) \times \dots \times K(a_{n-1}, r) \times (K(a_n, r) \setminus \{0\})$$

is obviously connected (cf. the proof of Proposition 1.9.7).

(d) If Ω is an open Reinhardt set such that $\Omega \setminus V_0$ is connected, then Ω itself is connected. In particular, if Ω is log-convex, then Ω is a domain.

(e) Let $X \subset \mathbb{R}^n$ be a fat domain and let $D := \operatorname{int} \overline{\exp X}$ (cf. (a)). Then:

- $0 \in D$ iff there exists an $x^0 \in X$ such that $x^0 + \mathbb{R}^n_- \subset X$.
- D is complete iff $X + \mathbb{R}^n_- = X$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ define

$$\mathbb{C}^n(\alpha) = \mathbb{C}(\alpha_1) \times \dots \times \mathbb{C}(\alpha_n),$$

where

$$\mathbb{C}(x) := \begin{cases} \mathbb{C} & \text{if } x \geq 0, \\ \mathbb{C}_* & \text{if } x < 0. \end{cases}$$

Note that $\mathbb{C}^n(\alpha) = \mathbb{C}^n((\operatorname{sgn} \alpha_1, \dots, \operatorname{sgn} \alpha_n))$. Let $\mathbb{C}^n(\Sigma) := \bigcap_{\alpha \in \Sigma} \mathbb{C}^n(\alpha)$ where $\Sigma \subset \mathbb{R}^n$. Observe that the function

$$\mathbb{C}^n(\alpha) \ni z \mapsto |z^\alpha| := |z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n} \in \mathbb{R}^n_+$$

is well defined (here $0^0 := 1$). Notice that in the case where $\alpha \in \mathbb{Z}^n$, $|z^\alpha|$ coincides with the absolute value of z^α. Let

$$\boldsymbol{D}_{\alpha,c} := \{z \in \mathbb{C}^n(\alpha) : |z^\alpha| < e^c\}, \quad \alpha \in \mathbb{R}^n,\ c \in \mathbb{R}.\text{[18]} \tag{1.5.1}$$

Observe that $\boldsymbol{D}_{\alpha,c}$ is a Reinhardt domain (EXERCISE). It is called an *elementary Reinhardt domain*. We put $\boldsymbol{D}_\alpha := \boldsymbol{D}_{\alpha,0}$. Observe that $\boldsymbol{D}_{\alpha,c} = \boldsymbol{D}_{\beta,d}$ iff $(\beta, d) = \mu(\alpha, c)$ for some $\mu > 0$.

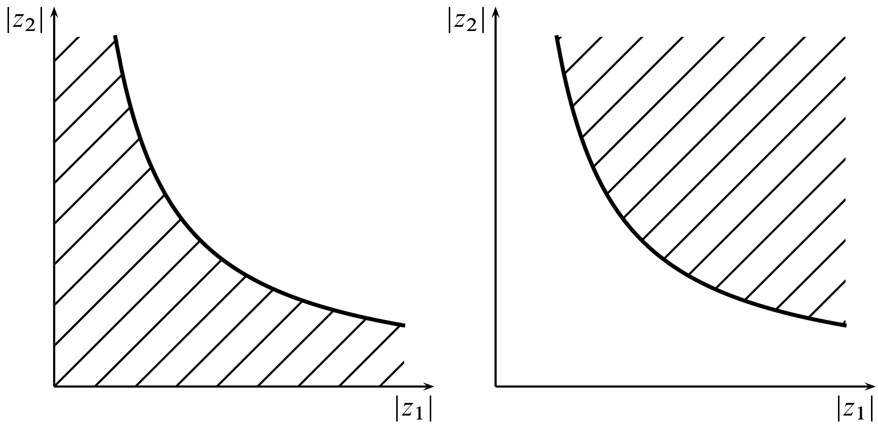

Figure 1.5.3. $\boldsymbol{D}_{(1,1)}$ and $\boldsymbol{D}_{(-1,-1)}$.

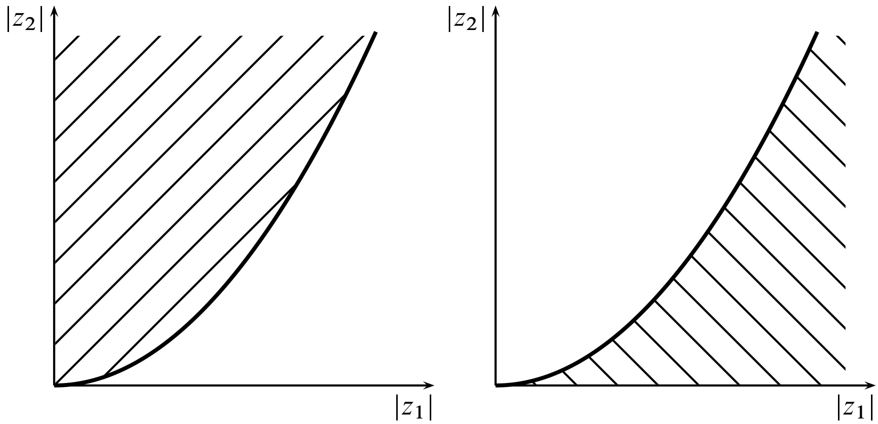

Figure 1.5.4. $\boldsymbol{D}_{(2,-1)}$ and $\boldsymbol{D}_{(-2,1)}$.

Remark 1.5.7. (a) Let $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}^n)_*$. For every $j \in \{1, \dots, n\}$ we have:

$$(\boldsymbol{D}_{\alpha,c})^{\frown(j)} = \boldsymbol{D}_{\alpha,c} \iff \boldsymbol{D}_{\alpha,c} \cap \boldsymbol{V}_j \neq \varnothing \iff \alpha_j \geq 0.$$

[18]Note that $\boldsymbol{D}_{0,c} = \begin{cases} \mathbb{C}^n & \text{if } c > 0, \\ \varnothing, & \text{if } c \leq 0. \end{cases}$

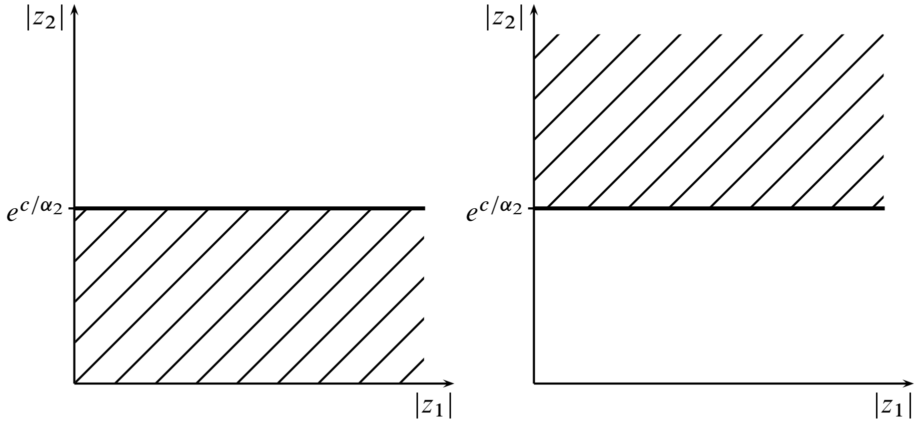

Figure 1.5.5. Elementary domains $\boldsymbol{D}_{\alpha,c}$ with ($\alpha_1 = 0, \alpha_2 > 0$) and ($\alpha_1 = 0, \alpha_2 < 0$).

In particular, $\boldsymbol{D}_{\alpha,c}$ is complete iff $\alpha \in \mathbb{R}^n_+$.

(b) Suppose that $\varnothing \neq D = \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c}$, where $A \subset \mathbb{R}^n \times \mathbb{R}$. Let

$$B := \{\alpha \in \mathbb{R}^n : \exists_{c\in\mathbb{R}} : (\alpha, c) \in A\}.$$

Then, for every $j \in \{1, \dots, n\}$, we have:

$$\widehat{D}^{(j)} = D \iff D \cap \boldsymbol{V}_j \neq \varnothing \iff \forall_{\alpha\in B} : \alpha_j \geq 0.$$

Indeed, in view of (a), we only need to observe that if $\alpha_j \geq 0$ for every $\alpha \in B$, then $\widehat{D}^{(j)} = D$. In fact,

$$\widehat{D}^{(j)} \subset \operatorname{int} \bigcap_{(\alpha,c)\in A} \widehat{\boldsymbol{D}_{\alpha,c}}^{(j)} = \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c} = D.$$

(c) $\log \boldsymbol{D}_{\alpha,c} = \boldsymbol{H}_{\alpha,c}$ (cf. Remark 1.4.1 (e)).

(d) If $\alpha \in \mathbb{R}^s_{>0} \times \mathbb{R}^{n-s}_{<0}$ with $0 \leq s \leq n$. Then

$$\boldsymbol{D}_{\alpha,c} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdots |z_s|^{\alpha_s} < e^c |z_{s+1}|^{-\alpha_{s+1}} \cdots |z_n|^{-\alpha_n}\}.$$ [19]

Consequently,

$$\begin{aligned}\overline{\boldsymbol{D}_{\alpha,c}} &= \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^{\alpha_1} \cdots |z_s|^{\alpha_s} \leq e^c |z_{s+1}|^{-\alpha_{s+1}} \cdots |z_n|^{-\alpha_n}\} \\ &= \{z \in \mathbb{C}^n(\alpha) : |z^\alpha| \leq e^c\} \cup \{z \in \boldsymbol{V}_0 : z_1 \cdots z_s = z_{s+1} \cdots z_n = 0\} \quad (1.5.2)\end{aligned}$$

(observe that if $s = 0$, then $\overline{\boldsymbol{D}_{\alpha,c}} \subset \mathbb{C}^n(\alpha)$). In particular, $\boldsymbol{D}_{\alpha,c}$ is fat for any $(\alpha, c) \in \mathbb{R}^n \times \mathbb{R}$.

[19] If $s = 0$, then $\boldsymbol{D}_{\alpha,c} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : 1 < e^c |z_1|^{-\alpha_1} \cdots |z_n|^{-\alpha_n}\}$.

Indeed, to prove (1.5.2) fix a point $b = (b_1, \dots, b_n) \in \mathbb{C}^n$ with

$$|b_1|^{\alpha_1} \cdots |b_s|^{\alpha_s} = e^c |b_{s+1}|^{-\alpha_{s+1}} \cdots |b_n|^{-\alpha_n}.$$

We consider the following three cases:

- $s \le n-1$ and $b_{s+1} \cdots b_n \ne 0$. Put $a(u) := (b_1, \dots, b_s, ub_{s+1}, \dots, ub_n)$, $u > 0$. Then $\lim_{u \to 1} a(u) = b$, $a(u) \in \mathbb{C}^n(\alpha)$, $a(u) \in \boldsymbol{D}_{\alpha,c}$ for $u > 1$, and $a(u) \notin \overline{\boldsymbol{D}_{\alpha,c}}$ for $0 < u < 1$.
- $s \ge 1$ and $b_1 \cdots b_s \ne 0$. Put $a(t) := (tb_1, \dots, tb_s, b_{s+1}, \dots, b_n)$, $t > 0$. Then $\lim_{t \to 1} a(t) = b$, $a(t) \in \mathbb{C}^n(\alpha)$, $a(t) \in \boldsymbol{D}_{\alpha,c}$ for $0 < t < 1$, and $a(t) \notin \overline{\boldsymbol{D}_{\alpha,c}}$ for $t > 1$.
- $1 \le s \le n-1$ and $b_1 \cdots b_s = b_{s+1} \cdots b_n = 0$. We may assume that $b_1 \cdots b_k \ne 0, b_{k+1} = \dots = b_s = 0\,(0 \le k \le s-1), b_{s+1} \cdots b_\ell \ne 0, b_{\ell+1} = \dots = b_n = 0$ $(s+1 \le \ell \le n-1)$. Put

  $$a(t,u) := (b_1, \dots, b_k, t, \dots, t, b_{s+1}, \dots, b_\ell, u, \dots, u), \quad t, u > 0.$$

  Then $\lim_{t,u \to 0} a(t,u) = b$, $a(t,u) \in \mathbb{C}^n(\alpha)$, $a(t,u) \in \boldsymbol{D}_{\alpha,c}$ if $t \ll u$, and $a(t,u) \notin \overline{\boldsymbol{D}_{\alpha,c}}$ if $u \ll t$.

(e) If $D = \operatorname{int} \bigcap_{(\alpha,c) \in A} \boldsymbol{D}_{\alpha,c}$, where $A \subset \mathbb{R}^n \times \mathbb{R}$, then

$$\log D = \operatorname{int} \bigcap_{(\alpha,c) \in A} \boldsymbol{H}_{\alpha,c}$$

and D is fat. In particular, D is log-convex.

Indeed, by (c) and Remark 1.5.6 (a), we get

$$\log D = \operatorname{int} \bigcap_{(\alpha,c) \in A} \log \boldsymbol{D}_{\alpha,c} = \operatorname{int} \bigcap_{(\alpha,c) \in A} \boldsymbol{H}_{\alpha,c}.$$

Moreover, by (d), we have

$$\begin{aligned}\operatorname{int} \overline{D} \subset \operatorname{int} \overline{\bigcap_{(\alpha,c) \in A} \boldsymbol{D}_{\alpha,c}} &\subset \operatorname{int} \bigcap_{(\alpha,c) \in A} \overline{\boldsymbol{D}_{\alpha,c}} \\ &\overset{20}{\subset} \operatorname{int} \bigcap_{(\alpha,c) \in A} \operatorname{int} \overline{\boldsymbol{D}_{\alpha,c}} = \operatorname{int} \bigcap_{(\alpha,c) \in A} \boldsymbol{D}_{\alpha,c} = D.\end{aligned}$$

For any Reinhardt domain $D \subset \mathbb{C}^n$ define its *fat hull* D^* as

$$D^* := \operatorname{int} \overline{D} = \operatorname{int} \overline{D \setminus V_0} = \operatorname{int} \overline{\exp \log D}. \tag{1.5.3}$$

[20] $\operatorname{int} \bigcap_i A_i \subset \operatorname{int}(\bigcap_i \operatorname{int} A_i)$.

Remark 1.5.8. Let $D \subset \mathbb{C}^n$ be a log-convex Reinhardt domain.

(a) We already know (cf. Remark 1.5.6 (a)) that D^* is a fat log-convex Reinhardt domain with $\log D^* = \log D$. In particular, $D^* \setminus D \subset \boldsymbol{V}_0$.

(b) If $\log D = \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{H}_{\alpha,c}$, where $A \subset \mathbb{R}^n \times \mathbb{R}$, then

$$D^* = \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c}.$$

(c) If $D^* \cap \boldsymbol{V}_j \neq \varnothing$, then $(D^*)\widehat{\ }^{(j)} = D^*$ (cf. Remark 1.5.7 (b)).

(d) If $D \cap \boldsymbol{V}_j \neq \varnothing$, then for every point $a = (a', a_j, a'') \in D \subset \mathbb{C}_*^{j-1} \times \mathbb{C} \times \mathbb{C}_*^{n-j}$, we have $(a', \lambda a_j, a'') \in D$, $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$ (use (c) and (a)). Note that the result may be not true for an arbitrary $a \in D$ – cf. Figure 1.5.6.

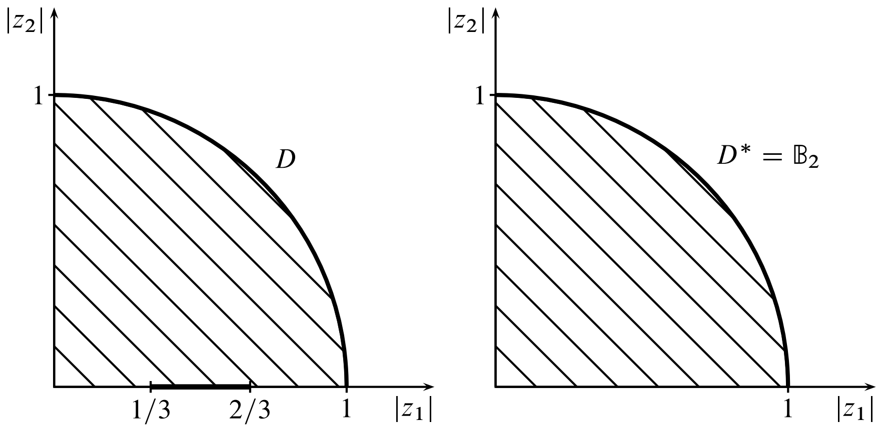

Figure 1.5.6. $D := \mathbb{B}_2 \setminus \{(z_1, 0) : 1/3 \le |z_1| \le 2/3\}$. If D is a log-convex Reinhardt domain, then $D^* \setminus D \subset \boldsymbol{V}_0$.

Remark 1.5.9. Frequently we will consider Reinhardt domains $D \neq \varnothing$ of the form

$$D = \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c},$$

where $A \subset (\mathbb{R}^n)_* \times \mathbb{R}$. Similarly as in Remark 1.4.1, we may always find the following minimal representation of D. Put

$$B := \operatorname{pr}_{\mathbb{R}^n}(A), \quad \boldsymbol{c}(\alpha) := \sup\{\log|z^\alpha| : z \in D\}, \ \alpha \in B.$$

Note that $\boldsymbol{c}(\alpha) \le \inf\{c : (\alpha, c) \in A\}$, $\alpha \in B$. Since each $\boldsymbol{D}_{\alpha,c}$ is fat (Remark 1.5.7 (d)), we get

$$D = \operatorname{int} \bigcap_{\alpha\in B} \boldsymbol{D}_{\alpha,\boldsymbol{c}(\alpha)}.$$

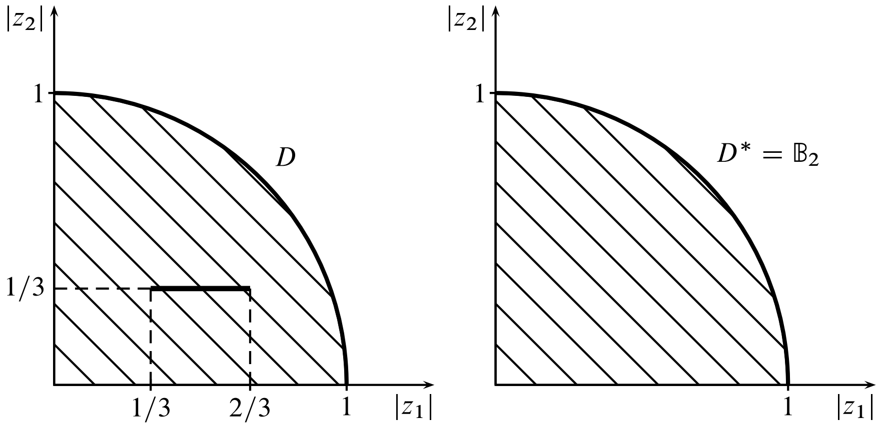

Figure 1.5.7. $D := \mathbb{B}_2 \setminus \{(z_1, z_2) : 1/3 \le |z_1| \le 2/3,\ |z_2| = 1/3\}$. If D is an arbitrary Reinhardt domain, then it may happen that $D^* \setminus D \not\subset \boldsymbol{V}_0$.

Put $B_0 := \{\alpha \in B : \partial D \cap \partial \boldsymbol{D}_{\alpha,\boldsymbol{c}(\alpha)} \cap \mathbb{C}_*^n \neq \varnothing\}$. Then

$$D = \operatorname{int} \bigcap_{\alpha \in B_0} \boldsymbol{D}_{\alpha,\boldsymbol{c}(\alpha)} =: D_0.$$

Indeed, since D and D_0 are fat, we only need to show that $D \cap \mathbb{C}_*^n = D_0 \cap \mathbb{C}_*^n$, which follows directly from Remark 1.4.1.

Definition 1.5.10. We say that a Reinhardt domain D satisfies the *Fu condition* (cf. [Fu 1994]) if for every $j \in \{1, \dots, n\}$ we have

$$(\partial D) \cap \boldsymbol{V}_j \neq \varnothing \Longrightarrow D \cap \boldsymbol{V}_j \neq \varnothing.$$

Remark 1.5.11. (a) D satisfies the Fu condition iff for every $j \in \{1, \dots, n\}$, either $D \cap \boldsymbol{V}_j \neq \varnothing$ or $\overline{D} \cap \boldsymbol{V}_j = \varnothing$. Consequently, after a permutation of variables, we may always assume that there exists $k = \mathfrak{F}(D) \in \{0, \dots, n\}$ with $D \cap \boldsymbol{V}_j \neq \varnothing$, $j = 1, \dots, k$, $\overline{D} \cap \boldsymbol{V}_j = \varnothing$, $j = k+1, \dots, n$.

(b) The elementary Reinhardt domain $\boldsymbol{D}_{\alpha,c}$ satisfies the Fu condition iff $\alpha \in \mathbb{R}_+^n$ or $\alpha \in \mathbb{R}_-^n$.

(c) The Reinhardt domain

$$T_\sigma := \{(z_1, z_2) \in \mathbb{D} \times \mathbb{D} : |z_1|^\sigma < |z_2|\}, \quad \sigma > 0,$$

does not satisfy the Fu condition.

In particular, the *Hartogs triangle*

$$T = T_1 = \{(z_1, z_2) \in \mathbb{D} \times \mathbb{D} : |z_1| < |z_2|\}$$

does not satisfy the Fu condition.

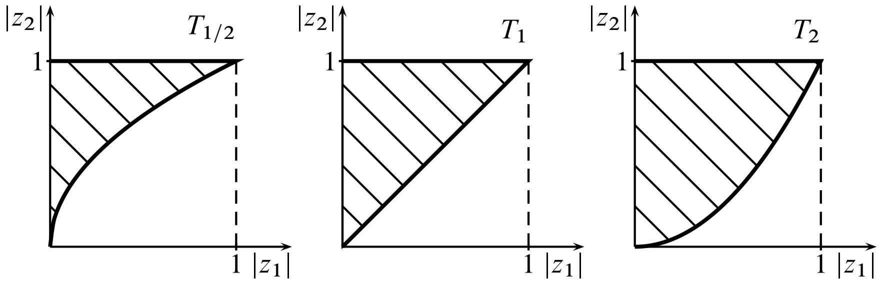

Figure 1.5.8. The Reinhardt domain $T_\sigma := \{(z_1, z_2) \in \mathbb{D}^2 : |z_1|^\sigma < |z_2|\}$, $\sigma > 0$, does not satisfy the Fu condition.

(c) One can prove (cf. [Fu 1994]) that the Fu condition is satisfied whenever ∂D is $\mathcal{C}^1$, i.e. for every $a \in \partial D$ there exist a neighborhood U of a and a $\mathcal{C}^1$ function $\rho\colon U \to \mathbb{R}$ such that:

- $U \cap D = \{z \in U : \rho(z) < 0\}$,
- $U \setminus \bar{D} = \{z \in U : \rho(z) > 0\}$,
- $\operatorname{grad} \rho \neq 0$ on U (cf. Definition 1.18.1).

Indeed, suppose that $a \in (\partial D) \cap V_j$, but $D \cap V_j = \varnothing$. We may assume that $j = n$. Let $U := \mathbb{P}(a, r)$ and ρ be as above. Write $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $U = \mathbb{P}(a', r) \times K(r) = U' \times U_n$. Since $(U' \times \{0\}) \cap D = \varnothing$, we conclude that $\rho(z', 0) \geq 0$, $z' \in U'$. Hence, since $\rho(a', 0) = 0$, we get $\frac{\partial \rho}{\partial z_j}(a) = 0$, $j = 1, \dots, n-1$. Thus $\frac{\partial \rho}{\partial z_n}(a) \neq 0$.

First consider the case $\frac{\partial \rho}{\partial x_n}(a) \neq 0$, where $z_n = x_n + i y_n$. We may assume that $\frac{\partial \rho}{\partial x_n}(a) < 0$ (Exercise). Then $\rho(a', t) < 0$ for $0 < t < t_0$. Since D is Reinhardt, we conclude that $\rho(a', -t) < 0$ for $0 < t < t_0$. Finally, $\frac{\partial \rho}{\partial x_n}(a', 0) = 0$; a contradiction.

The case where $\frac{\partial \rho}{\partial y_n}(a) \neq 0$ is similar – Exercise.

(d) Notice that the Fu condition is not invariant under biholomorphic mappings. For example, $\mathbb{D}_*$ and $\mathbb{C} \setminus \bar{\mathbb{D}}$ (Exercise).

Definition 1.5.12 (Algebraic mappings). For a matrix $A = \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix} \in \mathbb{GL}(n, \mathbb{Z})$[21] and $a = (a_1, \dots, a_n) \in \mathbb{C}_*^n$, let

$$\Phi_{a,A}\colon \mathbb{C}^n(A) \to \mathbb{C}^n, \quad \Phi_{a,A}(z) := (a_1 z^{\alpha^1}, \dots, a_n z^{\alpha^n}),$$

where $\mathbb{C}^n(A) := \mathbb{C}^n(\alpha^1) \cap \dots \cap \mathbb{C}^n(\alpha^n)$. We put $\Phi_A := \Phi_{\mathbf{1},A}$. Any mapping of the form $\Phi_{a,A}$ is called an *algebraic mapping*. We say that two Reinhardt domains

[21] $\mathbb{GL}(n, \mathbb{Z}) := \{A \in \mathbb{M}(n \times n; \mathbb{Z}) : |\det A| = 1\}$.

are *algebraically equivalent* ($D \overset{\text{alg}}{\simeq} G$) if there exists an algebraic mapping $\Phi_{a,A}$ such that $D \subset \mathbb{C}^n(A)$ and $\Phi_{a,A}$ maps bijectively D onto G.

Remark 1.5.13. Observe that:

(a) For any $A, B \in \mathbb{GL}(n, \mathbb{Z})$ and $a, b \in \mathbb{R}^n$, we have $\Phi_{a,A} \circ \Phi_{b,B} = \Phi_{c,C}$ on $\mathbb{C}^n_*$, where $C := AB$ and $c := \Phi_{a,A}(b)$.

(b) $\Phi_{a,A}|_{\mathbb{C}^n_*} : \mathbb{C}^n_* \to \mathbb{C}^n_*$ is bijective and $(\Phi_{a,A}|_{\mathbb{C}^n_*})^{-1} = \Phi_{b,A^{-1}}|_{\mathbb{C}^n_*}$, where $\Phi_{a,A}(b) = \mathbf{1}$.

(c) Notice that in general $\Phi_{a,A}(\mathbb{C}^n(A)) \not\subset \mathbb{C}^n(A^{-1})$. Take for example $A := \left[\begin{smallmatrix} 1 & 1 \\ 3 & 4 \end{smallmatrix}\right] \in \mathbb{GL}(2, \mathbb{Z})$. Then $\mathbb{C}^2(A) = \mathbb{C}^2$, $A^{-1} = \left[\begin{smallmatrix} 4 & -1 \\ -3 & 1 \end{smallmatrix}\right]$, $\mathbb{C}^2(A^{-1}) = \mathbb{C}^2_*$, and $\Phi_A(\mathbb{C}^2) = \mathbb{C}^2_* \cup \{(0,0)\}$.

Directly from Lemma 1.4.12 we get

Lemma 1.5.14. *Let $D \subset \mathbb{C}^n$ be a log-convex Reinhardt domain. Then the following conditions are equivalent:*

(i) $\boldsymbol{E}(\log D) = \{0\}$ *(cf. Definition 1.4.5);*

(ii) *there exist a non-singular matrix* $A = \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix} \in \mathbb{M}(n \times n, \mathbb{Z})$ *and a vector* $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ *such that* $D \setminus V_0 \subset \boldsymbol{D}_{\alpha^1, c_1} \cap \dots \cap \boldsymbol{D}_{\alpha^n, c_n}$;[22]

(iii) *there exists a matrix $A \in \mathbb{GL}(n, \mathbb{Z})$ such that $D \subset \mathbb{C}^n(A)$ and $\Phi_A(D)$ is bounded.*

Lemma 1.5.15. *Let $D \subset \mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ be a log-convex Reinhardt domain. Then the following conditions are equivalent:*

(i) *there exists a point $(b', 0) \in \bar{D} \cap (\mathbb{C}^{n-1}_* \times \{0\})$;*

(ii) *for any point $(a', a_n) \in D \cap \mathbb{C}^n_*$ we have $\{(a', \lambda a_n) : 0 < |\lambda| \le 1\} \subset D$.*

Observe that the lemma implies Remark 1.5.8 (d).

Proof. The implication (i) $\Rightarrow$ (ii) follows directly from Lemma 1.4.13. The converse implication is obvious. $\square$

We come back to characterizations of the domain of convergence of a power series

$$S = \sum_{\alpha \in \mathbb{Z}^n_+} a_\alpha z^\alpha$$

from § 1.3. There we have defined three sets $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ (Definition 1.3.3) and observed that the sets $\mathfrak{B}$, $\mathfrak{C}$, and $\mathfrak{D}$ are complete Reinhardt.

[22] In particular, $D \subset \mathbb{C}^n(\alpha^1) \cap \dots \cap \mathbb{C}^n(\alpha^n)$.

Proposition 1.5.16. *The set $\mathfrak{B}$ is log-convex. Consequently, since*

$$\log \mathfrak{D} = \log \operatorname{int} \mathfrak{B} = \operatorname{int}(\log \mathfrak{B}),$$

the domain of convergence $\mathfrak{D}$ is also log-convex.

Proof. Take $x, y \in \mathfrak{B} \cap \mathbb{R}^n_{>0}$. Let $C > 0$ be such that $|a_\alpha x^\alpha| \le C$, $|a_\alpha y^\alpha| \le C$, $\alpha \in \mathbb{Z}^n_+$. Then for every $t \in [0,1]$, we have

$$|a_\alpha (x_1^t y_1^{1-t})^{\alpha_1} \cdots (x_n^t y_n^{1-t})^{\alpha_n}| \le |a_\alpha x^\alpha|^t |a_\alpha y^\alpha|^{1-t} \le C, \quad \alpha \in \mathbb{Z}^n_+. \qquad \square$$

Example 1.5.17. There exists a power series S such that $\mathfrak{D}_S = \mathbb{B}_n \subset \mathbb{C}^n$. We will see later in Proposition 1.11.11 that, using some Baire category argument, one can prove that there exist many power series S with $\mathfrak{D}_S = \mathbb{B}_n$. Here the problem is to find a concrete one.

Indeed, let $\{\xi_1, \xi_2, \dots\} \subset \partial\mathbb{B}_n$ be an arbitrary countable set which is dense in $\partial\mathbb{B}_n$ (Exercise: find such a set). Define

$$S := \sum_{\nu \in (\mathbb{Z}^n_+)_*} \frac{|\nu|!\, \bar{\xi}^\nu_{|\nu|}}{\nu!} z^\nu.$$

Notice that S is obtained from the series

$$\sum_{k=1}^\infty \langle z, \xi_k\rangle^k = \sum_{k=1}^\infty \Big(\sum_{j=1}^n z_j \bar{\xi}_{k,j}\Big)^k = \sum_{k=1}^\infty \sum_{\nu \in \mathbb{Z}^n_+ :\, |\nu| = k} \frac{k!}{\nu!} \bar{\xi}^\nu_k z^\nu.$$

To prove that $\mathbb{B}_n \subset \mathfrak{D}_S$, observe that

$$\left| \frac{|\nu|!\, \bar{\xi}^\nu_{|\nu|}}{\nu!} z^\nu \right| \le \langle \boldsymbol{R}(z), \boldsymbol{R}(\xi_{|\nu|})\rangle^{|\nu|} \le \|z\|^{|\nu|}, \quad z \in \mathbb{B}_n,\ \nu \in (\mathbb{Z}^n_+)_*.$$

Since $\mathfrak{D}_S$ is fat (Remark 1.3.5 (d)), we only need to show that $\mathfrak{D}_S \subset \overline{\mathbb{B}}_n$. Suppose that $\mathfrak{D}_S \setminus \overline{\mathbb{B}}_n \neq \varnothing$. Then there exist k_0 and $t > 1$ such that $a := t\xi_{k_0} \in \mathfrak{D}_S$. Let $C > 0$ be such that

$$\left| \frac{|\nu|!\, \bar{\xi}^\nu_{|\nu|}}{\nu!} a^\nu \right| \le C, \quad \nu \in (\mathbb{Z}^n_+)_*.$$

Put $N(k) := \#\{\nu \in \mathbb{Z}^n_+ : |\nu| = k\} = \binom{k+n}{n}$. Then

$$|\langle \xi_k, \xi_{k_0}\rangle^k| \le \sum_{\nu \in \mathbb{Z}^n_+ :\, |\nu| = k} \frac{k!}{\nu!} \boldsymbol{R}(\xi_k)^\nu \boldsymbol{R}(\xi_{k_0})^\nu \le N(k) \frac{C}{t^k}, \quad k \in \mathbb{N}.$$

Hence

$$1 = \limsup_{k \to +\infty} |\langle \xi_k, \xi_{k_0}\rangle| \le \lim_{k \to +\infty} (N(k)C)^{1/k} \frac{1}{t} = \frac{1}{t} < 1;$$

a contradiction.

Exercise* 1.5.18. Given a complex norm $N\colon \mathbb{C}^n \to \mathbb{R}_+$ such that $N(z) = N(\boldsymbol{R}(z))$, $z \in \mathbb{C}^n$, decide whether there exists a power series S such that $\mathfrak{D}_S = \{z \in \mathbb{C}^n : N(z) < 1\}$.

1.6 Domains of convergence of Laurent series

Consider an (*n-fold*) *Laurent series*

$$S = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha \quad (z \in \mathbb{C}^n),$$

where $(a_\alpha)_{\alpha \in \mathbb{Z}^n} \subset \mathbb{C}$. Put $\Sigma(S) := \{\alpha \in \mathbb{Z}^n : a_\alpha \neq 0\}$. Observe that in the case where $\Sigma(S) \subset \mathbb{Z}^n_+$, the series S reduces to a power series (cf. § 1.3). Similarly as in the case of power series we introduce the following sets.

Definition 1.6.1. Given a Laurent series S, put

$$\mathfrak{B} = \mathfrak{B}_S := \Big\{z \in \mathbb{C}^n(\Sigma(S)) : \sup_{\alpha \in \Sigma(S)} |a_\alpha z^\alpha| < +\infty\Big\},$$

$$\mathfrak{C} = \mathfrak{C}_S := \Big\{z \in \mathbb{C}^n(\Sigma(S)) : \sum_{\alpha \in \Sigma(S)} |a_\alpha z^\alpha| < +\infty\Big\},$$ [23]

$$\mathfrak{D} = \mathfrak{D}_S := \operatorname{int} \mathfrak{C}.$$

Clearly $\mathfrak{D} \subset \mathfrak{C} \subset \mathfrak{B}$. The set $\mathfrak{D}$ is traditionally called the *domain of convergence of* the Laurent series S.[24]

It is clear that $\mathfrak{D}_S$ is an open Reinhardt set [25]. Put

$$\mathfrak{B}^1 := \{a \in \mathbb{C}^n : \exists_{\substack{U \subset \mathbb{C}^n(\Sigma(S))\\ a \in U \text{ - open}}} \exists_{C>0} : \|a_\alpha z^\alpha\|_U \le C,\ \alpha \in \Sigma(S)\},$$

$$\mathfrak{B}^2 := \{a \in \mathbb{C}^n : \exists_{\substack{U \subset \mathbb{C}^n\\ a \in U \text{ - open}}} \exists_{C>0} : |a_\alpha| \le \frac{C}{r^\alpha},\ r \in \mathbb{R}^n_{>0} \cap \boldsymbol{R}(U),\ \alpha \in \mathbb{Z}^n\},$$

$$\mathfrak{B}^3 := \{a \in \mathbb{C}^n : \exists_{\substack{U \subset \mathbb{C}^n(\Sigma(S))\\ a \in U \text{ - open}}} \exists_{C>0} \exists_{\theta \in (0,1)} : \|a_\alpha z^\alpha\|_U \le C\theta^{|\alpha|},\ \alpha \in \Sigma(S)\},$$

where $\|\varphi\|_A := \sup\{|\varphi(z)| : z \in A\}$. It is clear that $\operatorname{int} \mathfrak{B} \supset \mathfrak{B}^1 = \mathfrak{B}^2 \supset \mathfrak{B}^3 \subset \operatorname{int} \mathfrak{C} = \mathfrak{D} \subset \operatorname{int} \mathfrak{B}$.

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[23] That is, $\sup\Big\{\sum_{\alpha \in A} |a_\alpha z^\alpha| : A \subset \Sigma(S),\ \#A < +\infty\Big\} < +\infty$. Observe that, by Proposition 1.2.10, $\mathfrak{C}_S := \{z \in \mathbb{C}^n(\Sigma) :$ the series $\sum_{\alpha \in \Sigma(S)} a_\alpha z^\alpha$ is absolutely summable$\}$.

[24] If $\Sigma(S) = \varnothing$, then we put $\mathfrak{B} = \mathfrak{C} := \mathbb{C}^n$.

[25] Proposition 1.6.5 (d) will show that $\mathfrak{D}_S$ is connected and, therefore, $\mathfrak{D}_S$ is really a domain.

Lemma 1.6.2. *Let $K \subset \mathbb{C}^n$ be a Reinhardt compact set and let $r > 0$. Put*

$$K^{(r)} := \bigcup_{a \in K} \overline{\mathbb{P}}(a, r)$$

and observe that $K^{(r)}$ is also a Reinhardt compact (Exercise). *Then there exists a $\theta \in (0, 1)$ such that for every $\alpha \in \mathbb{R}^n$ with $K^{(r)} \subset \mathbb{C}^n(\alpha)$ we have*

$$\max_{z \in K} |z^\alpha| \le \theta^{|\alpha|} \max_{z \in K^{(r)}} |z^\alpha|,$$

where $|\alpha| := |\alpha_1| + \dots + |\alpha_n|$.

Proof. Observe that if $z = (z_1, \dots, z_n) \in K$ and $\alpha_{j_0} < 0$, then $|z_{j_0}| > r$. Moreover, there exists $0 < \theta < 1$ such that $|z_j|/\theta \le |z_j| + r$, $j = 1, \dots, n$, for any $z = (z_1, \dots, z_n) \in K$ (Exercise). Consequently, for $z \in K$, we have

$$\begin{aligned}(1/\theta)^{|\alpha|}|z^\alpha| &= \prod_{j:\,\alpha_j \ge 0} (|z_j|/\theta)^{\alpha_j} \prod_{j:\,\alpha_j < 0} (|z_j|\theta)^{\alpha_j} \\ &\le \prod_{j:\,\alpha_j \ge 0} (|z_j| + r)^{\alpha_j} \prod_{j:\,\alpha_j < 0} (|z_j| - r)^{\alpha_j} \le \sup_{w \in K^{(r)}} |w^\alpha|. \end{aligned}$$

□

Lemma 1.6.3. $\mathfrak{B}^1 = \mathfrak{B}^2 = \mathfrak{B}^3 = \operatorname{int} \mathfrak{B} = \mathfrak{D}$.

Proof. To prove that $\mathfrak{B}^1 = \mathfrak{B}^2 = \mathfrak{B}^3$ we only need to show that $\mathfrak{B}^1 \subset \mathfrak{B}^3$. Let a, U, and C be as in the definition of $\mathfrak{B}^1$. We may assume that U is Reinhardt. Let $V \Subset U$ be a Reinhardt neighborhood of a and let $r > 0$ be such that $\overline{V}^{(r)} \subset U$. Now, we apply Lemma 1.6.2 with $K := \overline{V}^{(r)}$.

It remains to show that $\operatorname{int} \mathfrak{B} \subset \mathfrak{B}^1$. Fix an $a = (a_1, \dots, a_n) \in \operatorname{int} \mathfrak{B}$ and small $\varepsilon \in (0, 1)$. For $\sigma = (\sigma_1, \dots, \sigma_n) \in \{-1, 1\}^n$ define

$$b(\sigma) = (b_1(\sigma), \dots, b_n(\sigma)),$$

where

$$b_j(\sigma) := \begin{cases} (1 + \varepsilon)a_j & \text{if } a_j \ne 0 \text{ and } \sigma_j = 1, \\ (1 - \varepsilon)a_j & \text{if } a_j \ne 0 \text{ and } \sigma_j = -1, \\ \varepsilon & \text{if } a_j = 0, \end{cases} \qquad j = 1, \dots, n.$$

Taking sufficiently small $\varepsilon \in (0, 1)$, we may assume that $b(\sigma) \in \mathfrak{B}$ for any σ. Let $C > 0$ be such that $|a_\alpha (b(\sigma))^\alpha| \le C$, $\alpha \in \Sigma(S)$, $\sigma \in \{-1, 1\}^n$. Put $U(\sigma) := U_1(\sigma) \times \dots \times U_n(\sigma)$, where

$$U_j(\sigma) := \begin{cases} K((1 + \varepsilon)|a_j|) & \text{if } a_j \ne 0 \text{ and } \sigma_j = 1, \\ \mathbb{C} \setminus \overline{K}(1 - \varepsilon)|a_j|) & \text{if } a_j \ne 0 \text{ and } \sigma_j = -1, \\ K(\varepsilon) & \text{if } a_j = 0, \end{cases} \qquad j = 1, \dots, n.$$

Observe that $U := \bigcap_{\sigma\in\{-1,1\}^n} U(\sigma)$ is a neighborhood of a. We will show that $|a_\alpha z^\alpha| \le C$, $z \in U$, $\alpha \in \Sigma(S)$. Take such z and $\alpha = (\alpha_1,\dots,\alpha_n)$ and let $\sigma := (\sigma_1,\dots,\sigma_n)$ with $\sigma_j := \begin{cases} 1 & \text{if } \alpha_j \ge 0 \\ -1 & \text{if } \alpha_j < 0 \end{cases}$. Then $|a_\alpha z^\alpha| \le |a_\alpha (b(\sigma))^\alpha| \le C$. □

Remark 1.6.4. Notice that in contrast to the case of power series, the domain of convergence of a Laurent series need not be fat. For example, if $S = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{1}{z^k}$, then $\mathfrak{D}_S = \mathbb{C}_*$.

Proposition 1.6.5. *Assume $\mathfrak{D}_S \neq \varnothing$. Then:*

(a) *The series S is locally normally summable in $\mathfrak{D}_S$. In particular, the function*

$$f(z) := \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha, \quad z \in \mathfrak{D}_S,$$

is well defined and continuous.[26]

(b) $$a_\alpha = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{P}(r)} \frac{f(\zeta)}{\zeta^{\alpha+\mathbf{1}}}\,d\zeta, \quad \alpha\in\mathbb{Z}^n,\ r\in\mathfrak{D}_S\cap\mathbb{R}^n_{>0},$$

where $\partial_0\mathbb{P}(a,r) := \partial K(a_1,r_1)\times\cdots\times\partial K(a_n,r_n)$, $\mathbf{1} := (1,\dots,1)\in\mathbb{N}^n$, and

$$\int_{\partial_0\mathbb{P}(r)} \varphi(\zeta)d\zeta := i^n \int_{[0,2\pi]^n} \varphi(r\cdot e^{i\theta})\, r_1 e^{i\theta_1}\cdots r_n e^{i\theta_n}\, d\Lambda_n(\theta).$$

Hence,

$$|a_\alpha| \le \frac{\|f\|_{\partial_0\mathbb{P}(r)}}{r^\alpha}, \quad \alpha\in\mathbb{Z}^n,\ r\in\mathfrak{D}_S\cap\mathbb{R}^n_{>0}.$$

Consequently, for any Reinhardt domain $U \Subset \mathfrak{D}_S$ we have the Cauchy inequalities

$$|a_\alpha| \le \frac{\|f\|_U}{r^\alpha}, \quad \alpha\in\mathbb{Z}^n,\ r\in U\cap\mathbb{R}^n_{>0}. \tag{1.6.1}$$

(c) *If $\mathfrak{D}_S \cap V_{j_0} \neq \varnothing$, then $\Sigma(S) \subset \mathbb{Z}^{j_0-1}\times\mathbb{Z}_+\times\mathbb{Z}^{n-j_0}$ and $\mathfrak{D}_S = \widehat{\mathfrak{D}_S}^{(j_0)}$. Consequently, if $\mathfrak{D}_S\cap V_j \neq \varnothing$, $j = 1,\dots,n$, then $\Sigma(S)\subset\mathbb{Z}^n_+$ and $\mathfrak{D}_S = \widehat{\mathfrak{D}_S}$, i.e. $\mathfrak{D}_S$ is a complete Reinhardt domain. In particular, if $0\in\mathfrak{D}_S$, then $\mathfrak{D}_S$ is a complete Reinhardt domain.*

(d) *$\mathfrak{D}_S$ is log-convex. In particular, $\mathfrak{D}_S$ is connected.*

Proof. (a) follows from Lemma 1.6.3.

(b) Since the series S is locally uniformly summable, we get

$$\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{P}(r)} \frac{f(\zeta)}{\zeta^{\alpha+\mathbf{1}}}\,d\zeta = \sum_{\beta\in\mathbb{Z}^n} a_\beta \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{P}(r)} \zeta^{\beta-\alpha-\mathbf{1}}\,d\zeta = a_\alpha.$$

[26]Notice that, in fact, f is holomorphic – cf. Theorem 1.7.19.

(c) To simplify notation assume that $j_0 = n$. Fix an $a \in \mathfrak{D}_S \cap V_n$ and let $U \Subset \mathfrak{D}_S$ be a Reinhardt neighborhood of a. By (1.6.1) we have

$$|a_\alpha| \le \frac{\|f\|_U}{r'^{\alpha'}} r_n^{-\alpha_n},$$
$$\alpha = (\alpha', \alpha_n) \in \mathbb{Z}^{n-1} \times \mathbb{Z},\ r = (r', r_n) \in U \cap \mathbb{R}^n_{>0} \subset \mathbb{R}^{n-1} \times \mathbb{R}.$$

Letting $r_n \to 0$, we conclude that $a_\alpha = 0$ if $\alpha_n < 0$. Moreover,

$$\|a_\alpha z^\alpha\|_{\widehat{U}^{(n)}} = \|a_\alpha z^\alpha\|_U, \quad \alpha \in \Sigma(S),$$

which implies that $\widehat{\mathfrak{D}_S}^{(n)} \subset \mathfrak{D}_S$.

(d) Take $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \log \mathfrak{D}_S$. Let

$$a := (e^{x_1}, \dots, e^{x_n}), \quad b := (e^{y_1}, \dots, e^{y_n}) \in \mathfrak{D}_S \cap \mathbb{R}^n_{>0}$$

and let $U_a, U_b \Subset \mathfrak{D}_S \cap \mathbb{C}^n_*$ be neighborhoods of a and b, respectively. By Lemma 1.6.3, there exist $C > 0$ and $0 < \theta < 1$ such that

$$\|a_\alpha z^\alpha\|_{U_a \cup U_b} \le C\theta^{|\alpha|}, \quad \alpha \in \mathbb{Z}^n \quad (\text{Exercise}).$$

Define

$$U := \{(e^{i\theta_1}|z_1|^{1-t}|w_1|^t, \dots, e^{i\theta_n}|z_n|^{1-t}|w_n|^t) :$$
$$(z_1, \dots, z_n) \in U_a,\ (w_1, \dots, w_n) \in U_b,\ (\theta_1, \dots, \theta_n) \in \mathbb{R}^n,\ t \in [0,1]\} \subset \mathbb{C}^n_*.$$

One can easily check that U is open and

$$\|a_\alpha z^\alpha\|_U \le C\theta^{|\alpha|}, \quad \alpha \in \mathbb{Z}^n.$$

Consequently, $U \subset \mathfrak{D}_S$. Since $[x, y] \subset \log U$, we conclude that $\mathfrak{D}_S$ is log-convex. □

Proposition 1.6.6. *Let $\alpha \in (\mathbb{R}^n)_*$, $c \in \mathbb{R}$, $r \in \mathbb{R}^n_{>0}$ be such that $r^\alpha = e^c$. Then the elementary Reinhardt domain $\boldsymbol{D}_{\alpha,c}$ is the domain of convergence of the Laurent series*

$$S = \sum_{\nu \in \mathbb{Z}^n} \frac{N(\nu)}{r^\nu} z^\nu,$$

where

$$N(\nu) := \#\{k \in \mathbb{Z}_+ : \lfloor k\alpha \rfloor = \nu\}, \quad \lfloor k\alpha \rfloor := (\lfloor k\alpha_1 \rfloor, \dots, \lfloor k\alpha_n \rfloor) \in \mathbb{Z}^n.$$

Observe that:

- S is obtained by grouping terms in the series $\sum_{k=0}^\infty \frac{z^{\lfloor k\alpha \rfloor}}{r^{\lfloor k\alpha \rfloor}}$.

- $\mathbb{C}^n(\lfloor k\alpha\rfloor) = \mathbb{C}^n(\alpha)$, $k \in \mathbb{Z}_+$; in particular, $\Sigma(S) = \mathbb{C}^n(\alpha)$.
- If $\alpha \in (\mathbb{R}^n_+)_*$, then S is a power series.
- If $\alpha \in \mathbb{Z}^n$, then $S = \sum_{k=0}^{\infty} \frac{1}{r^{k\alpha}} z^{k\alpha}$.

Proof. We may assume that $\alpha \in \mathbb{R}^n_*$. Moreover, using the biholomorphism $\mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (z_1/r_1, \dots, z_n/r_n)$ we may reduce the proof to the case where $r_1 = \dots = r_n = 1$ ($c = 0$). Notice that

$$\lim_{k\to+\infty} \frac{\lfloor k\alpha\rfloor}{k} = \alpha.$$

Hence the classical Cauchy criterion implies that the series $\sum_{k=0}^{\infty} z^{\lfloor k\alpha\rfloor}$ is absolutely convergent in $\boldsymbol{D}_\alpha$. Using Theorem 1.2.7, we conclude that $\boldsymbol{D}_\alpha \subset \operatorname{int} \mathfrak{C}_S = \mathfrak{D}_S$.

Conversely, let $U \Subset \mathfrak{D}_S$ be an arbitrary Reinhardt domain. By Lemma 1.6.3, there exist $C > 0$, $\theta \in (0, 1)$ such that

$$N(\nu)|z^\nu| \le C\theta^{|\nu|}, \quad z \in U, \ \nu \in \mathbb{Z}^n.$$

Therefore,

$$|z^{\lfloor k\alpha\rfloor/k}| \le (N(\lfloor k\alpha\rfloor)|z^{\lfloor k\alpha\rfloor}|)^{1/k} \le (C\theta^{|\lfloor k\alpha\rfloor|})^{1/k}, \quad z \in U, \ k \in \mathbb{N}.$$

Letting $k \to +\infty$ we get $|z^\alpha| \le \theta^{|\alpha|} < 1$, $z \in U$, and, consequently, $U \subset \boldsymbol{D}_\alpha$. □

Proposition 1.6.7. *Let $S_j = \sum_{\alpha\in\mathbb{Z}^n} a^j_\alpha z^\alpha$ be a Laurent series, $j = 1, \dots, m$, such that $\mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m} \ne \varnothing$.*[27] *For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^m$, define*

$$S(\lambda) = \lambda_1 S_1 + \dots + \lambda_m S_m := \sum_{\alpha\in\mathbb{Z}^n} (\lambda_1 a^1_\alpha + \dots + \lambda_m a^m_\alpha) z^\alpha.$$

Then there exists a set $C \subset \mathbb{C}^n$ such that

(*) *C is the union of a countable family of complex $(m-1)$-dimensional vector subspaces of $\mathbb{C}^m$ and*

$$\mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m} \overset{\text{(L)}}{\subset} \mathfrak{D}_{S(\lambda)} \overset{\text{(R)}}{\subset} \operatorname{int} \bar{\mathfrak{D}}_{S_1} \cap \dots \cap \operatorname{int} \bar{\mathfrak{D}}_{S_m}, \quad \lambda \in \mathbb{C}^m \setminus C.$$

In particular, if $\mathfrak{D}_{S_j}$ is fat (e.g. S_j is a power series – cf. Remark 1.3.5 (d)*), $j = 1, \dots, m$, then*

$$\mathfrak{D}_{S(\lambda)} = \mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m}, \quad \lambda \in \mathbb{C}^m \setminus C.$$

[27] Note that:

- By Proposition 1.6.5 (d), $\mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m}$ is log-convex and, consequently, it is a domain.
- If S_j is a power series with $\mathfrak{D}_{S_j} \ne \varnothing$, $j = 1, \dots, m$, then obviously $0 \in \mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m} \ne \varnothing$.

Proof. First observe that the inclusion (L) holds for every $\lambda \in \mathbb{C}^m$.

To prove that there exists a set $C \subset \mathbb{C}^m$ with (*) such that (R) is true for $\lambda \in \mathbb{C}^m \setminus C$, it suffices to show that there exists a set C with (*) such that

$$\mathfrak{D}_{S(\lambda)} \cap \mathbb{Q}_*^{2n} \subset \mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m}, \quad \lambda \in \mathbb{C}^m \setminus C,$$

or equivalently,

$$\mathbb{Q}_*^{2n} \setminus (\mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m}) \subset \mathbb{C}^n \setminus \mathfrak{D}_{S(\lambda)}, \quad \lambda \in \mathbb{C}^m \setminus C.$$

We only need to show that for every $b \in \mathbb{C}_*^n \setminus (\mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m})$ the vector space

$$V(b) := \{\lambda \in \mathbb{C}^m : b \in \mathfrak{D}_{S(\lambda)}\}$$

has dimension $\leq m-1$. To prove that $\dim V(b) \leq m-1$, suppose that for a $b \in \mathbb{C}_*^n \setminus (\mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m})$ there exist $\lambda^1, \dots, \lambda^m \in V(b)$ such that the matrix $P := [\lambda_k^j]$ is non-singular. Let $U \Subset \mathbb{C}_*^n \cap \mathfrak{D}_{S(\lambda^1)} \cap \dots \cap \mathfrak{D}_{S(\lambda^m)}$ be a Reinhardt neighborhood of b. By Remark 1.3.5 (d), there exist $C > 0$, $\theta \in (0,1)$ such that $|A^j(z)| \leq C\theta^{|\alpha|}$, where

$$A^j(z) := \lambda_1^j a_\alpha^1 z^\alpha + \dots + \lambda_m^j a_\alpha^m z^\alpha, \quad z \in U, \ \alpha \in \mathbb{Z}^n, \ j = 1, \dots, m.$$

Hence, by the Cramer formulas, we have

$$a_\alpha^j z^\alpha = q_1^j A^1(z) + \dots + q_m^j A^m(z), \quad j = 1, \dots, m,$$

where $Q = [q_k^j] := P^{-1}$. Consequently, there exists a $C' > 0$ such that

$$|a_\alpha^j z^\alpha| \leq C'\theta^{|\alpha|}, \quad z \in U, \ \alpha \in \mathbb{Z}^n, \ j = 1, \dots, m,$$

which implies that $b \in \mathfrak{D}_{S_1} \cap \dots \cap \mathfrak{D}_{S_m}$; a contradiction. $\square$

From Propositions 1.6.6 and 1.6.7 one immediately obtains the following

Corollary 1.6.8. *For any $\alpha^j \in (\mathbb{R}^n)_*$ (resp. $(\mathbb{R}_+^n)_*$), $c_j \in \mathbb{R}$, $j = 1, \dots, m$, there exists a Laurent (resp. power) series whose domain of convergence coincides with $\boldsymbol{D}_{\alpha^1, c_1} \cap \dots \cap \boldsymbol{D}_{\alpha^m, c_m}$.*

Exercise 1.6.9. Find (effectively) a power series whose domain of convergence equals

$$\{(z_1, z_2) \in \mathbb{D}^2 : 2|z_1 z_2| < 1\}.$$

1.7 Holomorphic functions

Definition 1.7.1. Let $\Omega \subset \mathbb{C}^n$ be open. A continuous mapping $f\colon \Omega \to \mathbb{C}^m$ is *holomorphic* on Ω ($f \in \mathcal{O}(\Omega, \mathbb{C}^m)$) if f is *separately holomorphic*, i.e. for any point $a = (a_1, \dots, a_n) \in \Omega$ and for any $k \in \{1, \dots, n\}$, the mapping

$$\lambda \mapsto f(a_1, \dots, a_{k-1}, \lambda, a_{k+1}, \dots, a_n)$$

is holomorphic near a_k; equivalently, the complex partial derivatives

$$\frac{\partial f_j}{\partial z_k}(z), \quad j = 1, \dots, m,\ k = 1, \dots, n,$$

exist at any point $z \in \Omega$. *Notice that in fact the continuity of f follows from the separate holomorphy – cf. Theorem* 1.7.13. Put $\mathcal{O}(\Omega) := \mathcal{O}(\Omega, \mathbb{C}) =$ the space of all holomorphic functions on Ω. Functions holomorphic on $\mathbb{C}^n$ are called *entire holomorphic functions*.

Exercise 1.7.2. (a) $\mathcal{O}(\Omega)$ is a complex algebra.

(b) Let $a = (a', a'') \in \Omega \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$, $\Omega' := \{z' \in \mathbb{C}^k : (z', a'') \in \Omega\}$. If $f \in \mathcal{O}(\Omega)$, then $f(\cdot, a'') \in \mathcal{O}(\Omega')$.

(c) Every polynomial of n complex variables is an entire function, i.e. $\mathcal{P}(\mathbb{C}^n) \subset \mathcal{O}(\mathbb{C}^n)$.

Proposition 1.7.3 (Cauchy integral formula). *If $f \in \mathcal{O}(\mathbb{P}(a,r)) \cap \mathcal{C}(\overline{\mathbb{P}}(a,r))$ with $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ and $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$, then*

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^n} \int_{\partial K(a_1,r_1)} \Big(\dots \Big(\int_{\partial K(a_n,r_n)} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)}\, d\zeta_n \Big) \dots \Big) d\zeta_1 \\ &= \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{P}(a,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z = (z_1, \dots, z_n) \in \mathbb{P}(a,r). \end{aligned} \tag{1.7.1}$$

Notice that for $z = (z_1, \dots, z_n) \in \mathbb{P}(a,r)$, the function

$$\partial K(a_1, r_1) \times \dots \times \partial K(a_n, r_n) \ni (\zeta_1, \dots, \zeta_n) \mapsto \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)}$$

is continuous and, therefore, by the Fubini theorem, the above integral is independent of the order of integration.

Proof. We apply induction on n. For $n = 1$ the result reduces to the classical Cauchy integral formula (cf. [Con 1973], Chapter IV, Theorem 5.4).

$n - 1 \rightsquigarrow n$: We may assume that $a = 0$. Fix a $z = (z', z_n) \in \mathbb{P}(r') \times K(r_n)$ ($r = (r', r_n)$). We have

$$f(z) = f(z', z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{\partial_0 \mathbb{P}(r')} \frac{f(\zeta', z_n)}{\zeta' - z'}\, d\zeta'. \tag{1.7.2}$$

Observe that $f(\zeta', \cdot) \in \mathcal{O}(K(r_n)) \cap \mathcal{C}(\overline{K}(r_n))$ for any $\zeta' \in \partial_0 \mathbb{P}(r')$.

Indeed, fix a $\zeta' \in \partial_0 \mathbb{P}(r')$ and let $\mathbb{P}(r') \ni \zeta'_\nu \to \zeta'$. Then

$$\mathcal{O}(K(r_n)) \ni f(\zeta'_\nu, \cdot) \to f(\zeta', \cdot)$$

uniformly on $K(r_n)$. Hence, by the Weierstrass theorem (cf. [Con 1973], Ch. VII, Theorem 2.1), $f(\zeta', \cdot) \in \mathcal{O}(K(r_n))$.

Consequently, by the classical Cauchy formula,

$$f(\zeta', z_n) = \frac{1}{2\pi i} \int_{\partial K(r_n)} \frac{f(\zeta', \zeta_n)}{\zeta_n - z_n} \, d\zeta_n,$$

which together with (1.7.2) gives (1.7.1). □

Exercise 1.7.4 (Cauchy integral formula). Observe that the following slightly generalized Cauchy integral formula is true (with the same proof).

Let $D_j \subset \mathbb{C}$ be a bounded domain whose boundary is a finite union of piecewise $\mathcal{C}^1$ Jordan curves with positive orientation with respect to D_j, $j = 1, \dots, n$. Put $D := D_1 \times \dots \times D_n$ and let $f \in \mathcal{O}(D) \cap \mathcal{C}(\overline{D})$. Then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D_1} \dots \int_{\partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \dots (\zeta_n - z_n)} \, d\zeta_1 \dots d\zeta_n,$$
$$z = (z_1, \dots, z_n) \in D.$$

Exercise 1.7.5. Let T be the Hartogs triangle (Remark 1.5.11 (c)) and let $f \in \mathcal{O}(T) \cap \mathcal{C}(\overline{T})$, $f(z,w) := z^2/w$. Prove that f is not a uniform limit of a sequence of functions $f_k \in \mathcal{O}(D_k)$, where D_k is a neighborhood of $\overline{T}$, $k = 1, 2, \dots$.

Compare this result with the theorem of Mergelyan in classical one-variable complex analysis (cf. [Rud 1974], Chapter 20). For more information see [Bed-For 1978].

Theorem 1.7.6. *Let $\Omega \subset \mathbb{C}^n$ be open and let $f \in \mathcal{O}(\Omega)$. Then:*

- *f has all complex derivatives in Ω.*
- *For any point $a \in \Omega$ and a polydisc $\mathbb{P}(a, r) \Subset \Omega$ ($r = (r_1, \dots, r_n)$), we have*

$$D^\alpha f(z) = \frac{\alpha!}{(2\pi i)^n} \int_{\partial_0 \mathbb{P}(a,r)} \frac{f(\zeta)}{(\zeta - z)^{\alpha + \mathbf{1}}} \, d\zeta, \quad z \in \mathbb{P}(a, r), \ \alpha \in \mathbb{Z}_+^n,$$

  the Taylor series $T_a f$ is locally uniformly summable in $\mathbb{P}(a, r)$, and

$$f(z) = T_a f(z), \quad z \in \mathbb{P}(a, r),$$

$$d(T_a f) \geq d_\Omega(a) := \sup\{\tau > 0 : \mathbb{P}(a, \tau) \subset \Omega\}, \quad a \in \Omega.$$

- *For a function $g \colon \Omega \to \mathbb{C}$ the following conditions are equivalent:*

(i) $g \in \mathcal{O}(\Omega)$;

(ii) *for every point* $a \in \Omega$ *there exist a power series* $\sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha (z-a)^\alpha$ *and a polydisc* $\mathbb{P}(a,r) \subset \Omega$ *such that the power series is locally uniformly summable in* $\mathbb{P}(a,r)$ *and*

$$g(z) = \sum_{\alpha\in\mathbb{Z}_+^n} a_\alpha (z-a)^\alpha, \quad z \in \mathbb{P}(a,r).$$

Proof. We may assume that $a = 0$ and $\mathbb{P}(r) \Subset \Omega$. Observe that for $(\zeta, z) \in \partial_0\mathbb{P}(r) \times \mathbb{P}(r)$,

$$\frac{1}{\zeta - z} = \sum_{\alpha\in\mathbb{Z}_+^n} \frac{z^\alpha}{\zeta^{\alpha+\mathbf{1}}},$$

and the series is locally normally summable. Hence, by the Cauchy integral formula (Proposition 1.7.3), we get

$$f(z) = \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{P}(r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{\alpha\in\mathbb{Z}_+^n}\left(\frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{P}(r)} \frac{f(\zeta)}{\zeta^{\alpha+\mathbf{1}}}\,d\zeta\right) z^\alpha,$$

$$z \in \mathbb{P}(r).$$

It remains to apply Proposition 1.3.12. □

Lemma 1.7.7. *Let* $f \in \mathcal{O}(\Omega)$ *with* $\mathbb{P}(a', r') \times \partial_0\mathbb{P}(a'', r'') \subset \Omega$ *(*$r = (r', r'') \in \mathbb{R}^k_{>0} \times \mathbb{R}^{n-k}_{>0}$*,* $a = (a', a'') \in \mathbb{C}^k \times \mathbb{C}^{n-k}$*). Define*

$$g(z) := \frac{1}{(2\pi i)^{n-k}}\int_{\partial_0\mathbb{P}(a'',r'')} \frac{f(z', \zeta)}{\zeta - z''}\,d\zeta, \quad z = (z', z'') \in \mathbb{P}(a,r). \tag{1.7.3}$$

Then $g \in \mathcal{O}(\mathbb{P}(a,r))$.

Proof. It is obvious that g is continuous. Let

$$F(z,\zeta) := \frac{f(z',\zeta)}{\zeta - z''}, \quad z = (z', z'') \in \mathbb{P}(a,r),\ \zeta \in \partial_0\mathbb{P}(a'', r'').$$

Observe that

$$\frac{\partial F}{\partial z_j}(z,\zeta) = \begin{cases} \frac{\frac{\partial f}{\partial z_j}(z',\zeta)}{\zeta - z''} & \text{if } j = 1,\dots,k, \\ \frac{f(z',\zeta)}{(\zeta - z'')^{e_j+\mathbf{1}}} & \text{if } j = k+1,\dots,n. \end{cases}$$

In particular, the function

$$\mathbb{P}(a,r) \times \partial_0\mathbb{P}(a'', r'') \ni (z,\zeta) \mapsto \frac{\partial F}{\partial z_j}(z,\zeta)$$

is continuous, $j = 1, \dots, n$. Consequently,

$$\frac{\partial g}{\partial z_j}(z) = \frac{1}{(2\pi i)^{n-k}} \int_{\partial_0 \mathbb{P}(a'',r'')} \frac{\partial F}{\partial z_j}(z,\zeta)\, d\zeta, \quad z \in \mathbb{P}(a,r),\ j = 1, \dots, n,$$

exist. □

Exercise 1.7.8. Try to generalize Lemma 1.7.7 and find "optimal" assumptions for a continuous function $f : \mathbb{P}(a', r') \times \partial_0 \mathbb{P}(a'', r'') \to \mathbb{C}$ under which the function g given by (1.7.3) is holomorphic on $\mathbb{P}(a, r)$.

Exercise 1.7.9. (a) Holomorphic functions are infinitely differentiable in the complex sense.

(b) If $f \in \mathcal{O}(\Omega)$, then $D^\alpha f \in \mathcal{O}(\Omega)$ for arbitrary $\alpha \in \mathbb{Z}_+^n$.

Proposition 1.7.10 (Identity principle). *Let $f, g \in \mathcal{O}(D)$, where $D \subset \mathbb{C}^n$ is a domain. Then the following conditions are equivalent:*

(i) $f \equiv g$*;*

(ii) *there exists an $a \in D$ such that $T_a f = T_a g$;*

(iii) $\operatorname{int}\{z \in D : f(z) = g(z)\} \neq \varnothing$.

Proof. Clearly (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii). Since D is connected, to prove the implication (ii) $\Rightarrow$ (i) it is sufficient to note that the set $D_0 := \{z \in D : T_z f = T_z g\}$ is non-empty open and closed in D. □

Exercise 1.7.11. (a) Let $D \subset \mathbb{C}^n$ be a domain such that $D \cap \mathbb{R}^n \neq \varnothing$. Show that if $f \in \mathcal{O}(D)$ is such that $f = 0$ in $D \cap \mathbb{R}^n$, then $f \equiv 0$.

(b) Let $D \subset \mathbb{C}^n$ be a domain and let $G := \{\bar{z} : z \in D\}$. Assume that $f \in \mathcal{O}(D \times G)$ is such that $f(z, \bar{z}) = 0$ for z in a neighborhood of a point $a \in D$. Prove that $f \equiv 0$.

Proposition 1.7.12. *Let $f : \Omega \to \mathbb{C}$. The following conditions are equivalent:*

(i) $f \in \mathcal{O}(\Omega)$*;*

(ii) *f is differentiable in the complex sense at any point of Ω;*

(iii) (Osgood theorem) *f is locally bounded and separately holomorphic in Ω (cf. Theorem 1.7.13).*

Proof. It is clear that (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) (cf. Proposition 1.3.12 and Theorem 1.7.6).

(iii) $\Rightarrow$ (ii): Suppose that $|f| \le C$ in $\mathbb{P}(a, r) \Subset \Omega$. Then, by the Schwarz lemma (cf. [Con 1973], Chapter VI, Lemma 2.1), we obtain

$$\begin{aligned} |f(z) - f(a)| &\le |f(z_1, z_2, \dots, z_n) - f(a_1, z_2, \dots, z_n)| + \cdots \\ &\qquad \cdots + |f(a_1, \dots, a_{n-1}, z_n) - f(a_1, \dots, a_{n-1}, a_n)| \\ &\le \frac{2C}{r}(|z_1 - a_1| + \cdots + |z_n - a_n|), \quad z \in \mathbb{P}(a, r), \end{aligned} \tag{1.7.4}$$

which shows that f is continuous. □

The following result illustrates the essential difference between real and complex analysis.

Theorem* 1.7.13 (Hartogs' theorem on separate holomorphy, cf. [Kra 1992]). *Let $\Omega \subset \mathbb{C}^n$ be open and let $f : \Omega \to \mathbb{C}$ be separately holomorphic, i.e. the partial complex derivative $\frac{\partial f}{\partial z_j}(z)$ exists for all $z \in \Omega$ and $j = 1, \dots, n$. Then $f \in \mathcal{O}(\Omega)$.*

Proposition 1.7.12 (ii) implies also

Proposition 1.7.14. *The composition of holomorphic mappings is holomorphic.*

Proposition 1.7.15. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain and let $f \in \mathcal{O}(D)$. Define*

$$a_\alpha(f, r) := \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{P}(r)} \frac{f(\zeta)}{\zeta^{\alpha+\mathbf{1}}} \, d\zeta, \quad \alpha \in \mathbb{Z}^n, \ r \in D \cap \mathbb{R}^n_{>0}.$$

Then:

(a) *For any $\alpha \in \mathbb{Z}^n$, the number $a_\alpha(f, r)$ is independent of $r \in D \cap \mathbb{R}^n_{>0}$. In particular, we define $a_\alpha = a^f_\alpha = a_\alpha(f) := a_\alpha(f, r)$.*

(b) *Consequently, $D \subset \mathfrak{D}_f$, where $\mathfrak{D}_f$ denotes the domain of convergence of the Laurent series $\sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha$; cf. Proposition* 1.6.5 (b).

(c)
$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha, \quad z \in D.$$

(d) *If $D \cap V_j \neq \varnothing$, $j = 1, \dots, n$ (in particular, if $0 \in D$, e.g. D is complete), then $a_\alpha = 0$ for all $\alpha \in \mathbb{Z}^n \setminus \mathbb{Z}^n_+$ (cf. Proposition* 1.6.5 (c)). *Consequently, if $0 \in D$, then $f(z) = T_0 f(z)$, $z \in D$.*

Proof. We apply induction on n. For $n = 1$ the result is well known (cf. [Con 1973], Chapter V). Assume that it is true for $n - 1$.

(a) Since D is connected, it suffices to show that any point $a \in D$ has a Reinhardt neighborhood U such that $a_\alpha(f, r)$ is independent of $r \in U \cap \mathbb{R}^n_{>0}$.

Let $U = \mathbb{A}^n(r^-, r^+) \subset D$ be an arbitrary annulus centered at 0[28] and let $r = (r', r_n)$, $s = (s', s_n) \in U \cap (\mathbb{R}^{n-1}_{>0} \times \mathbb{R}_{>0})$, $\alpha \in \mathbb{Z}^n$. Write $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$.

[28] $\mathbb{A}^n(r^-, r^+) := \mathbb{A}(r_1^-, r_1^+) \times \cdots \times \mathbb{A}(r_n^-, r_n^+)$, $r^- = (r_1^-, \dots, r_n^-)$, $r^+ = (r_1^+, \dots, r_n^+)$, $-\infty \le r_j^- < r_j^+ \le +\infty$, $r_j^+ > 0$, $\mathbb{A}(r_j^-, r_j^+) := \{z \in \mathbb{C} : r_j^- < |z| < r_j^+\}$, $j = 1, \dots, n$.

Then, using the inductive assumption, we get

$$\begin{aligned}
a_\alpha(f,s) &= \frac{1}{(2\pi i)^n}\int_{\partial_0\mathbb{P}^n(s)} \frac{f(\zeta)}{\zeta^{\alpha+\mathbf{1}}}\,d\zeta\\
&= \frac{1}{2\pi i}\int_{\partial K(s_n)}\left(\frac{1}{(2\pi i)^{n-1}}\int_{\partial_0\mathbb{P}^{n-1}(s')}\frac{f(\zeta',\zeta_n)}{\zeta'^{\alpha'+\mathbf{1}}}\,d\zeta'\right)\frac{d\zeta_n}{\zeta_n^{\alpha_n+1}}\\
&= \frac{1}{2\pi i}\int_{\partial K(s_n)} a_{\alpha'}(f(\cdot,\zeta_n),s')\frac{d\zeta_n}{\zeta_n^{\alpha_n+1}}\\
&= \frac{1}{2\pi i}\int_{\partial K(s_n)} a_{\alpha'}(f(\cdot,\zeta_n),r')\frac{d\zeta_n}{\zeta_n^{\alpha_n+1}} = a_\alpha(f,(r',s_n)).
\end{aligned}$$

The same argument with respect to the last variable shows that

$$a_\alpha(f,(r',s_n)) = a_\alpha(f,r).$$

(c) Fix $U := \mathbb{A}^n(r^-,r^+) \subset D$. By the inductive assumption, using Theorem 1.2.7, for every $z = (z',z_n) \in U \subset \mathbb{C}^{n-1}\times\mathbb{C}$, we get:

$$\begin{aligned}
f(z) &= \sum_{\alpha_n\in\mathbb{Z}} a_{\alpha_n}(f(z',\cdot))z_n^{\alpha_n} = \sum_{\alpha_n\in\mathbb{Z}}\Big(\frac{1}{2\pi i}\int_{\partial K(r_n)}\frac{f(z',\zeta_n)}{\zeta_n^{\alpha_n+1}}\,d\zeta_n\Big)z_n^{\alpha_n}\\
&= \sum_{\alpha_n\in\mathbb{Z}}\Big(\frac{1}{2\pi i}\int_{\partial K(r_n)}\frac{1}{\zeta_n^{\alpha_n+1}}\Big(\sum_{\alpha'\in\mathbb{Z}^{n-1}} a_{\alpha'}(f(\cdot,\zeta_n))z'^{\alpha'}\Big)\,d\zeta_n\Big)z_n^{\alpha_n}\\
&= \sum_{\alpha\in\mathbb{Z}^n}\Big(\frac{1}{2\pi i}\int_{\partial K(r_n)}\frac{a_{\alpha'}(f(\cdot,\zeta_n))}{\zeta_n^{\alpha_n+1}}\,d\zeta_n\Big)z^\alpha = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha(f)z^\alpha. \qquad \square
\end{aligned}$$

Corollary 1.7.16 (Cauchy inequalities). *If $f \in \mathcal{O}(\mathbb{P}(a,r)) \cap \mathcal{C}(\overline{\mathbb{P}}(a,r))$, then*

$$|D^\alpha f(a)| \le \frac{\alpha!}{r^\alpha}\|f\|_{\partial_0\mathbb{P}(a,r)}, \quad \alpha\in\mathbb{Z}_+^n.$$

Similarly as in the case of one complex variable, the following results are easy consequences of the Cauchy inequalities (Exercise).

Proposition 1.7.17 (Liouville theorem). *Let $f \in \mathcal{O}(\mathbb{C}^n)$, $k \in \mathbb{Z}_+$. Then the following conditions are equivalent:*

(i) *f is a polynomial of degree $\le k$;*

(ii) $\exists_{C,R_0>0} : |f(z)| \le C\|z\|^k$ *for* $\|z\| \ge R_0$.

Corollary 1.7.18. *For an arbitrary compact $K \subset \Omega$ and a polyradius r such that $K^{(r)} \subset \Omega$ we have*

$$\|D^\alpha f\|_K \le \frac{\alpha!}{r^\alpha}\|f\|_{K^{(r)}}, \quad f\in\mathcal{O}(\Omega),\ \alpha\in\mathbb{Z}_+^n,$$

where $K^{(r)} := \bigcup_{a\in K}\overline{\mathbb{P}}(a,r)$.

Hence, using Proposition 1.7.12, we get

Theorem 1.7.19 (Weierstrass theorem). *If $\mathcal{O}(\Omega) \ni f_\nu \to f$ locally uniformly on Ω, then $f \in \mathcal{O}(\Omega)$ and $D^\alpha f_\nu \to D^\alpha f$ locally uniformly on Ω for any $\alpha \in \mathbb{Z}_+^n$.*

Proposition 1.7.20. *Let $D \subset \mathbb{C}^n$ be a domain and let $f \in \mathcal{O}(D)$, $f \not\equiv \text{const.}$ Then f is an open mapping.*

Proof. Fix an $a \in D$. By the identity principle, there exists an $X \in \mathbb{C}^n$ such that the function

$$S \ni \lambda \overset{g}{\mapsto} f(a + \lambda X)$$

is not constant, where S denotes the connected component of the set

$$\{\lambda \in \mathbb{C} : a + \lambda X \in D\}$$

that contains 0. Then g is an open mapping (cf. [Con 1973], Chapter IV, Theorem 7.5) and, consequently, $f(U)$ is open for any open neighborhood U of a. $\square$

The above proposition implies in particular the following

Proposition 1.7.21 (Maximum principle). *Let $D \subset \mathbb{C}^n$ be a domain and let $f \in \mathcal{O}(D)$, $f \not\equiv \text{const.}$ Then:*

(a) *$|f|$ does not attain local maxima in D.*

(b) *If, moreover, D is bounded, then*

$$|f(z)| < \sup\{\limsup_{D \ni z \to \zeta} |f(z)| : \zeta \in \partial D\}, \quad z \in D.$$

Lemma 1.7.22. *For any compact $K \subset \Omega$ and $r = (r_1, \dots, r_n)$ such that $K^{(r)} \subset \Omega$ we have*

$$\|f\|_K \le \frac{1}{(\pi r_1^2) \dots (\pi r_n^2)} \int_{K^{(r)}} |f| \, d\Lambda_{2n}, \quad f \in \mathcal{O}(\Omega).$$

Observe that $(\pi r_1^2) \dots (\pi r_n^2) = \Lambda_{2n}(\mathbb{P}(r))$.

Proof. Fix an $f \in \mathcal{O}(\Omega)$. It suffices to prove that

$$f(a) = \frac{1}{(\pi r_1^2) \dots (\pi r_n^2)} \int_{\mathbb{P}(a,r)} f \, d\Lambda_{2n}, \quad a \in K.$$

By the Cauchy integral formula, for every $a \in K$ we have

$$\begin{aligned}
&(\tfrac{1}{2} r_1^2) \dots (\tfrac{1}{2} r_n^2) f(a) \\
&\quad = \Big(\int_{[0,r]} \tau^{\mathbf{1}} \, d\Lambda_n(\tau) \Big) \Big(\frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} f(a + \tau \cdot e^{i\theta}) d\Lambda_n(\theta) \Big) \\
&\quad = \frac{1}{(2\pi)^n} \int_{\mathbb{P}(a,r)} f \, d\Lambda_{2n}.
\end{aligned}$$

$\square$

Lemma 1.7.23. *Assume that a family $\mathcal{F} \subset \mathcal{O}(\Omega)$ is locally uniformly bounded in Ω. Then $\mathcal{F}$ is equicontinuous.*

Proof. Fix a $\mathbb{P}(a,r) \Subset \Omega$. Set $C := \sup_{f\in\mathcal{F}} \|f\|_{\mathbb{P}(a,r)}$. Now, using (1.7.4), we get

$$|f(z) - f(a)| \le \frac{2C}{r}(|z_1 - a_1| + \cdots + |z_n - a_n|), \quad f \in \mathcal{F},\ z \in \mathbb{P}(a,r). \qquad \square$$

Having Lemma 1.7.23, the reader is asked to repeat the proof of the classical (one-dimensional) Montel theorem (cf. [Con 1973], Chapter VII, theorem 2.9) to obtain

Theorem 1.7.24 (Montel theorem). *Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ be a family locally uniformly bounded in Ω. Then for arbitrary sequence $(f_\nu)_{\nu=1}^\infty \subset \mathcal{F}$ there exists a subsequence which converges locally uniformly to a holomorphic function on Ω.*

Theorem 1.7.25 (Vitali theorem). *Let $D \subset \mathbb{C}^n$ be a domain and let a sequence $(f_\nu)_{\nu=1}^\infty \subset \mathcal{O}(D)$ be locally uniformly bounded and pointwise convergent on a non-empty open subset $U \subset D$. Then the sequence $(f_\nu)_{\nu=1}^\infty$ is convergent locally uniformly in D.*

Proof. (The reader is asked to complete details.) Similarly as in the case of one complex variable (cf. [Con 1973], Chapter VII), the main difficulty is to show that the sequence $(f_\nu)_{\nu=1}^\infty$ is pointwise convergent in all of D. Let

$$D_0 := \{a \in D : (f_\nu)_{\nu=1}^\infty \text{ is pointwise convergent in a neighborhood of } a\}.$$

The set D_0 is non-empty and open. It is sufficient to show that it is closed in D. Fix an accumulation point $b \in D$ of D_0. Let $\mathbb{P}(b,r) \subset D$. For $a \in D_0 \cap \mathbb{P}(b,r)$ and $X \in \mathbb{C}^n$, $X \neq 0$, let $S_{a,X}$ be the connected component of

$$D \cap \{a + \lambda X : \lambda \in \mathbb{C}\}$$

with $0 \in S_{a,X}$. By the classical one-dimensional Vitali theorem, the sequence $(f_\nu)_{\nu=1}^\infty$ is pointwise convergent in $S_{a,X}$ and, consequently, in $\bigcup_{\substack{a\in D_0\cap\mathbb{P}(b,r)\\ X\in(\mathbb{C}^n)_*}} S_{a,X}$. It remains to observe that the latter set is a neighborhood of b. $\square$

A bijective holomorphic mapping $f : \Omega \to \Omega'$ (where Ω and Ω' are open in $\mathbb{C}^n$) is called *biholomorphic* ($f \in \operatorname{Bih}(\Omega, \Omega')$) if f^{-1} is also holomorphic.

Using the classical inverse mapping theorem (in $\mathbb{R}^{2n}$) and Exercise 1.3.11, we get

Theorem 1.7.26 (Inverse mapping theorem). *Let $f = (f_1, \dots, f_n) : \Omega \to \mathbb{C}^n$ be a holomorphic mapping with*

$$Jf(a) := \det\left[\frac{\partial f_j}{\partial z_k}(a)\right]_{j,k=1,\dots,n} \neq 0$$

for some $a \in \Omega$. Then there exists an open neighborhood U of a ($U \subset \Omega$) such that $f(U)$ is an open set and $f|_U : U \to f(U)$ is biholomorphic.

Recall (cf. [Con 1973]) that in the case $n = 1$, for a holomorphic mapping $f : \Omega \to \mathbb{C}$, the following conditions are equivalent:

- $f(\Omega)$ is open and $f : \Omega \to f(\Omega)$ is biholomorphic (conformal);
- f is injective and $f'(z) \neq 0$, $z \in \Omega$;
- f is injective.

Notice that the result remains true for $n \geq 2$ (with a much more difficult proof).

Theorem* 1.7.27 (Cf. [Nar 1971], p. 86). *Let $\Omega \subset \mathbb{C}^n$ be open and let $f = (f_1, \dots, f_n) : \Omega \to \mathbb{C}^n$ be holomorphic. Then the following conditions are equivalent:*

(i) *$f(\Omega)$ is open and $f : \Omega \to f(\Omega)$ is biholomorphic;*

(ii) *f is injective and $Jf(z) \neq 0$, $z \in \Omega$;*

(iii) *f is injective.*

Theorem 1.7.28 (Hurwitz-type theorem). *Let $\Omega \subset \mathbb{C}^n$ be open, $a \in \Omega$, and let $f, f_k : \Omega \to \mathbb{C}^n$, $k \in \mathbb{N}$, be holomorphic mappings with $f_k \to f$ uniformly on Ω. Assume that $f(a) = 0$ and* $\det f'(a) \neq 0$. *Then there exist an open neighborhood $U \subset \Omega$ of a and a $k_0 \in \mathbb{N}$ such that $0 \in f_k(U)$, $k \geq k_0$.*

Proof. (The reader is asked to complete details.) First observe that the proof of the inverse mapping theorem (in the real case) implies the following:

Let $g : \Omega \to \mathbb{C}^n$ be a holomorphic mapping with $\det g'(a) \neq 0$ and let $r > 0$ be such that

$$\det g'(z) \neq 0, \quad \|g'(z) - g'(a)\| \leq \frac{1}{2} \frac{1}{\|(g'(a))^{-1}\|}, \quad z \in \overline{\mathbb{B}}(a, r) \subset \Omega.$$

Then $\mathbb{B}(g(a), \rho) \subset g(\mathbb{B}(a, r))$ with

$$\rho := \frac{r}{2} \frac{1}{\|g'(a)\|}.$$

Using the above remark and the Weierstrass Theorem 1.7.19, we find $r, s > 0$, and $k_0 \in \mathbb{N}$ such that $\mathbb{B}(g(a), s) \subset g(\mathbb{B}(a, r))$ for $g \in \{f, f_{k_0+1}, f_{k_0+2}, \dots\}$. Since $0 = \|f(a)\| < s$, we may assume that $\|f_k(a)\| < s$ for $k \geq k_0$, which shows that $0 \in \mathbb{B}(f_k(a), s) \subset f(\mathbb{B}(a, r))$ for $k \geq k_0$. $\square$

1.8 Balanced domains

Sometimes it is convenient to consider a wider class of domains than complete Reinhardt ones.

Definition 1.8.1. We say that a domain $D \subset \mathbb{C}^n$ is *balanced* (*complete circular*) if $\lambda z \in D$ for every $z \in D$ and $\lambda \in \overline{\mathbb{D}}$.

Observe that every balanced domain is starlike. Let h_D denote the Minkowski function of D (cf. Definition 1.4.14).

Exercise 1.8.2. (a) (Cf. Exercise 1.4.16.) Let $D \subset \mathbb{C}^n$ be a balanced domain and let $h\colon \mathbb{C}^n \to \mathbb{R}_+$. Then the following conditions are equivalent:

(i) $h = h_D$;

(ii) h is upper semicontinuous, $D = \{z \in \mathbb{C}^n : h(z) < 1\}$, and

$$h(\lambda z) = |\lambda| h(z), \quad z \in \mathbb{C}^n,\ \lambda \in \mathbb{C};$$

(b) Let $q\colon \mathbb{C}^n \to \mathbb{R}_+$ be a $\mathbb{C}$-seminorm (cf. § 1.10) and let $B := \{z \in \mathbb{C}^n : q(z) < 1\}$. Then $h_B = q$.

Lemma 1.8.3. *Let $D \subset \mathbb{C}^n$ be a complete Reinhardt domain.*[29] *Then*

$$h_D(\lambda \cdot z) \le h_D(z), \quad z \in \mathbb{C}^n,\ \lambda \in \overline{\mathbb{D}}^n \tag{1.8.1}$$

(in particular, $h_D(\lambda \cdot z) = h_D(z)$, $z \in \mathbb{C}^n$, $\lambda \in \mathbb{T}^n$) and h_D is continuous.

Consequently, if $h\colon \mathbb{C}^n \to \mathbb{R}_+$ is an upper semicontinuous function such that

- *$h(\lambda z) = |\lambda| h(z)$, $z \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$,*
- *$h(\lambda \cdot z) \le h(z)$, $z \in \mathbb{C}^n$, $\lambda \in \overline{\mathbb{D}}^n$,*

then h must be continuous.

Proof. The proof of (1.8.1) is left as an Exercise. To prove that h_D is continuous it suffices to show that h_D is lower semicontinuous at any point $a \in \mathbb{C}^n$ such that $h_D(a) > 0$. Fix such an $a = (a_1, \dots, a_n)$. We may assume that $a_1 \cdots a_s \ne 0$, $a_{s+1} = \cdots = a_n = 0$ for some $1 \le s \le n$. Fix a $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, put $m := \min\{|z_j/a_j| : j = 1, \dots, s\}$, and let $\lambda_j \in \overline{\mathbb{D}}$ be such that $\lambda_j z_j / a_j = m$, $j = 1, \dots, s$. Then

$$\begin{aligned} m h_D(a) &= h_D(m a_1, \dots, m a_s, 0, \dots, 0) \\ &= h_D(\lambda_1 z_1, \dots, \lambda_s z_s, 0 z_{s+1}, \dots, 0 z_n) \le h_D(z). \end{aligned}$$

Consequently,

$$\min\{|z_j/a_j| : j = 1, \dots, s\} \cdot h_D(a) \le h_D(z), \quad z \in \mathbb{C}^n,$$

which implies that $\liminf_{z \to a} h_D(z) = h_D(a)$. □

[29] In particular, D is balanced.

Proposition 1.8.4. *Let $D \subset \mathbb{C}^n$ be a balanced domain and let $f \in \mathcal{O}(D)$. Then*

$$f(z) = \sum_{k=0}^{\infty} Q_k(z), \quad z \in D, \tag{1.8.2}$$

where

$$Q_k(z) := \frac{1}{k!} f^{(k)}(0)(z) = \sum_{\alpha \in \mathbb{Z}_+^n:\, |\alpha|=k} \frac{1}{\alpha!} D^\alpha f(0) z^\alpha, \quad z \in \mathbb{C}^n;$$

observe that $Q_k : \mathbb{C}^n \to \mathbb{C}$ is a homogeneous polynomial of degree k. Moreover, for any compact $K \subset D$ there exist $C > 0$ and $\theta \in (0,1)$ such that

$$\|Q_k\|_K \le C\theta^k, \quad k \in \mathbb{Z}_+.$$

In particular, the series converges locally normally in D.

Proof. Take an $a \in D \setminus \{0\}$. The function

$$K(1/h_D(a)) \ni \lambda \overset{\varphi_a}{\longmapsto} f(\lambda a)$$ [30]

is holomorphic. Hence

$$f(a) = \varphi_a(1) = \sum_{k=0}^{\infty} \frac{1}{k!} \varphi_a^{(k)}(0) = \sum_{k=0}^{\infty} Q_k(a).$$

Thus the formula (1.8.2) is true (and the series is pointwise convergent in D). It remains to prove the estimate.

Take a compact $K \subset D$. Let $\theta \in (0,1)$ be such that

$$L := \{\lambda z : |\lambda| \le 1/\theta,\ z \in K\} \subset D.$$

Then, for any $a \in K$, by the one-dimensional Cauchy inequalities, we get

$$|Q_k(a)| = \frac{1}{k!} |\varphi_a^{(k)}(0)| \le \|\varphi_a\|_{K(1/\theta)} \theta^k \le \|f\|_L \theta^k, \quad k \in \mathbb{Z}_+. \qquad \square$$

Exercise 1.8.5. Let $f \in \mathcal{O}(\mathbb{C}^n)$ be such that $f(a', \cdot)$ is a polynomial for every $a' \in \mathbb{C}^{n-1}$. Prove that f is a polynomial.

Hint. Write $f(z) = \sum_{k=0}^{\infty} P_k(z)$, $z \in \mathbb{C}^n$, where P_k is a homogeneous polynomial of degree k. We have to show that there exists a k_0 such that int $P_k^{-1}(0) \neq \varnothing$ for $k \ge k_0$. Define $A'_k := \{a' \in \mathbb{C}^{n-1} : \forall_{\ell \ge k} :\ P_\ell(a', \cdot) \equiv 0\}$. Then A'_k is closed and $A'_k \nearrow \mathbb{C}^{n-1}$. Hence, by Baire's theorem,[31] there exists a k_0 with int $A'_{k_0} \neq \varnothing$.

[30] $K(1/0) := \mathbb{C}$.

[31] **Theorem** (Baire). *Let (X, ρ) be a complete metric space. Assume that $X = \bigcup_{k=1}^{\infty} A_k$. Then there exists a k_0 such that* int $\overline{A}_{k_0} \neq \varnothing$.

1.9 Extension of holomorphic functions

We move to problems related to extendibility of holomorphic function.

Theorem 1.9.1 (Hartogs extension theorem). *Let $D \subset \mathbb{C}^n$ be a domain, $n \geq 2$, and let $K \subset D$ be a compact set such that $D \setminus K$ is connected. Then $\mathcal{O}(D \setminus K) = \mathcal{O}(D)|_{D \setminus K}$, i.e. any function $f \in \mathcal{O}(D \setminus K)$ extends holomorphically to D.*

Notice that the above result does not hold for $n = 1$, e.g. $f(z) := 1/z$, $z \in \mathbb{C}_*$.

Proof. First consider a special case where $D = D' \times K(r)$. Suppose that $K \subset K' \times \overline{K}(\theta_0 r)$, where $K' \Subset D'$ and $0 < \theta_0 < 1$. Fix a function $f \in \mathcal{O}(D \setminus K)$ and define

$$\tilde{f}(z) := \frac{1}{2\pi i} \int_{\partial K(\theta r)} \frac{f(z', \zeta)}{\zeta - z_n} \, d\zeta, \quad z = (z', z_n) \in D' \times K(\theta r), \ \theta_0 < \theta < 1.$$

By the Cauchy theorem (cf. [Con 1973], Chapter IV, Theorem 5.7), $\tilde{f}(z)$ is independent of $\theta \in (\theta_0, 1)$ with $z_n \in K(\theta r)$. By Lemma 1.7.7, $\tilde{f} \in \mathcal{O}(D)$. Observe that $\tilde{f}(z', z_n) = f(z', z_n)$ if $z' \in D' \setminus K'$. Hence, by the identity principle, $\tilde{f} = f$ in $D \setminus K$.

Sketch of the general case (details are left to the reader, cf. e.g. [Sob 2003]): Fix an $f \in \mathcal{O}(D \setminus K)$. Consider the family $\mathfrak{F}$ of all pairs (C, Ω), where

- $C \Subset \mathrm{pr}_{\mathbb{C}^{n-1}}(D)$ is a convex domain.
- $\Omega = G_1 \cup \dots \cup G_N \subset \mathbb{C}$ is an open subset being a finite union of domains such that ∂G_j is a finite union of Jordan $\mathcal{C}^1$-curves with positive orientation with respect to G_j (cf. Remark 1.7.4), $j = 1, \dots, N$, and $\overline{G}_j \cap \overline{G}_k = \varnothing$ for $j \neq k$.
- $K \cap (C \times \mathbb{C}) \subset C \times \Omega \Subset D$. Define

$$\tilde{f}_{C,\Omega}(z) := \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(z', \zeta)}{\zeta - z_n} \, d\zeta, \quad z = (z', z_n) \in C \times \Omega \in \mathfrak{F}.$$

  Then $\tilde{f}_{C,\Omega} \in \mathcal{O}(C \times \Omega)$ and $\tilde{f}_{C,\Omega} = f|_{C \times \Omega}$ if $C \subset p(D) \setminus p(K)$. It is clear that $\bigcup_{(C,\Omega) \in \mathfrak{F}} C \times \Omega = D \setminus K$. It remains to observe that the family $(\tilde{f}_{C,\Omega})_{(C,\Omega) \in \mathfrak{F}}$ defines one function in $D \setminus K$. □

See [Jar-Pfl 2000], Theorem 2.6.6, for a different proof based on the $\bar{\partial}$-techniques.

Corollary 1.9.2. *For $n \geq 2$ the zeros of holomorphic functions are not isolated.*

Proof. Suppose that $f \in \mathcal{O}(\mathbb{P}(a, r))$, $n \geq 2$, $f(a) = 0$, and $f(z) \neq 0$ for $z \neq a$. Then, by Hartogs' extension theorem, the function $1/f$ would extend holomorphically onto $\mathbb{P}(a, r)$; a contradiction. □

Notice the fundamental difference between the cases $n = 1$ and $n \geq 2$. This is one of the main reasons why the theory of several complex variables is not a straightforward generalization of the one-dimensional case.

Definition 1.9.3. A set $M \subset \mathbb{C}^n$ is called *thin* if for every point $a \in M$ there exist a polydisc $\mathbb{P}(a, r)$ and a function $\varphi \in \mathcal{O}(\mathbb{P}(a, r))$, $\varphi \not\equiv 0$, such that $M \cap \mathbb{P}(a, r) \subset \varphi^{-1}(0)$.

Remark 1.9.4. (a) If M is thin, then $\operatorname{int} M = \varnothing$.

(b) If M is thin and $N \subset M$, then N is thin.

(c) If M_1, M_2 are thin, then $M_1 \cup M_2$ is thin.

(d) If $\varphi \in \mathcal{O}(D)$, $\varphi \not\equiv 0$, where $D \subset \mathbb{C}^n$ is a domain, then $\varphi^{-1}(0)$ is thin. In particular, $V_0, V_1, \dots, V_n$ are thin.

(e) If M is thin, then $M \times \mathbb{C}^m$ is thin.

Lemma 1.9.5. *Let $\varphi \in \mathcal{O}(\mathbb{P}(r))$, $\varphi(0) = 0$, $\varphi \not\equiv 0$. Then, after a suitable linear change of coordinates, we have $\varphi(0', \cdot) \not\equiv 0$.*

Proof. By Theorem 1.7.6, the function φ may be expanded into a series of homogeneous polynomials

$$\varphi(z) = T_0\varphi(z) = \sum_{j=0}^{\infty} \Big(\sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha \varphi(0) z^\alpha \Big) = \sum_{j=k}^{\infty} Q_j(z), \quad z \in \mathbb{P}(r),$$

with $Q_k \not\equiv 0$ (see also Proposition 1.8.4). In particular, the set $V := Q_k^{-1}(0)$ is thin. Observe that for every $X \notin V$, $\|X\| = 1$, the function

$$K(r) \ni \lambda \xmapsto{\Phi_X} \varphi(\lambda X)$$

is not identically zero. Consequently, after a linear change of coordinates $L\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $L(e_n) = X$, we have $(\varphi \circ L)(0', z_n) = \varphi(L(z_n e_n)) = \varphi(z_n X) = \Phi_X(z_n)$. $\square$

Exercise 1.9.6. Let $\varphi_k \in \mathcal{O}(\mathbb{P}(r))$, $\varphi_k(0) = 0$, $\varphi_k \not\equiv 0$, $k \in \mathbb{N}$. Then, after a suitable linear change of coordinates, we have $\varphi_k(0', \cdot) \not\equiv 0$, $k \in \mathbb{N}$.

Hint. Use Baire's theorem.

Proposition 1.9.7. *Let $D \subset \mathbb{C}^n$ be a domain and let $M \subset D$ be a thin set. Then the set $D \setminus M$ is connected.*

Proof. First observe that it suffices to prove that every point $a \in D$ has a convex neighborhood $U_a \subset D$ such that $U_a \setminus M$ is arcwise connected (cf. Remark 1.5.6 (c)).

Indeed, suppose for a moment that this is true and take arbitrary two different points $a, b \in D \setminus M$. Let $\gamma\colon [0, 1] \to D$ be an arbitrary curve with $\gamma(0) = a$,

$\gamma(1) = b$. For every $t \in [0,1]$ the point $\gamma(t)$ has a convex neighborhood $U_{\gamma(t)}$ such that $U_{\gamma(t)} \setminus M$ is connected. One can select a chain of neighborhoods $U_{\gamma(t_0)}, \dots, U_{\gamma(t_N)}$, $0 = t_0 < \dots < t_N = 1$, $U_{\gamma(t_{i-1})} \cap U_{\gamma(t_i)} \neq \varnothing$, $i = 1, \dots, N$. Fix arbitrary points $c_i \in U_{\gamma(t_{i-1})} \cap U_{\gamma(t_i)} \setminus M$, $i = 1, \dots, N$. Now we connect a with c_1 in $U_{\gamma(t_0)} \setminus M$. Next, we connect c_1 with c_2 in $U_{\gamma(t_1)} \setminus M$, etc. Finally, we connect c_N with b in $U_{\gamma(t_N)} \setminus M$.

Fix an $a \in D$. We may assume that $a = 0$ and that $\mathbb{P}(r) \cap M \subset \varphi^{-1}(0)$, where $\mathbb{P}(r) \Subset D$, $\varphi \in \mathcal{O}(\overline{\mathbb{P}}(r))$, $\varphi \not\equiv 0$. Using Lemma 1.9.5, we easily reduce the situation (Exercise) to the case where $\varphi(0', \cdot) \not\equiv 0$, $\varphi(0', z_n) \neq 0$ for $0 < |z_n| \leq r_n$, and $\varphi(z', z_n) \neq 0$ for $z' \in \overline{\mathbb{P}}(r')$, $s_n \leq |z_n| \leq r_n$ for some $0 < s_n < r_n$.

Observe that for every $z' \in \mathbb{P}(r')$, the function $\varphi(z', \cdot)$ has a finite number of zeros in $K(r_n)$ and, consequently, the fiber $F_{z'} := \{z_n \in K(r_n) : (z', z_n) \notin M\}$ is connected. Fix $\zeta \in \mathbb{A}(s_n, r_n)$.

Take two points $u = (u', u_n)$, $v = (v', v_n) \in \mathbb{P}(r) \setminus M$. First we connect $u = (u', u_n)$ with (u', ζ) in the fiber $F_{u'}$. Next, we connect (u', ζ) with (v', ζ) by a segment (which is obviously contained in $\mathbb{P}(r) \setminus M$), and finally, we connect (v', ζ) with $v = (v', v_n)$ in the fiber $F_{v'}$. $\square$

The classical Riemann theorem on removable singularities (cf. [Con 1973], Chapter V, Theorem 3.8) generalizes to several complex variables as follows.

Theorem 1.9.8 (Riemann removable singularities theorem). *Let D be a domain in $\mathbb{C}^n$ and let $M \subset D$ be thin and closed in D. Then every function $f \in \mathcal{O}(D \setminus M)$ which is locally bounded in D (i.e. every point $a \in D$ has a neighborhood U_a such that f is bounded in $U_a \setminus M$) extends holomorphically to D.*

Proof. Fix a function $f \in \mathcal{O}(D \setminus M)$ such that f is locally bounded on D. Observe that the problem of continuation across M is local.

In fact, if every point $a \in D$ admits a convex neighborhood U_a and a function $\tilde{f}_a \in \mathcal{O}(U_a)$ such that $\tilde{f}_a = f$ in $U_a \setminus M$, then by Remark 1.9.4 (a), the function $\tilde{f}$ defined as $\tilde{f} := \tilde{f}_a$ in U_a gives the required extension.

Fix an $a \in D$. We may assume (cf. the proof of Proposition 1.9.7) that $a = 0 \in M$ and $M \cap \mathbb{P}(r) \subset \varphi^{-1}(0)$, where $\mathbb{P}(r) \Subset D$, $\varphi \in \mathcal{O}(\overline{\mathbb{P}}(r))$, and $\varphi(0', z_n) \neq 0$, $0 < |z_n| \leq r_n$. Suppose that $\varphi(0', \cdot)$ has zero of order p at $z_n = 0$ ($p \in \mathbb{N}$).

Let $\varepsilon := \min\{|\varphi(0', z_n)| : |z_n| = r_n\}$. Shrinking r' (with fixed r_n) we may assume that $|\varphi(z', z_n) - \varphi(0', z_n)| < \varepsilon$ for $z' \in \mathbb{P}(r')$, $|z_n| = r_n$. Now, by the Rouché theorem (cf. [Con 1973], Chapter V, Theorem 3.8), for every $z' \in \mathbb{P}(r')$ the function $\varphi(z', \cdot)$ has exactly p zeros (counted with multiplicities) in the disc $K(r_n)$, say $\xi_1(z'), \dots, \xi_p(z')$. Note that $\varphi(z', z_n) \neq 0$, $z_n \in \partial K(r_n)$. In particular, for every $z' \in \mathbb{P}(r')$ the function $f(z', \cdot)$ is holomorphic in $\overline{K}(r_n) \setminus \{\xi_1(z'), \dots, \xi_p(z')\}$ and locally bounded in $K(r_n)$. Hence, by the classical (one-dimensional) Riemann theorem on removable singularities, $f(z', \cdot)$ extends holomorphically to a function $\widetilde{f(z', \cdot)} \in \mathcal{O}(K(r_n))$. Let $\tilde{f}(z', z_n) := \widetilde{f(z', \cdot)}(z_n)$, $(z', z_n) \in \mathbb{P}(r)$. By the

Cauchy integral formula, we have

$$\tilde{f}(z', z_n) = \frac{1}{2\pi i} \int_{\partial K(r_n)} \frac{f(z', \zeta)}{\zeta - z_n}\, d\zeta, \quad (z', z_n) \in \mathbb{P}(r).$$

By Lemma 1.7.7, $\tilde{f} \in \mathcal{O}(\mathbb{P}(r))$. It is clear that $\tilde{f} = f$ in $\mathbb{P}(r) \setminus M$. □

Corollary 1.9.9. *Suppose that $D \subset \mathbb{C}^n$ is a log-convex Reinhardt domain. Then $\mathcal{H}^\infty(D^*)|_D = \mathcal{H}^\infty(D)$ (cf. (1.5.3)), where $\mathcal{H}^\infty(\Omega)$ denotes the space of all bounded holomorphic functions on Ω (it is a Banach algebra with the supremum norm – cf. Example 1.10.7* (c)*). More precisely, the restriction mapping*

$$\mathcal{H}^\infty(D^*) \ni f \mapsto f|_D \in \mathcal{H}^\infty(D)$$

is an algebraic and topological isomorphism (cf. Proposition 1.9.12).

Exercise 1.9.10. Observe that the above Riemann theorem gives an alternative proof of Proposition 1.9.7 for the case where M is relatively closed.

The next results present a class of thin sets $M \subset D$ such that *every* function holomorphic in $D \setminus M$ extends to D.

Proposition 1.9.11. *Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a domain and let $M \subset D$ be closed in D. Assume that for every $a \in M$ there exist an open neighborhood $U \subset D$ and $\varphi_1, \varphi_2 \in \mathcal{O}(U)$ for which $M \cap U \subset \varphi^{-1}(0) \cap \varphi_2^{-1}(0)$ and*

$$\operatorname{rank}\left[\frac{\partial \varphi_j}{\partial z_k}(z)\right]_{j=1,2,\ k=1,\dots,n} = 2, \quad z \in U.^{32}$$

Then every function $f \in \mathcal{O}(D \setminus M)$ extends holomorphically to D.

Proof. As in the Riemann theorem, it suffices to extend f locally. Fix an $a \in M$ and let U, φ_1, φ_2 be as above. We may assume that $a = 0$ and

$$\det\left[\frac{\partial \varphi_j}{\partial z_k}(0)\right]_{j,k=1,2} \neq 0.$$

Consider the mapping

$$U \ni z \overset{\Phi}{\mapsto} (\varphi_1(z), \varphi_2(z), z_3, \dots, z_n).$$

Then $J\Phi(0) \neq 0$ and, consequently, by the inverse mapping theorem (Theorem 1.7.26), we may assume (shrinking U if necessary) that $\Phi\colon U \to \Phi(U) =: V$ is biholomorphic. Put

$$N := \{w \in V : w_1 = w_2 = 0\}, \quad g := f \circ \Phi^{-1}|_{V \setminus N}.$$

[32] For example, $M \subset V_j \cap V_k$ with $j \neq k$.

We only need to extend g to V. The case $n = 2$ follows directly from the Hartogs extension theorem (Theorem 1.9.1).

Thus we may assume that $n \geq 3$, $U = \mathbb{P}(r) \Subset D$, $\varphi_j(z) = z_j$, $z \in U$, $j = 1, 2$. Write $z = (z', z'') \in \mathbb{C}^2 \times \mathbb{C}^{n-2}$. For each $z'' \in \mathbb{P}(r'')$, the function $f(\cdot, z'')$ is holomorphic in $\mathbb{P}(r') \setminus \{0'\}$. By the Hartogs extension theorem, $f(\cdot, z'')$ extends to a function $\widetilde{f(\cdot, z'')} \in \mathcal{O}(\mathbb{P}(r'))$. Put $\tilde{f}(z) := \widetilde{f(\cdot, z'')}(z')$, $z = (z', z'') \in \mathbb{P}(r)$. It remains to prove that $\tilde{f} \in \mathcal{O}(\mathbb{P}(r))$. Observe that

$$\tilde{f}(z) = \frac{1}{(2\pi i)^2} \int_{\partial_0 \mathbb{P}(r')} \frac{f(\zeta, z'')}{\zeta - z'}\, d\zeta, \quad z = (z', z'') \in \mathbb{P}(r).$$

Hence, by Lemma 1.7.7, $\tilde{f} \in \mathcal{O}(\mathbb{P}(r))$. □

Now, Corollary 1.9.9 may be extended to the following more general result.

Proposition 1.9.12. *Let $D \subset \mathbb{C}^n$ be a log-convex Reinhardt domain. Put $M := D^* \setminus D \subset V_0$. Define M^r to be the set of all $a = (a_1, \dots, a_n) \in M$ such that there exists exactly one $j \in \{1, \dots, n\}$ with $a_j = 0$. Let $f \in \mathcal{O}(D)$ be a* function of slow growth *near M^r, i.e. every point $a \in M^r$ has an open neighborhood U_a such that $U_a \setminus V_0 \subset D$ and*

$$(\operatorname{dist}(z, \partial D))^N |f(z)| \leq C, \quad z \in U_a \setminus V_0$$ [33]

for some constants $0 \leq N < 1$ and $C > 0$, which may depend on f and a. Then f extends holomorphically to D^* (cf. Corollary 1.11.4).

Proof. Fix $a \in M^r$, U_a, N, and C as above. We may assume that $a_n = 0$, $U_a = U' \times U_n \subset \mathbb{C}_*^{n-1} \times \mathbb{C}$, and $\operatorname{dist}(z, \partial D) \geq |z_n|$, $z = (z', z_n) \in U_a$, $z_n \neq 0$. Write

$$f(z', z_n) = \sum_{k=-\infty}^{\infty} f_k(z') z_n^k, \quad z = (z', z_n) \in U_a = U' \times U_n,\ z_n \neq 0.$$

By the Cauchy inequalities we get $|f_k(z')| \leq C|z_n|^{-N-k}$, $(z', z_n) \in U_a$, $z_n \neq 0$. Letting $z_n \to 0$, we conclude that $f_k \equiv 0$ for $k < 0$. Thus, for every $z' \in U'$, the function $f(z', \cdot)$ extends holomorphically to $U_n = K(r_n)$. By the Cauchy integral formula, the extension is given by the formula

$$\tilde{f}(z) = \frac{1}{2\pi i} \int_{\partial K(s_n)} \frac{f(z', \zeta)}{\zeta - z_n} d\zeta, \quad z = (z', z_n) \in U' \times K(s_n),\ 0 < s_n < r_n.$$

Using Lemma 1.7.7, we conclude that $\tilde{f}$ is holomorphic in U_a.

Consequently, f extends holomorphically to the domain $D^* \setminus M^s$, where $M^s := M \setminus M^r$. Now, by Proposition 1.9.11, we conclude that f extends holomorphically to D^*. □

[33] $\operatorname{dist}(z, A) := \inf\{\|z - \zeta\| : \zeta \in A\}$, $z \in \mathbb{C}^n \supset A$.

Remark* 1.9.13. Proposition 1.9.11 remains true in a much more general context, namely for analytic sets $M \subset D$ with $\dim M \le n-2$.

A set $M \subset \Omega$ is an *analytic subset of* Ω if for any point $a \in \Omega$ there exist a neighborhood $U_a \subset \Omega$ and a finite family $\mathcal{F}_a \subset \mathcal{O}(U_a)$ such that $M \cap U_a = \bigcap_{f \in \mathcal{F}_a} f^{-1}(0)$. Note that M is closed in Ω (Exercise).

A point $a \in M$ is *regular* ($a \in \operatorname{Reg}(M)$) if there exists a neighborhood $U_a \subset \Omega$ such that $M \cap U_a$ is a complex manifold.[34] Points from $\operatorname{Sing}(M) := M \setminus \operatorname{Reg}(M)$ are called *singular*. Observe that if $n = 1$, then $\operatorname{Sing}(M) = \varnothing$.

Obviously, the set $\operatorname{Reg}(M)$ is open in M and $\operatorname{Sing}(M)$ is closed in Ω. One can prove that (all details may be found e.g. in [Chi 1989]):

- $\dim M = 0$ iff M is discrete.
- The set $\operatorname{Reg}(M)$ is dense in M and, consequently, the set $\operatorname{Sing}(M)$ is nowhere dense in M. Thus, we can define the *dimension of* M *at a point* $a \in M$: $\dim_a M := \limsup_{\operatorname{Reg}(M) \ni z \to a} \dim_z M$ and the (global) *dimension of* M: $\dim M := \max_{a \in M} \dim_a M$.
- $\operatorname{Sing}(M)$ is an analytic subset of Ω and $\dim_z \operatorname{Sing}(M) < \dim_z M$, $z \in \operatorname{Sing}(M)$.

Now, we come back to a generalization of Proposition 1.9.11.

Proposition* 1.9.14. *Let* M *be an analytic subset of a domain* $D \subset \mathbb{C}^n$ *such that* $\dim M \le n-2$. *Then* $\mathcal{O}(D \setminus M) = \mathcal{O}(D)|_{D \setminus M}$.

Proof. By Proposition 1.9.11 any function $f \in \mathcal{O}(D \setminus M)$ extends holomorphically to $D \setminus \operatorname{Sing}(M)$. Repeating the same procedure gives a holomorphic extension to $D \setminus \operatorname{Sing}(\operatorname{Sing}(M))$. Since $\dim \operatorname{Sing}(M) < \dim M$, the procedure leads after a finite number of steps to a holomorphic extension to $D \setminus N$ with $\dim N \le 0$. If $N \ne \varnothing$, then N is discrete and we apply (locally) the Hartogs extension theorem (Theorem 1.9.1). □

1.10 Natural Fréchet spaces

First, let us recall the following general definitions.

Let $\mathcal{F}$ be a complex vector space. A mapping $q : \mathcal{F} \to \mathbb{R}_+$ is a *seminorm* ($\mathbb{C}$-*seminorm*) if:

- $q(0) = 0$,

[34] A relatively closed subset N of an open set $U \subset \mathbb{C}^n$ is a *complex manifold* if:

- either N is an open subset of U (and, consequently, N is the union of a family of connected components of U; in this case we put $\dim_a N := n$, $a \in N$),
- or every point $a \in N$ has a neighborhood $V_a \subset U$ such that $N \cap V_a = \varphi^{-1}(0)$, where $\varphi \in \mathcal{O}(V_a, \mathbb{C}^{n-d})$ and $\operatorname{rank}\left[\frac{\partial \varphi_j}{\partial z_k}(z)\right]_{\substack{j=1,\dots,n-d \\ k=1,\dots,n}} = n-d$, $z \in V_a$ (in particular, in this case N is thin; we put $\dim_a N := d$).

Moreover, we put $\dim \varnothing := -1$.

- $q(\lambda f) = |\lambda| q(f), \quad \lambda \in \mathbb{C},\ f \in \mathcal{F}$,
- $q(f + g) \le q(f) + q(g), \quad f, g \in \mathcal{F}$.

Notice, any $\mathbb{C}$-norm $\|\ \|\colon \mathcal{F} \to \mathbb{R}_+$ is obviously a $\mathbb{C}$-seminorm.

Observe that for any finite family I of seminorms on $\mathcal{F}$, the function

$$\max I := \max\{q : q \in I\}$$

is also a seminorm. For any seminorm q let

$$B_q(f_0, r) := \{f \in \mathcal{F} : q(f - f_0) < r\}$$

be the *open ball centered at $f_0 \in \mathcal{F}$ with radius $r > 0$*.

Given a non-empty family Q of seminorms on $\mathcal{F}$, we introduce on $\mathcal{F}$ a *topology generated by Q*. Namely, we say that a set $U \subset \mathcal{F}$ is *open* if for every $f_0 \in U$ there exist a finite set $I \subset Q$ and an $r > 0$ such that

$$B_{\max I}(f_0, r) \subset U.$$

Directly from the above definition it follows that the family $\mathcal{T}(Q)$ of all open sets is a topology on $\mathcal{F}$ (Exercise).

We say that two families of seminorms Q_1, Q_2 on $\mathcal{F}$ are *equivalent* if $\mathcal{T}(Q_1) = \mathcal{T}(Q_2)$.

Below we collect (in form of an exercise) some basic properties of $\mathcal{T}(Q)$ (cf. [Sch 1970], [Trè 1967]).

Exercise 1.10.1. (a) For an arbitrary finite set $I \subset Q$, $f_0 \in \mathcal{F}$, and $r > 0$, the open ball $B_{\max I}(f_0, r)$ is open in $\mathcal{T}(Q)$.

(b) Let X be a topological space. A mapping $\varphi\colon X \to \mathcal{F}$ is continuous at a point $x_0 \in X$ iff for any $q \in Q$ and $\varepsilon > 0$ there exists a neighborhood $V \subset X$ of x_0 such that $\varphi(V) \subset B_q(\varphi(x_0), \varepsilon)$.

(c) Any seminorm $q \in Q$ is continuous in the topology $\mathcal{T}(Q)$.

(d) The addition $\mathcal{F} \times \mathcal{F} \ni (f, g) \mapsto f + g \in \mathcal{F}$ and multiplication $\mathbb{C} \times \mathcal{F} \ni (\lambda, f) \mapsto \lambda f \in \mathcal{F}$ are continuous.

(e) Let $\mathcal{F}_i$ be a complex vector space endowed with a topology $\mathcal{T}_i = \mathcal{T}(Q_i)$ generated by a family Q_i of seminorms on $\mathcal{F}_i$, $i = 1, 2$. Let $L\colon \mathcal{F}_1 \to \mathcal{F}_2$ be a $\mathbb{C}$-linear mapping. Then L is continuous iff

$$\forall_{q \in Q_2}\ \exists_{\substack{I \subset Q_1 \\ I \text{ finite}}}\ \exists_{C>0} :\ q \circ L \le C \max I.$$

(f) Two families of seminorms Q_1, Q_2 on $\mathcal{F}$ are equivalent iff

$$\forall_{q \in Q_i}\ \exists_{\substack{I \subset Q_{3-i} \\ I \text{ finite}}}\ \exists_{C>0} :\ q \le C \max I, \quad i = 1, 2.$$

(g) Any family Q of seminorms is equivalent to the family

$$\{\max I : I \subset Q,\ I \text{ finite}\}.$$

(h) For any countable family of seminorms there exists an equivalent countable family of seminorms $\{q_k : k = 1, 2, \dots\}$ such that $q_k \le q_{k+1}$, $k = 1, 2, \dots$.

(i) Any family Q of seminorms is equivalent to the following maximal family:

$$Q_{\max} := \{q : q \text{ is a continuous seminorm on } \mathcal{F} \text{ in the sense of } \mathcal{T}(Q)\}.$$

(j) The topology $\mathcal{T}(Q)$ is Hausdorff iff $\bigcap_{q \in Q} q^{-1}(0) = \{0\}$.

(k) If $Q = \{q_k : k = 1, 2, \dots\}$ is a countable family of seminorms with $\bigcap_{k=1}^{\infty} q_k^{-1}(0) = \{0\}$, then the topology $\mathcal{T}(Q)$ is given by the distance

$$\rho(f, g) = \rho_Q(f, g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{q_k(f - g)}{1 + q_k(f - g)}, \quad f, g \in \mathcal{F}. \tag{1.10.1}$$

(l) If Q is as in (k), then a sequence $(f_\nu)_{\nu=1}^{\infty} \subset \mathcal{F}$ is a Cauchy sequence in $(\mathcal{F}, \rho)$ (where ρ is given by (1.10.1)) iff

$$\forall_{\varepsilon > 0}\ \forall_{k \in \mathbb{N}}\ \exists_{\nu_0 \in \mathbb{N}}\ \forall_{\mu, \nu \ge \nu_0} : q_k(f_\mu - f_\nu) \le \varepsilon.$$

In particular, $(f_\nu)_{\nu=1}^{\infty} \subset \mathcal{F}$ remains a Cauchy sequence in $(\mathcal{F}, \rho')$, where ρ' is the distance corresponding to a sequence $Q' = \{q'_k : k = 1, 2, \dots\}$ with $\mathcal{T}(Q) = \mathcal{T}(Q')$.

(m)* The topology $\mathcal{T}(Q)$ is metrizable iff there exists an equivalent countable family of seminorms Q_0 such that $\bigcap_{q \in Q_0} q^{-1}(0) = \{0\}$.

Definition 1.10.2. Let $\mathcal{F}$ be a complex vector space endowed with the topology generated by a countable family of seminorms $Q = \{q_1, q_2, \dots\}$ with $\bigcap_{k=1}^{\infty} q_k^{-1}(0) = \{0\}$. We say that $\mathcal{F}$ is a *Fréchet space* if the metric space $(\mathcal{F}, \rho_Q)$ is complete (cf. Exercise 1.10.1 (k,ℓ)).

Definition 1.10.3. Let $\mathcal{F}$ be a Fréchet space with the topology $\mathcal{T} = \mathcal{T}(Q)$. A set $A \subset \mathcal{F}$ is said to be *bounded* if the set $q(A) \subset \mathbb{R}_+$ is bounded for any $q \in Q$.[35]

The following property of Fréchet spaces will play an important role in the sequel.

Theorem 1.10.4 (Banach theorem, cf. [Gof-Ped 1965], § 5.8). *Let $\mathcal{F}_1, \mathcal{F}_2$ be Fréchet spaces and let $L : \mathcal{F}_1 \to \mathcal{F}_2$ be an injective continuous linear mapping. Then either L is surjective (and then L^{-1} is also continuous) or the image $L(\mathcal{F}_1)$ is of the first Baire category in $\mathcal{F}_2$.*[36]

We will be only interested in special Fréchet spaces $\mathcal{F} \subset \mathcal{O}(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is open (cf. Chapter 3).

[35] Notice that this property is independent of the family Q with $\mathcal{T} = \mathcal{T}(Q)$.

[36] A subset A of a topological space X is said to be of the *first Baire category* if $A = \bigcup_{k=1}^{\infty} A_k$, where each set A_k is *nowhere dense*, i.e. $\operatorname{int} \overline{A}_k = \varnothing$.

Definition 1.10.5. Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ be a vector subspace endowed with a Fréchet space topology $\mathcal{T} = \mathcal{T}(Q)$. We say that $\mathcal{F}$ is a *natural Fréchet space* if for any sequence $(f_k)_{k=1}^{\infty} \subset \mathcal{F}$ and $f_0 \in \mathcal{F}$,

$$\text{if } f_k \to f_0 \text{ in the sense of } \mathcal{T}, \text{ then } f_k \to f_0 \text{ locally uniformly in } \Omega \tag{1.10.2}$$

(see also (1.10.4)). In the case where $\mathcal{F}$ is a Banach (resp. Hilbert) space, we say that $\mathcal{F}$ *is a natural Banach* (*resp. Hilbert*) *space*.

Remark 1.10.6. (a) Let $\mathcal{F} \subset \mathcal{O}(\Omega)$ be a vector subspace endowed with a Fréchet space topology $\mathcal{T} = \mathcal{T}(Q)$. Then $\mathcal{F}$ is a natural Fréchet space iff if for any sequence $(f_k)_{k=1}^{\infty} \subset \mathcal{F}$ and $f_0 \in \mathcal{F}$,

$$\text{if } f_k \to f_0 \text{ in the sense of } \mathcal{T}, \text{ then } f_k \to f_0 \text{ pointwise in } \Omega. \tag{1.10.3}$$

Indeed, suppose that (1.10.3) is satisfied. Let $\mathcal{T}' = \mathcal{T}(Q')$ denote the topology generated by the family $Q' := Q \cup Q''$, where Q'' stands for the family of all seminorms of the form

$$\mathcal{F} \ni f \mapsto \|f\|_K := \sup_K |f|, \quad K \Subset \Omega;$$

cf. Example 1.10.7 (a). In other words, $f_k \to f_0$ in $\mathcal{T}'$ iff $f_k \to f_0$ in $\mathcal{T}$ and $f_k \to f_0$ locally uniformly in Ω. Condition (1.10.3) guarantees that $(\mathcal{F}, \mathcal{T}')$ is a Fréchet space. The identity operator $\mathrm{id}\colon (\mathcal{F}, \mathcal{T}') \to (\mathcal{F}, \mathcal{T})$ is obviously a continuous bijection. Now, the Banach Theorem 1.10.4 implies that its inverse is continuous, which gives (1.10.2).

(b) $\boxed{?}$ Surprisingly, we do not know any example of a Fréchet space $(\mathcal{F}, \mathcal{T})$ with $\mathcal{F} \subset \mathcal{O}(\Omega)$ such that $\mathcal{F}$ is not natural. $\boxed{?}$

Many classical spaces of holomorphic functions have structures of natural Fréchet spaces.

Example 1.10.7 (Natural Fréchet spaces). The reader is asked to complete all details.

(a) The whole space $\mathcal{O}(\Omega)$ endowed with the topology τ_Ω of locally uniform convergence is a natural Fréchet space (cf. Theorem 1.7.19). More precisely, $\tau_\Omega := \mathcal{T}(Q)$, where Q is the following family of seminorms

$$\mathcal{O}(\Omega) \ni f \mapsto \|f\|_K := \sup_K |f|, \quad K \Subset \Omega.$$

Observe that Q is equivalent to every family $(\|\ \|_{K_j})_{j=1}^{\infty}$, where $(K_j)_{j=1}^{\infty}$ is an arbitrary sequence of compact subsets of Ω with $K_j \subset \operatorname{int} K_{j+1}$, $\bigcup_{j=1}^{\infty} K_j = \Omega$.

Notice that condition (1.10.2) means that the inclusion operator

$$(\mathcal{F}, \mathcal{T}) \to (\mathcal{O}(\Omega), \tau_\Omega) \tag{1.10.4}$$

is continuous.

(b) The space $\mathcal{H}^\infty(\Omega)$ of all *bounded holomorphic functions on* Ω endowed with the topology of uniform convergence (i.e. the topology induced by the supremum norm $\| \ \|_\Omega$) is a natural Banach space. Notice that in fact $\mathcal{H}^\infty(\Omega)$ is a Banach algebra.

(c) The space $L_h^p(\Omega) := \mathcal{O}(\Omega) \cap L^p(\Omega)$ of all *p-integrable holomorphic functions on* Ω endowed with the L^p-topology (i.e. the topology induced by the L^p-norm $\| \ \|_{L^p(\Omega)}$) is a natural Banach space, where $L^p(\Omega)$ is taken w.r.t. the Lebesgue measure Λ_{2n} in $\mathbb{C}^n$ ($1 \le p \le +\infty$). Obviously, $L_h^\infty(\Omega) = \mathcal{H}^\infty(\Omega)$.

To prove that $L_h^p(\Omega)$ is a natural Banach space we only need to show that the topology induced by $L^p(\Omega)$ on $L_h^p(\Omega)$ is stronger than the topology of locally uniform convergence. By Lemma 1.7.22, we get

$$\|f\|_K \le \frac{1}{(\pi r^2)^n} \int_{K^{(r)}} |f| \, d\Lambda_{2n}, \quad f \in \mathcal{O}(\Omega), \ K \Subset \Omega, \ 0 < r < d_\Omega(K). \tag{1.10.5}$$

Hence, by the Hölder inequality,

$$\|f\|_K \le \frac{\Lambda_{2n}^{1/q}(K^{(r)})}{(\pi r^2)^n} \|f\|_{L^p(\Omega)}, \quad f \in L_h^p(\Omega), \ K \Subset \Omega, \ 0 < r < d_\Omega(K),$$

where $1/p + 1/q = 1$.

- If D is a Reinhardt domain, $f \in L_h^p(D)$, $f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha^f z^\alpha$, then

$$\{z^\alpha : \alpha \in \Sigma(f)\} \subset L_h^p(D), \quad \|a_\alpha^f z^\alpha\|_{L^p(D)} \le \|f\|_{L^p(D)}, \quad \alpha \in \Sigma(f),$$

where $\Sigma(f) := \{\alpha \in \mathbb{Z}^n : a_\alpha^f \neq 0\}$. Indeed,

$$\begin{aligned}
&\int_D |a_\alpha^f z^\alpha|^p \, d\Lambda_{2n}(z) \\
&\overset{\text{Prop. 1.7.15}}{=} (2\pi)^n \int_{\boldsymbol{R}(D)} \left| \frac{1}{(2\pi i)^n} \int_{\partial_0 \mathbb{P}(r)} \frac{f(\zeta)}{\zeta^{\alpha+\mathbf{1}}} \, d\zeta \right|^p r^{p\alpha+\mathbf{1}} \, d\Lambda_n(r) \\
&\le (2\pi)^{n(1-p)} \int_{\boldsymbol{R}(D)} \left(\int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})| \, d\Lambda_n(\theta) \right)^p r^{\mathbf{1}} \, d\Lambda_n(r) \\
&\overset{\text{Hölder ineq.}}{\le} \int_{\boldsymbol{R}(D)} \int_{[0,2\pi]^n} |f(r \cdot e^{i\theta})|^p \, d\Lambda_n(\theta) r^{\mathbf{1}} \, d\Lambda_n(r) \\
&= \int_D |f|^p \, d\Lambda_{2n}, \quad \alpha \in \Sigma(f).
\end{aligned}$$

- The space $L_h^2(\Omega)$ with the scalar product

$$L^2(\Omega) \times L^2(\Omega) \ni (f, g) \mapsto \langle f, g \rangle_{L^2(\Omega)} := \int_\Omega f \bar{g} \, d\Lambda_{2n}$$

is a natural Hilbert space. Moreover, if $D \subset \mathbb{C}^n$ is a Reinhardt domain, then:

– The functions $\{z^\alpha : \alpha \in \mathbb{Z}^n,\ z^\alpha \in L^2_h(D)\}$ are pairwise orthogonal in $L^2_h(D)$.

– If $f \in L^2_h(D)$, $f(z) = \sum_{\alpha \in \mathbb{Z}^n} a^f_\alpha z^\alpha$, then

$$\|f\|^2_{L^2(D)} = \sum_{\alpha \in \Sigma(f)} \|a_\alpha z^\alpha\|^2_{L^2(D)}.$$

Indeed, if $z^\alpha, z^\beta \in L^2_h(D)$, then, using polar coordinates, we get

$$\begin{aligned} \langle z^\alpha, z^\beta \rangle_{L^2(D)} &= \int_D z^\alpha \bar{z}^\beta \, d\Lambda_{2n}(z) \\ &= \int_{\boldsymbol{R}(D)} r^{\alpha+\beta+\mathbf{1}} d\Lambda_n(r) \cdot \int_{[0,2\pi]^n} e^{i\langle \alpha-\beta,\theta\rangle} d\Lambda_n(\theta) \\ &= \int_{\boldsymbol{R}(D)} r^{\alpha+\beta+\mathbf{1}} d\Lambda_n(r) \cdot \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ (2\pi)^n & \text{if } \alpha = \beta. \end{cases} \end{aligned}$$

Recall that the Laurent series $\sum_{\alpha \in \mathbb{Z}^n} a^f_\alpha z^\alpha$ is locally uniformly summable in D. Hence

$$\begin{aligned} \|f\|^2_{L^2(D)} &= \sup_{\substack{U \Subset D \\ U \text{ is a Reinhardt domain}}} \int_U \sum_{\alpha,\beta \in \Sigma(f)} a^f_\alpha \bar{a}^f_\beta z^\alpha \bar{z}^\beta \, d\Lambda_{2n}(z) \\ &= \sup_{\substack{U \Subset D \\ U \text{ is a Reinhardt domain}}} \sum_{\alpha,\beta \in \Sigma(f)} a^f_\alpha \bar{a}^f_\beta \int_U z^\alpha \bar{z}^\beta \, d\Lambda_{2n}(z) \\ &= \sup_{\substack{U \Subset D \\ U \text{ is a Reinhardt domain}}} \sum_{\alpha \in \Sigma(f)} \|a^f_\alpha z^\alpha\|^2_{L^2(U)} = \sum_{\alpha \in \Sigma(f)} \|a^f_\alpha z^\alpha\|^2_{L^2(D)}. \end{aligned}$$

(d) The space $\mathcal{A}(\Omega) := \mathcal{O}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ with the topology generated by the seminorms

$$\mathcal{A}(\Omega) \ni f \mapsto \|f\|_K, \quad K \Subset \bar{\Omega},$$

is a natural Fréchet space.

We only need to observe that the above family of seminorms is equivalent to every family $(\|\ \|_{\bar{\Omega} \cap K_j})_{j=1}^\infty$, where $(K_j)_{j=1}^\infty$ is an arbitrary sequence of compact subsets of $\mathbb{C}^n$ with $K_j \subset \operatorname{int} K_{j+1}$, $\bigcup_{j=1}^\infty K_j = \mathbb{C}^n$.

Observe that if Ω is bounded, then $\mathcal{A}(\Omega)$ is a natural Banach space; in fact, in this case, $\mathcal{A}(\Omega)$ is a closed subalgebra of $\mathcal{H}^\infty(\Omega)$.

(e) The space

$$\mathcal{H}^\infty_{\text{loc}}(\Omega) := \{f \in \mathcal{O}(\Omega) : \forall_{K \Subset \bar{\Omega}} : \|f\|_{K \cap \Omega} < +\infty\}$$

endowed with the seminorms

$$\mathcal{H}^\infty_{\text{loc}}(\Omega) \ni f \mapsto \|f\|_{K \cap \Omega}, \quad K \Subset \bar{\Omega},$$

is a natural Fréchet space.

EXERCISE. $\mathcal{H}^\infty_{\mathrm{loc}}(\Omega) = \{f \in \mathcal{O}(\Omega) : \forall_{r>0} : \|f\|_{\mathbb{B}(r)\cap\Omega} < +\infty\}$ and the Fréchet topology of $\mathcal{H}^\infty_{\mathrm{loc}}(\Omega)$ is given by the seminorms $\mathcal{H}^\infty_{\mathrm{loc}}(\Omega) \ni f \mapsto \|f\|_{\mathbb{B}(r)\cap\Omega}$, $r > 0$.

Observe that $\mathcal{A}(\Omega)$ is a closed subalgebra of $\mathcal{H}^\infty_{\mathrm{loc}}(\Omega)$. Moreover, if Ω is bounded, then $\mathcal{H}^\infty_{\mathrm{loc}}(\Omega) = \mathcal{H}^\infty(\Omega)$.

(f) The space

$$\mathcal{O}^{(k)}(\Omega, \delta) := \{f \in \mathcal{O}(\Omega) : \|\delta^k f\|_\Omega < +\infty\} \quad (k \geq 0)$$

of all *δ-tempered holomorphic functions on Ω of degree $\leq k$* with the norm

$$\mathcal{O}^{(k)}(\Omega, \delta) \ni f \mapsto \|\delta^k f\|_\Omega$$

is a natural Banach space, where the *weight* $\delta \colon \Omega \to (0, 1]$ is an arbitrary continuous function. Note that $\mathcal{O}^{(0)}(\Omega, \delta) = \mathcal{H}^\infty(\Omega)$ and $\mathcal{O}^{(k)}(\Omega, \delta) \subset \mathcal{O}^{(k')}(\Omega, \delta)$, $k \leq k'$. From a certain point of view, the most important is the weight function $\delta = \delta_\Omega$ given by the formula

$$\delta_\Omega(z) := \min\left\{\rho_\Omega(z), \frac{1}{\sqrt{1 + \|z\|^2}}\right\}, \quad z \in \Omega,$$

where $\rho_\Omega(a) := \sup\{r > 0 : \mathbb{B}(a, r) \subset \Omega\}$, $a \in \Omega$, denotes the Euclidean distance function to $\partial\Omega$; $\rho_{\mathbb{C}^n} \equiv +\infty$, $\delta_{\mathbb{C}^n} = \frac{1}{\sqrt{1+\|\ \|^2}} =: \delta_0$. Functions from the space $\mathcal{O}^{(k)}(\Omega) := \mathcal{O}^{(k)}(\Omega, \delta_\Omega)$ are called *holomorphic functions with polynomial growth of degree $\leq k$*. By the Liouville theorem, Proposition 1.7.17, the space $\mathcal{O}^{(k)}(\mathbb{C}^n)$ coincides with the space $\mathcal{P}_{\lfloor k \rfloor}(\mathbb{C}^n)$ of all complex polynomials of degree $\leq \lfloor k \rfloor$.

(g) Let $\delta \colon \Omega \to (0, 1]$ be a function such that:

- $\delta \leq \rho_\Omega$,
- $|\delta(z') - \delta(z'')| \leq \|z' - z''\|$, $z' \in \Omega$, $z'' \in \mathbb{B}(z', \rho_\Omega(z'))$ (for example, $\delta = \delta_\Omega$).
  Then

$$\|\delta^{(k+2n)/p} f\|_\Omega \leq \mathrm{const}(n, k, p)\left(\int_\Omega |f|^p \delta^k \, d\Lambda_{2n}\right)^{1/p}, \qquad k \geq 0, \ p \geq 1, \ f \in \mathcal{O}(\Omega).$$

In particular, $L^2_h(\Omega) \subset \mathcal{O}^{(n)}(\Omega, \delta)$.

Indeed, fix k, p, f, and $a \in \Omega$. By (1.10.5) with $K := \{a\}$ and $r := \frac{\delta(a)}{2\sqrt{n}} \leq \frac{1}{2} d_\Omega(a)$, we get

$$|f(a)| \leq \frac{\Lambda_{2n}^{1/q}(\mathbb{P}(a, r))}{(\pi r^2)^n}\left(\int_{\mathbb{P}(a,r)} |f|^p \, d\Lambda_{2n}\right)^{1/p},$$

where $1/p + 1/q = 1$. Observe that $\delta(z) \geq \delta(a) - \|z-a\| \geq \frac{1}{2}\delta(a)$, $z \in \mathbb{P}(a,r)$. Consequently,

$$\begin{aligned}
\delta^{(k+2n)/p}(a)|f(a)| &\leq \left(\delta^{k+2n}(a)\frac{1}{(\pi r^2)^n}\int_{\mathbb{P}(a,r)} |f|^p \, d\Lambda_{2n}\right)^{1/p} \\
&\leq \left(2^k \frac{1}{(\pi(\frac{1}{2\sqrt{n}})^2)^n}\int_{\mathbb{P}(a,r)} |f|^p \delta^k \, d\Lambda_{2n}\right)^{1/p} \\
&\leq \left(2^{k+2n}(n/\pi)^n \int_{\Omega} |f|^p \delta^k \, d\Lambda_{2n}\right)^{1/p}.
\end{aligned}$$

(h) Let $(\mathcal{F}_i)_{i\in I}$ be a countable family of natural Fréchet spaces in $\mathcal{O}(\Omega)$. Let $\mathcal{T}(Q_i)$ denote the topology of $\mathcal{F}_i$ generated by a family Q_i of seminorms, $i \in I$. Put

$$\mathcal{F} := \bigcap_{i\in I} \mathcal{F}_i.$$

Then $\mathcal{F}$ endowed with the topology $\mathcal{T}(Q)$, where $Q := \bigcup_{i\in I} Q_i|_{\mathcal{F}}$, is a natural Fréchet space.

In particular, we introduce the following natural Fréchet spaces:

$$\begin{aligned}
L_h^{\diamond}(\Omega) &:= \bigcap_{1\leq p\leq +\infty} L_h^p(\Omega), \text{ [37]} \\
\mathcal{O}^{(0+)}(\Omega,\delta) &:= \bigcap_{k>0} \mathcal{O}^{(k)}(\Omega,\delta) = \bigcap_{\nu=1}^{\infty} \mathcal{O}^{(1/\nu)}(\Omega,\delta), \\
\mathcal{O}^{(0+)}(\Omega) &:= \mathcal{O}^{(0+)}(\Omega,\delta_\Omega).
\end{aligned}$$

Note that:

- $L_h^{\diamond}(\Omega) = \mathcal{H}^\infty(\Omega)$ iff $\Lambda_{2n}(\Omega) < +\infty$.
- $\mathcal{H}^\infty(\Omega) \subset \mathcal{O}^{(0+)}(\Omega,\delta)$ and the inclusion $\mathcal{H}^\infty(\Omega) \to \mathcal{O}^{(0+)}(\Omega,\delta)$ is continuous.

(i) Let $A \subset \mathbb{Z}_+^n$, $0 \in A$, and let $(\mathcal{F}_\alpha)_{\alpha\in A}$ be a family of natural Fréchet spaces in $\mathcal{O}(\Omega)$. Let $\mathcal{T}(Q_\alpha)$ denote the topology of $\mathcal{F}_\alpha$ generated by a family Q_α of seminorms, $\alpha \in A$. Define

$$\mathcal{F} = \mathcal{F}_A := \{f \in \mathcal{F}_0 : D^\alpha f \in \mathcal{F}_\alpha,\ \alpha \in A\}.$$

[37] Observe that $L_h^{\diamond}(\Omega) = L_h^1(\Omega) \cap \mathcal{H}^\infty(\Omega)$. In fact, if $f \in L_h^1(\Omega) \cap \mathcal{H}^\infty(\Omega)$, then for every $1 < p < \infty$ we get

$$\int_\Omega |f|^p d\Lambda_{2n} \leq \|f\|_\Omega^{p-1} \int_\Omega |f| d\Lambda_{2n} \leq (\max\{\|f\|_{L^1(\Omega)}, \|f\|_\Omega\})^p.$$

Then $\mathcal{F}$ endowed with the topology generated by the seminorms

$$\mathcal{F} \ni f \mapsto q(D^\alpha f), \quad q \in Q_\alpha,\ \alpha \in A,$$

is a natural Fréchet space. In particular, for $k \in \mathbb{Z}_+ \cup \{\infty\}$, we define:

$$\begin{aligned}
\mathcal{H}^{\infty,k}(\Omega) : &= \{f \in \mathcal{O}(\Omega) : \forall_{\alpha\in\mathbb{Z}_+^n:|\alpha|\le k} : \ D^\alpha f \in \mathcal{H}^\infty(\Omega)\},\\
L_h^{p,k}(\Omega) : &= \{f \in \mathcal{O}(\Omega) : \forall_{\alpha\in\mathbb{Z}_+^n:\,|\alpha|\le k} : \ D^\alpha f \in L_h^p(\Omega)\},\\
\mathcal{A}^k(\Omega) : &= \{f \in \mathcal{O}(\Omega) : \forall_{\alpha\in\mathbb{Z}_+^n:\,|\alpha|\le k} : \ D^\alpha f \in \mathcal{A}(\Omega)\},\\
\mathcal{H}_{\mathrm{loc}}^{\infty,k}(\Omega) : &= \{f \in \mathcal{O}(\Omega) : \forall_{\alpha\in\mathbb{Z}_+^n:\,|\alpha|\le k} : \ D^\alpha f \in \mathcal{H}_{\mathrm{loc}}^\infty(\Omega)\},\\
L_h^{\diamond,k}(\Omega) : &= \{f \in \mathcal{O}(\Omega) : \forall_{\alpha\in\mathbb{Z}_+^n:\,|\alpha|\le k} : \ D^\alpha f \in L_h^\diamond(\Omega)\}.
\end{aligned}$$

Moreover, let

$$\mathcal{H}^{\infty,S}(\Omega) := \{f \in \mathcal{O}(\Omega) : \forall_{\alpha\in S} : \ D^\alpha f \in \mathcal{H}^\infty(\Omega)\}, \quad \varnothing \neq S \subset \mathbb{Z}_+^n.$$

Observe that: $\mathcal{H}^{\infty,k}(\Omega) = L_h^{\infty,k}(\Omega)$, $\mathcal{H}^{\infty,0}(\Omega) = \mathcal{H}^\infty(\Omega)$, $L_h^{p,0}(\Omega) = L_h^p(\Omega)$, $\mathcal{A}^0(\Omega) = \mathcal{A}(\Omega)$, $\mathcal{H}_{\mathrm{loc}}^{\infty,0}(\Omega) = \mathcal{H}_{\mathrm{loc}}^\infty(\Omega)$, $L_h^{\diamond,0}(\Omega) = L_h^\diamond(\Omega)$.

(j) The space $\mathcal{S} := \mathcal{H}^{\infty,k}(\Omega)$ endowed with the norm

$$\|f\|_{\mathcal{S}} = \|f\|_{\mathcal{H}^{\infty,k}(\Omega)} := 2^k \max\{\|D^\alpha f\|_\Omega : |\alpha| \le k\}$$

is a natural Banach algebra.

Indeed, for $f, g \in \mathcal{S}$, using the Leibniz formula, we have:

$$\begin{aligned}
\|fg\|_{\mathcal{S}} = 2^k \max_{|\alpha|\le k} \|D^\alpha(fg)\|_\Omega &\le 2^k \max_{|\alpha|\le k} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \|D^\beta f\|_\Omega \|D^{\alpha-\beta} g\|_\Omega\\
&\le \|f\|_{\mathcal{S}} \|g\|_{\mathcal{S}} \frac{1}{2^k} \max_{|\alpha|\le k} \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \overset{\text{EXERCISE}}{\le} \|f\|_{\mathcal{S}} \|g\|_{\mathcal{S}}.
\end{aligned}$$

(k) Let $(\mathcal{S}, \|\ \|_S)$ be a natural Banach algebra in $\mathcal{O}(\Omega)$. Then $\mathcal{S} \subset \mathcal{H}^\infty(\Omega)$ and $\|f\|_\Omega \le \|f\|_{\mathcal{S}}$, $f \in \mathcal{S}$.

Indeed, since the identity operator $(\mathcal{S}, \|\ \|_S) \to (\mathcal{O}(\Omega), \tau_\Omega)$ is continuous (cf. (a)), for every compact $K \subset \Omega$ there exists a constant C_K such that

$$\|f\|_K \le C_K \|f\|_{\mathcal{S}}, \quad f \in \mathcal{S}.$$

Since $(\mathcal{S}, \|\ \|_S)$ is a Banach algebra, we get $\|f\|_K^k = \|f^k\|_K \le C_K \|f^k\|_{\mathcal{S}} \le \|f\|_{\mathcal{S}}^k$, $f \in \mathcal{S}$, $k \in \mathbb{N}$. Consequently, $\|f\|_K \le (C_K)^{1/k} \|f\|_{\mathcal{S}}$, $f \in \mathcal{S}$, $k \in \mathbb{N}$. Letting $k \to +\infty$, we conclude that $\|f\|_K \le \|f\|_{\mathcal{S}}$, $f \in \mathcal{S}$, which directly implies the required result.

1.11 Domains of holomorphy

We already know that there exist pairs of domains $D \subsetneq \widetilde{D} \subset \mathbb{C}^n$ such that $\mathcal{O}(\widetilde{D})|_D = \mathcal{O}(D)$ (cf. the Hartogs extension theorem) or $\mathcal{H}^\infty(\widetilde{D})|_D = \mathcal{H}^\infty(D)$ (cf. Riemann Theorem 1.9.8). In the first case D is not a domain of existence with respect to $\mathcal{O}(D)$, in the second – with respect to $\mathcal{H}^\infty(D)$.

More generally, let $D \subset \mathbb{C}^n$ be a domain and let $\varnothing \neq \mathcal{S} \subset \mathcal{O}(D)$. We are interested in the characterization of those domains D which are maximal domains of existence of functions from $\mathcal{S}$ (cf. [Jar-Pfl 2000], § 1.7).

Definition 1.11.1. We say that D is an *$\mathcal{S}$-domain of holomorphy* if

$$d_D(a) = \inf\{d(T_a f) : f \in \mathcal{S}\}, \quad a \in D; \text{[38]}$$

equivalently, for any $r > d_D(a)$ there exists an $f \in \mathcal{S}$ such that $d(T_a f) < r$.

Note that the whole space $\mathbb{C}^n$ is an $\mathcal{S}$-domain of holomorphy for any $\varnothing \neq \mathcal{S} \subset \mathcal{O}(\mathbb{C}^n)$.

If $\mathcal{S} = \{f\}$, then we say that D is a *domain of existence of f*.

If $\mathcal{S} = \mathcal{O}(D)$, then we say that D is a *domain of holomorphy*.

Suppose that we have assigned to each domain D a family $\mathcal{F}(D) \subset \mathcal{O}(D)$ (e.g. $D \to \mathcal{H}^\infty(D)$, $D \to L_h^p(D)$). Then, instead of saying that D is an $\mathcal{F}(D)$-domain of holomorphy, we shortly say that D is an *$\mathcal{F}$-domain of holomorphy* (e.g. $\mathcal{H}^\infty$-domain of holomorphy, L_h^p-domain of holomorphy).

Obviously, if D is an $\mathcal{S}$-domain of holomorphy, then D is a $\mathcal{T}$-domain of holomorphy for any family $\mathcal{T}$ with $\mathcal{S} \subset \mathcal{T} \subset \mathcal{O}(D)$. In particular, any $\mathcal{S}$-domain of holomorphy is a domain of holomorphy.

Proposition 1.11.2. *Let $D \subset \mathbb{C}^n$ be a domain and let $\varnothing \neq \mathcal{S} \subset \mathcal{O}(D)$. Then D is an $\mathcal{S}$-domain of holomorphy iff*

(*) *there are no domains D_0, $\widetilde{D} \subset \mathbb{C}^n$ with $\varnothing \neq D_0 \subset D \cap \widetilde{D}$, $\widetilde{D} \not\subset D$, such that for each $f \in \mathcal{S}$ there exists an $\tilde{f} \in \mathcal{O}(\widetilde{D})$ with $\tilde{f} = f$ on D_0.*[39]

Proof. Suppose that (*) is satisfied, but D is not an $\mathcal{S}$-domain of holomorphy. Then there exist $a \in D$ and $r > d_D(a) =: r_0$ such that $d(T_a f) \geq r$ for any $f \in \mathcal{S}$. Put $D_0 := \mathbb{P}(a, r_0)$, $\widetilde{D} := \mathbb{P}(a, r)$, and $\tilde{f}(z) := T_a f(z)$, $z \in \widetilde{D}$; a contradiction.

Conversely, suppose that D is an $\mathcal{S}$-domain of holomorphy, but (*) is not fulfilled. Let D_0, $\widetilde{D}$ be as in (*). By the identity principle (Proposition 1.7.10) we may assume that D_0 is a connected component of $D \cap \widetilde{D}$. Then there exists an $a \in D_0$ such that $d_D(a) < d_{\widetilde{D}}(a)$ (Exercise). Consequently, for any $f \in \mathcal{S}$ we get $d(T_a f) = d(T_a \tilde{f}) \geq d_{\widetilde{D}}(a) > d_D(a)$; a contradiction. $\square$

[38] Recall that $d(T_a f) \geq d_D(a)$, $a \in D$, $f \in \mathcal{O}(D)$ (Theorem 1.7.6).

[39] Notice that if D is fat, then $\widetilde{D} \not\subset \overline{D}$.

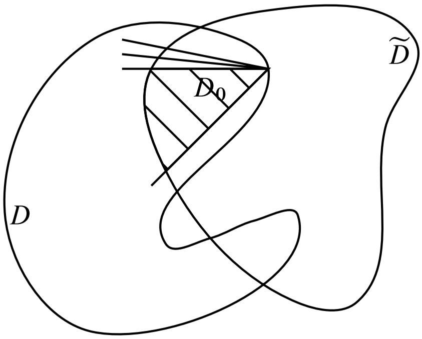

Figure 1.11.1. For each $f \in \mathcal{S}$ there exists an $\tilde{f} \in \mathcal{O}(\widetilde{D})$ with $\tilde{f} = f$ on D_0.

Remark 1.11.3. (a) Let D_0, $\widetilde{D}$ be as in Proposition 1.11.2 (*). First observe that $\tilde{f}$ is uniquely determined by f. Put $\widetilde{S} := \{\tilde{f} : f \in \mathcal{S}\}$. Then the *extension operator*

$$\mathcal{S} \ni f \mapsto \tilde{f} \in \widetilde{\mathcal{S}}$$

is bijective. Observe that:

- $\widetilde{(\mu f)} = \mu \tilde{f}$, provided that $f, \mu f \in \mathcal{S}$ $(\mu \in \mathbb{C})$,
- $\widetilde{f + g} = \tilde{f} + \tilde{g}$, provided that $f, g, f + g \in \mathcal{S}$,
- $\widetilde{D^\alpha f} = D^\alpha \tilde{f}$, provided that $f, D^\alpha f \in \mathcal{S}$ $(\alpha \in \mathbb{Z}_+^n)$.

In particular,

- if $\mathcal{S}$ is a vector space (resp. an algebra), then so is $\widetilde{\mathcal{S}}$ and the above extension operator is an algebraic isomorphism,
- if $\mathcal{S}$ is stable under differentiation (i.e. $f \in \mathcal{S} \Rightarrow \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \in \mathcal{S}$), then so is $\widetilde{\mathcal{S}}$.

(b) Let D_0, $\widetilde{D}$ be as in Proposition 1.11.2 (*). Observe that we do not require $\tilde{f} = f$ on $\widetilde{D} \cap D$ but only on D_0. It may happen that $\tilde{f} \not\equiv f$ on the whole of $\widetilde{D} \cap D$. Take for example $D := \mathbb{C} \setminus (-\infty, 0]$ and $\mathcal{S} := \{\mathrm{Log}\}$, where Log stands for the principal branch of the logarithm $(\mathrm{Log}\, 1 = 0)$. Put $\widetilde{D} := \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$, $D_0 := \{z \in \mathbb{C} : \mathrm{Re}\, z < 0,\ \mathrm{Im}\, z > 0\}$. Then the function Log extends to an $\tilde{f} \in \mathcal{O}(\widetilde{D})$ with $\tilde{f} = \mathrm{Log}$ on D_0 but not on $\widetilde{D} \cap D$ (Exercise), which leads to a non-univalent extension.

It is natural to ask whether such an example is possible in the case where $\mathcal{S}$ contains more functions, in particular, $\mathcal{S} = \mathcal{O}(D)$. Below we will see that for $n = 1$ such an example with $\mathcal{S} = \mathcal{O}(D)$ is impossible. However, for $n \geq 2$ there are such situations (cf. [Jar-Pfl 2000], p. 1).

Theorem 1.12.4 will show that if D is a Reinhardt domain and $\mathcal{S}$ is invariant under rotations of variables, then $\tilde{f} = f$ on the whole of $\widetilde{D} \cap D$. Thus, in the category of Reinhardt domains the above phenomena do not occur.

(c) Any domain $D \subsetneq \mathbb{C}^1$ is an $\mathcal{S}$-domain of holomorphy, where

$$\mathcal{S} := \Big\{D \ni z \mapsto \frac{1}{z-a} : a \notin D\Big\}.$$

(d) Any fat domain $D \subsetneq \mathbb{C}^1$ is an $\mathcal{S}$-domain of holomorphy, where

$$\mathcal{S} := \Big\{D \ni z \mapsto \frac{1}{z-a} : a \notin \bar{D}\Big\}.$$

In particular, any fat domain $D \subset \mathbb{C}^1$ is an $\mathcal{H}^\infty(D) \cap \mathcal{O}(\bar{D})$-domain of holomorphy.[40]

(e) Let D_i be an $\mathcal{S}_i$-domain of holomorphy, $i \in I$, and let D be a connected component of $\operatorname{int} \bigcap_{i \in I} D_i$. Then D is an $\mathcal{S}$-domain of holomorphy with

$$\mathcal{S} := \bigcup_{i \in I} \mathcal{S}_i|_D.$$

In particular, if D_i is a domain of holomorphy for every $i \in I$, then D is a domain of holomorphy.

Indeed,

$$\begin{aligned} d_D(a) = \inf\{d_{D_i}(a) : i \in I\} &= \inf\big\{\inf\{d(T_a f) : f \in \mathcal{S}_i\} : i \in I\big\} \\ &= \inf\{d(T_a f) : f \in \mathcal{S}\}, \quad a \in D. \end{aligned}$$

(f) Let $D_j \subset \mathbb{C}^{n_j}$ be an $\mathcal{S}_j$-domain of holomorphy, $j = 1, \dots, N$. Then $D := D_1 \times \cdots \times D_N$ is an $\mathcal{S}$-domain of holomorphy with

$$\mathcal{S} := \{f \circ \mathrm{pr}_{D_j} : f \in \mathcal{S}_j,\ j = 1, \dots, N\},$$

where $\mathrm{pr}_{D_j} : D_1 \times \cdots \times D_N \to D_j$ is the standard projection, $j = 1, \dots, N$. In particular, if $D_j \subset \mathbb{C}^{n_j}$ is a domain of holomorphy, $j = 1, \dots, N$, then $D_1 \times \cdots \times D_N$ is a domain of holomorphy.

Indeed,

$$\begin{aligned} d_D(a) &= \min\{d_{D_j}(a_j) : j = 1, \dots, N\} \\ &= \min\big\{\inf\{d(T_{a_j} f) : f \in \mathcal{S}_j\} : j = 1, \dots, N\big\} \\ &= \inf\{d(T_{(a_1,\dots,a_N)} f) : f \in \mathcal{S}\}, \quad a = (a_1, \dots, a_N) \in D. \end{aligned}$$

[40] Recall that $\mathcal{O}(\bar{D}) := \bigcup_{\substack{\bar{D} \subset U \subset \mathbb{C} \\ U \text{ open}}} \mathcal{O}(U)|_D$.

(g) Let D be a domain of holomorphy, let $f = (f_1, \dots, f_N) \in \mathcal{O}(D, \mathbb{C}^N)$, and let G be a connected component of the set

$$f^{-1}(\mathbb{D}^N) = \{z \in D : |f_j(z)| < 1,\ j = 1, \dots, N\}.$$

Then G is a domain of holomorphy.

Indeed, take an $a \in G$. If $d_G(a) = d_D(a)$, then

$$\begin{aligned} d_G(a) = d_D(a) &= \inf\{d(T_a f) : f \in \mathcal{O}(D)\} \\ &\geq \inf\{d(T_a f) : f \in \mathcal{O}(G)\} \geq d_G(a). \end{aligned}$$

If $r := d_G(a) < d_D(a)$, then there exists a point $b \in \partial G \cap \partial \mathbb{P}(a, r)$. Consequently, there exists a $j \in \{1, \dots, N\}$ with $|f_j(b)| = 1$. Hence the function $g := 1/(f_j - f_j(b))$ is holomorphic in G and $d(T_a g) = r$.

(h) Let D be a domain of holomorphy and let $f_0 \in \mathcal{O}(D)$, $f_0 \not\equiv 0$. Then $G := D \setminus f_0^{-1}(0)$ is a domain of holomorphy.[41] In particular, if $D \subset \mathbb{C}^n$ is a domain of holomorphy, then $D \setminus (V_{i_1} \cup \dots \cup V_{i_k})$ is a domain of holomorphy for any $1 \leq i_1 < \dots < i_k \leq n$.

Indeed, take an $a \in G$. The case $d_G(a) = d_D(a)$ is the same as in (g). If $r := d_G(a) < d_D(a)$, then there exists a $b \in f_0^{-1}(0) \cap \partial \mathbb{P}(a, r)$. Thus the function $g := 1/f_0$ is holomorphic in G and $d(T_a g) = r$.

(i) Observe that if $G := D \setminus F$, where $F \neq \varnothing$ is a closed thin subset of D, then, by the Riemann removable singularity theorem (Theorem 1.9.8), $\mathcal{H}^\infty(G) = \mathcal{H}^\infty(D)|_G$ and, consequently, G is not an $\mathcal{H}^\infty(G)$-domain of holomorphy.

(j) Assume that D is not a domain of holomorphy and let $\widetilde{D}$ be as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{O}(D)$. Then $\tilde{f}(\widetilde{D}) \subset f(D)$, $f \in \mathcal{O}(D)$.

Indeed, suppose that there exists a $b \in \tilde{f}(\widetilde{D}) \setminus f(D)$. Then the function $g := 1/(f - b)$ is holomorphic in D and $g \cdot (f - b) \equiv 1$. Hence, by the identity principle, $\tilde{g} \cdot (\tilde{f} - b) \equiv 1$; a contradiction.

(k) Assume that D is not an $\mathcal{H}^\infty$-domain of holomorphy and let D_0, $\widetilde{D}$ be as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{H}^\infty(D)$. Then $\|\tilde{f}\|_{\widetilde{D}} \leq \|f\|_D$, $f \in \mathcal{H}^\infty(D)$ (Exercise).

(l) Let D be a domain in $\mathbb{C}^n$. Assume that for any point $a \in \partial D$ there exists a function $f_a \in \mathcal{O}(D, \mathbb{D})$ with $\lim_{D \ni z \to a} |f_a(z)| = 1$. Then D is an $\mathcal{H}^\infty(D)$ domain of holomorphy.

Indeed, suppose that D is not an $\mathcal{H}^\infty$-domain of holomorphy and let D_0 and $\widetilde{D}$ be as in Proposition 1.11.2 (*). We may assume that D_0 is a connected component of $D \cap \widetilde{D}$. Take an $a \in \widetilde{D} \cap \partial D_0$ and let f_a be as above. Then, $|\tilde{f}_a| \leq 1$ in $\widetilde{D}$ (cf. (k)) and $|\tilde{f}_a(a)| = \lim_{D_0 \ni z \to a} |f_a(z)| = 1$. Consequently, by the maximum principle, $|\tilde{f}_a| \equiv 1$. In particular, $|f_a| = 1$ on D_0; a contradiction.

(m) Any convex domain $D \subset \mathbb{C}^n$ is an $\mathcal{H}^\infty$-domain of holomorphy.

[41] Recall that, by Proposition 1.9.7, G is connected.

Indeed, take a convex domain $D \subsetneq \mathbb{C}^n$ and fix an $a \in \partial D$. Since D is convex, there exists an affine function $\ell : \mathbb{C}^n \to \mathbb{R}$ such that $\ell < 0$ on D and $\ell(a) = 0$. Suppose that $\ell(z) = b_0 + \sum_{j=1}^n (b_j x_j + c_j y_j)$, $z = (x_1 + iy_1, \dots, x_n + iy_n)$, where $b_0, \dots, b_n, c_1, \dots, c_n \in \mathbb{R}$. Define $L(z) := b_0 + \sum_{j=1}^n (b_j - ic_j) z_j$. Obviously $\ell = \operatorname{Re} L$. Let $f_0 := e^L$. Then $|f_0| = e^{\operatorname{Re} L} = e^{\ell} < 1$ on D and $|f_0(a)| = 1$. It remains to apply (l).

(n) Suppose that $\mathcal{S}$ is a natural Fréchet space (cf. Definition 1.10.5). Let $\mathcal{T}(Q)$ be the topology of $\mathcal{F}$ generated by a family Q of seminorms.

Assume that D is not an $\mathcal{S}$-domain of holomorphy and let D_0, $\widetilde{D}$ be as in Proposition 1.11.2 (*). Let $\widetilde{\mathcal{S}} := \{\tilde{f} : f \in \mathcal{S}\} \subset \mathcal{O}(\widetilde{D})$. We endow the space $\widetilde{\mathcal{S}}$ with a topology $\mathcal{T}(\widetilde{Q})$ generated by the following family $\widetilde{Q}$ of seminorms:

$$\widetilde{\mathcal{S}} \ni \tilde{f} \mapsto q(f), \quad q \in Q,$$
$$\widetilde{\mathcal{S}} \ni \tilde{f} \mapsto \|\tilde{f}\|_{\widetilde{K}}, \quad \widetilde{K} \Subset \widetilde{D}.$$

Notice that $\tilde{f}_\nu \to \tilde{f}$ in the sense of $\mathcal{T}(\widetilde{Q})$ iff $\tilde{f}_\nu \to \tilde{f}$ locally uniformly on $\widetilde{D}$ and $f_\nu \to f$ in the sense of $\mathcal{T}(Q)$. Observe that $\widetilde{\mathcal{S}}$ is a Fréchet space.

Indeed, if $(\tilde{f}_\nu)_{\nu=1}^\infty$ is a Cauchy sequence in $\widetilde{\mathcal{S}}$, then $(f_\nu)_{\nu=1}^\infty$ is a Cauchy sequence in $\mathcal{S}$ and $(\tilde{f}_\nu)_{\nu=1}^\infty$ is a Cauchy sequence in $\mathcal{O}(\widetilde{D})$ in the topology of locally uniform convergence. Hence there exist functions $f_0 \in \mathcal{S}$ and $g_0 \in \mathcal{O}(\widetilde{D})$ such that $f_\nu \to f_0$ in $\mathcal{S}$ and $\tilde{f}_\nu \to g_0$ locally uniformly in $\widetilde{D}$. Since $\mathcal{S}$ is a natural Fréchet space, we conclude that $f_\nu \to f_0$ locally uniformly on D. In particular, $f_0 = g_0$ on D_0. Thus $g_0 = \tilde{f}_0$ and, finally, $\tilde{f}_\nu \to \tilde{f}_0$ in $\widetilde{\mathcal{S}}$.

The mapping $\widetilde{\mathcal{S}} \ni \tilde{f} \to f \in \mathcal{S}$ is obviously continuous. Since $\widetilde{\mathcal{S}}$ is a Fréchet space, the Banach theorem (Theorem 1.10.4) implies that the above operator is a topological isomorphism, i.e. for each compact $\widetilde{K} \subset \widetilde{D}$ there exist a finite set $I \subset Q$ and $c > 0$ such that

$$\|\tilde{f}\|_{\widetilde{K}} \le c \max I(f), \quad f \in \mathcal{S}.$$

In particular, if $\mathcal{S}$ is a natural Banach space with a norm $\| \ \|_{\mathcal{S}}$, then for every compact $\widetilde{K} \subset \widetilde{D}$ there exists a constant $c > 0$ such that

$$\|\tilde{f}\|_{\widetilde{K}} \le c \|f\|_{\mathcal{S}}, \quad f \in \mathcal{S}.$$

In the special case where $\mathcal{S}$ is a natural Banach algebra, we get more. Namely, $\|\tilde{f}\|_{\widetilde{D}} \le \|f\|_{\mathcal{S}}$, $f \in \mathcal{S}$ (Exercise — cf. Example 1.10.7 (k)).

(o) Let $\mathcal{S}$, D_0, $\widetilde{D}$ be as above. By virtue of (n), if $\mathcal{S}$ is a closed subspace of $\mathcal{O}(D)$ (in the topology of locally uniform convergence in D), then for each compact $\widetilde{K} \subset \widetilde{D}$ there exist a compact $K \subset D$ and a constant $c > 0$ such that

$$\|\tilde{f}\|_{\widetilde{K}} \le c \|f\|_K, \quad f \in \mathcal{S}.$$

(p) Let $\mathcal{S}$, D_0, $\widetilde{D}$ be as above. In the special case, if $\mathcal{S}$ is a closed subalgebra of $\mathcal{O}(D)$, then for each compact $\widetilde{K} \subset \widetilde{D}$ there exists a compact $K \subset D$ such that

$$\|\tilde{f}\|_{\widetilde{K}} \le \|f\|_K, \quad g \in \mathcal{S} \quad \text{(Exercise)}.$$

Proposition 1.9.12 implies the following result.

Corollary 1.11.4. (a) *If a Reinhardt domain $D \subset \mathbb{C}^n$ is an $\mathcal{H}^\infty_{\mathrm{loc}}(D)$-domain of holomorphy, then D is fat.*

(b) *If a Reinhardt domain $D \subset \mathbb{C}^n$ is an $\mathcal{O}^{(N)}$-domain of holomorphy with $0 \le N < 1$, then D is fat.*

Remark 1.11.5. Let $T_\sigma := \{(z_1, z_2) \in \mathbb{D} \times \mathbb{D} : |z_1|^\sigma < |z_2|\}$, $\sigma = p/q \in \mathbb{Q}_{>0}$.

(a) First observe that T_σ is an $\mathcal{H}^\infty$-domain of holomorphy. Although it follows from the general results (Theorem 3.4.1), here we give a direct elementary proof.

Suppose that D_0, $\widetilde{D}$ are as in Proposition 1.11.2 (*) with $D = T_\sigma$ and $\mathcal{S} = \mathcal{H}^\infty(T_\sigma)$. Since $\mathbb{D} \times \mathbb{D}$ is obviously an $\mathcal{H}^\infty$-domain of holomorphy, we conclude that $\widetilde{D} \subset \mathbb{D} \times \mathbb{D}$. Let $f(z) := z_1^p/z_2^q$, $z = (z_1, z_2) \in T_\sigma$. Then $f \in \mathcal{H}^\infty(T_\sigma)$ and, therefore, there exists an $\tilde{f} \in \mathcal{O}(\widetilde{D})$ such that $\tilde{f} = f$ on D_0 and $\|\tilde{f}\|_{\widetilde{D}} \le \|f\|_{T_\sigma} \le 1$ (Remark 1.11.3 (k)). Consequently, $z_2^q \tilde{f}(z) = z_1^p$, $z = (z_1, z_2) \in \widetilde{D}$. Let $b = (b_1, b_2) \in \partial T_\sigma \cap \widetilde{D}$. If $b_2 \ne 0$, then $\tilde{f}(z) = z_1^p/z_2^q$ for $z = (z_1, z_2)$ in an open neighborhood $U \subset \widetilde{D}$ of b. Then $|z_1^p/z_2^q| \le 1$ in U and, by the maximum principle, $U \subset T_\sigma$; a contradiction. If $b = 0$, then $\tilde{f}$ is holomorphic in a small polydisc $\mathbb{P}(r) \subset \widetilde{D}$, $\tilde{f}(z) = \sum_{j,k=0}^\infty a_{j,k} z_1^j z_2^k$, $z = (z_1, z_2) \in \mathbb{P}(r)$. Consequently, $\sum_{j,k=0}^\infty a_{j,k} z_1^j z_2^{k+q} = z_1^p$, $(z_1, z_2) \in \mathbb{P}(r)$, which is impossible.

(b) The mapping $\mathbb{C} \times \mathbb{C}_* \ni (z_1, z_2) \mapsto (z_1, z_1/z_2) \in \mathbb{C} \times \mathbb{C}$ maps biholomorphically the Hartogs triangle T onto $\mathbb{D} \times \mathbb{D}_*$. Observe that $\mathbb{D} \times \mathbb{D}_*$ is not an $\mathcal{H}^\infty$-domain of holomorphy. In particular, the notion of an $\mathcal{H}^\infty$-domain of holomorphy is not invariant under biholomorphic mappings.

Proposition 1.11.6. *Let $D = \mathfrak{D}_S \ne \varnothing$ be the domain of convergence of a Laurent series*

$$S = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha.$$

Then D is a domain of holomorphy.

Proof. Suppose that $\widetilde{D}$, D_0 are as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{O}(D)$.

Put $f_\alpha(z) := a_\alpha z^\alpha$, $z \in D$, with $\alpha \in \Sigma(S)_*$. Observe that $\widetilde{D} \subset \mathbb{C}^n(\Sigma(S))$ and $\tilde{f}_\alpha(z) = a_\alpha z^\alpha$, $z \in \widetilde{D}$.

Indeed, by Remark 1.11.3 (c), (f), $\mathbb{C}^n(\Sigma(S))$ is a domain of holomorphy. Obviously, $\mathcal{O}(\mathbb{C}^n(\Sigma(S)))|_D \subset \mathcal{O}(D)$. Hence $\widetilde{D} \subset \mathbb{C}^n(\Sigma(S))$.

To get a contradiction we are going to show that $\widetilde{D} \subset \mathfrak{D}_S = D$. Suppose that there exists an $a \in \widetilde{D} \setminus D$ and let $\widetilde{K} := \overline{\mathbb{B}}(a, r) \subset \widetilde{D}$. By Remark 1.11.3 (p) with $\mathcal{S} = \mathcal{O}(D)$, there exists a compact $K \subset D$ such that $\|\tilde{f}\|_{\widetilde{K}} \le \|f\|_K$ for any $f \in \mathcal{O}(D)$. By Lemma 1.6.3, there exist $C > 0$ and $\theta \in (0,1)$ such that

$$\|a_\alpha z^\alpha\|_K \le C\theta^{|\alpha|}, \quad \alpha \in \Sigma(S).$$

Consequently,

$$\|a_\alpha z^\alpha\|_{\widetilde{K}} \le C\theta^{|\alpha|}, \quad \alpha \in \Sigma(S).$$

Thus int $\widetilde{K} \subset \mathfrak{D}_S = D$; a contradiction. □

Proposition 1.11.7. *For any $\alpha \in (\mathbb{R}^n)_*$ and $c \in \mathbb{R}$, the elementary Reinhardt domain*

$$\boldsymbol{D}_{\alpha,c} = \{z \in \mathbb{C}^n(\alpha) : |z^\alpha| < e^c\}$$

is a domain of holomorphy.

Proof. Use Propositions 1.6.6 and 1.11.6. □

Remark 1.11.8. (a) Observe that it is much easier to prove that $\boldsymbol{D}_{\alpha,c}$ is locally a domain of holomorphy, i.e. every $a = (a_1, \dots, a_n) \in \partial\boldsymbol{D}_{\alpha,c}$ has an open neighborhood U such that each connected component of $U \cap \boldsymbol{D}_{\alpha,c}$ is a domain of holomorphy.

Indeed, if $a \in \mathbb{C}_*^n \cap \partial\boldsymbol{D}_{\alpha,c}$, then let $U := \mathbb{P}(a,r) \subset \mathbb{C}_*^n$ and let $f(z) := f_1(z_1)\cdots f_n(z_n)$, $z = (z_1, \dots, z_n) \in U$, where $f_j \in \mathcal{O}(K(a_j, r))$ is a holomorphic branch of the α_j-power, $j = 1, \dots, n$. Then $U \cap \boldsymbol{D}_{\alpha,c} = \{z \in U : |f(z)| < e^c\}$ and we may apply Remark 1.11.3 (g).

If $a \in V_0 \cap \partial\boldsymbol{D}_{\alpha,c}$, then let $U := \mathbb{P}(a,r) \subset \mathbb{C}^n$ be arbitrary. Suppose that a connected component D of $U \cap \boldsymbol{D}_{\alpha,c}$ is not a domain of holomorphy. Let D_0, $\widetilde{D}$ be as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{O}(D)$. Since U is a domain of holomorphy, we have $\widetilde{D} \subset U$. We may assume that D_0 is a connected component of $D \cap \widetilde{D}$. The first part of the proof shows that $\partial D_0 \cap \widetilde{D} \subset V_0$.

Thus, it suffices to show that for any point $b \in V_0 \cap \partial\boldsymbol{D}_{\alpha,c}$ there exists a function $f \in \mathcal{O}(\boldsymbol{D}_{\alpha,c})$ which cannot be continued through b. We may assume that $\alpha_1, \dots, \alpha_s > 0$, $\alpha_{s+1}, \dots, \alpha_n < 0$, $1 \le s \le n-1$, $b_1 \cdots b_s = b_{s+1} \cdots b_n = 0$, $b_n = 0$ (cf. Remark 1.5.7 (d)). Consequently, one can take $f(z) := 1/z_n$.

(b) One should mention the following general result which will follow from Theorems 1.15.5 (viii) and 1.16.1.

Theorem* 1.11.9. *Let $D \subset \mathbb{C}^n$ be a domain. Then D is a domain of holomorphy iff D is locally a domain of holomorphy, i.e. every point $a \in \partial D$ has a neighborhood U such that each connected component of $U \cap D$ is a domain of holomorphy.*

Lemma 1.11.10. *D is an $\mathcal{S}$-domain of holomorphy iff there exists a dense subset $A \subset D$ such that $d_D(a) = \inf\{d(T_a f) : f \in \mathcal{S}\}$, $a \in A$.*

Proof. Let $a \in D$ and let $r > r_0 > d_D(a)$. Suppose that $d(T_a f) \ge r$ for any $f \in \mathcal{S}$. Then there exists an open neighborhood $U \subset D$ of a such that $d(T_b f) \ge r_0 > d_D(b)$ for all $f \in \mathcal{S}$ and $b \in U$; a contradiction. □

Proposition 1.11.11. *Let $\mathcal{S} \subset \mathcal{O}(D)$ be a natural Fréchet space (Definition 1.10.5). Then the following conditions are equivalent:*

(i) *D is an $\mathcal{S}$-domain of holomorphy;*

(ii) *the set $\mathcal{S} \setminus \mathfrak{N}(\mathcal{S})$, where*

$$\mathfrak{N}(\mathcal{S}) := \{f \in \mathcal{S} : D \text{ is the domain of existence of } f\},$$

is of the first Baire category in $\mathcal{S}$;[42]

(iii) $\mathfrak{N}(\mathcal{S}) \neq \varnothing$.

Proof. Obviously, (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): For $a \in D$ and $r > 0$ let $\mathcal{S}_{a,r} := \{f \in \mathcal{S} : d(T_a f) \geq r\}$. It is clear that $\mathcal{S}_{a,r}$ is a vector subspace of $\mathcal{S}$. Let $\mathcal{T}(Q)$ be the topology of $\mathcal{S}$. We endow the space $\mathcal{S}_{a,r}$ with a topology $\mathcal{T}(Q_{a,r})$, where $Q_{a,r}$ is the following family of seminorms:

$$\begin{aligned} \mathcal{S}_{a,r} \ni f &\mapsto q(f), & q &\in Q, \\ \mathcal{S}_{a,r} \ni f &\mapsto \|T_a f\|_K, & K &\Subset \mathbb{P}(a,r). \end{aligned}$$

One can easily verify that $\mathcal{S}_{a,r}$ endowed with this topology is a Fréchet space (cf. Remark 1.11.3 (n)).

The inclusion $\mathcal{S}_{a,r} \to \mathcal{S}$ is obviously continuous. Hence, by the Banach theorem (Theorem 1.10.4), either $\mathcal{S}_{a,r} = \mathcal{S}$ or $\mathcal{S}_{a,r}$ is of the first Baire category in $\mathcal{S}$. Since D is an $\mathcal{S}$-domain of holomorphy, $\mathcal{S}_{a,r}$ is of the first category if $r > d_D(a)$.

Now let $A \subset D$ be countable and dense in D. Put

$$\mathcal{S}_0 := \bigcup_{a \in A, k \in \mathbb{N}} \mathcal{S}_{a, d_D(a)+1/k}.$$

Then $\mathcal{S}_0$ is of the first Baire category in $\mathcal{S}$. Finally, by Lemma 1.11.10, we get $\mathcal{S} \setminus \mathfrak{N}(\mathcal{S}) = \mathcal{S}_0$. □

Exercise 1.11.12. Let $D_j \subset \mathbb{C}^n$ be a domain of existence of a function $f_j \in \mathcal{O}(D_j)$, $j = 1, \dots, N$. Assume that G is a connected component of $D_1 \cap \dots \cap D_N \neq \varnothing$. For $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$ let $F_\lambda := \lambda_1 f_1|_G + \dots + \lambda_N f_N|_G \in \mathcal{O}(G)$. Prove that there exists a $\lambda_0 \in \mathbb{C}^N$ such that G is the domain of holomorphy of F_{λ_0}.

Hint. Let $\mathcal{S} := \{F_\lambda : \lambda \in \mathbb{C}^N\} \subset \mathcal{O}(G)$. Observe that G is an $\mathcal{S}$-domain of holomorphy and $\mathcal{S}$ is a natural Fréchet space in $\mathcal{O}(G)$. Next use Proposition 1.11.11.

The following result gives the full geometric characterization of Reinhardt domains of holomorphy.

[42] Roughly speaking, if D is an $\mathcal{S}$-domain of holomorphy, then "almost all" functions $f \in \mathcal{S}$ are not holomorphically continuable beyond D.

Theorem 1.11.13. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain. Then the following conditions are equivalent:*

(i) *D is a domain of holomorphy;*

(ii) *D satisfies the following two geometric conditions:*

  (ii)$_1$ $\log D$ *is convex,*

  (ii)$_2$ *D is* relatively complete, *i.e. for every $j \in \{1,\dots,n\}$, if $D \cap V_j \neq \varnothing$, then $\widehat{D}^{(j)} \subset D$;*[43]

(iii) *D satisfies the following two geometric conditions:*

  (iii)$_1$ $\log D$ *is convex,*

  (iii)$_2$ *D is* weakly relatively complete, *that is, for every $j \in \{1,\dots,n\}$, if $D \cap V_j \neq \varnothing$, then $(a', 0, a'') \in D$ for any $(a', a_j, a'') \in D \subset \mathbb{C}^{j-1} \times \mathbb{C} \times \mathbb{C}^{n-j}$;*

(iv) *D is log-convex and $D = D^* \setminus M$, where D^* was defined in (1.5.3) and*

$$M = M(D) := \bigcup_{\substack{j\in\{1,\dots,n\}:\\ D\cap V_j=\varnothing}} V_j.\text{[44]}$$

In particular:

- *If D is a Reinhardt domain of holomorphy such that $D \cap V_j \neq \varnothing$, $j = 1,\dots,n$ (e.g. $0 \in D$), then D must be a complete Reinhardt domain.*
- *D is a fat domain of holomorphy iff D is log-convex and $D = D^*$.*

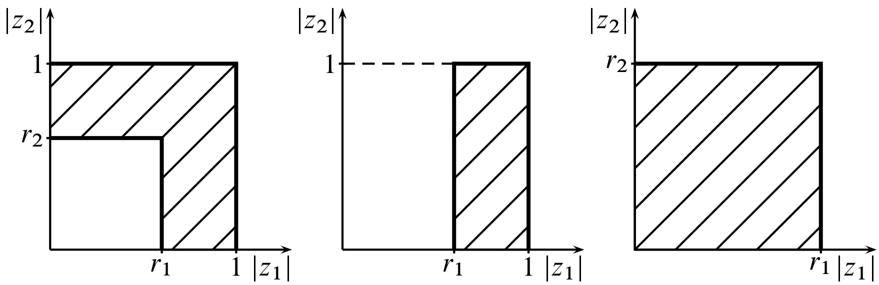

Figure 1.11.2. Which of the above domains are relatively complete ?

Proof. (i) $\Rightarrow$ (ii): Let $f \in \mathcal{O}(D)$, $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha$, $z \in D$, be such that D is the domain of existence of f (cf. Proposition 1.11.11). Then $D = \mathfrak{D}_f$,

[43]Consequently, if D is a Reinhardt domain of holomorphy, then for any $j \in \{1,\dots,n\}$ we have: $D \cap V_j \neq \varnothing \iff \widehat{D}^{(j)} = D$. Observe that condition (ii)$_2$ is automatically satisfied if $D \subset \mathbb{C}^n_*$ or if D is complete.

[44]Recall that D^* is log-convex and relatively complete (Remark 1.5.8).

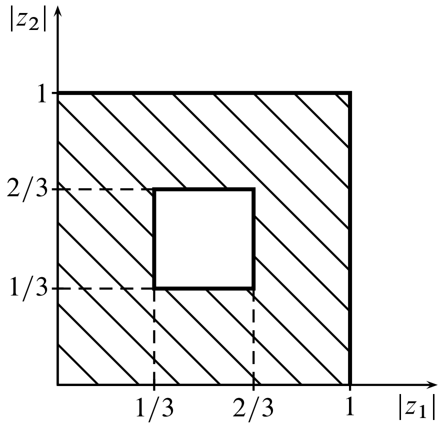

Figure 1.11.3. The domain $D := \mathbb{D}^2 \setminus (\bar{\mathbb{A}}(1/3, 2/3) \times \bar{\mathbb{A}}(1/3, 2/3))$ is weakly relatively complete, but not relatively complete.

where $\mathfrak{D}_f$ denotes the domain of convergence of the Laurent series $\sum_{\alpha\in\mathbb{Z}^n} a_\alpha z^\alpha$. Consequently, the result follows directly from Proposition 1.6.5 (c), (d).

The implication (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv): Since $D^* \setminus D \subset V_0$ (Remark 1.5.8 (a)), we only need to show that $D^* \setminus M \subset D$. Take a point $a = (a_1, \dots, a_n) \in D^* \setminus M$. Since $D^* \setminus V_0 = D \setminus V_0$, we may assume that $a \in V_0 \setminus M$, say $a_1 = \dots = a_s = 0$, $a_{s+1} \cdots a_n \neq 0$ for some $1 \le s \le n$. Since $a \notin M$, we conclude that $D \cap V_j \neq \varnothing$, $j = 1, \dots, s$. It is clear that for sufficiently small $\varepsilon > 0$ the point $b = (\varepsilon, \dots, \varepsilon, a_{s+1}, \dots, a_n)$ belongs to $D^* \setminus V_0 = D \setminus V_0$. Now, using (iii)$_2$ (with respect to all $j \in \{1, \dots, s\}$), we see that $a = (0, \dots, 0, a_{s+1}, \dots, a_n) \in D$.

(iv) $\Rightarrow$ (i): Since $\log D$ is convex, there exists a family $A \subset \mathbb{R}^n \times \mathbb{R}$ such that $\log D = \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{H}_{\alpha,c}$. Then $D^* := \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c}$ (Remark 1.5.8 (b)). By Proposition 1.11.7, each domain $\boldsymbol{D}_{\alpha,c}$ is a domain of holomorphy. Consequently, D^* is a domain of holomorphy (cf. Remark 1.11.3 (e)). Now, we may use Remark 1.11.3 (h). $\square$

Corollary 1.11.14. *If D is a Reinhardt domain of holomorphy with the Fu condition, then D is fat* (*cf. Remark* 1.13.11 (b)).

Note that the Hartogs triangle is a fat Reinhardt domain of holomorphy without the Fu condition.

Corollary 1.11.15. *If $(D_k)_{k=1}^\infty$ is a sequence of Reinhardt domains of holomorphy with $D_k \subset D_{k+1}$, then $D := \bigcup_{k=1}^\infty D_k$ is a Reinhardt domain of holomorphy.*

Corollary 1.11.16. *Let $D \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ be a Reinhardt domain of holomorphy. Then:*

(a) $\mathrm{pr}_{\mathbb{C}^k}(D)$ *is a Reinhardt domain of holomorphy in* $\mathbb{C}^k$.

(b) *For any* $(a, b) \in D \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ *the set* $D' := \{z \in \mathbb{C}^k : (z, b) \in D\}$ *is a Reinhardt domain of holomorphy.*

Proof. (a) Use Theorem 1.11.13 (ii) (Exercise).

(b) It is clear that D' is k-circled and relatively complete. It remains to show that D' is log-convex (then D' must be a domain). This is clear if $b \in \mathbb{C}_*^{n-k}$. Suppose that $b = (b_{k+1}, \dots, b_s, 0, \dots, 0)$ with $k + 1 \le s \le n - 1$, where $b_{k+1} \cdots b_s \ne 0$. Take $p = (p_1, \dots, p_k)$, $q = (q_1, \dots, q_k) \in D' \cap \mathbb{R}^k_{>0}$ and let Γ be the hyperbolic segment between p and q,

$$\Gamma := \{(p_1^{1-t} q_1^t, \dots, p_k^{1-t} q_k^t) : t \in [0, 1]\}.$$

We want to show that $\Gamma \times \{b\} \subset D$. Let

$$U_p = U_p^1 \times \cdots \times U_p^s \times \mathbb{P}_{n-s}(\varepsilon), \quad U_q = U_q^1 \times \cdots \times U_q^s \times \mathbb{P}_{n-s}(\varepsilon) \subset D$$

be Reinhardt neighborhoods of (p, b) and (q, b), respectively. Then

$$\Gamma \times \{(b_{k+1}, \dots, b_s)\} \times K_*(\varepsilon) \times \cdots \times K_*(\varepsilon) \subset D.$$

Consequently, the relative completeness of D implies that $\Gamma \times \{b\} \subset D$. □

Remark 1.11.17. Notice that for a general domain of holomorphy $D \subset \mathbb{C}^n$, the projection $\mathrm{pr}_{\mathbb{C}^k}(D)$ need not be a domain of holomorphy – cf. e.g. [Pfl1978], [Kas 1980], [Shc 1982], [Joi 2000].

Proposition 1.11.18. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain. Then the following conditions are equivalent:*

(i) *D is a domain of holomorphy;*

(ii) *for any $a = (a_1, \dots, a_n) \in \mathbb{R}^n_{>0}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}^n)_*$, and $\beta = (\beta_1, \dots, \beta_n) \in \{e_1, \dots, e_n\} \subset \mathbb{Z}^n_+$, the set*

$$D_{a,\alpha,\beta} := \{(\lambda, \mu) \in (\mathbb{C}(\alpha_1) \cap \cdots \cap \mathbb{C}(\alpha_n)) \times \mathbb{C} : \\ (a_1 |\lambda|^{\alpha_1} |\mu|^{\beta_1}, \dots, a_n |\lambda|^{\alpha_n} |\mu|^{\beta_n}) \in D\}$$

is a Reinhardt domain of holomorphy (provided that $D_{a,\alpha,\beta} \ne \varnothing$).

Notice the special role played by two-dimensional Reinhardt domains (cf. Proposition 1.15.9).

Proof. Define $\Phi_{a,\alpha,\beta} : (\mathbb{C}(\alpha_1) \cap \cdots \cap \mathbb{C}(\alpha_n)) \times \mathbb{C} \to \mathbb{C}^n$,

$$\Phi_{a,\alpha,\beta}(\lambda, \mu) := (a_1 |\lambda|^{\alpha_1} |\mu|^{\beta_1}, \dots, a_n |\lambda|^{\alpha_n} |\mu|^{\beta_n}).$$

First observe that $D_{a,\alpha,\beta} = \Phi_{a,\alpha,\beta}^{-1}(D)$ is a Reinhardt open set and

$$\log D_{a,\alpha,\beta} = \{(t,u) \in \mathbb{R}^2 : \log a + t\alpha + u\beta \in \log D\}.$$

It is clear that D is log-convex iff each $D_{a,\alpha,\beta}$ is log-convex.

It remains to discuss the relative completeness. First assume that each $D_{a,\alpha,\beta}$ is relatively complete. We will prove that D is weakly relatively complete. Suppose that $D \cap V_j \neq \varnothing$. We may assume that $j = n$. Fix a $b = (b', 0) \in D \cap (\mathbb{R}^{n-1}_{>0} \times \{0\})$. Take a point $c = (c', c_n) \in D \cap \mathbb{R}^n_+$, $c' \neq b'$, $c_n > 0$. We want to prove that $(c', 0) \in D$. We may assume that $c' = (\underbrace{0, \dots, 0}_{s}, c_{s+1}, \dots, c_{n-1})$ with $c_{s+1}, \dots, c_{n-1} > 0$ for some $0 \leq s \leq n-1$.

First, consider the case $s = 0$. Define

$$a := (b', c_n), \qquad \alpha_j := \frac{\log(c_j/b_j)}{\log 2}, \quad j = 1, \dots, n-1,$$
$$\alpha := (\alpha', 0) \in \mathbb{R}^n, \quad \beta := e_n.$$

Since $\Phi_{a,\alpha,\beta}(1,0) = b$ and $\Phi_{a,\alpha,\beta}(2,1) = c$, we conclude that $(1,0), (2,1) \in D_{a,\alpha,\beta}$. Thus $(2,0) \in D_{a,\alpha,\beta}$ and, consequently, $(c',0) = \Phi_{a,\alpha,\beta}(2,0) \in D$.

Now, let $s > 0$ and suppose that $(c', 0) \notin D$. Define

$$a := (\underbrace{1, \dots, 1}_{s}, c_{s+1}, \dots, c_n), \quad \alpha := (\underbrace{1, \dots, 1}_{s}, 0, \dots, 0), \quad \beta := e_n.$$

Then $\Phi_{a,\alpha,\beta}(0,1) = c \in D$, $\Phi_{a,\alpha,\beta}(0,0) = (c',0) \notin D$. Thus $(0,1) \in D_{a,\alpha,\beta}$ and $(0,0) \notin D_{a,\alpha,\beta}$. By the first part of the proof we know that $\Phi_{a,\alpha,\beta}(\varepsilon, 0) = (\underbrace{\varepsilon, \dots, \varepsilon}_{s}, c_{s+1}, \dots, c_{n-1}, 0) \in D$ for $0 < \varepsilon \ll 1$. So $(\varepsilon, 0) \in D_{a,\alpha,\beta}$ for $0 < \varepsilon \ll 1$. Consequently, since $D_{a,\alpha,\beta}$ is a domain of holomorphy, we conclude (cf. Theorem 1.11.13) that $(0,0) \in D_{a,\alpha,\beta}$; a contradiction.

Conversely, assume that D is relatively complete. We will prove that each $D_{a,\alpha,\beta}$ is weakly relatively complete. Suppose that $(\lambda_0, 0), (\lambda_1, \mu_1) \in D_{a,\alpha,\beta}$ with $\mu_1 \neq 0$. We want to show that $(\lambda_1, 0) \in D_{a,\alpha,\beta}$. After a permutation of variables, we may assume that $\beta = e_n$. The points

$$b := \Phi_{a,\alpha,\beta}(\lambda_0, 0) = (a_1|\lambda_0|^{\alpha_1}, \dots, a_{n-1}|\lambda_0|^{\alpha_{n-1}}, 0),$$
$$c := \Phi_{a,\alpha,\beta}(\lambda_1, \mu_1) = (a_1|\lambda_1|^{\alpha_1}, \dots, a_{n-1}|\lambda_1|^{\alpha_{n-1}}, a_n|\lambda_1|^{\alpha_n}|\mu_1|)$$

belong to D. Hence $(a_1|\lambda_1|^{\alpha_1}, \dots, a_{n-1}|\lambda_1|^{\alpha_{n-1}}, 0) = \Phi_{a,\alpha,\beta}(\lambda_1, 0) \in D$, which means that $(\lambda_1, 0) \in D_{a,\alpha,\beta}$.

Now suppose that $(0, \mu_0), (\lambda_1, \mu_1) \in D_{a,\alpha,\beta}$ with $\lambda_1 \neq 0$. We want to show that $(0, \mu_1) \in D_{a,\alpha,\beta}$. Observe that $\alpha_1, \dots, \alpha_n \geq 0$. We may assume that $\alpha_1, \dots, \alpha_s > 0$, $\alpha_{s+1} = \dots = \alpha_n = 0$, $1 \le s \le n$. Thus the points

$$\begin{aligned} b &:= \Phi_{a,\alpha,\beta}(0, \mu_0) = (0, \dots, 0, a_{s+1}|\mu_0|^{\beta_{s+1}}, \dots, a_n|\mu_0|^{\beta_n}), \\ c &:= \Phi_{a,\alpha,\beta}(\lambda_1, \mu_1) \\ &\;= (a_1|\lambda_1|^{\alpha_1}|\mu_1|^{\beta_1}, \dots, a_s|\lambda_1|^{\alpha_s}|\mu_1|^{\beta_s}, a_{s+1}|\mu_1|^{\beta_{s+1}}, \dots, a_n|\mu_1|^{\beta_n}) \end{aligned}$$

belong to D. The relative completeness of D implies that

$$\Phi_{a,\alpha,\beta}(0, \mu_1) = (0, \dots, 0, a_{s+1}|\mu_1|^{\beta_{s+1}}, \dots, a_n|\mu_1|^{\beta_n}) \in D.$$

Thus $(0, \mu_1) \in D_{a,\alpha,\beta}$. □

Remark 1.11.19. In [Lan-Spi 1995] the reader may find another geometric characterization of Reinhardt domains of holomorphy in $\mathbb{C}^2$.

1.12 Envelopes of holomorphy

As we already mentioned in § 1.11, there exist pairs of domains $D \subsetneq \widetilde{D} \subset \mathbb{C}^n$ such that $\mathcal{O}(\widetilde{D})|_D = \mathcal{O}(D)$ or $\mathcal{H}^\infty(\widetilde{D})|_D = \mathcal{H}^\infty(D)$. So far we were concentrated on characterization of those (Reinhardt) domains $D \subset \mathbb{C}^n$ which are domains of existence with respect to the family $\mathcal{O}(D)$ of all functions holomorphic on D. In the present section we make a step further and answer a more general question whether for a given (Reinhardt) domain $D \subset \mathbb{C}^n$ and a family $\varnothing \ne \mathcal{S} \subset \mathcal{O}(D)$ there exists a maximal domain $\widetilde{D} \subset \mathbb{C}^n$ such that every function from $\mathcal{S}$ extends holomorphically to $\widetilde{D}$ (cf. [Jar-Pfl 2000], § 1.7, for the general theory of holomorphic extension).

Definition 1.12.1. Let $D \subset \mathbb{C}^n$ be a domain and let $\varnothing \ne \mathcal{S} \subset \mathcal{O}(D)$. We say that a domain $\widetilde{D} \subset \mathbb{C}^n$ is an *$\mathcal{S}$-envelope of holomorphy* if

- $D \subset \widetilde{D}$,
- for any $f \in \mathcal{S}$ there exists an $\tilde{f} \in \mathcal{O}(\widetilde{D})$ with $\tilde{f} = f$ on D (notice that $\tilde{f}$ is uniquely determined by f),
- $\widetilde{D}$ is an $\widetilde{\mathcal{S}}$-domain of holomorphy with $\widetilde{\mathcal{S}} := \{\tilde{f} : f \in \mathcal{S}\}$ (cf. Definition 1.11.1).

In the case $\mathcal{S} = \mathcal{O}(D)$ we say that $\widetilde{D}$ is an *envelope of holomorphy*.

Remark 1.12.2. (a) If $D_1 \subset D_2 \subset \mathbb{C}^n$ are domains and $\widetilde{D}_j$ is an $\mathcal{S}_j$-envelope of holomorphy with respect to a family $\mathcal{S}_j \subset \mathcal{O}(D_j)$, $j = 1, 2$, with $\mathcal{S}_2|_{D_1} \subset \mathcal{S}_1$, then $\widetilde{D}_1 \subset \widetilde{D}_2$.

In particular, the $\mathcal{S}$-envelope of holomorphy $\widetilde{D}$ is uniquely determined. We write $\widetilde{D} = \mathcal{E}(D, \mathcal{S})$. Let $\mathcal{E}(D) := \mathcal{E}(D, \mathcal{O}(D))$.

Indeed, we know that $D_j \subset \widetilde{D}_j$ and for every $f_j \in \mathcal{S}_j$ there exists an $\tilde{f}_j \in \mathcal{O}(\widetilde{D}_j)$ with $\tilde{f}_j = f_j$ on D_j. Moreover, $\widetilde{D}_j$ is an $\widetilde{\mathcal{S}}_j$-domain of holomorphy, where $\widetilde{\mathcal{S}}_j := \{\tilde{f}_j : f_j \in \mathcal{S}_j\}$, $j = 1, 2$. Suppose that $\widetilde{D}_1 \not\subset \widetilde{D}_2$. Then every function

$\tilde{f}_2 \in \widetilde{\mathcal{S}}_2$ extends holomorphically to $\widetilde{D}_1$ (to $(f_2|_{D_1})\tilde{}_1$) with $(f_2|_{D_1})\tilde{}_1 = f_2|_{D_1} = \tilde{f}_2$ on $D_1 \subset \widetilde{D}_1 \cap \widetilde{D}_2$. Consequently, $\widetilde{D}_2$ is not an $\widetilde{\mathcal{S}}_2$-domain of holomorphy; a contradiction.

(b) In general, the $\mathcal{S}$-envelope of holomorphy (in the sense of the above definition) need not exist; take, for example, $D := \mathbb{D}$, $\mathcal{S} := \{\mathrm{Log}(z+1)\}$ (Exercise).

(c) There are also examples of domains $D \subset \mathbb{C}^n$ ($n \geq 2$) such that $\mathcal{E}(D)$ does not exist (see e.g. the Shabat example in [Jar-Pfl 2000], p. 1). The interested reader may consult also [Vla 1966] (to find relations between envelopes of holomorphy and theoretical physics) and [Jup 2006].

(d) Let $D \subset \mathbb{C}^n$ be a *starlike* domain (i.e. $tz \in D$ for every $z \in D$ and $t \in [0,1]$) and let $\mathcal{S} \subset \mathcal{O}(D)$ be such that, for any $f \in \mathcal{S}$ and $t \in (0,1]$, the function $D \ni z \mapsto f(tz)$ belongs to $\mathcal{S}$. Then $\mathcal{E}(D,\mathcal{S})$ exists and is a starlike domain in $\mathbb{C}^n$ (cf. [Jar-Pfl 2000], Remark 1.9.6 (a)).

(e) Let $D \subset \mathbb{C}^n$ be a balanced domain and let $\mathcal{S} \subset \mathcal{O}(D)$ be such that, for any $f \in \mathcal{S}$ and $\lambda \in \overline{\mathbb{D}} \setminus \{0\}$, the function $D \ni z \mapsto f(\lambda z)$ belongs to $\mathcal{S}$. Then $\mathcal{E}(D,\mathcal{S})$ exists and is a balanced domain in $\mathbb{C}^n$ (cf. [Jar-Pfl 2000], Remark 1.9.6 (f)).

(f) There exists a *circular* domain $D \subset \mathbb{C}^2$ (i.e. $\lambda z \in D$ for every $z \in D$ and $\lambda \in \mathbb{T}$) such that $\mathcal{E}(D)$ does not exist (cf. [Cas-Tra 1991], see also [Jar-Pfl 2000], Example 3.1.20).

Remark 1.12.3. Notice that for an arbitrary domain $D \subset \mathbb{C}^n$ and $\mathcal{S} \subset \mathcal{O}(D)$, the $\mathcal{S}$ envelope of holomorphy always exists in the category of Riemann domains – cf. [Jar-Pfl 2000].

In the case of Reinhardt domains we have the following existence theorem.

Theorem 1.12.4. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain and let $\mathcal{S} \subset \mathcal{O}(D)$ be such that*

$$\{f \circ \boldsymbol{T}_\lambda : f \in \mathcal{S},\ \lambda \in \mathbb{T}^n\} = \mathcal{S}. \tag{1.12.1}$$

Let $\widetilde{D} := \operatorname{int} \bigcap_{f \in \mathcal{S}} \mathfrak{D}_f$, where $\mathfrak{D}_f$ denotes the domain of convergence of the Laurent series of f (observe that $\widetilde{D}$ is a Reinhardt log-convex open set; in particular, by Remark 1.5.6 (d)*, $\widetilde{D}$ is connected). Then $\widetilde{D} = \mathcal{E}(D,\mathcal{S})$. If $D \cap V_j \neq \varnothing$, $j = 1,\dots,n$ (e.g. $0 \in D$), then $\mathcal{E}(D,\mathcal{S})$ is a complete Reinhardt domain.*

Observe that all classical spaces of holomorphic functions (e.g. $\mathcal{H}^{\infty,k}(D)$, $\mathcal{A}^k(D)$, $L_h^{p,k}(D)$, $\mathcal{O}^{(k)}(D)$) satisfy (1.12.1). In the case where $\mathcal{S}$ does not satisfy (1.12.1), it is possible that the envelope $\mathcal{E}(D,\mathcal{S})$ exists, but is not a Reinhardt domain, e.g. $D = \mathbb{D}$, $\mathcal{S} := \{1/(z-1)\}$.

Proof. For an arbitrary function $f \in \mathcal{S}$ consider its Laurent expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha^f z^\alpha, \quad z \in D$$

(cf. Proposition 1.7.15 (c)). Obviously, $a_\alpha^{f\circ \boldsymbol{T}_\lambda} = a_\alpha^f \lambda^\alpha$ and $\Sigma(f\circ \boldsymbol{T}_\lambda) = \Sigma(f)$, $\lambda \in \mathbb{T}^n$. The function $\tilde f(z) := \sum_{\alpha\in\mathbb{Z}^n} a_\alpha^f z^\alpha$, $z \in \mathfrak{D}_f$, gives a holomorphic extension of f to $\mathfrak{D}_f$. Observe that $\mathfrak{D}_{f\circ\boldsymbol{T}_\lambda} = \mathfrak{D}_f$ and $\widetilde{f\circ\boldsymbol{T}_\lambda} = \tilde f\circ\boldsymbol{T}_\lambda$, $\lambda\in\mathbb{T}^n$. Thus, every function $f\in\mathcal{S}$ has a holomorphic extension $\tilde f|_{\widetilde D}$ to $\widetilde D$. Notice that if $D\cap \boldsymbol{V}_j \neq \varnothing$, $j = 1,\dots,n$, then each $\mathfrak{D}_f$ is complete and, consequently, $\widetilde D$ is a complete Reinhardt domain. It remains to show that $\widetilde D$ is an $\widetilde{\mathcal{S}}$-domain of holomorphy. Observe that in the case where $\mathcal{S} = \mathcal{O}(D)$, the result follows directly from Proposition 1.11.6 and Remark 1.11.3 (e).

Suppose that $a\in\widetilde D$ is such that $d(T_a\tilde f) \geq s > d_{\widetilde D}(a) =: r$ for any $f\in\mathcal{S}$. Define a new Reinhardt domain

$$G := \bigcup_{\lambda\in\mathbb{T}^n} \boldsymbol{T}_\lambda(\mathbb{P}(a,s)) = \bigcup_{\lambda\in\mathbb{T}^n} \mathbb{P}(\lambda\cdot a, s).$$

For $f\in\mathcal{S}$ let $\hat f(z) := T_a\tilde f(z)$, $z\in\mathbb{P}(a,s)$. Notice that $(\widehat{f\circ\boldsymbol{T}_\lambda})\circ\boldsymbol{T}_\lambda^{-1} = \tilde f$ on $\mathbb{P}(\lambda\cdot a, r)$. Moreover,

$$(\widehat{f\circ\boldsymbol{T}_\lambda})\circ\boldsymbol{T}_\lambda^{-1} = (\widehat{f\circ\boldsymbol{T}_\mu})\circ\boldsymbol{T}_\mu^{-1} \quad\text{on}\quad \mathbb{P}(\lambda\cdot a, s)\cap\mathbb{P}(\mu\cdot a, s),\ \lambda,\mu\in\mathbb{T}^n.$$

Indeed, first observe that $\mathbb{P}(\lambda\cdot a,s)\cap\mathbb{P}(\mu\cdot a,s)$ is convex and, therefore, connected.

- If $\mathbb{P}(\lambda\cdot a,r)\cap\mathbb{P}(\mu\cdot a,r)\neq\varnothing$, then the equality follows easily from the identity principle.
- If $\mathbb{P}(\lambda\cdot a,s)\cap\mathbb{P}(\mu\cdot a,s)\neq\varnothing$ but $\mathbb{P}(\lambda\cdot a,r)\cap\mathbb{P}(\mu\cdot a,r)=\varnothing$, then we proceed as follows.

For each $k\in\{1,\dots,n\}$, take ζ_k^j $(j = 1,\dots,N)$ on the shorter arc of $\mathbb{T}$ determined by λ_k and μ_k in such a way that $\zeta_k^1 = \lambda_k$, $\zeta_k^N = \mu_k$, and

$$|\zeta_k^j - \zeta_k^{j+1}||a_k| < 2r, \quad j = 1,\dots,N-1.$$

Then $\left|\frac{\lambda_k+\mu_k}{2}a_k - \zeta_k^j a_k\right| \leq \frac{|\lambda_k-\mu_k|}{2}|a_k| < s$ and consequently,

$$\frac{\lambda+\mu}{2}\cdot a \in \bigcap_{j=1}^N \mathbb{P}(\zeta^j\cdot a, s).$$

Thus, we have found $\zeta^1,\dots,\zeta^N\in\mathbb{T}^n$ such that

- $\zeta^1 = \lambda$, $\zeta^N = \mu$,
- $\mathbb{P}(\zeta^j\cdot a, r)\cap\mathbb{P}(\zeta^{j+1}\cdot a, r)\neq\varnothing$, $j = 1,\dots,N-1$,
- $\bigcap_{j=1}^N \mathbb{P}(\zeta^j\cdot a, s)\neq\varnothing$,

which permits us to apply successively the previous case and the identity principle.

Consequently, the function $\tilde{\tilde f}\colon G\to\mathbb{C}$ defined by the formula

$$\tilde{\tilde f}(z) := (\widehat{f\circ\boldsymbol{T}_\lambda})\circ\boldsymbol{T}_\lambda^{-1}(z), \quad z\in\mathbb{P}(\lambda\cdot a, s),$$

is well defined. Since $\tilde{\tilde{f}} = \tilde{f}$ in $\mathbb{P}(a, r)$, the Laurent series of $\tilde{\tilde{f}}$ in G coincides with $\sum_{\alpha\in\mathbb{Z}^n} a_\alpha^f z^\alpha$ (EXERCISE), which implies that $G \subset \widetilde{D}$; a contradiction. □

Corollary 1.12.5. *Let $D \subset \mathbb{C}^n$ be a bounded Reinhardt domain of holomorphy and let U be any domain of holomorphy with $\overline{D} \subset U$. Then there exists a Reinhardt domain of holomorphy D' such that $\overline{D} \subset D' \subset U$.*

In particular, if $\overline{D}$ has a neighborhood basis consisting of domains of holomorphy, then $\overline{D}$ has a neighborhood basis consisting of Reinhardt domains of holomorphy.

Proof. Let $2r := d_U(\overline{D}) > 0$. Then

$$G := \bigcup_{z\in D} \mathbb{P}(z, r)$$

is a Reinhardt domain with $\overline{D} \subset G \subset U$. Let $D' := \mathcal{E}(G)$ be the envelope of holomorphy of G. Then G is a Reinhardt domain of holomorphy (Theorem 1.12.4) and $\overline{D} \subset G \subset D' \subset \mathcal{E}(U) = U$ (Remark 1.12.2 (a)). □

Proposition 1.12.6. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain and let $\mathcal{S} \subset \mathcal{O}(D)$ satisfy* (1.12.1).

(a) *If D is an $\mathcal{S}$-domain of holomorphy, then*

$$D = \operatorname{int} \bigcap_{f\in\mathcal{S}} \mathfrak{D}_f,$$

where $\mathfrak{D}_f$ denotes the domain of convergence of the Laurent series of f.

(b) *Let $\varnothing \neq \mathcal{S} \subset \mathcal{H}^\infty(D)$ be such that $\mathcal{S} = \mathbb{R}_{>0} \cdot \mathcal{S}$. If D is an $\mathcal{S}$-domain of holomorphy, then*

$$D = \operatorname{int} \bigcap_{\substack{f\in\mathcal{S},\ \|f\|_D=1 \\ \alpha\in\Sigma(f)_*}} \{z \in \mathbb{C}^n(\alpha) : |a_\alpha^f z^\alpha| < 1\},$$

where

$$f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha^f z^\alpha,\ z \in D, \quad \Sigma(f) = \{\alpha \in \mathbb{Z}^n : a_\alpha^f \neq 0\}.$$

(c)

$$\mathcal{E}(D, \mathcal{H}^\infty(D)) = \operatorname{int} \bigcap_{\substack{f\in\mathcal{H}^\infty(D),\ \|f\|_D=1 \\ \alpha\in\Sigma(f)_*}} \{z \in \mathbb{C}^n(\alpha) : |a_\alpha^f z^\alpha| < 1\}.$$

Proof. (a) follows directly from the proof of Theorem 1.12.4.

(b) By Proposition 1.6.5 (b), for every $f \in \mathcal{H}^\infty(D)$, we have:

$$D \subset \operatorname{int} \bigcap_{\alpha \in \Sigma(f)_*} \{z \in \mathbb{C}^n(\alpha) : |a_\alpha^f z^\alpha| < \|f\|_D\} \subset \mathfrak{D}_f.$$

Hence, using Theorem 1.12.4, we get

$$\begin{aligned} D &\subset \operatorname{int} \bigcap_{f \in \mathcal{S},\, \|f\|_D = 1} \operatorname{int} \bigcap_{\alpha \in \Sigma(f)_*} \{z \in \mathbb{C}^n(\alpha) : |a_\alpha^f z^\alpha| < 1\} \\ &\subset \operatorname{int} \bigcap_{f \in \mathcal{S},\, \|f\|_D = 1} \mathfrak{D}_f = D \quad (\text{EXERCISE}). \end{aligned}$$

(c) The proof of Theorem 1.12.4 shows that

$$\widetilde{D} := \mathcal{E}(D, \mathcal{H}^\infty(D)) = \operatorname{int} \bigcap_{\substack{f \in \mathcal{H}^\infty(D) \\ \|f\|_D = 1}} \mathfrak{D}_f.$$

Let $\tilde{f}$ denote the holomorphic extension of f to $\widetilde{D}$. Recall that $\|\tilde{f}\|_{\widetilde{D}} = \|f\|_D$, $f \in \mathcal{H}^\infty(D)$ (Remark 1.11.3 (k)). Hence, by (b), we have

$$\begin{aligned} \widetilde{D} &= \operatorname{int} \bigcap_{\substack{\tilde{f} \in \mathcal{H}^\infty(\widetilde{D}),\, \|\tilde{f}\|_{\widetilde{D}} = 1 \\ \alpha \in \Sigma(\tilde{f})_*}} \{z \in \mathbb{C}^n(\alpha) : |a_\alpha^{\tilde{f}} z^\alpha| < 1\} \\ &= \operatorname{int} \bigcap_{\substack{f \in \mathcal{H}^\infty(D),\, \|f\|_D = 1 \\ \alpha \in \Sigma(f)_*}} \{z \in \mathbb{C}^n(\alpha) : |a_\alpha^f z^\alpha| < 1\}. \end{aligned}$$ □

Remark 1.12.7. Let $D \subset \mathbb{C}^n$ be a log-convex Reinhardt domain. Then

$$\mathcal{E}(D) = D^* \setminus M(D).$$ [45]

Indeed, Theorem 1.11.13 and Remark 1.11.3 (h) show that $D^* \setminus M(D)$ is a domain of holomorphy containing D. Consequently, $G := \mathcal{E}(D) \subset D^* \setminus M(D)$. Using once again Theorem 1.11.13 (iv), we get $G = G^* \setminus M(G) \supset D^* \setminus M(D)$.

Proposition 1.12.8. *Let $F : G \to D$ be a biholomorphic mapping between two domains $D, G \subset \mathbb{C}^n$. Assume that $\widetilde{G} := \mathcal{E}(G)$ and $\widetilde{D} := \mathcal{E}(D)$ exist (e.g. G and D are Reinhardt domains). Then F extends to a biholomorphic mapping $\widetilde{F} : \widetilde{G} \to \widetilde{D}$.*

[45] Cf. Theorem 1.11.13 (iv).

Proof. Let $\widetilde{F}\colon \widetilde{G} \to \mathbb{C}^n$ denote the holomorphic extension of F. Observe that $\det \widetilde{F}' = \widetilde{\det F'}$. In particular, by Remark 1.11.3 (j), $\det \widetilde{F}'(z) \neq 0$, $z \in \widetilde{G}$, which shows that $\widetilde{F}$ is locally biholomorphic. We only need to show that $\widetilde{F}(\widetilde{G}) \subset \widetilde{D}$ (then $\widetilde{F^{-1}} \circ \widetilde{F} = \mathrm{id}_{\widetilde{D}}$, where $\widetilde{F^{-1}}$ denotes the holomorphic extension of F^{-1} to $\widetilde{D}$ and, consequently, exchanging the roles of G and D finishes the proof). Suppose that $\widetilde{F}(\widetilde{G}) \not\subset \widetilde{D}$ and let $b \in \widetilde{G}$ be such that $\widetilde{F}(b) \notin \widetilde{D}$. Let Ω be the connected component of $\widetilde{F}^{-1}(\widetilde{D})$ containing G. Then, by the identity principle, $\widetilde{F^{-1}} \circ \widetilde{F} = \mathrm{id}_{\Omega}$. Fix an $a \in G$ and let $\gamma\colon [0,1] \to \widetilde{G}$ be a curve with $\gamma(0) = a$, $\gamma(1) = b$. Let $t_0 = \sup\{t \in [0,1] : \gamma([0,t]) \subset \Omega\}$, $c := \gamma(t_0)$. Observe that $\widetilde{F}(c) \in \partial\widetilde{D}$. Since $\widetilde{F}$ is locally biholomorphic, there exists a connected open neighborhood $U \subset \widetilde{G}$ of c such that $\widetilde{F}|_U : U \to \widetilde{F}(U) =: V$ is biholomorphic. Take an arbitrary function $g \in \mathcal{O}(\widetilde{D})$. Then the function $g \circ F$ is holomorphic on G and, therefore, extends to $\widetilde{g \circ F} \in \mathcal{O}(\widetilde{G})$. Observe that, by the identity principle, $\widetilde{g \circ F} = g \circ \widetilde{F}$ on Ω (because we have equality on G). Define $\tilde{g} := \widetilde{g \circ F} \circ (\widetilde{F}|_U)^{-1} \in \mathcal{O}(V)$. Then for $w = \widetilde{F}(z) \in \widetilde{F}(U \cap \Omega) \subset V \cap \widetilde{D}$ we get

$$\tilde{g}(w) = \tilde{g} \circ \widetilde{F}(z) = \widetilde{g \circ F} \circ (\widetilde{F}|_U)^{-1} \circ \widetilde{F}(z) = \widetilde{g \circ F}(z) = g \circ \widetilde{F}(z) = g(w).$$

Consequently, $\widetilde{D}$ is not a domain of holomorphy; a contradiction. □

1.13 Holomorphic convexity

The idea of holomorphic convexity has its roots in the following well-known characterization of convex domains in $\mathbb{R}^m$, namely, an open set $U \subset \mathbb{R}^m$ is convex iff for every compact $K \subset U$ the set

$$\{x \in U : \forall_{a \in \mathbb{R}^m} : \langle x, a\rangle \le \max_{y \in K} \langle y, a\rangle\} = \{x \in U : \forall_{\substack{L:\mathbb{R}^m \to \mathbb{R} \\ L \text{ is linear}}} : L(x) \le \max_K L\}$$

is compact.

Definition 1.13.1. Let $D \subset \mathbb{C}^n$ be a domain and let $\varnothing \neq \mathcal{S} \subset \mathcal{O}(D)$. We say that D is *$\mathcal{S}$-convex* if for every compact $K \subset D$ the set

$$\widehat{K}_{\mathcal{S}} := \{z \in D : \forall_{f \in \mathcal{S}} : |f(z)| \le \|f\|_K\}$$

is compact. In the case where $\mathcal{S} = \mathcal{O}(D)$ we say that D is *holomorphically convex*. Suppose that we have assigned to each domain D a family $\mathcal{F}(D) \subset \mathcal{O}(D)$ (e.g. $D \to \mathcal{H}^\infty(D)$). Then, instead of saying that D is $\mathcal{F}(D)$-convex, we shortly say that D is *$\mathcal{F}$-convex* (e.g. $\mathcal{H}^\infty$-convex).

Exercise 1.13.2. Prove the following:

(a) If $K_1 \subset K_2 \Subset D$ and $\mathcal{S}_1 \subset \mathcal{S}_2$, then $(\widehat{K_1})_{\mathcal{S}_2} \subset (\widehat{K_2})_{\mathcal{S}_1}$.

(b) The set $\widehat{K}_{\mathcal{S}}$ is closed in D.

(c) If $z_1, \dots, z_n \in \mathcal{S}$, then $\widehat{K}_{\mathcal{S}}$ is bounded.

(d) $\|f\|_{\widehat{K}_{\mathcal{S}}} = \|f\|_K$, $f \in \mathcal{S}$. In particular, $\widehat{K}_{\mathcal{S}}$ if is compact, then $\widehat{(\widehat{K}_{\mathcal{S}})}_{\mathcal{S}} = \widehat{K}_{\mathcal{S}}$.

(e) $\widehat{K}_{\mathcal{S}} = \widehat{K}_{\mathcal{G}}$, where $\mathcal{G}$ denotes the closure in $\mathcal{O}(D)$ of the family

$$\{af^k : a \in \mathbb{C},\ f \in \mathcal{S},\ k \in \mathbb{N}\}.$$

(f) If $F\colon D \to D'$ is biholomorphic, then $\widehat{F(K)}_{\mathcal{O}(D')} = F(\widehat{K}_{\mathcal{O}(D)})$ for any compact $K \subset D$. In particular, D is holomorphically convex iff D' is holomorphically convex.

Remark 1.13.3. (a) By Proposition 1.7.15 and Exercise 1.13.2 (e), if D is a Reinhardt domain, then $\widehat{K}_{\mathcal{O}(D)} = \widehat{K}_{\mathcal{S}}$, where

$$\mathcal{S} := \Big\{f \in \mathcal{O}(D) : f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha^f z^\alpha,\ z \in D,\ \#\Sigma(f) < \infty\Big\},$$

where $\Sigma(f) := \{\alpha \in \mathbb{Z}^n : a_\alpha^f \neq 0\}$. In particular, if D is a complete Reinhardt domain, then $\widehat{K}_{\mathcal{O}(D)} = \widehat{K}_{\mathcal{P}(\mathbb{C}^n)|_D}$. See also Proposition 1.13.7.

(b) D is $\mathcal{S}$-convex iff there exists a sequence $(K_\nu)_{\nu=1}^\infty$ of compact subsets of D such that $\widehat{(K_\nu)}_{\mathcal{S}} = K_\nu \subset \operatorname{int} K_{\nu+1}$ for any ν and $D = \bigcup_{\nu=1}^\infty K_\nu$.

Indeed, the implication ($\Leftarrow$) is obvious. To prove ($\Rightarrow$), let $(L_j)_{j=1}^\infty$ be an arbitrary sequence of compact sets such that $L_j \subset \operatorname{int} L_{j+1}$ and $D = \bigcup_{j=1}^\infty L_j$. Put $K_1 := \widehat{(L_1)}_{\mathcal{S}}$. Since $D = \bigcup_{j=1}^\infty \operatorname{int} L_j$, there exists a $j_2 > 1$ such that $K_1 \subset \operatorname{int} L_{j_2}$. Put $K_2 := \widehat{(L_{j_2})}_{\mathcal{S}}$. Now take a $j_3 > j_2$ such that $K_2 \subset \operatorname{int} L_{j_3}$ etc.

Exercise 1.13.4. Let $(K_j)_{j=1}^\infty$ be an arbitrary sequence of compact subsets of a domain $D \subset \mathbb{C}^n$ such that $K_j \subset \operatorname{int} K_{j+1}$ and $D = \bigcup_{j=1}^\infty K_j$. Let $A \subset D$ be an infinite set without accumulation points in D. Prove that there exist sequences $(a_k)_{k=1}^\infty \subset A$ and $(j_k)_{k=1}^\infty \subset \mathbb{N}$, $j_k < j_{k+1}$, such that $a_k \in K_{j_{k+1}} \setminus K_{j_k}$, $k \in \mathbb{N}$.

Theorem 1.13.5 (Holomorphic convexity). *Let $D \subset \mathbb{C}^n$. Then the following conditions are equivalent:*

(i) *D is a domain of holomorphy;*

(ii) *D is holomorphically convex;*

(iii) *$d_D(\widehat{K}_{\mathcal{O}(D)}) = d_D(K)$ for every compact set $K \subset D$, where $d_D(A) := \inf\{d_D(z) : z \in A\}$, $A \subset D$;*

(iv) *$d_D(\widehat{K}_{\mathcal{O}(D)}) > 0$ for every compact set $K \subset D$;*

(v) *For every infinite subset $A \subset D$ without accumulation points in D, there exists a function $f \in \mathcal{O}(D)$ such that $\sup_A |f| = +\infty$.*

Proof. The implications (ii) $\Leftrightarrow$ (iv), (iii) $\Rightarrow$ (iv) are elementary (Exercise). The implication (v) $\Rightarrow$ (i) follows from Remark 1.11.3 (p) (Exercise).

(ii) $\Rightarrow$ (v): By Remark 1.13.3 (b) there exists a sequence $(K_\nu)_{\nu=1}^\infty$ of compact subsets of D such that $\widehat{(K_\nu)}_{\mathcal{O}(D)} = K_\nu \subset \operatorname{int} K_{\nu+1}$ and $\bigcup_{\nu=1}^\infty K_\nu = D$. Using Exercise 1.13.4, we may assume that there is a sequence $(a_\nu)_{\nu=1}^\infty \subset A$ such that $a_\nu \in K_{\nu+1} \setminus K_\nu$, $\nu \geq 1$. Since $a_1 \notin K_1$ and $K_1 = (\widehat{K}_1)_{\mathcal{O}(D)}$, there exists a function $f_1 \in \mathcal{S}$ such that $|f_1(a_1)| > \|f_1\|_{K_1}$. Replacing f_1 by $(af_1)^N$ with suitable $a > 0$ and $N \in \mathbb{N}$, we may assume that $|f_1(a_1)| \geq 1$, and $\|f_1\|_{K_1} \leq 1/2$. Repeating the above argument for the remaining a_ν's, we find a sequence $(f_\nu)_{\nu=1}^\infty \subset \mathcal{O}(D)$ such that $|f_\nu(a_\nu)| \geq \nu + \sum_{\mu=1}^{\nu-1} |f_\mu(a_\nu)|$ and $\|f_\nu\|_{K_\nu} \leq 1/2^\nu$. Now put $f := \sum_{\nu=1}^\infty f_\nu$. The series is locally normally convergent in D. Hence $f \in \mathcal{O}(D)$. Moreover, $|f(a_\nu)| \geq \nu$ for every ν (Exercise).

(i) $\Rightarrow$ (iii): Suppose that for some $a \in \widehat{K}_{\mathcal{O}(D)}$ we have $d_D(a) < d_D(K) =: r$. Let $0 < s < r$. By the Cauchy inequalities we obtain

$$\|D^\alpha f\|_K \leq \frac{\alpha!}{s^{|\alpha|}} \|f\|_{K^{(s)}}, \quad f \in \mathcal{O}(D).$$

Hence we get

$$|D^\alpha f(a)| \leq \frac{\alpha!}{s^{|\alpha|}} \|f\|_{K^{(s)}}, \quad f \in \mathcal{O}(D).$$

In particular, $d(T_a f) \geq s$ and hence $d(T_a f) \geq r$, $f \in \mathcal{O}(D)$. Finally, since D is a domain of holomorphy, we conclude that $\mathbb{P}(a, r) \subset D$; a contradiction. $\square$

Exercise* 1.13.6. Let $D \subset \mathbb{C}^n$ be holomorphically convex and let $A \subset D$ be an infinite set without accumulation points in D. Prove that there exists an $f \in \mathcal{O}(D)$, $f \not\equiv 0$, such that $\sup\{\operatorname{ord}_a f : a \in A\} = +\infty$, where $\operatorname{ord}_a f$ denotes the order of zero of f at a.

Hint. Try to find f as an infinite product.

Proposition 1.13.7. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain. Then for every Reinhardt compact set $K \subset D$ we have $\widehat{K}_{\mathcal{O}(D)} = \widehat{K}_{\mathcal{S}}$, where*

$$\mathcal{S} := \{z^\alpha|_D : \alpha \in \mathbb{Z}^n \text{ is such that } D \subset \mathbb{C}^n(\alpha)\}.$$

Observe that if D is a complete Reinhardt domain, then $\mathcal{S} = \{z^\alpha|_D : \alpha \in \mathbb{Z}_+^n\}$.

Proof. We already know (Remark 1.13.3 (a)) that $\widehat{K}_{\mathcal{O}(D)} = \widehat{K}_{\mathcal{S}_0}$, where

$$\mathcal{S}_0 := \Big\{ \sum_{\substack{\alpha \in \mathbb{Z}^n \\ |\alpha| \leq N}} a_\alpha z^\alpha|_D : N \in \mathbb{N},\ a_\alpha \neq 0 \Rightarrow D \subset \mathbb{C}^n(\alpha) \Big\}.$$

We only need to show that $\widehat{K}_{\mathcal{S}} \subset \widehat{K}_{\mathcal{S}_0}$. To this aim, fix a point $a \in \widehat{K}_{\mathcal{S}}$ and a function $f = \sum_{\substack{\alpha \in \mathbb{Z}^n \\ |\alpha| \leq N}} a_\alpha^f z^\alpha|_D \in \mathcal{S}_0$. The Cauchy inequalities imply that $\|a_\alpha^f z^\alpha\|_K \leq$

$\|f\|_K$, $\alpha \in \Sigma(f)$. Put $C(N) := \#\{\alpha \in \mathbb{Z}^n : |\alpha| \le N\}$. Then

$$|f(a)| \le C(N) \max_{\alpha \in \Sigma(f)} |a^f_\alpha a^\alpha| \le C(N) \max_{\alpha \in \Sigma(f)} \|a^f_\alpha z^\alpha\|_K \le C(N)\|f\|_K.$$

Putting f^k instead of f gives

$$|f^k(a)| \le C(kN)\|f\|_K^k, \quad k \in \mathbb{N}.$$

Hence,

$$|f(a)| \le (C(kN))^{1/k}\|f\|_K.$$

It remains to observe that $(C(kN))^{1/k} \to 1$ when $k \to +\infty$ (Exercise). $\square$

Exercise 1.13.8. Prove that $C(N) = \sum_{k=0}^n \binom{n}{k}\binom{N+n-k}{n}$.

Exercise 1.13.9. Let $D \subset \mathbb{C}^n$ be a balanced domain (Definition 1.8.1). Using Proposition 1.8.4 prove that for every balanced compact set $K \subset D$ we have $\widehat{K}_{\mathcal{O}(D)} = \widehat{K}_{\mathcal{S}}$, where

$$\mathcal{S} := \{Q|_D : Q \in \mathcal{P}(\mathbb{C}^n),\ Q \text{ is a homogeneous polynomial}\}.$$

Definition 1.13.10. We say that a Reinhardt domain $D \subset \mathbb{C}^n$ satisfies the *weak Fu condition* if for every $j \in \{1, \dots, n\}$ the following implication holds:

$$\bar{D} \cap V_j \setminus \Big(\bigcup_{k \ne j} V_k\Big) \ne \varnothing \Longrightarrow D \cap V_j \ne \varnothing.$$

Remark 1.13.11. (a) It is clear that if D satisfies the Fu condition, then D satisfies the weak Fu condition. The domain

$$T_\sigma = \{(z_1, z_2) \in \mathbb{D} \times \mathbb{D} : |z_1|^\sigma < |z_2|\}, \quad \sigma > 0,$$

satisfies the weak Fu condition but does not satisfy the Fu condition.

(b) If a Reinhardt domain of holomorphy $D \subset \mathbb{C}^n$ satisfies the weak Fu condition, then D is fat (cf. Corollary 1.11.14).

Indeed, it follows from Theorem 1.11.13 that $D = D^* \setminus M$, where $M := \bigcup_{j \in I} V_j$, $I := \{j : V_j \cap D = \varnothing\}$. It remains to observe that $M \cap D^* \ne \varnothing \Leftrightarrow \exists_{j \in I} : D^* \cap V_j \setminus \bigcup_{k \ne j} V_k \ne \varnothing$.

Remark 1.13.12. Let $D \subset \mathbb{C}^n$ be a domain and let $\varnothing \ne \mathcal{S} \subset \mathcal{O}(D)$. It is natural to ask whether D is an $\mathcal{S}$-domain of holomorphy iff D is $\mathcal{S}$-convex.

Consider, for example, the case where $\mathcal{S} = \mathcal{H}^\infty(D)$.

(a) If $D \subset \mathbb{C}$ is a bounded domain, then D is an $\mathcal{H}^\infty$-domain of holomorphy iff D is $\mathcal{H}^\infty$-convex (cf. [Ahe-Sch 1975], see also [Jar-Pfl 2000], Theorem 4.1.1).

(b) Let $T_\sigma := \{(z_1, z_2) \in \mathbb{D}^2 : |z_1|^\sigma < |z_2|\}, \sigma = p/q \in \mathbb{Q}_{>0}$. Recall T_σ is an $\mathcal{H}^\infty$-domain of holomorphy (Remark 1.11.5 (a)). Moreover, T_σ is not $\mathcal{H}^\infty$-convex (also for arbitrary $\sigma > 0$).

Indeed, let

$$K := \{(0, z_2) : |z_2| = 1/2\} \subset T_\sigma.$$

Then, using the one-dimensional Riemann theorem on removable singularities, we get $\{(0, z_2) : 0 < |z_2| \le 1/2\} \subset \widehat{K}_{\mathcal{H}^\infty(T_\sigma)}$ (EXERCISE), which implies that $\widehat{K}_{\mathcal{H}^\infty(T_\sigma)}$ is not compact.

Observe that for any compact $K \subset T_\sigma$ we have $\overline{\widehat{K}_{\mathcal{H}^\infty(T_\sigma)}} \cap \partial T_\sigma \subset \{(0,0)\}$.

For, let $f(z_1, z_2) := z_1^p/z_2^q$, $f \in \mathcal{H}^\infty(T_\sigma)$. Suppose that there exists a sequence $\widehat{K}_{\mathcal{H}^\infty(T_\sigma)} \ni b_k \to b \in (\partial T_\sigma) \setminus \{(0,0)\}$. If $|b_2| = 1$, then $1 = \lim_{k\to+\infty} |b_{k,2}| \le \|z_2\|_K < 1$; a contradiction. If $|b_1|^\sigma = |b_2| < 1$, then $1 = \lim_{k\to+\infty} |f(b_k)| \le \|f\|_K < 1$; a contradiction.

(c) N. Sibony in [Sib 1975] constructed an example of a fat domain $D \subsetneq \mathbb{D} \times \mathbb{D}$ such that D is $\mathcal{H}^\infty$-convex, but $\mathcal{H}^\infty(D) = \mathcal{H}^\infty(\mathbb{D} \times \mathbb{D})|_D$; in particular, D is not an $\mathcal{H}^\infty$-domain of holomorphy.

(d) Let $D \subset \mathbb{C}^n$ be a Reinhardt $\mathcal{H}^\infty$-convex domain. Then D satisfies the weak Fu condition (in particular, D is fat).

Indeed, suppose that $\bar{D} \cap V_j \setminus \bigcup_{k \ne j} V_k \ne \varnothing$ and $D \cap V_j = \varnothing$. We may assume that $j = n$. Then, by Lemma 1.5.15, for every $a = (a', a_n) \in D \cap \mathbb{C}_*^n$, the set $\{a'\} \times (\bar{K}(\varepsilon) \setminus \{0\})$ is contained in D for an $\varepsilon > 0$. Let $K := \{a'\} \times \partial K(\varepsilon) \Subset D$. Then (cf. (b)) $\{a'\} \times K_*(\varepsilon) \subset \widehat{K}_{\mathcal{H}^\infty(D)} \Subset D$;[46] a contradiction.

Proposition 1.13.13. *Let $D \subset \mathbb{C}^n$ be a log-convex Reinhardt domain.*

(a) *If D is L_h^2-convex, then D satisfies the weak Fu condition (in particular, D is fat).*

(b) *If $L_h^2(D) \ne \{0\}$ (in particular, if D is L_h^2-convex), then $\boldsymbol{E}(\log D) = \{0\}$ (cf. Lemma 1.5.14).*

We need the following two lemmas.

Lemma 1.13.14. *Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^m$ be arbitrary domains, and let $f \in L_h^p(D \times G)$ $(1 \le p < +\infty)$. Then $f(z, \cdot) \in L_h^p(G)$ for every $z \in D$.*

Proof. Take a $z_0 \in D$ and let $\mathbb{P}(z_0, r) \Subset D$. Then, by Lemma 1.7.22 (with $K := \{z_0\}$) in the case where $p \in \mathbb{N}$, or by Proposition 1.14.14 (with $u := |f|^p$) in the general case, we get

$$\begin{aligned} |f(z_0, w)|^p &\le \frac{1}{(\pi r^2)^n} \int_{\mathbb{P}(z_0,r)} |f(z, w)|^p \, d\Lambda_{2n}(z) \\ &\le \frac{1}{(\pi r^2)^n} \int_D |f(z, w)|^p \, d\Lambda_{2n}(z), \quad w \in G. \end{aligned}$$

[46] Recall that $K_*(r) = K(r) \setminus \{0\}$.

Consequently, by the Fubini theorem,

$$\begin{aligned}\int_G |f(z_0,w)|^p \, d\Lambda_{2m}(w) &\le \frac{1}{(\pi r^2)^n} \int_G \Big(\int_D |f(z,w)|^p \, d\Lambda_{2n}(z) \Big) d\Lambda_{2m}(w) \\ &= \frac{1}{(\pi r^2)^n} \int_{D\times G} |f(z,w)|^p \, d\Lambda_{2(n+m)}(z,w),\end{aligned}$$

i.e. $f(z_0,\cdot) \in L_h^p(G)$. □

Lemma 1.13.15. *Let $f \in L_h^2(\mathbb{D}_*)$. Then f extends holomorphically to $\mathbb{D}$.*

Proof. Write $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$, $z \in \mathbb{D}_*$. Then, by Example 1.10.7 (c), we get

$$2\pi |a_k|^2 \int_0^1 r^{2k+1} dr = \|a_k z^k\|^2_{L^2(\mathbb{D}_*)} \le \|f\|^2_{L^2(\mathbb{D}_*)}, \quad k \in \Sigma(f).$$

Consequently, $\Sigma(f) \subset \mathbb{Z}_+$. □

Proof of Proposition 1.13.13. (a) We argue as in Remark 1.13.12 (d). Suppose that $\bar{D} \cap (\mathbb{C}_*^{n-1} \times \{0\}) \ne \varnothing$ and $D \cap V_n = \varnothing$. By Lemma 1.5.15, for every $a = (a', a_n) \in D \cap \mathbb{C}_*^n$ there exists an $\varepsilon > 0$ so small that $\mathbb{P}(a',\varepsilon) \times (\bar{K}(\varepsilon) \setminus \{0\}) \subset D$. By Lemma 1.13.14, $f(a',\cdot) \in L_h^2(K_*(\varepsilon))$. Put $K := \{a'\} \times \partial K(\varepsilon) \subset D$. Then, by Lemma 1.13.15, $\{a'\} \times K_*(\varepsilon) \subset \hat{K}_{L_h^2(D)} \Subset D$; a contradiction.

(b) Put $X := \log D$. Suppose that $F := \boldsymbol{E}(X) \ne \{0\}$. Let $f \in L_h^2(D)$, $f \not\equiv 0$, $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha^f z^\alpha$. By Example 1.10.7 (c), there exists an $\alpha^0 \in \Sigma(f)$ such that $z^{\alpha^0} \in L_h^2(D)$. Recall (Remark 1.4.7 (f)) that $X = Y + F$, where $Y \subset F^\perp$. Write $\mathbb{R}^n \ni x = x' + x'' \in F^\perp + F$. Then, using the Fubini theorem, we obtain

$$\begin{aligned}\|z^{\alpha^0}\|^2_{L^2(D)} &= \int_D |z^{\alpha^0}|^2 d\Lambda_{2n}(z) \\ &= (2\pi)^n \int_{\boldsymbol{R}(D)} r^{2\alpha^0+\boldsymbol{1}} d\Lambda_n(r) \overset{r=e^x}{=} (2\pi)^n \int_X e^{\langle x, 2\alpha^0+\boldsymbol{2}\rangle} d\Lambda_n(x) \\ &= (2\pi)^n \int_Y e^{\langle x', 2\alpha^0+\boldsymbol{2}\rangle} d\Lambda_{F^\perp}(x') \int_F e^{\langle x'', 2\alpha^0+\boldsymbol{2}\rangle} d\Lambda_F(x'') \\ &= (2\pi)^n \int_Y e^{\langle x', 2\alpha^0+\boldsymbol{2}\rangle} d\Lambda_{F^\perp}(x') \cdot (+\infty) = +\infty;\end{aligned}$$

a contradiction. □

Example 1.13.16. Let $T_\sigma := \{(z_1,z_2) \in \mathbb{D}^2 : |z_1|^\sigma < |z_2|\}$, $\sigma \in \mathbb{Q}_{>0}$. Then T_σ is L_h^p-convex iff $1 \le p < 2(1 + 1/\sigma)$. In particular, the Hartogs triangle T is L_h^p-convex iff $1 \le p < 4$.

Indeed, since $\mathcal{H}^\infty(T_\sigma) \subset L^p_h(T_\sigma)$, Remark 1.13.12 (b) implies that $\overline{\widehat{K}_{L^p_h(T_\sigma)}} \cap \partial T_\sigma \subset \{(0,0)\}$ for any compact $K \subset T_\sigma$.

To simplify notation put $\tau := 1/\sigma$. First we will find a criterion for the function z^α ($\alpha \in \mathbb{Z}^2$) to be in the space $L^p_h(T_\sigma)$. We have

$$\begin{aligned}
\int_{T_\sigma} |z^\alpha|^p d\Lambda_4(z) &= (2\pi)^2 \int_{\boldsymbol{R}(T_\sigma)} r^{p\alpha+\boldsymbol{1}} d\Lambda_2(r) \\
&= (2\pi)^2 \int_0^1 \Big(\int_0^{r_2^\tau} r_1^{p\alpha_1+1} dr_1 \Big) r_2^{p\alpha_2+1} dr_2 \\
(\text{if } p\alpha_1+1>-1) \quad &= (2\pi)^2 \int_0^1 \frac{r_2^{(p\alpha_1+2)\tau}}{p\alpha_1+2} r_2^{p\alpha_2+1} dr_2 \\
(\text{if } (p\alpha_1+2)\tau+p\alpha_2+1>-1) \quad &= \frac{(2\pi)^2}{(p\alpha_1+2)\tau+p\alpha_2+2}.
\end{aligned}$$

Thus $z^\alpha \in L^p_h(T_\sigma) \Leftrightarrow \alpha_1 > -2/p,\ p(\alpha_1+\alpha_2) > -2(1+\tau)$. In particular, $1/z_2 \in L^p_h(T_\sigma) \Leftrightarrow p < 2(1+\tau)$. Observe that the function $1/z_2$ explodes at zero. Hence T_σ is $L^p_h(T_\sigma)$-convex for all $1 \le p < 2(1+\tau)$.

Suppose that T_σ is $L^{2(1+\tau)}_h$-convex. To get a contradiction it suffices to prove that for every function $f \in L^{2(1+\tau)}_h(T_\sigma)$, the function $f(0,\cdot)$ extends holomorphically to $\mathbb{D}$. Write $f(z) = \sum_{\alpha\in\mathbb{Z}^2} a_\alpha z^\alpha$, $z \in T_\sigma$. By Example 1.10.7 (c) and the above criterion (with $p = 2(1+\tau)$), we know that

$$\begin{aligned}
\Sigma(f) &\subset \{\alpha \in \mathbb{Z}^2 : \alpha_1 > -1/(1+\tau),\ \alpha_1+\alpha_2 > -1\} \\
&= \{\alpha \in \mathbb{Z}_+ \times \mathbb{Z} : \alpha_1+\alpha_2 \ge 0\}.
\end{aligned}$$

Hence

$$f(0,z_2) = \sum_{\alpha_2\in\mathbb{Z}_+} a_{0,\alpha_2} z_2^{\alpha_2}, \quad z_2 \in \mathbb{D}_*,$$

which implies that the function $f(0,\cdot)$ extends holomorphically to $\mathbb{D}$.

Remark 1.13.17. (a) Proposition 1.13.13 and Theorem 3.6.4 will show that *every L^2_h-convex Reinhardt domain $D \subset \mathbb{C}^n$ is an L^2_h-domain of holomorphy.*

(b) Notice that the following general result is true (cf. [Irg 2002], Theorem IV.1): *Any bounded L^2_h-convex domain $D \subset \mathbb{C}^n$ is an L^2_h-domain of holomorphy.*

Proposition 1.13.18 ([Pfl 1984]). *Let $D \subset \mathbb{C}^n$ be an arbitrary domain and let $a \in D$. Put*

$$\mathcal{F}_a(D) := \{f \in \mathcal{O}(D,\mathbb{D}) : f(a) = 0\}.$$

Then the following conditions are equivalent:

(i) *for any infinite set* $A \subset D$ *without accumulation points in* D *we have*

$$\sup\{|f(b)| : f \in \mathcal{F}_a(D),\ b \in A\} = 1;$$

(ii) *for any infinite set* $A \subset D$ *without accumulation points in* D*, there exists a function* $f_0 \in \mathcal{F}_a(D)$ *such that* $\sup\{|f_0(b)| : b \in A\} = 1$.

Observe that (ii) implies that D is $\mathcal{H}^\infty$-convex. Indeed, suppose that $A \subset \widehat{K}_{\mathcal{H}^\infty(D)}$ has no accumulation points and let f_0 be as in (ii). Then $\sup\{|f_0(b)| : b \in A\} \le \|f_0\|_K < 1$; a contradiction.

Proof. (i) $\Rightarrow$ (ii): Take sequences $(b^k)_{k=1}^\infty \subset A$ and $(f_k)_{k=1}^\infty \subset \mathcal{F}_a(D)$ such that $f_k(b^k) \ge 1 - 1/2^{2k}$, $k = 1, 2, \dots$. Put

$$g_k := \frac{1 + f_k}{1 - f_k}.$$

Then $g_k \in \mathcal{O}(D, \mathbb{H}^+)$ and $g_k(a) = 1$, where $\mathbb{H}^+ := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. Let

$$g := \sum_{k=1}^\infty \frac{1}{2^k} g_k.$$

Let

$$M_D(z) := \sup\{|f(z)| : f \in \mathcal{F}_a(D)\}$$

(cf. Example 4.2.3). Observe that, by Lemma 1.7.23, M_D is continuous, and by the Montel theorem (Theorem 1.7.24), $M_D < 1$. Hence, for any compact set $K \subset D$ there exists $\theta = \theta(K) < 1$ such that $\|f_k\|_K \le \theta$, $k = 1, 2, \dots$. Consequently, $\|g_k\|_K \le 2/(1-\theta)$, $k = 1, 2, \dots$. It follows that the series is convergent locally uniformly in D and so $g \in \mathcal{O}(D, \mathbb{H}^+)$. Note that $g(a) = 1$. We have

$$|g(b^k)| \ge |\operatorname{Re} g(b^k)| \ge \frac{1}{2^k} g_k(b^k) \ge 2^k \to +\infty.$$

Now, put

$$f := \frac{g - 1}{g + 1}.$$

Then $f \in \mathcal{F}_a(D)$ and

$$|f(b^k)| \ge \frac{|g(b^k)| - 1}{|g(b^k)| + 1} \ge \frac{2^k - 1}{2^k + 1} \to 1. \qquad \square$$

Theorem 1.13.19 ([Pfl 1984], [Fu 1994]). *Let* $D \subset \mathbb{C}^n$ *be a bounded Reinhardt domain of holomorphy satisfying the Fu condition. Then for any points* $a \in D$*,* $b \in \partial D$ *and for any sequence* $D \ni b^k \to b$*, there exists a sequence* $(f_k)_{k=1}^\infty \subset \mathcal{O}(D, \mathbb{D})$ *such that* $f_k(a) = 0$ *and* $|f_k(b^k)| \to 1$.

In particular, by the remark after Proposition 1.13.18, D *is* $\mathcal{H}^\infty$*-convex.*

Proof. We may assume that $D \subset \mathbb{D}^n$. Since D satisfies the Fu condition, we may assume that $D \cap \boldsymbol{V}_j \neq \varnothing$, $j = 1, \dots, s$, and $\overline{D} \cap \boldsymbol{V}_j = \varnothing$, $j = s+1, \dots, n$, for some $0 \le s \le n$. Thus there exists an $\eta_0 \in (0,1)$ such that

$$D \subset G := \{z \in \mathbb{D}^n : |z_j| > \eta_0, \ j = s+1, \dots, n\}.$$

Fix $a \in D$, $b \in \partial D$, and a sequence $D \ni b^k \to b$.

First consider the case where $b \in \mathbb{C}_*^n$. We may assume that $b \in \mathbb{R}_{>0}^n$. Let

$$U := \{z \in \mathbb{C}_*^n : |\log|z_j|| < 2|\log b_j|, \ j = 1, \dots, n\};$$

U is an open neighborhood of b. We may assume that $b^k \in U$, $k \in \mathbb{N}$.

First observe that it suffices to prove that

(*) there exists a sequence $(\varphi_k)_{k=1}^\infty \subset \mathcal{O}(D, \mathbb{D})$ such that $|\varphi_k(b^k)| \to 1$ and $\sup\{|\varphi_k(a)| : k \in \mathbb{N}\} \le 1/2$.

Indeed, suppose that we have found such a sequence. For $c \in \mathbb{D}$ let

$$h_c(\lambda) := \frac{\lambda - c}{1 - \bar{c}\lambda}, \quad \lambda \in \mathbb{C} \setminus \{1/\bar{c}\};$$

$h_c|_{\mathbb{D}}$ is a Möbius automorphism of $\mathbb{D}$ with $h_c(c) = 0$. Define $f_k := h_{\varphi_k(a)} \circ \varphi_k$. Then, obviously, $f_k \in \mathcal{O}(D, \mathbb{D})$ and $f_k(a) = 0$. To show that $|f_k(b^k)| \to 1$, take an arbitrary convergent subsequence $|f_{k_\ell}(b^{k_\ell})| \to t_0 \in [0,1]$. We may assume that $\varphi_{k_\ell}(b^{k_\ell}) \to c_0 \in \partial\mathbb{D}$, $\varphi_{k_\ell}(a) \to c_1 \in \mathbb{D}$. Then

$$t_0 = \lim_{\ell \to +\infty} |f_{k_\ell}(b^{k_\ell})| = \lim_{\ell \to +\infty} \left| \frac{\varphi_{k_\ell}(b^{k_\ell}) - \varphi_{k_\ell}(a)}{1 - \overline{\varphi_{k_\ell}(a)} \cdot \varphi_{k_\ell}(b^{k_\ell})} \right| = \left| \frac{c_0 - c_1}{1 - \bar{c}_1 c_0} \right| = 1.$$

Since $X := \log D$ is convex and $x_0 := (\log|b_1|, \dots, \log|b_n|) \in \partial X$, there exists a vector $\alpha \in (\mathbb{R}^n)_*$ such that $X \subset H_\alpha^{x_0}$. Put $c := \langle x_0, \alpha \rangle$. Observe that $D \subset \boldsymbol{D}_{\alpha,c}$. Since $D \cap V_j \neq \varnothing$, $j = 1, \dots, s$, we conclude that $\alpha_1, \dots, \alpha_s \ge 0$ (Exercise). We may assume that $\alpha_1 = \dots = \alpha_t = 0, \alpha_{t+1}, \dots, \alpha_s > 0, 0 \le t \le s$.

Take an arbitrary $\varepsilon > 0$. By the Kronecker theorem,[47] there exist sequences $(p_{\nu,j})_{\nu=1}^\infty \subset \mathbb{Z}$, $j = 1, \dots, n$, $(q_\nu)_{\nu=1}^\infty \subset \mathbb{N}$ such that

$$|p_{\nu,j} - q_\nu \alpha_j| \le \varepsilon, \quad \operatorname{sgn} p_{\nu,j} = \operatorname{sgn} \alpha_j, \quad j = 1, \dots, n, \ q_\nu \to +\infty.$$

---

[47] There are the following two equivalent formulations of the Kronecker theorem (cf. [Har-Wri 1979], Theorems 442 and 444).

Theorem. *Assume that* $\alpha_1, \dots, \alpha_n, 1$ *are linearly independent over* $\mathbb{Q}$*. Let* $\mu_1, \dots, \mu_n \in \mathbb{R}$, $\varepsilon > 0$ *and* $C > 0$ *be arbitrary. Then there exist* $p_1, \dots, p_n, q \in \mathbb{Z}$ *such that* $q \ge C$ *and* $|q\alpha_j - p_j - \mu_j| \le \varepsilon$, $j = 1, \dots, n$.

In particular, the set

$$\{(q\alpha_1 - \lfloor q\alpha_1 \rfloor, \dots, q\alpha_n - \lfloor q\alpha_n \rfloor) : q \in \mathbb{N}\}$$

is dense in $[0,1]^n$.

For example, the set $\{e^{i\ell\alpha 2\pi} : \ell \in \mathbb{N}\}$ *is dense in* $\mathbb{T}$ *when* $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Put

$$\psi_{\varepsilon,\nu}(z) := e^{-q_\nu c} z_1^{p_{\nu,1}} \cdots z_n^{p_{\nu,n}}, \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n(\alpha).$$

Then

$$\log|\psi_{\varepsilon,\nu}(z)| = q_\nu\Big(-c + \sum_{j=t+1}^{n} \alpha_j \log|z_j|\Big) + \sum_{j=t+1}^{n} (p_{\nu,j} - q_\nu\alpha_j)\log|z_j|,$$
$$z \in \mathbb{C}^t \times \mathbb{C}_*^{n-t}.$$

In particular, if $z \in U$, then

$$\log|\psi_{\varepsilon,\nu}(z)| \geq q_\nu\Big(-c + \sum_{j=t+1}^{n} \alpha_j \log|z_j|\Big) - \varepsilon M_b,$$

where $M_b := 2\sum_{j=t+1}^{n} |\log b_j|$ (note that M_b depends only on b). In other words,

$$|\psi_{\varepsilon,\nu}(z)| \geq (e^{-c}|z^\alpha|)^{q_\nu} e^{-\varepsilon M_b}, \quad z \in U.$$

Consequently, letting $D \cap U \ni z \to b$, we conclude that $\|\psi_{\varepsilon,\nu}\|_D \geq e^{-\varepsilon M_b}$. Let $\varphi_{\varepsilon,\nu} := \psi_{\varepsilon,\nu}/\|\psi_{\varepsilon,\nu}\|_D$. To estimate $\varphi_{\varepsilon,\nu}(a)$ we argue as follows. There are two cases.

- There exists a $j_0 \in \{t+1, \dots, s\}$ such that $a_{j_0} = 0$: Then, obviously, $\varphi_{\varepsilon,\nu}(a) = 0$.
- $a_{t+1} \cdots a_s \neq 0$: Then $a \in \mathbb{C}^t \times \mathbb{C}_*^{n-t}$ and

$$\log|\psi_{\varepsilon,\nu}(a)| \leq q_\nu\Big(-c + \sum_{j=t+1}^{n} \alpha_j \log|a_j|\Big) + \varepsilon M_a,$$

where $M_a := \sum_{j=t+1}^{n} |\log|a_j||$ (M_a depends only on a). Thus

$$|\psi_{\varepsilon,\nu}(a)| \leq (e^{-c}|a^\alpha|)^{q_\nu} e^{\varepsilon M_a}$$

and hence

$$|\varphi_{\varepsilon,\nu}(a)| \leq (e^{-c}|a^\alpha|)^{q_\nu} e^{\varepsilon(M_a+M_b)}.$$

Theorem. *Assume that $\alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$. Let $\mu_1, \dots, \mu_n \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$ be arbitrary. Then there exist $p_1, \dots, p_n \in \mathbb{Z}$, $q \in \mathbb{R}$, such that $q \geq C$ and $|q\alpha_j - p_j - \mu_j| \leq \varepsilon$, $j = 1, \dots, n$.*

In the case $\mu_1 = \dots = \mu_n = 0$, as a direct consequence one obtains the following approximation theorem (Exercise).

Theorem. *Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}$, $\varepsilon > 0$ and $C > 0$ be arbitrary. Then there exist $p_1, \dots, p_n, q \in \mathbb{Z}$ such that $q \geq C$ and $|q\alpha_j - p_j| \leq \varepsilon$, $\operatorname{sgn} p_j = \operatorname{sgn}\alpha_j$, $j = 1, \dots, n$.*

Recall that $q_\nu \to +\infty$. Consequently, we may assume that $|\varphi_{\varepsilon,\nu}(a)| \le 1/2$, $\nu \in \mathbb{N}$.

Now, we are going to estimate $\|\psi_{\varepsilon,\nu}\|_D$ from above. There are two cases:

- $t < s$: Then we have

$$\log|\psi_{\varepsilon,\nu}(z)| \le \sum_{j=t+1}^{n} \varepsilon|\log|z_j|| \le \varepsilon((s-t)|\log\eta| + (n-s)|\log\eta_0|) =: \varepsilon M_\eta,$$
$$z \in A_\eta \cap (\mathbb{C}^t \times \mathbb{C}_*^{n-t}),$$

where

$$A_\eta := \{z \in G \cap \boldsymbol{D}_{\alpha,c} : |z_j| \ge \eta,\ j = t+1,\dots,s\}$$

and $0 < \eta < 1$ is so small that $\{z \in G : |z_j| < \eta,\ j = t+1,\dots,s\} \Subset \boldsymbol{D}_{\alpha,c}$. The maximum principle implies that $|\psi_{\varepsilon,\nu}| \le e^{\varepsilon M_\eta}$ on $G \cap \boldsymbol{D}_{\alpha,c} \supset D$.

- $t = s$: Then

$$\log|\psi_{\varepsilon,\nu}(z)| \le \varepsilon(n-s)|\log\eta_0|, \quad z \in D \cap (\mathbb{C}^t \times \mathbb{C}_*^{n-t}).$$

Thus,
$$|\psi_{\varepsilon,\nu}(z)| \le e^{\varepsilon M_0}, \quad z \in D,$$
where M_0 is independent of ε and ν. Consequently, if $z \in U$, then

$$|\varphi_{\varepsilon,\nu}(z)| \ge (e^{-c}|z^\alpha|)^{q_\nu} e^{-\varepsilon(M_b+M_0)}, \quad \nu = 1,2,\dots.$$

Suppose that (*) is not true, i.e. there exists a $\theta \in (0,1)$ such that

$$\sup\{|\varphi(b^k)| : \varphi \in \mathcal{O}(D,\mathbb{D}),\ |\varphi(a)| \le 1/2\} \le \theta, \quad k = 1,2,\dots.$$

Then
$$(e^{-c}|(b^k)^\alpha|)^{q_\nu} e^{-\varepsilon(M_b+M_0)} \le \theta, \quad \varepsilon > 0,\ \nu,\ k = 1,2,\dots.$$
Fixing ε and ν, and next letting $k \to +\infty$, we get

$$e^{-\varepsilon(M_b+M_0)} \le \theta, \quad \varepsilon > 0;$$

a contradiction.

Now, consider the case where $b_1 \dots b_n = 0$. Observe that the case $b = 0$ is excluded – then $s = n$ and, consequently, D is complete, which gives a contradiction (because $b \in \partial D$).

We may assume that $b_1 = \dots = b_r = 0$, $b_{r+1}\dots b_n \ne 0$, $1 \le r \le s$. Let $D' := \{z' \in \mathbb{C}^{n-r} : (0,\dots,0,z') \in D\} = \pi(D)$, where $\pi : \mathbb{C}^n \to \mathbb{C}^{n-r}$, $\pi(z) := (z_{r+1},\dots,z_n)$. Observe that D' is a bounded Reinhardt domain of holomorphy with the Fu condition and $\pi(b) \in \partial D'$. We repeat the above argument with $\pi(a), \pi(b), \pi(b^k)$. We get a sequence $f_k' \in \mathcal{O}(D',\mathbb{D})$, $k \in \mathbb{N}$, with $f_k'(\pi(a)) = 0$, $|f_k'(\pi(b^k))| \to 1$. Now we only need to define $f_k := f_k' \circ \pi$, $k \in \mathbb{N}$. $\square$

Remark 1.13.20. Let us summarize what we have proved so far. For a Reinhardt domain of holomorphy $D \subset \mathbb{C}^n$, consider the following conditions:

(1) D is bounded.
(2) D is fat.
(3) D satisfies the weak Fu condition.
(4) D satisfies the Fu condition.
(5) $\boldsymbol{E}(\log D) = \{0\}$.
(6) D is $\mathcal{H}^\infty$-convex.
(7) D is L^2_h-convex.

Then:

- (1) + (6) $\Rightarrow$ (7).
- (1) $\Rightarrow$ (5).
- (4) $\Rightarrow$ (3) $\Rightarrow$ (2) (Remark 1.13.11 (b)).
- (6) $\Rightarrow$ (3) (Remark 1.13.12 (d)).
- (7) $\Rightarrow$ (3) + (5) (Proposition 1.13.13).
- (1) + (4) $\Rightarrow$ (6) (Theorem 1.13.19).
- (1) + (3) + (7) $\not\Rightarrow$ (6) ($D := T$; Remark 1.13.12 (b), Example 1.13.16).
- $\boxed{?}$ (1) + (3) $\Rightarrow$ (7). $\boxed{?}$

1.14 Plurisubharmonic functions

Our experiences so far have shown that complex analysis has some relations to convex analysis in the real sense. Convex functions of one real variable may be understood as "sub-affine" functions. Affine functions of one real variable are solutions of the equation $u'' = 0$. This equation in the case of several real variables corresponds to the Laplace equation $\Delta u = 0$. Consequently, the harmonic functions of n real variables correspond in some sense to the affine functions of one real variable. Thus, it is natural to introduce subharmonic functions and, finally, plurisubharmonic functions of n complex variables (as those functions that are subharmonic on every complex affine line) (cf. [Rad 1937], [Vla 1966], [Hay-Ken 1976], [Kli 1991], [Ran 1995], [Jar-Pfl 2000]). *We assume that the reader is familiar with basic properties of subharmonic functions* (*in* $\mathbb{C}$).

Let $\Omega \subset \mathbb{C}^n$ be open. For $u \colon \Omega \to \mathbb{R}_{-\infty}$, $a \in \Omega$, and $X \in \mathbb{C}^n$, we define

$$\Omega_{a,X} := \{\lambda \in \mathbb{C} : a + \lambda X \in \Omega\}, \quad \Omega_{a,X} \ni \lambda \overset{u_{a,X}}{\longmapsto} u(a + \lambda X).$$

Definition 1.14.1. A function $u \colon \Omega \to \mathbb{R}_{-\infty}$ is called *plurisubharmonic* (briefly *psh*; $u \in \mathcal{PSH}(\Omega)$) if

- u is upper semicontinuous on Ω ($u \in \mathcal{C}^{\uparrow}(\Omega)$) (cf. p. 28),
- for every $a \in \Omega$ and $X \in \mathbb{C}^n$ the function $u_{a,X}$ is subharmonic in a neighborhood of zero.

We say that a function $u\colon \Omega \to \mathbb{R}_+$ is *logarithmically plurisubharmonic* (*log-psh*) if $\log u \in \mathcal{PSH}(\Omega)$.

Exercise 1.14.2. (a) Let $L\colon \mathbb{C}^n \to \mathbb{R}$ be an $\mathbb{R}$-linear mapping. Decide whether $L \in \mathcal{PSH}(\mathbb{C}^n)$.

(b) Prove that every complex seminorm $q\colon \mathbb{C}^n \to \mathbb{R}_+$ is plurisubharmonic.

Remark 1.14.3. Directly from the theory of subharmonic functions one gets the following properties of psh functions (Exercise).

(a) For an upper semicontinuous function $u\colon \Omega \to \mathbb{R}_{-\infty}$ the following conditions are equivalent:

(i) $u \in \mathcal{PSH}(\Omega)$;

(ii) $\forall_{a\in\Omega}\ \forall_{X\in\mathbb{C}^n\colon \|X\|_\infty=1}\ \exists_{0<R\le d_\Omega(a)}$:

$$u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a+re^{i\theta}X)\,d\theta, \quad 0<r<R;$$

(iii) $\forall_{a\in\Omega}\ \forall_{X\in\mathbb{C}^n\colon \|X\|_\infty=1}\ \exists_{0<R\le d_\Omega(a)}$:

$$u(a) \le \frac{1}{\pi r^2}\int_{K(r)} u(a+\zeta X)\,d\Lambda_2(\zeta), \quad 0<r<R;$$

(iv) $\forall_{a\in\Omega}\ \forall_{X\in\mathbb{C}^n\colon \|X\|_\infty=1}\ \exists_{0<R\le d_\Omega(a)}\ \forall_{0<r<R}\ \forall_{f\in\mathcal{P}(\mathbb{C})}$:[48] if $u(a+\lambda X) \le \operatorname{Re} f(\lambda)$ for $|\lambda| = r$, then $u(a) \le \operatorname{Re} f(0)$;

(v) $\forall_{a\in\Omega}\ \forall_{X\in\mathbb{C}^n\colon \|X\|_\infty=1}\ \exists_{0<R\le d_\Omega(a)}\ \forall_{0<r<R}\ \forall_{h\in\mathcal{H}(K(r))\cap\mathcal{C}(\bar{K}(r))}$: if $u_{a,X}(\lambda) \le h(\lambda)$ for $|\lambda| = r$, then $u(a) \le h(0)$ ($\mathcal{H}(U)$ denotes the space of all real-valued harmonic functions on U);

(vi) for any $a \in \Omega$ and $X \in \mathbb{C}^n$ the function $u_{a,X}$ is subharmonic in $\Omega_{a,X}$.

(b) $\mathcal{PSH}(\Omega) + \mathcal{PSH}(\Omega) = \mathcal{PSH}(\Omega)$, $\quad \mathbb{R}_{>0}\cdot\mathcal{PSH}(\Omega) = \mathcal{PSH}(\Omega)$.

(c) $|f|$ is log-psh on Ω for any $f \in \mathcal{O}(\Omega)$.

(d) If $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega)$ and $u_\nu \searrow u$ pointwise on Ω, then $u \in \mathcal{PSH}(\Omega)$.

In particular, if $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega)$ and $u_\nu \le 0$, $\nu \in \mathbb{N}$, then $\sum_{\nu=1}^\infty u_\nu \in \mathcal{PSH}(\Omega)$.

(e) If $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega,\mathbb{R})$ and $u_\nu \to u$ locally uniformly in Ω, then $u \in \mathcal{PSH}(\Omega)$.

(f) If $u_1,\dots,u_N \in \mathcal{PSH}(\Omega)$, then $\max\{u_1,\dots,u_N\} \in \mathcal{PSH}(\Omega)$.

(g) (Liouville type theorem) If $u \in \mathcal{PSH}(\mathbb{C}^n)$ and $\sup_{\mathbb{C}^n} u < +\infty$, then $u \equiv \text{const}$.

(h) Let $I \subset \mathbb{R}$ be an open interval and let $\varphi\colon I \to \mathbb{R}$ be convex and increasing. Then $\varphi\circ u \in \mathcal{PSH}(\Omega)$ for every $u \in \mathcal{PSH}(\Omega)$ with $u(\Omega) \subset I$. Consequently:

[48] Recall that $\mathcal{P}(\mathbb{C})$ denotes the space of all complex polynomials of one complex variable.

- If $u \in \mathcal{PSH}(\Omega)$, then $e^u \in \mathcal{PSH}(\Omega)$ (in particular, any log-psh function is psh).
- If $u \in \mathcal{PSH}(\Omega, \mathbb{R}_+)$, then $u^p \in \mathcal{PSH}(\Omega)$ for every $p \geq 1$.

(i) If u_1, u_2 are log-psh, then $u_1 + u_2$ is log-psh.

Proposition 1.14.4 (Maximum principle). *Let $D \subset \mathbb{C}^n$ be a domain and let $u \in \mathcal{PSH}(D)$. If $u \leq u(a)$ for some $a \in D$, then $u \equiv u(a)$.*

In particular, if $D \subset \mathbb{C}^n$ is a bounded domain, $u \in \mathcal{PSH}(D)$, and $u \not\equiv$ const, then

$$u(z) < \sup\Big\{\limsup_{D \ni w \to \zeta} u(w) : \zeta \in \partial D\Big\}, \quad z \in D.$$

Proof. Let $D_0 := \{x \in D : u(x) = u(a)\}$. Observe that the set

$$D \setminus D_0 = \{x \in D : u(x) < u(a)\}$$

is open and, therefore, D_0 is closed in D. Let $z_0 \in D_0$. Applying the maximum principle (for subharmonic functions) to each of the functions $u_{z_0,X}$ with $\|X\|_\infty = 1$, we conclude that $\mathbb{P}(z_0, d_D(z_0)) \subset D_0$. Thus D_0 is open and therefore $D = D_0$. $\square$

If $u\colon \Omega \to \mathbb{R}$ is twice $\mathbb{R}$-differentiable at a point $a \in \Omega$, then we define the *Levi form of u at a*:

$$\mathcal{L}u(a; X) := \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) X_j \bar{X}_k, \quad X = (X_1, \dots, X_n) \in \mathbb{C}^n. \tag{1.14.1}$$

Notice that $\mathcal{L}(\| \|^2)(a; X) = \|X\|^2$ for any $a, X \in \mathbb{C}^n$. Observe that

$$\mathcal{L}u(a; X) = \frac{\partial^2 u_{a,X}}{\partial \lambda \partial \bar{\lambda}}(0).$$

Consequently, we have the following result:

Proposition 1.14.5. *Let $u \in \mathcal{C}^2(\Omega, \mathbb{R})$. Then*

$$u \in \mathcal{PSH}(\Omega) \iff \forall_{a \in \Omega,\, X \in \mathbb{C}^n} : \ \mathcal{L}u(a; X) \geq 0.$$

Exercise 1.14.6. Assume that $I \subset \mathbb{R}$ is an open interval, $u \in \mathcal{C}^2(\Omega, \mathbb{R})$, $u(\Omega) \subset I$, and $\varphi \in \mathcal{C}^2(I, \mathbb{R})$. Prove that

$$\mathcal{L}(\varphi \circ u)(a; X) = \varphi''(u(a)) \Big| \sum_{j=1}^{n} \frac{\partial u}{\partial z_j}(a) X_j \Big|^2 + \varphi'(u(a)) \mathcal{L}u(a; X)$$

for $a \in \Omega$, $X \in \mathbb{C}^n$.

Notice that the above formula and Proposition 1.14.5 give a direct proof of Remark 1.14.3 (h) for the case where u and φ are of class $\mathcal{C}^2$.

Exercise 1.14.7. Let $F\colon \Omega' \to \Omega$ be holomorphic, where $\Omega' \subset \mathbb{C}^m$ is open. Prove that for $u \in \mathcal{C}^2(\Omega, \mathbb{R})$ we have

$$\mathcal{L}(u \circ F)(b; Y) = \mathcal{L}u(F(b); F'(b)(Y)), \quad b \in \Omega',\ Y \in \mathbb{C}^m.$$

Consequently, if $u \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^2(\Omega, \mathbb{R})$, then $u \circ F \in \mathcal{PSH}(\Omega')$; cf. Proposition 1.14.34.

Definition 1.14.8. A function $u \in \mathcal{C}^2(\Omega, \mathbb{R})$ is called *strictly plurisubharmonic* if

$$\mathcal{L}u(a; X) > 0, \quad a \in \Omega,\ X \in (\mathbb{C}^n)_*.$$

The following proposition gives a very useful tool for constructing new psh functions.

Proposition 1.14.9. *Let $G \subset \Omega \subset \mathbb{C}^n$ be open and let $v \in \mathcal{PSH}(G)$, $u \in \mathcal{PSH}(\Omega)$. Assume that*

$$\limsup_{G \ni z \to \zeta} v(z) \le u(\zeta), \quad \zeta \in \Omega \cap \partial G.$$

Put

$$\tilde{u}(z) := \begin{cases} \max\{v(z), u(z)\} & \text{for } z \in G, \\ u(z) & \text{for } z \in \Omega \setminus G. \end{cases}$$

Then $\tilde{u} \in \mathcal{PSH}(\Omega)$.

Proof. It is clear that $\tilde{u} \in \mathcal{C}^{\uparrow}(\Omega)$. Obviously $\tilde{u}$ is psh on $\Omega \setminus \partial G$. Take a point $a \in \Omega \cap \partial G$, a vector $X \in \mathbb{C}^n$ with $\|X\|_\infty = 1$, and $0 < r < d_\Omega(a)$. Then

$$\tilde{u}(a) = u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u_{a,X}(re^{i\theta})\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}_{a,X}(re^{i\theta})\, d\theta$$

and we apply Remark 1.14.3 (a). □

Exercise 1.14.10 ([Hay 1989]). Let $\mathbb{H}^- := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$, $b < 0$, and $M < 0$. Moreover, let $u \in \mathcal{SH}(\mathbb{H}^-)$, $u < 0$, and $u(\lambda) \le M$ for all λ with $\operatorname{Re}\lambda = b$. Then $u \le M$ on $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le b\}$.

Hint: Use Proposition 1.14.9 and Remark 1.14.3 (g).

Our next aim is to find some characterizations of psh functions in terms of mean value inequalities. Let $a = (a_1, \dots, a_n) \in \mathbb{C}^n$, $r = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$.

If $u\colon \partial_0\mathbb{P}(a,r) \to \mathbb{R}_{-\infty}$[49] is bounded from above and measurable,[50] then we define

$$\begin{aligned}
&\boldsymbol{P}(u;a,r;z)\\
&\quad := \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n}\Big(\prod_{j=1}^{n}\frac{r_j^2-|z_j-a_j|^2}{|r_je^{i\theta_j}-(z_j-a_j)|^2}\Big)u(a+r\cdot e^{i\theta})\,d\Lambda_n(\theta),\\
&\qquad\qquad z=(z_1,\dots,z_n)\in\mathbb{P}(a,r),\\
&\boldsymbol{J}(u;a,r) := \boldsymbol{P}(u;a,r;a) = \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} u(a+r\cdot e^{i\theta})\,d\Lambda_n(\theta).
\end{aligned}$$

If $u\colon \mathbb{P}(a,r) \to \mathbb{R}_{-\infty}$ is bounded from above and measurable, then we define

$$\boldsymbol{A}(u;a,r) := \frac{1}{(\pi r_1^2)\dots(\pi r_n^2)}\int_{\mathbb{P}(a,r)} u\,d\Lambda_{2n} = \frac{1}{\pi^n}\int_{\mathbb{D}^n} u(a+r\cdot w)\,d\Lambda_{2n}(w).$$

Exercise 1.14.11. Let $\Omega\subset\mathbb{C}^n$ be open and let $u\colon \Omega\to\mathbb{R}_{-\infty}$ be upper semicontinuous.

(a) Prove that the functions

$$\begin{aligned}
&\{(z,r)\in\Omega\times\mathbb{R}^n_{>0} : \partial_0\mathbb{P}(z,r)\subset\Omega\}\ni(z,r)\mapsto \boldsymbol{J}(u;z,r),\\
&\{(z,r)\in\Omega\times\mathbb{R}^n_{>0} : \overline{\mathbb{P}}(z,r)\subset\Omega\}\ni(z,r)\mapsto \boldsymbol{A}(u;z,r),\\
&\{(z,r,X)\in\Omega\times\mathbb{R}_{>0}\times\mathbb{C}^n : z+r\mathbb{T}\cdot X\subset\Omega\}\ni(z,r,X)\\
&\qquad\mapsto \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta}X)\,d\theta
\end{aligned}$$

are upper semicontinuous (in particular, measurable).

Hint. Use Fatou's lemma.

(b) Prove that

$$\begin{aligned}
\boldsymbol{A}(u;a,r) &= \frac{2}{r_1^2}\dots\frac{2}{r_n^2}\int_0^{r_1}\dots\int_0^{r_n}\boldsymbol{J}(u;a,(\tau_1,\dots,\tau_n))\tau_1\dots\tau_n d\tau_1\dots\,d\tau_n\\
&= 2^n\int_0^1\dots\int_0^1 \boldsymbol{J}(u;a,(\tau_1r_1,\dots,\tau_nr_n))\tau_1\dots\tau_n\,d\tau_1\dots\,d\tau_n.
\end{aligned}$$

Proposition 1.14.12. *Let $\Omega\subset\mathbb{C}^n$ be open and let $u\in\mathcal{PSH}(\Omega)$, $a\in\Omega$. Then*

$$\boldsymbol{J}(u;a,r)\searrow u(a)\ \text{ when } r\searrow 0,\quad \boldsymbol{A}(u;a,r)\searrow u(a)\ \text{ when } r\searrow 0.$$

[49] Recall that $\partial_0\mathbb{P}(a,r) := \partial K(a_1,r_1)\times\dots\times\partial K(a_n,r_n)$.

[50] That is, the function $[0,2\pi)^n\ni\theta\mapsto u(a+r\cdot e^{i\theta})$ is Lebesgue measurable.

Proof. By Exercise 1.14.11 (b) it is enough to consider only $\boldsymbol{J}(u;a,\cdot)$. First, we prove that $\boldsymbol{J}(u;a,r') \le \boldsymbol{J}(u;a,r'')$ for $r' = (r'_1,\dots,r'_n)$, $r'' = (r''_1,\dots,r''_n)$, $0 < r'_j \le r''_j < d_D(a)$, $j = 1,\dots,n$.

The case $n = 1$ is well known (cf. [Vla 1966], Chapter 2, § 8). Hence

$$\boldsymbol{J}(u(z',\cdot,z'');a_j,r'_j) \le \boldsymbol{J}(u(z',\cdot,z'');a_j,r''_j), \quad (z',a_j,z'') \in \mathbb{P}(a,d_D(a)),\\ j = 1,\dots,n.$$

Consequently, using a finite induction, one can easily get the required inequality.

By Fatou's lemma we have

$$u(a) \le \lim_{r\to 0} \boldsymbol{J}(u;a,r) \le \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} \limsup_{r\to 0} u(a + r\cdot e^{i\theta})\,d\theta \le u(a),$$

which proves that $\boldsymbol{J}(u;a,r) \searrow u(a)$ when $r \searrow 0$. □

Proposition 1.14.13. *Let $u_1, u_2 \in \mathcal{PSH}(\Omega)$. If $u_1 = u_2$ Λ_{2n}-almost everywhere in Ω, then $u_1 \equiv u_2$.*

Proof. Fix an $a \in \Omega$. Since $u_1 = u_2$ Λ_{2n}-almost everywhere, we get

$$\boldsymbol{A}(u_1;a,r) = \boldsymbol{A}(u_2;a,r), \quad 0 < r < d_D(a).$$

Hence, by Proposition 1.14.12, $u_1(a) = u_2(a)$. □

Proposition 1.14.14. *Let $\Omega \subset \mathbb{C}^n$ be open, let $u \in \mathcal{PSH}(\Omega)$, and let $\overline{\mathbb{P}}(a,r) \subset \Omega$ $(r = (r_1,\dots,r_n) \in \mathbb{R}^n_{>0})$. Then*

$$u(z) \le \boldsymbol{P}(u;a,r;z), \quad z \in \mathbb{P}(a,r), \tag{1.14.2}$$

$$u(a) \le \boldsymbol{J}(u;a,r), \tag{1.14.3}$$

$$u(a) \le \boldsymbol{A}(u;a,r). \tag{1.14.4}$$

Proof. Inequality (1.14.2) is well known for $n = 1$. In particular,

$$u(w',z_j,w'') \le \boldsymbol{P}(u(w',\cdot,w'');a_j,r_j;z_j), \quad (w',z_j,w'') \in \mathbb{P}(a,r),\\ j = 1,\dots,n.$$

Hence, after finite induction, we get (1.14.2).

Inequality (1.14.3) follows directly from (1.14.2).

Inequality (1.14.4) follows from (1.14.3) and Exercise 1.14.11 (b). □

Proposition 1.14.15. *Let $D \subset \mathbb{C}^n$ be a domain. If $u \in \mathcal{PSH}(D)$ and $u \not\equiv -\infty$, then $u \in L^1(D,\mathrm{loc})$; in particular, $\Lambda_{2n}(u^{-1}(-\infty)) = 0$.*

Proof. Suppose that there exists a point $a \in D$ such that $\int_U u \, d\Lambda_{2n} = -\infty$ for every neighborhood U of a. Let $2r := d_D(a)$. Observe that $\int_{\mathbb{P}(z,r)} u \, d\Lambda_{2n} = -\infty$ for any $z \in \mathbb{P}(a,r)$. Consequently,

$$u(z) \le \boldsymbol{A}(u; z, r) = -\infty, \quad z \in \mathbb{P}(a,r).$$

Hence $u = -\infty$ in $\mathbb{P}(a,r)$. Let

$$D_0 := \{z \in D : u = -\infty \text{ in a neighborhood of } z\}.$$

We have proved that $D_0 \neq \varnothing$. The same method of proof shows that D_0 is closed in D. Thus $D_0 = D$ – a contradiction. □

Proposition 1.14.16. *If a family $(u_i)_{i \in I} \subset \mathcal{PSH}(\Omega)$ is locally bounded from above, then the function $u := (\sup_{i \in I} u_i)^*$ is psh in Ω.*

Here v^* denotes the *upper semicontinuous regularization* of the function v, $v^*(z) := \limsup_{w \to z} v(w)$, $z \in \Omega$.[51]

Proof. Take $a \in \Omega$, $X \in \mathbb{C}^n$, $\|X\|_\infty = 1$, and let $\overline{\mathbb{P}}(a,2r) \subset \Omega$. Then we have

$$\sup_{i \in I} u_i(z) \le \sup_{i \in I} \frac{1}{2\pi} \int_0^{2\pi} u_i(z + re^{i\theta} X) \, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta} X) \, d\theta, \qquad z \in \mathbb{P}(a,r).$$

By Exercise 1.14.11 (a), the right-hand side is an upper semicontinuous function of z. In particular,

$$u(a) \le \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta} X) \, d\theta.$$ □

Proposition 1.14.17. *If a sequence $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega)$ is locally bounded from above, then the function $u := (\limsup_{\nu \to \infty} u_\nu)^*$ is psh on Ω.*

Proof. Use the same method as in the proof of Proposition 1.14.16 (Exercise). □

Definition 1.14.18. A set $M \subset \mathbb{C}^n$ is called *pluripolar* if any point $a \in M$ has a connected neighborhood U_a and a function $v_a \in \mathcal{PSH}(U_a)$ with $v_a \not\equiv -\infty$, $M \cap U_a \subset v_a^{-1}(-\infty)$.

We say that a pluripolar set $M \subset \mathbb{C}^n$ is *locally complete* if any point $a \in M$ has a connected neighborhood U_a and a function $v_a \in \mathcal{PSH}(U_a)$ with $v_a \not\equiv -\infty$, $M \cap U_a = v_a^{-1}(-\infty)$.

[51] Notice that in general $v^* : \Omega \to [-\infty, +\infty]$ is upper semicontinuous on Ω. If v is locally bounded from above, then $v^* : \Omega \to \mathbb{R}_{-\infty}$ and

$$v^* = \inf\{\varphi : \varphi \in \mathcal{C}^{\uparrow}(\Omega, \mathbb{R}),\ v \le \varphi\} = \inf\{\varphi : \varphi \in \mathcal{C}(\Omega, \mathbb{R}),\ v \le \varphi\}.$$

By Proposition 1.14.15, if M is pluripolar, then $\Lambda_{2n}(M) = 0$. It is clear that any thin set $M \subset \Omega$ (Definition 1.14.5) is pluripolar.

The problem of whether an arbitrary pluripolar set can be described by one global psh function (cf. [Lel 1957]) was open during many years and was finally solved by B. Josefson in 1978.

Theorem* 1.14.19 (Josefson theorem; cf. [Jos 1978], see also [Jar-Pfl 2000], Theorem 2.1.39). *If $M \subset \mathbb{C}^n$ is pluripolar, then there exists a $v \in \mathcal{PSH}(\mathbb{C}^n)$, $v \not\equiv -\infty$, such that $M \subset v^{-1}(-\infty)$.*

In the case of locally complete pluripolar sets an analogous result was proved by M. Coltoiu in 1990.

Theorem* 1.14.20 ([Col 1990]). *Let $D \subset \mathbb{C}^n$ be a domain of holomorphy and let $M \subset D$ be a relatively closed locally complete pluripolar set. Then there exists a $v \in \mathcal{PSH}(D)$, $v \not\equiv -\infty$, such that $M = v^{-1}(-\infty)$.*

Example 1.14.21 ([Wie 2000]). Let $M := \mathbb{T} \times \{0\}$. Then M is pluripolar (Exercise), but M is not complete pluripolar.

Indeed, suppose that M is complete pluripolar and let $v \in \mathcal{PSH}(\mathbb{C}^2)$ be such that $M = v^{-1}(-\infty)$, $v \not\equiv -\infty$ (Theorem 1.14.20). Then $\mathbb{T} \subset \{z \in \mathbb{C} : v(z, 0) = \infty\}$. Hence $v(\cdot, 0) \equiv -\infty$ (Exercise) and so $\mathbb{C} \times \{0\} \subset v^{-1}(-\infty)$; a contradiction.

Proposition 1.14.22. *Let $M_j \subset \mathbb{C}^n$ be pluripolar, $j \in \mathbb{N}$. Then $M := \bigcup_{j=1}^{\infty} M_j$ is pluripolar.*

Proof. By Josefson's theorem (Theorem 1.14.19), for each $j \in \mathbb{N}$ there exists a $v_j \in \mathcal{PSH}(\mathbb{C}^n)$, $v_j \not\equiv -\infty$, such that $M_j \subset v_j^{-1}(-\infty)$. Since, for each j the set $v_j^{-1}(-\infty)$ is of zero measure, there exists a point $b \in \mathbb{D}^n$ such that $v_j(b) > -\infty$ for all j. We may assume that $v_j \le 0$ on $\mathbb{P}(j)$ and $v_j(b) \ge -2^{-j}$, $j \in \mathbb{N}$.[52] Define $v := \sum_{j=1}^{\infty} v_j$. Then $v \in \mathcal{PSH}(\mathbb{C}^n)$ (cf. Remark 1.14.3 (d)), $v(b) \ge -1$, and $M \subset v^{-1}(-\infty)$. □

Proposition 1.14.23. *Let $\Omega \subset \mathbb{C}^n$ be open and let a sequence $(u_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{PSH}(\Omega)$ be locally bounded from above.*

(a) *Put $u := \sup_{\nu\in\mathbb{N}} u_\nu$. Then the set $\{z \in \Omega : u(x) < u^*(x)\}$ is of zero measure.*[53]

(b) *Put $u := \limsup_{\nu\to+\infty} u_\nu$. Then the set $\{z \in \Omega : u(z) < u^*(z)\}$ is of zero measure.*

[52] It suffices to substitute v_j by a function of the form $\varepsilon_j(v_j - c_j)$ with $c_j := \sup_{\mathbb{P}(j)} v_j$ and an appropriate $\varepsilon_j > 0$.

[53] The result remains true in the case where $u := \sup_{i\in I} u_i$ (with arbitrary I) and $(u_i)_{i\in I} \subset \mathcal{PSH}(\Omega)$ is locally bounded from above – cf. [Jar-Pfl 2000], Prop. 2.1.38(a).

Proof. (a) Observe that the function u is measurable. To prove that $u = u^*$ a.e., it suffices to show that $\boldsymbol{A}(u; a, r) = \boldsymbol{A}(u^*; a, r)$ for any $a \in \Omega$ and $0 < r < d_\Omega(a)$. Fix a and r as above. We have

$$u(z) \le \boldsymbol{P}(u; a, \tau; z), \quad z \in \mathbb{P}(a, \tau),\ 0 < \tau < r.$$

Hence

$$u^*(z) \le \boldsymbol{P}(u; a, \tau; z), \quad z \in \mathbb{P}(a, \tau),\ 0 < \tau < r.$$

Observe that

$$\boldsymbol{P}(\boldsymbol{P}(u; a, \tau; \cdot); a, \tau'; z) = \boldsymbol{P}(u; a, \tau; z), \quad z \in \mathbb{P}(a, \tau'),\ 0 < \tau' < \tau < r.$$

Thus

$$\boldsymbol{P}(u^*; a, \tau; z) \le \boldsymbol{P}(u; a, \tau, z), \quad z \in \mathbb{P}(a, \tau),\ 0 < \tau < r.$$

In particular, $\boldsymbol{J}(u^*; a, \tau) \le \boldsymbol{J}(u; a, \tau)$, $0 < \tau < r$. Consequently, $\boldsymbol{A}(u; a, r) = \boldsymbol{A}(u^*; a, r)$.

(b) Let $v_k := \sup_{\nu \ge k} u_\nu$, $k \in \mathbb{N}$. Then $v_k \searrow u$ and

$$\{z \in \Omega : u(z) < u^*(z)\} \subset \bigcup_{k=1}^{\infty} \{z \in \Omega : v_k(z) < v_k^*(z)\}$$

and we apply (a). $\square$

In fact the following more general result is true.

Theorem* 1.14.24 (Bedford–Taylor theorem; cf. [Kli 1991], Theorem 4.7.6).

(a) *Assume that a family $(u_i)_{i \in I} \subset \mathcal{PSH}(\Omega)$ is locally bounded from above. Put $u := \sup_{i \in I} u_i$. Then the set $\{z \in \Omega : u(z) < u^*(z)\}$ is pluripolar.*

(b) *Assume that a sequence $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega)$ is locally bounded from above. Put $u := \limsup_{\nu \to +\infty} u_\nu$. Then the set $\{z \in \Omega : u(z) < u^*(z)\}$ is pluripolar.*

Proposition 1.14.25 (Removable singularities of psh functions). *Let M be a closed pluripolar subset of Ω.*

(a) *Let $u \in \mathcal{PSH}(\Omega \setminus M)$ be locally bounded from above in Ω.*[54] *Define*

$$\tilde{u}(z) := \limsup_{\Omega \setminus M \ni w \to z} u(w), \quad z \in \Omega$$

(*notice that $\tilde{u}$ is well defined*). *Then $\tilde{u} \in \mathcal{PSH}(\Omega)$.*

(b) *For every function $u \in \mathcal{PSH}(\Omega)$ we have*

$$u(z) = \limsup_{\Omega \setminus M \ni w \to z} u(w), \quad z \in \Omega.$$

(c) *If Ω is a domain, then the set $\Omega \setminus M$ is connected.*

[54] That is every point $a \in \Omega$ has a neighborhood V_a such that u is bounded from above in $V_a \setminus M$.

Proof. (a) The result has a local character. Thus we may assume $\Omega = D$ is connected, $u \le 0$ in $D \setminus M$, and $M \subset v^{-1}(-\infty)$ with $v \in \mathcal{PSH}(D)$, $v \le 0$, $v \not\equiv -\infty$. Put

$$u_\nu := \begin{cases} u + (1/\nu)v & \text{on } D \setminus M, \\ -\infty & \text{on } M, \end{cases} \qquad \nu \in \mathbb{N}.$$

Then $u_\nu \in \mathcal{PSH}(D)$, $\nu \in \mathbb{N}$ (EXERCISE). Put $u_0 = \sup_{\nu\in\mathbb{N}} u_\nu$. Observe that $u_0 = u$ on $D \setminus P$ and $u_0 = -\infty$ on P, where $P := v^{-1}(-\infty)$ (P is pluripolar). By Proposition 1.14.16, $(u_0)^* \in \mathcal{PSH}(D)$. By Proposition 1.14.23 (a), the set $A := \{z \in D : u_0(z) \le (u_0)^*(z)\}$ is of zero measure. Then $(u_0)^* = u_0 = u$ on $D \setminus (P \cup A)$. Hence, by Proposition 1.14.13, $(u_0)^* = u$ on $D \setminus M$.

It remains to prove that $(u_0)^* = \tilde{u}$. Obviously, $(u_0)^* = u = \tilde{u}$ on $D \setminus M$. Take an $a \in M$. Then

$$\begin{aligned}
\tilde{u}(a) &= \limsup_{D\setminus M \ni z \to a} u(z) = \limsup_{D\setminus M \ni z \to a} (u_0)^*(z) \le \limsup_{z\to a} (u_0)^*(z) = (u_0)^*(a) \\
&= \limsup_{z\to a} u_0(z) \le \limsup_{D\setminus P \ni z \to a} u_0(z) = \limsup_{D\setminus P \ni z \to a} u(z) \\
&\le \limsup_{D\setminus M \ni z \to a} u(z) = \tilde{u}(a).
\end{aligned}$$

(b) Let

$$\tilde{u}(z) := \limsup_{\Omega\setminus M \ni w \to z} u(w), \qquad z \in \Omega.$$

By (a), $\tilde{u} \in \mathcal{PSH}(\Omega)$. Moreover, $\tilde{u} = u$ on $\Omega \setminus M$. Now, since $\Lambda_{2n}(M) = 0$, we use Proposition 1.14.13.

(c) Suppose that $\Omega \setminus M = U_1 \cup U_2$, where U_1 and U_2 are disjoint and non–empty open sets. Then, in view of (a), the function $u(z) := j$ for $z \in U_j$ would extend to a psh function on Ω, which contradicts the maximum principle. $\square$

Definition 1.14.26. Let Ω be an open subset of $\mathbb{C}^n$. A function $u \in \mathcal{C}^2(\Omega, \mathbb{R})$ is *pluriharmonic* on Ω ($u \in \mathcal{PH}(\Omega)$) if

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) = 0, \qquad z \in \Omega,\ j,k = 1,\dots,n. \tag{1.14.5}$$

Remark 1.14.27. (a) If $n = 1$, then $\mathcal{PH}(\Omega) = \mathcal{H}(\Omega)$.

(b) $\mathcal{PH}(\Omega)$ is a vector space; $\mathcal{PH}(\Omega) \subset \mathcal{PSH}(\Omega)$.

(c) For a function $u \in \mathcal{C}^2(\Omega, \mathbb{R})$ the following conditions are equivalent:

(i) $u \in \mathcal{PH}(\Omega)$;

(ii) $u_{a,X} \in \mathcal{H}(\Omega_{a,X})$ for any $a \in \Omega$ and $X \in \mathbb{C}^n$;

(iii) $\mathcal{L}u(a; X) = 0$ for any $a \in \Omega$ and $X \in \mathbb{C}^n$.

(d) Condition (1.14.5) is equivalent to the following system of equations

$$\frac{\partial^2 u}{\partial x_j \partial y_k}(z) = \frac{\partial^2 u}{\partial x_k \partial y_j}(z), \quad \frac{\partial^2 u}{\partial x_j \partial x_k}(z) + \frac{\partial^2 u}{\partial y_j \partial y_k}(z) = 0, \tag{1.14.6}$$
$$z \in \Omega,\ j,k = 1,\dots,n.$$

In particular,

$$\frac{\partial^2 u}{\partial x_j^2}(z) + \frac{\partial^2 u}{\partial y_j^2}(z) = 0, \quad z \in \Omega,\ j = 1,\dots,n,$$

which shows that $\mathcal{PH}(\Omega) \subset \mathscr{H}(\Omega) \subset \mathcal{C}^\infty(\Omega)$.

(e) If $f = u + iv \in \mathcal{O}(\Omega)$, then $u \in \mathcal{PH}(\Omega)$.

Proposition 1.14.28. *If $D \subset \mathbb{C}^n$ is a starlike domain with respect to a point $a \in D$, then for any $u \in \mathcal{PH}(D)$ there exists an $f \in \mathcal{O}(D)$ such that $u = \operatorname{Re} f$.*

In particular, any pluriharmonic function is locally the real part of a holomorphic function.

Proof. (Cf. Remark 1.19.8.) We may assume that $a = 0$. Define

$$v(z) := -i \int_0^1 \sum_{j=1}^n \Big(z_j \frac{\partial u}{\partial z_j}(tz) - \bar z_j \frac{\partial u}{\partial \bar z_j}(tz) \Big) dt, \quad z \in D.$$

Then $f := u + iv \in \mathcal{C}^1(D)$ and using (1.14.5) we get

$$\begin{aligned}
\frac{\partial f}{\partial \bar z_k}(z) &= \frac{\partial u}{\partial \bar z_k} + \int_0^1 \Big(\sum_{j=1}^n \Big(z_j \frac{\partial^2 u}{\partial \bar z_k \partial z_j}(tz)t - \bar z_j \frac{\partial^2 u}{\partial \bar z_k \partial \bar z_j}(tz)t \Big) - \frac{\partial u}{\partial \bar z_k}(tz) \Big) dt \\
&= \frac{\partial u}{\partial \bar z_k} - \int_0^1 \Big(t \sum_{j=1}^n \Big(z_j \frac{\partial^2 u}{\partial z_j \partial \bar z_k}(tz) + \bar z_j \frac{\partial^2 u}{\partial \bar z_j \partial \bar z_k}(tz) \Big) + \frac{\partial u}{\partial \bar z_k}(tz) \Big) dt \\
&= \frac{\partial u}{\partial \bar z_k}(z) - \int_0^1 \frac{d}{dt} \Big(t \frac{\partial u}{\partial \bar z_k}(tz) \Big) dt = 0, \quad k = 1,\dots,n. \qquad \square
\end{aligned}$$

Corollary 1.14.29. *Let $\Omega_j \subset \mathbb{C}^{n_j}$ be open, $j = 1, 2$, and let $F \in \mathcal{O}(\Omega_1, \Omega_2)$. Then $u \circ F \in \mathcal{PH}(\Omega_1)$ for any $u \in \mathcal{PH}(\Omega_2)$.*

Proposition 1.14.25 implies the following important corollary.

Corollary 1.14.30. *Let M be a closed pluripolar subset of Ω.*

(a) *Let $u \in \mathcal{PH}(\Omega \setminus M)$ be locally bounded in Ω. Then u extends pluriharmonically to Ω.*

(b) *Let $f \in \mathcal{O}(\Omega \setminus M)$ be locally bounded in Ω. Then f extends holomorphically to Ω.*

Proof. Since $u \in \mathcal{PSH}(\Omega \setminus M)$ and u is locally bounded from above in Ω, Proposition 1.14.25 implies that u extends to a function $\tilde{u}_+ \in \mathcal{PSH}(\Omega)$. We can repeat the same for the function $-u$. Thus $-u$ extends to a function $\tilde{u}_- \in \mathcal{PSH}(\Omega)$. Then $\tilde{u}_+ + \tilde{u}_- \in \mathcal{PSH}(\Omega)$ and $\tilde{u}_+ + \tilde{u}_- = u + (-u) = 0$ on $\Omega \setminus M$. Hence, by Proposition 1.14.13, $\tilde{u}_+ + \tilde{u}_- \equiv 0$, which implies that u extends to a function $\tilde{u} \in \mathcal{C}(\Omega)$.

By Proposition 1.14.14, for any $a \in M$ and $0 < r < d_\Omega(a)$, we get $\tilde{u}(z) = \boldsymbol{P}(\tilde{u}; a, r; z)$, $z \in \mathbb{P}(a, r)$. In particular, $\tilde{u}$ is of class $\mathcal{C}^\infty$ in Ω. Since the interior of M is empty, we see that $\tilde{u}$ must be pluriharmonic in Ω.

(b) follows from (a) – Exercise. □

Proposition 1.14.31 (Hartogs lemma). *Let $(u_\nu)_{\nu=1}^\infty \subset \mathcal{PSH}(\Omega)$ be a sequence locally bounded from above. Assume that for some $m \in \mathbb{R}$,*

$$\limsup_{\nu\to+\infty} u_\nu \le m.$$

Then for every compact subset $K \subset \Omega$ and for every $\varepsilon > 0$, there exists a ν_0 such that

$$\max_K u_\nu \le m + \varepsilon, \quad \nu \ge \nu_0.$$

Notice that the above result gives a tool to prove Theorem 1.7.13.

Proof. Take an $\varepsilon > 0$. It is sufficient to show that for every $a \in \Omega$ there exist $\delta(a) > 0$ and $\nu(a)$ such that $u_\nu \le m + \varepsilon$ in $\mathbb{P}(a, \delta(a))$ for $\nu \ge \nu(a)$. Fix a and $0 < R < d_\Omega(a)/2$. We may assume that $u_\nu \le 0$ in $\overline{\mathbb{P}}(a, 2R)$ for any $\nu \ge 1$, and $m < 0$. Let $0 < \delta < R/2$. Then

$$\begin{aligned}
\limsup_{\nu\to+\infty} \sup_{z\in\mathbb{P}(a,\delta)} u_\nu(z) &\le \limsup_{\nu\to+\infty} \sup_{z\in\mathbb{P}(a,\delta)} \boldsymbol{A}(u_\nu; z, R+\delta) \\
&\le \limsup_{\nu\to+\infty} \frac{R^{2n}}{(R+\delta)^{2n}} \boldsymbol{A}(u_\nu; a, R) \\
&\le \frac{R^{2n}}{(R+\delta)^{2n}} \boldsymbol{A}(\limsup_{\nu\to+\infty} u_\nu; a, R) \\
&\le \frac{R^{2n}}{(R+\delta)^{2n}} \boldsymbol{A}(m; a, R) \le \frac{R^{2n}}{(R+\delta)^{2n}} m < m + \varepsilon,
\end{aligned}$$

provided that δ is sufficiently small. □

Recall that smooth psh functions may be easily described by properties of their Levi forms (Proposition 1.14.5). Thus it is important to be able to approximate (at least locally) a given psh function by smooth psh functions. The required approximation may be given by the following procedure.

Let $\Phi(z_1,\dots,z_n) := \Psi(z_1)\cdots\Psi(z_n)$, $z = (z_1,\dots,z_n) \in \mathbb{C}^n$, where $\Psi \in \mathcal{C}_0^\infty(\mathbb{C},\mathbb{R}_+)$ is such that:

- $\operatorname{supp}\Psi = \overline{\mathbb{D}}$,
- $\Psi(z) = \Psi(|z|)$, $z \in \mathbb{C}$,
- $\int \Psi\, d\Lambda_2 = 1$.

Exercise 1.14.32. Find an effective formula for a Ψ with the above properties.

Put

$$\Phi_\varepsilon(z) := \frac{1}{\varepsilon^{2n}}\Phi\Big(\frac{z}{\varepsilon}\Big), \quad z \in \mathbb{C}^n,\ \varepsilon > 0.$$

Notice that:

- $\Phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{C}^n,\mathbb{R}_+)$,
- $\operatorname{supp}\Phi_\varepsilon = \overline{\mathbb{P}}(\varepsilon)$,
- $\Phi_\varepsilon \circ \boldsymbol{T}_\zeta = \Phi_\varepsilon$, $\zeta \in \mathbb{T}^n$,
- $\int_{\mathbb{C}^n} \Phi_\varepsilon\, d\Lambda_{2n} = 1$.

Let

$$\Omega_\varepsilon := \{z \in \Omega : d_\Omega(z) > \varepsilon\}, \quad \varepsilon > 0.$$

For every function $u \in L^1(\Omega,\mathrm{loc})$, define

$$\begin{aligned} u_\varepsilon(z) :&= \int_\Omega u(w)\Phi_\varepsilon(z-w)\, d\Lambda_{2n}(w) \\ &= \int_{\mathbb{D}^n} u(z+\varepsilon w)\Phi(w)\, d\Lambda_{2n}(w), \quad z \in \Omega_\varepsilon. \end{aligned} \tag{1.14.7}$$

The function u_ε is called the *ε-regularization* of u.

Proposition 1.14.33. *If $u \in \mathcal{PSH}(\Omega)$, $u \not\equiv -\infty$, then $u_\varepsilon \in \mathcal{PSH}(\Omega_\varepsilon) \cap \mathcal{C}^\infty(\Omega_\varepsilon)$ and $u_\varepsilon \searrow u$ pointwise in Ω when $\varepsilon \searrow 0$.*

Proof. It is clear that $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon)$. Take an $a \in \Omega_\varepsilon$. By the second part of (1.14.7) we get

$$u_\varepsilon(a) = (2\pi)^n \int_0^1 \dots \int_0^1 \boldsymbol{J}(u;a,\varepsilon(\tau_1,\dots,\tau_n))\Phi(\tau_1,\dots,\tau_n)\tau_1\dots\tau_n\, d\tau_1\dots\, d\tau_n.$$

Consequently, by Proposition 1.14.12, $u_\varepsilon \searrow u$. It remains to prove that u_ε is psh. We will apply Remark 1.14.3 (a). Fix $a \in \Omega_\varepsilon$, $X \in \mathbb{C}^n$, $\|X\|_\infty = 1$, and $0 < r < d_{\Omega_\varepsilon}(a)$. Then

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} u_\varepsilon(a + re^{i\theta}X)\, d\theta \\ &\qquad = \int_{\mathbb{D}^n}\Big(\frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta}X + \varepsilon w)\, d\theta\Big)\Phi(w)\, d\Lambda_{2n}(w) \\ &\qquad \ge \int_{\mathbb{D}^n} u(a+\varepsilon w)\Phi(w)\, d\Lambda_{2n}(w) = u_\varepsilon(a). \end{aligned}$$

□

Proposition 1.14.34. *Let $\Omega' \subset \mathbb{C}^n$ be open and let $F \in \mathcal{O}(\Omega', \Omega)$. Then $u \circ F \in \mathcal{PSH}(\Omega')$ for any $u \in \mathcal{PSH}(\Omega)$.*

Proof. We may assume that $u \in L^1(\Omega, \mathrm{loc})$. We already know that the result holds if $u \in \mathcal{C}^2(\Omega)$ (Exercise 1.14.7).

Let u_ε denote the ε-regularization of u. Put $\Omega'_\varepsilon := F^{-1}(\Omega_\varepsilon)$. Then $u_\varepsilon \circ F \in \mathcal{PSH}(\Omega'_\varepsilon)$ and $u_\varepsilon \circ F \searrow u \circ F$. Consequently, $u \circ F \in \mathcal{PSH}(\Omega)$. □

Corollary 1.14.35. *Let $u\colon \Omega \to \mathbb{R}_{-\infty}$ be upper semicontinuous. Then u is psh on Ω iff for any analytic disc $\varphi\colon \mathbb{D} \to \Omega$ the function $u \circ \varphi$ is subharmonic in $\mathbb{D}$.*

Lemma 1.14.36. *Let $\Omega \subset \mathbb{C}^n$ be open and let $u \in \mathcal{PSH}(\Omega)$, $u \geq 0$. Then u is log-psh iff for any $a \in \mathbb{C}$ and $j \in \{1, \dots, n\}$ the function*

$$\Omega \ni z \xmapsto{v_{a,j}} |e^{az_j}|u(z)$$

is psh.

Proof. We only need to prove that if $v_{a,j}$ is psh (for any a and j), then $\log u$ is psh. By definition, we have to check that for any $z_0 \in \Omega$ and $X \in (\mathbb{C}^n)_*$, the function $\lambda \mapsto \log u(z_0 + \lambda X)$ is subharmonic (in the region where it is defined), equivalently (cf. [Vla 1966], Chapter 2, § 15), we have to prove that for any $w_0 \in \mathbb{C}$, the function

$$\lambda \xmapsto{\varphi} |e^{w_0\lambda}|u(z_0 + \lambda X)$$

is subharmonic. Let k be such that $X_k \neq 0$. Put $a := w_0/X_k$. Then $\varphi(\lambda) = |e^{-az_{0,k}}|v_{a,k}(z_0 + \lambda X)$. Thus φ is subharmonic provided $v_{a,k}$ is psh. □

Proposition 1.14.37. (a) *Any $\mathbb{C}$-seminorm $q\colon \mathbb{C}^n \to \mathbb{R}_+$ is log-psh.*

(b) *Let $h\colon \mathbb{C}^n \to \mathbb{R}_+$ be such that*

$$h(\lambda z) = |\lambda| h(z), \quad \lambda \in \mathbb{C},\ z \in \mathbb{C}^n.$$

Then h is psh iff h is log-psh.

Proof. (a) By Exercise 1.14.2 (b) we have $q \in \mathcal{PSH}(\mathbb{C}^n)$. Now we can apply Lemma 1.14.36 because $|e^{az_j}|q(z) = q(e^{az_j}z)$ and the right-hand side is psh by Proposition 1.14.34.

(b) follows from the proof of (a). □

Exercise 1.14.38. Let $\Omega \subset \mathbb{C}^n$ be open and let $u \in \mathcal{C}^2(\Omega, \mathbb{R})$. Prove that

$$4\mathcal{L}u((x + iy); (a + ib)) = \mathcal{H}u((x, y); (a, b)) + \mathcal{H}u((x, y); (b, -a)),$$
$$x + iy \in \Omega,\ a + ib \in \mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n,$$

where $\mathcal{H}$ denotes the *real Hessian*: if $U \subset \mathbb{R}^N$ is open and $v \in \mathcal{C}^2(U, \mathbb{R})$, then

$$\mathcal{H}v(x;\xi) := \sum_{j,k=1}^{N} \frac{\partial^2 v}{\partial x_j \partial x_k}(x)\xi_j\xi_k, \quad x \in U,\ \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N. \quad (1.14.8)$$

Proposition 1.14.39. *Let U be a domain in $\mathbb{R}^n$ and let $v \colon U \to \mathbb{R}_{-\infty}$. Define*

$$\widetilde{U} := U + i\mathbb{R}^n \subset \mathbb{C}^n, \qquad \tilde{v}(x + iy) := v(x), \quad x + iy \in \widetilde{U}.$$

Then $\tilde{v} \in \mathcal{PSH}(\widetilde{U})$ iff v is convex on U.[55]

Proof. First consider the case where v is of class $\mathcal{C}^2$. By Exercise 1.14.38 we get

$$4\mathcal{L}\tilde{v}(x + iy; a + ib) = \mathcal{H}v(x; a) + \mathcal{H}v(x; b),$$

which, of course, implies the required result.

In the general case, assume that $\tilde{v}$ is psh and let $(\tilde{v})_\varepsilon$ denote the ε-regularization of $\tilde{v}$. Observe that $(\widetilde{U})_\varepsilon + i\mathbb{R}^n = (\widetilde{U})_\varepsilon$. Hence $(\widetilde{U})_\varepsilon = U^\varepsilon + i\mathbb{R}^n$ for an open set $U^\varepsilon \subset \mathbb{R}^n$ (Exercise). Moreover,

$$\begin{aligned}(\tilde{v})_\varepsilon(z + it) &= \int_{\mathbb{D}^n} \tilde{v}(z + it + \varepsilon w)\Phi(w)\, d\Lambda_{2n}(w) \\ &= \int_{\mathbb{D}^n} \tilde{v}(z + \varepsilon w)\Phi(w)\, d\Lambda_{2n}(w) = (\tilde{v})_\varepsilon(z), \quad z \in (\widetilde{U})_\varepsilon,\ t \in \mathbb{R}^n.\end{aligned}$$

Hence, $(\tilde{v})_\varepsilon(x + iy) = v^\varepsilon(x)$, $x + iy \in (\widetilde{U})_\varepsilon$, where $v^\varepsilon \colon U^\varepsilon \to \mathbb{R}$. Note that $v^\varepsilon \searrow v$. By the first part of the proof, v^ε is convex in U^ε for any $\varepsilon > 0$. Consequently, v is convex (Exercise).

Conversely, assume that v is convex and let v_ε be the ε-regularization of v (in $\mathbb{R}^n$):

$$v_\varepsilon(x) := \int_{\mathbb{B}(1)} v(x + \varepsilon y)\Theta(y)\, d\Lambda_n(y), \quad x \in U_\varepsilon := \{x \in U : \mathbb{B}(x, \varepsilon) \subset U\},$$

where Θ is a "regularization" function in $\mathbb{R}^n$.[56] Put $\widetilde{U_\varepsilon} := U_\varepsilon + i\mathbb{R}^n \subset \mathbb{C}^n$, $\widetilde{v_\varepsilon}(x + iy) := v_\varepsilon(x)$, $x + iy \in \widetilde{U_\varepsilon}$. Note that $\widetilde{v_\varepsilon} \searrow \tilde{v}$. By the first part of the proof, $\widetilde{v_\varepsilon}$ is psh in $\widetilde{U_\varepsilon}$ for any $\varepsilon > 0$. Consequently, $\tilde{v}$ is psh in $\widetilde{U}$. $\square$

[55] That is, if $[x, y] \subset U$, then $v(tx + (1-t)y) \le tv(x) + (1-t)v(y)$, $t \in [0, 1]$. If $v \in \mathcal{C}^2(U, \mathbb{R})$, then v is convex iff $\mathcal{H}v(x;\xi) \ge 0$ for any $x \in U$ and $\xi \in \mathbb{R}^n$.

[56] That is $\Theta \in \mathcal{C}_0^\infty(U, \mathbb{R}_+)$, $\operatorname{supp}\Theta = \overline{\mathbb{B}}(1)$, $\int_{\mathbb{B}(1)} \Theta\, d\Lambda_n = 1$, and $\Theta(x) = \Theta(|x_1|, \dots, |x_n|)$ for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. It is known that if v is convex in U, then v_ε is convex in U_ε, $v_\varepsilon \in \mathcal{C}^\infty(U_\varepsilon)$, and $v_\varepsilon \searrow v$.

Proposition 1.14.40. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain and let $u\colon D \to \mathbb{R}_{-\infty}$ be such that*

$$u(z_1, \dots, z_n) = u(|z_1|, \dots, |z_n|), \quad (z_1, \dots, z_n) \in D.$$

Put

$$\tilde{u}(r_1, \dots, r_n) := u(r_1, \dots, r_n), \quad (r_1, \dots, r_n) \in \boldsymbol{R}(D),$$
$$\tilde{\tilde{u}}(x) := u(e^x), \quad x \in \log D.$$

For any $I = (i_1, \dots, i_k)$ with $1 \le i_1 < \dots < i_k \le n$ $(0 \le k \le n-1)$, let D_I denote the intersection of D with the $(n-k)$-dimensional subspace $\{z_{i_j} = 0 : j = 1, \dots, k\}$. We identify D_I with a Reinhardt open set in $\mathbb{C}^{n-k}$. Let u_I denote the restriction of u to D_I.

(a) *In the case where $D \subset \mathbb{C}_*^n$ we get: $u \in \mathcal{PSH}(D)$ iff $\tilde{\tilde{u}}$ is convex.*

(b) *In the general case, $u \in \mathcal{PSH}(D)$ iff*

(i) *$\tilde{u}$ is upper semicontinuous on $\boldsymbol{R}(D)$,*

(ii) *for any $k = 0, \dots, n-1$ and $I = (i_1, \dots, i_k)$ with $1 \le i_1 < \dots < i_k \le n$, the function $\widetilde{\widetilde{u_I}}$ is convex on $\log D_I$,*

(iii) *for any $j = 1, \dots, n$, if*

$$\{(r_1^0, \dots, r_{j-1}^0)\} \times [0, \delta_j) \times \{(r_{j+1}^0, \dots, r_n^0)\} \subset \boldsymbol{R}(D),$$

then the function

$$[0, \delta_j) \ni r_j \mapsto \tilde{u}(r_1^0, \dots, r_{j-1}^0, r_j, r_{j+1}^0, \dots, r_n^0)$$

is increasing.

Proof. (a) First assume that $u \in \mathcal{C}^2(D, \mathbb{R})$. Recall that $u(z) = u(e^{i\theta} \cdot z)$ for any $z \in D$ and $\theta \in \mathbb{R}^n$. Hence $\mathcal{L}u(z; X) = \mathcal{L}u(e^{i\theta} \cdot z; e^{i\theta} \cdot X)$ for any $z \in D$, $\theta \in \mathbb{R}^n$, and $X \in \mathbb{C}^n$. Consequently, for any $z \in D$ and $\theta \in \mathbb{R}^n$ we get:

$$\forall_{X \in \mathbb{C}^n} : \mathcal{L}u(z; X) \ge 0 \iff \forall_{X \in \mathbb{C}^n} : \mathcal{L}u((|z_1|, \dots, |z_n|); X) \ge 0.$$

One easily checks that

$$4\mathcal{L}u(z; X) = \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{u}}{\partial r_j \partial r_k}(r) \frac{\bar{z}_j}{r_j} \frac{z_k}{r_k} X_j \bar{X}_k + \sum_{j=1}^{n} \frac{\partial \tilde{u}}{\partial r_j}(r) \frac{1}{r_j} |X_j|^2,$$
$$z = (z_1, \dots, z_n) \in D,\ r = (r_1, \dots, r_n) := (|z_1|, \dots, |z_n|).$$

In particular,

$$\begin{aligned}
4\mathcal{L}u(e^x;X) &= 4\operatorname{Re}(\mathcal{L}u(e^x;X)) \\
&= \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{u}}{\partial r_j \partial r_k}(e^x)\operatorname{Re}(X_j\bar{X}_k) + \sum_{j=1}^{n}\frac{\partial \tilde{u}}{\partial r_j}(e^x)e^{-x_j}|X_j|^2 \\
&= \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{u}}{\partial r_j \partial r_k}(e^x)(a_ja_k + b_jb_k) + \sum_{j=1}^{n}\frac{\partial \tilde{u}}{\partial r_j}(e^x)e^{-x_j}(a_j^2+b_j^2), \\
&\qquad\qquad x \in \log D,\ X = a+ib \in \mathbb{C}^n.
\end{aligned}$$

On the other hand,

$$\mathcal{H}\tilde{\tilde{u}}(x;\xi) = \sum_{j,k=1}^{n} \frac{\partial^2 \tilde{u}}{\partial r_j \partial r_k}(e^x)e^{x_j}e^{x_k}\xi_j\xi_k + \sum_{j=1}^{n}\frac{\partial \tilde{u}}{\partial r_j}(e^x)e^{x_j}\xi_j^2, \qquad x \in \log D,\ \xi \in \mathbb{R}^n.$$

Finally,

$$4\mathcal{L}u(e^x; a+ib) = \mathcal{H}\tilde{\tilde{u}}(x, e^{-x}\cdot a) + \mathcal{H}\tilde{\tilde{u}}(x, e^{-x}\cdot b), \qquad x \in \log D,\ X = a+ib \in \mathbb{C}^n,$$

which implies the required relation.

Now, let u be arbitrary and assume that u is psh. Let u_ε denote the ε-regularization of u (u_ε is psh and $u_\varepsilon \searrow u$). Observe that D_ε is Reinhardt and for any $z \in D_\varepsilon$ and $\theta \in \mathbb{R}^n$ we get

$$\begin{aligned}
u_\varepsilon(e^{i\theta}\cdot z) &= \int_{\mathbb{D}^n} u(e^{i\theta}\cdot z + \varepsilon w)\Phi(w)\, d\Lambda_{2n}(w) \\
&= \int_{\mathbb{D}^n} u(e^{i\theta}\cdot z + \varepsilon e^{i\theta}\cdot w)\Phi(w)\, d\Lambda_{2n}(w) \\
&= \int_{\mathbb{D}^n} u(z + \varepsilon w)\Phi(w)\, d\Lambda_{2n}(w) = u_\varepsilon(z).
\end{aligned}$$

Thus $u_\varepsilon(z_1,\dots,z_n) = u_\varepsilon(|z_1|,\dots,|z_n|)$ for any $(z_1,\dots,z_n) \in D_\varepsilon$. By the first part of the proof, $\tilde{\tilde{u}}_\varepsilon$ is convex in $\log(D_\varepsilon)$. Since $\tilde{\tilde{u}}_\varepsilon \searrow \tilde{\tilde{u}}$, we conclude that $\tilde{\tilde{u}}$ is convex.

Conversely, assume that $v := \tilde{\tilde{u}}$ is convex in $G := \log D$. Let v_ε denote the standard ε-regularization of v:

$$v_\varepsilon(x) := \int_{\mathbb{B}(1)} v(x+\varepsilon y)\Theta(y)\, d\Lambda_n(y), \quad x \in G_\varepsilon := \{x \in G : \mathbb{B}(x,\varepsilon) \subset G\}$$

(v_ε is convex in G_ε and $v_\varepsilon \searrow v$). Define

$$D^\varepsilon := \{(z_1,\dots,z_n) \in \mathbb{C}^n : (\log|z_1|,\dots,\log|z_n|) \in G_\varepsilon\},$$
$$u^\varepsilon(z) := v_\varepsilon(\log|z_1|,\dots,\log|z_n|), \quad (z_1,\dots,z_n) \in D^\varepsilon.$$

Observe that $D^\varepsilon \nearrow D$ and $u^\varepsilon \searrow u$. By the first part of the proof, $u^\varepsilon \in \mathcal{PSH}(D^\varepsilon)$. Hence $u \in \mathcal{PSH}(D)$.

(b) First assume that $u \in \mathcal{PSH}(D)$. Then it is clear that (i) is satisfied. Observe that $\log D = \log(D \cap \mathbb{C}_*^n)$. Hence, by (a), condition (ii) is satisfied for $k = 0$. For $k = 1,\dots,n-1$, the function u_I is psh on D_I. Consequently, we can repeat the above argument and so (ii) is satisfied for any k. To prove (iii) observe that the function

$$K(\delta_j) \ni \lambda \overset{v}{\mapsto} u(r_1^0,\dots,r_{j-1}^0,\lambda,r_{j+1}^0,\dots,r_n^0)$$

is well defined, radial, and subharmonic. In particular, the function

$$[0,\delta_j) \ni r_j \mapsto \boldsymbol{J}(v;0,r_j) = \tilde{u}(r_1^0,\dots,r_{j-1}^0,r_j,r_{j+1}^0,\dots,r_n^0)$$

is increasing. Now, assume that (i), (ii), and (iii) are satisfied. Then obviously u is upper continuous on D. Moreover, by (a), u is psh on $D \cap \mathbb{C}_*^n$ and, more generally, each function u_I is psh in $D_I \cap \mathbb{C}_*^{n-k}$, $I = (i_1,\dots,i_k)$.

Take an $a = (a_1,\dots,a_n) \in D$ with $a_1 \cdots a_n = 0$ and $X = (X_1,\dots,X_n) \in \mathbb{C}^n$ with $\|X\|_\infty = 1$. We want to prove that

$$u(a) \le \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta}X)\,d\theta$$

for sufficiently small $r > 0$. We may assume that $a_1 \cdots a_\ell \neq 0$, $a_{\ell+1} = \cdots = a_n = 0$ with $0 \le \ell \le n-1$. Let $0 < r < d_D(a)$ be so small that $z_1 \cdots z_\ell \neq 0$ for $(z_1,\dots,z_n) \in a + r\overline{\mathbb{D}}X$. Take $I := (\ell+1,\dots,n)$. Recall that u_I is psh in $D_I \cap (\mathbb{C}_*)^\ell$ (if $\ell \ge 1$). Hence, using (iii), we get

$$\begin{aligned} u(a) &\le \frac{1}{2\pi}\int_0^{2\pi} u(a_1 + re^{i\theta}X_1,\dots,a_\ell + re^{i\theta}X_\ell,0,\dots,0)\,d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} u(a_1 + re^{i\theta}X_1,\dots,a_\ell + re^{i\theta}X_\ell, re^{i\theta}X_{\ell+1},\dots,re^{i\theta}X_n)\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta}X)\,d\theta. \end{aligned}$$

□

1.15 Pseudoconvexity

As we said at the beginning of § 1.14, plurisubharmonic functions may be considered as a counterpart of convex functions. Recall that a domain $D \subset \mathbb{R}^k$ is convex iff the function $-\log \operatorname{dist}(\cdot, \partial D)$ is convex (cf. [Vla 1966], Chapter 2, § 13). This suggests that we consider the class of those domains $D \subset \mathbb{C}^n$ for which the function $-\log \operatorname{dist}(\cdot, \partial D)$ is plurisubharmonic. We are led to the following definition.

Definition 1.15.1. An open set $\Omega \subset \mathbb{C}^n$ is called *pseudoconvex* if

$$-\log d_\Omega \in \mathcal{PSH}(\Omega).$$ [57]

Notice that $\mathbb{C}^n$ is pseudoconvex (because $-\log d_{\mathbb{C}^n} \equiv -\infty$). Moreover, Ω is pseudoconvex iff each of its connected components is pseudoconvex.

Remark 1.15.2. (a) Every domain $D \subset \mathbb{C}$ is pseudoconvex.

Indeed, if $D \subsetneq \mathbb{C}$, then $d_D(z) = \inf\{|z-\zeta| : \zeta \notin D\}$, $z \in D$. Hence

$$-\log d_D(\cdot) = \sup\left\{\log\left|\frac{1}{\cdot - \zeta}\right| : \zeta \notin D\right\} \in \mathcal{SH}(D)$$

(cf. Remark 1.14.3 (c) and Proposition 1.14.16).

(b) If $(D_i)_{i\in I}$ is a family of pseudoconvex open subsets of $\mathbb{C}^n$, then the open set $\Omega := \operatorname{int} \bigcap_{i\in I} D_i$ is pseudoconvex.

Indeed, we have $d_\Omega(z) = \inf\{d_{D_i}(z) : i \in I\}$, $z \in \Omega$ (Exercise). Hence, by Proposition 1.14.16, $-\log d_\Omega \in \mathcal{PSH}(\Omega)$.

(c) If $(D_k)_{k=1}^\infty$ is a sequence of pseudoconvex domains in $\mathbb{C}^n$ such that $D_k \subset D_{k+1}$, $k \in \mathbb{N}$, then $D := \bigcup_{k=1}^\infty D_k$ is pseudoconvex.

Indeed, since $-\log d_{D_k} \searrow -\log d_D$, we only need to use Remark 1.14.3 (d).

(d) If D_j is a pseudoconvex subset of $\mathbb{C}^{n_j}$, $j = 1, \dots, N$, then

$$D := D_1 \times \cdots \times D_N$$

is pseudoconvex in $\mathbb{C}^{n_1+\cdots+n_N}$. In particular, for any domains $D_1, \dots, D_n \subset \mathbb{C}$, the domain $D := D_1 \times \cdots \times D_n$ is pseudoconvex in $\mathbb{C}^n$.

Indeed, we have

$$d_D(z_1, \dots, z_n) = \min\{d_{D_j}(z_j) : j = 1, \dots, N\}, \quad (z_1, \dots, z_n) \in D.$$

Hence, by Remark 1.14.3 (f), $-\log d_D \in \mathcal{PSH}(D)$.

For a domain $D \subset \mathbb{C}^n$, put

$$\delta_{D,X}(a) := \sup\{r > 0 : a + K(r)\cdot X \subset D\}, \quad a \in D,\ X \in \mathbb{C}^n.$$

Observe that:

- If $n = 1$, then $\delta_{D,X}(z) = d_D(z)/|X|$, $z \in D$.
- $\delta_{D,X}(a+\lambda X) = d_{D_{a,X}}(\lambda)$, $\lambda \in D_{a,X}$, where $D_{a,X} := \{\lambda \in \mathbb{C} : a+\lambda X \in D\}$.

[57] Observe that the function d_Ω is continuous.

Exercise 1.15.3. The function

$$D \times \mathbb{C}^n \ni (a, X) \mapsto \delta_{D,X}(a) \in (0, +\infty]$$

is lower semicontinuous.

Given a $\mathbb{C}$-norm $q\colon \mathbb{C}^n \to \mathbb{R}_+$, define

$$d_{D,q}(a) := \sup\{r > 0 : B_q(a, r) \subset D\}, \quad a \in D,$$

where $B_q(a, r) := \{z \in \mathbb{C}^n : q(z - a) < r\}$. Obviously, $d_{D,\|\,\|_\infty} = d_D$. Notice that the function $d_{D,q}$ is continuous.

Exercise 1.15.4. $d_{D,q} = \inf\{\delta_{D,X} : X \in \mathbb{C}^n,\ q(X) = 1\}$.

For a compact $K \subset D$ and a family $\mathcal{S} \subset \mathcal{PSH}(D)$ let

$$\widetilde{K}_{\mathcal{S}} := \{z \in D : \forall_{u \in \mathcal{S}} :\ u(z) \le \max_K u\}.$$

By Remark 1.14.3 (c), $\widetilde{K}_{\mathcal{PSH}(D)} \subset \widehat{K}_{\mathcal{O}(D)}$. Moreover, the set $\widetilde{K}_{\mathcal{PSH}(D) \cap \mathcal{C}(D)}$ is relatively closed in D.

A function $u\colon D \to \mathbb{R}$ is called an *exhaustion function* if for any $t \in \mathbb{R}$ the set $\{z \in D : u(z) \le t\}$ is relatively compact in D.

The next result contains various equivalent descriptions of pseudoconvexity.

Theorem 1.15.5. *Let D be an open subset of $\mathbb{C}^n$. Then the following conditions are equivalent:*

(i) $-\log \delta_{D,X} \in \mathcal{PSH}(D)$ *for every* $X \in \mathbb{C}^n$*;*

(ii) $-\log d_{D,q} \in \mathcal{PSH}(D)$ *for every* $\mathbb{C}$*-norm* q*;*

(iii) *D is pseudoconvex;*

(iv) *there exists an exhaustion function* $u \in \mathcal{PSH}(D) \cap \mathcal{C}(D)$*;*

(v) *there exists an exhaustion function* $u \in \mathcal{PSH}(D)$*;*

(vi) $\widetilde{K}_{\mathcal{PSH}(D) \cap \mathcal{C}(D)}$ *is compact in D for every compact* $K \subset D$*;*

(vii) $\widetilde{K}_{\mathcal{PSH}(D)}$ *is relatively compact in D for every compact* $K \subset D$*;*

(viii) *every point* $a \in \partial D$ *has an open neighborhood* U_a *such that* $U_a \cap D$ *is pseudoconvex, i.e. D is* locally pseudoconvex*;*

(ix) (Kontinuitätssatz) *for every sequence* $\varphi_k \in \mathcal{C}(\overline{\mathbb{D}}, D) \cap \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$*,* $k = 1, 2, \dots,$ *if* $\bigcup_{k=1}^\infty \varphi_k(\mathbb{T}) \Subset D$*, then* $\bigcup_{k=1}^\infty \varphi_k(\overline{\mathbb{D}}) \Subset D$*;*

(x) (Kontinuitätssatz) *for every continuous mapping* $\varphi\colon [0, 1] \times \overline{\mathbb{D}} \to \mathbb{C}^n$*, if* $\varphi(t, \cdot) \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$*,* $t \in [0, 1)$*, and* $\varphi(([0, 1) \times \overline{\mathbb{D}}) \cup (\{1\} \times \mathbb{T})) \subset D$*, then* $\varphi(\{1\} \times \overline{\mathbb{D}}) \subset D$*.*

Proof. The case $D = \mathbb{C}^n$ is elementary. Thus we may assume that $D \subsetneq \mathbb{C}^n$.

(i) $\Rightarrow$ (ii) follows from Exercise 1.15.4 and Proposition 1.14.16.

The implication (ii) $\Rightarrow$ (iii) is trivial.

For the proof of (iii) $\Rightarrow$ (iv) take $u(z) := \max\{-\log d_D(z), \|z\|\}$, $z \in D$.

The implications (iv) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (vii) are trivial.

(v) $\Rightarrow$ (vii): If u is as in (v), then

$$\widetilde{K}_{\mathcal{PSH}(D)} \subset \{z \in D : u(z) \le \max_K u\} \Subset D.$$

In the same way we check that (iv) $\Rightarrow$ (vi).

(vii) $\Rightarrow$ (i): (This is the main part of the proof.) Fix $a \in D$, $X, Y \in (\mathbb{C}^n)_*$. We want to show that the function

$$D_{a,Y} \ni \lambda \to -\log \delta_{D,X}(a + \lambda Y)$$

is subharmonic.

First consider the case where X and Y are linearly dependent. We may assume that $X = Y$. Since $\delta_{D,X}(a + \lambda X) = d_{D_{a,X}}(\lambda)$, $\lambda \in D_{a,X}$, we can use Remark 1.15.2 (a).

Now assume that X, Y are linearly independent. It is sufficient to prove (cf. Remark 1.14.3 (a) (iv)) that if $\overline{K}(r) \subset D_{a,Y}$, and if $p \in \mathcal{P}(\mathbb{C})$ is such that

$$-\log \delta_{D,X}(a + \lambda Y) \le \operatorname{Re} p(\lambda), \quad \lambda \in \partial K(r),$$

then the same inequality holds for all $\lambda \in K(r)$. In other words, if

$$\delta_{D,X}(a + \lambda Y) \ge e^{-\operatorname{Re} p(\lambda)}, \quad \lambda \in \partial K(r),$$

then the same is true for all $\lambda \in K(r)$. Thus we have to show that if

$$a + \lambda Y + K(|e^{-p(\lambda)}|) \cdot X \subset D, \quad \lambda \in \partial K(r),$$

then the same inclusion holds for all $\lambda \in K(r)$.

For $0 \le \theta < 1$ let

$$K_\theta := \{a + \lambda Y + \overline{K}(\theta |e^{-p(\lambda)}|) \cdot X : \lambda \in \partial K(r)\},$$
$$M_\theta := \{a + \lambda Y + \overline{K}(\theta |e^{-p(\lambda)}|) \cdot X : \lambda \in \overline{K}(r)\}.$$

Observe that K_θ and M_θ are compact and $K_{\theta'} \subset K_{\theta''}$, $M_{\theta'} \subset M_{\theta''}$ for $0 \le \theta' < \theta'' < 1$. Our problem is to show that if $K_\theta \subset D$ for all $0 \le \theta < 1$, then $M_\theta \subset D$ for all $0 \le \theta < 1$. Thus assume that $K_\theta \subset D$ for all $0 \le \theta < 1$ and let $I_0 := \{\theta \in [0, 1) : M_\theta \subset D\}$.

Notice that $M_0 = a + \overline{K}(r)Y \subset D$. Hence $I_0 \ne \varnothing$. Suppose that $\theta_0 \in I_0$. Since M_{θ_0} is compact, there exists a $\theta \in (\theta_0, 1)$ such that $M_\theta \subset D$. Consequently,

I_0 is open. It remains to prove that I_0 is closed in $[0,1)$, i.e. if $M_\theta \subset D$ for $0 < \theta < \theta_0 < 1$, then $M_{\theta_0} \subset D$.

Fix $0 < \theta < \theta_0$. Observe that

$$M_\theta = \{a + \lambda Y + \zeta e^{-p(\lambda)} X : |\lambda| \le r, |\zeta| \le \theta\} \Subset D.$$

Take a $u \in \mathcal{PSH}(D)$ and define

$$v_\zeta(\lambda) := u(a + \lambda Y + \zeta e^{-p(\lambda)} X), \quad \zeta \in \bar{K}(\theta), \ \lambda \in \bar{K}(r).$$

Then v_ζ is subharmonic and, therefore, the maximum principle gives

$$v_\zeta(\lambda) \le \max_{\partial K(r)} v_\zeta \le \max_{K_\theta} u \le \max_{K_{\theta_0}} u, \quad \lambda \in \bar{K}(r).$$

Consequently, $M_\theta \subset (\widetilde{K_{\theta_0}})_{\mathcal{PSH}(D)} \Subset D$ for any $0 < \theta < \theta_0$ and hence $M_{\theta_0} \subset D$.

The implication (iii) $\Rightarrow$ (viii) is trivial.

(viii) $\Rightarrow$ (iv): For an $a \in \partial D$ let U_a be a neighborhood of a such that $U_a \cap D$ is pseudoconvex. Clearly, there exists a smaller neighborhood $V_a \subset U_a$ such that $d_D = d_{U_a \cap D}$ in $V_a \cap D$ (EXERCISE). In particular, $-\log d_D \in \mathcal{PSH}(V_a \cap D)$. Consequently, there exists a closed set $F \subset \mathbb{C}^n$ such that $F \subset D$ and $-\log d_D \in \mathcal{PSH}(D \setminus F)$. Let

$$\varphi_0(t) := \max\{-\log d_D(z) : z \in F, \ \|z\| \le t\}, \quad t \in \mathbb{R}$$

(with $\max \varnothing = -\infty$). One can easily prove (EXERCISE) that there exists an increasing convex function $\varphi\colon \mathbb{R} \to \mathbb{R}_+$ such that $\varphi(t) > \max\{t, \varphi_0(t)\}$, $t \in \mathbb{R}$. Put

$$u(z) := \max\{-\log d_D(z), \varphi(\|z\|)\}, \quad z \in D.$$

The function u is obviously continuous. Since $\varphi(\|z\|) > -\log d_D(z)$ for z in a neighborhood of F, the function u is plurisubharmonic in D (cf. Remark 1.14.3 (h)). Moreover,

$$\{z \in D : u(z) \le t\} \subset \{z \in D : d_D(z) \ge e^{-t}, \ \|z\| \le t\} \Subset D, \quad t \in \mathbb{R}.$$

(vii) $\Rightarrow$ (ix): Put $K := \overline{\bigcup_{k=1}^\infty \varphi_k(\mathbb{T})}$. It suffices to show that $\varphi_k(\bar{\mathbb{D}}) \subset \widetilde{K}_{\mathcal{PSH}(D)}$, $k = 1, 2, \dots$. Let $u \in \mathcal{PSH}(D)$. Then, for every k, the function $u \circ \varphi_k$ is subharmonic in $\mathbb{D}$ (Proposition 1.14.34) and upper semicontinuous on $\bar{\mathbb{D}}$. In particular, by the maximum principle we have

$$\max_{\varphi_k(\bar{\mathbb{D}})} u = \max_{\bar{\mathbb{D}}} u \circ \varphi_k = \max_{\mathbb{T}} u \circ \varphi_k = \max_{\varphi_k(\mathbb{T})} u \le \max_K u, \quad k = 1, 2, \dots.$$

(ix) $\Rightarrow$ (x): Since φ is continuous and $\varphi(\{1\} \times \mathbb{T}) \subset D$, there exists a $\theta \in (0,1)$ such that $K := \varphi([\theta, 1] \times \mathbb{T}) \Subset D$. Take $\theta \le t_k \nearrow 1$ and let $\varphi_k(\lambda) := \varphi(t_k, \lambda)$,

$\lambda \in \overline{\mathbb{D}}$. Then $\varphi_k \in \mathcal{C}(\overline{\mathbb{D}}, D) \cap \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$, $k = 1, 2, \dots$, and $\bigcup_{k=1}^{\infty} \varphi_k(\mathbb{T}) \subset K \Subset D$. Consequently, by (ix), we conclude that $L := \bigcup_{k=1}^{\infty} \varphi_k(\overline{\mathbb{D}}) \Subset D$. In particular, $\varphi(\{1\} \times \mathbb{T}) \subset \overline{L} \subset D$.

(x) $\Rightarrow$ (iii): We keep all the notations from the proof of the implication (vii) $\Rightarrow$ (i). Recall that the only problem is to show that the set I_0 is closed in $[0, 1)$. Suppose that $[0, \theta_0) \subset I_0$. Fix a $\zeta \in \overline{\mathbb{D}}$, and define

$$\varphi(t, \lambda) := a + r\lambda Y + t\theta_0\zeta e^{-p(r\lambda)} X, \quad t \in [0, 1],\ \lambda \in \mathbb{C}.$$

To prove that $M_{\theta_0} \subset D$, we have to show that $\varphi(\{1\} \times \overline{\mathbb{D}}) \subset D$. Observe that φ is continuous, $\varphi(t, \cdot)$ is holomorphic, $\varphi([0, 1) \times \overline{\mathbb{D}}) \subset \bigcup_{\theta \in [0, \theta_0)} M_\theta \subset D$, and $\varphi(\{1\} \times \mathbb{T}) \subset K_{\theta_0} \subset D$. Thus, by (x), $\varphi(\{1\} \times \overline{\mathbb{D}}) \subset D$, which finishes the proof. □

Corollary 1.15.6. *Let $F : D \to D'$ be biholomorphic. Then D is pseudoconvex iff D' is pseudoconvex.*

Proof. Use Theorem 1.15.5 (v) and Proposition 1.14.34. □

Corollary 1.15.7. *Any holomorphically convex domain $D \subset \mathbb{C}^n$ is pseudoconvex. In particular, any convex domain is pseudoconvex.*

It is natural to ask whether the converse implication is also true. This is the famous *Levi Problem*, which will be discussed in § 1.16.

Corollary 1.15.8. *If a domain $D \subset \mathbb{C}^n$ is pseudoconvex, then for any complex affine subspace $H \subset \mathbb{C}^n$, the open set $D \cap H$ (which is identified with an open subset of $\mathbb{C}^k$, $k = \dim H$) is pseudoconvex.*

Proposition 1.15.9. *A domain $D \subset \mathbb{C}^n$ is pseudoconvex iff for arbitrary $a \in D$, $X, Y \in \mathbb{C}^n$, the open set*

$$D_{a,X,Y} := \{(\mu, \lambda) \in \mathbb{C}^2 : a + \mu X + \lambda Y \in D\}$$

is pseudoconvex.

Proof. By Corollary 1.15.8, $D_{a,X,Y}$ is pseudoconvex provided D is pseudoconvex.

Assume that each $D_{a,X,Y}$ is pseudoconvex, i.e. $-\log \delta_{D_{a,X,Y},\xi}$ is plurisubharmonic in $D_{a,X,Y}$ for any a, X, Y, and $\xi \in \mathbb{C}^2$. Observe that

$$\delta_{D,X}(a + \lambda Y) = \delta_{D_{a,X,Y},(1,0)}(0, \lambda),$$

which implies that $-\log \delta_{D,X}$ is plurisubharmonic in D for any X. □

Corollary 1.15.10. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain and let $u \in \mathcal{PSH}(D)$. Then the open set $\Omega := \{x \in D : u(x) < 0\}$ is pseudoconvex.*

Proof. First assume that u is additionally continuous. Let $K \subset \Omega$ be a compact set and let $\varepsilon > 0$ be such that $u \le -\varepsilon$ on K. Then

$$\widetilde{K}_{\mathcal{PSH}(\Omega)} \subset \{z \in D : u(z) \le -\varepsilon\} \cap \widetilde{K}_{\mathcal{PSH}(D)} \Subset \Omega.$$

Now, let u be arbitrary. Let u_ε denote the ε-regularization of u (cf. Proposition 1.14.33). By the first part of the proof (applied to the function $-\log d_D + \log \varepsilon$) the open set

$$D_\varepsilon = \{z \in D : d_D(z) > \varepsilon\} = \{z \in D : -\log d_D(z) + \log \varepsilon < 0\}$$

is pseudoconvex. Further, for each ε, the open set $\Omega^\varepsilon := \{z \in D_\varepsilon : u_\varepsilon(z) < 0\}$ is pseudoconvex. Observe that $\Omega^\varepsilon \nearrow \Omega$ when $\varepsilon \searrow 0$. It remains to use Remark 1.15.2 (c). □

Proposition 1.15.11. *Let $D \subset \mathbb{C}^n$ be a balanced domain and let $h = h_D$ be its Minkowski function. Then D is pseudoconvex iff $h \in \mathcal{PSH}(\mathbb{C}^n)$ iff $\log h \in \mathcal{PSH}(\mathbb{C}^n)$ (cf. Proposition* 1.14.37 (b)).

Proof. If $\log h \in \mathcal{PSH}(\mathbb{C}^n)$, then D is pseudoconvex by Corollary 1.15.10. Observe that

$$\delta_{D,X}(0) = 1/h(X), \quad X \in \mathbb{C}^n.$$

Consequently, if D is pseudoconvex, then $\log h$ is psh by Theorem 1.15.5 (i). □

Proposition 1.15.12 (Siciak's example – cf. [Sic 1982]). *For any $n \ge 2$ there exists a psh function $h\colon \mathbb{C}^n \to \mathbb{R}_+$, $h \not\equiv 0$, with $h(\lambda z) = |\lambda| h(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$,*[58] *such that the set $h^{-1}(0)$ is dense in $\mathbb{C}^n$.*

In particular, the balanced domain $D := \{z \in \mathbb{C}^n : h(z) + \|z\| < 1\} \subset \mathbb{B}_n$ is a pseudoconvex domain with irregular Minkowski function.

Proof. We write $\mathbb{Q}^{2n-2} = \{r_j : j \in \mathbb{N}\} \subset \mathbb{C}^{n-1}$ and we define the linear functionals $L_j : \mathbb{C}^n \to \mathbb{C}$ by $L_j(z) := \langle z, (1, r_j)\rangle$. Put $V_j := L_j^{-1}(0)$, $V := \bigcup_{j=1}^\infty V_j$. Then we define a sequence of psh functions by

$$h_j := \left(\frac{|L_1 \cdots L_j|}{\|L_1 \cdots L_j\|_{\mathbb{B}_n}}\right)^{1/j}.$$

Observe that

$$h_j \ge 0, \quad h_j(\lambda z) = |\lambda| h_j(z), \ \lambda \in \mathbb{C}, \ z \in \mathbb{C}^n, \quad h_j|_{V_1 \cup \dots \cup V_j} \equiv 0.$$

Moreover, by the maximum principle, there are points $z_j \in \partial\mathbb{B}_n$ such that $h_j(z_j) = 1$. By the Hartogs Lemma for psh functions (Proposition 1.14.31), it turns out that there

[58] Notice that, by Proposition 1.14.37 (b), $\log h \in \mathcal{PSH}(\mathbb{C}^n)$.

is a point z^*, $\|z^*\| < 2$, with $\limsup_{j\to\infty} h_j(z^*) \geq 2/3$. So taking an appropriate subsequence $(h_{j_\nu})_\nu \subset (h_j)_j$ with $h_{j_\nu}(z^*) \geq 1/2$ and defining

$$h(z) := \prod_{\nu=1}^{\infty} (h_{j_\nu}(z))^{2^{-\nu}} = \exp\Big(\sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \log |h_{j_\nu}(z)|\Big), \quad z \in \mathbb{C}^n,$$

we obtain a psh function on $\mathbb{C}^n$ (Exercise) with

$$h|_V = 0, \quad h(z^*) \geq \frac{1}{2}, \quad h(\lambda z) = |\lambda| h(z),\ \lambda \in \mathbb{C},\ z \in \mathbb{C}^n. \qquad \square$$

The following lemma will be used in the proofs of the next propositions.

Lemma 1.15.13. *Let $\varnothing \neq \Sigma \subset (\mathbb{R}^n)_*$ and let $(c_\alpha)_{\alpha\in\Sigma} \subset \mathbb{R}_{>0}$ be such that*

$$\sup\{c_\alpha^{1/|\alpha|} : \alpha \in \Sigma\} < +\infty.[59] \tag{1.15.1}$$

Define

$$u(z) := \sup\{|c_\alpha z^\alpha|^{1/|\alpha|} : \alpha \in \Sigma\}, \quad z \in \mathbb{C}^n(\Sigma).$$

In the case where the set $\Sigma \subset \mathbb{Z}^n$ is unbounded put

$$v(z) := \limsup_{|\alpha|\to+\infty} |c_\alpha z^\alpha|^{1/|\alpha|}, \quad z \in \mathbb{C}^n(\Sigma).$$

Then:

(a) *the family $(|c_\alpha z^\alpha|^{1/|\alpha|})_{\alpha\in\Sigma}$ is locally bounded in $\mathbb{C}^n(\Sigma)$;*

(b) *$u^*, v^* \in \mathcal{PSH}(\mathbb{C}^n(\Sigma))$;*

(c) *u^*, v^* are invariant with respect to n-rotations;*[60]

(d) *$u, v \in \mathcal{C}(\mathbb{C}^n_*)$;*

(e) *$D_{u^*} := \{z \in \mathbb{C}^n(\Sigma) : u^*(z) < 1\} = \operatorname{int} \bigcap_{\alpha\in\Sigma} \{z \in \mathbb{C}^n(\Sigma) : c_\alpha |z^\alpha| < 1\}$;*

(f) *$D_{v^*} := \{z \in \mathbb{C}^n(\Sigma) : v^*(z) < 1\} = \bigcup_{\nu=1}^\infty (\operatorname{int} \bigcap_{\substack{\alpha\in\Sigma \\ |\alpha|\geq\nu}} \{z \in \mathbb{C}^n(\Sigma) : c_\alpha |z^\alpha| < 1\})$;*

(g) *if $D_{u^*} \neq \varnothing$, then $\bar{D}_{u^*} \cap \mathbb{C}^n_* = \{z \in \mathbb{C}^n_* : u(z) \leq 1\}$;*

(h) *if $D_{v^*} \neq \varnothing$, then $\bar{D}_{v^*} \cap \mathbb{C}^n_* = \{z \in \mathbb{C}^n_* : v(z) \leq 1\}$;*

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[59] Observe that (1.15.1) is satisfied if

$$D := \operatorname{int} \bigcap_{\alpha\in\Sigma} \{z \in \mathbb{C}^n(\Sigma) : c_\alpha |z^\alpha| < 1\} \neq \varnothing.$$

Indeed, take an $a = (a_1, \dots, a_n) \in D \setminus V_0$ and let $C := \max\{|a_j|,\ 1/|a_j| : j = 1, \dots, n\}$. Then $c_\alpha^{1/|\alpha|} \leq |a^{-\alpha}|^{1/|\alpha|} \leq 1/C$, $\alpha \in \Sigma$.

[60] That is, $\varphi \circ T_\zeta = \varphi$, $\zeta \in \mathbb{T}^n$, $\varphi \in \{u^*, v^*\}$.

(i) *if* $\Sigma \subset \mathbb{R}^n_+$, *then for* $h \in \{u^*, v^*\}$ *we have*

$$h(\lambda z) = |\lambda| h(z), \quad \lambda \in \mathbb{C},\ z \in \mathbb{C}^n, \tag{1.15.2}$$

$$h(\lambda \cdot z) \le h(z), \qquad \lambda \in \overline{\mathbb{D}}^n,\ z \in \mathbb{C}^n; \tag{1.15.3}$$

in particular, by Lemma 1.8.3, $u^* = h_{D_{u^*}}$, $v^* = h_{D_{v^*}}$.

Proof. To simplify notation assume that $\mathbb{C}^n(\Sigma) = \mathbb{C}^s \times \mathbb{C}^{n-s}_*$ for some $0 \le s \le n$. In particular, $\Sigma \subset \mathbb{R}^s_+ \times \mathbb{R}^{n-s}$. Assume that $c_\alpha^{1/|\alpha|} \le C_0$, $\alpha \in \Sigma$.

(a) If $z = (z_1, \dots, z_n) \in \mathbb{C}^n(\Sigma)$ and $|z_j| \le C$, $j = 1, \dots, n$, $|z_j| \ge 1/C$, $j = s+1, \dots, n$, then $|c_\alpha z^\alpha|^{1/|\alpha|} \le C_0 C$, $\alpha \in \Sigma$.

(b) follows from Propositions 1.14.16 and 1.14.17.

(c) is obvious.

(d) Fix a point $a \in \mathbb{C}^n_*$ and let $\mathbb{P}(a,r) \Subset \mathbb{C}^n_*$. Let $f_\alpha \in \mathcal{O}(\mathbb{P}(a,r))$ be an arbitrary branch of the function $c_\alpha^{1/|\alpha|} z_1^{\alpha_1/|\alpha|} \cdots z_n^{\alpha_n/|\alpha|}$, $\alpha \in \Sigma$. Then the family $(f_\alpha)_{\alpha\in\Sigma}$ is uniformly bounded in $\mathbb{P}(a,r)$ and, consequently, it is equicontinuous (Lemma 1.7.23). In particular, the family $(|c_\alpha z^\alpha|^{1/|\alpha|})_{\alpha\in\Sigma}$ is equicontinuous in $\mathbb{P}(a,r)$. Hence the functions u and v are continuous in $\mathbb{P}(a,r)$ (Exercise).

(e) If $u^*(a) < 1$, then there exist a Reinhardt neighborhood $U \subset \mathbb{C}^n(\Sigma)$ of a and $0 < \theta < 1$ such that $|c_\alpha z^\alpha|^{1/|\alpha|} < \theta$, $z \in U$, $\alpha \in \Sigma$. Consequently,

$$U \subset \operatorname{int} \bigcap_{\alpha\in\Sigma} \{z \in \mathbb{C}^n(\Sigma) : c_\alpha |z^\alpha| < 1\}.$$

Conversely, if U is a Reinhardt neighborhood of a such that $c_\alpha |z^\alpha| < 1$, $z \in U$, $\alpha \in \Sigma$, then take a Reinhardt neighborhood $V \Subset U$ of a and an $r > 0$ such that $\overline{V}^{(r)} \subset U$.[61] By Lemma 1.6.2, there exists $0 < \theta < 1$ such that

$$c_\alpha \sup_{z\in V} |z|^\alpha \le \theta^{|\alpha|} c_\alpha \max_{z\in\overline{V}^{(r)}} |z|^\alpha \le \theta^{|\alpha|} c_\alpha \sup_{z\in U} |z|^\alpha \le \theta^{|\alpha|}, \quad \alpha \in \Sigma,$$

which implies that $u^*(a) \le \theta < 1$.

(f) Put

$$G := \bigcup_{\nu=1}^{\infty} \Big(\operatorname{int} \bigcap_{\alpha\in\Sigma,\ |\alpha|\ge\nu} \{z \in \mathbb{C}^n(\Sigma) : c_\alpha |z^\alpha| < 1\} \Big).$$

If $a \in G$, then there exist a Reinhardt neighborhood $U \subset \mathbb{C}^n(\Sigma)$ of a and $\nu_0 \in \mathbb{N}$ such that $c_\alpha |z^\alpha| < 1$, $z \in U$, $\alpha \in \Sigma$, $|\alpha| \ge \nu_0$. Consequently, by (e), $w^*(a) < 1$, where $w(z) := \sup\{|c_\alpha z^\alpha|^{1/|\alpha|} : \alpha \in \Sigma,\ |\alpha| \ge \nu_0\}$, $z \in \mathbb{C}^n(\Sigma)$. Hence $v^*(a) \le w^*(a) < 1$.

Conversely, assume that $v^*(a) < \theta' < \theta < 1$ for some $a \in \mathbb{C}^n(\Sigma)$ and let $V \Subset U \Subset \mathbb{C}^n(\Sigma)$ be neighborhoods of a with $v(z) < \theta'$, $z \in U$. Then, by the

[61] Recall that $A^{(r)} := \bigcup_{a\in A} \mathbb{P}(a,r)$.

Hartogs lemma (Proposition 1.14.31), there exists a $\nu_0 \in \mathbb{N}$ such that $|c_\alpha z^\alpha|^{1/|\alpha|} < \theta$, $z \in V$, $\alpha \in \Sigma$, $|\alpha| \geq \nu_0$. Consequently, $V \subset G$.

(g) The only problem is to show that if $b \in \mathbb{R}^n_{>0}$ is such that $u(b) = 1$, then $b \in \bar{D}_{u^*}$. Fix an $a \in D_{u^*} \cap \mathbb{R}^n_{>0}$. Put $z(t) := (a_1^{1-t} b_1^t, \dots, a_n^{1-t} b_n^t)$, $\varphi(t) := u(z(t))$, $t \in [0, 1]$. Then φ is continuous, $\varphi(0) = u(a) < \theta < 1$ and $\varphi(1) = u(b) = 1$. We only need to prove that $\varphi(t) < 1$, $t \in (0, 1)$. Suppose that $\varphi(t_0) = 1$ for some $t_0 \in (0, 1)$. Take $0 < \varepsilon < 1 - \theta^{1-t_0}$ and let $\alpha \in \Sigma$ be such that $(c_\alpha (z(t_0))^\alpha)^{1/|\alpha|} > 1 - \varepsilon$. Thus

$$\Big((c_\alpha a^\alpha)^{1/|\alpha|}\Big)^{1-t_0} \Big((c_\alpha b^\alpha)^{1/|\alpha|}\Big)^{t_0} > 1 - \varepsilon.$$

Since $(c_\alpha a^\alpha)^{1/|\alpha|} < \theta$, we conclude that

$$\theta^{1-t_0} \Big((c_\alpha b^\alpha)^{1/|\alpha|}\Big)^{t_0} > 1 - \varepsilon > \theta^{1-t_0}.$$

Hence, $u(b) \geq (c_\alpha b^\alpha)^{1/|\alpha|} > 1$; a contradiction.

(h) Take a $b \in \mathbb{R}^n_{>0}$ such that $v(b) = 1$. Fix an $a \in D_{v^*} \cap \mathbb{R}^n_{>0}$. Put $z(t) := (a_1^{1-t} b_1^t, \dots, a_n^{1-t} b_n^t)$, $\varphi(t) := v(z(t))$, $t \in [0, 1]$; φ is continuous, $\varphi(0) = v(a) < \theta < 1$ and $\varphi(1) = v(b) = 1$. We want to prove that $\varphi(t) < 1$, $t \in (0, 1)$. Suppose that $\varphi(t_0) = 1$ for some $t_0 \in (0, 1)$. Take $0 < \varepsilon < 1 - \theta^{1-t_0}$. Then there exists a sequence $(\alpha(k))_{k=1}^\infty \subset \Sigma$ such that $|\alpha(k)| \to +\infty$ and $(c_{\alpha(k)} (z(t_0))^{\alpha(k)})^{1/|\alpha(k)|} > 1 - \varepsilon$. Thus

$$\Big((c_{\alpha(k)} a^{\alpha(k)})^{1/|\alpha(k)|}\Big)^{1-t_0} \Big((c_{\alpha(k)} b^{\alpha(k)})^{1/|\alpha(k)|}\Big)^{t_0} > 1 - \varepsilon, \quad k \in \mathbb{N}.$$

Since $\limsup_{k\to+\infty} (c_{\alpha(k)} a^{\alpha(k)})^{1/|\alpha(k)|} < \theta$, $k \in \mathbb{N}$, we conclude that there exists a $k_0 \in \mathbb{N}$ such that

$$\theta^{1-t_0} \Big((c_{\alpha(k)} b^{\alpha(k)})^{1/|\alpha(k)|}\Big)^{t_0} > 1 - \varepsilon > \theta^{1-t_0}, \quad k \geq k_0.$$

Hence, $v(b) \geq ((1 - \varepsilon)\theta^{t_0 - 1})^{1/t_0} > 1$; a contradiction.

(i) We have $s = n$, i.e. $\mathbb{C}^n(\Sigma) = \mathbb{C}^n$. It is obvious that u and v satisfy (1.15.2) and (1.15.3). Moreover, $v(z) \leq u(z) \leq C_0 \|z\|_\infty$, $z \in \mathbb{C}^n$. In particular, u and v are continuous at 0 and $u^*(0) = v^*(0) = 0$. To prove that u^* (resp. v^*) satisfies (1.15.2) and (1.15.3) we need the following general observation.

If a function $h : \mathbb{C}^n \to \mathbb{R}_+$ with $h(z) \leq C_0 \|z\|_\infty$, $z \in \mathbb{C}^n$, satisfies (1.15.2) and (1.15.3), then so does h^*.

Condition (1.15.2) with $\lambda \neq 0$ and condition (1.15.3) with $\lambda \in \bar{\mathbb{D}}^n \setminus V_0$ are elementary. It remains to check (1.15.3) with $\lambda \in (\bar{\mathbb{D}}^n \cap V_0)_*$. Fix such a λ. We may assume that $\lambda_1 \cdots \lambda_r \neq 0$, $\lambda_{r+1} = \dots = \lambda_n = 0$ with $1 \leq r \leq n - 1$. Fix an $a \in (\mathbb{C}^n)_*$. We may assume that $a_{r+1} = \dots = a_t = 0$ and $a_{t+1} \cdots a_n \neq 0$ with $r \leq t \leq n$. Observe that if $\mathbb{C}^n \ni z \to \lambda \cdot a$, then $z = \mu(z) \cdot w(z)$, where

$$w(z) := (z_1/\lambda_1, \dots, z_r/\lambda_r, z_{r+1}, \dots, z_t, a_{t+1}, \dots, a_n) \to a,$$

$$\mu(z) := (\lambda_1, \dots, \lambda_r, 1, \dots, 1, z_{t+1}/a_{t+1}, \dots, z_n/a_n) \\ \to \mu^0 := (\lambda_1, \dots, \lambda_r, 1, \dots, 1, 0, \dots, 0).$$

In particular, $\mu(z) \in \bar{\mathbb{D}}^n$ for z near $\lambda \cdot a$. Consequently,

$$h^*(\lambda \cdot a) = \limsup_{z \to \lambda \cdot a} h(z) \le \limsup_{\substack{w \to a \\ \bar{\mathbb{D}}^n \ni \mu \to \mu^0}} h(\mu \cdot w) \le \limsup_{w \to a} h(w) = h^*(a). \qquad \square$$

Remark 1.15.14. (a) The functions u, v need not be continuous on the whole $\mathbb{C}^n(\Sigma)$. For example:

- $n := 2$, $\Sigma := \{(t, 1-t) : t \in (0,1)\}$, $c_\alpha := 1$, $\alpha \in \Sigma$. Then $\mathbb{C}^2(\Sigma) = \mathbb{C}^2$ and $u(z_1, z_2) = \begin{cases} 0 & \text{if } z_1 z_2 = 0 \\ \max\{|z_1|, |z_2|\} & \text{if } z_1 z_2 \neq 0 \end{cases}$.
- $n := 2$, $\Sigma := \mathbb{N}^2$, $c_\alpha := 1$, $\alpha \in \Sigma$. Then v coincides with the above u.

(b) If $D_{u^*} = \varnothing$ (resp. $D_{v^*} = \varnothing$), then the formula in (g) (resp. (h)) need not be true. For example:

- $n := 1$, $\Sigma := \{-1, 1\}$, $c_\alpha := 1$, $\alpha \in \Sigma$. Then $\mathbb{C}(\Sigma) = \mathbb{C}_*$ and $u(z) = \max\{|z|, 1/|z|\}$. Hence $D_{u^*} = \varnothing$, but $\{z \in \mathbb{C}_* : u^*(z) \le 1\} = \mathbb{T}$.
- $n := 1$, $\Sigma := \mathbb{Z}_*$, $c_\alpha := 1$, $\alpha \in \Sigma$. Then v coincides with the above u.

Proposition 1.15.15. *Consider a Laurent series $f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha$ whose domain of convergence $\mathfrak{D}$ is non-empty and the set $\Sigma := \{\alpha \in \mathbb{Z}^n : a_\alpha \neq 0\}$ is unbounded.*[62] *Put*

$$v(z) := \limsup_{|\alpha| \to +\infty} |a_\alpha z^\alpha|^{1/|\alpha|}, \quad z \in \mathbb{C}^n(\Sigma_*).$$[63]

Then $\mathfrak{D} = \{z \in \mathbb{C}^n(\Sigma_*) : v^*(z) < 1\} =: \mathfrak{D}_0$.

Moreover, if the function $f(z) := \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha$, $z \in \mathfrak{D}$, is bounded and $\|f\|_{\mathfrak{D}} \le 1$, then $\mathfrak{D} = \{z \in \mathbb{C}^n(\Sigma_*) : u^*(z) < 1\} =: \mathfrak{D}_1$, where

$$u(z) := \sup\{|a_\alpha z^\alpha|^{1/|\alpha|} : \alpha \in \Sigma_*\}, \quad z \in \mathbb{C}^n(\Sigma_*).$$

Proof. $\mathfrak{D} \subset \mathfrak{D}_0$: For every open set $U \Subset \mathfrak{D}$ there exist $C > 0$ and $0 < \theta < 1$ such that $\|a_\alpha z^\alpha\|_U \le C\theta^{|\alpha|}$, $\alpha \in \Sigma$ (Proposition 1.6.5 (d)). Hence $v \le \theta$ on U, and therefore $v^* \le \theta$ on U. Thus $\mathfrak{D} \subset \mathfrak{D}_0$.

$\mathfrak{D}_0 \subset \mathfrak{D}$: First observe that the family $(|a_\alpha z^\alpha|^{1/|\alpha|})_{\alpha \in \Sigma_*} \subset \mathcal{PSH}(\mathbb{C}^n(\Sigma_*))$ is locally bounded (Exercise – use Lemma 1.15.13 (a)).

By Lemma 1.15.13 (f), every point $a \in \mathfrak{D}_0$ has a neighborhood $U \Subset \mathbb{C}^n(\Sigma)$ for which there exist $0 < \theta < 1$ and $\nu_0 \in \mathbb{N}$ such that $\|a_\alpha z^\alpha\|_U^{1/|\alpha|} \le \theta$, $\alpha \in \Sigma$, $|\alpha| \ge k_0$. Then $U \subset \mathfrak{D}$, and finally, $\mathfrak{D}_0 \subset \mathfrak{D}$.

Since $v \le u$, we get $\mathfrak{D}_1 \subset \mathfrak{D}_0 = \mathfrak{D}$. By the Cauchy inequalities, we have $\|a_\alpha z^\alpha\|_{\mathfrak{D}} \le \|f\|_{\mathfrak{D}} \le 1$, $\alpha \in \Sigma$. Hence, by Lemma 1.6.2, $\mathfrak{D} \subset \mathfrak{D}_1$. $\square$

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[62] Notice that Σ is finite iff $f(z) = P(z)/z^\gamma$, where P is a polynomial and $\gamma \in \mathbb{Z}_+^n$.

[63] Note that if $n = 1$, then $v(z) = \max\{|z|/R^+, R^-/|z|\}$, where R^- and R^+ are given by (1.1.2).

Proposition 1.15.16. *Let $\varnothing \neq D \subsetneq \mathbb{C}^n$ be a Reinhardt domain. Then the following conditions are equivalent:*

(i) *D is a fat domain of holomorphy;*

(ii) *there exist $\Sigma \subset (\mathbb{R}^n)_*$ and a family $(c_\alpha)_{\alpha\in\Sigma} \subset \mathbb{R}_{>0}$ with (1.15.1) such that*

$$D = \{z \in \mathbb{C}^n(\Sigma) : u^*(z) < 1\},$$

where $u(z) := \sup\{|c_\alpha z^\alpha|^{1/|\alpha|} : \alpha \in \Sigma\}$, $z \in \mathbb{C}^n(\Sigma)$;

(ii′) *there exist $\Sigma \subset (\mathbb{R}^n)_*$ and a family $(c_\alpha)_{\alpha\in\Sigma} \subset \mathbb{R}_{>0}$ such that*

$$D = \operatorname{int} \bigcap_{\alpha\in\Sigma} \{z \in \mathbb{C}^n(\Sigma) : c_\alpha |z^\alpha| < 1\}.$$

Proof. The equivalence (ii) $\Leftrightarrow$ (ii′) follows from Lemma 1.15.13 (e). The equivalence (ii′) $\Leftrightarrow$ (i) is a consequence of Remark 1.5.8 (b), Theorem 1.11.13, and the footnote in (1.15.1). □

1.16 Levi problem

Recall that in Corollary 1.15.7 a necessary geometric condition is given for a domain $D \subset \mathbb{C}^n$ to be a domain of holomorphy. This was already observed by E. E. Levi at the beginning of the last century. He even asked whether the converse implication remains true. This is the famous Levi problem which waited a long time for its answer. In the middle of the last century Oka proved the converse. In the meantime, different proofs for this fact have been given based on sheaf and cohomology theory, on the $\bar\partial$-problem, or on integral representation formulas for holomorphic functions. For more details the reader is referred to the books quoted at the end of this book (e.g. [Gra-Fri 2002], [Hör 1990], [Kra 1992], [Ran 1986]).

Theorem* 1.16.1. *Let $D \subset \mathbb{C}^n$ be an arbitrary domain. Then D is pseudoconvex iff D is a domain of holomorphy.*

Here we restrict our discussion to the case of Reinhardt domains.

Exercise 1.16.2. Prove that the complete Reinhardt domain

$$D := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| + |z_2|^{1/4} < 1\}$$

is pseudoconvex and a domain of holomorphy (without Theorem 1.16.1 and Proposition 1.16.3). Notice that D is not convex (cf. Exercise 1.18.7).

Proposition 1.16.3. *Any pseudoconvex Reinhardt domain $D \subset \mathbb{C}^n$ is logarithmically convex.*

Proof. Note that the function d_D satisfies the following conditions

- $d_D(z) = d_D(\lambda \cdot z) = d_D(|z_1|, \dots, |z_n|), \quad z \in D, \lambda \in \mathbb{T}^n$,
- $\lim d_D(z) = 0$ if $D \ni z \to z^0$, $z^0 \in \partial D$.

Put $u\colon \log D \to \mathbb{R}$,

$$u(x) := -\log d_D(e^x).$$

Then, in virtue of Proposition 1.14.40, u is a convex function with $u(x) \to \infty$ if $\log D \ni x \to x^0 \in \partial(\log D)$.

Now let us assume that $\log D$ is not convex. So we may fix two points $a, b \in \log D$ with $[a, b] \not\subset \log D$. Recall that $\log D$ is connected. Therefore, we may choose a continuous curve $\gamma\colon [0,1] \to \log D$ with $\gamma(0) = a$, $\gamma(1) = b$. Put

$$t_0 := \sup\{t \in [0,1] : [a, \gamma(t)] \subset \log D\}.$$

Then $t_0 \in (0,1)$ and $[a, \gamma(t_0)] \not\subset \log D$. Fix a point $x^0 \in [a, \gamma(t_0)] \setminus \log D$. Obviously, $x^0 \in \partial(\log D)$. Let $(0, t_0) \ni t_j \nearrow t_0$. Then by construction there exist points $x^j \in [a, \gamma(t_j)] \subset \log D$ such that $x^j \to x^0$. Hence, $u(x^j) \to \infty$.

On the other hand, $u|_{\gamma([0,1])} \le c$ for a suitable $c \in \mathbb{R}$. Applying that u is convex we see that $u(x^j) \le c$, $j \in \mathbb{N}$; a contradiction. $\square$

Now we pass to the solution of the Levi problem for the class of Reinhardt domains.

Theorem 1.16.4. *Any pseudoconvex Reinhardt domain $D \subset \mathbb{C}^n$ is a domain of holomorphy.*

Proof. Because of the former Proposition 1.16.3 it remains only to prove that D is weakly relatively complete (see Theorem 1.11.13). Assume the contrary. Without loss of generality we may suppose that $V_1 \cap D \ne \varnothing$ and there exists a point $z^0 = (z_1^0, \tilde{z}^0) \in D$ with $(0, \tilde{z}^0) \notin D$.

For a moment let us assume that $z_2^0 \cdots z_n^0 \ne 0$. Then $z_1^0 \ne 0$. By assumption there is a point $a = (a_1, \tilde{a}) \in D \setminus V_0$ such that $\mathbb{D}a_1 \times \{\tilde{a}\} \subset D$. Then we may connect the points z^0 and a in $D \setminus V_0$, i.e. we choose a continuous curve $\gamma = (\gamma_1, \tilde{\gamma})\colon [0,1] \to D \setminus V_0$ with $\gamma(0) = a$ and $\gamma(1) = z^0$. Applying that D is logarithmically convex we get $\mathbb{D}_*\gamma_1(t) \times \{\tilde{\gamma}(t)\} \subset D$. Put

$$t_0 := \sup\{t \in [0,1] : (0, \tilde{\gamma}(t)) \in D\} \in (0,1].$$

Hence, $(0, \tilde{\gamma}(t_0)) \in \partial D$. Note that $-\log d_D(\gamma(t)) \le c$ for a suitable $c \in \mathbb{R}_+$, $t \in [0,1]$. Therefore, in virtue of the maximum principle for subharmonic functions, it follows that $-\log d_D(0, \tilde{\gamma}(t)) \le c$ which contradicts the fact that $d_D(0, \tilde{\gamma}(t)) \to 0$ if $t \nearrow t_0$.

So it remains to discuss z^0 with $z_2^0 = \cdots = z_k^0 = 0$ for a suitable k, $2 \le k \le n$, and $z_{k+1}^0 \cdots z_n^0 \ne 0$. We have $V_j \cap D \ne \varnothing$, $1 \le j \le k$.

Put

$$D' := \{(w_1, \widehat{w}) \in \mathbb{C} \times \mathbb{C}^{n-k} : (w_1, 0, \dots, 0, \widehat{w}) \in D\}.$$

Obviously, D' is an open Reinhardt set.

We claim first that D' is connected. Indeed, fix two points $a = (a_1, \hat{a}), b = (b_1, \hat{b}) \in D' \cap \mathbb{C}_*^{n-k+1}$. It suffices to connect these points in D' (Exercise). Let $s \in (0,1)$ be such that $a^* := (a_1, s, \dots, s, \hat{a}) \in D$ and $b^* := (b_1, s, \dots, s, \hat{b}) \in D$. Choose a curve $\gamma : [0,1] \to D \setminus V_0$ with $\gamma(0) = a^*$ and $\gamma(1) = b^*$. Then $(\gamma_1, 0, \dots, 0, \hat{\gamma})$ connects $(a_1, 0, \dots, 0, \hat{a})$ and $(b_1, 0, \dots, 0, \hat{b})$ in D. Otherwise, there is a $t_0 \in (0,1)$ such that

$$(\gamma_1, 0, \dots, 0, \hat{\gamma})(t) \in D, \ \ 0 \le t < t_0, \quad \text{and} \quad (\gamma_1, 0, \dots, 0, \hat{\gamma})(t_0) \in \partial D.$$

Observe that $-\log d_D|_{\gamma([0,1])} \le c$ for a suitable $c \in \mathbb{R}_+$. Moreover, using successively the logarithmic convexity, there is an $r \in (0,1)$,

$$r < \min\{\inf\{|\gamma_j(t)| : t \in [0,1]\} : 2 \le j \le k\},$$

such that

$$\{\gamma_1(t)\} \times K_*(r) \times \cdots \times K_*(r) \times \{\hat{\gamma}(t)\} \subset D, \quad 0 \le t < t_0.$$

Hence, by the maximum principle for psh functions,

$$-\log d_D(\gamma_1(t), 0, \dots, 0, \hat{\gamma}(t)) \le c, \quad 0 \le t < t_0,$$

which contradicts the property of t_0. In particular, $(\gamma_1, \hat{\gamma})$ connects a and b in D'.

By assumption, D is pseudoconvex. Therefore, there exists an exhausting function $u \in \mathcal{PSH}(D)$. Put $u' : D' \to \mathbb{R}_{-\infty}$, $u'(w_1, \widehat{w}) := u(w_1, 0, \dots, 0, \widehat{w})$. Obviously, $u' \in \mathcal{PSH}(D')$ and u' is an exhausting function of D'. Hence, D' is pseudoconvex. So we may apply the previous case in order to conclude that with $(z_1^0, z_{k+1}^0, \dots, z_n^0) =: (z_1^0, \hat{z}^0) \in D'$ it follows that $(0, \hat{z}^0) \in D'$ or $(0, \tilde{z}^0) \in D$; a contradiction. $\square$

1.17 Hyperconvexity

The following class of open sets with "good" psh exhaustion functions will be useful in the sequel.

Definition 1.17.1. We say that an open set $\Omega \subset \mathbb{C}^n$ is *hyperconvex* if there exists a function $u \in \mathcal{PSH}(\Omega)$, $u < 0$, such that

$$\{z \in \Omega : u(z) < t\} \Subset \Omega, \quad t < 0. \tag{1.17.1}$$

Let $\Omega \subset \mathbb{C}^n$ be open and let $N \subset \Omega$. Define the *relative extremal function*

$$h_{N,\Omega} := \sup\{u : \ u \in \mathcal{PSH}(\Omega),\ u \le 1 \text{ on } \Omega,\ u \le 0 \text{ on } N\}.$$

The function $h^*_{N,\Omega}$ is called the *regularized relative extremal function* or *plurisubharmonic measure of N relative to Ω*. Observe that $0 \le h_{N,\Omega} \le h^*_{N,\Omega} \le 1$ and $h^*_{N,\Omega} \in \mathcal{PSH}(\Omega)$ (cf. Proposition 1.14.16). It is clear that $h^*_{N,\Omega} = 0$ on $\operatorname{int} N$.

Example 1.17.2. The Hartogs triangle $T := \{z \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$ (see Remark 1.5.11) is not hyperconvex.

In fact, suppose the contrary. Then there exists a $u \in \mathcal{PSH}(T)$, $u \le 0$ satisfying (1.17.1). Then $u(0,\cdot)$ is negative subharmonic on $\mathbb{D}_*$. Therefore, in virtue of Proposition 1.14.25, it extends to a subharmonic function on the whole $\mathbb{D}$. Applying the maximum principle for subharmonic functions we get

$$u(0,z_2) \le \sup\{u(0,\lambda) : |\lambda| = 1/2\} =: t_0 < 0, \quad 0 < |z_2| < 1/2.$$

Fix a $t_1 \in (t_0, 0)$. Then

$$\{0\} \times K_*(1/2) \subset \{z \in T : u(z) < t_1\} \Subset T,$$

which contradicts (1.17.1).

Exercise 1.17.3. Let

$$T_{\sigma_1,\sigma_2} := \{(z_1,z_2) \in \mathbb{D}^2 : |z_1|^{\sigma_1} < |z_2| < |z_1|^{\sigma_2}\}, \quad \sigma_1 > \sigma_2 \ge 0.$$

Check whether T_{σ_1,σ_2} is hyperconvex.

Proposition 1.17.4. *Let $D \subset \mathbb{C}^n$ be a domain. Then D is hyperconvex iff there exists a continuous function $u \in \mathcal{PSH}(D, \mathbb{R}_-)$ with* (1.17.1).

Proof. (Cf. [Zah 1974].) Let $u\colon D \to [-\infty, 0)$ be a psh function with (1.17.1). We will construct a continuous psh function $v_0\colon D \to (-\infty, 0)$ with (1.17.1). Fix a ball $K := \overline{\mathbb{B}}(a,r) \subset D$ and let $v := h^*_{K,D}$. Recall that $v \in \mathcal{PSH}(D)$ and $v = 0$ in $\mathbb{B}(a,r)$. The maximum principle implies that $v(z) < 1$ for any $z \in D$.

By the Oka theorem for subharmonic functions (cf. [Vla 1966], Chapter 2, § 9), for any point $b \in \partial\mathbb{B}(a,r)$ we get $v(b) = \lim_{[0,1)\ni t\to 1} v(a + t(b-a)) = 0$. Thus $v = 0$ on K.

Fix a $t_0 > 0$ such that $u \le -t_0$ on K and put $u_0 := (1/t_0)u + 1$. Then $u_0 \in \mathcal{PSH}(D)$, $u_0 \le 1$, and $u_0 \le 0$ on K. Hence $u_0 \le h_{K,D} \le v$. Consequently, the function $v_0 := v - 1$ satisfies (1.17.1). We will show that v is continuous (then v_0 satisfies all the required conditions).

For $\alpha \in (0,1)$ let $D_\alpha := \{z \in D : v(z) < \alpha\}$. Notice that $K \subset D_\alpha \Subset D$ and $D_\alpha \nearrow D$ when $\alpha \nearrow 1$. Moreover, $h^*_{K,D_\alpha} = 0$ on K (use the same argument as above). Observe that $\alpha h^*_{K,D_\alpha} \le v$ on D_α. Indeed, define

$$h := \begin{cases} \max\{\alpha h^*_{K,D_\alpha}, v\} & \text{on } D_\alpha, \\ v & \text{on } D \setminus D_\alpha. \end{cases}$$

Then

$$\limsup_{D_\alpha \ni z \to \zeta} \alpha h^*_{K,D}(z) \le \alpha \le v(\zeta), \quad \zeta \in D \cap \partial D_\alpha.$$

Hence, by Proposition 1.14.9, $h \in \mathcal{PSH}(D)$. Obviously $h \le 1$ on D and $h = 0$ on K. Thus $h \le h_{K,D} \le v$. In particular, $\alpha h^*_{K,D_\alpha} \le h \le v$ on D_α.

Fix a point $z_0 \in D$. We want to prove that v is continuous at z_0. Let $\beta(\alpha) := \max_{\overline{D}_\alpha} v$. Observe that $\alpha \le \beta(\alpha) < 1$. In particular, $\beta(\alpha) \to 1$ when $\alpha \to 1$. Fix an $\eta > 0$ and let $\alpha = \alpha(\eta) \in (0, 1)$ be such that $z_0 \in D_\alpha$ and $\beta(\alpha)/\alpha - 1 \le \eta$. Let $(v_\varepsilon)_{0<\varepsilon\le\varepsilon_0}$ be a family of $\mathcal{C}^\infty$ psh functions defined in a neighborhood Ω of $\overline{D}_\alpha$, $\Omega \subset D$, such that $v_\varepsilon \searrow v$ on Ω when $\varepsilon \searrow 0$ (Proposition 1.14.33). Take an $\varepsilon > 0$ such that for $w := v_\varepsilon \in \mathcal{PSH}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ we have $w \ge v$ on Ω, $w \le \eta$ on K, and $w \le \beta(\alpha) + \eta$ on $\overline{D}_\alpha$. Consequently,

$$(w - \eta)/\beta(\alpha) \le h^*_{K,D_\alpha} \quad \text{on } D_\alpha.$$

Hence,

$$\begin{aligned} 0 \le w - v &\le \beta(\alpha) h^*_{K,D_\alpha} + \eta - v \\ &\le (\beta(\alpha)/\alpha - 1)v + \eta \le \beta(\alpha)/\alpha - 1 + \eta \le 2\eta \quad \text{on } D_\alpha. \end{aligned}$$

Now, by the continuity of w, there exists a neighborhood U of z_0, $U \subset D_\alpha$, such that $|w(z) - w(z_0)| \le \eta$ for $z \in U$. Finally, $|v(z) - v(z_0)| \le 5\eta$ for $z \in U$. $\square$

There is the following localization result for hyperconvexity.

Theorem 1.17.5. *Let $D \subset \mathbb{C}^n$ be a domain. Then the following conditions are equivalent:*

(i) *D is hyperconvex;*

(ii) *for any boundary point $a \in \partial D$ (including $a = \infty$ when D is unbounded) there exists a $u \in \mathcal{PSH}(D)$, $u < 0$, such that $\lim_{D \ni z \to a} u(z) = 0$.*

Every function u as in (ii) is called a *weak psh barrier* for a.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i): Fix $K := \overline{\mathbb{B}}(a_0, r) \subset D$ and put $u := h^*_{K,D} - 1$. Then $u \in \mathcal{PSH}(D)$, $u < 0$ (see the proof for Theorem 1.17.4). Assume that (1.17.1) is not fulfilled. Then there are $a \in \partial D$, $t < 0$, and a sequence of points $(z_j)_j \subset D$ with $z_j \to a$ and $u(z_j) \le t$. Choose a weak psh barrier function $v \in \mathcal{PSH}(D)$, $v < 0$, with $v(z) \to 0$ if $D \ni z \to a$. In particular, $v(z_j) \to 0$. Note that $\sup\{v(z) : z \in K\} =: -\tau < 0$. Then $\tilde{v} := \frac{v}{\tau} + 1 \in \mathcal{PSH}(D)$, $\tilde{v} \le 1$, $\tilde{v}|_K \le 0$, and $\tilde{v}(z_j) \to 1$. Hence, $\tilde{v} \le h^*_{K,D} = u + 1$. Therefore, $t \ge u(z_j) \ge \tilde{v}(z_j) - 1 \to 0$; a contradiction. $\square$

In the case of a Reinhardt domain an even stronger version of Theorem 1.17.5 is true.

Proposition 1.17.6. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Assume that for any $a \in \partial D \cap V_0$*[64] *(including $a = \infty$ if D is unbounded) there exists a weak psh barrier function. Then D is hyperconvex.*

Proof. Let $a \in \partial D \setminus V_0$. Put $\xi := \log a$. Then ξ is a boundary point of the convex domain $\log D$. Therefore, we may take a real linear functional $L\colon \mathbb{R}^n \to \mathbb{R}$, $L(x) = \sum_{j=1}^n \alpha_j x_j$, such that $L(x) < L(\xi)$, $x \in \log D$. Put $u\colon \mathbb{C}_*^n \to \mathbb{R}$, $u(z) := \sum_{j=1}^n \alpha_j \log|z_j|$. Then $v := u - u(a) \in \mathcal{PSH}(\mathbb{C}_*^n) \cap \mathcal{C}(\mathbb{C}_*^n)$, $v < 0$ on $D \cap \mathbb{C}_*^n$, such that $\lim_{D \ni z \to a} v(z) = 0$. Observe that v is locally bounded from above on $D \setminus V_0$. In virtue of Theorem 1.14.25, v extends to a psh function on D, which is everywhere negative (use the maximum principle). Hence this extension gives a weakly psh barrier at a.

Hence, using the assumption, Theorem 1.17.5 completes the proof. □

In the general theory, hyperconvex domains do not contain non-trivial entire holomorphic curves.

Definition 1.17.7. A domain $D \subset \mathbb{C}^n$ is called *Brody hyperbolic* if any $\varphi \in \mathcal{O}(\mathbb{C}, D)$ is identically constant.

Exercise 1.17.8. Observe that any elementary Reinhardt domain $\boldsymbol{D}_{\alpha,c}$ is not Brody hyperbolic.

Proposition 1.17.9. *Let $D \subset \mathbb{C}^n$ be a hyperconvex domain. Then D is Brody hyperbolic.*

Proof. Let $u \in \mathcal{PSH}(D)$ denote the negative exhaustion function from the definition of hyperconvexity. If $\varphi \in \mathcal{O}(\mathbb{C}, D)$, then $v := u \circ \varphi \in \mathcal{SH}(\mathbb{C})$ and $v \le 0$. Hence, in virtue of the Liouville theorem for subharmonic functions (see 1.14.3 (g)), $v \equiv c \in \mathbb{R}$. Applying (1.17.1) implies that $\varphi(\mathbb{C})$ is bounded and therefore, φ is a constant function according to the classical Liouville theorem for holomorphic functions. □

In the case of Reinhardt domains there are the following results for Brody hyperbolic domains. We begin with a direct consequence of Lemma 1.5.14 (iii).

Lemma 1.17.10. *Let $D \subset \mathbb{C}^n$ be a Brody hyperbolic Reinhardt domain of holomorphy. Then there exist a matrix $A := \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix} \in \mathbb{GL}(n, \mathbb{Z})$ and a vector $c \in \mathbb{R}^n$ such that $D \subset \boldsymbol{D}_{\alpha^1,c_1} \cap \dots \cap \boldsymbol{D}_{\alpha^n,c_n}$.*

[64] Recall that $V_0 = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0\}$.

Proof. We only need to observe that an affine line $L = a + \mathbb{R}b$, $b \neq 0$, is contained in $\log D$ iff the entire curve

$$\mathbb{C} \ni \lambda \mapsto (e^{a_1}e^{\lambda b_1}, \dots, e^{a_n}e^{\lambda b_n})$$

has its image in D. $\square$

Theorem 1.17.11. *Let $D \subset \mathbb{C}^n$ be a Brody hyperbolic Reinhardt domain of holomorphy. Then D is algebraically equivalent to a bounded domain (cf. Definition 1.5.12), i.e. there exists a matrix $A := \begin{bmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{bmatrix} \in \mathbb{GL}(n, \mathbb{Z})$ such that $D \subset \mathbb{C}^n(A)$ and Φ_A maps biholomorphically D onto a bounded Reinhardt domain of holomorphy, where*

$$\Phi_A : D \to \mathbb{C}^n, \quad \Phi_A(z) := (z^{\alpha^1}, \dots, z^{\alpha^n}), \quad z \in D.$$

Proof. The proof is done by induction. For $n = 1$ the only unbounded Reinhardt domains in $\mathbb{C}$ are $\mathbb{C}$, $\mathbb{C}_*$, and $\mathbb{A}(r, \infty)$ with $r > 0$. The first two are not Brody hyperbolic. The annulus can be algebraically mapped by $z \mapsto 1/z$ onto a bounded Reinhardt domain.

Now let $n > 1$ and assume that the theorem is true for all lower dimensions. If $D \subset \mathbb{C}^n_*$, then Remark 1.5.13 (b) and Lemma 1.17.10 apply.[65]

In order to discuss the remaining case let us assume, without loss of generality, that $D \cap V_n \neq \varnothing$. Then, in virtue of Corollary 1.11.16, $\widetilde{D} := \mathrm{pr}_{\mathbb{C}^{n-1}}(D) = \mathrm{pr}_{\mathbb{C}^{n-1}}(D \cap V_n)$ is a Reinhardt domain of holomorphy in $\mathbb{C}^{n-1}$. Moreover, it is easily seen that $\widetilde{D}$ is Brody hyperbolic. Applying the induction hypothesis we find a matrix $\tilde{A} \in \mathbb{GL}(n-1, \mathbb{Z})$ with the same properties as above. So $\Phi_{\tilde{A}}$ is defined on $\widetilde{D}$ and maps $\widetilde{D}$ biholomorphically onto its image $\Phi_{\tilde{A}}(\widetilde{D})$, a bounded Reinhardt domain of holomorphy. Now put

$$A := \begin{bmatrix} \tilde{A} & 0 \\ 0 & 1 \end{bmatrix}.$$

Then $A \in \mathbb{GL}(n, \mathbb{Z})$ and Φ_A maps D biholomorphically onto a Brody hyperbolic Reinhardt domain of holomorphy $\Phi_A(D)$ which is contained in $\mathbb{P}_{n-1}(r) \times \mathbb{C}$ (Exercise).

Therefore, we may assume from the very beginning that D satisfies

$$D \cap V_n \neq \varnothing, \quad D \subset \mathbb{P}_{n-1}(r) \times \mathbb{C}$$

for some $r > 0$. Without loss of generality let

$$D \cap V_j \neq \varnothing,\ j = 1, \dots, k, \quad D \cap V_j = \varnothing,\ j = k+1, \dots, n-1,$$

where $k \in \{0, \dots, n-1\}$.

[65] Notice that, in general, if A is as in Lemma 1.17.10, then $(\Phi_A)|_D$ need not be biholomorphic. For example, if A is as in Remark 1.5.13 (c), then $\mathbb{D}^2 \subset \{|z_1 z_2| < 1,\ |z_1^3 z_2^4| < 1\}$, but $\Phi_A(0,0) = (0,0) \notin \mathbb{C}^n(A^{-1})$.

Then we find an $\alpha = (\alpha_1, \dots, \alpha_n) = (0, \alpha'') \in \mathbb{Z}^k \times \mathbb{Z}^{n-k}$, $\alpha_n \neq 0$, and an $m \in \mathbb{R}_+$ such that

$$|z^\alpha| \leq m, \quad z \in D \cap (\mathbb{C}^{n-1} \times \mathbb{C}_*).$$

Indeed, if $k = 0$, then Lemma 1.17.10 applies directly (we take $\alpha := \alpha^j$, where j is such that $\alpha_n^j \neq 0$). Now let $k \geq 1$. Put

$$D'' := \mathrm{pr}_{\mathbb{C}^{n-k}}(D) = \mathrm{pr}_{\mathbb{C}^{n-k}}\Big(D \cap \bigcap_{j=1}^{k} V_j\Big);$$

recall that D is relatively complete. Then D'' is a Brody hyperbolic Reinhardt domain of holomorphy. In virtue of Lemma 1.17.10, we find $\alpha'' = (\alpha_{k+1}, \dots, \alpha_n) \in \mathbb{Z}^{n-k}$, $\alpha_n \neq 0$, and $m \in \mathbb{R}_+$ such that

$$|(z'')^{\alpha''}| \leq m, \quad z'' \in D'',\ z_n \neq 0.$$

Then, setting $\alpha := (0, \dots, 0, \alpha'')$ completes the argument.

In a last step put

$$\beta := (\beta_1, \dots, \beta_{n-1}), \quad \beta_j := \begin{cases} 0 & \text{if } j \leq k, \\ \lfloor \frac{\alpha_j}{|\alpha_n|} \rfloor + 1 & \text{if } k < j < n. \end{cases}$$

Note that $s_j := \beta_j |\alpha_n| - \alpha_j \geq 0$, $j = k+1, \dots, n-1$.

Then, for $z = (z', z'') \in D$, $z_n \neq 0$, we get

$$|z^\beta| \leq |(z'')^{\alpha''}|^{1/|\alpha_n|} |z_{k+1}|^{s_{k+1}} \dots |z_{n-1}|^{s_{n-1}} \leq m^{1/|\alpha_n|} r^{n-k-1}.$$

By continuity, this estimate remains true on D. Finally, we introduce

$$A := \begin{bmatrix} \mathbb{I}_{n-1} & 0 \\ \beta & 1 \end{bmatrix} \in \mathbb{M}(n \times n; \mathbb{Z}).$$ [66]

Note that $\det A = 1$, Φ_A is defined on D with a non-vanishing Jacobian, and Φ_A is injective on D. Hence, Φ_A gives an algebraic biholomorphic mapping from D onto its image $\Phi_A(D)$ which is a bounded Reinhardt domain of holomorphy. $\square$

Based on the former theorem we have the following result for Brody hyperbolic Reinhardt domains of holomorphy.

Proposition 1.17.12. *Let $D \subset \mathbb{C}^n$ be a Brody hyperbolic Reinhardt domain of holomorphy. Then the following conditions are equivalent:*

[66] $\mathbb{I}_k$ denotes the $(k \times k)$-dimensional unit matrix.

(i) *D is algebraically equivalent to an unbounded Reinhardt domain of holomorphy;*

(ii) *D is algebraically equivalent to a bounded Reinhardt domain of holomorphy G which does not satisfy the Fu condition.*

Proof. (ii) $\Rightarrow$ (i): We may assume that D is bounded and does not satisfy the Fu condition. Therefore, $V_j \cap D = \varnothing$ and $\bar{D} \cap V_j \neq \varnothing$ for a certain j. Take simply the following map

$$F \colon D \to \mathbb{C}^n, \quad F(z) := (z_1, \dots, z_{j-1}, \tfrac{1}{z_j}, z_{j+1}, \dots, z_n).$$

Then $G := F(D)$ is an unbounded Reinhardt domain of holomorphy and F is an algebraic biholomorphism from D onto G.

(i) $\Rightarrow$ (ii): Without loss of generality, we assume that D is unbounded. In virtue of Theorem 1.17.11 there are a matrix $A = [a_{j,k}]_{1\le j,k\le n} \in \mathbb{GL}(n, \mathbb{Z})$ and a bounded Reinhardt domain G such that $\Phi_A \colon G \to D$ is biholomorphic. Obviously, G is a Reinhardt domain of holomorphy.

Suppose that G does satisfy the Fu condition. So we may assume that

$$V_j \cap G \neq \varnothing,\ j = 1, \dots, k, \quad \text{and} \quad \bar{G} \cap V_j = \varnothing,\ j = k+1, \dots, n,$$

where k is a suitable number in $\{0, \dots, n\}$. Therefore, $a_{r,j} \ge 0$ if $j \in \{1, \dots, k\}$ and $1 \le r \le n$. Moreover, there is an $m > 0$ such that $|z_j| \ge m$ if $z \in G$ and $j = k+1, \dots, n$. Denote now by a^r the r-th row of A. Then

$$|(\Phi_A(z))_r| = |z^{a^r}| = |z_1^{a_{r,1}} \cdots z_n^{a_{r,n}}| \le m' \quad \text{if } z \in G,\ 1 \le r \le n,$$

where m' is a suitable real number. So, $\Phi_A(G) = D$ is bounded; a contradiction. □

Now we are able to present a complete description of hyperconvex Reinhardt domains. Before presenting the result we need an additional definition which sharpens the notion of hyperconvexity.

Definition 1.17.13. A domain (resp. a Reinhardt domain) $D \subset \mathbb{C}^n$ is said to be *strictly hyperconvex* (resp. *strictly R-hyperconvex*) if there exist a domain (resp. Reinhardt domain) D' and a function $u \in \mathcal{PSH}(D') \cap \mathcal{C}(D')$ (resp. $u \in \mathcal{PSH}(D') \cap \mathcal{C}(D')$ with $u(z) = u(|z_1|, \dots, |z_n|)$, $z \in D'$) such that

- $D \subset D'$, $u < 1$ on D',
- $D = \{z \in D' : u(z) < 0\}$,
- $D_t := \{z \in D' : u(z) < t\} \Subset D'$ and D_t is connected, $0 < t < 1$.

Obviously, every strictly R-hyperconvex Reinhardt domain is strictly hyperconvex.

Remark 1.17.14. Let D, D', and u be as in Definition 1.17.13.

(a) Then D, D_t, and D' are pseudoconvex domains (see Theorem 1.15.5) with $D \Subset D_t \Subset D'$, $0 < t < 1$.

(b) D is fat. Otherwise there exists a point $a \in \operatorname{int} \overline{D} \setminus D$; in particular, $a \in \partial D$. Hence, $u(a) = 0$. On the other hand, there is an $r > 0$ such that $\mathbb{P}(a,r) \subset \overline{D}$ and, therefore, $u \le 0$ on $\mathbb{P}(a,r)$ (use that u is continuous). Then the maximum principle leads to $u|_{\mathbb{P}(a,r)} = 0$. So, $\mathbb{P}(a,r) \subset \partial D$; a contradiction.

(c) $\overline{D}$ has a Stein neighborhood basis.[67] Indeed, let U be an open set containing $\overline{D}$. Choose another open set V with $\overline{D} \subset V \Subset U \cap D'$. Put

$$t_0 := \inf\{u(z) : z \in D' \setminus V\} \in (0,1).$$

Then D_{t_0} is pseudoconvex with $D_{t_0} \subset V$. Applying the solution of the general Levi problem, D_{t_0} is a domain of holomorphy.[68]

(d) If D is even strictly R-hyperconvex, then $\overline{D}$ has a neighborhood basis of Reinhardt domains of holomorphy. Observe that here the D_t's are pseudoconvex Reinhardt domains and therefore domains of holomorphy (see Theorem 1.16.4).

(e) Using Corollary 1.12.5 and (c) we see if D is a strictly hyperconvex Reinhardt domain, then $\overline{D}$ has a neighborhood basis consisting of Reinhardt domains of holomorphy.

Exercise 1.17.15. Prove that $\overline{T}$ (T is the Hartogs triangle) has no Stein neighborhood basis. Recall that T is fat and does not satisfy the Fu condition

Now we are in the position to present the full characterization of hyperconvex Reinhardt domains (cf. [HDT 2003], [Zwo 2000]).

Theorem 1.17.16. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain. Then the following conditions are equivalent:*

(i) *D is hyperconvex;*

(ii) *D is bounded, pseudoconvex, and satisfies the Fu condition;*

(iii) *D is strictly R-hyperconvex;*

(iv) *D is bounded, fat, and $\overline{D}$ has a neighborhood basis of Reinhardt domains of holomorphy;*

(v) *D is $\mathcal{H}^\infty$-convex.*

Proof. First note that (iii) $\Rightarrow$ (iv) has been shown in Remark 1.17.14 and (ii) $\Rightarrow$ (v) follows from Theorem 1.13.19.

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[67] A compact set $K \subset \mathbb{C}^n$ has a *Stein neighborhood basis* if any open set U, $K \subset U$, contains a domain of holomorphy G with $K \subset G$. Domains of holomorphy are often called *Stein domains* in honor of Karl Stein.

[68] Let us emphasize that here we have used the general solution of the Levi problem (Theorem 1.16.1) although this has not and will not be proved in this book.

(i) $\Rightarrow$ (ii): Suppose the contrary. Then either D is unbounded or D is bounded and does not satisfy the Fu condition. Recall that D is Brody hyperbolic, since it is hyperconvex (see Proposition 1.17.9). If D is unbounded, then D is biholomorphic to a bounded Reinhardt domain of holomorphy that does not fulfill the Fu condition (use Proposition 1.17.12). So, without loss of generality, we may assume from the very beginning that D is bounded and does not satisfy the Fu condition. We may also assume that there are $k, \ell \in \mathbb{N}$, $1 \le k \le \ell \le n$, such that

$$\begin{gathered} \bar{D} \cap V_j \neq \varnothing, \quad D \cap V_j = \varnothing, \quad 1 \le j \le k, \\ \bar{D} \cap V_j = \varnothing, \quad k+1 \le j \le \ell, \quad D \cap V_j \neq \varnothing, \quad \ell+1 \le j \le n. \end{gathered}$$

Assume that $\ell < n$. Put $\widetilde{D} := \{z \in \mathbb{C}^\ell : (z, 0, \dots, 0) \in D\}$. Then $\widetilde{D}$ is a hyperconvex bounded Reinhardt domain of holomorphy not satisfying the Fu condition. Hence for the further argument we may assume that $\ell = n$. Therefore, $D \subset \mathbb{C}_*^n$. Moreover, we may assume that $\mathbf{1} \in D$. Recall that $\log D$ is convex, unbounded, and $\log D \subset \{x \in \mathbb{R}^n : x_j < r,\ j = 1, \dots, n\}$ for a suitable $r \in \mathbb{R}_{>0}$. Thus we find a sequence of points $x^j \in \log D$ with $\|x^j\| \to \infty$. Let $h = h_{\log D}$ be the Minkowski function of $\log D$ (Definition 1.4.14). Then $h(x^j) < 1$, $j \in \mathbb{N}$. Therefore, $h(\frac{x^j}{\|x^j\|}) < \frac{1}{\|x^j\|} \to 0$. Using the compactness of the unit sphere we find a vector v, $\|v\| = 1$, with $h(v) = 0$ which implies that $\mathbb{R}_+ v \subset \log D$. In particular, $v_j \le 0$, $1 \le j \le n$. Put $\alpha := -v$. Then

$$(e^t, e^{t\alpha_2}, \dots, e^{t\alpha_n}) \in D, \quad t < 0.$$

In particular,

$$\{(e^\lambda, e^{\lambda\alpha_2}, \dots, e^{\lambda\alpha_\ell}, 1, \dots, 1) : \lambda \in \mathbb{C},\ \operatorname{Re}\lambda < 0\} \subset D.$$

Now, let $u \in \mathcal{PSH}(D)$, $u < 0$, be a psh exhaustion function for D according to the definition of hyperconvexity. Put $\hat{u} \colon D \to [-\infty, 0)$,

$$\hat{u}(z) := \sup\{u(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) : \theta_j \in \mathbb{R},\ j = 1, \dots, n\}.$$

Using the compactness of the n-dimensional torus it turns out that $\hat{u}$ is semicontinuous from above on D, $\hat{u} \in \mathcal{PSH}(D)$, $\hat{u} < 0$, and satisfies (1.17.1) (Exercise). Finally, define $\tilde{u} \colon \mathbb{D}_* \to [-\infty, 0)$,

$$\tilde{u}(\lambda) := \hat{u}(|\lambda|, |\lambda|^{\alpha_2}, \dots, |\lambda|^{\alpha_n}).$$

Note that the functions $\mathbb{D}_* \ni \lambda \mapsto \lambda^{\alpha_j}$ are locally holomorphic. Therefore, $\tilde{u}$ is subharmonic on $\mathbb{D}_*$ and negative. Hence, it extends to a subharmonic function u^* on $\mathbb{D}$. Applying property (1.17.1) leads to $u^*(0) = 0$. Then, by the maximum principle, $\tilde{u}$ is identically 0; a contradiction.

(ii) $\Rightarrow$ (iii): Without loss of generality we may assume that the point $\mathbf{1} = (1,\dots,1) \in D$, $D \cap V_j \neq \varnothing$, $1 \le j \le k$, and $D \cap V_j = \varnothing$, $k+1 \le j \le n$ with a $k \in \{0,\dots,n\}$. Since D satisfies the Fu condition we have $D \Subset \widetilde{D} := \mathbb{C}^n \setminus \bigcup_{j=k+1}^n V_j$. Moreover, $\widetilde{D}$ is a Reinhardt domain of holomorphy.

By assumption, D is a Reinhardt domain of holomorphy. Thus, $\log D$ is convex and $0 \in \log D$ (recall that $\mathbf{1} \in D$). Let $h = h_{\log D}$. Then h is continuous and convex (Exercise 1.4.16). Applying Theorem 1.14.40, the function $u\colon \mathbb{C}_*^n \to \mathbb{R}$, $u(z) := h(\log|z_1|,\dots,\log|z_n|) - 1$, belongs to $\mathcal{PSH}(\mathbb{C}_*^n) \cap \mathcal{C}(\mathbb{C}_*^n)$. In particular, u is defined on $\widetilde{D} \setminus V_0$

If $k = 0$, then $\widetilde{D} = \mathbb{C}_*^n$. So $u \in \mathcal{PSH}(\widetilde{D}) \cap \mathcal{C}(\widetilde{D})$, $D = \{z \in \widetilde{D} : u(z) < 0\}$, and $D_t := \{z \in \widetilde{D} : u(z) < t\} \Subset \widetilde{D}$ is a Reinhardt domain for all $t > 0$. Taking $D' := D_1$ shows that D is strictly R-hyperconvex.

Now assume that $k \neq 0$. We like to show that u extends to a psh function on $\widetilde{D}$. In virtue of Theorem 1.14.25, it suffices to show that u is locally bounded from above on $\widetilde{D}$. Indeed, let $a \in \widetilde{D} \cap V_0$. Without loss of generality, let $a = (0, a'') \in \mathbb{C}^s \times \mathbb{C}_*^{n-s}$, $s \in \{1,\dots,k\}$. Then $\widehat{D}_s := \operatorname{pr}_{\mathbb{C}^{n-s}}(D)$ is again a Reinhardt domain of holomorphy containing $\mathbf{1}_{n-s}$, and thus a neighborhood of $\mathbf{1}_{n-s}$. Therefore, $\log \widehat{D}_s$ is a convex domain in $\mathbb{R}^{n-s}$. Moreover, we find an $\ell \in \mathbb{N}$ such that

$$\tfrac{1}{\ell}(\log|a_{s+1}|,\dots,\log|a_n|) \in \log \widehat{D}_s.$$

Put $b_j := |a_j|^{1/\ell} > 0$, $j = s+1,\dots,n$. Hence, we conclude that $b'' := (b_{s+1},\dots,b_n) \in \widehat{D}_s$, which means there is a point $c' \in \mathbb{C}^s$ with $(c', b'') \in D$. Now we use that D cuts all the first k axes and get $(0, b'') \in D$. Therefore, we find a positive ε such that

$$U := \mathbb{P}_s(\varepsilon) \times \mathbb{A}^{n-s}(r^-, r^+) \subset D,$$

where

$$r^- := (1+\varepsilon)^{-1/\ell}(b_{s+1},\dots,b_n), \quad r^+ := (1+\varepsilon)^{1/\ell}(b_{s+1},\dots,b_n).$$

Obviously,

$$V := \mathbb{P}_s(\varepsilon^\ell) \times \mathbb{A}^{n-s}(\rho^-, \rho^+),$$

where

$$\rho^- := (1+\varepsilon)^{-\ell}(|a_{s+1}|,\dots,|a_n|), \quad \rho^+ := (1+\varepsilon)^{\ell}(|a_{s+1}|,\dots,|a_n|),$$

is a neighborhood of a. Take a $z \in V \setminus V_0$ and put $\xi := (\log|z_1|,\dots,\log|z_n|)$. Then, by construction, $\xi/\ell \in \log U \subset D$. Therefore, $u(z) = h(\xi) - 1 \le \ell - 1$. Hence, u is bounded on $V \setminus V_0$. Since the point a was arbitrary, we know that u extends to a psh function $\tilde{u}$ on $\widetilde{D}$.

It remains to show that $\tilde{u}$ is continuous at all points in $\widetilde{D} \cap V_0$. Without loss of generality fix as above an $a = (0, a'') \in D' \cap V_0 \cap (\mathbb{C}^s \times \mathbb{C}_*^{n-s})$ with an $s \in \{1, \dots, k\}$. Repeating the previous argument we conclude that

$$\tilde{u}(a) \leq \tilde{h}(\log |a_{s+1}|, \dots, \log |a_n|) - 1,$$

where $\tilde{h} := h_{\log(\mathrm{pr}_{\mathbb{C}^{n-s}}(D))}$. It suffices to show that $\tilde{u}$ is lower semicontinuous at a. Suppose that there is a constant $c < \tilde{u}(a)$ and a sequence of points $z^j \in \widetilde{D}$ with $z^j \to a$ such that $\tilde{u}(z^j) \leq c$. Without loss of generality we may assume that $z^j \in D \cap \mathbb{C}_*^n$. Hence

$$\begin{aligned}\tilde{h}(\log |z_{s+1}^j|, \dots, \log |z_n^j|) &\leq h(\log |z_1^j|, \dots, \log |z_n^j|) = u(z^j) + 1 \\ &< c + 1 < \tilde{u}(a) + 1 \leq \tilde{h}(\log |a_{s+1}|, \dots, \log |a_n|).\end{aligned}$$

Using the continuity of $\tilde{h}$ leads to a contradiction. So $\tilde{u}$ is continuous in a. Since the point a was arbitrarily chosen, we have shown that $\tilde{u}$ is continuous on the whole of $\widetilde{D}$.

Note that $D \setminus V_0 = \{z \in \widetilde{D} \setminus V_0 : u(z) < 1\}$. Therefore, using the maximum principle, we have $\tilde{u} < 0$ on D. Moreover, the continuity of $\tilde{u}$ implies $D = \{z \in \widetilde{D} : \tilde{u}(z) < 0\}$.

For $t > 0$ put $\widetilde{D}_t := \{z \in \widetilde{D} : \tilde{u}(z) < t\}$. Note that $\widetilde{D}_t$ is a Reinhardt open set, $\widetilde{D}_t \Subset \widetilde{D}$, and $\widetilde{D}_t \setminus V_0$ is connected. Applying Remark 1.5.6 (d) we conclude that $\widetilde{D}_t$ is connected. With $D' := \widetilde{D}_1$ it follows that D is strictly R-hyperconvex also in the case $k \neq 0$.

(iv) $\Rightarrow$ (i): Observe that $\bar{D}$ has a Stein neighborhood basis and D is fat. Therefore, $D = \operatorname{int} \bar{D} = \operatorname{int} \bigcap D_j$ for a certain decreasing sequence of Reinhardt domains of holomorphy D_j. Hence, D is a domain of holomorphy.

Now, in view of Proposition 1.17.6, we will study boundary points of D which belong to V_0. So let $a \in \partial D \cap V_0$. First assume that $0 \in \partial D$. Fix a point $b = (r, \dots, r) \in D$ with a certain positive r. Note that $0 \in D_j$, $j \in \mathbb{N}$. Therefore, $\mathbb{P}(r) \subset D_j$, $j \in \mathbb{N}$. Recall that D is fat. So $\mathbb{P}(r) \subset \operatorname{int} \bar{D} = D$; a contradiction.

Therefore, we may assume that $a = (0, a'') \in \mathbb{C}^k \times \mathbb{C}_*^{n-k}$ with a suitable k, $1 \leq k < n$. Denote by D'' the projection of D to $\mathbb{C}^{n-k}$, i.e.

$$D'' := \mathrm{pr}_{\mathbb{C}^{n-k}}(D).$$

Clearly, D'' is a Reinhardt domain of holomorphy in $\mathbb{C}^{n-k}$ (see Corollary 1.11.16). Assume for a moment that $a'' \in \partial D'' \cap \mathbb{C}^{n-k}$. Then, using the proof of Proposition 1.17.6, there is a weak psh barrier function $u \in \mathcal{PSH}(D'')$ for a'', i.e. $u < 0$ and $u(z'') \to 0$ if $D'' \ni z'' \to a''$. In such a case, the function $v : D \to [-\infty, 0)$, $v(z', z'') := u(z'')$, delivers a weak psh barrier function for a.

To see that $a'' \in \partial D'' \cap \mathbb{C}^{n-k}$ it remains to verify that D does not contain any point "over" a'', i.e. $D' := \{z' \in \mathbb{C}^k : (z', a'') \in D\} = \varnothing$.

Suppose the contrary i.e. $D' \neq \varnothing$. Then we choose a point $b = (b', a'') \in D \bigcap \mathbb{C}_*^n$ and a small positive number ε with $\{b'\} \times \mathbb{P}_{n-k}(a'', \varepsilon) \subset D$. Note that $D_j \cap V_s \neq \varnothing$, $1 \leq s \leq k$. Therefore, applying the relative completeness of D_j, we have $P := \mathbb{P}_k(r) \times \mathbb{P}_{n-k}(a'', \varepsilon) \subset D_j$, $j \in \mathbb{N}$, where $r := \min\{|b_\ell| : 1 \leq \ell \leq k\}$. Hence, $P \subset \operatorname{int} \overline{D} = D$. In particular, $a \in D$; a contradiction.

(v) $\Rightarrow$ (ii): We only have to note that any $\mathcal{H}^\infty$-convex domain is Brody hyperbolic (use Liouville's theorem). Then one may follow the argument in the proof before as follows: fix an $f \in \mathcal{H}^\infty(D)$ and put $u := |f|$ on D. Using the notation from before, $\tilde{u}$ extends to a psh function on $\mathbb{D}$. Therefore, by the maximum principle,

$$\begin{aligned} \left| f(|\lambda|, |\lambda|^{\alpha_2}, \dots, |\lambda|^{\alpha_n}) \right| &\leq \tilde{u}(\lambda) \leq \sup_{|\zeta|=1/2} \tilde{u}(\zeta) \\ &\leq \sup_{z \in \partial_0 \mathbb{P}(r)} |f(z)|, \quad 0 < |\lambda| \leq 1/2, \end{aligned}$$

where $r := (2^{-1}, 2^{-\alpha_2}, \dots, 2^{-\alpha_n})$; a contradiction. □

In connection with the Montel theorem there is the following notion.

Definition 1.17.17. A domain $D \subset \mathbb{C}^n$ is called *taut* if any sequence $(f_j)_{j \in \mathbb{N}} \subset \mathcal{O}(\mathbb{D}, D)$ allows a subsequence $(f_{j_k})_k$, which diverges compactly (i.e. for any compact sets $K \subset D$ and $L \subset \mathbb{D}$ there is a $j_{K,L}$ such $f_j(L) \cap K = \varnothing$, $j \geq j_{K,L}$), or a subsequence $(f_{j_\ell})_\ell$ with $f_{j_\ell} \to f \in \mathcal{O}(\mathbb{D}, D)$ locally uniformly on $\mathbb{D}$.

Exercise 1.17.18. When is a planar domain $D \subset \mathbb{C}$ taut ?

Example 1.17.19. The Hartogs triangle T is taut.

Indeed, let $\varphi_k = (\varphi_{k,1}, \varphi_{k,2}) : \mathbb{D} \to T$, $\varphi_k \to \varphi_0$ locally uniformly in $\mathbb{D}$ with $\varphi_0 = (\varphi_{0,1}, \varphi_{0,2}) \in \mathcal{O}(\mathbb{D}, \overline{T})$, $\varphi_{k,1}/\varphi_{k,2} \to \psi$ locally uniformly in $\mathbb{D}$ with $\psi \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$. Note that $\varphi_{0,1} \equiv \psi \cdot \varphi_{0,2}$. By the Hurwitz theorem (cf. [Con 1973], Chapter VI, Theorem 2.5), either $\varphi_{0,2} \equiv 0$ or $\varphi_{0,2}$ has no zeroes. In the first case $\varphi_0(\mathbb{D}) = \{(0,0)\} \subset \partial T$. In the second case, either $\varphi_{0,2}(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$ (and then $\varphi_0(\mathbb{D}) \subset \partial T$), or $\varphi_{0,2}(\mathbb{D}) \subset \mathbb{D}$. In the latter case, either $\psi(\mathbb{D}) \cap \mathbb{T} \neq \varnothing$ (and then $\varphi_0(\mathbb{D}) \subset \partial T$), or $\varphi_0(\mathbb{D}) \subset T$.

Exercise 1.17.20. Decide whether the domain

$$T_\sigma := \{(z_1, z_2) \in \mathbb{D}^2 : |z_1|^\sigma < |z_2|\} \quad (\sigma > 0)$$

is taut.

Hint. Use the maximum principle for subharmonic functions.

Using Theorem 1.15.5(x) it is easy to solve the following exercise.

Exercise 1.17.21. Prove that any taut domain is pseudoconvex.

On the other hand it turns out that hyperconvexity implies tautness.

Theorem 1.17.22. *Any hyperconvex bounded domain $D \subset \mathbb{C}^n$ is taut.*

Proof. Let us start with a sequence $(f_j)_j \subset \mathcal{O}(\mathbb{D}, D)$. By the Montel theorem we may assume that $f_j \to f \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ locally uniformly. We have to show that $f \in \mathcal{O}(\mathbb{D}, D)$. Otherwise, $f(\lambda_0) \in \partial D$ for a $\lambda_0 \in \mathbb{D}$. Note that all values of f are in $\bar{D}$.

By assumption there is a continuous function $u \in \mathcal{PSH}(D)$, $u < 0$, satisfying (1.17.1). Putting $u := 0$ on ∂D, we extend continuously u to $\bar{D}$. Then $u \circ f_j \to u \circ f$ locally uniformly on $\mathbb{D}$. Hence, $u \circ f \in \mathcal{PSH}(\mathbb{D})$ with $u \leq 1$. Observe that $u \circ f(\lambda_0) = 1$. Therefore the maximum principle gives the contradiction. ☐

For Reinhardt domains of holomorphy we even have the following characterization of taut domains.

Theorem 1.17.23. *Let $D \subset \mathbb{C}^n$ be a taut Reinhardt domain of holomorphy. Then D is algebraically equivalent to a bounded domain.*

Remark 1.17.24. It has to be pointed out that the converse statement is also true; its proof will be given later in Theorem 4.7.2.

The proof of Theorem 1.17.23 will use the following lemma.

Lemma 1.17.25. *Any taut domain $D \subset \mathbb{C}^n$ is Brody hyperbolic.*

Proof. Otherwise there exists $\varphi \in \mathcal{O}(\mathbb{C}, D)$ which is not identically constant. Put $\varphi_j : \mathbb{D} \to D$, $\varphi_j(\lambda) := \varphi(j\lambda)$. Since $\varphi_j(0) = \varphi(0)$, no subsequence diverges locally uniformly. Assume there is a subsequence $(\varphi_{j_k})_k$ with $\varphi_{j_k} \to f \in \mathcal{O}(\mathbb{D}, D)$ locally uniformly. Then $|\varphi_{j_k}(\lambda) - f(\lambda)| \leq 1$, $|\lambda| \leq 1/2$, if $k \geq k_0$. Therefore,

$$|\varphi_{j_k}(\lambda)| \leq |f(\lambda)| + |\varphi_{j_k}(\lambda) - f(\lambda)| \leq \|f\|_{(1/2)\mathbb{D}} + 1 =: C, \quad |\lambda| \leq 1/2,\ k \geq k_0.$$

Hence, φ is bounded and so identically constant; a contradiction. ☐

Proof of Theorem 1.17.23. The proof follows directly from Lemma 1.17.25 and Theorem 1.17.11. ☐

1.18* Smooth pseudoconvex domains

This section collects terminology and basic results related to the pseudoconvexity of smooth domains (proofs and details may be found e.g. in [Jar-Pfl 2000], § 2.2). The reader may skip this section during the first reading.

Definition 1.18.1. Let $D \subset \mathbb{C}^n$ be a bounded domain. We say that ∂D is *smooth of class* $\mathcal{C}^k$ (or $\mathcal{C}^k$*-smooth*) *in a neighborhood of a point* $a \in \partial D$ if there exist an open neighborhood U of a and a function $u \in \mathcal{C}^k(U, \mathbb{R})$ such that

$$D \cap U = \{z \in U : u(z) < 0\}, \tag{1.18.1}$$

$$U \setminus \bar{D} = \{z \in U : u(z) > 0\}, \tag{1.18.2}$$

$$\operatorname{grad} u(z) \neq 0 \ \text{ for } \ z \in U \cap \partial D; \tag{1.18.3}$$

here $k \in \mathbb{N} \cup \{\infty\} \cup \{\omega\}$, where $u \in \mathcal{C}^\omega$ means that u is real analytic.

The function u is called a *local defining function for* D.

Observe that if $u \in \mathcal{C}^k(U, \mathbb{R})$ satisfies (1.18.1) and $\operatorname{grad} u(z) \neq 0$ for all $z \in U$, then u satisfies (1.18.2) in a sufficiently small neighborhood of a, i.e. u is a local defining function in a suitable neighborhood of a.

We say that D is $\mathcal{C}^k$*-smooth* or *has a* $\mathcal{C}^k$*-smooth boundary* if ∂D is $\mathcal{C}^k$-smooth at any point $a \in \partial D$.

Put

$$T_b^{\mathbb{C}}(\partial D) := \Big\{X \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial u}{\partial z_j}(b) X_j = 0\Big\}, \quad b \in U \cap \partial D.$$

The complex space $T_b^{\mathbb{C}}(\partial D)$ is called the *complex tangent space to* ∂D *at* b; notice that the condition

$$\sum_{j=1}^{n} \frac{\partial u}{\partial z_j}(z) X_j = 0$$

means that $X \perp \operatorname{grad} u(b)$ in the sense of the Hermitian scalar product in $\mathbb{C}^n$. The definition of $T_b^{\mathbb{C}}(\partial D)$ is independent of u (this will follow from Proposition 1.18.2 (b)). Observe that if $n = 1$, then $T_b^{\mathbb{C}}(\partial D) = \{0\}$.

Proposition* 1.18.2 ([Jar-Pfl 2000], Proposition 2.2.3). *Let* $D \subset \mathbb{C}^n$ *be a bounded domain,* $a \in \partial D$*, and let* U *be an open neighborhood of* a*.*

(a) *Let* $u_1, u_2 \in \mathcal{C}^k(U, \mathbb{R})$ *be two local defining functions* ($k \in \mathbb{N} \cup \{\infty\}$). *Then* $u_2 = v u_1$ *with* $v \in \mathcal{C}^{k-1}(U, \mathbb{R}_{>0})$.

(b) *The space* $T_b^{\mathbb{C}}(\partial D)$ *is independent of the local defining function* $u \in \mathcal{C}^1(U, \mathbb{R})$*,* $b \in U \cap \partial D$.

(c) *Let* $u_1, u_2 \in \mathcal{C}^k(U, \mathbb{R})$*,* $k \geq 2$*, be two local defining functions with* $u_2 = v u_1$*, where* $v \in \mathcal{C}^{k-1}(U, \mathbb{R}_{>0})$ *is as in* (a)*. Then*

$$\mathcal{L}u_2(b; X) = v(b)\mathcal{L}u_1(b; X), \quad b \in U \cap \partial D, \ X \in T_b^{\mathbb{C}}(\partial D),$$

where $\mathcal{L}$ *denotes the Levi form* (*cf.* (1.14.1)).

(d) *Let* $k \in \mathbb{N} \cup \{\infty\}$*. Then the following conditions are equivalent:*

(i) D *is* $\mathcal{C}^k$*-smooth;*

(ii) *there exists a function* $u \in \mathcal{C}^k(\mathbb{C}^n, \mathbb{R})$ *satisfying* (1.18.1), (1.18.2), (1.18.3) *with* $U := \mathbb{C}^n$.

The above function u is called a *global defining function for* D.

Proposition* 1.18.3 ([Jar-Pfl 2000], Proposition 2.2.23). *Let* $D \subset \mathbb{C}^n$ *be a bounded* $\mathcal{C}^2$*-smooth domain. Then* D *is pseudoconvex iff for any local defining function* $u \in \mathcal{C}^2(V, \mathbb{R})$ *we have:*

$$\mathcal{L}u(b; X) \geq 0, \quad b \in V \cap \partial D, \ X \in T_b^{\mathbb{C}}(\partial D) \quad (\textit{Levi condition}).$$

Notice that by Proposition 1.18.2 (c), the Levi condition is independent of u.

Definition 1.18.4. Let $D \subset \mathbb{C}^n$ be a bounded domain. We say that ∂D is *strongly pseudoconvex in a neighborhood of a point* $a \in \partial D$ if there exist an open neighborhood U of a and a local defining function $u \in \mathcal{C}^2(U, \mathbb{R})$ such that

$$\mathcal{L}u(b; X) > 0, \quad b \in U \cap \partial D, \ X \in (T_b^{\mathbb{C}}(\partial D))_*. \tag{1.18.4}$$

Observe that by Proposition 1.18.2 (c), the definition is independent of u.

We say that D is *strongly pseudoconvex* if ∂D is strongly pseudoconvex at any point $a \in \partial D$.

Remark 1.18.5. (a) Obviously, if $n = 1$, then any $\mathcal{C}^2$-smooth domain $D \Subset \mathbb{C}$ is strongly pseudoconvex.

(b) The notion of the strong pseudoconvexity is invariant under local biholomorphic mappings (Exercise).

(c) We will see (Proposition 1.18.8 (a)) that any strongly pseudoconvex domain is hyperconvex and, consequently, pseudoconvex.

(d) Recall that a bounded domain $D \subset \mathbb{C}^n$ is said to be *strongly convex at a point* $a \in \partial D$ if there exist an open neighborhood U of a and a local defining function $u \in \mathcal{C}^2(U, \mathbb{R})$ for D such that

$$\mathcal{H}u(z; \xi) > 0, \quad z \in U \cap \partial D, \ \xi \in (T_z^{\mathbb{R}}(\partial D))_*,$$

where $\mathcal{H}$ denotes the real Hessian (cf. (1.14.8)) and $T_z^{\mathbb{R}}(\partial D)$ is the real $(2n-1)$-dimensional tangent space to ∂D at z. The definition is independent of u.

In particular, any strongly convex domain $D \subset \mathbb{C}^n$ is strongly pseudoconvex (Exercise).

Proposition* 1.18.6 ([Jar-Pfl 2000], Proposition 2.2.5). *Let* $D \subset \mathbb{C}^n$ *be a bounded domain.*

(a) *Assume that* ∂D *is* $\mathcal{C}^2$*-smooth at* $a \in \partial D$. *Let* U *be an open neighborhood of* a *and let* $u \in \mathcal{C}^2(U, \mathbb{R})$ *be strictly psh with* (1.18.1) *and* (1.18.2). *Then* u *satisfies* (1.18.3). *In particular,* u *is a local defining function.*

(b) *Let U be an open neighborhood of ∂D and let $u \in \mathcal{C}^k(U, \mathbb{R})$, $k \geq 2$, be a local defining function with* (1.18.4)*. Then there exists a $c > 0$ such that for the function $u_c := \frac{1}{c}(e^{cu} - 1)$ we have:*

$$\mathcal{L}u_c(b; X) > 0, \quad b \in \partial D, \ X \in (\mathbb{C}^n)_*.$$

In particular, u_c is strictly psh in a neighborhood of ∂D (notice that u_c is a local $\mathcal{C}^k$-defining function).

(c) *For $k \geq 2$, the following conditions are equivalent:*

(i) *D is $\mathcal{C}^k$-smooth and strongly pseudoconvex;*

(ii) *there exist an open neighborhood U of ∂D and a strictly psh function $u \in \mathcal{C}^k(U, \mathbb{R})$ with* (1.18.1) *and* (1.18.2)*.*

With respect to (b) and (c) compare [For 1979] and [Beh 1985] for the case of general pseudoconvex domains.

Exercise 1.18.7 (Complex ellipsoids; cf. [Jar-Pfl 1993], § 8.4). For $n \geq 2$, $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$, define the *complex ellipsoid*

$$\mathbb{E}_p := \Big\{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{n} |z_j|^{2p_j} < 1\Big\}. \tag{1.18.5}$$

Obviously $\mathbb{E}_{\mathbf{1}} = \mathbb{B}_n$. Prove the following properties of $\mathbb{E}_p$.

(a) $\mathbb{E}_p \subset \mathbb{D}^n$ is a complete Reinhardt domain of holomorphy (use Theorem 1.11.13).

(b) $\mathbb{E}_p$ is convex iff $p_1, \dots, p_n \geq 1/2$.

(c) $\mathbb{E}_p$ is geometrically strictly convex[69] if and only if $p_1, \dots, p_n \geq 1/2$ and $\#\{j : p_j = 1/2\} \leq 1$.

(d) $\partial\mathbb{E}_p$ is C^ω-smooth and strongly pseudoconvex at all points $z \in \partial\mathbb{E}_p \cap \mathbb{C}^n_*$.

(e) If $p_1, \dots, p_n > 1/2$, then $\mathbb{E}_p$ is strongly convex at all points $z \in \partial\mathbb{E}_p \cap \mathbb{C}^n_*$.

(f) $\mathbb{E}_p$ is $\mathcal{C}^1$-smooth iff $p_1, \dots, p_n > 1/2$.

(g) $\mathbb{E}_p$ is $\mathcal{C}^2$-smooth iff $p_1, \dots, p_n \geq 1$.

(h) For $p_1, \dots, p_n \geq 1$ the following conditions are equivalent:

(i) $\mathbb{E}_p$ is strongly convex;

(ii) $\mathbb{E}_p$ is strongly pseudoconvex;

(iii) $\mathbb{E}_p = \mathbb{B}_n$ (i.e. $p_1 = \dots = p_n = 1$).

Determine the interrelations between regularity of the Minkowski function $h_{\mathbb{E}_p}$ (cf. Definition 1.8.1) and p.

[69] That is, if $a, b \in \partial\mathbb{E}_p$, $a \neq b$, then $\{a + t(b - a) : 0 < t < 1\} \subset \mathbb{E}_p$.

Proposition* 1.18.8 ([Jar-Pfl 2000], Proposition 2.2.25). *Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex domain.*

(a) *If D is $\mathcal{C}^k$-smooth ($k \geq 2$), then there exist an open neighborhood U of $\bar{D}$ and a strictly psh defining function $u \in \mathcal{C}^k(U, \mathbb{R})$. In particular, any strongly pseudoconvex open set is hyperconvex.*

(b) *For any open neighborhood U of $\bar{D}$ there exists a strongly pseudoconvex $\mathcal{C}^\infty$-smooth open set D' such that $\bar{D} \subset D' \subset U$.*

1.19* Complete Kähler metrics

Following H. Grauert [Gra 1956], we will study complete Kähler metrics on a Reinhardt domain D in $\mathbb{C}^n$ and interrelations between the existence of such a metric and holomorphic convexity of D. Before explaining details let us introduce the notion of a Hermitian metric on D.

Definition 1.19.1. Let $D \subset \mathbb{C}^n$ be a domain. A system $\boldsymbol{g} = (g_{\nu,\mu})_{1 \leq \nu,\mu \leq n}$ of continuous functions $g_{\nu,\mu} \colon D \to \mathbb{C}$ is a *Hermitian metric* (resp. *pseudometric*) on D if $g_{\nu,\mu} = \overline{g_{\mu,\nu}}$ for all ν, μ and

$$\boldsymbol{g}(z; X) := \sum_{\nu,\mu=1}^{n} g_{\nu,\mu}(z) X_\nu \bar{X}_\mu > 0 \ \text{(resp. } \geq 0), \quad z \in D, \ X \in \mathbb{C}^n, X \neq 0.$$

If $X = 0$, then $\boldsymbol{g}(z; X) = 0$. Observe that $\overline{\boldsymbol{g}(z; X)} = \boldsymbol{g}(z; X)$.

Given a Hermitian pseudometric $\boldsymbol{g}$ on D as above and a piecewise $\mathcal{C}^1$-curve $\gamma \colon [0, 1] \to D$. Then the $\boldsymbol{g}$*-length* $L_{\boldsymbol{g}}(\gamma)$ of γ is defined as

$$L_{\boldsymbol{g}}(\gamma) := \int_0^1 \sqrt{\boldsymbol{g}(\gamma(t); \gamma'(t))}\, dt.$$

Having the notion of the $\boldsymbol{g}$-length of a curve we introduce the $\boldsymbol{g}$*-pseudodistance* $d_{\boldsymbol{g}}$ [70] between two points of D. Namely, we put

$$d_{\boldsymbol{g}}(z_1, z_2) := \inf\{L_{\boldsymbol{g}}(\gamma) : \gamma \in \widehat{\mathcal{C}}^1([0, 1], D), \ \gamma(0) = z_1, \ \gamma(1) = z_2\}, \qquad z_1, z_2 \in D,$$ [71]

where $\widehat{\mathcal{C}}^1([0, 1], D)$ denotes the set of all piecewise $\mathcal{C}^1$-curves in D.

[70] A pseudodistance d on D is a function $d \colon D \times D \to \mathbb{R}_+$ such that $d(z, z) = 0$, $d(z, w) = d(w, z)$, and $d(z, w) \leq d(z, u) + d(u, w)$ for arbitrary $z, w, u \in D$. It is a distance if, in addition, $d(z, w) > 0$ for $z \neq w$.

[71] Note that any two points in D can be connected even by a $\mathcal{C}^\infty$-curve in D. Moreover, observe that $d_{\boldsymbol{g}}(z_1, z_2) = \inf\{L_{\boldsymbol{g}}(\gamma)\}$, where the infimum is taken over all $\mathcal{C}^\infty$-curves in D connecting z_1, z_2 (Exercise).

If $\boldsymbol{g}$ is a Hermitian metric on D, then $d_{\boldsymbol{g}}$ is a distance (EXERCISE). We say that $\boldsymbol{g}$ is a *complete Hermitian metric* on D iff $\boldsymbol{g}$ is a Hermitian metric on D and $B_{\boldsymbol{g}}(a,r) := B_{d_{\boldsymbol{g}}}(a,r) = \{z \in D : d_{\boldsymbol{g}}(a,z) < r\} \Subset D$ for any $a \in D$ and any positive $r \in \mathbb{R}$. This definition means that boundary points of D have "infinite distance" from inner points of D.

In the sequel we deal with special Hermitian metrics, the so-called Kähler metrics.

Definition 1.19.2. Let $\boldsymbol{g} = (g_{\nu,\mu})$ be a $\mathcal{C}^\infty$-Hermitian metric (resp. pseudometric) on D, i.e. all the $g_{\nu,\mu} \in \mathcal{C}^\infty(D)$.

(a) $\boldsymbol{g}$ is said to be a *Kähler metric* (resp. *pseudometric*) if the functions $g_{\nu,\mu}$ fulfill the following relations:

$$\frac{\partial g_{\nu,\mu}}{\partial z_j} = \frac{\partial g_{j,\mu}}{\partial z_\nu} \text{ (and then also } \frac{\partial g_{\nu,\mu}}{\partial \bar{z}_j} = \frac{\partial g_{\nu,j}}{\partial \bar{z}_\mu}), \quad 1 \le j,\nu,\mu \le n.$$

(b) $\boldsymbol{g}$ is a $\mathcal{C}^\omega$-*Kähler metric* if $\boldsymbol{g}$ is Kähler and $g_{\nu,\mu} \in \mathcal{C}^\omega(D)$, $1 \le \nu,\mu \le n$.

(c) $\boldsymbol{g}$ is said to be a *complete Kähler metric* if it is a complete Hermitian one.

Example 1.19.3. (a) Let $D \subset \mathbb{C}^n$ be a domain. Assume that $u\colon D \to \mathbb{R}$ is a $\mathcal{C}^k$-function, $k \in \{\infty, \omega\}$, which is strictly psh on D. Then $g_{\nu,\mu} := \frac{\partial^2 u}{\partial z_\nu \partial \bar{z}_\mu}$ gives a $\mathcal{C}^k$-Kähler metric on D (EXERCISE).

(b) Put $g_{\nu,\mu} := \delta_{\nu,\mu}$.[72] Then $\boldsymbol{g} := (g_{\nu,\mu})$ gives the Euclidean metric on $\mathbb{C}^n$; it is a complete Kähler metric on $\mathbb{C}^n$.

Let $f = (f_1,\dots,f_m)\colon G \to D$, $G \subset \mathbb{C}^n$, $D \subset \mathbb{C}^m$, be a holomorphic mapping. Assume that $\boldsymbol{g} = (g_{\nu,\mu})$ is a Kähler metric on D. Define for $z \in G$:

$$\tilde{g}_{\nu,\mu}(z) := \sum_{k,j=1}^{m} g_{k,j}(f(z))\frac{\partial f_k}{\partial z_\nu}(z)\frac{\partial \bar{f}_j}{\partial \bar{z}_\mu}(z), \quad 1 \le \nu,\mu \le n,$$

i.e. $\tilde{\boldsymbol{g}}(z;X) = \boldsymbol{g}(f(z); f'(z)X)$. Then it is easily seen that $\tilde{\boldsymbol{g}} := (\tilde{g}_{\nu,\mu})$ is a Kähler pseudometric on D (EXERCISE). We write $f^{-1}(\boldsymbol{g}) := \tilde{\boldsymbol{g}}$ and say that $\tilde{\boldsymbol{g}}$ is the *pullback* of $\boldsymbol{g}$ via f.

Moreover, let $\gamma\colon [0,1] \to G$ be a piecewise $\mathcal{C}^1$-curve. Then $f \circ \gamma$ gives a piecewise $\mathcal{C}^1$-curve in D and $L_{\boldsymbol{g}}(f \circ \gamma) = L_{f^{-1}(\boldsymbol{g})}(\gamma)$ and therefore,

$$d_{f^{-1}(\boldsymbol{g})}(a,b) \ge d_{\boldsymbol{g}}(f(a), f(b)), \quad a,b \in G.$$

There is the following equivalent description for a $\mathcal{C}^\infty$-Hermitian metric $\boldsymbol{g}$ to be Kähler.

Theorem 1.19.4. *Let $D \subset \mathbb{C}^n$ be as above and $\boldsymbol{g}$ a $\mathcal{C}^\infty$-Hermitian metric on D. Then the following properties are equivalent:*

[72] As usual, $\delta_{\nu,\mu}$ means the Kronecker symbol.

(i) $\boldsymbol{g}$ *is Kähler;*

(ii) *for any point* $a \in D$ *there exist a polydisc* $P = \mathbb{P}(a,r) \subset D$ *and a strictly psh function* $U \in \mathcal{C}^\infty(P,\mathbb{R})$ *such that*

$$g_{\nu,\mu}(z) = \frac{\partial^2 U}{\partial z_\nu \partial \bar{z}_\mu}(z), \quad z \in P,\ 1 \le \nu,\mu \le n.$$

The function U in Theorem 1.19.4 is a *local potential* of $\boldsymbol{g}$.

The proof of Theorem 1.19.4 needs some preparations. First, we recall the Poincaré lemma from an analysis course.

Lemma 1.19.5. *Let* $G \subset \mathbb{R}^N$ *be a convex domain and let* $(f_{i,j})_{1\le i<j\le N} \subset \mathcal{C}^\infty(G,\mathbb{R})$ *(resp.* $(f_j)_{1\le j\le N} \subset \mathcal{C}^\infty(G,\mathbb{R})$*). Assume the following integrability conditions*

$$\frac{\partial f_{k,\ell}}{\partial x_j} - \frac{\partial f_{j,\ell}}{\partial x_k} + \frac{\partial f_{j,k}}{\partial x_\ell} = 0, \quad 1 \le j < k < \ell \le N$$
$$\Big(\textit{resp.}\ \frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j}, \quad 1 \le j,k \le N\Big)$$

on G*. Then there are* $\mathcal{C}^\infty(G,\mathbb{R})$*-functions* g_j, $1 \le j \le N$, *(resp. a* $\mathcal{C}^\infty(G,\mathbb{R})$*-function* g*) with*

$$\frac{\partial g_k}{\partial x_j} - \frac{\partial g_j}{\partial x_k} = f_{j,k}, \quad 1 \le j < k \le N \quad \Big(\textit{resp.}\ \frac{\partial g}{\partial x_j} = f_j, \quad 1 \le j \le N\Big).$$

Note that this result is often formulated in the language of differential forms, e.g. if the 2-form $\alpha := \sum_{1\le i<j\le n} f_{i,j}\,dx_i \wedge dx_j$ is d-closed, i.e. $d\alpha = 0$, then there exists a 1-form $\beta = \sum_{j=1}^n g_j\,dx_j$ with $d\beta = \alpha$. The reader is asked to recall or to study the meaning of differential forms.

Moreover, there are similar results for the complex case. We only formulate that one which is needed in the sequel.

Lemma 1.19.6. *Let* $a \in \mathbb{C}^n$, $r \in (0,\infty)$, *and* $\alpha_j \in \mathcal{C}^\infty(\mathbb{P}(a,r))$, $1 \le j \le n$. *Assume the following integrability conditions on* $\mathbb{P}(a,r)$*:*

$$\frac{\partial \alpha_j}{\partial \bar{z}_k} = \frac{\partial \alpha_k}{\partial \bar{z}_j}, \quad 1 \le j,k \le n.$$

Then, for any $r' \in (0,r)$*, there is a function* $f \in \mathcal{C}^\infty(\mathbb{P}(a,r'))$ *such that*

$$\frac{\partial f}{\partial \bar{z}_j} = \alpha_j \quad \textit{on } \mathbb{P}(a,r'), \quad 1 \le j \le n.$$

Proof. To prove the lemma we slightly reformulate its statement as follows.

For $m = 1, \dots, n+1$, *the following is true:*

$(*)_m$ *given positive numbers* $r' < r$ *and* $\alpha_j \in \mathcal{C}^\infty(\mathbb{P}(a,r))$, $j = 1, \dots, n$, *with* $\alpha_m = \dots = \alpha_n = 0$ *such that the integrability conditions are satisfied, then there exists an* $f \in \mathcal{C}^\infty(\mathbb{P}(a,r'))$ *with* $\frac{\partial f}{\partial \bar{z}_j} = \alpha_j$ *on* $\mathbb{P}(a,r')$, $1 \le j \le n$.

Note that $(*)_{n+1}$ is exactly the statement of the lemma.

To prove this new formulation we proceed by induction on m. We may assume $a = 0$ (Exercise). The case $m = 1$ is obviously true; take just the function $f = 0$.

Now let us assume that $(*)_m$ is true for some $m \le n$. Fix positive numbers $r' < r$ and functions $\alpha_j \in \mathcal{C}^\infty(\mathbb{P}(r))$, $1 \le j \le n$, with $\alpha_{m+1} = \dots = \alpha_n = 0$, such that the integrability conditions are satisfied. Choose numbers r_1, r_2 with $r' < r_1 < r_2 < r$ and a cut-off function $\chi \in \mathcal{C}^\infty(\mathbb{C})$ such that $\chi|_{K(r_1)} \equiv 1$ and $\chi(\lambda) = 0$ if $|\lambda| \ge r_2$.

For $z = (z', z_m, z'') \in \mathbb{P}(r) \subset \mathbb{C}^{m-1} \times \mathbb{C} \times \mathbb{C}^{n-m}$, the function

$$\mathbb{C} \ni \lambda \mapsto \begin{cases} \chi(\lambda)\alpha_m(z', \lambda, z'') & \text{if } |\lambda| < r, \\ 0 & \text{if } |\lambda| \ge r \end{cases}$$

is in $\mathcal{C}^\infty(\mathbb{C})$ and has a compact support. Therefore,

$$g\colon \mathbb{P}(r) \to \mathbb{C}, \quad g(z) := -\frac{1}{\pi} \int_{\mathbb{C}} \chi(\lambda) \frac{\alpha_m(z', \lambda, z'')}{\lambda - z_m} d\xi d\eta, \quad \lambda = \xi + i\eta,$$

belongs to $\mathcal{C}^\infty(\mathbb{P}(r))$. Moreover, we have

- $\frac{\partial g}{\partial \bar{z}_m} = \chi(z_m)\alpha_m(z) = \alpha_m(z)$, when $z \in \mathbb{P}(r_1)$,[73]
- $\frac{\partial g}{\partial \bar{z}_j}(z) = 0$, when $z \in \mathbb{P}(r)$ and $j > m$ (use the assumption and the integrability conditions).

Instead of dealing with α_j we are going to study the functions

$$\tilde{\alpha}_j := \alpha_j - \frac{\partial g}{\partial \bar{z}_j} \in \mathcal{C}^\infty(\mathbb{P}(r)), \quad 1 \le j \le n.$$

Note that this new system of functions fulfills the conditions of $(*)_m$ on $\mathbb{P}(r_1)$. Therefore, by the induction hypothesis, there exists an $\tilde{f} \in \mathcal{C}^\infty(\mathbb{P}(r'))$ such that

$$\frac{\partial \tilde{f}}{\partial \bar{z}_j} = \tilde{\alpha}_j, \quad 1 \le j \le n$$

on $\mathbb{P}(r')$. Setting $f := \tilde{f} + g$ on $\mathbb{P}(r')$ leads to a function solving all required differential equations. $\square$

[73] Recall from a one complex variable course that if $h \in \mathcal{C}_0^\infty(\mathbb{C})$, then the function $v(\lambda) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{h(\zeta)}{\lambda - \zeta} d\xi d\eta$, $\zeta = \xi + i\eta$, belongs to $\mathcal{C}^\infty(\mathbb{C})$ and satisfies $\frac{\partial v}{\partial \bar{\lambda}} = h$ on $\mathbb{C}$ (Exercise).

Remark 1.19.7. In fact, the above lemma is still true for $r' = r$. In this strong form the result is due to Dolbeault; it is Dolbeault's lemma (see e.g. [Hör 1990]). As before, it is mostly formulated with the help of a differential form. So we repeat our suggestion from above for the interested reader to study differential forms.

Proof of Theorem 1.19.4. Obviously, we only have to prove (i) $\Rightarrow$ (ii). Write

$$g_{\nu,\mu} = \alpha_{\nu,\mu} + i\beta_{\nu,\mu}$$

with real-valued functions $\alpha_{\nu,\mu}$ and $\beta_{\nu,\mu}$. Note that $\alpha_{j,k} = \alpha_{k,j}$ and $\beta_{j,k} = -\beta_{k,j}$. Now fix an $a \in D$ and a polydisc $\mathbb{P}(a, r) \subset D$. Then we introduce the following system of functions $(f_{j,k})_{1 \le j < k \le 2n} \subset \mathcal{C}^\infty(D, \mathbb{R})$, where

$$f_{j,k} := \begin{cases} -\beta_{j,k} & \text{if } 1 \le j < k \le n, \\ -\beta_{j-n,k-n} & \text{if } n+1 \le j < k \le n+n, \\ \alpha_{j,k-n} & \text{if } 1 \le j \le n,\ n+1 \le k \le n+n. \end{cases}$$

Then the integrability conditions in Lemma 1.19.5 (with $N = 2n$) are satisfied. Indeed, we have to show the following four equations.

$$\begin{aligned}
&\text{(i)} && \frac{\partial \alpha_{k,\ell}}{\partial x_j} - \frac{\partial \alpha_{j,\ell}}{\partial x_k} - \frac{\partial \beta_{j,k}}{\partial y_\ell} = 0, && 1 \le j < k \le n,\ 1 \le \ell \le n, \\
&\text{(ii)} && \frac{\partial \alpha_{j,\ell}}{\partial y_k} - \frac{\partial \alpha_{j,k}}{\partial y_\ell} + \frac{\partial \beta_{k,\ell}}{\partial x_j} = 0, && 1 \le j \le n,\ 1 \le k < \ell \le n, \\
&\text{(iii)} && \frac{\partial \beta_{k,\ell}}{\partial x_j} - \frac{\partial \beta_{j,\ell}}{\partial x_k} + \frac{\partial \beta_{j,k}}{\partial x_\ell} = 0, && 1 \le j < k < \ell \le n, \\
&\text{(iv)} && \frac{\partial \beta_{k,\ell}}{\partial y_j} - \frac{\partial \beta_{j,\ell}}{\partial y_k} + \frac{\partial \beta_{j,k}}{\partial y_\ell} = 0, && 1 \le j < k < \ell \le n.
\end{aligned}$$

To do so recall the Kähler conditions from Definition 1.19.2. Separating them into the real and imaginary parts we have

$$\begin{aligned}
&(k'_{j,\mu,\nu}) && \frac{\partial \alpha_{j,\mu}}{\partial x_\nu} + \frac{\partial \beta_{j,\mu}}{\partial y_\nu} - \left(\frac{\partial \alpha_{\nu,\mu}}{\partial x_j} + \frac{\partial \beta_{\nu,\mu}}{\partial y_j}\right) = 0, && 1 \le j, \nu, \mu \le n, \\
&(k''_{j,\nu,\mu}) && -\frac{\partial \alpha_{j,\mu}}{\partial y_\nu} + \frac{\partial \beta_{j,\mu}}{\partial x_\nu} - \left(\frac{\partial \beta_{\nu,\mu}}{\partial x_j} - \frac{\partial \alpha_{\nu,\mu}}{\partial y_j}\right) = 0, && 1 \le j, \nu, \mu \le n.
\end{aligned}$$

Inserting $(k'_{k,\ell,j})$ into (i) (resp. $(k''_{k,\ell,j})$ into (ii)) we see that equation (i) (resp. (ii)) is the same as (iv) (resp. (iii)). So what remains is to verify (iii) and (iv). Note that

$$\begin{aligned} 0 &= \frac{\partial \alpha_{k,\ell}}{\partial x_j} + \frac{\partial \beta_{k,\ell}}{\partial y_j} - \left(\frac{\partial \alpha_{j,\ell}}{\partial x_k} + \frac{\partial \beta_{j,\ell}}{\partial y_k}\right) \\ &\quad - \left(\frac{\partial \alpha_{k,j}}{\partial x_\ell} + \frac{\partial \beta_{k,j}}{\partial y_\ell} - \left(\frac{\partial \alpha_{\ell,j}}{\partial x_k} + \frac{\partial \beta_{\ell,j}}{\partial y_k}\right)\right) \\ &\quad + \frac{\partial \alpha_{j,k}}{\partial x_\ell} + \frac{\partial \beta_{j,k}}{\partial y_\ell} - \left(\frac{\partial \alpha_{\ell,k}}{\partial x_j} + \frac{\partial \beta_{\ell,k}}{\partial y_j}\right) \\ &= 2\left(\frac{\partial \beta_{k,\ell}}{\partial y_j} + \frac{\partial \beta_{j,k}}{\partial y_\ell} - \frac{\partial \beta_{j,\ell}}{\partial y_k}\right). \end{aligned}$$

Hence (iv) is true. In the same way the reader may verify (iii).

By Lemma 1.19.5 there exist $2n$ functions $\varphi_j, \psi_j \in \mathcal{C}^\infty(\mathbb{P}(a,r))$ satisfying the following equations:

$$\begin{aligned} \frac{\partial \varphi_k}{\partial x_j} - \frac{\partial \varphi_j}{\partial x_k} &= f_{j,k} = -\beta_{j,k}, && 1 \le j < k \le n, \\ \frac{\partial \psi_k}{\partial y_j} - \frac{\partial \psi_j}{\partial y_k} &= f_{n-j,n-k} = -\beta_{j,k}, && 1 \le j < k \le n, \\ \frac{\partial \psi_j}{\partial x_k} - \frac{\partial \varphi_k}{\partial y_j} &= f_{k,n+j} = \alpha_{k,j}, && 1 \le j,k \le n. \end{aligned}$$

Put $g_j := \varphi_j + i\psi_j \in \mathcal{C}^\infty(\mathbb{P}(a,r),\mathbb{C})$, $1 \le j \le n$. Then $(g_j)_{1\le j\le n}$ satisfies the integrability criterion in Lemma 1.19.6. Indeed, we have

$$\begin{aligned} &2\frac{\partial g_k}{\partial \bar z_j} - 2\frac{\partial g_j}{\partial \bar z_k} \\ &\quad= \frac{\partial \varphi_k}{\partial x_j} - \frac{\partial \psi_k}{\partial y_j} - \frac{\partial \varphi_j}{\partial x_k} + \frac{\partial \psi_j}{\partial y_k} + i\left(\frac{\partial \psi_k}{\partial x_j} + \frac{\partial \varphi_k}{\partial y_j} - \frac{\partial \varphi_j}{\partial y_k} - \frac{\partial \psi_j}{\partial x_k}\right) = 0, \\ &\qquad\qquad 1 \le j < k \le n. \end{aligned}$$

Now fix an $r' \in (0,r)$. Then, in virtue of Lemma 1.19.6, we find a function $h \in \mathcal{C}^\infty(\mathbb{P}(a,r'),\mathbb{C})$ satisfying

$$\frac{\partial h}{\partial \bar z_j} = g_j = \varphi_j + i\psi_j, \quad 1 \le j \le n.$$

Writing $h = h_1 + ih_2$ we have

$$2\varphi_j = \frac{\partial h_1}{\partial x_j} - \frac{\partial h_2}{\partial y_j} \text{ and } 2\psi_j = \frac{\partial h_2}{\partial x_j} - \frac{\partial h_1}{\partial y_j}, \quad 1 \le j \le n.$$

Combining the facts we collected so far, we get

$$\beta_{j,k} = \frac{\partial \varphi_j}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_j} = \frac{1}{4}\left(\frac{\partial^2}{\partial x_j \partial y_k} - \frac{\partial^2}{\partial x_k \partial y_j}\right)(2h_2), \quad 1 \le j < k \le n,$$

and
$$\alpha_{j,k} = \frac{1}{4}\Big(\frac{\partial^2}{\partial x_j \partial x_k} + \frac{\partial^2}{\partial y_k \partial y_j}\Big)(2h_2), \quad 1 \le j,k \le n.$$
Summarizing, the function $U := 2h_2 \in \mathcal{C}^\infty(\mathbb{P}(a,r'),\mathbb{R})$ fulfills
$$\frac{\partial^2 U}{\partial z_j \partial \bar{z}_k} = g_{j,k}, \quad 1 \le j,k \le n;$$
Hence U is a local potential around the point a. □

Remark 1.19.8. Another application of the Poincaré lemma deals with pluriharmonic functions (cf. Definition 1.14.26). Namely, using Lemma 1.19.5, we give another proof of Proposition 1.14.28.

Suppose that $D \subset \mathbb{C}^n$ is a convex domain and $u \in \mathcal{PH}(D)$. Then there exists an $f \in \mathcal{O}(D)$ with $\operatorname{Re} f = u$.

Proof. First, recall that u is a harmonic function and therefore $u \in \mathcal{C}^\infty(D)$. Obviously, the system $(f_j)_{1\le j\le 2n}$ with
$$f_j := \begin{cases} -\frac{\partial u}{\partial y_j} & \text{if } 1 \le j \le n, \\ \frac{\partial u}{\partial x_{j-n}} & \text{if } n+1 \le j \le 2n, \end{cases}$$
satisfies the integrability conditions from Lemma 1.19.5 (cf. (1.14.6)). Therefore, in virtue of Lemma 1.19.5, we find a $v \in \mathcal{C}^\infty(D,\mathbb{R})$ such that
$$\frac{\partial v}{\partial x_j} = -\frac{\partial u}{\partial y_j}, \quad \frac{\partial v}{\partial y_j} = \frac{\partial u}{\partial x_j}, \quad 1 \le j \le n.$$
Putting $f := u + iv$ gives a $\mathcal{C}^\infty(D)$-function which satisfies the Cauchy–Riemann equations. Hence f is the holomorphic function whose existence is claimed in the theorem. □

The following result connects the notion of holomorphic convexity with the one of a complete Kähler metric.

Theorem 1.19.9 ([Gra 1956]). *Every holomorphically convex domain D in $\mathbb{C}^n$ carries a complete $\mathcal{C}^\omega$-Kähler metric.*

Proof. By Remark 1.13.3 (b), there exists a sequence $(L_j)_{j=1}^\infty$ of holomorphically convex compact subsets of D such that $L_j \subset \operatorname{int} L_{j+1}$ and $D = \bigcup_{j=1}^\infty L_j$. Fix an index j. Then, for any point $z \in \partial L_{j+1}$ one may choose a function $f_z \in \mathcal{O}(D)$ satisfying $\|f_z\|_{L_j} < 1 < |f_z(z)|$.[74] Since ∂L_{j+1} is compact there are points $z_k \in \partial L_{j+1}$, $k = 1,\dots,k(j)$, and open neighborhoods $V_{j,k} = V_{j,k}(z_k) \subset D$

[74] Use that L_j is holomorphically convex.

such that $\partial L_{j+1} \subset \bigcup_{k=1}^{k(j)} V_{j,k}$ and $\inf_{V_{j,k}} |f_{j,k}| > 1$, $k = 1, \dots, k(j)$, where $f_{j,k} := f_{z_k}$. Since $\|f_{j,k}\|_{L_j} < 1$, we may choose an exponent $\varkappa_j \in \mathbb{N}$ such that $\|\tilde{f}_{j,k}\|^2_{L_j} < \frac{1}{2^j k(j)}$, where $\tilde{f}_{j,k} := j f_{j,k}^{\varkappa_j}$. In a next step we discuss the series

$$u := \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{k(j)} \tilde{f}_{j,k} \overline{\tilde{f}_{j,k}} \Big).$$

In virtue of the above construction it is clear that this series converges locally uniformly on D. Reading this sequence on $D \times \overline{D}$ as

$$u(z,w) = \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{k(j)} \tilde{f}_{j,k}(z) \overline{\tilde{f}_{j,k}}(\overline{w}) \Big)$$

shows that the series, in fact, gives a holomorphic function on $D \times \overline{D}$. In virtue of the Weierstrass theorem (Theorem 1.7.19), we finally obtain

$$g_{\nu,\mu}(z) := \frac{\partial^2 u}{\partial z_\nu \partial \bar{z}_\mu}(z, \bar{z}) = \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{k(j)} \frac{\partial \tilde{f}_{j,k}(z)}{\partial z_\nu} \frac{\partial \overline{\tilde{f}_{j,k}}(z)}{\partial \bar{z}_\mu} \Big).$$

Observe that the $g_{\nu,\mu}$'s are real analytic functions on D which define a Hermitian pseudometric $\boldsymbol{g}$. Put $\hat{g}_{\nu,\mu}(z) := g_{\nu,\mu}(z) + \delta_{\nu,\mu}$. Then the $\hat{g}_{\nu,\mu}$'s define a $\mathcal{C}^\omega$-Kähler metric $\hat{\boldsymbol{g}}$ on D, i.e. $\hat{\boldsymbol{g}}(z;X) = \boldsymbol{g}(z;X) + \|X\|^2$.

What remains is to show that this metric is a complete one on D. Note that $d_{\boldsymbol{g}} \le d_{\hat{\boldsymbol{g}}}$. Therefore, $B_{\hat{\boldsymbol{g}}}(a,r) \subset B_{\boldsymbol{g}}(a,r)$, $a \in D$, $r > 0$. Hence, it suffices to show that $B_{\boldsymbol{g}}(a,r)$ lies relatively compact in D.

In fact, fix $a \in D$ and $r > 0$. We may assume that $a \in \operatorname{int} L_1$. Assume now that there is a point $b \in \mathbb{B}(a,r) \setminus L_{s+1}$. Take an arbitrary $\mathcal{C}^1$-curve $\gamma\colon [0,1] \to D$ with $\gamma(0) = a$, $\gamma(1) = b$. Then there exists a $t_0 \in (0,1)$ with the following properties: $\gamma(t) \in \operatorname{int} L_{s+1}$, $0 \le t < t_0$, and $\gamma(t_0) \in \partial L_{s+1}$. We find an index j_b, $1 \le j_b \le k(s)$, such that $\gamma(t_0) \in V_{s,j_b}$. Therefore, $|f_{s,j_b}(\gamma(t_0))| > 1$. Thus, by definition, $|\tilde{f}_{s,j_b}(\gamma(t_0))| > s$. On the other hand, recall that $|\tilde{f}_{s,j_b}(\gamma(0))| < \sqrt{\frac{1}{2k(1)}} =:$ const. Thus, $\tilde{\gamma} := \tilde{f}_{s,j_b} \circ \gamma\colon [0,t_0] \to \mathbb{C}$ defines a $\mathcal{C}^1$-curve in the complex plane; in particular,

$$\int_0^{t_0} \Big| \frac{d\tilde{\gamma}(t)}{dt} \Big| dt \ge |\tilde{\gamma}(t_0) - \tilde{\gamma}(0)| \ge s - \text{const}\,.$$

It remains to estimate $L_{\boldsymbol{g}}(\gamma)$ from below. In fact, we have

$$L_{\boldsymbol{g}}(\gamma) \ge \int_0^{t_0} \sqrt{\boldsymbol{g}(\gamma(t);\gamma'(t))}dt \ge \int_0^{t_0} \|\tilde{f}'_{s,j_b}(\gamma(t))\gamma'(t)\| \ge \int_0^{t_0} \Big| \frac{d\tilde{\gamma}(t)}{dt} \Big| dt.$$

Hence, $d_{\boldsymbol{g}}(a,b) \ge s - \text{const} \to \infty$ when $s \to \infty$. Therefore, $B_{\boldsymbol{g}}(a,r)$ is contained in some L_σ; in particular, it is a relatively compact subset of D.

Hence the proof is completed. □

In virtue of Theorem 1.13.5 we have the following consequence.

Corollary 1.19.10. *Any Reinhardt domain of holomorphy carries a complete $\mathcal{C}^\omega$-Kähler metric.*

In a next step we discuss some examples of domains carrying a complete $\mathcal{C}^\omega$-Kähler metric.

Lemma 1.19.11. *Let $D := (\mathbb{C}^n)_*$. Then D carries a complete $\mathcal{C}^\omega$-Kähler metric.*

Proof. Put $u\colon (0,\infty) \to \mathbb{R}$, $u(t) := \frac{t-1}{t\log t}$. Then u is a real analytic function satisfying the following properties (Exercise):

- $u(t) > 0, \quad t \in (0,\infty)$,
- $\int_0^1 tu^2(t)dt =: d \in \mathbb{R}_+$,
- $\int_0^1 u(t)dt = \infty$.

Moreover, put $v(t) := \int_0^t \tau u^2(\tau)d\tau$, $t \in (0,\infty)$, and let U be a primitive of $\tau \mapsto \frac{v(\tau)}{\tau}$ on $(0,\infty)$. Finally, set $h\colon D \to \mathbb{R}$, $h(z) := U(\|z\|^2)$, and define $g_{\nu,\mu} := \frac{\partial^2 h}{\partial z_\nu \partial \bar z_\mu}$. Then the $g_{\nu,\mu}$ are real analytic functions on D and they give a Kähler pseudometric $\boldsymbol{g} = (g_{\nu,\mu})_{1\le\nu,\mu\le n}$.

Indeed,

$$g_{\nu,\mu}(z) = U''(\|z\|^2)z_\mu \bar z_\nu + U'(\|z\|^2)\delta_{\nu,\mu}, \quad z \in D,\ \nu,\mu = 1,\dots,n.$$

Therefore, applying the Schwarz inequality,

$$\boldsymbol{g}(z;X) = \Big(u^2(\|z\|^2) - \frac{v(\|z\|^2)}{\|z\|^2}\Big)\Big|\sum_{\nu=1}^n \bar z_\nu X_\nu\Big|^2 + \frac{v(\|z\|^2)}{\|z\|^2}\|X\|^2 \ge 0, \qquad z \in D,\ X \in \mathbb{C}^n.$$

It remains to modify $\boldsymbol{g}$ to obtain a Kähler metric on D. We simply set $\tilde{\boldsymbol{g}} = (\tilde g_{\nu,\mu})$, where $\tilde g_{\nu,\mu} := g_{\nu,\mu} + \delta_{\nu,\mu}$.

With respect to $\tilde{\boldsymbol{g}}$ we have $d_{\tilde{\boldsymbol{g}}}(a,b) \ge \|a-b\|$, $a,b \in D$. Hence, $\tilde{\boldsymbol{g}}$ is "complete at infinity". To discuss the behavior of $\tilde{\boldsymbol{g}}$ near the origin, fix points $a,b \in D$ with $\|b\| < 1 < \|a\|$. Let $\gamma\colon [0,1] \to D$ be a $\mathcal{C}^1$-curve in D with $\gamma(1) = b$ and $\gamma(0) = a$. Since γ has to pass $\partial\mathbb{B}$, it suffices to consider a γ satisfying $\|\gamma(0)\| = 1$

and $\|\gamma(t)\| \le 1$, $0 \le t \le 1$. Then

$$\begin{aligned} L_{\tilde{g}}(\gamma) \ge L_{g}(\gamma) &\ge \int_0^1 u(\|\gamma(t)\|^2)\Big|\sum_{\nu,\mu=1}^n \bar{\gamma}_\nu(t)\gamma'_\mu(t)\Big|dt \\ &\ge \frac{1}{2}\Big|\int_0^1 u(\|\gamma(t)\|^2) 2\,\mathrm{Re}\Big(\sum_{\nu,\mu=1}^n \bar{\gamma}_\nu(t)\gamma'_\mu(t)\Big)dt\Big| \\ &\ge \frac{1}{2}\int_{\|\gamma(1)\|^2}^{\|\gamma(0)\|^2} u(t)dt \ge \frac{1}{2}\int_{\|b\|^2}^1 u(t)dt. \end{aligned}$$

So we obtain

$$d_{\tilde{g}}(a,b) \ge \frac{1}{2}\int_{\|b\|^2}^1 u(t)dt \xrightarrow[b\to 0]{} \infty.$$

Hence, $\hat{g}$ is a complete Kähler metric on D. □

Using the metric we found in Lemma 1.19.11, it is possible to generalize Lemma 1.19.11 in the following form.

Theorem 1.19.12. *Let $D \subset \mathbb{C}^n$ be a holomorphically convex domain and let $f_1,\dots,f_k \in \mathcal{O}(D)$. Define $A := \{z \in D : f_1(z) = \dots = f_k(z) = 0\}$. Then $D \setminus A$ carries a complete $\mathcal{C}^\omega$-Kähler metric.*

Proof. Observe that $f = (f_1,\dots,f_k) : D \setminus A \to \mathbb{C}^k \setminus \{0\}$ defines a holomorphic mapping. Therefore, we have the Kähler pseudometric $\tilde{g} := f^{-1}(g)$, where g denotes the complete Kähler metric on $\mathbb{C}^k_*$ from Lemma 1.19.11. In virtue of Theorem 1.19.9, we may take a complete Kähler metric $\hat{g}$ on D. Then $h_{\nu,\mu} := \tilde{g}_{\nu,\mu} + \hat{g}_{\nu,\mu}$ leads to the Kähler metric $h = (h_{\nu,\mu})$ on $D \setminus A$ we are looking for.

In fact, recall that $d_h(a,b) \ge d_{\tilde{g}}(a,b) \ge d_g(f(a), f(b))$, $a,b \in D \setminus A$.

Suppose that there is a point $z' \in D \setminus A$ and a sequence $(a_j)_j \subset D \setminus A$ that converges to a boundary point a of $D \setminus A$. In case that $a \in \partial D$ we have $d_h(z',a_j) \ge d_{\hat{g}}(z',a_j) \to \infty$ when $j \to \infty$. Or we have that $a \in A$ and then $d_h(z',a_j) \ge d_g(f(z'), f(a_j)) \to \infty$. Hence, h is a complete Kähler metric on $D \setminus A$. □

Remark 1.19.13. (a) Recall that $D := (\mathbb{C}^n)_*$, $n \ge 2$, is not holomorphically convex (Exercise). Nevertheless, Lemma 1.19.11 shows that D carries a complete Kähler metric. Therefore, the converse of the statement in Theorem 1.19.9 is, in general, not true.

(b) In the case of a Reinhardt domain D of holomorphy we know that D and also $D \setminus V_0$ carry complete Kähler metrics.

(c) In [Gra 1956] it is shown that for any domain of holomorphy $D \subset \mathbb{C}^n$ and any analytic subset A of D,[75] the domain $D \setminus A$ carries a complete Kähler metric.

[75] For a definition see Remark 1.9.13.

Moreover, if D has a $\mathcal{C}^{\omega}$-boundary, then we have the following characterization: D is a domain of holomorphy if and only if there is a complete Kähler metric on D.

(d) In [Ohs 1980a], T. Ohsawa has generalized the above result by H. Grauert for domains with $\mathcal{C}^1$-boundary. Hence, in the category of domains with a $\mathcal{C}^1$-boundary, there is a complete description of domains of holomorphy in terms of complete Kähler metrics.

We start to discuss the consequences of the existence of a complete Kähler metric in case of Reinhardt domains.

Theorem 1.19.14. *Let D be a Reinhardt domain in $\mathbb{C}^n$. Assume that there is a complete Kähler metric on D. Then D is logarithmically convex.*

Proof. Take a complete Kähler metric $\boldsymbol{g} = (g_{\nu,\mu})$ on D. Put

$$\tilde{g}_{\nu,\mu}(z) := \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} g_{\nu,\mu}(e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)e^{i(\theta_\nu-\theta_\mu)}d\theta_1 \dots d\theta_n, \quad z \in D.$$

An easy calculation shows that $\tilde{\boldsymbol{g}} := (\tilde{g}_{\nu,\mu})$ is again a Kähler metric on D (EXERCISE).

Now let $\gamma \colon [0,1] \to D$ be a $\mathcal{C}^1$-curve. Then

$$\begin{aligned}
L_{\tilde{\boldsymbol{g}}}(\gamma) &= \int_0^1 \sqrt{\tilde{\boldsymbol{g}}(\gamma(t);\gamma'(t))}dt \\
&= \int_0^1 \left(\left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \boldsymbol{g}(\gamma_\theta(t);\gamma_\theta'(t))d\theta_1 \dots d\theta_n\right)^{1/2} dt \\
&\ge \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^1 \sqrt{\boldsymbol{g}(\gamma_\theta(t);\gamma_\theta'(t))}dt\,d\theta_1 \dots d\theta_n \\
&\ge \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} L_{\boldsymbol{g}}(\gamma_\theta)d\theta_1 \dots d\theta_n \\
&\ge \inf\{d_{\boldsymbol{g}}(z,w) : z \in \mathbb{T}_{\gamma(0)}, w \in \mathbb{T}_{\gamma(1)}\} = d_{\boldsymbol{g}}(\mathbb{T}_{\gamma(0)}, \mathbb{T}_{\gamma(1)}),
\end{aligned}$$

where $\mathbb{T}_a := \{(\zeta_1 a_1, \dots, \zeta_n a_n) : \zeta_1, \dots, \zeta_n \in \mathbb{T}\}$ denotes the n-dimensional torus through $a \in \mathbb{C}^n$ and

$$\gamma_\theta(t) := (\gamma_{\theta,1}, \dots, \gamma_{\theta,n})(t) := (e^{i\theta_1}\gamma_1(t), \dots, e^{i\theta_n}\gamma_n(t)).$$ [76]

Using the last inequality we continue proving that $\tilde{\boldsymbol{g}}$ is complete on D. Fix points $a, a_j \in D$ with $a_j \to \partial D$ (or $a_j \to \infty$ (if possible)). Then $d_{\tilde{\boldsymbol{g}}}(a,a_j) \ge$

[76](a) Observe that $\theta \mapsto L_{\boldsymbol{g}}(\gamma_\theta)$ is continuous. (b) The first inequality is a consequence of the Schwarz inequality (EXERCISE).

$d_{\boldsymbol{g}}(\mathbb{T}_a, \mathbb{T}_{a_j})$ for any j. Thus, for suitable $b_j \in \mathbb{T}_a$ and $c_j \in \mathbb{T}_{a_j}$, it follows that $d_{\tilde{\boldsymbol{g}}}(a, a_j) + 1 \geq d_{\boldsymbol{g}}(b_j, c_j)$. We may assume that $c_j \to \partial D$ (or $c_j \to \infty$) and $d_{\tilde{\boldsymbol{g}}}(a, a_j) \geq d_{\boldsymbol{g}}(a, c_j) - d_{\boldsymbol{g}}(b_j, a) - 1 \geq d_{\boldsymbol{g}}(a, c_j) - M$ for a suitable number M (observe that $d_{\boldsymbol{g}}(a, \cdot)$ is continuous on the compact torus $\mathbb{T}_a$). Therefore, applying that $\boldsymbol{g}$ is complete, we get $d_{\tilde{\boldsymbol{g}}}(a, a_j) \to \infty$, i.e. $\tilde{\boldsymbol{g}}$ is a complete Kähler metric on D.

Now, take the pullback $\boldsymbol{h}$ of $\tilde{\boldsymbol{g}}$ via the holomorphic mapping

$$\Phi\colon T := \log D + i\,\mathbb{R}^n \to D, \quad \Phi(w) := (e^{w_1}, \dots, e^{w_n}),$$

i.e. $\boldsymbol{h} := \Phi^{-1}(\tilde{\boldsymbol{g}}) = (h_{\nu,\mu})$. In particular, $h_{\nu,\mu}(w) = \tilde{g}_{\nu,\mu}(\Phi(w))e^{w_\nu}e^{\bar{w}_\mu} = h_{\nu,\mu}(u)$ when $w = u + iv \in T$, i.e. the functions $h_{\nu,\mu}$ depend only on the variable u. Exploiting the Kähler conditions for $\boldsymbol{h}$ we see that $\frac{\partial h_{\nu,\mu}}{\partial u_j} = \frac{\partial h_{j,\mu}}{\partial u_\nu}$ on T. So we obtain n closed one-forms on $\log D$,[77] namely $\alpha_\mu := \sum_{\nu=1}^n h_{\nu,\mu} du_\nu$, $1 \leq \mu \leq n$ (cf. Lemma 1.19.5).

Suppose now that $\log D$ is not convex. Then one may choose points $u', u'' \in \log D$ such that their connecting segment $[u', u'']$ is not contained in $\log D$. Applying that $\log D$ is connected there is a $\mathcal{C}^1$-curve $\gamma\colon [0,1] \to \log D$ connecting u' with u''. Then there exists $t_0 \in (0,1]$ such that $[u', \gamma(t)] \subset \log D$, $0 \leq t < t_0$, but $[u', \gamma(t_0)] \not\subset \log D$. Take an $s_0 \in (0,1)$ with $\gamma_{t_0}(s) := u' + s(\gamma(t_0) - u') \in \log D$, $0 \leq s < s_0$, and $\gamma_{t_0}(s_0) \in \partial \log D$. Observe that $\Phi(\gamma_{t_0}(s_0)) \in \partial D$. Then, setting $X := \gamma'_{t_0}(0)$, the Hölder inequality leads to

$$\begin{aligned}\sum_{\mu=1}^n X_\mu \int_{\gamma_{t_0}|_{[0,s]}} \alpha_\mu &\geq \big(L_{\boldsymbol{h}}(\gamma_{t_0}|_{[0,s]})\big)^2 \\ &\geq d_{\boldsymbol{h}}(\gamma_{t_0}(0), \gamma_{t_0}(s))^2 \geq d_{\tilde{\boldsymbol{g}}}(\Phi(\gamma_{t_0}(0)), \Phi(\gamma_{t_0}(s)))^2 \underset{s \nearrow s_0}{\longrightarrow} \infty,\end{aligned}$$

since $\tilde{\boldsymbol{g}}$ is a complete metric on D.

For $0 \leq t < t_0$, put $\gamma_t\colon [0,1] \to \log D$, $\gamma_t(s) := u' + s(\gamma(t) - u')$ and $X(t) := \gamma(t) - u'$. Note that $|\sum_{\mu=1}^n \int_{\gamma|_{[0,t_0]}} \alpha_\mu| \leq M$, $M > 2$.

Now, choose $s \in [0, s_0)$ near s_0 such that $\sum_{\mu=1}^n X_\mu \int_{\gamma_{t_0}|_{[0,s]}} \alpha_\mu \geq 2M + 1$.

[77] A *one-form* (or a *Pfaffian form*) α on a domain $\Omega \subset \mathbb{R}^n$ can be thought as an n-tuple $(f_1, \dots, f_n)$ of continuous functions $f_j : \Omega \to \mathbb{C}$ written in the form $\alpha = \sum_{j=1}^n f_j\, dx_j$. Such one-forms can be integrated along $\mathcal{C}^1$-curves $\gamma\colon [a,b] \to \Omega$ defining

$$\int_\gamma \alpha := \int_a^b \sum_{j=1}^n f_j(\gamma(t))\gamma'_j(t)dt.$$

In case that all $f_j \in \mathcal{C}^1(\Omega)$, α is called *closed* if $\frac{\partial f_j}{\partial x_k} = \frac{\partial f_k}{\partial x_j}$, $1 \leq j, k \leq n$.

Then, for a t, $t < t_0$, sufficiently near t_0, it follows that

$$2M \le \sum_{\mu=1}^{n} X_\mu \int_{\gamma_{t_0}|_{[0,s]}} \alpha_\mu - 1 \le \sum_{\mu=1}^{n} X_\mu(t) \int_{\gamma_t|_{[0,s]}} \alpha_\mu = L_{\boldsymbol{h}}(\gamma_t|_{[0,s]})$$
$$\le L_{\boldsymbol{h}}(\gamma_t) = \sum_{\mu=1}^{n} X_\mu(t) \int_{\gamma_t} \alpha_\mu \overset{(*)}{=} \Big|\sum_{\mu=1}^{n} X_\mu(t) \int_{\gamma|_{[0,t]}} \alpha_\mu\Big| \le M + 1;$$

a contradiction. Observe that equality $(*)$ is a consequence of the fact that the curves γ_t and $\gamma|_{[0,t]}$ are homotopic (EXERCISE) and the one-forms α_μ are closed. Hence, the Stokes theorem applies and gives $(*)$. $\square$

Thus Theorem 1.19.14 shows that a Reinhardt domain with a complete Kähler metric is almost holomorphically convex. Hence, by Remark 1.12.7, we get

Corollary 1.19.15. *Let D be a Reinhardt domain in $\mathbb{C}^n$. Assume that there is a complete Kähler metric on D. Then $D^* \setminus M(D)$ is the envelope of holomorphy of D.*

In fact a stronger result holds. In order to be able to formulate it we need the following definition.

Definition 1.19.16. For a Reinhardt domain $D \subset \mathbb{C}^n$, let $\widehat{D}$ be the set of all points $a \in \mathbb{C}^n$ such that there is a neighborhood $U = U(a)$ with

$$U \setminus \bigcup_{\substack{V_{i_1}\cap\dots\cap V_{i_k}\cap D=\varnothing,\\ 1\le i_1<\dots<i_k\le n,\\ 2\le k\le n}} V_{i_1} \cap \dots \cap V_{i_k} \subset D.$$

Exercise 1.19.17. (a) $\widehat{D}$ is a Reinhardt domain containing D.

(b) $\widehat{D} \subset D^* \setminus M(D)$.

(c) Find a log-convex Reinhardt domain $D \subset \mathbb{C}^2$ such that $\widehat{D} \subsetneq D^* \setminus M(D)$.

With this notion in mind there is the following result which we present here without giving a proof.

Theorem* 1.19.18 ([Gra 1956]). *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain which carries a complete Kähler metric. Then $\widehat{D}$ is holomorphically convex. Consequently, $\widehat{D} = D^* \setminus M(D)$.*

In particular, if $D \subset \mathbb{C}^2$ is a Reinhardt domain with a complete Kähler metric, then D or $D \cup \{0\}$ is a domain of holomorphy.

We like to mention that one main step of the proof of Theorem 1.19.18 consists in proving the following intermediate result:

Suppose that $D \subset \mathbb{C}^n$ is a Reinhardt domain carrying a complete Kähler metric. Assume further that $V_j \cap D \neq \varnothing$. Then $(a', 0, a'') \in D$ for every (a', a_j, a'') in $D \cap (\mathbb{C}_*^{j-1} \times \mathbb{C} \times \mathbb{C}_*^{n-j})$.[78]

Remark 1.19.19. (a) Theorem 1.19.18 can be thought as a special case of the following general result (cf. [Die-Pfl 1981]): Let $D \subset \mathbb{C}^n$ be a domain that carries a complete Kähler metric. Moreover assume that D is locally a domain of holomorphy at any point $a \in A := \operatorname{int} \overline{D} \setminus D$.[79] Then D is a domain of holomorphy.

As in the Reinhardt case this result shows that the obstruction for D to be a domain of holomorphy lies in the nature of a certain thin complement $\operatorname{int} \overline{D} \setminus D$.

(b) In [Ohs 1980a], T. Ohsawa proved the following result: Let $D \subset \mathbb{C}^n$ be a domain and $A \subset D$ a closed real $\mathcal{C}^1$-submanifold of D of real dimension $2n-2$. Assume that $D \setminus A$ carries a complete Kähler metric. Then A is necessarily a complex submanifold. In particular, the real dimension of A is even. Observe that here the real codimension $d := 2n - (2n-2)$ of A is by assumption exactly 2.

(c) For the higher codimensional case, the following modification of the former theorem is due to [Die-For 1982]: Let D and A be as in (b). Moreover, assume that A is now a real analytic closed submanifold of dimension $2n-d$, $d \geq 3$, such that $D \setminus A$ has a complete Kähler metric. Then A is an analytic set. In particular, A has an even real dimension.

(d) Surprisingly, the condition on A to be real analytic cannot be weakened, for example, to be only of type $\mathcal{C}^\infty$. In fact, there is the following result (see [Die-For 1982]): There is a closed $\mathcal{C}^\infty$-submanifold A of $\mathbb{B}_n$ of real dimension $2n-d$, $d \geq 3$, which is not an analytic set but, nevertheless, $D \setminus A$ allows a complete Kähler metric.

(e) Observe that for a real closed $\mathcal{C}^1$-submanifold A in a domain D of real dimension $2n-2$ the following is true: A is analytic if and only if A is nowhere linearly generated (for a point $a \in A$, A is called *linearly generated at a* if the smallest complex linear subspace of $\mathbb{C}^n$ containing the real tangent space of A at a coincides with $\mathbb{C}^n$). This observation allows the following generalization of (b) (cf. [Die-For 1984]): Let A be a closed real $\mathcal{C}^\infty$-submanifold of a domain $D \subset \mathbb{C}^n$ of real dimension $2n-d$, $d \geq 3$, such that $D \setminus A$ admits a complete Kähler metric. Then A is nowhere linearly generated.

[78] Compare the notion of weak relative completeness (cf. Theorem 1.11.13).

[79] That is, there is a neighborhood $U = U(a)$ such that any connected component of $D \cap U$ is a domain of holomorphy.

Chapter 2
Biholomorphisms of Reinhardt domains

2.1 Introduction

Let $G, D \subset \mathbb{C}^n$ be domains. Recall that $\mathrm{Bih}(G, D)$ denotes the set of all biholomorphic mappings $F : G \to D$. For $a \in G$, $b \in D$, put

$$\mathrm{Bih}_{a,b}(G, D) := \{F \in \mathrm{Bih}(G, D) : F(a) = b\}.$$

Define

$$\mathrm{Aut}(G) := \mathrm{Bih}(G, G),\ \ \mathrm{Aut}_{a,b}(G) := \mathrm{Bih}_{a,b}(G, G),\ \ \mathrm{Aut}_a(G) := \mathrm{Aut}_{a,a}(G).$$

Recall that $\mathrm{Aut}(G)$ with the operation

$$\mathrm{Aut}(G) \times \mathrm{Aut}(G) \ni (\Phi, \Psi) \to \Psi \circ \Phi \in \mathrm{Aut}(G)$$

is a group and $\mathrm{Aut}_a(G)$ is a subgroup of $\mathrm{Aut}(G)$. Observe that if $F \in \mathrm{Bih}(G, D)$, then the mapping

$$\mathrm{Aut}(G) \ni \Phi \xrightarrow{\Xi_F} F \circ \Phi \circ F^{-1} \in \mathrm{Aut}(D)$$

is a group isomorphism. Moreover, if $F \in \mathrm{Bih}_{a,b}(G, D)$, then $\Xi_F(\mathrm{Aut}_a(G)) = \mathrm{Aut}_b(D)$.

From the point of view of the theory of holomorphic functions, domains which are biholomorphic ($G \overset{\text{bih}}{\simeq} D$) may be identified – thus it is important to know when $\mathrm{Bih}(G, D) \neq \varnothing$ or (more precisely) when $\mathrm{Bih}_{a,b}(G, D) \neq \varnothing$.

On the other hand, the group $\mathrm{Aut}(G)$ characterizes the holomorphic geometry of G and, therefore, it is important to describe the structures of $\mathrm{Aut}(G)$ and $\mathrm{Aut}_{a,b}(G)$.

Definition 2.1.1. We say that a domain G is *homogeneous* if the group $\mathrm{Aut}(G)$ *acts transitively on* G, which means that $\mathrm{Aut}_{a,b}(G) \neq \varnothing$ for any $a, b \in G$, i.e. for any $a, b \in G$ there exists a $\Phi \in \mathrm{Aut}(G)$ with $\Phi(a) = b$.

We say that a domain G is *symmetric at a point* $a \in G$ if there exists a $\Phi \in \mathrm{Aut}_a(G)$ such that $\Phi^2 = \mathrm{id}$ and a is an isolated point of the fixed point set

$$\mathrm{Fix}(\Phi) := \{z \in G : \Phi(z) = z\}.$$

We say that G is *symmetric* if G is symmetric at every point $a \in G$.

Remark 2.1.2. (a) G is homogeneous iff there exists a point $a_0 \in G$ such that $\operatorname{Aut}_{a_0,b}(G) \neq \varnothing$ for every $b \in G$.

(b) The notion of a homogeneous (resp. symmetric) domain is invariant under biholomorphic mappings.

Indeed, let $F \in \operatorname{Bih}(G, D)$ and assume that D is homogeneous. Fix $a, b \in G$. Let $\Phi \in \operatorname{Aut}_{F(a),F(b)}(D)$. Then $F^{-1} \circ \Phi \circ F \in \operatorname{Aut}_{a,b}(G)$.

In the case where D is symmetric, $a \in G$, and $\Phi \in \operatorname{Aut}_{F(a)}(D)$ is such that $\Phi^2 = \mathrm{id}$ and the point $F(a)$ is isolated in $\operatorname{Fix}(\Phi)$, then $\Psi := F^{-1} \circ \Phi \circ F \in \operatorname{Aut}_a(G)$, $\Psi^2 = \mathrm{id}$, and the point a is isolated in $F^{-1}(\operatorname{Fix}(\Phi)) = \operatorname{Fix}(\Psi)$.

(c) If $G_j \subset \mathbb{C}^{n_j}$ is homogeneous (resp. symmetric), $j = 1, 2$, then $G_1 \times G_2$ is homogeneous (resp. symmetric).

(d) If G is homogeneous and symmetric at a point $a_0 \in G$, then G is symmetric.

Indeed, let $\Phi \in \operatorname{Aut}_{a_0}(G)$ be such that $\Phi^2 = \mathrm{id}$ and a_0 is an isolated point of the set $\operatorname{Fix}(\Phi)$. Take a $b \in G$ and let $F \in \operatorname{Aut}_{a_0,b}(G)$. Put $\Psi := F \circ \Phi \circ F^{-1} \in \operatorname{Aut}_b(G)$. Then $\Psi^2 = \mathrm{id}$ and $\operatorname{Fix}(\Psi) = F(\operatorname{Fix}(\Phi))$.

(e) Let G be a homogeneous domain with $0 \in G$. By (d), if G is symmetric with respect to 0 in the geometric sense (i.e. $z \in G \Rightarrow -z \in G$), then G is symmetric in the sense of Definition 2.1.1.

(f) If G is homogeneous and $\operatorname{Bih}(G, D) \neq \varnothing$, then $\operatorname{Bih}_{a,b}(G, D) \neq \varnothing$ for all points $a \in G$, $b \in D$.

Theorem 2.1.3. *Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain. Then D is a domain of holomorphy.*

Proof. Suppose D_0 and $\widetilde{D}$ to be as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{O}(D)$. We may assume that D_0 is a connected component of $D \cap \widetilde{D}$. Fix points $a \in D_0$, $b \in (\partial D_0) \cap \widetilde{D}$. Let $(b_k)_{k=1}^{\infty} \subset D_0$ be such that $b_k \to b$. Since D is homogeneous, there exists a $\Phi_k \in \operatorname{Aut}_{a,b_k}(D)$, $k \in \mathbb{N}$. Since D is bounded, we may assume that $\Phi_k \to \Phi$ locally uniformly in D, where $\Phi\colon D \to \overline{D}$ is a holomorphic mapping with $\Phi(a) = b$. We are going to show that $J\Phi(a) \neq 0$.

Indeed, put $\Psi_k := \Phi_k^{-1} \in \operatorname{Aut}_{b_k,a}(D)$. Let $\widetilde{\Psi}_k\colon \widetilde{D} \to \mathbb{C}^n$ be the holomorphic extension of Ψ_k with $\widetilde{\Psi}_k = \Psi_k$ on D_0, $k \in \mathbb{N}$. Since D is bounded, we may assume that $\Psi_k \to \Psi$ locally uniformly in D, where $\Psi\colon D \to \overline{D}$ is a holomorphic mapping. Let $\widetilde{\Psi}$ be the holomorphic extensions of Ψ. Since the extension operator is continuous (cf. Remark 1.11.3 (p)), we conclude that $\widetilde{\Psi}_k \to \widetilde{\Psi}$ locally uniformly in $\widetilde{D}$. Let $U \Subset D_0$ be a connected neighborhood of a such that $\Phi(U) \Subset \widetilde{D}$. We may assume that $\Phi_k(U) \subset \widetilde{D}$, $k \in \mathbb{N}$. Thus $\Phi_k(U)$ is a domain contained in $D \cap \widetilde{D}$ with $b_k = \Phi_k(a) \in \Phi_k(U)$. Hence $\Phi_k(U) \subset D_0$. Consequently, for $z \in U$, we obtain

$$z = \Psi_k(\Phi_k(z)) = \widetilde{\Psi}_k(\Phi_k(z)) \to \widetilde{\Psi}(\Phi(z))$$

and, therefore, $J\Phi(z) \neq 0$, $z \in U$. Hence, we may assume that $\Phi|_U\colon U \to V$ is biholomorphic, where $V := \Phi(U)$ is an open neighborhood of b.

Recall that $V \subset \bar{D}$. If D is fat, then we have a contradiction. In the general case we argue as follows. Take a Euclidean ball $B \Subset U$ centered at a. For a compact $K \subset \mathbb{C}^n$ let $\rho(b, K) := \inf\{\|b - w\| : w \in K\}$. Since $b \notin \Phi_k(\bar{B})$, we get $\rho(b, \Phi_k(\partial B)) = \rho(b, \Phi_k(\bar{B}))$, $k \in \mathbb{N}$ (Exercise). Then

$$\begin{aligned} 0 < \rho(b, \partial\Phi(\bar{B})) = \rho(b, \Phi(\partial B)) &= \lim_{k\to+\infty} \rho(b, \Phi_k(\partial B)) \\ &= \lim_{k\to+\infty} \rho(b, \Phi_k(\bar{B})) = \rho(b, \Phi(\bar{B})) = 0; \end{aligned}$$

a contradiction. □

Exercise 2.1.4. Prove that $(\mathbb{C}^2)_*$ is homogeneous, but is not a domain of holomorphy.

Exercise 2.1.5 (Examples of groups of automorphisms of planar domains).

(a) $\operatorname{Aut}(\mathbb{C}) = \{\mathbb{C} \ni z \mapsto az + b \in \mathbb{C} : (a, b) \in \mathbb{C}_* \times \mathbb{C}\}$,
$\operatorname{Aut}_0(\mathbb{C}) = \{\mathbb{C} \ni z \mapsto az \in \mathbb{C} : a \in \mathbb{C}_*\}$.

In particular:

- $\operatorname{Aut}(\mathbb{C})$ acts transitively on $\mathbb{C}$;
- $\operatorname{Aut}(\mathbb{C})$ depends on 4 real parameters;
- $\operatorname{Aut}_0(\mathbb{C})$ depends on 2 real parameters.

(b) Given an $a \in \mathbb{D}$, put

$$h_a(z) := \frac{z - a}{1 - \bar{a}z}, \quad z \in \mathbb{C} \setminus \{1/\bar{a}\} \supset \bar{\mathbb{D}}. \tag{2.1.1}$$

Observe that $h_a \in \operatorname{Aut}(\mathbb{D})$, $h_a(a) = 0$, and $h_a^{-1} = h_{-a}$. Then:

$$\begin{aligned} \operatorname{Aut}(\mathbb{D}) &= \{\zeta h_a : \zeta \in \mathbb{T},\ a \in \mathbb{D}\}, \\ \operatorname{Aut}_0(\mathbb{D}) &= \{\zeta h_0 : \zeta \in \mathbb{T}\}. \end{aligned}$$

In particular:

- $\operatorname{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$;
- $\operatorname{Aut}(\mathbb{D})$ depends on 3 real parameters;
- $\operatorname{Aut}_0(\mathbb{D})$ depends on 1 real parameter.

(c) Let $A = A(R) := \{z \in \mathbb{C} : 1/R < |z| < R\}$. Observe that every annulus $\{z \in \mathbb{C} : r_1 < |z| < r_2\}$ with $0 < r_1 < r_2 < +\infty$ is biholomorphic to $A(R)$ with $R := \sqrt{r_2/r_1}$. Then

$$\operatorname{Aut}(A) = \{A \ni z \mapsto \zeta z \in A : \zeta \in \mathbb{T}\} \cup \{A \ni z \mapsto \zeta/z \in A : \zeta \in \mathbb{T}\}.$$

In particular:

- $\operatorname{Aut}(A)$ does not act transitively on A;

- $\mathrm{Aut}(A)$ depends on 1 real parameter.[1]

(d) $\mathrm{Aut}(\mathbb{C}_* \times \mathbb{C}_*)$: The following mappings $F: \mathbb{C}_*^2 \to \mathbb{C}_*^2$ are biholomorphic:

- $F(z_1, z_2) := (z_1 e^{n_2 f(z_1^{n_1} z_2^{n_2})}, z_2 e^{-n_1 f(z_1^{n_1} z_2^{n_2})})$, where $n_1, n_2 \in \mathbb{Z}$ and $f \in \mathcal{O}(\mathbb{C}_*)$;
- $F(z_1, z_2) := (z_1^{a_{1,1}} z_2^{a_{1,2}}, z_1^{a_{2,1}} z_2^{a_{2,2}})$, where $a_{j,k} \in \mathbb{Z}$ with $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = \pm 1$;
- $F(z_1, z_2) := (c_1 z_1, c_2 z_2)$, where $c_1, c_2 \in \mathbb{C}_*$.

The full description of $\mathrm{Aut}(\mathbb{C}_*^2)$ seems to be not known; $\boxed{?}$ it is conjectured that the mappings above generate $\mathrm{Aut}(\mathbb{C}_*^2)$. $\boxed{?}$

(e) $\mathrm{Aut}(\mathbb{C} \times \mathbb{C}_*)$: Due to [Nis 1986] the following theorem is true.

Let $F \in \mathrm{Aut}(\mathbb{C}^2)$ *with* $F_2(\cdot, 0) = 0$ (*in particular,* $F|_{\mathbb{C}\times\mathbb{C}_*} \in \mathrm{Aut}(\mathbb{C} \times \mathbb{C}_*)$) *and* $\det F' = c \in \mathbb{C}$. *Then*

$$F(z_1, z_2) = (c z_1 e^{-\alpha(z_1 z_2, z_2)} + \beta(z_1 z_2, z_2), z_2 e^{\alpha(z_1 z_2, z_2)}),$$

where $\alpha, \beta \in \mathcal{O}(\mathbb{C}^2)$.

$\boxed{?}$ According to our knowledge the full description of $\mathrm{Aut}(\mathbb{C} \times \mathbb{C}_*)$ remains still open. $\boxed{?}$

(f) $\mathrm{Aut}(\mathbb{C}^2)$: Recall that the set of holomorphic automorphisms of $\mathbb{C}$ is quite simple. In contrast, for $n > 1$ the situation for $\mathrm{Aut}(\mathbb{C}^n)$ is much more complicated. There are, for example, automorphisms F of the following types:

- $F(z_1, z_2) := (z_1, z_2 + f(z_1))$, where $f \in \mathcal{O}(\mathbb{C})$; mappings of this type are usually called *shears*;
- $F(z_1, z_2) := (z_1, e^{h(z_1)} z_2 + f(z_2))$, where $f, h \in \mathcal{O}(\mathbb{C})$; mappings of this type are the so–called *overshears*;
- $F(z_1, z_2) := (z_1 e^{z_1 z_2}, z_2 e^{z_1 z_2})$.

It is known that mappings of the third type do not belong to the group of automorphisms generated by the overshears.

Moreover, the following result shows how complicated $\mathrm{Aut}(\mathbb{C}^n)$ may be.

Theorem 2.1.6 ([FMV 2006]). *Let* $n, k \in \mathbb{N}$, $n \geq 2$ *and let* $a_1, \dots, a_k \in \mathbb{C}^n$ *be pairwise distinct points. Then there is a polynomial automorphism* $F \in \mathrm{Aut}(\mathbb{C}^n)$ *such that* $\mathrm{Fix}(F) = \{a_j : j = 1, \dots, k\}$.

Proof. (Details are left to the reader as an EXERCISE.) Let $a_j = (a_j', a_j'') \in \mathbb{C} \times \mathbb{C}^{n-1}$. One may assume that the numbers a_j' are all distinct (take a suitable invertible linear transformation). Denote by $f_j: \mathbb{C} \to \mathbb{C}$ the Lagrange interpolation polynomial for $a_1', \dots, a_k'$ and $a_{1,j}'', \dots, a_{k,j}''$, $j = 2, \dots, n$, i.e. $f :=$

[1] The description of $\mathrm{Aut}(A)$ in the case where $A := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ is a degenerated annulus (i.e. $r_1 = 0$ or/and $r_2 = +\infty$) is left for the reader. In particular,

$$\mathrm{Aut}(\mathbb{C}_*) = \{\mathbb{C}_* \ni z \mapsto az \in \mathbb{C}_* : a \in \mathbb{C}_*\} \cup \{\mathbb{C}_* \ni z \mapsto a/z \in \mathbb{C}_* : a \in \mathbb{C}_*\}.$$

$(f_2, \dots, f_n)$ is a polynomial mapping with $f(a'_j) = a''_j$. Then $\widetilde{F}\colon \mathbb{C}^n \to \mathbb{C}^n$, $\widetilde{F}(z) = \widetilde{F}(z_1, z') := (z_1, z' + f(z_1))$, is a polynomial automorphism of $\mathbb{C}^n$ with $\widetilde{F}(a'_j, 0) = a_j$.

In a second step, consider the polynomial automorphism $G\colon \mathbb{C}^n \to \mathbb{C}^n$

$$G(z) := (z_1 + z_2 + (z_1 - a'_1) \cdots (z_1 - a'_k), z_2 + (z_1 - a'_1) \cdots (z_1 - a'_k), iz_3, \dots, iz_n).$$

Then $\operatorname{Fix}(G) = \{(a'_1, 0), \dots, (a'_k, 0)\}$. Finally, put $F := \widetilde{F} \circ G \circ \widetilde{F}^{-1}$ and observe that F has all the required properties. $\square$

From now on we will be concentrated on bounded domains in $\mathbb{C}^n$.

Theorem 2.1.7 (Cartan). *Let $G \subset \mathbb{C}^n$ be a bounded domain, let $a \in G$, and let $\Phi\colon G \to G$ be a holomorphic mapping such that $\Phi(a) = a$ and $\Phi'(a) = \mathrm{id}$. Then $\Phi = \mathrm{id}$.*

Notice that the assumption that G is bounded is essential – take for instance $G := \mathbb{C}$, $a := 0$, $\Phi(z) := z(1 - z)$.

Proof. We may assume that $a = 0$. Suppose that $\Phi \not\equiv \mathrm{id}$. Fix $r, R > 0$ such that $\mathbb{P}(r) \subset D \subset \mathbb{P}(R)$. We have

$$\Phi(z) = \sum_{k=0}^{\infty} Q_k(z), \quad z \in \mathbb{P}(r),$$

where $Q_k\colon \mathbb{C}^n \to \mathbb{C}^n$ is a homogeneous polynomial mapping of degree k (cf. Proposition 1.8.4). We know that $Q_0 = 0$ and $Q_1 = \mathrm{id}$. Let $k_0 \geq 2$ be such that $Q_2 = \cdots = Q_{k_0-1} = 0$, $Q_{k_0} \not\equiv 0$. Denote by Φ^ν the ν-th iterate of the mapping Φ, i.e. $\Phi^0 := \mathrm{id}$, $\Phi^{\nu+1} := \Phi^\nu \circ \Phi$. Then

$$\Phi^\nu(z) = z + \nu Q_{k_0}(z) + \sum_{k=k_0+1}^{\infty} Q_{\nu,k}(z), \quad z \in \mathbb{P}(r),$$

where $Q_{\nu,k}\colon \mathbb{C}^n \to \mathbb{C}^n$ is a homogeneous polynomial of degree k. To prove this formula use induction on ν. Suppose that the formula is true for a $\nu \geq 1$. Let $\delta > 0$ be such that $\Phi(\mathbb{P}(\delta)) \subset \mathbb{P}(r)$. Then for $z \in \mathbb{P}(\delta)$ we get (Exercise)

$$\begin{aligned}
\Phi^{\nu+1}(z) = \Phi^\nu(\Phi(z)) &= \Phi(z) + \nu Q_{k_0}(\Phi(z)) + \sum_{k=k_0+1}^{\infty} Q_{\nu,k}(\Phi(z)) \\
&= z + \sum_{k=k_0}^{\infty} Q_k(z) + \nu Q_{k_0}(z + \text{higher order terms}) \\
&\quad + \sum_{k=k_0+1}^{\infty} Q_{\nu,k}(z + \text{higher order terms})
\end{aligned}$$

$$= z + (\nu + 1)Q_{k_0}(z) + \sum_{k=k_0+1}^{\infty} Q_{\nu+1,k}(z).$$

It remains to use the identity principle to conclude that the formula holds on the whole of $\mathbb{P}(r)$.

Hence, by the Cauchy inequalities, for any $z \in \mathbb{P}(r)$ we get

$$|\nu(Q_{k_0})_j(z)| \le \max\{|(\Phi^\nu)_j(\zeta z)| : \zeta \in \mathbb{T}\} \le R, \quad j = 1, \dots, n.$$

Letting $\nu \to +\infty$, we obtain $Q_{k_0} \equiv 0$; a contradiction. □

Proposition 2.1.8 (Cartan). *Let $G, D \subset \mathbb{C}^n$ be circular domains*[2] *with $0 \in G$, $0 \in D$, such that G is bounded, and let $F \in \mathrm{Bih}_{0,0}(G, D)$. Then F is a linear isomorphism.*

Notice that the assumption that G is bounded is essential – take for instance $G = D = \mathbb{C}^2$, $F(z_1, z_2) := (z_1 + f(z_2), z_2)$, where $f \in \mathcal{O}(\mathbb{C})$ is a nonlinear entire function with $f(0) = 0$; cf. Example 2.1.5 (f).

Proof. For $\zeta \in \mathbb{T}$ put $H_\zeta(z) := F^{-1}((1/\zeta)F(\zeta z))$, $z \in G$. Then $H_\zeta \in \mathrm{Aut}_0(G)$ and $H'_\zeta(0) = \mathrm{id}$. Therefore, by Theorem 2.1.7, $H_\zeta = \mathrm{id}$, i.e. $F(\zeta z) = \zeta F(z)$, $z \in G$, $\zeta \in \mathbb{T}$. Expand F into a series of homogeneous polynomials in a polydisc $\mathbb{P}(r) \subset G$:

$$F(z) = \sum_{k=1}^{\infty} Q_k(z), \quad z \in \mathbb{P}(r).$$

Then

$$F(z) = \sum_{k=1}^{\infty} \zeta^{k-1} Q_k(z), \quad z \in \mathbb{P}(r), \ \zeta \in \mathbb{T}.$$

This means that $Q_k = 0$ in $\mathbb{P}(r)$ for $k \ge 2$ (Exercise), and so, by the identity principle, $F \equiv Q_1$. Therefore F is a linear mapping. Since F is biholomorphic, it must be a linear isomorphism. □

Proposition 2.1.9. *Let $\| \ \|_j$ be a $\mathbb{C}$-norm in $\mathbb{C}^{n_j}$, let*

$$B_j := \{z \in \mathbb{C}^{n_j} : \|z\|_j < 1\}, \quad j = 1, 2,$$

and let $F : B_1 \to B_2$ be a holomorphic mapping with $F(0) = 0$. Then $\|F(z)\|_2 \le \|z\|_1$, $z \in B_1$.

In particular, if $F \in \mathrm{Bih}_{0,0}(B_1, B_2)$, then F is a linear isomorphism (*Proposition* 2.1.8) *and $\|F(z)\|_2 = \|z\|_1$, $z \in B_1$.*

[2] Recall that a domain $G \subset \mathbb{C}^n$ is circular if $\zeta z \in G$ for every $z \in G$ and $\zeta \in \mathbb{T}$. Obviously, any Reinhardt domain is circular.

Proof. We may assume that $F \not\equiv 0$. Fix a $z_0 \in (B_1)_*$ with $b := F(z_0) \neq 0$. Let $L\colon \mathbb{C}^{n_2} \to \mathbb{C}$ be a $\mathbb{C}$-linear mapping with

$$|L(b)| = \|b\|_2, \quad |L(w)| \le \|w\|_2, \quad w \in \mathbb{C}^{n_2}.$$ [3]

Consider the holomorphic mapping $\varphi(\lambda) := L(F(\lambda z_0))$, $|\lambda| < 1/\|z_0\|_1$. Then, by the classical Schwarz lemma, we obtain $|\varphi(\lambda)| \le |\lambda|\|z_0\|_1$, $|\lambda| < 1/\|z_0\|_1$. In particular, for $\lambda = 1$, we get $\|F(z_0)\|_2 \le \|z_0\|_1$. □

Remark 2.1.2 (f) and the above proposition imply immediately the following

Corollary 2.1.10. *Let $\|\ \|_j$ be a $\mathbb{C}$-norm in $\mathbb{C}^n$ and let*

$$B_j := \{z \in \mathbb{C}^n : \|z\|_j < 1\}, \quad j = 1, 2.$$

Assume that B_1 is homogeneous. Then $\mathrm{Bih}(B_1, B_2) \neq \varnothing$ *iff there exists a $\mathbb{C}$-linear isomorphism $F\colon \mathbb{C}^n \to \mathbb{C}^n$ with $\|F(z)\|_2 = \|z\|_1$, $z \in \mathbb{C}^n$ (equivalently, $F(B_1) = B_2$).*

Notice that the following general result is true.

Theorem* 2.1.11 ([Kau-Upm 1976]). *Let $D_1, D_2 \subset \mathbb{C}^n$ be arbitrary bounded balanced pseudoconvex domains such that* $\mathrm{Bih}(D_1, D_2) \neq \varnothing$. *Then there exists an $F \in$* $\mathrm{Bih}(D_1, D_2)$ *with $F(0) = 0$.*

The proof of the above result is based on techniques from Lie groups, i.e. it is beyond of the scope of our book, so we have to skip it. $\boxed{?}$ Is there a more direct proof which is based on techniques presented so far? $\boxed{?}$

Example 2.1.12 (Elementary homogeneous domains in $\mathbb{C}^n$). (All details are left to the reader as an Exercise.)

(a) *Unit polydisc $\mathbb{D}^n$.*

$$\begin{aligned}\mathrm{Aut}(\mathbb{D}^n) &= \{\mathbb{D}^n \ni z \mapsto (\zeta_1 h_{a_1}(z_{\sigma(1)}), \dots, \zeta_n h_{a_n}(z_{\sigma(n)})) \in \mathbb{D}^n : \\ &\qquad\qquad \zeta_j \in \mathbb{T},\ a_j \in \mathbb{D},\ j = 1, \dots, n,\ \sigma \in \mathfrak{S}_n\} =: \mathfrak{G}, \\ \mathrm{Aut}_0(\mathbb{D}^n) &= \{\mathbb{D}^n \ni z \mapsto (\zeta_1 z_{\sigma(1)}, \dots, \zeta_n z_{\sigma(n)}) \in \mathbb{D}^n : \\ &\qquad\qquad \zeta_j \in \mathbb{T},\ j = 1, \dots, n,\ \sigma \in \mathfrak{S}_n\} =: \mathfrak{G}_0,\end{aligned}$$

where $\mathfrak{S}_n$ denotes the group of all permutations of n-elements. In particular:

- the group $\mathrm{Aut}(\mathbb{D}^n)$ acts transitively on $\mathbb{D}^n$; $\mathbb{D}^n$ is homogeneous and symmetric (Remark 2.1.2 (e));

[3] Let $V := \mathbb{C}b \subset \mathbb{C}^{n_2}$ and let $L_0\colon V \to \mathbb{C}$ be given by the formula $L_0(\lambda b) := \lambda\|b\|_2$, $\lambda \in \mathbb{C}$. Observe that $L_0(b) = \|b\|$ and $|L(w)| = \|w\|_2$, $w \in V$. Let $P\colon \mathbb{C}^{n_2} \to V$ be the orthogonal projection with respect to the standard Hermitian scalar product in $\mathbb{C}^{n_2}$. Put $L := L_0 \circ P\colon \mathbb{C}^{n_2} \to \mathbb{C}$. Observe that L is a $\mathbb{C}$-linear mapping with $L(b) = L_0(b) = \|b\|$ and $|L(w)| = |L_0(P(w))| = \|P(w)\|_2 \le \|w\|_2$, $w \in \mathbb{C}^{n_2}$.

- the group $\mathrm{Aut}(\mathbb{D}^n)$ depends on $d(n) := 3n$ real parameters;
- the group $\mathrm{Aut}_0(\mathbb{D}^n)$ depends on $d_0(n) := n$ real parameters.

Indeed, it is easy to see that $\mathfrak{G}$ is a subgroup of $\mathrm{Aut}(\mathbb{D}^n)$, $\mathfrak{G}_0$ is a subgroup of $\mathrm{Aut}_0(\mathbb{D}^n)$, and $\mathfrak{G}$ acts transitively on $\mathbb{D}^n$. We only need to show that $\mathrm{Aut}_0(\mathbb{D}^n) \subset \mathfrak{G}_0$. By Propositions 2.1.8, 2.1.9, any automorphism $\Phi \in \mathrm{Aut}_0(\mathbb{D}^n)$ is a linear isomorphism with $\|\Phi(z)\|_\infty = \|z\|_\infty$, $z \in \mathbb{C}^n$. Let $[\Phi_{j,k}]_{j,k=1,\dots,n} \in \mathbb{GL}(n,\mathbb{C})$ denote the matrix representation of Φ. We have

$$\max\Big\{\Big|\sum_{k=1}^{n} \Phi_{j,k} z_k\Big| : j = 1,\dots,n\Big\} = \|z\|_\infty, \quad z = (z_1,\dots,z_n) \in \mathbb{C}^n.$$

In particular,

$$\begin{aligned} \max\{|\Phi_{1,k}|,\dots,|\Phi_{n,k}|\} = 1, &\quad k = 1,\dots,n, \\ |\Phi_{j,1}| + \dots + |\Phi_{j,n}| \le 1, &\quad j = 1,\dots,n. \end{aligned}$$

Thus the matrix $[\Phi_{j,k}]$ has in each row, and each column, exactly one nonzero element (which must have absolute value 1). This means that $\Phi \in \mathfrak{G}_0$.

(b) *Unit Euclidean ball* $\mathbb{B}_n$. For $a \in (\mathbb{B}_n)_*$, let

$$h_a(z) := \frac{1}{\|a\|^2} \frac{\sqrt{1-\|a\|^2}(\|a\|^2 z - \langle z,a\rangle a) - \|a\|^2 a + \langle z,a\rangle a}{1 - \langle z,a\rangle},$$
$$z \in \mathbb{C}^n \setminus \{\langle z,a\rangle = 1\} \supset \overline{\mathbb{B}}_n,$$

where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian complex scalar product in $\mathbb{C}^n$. Let, moreover, $h_0 := \mathrm{id}$. Observe that in the case where $n = 1$ the above definition of h_a agrees with that from (2.1.1). Denote by $\mathbb{U}(n)$ the group of all unitary automorphisms of $\mathbb{C}^n$.[4] Under above notation we have:

$$\begin{aligned} \mathrm{Aut}(\mathbb{B}_n) &= \{U \circ h_a : U \in \mathbb{U}(n),\ a \in \mathbb{B}_n\}, \\ \mathrm{Aut}_0(\mathbb{B}_n) &= \mathbb{U}(n). \end{aligned}$$

In particular:

- the group $\mathrm{Aut}(\mathbb{B}_n)$ acts transitively on $\mathbb{B}_n$; $\mathbb{B}_n$ is homogeneous and symmetric;
- the group $\mathrm{Aut}(\mathbb{B}_n)$ depends on $b(n) := n^2 + 2n$ real parameters;
- the group $\mathrm{Aut}_0(\mathbb{B}_n)$ depends on $b_0(n) := n^2$ real parameters.

[4] A $\mathbb{C}$-linear mapping $L\colon \mathbb{C}^n \to \mathbb{C}^n$ is *unitary* if one of the following equivalent conditions is satisfied:
- $\langle L(z'), L(z'')\rangle = \langle z', z''\rangle$, $z', z'' \in \mathbb{C}^n$;
- $\|L(z)\| = \|z\|$, $z \in \mathbb{C}^n$;
- $LL^* = L^*L = \mathbb{I}_n$ (L is identified with its matrix representation; $L^* := \overline{L}^t$).

The group $\mathbb{U}(n)$ depends on n^2 real parameters.

Indeed, the fact that $\mathrm{Aut}_0(\mathbb{B}_n) = \mathbb{U}(n)$ follows immediately from Propositions 2.1.8 and 2.1.9. Since $h_a(a) = 0$, we only need to prove that $h_a \in \mathrm{Aut}(\mathbb{B}_n)$. Fix an $a \in (\mathbb{B}_n)_*$. Direct calculations show that

$$1 - \langle h_a(z), h_a(w)\rangle = \frac{(1-\langle a,a\rangle)(1-\langle z,w\rangle)}{(1-\langle z,a\rangle)(1-\langle a,w\rangle)}, \quad z,w \in \overline{\mathbb{B}}_n.$$

Taking $w = z$, we conclude that $h_a(\mathbb{B}_n) \subset \mathbb{B}_n$ and $h_a(\partial\mathbb{B}_n) \subset \partial\mathbb{B}_n$. In particular, $h_a \circ h_{-a}$ is well defined in a neighborhood of $\overline{\mathbb{B}}_n$. Using once again direct calculations, we prove that $h_a \circ h_{-a} = \mathrm{id}$. Hence $h_a \in \mathrm{Aut}(\mathbb{B}_n)$ and $h_a^{-1} = h_{-a}$.

(c) *Unit Lie ball* $\mathbb{L}_n$. Let

$$\mathbb{L}_n := \{z \in \mathbb{C}^n : L_n(z) < 1\} = \{z \in \mathbb{B}_n : 2\|z\|^2 - |\langle z,\bar{z}\rangle|^2 < 1\},$$

where

$$\begin{aligned} L_n(z) :&= (\|z\|^2 + \sqrt{\|z\|^4 - |\langle z,\bar{z}\rangle|^2})^{1/2} \\ &= (\|x\|^2 + \|y\|^2 + 2\sqrt{\|x\|^2\|y\|^2 - \langle x,y\rangle^2})^{1/2}, \\ &\qquad\qquad z = x + iy \in \mathbb{R}^n + i\mathbb{R}^n \simeq \mathbb{C}^n. \end{aligned}$$

The *Lie norm* L_n is the maximal $\mathbb{C}$-norm $q\colon \mathbb{C}^n \to \mathbb{R}_+$ with $q(x) = \|x\|$ for all $x \in \mathbb{R}^n \simeq \mathbb{R}^n + i0$[5] (cf. Exercise 2.1.14). Observe that:

- $\mathbb{L}_1 = \mathbb{D}$.
- $\mathbb{L}_2 = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1 + iz_2| < 1,\ |z_1 - iz_2| < 1\} \overset{\text{bih}}{\simeq} \mathbb{D}^2$.
- For $n \geq 2$ the ball $\mathbb{L}_n$ is not Reinhardt.

One can prove that for any $a \in \mathbb{L}_n$ there exists an $h_a \in \mathrm{Aut}(\mathbb{L}_n)$ such that $h_a(a) = 0$ (cf. [Hua 1963], p. 86–87; for $n \geq 3$ the proof is heavily "technical" and, therefore, we skip it); in the case where $n = 1$ the above mapping h_a agrees with that in (2.1.1).

Under the above notation we have:

$$\begin{aligned} \mathrm{Aut}(\mathbb{L}_n) &= \{\zeta A \circ h_a : A \in \mathbb{O}(n),\ \zeta \in \mathbb{T},\ a \in \mathbb{L}_n\}, \\ \mathrm{Aut}_0(\mathbb{L}_n) &= \{\zeta A : \zeta \in \mathbb{T},\ A \in \mathbb{O}(n)\} =: \mathfrak{G}_0, \end{aligned}$$

where $\mathbb{O}(n) :=$ the group of all orthogonal[6] isomorphisms $A\colon \mathbb{R}^n \to \mathbb{R}^n$ acting on $\mathbb{C}^n$ according to the formula $\mathbb{C}^n \ni x + iy \mapsto A(x) + iA(y) \in \mathbb{C}^n$.

---

[5] Recall that $\|\ \|$ stands for the Euclidean norm.

[6] An $\mathbb{R}$-linear mapping $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal* if one of the following equivalent conditions is satisfied:

- $\langle A(x'), A(x'')\rangle = \langle x', x''\rangle$, $x', x'' \in \mathbb{R}^n$;
- $\|A(x)\| = \|x\|$, $x \in \mathbb{R}^n$;
- $AA^t = A^tA = \mathbb{I}_n$ (A is identified with its matrix representation).

The group $\mathbb{O}(n)$ depends on $\binom{n}{2}$ real parameters.

In particular:

- the group $\mathrm{Aut}(\mathbb{L}_n)$ acts transitively on $\mathbb{L}_n$; $\mathbb{L}_n$ is homogeneous and symmetric;
- the group $\mathrm{Aut}(\mathbb{L}_n)$ depends on $\ell(n) := \binom{n}{2} + 2n + 1$ real parameters;
- the group $\mathrm{Aut}_0(\mathbb{L}_n)$ depends on $\ell_0(n) := \binom{n}{2} + 1$ real parameters.

Indeed, since $\mathfrak{G}_0$ is obviously contained in $\mathrm{Aut}_0(\mathbb{L}_n)$, we only need to prove that any automorphism $\Phi \in \mathrm{Aut}_0(\mathbb{L}_n)$ belongs to $\mathfrak{G}_0$. We already know that Φ is $\mathbb{C}$-linear and $L_n \circ \Phi \equiv L_n$ (Proposition 2.1.9). As always, we identify Φ with its matrix representation. Write $\Phi = A + iB$, where $A, B \in \mathbb{M}(n \times n, \mathbb{R})$. Then the identity $L_n \circ \Phi \equiv L_n$ implies that

$$\begin{aligned} &\|Ax - By\|^2 + \|Ay + Bx\|^2 \\ &\qquad + 2\sqrt{\|Ax - By\|^2\|Ay + Bx\|^2 - \langle Ax - By, Ay + Bx\rangle^2} \\ &\quad = \|x\|^2 + \|y\|^2 + 2\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}, \quad x + iy \in \mathbb{C}^n. \end{aligned} \tag{2.1.2}$$

Suppose that we already know that A, B are $\mathbb{R}$-linearly dependent. Then we write $A + iB = \zeta P$ with $\zeta \in \mathbb{T}$ and $P \in \mathbb{M}(n \times n, \mathbb{R})$. Putting in (2.1.2) $y = 0$, we get $\|Px\| = \|x\|$, $x \in \mathbb{R}^n$, which shows that $P \in \mathbb{O}(n)$.

Thus, it suffices to show that A, B are $\mathbb{R}$-linearly dependent. We may assume that $A \neq 0$ and $B \neq 0$. Put $U := \mathrm{Ker}\, A$, $V := \mathrm{Ker}\, B$. Suppose that U and V are non-zero. Then identity (2.1.2) implies that

$$\|Ay + Bx\|^2 = \|x\|^2 + \|y\|^2 + 2\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}, \quad (x, y) \in U \times V.$$

In particular, if $y = 0$, we get $\|Bx\| = \|x\|$, $x \in U$. Similarly, $\|Ay\| = \|y\|$, $y \in V$. Consequently,

$$\langle Ay, Bx\rangle = \sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}, \quad (x, y) \in U \times V.$$

Since the left-hand side is bilinear, we conclude that either U or V is trivial.

Suppose, for instance, that A is non-singular.

Observe that for $x + iy \neq 0$, the right-hand side of (2.1.2) is differentiable iff $\|x\|^2\|y\|^2 \neq \langle x, y\rangle^2$, i.e. x and y are $\mathbb{R}$-linearly independent. Consequently, for $x + iy \neq 0$ we get: x and y are $\mathbb{R}$-linearly dependent iff $Ax - By$ and $Ay + Bx$ are $\mathbb{R}$-linearly dependent.

Take an arbitrary $x \in (\mathbb{R}^n)_*$ and let $y := \alpha x$ with $\alpha \in \mathbb{R}$. Suppose that $Ax - By = \beta(Ay + Bx)$ for some $\beta \in \mathbb{R}$. Hence $(1 - \alpha\beta)Ax = (\alpha + \beta)Bx$, and consequently, Ax and Bx must be $\mathbb{R}$-linearly dependent.

In the remaining case we have $Bx = -\alpha Ax$.

Thus Ax and Bx are $\mathbb{R}$-linearly dependent for any $x \in \mathbb{R}^n$. Put $C := A^{-1}B$. We only need to show that there exists a $\gamma \in \mathbb{R}$ such that $C = \gamma \mathbb{I}_n$. Fix $x, y \in (\mathbb{R}^n)_*$ and let $\gamma(x), \gamma(y) \in \mathbb{R}$ be such that $Cx = \gamma(x)x$, $Cy = \gamma(y)y$. We want to prove that $\gamma(x) = \gamma(y)$. If x and y are $\mathbb{R}$-linearly independent, then $x + y \neq 0$ and $C(x+y) = \gamma(x+y)(x+y)$, which directly implies that $\gamma(x) = \gamma(y) = \gamma(x+y)$. If x and y are $\mathbb{R}$-linearly dependent, then the result is obvious.

(d) Observe that

$$\begin{gathered} d(1) = \ell(1) = b(1) = 3, \quad d(2) = \ell(2) = 6 < 8 = b(2), \\ d(n) < \ell(n) < b(n), \quad n \geq 3, \\ d_0(1) = \ell_0(1) = b_0(1) = 1, \quad d_0(2) = \ell_0(2) = 2 < 4 = b_0(2), \\ d_0(n) < \ell_0(n) < b_0(n), \quad n \geq 3. \end{gathered}$$

Consequently, it is intuitively clear that $\operatorname{Bih}(\mathbb{D}^n, \mathbb{B}_n) = \varnothing$ for $n \geq 2$ and that $\operatorname{Bih}(\mathbb{D}^n, \mathbb{L}_n) = \operatorname{Bih}(\mathbb{B}_n, \mathbb{L}_n) = \varnothing$ for $n \geq 3$.

A precise proof will be presented in Theorem 2.1.17.

Exercise 2.1.13. Let $N: \mathbb{C}^n \to \mathbb{R}_+$ be an arbitrary complex norm, $n \geq 2$, and let $\mathcal{B} := \{z \in \mathbb{C}^n : N(z) < 1\}$, $\mathcal{B}(r) := \{z \in \mathcal{B} : N(z) < r\}$ and $\mathcal{A} = \mathcal{A}(r) := \{z \in \mathcal{B} : r < N(z)\}$, $0 < r < 1$.

(a) Using Hartogs Theorem 1.9.1 prove that

$$\operatorname{Aut}(\mathcal{A}) = \{\Phi|_{\mathcal{A}} : \Phi \in \operatorname{Aut}(\mathcal{B}) : \Phi(\mathcal{B}(r)) = \mathcal{B}(r)\}.$$

(b) Using Example 2.1.12 (a), (b) prove that in the case where $N \in \{\| \,\|, \| \,\|_\infty\}$ we have

$$\operatorname{Aut}(\mathcal{A}) = \{\Phi|_{\mathcal{A}} : \Phi \in \operatorname{Aut}_0(\mathcal{B})\}$$

(notice that the above relation holds in fact for an arbitrary norm N – cf. Example 4.2.43).

Exercise 2.1.14 (Maximal norm). Let

$$\mathcal{F}_{\max} := \{q : \; q: \mathbb{C}^n \to \mathbb{R}_+ \text{ is a } \mathbb{C}\text{-norm}, \; \forall_{x \in \mathbb{R}^n} : q(x) = \|x\|\}.$$

Then

$$L_n = \sup\{q : q \in \mathcal{F}_{\max}\}. \tag{2.1.3}$$

Complete the following sketch of the proof of (2.1.3) based on [Dru 1974].

Step 1. Define

$$\| \,\|_{\max} := \sup\{q : q \in \mathcal{F}_{\max}\}.$$

Then $\| \,\|_{\max} \in \mathcal{F}_{\max}$ and

$$\|x + iy\| \leq \|z\|_{\max} \leq \|x\| + \|y\|, \quad z = x + iy \in \mathbb{C}^n.$$

Step 2. Let $\Phi(x) := \langle a, x\rangle + i\langle b, x\rangle$, $x \in \mathbb{R}^n$, where $a, b \in \mathbb{R}^n$. Then, using Lagrange's multipliers, we get

$$\begin{aligned}\|\Phi\| : &= \sup\{|\Phi(x)| : x \in \mathbb{R}^n,\ \|x\| = 1\}\\ &= \sup\{|\Phi(x)| : x \in \mathbb{R}a + \mathbb{R}b,\ \|x\| = 1\}\\ &= \Big(\frac{1}{2}\Big(A + C + \sqrt{(A-C)^2 + 4B^2}\Big)\Big)^{1/2},\end{aligned}$$

where $A := \|a\|^2$, $B := \langle a, b\rangle$, $C := \|b\|^2$.

Step 3. In virtue of the Hahn–Banach theorem, we have

$$\begin{aligned}\|z\|_{\max} &= \sup\{|\widetilde{\Phi}(z)| : \ \widetilde{\Phi} \colon \mathbb{C}^n \to \mathbb{C} \text{ is } \mathbb{C}\text{-linear},\ \|\widetilde{\Phi}|_{\mathbb{R}^n}\| \le 1\}\\ &= \sup\{|\Phi(x) + i\Phi(y)| : \Phi \colon \mathbb{R}^n \to \mathbb{C} \text{ is } \mathbb{R}\text{-linear},\ \|\Phi\| = 1\}\end{aligned}$$

for $z = x + iy \in \mathbb{C}^n$.

Step 4. Using Steps 2, 3, and the Lagrange's multipliers method, we get

$$\begin{aligned}\|z\|^2_{\max} &= \sup\{|\langle a, x\rangle + i\langle b, x\rangle + i(\langle a, y\rangle + i\langle b, y\rangle)|^2 :\\ &\qquad a, b \in \mathbb{R}^n,\ A + C + \sqrt{(A-C)^2 + 4B^2} = 2\}\\ &= \sup\{(\langle a, x\rangle - \langle b, y\rangle)^2 + (\langle a, y\rangle + \langle b, x\rangle)^2 :\\ &\qquad a, b \in \mathbb{R}x + \mathbb{R}y,\ A + C + \sqrt{(A-C)^2 + 4B^2} = 2\}\end{aligned}$$

for $z = x + iy \in \mathbb{C}^n$, where A, B, C are as in Step 2.

Step 5. The function

$$L_n(z) = (\|x\|^2 + \|y\|^2 + 2\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2})^{1/2}, \quad z = x + iy \in \mathbb{C}^n,$$

has the following properties:

- $L_n(\lambda z) = |\lambda| L_n(z)$, $z \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$,
- $L_n(x) = \|x\|$, $x \in \mathbb{R}^n$,
- if $\langle x, y\rangle = 0$, then $L_n(x + iy) = \|x\| + \|y\|$.

Step 6. Every $z = x + iy \in \mathbb{C}^n$ may be written in the form $z = \zeta(x' + iy')$, where $\zeta \in \mathbb{T}$ and $\langle x', y'\rangle = 0$.

Step 7. Using Steps 5 and 6, we conclude that the proof reduces to the equality $\|x + iy\|_{\max} = \|x\| + \|y\|$ for all $x, y \in \mathbb{R}^n$ with $\langle x, y\rangle = 0$. Fix such x, y. We may assume that x, y are linearly independent. Observe that, in fact, we have to prove that $\|x + iy\|_{\max} \ge \|x\| + \|y\|$. Define $a := x/\|x\|$, $b := -y/\|y\|$. Obviously, $a, b \in \mathbb{R}x + \mathbb{R}y$ and $A = C = 1$, $B = 0$. Thus $A + C + \sqrt{(A-C)^2 + 4B^2} = 2$. Consequently,

$$\|x + iy\|^2_{\max} \ge |\langle a, x\rangle + i\langle b, x\rangle + i(\langle a, y\rangle + i\langle b, y\rangle)|^2 = (\|x\| + \|y\|)^2.$$

The proof of (2.1.3) is completed.

Exercise 2.1.15 (Minimal norm). Let

$$\mathcal{F}_{\min} := \{q \in \mathcal{F}_{\max} : \forall_{z \in \mathbb{C}^n} : \ q(z) \le \|z\|\},$$

where $\mathcal{F}_{\max}$ is as in Exercise 2.1.14. Then the *minimal norm*

$$\|z\|_{\min} := \inf\{q : q \in \mathcal{F}_{\min}\}$$

is well defined, $\| \ \|_{\min} \in \mathcal{F}_{\min}$, and

$$\begin{aligned}\|z\|_{\min} &= \frac{1}{\sqrt{2}}\sqrt{\|z\|^2 + |\langle z, \bar{z}\rangle|^2} \\ &= \left(\frac{1}{2}\left(\|x\|^2 + \|y\|^2 + \sqrt{(\|x\|^2 - \|y\|^2)^2 + 4\langle x, y\rangle^2}\right)\right)^{1/2}, \\ &= \max\{|\langle z, a\rangle| : a \in \mathbb{R}^n, \ \|a\| = 1\}, \quad z = x + iy \in \mathbb{C}^n.\end{aligned}$$

Notice that an analogous result is true for the complexification $\mathcal{H}_{\mathbb{C}}$ of a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ – [Ava 1997].

Complete the following sketch of the proof based on [Hah-Pfl 1988].

Step 1. $\|z\|_{\min} \ge \max\{\|x\|, \ \|y\|\}$, $z = x + iy \in \mathbb{C}^n$.

Let $q \in \mathcal{F}_{\min}$, $z = x + iy$. The case where x and y are linearly dependent is elementary. Thus assume that x and y are linearly independent. Define

$$p(\xi, \eta) := q\left(\xi\frac{x}{\|x\|} + i\eta\frac{y}{\|y\|}\right), \quad (\xi, \eta) \in \mathbb{R}^2.$$

Then $p\colon \mathbb{R}^2 \to \mathbb{R}_+$ is an $\mathbb{R}$-norm, $p(\pm t, 0) = p(0, \pm t) = t$, $t > 0$, and

$$p(\xi, \eta) \le \left\|\xi\frac{x}{\|x\|} + i\eta\frac{y}{\|y\|}\right\| = \sqrt{\xi^2 + \eta^2}, \quad (\xi, \eta) \in \mathbb{R}^2.$$

In particular,

$$\mathbb{D} \subset \{(\xi, \eta) \in \mathbb{R}^2 : p(\xi, \eta) < 1\} =: B,$$
$$B \cap \{\eta = 0\} = (-1, 1) \times \{0\}, \quad B \cap \{\xi = 0\} = \{0\} \times (-1, 1).$$

Hence $B \subset (-1, 1) \times (-1, 1)$. Consequently, $p(\xi, \eta) \ge \max\{|\xi|, \ |\eta|\}$, $(\xi, \eta) \in \mathbb{R}^2$. In particular,

$$q(x + iy) = p(\|x\|, \ \|y\|) \ge \max\{\|x\|, \ \|y\|\}.$$

Step 2.

$$\|z\|_{\min} = \max\{\|x \sin\theta + y\cos\theta\| : \theta \in [0, 2\pi]\}, \quad z = x + iy \in \mathbb{C}^n.$$

The right-hand side defines a norm from the family $\mathcal{F}_{\min}$, which gives the inequality '$\le$'. To prove the opposite inequality, take any $q \in \mathcal{F}_{\min}$. Then, by Step 1, we get

$$q(x+iy) = q(e^{i\theta}(x+iy)) = q(x\cos\theta - y\sin\theta + i(x\sin\theta + y\cos\theta)) \\ \ge \|x\sin\theta + y\cos\theta\|.$$

Step 3. By Step 2,

$$\|z\|_{\min}^2 = \min\{\|y\|^2 + (\|x\|^2 - \|y\|^2)\sin^2\theta + \langle x, y\rangle \sin(2\theta) : \theta \in [0, 2\pi]\} \\ = \frac{1}{2}\Big(\|x\|^2 + \|y\|^2 + \sqrt{(\|x\|^2 - \|y\|^2)^2 + 4\langle x, y\rangle^2}\Big), \ z = x + iy \in \mathbb{C}^n.$$

Step 4. Using Lagrange's multipliers gives

$$\max\{|\langle z, a\rangle| : a \in \mathbb{R}^n,\ \|a\| = 1\} = \|z\|_{\min}, \quad z = x + iy \in \mathbb{C}^n.$$

Remark 2.1.16. Notice that:

- $\max\{\|x\|, \|y\|\} \le \|z\|_{\min} \le \|z\|$, $z = x + iy \in \mathbb{C}^n$.
- $\|z\|_{\min} = \|z\|$ iff x, y are linearly dependent. In particular, if $n = 1$, then $\|z\|_{\min} = |z|$, $z \in \mathbb{C}$.
- $\|z\|_{\min} = \max\{\|x\|, \|y\|\}$ iff $\langle x, y\rangle = 0$.
- $\inf\{q : q \in \mathcal{F}_{\max}\}$ is not a norm.

Indeed, let $q_\varepsilon(z) := (1-\varepsilon)|z_1 + iz_2| + \varepsilon|z_1 - iz_2|$, $z = (z_1, z_2) \in \mathbb{C}^2$. Then $q_\varepsilon \in \mathcal{F}_{\max}$, but $q_\varepsilon(1, i) = 2\varepsilon \to 0$ (cf. [Hah-Pfl 1988]).

- The *minimal ball* $\mathbb{M}_n := \{z \in \mathbb{C}^n : \|z\|_{\min} < 1\}$ is not a Reinhardt domain for $n \ge 2$.
- If $n = 2$, then

$$\|z\|_{\min} = \left|\frac{z_1 - iz_2}{2}\right| + \left|\frac{z_1 + iz_2}{2}\right|, \quad z = (z_1, z_2) \in \mathbb{C}^2.$$

In particular, $\mathbb{M}_2$ is biholomorphic to the domain $\{w \in \mathbb{C}^2 : |w_1| + |w_2| < 1\}$.

- $\operatorname{Aut}(\mathbb{M}_n) = \{\zeta A : \zeta \in \mathbb{T},\ A \in \mathbb{O}(n)\}$ (cf. [Kim 1991], [Zwo 1996]). In particular, $\mathbb{M}_n$ is neither homogeneous nor symmetric. The group depends on $\binom{n}{2} + 1$ real parameters. Consequently, $\operatorname{Bih}(\mathbb{M}_n, D) = \varnothing$ for $D \in \{\mathbb{D}^n, \mathbb{B}_n, \mathbb{L}_n\}$, $n \ge 2$.

The following theorem is a generalization of the famous *Poincaré theorem* saying that $\operatorname{Bih}(\mathbb{B}_n, \mathbb{D}^n) = \varnothing$ for $n \ge 2$.

Theorem 2.1.17. *Let* $2 \le n = n_1 + \dots + n_k = m_1 + \dots + m_\ell$, $B_\mu \in \{\mathbb{B}_{n_\mu}, \mathbb{L}_{n_\mu}\}$, $\mu = 1, \dots, k$, $B'_\nu \in \{\mathbb{B}_{m_\nu}, \mathbb{L}_{m_\nu}\}$, $\nu = 1, \dots, \ell$. *Assume that if* $B_\mu = \mathbb{L}_{n_\mu}$ (*resp.* $B'_\nu = \mathbb{L}_{m_\nu}$), *then* $n_\mu \ge 3$ (*resp.* $m_\nu \ge 3$) *– cf. Example* 2.1.12 (c). *Then*

$$\operatorname{Bih}(B_1 \times \dots \times B_k, B'_1 \times \dots \times B'_\ell) \ne \varnothing$$

iff $\ell = k$ *and there exists a permutation* $\sigma \in \mathfrak{S}_k$ *such that* $m_{\sigma(\mu)} = n_\mu$ *and* $B'_{\sigma(\mu)} = B_\mu$, $\mu = 1, \dots, k$.

Moreover, every biholomorphic mapping $F\colon B_1 \times \dots \times B_k \to B'_1 \times \dots \times B'_k$ *is, up to a permutation of* $B'_1, \dots, B'_k$, *of the form*

$$F(z) = (F_1(z_1), \dots, F_k(z_k)), \quad z = (z_1, \dots, z_k) \in B_1 \times \dots \times B_k,$$

where $F_\mu \in \operatorname{Aut}(B_\mu)$, $\mu = 1, \dots, k$.

Remark 2.1.18. (a) In the case where $B_\mu = \mathbb{B}_{n_\mu}$, $\mu = 1, \dots, k$, $B'_\nu = \mathbb{B}_{m_\nu}$, $\nu = 1, \dots, \ell$, Theorem 2.1.17 shows that $\operatorname{Bih}(\mathbb{B}_{n_1} \times \dots \times \mathbb{B}_{n_k}, \mathbb{B}_{m_1} \times \dots \times \mathbb{B}_{m_\ell}) \neq \varnothing$ iff $\ell = k$ and there exists a permutation $\sigma \in \mathfrak{S}_k$ with $m_{\sigma(\mu)} = n_\mu$, $\mu = 1, \dots, k$.

(b) In particular, in the case where $k = 1$, $B_1 = \mathbb{B}_n$, $\ell = n \geq 2$, Theorem 2.1.17 reduces to the Poincaré theorem.

(c) In the case $k = \ell = n$, Theorem 2.1.17 reduces to the description of $\operatorname{Aut}(\mathbb{D}^n)$ given in Example 2.1.12 (a).

(d) In the case $k = 1$, $B_1 = \mathbb{L}_n$, $B'_\nu = \mathbb{B}_{m_\nu}$, $\nu = 1, \dots, \ell$, Theorem 2.1.17 shows that $\operatorname{Bih}(\mathbb{L}_n, \mathbb{B}_{m_1} \times \dots \times \mathbb{B}_{m_\ell}) = \varnothing$ for $n \geq 3$ (cf. Example 2.1.12 (c)).

Proof of Theorem 2.1.17. (The main idea of the proof is due to W. Jarnicki.) Since B_μ is homogeneous, $\mu = 1, \dots, k$, Remark 2.1.2 (c) implies that the domain $B_1 \times \dots \times B_k$ is also homogeneous. Now, by Corollary 2.1.10, $\operatorname{Bih}(B_1 \times \dots \times B_k, B'_1 \times \dots \times B'_\ell) \neq \varnothing$ iff there exists a $\mathbb{C}$-linear isomorphism $F = (F_1, \dots, F_\ell)\colon \mathbb{C}^n \to \mathbb{C}^n = \mathbb{C}^{m_1} \times \dots \times \mathbb{C}^{m_\ell}$ such that

$$\max\{\|F_1(z)\|'_1, \dots, \|F_\ell(z)\|'_\ell\} = \max\{\|z_1\|_1, \dots, \|z_k\|_k\},$$
$$z = (z_1, \dots, z_k) \in \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_k},$$

where

$$\| \; \|_\mu := \begin{cases} \| \; \| = \text{Euclidean norm} & \text{if } B_\mu = \mathbb{B}_{n_\mu}, \\ L_{n_\mu} = \text{Lie norm} & \text{if } B_\mu = \mathbb{L}_{n_\mu}, \end{cases} \quad \mu = 1, \dots, k,$$

$$\| \; \|'_\nu := \begin{cases} \| \; \| = \text{Euclidean norm} & \text{if } B'_\nu = \mathbb{B}_{m_\nu}, \\ L_{m_\nu} = \text{Lie norm} & \text{if } B'_\nu = \mathbb{L}_{m_\nu}, \end{cases} \quad \nu = 1, \dots, \ell.$$

First consider the set $A_p \subset \mathbb{C}^p$, $p \geq 3$, on which the Lie norm L_p is not real analytic, i.e.

$$\begin{aligned} A_p &:= \{w \in \mathbb{C}^p : \|w\|^4 = |\langle w, \bar{w}\rangle|^2\} \\ &= \{w = (w_1, \dots, w_p) \in \mathbb{C}^p : w_i \bar{w}_j \in \mathbb{R},\ i, j = 1, \dots, p\} \quad (\textsc{Exercise}). \end{aligned}$$

Observe that A_p is closed and $(A_p)_* = \bigcup_{i=1}^p M_i$, where $M_i := \psi_i(\mathbb{C}_* \times \mathbb{R}^{p-1})$,

$$\psi_i(\zeta, t_1, \dots, t_{p-1}) := (t_1/\bar{\zeta}, \dots, t_{i-1}/\bar{\zeta}, \zeta, t_i/\bar{\zeta}, \dots, t_p/\bar{\zeta});$$

ψ_i is a real analytic mapping. In particular, $\mathcal{H}^{p+1}(\psi_i(K)) < +\infty$ for every compact $K \subset \mathbb{C}_* \times \mathbb{R}^{p-1}$, where $\mathcal{H}^{p+1}$ denotes the $(p+1)$-Hausdorff measure in $\mathbb{R}^{2p}$ (Exercise). Note that $p+1 < 2p-1$. Consequently, A_p is a countable union of compact sets with finite $(p+1)$-dimensional Hausdorff measure ([Fed 1969], § 2.10) and therefore $\mathbb{C}^p \setminus A_p$ is connected (see [Rud 1980], Theorem 14.4.5, [Jar-Pfl 2000], p. 226).

Now, let $C := \{(z,w) \in \mathbb{C}^p \times \mathbb{C}^q : N_1(z) = N_2(w)\}$, where N_1 (resp. N_2) stands for the Euclidean or Lie norm in $\mathbb{C}^p$ (resp. $\mathbb{C}^q$). If N_1 (resp. N_2) is the Lie norm, then we assume that $p \geq 3$ (resp. $q \geq 3$). Then C is nowhere dense.

Indeed, define $S_1 \subset \mathbb{C}^p$, $S_2 \subset \mathbb{C}^q$,

$$S_1 := \begin{cases} \{0\} & \text{if } N_1 = \|\ \|, \\ A_p & \text{if } N_1 = L_p, \end{cases} \qquad S_2 := \begin{cases} \{0\} & \text{if } N_2 = \|\ \|, \\ A_q & \text{if } N_2 = L_q. \end{cases}$$

Then $S := (S_1 \times \mathbb{C}^q) \cup (\mathbb{C}^p \times S_2)$ is a closed set being a countable union of compact sets with finite t-dimensional Hausdorff measure where $t < 2(p+q)-1$. Hence $\mathbb{C}^p \times \mathbb{C}^q \setminus S$ is connected. Suppose that $\operatorname{int} C \neq \varnothing$. Then, by the identity principle for real analytic functions, $\mathbb{C}^p \times \mathbb{C}^q \setminus S \subset C$. Therefore, by continuity, $C = \mathbb{C}^p \times \mathbb{C}^q$; a contradiction.

Thus, for every $\mu' \neq \mu''$ and $\nu' \neq \nu''$, the sets

$$\{(z_1,\dots,z_k) \in \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_k} : \|z_{\mu'}\|_{\mu'} = \|z_{\mu''}\|_{\mu''}\},$$
$$\{(w_1,\dots,w_\ell) \in \mathbb{C}^{m_1} \times \dots \times \mathbb{C}^{m_\ell} : \|w_{\nu'}\|'_{\nu'} = \|w_{\nu''}\|'_{\nu''}\}$$

are nowhere dense. Consequently, since F is homeomorphic, the set

$$\bigcup_{\nu' \neq \nu''} \{z \in \mathbb{C}^n : \|F_{\nu'}(z)\|'_{\nu'} = \|F_{\nu''}(z)\|'_{\nu''}\} \cup \bigcup_{\mu' \neq \mu''} \{z \in \mathbb{C}^n : \|z_{\mu'}\|_{\mu'} = \|z_{\mu''}\|_{\mu''}\}$$

is nowhere dense. In particular, for every $j \in \{1,\dots,k\}$ there exist a non-empty open set $\Omega_j \subset \mathbb{C}^n$ and an $s \in \{1,\dots,\ell\}$ such that

$$\|F_s(z)\|'_s = \|z_j\|_j, \quad z \in \Omega_j.$$

Let

$$T_1 := \begin{cases} \{0\} & \text{if } \|\ \|_j = \|\ \|, \\ A_{n_j} & \text{if } \|\ \|_j = L_{n_j}, \end{cases} \qquad T_2 := \begin{cases} \{0\} & \text{if } \|\ \|'_s = \|\ \|, \\ A_{m_s} & \text{if } \|\ \|'_s = L_{m_s}. \end{cases}$$

Then

$$\begin{aligned} T := {}& (\mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_{j-1}} \times T_1 \times \mathbb{C}^{n_{j+1}} \times \dots \times \mathbb{C}^{n_k}) \\ & \cup F^{-1}(\mathbb{C}^{m_1} \times \dots \times \mathbb{C}^{m_{s-1}} \times T_2 \times \mathbb{C}^{m_{s+1}} \times \dots \times \mathbb{C}^{m_\ell}) \end{aligned}$$

is a closed set being a countable union of compact sets with finite t-dimensional Hausdorff measure where $t < 2n - 1$. Hence $\mathbb{C}^n \setminus S$ is connected and, by the identity principle for real analytic functions, we conclude that

$$\|F_s(z)\|'_s = \|z_j\|_j, \quad z \in \mathbb{C}^n. \tag{2.1.4}$$

In particular, $s =: \sigma(j)$ is uniquely determined. Moreover, $\sigma(j') \neq \sigma(j'')$ for $j' \neq j''$. Hence $k \leq \ell$.

Since the Euclidean norm in $\mathbb{C}^p$ is real analytic on $(\mathbb{C}^p)_*$, but the Lie norm is real analytic only on $\mathbb{C}^p \setminus A_p$, we conclude that both norms $\| \ \|_j$ and $\| \ \|'_s$ must be of the same type (i.e. both must be Euclidean or both must be Lie). Moreover, $n - m_s = \dim \operatorname{Ker} F_s = n - n_j$, which implies that $m_s = n_j$ and $B'_s = B_j$. It is also clear that F_s depends only on z_j, i.e. $F_s(z) = U_j(z_j)$, where $U_j : \mathbb{C}^{n_j} \to \mathbb{C}^{n_j}$ is a linear isomorphism. Condition (2.1.4) guarantees that $U_j \in \operatorname{Aut}(B_j)$. Finally, $k = \ell$ because $m_{\sigma(1)} + \cdots + m_{\sigma(k)} = n_1 + \cdots + n_k = n$. □

Exercise 2.1.19. Let $B_1, \dots, B_k$ be as in Theorem 2.1.17. Find a generalization of Example 2.1.12 (a) and characterize the group $\operatorname{Aut}(B_1 \times \cdots \times B_k)$ in terms of $\operatorname{Aut}(B_1), \dots, \operatorname{Aut}(B_k)$.

The phenomenon described in Theorem 2.1.17 appears also under other assumptions. Recall, for example, the following classical general result.

Theorem* 2.1.20 ([Nar 1971], p. 77). *Let D_j, G_j be bounded domains in $\mathbb{C}^{n_j}$ such that there is no non-constant holomorphic curve $\varphi\colon \mathbb{D} \to \partial G_j$, $j = 1, \dots, k$. Then any biholomorphic mapping*

$$\Psi\colon D_1 \times \cdots \times D_k \to G_1 \times \cdots \times G_k$$

is, up to a permutation of $G_1, \dots, G_k$, of the form

$$\Psi(z_1, \dots, z_k) = (\widetilde{\Psi}_1(z_1), \dots, \widetilde{\Psi}_k(z_k)), \quad (z_1, \dots, z_k) \in D_1 \times \cdots \times D_k,$$

where $\widetilde{\Psi}_j \in \operatorname{Bih}(D_j, G_j)$, $j = 1, \dots, k$.

Notice that the above theorem applies for instance to complex ellipsoids (in particular, to Euclidean balls), which is a direct consequence of the following lemma.

Lemma 2.1.21. *If $p \in \mathbb{N}^n$, then there is no non-constant holomorphic curve $\varphi\colon \mathbb{D} \to \partial\mathbb{E}_p$.*

Proof. Suppose that $\varphi = (\varphi_1, \dots, \varphi_n)\colon \mathbb{D} \to \partial\mathbb{E}_p$ is holomorphic.

First consider the case $p = \mathbf{1}$. Then $\|\varphi\| \equiv 1$. Obviously, $|\varphi_j| \leq 1$, $j = 1, \dots, n$. Composing φ with a rotation we may assume that $\varphi(0) = (1, 0, \dots, 0)$. Then, by the maximum principle, $\varphi_1 \equiv 1$. Consequently, $\varphi_j \equiv 0$, $j = 2, \dots, n$.

For arbitrary p we only need to observe that $(\varphi_1^{p_1}, \dots, \varphi_n^{p_n})\colon \mathbb{D} \to \partial\mathbb{B}_n$. □

Remark 2.1.22. On the other hand, the Lie ball $\mathbb{L}_n$, $n \geq 2$, does not satisfy the condition from Theorem 2.1.20.

Indeed,
$$\mathbb{D} \ni \lambda \mapsto (\tfrac{1}{2}(\lambda + 1), \tfrac{1}{2i}(\lambda - 1), 0, \dots, 0) \in \partial\mathbb{L}_n.$$
In particular, Theorem 2.1.20 does not imply Theorem 2.1.17.

2.2* Cartan theory

Summarizing, the previous results show that for $n \leq 3$ we have the following bounded homogeneous domains (which are not biholomorphically equivalent).

n	domain
1	$\mathbb{D}$
2	$\mathbb{B}_2$, $\mathbb{D}^2$
3	$\mathbb{B}_3$, $\mathbb{D}^3$, $\mathbb{D} \times \mathbb{B}_2$, $\mathbb{L}_3$

Theorem* 2.2.1 ([Car 1935]). *If $n \leq 3$, then any bounded homogeneous domain $G \subset \mathbb{C}^n$ is biholomorphic to one of the above canonical homogeneous domains. In particular, any bounded homogeneous domain $G \subset \mathbb{C}^n$ with $n \leq 3$ is symmetric (Remark 2.1.2 (c), Example 2.1.12).*

The first example of a 4-dimensional homogeneous non-symmetric bounded domain was given by I. Piatetsky-Shapiro in [Pia-Sha 1959].

Theorem* 2.2.2 ([Car 1935]). *Every bounded symmetric domain is homogeneous. Moreover, every bounded symmetric domain $G \subset \mathbb{C}^n$ is biholomorphic to a Cartesian product of canonical symmetric domains belonging to the following six* Cartan types*:*

- $n = pq$, $1 \leq p \leq q$, $\boldsymbol{I}_{p,q} := \{Z \in \mathbb{M}(p \times q, \mathbb{C}) : \mathbb{I}_p - ZZ^* > 0\}$;[7] *observe that* $\boldsymbol{I}_{1,n} \simeq \mathbb{B}_n$;
- $n = \binom{p}{2}$, $p \geq 2$, $\boldsymbol{II}_p := \{Z \in \mathbb{M}(p \times p, \mathbb{C}) : Z^t = -Z,\ \mathbb{I}_p - ZZ^* > 0\}$;
- $n = \binom{p+1}{2}$, $p \geq 1$, $\boldsymbol{III}_p := \{Z \in \mathbb{M}(p \times p, \mathbb{C}) : Z^t = Z,\ \mathbb{I}_p - ZZ^* > 0\}$;
- $\boldsymbol{IV}_n := \mathbb{L}_n$.

The domains of types $\boldsymbol{I}$–$\boldsymbol{IV}$ are called classical. *They are balanced – cf. Definition 1.4.14.*

- $n = 16$, *an exceptional domain* $\boldsymbol{V}_{16}$;
- $n = 27$, *an exceptional domain* $\boldsymbol{VI}_{27}$.

[7] For $A \in \mathbb{M}(m \times m, \mathbb{C})$, "$A > 0$" means that A is *positive definite*, i.e. $X^t A \overline{X} = \sum_{j,k=1}^{m} a_{j,k} X_j \overline{X}_k > 0$ for all $X \in (\mathbb{C}^m)_*$.

The above Cartan domains are not biholomorphically equivalent except for the following cases:

(a) $\boldsymbol{I}_{2,2} \overset{\text{bih}}{\simeq} \mathbb{L}_4$,

(b) $\boldsymbol{II}_2 \simeq \mathbb{D}$, $\boldsymbol{II}_3 \overset{\text{bih}}{\simeq} \mathbb{B}_3$, $\boldsymbol{II}_4 \overset{\text{bih}}{\simeq} \mathbb{L}_6$; *thus type* $\boldsymbol{II}$ *is essential only for* $p \geq 5$,

(c) $\boldsymbol{III}_1 \simeq \mathbb{D}$, $\boldsymbol{III}_2 \overset{\text{bih}}{\simeq} \mathbb{L}_3$; *thus type* $\boldsymbol{III}$ *is essential only for* $p \geq 2$,

(d) $\mathbb{L}_1 = \mathbb{D}$, $\mathbb{L}_2 \overset{\text{bih}}{\simeq} \mathbb{D}^2$; *thus type* $\boldsymbol{IV}$ *is essential only for* $n \geq 5$.

Let $\psi(n)$ *denote the number of biholomorphically non-equivalent canonical bounded symmetric domains* $G \subset \mathbb{C}^n$ *(from the above list) and let* $\Psi(n)$ *be the number of non-equivalent bounded symmetric domains* $G \subset \mathbb{C}^n$ *(which are biholomorphic to Cartesian products of canonical symmetric domains). The following table describes the situation for* $1 \leq n \leq 30$.

n	$\psi(n)$	$\boldsymbol{I}$	$\boldsymbol{II}$	$\boldsymbol{III}$	$\boldsymbol{IV}$	$\boldsymbol{V}/\boldsymbol{VI}$	$\Psi(n)$
1	1	$\mathbb{D}$	$(\boldsymbol{II}_2 \simeq \mathbb{D})$	$(\boldsymbol{III}_1 = \mathbb{D})$	$(\mathbb{L}_1 = \mathbb{D})$		1
2	1	$\mathbb{B}_2$			$(\mathbb{L}_2 \simeq \mathbb{D}^2)$		2
3	2	$\mathbb{B}_3$	$(\boldsymbol{II}_3 \simeq \mathbb{B}_3)$	$\boldsymbol{III}_2$	$(\mathbb{L}_3 \simeq \boldsymbol{III}_2)$		4
4	2	$\mathbb{B}_4, \boldsymbol{I}_{2,2}$			$(\mathbb{L}_4 \simeq \boldsymbol{I}_{2,2})$		7
5	2	$\mathbb{B}_5$			$\mathbb{L}_5$		11
6	4	$\mathbb{B}_6, \boldsymbol{I}_{2,3}$	$(\boldsymbol{II}_4 \simeq \mathbb{L}_6)$	$\boldsymbol{III}_3$	$\mathbb{L}_6$		21
7	2	$\mathbb{B}_7$			$\mathbb{L}_7$		31
8	3	$\mathbb{B}_8, \boldsymbol{I}_{2,4}$			$\mathbb{L}_8$		51
9	3	$\mathbb{B}_9, \boldsymbol{I}_{3,3}$			$\mathbb{L}_9$		80
10	5	$\mathbb{B}_{10}, \boldsymbol{I}_{2,5}$	$\boldsymbol{II}_5$	$\boldsymbol{III}_4$	$\mathbb{L}_{10}$		126
11	2	$\mathbb{B}_{11}$			$\mathbb{L}_{11}$		187
12	4	$\mathbb{B}_{12}, \boldsymbol{I}_{2,6}, \boldsymbol{I}_{3,4}$			$\mathbb{L}_{12}$		292
13	2	$\mathbb{B}_{13}$			$\mathbb{L}_{13}$		427
14	3	$\mathbb{B}_{14}, \boldsymbol{I}_{2,7}$			$\mathbb{L}_{14}$		638
15	5	$\mathbb{B}_{15}, \boldsymbol{I}_{3,5}$	$\boldsymbol{II}_6$	$\boldsymbol{III}_5$	$\mathbb{L}_{15}$		935
16	5	$\mathbb{B}_{16}, \boldsymbol{I}_{2,8}, \boldsymbol{I}_{4,4}$			$\mathbb{L}_{16}$	$\boldsymbol{V}_{16}$	1371
17	2	$\mathbb{B}_{17}$			$\mathbb{L}_{17}$		1960
18	4	$\mathbb{B}_{18}, \boldsymbol{I}_{2,9}, \boldsymbol{I}_{3,6}$			$\mathbb{L}_{18}$		2843
19	2	$\mathbb{B}_{19}$			$\mathbb{L}_{19}$		4024
20	4	$\mathbb{B}_{20}, \boldsymbol{I}_{2,10}, \boldsymbol{I}_{4,5}$			$\mathbb{L}_{20}$		5724
21	5	$\mathbb{B}_{21}, \boldsymbol{I}_{3,7}$	$\boldsymbol{II}_7$	$\boldsymbol{III}_6$	$\mathbb{L}_{21}$		8046
22	3	$\mathbb{B}_{22}, \boldsymbol{I}_{2,11}$			$\mathbb{L}_{22}$		11303
23	2	$\mathbb{B}_{23}$			$\mathbb{L}_{23}$		15687

n	$\psi(n)$	$\boldsymbol{I}$	$\boldsymbol{II}$	$\boldsymbol{III}$	$\boldsymbol{IV}$	$\boldsymbol{V}/\boldsymbol{VI}$	$\Psi(n)$
24	5	$\mathbb{B}_{24}, \boldsymbol{I}_{2,12}, \boldsymbol{I}_{3,8}, \boldsymbol{I}_{4,6}$			$\mathbb{L}_{24}$		21840
25	3	$\mathbb{B}_{25}, \boldsymbol{I}_{5,5}$			$\mathbb{L}_{25}$		30058
26	3	$\mathbb{B}_{26}, \boldsymbol{I}_{2,13}$			$\mathbb{L}_{26}$		41366
27	4	$\mathbb{B}_{27}, \boldsymbol{I}_{3,9}$			$\mathbb{L}_{27}$	$\boldsymbol{VI}_{27}$	56525
28	6	$\mathbb{B}_{28}, \boldsymbol{I}_{2,14}, \boldsymbol{I}_{4,7}$	$\boldsymbol{II}_8$	$\boldsymbol{III}_7$	$\mathbb{L}_{28}$		77126
29	2	$\mathbb{B}_{29}$			$\mathbb{L}_{29}$		104490
30	5	$\mathbb{B}_{30}, \boldsymbol{I}_{2,15}, \boldsymbol{I}_{3,10}, \boldsymbol{I}_{5,6}$			$\mathbb{L}_{30}$		141526

Remark 2.2.3. The biholomorphisms in (a)–(c) are given by the following formulas:

(a)

$$\mathbb{L}_4 \ni (z_1, z_2, z_3, z_4) \mapsto \begin{bmatrix} z_1 + iz_4 & iz_2 + z_3 \\ iz_2 - z_3 & z_1 - iz_4 \end{bmatrix} \in \boldsymbol{I}_{2,2} \quad (\textsc{Exercise}).$$

(b)

$$\mathbb{B}_3 \ni (z_1, z_2, z_3) \mapsto \begin{bmatrix} 0 & z_1 & z_2 \\ -z_1 & 0 & z_3 \\ -z_2 & -z_3 & 0 \end{bmatrix} \in \boldsymbol{II}_3 \quad (\textsc{Exercise}),$$

and (cf. [Mor 1956])

$$\mathbb{L}_6 \ni (z_1, \dots, z_6) \mapsto \begin{bmatrix} 0 & z_1 + iz_2 & z_3 + iz_4 & z_5 + iz_6 \\ -(z_1 + iz_2) & 0 & z_5 - iz_6 & -z_3 + iz_4 \\ -(z_3 + iz_4) & -(z_5 - iz_6) & 0 & z_1 - iz_2 \\ -(z_5 + iz_6) & -(-z_3 + iz_4) & -(z_1 - iz_2) & 0 \end{bmatrix} \in \boldsymbol{II}_4.$$

(c)

$$\mathbb{L}_3 \ni (z_1, z_2, z_3) \mapsto \begin{bmatrix} z_1 + iz_3 & z_2 \\ z_2 & z_1 - iz_3 \end{bmatrix} \in \boldsymbol{III}_2.$$

Exercise 2.2.4. Find a formula for $\Psi(n)$.

Remark 2.2.5. Let us mention the following two results related to various characterizations of a bounded domain $D \subset \mathbb{C}^n$ by its automorphism group $\operatorname{Aut}(D)$.

(a) Assume that $b \in \partial D$ is a point such that ∂D is strongly pseudoconvex at b (such a point always exists if $\partial D \in \mathcal{C}^2$). Moreover, assume that there exist a compact $K \subset D$ and sequences $(a_k)_{k=1}^\infty \subset K$, $(\Phi_k)_{k=1}^\infty \subset \operatorname{Aut}(D)$ such that $\Phi_k(a_k) \to b$. Then $D \overset{\text{bih}}{\simeq} \mathbb{B}_n$ ([Ros 1979]).

(b) We say that a bounded domain $D \subset \mathbb{C}^n$ has *piecewise* $\mathcal{C}^k$*-boundary* if ∂D is a topological $(2n-1)$-dimensional manifold, and there exist an open neighborhood U of ∂D and $\rho \in \mathcal{C}^k(U, \mathbb{R}^m)$ (with some m) such that $\rho_1 \cdots \rho_m = 0$ on ∂D and for every $1 \le j_1 < \cdots < j_\ell \le m$ $(1 \le \ell \le m)$ we have $d\rho_{j_1} \wedge \cdots \wedge d\rho_{j_\ell}(z) \neq 0$ for all $z \in U$ with $\rho_{j_1}(z) = \cdots = \rho_{j_\ell}(z) = 0$.

Let $D \subset \mathbb{C}^n$ be a bounded homogeneous domain with piecewise $\mathcal{C}^2$-boundary. Then $D \overset{\text{bih}}{\simeq} \mathbb{B}_{n_1} \times \cdots \times \mathbb{B}_{n_p}$ ([Pin 1982]). In particular, every bounded homogeneous domain $D \subset \mathbb{C}^n$ with smooth boundary is biholomorphically equivalent to $\mathbb{B}_n$.

(c) In virtue of Theorem 2.1.17 (and Remark 2.1.18), the above result implies that the boundary of $\mathbb{L}_n$ ($n \geq 3$) is not piecewise $\mathcal{C}^2$.

Exercise* 2.2.6. Prove directly (without using Remark 2.2.5 (c)) that the boundary of $\mathbb{L}_n$ ($n \geq 3$) is not piecewise $\mathcal{C}^2$.

2.3 Biholomorphisms of bounded complete Reinhardt domains in $\mathbb{C}^2$

Let us look more thoroughly into the problem of biholomorphic classification of bounded Reinhardt domains $D \subset \mathbb{C}^2$ with $V_j \cap D \neq \varnothing$, $j = 1, 2$. The case where $D_1, D_2 \subset \mathbb{C}^2$ are bounded convex complete Reinhardt domains was first considered by K. Reinhardt in [Rei 1921]. The general case was completely solved by P. Thullen in [Thu 1931] (see also [Car 1931]).

Observe that, by Proposition 1.12.8, we may always assume that D_1, D_2 are bounded complete Reinhardt domains of holomorphy. By rescaling variables, we may reduce the situation to the case where D_j is *normalized*, i.e.

$$\{z \in \mathbb{C} : (z_1, 0) \in D_j\} = \{z_2 \in \mathbb{C} : (0, z_2) \in D_j\} = \mathbb{D}, \quad j = 1, 2. \tag{2.3.1}$$

In particular, $D \subset \mathbb{D}^2$.

Lemma 2.3.1. *Let $D \subset \mathbb{C}^2$ be a normalized complete Reinhardt domain of holomorphy. Then*

$$D = \{(z_1, z_2) \in \mathbb{D}^2 : |z_1| < R(|z_2|)\},$$

where $R = R_D : [0, 1) \to (0, 1]$ is a continuous function with $R(0) = 1$.

Proof. Since $\log D$ is convex, we conclude (Exercise) that for every $u \in (0, 1)$ there exists exactly one $t =: R(u) \in (0, 1)$ such that $(t, u) \in \partial \boldsymbol{R}(D)$ (recall that $\boldsymbol{R}(D) := \{(|z_1|, |z_2|) : (z_1, z_2) \in D\}$). It remains to observe that the function $R : [0, 1) \to (0, 1]$ is continuous (Exercise). $\square$

Example 2.3.2. Let

$$\mathbb{E}_p = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2p_1} + |z_2|^{2p_2} < 1\}, \quad p = (p_1, p_2) \in \mathbb{R}^2_{>0},$$

be a complex ellipsoid; cf. (1.18.5). If ($p_1 = 1$, $p_2 \neq 1$) or ($p_1 \neq 1$, $p_2 = 1$), then $\mathbb{E}_p$ is traditionally called a *Thullen domain*.

The domain $\mathbb{E}_p$ is a normalized complete Reinhardt domain of holomorphy (cf. Exercise 1.18.7). Notice that

$$R_{\mathbb{E}_p}(t) := (1 - t^{2p_2})^{1/(2p_1)}, \quad t \in [0, 1).$$

Exercise 2.3.3. Determine the function R_D for the domain

$$D := \{(z_1, z_2) \in \mathbb{D}^2 : |z_1|^{\alpha_1}|z_2|^{\alpha_2} < \theta\},$$

where $\alpha_1, \alpha_2 > 0$, $0 < \theta < 1$.

Recall that for $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$, we have put $\boldsymbol{T}_\zeta(z) = \zeta \cdot z = (\zeta_1 z_1, \zeta_2 z_2)$, $z = (z_1, z_2) \in \mathbb{C}^2$. Let $\boldsymbol{S}(z_1, z_2) := (z_2, z_1)$, $\boldsymbol{T}_\zeta^* := \boldsymbol{T}_\zeta \circ \boldsymbol{S}$.

The following three results due to P. Thullen [Thu 1931] give the full characterization of biholomorphic equivalence of bounded normalized complete Reinhardt domains of holomorphy in $\mathbb{C}^2$.

Theorem 2.3.4. *Let $\alpha > 0$, $\alpha \neq 1$. Then the group* $\operatorname{Aut}(\mathbb{E}_{(\alpha,1)})$ *coincides with the set of all mappings of the form*

$$\mathbb{E}_{(\alpha,1)} \ni z \overset{\Psi_{c,\zeta}}{\longmapsto} \left(\zeta_1 z_1 \left(\frac{1 - |c|^2}{(1 - \bar{c} z_2)^2}\right)^{\frac{1}{2\alpha}}, \zeta_2 h_c(z_2)\right) \in \mathbb{E}_{(\alpha,1)}, \tag{2.3.2}$$

where $c \in \mathbb{D}$, $(\zeta_1, \zeta_2) \in \mathbb{T}^2$, and the branch of the power $(\ )^{1/(2\alpha)}$ may be arbitrarily chosen.

In particular, $\mathbb{E}_{(\alpha,1)}$, $\alpha \neq 1$, is not homogeneous. Observe that $\operatorname{Aut}(\mathbb{E}_{(\alpha,1)})$ $(\alpha \neq 1)$ depends on four real parameters (cf. Example 2.1.12 (d)).

Theorem 2.3.5. *Let $D \subset \mathbb{C}^2$ be a normalized bounded complete Reinhardt domain of holomorphy.*

(a) *The following conditions are equivalent:*

(i) $\operatorname{Aut}(D)$ *acts transitively on D;*

(ii) *either $D = \mathbb{D}^2$ or $D = \mathbb{B}_2$ (cf. Example* 2.1.12 (a), (b), (d), *Theorem* 2.1.17).

(b) *Assume that $D \notin \{\mathbb{D}^2, \mathbb{B}_2\}$. Then the following conditions are equivalent:*

(i) *there exist $b \in \mathbb{D}_*$ and $\Phi_b \in \operatorname{Aut}(D)$ such that*

$$\Phi_b(z) = (z_1 f_b(z_2), m_b(z_2)), \quad z = (z_1, z_2) \in D,$$

where $f_b \in \mathcal{O}^*(\mathbb{D})$[8] *and $m_b \in \operatorname{Aut}(\mathbb{D})$, $m_b(b) = 0$ (note that $\Phi_b(0, b) = (0, 0)$);*

(ii) *for every $c \in \mathbb{D}$ there exists a $\Phi_c \in \operatorname{Aut}(D)$ such that*

$$\Phi_c(z) = (z_1 f_c(z_2), m_c(z_2)), \quad z = (z_1, z_2) \in D,$$

where $f_c \in \mathcal{O}^*(\mathbb{D})$ and $m_c \in \operatorname{Aut}(\mathbb{D})$, $m_c(c) = 0$ ($\Phi_c(0, c) = (0, 0)$);

[8] $\mathcal{O}^*(G) := \{f \in \mathcal{O}(G) : f(z) \neq 0,\ z \in G\}$.

(iii) $D = \mathbb{E}_{(\alpha,1)}$ *for some* $\alpha \neq 1$.

(c) *The following conditions are equivalent:*

(i) *there exists an* $a \in D$ *such that* $\operatorname{Aut}(D) = \operatorname{Aut}_a(D)$*;*

(ii) $\operatorname{Aut}(D) = \operatorname{Aut}_0(D)$*;*

(iii) *any automorphism* $\Phi \in \operatorname{Aut}(D)$ *is of the form* $\boldsymbol{T}_\zeta$ *or* $\boldsymbol{T}_\zeta^*$ *with* $\zeta \in \mathbb{T}^2$ *(the second case is possible only if* $\boldsymbol{S}(D) = D$.

Obviously, (b) may be formulated also with respect to the second variable.

Theorem 2.3.6. *Let* $D_1, D_2 \subset \mathbb{C}^2$ *be normalized bounded complete Reinhardt domains of holomorphy and let* $F \in \operatorname{Bih}(D_1, D_2)$*. Then we have the following possibilities:*

- $D_1 = D_2 = \mathbb{D}^2$*;*
- $D_1 = D_2 = \mathbb{B}_2$*;*
- $D_1 = D_2 = \mathbb{E}_{(\alpha,1)}$ *with* $\alpha \neq 1$ *and* $F = \Psi_{c,\zeta}$ *for some* $c \in \mathbb{D}$ *and* $\zeta \in \mathbb{T}^2$*, where* $\Psi_{c,\zeta}$ *is as in* (2.3.2)*;*
- $D_1 = \mathbb{E}_{(\alpha,1)}$*,* $D_2 = \mathbb{E}_{(1,\alpha)}$ *with* $\alpha \neq 1$ *and* $F = \boldsymbol{S} \circ \Psi_{c,\zeta}$*, for some* $c \in \mathbb{D}$ *and* $\zeta \in \mathbb{T}^2$*, where* $\Psi_{c,\zeta}$ *is as in* (2.3.2)*;*
- $D_1 = D_2 = \mathbb{E}_{(1,\alpha)}$ *with* $\alpha \neq 1$ *and* $F = \boldsymbol{S} \circ \Psi_{c,\zeta} \circ \boldsymbol{S}$ *for some* $c \in \mathbb{D}$ *and* $\zeta \in \mathbb{T}^2$*, where* $\Psi_{c,\zeta}$ *is as in* (2.3.2)*;*
- *in all remaining cases, either* $D_2 = D_1$ *and* $F = \boldsymbol{T}_\zeta$ *with* $\zeta \in \mathbb{T}^2$*, or* $D_2 = \boldsymbol{S}(D_1)$ *and* $F = \boldsymbol{T}_\zeta^*$ *with* $\zeta \in \mathbb{T}^2$*; in particular,* $F \in \operatorname{Bih}_{0,0}(D_1, D_2)$.

Our method of proof of the Thullen theorems needs the notion of the so-called *Wu ellipsoid* introduced by H. Wu in [Wu 1993], see also [Joh 1948] and [Jar-Pfl 2005], § 1.2.6.

Exercise 2.3.7. Let $L, L_1, L_2 \in \mathbb{GL}(n, \mathbb{C})$. Then:

(a) $L^{-1}(\mathbb{B}_n) = \{z \in \mathbb{C}^n : z^t M \bar{z} < 1\}$, where $M := L^t \bar{L}$, and therefore, $L^{-1}(\mathbb{B}_n)$ is given by the Hermitian scalar product $(z, w) \mapsto z^t M \bar{w}$.

(b) $\Lambda_{2n}(L^{-1}(\mathbb{B}_n)) = \frac{\Lambda_{2n}(\mathbb{B}_n)}{|\det L|^2}$.

(c) $L^{-1}(\mathbb{B}_n)$ is a Reinhardt domain iff $L^{-1}(\mathbb{B}_n) = \{z \in \mathbb{C}^n : r \cdot z \in \mathbb{B}_n\}$ for some $r \in \mathbb{R}^n_{>0}$.

(d) $L_1^{-1}(\mathbb{B}_n) = L_2^{-1}(\mathbb{B}_n)$ iff $L_2 \circ L_1^{-1} \in \mathbb{U}(n)$.

Lemma 2.3.8. *For every bounded domain* $\varnothing \neq D \subset \mathbb{C}^n$*, the family*

$$\{L^{-1}(\mathbb{B}_n) : L \in \mathbb{GL}(n, \mathbb{C}),\ D \subset L^{-1}(\mathbb{B}_n)\}^{9}$$

contains exactly one domain (the Wu ellipsoid) $\mathbb{E}(D)$ *with minimal volume.*

[9] Observe that the family is not empty because D is bounded.

Proof. Let $\mathcal{F}(D) := \{L \in \mathbb{GL}(n,\mathbb{C}) : D \subset L^{-1}(\mathbb{B}_n)\}$. By Exercise 2.3.7 (b), we want to maximize the function $\mathcal{F}(D) \ni L \mapsto |\det L|$. Let B be the smallest balanced domain containing D, $B = \operatorname{int}\bigcap_{\substack{G\supset D\\ G \text{ is balanced}}} G$. Then $B \subset L^{-1}(\mathbb{B}_n)$ for every $L \in \mathcal{F}(D)$. Hence $\|L(z)\| \le h_B(z)$, $z \in \mathbb{C}^n$, where h_B stands for the Minkowski function of B. In particular, the set $\mathcal{F}(D)$ is bounded in $\mathbb{M}(n\times n,\mathbb{C})$. Consequently, there exists an $L_0 \in \mathcal{F}(D)$ such that $|\det L_0| = C := \sup\{|\det L| : L \in \mathcal{F}(D)\}$ (Exercise).

Let $\mathcal{F}_0(D) := \{L \in \mathcal{F}(D) : |\det L| = C\}$. We have to show that $M := L_2 \circ L_1^{-1} \in \mathbb{U}(n)$ for any $L_1, L_2 \in \mathcal{F}_0(D)$ (Exercise 2.3.7 (d)). We may assume that $\det L_1 = \det L_2 = C$. Suppose that the Hermitian matrix $M^t\bar{M} \neq \mathbb{I}_n$. Then we can write $M^t\bar{M} = P\Delta P^{-1}$, where $P \in \mathbb{U}(n)$ and Δ is a diagonal matrix with elements $d_1,\dots,d_n > 0$. Since $\det M = 1$, we have $d_1\cdots d_n = 1$. Moreover, since $M^t\bar{M} \neq \mathbb{I}_n$, we conclude that $d_j \neq 1$ for at least one j. Observe that

$$\begin{aligned}\frac{1}{2}z^t(L_1^t\bar{L}_1 + L_2^t\bar{L}_2)\bar{z} &= \frac{1}{2}(\|L_1(z)\|^2 + \|L_2(z)\|^2)\\ &\le \frac{1}{2}(h_B^2(z) + h_B^2(z)) = h_B^2(z), \quad z \in \mathbb{C}^n.\end{aligned}$$

Write $\frac{1}{2}(L_1^t\bar{L}_1 + L_2^t\bar{L}_2) = L^t\bar{L}$ with $L \in \mathbb{GL}(n,\mathbb{C})$. The above inequality shows that $L \in \mathcal{F}(D)$. Thus $|\det L| \le C$. On the other hand we have

$$\begin{aligned}|\det L|^2 &= \frac{1}{2^n}\det(L_1^t\bar{L}_1 + L_2^t\bar{L}_2)\\ &= \frac{1}{2^n}\det(L_1^t)\det(\mathbb{I}_n + (L_1^t)^{-1}L_2^t\bar{L}_2(\bar{L}_1)^{-1})\det(\bar{L}_1)\\ &= \frac{1}{2^n}C^2\det(\mathbb{I}_n + M^t\bar{M})\\ &= \frac{1}{2^n}C^2\det(P)\det(\mathbb{I}_n + P^{-1}M^t\bar{M}P)\det(P^{-1})\\ &= \frac{1}{2^n}C^2\det(\mathbb{I}_n + \Delta) = C^2\frac{1+d_1}{2}\cdots\frac{1+d_n}{2} > C^2\sqrt{d_1\dots d_n} = C^2;\end{aligned}$$

a contradiction. □

Remark 2.3.9. (a) If $A \in \mathbb{GL}(n,\mathbb{C})$, then for every bounded domain $D \subset \mathbb{C}^n$ we have $A(\mathbb{E}(D)) = \mathbb{E}(A(D))$. In particular, if $A(D) = D$, then $A(\mathbb{E}(D)) = \mathbb{E}(D)$.

(b) If D is a bounded Reinhardt domain, then $\mathbb{E}(D) = \{z \in \mathbb{C}^n : r\cdot z \in \mathbb{B}_n\}$ for some $r \in \mathbb{R}^n_{>0}$.

Lemma 2.3.10. *Let $D_1, D_2 \subset \mathbb{C}^n$ be bounded Reinhardt domains with $0 \in D_1 \cap D_2$. Let $F \in \mathrm{Bih}_{0,0}(D_1, D_2)$. Then*

$$F(z) = \rho^{-1}\cdot U(r\cdot z),$$

where $U \in \mathbb{U}(n)$, $r, \rho \in \mathbb{R}^n_{>0}$, and $\rho^{-1} := (1/\rho_1,\dots,1/\rho_n)$.

Proof. By Remark 2.3.9 (b),

$$\mathbb{E}(D_1) = \{z \in \mathbb{C}^n : r \cdot z \in \mathbb{B}_n\}, \quad \mathbb{E}(D_2) = \{z \in \mathbb{C}^n : \rho \cdot z \in \mathbb{B}_n\},$$

for some $r, \rho \in \mathbb{R}^n_{>0}$.

By Proposition 2.1.8, $F \in \mathbb{GL}(n, \mathbb{C})$. By Remark 2.3.9 (a), $F(\mathbb{E}(D_1)) = \mathbb{E}(D_2)$. Consequently, the linear isomorphism

$$\mathbb{C}^n \ni z \overset{U}{\longmapsto} \rho \cdot F(r^{-1} \cdot z) \in \mathbb{C}^n$$

maps $\mathbb{B}_n$ onto $\mathbb{B}_n$, and therefore, it belongs to $\mathbb{U}(n)$. □

Proposition 2.3.11. *Let $D_1, D_2 \subset \mathbb{C}^2$ be bounded complete Reinhardt domains of holomorphy and let $F \in \mathrm{Bih}_{0,0}(D_1, D_2)$. Then:*

- *either $D_j = \mathbb{E}(D_j)$, $j = 1, 2$, and F has the form from Lemma 2.3.10, or*
- *$D_j \subsetneq \mathbb{E}(D_j)$, $j = 1, 2$, and $F(z_1, z_2) = (r_1\zeta_1 z_{\sigma(1)}, r_2\zeta_2 z_{\sigma(2)})$, where $(r_1, r_2) \in \mathbb{R}^2_{>0}$, $(\zeta_1, \zeta_2) \in \mathbb{T}^2$, $\sigma \in \mathfrak{S}_2$.*

Proof. Using Remark 2.3.9 (b) and Lemma 2.3.10, we may assume that, after rescaling variables, we have $D_j \subsetneq \mathbb{E}(D_j) = \mathbb{B}_2$, $j = 1, 2$, and $F \in \mathbb{U}(n)$. Next, by permuting and rotating variables, we may also assume that

$$F(z_1, z_2) = (z_1 \cos\alpha + z_2 \sin\alpha, -z_1 \sin\alpha + z_2 \cos\alpha)$$

with $\alpha \in [0, \pi/2)$ (Exercise). We have to prove that $\alpha = 0$. Suppose that $\alpha \in (0, \pi/2)$ and consider the following construction.

Take a point $x^0 = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2_+ \cap \partial D_1$ ($r > 0$, $0 \le \theta \le \pi/2$). Fix an arbitrary $\varphi = (\varphi_1, \varphi_2) \in \mathbb{R}^2$ and consider the point

$$(e^{i\varphi_1} F_1(x^0), e^{i\varphi_2} F_2(x^0)) \in \partial D_2.$$

Put

$$(y_1(\varphi), y_2(\varphi)) := F^{-1}(e^{i\varphi_1} F_1(x^0), e^{i\varphi_2} F_2(x^0)) \in \partial D_1$$

and, finally, let

$$\xi(\varphi) = (\xi_1(\varphi), \xi_2(\varphi)) := (|y_1(\varphi)|, |y_2(\varphi)|) \in \mathbb{R}^2_+ \cap \partial D_1.$$

Direct calculations give

$$\xi_1(\varphi) = r\sqrt{\cos^2\theta + \frac{1}{2}\sin 2(\theta - \alpha) \cdot \sin 2\alpha \cdot (1 - \cos(\varphi_1 - \varphi_2))},$$
$$\xi_2(\varphi) = r\sqrt{\sin^2\theta - \frac{1}{2}\sin 2(\theta - \alpha) \cdot \sin 2\alpha \cdot (1 - \cos(\varphi_1 - \varphi_2))}.$$

Put $d(\theta) := \sin 2(\theta - \alpha) \cdot \sin 2\alpha \in (-1, 1)$. We have proved that for any point $x^0 = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2_+ \cap \partial D_1$, the points

$$x(t) := \Big(r\sqrt{\cos^2\theta + d(\theta)t},\ r\sqrt{\sin^2\theta - d(\theta)t}\Big), \quad 0 \le t \le 1,$$

belong to $\mathbb{R}^2_+ \cap \partial D_1$. Observe that $x(0) = x^0$ and $\|x(t)\| = r = \|x^0\|$. Consequently, if $\theta \neq \alpha$, then the boundary of $\mathbb{R}^2_+ \cap D_1$ contains the arc

$$I(x^0) := \{x(t) : 0 \le t \le 1\}.$$

Thus, there exist $r_-, r_+ > 0$ such that

$$\{(\rho\cos\psi, \rho\sin\psi) : \ (\rho = r_-,\ 0 \le \psi < \alpha) \text{ or } (\rho = r_+,\ \alpha < \psi \le \pi/2)\} \\ \subset \mathbb{R}^2_+ \cap \partial D_1.$$

Finally, using the completeness of D_1, we get $D_1 = \mathbb{B}_2$ – a contradiction. □

Corollary 2.3.12. *Let $D_1, D_2 \subset \mathbb{C}^2$ be normalized bounded complete Reinhardt domains of holomorphy and let $F \in \mathrm{Bih}_{0,0}(D_1, D_2)$. Then either $D_1 = D_2 = \mathbb{B}_2$ or $F = \boldsymbol{T}_\zeta$ (and $D_2 = D_1$) or $F = \boldsymbol{T}^*_\zeta$ (and $D_2 = \boldsymbol{S}(D_1)$) for some $\zeta \in \mathbb{T}^2$.*

Proof. The result follows from Proposition 2.3.11 and the fact that $D_j = \mathbb{E}(D_j)$ iff $D_j = \mathbb{B}_2$ (because D_j is normalized), $j = 1, 2$. □

Proposition 2.3.13. *Let $D_1, D_2 \subset \mathbb{C}^2$ be normalized bounded complete Reinhardt domains of holomorphy, $D_2 \neq \mathbb{B}_2$, and let $F \in \mathrm{Bih}(D_1, D_2)$ be such that $F(0, b) = (0, 0)$ with $b \neq 0$. Then either*

$$F(z) = (z_1 f(z_2), m(z_2)), \quad z = (z_1, z_2) \in D_1, \tag{2.3.3}$$

or

$$F(z) = (m(z_2), z_1 f(z_2)), \quad z = (z_1, z_2) \in D_1, \tag{2.3.4}$$

where $m \in \mathrm{Aut}(\mathbb{D})$, $m(b) = 0$, and $f \in \mathcal{O}^*(\mathbb{D})$.

Proof. For an arbitrary $\lambda \in \mathbb{T}$, let $F_\lambda := F \circ \boldsymbol{T}_{(\lambda,1)} \circ F^{-1} \in \mathrm{Aut}_0(D_2)$. By Corollary 2.3.12, there exists a $\xi(\lambda) \in \mathbb{T}^2$ such that either $F_\lambda = \boldsymbol{T}_{\xi(\lambda)}$ or $F_\lambda = \boldsymbol{T}^*_{\xi(\lambda)}$.

- The case where $F_\lambda = \boldsymbol{T}_{\xi(\lambda)}$ for uncountably many $\lambda \in \mathbb{T}$. We have

$$F(\lambda z_1, z_2) = (\xi_1(\lambda)F_1(z), \xi_2(\lambda)F_2(z)), \quad z \in D_1,\ \lambda \in I \subset \mathbb{T},$$

where I is uncountable. Observe that if $\xi(\lambda) = (1, 1)$, then $F(\lambda z_1, z_2) = F(z)$, $z \in D_1$, which implies that $\lambda = 1$. Thus at least one of the sets

$$I_1 := \{\lambda \in I : \xi_1(\lambda) \neq 1\}, \quad I_2 := \{\lambda \in I : \xi_1(\lambda) = 1,\ \xi_2(\lambda) \neq 1\}$$

is uncountable.

Case 1. I_1 is uncountable. We have

$$(F_1(0,z_2), F_2(0,z_2)) = (\xi_1(\lambda)F_1(0,z_2), \xi_2(\lambda)F_2(0,z_2)), \quad z_2 \in \mathbb{D},\ \lambda \in I_1.$$

Hence $F_1(0,\cdot) \equiv 0$, i.e. $F_1(z) = z_1 G_1(z)$ with $G_1 \in \mathcal{O}(D_1)$. Since $F_1(0,\cdot) \equiv 0$ and F is a biholomorphism, we conclude that $F_2(0,\cdot) = m \in \operatorname{Aut}(\mathbb{D})$, $m(b) = 0$. Consequently, $F_2(z) - m(z_2) = z_1 G_2(z)$ with $G_2 \in \mathcal{O}(D_1)$. Thus

$$\begin{aligned}&(\lambda z_1 G_1(\lambda z_1, z_2), \lambda z_1 G_2(\lambda z_1, z_2) + m(z_2))\\ &\quad = (\xi_1(\lambda) z_1 G_1(z), \xi_2(\lambda) z_1 G_2(z) + \xi_2(\lambda) m(z_2)), \quad z = (z_1,z_2) \in D_1,\ \lambda \in I_1.\end{aligned}$$

Taking $z_1 = 0$, we get $m \equiv \xi_2(\lambda) m$, so $\xi_2(\lambda) = 1$ for $\lambda \in I_1$. Hence

$$\lambda G_1(\lambda z_1, z_2) = \xi_1(\lambda) G_1(z), \quad \lambda G_2(\lambda z_1, z_2) = G_2(z).$$

First consider the second equation. In terms of the power series expansion of G_2 we get

$$\sum_{j,k=0}^{\infty} G_{2,j,k}(\lambda^{j+1} - 1) z_1^j z_2^k = 0, \quad z = (z_1,z_2) \in D_1,\ \lambda \in I_1.$$

Since I_1 is uncountable, there exists a $\lambda \in I_1$ with $\lambda^{j+1} \neq 1$, $j = 0,1,2,\dots$. Hence $G_2 \equiv 0$.

Now we come back to the first equation. We have

$$0 \neq \det \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} \\ \frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} \end{bmatrix}(0,z_2) = \det \begin{bmatrix} G_1(0,z_2) & 0 \\ 0 & m'(z_2) \end{bmatrix} = G_1(0,z_2) m'(z_2).$$

Hence $G_1(0,z_2) \neq 0$, $z_2 \in \mathbb{D}$. We have $\lambda G_1(0,z_2) = \xi_1(\lambda) G_1(0,z_2)$. Hence $\xi_1(\lambda) = \lambda$, $\lambda \in I_1$. Therefore, $G_1(\lambda z_1, z_2) = G_1(z)$, $z = (z_1,z_2) \in D_1$, $\lambda \in I_1$. By a power series argument we see that G_1 depends only on z_2. Finally,

$$F(z) = (z_1 f(z_2), m(z_2)), \quad z = (z_1,z_2) \in D_1,$$

where $f \in \mathcal{O}^*(\mathbb{D})$ and $m \in \operatorname{Aut}(\mathbb{D})$, $m(b) = 0$.

Case 2. I_2 is uncountable. We have $F_1(\lambda z_1, z_2) = F_1(z)$, $z = (z_1,z_2) \in D_1$, $\lambda \in I_2$, which implies that F_1 depends only on z_2. Hence $F_1(z) = m(z_2)$ with $m(b) = 0$. Furthermore, $F_2(0,z_2) = \xi_2(\lambda) F_2(0,z_2)$, which implies that $F_2(0,\cdot) \equiv 0$. Consequently, $m \in \operatorname{Aut}(\mathbb{D})$ and $F_2(z) = z_1 G_2(z)$ with $G_2 \in \mathcal{O}(D_1)$. Moreover,

$$0 \neq \det \begin{bmatrix} \frac{\partial F_1}{\partial z_1} & \frac{\partial F_1}{\partial z_2} \\ \frac{\partial F_2}{\partial z_1} & \frac{\partial F_2}{\partial z_2} \end{bmatrix}(0,z_2) = \det \begin{bmatrix} 0 & m'(z_2) \\ G_2(0,z_2) & 0 \end{bmatrix} = -G_2(0,z_2) m'(z_2),$$

which says that $G_2(0, z_2) \neq 0$, $z_2 \in \mathbb{D}$. Since $\lambda G_2(0, z_2) = \xi_2(\lambda) G_2(0, z_2)$, we get $\xi_2(\lambda) = \lambda$, $\lambda \in I_2$. Therefore, $G_2(\lambda z_1, z_2) = G_2(z)$, $z = (z_1, z_2) \in D_1$, $\lambda \in I_2$. Hence G_2 depends only on z_2. Finally,

$$F(z) = (m(z_2), z_1 f(z_2)), \quad z = (z_1, z_2) \in D_1,$$

where $f \in \mathcal{O}^*(\mathbb{D})$ and $m \in \operatorname{Aut}(\mathbb{D})$, $m(b) = 0$.

- The case where $F_\lambda = \boldsymbol{T}^*_{\xi(\lambda)}$ for uncountably many $\lambda \in \mathbb{T}$. We have

$$F(\lambda z_1, z_2) = (\xi_1(\lambda) F_2(z), \xi_2(\lambda) F_1(z)), \quad z \in D_1, \ \lambda \in I \subset \mathbb{T},$$

where I is uncountable. In particular,

$$F_1(0, z_2) = \xi_1(\lambda) F_2(0, z_2), \quad F_2(0, z_2) = \xi_2(\lambda) F_1(0, z_2).$$

Observe that $F_1(0, \cdot) \not\equiv 0$ (even more, if $F_1(0, c) = 0$ for some $c \in \mathbb{D}$, then $F_2(0, c) = 0$ and hence $F(0, c) = (0, 0)$, which implies that $c = b$). Consequently, $\xi_2(\lambda)\xi_1(\lambda) = 1$. We get

$$F_1(\lambda z_1, z_2) = \xi_1(\lambda) F_2(z_1, z_2) = \xi_1(\lambda)\xi_2(\lambda) F_1((1/\lambda) z_1, z_2) = F_1((1/\lambda) z_1, z_2), \quad (z_1, z_2) \in D_1, \ \lambda \in I.$$

Thus F_1 and F_2 depend only on z_2 – a contradiction. □

Proposition 2.3.14. *Let $D \subset \mathbb{C}^2$ be a normalized bounded complete Reinhardt domain of holomorphy.*

(a) *Assume that for a $b \in \mathbb{D}_*$ there exists a $\Phi_b \in \operatorname{Aut}(D)$ of the form*

$$\Phi_b(z) = (z_1 f_b(z_2), m_b(z_2)), \quad z = (z_1, z_2) \in D, \tag{2.3.5}$$

where $m_b \in \operatorname{Aut}(\mathbb{D})$, $m(b) = 0$, and $f_b \in \mathcal{O}^*(\mathbb{D})$. Then for any $c \in \mathbb{D}$ there exists a $\Phi_c \in \operatorname{Aut}(D)$ of the form

$$\Phi_c(z) = (z_1 f_c(z_2), m_c(z_2)), \quad z = (z_1, z_2) \in D,$$

where $m_c \in \operatorname{Aut}(\mathbb{D})$, $m(c) = 0$, and $f_c \in \mathcal{O}^*(\mathbb{D})$.

Moreover, either $D = \mathbb{D}^2$ or $D = \mathbb{B}_2$ or $D = \mathbb{E}_{(\alpha,1)}$ with $\alpha \neq 1$ and

$$\Phi_b(z_1, z_2) = \left(\zeta_1 z_1 \left(\frac{1 - |b|^2}{(1 - \bar{b} z_2)^2}\right)^{\frac{1}{2\alpha}}, \zeta_2 h_b(z_2)\right), \quad z = (z_1, z_2) \in \mathbb{E}_{(\alpha,1)},$$

where $(\zeta_1, \zeta_2) \in \mathbb{T}^2$.

(b) *Assume that for a $b \in \mathbb{D}_*$ there exists a $\Phi_b \in \operatorname{Aut}(D)$ of the form*

$$\Phi_b(z) = (m_b(z_2), z_1 f_b(z_2)), \quad z = (z_1, z_2) \in D, \tag{2.3.6}$$

where $m_b \in \operatorname{Aut}(\mathbb{D})$, $m_b(b) = 0$, *and* $f_b \in \mathcal{O}^*(\mathbb{D})$. *Then there exist a point* $a \in \mathbb{D}_*$ *and an automorphism* $\Phi_a^* \in \operatorname{Aut}(D)$ *of the form*

$$\Phi_a^*(z) = (m_a^*(z_1), z_2 f_a^*(z_1)), \quad z = (z_1, z_2) \in D,$$

where $m_a^* \in \operatorname{Aut}(\mathbb{D})$, $m_a^*(a) = 0$, $f_a^* \in \mathcal{O}^*(\mathbb{D})$. *Consequently, after permutation of variables, we are in the situation as in* (a).

Proof. (a) *Step* 1. Let $\Psi := \Phi_b^{-1}$. Observe that $\Psi(z) = (z_1 g(z_2), k(z_2))$, $z = (z_1, z_2) \in D$, where $k = m_b^{-1} \in \operatorname{Aut}(\mathbb{D})$, $k(0) = b$, and $g = 1/f_b \circ k \in \mathcal{O}^*(\mathbb{D})$. Define $\Psi_{\eta,\lambda} := \boldsymbol{T}_{(1,\eta)} \circ \Phi_b \circ \boldsymbol{T}_{(1,\lambda)} \circ \Psi \in \operatorname{Aut}(D)$, $\eta, \lambda \in \mathbb{T}$. Observe that the required Φ_c exists for every $c \in \mathbb{D}$ such that there exist $\eta, \zeta \in \mathbb{T}$ with $\Psi_{\eta,\zeta}(0,0) = (0, c)$, and then

$$\begin{aligned}
\Phi_c(z_1, z_2) &:= \Psi_{\eta,\lambda}^{-1}(z_1, z_2) = \Phi_b \circ \boldsymbol{T}_{(1,1/\lambda)} \circ \Psi \circ \boldsymbol{T}_{(1,1/\eta)}(z_1, z_2) \\
&= \Phi_b \circ \boldsymbol{T}_{(1,1/\lambda)} \circ \Psi(z_1, (1/\eta)z_2) \\
&= \Phi_b \circ \boldsymbol{T}_{(1,1/\lambda)}(z_1 g((1/\eta)z_2), k((1/\eta)z_2)) \\
&= \Phi_b(z_1 g((1/\eta)z_2), (1/\lambda)k((1/\eta)z_2)) \\
&= (z_1 g((1/\eta)z_2) f_b((1/\lambda)k((1/\eta)z_2)), m_b((1/\lambda)k((1/\eta)z_2))) \\
&=: (z_1 f_c(z_2), m_c(z_2)), \quad (z_1, z_2) \in D.
\end{aligned}$$

Thus, every point $c \in \mathbb{D}$ such that there exists a $\lambda \in \mathbb{T}$ with $|c| = |m(\lambda b)|$ is "accessible". Direct calculations show (EXERCISE) that

$$\{|m_b(\lambda b)| : \lambda \in \mathbb{T}\} = \left(0, \frac{2|b|}{1 + |b|^2}\right) =: (0, r_1).$$

Observe that $r_1 > |b|$.

Repeating the above procedure with Φ_b substituted by Φ_c with $0 < |c| < r_1$ leads to a new set of accessible points of the form $\{d \in \mathbb{D} : 0 < |d| < r_2\}$ with $r_2 := \frac{2r_1}{1+r_1^2} > r_1$. Let $r_{n+1} := \frac{2r_n}{1+r_n^2}$. It remains to observe that $r_n \nearrow 1$ (EXERCISE).

Step 2. Write

$$D = \{(z_1, z_1) \in \mathbb{D}^2 : |z_1| < R(|z_2|)\},$$

where $R\colon [0, 1) \to (0, 1]$ is a continuous function with $R(0) = 1$ (Lemma 2.3.1). Let $z^0 = (z_1^0, z_2^0) \in \partial D \cap (\overline{\mathbb{D}} \times \mathbb{D})$, $|z_1^0| = R(|z_2^0|)$. Let Φ_c be as in Step 1. Then $\Phi_c(z^0) \in \partial D \cap (\overline{\mathbb{D}} \times \mathbb{D})$ and therefore $|z_1^0 f_c(z_2^0)| = R(|h_c(z_2^0)|)$, which gives the equation

$$R(|z_2|)|f_c(z_2)| = R(|h_c(z_2)|), \quad z_2, c \in \mathbb{D}. \tag{2.3.7}$$

Take $c = re^{i\theta}$, $z_2 = re^{i(t+\theta)}$. Observe that the function

$$\mathbb{R} \ni t \overset{\varphi_r}{\longmapsto} R\Big(r\Big|\frac{e^{it}-1}{1-r^2e^{it}}\Big|\Big) = R(r)|f_c(re^{i(t+\theta)})|$$

is of class $\mathcal{C}^\infty$. Consequently, R is $\mathcal{C}^\infty$ on $(0, 1)$.

Indeed, write

$$\varphi_r(t) = R \circ \psi_r(\cos t), \quad t \in \mathbb{R},$$

where

$$\psi_r(u) := r\Big(\frac{2-2u}{1+r^4-2r^2u}\Big)^{1/2}, \quad u \in (-1, 1).$$

A short calculation shows that $\psi_r'(u) < 0$, $u \in (-1, 1)$. Moreover,

$$\psi_r((-1, 1)) = \Big(0, \frac{2r}{1+r^2}\Big) := I_r.$$

Consequently, R is $\mathcal{C}^\infty(I_r)$. Letting $r \nearrow 1$, we conclude that $R \in \mathcal{C}^\infty(0, 1)$.

Step 3. Put

$$U(t) := \log R(t), \quad Q(t) := U''(t) + (1/t)U'(t), \quad t \in (0, 1).$$

We have

$$Q(|z|) = \Delta U(|z|), \quad z \in \mathbb{D}_*,$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (Exercise). Moreover, since $\log |f_c|$ is a harmonic function ([Con 1973], Chapter X), we get

$$\begin{aligned} Q(|z|) &= \Delta \log R(|z|) + \Delta \log |f_c(z)| = \Delta \log R(|h_c(z)|) \\ &= \Big|\frac{1-|c|^2}{(1-\bar{c}z)^2}\Big|^2 Q\Big(\Big|\frac{z-c}{1-\bar{c}z}\Big|\Big), \quad z \in \mathbb{D}_*,\ c \in \mathbb{D} \setminus \{z\} \quad \text{(Exercise)}. \end{aligned} \tag{2.3.8}$$

We are going to determine the function R.

- First consider the special case $Q \equiv 0$. Then the equation

$$U''(t) + (1/t)U'(t) = 0$$

gives $U(t) = C_0 \log t + \log C_1$, and hence $R(t) = C_1 t^{C_0}$, $0 < t < 1$. The continuity of R and condition $R(0) = 1$ imply that $C_0 = 0$, $C_1 = 1$, i.e. $R \equiv 1$. Consequently, in this case we get $D = \mathbb{D}^2$.

- Now, consider the case where $Q \not\equiv 0$. Observe that if $Q(t_0) \neq 0$, then for every $t \in (0, 1)$ there exists a $c \in \mathbb{D}$ such that $|h_c(t_0)| = t$. Hence, using (2.3.8), we conclude that $Q(t) \neq 0$.

Observe that if Q_1, Q_2 are two functions of this type, then, using (2.3.8), we get
$$\frac{Q_1(t)}{Q_2(t)} = \frac{Q_1(|\frac{z-c}{1-\bar{c}z}|)}{Q_2(|\frac{z-c}{1-\bar{c}z}|)}, \quad t = |z| \in (0,1),\ c \in \mathbb{D} \setminus \{z\}.$$
Fix a $t_0 \in (0,1)$. We have already observed that for any $t \in (0,1)$ there exists a $c \in \mathbb{D}$ such that $|h_c(t_0)| = t$. Then $Q_1(t_0)/Q_2(t_0) = Q_1(t)/Q_2(t)$. Consequently, $Q_1/Q_2 = \text{const}$.

Step 4. The domain $D := \mathbb{E}_{(\alpha,1)}$ satisfies the assumption of (a).

Indeed, for any $b \in \mathbb{D}$ and $(\zeta_1, \zeta_2) \in \mathbb{T}^2$, the mapping
$$F(z_1, z_2) = \left(\zeta_1 z_1 \left(\frac{1-|b|^2}{(1-\bar{b}z_2)^2}\right)^{\frac{1}{2\alpha}}, \zeta_2 h_b(z_2)\right),$$
is an automorphism of $\mathbb{E}_{(\alpha,1)}$ with $F(0,b) = (0,0)$ (EXERCISE).

Direct calculations show that, for $R(t) = (1-t^2)^{1/(2\alpha)}$, the corresponding function Q has the form
$$Q(t) = -\frac{2}{\alpha(1-t^2)^2} \quad \text{(EXERCISE)}.$$

Step 5. By Steps 3 and 4, for any domain with $Q \not\equiv 0$ we have
$$Q(t) = -\frac{2}{C(1-t^2)^2},$$
where $C \in \mathbb{R}_*$ is a constant. Hence $U(t) = \frac{1}{2C}\log(1-t^2) + \log C_1$, and so $R(t) = C_1(1-t^2)^{1/(2C)}$, $0 < t < 1$. The condition $R(0) = 1$ implies that $C_1 = 1$. Since D is bounded, we have $C \geq 0$. Thus $D = \mathbb{E}_{(C,1)}$.

Step 6. Observe that if $D = \mathbb{E}_{(\alpha,1)}$ with $\alpha \neq 1$, then
$$\Phi_b(z_1, z_2) = \left(\zeta_1 z_1 \left(\frac{1-|b|^2}{(1-\bar{b}z_2)^2}\right)^{\frac{1}{2\alpha}}, \zeta_2 h_b(z_2)\right), \quad z = (z_1, z_2) \in \mathbb{E}_{(\alpha,1)},$$
where $(\zeta_1, \zeta_2) \in \mathbb{T}^2$.

Indeed, since
$$1 - |h_b(z_2)|^2 = \frac{(1-|b|^2)(1-|z_2|^2)}{|1-\bar{b}z_2|^2}, \quad z_2 \in \mathbb{D}, \tag{2.3.9}$$
we get (cf. (2.3.7))
$$|f_b(z_2)| = \frac{R(|h_b(z_2)|)}{R(|z_2|)} = \left(\frac{1-|h_b(z_2)|^2}{1-|z_2|^2}\right)^{\frac{1}{2\alpha}} = \left(\frac{1-|b|^2}{|1-\bar{b}z_2|^2}\right)^{\frac{1}{2\alpha}}, \quad z_2 \in \mathbb{D}.$$

(b) Let $\Psi := \Phi_b^{-1}$. Observe that $\Psi(z) = (z_2 g(z_1), k(z_1))$, $z = (z_1, z_2) \in D$, where $k = m_b^{-1} \in \operatorname{Aut}(\mathbb{D})$, $k(0) = b$, and $g \in \mathcal{O}^*(\mathbb{D})$. Fix $\lambda \in \mathbb{T} \setminus \{1\}$ and put $a := m_b(\lambda b) \in \mathbb{D}_*$. Then

$$\begin{aligned}\Phi_b \circ \boldsymbol{T}_{(1,1/\lambda)} \circ \Psi(a, 0) &= \Phi_b \circ \boldsymbol{T}_{(1,1/\lambda)}(0, k(a)) \\ &= \Phi_b \circ \boldsymbol{T}_{(1,1/\lambda)}(0, \lambda b) = \Phi_b(0, b) = (0, 0).\end{aligned}$$

Moreover,

$$\begin{aligned}\Phi_a^*(z_1, z_2) := \Phi_b \circ \boldsymbol{T}_{(1,1/\lambda)} \circ \Psi(z_1, z_2) &= \Phi_b \circ \boldsymbol{T}_{(1,1/\lambda)}(z_2 g(z_1), k(z_1)) \\ &= \Phi_b(z_2 g(z_1), (1/\lambda)k(z_1)) \\ &= (m_b((1/\lambda)k(z_1)), z_2 g(z_1) f((1/\lambda)k(z_1))) \\ &=: (m_a^*(z_1), z_2 f_a^*(z_1)), \quad (z_1, z_2) \in D. \qquad \square\end{aligned}$$

Proof of Theorem 2.3.4. We already know (cf. the proof of Proposition 2.3.14 (a), Steps 4 and 6) that

$$\begin{aligned}\operatorname{Aut}(\mathbb{E}_{(\alpha,1)}) \supset \Big\{\mathbb{E}_{(\alpha,1)} \ni z \overset{\Psi_{c,\zeta}}{\longmapsto} \Big(\zeta_1 z_1 \Big(\frac{1 - |c|^2}{(1 - \bar{c} z_2)^2}\Big)^{\frac{1}{2\alpha}}, \zeta_2 h_c(z_2)\Big) \in \mathbb{E}_{(\alpha,1)} : \\ c \in \mathbb{D},\ (\zeta_1, \zeta_2) \in \mathbb{T}^2\Big\} \\ = \{\Phi \in \operatorname{Aut}(\mathbb{E}_{(\alpha,1)}) : \Phi \text{ has form (2.3.5)}\} =: \mathfrak{G}.\end{aligned}$$

Moreover, $\mathfrak{G}$ is a subgroup of $\operatorname{Aut}(\mathbb{E}_{(\alpha,1)})$ (Exercise).

Fix a $\Phi \in \operatorname{Aut}(\mathbb{E}_{(\alpha,1)})$. If $\Phi(0,0) = (0,0)$, then, by Corollary 2.3.12, either $\Phi = \boldsymbol{T}_\zeta$ or $\Phi = \boldsymbol{T}_\zeta^*$. The second case is impossible because $\mathbb{E}_{(\alpha,1)}$ is not symmetric. Hence $\Phi \in \mathfrak{G}$.

Now, assume that $\Phi(0,0) = (a,b) \neq (0,0)$. Then $F := \Psi_{b,\boldsymbol{1}} \circ \Phi \in \operatorname{Aut}(\mathbb{E}_{(\alpha,1)})$ and $F(0,0) = (c, 0)$ for some $c \in \mathbb{D}$. If $c = 0$, then $F \in \mathfrak{G}$ and hence $\Phi \in \mathfrak{G}$.

Suppose that $c \neq 0$. Put $G := \boldsymbol{S} \circ F^{-1} \circ \boldsymbol{S} \in \operatorname{Aut}(\mathbb{E}_{(1,\alpha)})$. By Proposition 2.3.13, G is either of the form (2.3.5) or (2.3.6). In the first case Proposition 2.3.14 (a) implies that $\mathbb{E}_{(1,\alpha)} = \mathbb{E}_{(\beta,1)}$, which is impossible. In the second case, since $G(\partial\mathbb{E}_{(1,\alpha)}) = \partial\mathbb{E}_{(1,\alpha)}$, we get

$$|h_c(z_2)|^2 + (1 - |z_2|^{2\alpha})^\alpha |f_c(z_2)|^{2\alpha} \equiv 1, \quad z_2 \in \mathbb{D}.$$

Hence

$$|f_c(z)|^{2\alpha} = \frac{1 - |h_c(z)|^2}{(1 - |z|^{2\alpha})^\alpha} \overset{(2.3.9)}{=} \frac{1 - |c|^2}{|1 - \bar{c} z|^2} \frac{1 - |z|^2}{(1 - |z|^{2\alpha})^\alpha}, \quad z \in \mathbb{D}.$$

Consequently, $\alpha = 1$; a contradiction.

Indeed, suppose that $0 < \alpha < 1$. Then for every $\zeta \in \mathbb{T}$ we get

$$\lim_{z\to\zeta} |f_c(z)|^{2\alpha} = \frac{1-|c|^2}{|1-\bar{c}\zeta|^2} \lim_{t\to 1-} \frac{1-t}{(1-t^\alpha)^\alpha} = 0.$$

Hence, by the maximum principle, $f_c \equiv 0$; a contradiction. If $\alpha > 1$, then

$$\lim_{z\to\zeta} \frac{1}{|f_c(z)|^{2\alpha}} = \frac{|1-\bar{c}\zeta|^2}{1-|c|^2} \lim_{t\to 1-} \frac{(1-t^\alpha)^\alpha}{1-t} = 0$$

and we have again a contradiction. □

Proof of Theorem 2.3.5. (a) Assume that D is homogeneous and $D \neq \mathbb{B}_2$. In particular, for any $b \in \mathbb{D}_*$ there exists a $\Phi_b \in \operatorname{Aut}(D)$ such that $\Phi_b(0,b) = (0,0)$. By Proposition 2.3.13, Φ_b is either of the form (2.3.5) or (2.3.6). Now, by Proposition 2.3.14, either $D = \mathbb{D}^2$ or $D = \mathbb{E}_{(\alpha,1)}$ or $D = \mathbb{E}_{(1,\alpha)}$ with $\alpha \neq 1$. By Theorem 2.3.4, the only homogeneous case is $D = \mathbb{D}^2$.

(b) follows from Proposition 2.3.14 (a) and Theorem 2.3.4.

(c) follows from Corollary 2.3.12 with $D_1 = D_2 = D$. □

Proposition 2.3.15. *Let $D \subset \mathbb{C}^2$ be a normalized bounded complete Reinhardt domain of holomorphy and let $\Phi \in \operatorname{Aut}(D)$ be such that $\Phi(0,0) = (a,b)$ with $ab \neq 0$. Then there exists a $\Psi \in \operatorname{Aut}(D)$ such that $\Psi(0,c) = (0,0)$ or $\Psi(c,0) = (0,0)$ for some $c \in \mathbb{D}_*$.*

Proof. Put $V_0 := \{(z_1,z_2) \in D : z_1 z_2 = 0\} = (\mathbb{D}\times\{0\}) \cup (\{0\}\times\mathbb{D})$, $V_* := V_0 \setminus \{(0,0)\} = (\mathbb{D}_*\times\{0\}) \cup (\{0\}\times\mathbb{D}_*)$. Suppose that the result is not true, i.e.

(*) $F(0,0) \notin V_*$ for every $F \in \operatorname{Aut}(D)$

(equivalently, $(0,0) \notin \Psi(V_*)$ for every $\Psi \in \operatorname{Aut}(D)$).

Define

$$\begin{aligned} &\Psi_\zeta := \Phi^{-1} \circ \boldsymbol{T}_\zeta \circ \Phi \in \operatorname{Aut}(D), \quad \zeta \in \mathbb{T}^2, \quad P(\zeta) := \Psi_\zeta(0,0),\\ &M := \{\boldsymbol{T}_\eta(P(\xi)) : \eta,\xi \in \mathbb{T}^2\}, \quad S(\zeta) := \Psi_\zeta^{-1}(V_0), \quad \zeta \in \mathbb{T}^2. \end{aligned}$$

Note that $P(\zeta) \notin V_0$ for all $\zeta \in \mathbb{T}^2 \setminus \{(1,1)\}$. Indeed, in view of (*), $P(\zeta) \in V_0$ iff $\Psi_\zeta(0,0) = (0,0)$, which means that $\boldsymbol{T}_\zeta \circ \Phi(0,0) = \Phi(0,0)$ and hence $\zeta = (1,1)$.

Moreover, $M \cap S(\zeta) = \{P(\zeta)\}$.

Indeed, it is clear that $P(\zeta) \in M \cap S(\zeta)$. Fix $\eta,\xi \in \mathbb{T}^2$ and suppose that $T_\eta(P(\xi)) \in S(\zeta)$, i.e. $\Psi_\zeta^{-1}(T_\eta(P(\xi))) = \Psi_\zeta^{-1} \circ T_\eta \circ \Psi_\xi(0,0) \in V_0$. By (*) we get $\Psi_\zeta^{-1}(T_\eta(P(\xi))) = (0,0)$. Thus $T_\eta(P(\xi)) = P(\zeta)$.

We are going to show that

(**) there exists a point $P(\zeta)$ which lies on a smooth 3-dimensional surface $N \subset M$.

Assume for a moment that (**) is already proved. Then the intersection $N \cap S(\zeta)$ cannot be a point, which leads to a contradiction.

Indeed, $N' := \Psi_\zeta(N)$ is also a smooth 3-dimensional surface. It suffices to prove that $N' \cap V_0 \neq \{(0,0)\}$. Assume that

$$N' = \{(x_1, y_1, x_2, \varphi(x_1, y_1, x_2)) : (x_1, y_1, x_2) \in U^3\},$$

where $U \subset \mathbb{R}$ is an open neighborhood of 0 and φ is a smooth function in U. Then $N' \cap V_0$ contains the curve $\{(0, 0, x_2, \varphi(0, 0, x_2)) : x_2 \in U\}$. All other cases are similar (Exercise).

We come back to (**). Consider the mapping

$$(0, 2\pi) \times (0, 2\pi) \ni \alpha \overset{f}{\mapsto} (|(P(e^{i\alpha}))_1|, |(P(e^{i\alpha}))_2|) \in \mathbb{R}^2_{>0}.$$

Put $W := f((0, 2\pi) \times (0, 2\pi))$. Observe that if W contains a smooth curve, then M contains a smooth 3-dimensional 2-circled surface.

Indeed, suppose that W contains a graph $u = \varphi(t)$, $t \in U$, where $U \subset \mathbb{R}_{>0}$ is open, φ is smooth in U and $\varphi(t) > 0$, $t \in U$. Then M contains the set

$$M' := \{(e^{i\beta} t, e^{i\gamma}\varphi(t)) : t \in U,\ \beta, \gamma \in \mathbb{R}\}.$$

Consider the mapping

$$U \times \mathbb{R}^2 \ni (t, \beta, \gamma) \overset{g}{\mapsto} (e^{i\beta} t, e^{i\gamma}\varphi(t)) = (t \cos\beta, t \sin\beta, \varphi(t)\cos\gamma, \varphi(t)\sin\gamma) \in \mathbb{R}^4$$

and calculate $g'(t, \beta, \gamma)$:

$$g'(t, \beta, \gamma) = \begin{bmatrix} \cos\beta & -t\sin\beta & 0 \\ \sin\beta & t\cos\beta & 0 \\ \varphi'(t)\cos\gamma & 0 & -\varphi(t)\sin\gamma \\ \varphi'(t)\sin\gamma & 0 & \varphi(t)\cos\gamma \end{bmatrix}.$$

Then $\operatorname{rank} g'(t, \beta, \gamma) = 3$ (Exercise), which implies that M' locally contains a smooth 3-dimensional surface.

Now we prove that W contains a smooth curve. The mapping f is real analytic (Exercise). If there exists an α with $\operatorname{rank} f'(\alpha) = 2$, then W contains an open set and, therefore, a curve. Thus we may assume that $\operatorname{rank} f'(\alpha) \le 1$, $\alpha \in (0, 2\pi) \times (0, 2\pi)$. Obviously, if $\operatorname{rank} f'(\alpha) = 1$ on a non-empty open set, then W contains a curve. It remains to exclude the case where $\operatorname{rank} f' \equiv 0$ on $(0, 2\pi) \times (0, 2\pi)$. Then f is constant. Thus $|P(\zeta)_1| = c_1 > 0$, $|P(\zeta)_2| = c_2 > 0$, $\zeta \in (\mathbb{T} \setminus \{1\})^2$. By continuity, $(0, 0) = (|P((1,1))_1|, |P((1,1))_2|) = (c_1, c_2)$; a contradiction. ☐

Proof of Theorem 2.3.6. Since $F \in \operatorname{Bih}(D_1, D_2)$, we see that D_1 is homogeneous iff D_2 is homogeneous. Thus, by Theorem 2.3.5 (a), $D_1 \in \{\mathbb{D}^2, \mathbb{B}_2\}$ iff $D_2 \in$

$\{\mathbb{D}^2, \mathbb{B}_2\}$. By the Poincaré Theorem 2.1.17, the only possible cases are $D_1 = D_2 = \mathbb{D}^2$ and $D_1 = D_2 = \mathbb{B}_2$.

Assume that D_j is not homogeneous, $j = 1, 2$. If there exists an $a \in D_1$ such that $\operatorname{Aut}(D_1) = \operatorname{Aut}_a(D_1)$, then $\operatorname{Aut}(D_2) = \operatorname{Aut}_{F(a)}(D_2)$. Consequently, by Theorem 2.3.5 (c), $a = F(a) = 0$ and by Corollary 2.3.12, $F = \boldsymbol{T}_\zeta$ (and $D_2 = D_1$) or $F = \boldsymbol{T}_\zeta^*$ (and $D_2 = \boldsymbol{S}(D_1)$) for some $\zeta \in \mathbb{T}^2$.

It remains to consider the case where D_j is not homogeneous and $\operatorname{Aut}(D_j) \neq \operatorname{Aut}_a(D_j)$ for any $a \in D_j$, $j = 1, 2$. Then, by Theorem 2.3.5 and Proposition 2.3.15, $D_1 = \mathbb{E}_p$, $D_2 = \mathbb{E}_q$ for some

$$p, q \in (\{1\} \times (\mathbb{R}_{>0} \setminus \{1\})) \cup ((\mathbb{R}_{>0} \setminus \{1\}) \times \{1\}).$$

In view of Theorem 2.3.4, we only need to prove that $p = q$ or $p = \boldsymbol{S}(q)$.

The case $F(0,0) = (0,0)$ follows from Corollary 2.3.12, so assume that $F(0,0) \neq (0,0)$.

In the case where $F(0,b) = (0,0)$ for some $b \in \mathbb{D}_*$ we use Proposition 2.3.13 and we conclude that F is either of the form (2.3.3) or (2.3.4). In fact, substituting D_2 by $\boldsymbol{S}(D_2)$, if necessary, we may assume that

$$F(z) = (z_1 f_b(z_2), m_b(z_2)), \quad z = (z_1, z_2) \in D_1,$$

where $f_b \in \mathcal{O}^*(\mathbb{D})$ and $m_b \in \operatorname{Aut}(\mathbb{D})$, $m_b(b) = 0$. Recall that

$$D_j = \{(z_1, z_2) \in \mathbb{D}^2 : |z_1| < R_j(|z_2|)\}, \quad j = 1, 2,$$

where

$$R_1(t) := (1 - t^{2p_2})^{1/(2p_1)}, \quad R_2(t) := (1 - t^{2q_2})^{1/(2q_1)}, \quad t \in [0, 1).$$

Since $F(\partial D_1 \cap (\bar{\mathbb{D}} \times \mathbb{D})) = \partial D_2 \cap (\bar{\mathbb{D}} \times \mathbb{D})$, we get

$$R_1(|z_2|)|f_b(z_2)| = R_2(|h_b(z_2)|), \quad z_2 \in \mathbb{D},$$

i.e.

$$(1 - |z|^{2p_2})^{1/(2p_1)}|f_b(z)| = (1 - |h_b(z)|^{2q_2})^{1/(2q_1)}, \quad z \in \mathbb{D},$$

and, consequently,

$$|f_b(z)| = \frac{(1 - |h_b(z)|^{2q_2})^{1/(2q_1)}}{(1 - |z|^{2p_2})^{1/(2p_1)}}, \quad z \in \mathbb{D}.$$

We have to consider the following three cases:

- $F \in \operatorname{Bih}(\mathbb{E}_{(\alpha,1)}, \mathbb{E}_{(\beta,1)})$. Then we have

$$|f_b(z)| = \left(\frac{1 - |b|^2}{|1 - \bar{b}z|^2}\right)^{\frac{1}{2\beta}} (1 - |z|^2)^{\frac{1}{2\beta} - \frac{1}{2\alpha}}, \quad z \in \mathbb{D}.$$

Since $f_b \in \mathcal{O}^*(\mathbb{D})$, letting $|z| \to 1$, we conclude that $\alpha = \beta$.

- $F \in \operatorname{Bih}(\mathbb{E}_{(\alpha,1)}, \mathbb{E}_{(1,\beta)})$. Then we have

$$|f_b(z)| = \left(\frac{1-|h_b(z)|^{2\beta}}{1-|h_b(z)|^2}\right)^{1/2} \left(\frac{1-|b|^2}{|1-\bar{b}z|^2}\right)^{1/2} (1-|z|^2)^{\frac{1}{2}-\frac{1}{2\alpha}}, \quad z \in \mathbb{D}.$$

Consequently, $\alpha = 1$; a contradiction.

- $F \in \operatorname{Bih}(\mathbb{E}_{(1,\alpha)}, \mathbb{E}_{(1,\beta)})$. Then we have

$$\begin{aligned}|f_b(z)| &= \left(\frac{1-|h_b(z)|^{2\beta}}{1-|z|^{2\alpha}}\right)^{1/2} = \left(\frac{1-|h_b(z)|^{2\beta}}{1-|h_b(z)|^2}\frac{1-|h_b(z)|^2}{1-|z|^2}\frac{1-|z|^2}{1-|z|^{2\alpha}}\right)^{1/2} \\ &= \left(\frac{1-|h_b(z)|^{2\beta}}{1-|h_b(z)|^2}\frac{1-|b|^2}{|1-\bar{b}z|^2}\frac{1-|z|^2}{1-|z|^{2\alpha}}\right)^{1/2}, \quad z \in \mathbb{D}.\end{aligned}$$

Letting $z \to \zeta \in \mathbb{T}$, we conclude that

$$\lim_{z\to\zeta} |f_b(z)| = \left(\beta \frac{1-|b|^2}{|1-\bar{b}\zeta|^2}\frac{1}{\alpha}\right)^{1/2} = \frac{\text{const}}{|1-\bar{b}\zeta|}, \quad \zeta \in \mathbb{T}.$$

Hence, by the identity principle, we get

$$f_b(z) = \frac{\eta}{1-\bar{b}z}, \quad z \in \mathbb{D},$$

with $\eta \in \mathbb{C}_*$. We have

$$|\eta|^2 \frac{1-|z|^{2\alpha}}{|1-\bar{b}z|^2} = 1-|h_b(z)|^{2\beta}, \quad z \in \mathbb{D}.$$

Since both sides of the above equality are real analytic functions on $\mathbb{C} \setminus \{1/\bar{b}\}$, we get

$$|\eta|^2 \frac{1-|z|^{2\alpha}}{|1-\bar{b}z|^2} = 1-|h_b(z)|^{2\beta}, \quad z \in \mathbb{C} \setminus \{1/\bar{b}\}.$$

Letting $z \to 1/\bar{b}$, we conclude that $\beta = 1$ (Exercise); a contradiction.

The case where $F(a,0) = (0,0)$ for some $a \in \mathbb{D}_*$ is analogous.

In the case where $F(a,b) = (0,0)$ for $ab \neq 0$ we may assume that $D_1 = \mathbb{E}_{(\alpha,1)}$. Then $\Psi_{b,\mathbf{1}}(a,b) = (a^*,0)$ ($\Psi_{b,\mathbf{1}}$ is as in Theorem 2.3.4). Consequently, $F \circ \Psi_{b,\mathbf{1}}^{-1}(a^*,0) = (0,0)$. Thus the problem is reduced to the previous situation, which implies that $D_2 = \mathbb{E}_{(\alpha,1)}$ and $F \circ \Psi_{b,\mathbf{1}}^{-1} = \Psi_{a^*,\zeta}$ for some $\zeta \in \mathbb{T}^2$. Finally, $F = \Psi_{b,\mathbf{1}} \circ \Psi_{a^*,\zeta} \in \operatorname{Aut}(\mathbb{E}_{(\alpha,1)})$. $\square$

2.4 Biholomorphisms of complete elementary Reinhardt domains in $\mathbb{C}^2$

Recall that a Reinhardt domain of the form

$$\boldsymbol{D}_{\alpha,c} = \{z \in \mathbb{C}^2 : |z_1|^{\alpha_1}|z_2|^{\alpha_2} < e^c\}, \quad \alpha = (\alpha_1, \alpha_2) \in (\mathbb{R}^2_+)_*, \; c \in \mathbb{R},$$

is a so-called elementary Reinhardt domain. Because of the restriction on the exponent α it is complete; moreover, it is a domain of holomorphy.

Remark 2.4.1. Observe that $\boldsymbol{D}_{\alpha,c}$ is algebraically equivalent (cf. Definition 1.5.12) to $\boldsymbol{D}_\alpha = \boldsymbol{D}_{\alpha,0}$ (Exercise). Therefore, we will only study domains of type $\boldsymbol{D}_\alpha$, $\alpha \in (\mathbb{R}^2_+)_*$. In fact, we will only consider the following three types of *normalized elementary Reinhardt domains*, namely:

- $\alpha_1\alpha_2 = 0$: then either $\boldsymbol{D}_\alpha = \boldsymbol{D}_{(1,0)} = \mathbb{D} \times \mathbb{C}$ or $\boldsymbol{D}_\alpha = \boldsymbol{D}_{(0,1)} = \mathbb{C} \times \mathbb{D}$ (obviously, both domains are biholomorphically equivalent);
- $\alpha_1\alpha_2 \neq 0$ and $\alpha_1/\alpha_2 = p/q$ with $p, q \in \mathbb{N}$, p, q relatively prime: then $\boldsymbol{D}_\alpha = \{z \in \mathbb{C}^2 : |z_1|^p|z_2|^q < 1\}$;
- $\alpha_1\alpha_2 \neq 0$ and $\alpha_1/\alpha_2 \notin \mathbb{Q}$: then $\boldsymbol{D}_\alpha = \boldsymbol{D}_{(t,1)}$ with $t := \alpha_1/\alpha_2 \in \mathbb{R}_{>0} \setminus \mathbb{Q}$.

Definition 2.4.2 (Cf. Definition 1.4.8). Let $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{R}^2_+)_*$.

(a) If $\alpha_1\alpha_2 = 0$ or $\alpha_1, \alpha_2 \in \mathbb{N}$, α_1, α_2 relatively prime, then the domain $\boldsymbol{D}_\alpha := \{z \in \mathbb{C}^2 : |z_1|^{\alpha_1}|z_2|^{\alpha_2} < 1\}$ is called an *elementary Reinhardt domain of rational type*.

(b) If $\alpha_1\alpha_2 \neq 0$ and $\alpha_1 \notin \mathbb{Q}$, $\alpha_2 = 1$, then $\boldsymbol{D}_\alpha = \{z \in \mathbb{C}^2 : |z_1|^{\alpha_1}|z_2| < 1\}$ is called an *elementary Reinhardt domain of irrational type*.

Remark 2.4.3. Let $\boldsymbol{D}_\alpha$ be an elementary Reinhardt domain. Then its logarithmic image contains the straight line $L := \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \alpha_1\xi_1 + \alpha_2\xi_2 = t\}$, $t < 0$, if $\alpha_1\alpha_2 \neq 0$, or $\{(\xi_1, \xi_2) : \xi_2 \in \mathbb{R}\}$, $\xi_1 < 0$, if $\alpha = (1, 0)$. Conversely, any unbounded complete Reinhardt domain of holomorphy $D \subsetneq \mathbb{C}^2$ whose logarithmic image contains a straight line is of the form $D = \boldsymbol{D}_{\alpha,c}$ (Exercise) and so biholomorphic to $\boldsymbol{D}_\alpha$.

Exercise 2.4.4. Let $\boldsymbol{D}_\alpha$, $\alpha \in \mathbb{N}^2$ (α_1, α_2 relatively prime). For an $f \in \mathcal{O}^*(\mathbb{D})$ put $\mathfrak{g}_f$,

$$\mathfrak{g}_f(z) := (z_1(f(z^\alpha))^{-\alpha_2}, z_2(f(z^\alpha))^{\alpha_1}), \quad z \in \boldsymbol{D}_\alpha.$$

Prove that $\mathfrak{g} := \{\mathfrak{g}_f : f \in \mathcal{O}^*(\mathbb{D})\}$ is a subgroup of $\operatorname{Aut}(\boldsymbol{D}_\alpha)$.

In the following theorem all automorphisms of an elementary Reinhardt domain, normalized as before, are described.

Theorem 2.4.5 ([Shi 1991], [Shi 1992]). *Let $\boldsymbol{D}_\alpha$ be a complete elementary Reinhardt domain.*

(a) *If* $\boldsymbol{D}_\alpha = \boldsymbol{D}_{(1,0)}$, *then*

$$\operatorname{Aut}(\boldsymbol{D}_\alpha) = \{\boldsymbol{D}_\alpha \ni z \mapsto (m(z_1), f(z_1)z_2 + g(z_1)) : \\ m \in \operatorname{Aut}(\mathbb{D}),\ f \in \mathcal{O}^*(\mathbb{D}),\ g \in \mathcal{O}(\mathbb{D})\}.$$

(a′) *If* $\boldsymbol{D}_\alpha = \boldsymbol{D}_{(0,1)}$, *then*

$$\operatorname{Aut}(D) = \boldsymbol{S} \circ \operatorname{Aut}(\boldsymbol{D}_{(1,0)}) \circ \boldsymbol{S}.$$ [10]

(b) *If* $\boldsymbol{D}_\alpha$ *is of rational type with* $\alpha_1, \alpha_2 \in \mathbb{N}$, *relatively prime, then*

$$\operatorname{Aut}(\boldsymbol{D}_\alpha) = \{\boldsymbol{T}_\zeta \circ \mathfrak{g}_f \circ \sigma : f \in \mathcal{O}^*(\mathbb{D}),\ \zeta \in \mathbb{T}^2,\ \sigma \in \mathcal{G}(\boldsymbol{D}_\alpha)\},$$

where

$$\mathcal{G}(\boldsymbol{D}_\alpha) := \begin{cases} \{\boldsymbol{S}, \operatorname{id}\} & \text{if } \alpha_1 = \alpha_2 = 1, \\ \{\operatorname{id}\} & \text{if } \alpha_1\alpha_2 \neq 1. \end{cases}$$

(c) *If* $\boldsymbol{D}_\alpha$ *is of irrational type* (*i.e.* $\alpha = (\alpha_1, 1)$, $\alpha_1 \notin \mathbb{Q}$), *then*

$$\operatorname{Aut}(\boldsymbol{D}_\alpha) = \{\boldsymbol{D}_\alpha \ni z \mapsto \boldsymbol{T}_\zeta(\delta^{-1}z_1, \delta^{\alpha_1}z_2) : \zeta \in \mathbb{T}^2,\ \delta > 0\}.$$

Exercise 2.4.6. Let $\boldsymbol{D}_\alpha$ be as in (b). Prove that $F \in \operatorname{Aut}(\boldsymbol{D}_\alpha)$ iff there exist $\zeta \in \mathbb{T}$, $f \in \mathcal{O}^*(\mathbb{D})$, and $A \in \mathbb{GL}(2, \mathbb{Z}) \cap \mathbb{M}(2 \times 2, \mathbb{Z}_+)$ with $\alpha A = \alpha$ such that $F = \boldsymbol{T}_{(\zeta,1)} \circ \mathfrak{g}_f \circ \Phi_A$ (for the definition of $\Phi_A := \Phi_{\boldsymbol{1},A}$ see Definition 1.5.12).

Moreover, the following equivalence result will be discussed.

Theorem 2.4.7 ([Shi 1991], [Shi 1992]). *Let* $\boldsymbol{D}_\alpha$ *and* $\boldsymbol{D}_\beta$ *be normalized complete elementary Reinhardt domains* (*in the sense of Remark* 2.4.1).

(a) *If* $\boldsymbol{D}_\alpha$ *is of rational and* $\boldsymbol{D}_\beta$ *of irrational type, then* $\boldsymbol{D}_\alpha$ *is not biholomorphically equivalent to* $\boldsymbol{D}_\beta$.

(b) $\boldsymbol{D}_{(1,0)}$ *and* $\boldsymbol{D}_\alpha$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, α_1, α_2 *relatively prime, are not biholomorphically equivalent.*

(c) *If* $\boldsymbol{D}_\alpha$ *and* $\boldsymbol{D}_\beta$ *are biholomorphically equivalent, then either* $\boldsymbol{D}_\alpha = \boldsymbol{D}_\beta$ *or* $\boldsymbol{D}_\alpha = \boldsymbol{S}(\boldsymbol{D}_\beta)$.

The proof will be based on the following notion of a Liouville foliation.

Definition 2.4.8. Let $D \subset \mathbb{C}^n$ be a domain. A system $(F_\eta)_{\eta \in A}$ (A a suitable index set) of sets $F_\eta \subset D$ is called a *holomorphic* (resp. *psh*) *Liouville foliation* of D if the following conditions are fulfilled:

- $F_{\eta_1} \cap F_{\eta_2} = \varnothing$ if $\eta_1 \neq \eta_2$,
- $D = \bigcup_{\eta \in A} F_\eta$,

[10] Recall that $\boldsymbol{S}(z, w) = (w, z)$.

- if $u \in \mathcal{H}^\infty(D)$ (resp. $u \in \mathcal{PSH}(D)$, bounded from above), then $u|_{F_\eta}$ is identically constant, $\eta \in A$,
- for $\eta_1, \eta_2 \in A$, $\eta_1 \neq \eta_2$, there exists a $u \in \mathcal{H}^\infty(D)$ (resp. an upper bounded $u \in \mathcal{PSH}(D)$) such that $u|_{F_{\eta_1}} \neq u|_{F_{\eta_2}}$.

Example 2.4.9 (A holomorphic Liouville foliation). Let $D = \boldsymbol{D}_\alpha \subset \mathbb{C}^2$ be an elementary Reinhardt domain of rational type.

If $\alpha = (1,0)$, put $F_\zeta := \{\zeta\} \times \mathbb{C}$, $\zeta \in \mathbb{D}$. Then $(F_\zeta)_{\zeta\in\mathbb{D}}$ is a holomorphic Liouville foliation of D. In fact, if $u \in \mathcal{H}^\infty(D)$, then, for $\zeta \in \mathbb{D}$, $u(\zeta,\cdot) \in \mathcal{H}^\infty(\mathbb{C})$. Hence, in virtue of the Liouville Theorem, it follows that $u|_{F_\zeta}$ is constant. Finally, observe that the function $D \ni z \mapsto z_1$ is a bounded holomorphic function on D which separates the fibers F_ζ.

If $\alpha_1\alpha_2 \neq 0$, put $F_\zeta := \{z \in D : z^\alpha = \zeta\}$, $\zeta \in \mathbb{D}$. Then, again, $(F_\zeta)_{\zeta\in\mathbb{D}}$ is a holomorphic Liouville foliation of D. In fact, we mention that for $\zeta \in \mathbb{D}\setminus\{0\}$ the map $\varphi_\zeta\colon \mathbb{C}_* \to D$, $\varphi_\zeta(\lambda) := (\lambda^{-\alpha_2}, \lambda^{\alpha_1}\tilde{\zeta})$, where $\tilde{\zeta}^{\alpha_2} = \zeta$, is holomorphic. Therefore, if $u \in \mathcal{H}^\infty(D)$, then $u \circ \varphi_\zeta \in \mathcal{H}^\infty(\mathbb{C}_*)$. Note that $\varphi_\zeta(\mathbb{C}_*) = F_\zeta$. Applying the Riemann removable singularity theorem and then the Liouville theorem, we conclude that $u|_{F_\zeta}$ is identically constant. In case of $\zeta = 0$ the fiber F_0 equals $(\{0\}\times\mathbb{C}) \cup (\mathbb{C}\times\{0\})$. By the same reasoning as above it is easily seen that if $u \in \mathcal{H}^\infty(D)$, then $u|_{F_0}$ is a constant function. Moreover, the bounded holomorphic function $g\colon D \to \mathbb{C}$, $g(z) := z^\alpha$, separates the different fibers.

In order to be able to present an elementary Reinhardt domain of irrational type as an example of a psh Liouville foliation we will need the following result due to Kronecker (cf. [Har-Wri 1979], see also p. 97).

Lemma 2.4.10. *Let $c \in \mathbb{R}\setminus\mathbb{Q}$, $b \in \mathbb{C}$. Moreover, put*

$$L_{c,b} := \{z \in \mathbb{C}^2 : cz_1 + z_2 = b\}$$

and $\Phi\colon \mathbb{C}^2 \to \mathbb{C}^2_*$, $\Phi(z) := (e^{2\pi z_1}, e^{2\pi z_2})$.

Then $\Phi(L_{c,b})$ is a dense subset of $F := \{z \in \mathbb{C}^2 : |z_1|^c|z_2| = e^{2\pi \operatorname{Re} b}\}$.

Proof. Take a point $z \in L_{c,b}$. Then $c\operatorname{Re} z_1 + \operatorname{Re} z_2 = \operatorname{Re} b$, hence $\Phi(L_{c,b}) \subset F$. On the other hand fix a point $z^0 \in F$. Then choose an $\omega \in \mathbb{C}$ with $e^{2\pi\omega} = z_2^0$. Setting $\zeta := (b-\omega)/c$ we have that $(\zeta,\omega) \in L_{c,b}$ and so $(\zeta + it, \omega - ict) \in L_{c,b}$, $t \in \mathbb{R}$. Then $\Phi(\zeta,\omega) = (z_1^0 e^{is}, z_2^0)$ for a suitable $s \in \mathbb{R}$. Moreover, it is well known (recall that the number c is irrational) that the set

$$\{\Phi(\zeta + it, \omega - ict) : t \in \mathbb{R}\} = \{(z_1^0 e^{i(s+t)}, z_2^0 e^{-ict}) : t \in \mathbb{R}\}$$

is dense in F. $\square$

Example 2.4.11 (A psh Liouville foliation). Let $D = \boldsymbol{D}_\alpha \subset \mathbb{C}^2$ ($\alpha = (\alpha_1, 1)$) be a normalized complete elementary Reinhardt domain of irrational type. Put

$F_t := \{z \in D : |z_1|^{\alpha_1}|z_2| = t\}$, $t \in [0,1)$. Then $(F_t)_{t\in[0,1)}$ is a psh Liouville foliation.

In fact, if $t = 0$, then F_0 is as in Example 2.4.9. Therefore, if $u \in \mathcal{PSH}(D)$ is bounded from above, then $u(0,\cdot) \in \mathcal{SH}(\mathbb{C})$ and $u(\cdot,0) \in \mathcal{SH}(\mathbb{C})$ are upper bounded and so, in virtue of the Liouville theorem for psh functions (see Remark 1.14.3 (g)), identically constant. Hence, $u|_{F_0}$ is a constant function.

Now let $t \in (0,1)$. Fix a $u \in \mathcal{PSH}(D)$ bounded from above. Using the holomorphic mapping $\Phi\colon \mathbb{C}^2 \to \mathbb{C}_*^2$, $\Phi(z) := (e^{2\pi z_1}, e^{2\pi z_2})$, we see that Φ maps the domain $\Omega := \{z \in \mathbb{C}^2 : \alpha_1 \operatorname{Re} z_1 + \operatorname{Re} z_2 < 0\}$ holomorphically onto $D \cap \mathbb{C}_*^2$. Thus $u \circ \Phi$ is a psh function on Ω which is bounded from above. Fixing a point $b = (e^{2\pi\beta_1}, e^{2\pi\beta_2}) \in F_t$ we define

$$L_{\alpha_1, \frac{1}{2\pi}\log t} := \{(\zeta,\omega) \in \mathbb{C}^2 : \alpha_1\zeta + \omega = \tfrac{1}{2\pi}\log t\} \subset \Omega.$$

Then $\mathbb{C} \ni \lambda \mapsto u \circ \Phi(\lambda, \frac{1}{2\pi}\log t - \alpha_1\lambda)$ is an upper bounded subharmonic function and therefore identically constant. Hence, u is constant on the Φ-image of $L_{\alpha_1, \frac{1}{2\pi}\log t}$ that is dense in F_t. Applying that u is upper semicontinuous we conclude that $u(b) \le u(p)$ for any $p \in F_t$. Changing the role of b and p we see that $u|_{F_t}$ is identically constant.

Finally, it remains to mention that $u\colon D \to \mathbb{R}$, $u(z) := |z_1|^{\alpha_1}|z_2|$, is bounded psh and separates different fibers.

In the sequel the following observation will serve as a basic argument.

Lemma 2.4.12. *Let $\Psi\colon D \to D'$ be a biholomorphic mapping between domains $D, D' \subset \mathbb{C}^n$. Assume that $(F_\alpha)_{\alpha\in A}$ (resp. $(F'_\beta)_{\beta\in B}$) is a holomorphic Liouville foliation of D (resp. D'). Then there exists a bijective map $\tau\colon A \to B$ such that $\Psi(F_\alpha) = F'_{\tau(\alpha)}$, $\alpha \in A$. The same result is true for psh Liouville foliations.*

Proof. We restrict ourselves to proving this lemma for holomorphic foliations. The analogous argument in the case of psh foliations is left as an Exercise.

In a first step we assume $D = D'$ and $\Psi = \mathrm{id}_D$. Observe that if $F_\alpha \cap F'_\beta \ne \varnothing$, then $F_\alpha = F'_\beta$. Indeed, suppose that both fibers are different. Then we may assume that $F_\alpha \setminus F'_\beta \ne \varnothing$. Fix points $p \in F_\alpha \cap F'_\beta$ and $q \in F_\alpha \setminus F'_\beta$. In view of the last condition in Definition 2.4.8 there is a bounded holomorphic function h on D with $h(p) \ne h(q)$. On the other hand, $p, q \in F_\alpha$, therefore, $h(p) = h(q)$; a contradiction. The remaining properties of Definition 2.4.8 then prove the lemma.

Now let D and D' be arbitrary. We have only to observe that $(\Psi(F_\alpha))_{\alpha\in A}$ defines a holomorphic Liouville foliation of D' (Exercise). Then the first step completes the proof. $\square$

Proof of Theorem 2.4.5 (a) *and* (a′). The proof will be based on the holomorphic foliation $(F_\zeta)_{\zeta\in\mathbb{D}}$, where $F_\zeta := \{\zeta\} \times \mathbb{C}$ (see Example 2.4.9). Let $\varphi \in \operatorname{Aut}(\boldsymbol{D}_\alpha)$.

In virtue of Lemma 2.4.12, there is a bijective mapping $\tau\colon \mathbb{D} \to \mathbb{D}$ such that $\varphi(F_\zeta) = F_{\tau(\zeta)}$, $\zeta \in \mathbb{D}$. Therefore, φ may be written in the form

$$\varphi(z) = (\varphi_1(z), \varphi_2(z)) = (\tau(z_1), \varphi_2(z)).$$

Therefore, $\tau \equiv \varphi_1(\cdot, 0)$ is a biholomorphic map from $\mathbb{D}$ to $\mathbb{D}$.

What remains is to describe the second component function. Let us fix a $\lambda_0 \in \mathbb{D}$; then $\varphi_2(\lambda_0, \cdot)$ is a biholomorphic map from $\mathbb{C}$ to $\mathbb{C}$, i.e. $\varphi_2(\lambda_0, w) = \gamma(\lambda_0)w + \delta(\lambda_0)$, $w \in \mathbb{C}$, where $\gamma(\lambda_0) \in \mathbb{C}_*$, $\delta(z_0) \in \mathbb{C}$. Now, we write φ_2 as its Hartogs series $\varphi_2(z) = \sum_{j=0}^\infty \gamma_j(z_1)z_2^j$, where $\gamma_j \in \mathcal{O}(\mathbb{D})$.[11] Then $\gamma_j(\lambda_0) = 0$, $j \geq 2$. Since λ_0 was arbitrarily chosen, we get $\varphi_2(z) = \gamma_0(z_1) + \gamma_1(z_1)z_2$. Observe that $\gamma_1 \in \mathcal{O}^*(\mathbb{D})$.

Obviously, any mapping given in the lemma is an automorphism of $\boldsymbol{D}_\alpha$.

It remains to mention that $\boldsymbol{S}$ gives a biholomorphic mapping between $\boldsymbol{D}_\alpha$ and $\mathbb{C} \times \mathbb{D}$. $\square$

To be able to continue the proof of Theorem 2.4.5, it is necessary to study another automorphism group.

The automorphism group of $\boldsymbol{D}_\alpha^*$. Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$, $\alpha \neq (0,0)$, α_1, α_2 relatively prime in the case where $\alpha_1\alpha_2 \neq 0$. We set

$$\boldsymbol{D}_\alpha^* := \{z \in \mathbb{C}_*^2 : |z_1|^{\alpha_1}|z_2|^{\alpha_2} < 1\} = \boldsymbol{D}_\alpha \setminus V_0.$$

An automorphism of $\boldsymbol{D}_\alpha^*$ is called an *algebraic* one if it is the restriction of an algebraic mapping $\Phi_{a,A}$ (cf. Definition 1.5.12).

Then we obtain the following automorphism groups.

Lemma 2.4.13.

$$\begin{aligned}\operatorname{Aut}(\boldsymbol{D}_{(1,0)}^*) &= \operatorname{Aut}(\mathbb{D}_* \times \mathbb{C}_*)\\ &= \{\boldsymbol{D}_{(1,0)}^* \ni z \mapsto (\eta z_1, f(z_1)z_2^\varepsilon) : \eta \in \mathbb{T}, f \in \mathcal{O}^*(\mathbb{D}_*),\ \varepsilon = \pm 1\}.\end{aligned}$$

Proof. Put $F_\zeta := \{\zeta\} \times \mathbb{C}_*$, $\zeta \in \mathbb{D}_*$. Then $(F_\zeta)_{\zeta \in \mathbb{D}_*}$ is a holomorphic Liouville foliation of $\boldsymbol{D}_{(1,0)}^*$ (Exercise). Let $\varphi \in \operatorname{Aut}(\boldsymbol{D}_{(1,0)}^*)$. In virtue of Lemma 2.4.12 there exists a bijective map $\tau\colon \mathbb{D}_* \to \mathbb{D}_*$ such that $\varphi(F_\zeta) = F_{\tau(\zeta)}, \zeta \in \mathbb{D}_*$. Observe that $\varphi_1(z_1, \cdot)$ is a bounded holomorphic function on the whole plane and, therefore, identically $\tau(z_1)$, i.e. $\varphi(z) = (\tau(z_1), \varphi_2(z))$. Since $\varphi_1(z_1, 0) = \tau(z_1)$, $z_1 \in \mathbb{D}_*$, the function τ is holomorphic and hence a biholomorphic map from $\mathbb{D}_*$ onto $\mathbb{D}_*$. Therefore, τ is a rotation of $\mathbb{D}_*$, i.e. $\tau(z_1) = \eta z_1$, where $|\eta| = 1$.

Using Laurent expansion we may write $\varphi_2(z) = \sum_{j=-\infty}^\infty c_j(z_1)z_2^j$, where $c_j \in \mathcal{O}(\mathbb{D}_*)$. In virtue of Lemma 2.4.12, $\varphi_2(z_1, \cdot)$ is a biholomorphic mapping

[11] Recall that $\gamma_j(z_1) = \frac{1}{2\pi i}\int_{\partial K(r)} \frac{f(z_1,\zeta)}{\zeta^{j+1}}d\zeta$, $z_1 \in \mathbb{D}$.

from $\mathbb{C}_*$ onto itself. Hence, $\varphi_2(z) = f(z_1)z_2^{\pm 1}$, $f(z_1) \neq 0$. Then the uniqueness of the Laurent expansion leads to $c_j = 0$, $j \neq \pm 1$, and $c_{-1} = f$, $c_1 = 0$ or $c_{-1} = 0$, $c_1 = f$, $z_1 \in \mathbb{D}_*$. So we are led to the following shape of φ_2, namely $\varphi_2(z) = f(z_1)z_2^{\pm 1}$, $z \in \boldsymbol{D}^*_{(1,0)}$, where $f \in \mathcal{O}^*(\mathbb{D}_*)$. □

In order to continue we observe that any domain $\boldsymbol{D}^*_\alpha$, $\alpha \in \mathbb{Z}^2_*$, α_1, α_2 relatively prime, is biholomorphically equivalent to $\boldsymbol{D}^*_{(1,0)}$. In fact, choose integers c, d such that $\alpha_1 d - \alpha_2 c = 1$ and define the following mapping $\varphi\colon \mathbb{C}^2_* \to \mathbb{C}^2_*$, $\varphi(z) = (z^\alpha, z_1^c z_2^d)$. Then φ gives a biholomorphic mapping from $\boldsymbol{D}^*_\alpha$ onto $\boldsymbol{D}^*_{(1,0)}$.

Corollary 2.4.14. *Let $\boldsymbol{D}^*_\alpha$, $\boldsymbol{D}^*_\beta$, $\alpha, \beta \in \mathbb{Z}^2_*$, where α_1, α_2, respectively β_1, β_2, are assumed to be relatively prime. Then $\boldsymbol{D}^*_\alpha$ and $\boldsymbol{D}^*_\beta$ are biholomorphically equivalent and a biholomorphic mapping is given by Φ_C, where $C \in \mathbb{GL}(2, \mathbb{Z})$ and $\beta C = \alpha$.*

Proof. Choose $c, d, \tilde{c}, \tilde{d} \in \mathbb{Z}$ with $\alpha_1 d - \alpha_2 c = \beta_1 \tilde{d} - \beta_2 \tilde{c} = 1$. Recall that Φ_A and Φ_B induces biholomorphic mappings from $\boldsymbol{D}^*_\alpha$ to $\boldsymbol{D}^*_{(1,0)}$ and from $\boldsymbol{D}^*_\beta$ to $\boldsymbol{D}^*_{(1,0)}$, where $A := \left[\begin{smallmatrix} \alpha_1 & \alpha_2 \\ c & d \end{smallmatrix}\right]$ and $B := \left[\begin{smallmatrix} \beta_1 & \beta_2 \\ \tilde{c} & \tilde{d} \end{smallmatrix}\right]$, respectively. Then set $C := B^{-1}A$. □

In the case where $\alpha \in \mathbb{Z}^2_*$, $\alpha_1\alpha_2 \neq 0$, α_1, α_2 relatively prime, we have the following description of the automorphism group of $\boldsymbol{D}^*_\alpha$.

Lemma 2.4.15. *Let $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2_*$ such that α_1, α_2 are relatively prime. Then*

$$\operatorname{Aut}(\boldsymbol{D}^*_\alpha) = \Big\{\boldsymbol{T}_\zeta \circ \mathfrak{g}_f \circ \Phi_P|_{\boldsymbol{D}^*_\alpha} : \zeta \in \mathbb{T}^2,\ f \in \mathcal{O}^*(\mathbb{D}_*),\ P \in \mathbb{GL}(2, \mathbb{Z}) \text{ with } \alpha P = \alpha\Big\}.$$ [12]

Proof. Let $\chi\colon \boldsymbol{D}^*_\alpha \to \boldsymbol{D}^*_{(1,0)}$ be the biholomorphic mapping from above, i.e. $\chi(z) = \Phi_A(z) = (z^\alpha, z_1^c z_2^d)$, where $c, d \in \mathbb{Z}$ with $\alpha_1 d - \alpha_2 c = 1$. Observe that $\chi^{-1}(z) = (z_1^d z_2^{-\alpha_2}, z_1^{-c} z_2^{\alpha_1})$. Then any automorphism φ of $\boldsymbol{D}^*_\alpha$ can be written in the form $\varphi = \chi^{-1} \circ \psi \circ \chi$, where $\psi \in \operatorname{Aut}(\boldsymbol{D}^*_{(1,0)})$. What remains is to apply Lemma 2.4.13 (Exercise).

Conversely, any map given in Lemma 2.4.15 belongs to $\operatorname{Aut}(\boldsymbol{D}^*_\alpha)$ (Exercise). □

Now we turn to the irrational case, i.e. $\alpha = (\alpha_1, 1)$, $\alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$. Here the method of proof has to be changed; one has to use a covering argument.

Lemma 2.4.16. *Let $\alpha = (\alpha_1, 1)$, $\alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$. Then*

$$\operatorname{Aut}(\boldsymbol{D}^*_\alpha) = \{\Phi_{\zeta,A} : \zeta \in \mathbb{C}^2_*,\ A \in \mathbb{GL}(2, \mathbb{Z}) \text{ such that } \Phi_{\zeta,A} \in \operatorname{Aut}(\boldsymbol{D}^*_\alpha)\},$$

where $\Phi_{\zeta,A}(z) := (\zeta_1 z_1^{a_{1,1}} z_2^{a_{1,2}}, \zeta_2 z_1^{a_{2,1}} z_2^{a_{2,2}})$. In particular, any automorphism of $\boldsymbol{D}^*_\alpha$ is an algebraic one.

[12] Recall that $\Phi_P(z) = (z_1^p z_2^q, z_1^r z_2^s)$. Moreover, note that $\mathfrak{g}_f$ is defined on $\boldsymbol{D}^*_\alpha$, $\alpha \in \mathbb{Z}^2$.

In order to prove Lemma 2.4.16 we need the following auxiliary considerations. Put $\Omega_\alpha := \{\xi \in \mathbb{R}^2 : \alpha_1\xi_1 + \xi_2 < 0\}$ and $T_\alpha := \Omega_\alpha + i\mathbb{R}^2$. Then we study the following holomorphic mapping

$$\Phi\colon T_\alpha \to \boldsymbol{D}_\alpha^*, \quad \Phi(\zeta) := (e^{2\pi\zeta_1}, e^{2\pi\zeta_2}).$$

Observe that (Exercise) Φ is a covering map, i.e. for any point $z \in \boldsymbol{D}_\alpha^*$ there is a suitable open neighborhood $U_z \subset \boldsymbol{D}_\alpha^*$ such that $\Phi^{-1}(U_z)$ is a union of pairwise disjoint open sets V_j, $j \in \mathbb{Z}$, such that $\Phi|_{V_j}$ is a biholomorphic mapping from V_j onto U_a. Moreover, it is easily seen that Ω_α and hence T_α is convex; in particular, it is simply connected. Therefore, T_α is the universal covering of $\boldsymbol{D}_\alpha^*$ (cf. [For 1981]).

Recall a few properties of the universal covering $\Phi\colon T_\alpha \to \boldsymbol{D}_\alpha^*$:

- For any simply connected domain $D \subset \mathbb{C}^2$, any holomorphic mapping $f\colon D \to \boldsymbol{D}_\alpha$, any point $z^0 \in D$, and any point $w^0 \in T_\alpha$ with $\Phi(w^0) = f(z^0)$ there exists a uniquely determined holomorphic function $\tilde{f}\colon D \to T_\alpha$ with $\tilde{f}(z^0) = w^0$ such that $\Phi \circ \tilde{f} = f$. $\tilde{f}$ is called the *lifting of f*. We advise the reader to look into general books on topology for this result.
- In particular, for any pair of points $w', w'' \in T_\alpha$ with $\Phi(w') = \Phi(w'')$ there is an $\hat{f} \in \operatorname{Aut}(T_\alpha)$ such that $\Phi \circ \hat{f} = \Phi$ and $\hat{f}(w') = w''$; $\hat{f}$ is uniquely defined.

In fact, $\hat{f}$ is uniquely defined since it is the lifting of $\Phi\colon T_\alpha \to \boldsymbol{D}_\alpha$. In virtue of the former property of the universal covering we find a holomorphic map $\hat{f}\colon T_\alpha \to T_\alpha$ with $\hat{f}(w') = w''$ such that $\Phi \circ \hat{f} = \Phi$. We have to show that $\hat{f} \in \operatorname{Aut}(T_\alpha)$. Changing the role of w' and w'' we also have a holomorphic $\hat{g}\colon T_\alpha \to T_\alpha$ with $\hat{g}(w'') = w'$ such that $\Phi = \Phi \circ \hat{g}$. Then $\Phi \circ (\hat{f} \circ \hat{g}) = \Phi$, $\Phi \circ (\hat{g} \circ \hat{f}) = \Phi$, $\hat{g} \circ \hat{f}(w') = w'$, and $\hat{f} \circ \hat{g}(w'') = w''$. Using again the first property of the universal covering we conclude that $\hat{f} \circ \hat{g} = \operatorname{id}|_{T_\alpha}$ and $\hat{g} \circ \hat{f} = \operatorname{id}|_{T_\alpha}$. Therefore, $\hat{f} \in \operatorname{Aut}(T_\alpha)$.

Moreover, it is easily seen that

$$\operatorname{Aut}^{\Phi}(T_\alpha) := \{\psi \in \operatorname{Aut}(T_\alpha) : \Phi \circ \psi = \Phi\} = \{\sigma_\eta : \eta \in \mathbb{Z}^2\},$$

where $\sigma_\eta(z) := z + i\eta$, $z \in T_\alpha$.

Now we are going to apply the above lifting properties for a given $\varphi \in \operatorname{Aut}(\boldsymbol{D}_\alpha^*)$. Then there is a lifting $\tilde{\varphi} \in \operatorname{Aut}(T_\alpha)$ such that $\Phi \circ \tilde{\varphi} = \varphi \circ \Phi$ (Exercise).

Moreover, for a fixed $\eta \in \mathbb{Z}^2$, $\tilde{\varphi} \circ \sigma_\eta \circ \tilde{\varphi}^{-1} \in \operatorname{Aut}^{\Phi}(T_\alpha)$. Therefore, we find an $\eta' \in \mathbb{Z}^2$ such that $\tilde{\varphi} \circ \tilde{\sigma}_\eta = \sigma_{\eta'} \circ \tilde{\varphi}$. It is easily seen that the mapping $\eta \to \eta'$ leads to a group isomorphism of $\mathbb{Z}^2$. Therefore, there exists a matrix $P \in \mathbb{GL}(2, \mathbb{Z})$ such that $\tilde{\varphi} \circ \sigma_\eta = \sigma_{\eta P} \circ \tilde{\varphi}$, $\eta \in \mathbb{Z}^2$.

So we are led to study the group of automorphisms of the domain T_α. We get the following lemma.

Lemma 2.4.17. *Let $\varphi \in \operatorname{Aut}(\boldsymbol{D}_\alpha^*)$. Assume that its lifting $\tilde{\varphi}\colon T_\alpha \to T_\alpha$ is a complex affine transformation, i.e. $\tilde{\varphi}(\zeta) = \zeta A + \beta$, where $A \in \mathbb{GL}(2, \mathbb{C})$ and*

$\beta \in \mathbb{C}^2$. *Then* φ *is of the form* $\varphi(z) = (a_1 z_1^{a_{1,1}} z_2^{a_{1,2}}, a_2 z_1^{a_{2,1}} z_2^{a_{2,2}}) = \Phi_{a,A}(z)$, *where* $A \in \mathbb{GL}(2,\mathbb{Z})$, $a = (a_1, a_2) \in \mathbb{C}_*^2$, *i.e.* φ *is an algebraic automorphism of* $\boldsymbol{D}_\alpha^*$.

Proof. From the discussion before we get

$$\tilde{\varphi} \circ \sigma_\eta(\zeta) = \sigma_{\eta P} \circ \tilde{\varphi}(\zeta), \quad \zeta \in T_\alpha, \eta \in \mathbb{Z}^2.$$

Using the form of $\tilde{\varphi}$ it follows that $A = P$. Finally, the equality $\varphi \circ \Phi = \Phi \circ \tilde{\varphi}$ leads to the form of φ claimed in the lemma. □

After all these preparations we proceed with the proof of Lemma 2.4.16.

Proof of Lemma 2.4.16. Let $\varphi \in \mathrm{Aut}(\boldsymbol{D}_\alpha^*)$ and $\tilde{\varphi} \in \mathrm{Aut}(T_\alpha)$ be its lifting. In virtue of Lemma 2.4.17 we have to show that $\tilde{\varphi}$ is a complex affine transformation. In a first step we will show that there are a $\tau \in \mathrm{Aut}(\mathbb{H}^-)$, an $f \in \mathcal{O}^*(\mathbb{H}^-)$, and an $h \in \mathcal{O}(\mathbb{H}^-)$ ($\mathbb{H}^- := \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$) such that

$$\tilde{\varphi}(\zeta) = \big(f(\zeta^*)\zeta_1 + h(\zeta^*),\ \tau(\zeta^*) - \alpha_1(f(\zeta^*) + h(\zeta^*))\big),$$

where $\zeta = (\zeta_1, \zeta_2) \in T_\alpha$ and (to simplify notation) $\zeta^* := \alpha_1\zeta_1 + \zeta_2$.

In fact, put $\psi : \mathbb{C}^2 \to \mathbb{C}^2$, $\psi(\zeta) := (\zeta_1, \zeta^*)$. Obviously, ψ is a biholomorphic mapping and $\psi|_{T_\alpha}$ maps T_α biholomorphically onto $\mathbb{C} \times \mathbb{H}^-$. Its inverse mapping is given by $\psi^{-1}(z, w) = (z, w - \alpha_1 z)$. Thus, $\mathrm{Aut}(T_\alpha) = \psi^{-1} \circ \mathrm{Aut}(\mathbb{C} \times \mathbb{H}^-) \circ \psi$.[13]

Moreover, let $g : \mathbb{H}^- \to \mathbb{D}$ be any biholomorphic map. Then

$$\tilde{g} : \mathbb{C} \times \mathbb{H}^- \to \mathbb{C} \times \mathbb{D}, \quad \tilde{g}(z, w) := (z, g(w)),$$

is also biholomorphic. So $\mathrm{Aut}(T_\alpha) = \psi^{-1} \circ \tilde{g}^{-1} \circ \mathrm{Aut}(\mathbb{C} \times \mathbb{D}) \circ \tilde{g} \circ \psi$. Then, in virtue of Theorem 2.4.5 (a), there are $\tilde{f} \in \mathcal{O}^*(\mathbb{D})$, $\tilde{h} \in \mathcal{O}(\mathbb{D})$, and $m \in \mathrm{Aut}(\mathbb{D})$ such that for $\zeta \in T_\alpha$ we get

$$\tilde{\varphi}(\zeta) = \psi^{-1} \circ \tilde{g}^{-1}\big(\tilde{f} \circ g(\zeta^*)\zeta_1 + \tilde{h} \circ g(\zeta^*),\ m \circ g(\zeta^*)\big),$$

and therefore, for $\zeta \in T_\alpha$,

$$\tilde{\varphi}(\zeta) = \big(\tilde{f} \circ g(\zeta^*)\zeta_1 + \tilde{h} \circ g(\zeta^*),\ g^{-1} \circ m \circ g(\zeta^*) - \alpha_1(\tilde{f} \circ g(\zeta^*)\zeta_1 + \tilde{h} \circ g(\zeta^*))\big),$$

which proves the above claim with $f := \tilde{f} \circ g$, $h := \tilde{h} \circ g$, and $\tau := g^{-1} \circ m \circ g$.

In virtue of Lemma 2.4.17 it remains to verify that $\tilde{\varphi}$ is a complex affine mapping.

Indeed, in virtue of the properties of the covering mappings we have a matrix $P = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in \mathbb{GL}(2, \mathbb{Z})$ such that for all pairs $(k, \ell) \in \mathbb{Z}^2$ the following identities are true:

$$\tilde{\varphi}(\zeta_1 + ik, \zeta_2 + i\ell) = \tilde{\varphi}(\zeta) + i(k, \ell)P, \quad \zeta \in T_\alpha.$$

[13] To be precise ψ should be understood as $\psi|_{T_\alpha}$.

In particular,

$$\begin{aligned}
&f(\zeta^* + i(\alpha_1 k + \ell))(\zeta_1 + ik) + h(\zeta^* + i(\alpha_1 k + \ell)) \\
&\qquad = f(\zeta^*)\zeta_1 + h(\zeta^*) + i(kp + \ell r), \ \tau(\zeta^* + i(\alpha_1 k + \ell)) \\
&\qquad\qquad - \alpha_1(f(\zeta^* + i(\alpha_1 k + \ell))(\zeta_1 + ik) + h(\zeta^* + i(\alpha_2 k + \ell)) \\
&\qquad = \tau(\zeta^*) - \alpha_1(f(\zeta^*)\zeta_1 + h(\zeta^*)) + i(kq + \ell s).
\end{aligned}$$

From these identities we deduce that f is a constant function.

In fact, fix a point $w_0 \in \mathbb{H}^-$. Then, applying the first of the above identities for points $(\zeta_1, w_0 - a_1\zeta_1) \in T_\alpha$, gives

$$\begin{aligned}
&f(w_0 + i(\alpha_1 k + \ell))(\zeta_1 + ik) + h(w_0 + i(\alpha_1 k + \ell)) \\
&\qquad = f(w_0)\zeta_1 + h(w_0) + i(kp + \ell r), \quad \zeta_1 \in \mathbb{C}.
\end{aligned}$$

Hence, $f(w_0 + i(\alpha_1 k + \ell)) = f(w_0) =: \lambda_0 \neq 0$, $k, \ell \in \mathbb{Z}$. Recall that the number α_1 is an irrational one. So the set $\{w_0 + i(\alpha_1 k + \ell)) : k, j \in \mathbb{Z}\}$ has an accumulation point in the plane. Then, in virtue of the identity theorem, it follows that $f \equiv \lambda_0$.

Applying the first of the above identities, we claim that h is a complex linear function.

Indeed, the above identity implies that

$$i\lambda_0 k + h(Z + i(\alpha_1 k + \ell)) = h(Z) + i(kp + \ell r)$$

for all $Z = \zeta^* \in \mathbb{H}^-$. Differentiation in direction of Z leads to

$$h'(Z + i(\alpha_1 k + \ell)) = h'(Z).$$

Fixing some $Z = Z_0 \in \mathbb{H}^-$ we have $h'(Z_0 + i(\alpha_1 k + \ell)) = h'(Z_0) := \mu_1$. So h' is constant, i.e. $h(Z) = \mu_1 Z + \mu_0$ for a suitable μ_0.

It remains to show that τ is a complex affine mapping.

In fact, using the second identity, we arrive at the following equality:

$$\begin{aligned}
&\tau(Z + i(\alpha_1 k + \ell)) - a_1(\lambda_0(\zeta_1 + ik) + \mu_1(Z + i(\alpha_1 k + \ell)) + \mu_0) \\
&\qquad = \tau(Z) - \alpha_1(\lambda_0\zeta_1 + \mu_1 Z + \mu_0) + i(kq + \ell s), \quad Z \in \mathbb{H}^-.
\end{aligned}$$

Again differentiation gives $\tau'(Z + i(\alpha_1 k + \ell)) = \tau'(Z)$, $Z \in \mathbb{H}^-$. As above, fixing $Z = Z_0$ and using the identity theorem, we arrive at $\tau' \equiv \tau'(Z_0) =: \tau_1 \neq 0$ (recall that τ is a biholomorphic mapping). As a consequence we conclude that $\tau(Z) = \tau_1 Z + \tau_0$ for a suitable τ_0.

Finally, rewriting $\tilde{\varphi}$, we see that

$$\tilde{\varphi}(\zeta) = (\lambda_0\zeta_1 + \mu_1(\zeta^*) + \mu_0, \tau_1(\zeta^*) + \tau_0 - \alpha_1(\lambda_0\zeta_1 + \mu_1(\zeta^*) + \mu_0)),$$

or

$$\tilde{\varphi}(\zeta) = \zeta \begin{bmatrix} \lambda_0 + \alpha_1\mu_1 & \alpha_1(\tau_1 - \lambda_0 - \mu_1\alpha_1) \\ \mu_1 & \tau_1 - \alpha_1\mu_1 \end{bmatrix} + (\mu_0, \tau_0 - \alpha_1\mu_0).$$

Obviously, the matrix is a non-singular one, and therefore, $\tilde{\varphi}$ is complex affine. Hence, Lemma 2.4.17 gives the end of the proof. □

Now we return to the proof of Theorem 2.4.5.

Proof of Theorem 2.4.5 (b). Obviously, any of the given mappings belongs to $\mathrm{Aut}(\boldsymbol{D}_\alpha)$. To prove the converse we will use the holomorphic Liouville foliation $(F_\zeta)_{\zeta\in\mathbb{D}}$, where $F_\zeta := \{z \in \mathbb{C}^2 : z^\alpha = \zeta\}$. Fix a $\varphi \in \mathrm{Aut}(\boldsymbol{D}_\alpha)$. Then, applying Lemma 2.4.12, there exists a bijective mapping $\tau\colon \mathbb{D} \to \mathbb{D}$ such that $\varphi(F_\zeta) = F_{\tau(\zeta)}$, $\zeta \in \mathbb{D}$. Observe that the fiber F_0 is the only one with a "singularity". So one concludes that $\tau(0) = 0$ (Exercise) which means that $\varphi|_{\boldsymbol{D}_\alpha^*}$ defines an automorphism of $\boldsymbol{D}_\alpha^*$. Using Lemma 2.4.15 shows that

$$\varphi(z) = (\zeta_1(f(z^\alpha))^{-\alpha_2} z_1^p z_2^q,\ \zeta_2(f(z^\alpha))^{\alpha_1} z_1^r z_2^s), \quad z \in \boldsymbol{D}_\alpha^*,$$

where $\zeta = (\zeta_1, \zeta_2) \in \mathbb{T}^2$, $f \in \mathcal{O}^*(\mathbb{D}_*)$, and $P = \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right] \in \mathbb{GL}(2,\mathbb{Z})$ with $\alpha P = \alpha$. Since $\alpha_j \in \mathbb{Z}_+$, one concludes that $P = \mathbb{I}_2$ if $\alpha_1\alpha_2 \neq 1$, and ($P = \mathbb{I}_2$ or $p = s = 0$, $q = r = 1$) if $\alpha_1 = \alpha_2 = 1$, which gives the description of σ.

We will only discuss the case when $\sigma = \mathrm{id}$ (the case when $\sigma = \boldsymbol{S}$ may be taken as an Exercise). Observe that $K := \frac{1}{2}\mathbb{D}_* \times \{1\} \Subset \boldsymbol{D}_\alpha$. Therefore, $\varphi_2(z_1, 1) = \zeta_2(f(z_1^{\alpha_1}))^{\alpha_1}$, $z_1 \in \frac{1}{2}\mathbb{D}_*$, is bounded. Applying the Riemann theorem of removable singularities we see that f extends holomorphically to $\mathbb{D}$. Taking into account that φ is bijective, it even follows that $f \in \mathcal{O}^*(\mathbb{D})$. Finally, a continuity argument leads to the description of φ on the whole of $\boldsymbol{D}_\alpha$. □

Finally, we discuss the case of normalized elementary Reinhardt domains of irrational type.

Proof of Theorem 2.4.5 (c). Here we use the psh Liouville foliation $(F_t)_{t\in[0,1)}$ from Example 2.4.11. Let $\varphi \in \mathrm{Aut}(\boldsymbol{D}_\alpha)$. Then there is a bijection $\tau\colon [0,1) \to [0,1)$ such that $\varphi(F_t) = F_{\tau(t)}$, $t \in [0,1)$. In particular, F_0 is homeomorphic to $F_{\tau(0)}$, which implies that $\tau(0) = 0$. So $\varphi|_{\boldsymbol{D}_\alpha^*} \in \mathrm{Aut}(\boldsymbol{D}_\alpha^*)$. Applying Lemma 2.4.16, $\varphi|_{\boldsymbol{D}_\alpha^*}$ is of the form

$$\varphi(z) = (\zeta_1 z_1^{a_{1,1}} z_2^{a_{1,2}}, \zeta_2 z_1^{a_{2,1}} z_2^{a_{2,2}}), \quad z \in \boldsymbol{D}_\alpha^*,$$

where $A = \left[\begin{smallmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{smallmatrix}\right] \in \mathbb{GL}(2,\mathbb{Z})$. Observing that the coordinate axes belong to $\boldsymbol{D}_\alpha$, it follows that $a_{i,j} \in \mathbb{Z}_+$ and, therefore $A = \mathbb{I}_2$, which implies Theorem 2.4.5 (c) (Exercise). □

Summarizing, Theorem 2.4.5 has been completely proved.

Now we turn to the proof of Theorem 2.4.7. We start with the discussion of bounded holomorphic functions on elementary Reinhardt domains. First, recall that an elementary Reinhardt domain $\boldsymbol{D}_\alpha$ of rational type carries at least one bounded holomorphic function that is not identically constant. On the other hand, we have

Lemma 2.4.18. *Any bounded holomorphic function on an elementary Reinhardt domain of irrational type is identically constant.*

Proof. Take an $f \in \mathcal{H}^\infty(\boldsymbol{D}_\alpha)$. Then $|f| \in \mathcal{PSH}(\boldsymbol{D}_\alpha)$. Therefore, in virtue of Example 2.4.11, $f|_{F_t}$ is identically equal to a constant s_t, where $(F_t)_{t\in[0,1)}$ denotes the psh Liouville foliation from that example. Recall that

$$F_t = \{z \in \boldsymbol{D}_\alpha : |z_1|^{\alpha_1}|z_2| = t\}.$$

In particular, $f(z_1, 1) = s_t$ whenever $|z_1|^{\alpha_1} = t$. Applying the identity theorem, it follows that $s_t = s$, $t \in [0, 1)$. Hence, $f \equiv s$ on $\boldsymbol{D}_\alpha$. □

Proof of Theorem 2.4.7. (a). In virtue of Lemma 2.4.18 and the remark before, it is clear that $\boldsymbol{D}_\alpha$ and $\boldsymbol{D}_\beta$ are not biholomorphically equivalent.

(b) Suppose that there is a biholomorphic map $\varphi\colon \boldsymbol{D}_{(0,1)} \to \boldsymbol{D}_\alpha$. Then, using the holomorphic Liouville foliation $(F_\zeta)_{\zeta\in\mathbb{D}}$ of $\boldsymbol{D}_\alpha$ and $(F'_\zeta)_{\zeta\in\mathbb{D}}$ of $\boldsymbol{D}_{(1,0)}$, respectively (see Example 2.4.9), there is a bijection $\tau\colon \mathbb{D} \to \mathbb{D}$ such that $\varphi(F_\zeta) = F'_{\tau(\zeta)}$, $\zeta \in \mathbb{D}$. In particular, $\varphi|_{F_{\zeta'}} = F'_0$. Using that F'_0 has a singularity at $(0, 0)$ we get, as before, a contradiction.

(c) Assume that $\boldsymbol{D}_\alpha$ and $\boldsymbol{D}_\beta$ are biholomorphically equivalent. In virtue of (a) there are two cases.

Case 1: Assume that $\boldsymbol{D}_\alpha$ and $\boldsymbol{D}_\beta$ are of rational type. Suppose that $\alpha = (1, 0)$, then $\beta = (1, 0)$ or $\beta = (0, 1)$ and a biholomorphic map is given either by the identity or by $\boldsymbol{S}$. Therefore we only have to discuss the case where $\alpha_1\alpha_2 \neq 0 \neq \beta_1\beta_2$, α_1, α_2, resp. β_1, β_2, relatively prime. Take a biholomorphic map $\varphi\colon \boldsymbol{D}_\alpha \to \boldsymbol{D}_\beta$. As in the proof before using holomorphic Liouville foliations we conclude that $\varphi|_{\boldsymbol{D}^*_\alpha} \in \operatorname{Aut}(\boldsymbol{D}^*_\alpha)$. Following Corollary 2.4.14 there is a biholomorphic mapping $\psi\colon \boldsymbol{D}^*_\beta \to \boldsymbol{D}^*_\alpha$ of the form $\psi = \Phi_A$, $A \in \mathbb{GL}(2, \mathbb{Z})$ with $\det A = 1$ and $\alpha A = \beta$. Hence, $\hat{\psi} := \varphi \circ \psi \in \operatorname{Aut}(\boldsymbol{D}^*_\beta)$. In virtue of Lemma 2.4.15, $\hat{\psi}$ may be written as

$$\hat{\psi}(z) = \boldsymbol{T}_\zeta \circ \mathfrak{g}_f \circ \Phi_P(z), \quad z \in \boldsymbol{D}^*_\beta.$$

Therefore,

$$\varphi|_{\boldsymbol{D}^*_\alpha} = \boldsymbol{T}_\zeta \circ \mathfrak{g}_f \circ \Phi_P \circ \Phi_{A^{-1}} = \boldsymbol{T}_\zeta \circ \mathfrak{g}_f \circ \Phi_{PA^{-1}}$$

and $\beta P A^{-1} = \alpha$. Thus,

$$\boldsymbol{T}_\zeta^{-1} \circ \varphi(z) = ((f(z^\beta))^{-\beta_2} z_1^p z_2^q, (f(z^\beta))^{\alpha_1} z_1^r z_2^s), \quad z \in \boldsymbol{D}^*_\alpha,$$

where $PA^{-1} =: \left[\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right]$. Observe that $\boldsymbol{T}_\zeta^{-1} \circ \varphi$ defines a biholomorphic mapping from $\boldsymbol{D}_\alpha$ onto $\boldsymbol{D}_\beta$.

Recall that the left-hand side is holomorphic on $\boldsymbol{D}_\alpha$. In particular, the functions $\mathbb{D}_* \ni \lambda \mapsto (f(\lambda^{\beta_1+\beta_2}))^{-\beta_2}\lambda^{p+q}$ and $\mathbb{D}_* \ni \lambda \mapsto (f(\lambda^{\beta_1+\beta_2}))^{\beta_1}\lambda^{r+s}$ extend holomorphically to $\mathbb{D}$. Therefore, f has a pole at 0, i.e. $f(\lambda) = \lambda^k \tilde{f}(\lambda)$, $\lambda \in \mathbb{D}_*$, where $\tilde{f} \in \mathcal{O}^*(\mathbb{D})$ and $k \in \mathbb{Z}$. Hence,

$$\boldsymbol{T}_\zeta^{-1} \circ \varphi(z) = \big((\tilde{f}(z^\beta))^{-\beta_2} z_1^{p-k\beta_1\beta_2} z_2^{q-k\beta_2^2},\\ (\tilde{f}(z^\beta))^{\beta_1} z_1^{r+k\beta_1^2} z_2^{s+k\beta_1\beta_2}\big), \quad z \in \boldsymbol{D}_\alpha^*.$$

Finally, we define an automorphism of $\boldsymbol{D}_\beta$, namely,

$$\chi(z) := \mathfrak{g}_{\tilde{f}}(z), \quad z \in \boldsymbol{D}_\beta.$$

Then, for $z \in \boldsymbol{D}_\alpha^*$, we get

$$\chi \circ \boldsymbol{T}_\zeta \circ \varphi(z) = (z_1^{p-k\beta_1\beta_2} z_2^{q-k\beta_2^2}, z_1^{r+k\beta_1^2} z_2^{s+k\beta_1\beta_2}).$$

Taking into account that the mapping on the left-hand side is holomorphic on $\boldsymbol{D}_\alpha$, it is easily seen that $\hat{\chi} := \chi \circ \boldsymbol{T}_\zeta \circ \varphi = \mathrm{id}\,|_{\boldsymbol{D}_\alpha}$ or $\hat{\chi} = \boldsymbol{S}|_{\boldsymbol{D}_\alpha}$. Hence, Case 1 is verified.

Case 2: Assume that $\boldsymbol{D}_\alpha$ and $\boldsymbol{D}_\beta$ are of irrational type, i.e. $\alpha = (\alpha_1, 1)$, $\beta = (\beta_1, 1)$, where $\alpha_1, \beta_1 \in \mathbb{R}_+ \setminus \mathbb{Q}$. Applying psh Liouville foliations, one gets that $\varphi|_{\boldsymbol{D}_\alpha^*} : \boldsymbol{D}_\alpha^* \to \boldsymbol{D}_\beta^*$ is a biholomorphic map. Following the proof of Lemma 2.4.16, one may show (Exercise) that $\varphi|_{\boldsymbol{D}_\alpha^*} = \Phi_{\zeta,A}|_{\boldsymbol{D}_\alpha^*}$. Applying now that the left mapping is holomorphic on $\boldsymbol{D}_\alpha$, it follows that either $\varphi(z) = \boldsymbol{T}_\zeta(z)$ or $\varphi(z) = \boldsymbol{T}_\zeta \circ \boldsymbol{S}(z)$ whenever $z \in \boldsymbol{D}_\alpha$.

In the first case observe that $|\zeta_1|^{\beta_1}|\zeta_2|^{\beta_2} = 1$. Then $\Phi_{\tilde{\zeta},\mathbb{I}_2}|_{\boldsymbol{D}_\beta} \in \mathrm{Aut}(\boldsymbol{D}_\beta)$, where $\tilde{\zeta} := (\zeta_1^{-1}, \zeta_2^{-1})$. Hence $\Phi_{\tilde{\zeta},\mathbb{I}_2} \circ \varphi = \mathrm{id}$ on $\boldsymbol{D}_\alpha$. A similar argument for the second case is left to the reader (Exercise). $\square$

Remark 2.4.19. An independent proof of Theorem 2.4.7 may be found in [Edi-Zwo 1999].

2.5* Miscellanea

Besides the problem of biholomorphic equivalence of Reinhardt domains $D_1, D_2 \subset \mathbb{C}^n$, one can try, for instance, to characterize all proper holomorphic mappings $F : D_1 \to D_2$.[14] In the remaining part of this chapter we collect several results related to this area of problems. More precisely:

[14] Recall that a mapping $f : X \to Y$ is *proper* if $f^{-1}(K)$ is compact for every compact $K \subset Y$. Every homeomorphism is proper.

- § 2.5.1 Biholomorphic equivalence of Reinhardt domains.
- § 2.5.2 Automorphisms of Reinhardt domains.
- § 2.5.3 Proper mappings.
- § 2.5.4 Non-compact automorphism groups.

In general, the methods of proofs of the presented results (based, for example, on the Lie theory or rescaling methods) are beyond the scope of the book. Nevertheless, we decided to put them here as illustrations of various streams of research. The results in this section may be also a starting point for further studies of the reader.

2.5.1 Biholomorphic equivalence of Reinhardt domains

It seems that in the category of Reinhardt domains one has

$$D_1 \overset{\text{bih}}{\simeq} D_2 \iff D_1 \overset{\text{alg}}{\simeq} D_2$$

(cf. Definition 1.5.12, Theorems 2.3.6, 2.4.7). Several particular cases are known (they were proved by methods based on the Lie theory). $\boxed{?}$ We like to point out that, unfortunately, we do not know any alternative methods of proof (without the Lie theory). $\boxed{?}$

Theorem* 2.5.1 ([Sun 1978]). *Let $D_j \subset \mathbb{C}^n$ be a bounded Reinhardt domain with $0 \in D_j$, $j = 1, 2$. Then $D_1 \overset{\text{bih}}{\simeq} D_2$ iff there exist $r_1, \dots, r_n > 0$ and a permutation $\sigma \in \mathfrak{S}_n$ such that*

$$D_2 = \{(r_1 z_{\sigma(1)}, \dots, r_n z_{\sigma(n)}) : (z_1, \dots, z_n) \in D_1\}.$$

In particular, $D_1 \overset{\text{bih}}{\simeq} D_2 \Leftrightarrow D_1 \overset{\text{alg}}{\simeq} D_2$.

Observe that the case $n = 2$ was already discussed in Theorem 2.3.6.

Definition 2.5.2. For $k = (k_1, \dots, k_s) \in \mathbb{N}^s$ and $p = (p_1, \dots, p_s) \in \mathbb{R}^s_{>0}$, let

$$\mathbb{E}_{k,p} := \Big\{(z_1, \dots, z_s) \in \mathbb{C}^{k_1} \times \dots \times \mathbb{C}^{k_s} : \sum_{j=1}^{s} \|z_j\|^{2p_j} < 1\Big\}$$

be the *generalized complex ellipsoid*.

In the case where $k_1 = \dots = k_s = 1$ the generalized complex ellipsoid reduces to the standard complex ellipsoid $\mathbb{E}_p$; cf. (1.18.5).

Theorem 2.5.1 implies the following classification theorem for generalized complex ellipsoids (the case where $p \in \mathbb{N}^s$, $q \in \mathbb{N}^t$ was solved in [Naru 1968]).

Theorem 2.5.3. *Let $\mathbb{E}_{k,p}$, $\mathbb{E}_{\ell,q} \subset \mathbb{C}^n$ be two generalized complex ellipsoids with:*
- $k = (k_1, \dots, k_s) \in \mathbb{N}^s$, $\ell = (\ell_1, \dots, \ell_t) \in \mathbb{N}^t$,
- $n = k_1 + \dots + k_s = \ell_1 + \dots + \ell_t$,
- $p_1 \le \dots \le p_s$, $q_1 \le \dots \le q_t$,
- $\#\{i \in \{1, \dots, s\} : p_i = 1\} \le 1$, $\#\{i \in \{1, \dots, t\} : q_i = 1\} \le 1$.

Then $\mathbb{E}_{k,p} \overset{\text{bih}}{\simeq} \mathbb{E}_{\ell,q}$ *iff* $s = t$, $k = \ell$, *and* $p = q$. *In particular,* $\mathbb{E}_p \overset{\text{bih}}{\simeq} \mathbb{E}_q$ *iff* $p = q$ *up to a permutation (cf. also* [Jar-Pfl 1993], *Theorem* 8.5.1).

Proof. Use Theorem 2.5.1 – Exercise. □

Let $D \subset \mathbb{C}^n$ be a Reinhardt domain satisfying the Fu condition. Recall (cf. Remark 1.5.11 (a)) that, after a permutation of variables, we may always assume that:

(*) there exists $k = \mathfrak{F}(D) \in \{0, \dots, n\}$ with $D \cap V_j \neq \varnothing$, $j = 1, \dots, k$, $\bar{D} \cap V_j = \varnothing$, $j = k + 1, \dots, n$.

Observe that if $0 \in D$, then $\mathfrak{F}(D) = n$.

Exercise 2.5.4. Let $T := \{(z_1, z_2) \in \mathbb{D}^2 : |z_1| < |z_2|\}$ be the Hartogs triangle and let $T^* := T \setminus (\{0\} \times \mathbb{D}) = \{(z_1, z_2) \in \mathbb{D}^2 : 0 < |z_1| < |z_2|\}$. Observe that $\mathfrak{F}(T) = 1$ and $\mathfrak{F}(T^*) = 0$. Prove that T and T^* are not biholomorphically equivalent.

Hint: Observe that $T \overset{\text{bih}}{\simeq} \mathbb{D} \times \mathbb{D}_*$ and $T^* \overset{\text{bih}}{\simeq} \mathbb{D}_* \times \mathbb{D}_*$.

Theorem* 2.5.5 ([Bar 1984]). *Let $D_1, D_2 \subset \mathbb{C}^n$ be bounded Reinhardt domains satisfying the Fu condition with (*). Then $D_1 \overset{\text{bih}}{\simeq} D_2$ iff $\mathfrak{F}(D_1) = \mathfrak{F}(D_2) =: k$ and D_1, D_2 are algebraically equivalent via a biholomorphism $\Phi_{r,A}$ such that*

$$a_{i,j} = \begin{cases} 1 & \text{if } i \le k \text{ and } j = \sigma(i), \\ 0 & \text{if } i \le k \text{ and } j \neq \sigma(i), \\ 0 & \text{if } i > k, \end{cases} \quad i = 1, \dots, n, \ j = 1, \dots, k,$$

where $\sigma \in \mathfrak{S}_k$.

Observe that if $\mathfrak{F}(D_1) = \mathfrak{F}(D_2) = n$, then $A(z) = (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ for a $\sigma \in \mathfrak{S}_n$. Thus the above result generalizes Theorem 2.5.1.

Theorem* 2.5.6 ([Shi 1988]). *Two bounded Reinhardt domains $D_1, D_2 \subset \mathbb{C}^n$ are biholomorphically equivalent iff they are algebraically equivalent.*

Theorem* 2.5.7 ([Kru 1988]). *Two hyperbolic (cf. § 4.7) Reinhardt domains $D_1, D_2 \subset \mathbb{C}^n$ are biholomorphically equivalent iff they are algebraically equivalent.*

Notice that any hyperbolic Reinhardt domain of holomorphy is algebraically equivalent to a bounded domain (see Theorem 4.7.2), so, in fact, Theorem 2.5.7 follows from Theorem 2.5.6.

Theorem* 2.5.8 ([Sol 2002]). *Two Reinhardt domains $D_1, D_2 \subset \mathbb{C}^2$ are biholomorphically equivalent iff they are algebraically equivalent.*

$\boxed{?}$ We do not know whether Theorem 2.5.8 remains true for $n \geq 3$. $\boxed{?}$

2.5.2 Automorphisms of Reinhardt domains

Theorem* 2.5.9 ([Naru 1968]). *If $p \in \mathbb{N}_2^s$, then $\operatorname{Aut}(\mathbb{E}_{k,p}) \simeq \mathbb{T}^n$.*

Theorem* 2.5.10 ([Lan 1984]). *Assume that $0 \leq k \leq n \geq 2$, $p \in \{1\}^k \times \mathbb{N}_2^{n-k}$. Then*

$$\operatorname{Aut}(\mathbb{E}_p) = \{F_{H,\zeta} : H \in \operatorname{Aut}(\mathbb{B}_k),\ \zeta \in \mathbb{T}^{n-k}\}, \tag{2.5.1}$$

where

$$F_{H,\zeta}(z)$$
$$:= \left(H(z'), \zeta_{k+1} z_{k+1} \left(\frac{1 - \|a'\|^2}{(1 - \langle z', a'\rangle)^2}\right)^{\frac{1}{2p_{k+1}}}, \dots, \zeta_n z_n \left(\frac{1 - \|a'\|^2}{(1 - \langle z', a'\rangle)^2}\right)^{\frac{1}{2p_n}}\right),$$

$z = (z', z_{k+1}, \dots, z_n) \in \mathbb{E}_p \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$, and $a' := H^{-1}(0')$.

In particular, the group $\operatorname{Aut}(\mathbb{E}_p)$ depends on $k^2 + k + n$ real parameters (cf. Example 2.1.12 (b)); if $k = 0$, then $\operatorname{Aut}(\mathbb{E}_p) \simeq \mathbb{T}^n$ (cf. Theorem 2.5.9).

Remark 2.5.11. Notice that in general, for arbitrary $p_{k+1}, \dots, p_n \in \mathbb{R}_{>0} \setminus \{1\}$, the set $\{F_{H,\zeta} : H \in \operatorname{Aut}(\mathbb{B}_k),\ \zeta \in \mathbb{T}^{n-k}\}$ is a subgroup of $\operatorname{Aut}(\mathbb{E}_p)$ (Exercise); cf. [Jar-Pfl 1993], Lemma 8.5.2. $\boxed{?}$ We do not know whether (2.5.1) remains true. $\boxed{?}$

Theorems 2.5.10, 2.1.20 and Lemma 2.1.21 imply the following

Example 2.5.12. Let $n = n_1 + \dots + n_k$, $0 \leq m_j \leq n_j$, $p^j \in \{1\}^{m_j} \times \mathbb{N}_2^{n_j - m_j}$, $j = 1, \dots, k$. Assume that if $n_j = 1$, then $m_j = 1$. Then the group

$$\operatorname{Aut}(\mathbb{E}_{p^1} \times \dots \times \mathbb{E}_{p^k})$$

depends on $d = n + \sum_{j=1}^k m_j(m_j + 1)$ real parameters. For instance, for arbitrary $p_1, p_2, p_3, p_4, p_1', p_2' \in \mathbb{N}_2$, we have:

$n = 2$:

d	$k = 1$	$k = 2$
2	$\mathbb{E}_{(p_1,p_2)}$	
4	$\mathbb{E}_{(1,p_2)}$	
6		$\mathbb{D}^2$
8	$\mathbb{B}_2$	

$n = 3$:

d	$k=1$	$k=2$	$k=3$
3	$\mathbb{E}_{(p_1,p_2,p_3)}$		
5	$\mathbb{E}_{(1,p_2,p_3)}$	$\mathbb{D}\times\mathbb{E}_{(p_1,p_2)}$	
7		$\mathbb{D}\times\mathbb{E}_{(1,p_2)}$	
9	$\mathbb{E}_{(1,1,p_3)}$		$\mathbb{D}^3$
11		$\mathbb{D}\times\mathbb{B}_2$	
15	$\mathbb{B}_3$		

$n = 4$:

d	$k=1$	$k=2$	$k=3$	$k=4$
4	$\mathbb{E}_{(p_1,p_2,p_3,p_4)}$	$\mathbb{E}_{(p_1,p_2)}\times\mathbb{E}_{(p_1',p_2')}$		
6	$\mathbb{E}_{(1,p_2,p_3,p_4)}$	$\mathbb{D}\times\mathbb{E}_{(p_1,p_2,p_3)}$, $\mathbb{E}_{(p_1,p_2)}\times\mathbb{E}_{(p_1',p_2')}$		
8		$\mathbb{D}\times\mathbb{E}_{(1,p_2,p_3)}$, $\mathbb{E}_{(1,p_2)}\times\mathbb{E}_{(1,p_2')}$	$\mathbb{D}^2\times\mathbb{E}_{(p_1,p_2)}$	
10	$\mathbb{E}_{(1,1,p_3,p_4)}$	$\mathbb{E}_{(p_1,p_2)}\times\mathbb{B}_2$	$\mathbb{D}^2\times\mathbb{E}_{(1,p_2)}$	
12		$\mathbb{D}\times\mathbb{E}_{(1,1,p_3)}$, $\mathbb{E}_{(1,p_2)}\times\mathbb{B}_2$		$\mathbb{D}^4$
14			$\mathbb{D}^2\times\mathbb{B}_2$	
16	$\mathbb{E}_{(1,1,1,p_4)}$	$\mathbb{B}_2\times\mathbb{B}_2$		
18		$\mathbb{D}\times\mathbb{B}_3$		
24	$\mathbb{B}_4$			

Remark 2.5.13. Let $D \subset \mathbb{C}^n$ be a hyperbolic Reinhardt domain such that $\operatorname{Aut}(D)$ depends on d real parameters. Recall that the group $\operatorname{Aut}(\mathbb{B}_n)$ depends on n^2+2n real parameters. There are the following general results (cf. [GIK 2000], [Isa 2007]):

- If $d > n^2+2$, then $D = \mathbb{B}_n$ up to rescaling of variables.
- If $D \not\simeq \mathbb{B}_n$, then $d \in [n, n^2+2]$ and d is of the same parity as n.
- $d = n^2+2$ iff $D = \mathbb{D}\times\mathbb{B}_{n-1}$ up to permutation and rescaling of variables.
- If $d = n^2$, then D is algebraically equivalent to one of the following domains:

(i) $\{z \in \mathbb{C}^n : r < \|z\| < R\}$, $0 \le r < R < +\infty$ (cf. Exercise 2.1.13);

(ii) $\mathbb{D}^3$ ($n = 3$);

(iii) $\mathbb{B}_2 \times \mathbb{B}_2$ ($n = 4$);

(iv) $\mathbb{E}_{(1,\dots,1,p_n)}$, $p_n \neq 1$ (cf. Theorem 2.5.10);

(v) $\{(z', z_n) \in \mathbb{B}_{n-1} \times \mathbb{C} : r(1 - \|z'\|^2)^\alpha < |z_n| < R(1 - \|z'\|^2)^\alpha\}$, $0 < r < R \le +\infty$, $\alpha \in \mathbb{R}$;

(vi) $\{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : re^{\alpha\|z'\|^2} < |z_n| < Re^{\alpha\|z'\|^2}\}$, $0 < r < R \le +\infty$, $\alpha \in \mathbb{R}_*$ and $(R = +\infty \Rightarrow \alpha > 0)$.

Some intermediate cases where $n^2 < d < n^2 + 2$ are also discussed in [GIK 2000] and [Isa 2007].

For $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$ and $1 \le s \le n-1$, define the *generalized Hartogs triangle*

$$\mathbb{F}_{p,s} := \Big\{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^{s} |z_j|^{2p_j} < \sum_{j=s+1}^{n} |z_j|^{2p_j} < 1\Big\}.$$

If $n = 2$, then $\mathbb{F}_{(1,1),1}$ is the standard Hartogs triangle (cf. Remark 1.5.11 (c)).

Theorem* 2.5.14 ([Lan 1989]). *Let $0 \le k \le n-1$, $p \in \{1\}^k \times \mathbb{N}_2^{n-k}$. Then*

$$\operatorname{Aut}(\mathbb{F}_{p,n-1}) = \{F_{H,\zeta} : H \in \operatorname{Aut}(\mathbb{B}_k),\ \zeta \in \mathbb{T}^{n-k}\}, \tag{2.5.2}$$

where, for $z = (z', z_{k+1}, \dots, z_n) \in \mathbb{F}_{p,n-1} \subset \mathbb{C}^k \times \mathbb{C}^{n-k}$ and $a' := H^{-1}(0')$, we put

$$F_{H,\zeta}(z) := \Big(z_n^{p_n} H(z'/z_n^{p_n}), \zeta_{k+1} z_{k+1}\Big(\frac{1 - \|a'\|^2}{(1 - \langle z'/z_n^{p_n}, a'\rangle)^2}\Big)^{\frac{1}{2p_{k+1}}}, \dots,$$
$$\zeta_{n-1} z_{n-1}\Big(\frac{1 - \|a'\|^2}{(1 - \langle z'/z_n^{p_n}, a'\rangle)^2}\Big)^{\frac{1}{2p_{n-1}}}, \zeta_n z_n\Big).$$

In particular, the group $\operatorname{Aut}(\mathbb{F}_{p,n-1})$ depends on $k^2 + k + n$ real parameters.

Remark 2.5.15. (a) To prove that $F_{H,\zeta}(\mathbb{F}_{p,n-1}) \subset \mathbb{F}_{p,n-1}$, note that

$$|(F_{H,\zeta})_n(z)|^{-2p_n} \sum_{j=1}^{n-1} |(F_{H,\zeta})_j(z)|^{2p_j}$$
$$= |z_n|^{-2p_n}\Big(|z_n^{2p_n}|\|H(z'/z_n^{p_n})\|^2 + \frac{1 - \|a'\|^2}{|1 - \langle z'/z_n^{p_n}, a'\rangle|^2} \sum_{j=k+1}^{n-1} |z_j|^{2p_j}\Big)$$
$$= 1 - \frac{(1 - \|a'\|^2)(1 - \|z'/z_n^{p_n}\|^2)}{|1 - \langle z'/z_n^{p_n}, a'\rangle|^2} + |z_n|^{-2p_n}\frac{1 - \|a'\|^2}{|1 - \langle z'/z_n^{p_n}, a'\rangle|^2} \sum_{j=k+1}^{n-1} |z_j|^{2p_j}$$
$$= 1 - \frac{1 - \|a'\|^2}{|1 - \langle z'/z_n^{p_n}, a'\rangle|^2}\Big(1 - |z_n|^{-2p_n}\Big(\|z'\|^2 + \sum_{j=k+1}^{n-1} |z_j|^{2p_j}\Big)\Big).$$

(b) Observe that in general, for arbitrary $p_{k+1}, \dots, p_{n-1} \in \mathbb{R}_{>0} \setminus \{1\}$, the set

$$\{F_{H,\zeta} : H \in \operatorname{Aut}(\mathbb{B}_k),\ \zeta \in \mathbb{T}^{n-k}\}$$

is a subgroup of $\operatorname{Aut}(\mathbb{F}_{p,n-1})$ (Exercise). $\boxed{?}$ We do not know whether (2.5.2) remains true. $\boxed{?}$

Theorem* 2.5.16 ([Che-Xu 2002]). *Let $2 \le s \le n-2$, $0 \le k \le s$, $0 \le \ell \le n-s$, $p \in \{1\}^k \times \mathbb{N}_2^{s-k} \times \{1\}^\ell \times \mathbb{N}_2^{n-s-\ell}$. Then*

$$\operatorname{Aut}(\mathbb{F}_{p,s}) = \{F_{H',H'',\zeta} : H' \in \mathbb{U}(k),\ H'' \in \mathbb{U}(\ell),\ \zeta \in \mathbb{T}^{n-k-\ell}\}, \tag{2.5.3}$$

where, for $z = (z', z_{k+1}, \dots, z_s, z'', z_{s+\ell+1}, \dots, z_n) \in \mathbb{F}_{p,s} \subset \mathbb{C}^k \times \mathbb{C}^{s-k} \times \mathbb{C}^\ell \times \mathbb{C}^{n-s-\ell}$, we put

$$F_{H',H'',\zeta}(z) := \big(H'(z'), \zeta_{k+1}z_{k+1}, \dots, \zeta_s z_s, H''(z''), \zeta_{s+\ell+1}z_{s+\ell+1}, \dots, \zeta_n z_n\big).$$

In particular, the group $\operatorname{Aut}(\mathbb{F}_{p,s})$ depends on $k^2 + \ell^2 + n - k - \ell$ real parameters.

Remark 2.5.17. Observe that in general, for arbitrary $p_{k+1}, \dots, p_s, p_{s+\ell+1}, \dots, p_n \in \mathbb{R}_{>0} \setminus \{1\}$, the set

$$\{F_{H,\zeta} : H \in \operatorname{Aut}(\mathbb{B}_k),\ \zeta \in \mathbb{T}^{n-k}\}$$

is a subgroup of $\operatorname{Aut}(\mathbb{F}_{p,s})$ (Exercise). $\boxed{?}$ We do not know whether (2.5.3) remains true. $\boxed{?}$

2.5.3 Proper mappings

Theorem* 2.5.18 ([Bar 1984]). *Let $D_1, D_2 \subset \mathbb{C}^n$ be bounded Reinhardt domains satisfying the Fu condition. Then any proper holomorphic mapping $F : D_1 \to D_2$ extends holomorphically to a neighborhood of $\bar{D}_1$.*

Theorem* 2.5.19 ([Lan 1984]). *Assume that $n \ge 2$. For arbitrary $p, q \in \mathbb{N}^n$ the following conditions are equivalent:*

(i) *there exists a proper holomorphic mapping $F : \mathbb{E}_p \to \mathbb{E}_q$;*

(ii) $(p_1/q_1, \dots, p_n/q_n) \in \mathbb{N}^n$.

Moreover, any proper holomorphic mapping $F : \mathbb{E}_p \to \mathbb{E}_q$ is, up to an automorphism of $\mathbb{E}_q$, of the form

$$F(z) = (z_1^{p_1/q_1}, \dots, z_n^{p_n/q_n}), \quad z = (z_1, \dots, z_n) \in \mathbb{E}_p.$$

In particular, any proper holomorphic mapping $F : \mathbb{E}_p \to \mathbb{E}_p$ is an automorphism (see [Ale 1977] for the case $\mathbb{E}_p = \mathbb{B}_n$).

Remark 2.5.20. The implication (ii) $\Rightarrow$ (i) is obvious and remains true for arbitrary $p, q \in \mathbb{R}^n_{>0}$. $\boxed{?}$ We do not know whether the implication (i) $\Rightarrow$ (ii) remains true. $\boxed{?}$

Theorem* 2.5.21 ([Lan 1989]). (a) *If $n \geq 3$, then for arbitrary $p, q \in \mathbb{N}^n$ the following conditions are equivalent:*

(i) *there exists a proper holomorphic mapping $F : \mathbb{F}_{p,n-1} \to \mathbb{F}_{q,n-1}$;*

(ii) $A := \{\ell \in \mathbb{N} : s_j := (\ell q_n - p_n)/q_j \in \mathbb{Z} : j = 1, \dots, n-1\} \neq \varnothing$ *and* $r_j := p_j/q_j \in \mathbb{N}$, $j = 1, \dots, n-1$.

Moreover, any proper holomorphic mapping $F : \mathbb{F}_{p,n-1} \to \mathbb{F}_{q,n-1}$ is, up to an automorphism of $\mathbb{F}_{q,n-1}$, of the form

$$F(z) = (z_1^{r_1} z_n^{s_1}, \dots, z_{n-1}^{r_{n-1}} z_n^{s_{n-1}}, z_n^{\ell}), \quad \ell \in A.$$

(b) *If $n = 2$, then for arbitrary $p, q \in \mathbb{N}^2$ the following conditions are equivalent:*

(i) *$F : \mathbb{F}_{p,1} \to \mathbb{F}_{q,1}$ is a proper holomorphic mapping;*

(ii) $F(z_1, z_2)$

$$= \begin{cases} (\zeta_1 z_2^{\ell q_2/q_1 - k p_2/p_1} z_1^k, \zeta_2 z_2^{\ell}) & \text{if } p_2/p_1 \notin \mathbb{N}, \\ & \ell q_2/q_1 - k p_2/p_1 \in \mathbb{Z}, \\ (\zeta_1 z_2^{\ell q_2/q_1} B(z_1 z_2^{-p_2/p_1}), \zeta_2 z_2^{\ell}) & \text{if } p_2/p_1 \in \mathbb{N},\ \ell q_2/q_1 \in \mathbb{N}, \end{cases}$$

where $\zeta_1, \zeta_2 \in \mathbb{T}$, $k, \ell \in \mathbb{N}$, and B is a finite Blaschke product.

Remark 2.5.22. (a) In the case $n \geq 3$ the implication (ii) $\Rightarrow$ (i) is elementary and remains true for arbitrary $p, q \in \mathbb{R}^n_{>0}$. Indeed

$$\begin{aligned} |F_n(z)|^{-2q_n} \sum_{j=1}^{n-1} |F_j(z)|^{2q_j} &= |z_n|^{-2\ell q_n} \sum_{j=1}^{n-1} |z_n|^{2 s_j q_j} |z_j|^{2 r_j q_j} \\ &= \sum_{j=1}^{n-1} |z_n|^{2(\ell q_n/q_j - p_n/q_j) q_j - 2\ell q_n} |z_j|^{2p_j} = |z_n|^{-2p_n} \sum_{j=1}^{n-1} |z_j|^{2p_j}. \end{aligned}$$

Observe that F is biholomorphic iff $\ell = r_1 = \dots = r_{n-1} = 1$ iff $p_j = q_j$, $j = 1, \dots, n-1$, and $(p_n - q_n)/p_j \in \mathbb{Z}$, $j = 1, \dots, n-1$. In particular, there are $p, q \in \mathbb{N}^n$ such that $\mathbb{F}_{p,n-1} \neq \mathbb{F}_{q,n-1}$ but $\mathbb{F}_{p,n-1} \simeq \mathbb{F}_{q,n-1}$. Take for instance $p_j = q_j$, $j = 1, \dots, n-1$, and $p_n \neq q_n$ such that $(p_n - q_n)/p_j \in \mathbb{Z}$, $j = 1, \dots, n-1$.

$\boxed{?}$ We do not know whether the implication (i) $\Rightarrow$ (ii) remains true. $\boxed{?}$

(b) In the case $n = 2$ the implication (ii) $\Rightarrow$ (i) is elementary and remains true for arbitrary $p, q \in \mathbb{R}^2_{>0}$. Indeed

$$|F_2(z)|^{-2q_2} |F_1(z)|^{2q_1} = \begin{cases} (|z_1||z_2|^{-p_2/p_1})^{2kq_1} & \text{if } p_2/p_1 \notin \mathbb{N}, \\ |B(z_1 z_2^{-p_2/p_1})|^{2q_1} & \text{if } p_2/p_1 \in \mathbb{N}. \end{cases}$$

Observe that F is biholomorphic iff

$$\begin{cases} \ell = k = 1, q_2/q - p_2/p_1 \in \mathbb{Z} & \text{if } p_2/p_1 \notin \mathbb{N}, \\ \ell = 1, B \in \mathrm{Aut}(\mathbb{D}), q_2/q_1 \in \mathbb{N} & \text{if } p_2/p_1 \in \mathbb{N}. \end{cases}$$

$\boxed{?}$ We do not know whether the implication (i) $\Rightarrow$ (ii) remains true. $\boxed{?}$

Theorem* 2.5.23. *Assume that $p \in \mathbb{N}^n$, $2 \le s \le n-2$.*

(a) ([Che-Xu 2001]) *The following conditions are equivalent:*

(i) *there exists a proper holomorphic mapping $F\colon \mathbb{F}_{p,s} \to \mathbb{F}_{p,s}$;*

(ii) *there exist permutations $\sigma \in \mathfrak{S}_s$ and $\delta \in \mathfrak{S}_{n-s}$ such that $p_{\sigma(j)}/p_j \in \mathbb{N}$, $j = 1, \dots, s$, $p_{s+\delta(k)}/p_{s+k} \in \mathbb{N}$, $k = 1, \dots, n-s$.*

(b) ([Che-Xu 2002]) *Any proper holomorphic mapping $F\colon \mathbb{F}_{p,s} \to \mathbb{F}_{p,s}$ is an automorphism* (*cf. Theorem* 2.5.16).

Let $\varphi \in \mathcal{C}^\infty([0,1], \mathbb{R}_+)$ be such that there exists an $h \in (0,1)$ for which

- $\varphi|_{[0,h]} \equiv 0$,
- $\varphi(1) = 1$,
- $\varphi' \ge 0$ and $\varphi'' \ge 0$ on $[0,1]$,
- $\varphi' > 0$ and $\varphi'' > 0$ on $(h,1)$.

Define

$$D_{\varphi,h} := \{(z_1,z_2) \in \mathbb{D}^2 : |z_1|^2 + \varphi(|z_2|^2) < 1\}.$$

Exercise 2.5.24. (a) $D_{\varphi,h}$ is a normalized (cf. (2.3.1)) bounded pseudoconvex complete Reinhardt domain with $(\partial\mathbb{D}) \times (h\overline{\mathbb{D}}) \subset \partial D_{\varphi,h}$.

(b) $D_{\varphi,h} \notin \{\mathbb{D}^2,\ \mathbb{E}_{(1,\alpha)}, \mathbb{E}_{(\alpha,1)},\ \alpha > 0\}$. Consequently, by the Thullen Theorem 2.3.6, every biholomorphic mapping $F\colon D_{\varphi_1,h_1} \to D_{\varphi_2,h_2}$ is of the form $F = \boldsymbol{T}_\zeta$ for some $\zeta \in \mathbb{T}^2$. Hence, $D_{\varphi_1,h_1} \overset{\text{bih}}{\simeq} D_{\varphi_2,h_2}$ iff $h_1 = h_2$ and $\varphi_1 = \varphi_2$.

(c) $D_{\varphi,h}$ is strongly pseudoconvex at a boundary point $a = (a_1,a_2) \in \partial D_{\varphi,h}$ iff $|a_2| > h$ (cf. § 1.18*). In particular, the set of weakly pseudoconvex boundary points is not contained in V_0.

Theorem* 2.5.25 ([Lan-Pat 1993]). *The following conditions are equivalent:*

(i) *there exists a proper holomorphic mapping $F\colon D_{\varphi_1,h_1} \to D_{\varphi_2,h_2}$;*

(ii) *there exist $m \in \mathbb{N}$ and $\zeta_1, \zeta_2 \in \mathbb{T}$ such that:*
   - $h_2 = h_1^m$, $\varphi_1(t) = \varphi_2(t^m)$, $t \in [0,1]$,
   - $F(z) = (\zeta_1 z_1, \zeta_2 z_2^m)$, $z = (z_1,z_2) \in D_{\varphi_1,h_1}$.

Theorem* 2.5.26 ([Lan 1994]). *Let $D \subset \mathbb{C}^2$ be a bounded smooth pseudoconvex complete Reinhardt domain whose weakly pseudoconvex boundary points are contained in V_0. Then any proper holomorphic mapping $F\colon D \to D$ is an automorphism.*

Theorem* 2.5.27 ([Lan-Pin 1995]). *Let D_1, $D_2 \subset \mathbb{C}^2$ be bounded pseudoconvex complete Reinhardt domains such that there exist a complex analytic variety W and an open neighborhood U of a point $a \in \partial D_1$ such that $W \cap U \subset \partial D_1$. Then any proper holomorphic mapping $F = (F_1, F_2)\colon D_1 \to D_2$ is such that F_1 and F_2 depend only on one variable.*

Moreover, if $D_1 = D_2$ is not a bidisc, then F has the form

$$F(z_1, z_2) = (\zeta_1 z_{\sigma(1)}, \zeta_2 z_{\sigma(2)}),$$

where $\zeta_1, \zeta_2 \in \mathbb{T}$, $\sigma \in \mathfrak{S}_2$.

Theorem* 2.5.28 ([Ber-Pin 1995], [Lan-Spi 1996], [Spi 1998]). *Let D_1, $D_2 \subset \mathbb{C}^2$ be bounded complete Reinhardt domains such that at least one of them is neither a bidisc nor a complex ellipsoid. Then any proper holomorphic mapping $F\colon D_1 \to D_2$ has the form*

$$F(z_1, z_2) = (c_1 z_{\sigma(1)}^{m_1}, c_2 z_{\sigma(2)}^{m_2}),$$

where $c_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, $j = 1, 2$, $\sigma \in \mathfrak{S}_2$.

Moreover, if $D_1 = D_2$, then F has the form

$$F(z_1, z_2) = (\zeta_1 z_{\sigma(1)}, \zeta_2 z_{\sigma(2)}),$$

where $\zeta_1, \zeta_2 \in \mathbb{T}$, $\sigma \in \mathfrak{S}_2$.

Remark 2.5.29. The full description of proper holomorphic mappings $F\colon D_1 \to D_2$, where $D_1, D_2 \subset \mathbb{C}^2$ are bounded Reinhardt domains, may be found in [Isa-Kru 2006].

For the case of proper holomorphic mappings between unbounded Reinhardt domains we mention the following result.

Theorem* 2.5.30 ([Edi-Zwo 1999]). *Let $\alpha, \beta \in \mathbb{Z}_+^2$ and let*

$$\mathbb{C}^2 \supset \boldsymbol{D}_\alpha \xrightarrow{F} \boldsymbol{D}_\beta \subset \mathbb{C}^2$$

be a proper holomorphic mapping. Then

$$F(z) = (H^{1/\beta_1}(z^\alpha) z_1^{k_1}, \zeta z_2^{k_2} H^{-1/\beta_2}(z^\alpha)),$$

or

$$F(z) = (H^{1/\beta_1}(z^\alpha) z_2^{\ell_1}, \zeta z_1^{\ell_2} H^{-1/\beta_2}(z^\alpha)),$$

where $H \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$, $\zeta \in \mathbb{T}$, and $k_1, k_2, \ell_1, \ell_2 \in \mathbb{Z}_+$ are such that

$$\alpha_2 \beta_1 k_1 = \alpha_1 \beta_2 k_2, \quad \alpha_1 \beta_1 \ell_1 = \alpha_2 \beta_2 \ell_2;$$

compare with Theorems 2.4.5, 2.4.7.

Theorem* 2.5.31 ([Din-Pri 1987]). *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain and let $F\colon D \to \mathbb{E}_p$ be a polynomial proper mapping with $p \in \mathbb{N}^n$. Then $F(z) = (z_1^{d_1}, \dots, z_n^{d_n})$ with $d_1, \dots, d_n \in \mathbb{N}$, up to action of $\mathbb{T}^n$ on D and an automorphism of $\mathbb{E}_p$.*[15] *Moreover, $D \overset{\text{bih}}{\simeq} \mathbb{E}_q$ with $q_j := d_j p_j$, $j = 1, \dots, n$.*

Theorem* 2.5.32 ([Din-Pri 1988]). *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain with $0 \in D$ and let $p \in \mathbb{N}^n$. Then the following conditions are equivalent:*

(i) *there exists a proper holomorphic mapping $F\colon D \to \mathbb{E}_p$;*
(ii) *there exists a proper polynomial mapping $F\colon D \to \mathbb{E}_p$.*

Theorem* 2.5.33 ([Din-Pri 1989]). *Let $D_1 \subset \mathbb{C}^n$ be a Reinhardt domain and let $D_2 \subset \mathbb{C}^n$ be a bounded simply connected strictly pseudoconvex domain with C^∞ boundary. Then any proper holomorphic map $F\colon D_1 \to D_2$ is, up to an automorphism of D_2, of the form $F(z) = (z_1^{d_1}, \dots, z_n^{d_n})$ with $d_1, \dots, d_n \in \mathbb{N}$.*

Remark 2.5.34. For $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{R}^2)_*$ and $0 < r^- < r^+ < +\infty$ let

$$D_{\alpha,r^-,r^+} := \{(z_1, z_2) \in \mathbb{C}^2(\alpha) : r^- < |z^\alpha| < r^+\}.$$

Recently Ł. Kosiński [Kos 2007] gave the full characterization of all proper holomorphic mappings $F\colon D_{\alpha,r^-,r^+} \to D_{\beta,R^-,R^+}$. More precisely, let

$$P_r := \mathbb{A}(1/r, r), \quad D_{\gamma,r} := \{(z_1, z_2) \in \mathbb{C}^2 : 1/r < |z_1||z_2|^\gamma < r\},$$
$$\gamma \in \mathbb{R} \setminus \mathbb{Q},\ r > 1.$$

One may prove (Exercise) that D_{α,r^-,r^+} is algebraically equivalent to a domain of one of the following three types:

$$\begin{aligned} &P_r \times \mathbb{C}, && \text{if } \alpha_1\alpha_2 = 0,\\ &P_r \times \mathbb{C}_*, && \text{if } \alpha_2/\alpha_1 \in \mathbb{Q}_*,\\ &D_{\gamma,r}, && \text{if } \gamma := \alpha_2/\alpha_1 \notin \mathbb{Q}. \end{aligned} \tag{2.5.4}$$

If D_1, D_2 are of type (2.5.4), then there are no proper holomorphic mappings $F\colon D_1 \to D_2$ except for the following four cases:

(1) $D_1 = P_r \times \mathbb{C}$, $D_2 = P_{r^m} \times \mathbb{C}$ ($m \in \mathbb{N}$), $F(z) = (\zeta z_1^{\varepsilon m}, P(z))$, where $\zeta \in \mathbb{T}$, $\varepsilon \in \{-1, 1\}$, $P(z) = \sum_{j=0}^N P_j(z_1) z_2^j$, $N \in \mathbb{N}$, $P_0, \dots, P_N \in \mathcal{O}(P_r)$, $P(z_1, \cdot) \not\equiv \text{const}$, $z_1 \in P_r$.

(2) $D_1 = P_r \times \mathbb{C}_*$, $D_2 = P_{r^m} \times \mathbb{C}$ ($m \in \mathbb{N}$), $F(z) = (\zeta z_1^{\varepsilon m}, z_2^{-k} P(z))$, where $\zeta \in \mathbb{T}$, $\varepsilon \in \{-1, 1\}$, $P(z) = \sum_{j=0}^N P_j(z_1) z_2^j$, $k, N \in \mathbb{N}$, $0 < k < N$, $P_0, \dots, P_N \in \mathcal{O}(P_r)$, $\sum_{j=0}^{k-1} |P_j(z_1)| > 0$, $\sum_{j=k+1}^N |P_j(z_1)| > 0$, $z_1 \in P_r$.

[15] That is, there exist $\zeta \in \mathbb{T}^n$ and $\Phi \in \operatorname{Aut}(\mathbb{E}_p)$ such that $\Phi \circ F \circ \boldsymbol{T}_\zeta(z) = (z_1^{d_1}, \dots, z_n^{d_n})$, $z \in D$.

(3) $D_1 = P_r \times \mathbb{C}_*$, $D_2 = P_{r^m} \times \mathbb{C}_*$ ($m \in \mathbb{N}$), $F(z) = (\zeta z_1^{\varepsilon m}, z_2^k g(z_1))$, where $\zeta \in \mathbb{T}$, $\varepsilon \in \{-1, 1\}$, $k \in \mathbb{Z}_*$, $g \in \mathcal{O}^*(P_r)$.

(4) $D_1 = D_{\gamma,r}$, $D_2 = D_{\delta,R}$ with

$$\frac{\log R}{\log r} = k_1 + \ell_1\delta, \quad \gamma\frac{\log R}{\log r} = k_2 + \ell_2\delta$$

for some $k = (k_1, k_2)$, $\ell = (\ell_1, \ell_2) \in \mathbb{Z}^2$, $F(z) = (az^{\varepsilon k}, bz^{\varepsilon \ell})$, $\varepsilon \in \{-1, 1\}$, $a, b \in \mathbb{C}$, $|a||b|^\delta = 1$.

2.5.4 Non-compact automorphism groups

Theorem* 2.5.35 ([Bed-Pin 1998] (see also [Bed-Pin 1988])). *Let $D \subset \mathbb{C}^2$ be a bounded domain with real analytic boundary such that* $\operatorname{Aut}(D)$ *is non-compact.*[16] *Then*

$$D \overset{\text{bih}}{\simeq} \mathbb{E}_{(1,m)} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\},$$

where $m \in \mathbb{N}$. In particular, D admits a proper holomorphic mapping onto $\mathbb{B}_2$.

We say that a bounded domain $D \subset \mathbb{C}^n$ with smooth boundary is of *finite type* if there exists an $m \in \mathbb{N}$ such that for every point $a \in \partial D$ and for every complex one-dimensional manifold V passing through a, the order of contact of ∂D and V does not exceed m, i.e. for any $a \in \partial D$ and $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ with $\varphi(0) = a$ we have $\operatorname{ord}_0(u \circ \varphi) \le m$ for any local defining function $u\colon U \to \mathbb{R}$ defined in a neighborhood U of a with $\varphi(\mathbb{D}) \subset U$.

Notice that Theorem 2.5.35 remains true if D is a pseudoconvex domain with smooth boundary of finite type.

Theorem* 2.5.36 ([Bed-Pin 1991]). *Let $D \subset \mathbb{C}^{n+1}$ be a bounded pseudoconvex domain with smooth boundary of finite type such that* $\operatorname{Aut}(D)$ *is non-compact. Assume that the Levi form of a defining function of D has rank at least $n-1$ at each boundary point. Then*

$$D \overset{\text{bih}}{\simeq} \mathbb{E}_{(1,\dots,1,m)} = \Big\{(z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C} : |w|^{2m} + \sum_{j=1}^n |z_j|^2 < 1\Big\},$$

where $m \in \mathbb{N}$.

Theorem* 2.5.37 ([Bed-Pin 1994]). *Let $D \subset \mathbb{C}^{n+1}$ be a convex bounded domain with smooth boundary of finite type such that* $\operatorname{Aut}(D)$ *is non-compact. Then there exist $m_1, \dots, m_n \in \mathbb{N}$ and $a_{\alpha,\beta} = \bar{a}_{\beta,\alpha} \in \mathbb{C}$ such that*

$$D \overset{\text{bih}}{\simeq} \Big\{(z, w) \in \mathbb{C}^n \times \mathbb{C} : |w|^2 + \sum a_{\alpha,\beta} z^\alpha \bar{z}^\beta < 1\Big\},$$

[16] That is, there exist a point $a \in D$ and a sequence $(\Phi_\nu)_{\nu=1}^\infty \subset \operatorname{Aut}(D)$ such that $\Phi_\nu(a) \to \partial D$.

where the sum is taken over all $\alpha, \beta \in \mathbb{Z}_+^n$ *with* $\alpha_1/m_1 + \cdots + \alpha_n/m_n = 1$ *and* $\beta_1/m_1 + \cdots + \beta_n/m_n = 1$.

Theorem* 2.5.38 ([FIK 1996a]). *Let* $D \subset \mathbb{C}^{n+1}$ *be a bounded Reinhardt domain with* $\mathcal{C}^\infty$*-smooth boundary such that* $\operatorname{Aut}(D)$ *is non-compact. Then*

$$D \overset{\text{bih}}{\simeq} \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : |w|^2 + P(|z_1|, \dots, |z_n|) < 1\},$$

up to permutation and rescaling of variables, where P *is a non-negative polynomial with real coefficients.*

The case where ∂D is only of class $\mathcal{C}^k$ was solved in [Isa-Kra 1997] – in this case P is a non-negative $\mathcal{C}^k$-function.

Theorem* 2.5.39 ([Isa-Kra 1998]). *Let* $D \subset \mathbb{C}^2$ *be a hyperbolic Reinhardt domain with* $\mathcal{C}^k$*-smooth boundary* ($k \geq 1$) *such that* $D \cap V_0 \neq \varnothing$ *and* $\operatorname{Aut}(D)$ *is non-compact. Then D is algebraically equivalent to one of the following three types of domains:*

- $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$, *where* $m < 0$ *or* $m > k$ *or* $m \in \mathbb{N}$*;*
- $\left\{(z_1, z_2) \in \mathbb{D} \times \mathbb{C} : \frac{1}{(1-|z_1|^2)^\alpha} < |z_2| < \frac{R}{(1-|z_1|^2)^\alpha}\right\}$, *where* $1 < R \leq +\infty$, $\alpha > 0$*;*
- $\{(z_1, z_2) \in \mathbb{C}^2 : \exp(\beta|z_1|^2) < |z_2| < R\exp(\beta|z_1|^2)\}$, *where* $1 < R \leq +\infty$, $\beta \in \mathbb{R}_*$ *and* ($R = +\infty \Rightarrow \beta > 0$).

Remark 2.5.40. (a) Some of the above results may give the impression that every domain with non-compact automorphism group is biholomorphic to a Reinhardt domain. This is not true – the following bounded pseudoconvex circular domain with real analytic boundary and non-compact automorphism group is not biholomorphic to any Reinhardt domain ([FIK 1996b]):

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^4 + |z_3|^4 + (\bar{z}_2 z_3 + \bar{z}_3 z_2)^2 < 1\}.$$

(b) In [KKS 2005] the reader may find a characterization of those "analytic polyhedra" in $\mathbb{C}^2$ whose automorphism groups are not compact.

(c) For general domains with non-compact automorphism groups the reader may contact the survey article [Isa-Kra 1999].

Chapter 3

Reinhardt domains of existence of special classes of holomorphic functions

3.1 General theory

Let D be a Reinhardt domain of holomorphy and let $\mathcal{S} \subset \mathcal{O}(D)$ be a natural Fréchet space (cf. § 1.10), e.g. $\mathcal{S} = \mathcal{H}^{\infty,k}(D)$, $\mathcal{A}^k(D)$, $L_h^{p,k}(D)$, $\mathcal{O}^{(N)}(D)$, $\mathcal{O}^{(0+)}(D)$; cf. Example 1.10.7. Our aim is to find geometric characterizations of those Reinhardt domains D which are $\mathcal{S}$-domains of holomorphy. We like to point out that such geometric characterizations are not known for more general classes of domains (e.g. balanced domains of holomorphy). Except for § 3.1, all results presented in this chapter are more elaborated and detailed versions of some results from [Jar-Pfl 2000], § 4.1.

Remark 3.1.1. Consider the case where $\mathcal{S} = \mathcal{O}^{(N)}(D)$ (Example 1.10.7 (f)).

(a) First recall some known general results. Let $G \subset \mathbb{C}^n$ be a domain of holomorphy (Reinhardt or not). Then:

- ([Jar-Pfl 2000], Corollary 4.3.9.) G is an $\mathcal{O}^{(2n+\varepsilon)}(G)$-domain of holomorphy for any $\varepsilon > 0$.
- ([Jar-Pfl 2000], Corollary 4.3.9.) If G is a bounded domain, then G is an $\mathcal{O}^{(n+\varepsilon)}(G)$-domain of holomorphy for any $\varepsilon > 0$.
- ([Jar-Pfl 2000], Theorem 4.2.7.) If G is bounded and fat, then G is an $L_h^2(G)$-domain of holomorphy; in particular, in this case G is an $\mathcal{O}^{(n)}(G)$-domain of holomorphy (cf. Example 1.10.7 (c), (f), (g)).

$\boxed{?}$ We do not know whether the above results are optimal, e.g. whether there exists a $\mu < n$ such that every bounded fat domain of holomorphy is an $\mathcal{O}^{(\mu)}$-domain of holomorphy. $\boxed{?}$

(b) In contrast to the above general situation, in the case where D is a Reinhardt domain of holomorphy, we are able to show that:

- D is an $\mathcal{O}^{(1)}$-domain of holomorphy (Theorem 3.4.4).
- If D is fat, then D is an $\mathcal{O}^{(\varepsilon)}$-domain of holomorphy for any $\varepsilon > 0$ (Theorem 3.4.3).

The following notion will be useful in the sequel.

Definition 3.1.2. Let $D \subset \mathbb{C}^n$ be a Reinhardt domain. We say that a natural Fréchet space $\mathcal{S} \subset \mathcal{O}(D)$ is *regular* if for every function $f \in \mathcal{S}$ with the Laurent expansion

$$f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha^f z^\alpha, \quad z \in D,$$

we have:

- $z^\alpha \in \mathcal{S}$, $\alpha \in \Sigma(f) = \{\alpha \in \mathbb{Z}^n : a_\alpha^f \neq 0\}$,
- the set $\{a_\alpha^f z^\alpha : \alpha \in \Sigma(f)\}$ is bounded in $\mathcal{S}$ (cf. Definition 1.10.3).

Remark 3.1.3. Observe that there are natural Fréchet spaces which are not regular. For example, $\mathcal{S} := \mathbb{C} \cdot f$, where $f \in \mathcal{O}(D)$ is not a "monomial" of the form cz^α.

Example 3.1.4 (Examples of regular natural Fréchet spaces). (a) $\mathcal{S} = \mathcal{O}(D)$.

Indeed, by the Cauchy inequalities, for any Reinhardt compact set $K \subset D$ we have $\|a_\alpha^f z^\alpha\|_K \leq \|f\|_K$, $\alpha \in \Sigma(f)$.

(b) $\mathcal{S} = \mathcal{H}_{\mathrm{loc}}^\infty(D)$.

Indeed, by the Cauchy inequalities, we have $\|a_\alpha^f z^\alpha\|_{\mathbb{B}(r)\cap D} \leq \|f\|_{\mathbb{B}(r)\cap D}$, $\alpha \in \Sigma(f)$, $r > 0$.

(c) $\mathcal{S} = \mathcal{H}^\infty(D)$.

Indeed, by the Cauchy inequalities we have $\|a_\alpha^f z^\alpha\|_D \leq \|f\|_D$, $\alpha \in \Sigma(f)$.

(d) $\mathcal{S} = \mathcal{O}^{(N)}(D)$ $(N > 0)$.

Indeed, the function δ_D is invariant under n-rotations. Hence, using once again the Cauchy inequalities, for $r \in D \cap \mathbb{R}_{>0}^n$, $\alpha \in \Sigma(f)$, we get

$$\delta_D^N(r)|a_\alpha^f r^\alpha| \leq \delta_D^N(r)\|f\|_{\partial_0\mathbb{P}(r)} = \|\delta_D^N f\|_{\partial_0\mathbb{P}(r)} \leq \|\delta_D^N f\|_D.$$

Thus $\|\delta_D^N a_\alpha^f z^\alpha\|_D \leq \|\delta_D^N f\|_D$, $\alpha \in \Sigma(f)$.

(e) $\mathcal{S} = L_h^p(D)$ $(1 \leq p < +\infty)$ (cf. Example 1.10.7 (c)).

(f) $\mathcal{S} = \mathcal{A}(D)$ in the case where D satisfies the Fu condition.

Indeed, in virtue of (b), we only need to observe that $z^\alpha \in \mathcal{C}(\bar{D})$, $\alpha \in \Sigma(f)$ (Exercise).

Remark 3.1.5. (a) Let $\mathcal{S}_j$ be a natural Fréchet space in $\mathcal{O}(D)$ with the topology $\mathcal{T}(Q_j)$ generated by a countable family Q_j of seminorms, $j \in \mathbb{N}$. Consider the space $\mathcal{S} := \bigcap_{j\in\mathbb{N}} \mathcal{S}_j$ with the topology generated by the family $\bigcup_{j=1}^\infty Q_j|_{\mathcal{S}}$. We know that $\mathcal{S}$ is also a natural Fréchet space in $\mathcal{O}(D)$ (Remark 1.10.7 (h)).

Observe that if each space $\mathcal{S}_j$ is regular, then $\mathcal{S}$ is regular.

In particular, the spaces $L_h^\diamond(D)$, $\mathcal{O}^{(0+)}(D)$ are regular.

(b) Let $A \subset \mathbb{Z}_+^n$, $0 \in A$, and suppose that $\mathcal{S}_\nu$ is a natural Fréchet space in $\mathcal{O}(D)$ with the topology $\mathcal{T}(Q_\nu)$ given by a countable family of seminorms Q_ν, $\nu \in A$. Define $\mathcal{S}_A := \{f \in \mathcal{O}(D) : D^\nu f \in \mathcal{S}_\nu,\ \nu \in A\}$. We know that $\mathcal{S}_A$ is a natural Fréchet space with the topology generated by the family of seminorms $\mathcal{S}_A \ni f \mapsto q(D^\nu f)$, $\nu \in A$, $q \in Q_\nu$ (Remark 1.10.7 (i)).

Observe that if each space $\mathcal{S}_\nu$ is regular, then $\mathcal{S}_A$ is regular.

Indeed, if $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha^f z^\alpha$, $z \in D$, then

$$D^\nu f(z) = \sum_{\alpha\in\Sigma(f)} D^\nu(a_\alpha^f z^\alpha) = \sum_{\alpha\in\Sigma(f),\ \binom{\alpha}{\nu}\neq 0} \nu!\binom{\alpha}{\nu} a_\alpha^f z^{\alpha-\nu}, \quad z \in D.$$

Hence, for every $\nu \in A$, the set $\{D^\nu(a^f_\alpha z^\alpha) : \alpha \in \Sigma(f)\}$ is bounded in $\mathcal{S}_\nu$, which implies that the set $\{a^f_\alpha z^\alpha : \alpha \in \Sigma(f)\}$ is bounded in $\mathcal{S}_A$.

In particular, the natural Fréchet spaces $\mathcal{H}^{\infty,k}(D)$, $\mathcal{H}^{\infty,k}_{\mathrm{loc}}(D)$, $L^{p,k}_h(D)$ and $L^{\diamond,k}_h(D)$ ($k \in \mathbb{N} \cup \{\infty\}$) are regular. Moreover, the space $\mathcal{A}^k(D)$ is regular provided that D satisfies the Fu condition.

Proposition 3.1.6. *Let $\varnothing \neq D \subsetneq \mathbb{C}^n$ be a fat Reinhardt domain and let $\mathcal{S} \subset \mathcal{O}(D)$ be a regular Fréchet space. Then the following conditions are equivalent:*

(i) *D is an $\mathcal{S}$-domain of holomorphy;*

(ii) *there exists an $f \in \mathcal{S}$, $f(z) = \sum_{\alpha \in \mathbb{Z}^n} a^f_\alpha z^\alpha$, $z \in D$, such that the set $\Sigma(f)$ is unbounded and*

$$D = \{z \in \mathbb{C}^n(\Sigma(f)) : v^*(z) < 1\},$$

where $v(z) := \limsup_{|\alpha| \to +\infty} |a^f_\alpha z^\alpha|^{1/|\alpha|}$, $z \in \mathbb{C}^n(\Sigma(f))$;

(ii′) *there exists an $f \in \mathcal{S}$, $f(z) = \sum_{\alpha \in \mathbb{Z}^n} a^f_\alpha z^\alpha$, $z \in D$, such that the set $\Sigma(f)$ is unbounded and*

$$D = \bigcup_{\nu=1}^{\infty} \Big(\operatorname{int} \bigcap_{\alpha \in \Sigma(f):\, |\alpha| \geq \nu} \{z \in \mathbb{C}^n(\Sigma(f)) : |a^f_\alpha z^\alpha| < 1\}\Big);$$

(iii) *there exist an unbounded set $\Sigma \subset (\mathbb{Z}^n)_*$ ($\Sigma \subset (\mathbb{Z}^n_+)_*$ if $0 \in D$) and $(c_\alpha)_{\alpha \in \Sigma} \subset \mathbb{R}_{>0}$ with (1.15.1)[1] such that:*
- *$D = \{z \in \mathbb{C}^n(\Sigma) : v^*(z) < 1\}$, where $v(z) := \limsup_{|\alpha| \to +\infty} |c_\alpha z^\alpha|^{1/|\alpha|}$, $z \in \mathbb{C}^n(\Sigma)$,*
- *$z^\alpha \in \mathcal{S}$, $\alpha \in \Sigma$, and the set $\{c_\alpha z^\alpha : \alpha \in \Sigma\}$ is bounded in $\mathcal{S}$;*

(iii′) *there exist an unbounded set $\Sigma \subset (\mathbb{Z}^n)_*$ ($\Sigma \subset (\mathbb{Z}^n_+)_*$ if $0 \in D$) and $(c_\alpha)_{\alpha \in \Sigma} \subset \mathbb{R}_{>0}$ with (1.15.1) such that:*
- *$D = \bigcup_{\nu=1}^{\infty} \big(\operatorname{int} \bigcap_{\alpha \in \Sigma:\, |\alpha| \geq \nu} \{z \in \mathbb{C}^n(\Sigma) : c_\alpha |z^\alpha| < 1\}\big)$,*
- *$z^\alpha \in \mathcal{S}$, $\alpha \in \Sigma$, and the set $\{c_\alpha z^\alpha : \alpha \in \Sigma\}$ is bounded in $\mathcal{S}$;*

(iv) *for every point $a \in \mathbb{C}^n_* \setminus \bar{D}$ there exist sequences $(\alpha(k))_{k=1}^{\infty} \subset (\mathbb{Z}^n)_*$ ($(\alpha(k))_{k=1}^{\infty} \subset (\mathbb{Z}^n_+)_*$ if $0 \in D$) and $(d(k))_{k=1}^{\infty} \subset \mathbb{R}_{>0}$ such that:*
- *$|\alpha(k)| \to +\infty$,*
- *$D \subset \mathbb{C}^n(\Sigma)$, where $\Sigma := \{\alpha(k) : k \in \mathbb{N}\}$,*
- *$z^{\alpha(k)} \in \mathcal{S}$, $k \in \mathbb{N}$, and the set $\{d(k) z^{\alpha(k)} : k \in \mathbb{N}\}$ is bounded in $\mathcal{S}$,*
- *$d(k)|a^{\alpha(k)}| \to +\infty$.*

[1] That is, $\sup\{c_\alpha^{1/|\alpha|} : \alpha \in \Sigma\} < +\infty$.

Observe that condition (iii′) gives an effective geometric characterization of $\mathcal{S}$-domains of holomorphy. Notice that the result need not be true for non-regular natural Fréchet spaces (cf. Remark 3.1.3).

Proof. The equivalences (ii) $\Leftrightarrow$ (ii′) and (iii) $\Leftrightarrow$ (iii′) follow from Lemma 1.15.13.

(i) $\Rightarrow$ (ii): By Proposition 1.11.11, there exists an $f \in \mathcal{S}$ such that D is the domain of existence of f. Let $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha^f z^\alpha$, $z \in D$, be the Laurent expansion of f. By Proposition 1.11.6 the domain of convergence $\mathfrak{D}_f$ of the above series is a domain of holomorphy. Thus $D = \mathfrak{D}_f$. In the case where $\Sigma(f)$ is finite we have $\mathfrak{D}_f = \mathbb{C}^n(\Sigma(f))$, which contradicts our assumption that $D \subsetneq \mathbb{C}^n$ is fat. Now, it remains to use Proposition 1.15.15.

(ii) $\Rightarrow$ (iii): Put $\Sigma := \Sigma(f)_*$, $c_\alpha := |a_\alpha^f|$. The regularity of $\mathcal{S}$ implies that the set $\{a_\alpha^f z^\alpha : \alpha \in \Sigma\}$ is bounded in $\mathcal{S}$.

(iii) $\Rightarrow$ (iv): Fix a point $a \in \mathbb{C}_*^n \setminus \bar{D}$. By Lemma 1.15.13 (h), $v(a) = v^*(a) > \eta > 1$. Thus, there exists a sequence $(\alpha(k))_{k=1}^\infty \subset \Sigma$ such that $|\alpha(k)| \to +\infty$ and $c_{\alpha(k)}|a^{\alpha(k)}| \geq \eta^{|\alpha(k)|} \to +\infty$, $k \to +\infty$.

(iv) $\Rightarrow$ (i): Suppose that D is not an $\mathcal{S}$-domain of holomorphy and let D_0, $\widetilde{D}$ be as in Proposition 1.11.2 (*). Since D is fat, we may assume that $\widetilde{D} \subset \mathbb{C}_*^n$ and that there exists a point $a \in \widetilde{D} \setminus \bar{D}$. Let $Q = \{q_i : i \in \mathbb{N}\}$ be a countable family of seminorms generating the topology of $\mathcal{S}$ with $q_i \leq q_{i+1}$, $i \in \mathbb{N}$. Since $\mathcal{S}$ is a natural Fréchet space, the extension operator $\mathcal{S} \ni g \mapsto \tilde{g} \in \mathcal{O}(\widetilde{D})$ is continuous (Remark 1.11.3 (n)). In particular, there exist $C > 0$ and $i_0 \in \mathbb{N}$ such that $|\tilde{g}(a)| \leq C q_{i_0}(g)$, $g \in \mathcal{S}$. Since the set $\{d(k)z^{\alpha(k)} : k \in \mathbb{N}\}$ is bounded in $\mathcal{S}$, there exists a constant $M > 0$ such that $q_{i_0}(d(k)z^{\alpha(k)}) \leq M$, $k \in \mathbb{N}$. In particular, $d(k)|a^{\alpha(k)}| \leq CM$, $k \in \mathbb{N}$; a contradiction. $\square$

Proposition 3.1.7. *Let $\varnothing \neq D \subsetneq \mathbb{C}^n$ be a Reinhardt domain and let $\mathcal{S} \subset \mathcal{H}^\infty(D)$ be a natural Banach algebra which is moreover regular* (*e.g.* $\mathcal{S} = \mathcal{H}^{\infty,k}(D)$ *with the norm* $\|f\|_{\mathcal{S}} := 2^k \max\{\|D^\nu f\|_D : |\nu| \leq k\}$; *cf. Example* 1.10.7 (j)). *Then the following conditions are equivalent:*

(i) *D is an $\mathcal{S}$-domain of holomorphy;*

(ii) *there exists an $f \in \mathcal{S}$, $\|f\|_{\mathcal{S}} \leq 1$, $\|f\|_D \leq 1$, $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha^f z^\alpha$, $z \in D$, such that the set $\Sigma(f)$ is unbounded and*

$$D = \{z \in \mathbb{C}^n(\Sigma(f)) : u^*(z) < 1\},$$

where $u(z) := \sup\{|a_\alpha^f z^\alpha|^{1/|\alpha|} : \alpha \in \Sigma(f)_*\}$, $z \in \mathbb{C}^n(\Sigma(f))$;

(ii′) *there exists an $f \in \mathcal{S}$, $\|f\|_{\mathcal{S}} \leq 1$, $\|f\|_D \leq 1$, $f(z) = \sum_{\alpha\in\mathbb{Z}^n} a_\alpha^f z^\alpha$, $z \in D$, such that the set $\Sigma(f)$ is unbounded and*

$$D = \operatorname{int} \bigcap_{\alpha\in\Sigma(f)} \{z \in \mathbb{C}^n(\Sigma(f)) : |a_\alpha^f z^\alpha| < 1\};$$

(iii) *there exist* $\Sigma \subset (\mathbb{Z}^n)_*$ $(\Sigma \subset (\mathbb{Z}^n_+)_*$ *if* $0 \in D)$ *and* $(c_\alpha)_{\alpha\in\Sigma} \subset \mathbb{R}_{>0}$ *with* (1.15.1) *such that:*
- $D = \{z \in \mathbb{C}^n(\Sigma) : u^*(z) < 1\}$, *where* $u(z) := \sup\{|c_\alpha z^\alpha|^{1/|\alpha|} : \alpha \in \Sigma\}$, $z \in \mathbb{C}^n(\Sigma)$,
- $z^\alpha \in \mathcal{S}$ *and* $\|c_\alpha z^\alpha\|_{\mathcal{S}} \le 1$, $\alpha \in \Sigma$;

(iii′) *there exist* $\Sigma \subset (\mathbb{Z}^n)_*$ $(\Sigma \subset (\mathbb{Z}^n_+)_*$ *if* $0 \in D)$ *and* $(c_\alpha)_{\alpha\in\Sigma} \subset \mathbb{R}_{>0}$ *with* (1.15.1) *such that:*
- $D = \operatorname{int}\bigcap_{\alpha\in\Sigma}\{z \in \mathbb{C}^n(\Sigma) : c_\alpha|z^\alpha| < 1\}$,
- $z^\alpha \in \mathcal{S}$ *and* $\|c_\alpha z^\alpha\|_{\mathcal{S}} \le 1$, $\alpha \in \Sigma$.

Proof. The equivalences (ii) $\Leftrightarrow$ (ii′) and (iii) $\Leftrightarrow$ (iii′) follow from Lemma 1.15.13.

(i) $\Rightarrow$ (ii): There exists an $f \in \mathcal{S}$ such that D is the domain of existence of f (Proposition 1.11.11). We may assume that $\|f\|_{\mathcal{S}} \le 1$, $\|f\|_D \le 1$. Since f is not holomorphically continuable beyond D, D coincides with the domain of convergence of the Laurent series $\sum_{\alpha\in\mathbb{Z}^n} a^f_\alpha z^\alpha$ of f. Observe that $\Sigma(f)$ is unbounded (because D is fat (cf. Corollary 1.11.4 (a))). It remains to apply the second part of Proposition 1.15.15.

The implication (ii) $\Rightarrow$ (iii) follows from the regularity of $\mathcal{S}$ (cf. the proof of Proposition 3.1.6).

(iii) $\Rightarrow$ (i): Notice that, by (iii′), D must be fat. Suppose that D is not an $\mathcal{S}$-domain of holomorphy and let D_0 and $\widetilde{D}$ be as usual. We may assume that $\widetilde{D} \subset \mathbb{C}^n(\Sigma)$. Since $\mathcal{S}$ is a natural Banach algebra, we have $\|\tilde{g}\|_{\widetilde{D}} \le \|g\|_{\mathcal{S}}$, $g \in \mathcal{S}$ (Remark 1.11.3 (n)). In particular, $\|c_\alpha z^\alpha\|^{1/|\alpha|}_{\widetilde{D}} \le 1$, $\alpha \in \Sigma$. Hence $u^* \le 1$ on $\widetilde{D}$. Since $\varnothing \ne D_0 \subset D \cap \widetilde{D}$, the maximum principle implies that $u^* < 1$ on $\widetilde{D}$. Thus $\widetilde{D} \subset D$; a contradiction. $\square$

Corollary 3.1.8. *Let* $D \subsetneq \mathbb{C}^n$ *be a Reinhardt domain and let* $k \in \mathbb{Z}_+$. *Then the following conditions are equivalent:*

(i) D *is an* $\mathcal{H}^{\infty,k}$*-domain of holomorphy;*

(ii)
$$D = \operatorname{int}\bigcap_{\alpha\in\Sigma} \boldsymbol{D}_{\alpha,c(\alpha)},$$
where $\Sigma \subset (\mathbb{Z}^n)_*$, $c\colon \Sigma \to \mathbb{R}$, *and*
$$2^k\beta!\left|\binom{\alpha}{\beta}\right|e^{-c(\alpha)}\|z^{\alpha-\beta}\|_D \le 1, \quad \alpha\in\Sigma,\ |\beta|\le k,\ \binom{\alpha}{\beta}\ne 0.$$

In particular, $\boldsymbol{D}_{\alpha,c}$ $(\alpha \ne 0)$ *is an* $\mathcal{H}^{\infty,k}$*-domain of holomorphy iff* $k = 0$ *and* $\alpha \in \mathbb{R}\cdot\mathbb{Z}^n$.

Exercise 3.1.9. Prove that $\mathcal{H}^{\infty,k}(\boldsymbol{D}_{\alpha,c}) \simeq \mathbb{C}$ for $k \ge 1$.

3.2 Elementary Reinhardt domains

This section is devoted to the most elementary case where $D = \boldsymbol{D}_{\alpha,c}$ is an elementary Reinhardt domain. The reader should consider the results below as an illustration of problems we will meet in the sequel.

Theorem 3.2.1 ([Jar-Pfl 1987]). *Let $D := \boldsymbol{D}_{\alpha,c}$ be an elementary Reinhardt domain with $\alpha \in (\mathbb{R}^n)_*$.*[2] *Then:*

(a) *For any $N > 0$ the domain D is an $\mathcal{O}^{(N)}$-domain of holomorphy.*

(b) *For every $k \in \mathbb{Z}_+$ the domain D is an $\mathcal{A}^k$-domain of holomorphy.*

(c) *The following conditions are equivalent:*

  (i) *D is an $\mathcal{H}^\infty$-domain of holomorphy;*

  (ii) *D is an $\mathcal{O}^{(0+)}$-domain of holomorphy;*

  (iii) *$\alpha \in \mathbb{R} \cdot \mathbb{Z}^n$.*

(d) *If $\alpha \notin \mathbb{R} \cdot \mathbb{Z}^n$, then $\mathcal{H}^\infty(D) \simeq \mathbb{C}$.*

(e) *If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $\alpha_1, \dots, \alpha_n$ are relatively prime, then the operator*

$$\mathcal{H}^\infty(\mathbb{D}) \ni g \mapsto \hat{g} \in \mathcal{H}^\infty(D),$$

where $\hat{g}(z) := g(e^{-c}z^\alpha)$, $z \in D$, defines a Banach algebra isomorphism

$$\mathcal{H}^\infty(\mathbb{D}) \xrightarrow{\Psi} \mathcal{H}^\infty(D).$$

(f) *If $\alpha \in \mathbb{R} \cdot \mathbb{Z}^n$, then $\mathcal{H}^\infty(D) \subsetneq \mathcal{O}^{(0+)}(D)$ and, consequently, $\mathcal{H}^\infty(D)$ is of the first Baire category in $\mathcal{O}^{(0+)}(D)$.*

(g) *$L^p_h(D) = \{0\}$, so D is never an L^p_h-domain of holomorphy, $1 \le p < +\infty$.*

Observe that the only problem in (e) is to prove that Ψ is surjective. Indeed, since $\mathbb{D}_* \subset \{e^{-c}z^\alpha : z \in D\}$, we see that $\|\hat{g}\|_D = \|g\|_{\mathbb{D}}$.

Proof. We may assume that $\alpha \in \mathbb{R}^n_*$ (Exercise). Moreover, we may assume that $\alpha_1, \dots, \alpha_s > 0$, $\alpha_{s+1}, \dots, \alpha_n < 0$ for some $0 \le s \le n$.

(a) Fix an $N > 0$ and suppose that D is not an $\mathcal{O}^{(N)}$-domain of holomorphy. Let $D_0, \tilde{D}$ be as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{O}^{(N)}(D)$. Since D is fat, we may assume that $\tilde{D} \subset \mathbb{C}^n_*$.

Put $\varepsilon := N/(3n)$. By the Kronecker theorem (cf. p. 97), there exist sequences

$$(p_{j,\nu})_{\nu=1}^\infty \subset \mathbb{N}, \quad j = 1, \dots, n, \qquad (q_\nu)_{\nu=1}^\infty \subset \mathbb{N}$$

such that

$$|p_{j,\nu} - q_\nu|\alpha_j|| \le \varepsilon, \quad j = 1, \dots, n, \quad q_\nu \to +\infty.$$

[2] That is, $D \subsetneq \mathbb{C}^n$. Recall that D is a fat log-convex domain with $\log D = \boldsymbol{H}_{\alpha,c}$.

We may assume that $q_\nu \ge \varepsilon(1/\alpha_1 + \cdots + 1/\alpha_s)$, $\nu = 1, 2, \ldots$. Put

$$g_\nu(z) := e^{-q_\nu c} z_1^{p_{1,\nu}} \cdots z_s^{p_{s,\nu}} \cdot z_{s+1}^{-p_{s+1,\nu}} \cdots z_n^{-p_{n,\nu}}, \quad z \in \mathbb{C}^n(\alpha).$$

Observe that

$$|g_\nu(z)|^{1/q_\nu} \to e^{-c}|z^\alpha| =: \theta(z), \quad z \in \mathbb{C}^n(\alpha) = \mathbb{C}^s \times \mathbb{C}_*^{n-s}.$$

Suppose for a moment that we already know that $\delta_D^N |g_\nu| \le 1$, $\nu = 1, 2, \ldots$. Then, by Remark 1.11.3 (n) (with $\mathcal{S} = \mathcal{O}^{(N)}(D)$), for every compact $\widetilde{K} \subset \widetilde{D}$ there exists a constant $C_{\widetilde{K}} > 0$ such that $\|g_\nu\|_{\widetilde{K}} \le C_{\widetilde{K}}$, $\nu = 1, 2, \ldots$. Hence $\theta(z) \le 1$ for $z \in \widetilde{K}$ and, consequently, $\theta(z) \le 1$ for $z \in \widetilde{D}$. The maximum principle for the plurisubharmonic function θ gives $\theta < 1$ on $\widetilde{D}$, which implies that $\widetilde{D} \subset D$; a contradiction.

We move to the proof of the estimate $\delta_D^N(z)|g_\nu(z)| \le 1$, $z \in D$, $\nu = 1, 2, \ldots$. Fix an $a \in D$. We may assume that $a \notin V_0$ (Exercise). Let $\eta := \theta(a) \in (0, 1)$. For $j \in \{1, \ldots, n\}$, put $\zeta^j := (a_1, \ldots, a_{j-1}, \eta^{-1/\alpha_j} a_j, a_{j+1}, \ldots, a_n) \in \partial D$. Consequently,

$$\rho_D(a) \le \|a - \zeta^j\| = |1 - \eta^{-1/\alpha_j}||a_j|.$$

Put $I := \{j \in \{1, \ldots, n\} : |a_j| \ge 1\}$. For $\nu \in \mathbb{N}$ we have

$$|g_\nu(a)| = \eta^{q_\nu} \frac{|g_\nu(a)|}{\eta^{q_\nu}} = \eta^{q_\nu} \frac{\prod_{j=1}^s |a_j|^{p_{j,\nu} - q_\nu \alpha_j}}{\prod_{j=s+1}^n |a_j|^{p_{j,\nu} + q_\nu \alpha_j}} \le \eta^{q_\nu} \left(\frac{\prod_{j \in I} |a_j|}{\prod_{j \notin I} |a_j|} \right)^\varepsilon$$

Finally,

$$\begin{aligned}
\delta_D^N(a)|g_\nu(a)| &\le \delta_0^{2n\varepsilon}(a) \rho_D^{n\varepsilon}(a) |g_\nu(a)| \\
&\le \eta^{q_\nu} \left(\delta_0^{2n}(a) \Big(\prod_{j=1}^n |1 - \eta^{-1/\alpha_j}||a_j| \Big) \frac{\prod_{j \in I} |a_j|}{\prod_{j \notin I} |a_j|} \right)^\varepsilon \\
&\overset{3}{\le} \eta^{q_\nu} \left(\eta^{-(1/\alpha_1 + \cdots + 1/\alpha_s)} \delta_0^{2n}(a) \prod_{j \in I} |a_j|^2 \right)^\varepsilon \\
&\le \eta^{q_\nu - (1/\alpha_1 + \cdots + 1/\alpha_s)\varepsilon} \le 1, \quad \nu = 1, 2, \ldots,
\end{aligned}$$

which finishes the proof of the estimate.

(b) Fix a $k \in \mathbb{Z}_+$ and suppose that D is not an $\mathcal{A}^k$-domain of holomorphy. Let $D_0, \widetilde{D}$ be as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{A}^k(D)$. We may assume that $\widetilde{D} \subset \mathbb{C}_*^n$. Take an $\varepsilon > 0$ so small that $\widetilde{D} \not\subset \boldsymbol{D}_{\alpha, c+\varepsilon} =: G$. By (a) (applied to $\boldsymbol{D}_{\alpha, c+\varepsilon}$) and Proposition 1.11.11, there exists an $f \in \mathcal{O}^{(1)}(G)$ such that G is the domain of existence of f. We are going to show that $z^{(3k+3)\mathbf{1}} f \in \mathcal{A}^k(D)$, which obviously will contradict the fact that f is not continuable beyond G.

[3] We have $|1 - \eta^{-1/\alpha_j}| \le \eta^{-1/\alpha_j}$ for $j = 1, \ldots, s$, and $|1 - \eta^{-1/\alpha_j}| \le 1$ for $j = s+1, \ldots, n$.

It suffices to prove that if $1 \le s \le n-1$, then

$$\lim_{\bar{D}\cap\mathbb{C}^n(\alpha)\ni z\to a} z^{(3k+3)\mathbf{1}-\sigma} D^\tau f(z) = 0, \quad a \in (\partial D)\setminus\mathbb{C}^n(\alpha),\text{[4]} \ |\sigma|+|\tau|\le k. \tag{3.2.1}$$

First observe that there exists a neighborhood U of the set $(\partial D)\setminus\mathbb{C}^n(\alpha)$ such that

$$d_G(z) \ge |z^{\mathbf{2}}|, \quad z \in U \cap \bar{D} \cap \mathbb{C}^n(\alpha). \tag{3.2.2}$$

Indeed, fix an $a = (a_1,\dots,a_n) \in (\partial D)\setminus\mathbb{C}^n(\alpha)$. Note that $a_1\cdots a_s = a_{s+1}\cdots a_n = 0$. We only need to prove that there exists a neighborhood U of a such that $\mathbb{P}(z,|z^{\mathbf{2}}|) \subset G$ for any $z \in U \cap D \setminus \boldsymbol{V}_0$ (Exercise). Let U be a neighborhood of a such that $|z^{\mathbf{2}-\boldsymbol{e}_j}| < 1$, $z \in U$, $j = 1,\dots,n$, and

$$\prod_{j=1}^{s}(1+|z^{\mathbf{2}-\boldsymbol{e}_j}|)^{\alpha_j}\prod_{j=s+1}^{n}(1-|z^{\mathbf{2}-\boldsymbol{e}_j}|)^{\alpha_j} < e^{\varepsilon}, \quad z \in U.$$

Then

$$\begin{aligned}&\prod_{j=1}^{s}(|z_j|+|z^{\mathbf{2}}|)^{\alpha_j}\prod_{j=s+1}^{n}(|z_j|-|z^{\mathbf{2}}|)^{\alpha_j}\\ &\qquad = |z_1|^{\alpha_1}\cdots|z_n|^{\alpha_n}\prod_{j=1}^{s}(1+|z^{\mathbf{2}-e_j}|)^{\alpha_j}\prod_{j=s+1}^{n}(1-|z^{\mathbf{2}-e_j}|)^{\alpha_j} < e^{c+\varepsilon},\end{aligned}$$

$z \in U \cap \boldsymbol{D}_\alpha$, which shows that $\mathbb{P}(z,|z^{\mathbf{2}}|) \subset G$, $z \in U \cap D \setminus \boldsymbol{V}_0$.

We need the following lemma.

Lemma 3.2.2. *Let $\Omega \subset \mathbb{C}^n$ be open and let $\delta\colon \Omega \to (0,1]$ be a function such that*

- $\delta \le \rho_\Omega$,
- $|\delta(z') - \delta(z'')| \le \|z'-z''\|$, $z' \in \Omega$, $z'' \in \mathbb{B}(z',\rho_\Omega(z'))$ (*cf. Example* 1.10.7 (g)).

Then

$$\|\delta^{N+|\tau|}D^\tau g\|_\Omega \le \tau!(\sqrt{n})^{|\tau|}2^{N+|\tau|}\|\delta^N g\|_\Omega, \quad N \ge 0,\ \tau \in \mathbb{Z}_+^n,\ g \in \mathcal{O}(\Omega).$$

Proof of Lemma 3.2.2. Fix N, τ, g, and $a \in \Omega$. Let $r := \frac{\delta(a)}{2\sqrt{n}} \le \frac{1}{2}d_\Omega(a)$. Observe that $\delta(z) \ge \frac12\delta(a)$, $z \in \mathbb{P}(a,r)$. By the Cauchy inequalities we get

$$\begin{aligned}\delta^{N+|\tau|}(a)|D^\tau g(a)| &\le \delta^{N+|\tau|}(a)\frac{\tau!}{r^{|\tau|}}\|g\|_{\mathbb{P}(a,r)} \le \tau!(2\sqrt{n})^{|\tau|}\delta^N(a)\|g\|_{\mathbb{P}(a,r)}\\ &\le \tau!(2\sqrt{n})^{|\tau|}2^N\|\delta^N g\|_{\mathbb{P}(a,r)} \le \tau!(\sqrt{n})^{|\tau|}2^{N+|\tau|}\|\delta^N g\|_\Omega,\end{aligned}$$

which finishes the proof of the lemma. □

[4] If $a \in (\partial D)\cap\mathbb{C}^n(\alpha)$, then $a \in G$ and, consequently, the function $z^{(3k+3)\mathbf{1}-\sigma}D^\tau f$ is obviously continuous at a. Observe that $(\partial D)\setminus\mathbb{C}^n(\alpha) \subset \bar{G}\setminus\mathbb{C}^n(\alpha) \subset \partial G$.

We come back to (3.2.1).

Fix an $a \in (\partial D) \setminus \mathbb{C}^n(\alpha)$. By Lemma 3.2.2, we get

$$\delta_G^{1+k} |D^\tau f| \le c_0 \text{ on } G, \quad |\tau| \le k,$$

where c_0 is a constant. Consequently, for $z \in U \cap \bar{D} \cap \mathbb{C}^n(\alpha)$, z near a (z should be so near a that $\delta_G(z) = \rho_G(z)$), using (3.2.2) we get

$$\begin{aligned} |z^{(3k+3)\mathbf{1}-\sigma} D^\tau f(z)| &\le c_0 |z^{(3k+3)\mathbf{1}-\sigma}| \delta_G^{-(1+k)}(z) \le c_1 |z^{(3k+3)\mathbf{1}-\sigma}| d_G^{-(1+k)}(z) \\ &\le c_1 |z^{(3k+3)\mathbf{1}-\sigma-2(1+k)\mathbf{1}}| = c_1 |z^{(k+1)\mathbf{1}-\sigma}| \xrightarrow[z \to a]{} 0, \end{aligned}$$

where c_1 is independent of z, which proves (3.2.1).

(c) The equivalence (i) $\Leftrightarrow$ (iii) follows from Corollary 3.1.8. The implication (i) $\Rightarrow$ (ii) is obvious. It remains to show that (ii) $\Rightarrow$ (iii).

Suppose that $\alpha \notin \mathbb{R} \cdot \mathbb{Z}^n$. Take an $f \in \mathcal{O}^{(0+)}(D)$,

$$f(z) = \sum_{\nu \in \Sigma(f)} a_\nu^f z^\nu, \quad z \in D.$$

To get a contradiction we will show that $f \equiv \text{const}$.

Suppose that there exists a $\nu \in \Sigma(f)_*$. Let $w \in \mathbb{R}^n$ be such that $w \perp \alpha$, $\|w\| = 1$, and $s := \langle w, \nu \rangle > 0$. Fix $0 < N < s$ and $x^0 \in \boldsymbol{H}_{\alpha,c}$. Note that $x^0 + tw \in \boldsymbol{H}_{\alpha,c}$, $t \in \mathbb{R}$. Put $r(t) := e^{x^0 + tw}$, $t \in \mathbb{R}$. Since $r(t) \in D$, the Cauchy inequalities imply (cf. Example 3.1.4 (d)):

$$\begin{aligned} |a_\nu^f| &\le \frac{\|f\|_{\partial_0 \mathbb{P}(r(t))}}{r(t)^\nu} \le \frac{\|\delta_D^N f\|_{\partial_0 \mathbb{P}(r(t))}}{r(t)^\nu \delta_D^N(r(t))} \\ &= e^{-\langle x^0, \nu \rangle} \frac{\|\delta_D^N f\|_D}{(e^{ts/N} \delta_D(r(t)))^N} \le e^{-\langle x^0, \nu \rangle} \frac{\|\delta_D^N f\|_D}{M^N}, \end{aligned}$$

where $M := \sup\{\delta_D(r(t)) e^{ts/N} : t \in \mathbb{R}\}$. It suffices to show that $M = +\infty$.

Suppose that $M < +\infty$. Put $T := \{t \in \mathbb{R}_{>0} : \rho_D(r(t)) > \delta_0(r(t))\}$. If $t \in T$, then

$$e^{-2ts/N} + \sum_{j=1}^n e^{2(x_j^0 + tw_j - ts/N)} \ge M^{-2}.$$

Consequently, T is bounded. Therefore, $\delta_D(r(t)) = \rho_D(r(t))$ for $t \ge t_0$. Now, we estimate ρ_D.

Let $d := \operatorname{dist}(x^0 + \mathbb{R}w, \partial \boldsymbol{H}_{\alpha,c}) = \operatorname{dist}(x^0, \partial \boldsymbol{H}_{\alpha,c})$. Fix a $t \in \mathbb{R}$ and $z \in (\partial D) \setminus V_0$ such that $\rho_D(r(t)) \ge \frac{1}{2}\|z - r(t)\|$. Write $|z_j| = r_j(t) e^{u_j}$ ($u_j \in \mathbb{R}$), $j = 1, \dots, n$. Note that $\|u\| \ge d$. Fix a $j = j(t)$ such that $|u_j| \ge d/\sqrt{n}$. Then we obtain

$$\rho_D(r(t)) \ge \tfrac{1}{2}|z_j - r_j(t)| \ge \tfrac{1}{2}||z_j| - r_j(t)| = \tfrac{1}{2} r_j(t) |e^{u_j} - 1| \ge d_0 r_j(t),$$

where $d_0 := \frac{1}{2}(1 - e^{-d/\sqrt{n}})$. Finally, choose j_0 such that there is a sequence $(t_k)_{k=1}^{\infty} \subset [t_0, +\infty)$ with $j(t_k) = j_0$ for all k and $\lim_{k\to\infty} t_k = \infty$. Then

$$M \geq \rho_D(r(t_k))e^{t_k s/N} \geq d_0 e^{x^0_{j_0} + t_k w_{j_0} + t_k \langle w,\nu\rangle/N} \xrightarrow[k\to\infty]{} \infty;$$

a contradiction.

(d) Let $f \in \mathcal{H}^\infty(D)$, $f(z) = \sum_{\nu\in\mathbb{Z}^n} a^f_\nu z^\nu$, $z \in D$. In view of Proposition 1.6.5 (b), for $\nu \in \Sigma(f)$, we have

$$\boldsymbol{H}_{\alpha,c} \subset \boldsymbol{H}_{\nu,c(\nu)}, \quad c(\nu) := \log \frac{\|f\|_D}{|a^f_\nu|}.$$

Consequently, $\Sigma(f) \subset (\mathbb{R}_+ \cdot \alpha) \cap \mathbb{Z}^n$. In particular, if $\alpha \notin \mathbb{R}\cdot\mathbb{Z}^n$, then $f \equiv \text{const}$.

(e) For $f = \sum_{\beta\in\mathbb{Z}^n} a^f_\beta z^\beta \in \mathcal{H}^\infty(D)$ define

$$g(\lambda) := \sum_{k=0}^{\infty} a^f_{k\alpha} e^{kc} \lambda^k, \quad \lambda \in \mathbb{D}.$$

Since $\mathbb{D}_* \subset \{e^{-c} z^\alpha : z \in D\} \subset \mathbb{D}$, for every $\lambda \in \mathbb{D}_*$ there exists a $z \in D$ such that $\lambda = e^{-c} z^\alpha$. We know that $\Sigma(f) \subset (\mathbb{R}_+ \cdot \alpha) \cap \mathbb{Z}^n$. Observe that in fact $\Sigma(f) \subset \mathbb{Z}_+ \cdot \alpha$ (because $\alpha_1, \dots, \alpha_n$ are relatively prime). Thus

$$\sum_{k=0}^{\infty} |a^f_{k\alpha} e^{kc} \lambda^k| = \sum_{k=0}^{\infty} |a^f_{k\alpha} e^{kc} (e^{-c} z^\alpha)^k| = \sum_{\beta\in\mathbb{Z}^n} |a^f_\beta z^\beta| < +\infty$$

and, therefore, g is well defined, $g \in \mathcal{O}(\mathbb{D})$, and $\hat{g}(z) = g(e^{-c} z^\alpha) = f(z)$, $z \in D$. Hence, $g \in \mathcal{H}^\infty(\mathbb{D})$ and $\Psi(g) = f$.

(f) Recall (Example 1.10.7 (j)) that the inclusion $\mathcal{H}^\infty(D) \to \mathcal{O}^{(0+)}(D)$ is continuous. Consequently, by the Banach theorem (Theorem 1.10.4), either $\mathcal{H}^\infty(D) = \mathcal{O}^{(0+)}(D)$ or $\mathcal{H}^\infty(D)$ is of first Baire category in the Fréchet space $\mathcal{O}^{(0+)}(D)$.

Define

$$f(z) := \operatorname{Log} \frac{1}{1 - e^{-c} z^\alpha}, \quad z \in D,$$

where Log stands for the principal branch of the logarithm. Obviously, f is holomorphic and unbounded. We are going to prove that $f \in \mathcal{O}^{(0+)}(D)$. Fix an $N > 0$. Then

$$|f(z)| \leq \pi + \log \frac{1}{1 - \theta(z)}, \quad z \in D,$$

where $\theta(z) := e^{-c}|z^\alpha|$, $z \in D$. In particular,

$$\delta^N_D(z)|f(z)| \leq \pi + \log 2 \ \text{ if } \ \theta(z) \leq 1/2.$$

Recall (cf. the proof of (a)) that

$$\rho_D(z) \le |1 - \theta^{-1/\alpha_n}(z)||z_n|, \quad z \in D.$$

Suppose that $1/2 < \theta(z) < 1$. Then

$$\rho_D(z) \le 2^\gamma(1 - \theta^\gamma(z))|z_n|,$$

where $\gamma := 1/|\alpha_n|$. Finally,

$$\begin{aligned}\delta_D^2(z)|f(z)|^{2/N} &\le \delta_0(z)2^\gamma(1-\theta^\gamma(z))|z_n|\Big(\pi + \log\frac{1}{1-\theta(z)}\Big)^{2/N}\\ &\le 2^\gamma(1-\theta^\gamma(z))\Big(\pi + \log\frac{1}{1-\theta(z)}\Big)^{2/N}.\end{aligned}$$

Since

$$\lim_{\theta\to 1-}(1-\theta^\gamma)\Big(\pi + \log\frac{1}{1-\theta}\Big)^{2/N} = 0 \quad (\text{Exercise}),$$

we conclude that $\delta_D^N f$ is bounded.

(g) Suppose that $L_h^p(D) \ne \{0\}$ for some $1 \le p < +\infty$. Then, by Example 3.1.4 (e), there exists a $\nu \in (\mathbb{Z}^n)_*$ such that $z^\nu \in L_h^p(D)$. On the other hand we have

$$\begin{aligned}\int_D |z^\nu|^p \, d\Lambda_{2n}(z) &= (2\pi)^n \int_{\boldsymbol{R}(D)} r^{p\nu+\mathbf{1}} \, d\Lambda_n(r)\\ &= (2\pi)^n \int_{\boldsymbol{H}_{\alpha,c}} e^{\langle x, p\nu+\mathbf{2}\rangle} \, d\Lambda_n(x).\end{aligned}$$

We may assume that $\alpha_n \ne 0$. Changing the variables

$$\boldsymbol{H}_{\alpha,c} \ni x = (x', x_n) \mapsto (x', \langle\alpha, x\rangle) \in \mathbb{R}^{n-1} \times (-\infty, c)$$

and, next, applying the Fubini theorem shows that the latter integral is infinite (Exercise); a contradiction. □

3.3 Maximal affine subspace of a convex set II

The present section is a continuation of § 1.4.

Definition 3.3.1. A log-convex Reinhardt domain $D \subset \mathbb{C}^n$ is of *rational* (resp. *irrational*) *type* if $\boldsymbol{E}(\log D)$ is of rational (resp. irrational) type.

Exercise 3.3.2 (Cf. Exercise 1.4.9). Let

$$D := \{(z_1, z_2) \in \mathbb{C}^2 : c|z_1|^\mu < |z_2| < d|z_1|^\mu\} \quad (c, d, \mu > 0).$$

Decide when D is of rational type.

Remark 3.3.3. Let $D \subset \mathbb{C}^n$ be a fat log-convex Reinhardt domain. Then D is of rational type iff $D = \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c}$, where $A \subset \mathbb{Z}^n \times \mathbb{R}$ (cf. Lemma 1.4.11 (iv), Remarks 1.5.7 (e), 1.5.8 (b)).

Definition 3.3.4. Let $X \subset \mathbb{R}^n$ be a convex domain. Define

$$\boldsymbol{F}(X) := (\boldsymbol{E}(X)^\perp \cap \mathbb{Q}^n)^\perp, \quad \boldsymbol{Z}(X) := X + \boldsymbol{F}(X),$$
$$\boldsymbol{K}_0(X) := X, \quad \boldsymbol{K}_j(X) := \boldsymbol{Z}(\boldsymbol{K}_{j-1}(X)), \quad j = 1, 2, \dots,$$
$$\boldsymbol{K}_\infty(X) := \bigcup_{j=0}^{\infty} \boldsymbol{K}_j(X), \quad \boldsymbol{M}(X) := \boldsymbol{E}(\boldsymbol{K}_\infty(X)).$$

Remark 3.3.5. (a) Recall that $\boldsymbol{E}(X) \subset \boldsymbol{F}(X)$ and $\boldsymbol{E}(X) = \boldsymbol{F}(X)$ iff X is of rational type (Remark 1.4.4 (c) (v)).

(b) $\boldsymbol{Z}(X)$ is a convex domain, $X \subset \boldsymbol{Z}(X)$, and $\boldsymbol{F}(X) \subset \boldsymbol{E}(\boldsymbol{Z}(X))$.In particular,

$$\boldsymbol{K}_0(X) \subset \boldsymbol{K}_1(X) \subset \boldsymbol{K}_2(X) \subset \cdots,$$
$$\boldsymbol{E}(\boldsymbol{K}_0(X)) \subset \boldsymbol{F}(\boldsymbol{K}_0(X)) \subset \boldsymbol{E}(\boldsymbol{K}_1(X)) \subset \boldsymbol{F}(\boldsymbol{K}_1(X)) \subset \boldsymbol{E}(\boldsymbol{K}_2(X)) \subset \cdots.$$

(c) $\boldsymbol{K}_j(X) = X + \boldsymbol{F}(\boldsymbol{K}_{j-1}(X))$, $j = 1, 2, \dots$.

(d) $\boldsymbol{K}_p(X) \subsetneq \boldsymbol{K}_{p+1}(X) \Leftrightarrow \boldsymbol{F}(\boldsymbol{K}_{p-1}(X)) \subsetneq \boldsymbol{E}(\boldsymbol{K}_p(X)) \subsetneq \boldsymbol{F}(\boldsymbol{K}_p(X))$. In particular, if $\boldsymbol{K}_p(X) = \boldsymbol{K}_{p+1}(X)$, then:

- $\boldsymbol{K}_p(X)$ is of rational type,
- $\boldsymbol{K}_p(X) = \boldsymbol{K}_j(X)$ for every $j \geq p + 1$,
- $\boldsymbol{K}_\infty(X) = \boldsymbol{K}_p(X)$.

(e) If $\boldsymbol{K}_0(X) \subsetneq \boldsymbol{K}_1(X) \subsetneq \cdots \subsetneq \boldsymbol{K}_p(X)$, then

$$\{0\} \subsetneq \boldsymbol{E}(\boldsymbol{K}_0(X)) \subsetneq \boldsymbol{F}(\boldsymbol{K}_0(X)) \subsetneq \cdots \subsetneq \boldsymbol{E}(\boldsymbol{K}_{p-1}(X)) \subsetneq \boldsymbol{F}(\boldsymbol{K}_{p-1}(X)).$$

Consequently, $\dim \boldsymbol{E}(X) \geq 1$ and $p \leq \lfloor \frac{n - \dim \boldsymbol{E}(X) + 1}{2} \rfloor \leq \lfloor \frac{n}{2} \rfloor$.

(f) If $X \subset Y$, then $\boldsymbol{Z}(X) \subset \boldsymbol{Z}(Y)$ and, consequently, $\boldsymbol{K}_\infty(X) \subset \boldsymbol{K}_\infty(Y)$.

(g) $\boldsymbol{K}_\infty(X) = \boldsymbol{K}(X) =$ the smallest convex domain of rational type containing X (cf. Remark 1.4.10).

Indeed, since $\boldsymbol{K}_\infty(X)$ is of rational type, the definition of $\boldsymbol{K}(X)$ implies that $\boldsymbol{K}(X) \subset \boldsymbol{K}_\infty(X)$. On the other hand, since $X \subset \boldsymbol{K}(X)$, we get $\boldsymbol{K}_\infty(X) \subset \boldsymbol{K}_\infty(\boldsymbol{K}(X)) = \boldsymbol{K}(X)$.

Example 3.3.6 (W. Jarnicki). There exists a convex domain $X \subset \mathbb{R}^n$ such that $\boldsymbol{K}_0(X) \subsetneq \boldsymbol{K}_1(X) \subsetneq \cdots \subsetneq \boldsymbol{K}_p(X)$ with $p = \lfloor \frac{n}{2} \rfloor$.

Indeed, let $r := \sqrt{2}$, $s := \sqrt{3}$. Consider the following basis $(b^1, \dots, b^n)$ of $\mathbb{R}^n$:

$$b^1 := (r, \underbrace{1, \dots, 1}_{n-1}),$$

$$b^{2k} := (\underbrace{0, \dots, 0}_{2k-1}, \underbrace{1, \dots, 1}_{n-2k+1}), \qquad k = 1, 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor,$$

$$b^{2k+1} := (\underbrace{0, \dots, 0}_{2k-1}, r, s, \underbrace{1, \dots, 1}_{n-2k-1}), \quad k = 1, 2, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Put

$$X := \left\{ \sum_{j=1}^{n} t_j b^j : \begin{cases} |t_{2j+1}| < t_{2j},\ j = 1, 2, \dots, k-1,\ |t_{2k}| < 1, & \text{when } n = 2k, \\ |t_{2j+1}| < t_{2j},\ j = 1, 2, \dots, k, & \text{when } n = 2k+1 \end{cases} \right\}.$$

Observe that X is convex. The equality $p = \lfloor \frac{n}{2} \rfloor$ follows directly from the identities below.

$(1)_\ell$: $\boldsymbol{K}_\ell(X) = X + \sum_{j=1}^{2\ell+1} \mathbb{R} b^j$, $\ell = 0, 1, 2, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor$,

$(2)_\ell$: $\boldsymbol{E}(\boldsymbol{K}_\ell(X)) = \sum_{j=1}^{2\ell+1} \mathbb{R} b^j$, $\ell = 0, 1, 2, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor$,

$(3)_\ell$: $\boldsymbol{F}(\boldsymbol{K}_\ell(X)) = \sum_{j=1}^{2\ell+2} \mathbb{R} b^j$, $\ell = 0, 1, 2, \dots, \left\lfloor \frac{n-2}{2} \right\rfloor$.

Since $\mathbb{R} b^1 \subset X$, we get $(1)_0$. We will use induction: $(1)_\ell \Rightarrow (2)_\ell \Rightarrow (3)_\ell \Rightarrow (1)_{\ell+1}$.

$(1)_\ell \Rightarrow (2)_\ell$: The inclusion "$\supset$" in $(2)_\ell$ follows trivially from $(1)_\ell$. Fix an $a = \sum_{j=1}^{n} t_j b^j \in \boldsymbol{E}(\boldsymbol{K}_\ell(X))$. Suppose that there exists a $j > 2\ell + 1$ with $t_j \neq 0$. Let j_0 be the smallest of such j's. We may assume that $t_{j_0} = -1$. Put $x := b^2 + b^4 + b^6 + \dots + b^{2q} = \sum_{j=1}^{n} x_j b^j$, where $q := \lfloor \frac{n-1}{2} \rfloor$. Then $x \in X \subset \boldsymbol{K}_\ell(X)$. Consider the vector $x + a \in \boldsymbol{K}_\ell(X)$. Using $(1)_\ell$, write $x + a = z + c$ with $z = \sum_{j=1}^{n} z_j b^j \in X$ and $c = \sum_{j=1}^{2\ell+1} c_j b^j$. Observe that $z_j = z_j + c_j = x_j + a_j$, $j = 2\ell+2, \dots, j_0$. In each of the following three cases we get a contradiction with the definition of X:

- j_0 is odd: Then $z_{j_0} = a_{j_0} = -1$, $z_{j_0-1} = x_{j_0-1} = 1$;
- $j_0 < n$ and j_0 is even: Then $z_{j_0} = 1 - 1 = 0$;
- $j_0 = n$ and j_0 is even: Then $z_{j_0} = a_n = -1$.

$(2)_\ell \Rightarrow (3)_\ell$: It suffices to observe that $(2)_\ell$ implies

$$\begin{aligned} &\boldsymbol{E}(\boldsymbol{K}_\ell(X))^\perp \cap \mathbb{Q}^n = \{a = (a_1, \dots, a_n) \in \mathbb{Q}^n : \langle a, b^1 \rangle = \dots = \langle a, b^{2\ell+1} \rangle = 0\} \\ &= \left\{ a \in \mathbb{Q}^n : \begin{cases} a_1 r + a_2 + \dots + a_n = 0 & \\ a_{2k} + \dots + a_n = 0 & \text{if } k \le \ell + 1/2 \\ a_{2k} r + a_{2k+1} s + a_{2k+2} + \dots + a_n = 0 & \text{if } k \le \ell \end{cases} \right\} \\ &= \{a \in \mathbb{Q}^n : a_1 = \dots = a_{2\ell+1} = 0, a_{2\ell+2} + a_{2\ell+3} + \dots + a_n = 0\}. \end{aligned}$$

$(3)_\ell \Rightarrow (1)_{\ell+1}$: The inclusion "$\subset$" in $(1)_{\ell+1}$ follows trivially from $(3)_\ell$. To prove the opposite inclusion, take $x = \sum_{j=1}^n x_j b^j \in X$ and $t = \sum_{j=1}^{2\ell+3} t_j b^j$. Define

$$\tilde{x} := x_1 b^1 + \sum_{j=1}^{\ell+1} ((|x_{2j}| + |t_{2j+1}|) b^{2j} + (x_{2j+1} + t_{2j+1}) b^{2j+1}) + \sum_{j=2\ell+4}^{n} x_j b^j \in X,$$

$$\tilde{t} := x_1 b^1 + \sum_{j=1}^{\ell+1} (x_{2j} - |x_{2j}| + t_{2j} - |t_{2j+1}|) b^{2j} \in \boldsymbol{F}(\boldsymbol{K}_\ell(X)).$$

Since $x + t = \tilde{x} + \tilde{t}$, we conclude that $x + t \in X + \boldsymbol{F}(\boldsymbol{K}_\ell(X)) \subset \boldsymbol{K}_\ell(X) + \boldsymbol{F}(\boldsymbol{K}_\ell(X)) = \boldsymbol{K}_{\ell+1}(X)$.

Exercise 3.3.7. Let X be the domain from the above example. Write X in the form

$$X = \boldsymbol{H}_{\alpha^1, c_1} \cap \dots \cap \boldsymbol{H}_{\alpha^{2p}, c_{2p}}, \quad p = \left\lfloor \frac{n}{2} \right\rfloor,$$

so that

$$\boldsymbol{K}_\ell(X) = \boldsymbol{H}_{\alpha^{2\ell+1}, c_{2\ell+1}} \cap \dots \cap \boldsymbol{H}_{\alpha^{2p}, c_{2p}}, \quad \ell = 0, 1, 2, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Hint. It suffices to choose α^j, c_j, $j = 1, \dots, 2p$, so that for every $x = \sum_{j=1}^n t_j b^j$ we have

$$\begin{aligned}
\langle x, \alpha^{2j-1} \rangle < c_{2j-1} &\Leftrightarrow t_{2j+1} < t_{2j}, & j &= 1, 2, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \\
\langle x, \alpha^{2j} \rangle < c_{2j} &\Leftrightarrow t_{2j+1} > -t_{2j}, & j &= 1, 2, \dots, \left\lfloor \frac{n-1}{2} \right\rfloor, \\
\langle x, \alpha^{n-1} \rangle < c_{n-1} &\Leftrightarrow t_n < 1, & n &= 2p, \\
\langle x, \alpha^{n} \rangle < c_{n} &\Leftrightarrow t_n > -1, & n &= 2p.
\end{aligned}$$

Proposition 3.3.8. *Let* $X = \boldsymbol{H}_{\alpha^1, c_1} \cap \dots \cap \boldsymbol{H}_{\alpha^N, c_N}$, *where* $\alpha^1, \dots, \alpha^N \in (\mathbb{R}^n)_*$ *and let* $r := \operatorname{rank}[\alpha^1, \dots, \alpha^N]$. *Further, let*

$$\begin{aligned}
\mathcal{I} &:= \{I = (i_1, \dots, i_r) : 1 \le i_1 < \dots < i_r \le N, \ \operatorname{rank}[\alpha^{i_1}, \dots, \alpha^{i_r}] = r\}, \\
A_I &:= \mathbb{Z}^n \cap (\mathbb{R}_+ \alpha^{i_1} + \dots + \mathbb{R}_+ \alpha^{i_r}), \quad I \in \mathcal{I}, \\
M_I &:= \{v \in \mathbb{R}^n : \langle v, \alpha \rangle = 0, \ \alpha \in A_I\} = A_I^\perp, \quad I \in \mathcal{I}, \\
M &:= \bigcap_{I \in \mathcal{I}} M_I = \Big(\bigcup_{I \in \mathcal{I}} A_I\Big)^\perp.
\end{aligned}$$

Then $\boldsymbol{M}(X) = M$ *and, consequently,* $\boldsymbol{K}(X) = X + M$. *In particular,* $\boldsymbol{K}(X) = \mathbb{R}^n$ *iff* $A_I = \{0\}$, $I \in \mathcal{I}$.

Proof. Observe that for every $I = (i_1, \dots, i_r) \in \mathcal{I}$ there exists an $x^I \in \mathbb{R}^n$ such that

$$\boldsymbol{H}_{\alpha^{i_k}, c_{i_k}} = \{x \in \mathbb{R}^n : \langle x - x^I, \alpha^{i_k} \rangle < 0\} = H^{x^I}_{\alpha^{i_k}}, \quad k = 1, \dots, r.$$

Put

$$Y_I := \operatorname{int} \bigcap_{\alpha \in A_I} H^{x^I}_{\alpha}, \quad X_I := \bigcap_{k=1}^{r} \boldsymbol{H}_{\alpha^{i_k}, c_{i_k}}, \quad I = (i_1, \dots, i_r) \in \mathcal{I}.$$

Obviously, Y_I is of rational type. Moreover, $X \subset X_I \subset Y_I$ and $\boldsymbol{E}(Y_I) = M_I$. Thus $\boldsymbol{M}(X) \subset \boldsymbol{M}(Y_I) = M_I$, $I \in \mathcal{I}$, which implies that $\boldsymbol{M}(X) \subset M$.

Now assume that $\boldsymbol{M}(X) \subsetneq \mathbb{R}^n$ and let

$$\boldsymbol{K}(X) = \operatorname{int} \bigcap_{(\beta, c) \in B} \boldsymbol{H}_{\beta, c},$$

where $B \subset (\mathbb{Z}^n)_* \times \mathbb{R}$. Then

$$\boldsymbol{M}(X) = \{v \in \mathbb{R}^n : \langle v, \beta \rangle = 0, \ \beta \in \operatorname{pr}_{\mathbb{R}^n}(B)\}.$$

To continue we need the following lemma.

Lemma 3.3.9. *Let* $X_j := \boldsymbol{H}_{\alpha^j, c_j} \subset \mathbb{R}^n$, $\alpha^j \in (\mathbb{R}^n)_*$, $c_j \in \mathbb{R}$, $j = 0, \dots, N$, $r := \operatorname{rank}[\alpha^1, \dots, \alpha^N]$, *and* $\varnothing \neq X_1 \cap \dots \cap X_N \subset X_0$. *Then there exist* $1 \le i_1 < \dots < i_r \le N$ *such that* $r = \operatorname{rank}[\alpha^{i_1}, \dots, \alpha^{i_r}]$ *and* $X_{i_1} \cap \dots \cap X_{i_r} \subset X_0$.

Proof of Lemma 3.3.9. We use induction on n. The case $n = 1$ is trivial. Assume that the result is true in $\mathbb{R}, \dots, \mathbb{R}^{n-1}$ ($n \ge 2$). We may assume that $N \ge 2$ and

$$\bigcap_{j \in \{1, \dots, k-1, k+1, \dots, N\}} X_j \not\subset X_0, \quad k = 1, \dots, N.^{5} \tag{3.3.1}$$

It suffices to prove that $r = N$. Consider the following two cases:

Case 1. $r < n$ (e.g. $N \le n - 1$): Let $F := \mathbb{R} \cdot \alpha^1 + \dots + \mathbb{R} \cdot \alpha^N$. Observe that $\dim F = r$ and $\alpha^0 \in F$ (because $X_1 \cap \dots \cap X_N \subset X_0$). We may assume that $F = \mathbb{R}^r \times \{0\}^{n-r}$. Let $\alpha^j = (\beta^j, 0) \in \mathbb{R}^r \times \{0\}^{n-r}$, $j = 0, \dots, N$. Consequently,

[5] Note that, after rejecting "superfluous" halfspaces, the rank of the new system of α's may be smaller. Nevertheless, if we find $1 \le i_1 < \dots < i_s \le N$, $s < r$, with $s = \operatorname{rank}[\alpha^{i_1}, \dots, \alpha^{i_s}]$ and $X_{i_1} \cap \dots \cap X_{i_s} \subset X_0$, then it suffices to take arbitrary $\alpha^{i_{s+1}}, \dots, \alpha^{i_r}$ so that $\operatorname{rank}[\alpha^{i_1}, \dots, \alpha^{i_r}] = r$.

$X_j = Y_j \times \mathbb{R}^{n-r}$, where Y_j is an open halfspace in $\mathbb{R}^r$ with $\beta^j \perp \partial Y_j$, $j = 0, \dots, N$. Clearly, $r = \operatorname{rank}[\beta^1, \dots, \beta^N]$. Moreover,

$$\bigcap_{j \in \{1,\dots,k-1,k+1,\dots,N\}} Y_j \not\subset Y_0, \quad k = 1, \dots, N.$$

Hence, by the inductive assumption, $r = N$.

Case 2. $r = n$: We may assume that $X_0 = \{x_n < 0\}$ and $(\alpha_1^N, \dots, \alpha_{n-1}^N) \neq 0$. Let $\alpha^j = (\beta^j, \alpha_n^j)$, $j = 0, \dots, N$. Observe that $\beta^N \neq 0$. Let

$$Y_j := \{y \in \mathbb{R}^{n-1} : (y, 0) \in X_j\} = \{y \in \mathbb{R}^{n-1} : \langle y, \beta^j \rangle < c_j\}, \quad j = 1, \dots, N,$$
$$Y_0 := \{y \in \mathbb{R}^{n-1} : (y, 0) \notin \overline{X}_N\} = \{y \in \mathbb{R}^{n-1} : \langle y, -\beta^N \rangle < -c_N\}.$$

Observe that

$$\varnothing \neq \bigcap_{j=1}^{N-1} Y_j \subset Y_0.$$ [6]

Let $s := \operatorname{rank}[\beta^1, \dots, \beta^{N-1}]$. By the inductive assumption, there exist $1 \le i_1 < \dots < i_s \le N-1$ such that $Y_{i_1} \cap \dots \cap Y_{i_s} \subset Y_0$. Consequently, $X_{i_1} \cap \dots \cap X_{i_s} \cap X_N \subset X_0$.[7] Hence $N = s + 1$. On the other hand $s \le n - 1 = r - 1$. Thus $r = N$. $\square$

Fix a $(\beta, c) \in B$. Since $X \subset \boldsymbol{H}_{\beta,c}$, Lemma 3.3.9 implies that there is an $I = (i_1, \dots, i_r) \in \mathcal{I}$ such that $X_I \subset \boldsymbol{H}_{\beta,c}$. Hence $\beta \in \boldsymbol{E}(\boldsymbol{H}_{\beta,c})^\perp \subset \boldsymbol{E}(X_I)^\perp = \mathbb{R}\alpha^{i_1} + \dots + \mathbb{R}\alpha^{i_r}$. Consequently, $\beta = \lambda_1 \alpha^{i_1} + \dots + \lambda_r \alpha^{i_r}$, $\lambda := (\lambda_1, \dots, \lambda_r) \in (\mathbb{R}^r)_*$. It remains to show that $\lambda \in \mathbb{R}_+^r$ (then $\operatorname{pr}_{\mathbb{R}^n}(B) \subset \bigcup_{I \in \mathcal{I}} A_I$, which implies that $M \subset \boldsymbol{M}(X)$).

Let $L \colon \mathbb{R}^n \to \mathbb{R}^r$ be given by

$$L(x) := (\langle x, \alpha^{i_1} \rangle, \dots, \langle x, \alpha^{i_r} \rangle), \quad x \in \mathbb{R}^n.$$

Then

$$L(X_I) = \{\xi = (\xi_1, \dots, \xi_r) \in \mathbb{R}^r : \xi_k < \langle x^I, \alpha^{i_k} \rangle, \ k = 1, \dots, r\}$$

and

$$L(\boldsymbol{H}_{\beta,c}) = \{\xi \in \mathbb{R}^r : \langle \xi, \lambda \rangle < 0\}.$$

Now, the inclusion $L(X) \subset L(\boldsymbol{H}_{\beta,c})$ implies that $\lambda \in \mathbb{R}_+^r$ (Exercise). $\square$

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[6] If $Y_1 \cap \dots \cap Y_{N-1} = \varnothing$, then $(X_1 \cap \dots \cap X_{N-1}) \cap \{x_n = 0\} = \varnothing$, which implies that $X_1 \cap \dots \cap X_{N-1} \subset X_0$; a contradiction. If $y \in Y_1 \cap \dots \cap Y_{N-1}$, then $(y, 0) \in (X_1 \cap \dots \cap X_{N-1}) \cap \{x_n = 0\}$. Consequently, $(y, 0) \notin X_N$. Thus $Y_1 \cap \dots \cap Y_{N-1} \subset \overline{Y}_0$, and finally, $Y_1 \cap \dots \cap Y_{N-1} \subset Y_0$.

[7] $(X_{i_1} \cap \dots \cap X_{i_s} \cap X_N) \cap \{x_n = 0\} = Y_{i_1} \cap \dots \cap Y_{i_s} \cap (\mathbb{R}^{n-1} \setminus \overline{Y}_N) = \varnothing$.

The following examples illustrate Proposition 3.3.8.

Example 3.3.10. Let $\alpha^1 := (\beta, \beta + 1, 0), \alpha^2 := \alpha^1 + \mathbf{1} = (\beta + 1, \beta + 2, 1)$, where $\beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$. Put

$$X := \boldsymbol{H}_{\alpha^1,0} \cap \boldsymbol{H}_{\alpha^2,0} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \beta x_1 + (\beta + 1)x_2 < 0,\\ (\beta + 1)x_1 + (\beta + 2)x_2 + x_3 < 0\}.$$

Then $\boldsymbol{K}(X) = \mathbb{R}^3$.

Indeed, $\operatorname{rank}[\alpha^1, \alpha^2] = 2$ and $A_{(1,2)} = \mathbb{Z}^3 \cap (\mathbb{R}_+ \cdot \alpha^1 + \mathbb{R}_+ \cdot \alpha^2) = \{0\}$.

Exercise 3.3.11. Let

$$\alpha^1 := (\alpha_1, \alpha_2, \alpha_3), \quad \alpha^2 := \mathbf{1} - \alpha^1 = (1 - \alpha_1, 1 - \alpha_2, 1 - \alpha_3),$$

where $0 < \alpha_j < 1$, $j = 1, 2, 3$, $\alpha_1 \neq \alpha_2$, $(\alpha_1 - \alpha_3)/(\alpha_1 - \alpha_2) \notin \mathbb{Q}$. Put

$$X := \boldsymbol{H}_{\alpha^1,0} \cap \boldsymbol{H}_{\alpha^2,0} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 < 0,\\ (1 - \alpha_1)x_1 + (1 - \alpha_2)x_2 + (1 - \alpha_3)x_3 < 0\}.$$

Then

$$\boldsymbol{K}(X) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 < 0\}.$$

3.4 $\mathcal{H}^\infty$-domains of holomorphy

In this section we characterize the most important class of special Reinhardt domains of holomorphy, namely $\mathcal{H}^\infty$-domains of holomorphy (cf. [Jar-Pfl 2000], § 4.1, for the general theory). Recall that we already presented a general characterization in Proposition 3.1.7.

Theorem 3.4.1 ([Jar-Pfl 1987]). *Let $D \subsetneq \mathbb{C}^n$ be a Reinhardt domain. Then the following conditions are equivalent:*

(i) *D is an $\mathcal{H}^\infty$-domain of holomorphy;*

(ii) *D is an $\mathcal{O}^{(0+)}$-domain of holomorphy;*

(iii) *D is a fat domain of holomorphy of rational type;*

(iv) *$D = \operatorname{int} \bigcap_{(\alpha,c) \in A} \boldsymbol{D}_{\alpha,c}$ for some $A \subset (\mathbb{Z}^n)_* \times \mathbb{R}$.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. The equivalence (iii) $\Leftrightarrow$ (iv) follows from Lemma 1.4.11. The implication (iv) $\Rightarrow$ (i) follows from Theorem 3.2.1 (c) and Remark 1.11.3 (e). So, we only need to prove that (ii) $\Rightarrow$ (iii).

It is clear that D is fat (cf. Proposition 1.9.12). Put $X := \log D$ and $F := \boldsymbol{F}(X)$ (cf. Definition 3.3.4). Assume that $\boldsymbol{E}(X)$ is not of rational type. Then $\dim F > \dim \boldsymbol{E}(X)$. Let $f \in \mathcal{O}^{(0+)}(D)$,

$$f(z) = \sum_{\nu \in \Sigma(f)} a_\nu z^\nu, \quad z \in D.$$

We will show that

$$\Sigma(f) \subset \boldsymbol{E}(X)^\perp. \tag{3.4.1}$$

Suppose for the moment that (3.4.1) is true. Then for $x \in X$, $v \in F$ we get

$$\sum_{\nu\in\Sigma(f)} a_\nu e^{\langle x+v,\nu\rangle} = \sum_{\nu\in\Sigma(f)} a_\nu e^{\langle x,\nu\rangle}.$$

Consequently, the series is summable in the domain $\exp(X+F)$. Since D is an $\mathcal{O}^{(0+)}$-domain of holomorphy, $\exp(X+F) \subset D$, and hence, $X+F \subset X$; a contradiction.

Now we are going to prove (3.4.1). Take a $\nu \notin \boldsymbol{E}(X)^\perp$. Choose $w \in \boldsymbol{E}(X)$, $\|w\| = 1$, such that $s := \langle w, \nu\rangle > 0$. Fix $0 < N < s$ and $x^0 \in X$. Put $r(t) := e^{x^0+tw}$, $t \in \mathbb{R}$. Now, we continue exactly as in the proof of Theorem 3.2.1 (c). $\square$

Proposition 3.4.2. *Let $D \subsetneq \mathbb{C}^n$ be a Reinhardt domain that is an $\mathcal{H}^\infty$-domain of holomorphy. Then $\mathcal{H}^\infty(D) \subsetneq \mathcal{O}^{(0+)}(D)$.*

Proof. In view of the proof of Theorem 3.2.1 (f), we only need to prove that there exist an $a \in (\partial D) \cap \mathbb{C}^n_*$ and a $\alpha \in (\mathbb{Z}^n)_*$ such that $D \subset \boldsymbol{D}_{\alpha,c}$ with $c := \log|a^\alpha|$.

Indeed, by Theorem 3.4.1, $D = \operatorname{int}\bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c}$ with $A \subset (\mathbb{Z}^n)_* \times \mathbb{R}$. It remains to use Remark 1.5.9. $\square$

Theorem 3.4.3. *Let $D \subsetneq \mathbb{C}^n$ be a fat Reinhardt domain of holomorphy and let $N > 0$. Then D is an $\mathcal{O}^{(N)}$-domain of holomorphy.*

Proof. Since $D = \operatorname{int}\bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c}$, where $A \subset (\mathbb{R}^n)_* \times \mathbb{R}$, and

$$\bigcup_{(\alpha,c)\in A} \mathcal{O}^{(N)}(\boldsymbol{D}_{\alpha,c})|_D \subset \mathcal{O}^{(N)}(D),$$

we only need to use Theorem 3.2.1 (a) and Remark 1.11.3 (e). $\square$

Theorem 3.4.4 ([HDT 2003]). *Every Reinhardt domain of holomorphy $D \subset \mathbb{C}^n$ is an $\mathcal{O}^{(1)}$-domain of holomorphy (cf. Theorem 3.4.3).*

Proof. By Theorem 1.11.13, $D = D^* \setminus M$, where D^* is a fat domain of holomorphy and $M := \bigcup_{j:\, V_j\cap D=\varnothing} V_j$. Suppose that D_0, $\widetilde{D}$ are as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{O}^{(1)}(D)$. We may assume that D_0 is a connected component of $D \cap \widetilde{D}$. Let $b \in \widetilde{D} \cap \partial D_0$. If $b \in \partial D^*$, then, by Theorem 3.4.3, $\widetilde{D} \subset D^*$. Thus $b \in D^* \cap M$, say $b \in V_j$. Then the function $f(z) := 1/z_j$ belongs to $\mathcal{O}^{(1)}(D)$ (recall that $V_j \cap D = \varnothing$). Since f does not extend through b, we get a contradiction. $\square$

Remark 3.4.5. Observe that if D is a non-fat Reinhardt domain of holomorphy, then by Proposition 1.9.12, D is not an $\mathcal{O}^{(N)}$-domain of holomorphy for $0 \le N < 1$.

Theorem 3.4.6 ([Jar-Pfl 1987]). *Let $D \subset \mathbb{C}^n$ be a Reinhardt log-convex domain. Then*

$$\mathcal{E}(D, \mathcal{H}^\infty(D)) = \operatorname{int} \overline{\exp \boldsymbol{K}(\log D))}$$

(*cf. Definition* 1.12.1).

Proof. Put $G_\ell := \mathcal{E}(D, \mathcal{H}^\infty(D))$, $G_r := \operatorname{int} \overline{\exp(\boldsymbol{K}(\log D))}$. Recall that

$$\boldsymbol{K}(\log D) = \boldsymbol{K}(\log G_r)$$

is the smallest convex domain of rational type that contains $X := \log D$. Note that both domains G_ℓ and G_r are fat and of rational type. Obviously, $G_r \subset G_\ell$. It remains to show that $\mathcal{H}^\infty(D) \subset \mathcal{O}(G_r)$. Fix an $f \in \mathcal{H}^\infty(D)$,

$$f(z) = \sum_{\alpha \in \boldsymbol{E}(X)^\perp \cap \mathbb{Z}^n} a_\alpha^f z^\alpha, \quad z \in D.$$

Looking at the definition of $\boldsymbol{Z}(X)$ and the form of the series it is clear that f extends to a bounded holomorphic function on $\exp(\boldsymbol{Z}(X))$.[8] Repeating this argument leads to a bounded extension of f on $\exp(\boldsymbol{K}(X))$. Finally, the Riemann theorem on removable singularities gives the extension to G_r. □

Example 3.4.7 (Cf. Example 3.3.10). For $\beta \in \mathbb{R}_{>0} \setminus \mathbb{Q}$, let $\alpha^1 := (\beta, \beta + 1, 0)$ and $\alpha^2 := (\beta + 1, \beta + 2, 1)$. Put

$$\begin{aligned} D &:= \boldsymbol{D}_{\alpha^1} \cap \boldsymbol{D}_{\alpha^2} \\ &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^\beta |z_2|^{\beta+1} < 1,\ |z_1|^{\beta+1} |z_2|^{\beta+2} |z_3| < 1\}. \end{aligned}$$

Then $\mathcal{H}^\infty(D) \simeq \mathbb{C}$.

Exercise 3.4.8 (Cf. Exercise 3.3.11). Let

$$\alpha^1 := (\alpha_1, \alpha_2, \alpha_3), \quad \alpha^2 := (1 - \alpha_1, 1 - \alpha_2, 1 - \alpha_3),$$

where $0 < \alpha_j < 1$, $j = 1, 2, 3$, $\alpha_1 \neq \alpha_2$, $(\alpha_1 - \alpha_3)/(\alpha_1 - \alpha_2) \notin \mathbb{Q}$. Put

$$\begin{aligned} D &:= \boldsymbol{D}_{\alpha^1} \cap \boldsymbol{D}_{\alpha^2} \\ &= \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^{\alpha_1} |z_2|^{\alpha_2} |z_3|^{\alpha_3} < 1,\ |z_1|^{1-\alpha_1} |z_2|^{1-\alpha_2} |z_3|^{1-\alpha_3} < 1\}. \end{aligned}$$

Then

$$\mathcal{E}(D, \mathcal{H}^\infty(D)) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1 z_2 z_3| < 1\}.$$

[8] For $x^0 \in X$, $v \in \boldsymbol{F}(X)$, and $\alpha \in \Sigma(f)$, we have $|a_\alpha^f| (e^{x^0 + v})^\alpha = |a_\alpha^f| e^{\langle x^0 + v, \alpha \rangle} = |a_\alpha^f| e^{\langle x^0, \alpha \rangle}$.

3.5 $\mathcal{A}^k$-domains of holomorphy

The next important space after the space $\mathcal{H}^\infty(D)$ is the space $\mathcal{A}^k(D)$, $0 \le k \le +\infty$. We begin with the case where $k \in \mathbb{Z}_+$.

Theorem 3.5.1 ([Jar-Pfl 1997]). *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain of holomorphy. Then the following conditions are equivalent:*

(i) *D is an $\mathcal{A}^k$-domain of holomorphy for every $k \in \mathbb{Z}_+$;*

(ii) *D is an $\mathcal{H}^{\infty,k}_{\mathrm{loc}}$-domain of holomorphy for every $k \in \mathbb{Z}_+$;*

(iii) *D is an $\mathcal{H}^{\infty}_{\mathrm{loc}}$-domain of holomorphy;*

(iv) *D is fat.*

In particular, if D is an $\mathcal{H}^\infty$-domain of holomorphy, then D is an $\mathcal{A}^k$-domain of holomorphy, $k \in \mathbb{Z}_+$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. The implication (iii) $\Rightarrow$ (iv) follows from Corollary 1.11.4 (a). The implication (iv) $\Rightarrow$ (i) follows from Remark 1.11.3 (e) and Theorem 3.2.1 (b). $\square$

We move to the case where $k = +\infty$.

Theorem 3.5.2 ([Jar-Pfl 1997]). *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain of holomorphy. Then the following conditions are equivalent:*

(i) *D is fat and satisfies the Fu condition;*

(ii) *D is an $\mathcal{H}^{\infty,\infty}_{\mathrm{loc}}$-domain of holomorphy;*

(iii) *D is an $\mathcal{A}^\infty$-domain of holomorphy;*

(iv) *D is an $\mathcal{O}(\bar{D})$-domain of holomorphy.*

Moreover, if D is an $\mathcal{H}^\infty$-domain of holomorphy, then each of the above conditions is equivalent to the following one:

(v) *D is an $\mathcal{H}^\infty(D) \cap \mathcal{O}(\bar{D})$-domain of holomorphy.*

We need the following two lemmas.

Lemma 3.5.3. *Let $D \subsetneq \mathbb{C}^n$ be a Reinhardt domain, $\mathcal{S} \in \mathcal{O}(D)$, and $k \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) *D is an $\mathcal{S}$-domain of holomorphy;*

(ii) *D is an $\mathcal{F}$-domain of holomorphy, where $\mathcal{F} := \{D^\beta f : f \in \mathcal{S},\ \beta \in S_k\}$, where $S_k := \{\beta \in \mathbb{Z}^n_+ : |\beta| = k\}$.*

Proof. The implication (ii) $\Rightarrow$ (i) is obvious (Remark 1.11.3 (a)).

(i) $\Rightarrow$ (ii): Suppose that D is not an $\mathcal{F}$-domain of holomorphy. Let $a \in D$ and $r > d_D(a)$ be such that each derivative $D^\beta f$ extends holomorphically to $\mathbb{P}(a,r)$, $\beta \in S_k$.

Observe that if $g \in \mathcal{O}(D)$ is such that each derivative $\frac{\partial g}{\partial z_j}$ extends to a function $g_j \in \mathcal{O}(\mathbb{P}(a,r))$, $j = 1,\dots,n$, then the function g itself extends holomorphically to $\mathbb{P}(a,r)$. Indeed, the extension may be given by the formula

$$\tilde{g}(z) = g(a) + \sum_{j=1}^{n}(z_j - a_j)\int_0^1 g_j(a + t(z-a))\,dt, \quad z \in \mathbb{P}(a,r) \quad (\text{Exercise}).$$

Consequently, using the above remark inductively, we easily conclude that every function $f \in \mathcal{S}$ extends holomorphically to $\mathbb{P}(a,r)$; a contradiction. □

Lemma 3.5.4. *Let $D \subsetneq \mathbb{C}^n$ be a Reinhardt domain. Assume that D is an $\mathcal{H}^{\infty,S}$-domain of holomorphy,*[9] *where S is such that there exists a $k_0 \in \mathbb{Z}_+$ such that $S_{k_0} \subset S$. Then*

$$D = \operatorname{int} \bigcap_{\substack{f \in \mathcal{H}^{\infty,S}(D)\\ \alpha \in \Sigma(f),\ \beta \in S\\ \alpha \neq \beta,\ \binom{\alpha}{\beta} \neq 0}} \left\{z \in \mathbb{C}^n(\alpha) : \left|a_\alpha^f \beta! \tbinom{\alpha}{\beta} z^{\alpha-\beta}\right| < \|D^\beta f\|_D\right\}.$$

Moreover, if $S = (\mathbb{Z}_+^n)_*$, then D satisfies the Fu condition.

Proof. By Lemma 3.5.3, D is an $\mathcal{F} := \{D^\beta f : f \in \mathcal{H}^{\infty,S}(D),\ \beta \in S\}$-domain of holomorphy. Observe that $\mathcal{F} \subset \mathcal{H}^\infty(D)$ is invariant under n-rotations (in the sense of (1.12.1)). Thus we may apply Proposition 1.12.6 (b) (Exercise).

Now, assume that $S = (\mathbb{Z}_+^n)_*$ and let $(\partial D) \cap V_{j_0} \neq \varnothing$ for some $j_0 \in \{1,\dots,n\}$. By Remark 1.5.7 (e), to prove that $\widehat{D}^{(j_0)} = D$ we only need to show that $\alpha_{j_0} - \beta_{j_0} \geq 0$ for any $\alpha \in \Sigma(f)$, $\beta \in S$ with $\alpha \neq \beta$, $\binom{\alpha}{\beta} \neq 0$.

Suppose that there exists $f \in \mathcal{H}^{\infty,S}(D)$ and $\alpha \in \Sigma(f)$ with $\alpha_{j_0} < 0$. Put $\beta = (\beta_1,\dots,\beta_n)$, $\beta_\nu := \max\{0,\alpha_\nu\}$, $\nu = 1,\dots,n$. Then $\beta \neq \alpha$, and $\binom{\alpha}{\beta} \neq 0$. Therefore

$$z^{\alpha-\beta} = \prod_{\substack{\nu \in \{1,\dots,n\}\\ \alpha_\nu < 0}} z_\nu^{\alpha_\nu}$$

is bounded on D. Consequently, $\alpha_{j_0} \geq 0$ because of $(\partial D) \cap V_{j_0} \neq \varnothing$; a contradiction. □

Now we are able to verify Theorem 3.5.2.

Proof of Theorem 3.5.2. We may assume that $D \subsetneq \mathbb{C}^n$. The implications (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are evident.

(ii) $\Rightarrow$ (i): Recall that D is fat (Corollary 1.11.4). Suppose that $\partial D \cap V_{j_0} \neq \varnothing$ for some $j_0 \in \{1,\dots,n\}$. Observe that for any $r > 0$ the open set $D_r := D \cap \mathbb{P}(r)$

[9] Recall that $\mathcal{H}^{\infty,S}(D) := \{f \in \mathcal{O}(D) : \forall_{\alpha \in S} : D^\alpha f \in \mathcal{H}^\infty(\Omega)\}$, $\varnothing \neq S \subset \mathbb{Z}_+^n$.

is fat and log-convex. We know that D_r is an $\mathcal{H}^{\infty,\mathbb{Z}_+^n}$-domain of holomorphy (cf. Remark 1.11.3 (e)). Hence, by Lemma 3.5.4, if $\bar{D}_r \cap V_{j_0} \neq \varnothing$, then $\widehat{D}_r^{(j_0)} = D_r$. Consequently, $\widehat{D}^{(j_0)} = D$.

(i) $\Rightarrow$ (iv) (resp. (i) $\Rightarrow$ (v) if D is an $\mathcal{H}^\infty$-domain of holomorphy)): Suppose that D is not an $\mathcal{O}(\bar{D})$-domain of holomorphy (resp. $\mathcal{H}^\infty(D) \cap \mathcal{O}(\bar{D})$-domain of holomorphy). Let D_0, $\widetilde{D}$ be as in Proposition 1.11.2 (*) with $\mathcal{S} = \mathcal{O}(\bar{D})$ (resp. $\mathcal{S} = \mathcal{H}^\infty(D) \cap \mathcal{O}(\bar{D})$). Recall (cf. Theorem 1.11.13 (b) (resp. 3.4.1 (iv))) that D can be written as

$$D = \operatorname{int} \bigcap_{(\alpha,c)\in A} \boldsymbol{D}_{\alpha,c},$$

where $A \subset (\mathbb{R}^n)_* \times \mathbb{R}$ (resp. $A \subset (\mathbb{Z}^n)_* \times \mathbb{R}$). Fix $(\alpha, c) \in A$ and $\varepsilon > 0$ such that $D \subset \boldsymbol{D}_{\alpha,c} \subset \boldsymbol{D}_{\alpha,c+\varepsilon}$ and $\widetilde{D} \not\subset \boldsymbol{D}_{\alpha,c+\varepsilon}$. Since $\boldsymbol{D}_{\alpha,c+\varepsilon}$ is a domain of holomorphy (resp. an $\mathcal{H}^\infty(\boldsymbol{D}_{\alpha,c+\varepsilon})$-domain of holomorphy), we only need to observe that, by Remark 1.5.11 (b), we have $\bar{D} \subset \overline{\boldsymbol{D}_{\alpha,c}} \subset \boldsymbol{D}_{\alpha,c+\varepsilon}$; a contradiction. $\square$

Example 3.5.5 ([Sib 1975]). Let $T = \{(z_1, z_2) \in \mathbb{D} \times \mathbb{D}_* : |z_1| < |z_2|\}$ be the Hartogs triangle. Then:

(a) T is an $\mathcal{A}^k$-domain of holomorphy for arbitrary $k \in \mathbb{Z}_+$ (cf. Theorem 3.5.1).

(b) T is not an $\mathcal{A}^\infty$-domain of holomorphy (cf. Theorem 3.5.2).

Exercise 3.5.6. Complete the following direct proof showing that T is an $\mathcal{A}^k(T)$-domain of holomorphy for every $k \in \mathbb{Z}_+$.

Fix a $k \in \mathbb{Z}_+$ and let D_0, $\widetilde{D}$ be as always with $D = T$ and $\mathcal{S} = \mathcal{A}^k(T)$. We may assume that $\widetilde{D} \not\subset \bar{T}$. Let $a = (a_1, a_2) \in (\widetilde{D} \setminus V_0) \setminus \bar{T}$. We have the following two cases:

- $|a_1| > 1$: Then the function $f(z_1, z_2) := 1/(z_1 - a_1)$ belongs to $\mathcal{O}(\bar{T}) \subset \mathcal{A}^k(T)$ and is not extendible to $\widetilde{D}$; a contradiction.
- $|a_2| < |a_1| \le 1$: Then the function $f(z_1, z_2) := z_1^{k+2}/(a_1 z_2 - a_2 z_1)$ belongs to $\mathcal{A}^k(T)$ and is not extendible to $\widetilde{D}$; a contradiction.

Exercise 3.5.7. Find a power series $f(z) = \sum_{k=0}^\infty a_k z^k$, $z \in \mathbb{D}$, such that $f \in \mathcal{A}^\infty(\mathbb{D}) \setminus \mathcal{O}(\bar{\mathbb{D}})$.

3.6 L_h^p-domains of holomorphy

Our aim in this section is to discuss the problem of geometric characterization of Reinhardt $\mathcal{S}$-domains of holomorphy $D \subset \mathbb{C}^n$ with $\mathcal{S} \subset L_h^p(D)$. Recall the general Proposition 3.1.6 and Theorem 3.2.1 (g) for the elementary Reinhardt domains.

The following lemma, which will be used in the proof of Theorem 3.6.4, is of independent interest.

Lemma 3.6.1. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain. Then $\mathcal{H}^{\infty,S_1}(D) \subset \mathcal{A}(D)$. Consequently, $\mathcal{H}^{\infty,k}(D) \subset \mathcal{A}^{k-1}(D)$ for $k \in \mathbb{N}$. In particular, $\mathcal{H}^{\infty,\infty}(D) \subset \mathcal{A}^{\infty}(D)$.*

Exercise 3.6.2. Find a fat domain $D \subset \mathbb{C} \setminus (-\infty, 0]$ such that

$$\operatorname{Log} \in \mathcal{H}^{\infty,S_1}(D) \setminus \mathcal{A}(D),$$

where Log stands for the principal branch of logarithm.

Before presenting the proof we recall a few well known facts from the theory of metric spaces. Let (X, ρ) be an arcwise connected metric space. For a continuous curve $\gamma\colon [a,b] \to X$ define its *ρ-length* $L_\rho(\gamma) \in [0, +\infty]$ by the formula

$$L_\rho(\gamma) := \sup\Big\{\sum_{j=1}^{N} \rho(\gamma(t_{j-1}), \gamma(t_j)) : a = t_0 < \cdots < t_N = b,\ N \in \mathbb{N}\Big\}.$$

Obviously, $\rho(\gamma(a), \gamma(b)) \le L_\rho(\gamma)$. One can easily prove (Exercise) that

$$L_\rho(\gamma) = L_\rho(\gamma|_{[a,c]}) + L_\rho(\gamma|_{[c,b]}), \quad a < c < b.$$

We define the *inner metric* ρ^i associated to ρ by the formula

$$\rho^i(x,y) := \inf\{L_\rho(\gamma) : \gamma \in \mathcal{C}([a,b], X),\ \gamma(a) = x,\ \gamma(b) = y\}, \quad x, y \in X.$$

Clearly, $\rho \le \rho^i$. One can easily prove (Exercise) that ρ^i is a metric.

In the case where X is a subdomain of E, where $(E, \|\ \|)$ is a normed space, with $\rho(x,y) = \|x - y\|$, $x, y \in X$, we have $\rho^i(x,y) = \|x-y\| = \rho(x,y)$ provided that the segment $[x,y]$ is contained in X. In particular, if $B(x^0, r) = \{x \in E : \|x - x^0\| < r\} \subset X$, then

$$\{x \in X : \rho^i(x^0, x) < r\} = B(x^0, r) \quad \text{(Exercise)}.$$

Nevertheless, one can easily find (Exercise) a bounded domain $X \subset E$ such that the metric ρ^i is unbounded.

One can prove (Exercise) that if X is a subdomain of $\mathbb{R}^m$, $f\colon X \to \mathbb{C}$ is Fréchet differentiable, and $\|f'(x)\| \le C$, $x \in X$, then

$$|f(x) - f(y)| \le C\rho^i(x,y), \quad x, y \in X.$$

Proof of Lemma 3.6.1. Since D has a univalent envelope of holomorphy (cf. Theorem 1.12.4), we may assume that D is a domain of holomorphy.[10] Since the

[10]Let $\widetilde{D}$ be the envelope of holomorphy of D. Then, by Lemma 3.5.3, $\mathcal{H}^{\infty,S_1}(\widetilde{D})|_D = \mathcal{H}^{\infty,S_1}(D)$. Moreover, $\mathcal{A}(\widetilde{D})|_D \subset \mathcal{A}(D)$.

case $D = \mathbb{C}^n$ is trivial, we may assume that $D \subsetneq \mathbb{C}^n$. Fix an $f \in \mathcal{H}^{\infty,S_1}(D)$. Obviously,

$$|f(z') - f(z'')| \le \big(\max_{\alpha\in S_1} \|D^\alpha f\|_D\big)\varXi_D(z', z''), \quad z', z'' \in D,$$

where $\varXi_D$ denotes the arc-length distance on D with respect to the ℓ^1-norm $\| \ \|_1$. To show that f extends continuously to $\overline{D}$ it suffices (Exercise) to prove that for any $a \in \partial D$ there are a constant $c > 0$ and a neighborhood U_a of a such that

$$\varXi_D(z', z'') \le c(\|z' - z''\| + \|p_J(z')\| + \|p_J(z'')\|), \quad z', z'' \in D \cap U_a,$$

where

$$J := (j_1, \dots, j_s), \quad 1 \le j_1 < \dots < j_s \le n,$$
$$\{j_1, \dots, j_s\} = \{j \in \{1, \dots, n\} : a_j = 0\},$$
$$p_J : \mathbb{C}^n \to \mathbb{C}^s, \quad p_J(z_1, \dots, z_n) := (z_{j_1}, \dots, z_{j_s}), \quad p_\varnothing := 0.$$

Fix a point $a \in \partial D$. We may assume that $J = \varnothing$ or $J = (1, \dots, s)$ $(1 \le s \le n)$. Take $z', z'' \in D$.

If $J = \varnothing$, then put $w' := z'$, $w'' := z''$.

If $J \ne \varnothing$, then put

$$w' := (|z'_1|, \dots, |z'_s|, z'_{s+1}, \dots, z'_n), \quad w'' := (|z''_1|, \dots, |z''_s|, z''_{s+1}, \dots, z''_n) \in D.$$

Obviously,

$$\varXi_D(z', w') \le 2\pi(|z'_1| + \dots + |z'_s|), \quad \varXi_D(z'', w'') \le 2\pi(|z''_1| + \dots + |z''_s|).$$

It remains to show that there is a constant $c' > 0$ and a neighborhood U_a such that

$$\varXi_D(w', w'') \le c'\|z' - z''\|, \quad z', z'' \in D \cap U_a.$$

Using continuity it suffices to consider only the case where $0 \ne |z'_j| \ne |z''_j| \ne 0$, $j = 1, \dots, n$. Let $L_1 = \dots = L_s = \operatorname{Log}$ be the principal branch of the logarithm and let $L_{s+1}, \dots, L_n$ be arbitrary branches of the logarithm defined in small neighborhoods of $a_{s+1}, \dots, a_n$, respectively. Define

$$\gamma = (\gamma_1, \dots, \gamma_n) \colon [0, 1] \to \mathbb{C}^n, \quad \gamma_j(t) := e^{(1-t)L_j(w'_j)+tL_j(w''_j)}, \quad j = 1, \dots, n.$$

Since D is logarithmically convex, $\gamma([0, 1]) \subset D$. What is left to be shown is that for each j there is a $c'_j > 0$ such that the length ℓ_j of γ_j is estimated by

$\ell_j \le c'_j|z'_j - z''_j|$ provided that z', z'' are near a. We have

$$\begin{aligned}\ell_j &= \int_0^1 |\gamma'_j(t)|\,dt = \int_0^1 |w'_j|^{1-t}|w''_j|^t |L_j(w'_j) - L_j(w''_j)|\,dt\\ &= \frac{|L_j(w'_j) - L_j(w''_j)|}{|\log|w'_j| - \log|w''_j||}\big||w'_j| - |w''_j|\big|\\ &\le \begin{cases} |z'_j - z''_j| & \text{if } j \le s,\\ \frac{||z'_j|-|z''_j||}{|\log|z'_j|-\log|z''_j||}|L_j(z'_j) - L_j(z''_j)| & \text{if } j \ge s+1.\end{cases}\end{aligned}$$

The case $j \le s$ is trivial. If $j \ge s+1$, then we only need to observe that

$$|L_j(z'_j) - L_j(z''_j)| \le c''_j|z'_j - z''_j|$$

in a small neighborhood of a_j and

$$\lim_{\substack{u,v\to|a_j|\\ 0<u\ne v>0}} \frac{|u-v|}{|\log u - \log v|} = |a_j|,$$

so the term $\frac{||z'_j|-|z''_j||}{|\log|z'_j|-\log|z''_j||}$ remains bounded near a_j.

The proof is completed. □

Remark 3.6.3. (a) Recall that if D is a bounded (Reinhardt or not) domain, then $\mathcal{A}^k(D) \subset \mathcal{H}^{\infty,k}(D)$. Consequently, if D is a bounded Reinhardt domain, then, by Lemma 3.6.1, $\mathcal{H}^{\infty,k}(D) \subset \mathcal{A}^{k-1}(D) \subset \mathcal{H}^{\infty,k-1}(D)$, $k \in \mathbb{N}$. In particular, if D is a bounded Reinhardt domain, then $\mathcal{H}^{\infty,\infty}(D) = \mathcal{A}^\infty(D)$.

(b) Let

$$\begin{aligned} D = T &= \{(z_1, z_2) \in \mathbb{D}\times\mathbb{D} : |z_1| < |z_2|\},\\ f_k(z) &:= \frac{z_1^{2k}}{z_2^k}, \quad z = (z_1, z_2) \in D,\ k \in \mathbb{N}.\end{aligned}$$

Then

$$f_k \in \mathcal{H}^{\infty,k}(D) \setminus \mathcal{A}^k(D) \subset \mathcal{A}^{k-1}(D) \setminus \mathcal{A}^k(D).$$

Indeed,

$$\frac{\partial^{p+q} f_k}{\partial z_1^p \partial z_2^q}(z_1, z_2) = p!\binom{2k}{p} q!\binom{-k}{q}\frac{z_1^{2k-p}}{z_2^{k+q}} =: c(k,p,q)\frac{z_1^{2k-p}}{z_2^{k+q}}.$$

Consequently, if $p + q \le k$, then

$$\left|\frac{\partial^{p+q} f_k}{\partial z_1^p \partial z_2^q}(z_1, z_2)\right| \le |c(k,p,q)||z_1^{k-(p+q)}| \le |c(k,p,q)|, \quad (z_1, z_2) \in D,$$

and so $f_k \in \mathcal{H}^{\infty,k}(D)$. Moreover, if $p+q=k$, then

$$\frac{\partial^{p+q} f_k}{\partial z_1^p \partial z_2^q}(z_1, z_2) = c(k,p,q)\Big(\frac{z_1}{z_2}\Big)^{2k-p},$$

which shows that $\frac{\partial^{p+q} f_k}{\partial z_1^p \partial z_2^q} \notin \mathcal{C}(\bar{D})$.

(c) In the case where D is an unbounded Reinhardt domain, Lemma 3.6.1 implies that $\mathcal{H}_{\mathrm{loc}}^{\infty,S_1}(D) \subset \mathcal{A}(D)$ if D satisfies the following condition:

(*) for every point $a \in \partial D$ there exists a bounded Reinhardt neighborhood U_a of a such that $D \cap U_a$ has a finite number of components.

In fact, take $f \in \mathcal{H}_{\mathrm{loc}}^{\infty,S_1}(D)$ and $a \in \partial D$. Let U_a be as in (*). If S is a connected component of $D \cap U_a$, then Lemma 3.6.1 shows that $f \in \mathcal{A}(S)$. Suppose that $D \cap U_a$ has a finite number of connected components, $D \cap U_a = S_1 \cup \dots \cup S_k$. Fix $a^0 \in D$ and $a^j \in S_j$, $j = 1, \dots, k$. Let $\gamma\colon [0,1] \to D$ be a curve connecting $a^0, a^1, \dots, a^k$ and let $r > 0$ be so big that $\gamma([0,1]) \cup (D \cap U_a) \subset \mathbb{P}(r)$. Then $D \cap U_a$ is contained in one connected component S of $D \cap \mathbb{P}(r)$. Hence, by the first part of the proof, $f \in \mathcal{A}(S) \subset \mathcal{A}(D \cap U_a)$.

In particular, if D satisfies (*), then $\mathcal{H}_{\mathrm{loc}}^{\infty,k}(D) \subset \mathcal{A}^{k-1}(D)$, $k \in \mathbb{N}$.

Observe that (*) is satisfied if D is log-convex – we take $U_a = \mathbb{P}(r_a)$ with sufficiently large $r_a > 0$ and observe that $D \cap \mathbb{P}(r_a)$ is log-convex and therefore connected (cf. Remark 1.5.6 (d)).

$\boxed{?}$ We do not know whether the inclusion $\mathcal{H}_{\mathrm{loc}}^{\infty,S_1}(D) \subset \mathcal{A}(D)$ is true for arbitrary (unbounded) Reinhardt domains. $\boxed{?}$

Theorem 3.6.4 ([Jar-Pfl 1997]). *Let $D \subsetneq \mathbb{C}^n$ be a Reinhardt domain of holomorphy. Then the following conditions are equivalent:*

(i) *for each $k \in \mathbb{Z}_+$ the domain D is an $L_h^{\diamond,k}$-domain of holomorphy;*[11]

(ii) *D is fat and there exists a $p \in [1,\infty)$ such that $L_h^p(D) \neq \{0\}$;*

(iii) *D is fat and $\boldsymbol{E}(\log D) = \{0\}$;*[12]

(iv) *for each $k \in \mathbb{Z}_+$ the domain D is an $L_h^{\diamond,k} \cap \mathcal{A}^k$-domain of holomorphy;*

(v) *D is fat and for every $a \notin \bar{D} \cup V_0$ there exist sequences $(c_j)_{j=1}^\infty \subset \mathbb{R}_{>0}$, $(\beta^j)_{j=1}^\infty \subset \mathbb{Z}^n$ ($(\beta^j)_{j=1}^\infty \subset \mathbb{Z}_+^n$ provided that D is complete) such that*

$$D \subset \mathbb{C}^n(\beta^j), \quad \|c_j z^{\beta^j}\|_{L^2(D)} \le 1, \ j \in \mathbb{N}, \quad \lim_{j\to\infty} c_j |a^{\beta^j}| = +\infty.$$

Proof. The implications (v) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) are obvious. The implication (i) $\Rightarrow$ (iv) follows directly from Lemma 3.6.1.

[11] Recall that $L_h^{\diamond,k}(D) = L_h^{1,k}(D) \cap \mathcal{H}^{\infty,k}(D)$ (cf. Example 1.10.7 (h)).
[12] Recall that $\boldsymbol{E}(\log D) = \{0\}$ for any bounded Reinhardt domain.

(iii) $\Rightarrow$ (v): Fix $a \notin \overline{D} \cup \boldsymbol{V}_0$ and $j \in \mathbb{N}$. Put $X := \log D$ and let

$$x^0 := (\log|a_1|, \dots, \log|a_n|).$$

Note that $x^0 \notin \overline{X}$. Since $\boldsymbol{E}(X) = \{0\}$, there exist linearly independent vectors α^1, ..., $\alpha^n \in \mathbb{Z}^n$ ($\alpha^1, \dots, \alpha^n \in \mathbb{Z}_+^n$ provided that D is complete) such that

$$X \subset \bigcap_{i=1}^{n} H_{\alpha^i}^{x^0} =: X_0;$$

cf. Lemma 1.4.11 (iii). Let $A := [\alpha_k^i]_{i,k=1,\dots,n} \in \mathbb{GL}(n,\mathbb{C})$. We may assume that $|\det A| \geq j^2\pi^n|a_1\cdots a_n|^2$. Put $\beta^j =: \alpha^1 + \dots + \alpha^n - \mathbf{1}$,

$$X_1 := \{\xi \in \mathbb{R}^n : \xi_i < \langle x^0, \alpha^i\rangle,\ i = 1,\dots,n\}.$$

Then we get

$$\begin{aligned}
\|z^{\beta^j}\|_{L^2(D)}^2 &= (2\pi)^n \int_{\boldsymbol{R}(D)} r^{2\beta^j+\mathbf{1}}\, d\Lambda_n(r) = (2\pi)^n \int_X e^{\langle x, 2(\beta^j+\mathbf{1})\rangle}\, d\Lambda_n(x)\\
&\leq (2\pi)^n \int_{X_0} e^{\langle x, 2(\beta^j+\mathbf{1})\rangle}\, d\Lambda_n(x) = (2\pi)^n \int_{X_0} e^{\langle x, 2(\alpha^1+\dots+\alpha^n)\rangle}\, d\Lambda_n(x)\\
&= \frac{(2\pi)^n}{|\det A|} \int_{X_1} e^{2\langle \xi, \mathbf{1}\rangle} d\Lambda_n(\xi) = \frac{\pi^n}{|\det A|} e^{2\langle x^0, \alpha^1+\dots+\alpha^n\rangle}\\
&= \frac{\pi^n}{|\det A|} |a^{\alpha^1+\dots+\alpha^n}|^2 \leq (1/j)^2 |a^{\beta^j}|^2.
\end{aligned}$$

Let $c_j := j|a^{-\beta^j}|$. Then

$$\|c_j z^{\beta^j}\|_{L^2(D)} \leq 1 \quad \text{and} \quad c_j|a^{\beta^j}| = j.$$

Since D is fat, we have $D \subset \bigcap_{i=1}^n \boldsymbol{D}_{\alpha^i, \langle \alpha^i, x^0\rangle} \subset \bigcap_{i=1}^n \mathbb{C}^n(\alpha^i) \subset \mathbb{C}^n(\beta^j)$ (EXERCISE), which finishes the proof of (v).

(ii) $\Rightarrow$ (iii): We argue as in the proof of Proposition 1.13.13 (b) (using Remark 3.1.4 (e) instead of Example 1.10.7 (c)) – EXERCISE.

(iii) $\Rightarrow$ (i): Fix a $k \in \mathbb{Z}_+$ and suppose that D is not an $L_h^{\diamond,k}$-domain of holomorphy. Then we find domains D_0, $\widetilde{D}$ such that $\varnothing \neq D_0 \subset D \cap \widetilde{D}$, $\widetilde{D} \not\subset D$, and for any $f \in L_h^{\diamond,k}(D)$ there exists an $\tilde{f} \in \mathcal{O}(\widetilde{D})$ with $\tilde{f} = f$ on D_0. Since D is fat, $\widetilde{D} \not\subset \overline{D}$. Moreover, we may assume that $\widetilde{D} \cap \boldsymbol{V}_0 = \varnothing$.

Since $\boldsymbol{E}(\log D) = \{0\}$ and D is fat, there are linearly independent vectors α^1, ..., $\alpha^n \in \mathbb{Z}^n$, $c_1, \dots, c_n \in \mathbb{R}$, and $\varepsilon > 0$ such that

$$D \subset G_0 := \bigcap_{j=1}^{n} \boldsymbol{D}_{\alpha^j, c_j} \subset \bigcap_{j=1}^{n} \boldsymbol{D}_{\alpha^j, c_j+\varepsilon} =: G_1,$$

and $\widetilde{D} \not\subset G_1$. Using the fact that the α^j's are linearly independent, we may assume that $c_1 = \cdots = c_n = 0$.[13]

Now, we fix an $a \in \widetilde{D} \setminus G_1$ and then a j_0 such that $|a^{\alpha^{j_0}}| \geq e^{\varepsilon}$. Moreover, we set $\alpha := \alpha^1 + \cdots + \alpha^n$. For $N \in \mathbb{N}$ we define

$$f_N(z) := \frac{z^{N\alpha}}{z^{\alpha^{j_0}} - a^{\alpha^{j_0}}}, \quad z \in G_1.$$

Obviously, $f_N \in \mathcal{O}(G_1)$. Observe that if $f_N \in L_h^{\diamond,k}(D)$, then

$$\tilde{f}_N(z)(z^{\alpha^{j_0}} - a^{\alpha^{j_0}}) = z^{N\alpha} \text{ on } \widetilde{D}$$

and we have a contradiction.

Consequently, it remains to prove that $f_N \in L_h^{\diamond,k}(D)$. Observe that $D^\beta f_N$ with $\beta \in \mathbb{Z}_+^n$, $|\beta| \leq k$, is a finite sum of terms of the form (Exercise)

$$d\,\frac{z^{N\alpha + \ell\alpha^{j_0} - \beta}}{(z^{\alpha^{j_0}} - a^{\alpha^{j_0}})^{\ell+1}},$$

where $d \in \mathbb{Z}$, $\ell \in \{0, \ldots, k\}$. Thus it suffices to find an N such that

$$\|z^{N\alpha - \beta}\|_{L^p(G_0)} \leq 1, \quad |\beta| \leq k,\ p \in \{1, +\infty\}.$$

Let $A := [\alpha^j_\ell]_{j,\ell=1,\ldots,n} \in \mathbb{GL}(n,\mathbb{C})$, $B := A^{-1}$. Put

$$T_j(x) := \sum_{\ell=1}^{n} B_{\ell,j} x_\ell, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ j = 1, \ldots, n.$$

If $p = 1$ and $\nu \in \mathbb{Z}^n$, then we have

$$\begin{aligned}\int_{G_0} |z^\nu|\, d\Lambda_{2n}(z) &= (2\pi)^n \int_{\log G_0} e^{\langle x, \nu + \mathbf{2}\rangle}\, d\Lambda_n(x) \\ &= (2\pi)^n \int_{\{\xi_1 < 0, \ldots, \xi_n < 0\}} e^{\langle B\xi, \nu + \mathbf{2}\rangle} |\det B|\, d\Lambda_n(\xi) \\ &= \frac{(2\pi)^n}{|\det A| T_1(\nu + \mathbf{2}) \cdots T_n(\nu + \mathbf{2})}\end{aligned}$$

provided that $T_j(\nu + \mathbf{2}) > 0$, $j = 1, \ldots, n$. In particular, if

$$T_j(\nu) \geq \frac{2\pi}{|\det A|^{1/n}} - T_j(\mathbf{2}), \quad j = 1, \ldots, n,$$

[13] Use the biholomorphic mapping $\mathbb{C}^n \ni (z_1, \ldots, z_n) \mapsto (r_1 z_1, \ldots, r_n z_n) \in \mathbb{C}^n$, where $r_1, \ldots, r_n > 0$ are such that $r_1^{\alpha_1^j} \cdots r_n^{\alpha_n^j} = e^{-c_j}$, $j = 1, \ldots, n$.

then $\|z^\nu\|_{L^1(D_0)} \le 1$. Hence, if $\nu = N\alpha - \beta$ and if

$$N \ge T_j(\beta) + \max\Big\{0, \frac{2\pi}{|\det A|^{1/n}} - T_j(\mathbf{2}) : j = 1, \dots, n, \ |\beta| \le k\Big\},$$

then $\|z^{N\alpha-\beta}\|_{L^1(G_0)} \le 1$ for arbitrary $|\beta| \le k$.

It remains to prove that $\|z^{N\alpha-\beta}\|_{\mathcal{H}^\infty(G_0)} \le 1$ for all $|\beta| \le k$. Fix such a β. Write $N\alpha - \beta = \lambda_1\alpha^1 + \dots + \lambda_n\alpha^n$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Then $\lambda_j = T_j(N\alpha - \beta) = N - T_j(\beta) \ge 0$, $j = 1, \dots, n$. Consequently, $\|z^{N\alpha-\beta}\|_{\mathcal{H}^\infty(G_0)} \le 1$. □

Remark 3.6.5. Notice that the following general result holds. If $D \subset \mathbb{C}^n$ is a bounded domain of holomorphy, then D is an L^2_h-domain of holomorphy iff $U \setminus D$ is not pluripolar (Definition 1.14.18) for any open set U such that $U \cap D \ne \varnothing$ (cf. [Pfl-Zwo 2002]).

Exercise 3.6.6. Using the above remark prove that every bounded fat Reinhardt domain of holomorphy is an L^2_h-domain of holomorphy.

Proposition 3.6.7 ([Jar-Pfl 1997]). *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain of holomorphy. Then the following conditions are equivalent:*

(i) *D is fat and there exist $0 \le m \le n$ and a permutation of coordinates such that $D = D' \times \mathbb{C}^{n-m}$ with $\boldsymbol{E}(\log D') = \{0\}$;*

(ii) *D is an $\mathcal{H}^{\infty,1}$-domain of holomorphy;*

(iii) *D is an $\mathcal{H}^{\infty,k}$-domain of holomorphy for any $k \in \mathbb{Z}_+$.*

Proof. (i) ⇒ (iii) follows from Theorem 3.6.4. The implication (iii) ⇒ (ii) is trivial.

(ii) ⇒ (i): Let $F := \boldsymbol{E}(\log D)$ and $m := \dim F^\perp$. We may assume that $e_1, \dots, e_s \in F^\perp$, $e_{s+1}, \dots, e_n \notin F^\perp$ for some $0 \le s \le m$.

The cases $m = 0$ and $m = n$ are trivial. Assume $1 \le m \le n - 1$. Since D is an $\mathcal{H}^{\infty,1}$-domain of holomorphy, Proposition 1.12.6 (b) implies that

$$D = \operatorname{int} \bigcap_{\substack{f \in \mathcal{H}^{\infty,1}(D),\ \|f\|_D = 1\\ \alpha \in \Sigma(f)_*}} \{z \in \mathbb{C}^n(\alpha) : |a^f_\alpha z^\alpha| < 1\}.$$

Thus

$$F^\perp = \operatorname{span}\{\alpha : \exists_{f \in \mathcal{H}^{\infty,1}(D)} : \ \alpha \in \Sigma(f)_*\}.$$

By Proposition 1.6.5 (b) we get

$$D \subset \operatorname{int} \bigcap_{\substack{f \in \mathcal{H}^{\infty,1}(D),\\ \alpha \in \Sigma(f),\ j \in \{1,\dots,n\}\\ \alpha \ne e_j,\ \alpha_j \ne 0}} \Big\{z \in \mathbb{C}^n(\alpha) : |a^f_\alpha \alpha_j z^{\alpha - e_j}| < \Big\|\frac{\partial f}{\partial z_j}\Big\|_D\Big\}.$$

Hence

$$F^\perp \supset \{\alpha - e_j : \exists_{f\in\mathcal{H}^{\infty,1}(D)}\ \exists_{j\in\{1,\dots,n\}} :\ \alpha\in\Sigma(f),\ \alpha\neq e_j,\ \alpha_j\neq 0\}.$$

Take an $f\in\mathcal{H}^{\infty,1}(D)$. If $\alpha\in\Sigma(f)$ is such that $\alpha\neq e_j$ and $\alpha_j\neq 0$, then $\alpha,\alpha-e_j\in F^\perp$ and, consequently, $e_j\in F^\perp$. Thus,

$$\Sigma(f)\subset\{e_{s+1},\dots,e_n\}\cup(\mathbb{Z}^s\times\{0\}^{n-s})$$

and, therefore, the Laurent expansion of f has the form

$$f(z)=\Big(\sum_{\beta\in\mathbb{Z}^s} a^f_{(\beta,0)}z'^\beta\Big)+a^f_{e_{s+1}}z_{s+1}+\dots+a^f_{e_n}z_n,$$
$$z=(z',z_{s+1},\dots,z_n)\in D\subset\mathbb{C}^s\times\mathbb{C}^{n-s}.$$

Since D is the domain of existence of $\mathcal{H}^{\infty,1}(D)$, we conclude that $D=D'\times\mathbb{C}^{n-s}$. Clearly, $F=\boldsymbol{E}(\log D')\times\mathbb{R}^{n-s}$. Hence $s=m$ and therefore $\boldsymbol{E}(\log D)=\{0\}$. $\square$

Proposition 3.6.8 ([Jar-Pfl 1997]). *Let $D\subsetneq\mathbb{C}^n$ be a Reinhardt domain. Then the following conditions are equivalent:*

(i) *D is an $\mathcal{H}^{\infty,S_1}$-domain of holomorphy;*

(ii) *there exist $A\subset(\mathbb{Z}^n)_*$ and functions $c_1,\dots,c_n\colon A\to\mathbb{R}$ such that*

$$D=\operatorname{int}\bigcap_{\substack{\alpha\in A,\ j\in\{1,\dots,n\}\\ \alpha\neq e_j,\ \alpha_j\neq 0}}\{z\in\mathbb{C}^n(\alpha):|z^{\alpha-e_j}|<e^{c_j(\alpha)}\}. \tag{3.6.1}$$

Proof. The implication (i) $\Rightarrow$ (ii) follows from Lemma 3.5.4. To prove that any domain D of the form (3.6.1) is an $\mathcal{H}^{\infty,S_1}$-domain of holomorphy, observe that

$$D=\operatorname{int}\bigcap_{\alpha\in A}\operatorname{int}\bigcap_{\substack{j\in\{1,\dots,n\}\\ \alpha\neq e_j,\ \alpha_j\neq 0}}\{z\in\mathbb{C}^n(\alpha):|z^{\alpha-e_j}|<e^{c_j(\alpha)}\}=:\operatorname{int}\bigcap_{\alpha\in A}G_\alpha.$$

Thus, it suffices to consider only the case where

$$G_\alpha=\bigcap_{\substack{j\in\{1,\dots,n\}\\ \alpha\neq e_j,\ \alpha_j\neq 0}}\{z\in\mathbb{C}^n(\alpha):|z^{\alpha-e_j}|<e^{c_j(\alpha)}\}.$$

We may assume that $\alpha_j\neq 0$, $j=1,\dots,n$ (otherwise $G_\alpha\simeq G'_\alpha\times\mathbb{C}^k$ and we consider a lower dimensional case). In particular, $\alpha\neq e_j$, $j=1,\dots,n$. Since G_α is fat, it is enough to prove that for any point $a\notin\overline{G}_\alpha\cup V_0$ there exists a function

$f \in \mathcal{H}^{\infty,S_1}(D)$ such that f cannot be continued across a. Fix such an a and let $j_0 \in \{1,\dots,n\}$ be such that $\delta := e^{c_{j_0}(\alpha)} - |a^{\alpha-e_{j_0}}| > 0$. Then the function

$$f(z) := \frac{z^\alpha}{z^{\alpha-e_{j_0}} - a^{\alpha-e_{j_0}}}, \quad z \in G_\alpha,$$

belongs to $\mathcal{H}^{\infty,S_1}(G_\alpha)$ and evidently cannot be continued across a. Indeed,

$$\frac{\partial f}{\partial z_j}(z) = \frac{\alpha_j z^{\alpha-e_j}}{z^{\alpha-e_{j_0}} - a^{\alpha-e_{j_0}}} - \frac{(\alpha-e_{j_0})_j z^{\alpha-e_j} z^{\alpha-e_{j_0}}}{(z^{\alpha-e_{j_0}} - a^{\alpha-e_{j_0}})^2}, \quad z \in G_\alpha,$$

and so

$$\left\| \frac{\partial f}{\partial z_j} \right\|_{G_\alpha} = \frac{\alpha_j e^{c_j(\alpha)}}{\delta} + \frac{(\alpha-e_{j_0})_j e^{c_j(\alpha)+c_{j_0}(\alpha)}}{\delta^2}. \qquad \square$$

Remark 3.6.9. There exists an $\alpha \in (\mathbb{Z}^n)_*$ such that $\mathcal{H}^{\infty,S_1}(G_\alpha) \not\subset \mathcal{H}^\infty(G_\alpha)$.

For example $\alpha := (1,-1)$, $c_1(\alpha) = c_2(\alpha) = 0$. Then

$$G_\alpha = \{(z_1,z_2) \in \mathbb{C}^2 : |z_2| > 1,\ |z_1| < |z_2|^2\},$$

and the unbounded function $f(z) := z_1/z_2$, $z = (z_1,z_2) \in G_\alpha$, belongs to $\mathcal{H}^{\infty,S_1}(G_\alpha)$ (Exercise).

$\boxed{?}$ The problem of characterizing circular (in particular, balanced) $\mathcal{F}$-domains of holomorphy is still open for many of the natural Fréchet spaces $\mathcal{F}$ we discussed so far. $\boxed{?}$ (Cf. [Sib 1975], [Sic 1982], [Sic 1984], [Sic 1985], [Jar-Pfl 1996] for positive results.)

Chapter 4
Holomorphically contractible families on Reinhardt domains

4.1 Introduction

Recall from Chapter 2 that $\operatorname{Bih}(\mathbb{D}^n, \mathbb{B}_n) = \operatorname{Bih}(\mathbb{B}_n, \mathbb{L}_n) = \varnothing$ for $n \geq 2$ and $\operatorname{Bih}(\mathbb{D}^n, \mathbb{L}_n) = \varnothing$ for $n \geq 3$ (see Theorem 2.1.17). In this chapter we will study other methods which may be useful to decide whether two given domains in $\mathbb{C}^n$ are not biholomorphically equivalent. The idea here is that two biholomorphically equivalent domains should have the same amount of functions of a special class (e.g. bounded holomorphic functions or psh functions with specific singularities) or geometric data (e.g. analytic discs through corresponding pairs of points). To give a rough idea of what we are going to deal with let us discuss again whether $\mathbb{D}^n$ and $\mathbb{B}_n$ are biholomorphically equivalent domains.

We introduce the following function:

$$\widehat{m}_D\colon D \to [0, \infty), \quad \widehat{m}_D(z) := \sup\{|f(z)| : f \in \mathcal{O}(D, \mathbb{D}),\ f(0) = 0\},$$

where D is a domain in $\mathbb{C}^n$ with $0 \in D$.

Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^m$ be domains, both containing the origin. Note that if $F \in \mathcal{O}(G, D)$, $F(0) = 0$, then (Exercise)

$$\widehat{m}_D(F(z)) \leq \widehat{m}_G(z), \quad z \in G. \tag{4.1.1}$$

In particular, if F is biholomorphic, then $\widehat{m}_D \circ F = \widehat{m}_G$.

In the case where $D \in \{\mathbb{D}^n, \mathbb{B}_n\}$ we get

$$\widehat{m}_D(z) = q_D(z), \quad z \in D, \tag{4.1.2}$$

where

$$q_D(z) := \begin{cases} \|z\|_\infty & \text{if } D = \mathbb{D}^n, \\ \|z\| & \text{if } D = \mathbb{B}_n, \end{cases} \quad z \in \mathbb{C}^n.$$

Indeed, let $f \in \mathcal{O}(D, \mathbb{D})$, $f(0) = 0$. Then, in virtue of Proposition 2.1.9, it follows that $|f(z)| \leq q_D(z)$, $z \in D$. Hence, $\widehat{m}_D \leq q_D$, $D \in \{\mathbb{D}^n, \mathbb{B}_n\}$.

On the other hand, in the case of $D = \mathbb{D}^n$ and $z \in \mathbb{D}^n \setminus \{0\}$ put

$$g_z\colon \mathbb{D}^n \to \mathbb{D}, \quad g_z(w) := w_j,$$

when $q_{\mathbb{D}^n}(z) = |z_j|$. Therefore, $\widehat{m}_{\mathbb{D}^n}(z) \geq |g_z(z)| = q_{\mathbb{D}^n}(z)$. Now let $D = \mathbb{B}_n$ and $z \in \mathbb{B}_n \setminus \{0\}$. Choose a rotation A_z such that $A_z z = (\tilde{z}_1, 0)$; in particular, $\|z\| = \|A_z z\| = |\tilde{z}_1|$. Put

$$g_z \colon \mathbb{B}_n \to \mathbb{D}, \quad g_z(w) := (A_z w)_1.$$

Obviously, $g_z \in \mathcal{O}(\mathbb{B}_n, \mathbb{D})$, $g_z(0) = 0$; hence $\widehat{m}_{\mathbb{B}_n}(z) \geq |g_z(z)| = \|z\| = q_{\mathbb{B}_n}(z)$. So the above equations are verified.

Exercise 4.1.1. Prove formula (4.1.2) for an arbitrary norm ball in $\mathbb{C}^n$.

Now let $F \colon \mathbb{B}_n \to \mathbb{D}^n$ be a biholomorphic mapping, $n \geq 2$. Using a Möbius transform $\varphi \colon \mathbb{D}^n \to \mathbb{D}^n$,

$$\varphi(z) := \left(\frac{F_1(z) - F_1(0)}{1 - \overline{F_1(0)} F_1(z)}, \dots, \frac{F_n(z) - F_n(0)}{1 - \overline{F_n(0)} F_n(z)} \right), \quad z \in \mathbb{D}^n,$$

we may even assume that $F(0) = 0$.

Then, by (4.1.1),

$$\widehat{m}_{\mathbb{D}^n}(F(z)) = \widehat{m}_{\mathbb{B}_n}(z), \quad z \in \mathbb{B}_n.$$

Therefore, we get the following equation:

$$\|F^{-1}(t, 1/2, \dots, 1/2)\| = \max\{t, 1/2\}, \quad t \in (0, 1).$$

Note that the left function is differentiable on $(0, 1)$, but, obviously, the right one is not; a contradiction.

Observe that in the above argument the number of bounded holomorphic functions on $\mathbb{D}^n$ and $\mathbb{B}_n$ is compared and this strategy finally has led to the result that both domains cannot be biholomorphically equivalent.

Instead of dealing with the function $\widehat{m}_D$, i.e. with the family of bounded holomorphic functions, we may take all analytic discs $\varphi \in \mathcal{O}(\mathbb{D}, D)$ in D through two given points, where D is a domain in $\mathbb{C}^n$ with $0 \in D$. We define

$$\hat{k}_D(z) := \inf\{r \in [0, 1) : \exists_{\varphi \in \mathcal{O}(\mathbb{D}, D)} : \varphi(0) = 0, \ \varphi(r) = z\}, \quad z \in D.$$ [1]

Remark 4.1.2. Let $G \subset \mathbb{C}^m$, $D \subset \mathbb{C}^n$ be domains both containing the origin. If $F \in \mathcal{O}(G, D)$, $F(0) = 0$, then (Exercise)

$$\hat{k}_D \circ F \leq \hat{k}_G; \tag{4.1.3}$$

see (4.1.1). In particular, if F is biholomorphic, then $\hat{k}_D \circ F = \hat{k}_G$.

[1] Note that $\hat{k}_D(0) = 0$.

We claim that $\hat{k}_D = q_D$, where $D \in \{\mathbb{D}^n, \mathbb{B}_n\}$.

Indeed, fix a $z \in D \setminus \{0\}$. Then $\varphi(\lambda) := \lambda \frac{z}{q_D(z)}$ gives a function $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = 0$ and $\varphi(q_D(z)) = z$. Therefore, $\hat{k}_D(z) \le q_D(z)$.

On the other hand, let $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = 0$ and $\varphi(r) = z$, $r > 0$. If $D = \mathbb{D}^n$, then using the Schwarz lemma for φ_j we get that $|z_j| = |\varphi_j(r)| \le r$, $j = 1, \dots, n$. Therefore, $q_{\mathbb{D}^n}(z) \le \hat{k}_{\mathbb{D}^n}(z)$. In the case when $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{B}_n)$ with $\varphi(0) = 0$ and $\varphi(r) = z$ we see that $q_{\mathbb{B}_n}(z) = q_{\mathbb{B}_n}(\varphi(r)) = r q_{\mathbb{B}_n}(\tilde{\varphi}(r))$, where $\varphi(\lambda) = \lambda\tilde{\varphi}(\lambda)$, $\lambda \in \mathbb{D}$, and $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$. Observe that $q_{\mathbb{B}_n}(\tilde{\varphi}(\lambda)) \le 1/|\lambda|$, $\lambda \in \mathbb{D} \setminus \{0\}$. Applying the maximum principle for the subharmonic function $q_{\mathbb{B}_n} \circ \tilde{\varphi}$ we obtain that $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{B}}_n)$. Hence, $q_{\mathbb{B}_n}(z) \le r$ and since r was arbitrarily chosen, we have $q_{\mathbb{B}_n}(z) \le \hat{k}_{\mathbb{B}_n}(z)$.

Observe that we may also use this geometric function to disprove the biholomorphic equivalence of $\mathbb{D}^n$ and $\mathbb{B}_n$ (Exercise).

Let us summarize what we have done so far. We have introduced a family

$$(\hat{d}_D)_{0 \in D \subset \mathbb{C}^n,\, n \in \mathbb{N}}$$

of functions $\hat{d}_D \colon D \to [0, \infty)$ ($\hat{d}_D \in \{\hat{m}_D, \hat{k}_D\}$) satisfying the following property:

(*) *for any domains $G \subset \mathbb{C}^m$, $0 \in G$, $D \subset \mathbb{C}^n$, $0 \in D$, and for any $F \in \mathcal{O}(G, D)$, $F(0) = 0$, we have $\hat{d}_D(F(z)) \le \hat{d}_G(z)$, $z \in G$.*

In particular, $\hat{d}_D(F(z)) = \hat{d}_G(z)$, $z \in G$, if $F \in \mathrm{Bih}_{0,0}(G, D)$. Moreover, these functions were explicitly described in terms of the geometry of D.

4.2 Holomorphically contractible families of functions

Let us begin with the following definition of a holomorphically contractible family which puts the functions of the introduction in a general context. The interested reader is referred to [Jar-Pfl 1993] and [Jar-Pfl 2005] for more information than is given in this chapter.

Definition 4.2.1. A family $(d_D)_D$ of functions $d_D \colon D \times D \to \mathbb{R}_+$, where D runs over all domains $D \subset \mathbb{C}^n$ (with arbitrary $n \in \mathbb{N}$), is said to be *holomorphically contractible* if the following two conditions are satisfied:

(A) $d_{\mathbb{D}}(a, z) = \boldsymbol{m}(a, z) = \left|\frac{z-a}{1-\bar{a}z}\right|$, $\quad a, z \in \mathbb{D}$ ($\boldsymbol{m}$ is the *Möbius distance*),

(B) for arbitrary domains $G \subset \mathbb{C}^m$, $D \subset \mathbb{C}^n$, any $F \in \mathcal{O}(G, D)$ works as a *contraction* with respect to d_G and d_D, i.e.

$$d_D(F(a), F(z)) \le d_G(a, z), \quad a, z \in G. \tag{4.2.1}$$

(Compare condition (B) and (*) from Section 4.1.)

Notice that there is another version of the definition of a holomorphically contractible family in which the normalization condition (A) is replaced by the condition

(A′) $d_{\mathbb{D}} = \boldsymbol{p}$, where $\boldsymbol{p} = \frac{1}{2}\log\frac{1+\boldsymbol{m}}{1-\boldsymbol{m}}$ is the *Poincaré distance* on $\mathbb{D}$.

Both definitions are obviously equivalent in the sense that $(d_D)_D$ fulfills (A) and (B) iff the family $(\tanh^{-1} d_D)_D$ satisfies (A′) and (B). In our opinion the normalization condition (A) is more handy in calculations.

Remark 4.2.2. (a) Recall that $\boldsymbol{m}$ and $\boldsymbol{p}$ are distances on $\mathbb{D}$.

(b) If $F \in \operatorname{Bih}(G, D)$, then $F^{-1} \in \mathcal{O}(D, G)$. Therefore,

$$d_D(F(a), F(z)) = d_G(a, z), \quad a, z \in G.$$

In particular, if $F \in \operatorname{Aut}(D)$, then $d_D(a, z) = d_D(F(a), F(z))$, $a, z \in D$.

(c) Moreover, if $D_j \subset \mathbb{C}^{n_j}$, $j = 1, 2$, are domains, then

$$d_{D_1\times D_2}((a_1, a_2), (b_1, b_2)) \geq \max\{d_{D_1}(a_1, b_1), d_{D_2}(a_2, b_2)\}, \tag{4.2.2}$$

whenever $(a_1, a_2), (b_1, b_2) \in D_1 \times D_2$ (Exercise, use (4.2.1) for the projection maps).

If in (4.2.2) always equality holds, then we say that $(d_D)_D$ satisfies the *product property*.

The following holomorphically contractible families of functions seem to be the most interesting ones.

Example 4.2.3 (Möbius pseudodistance).

$$\begin{aligned}\boldsymbol{m}_D(a, z) :&= \sup\{\boldsymbol{m}(f(a), f(z)) : f \in \mathcal{O}(D, \mathbb{D})\}\\ &= \sup\{|f(z)| : f \in \mathcal{O}(D, \mathbb{D}),\ f(a) = 0\}, \quad (a, z) \in D \times D;\end{aligned}$$

the function $\boldsymbol{c}_D := \tanh^{-1} \boldsymbol{m}_D$ is called the *Carathéodory pseudodistance*.

Indeed, to prove (B) it suffices to note that for $F \in \mathcal{O}(G, D)$ and $f \in \mathcal{O}(D, \mathbb{D})$ one has $f \circ F \in \mathcal{O}(G, \mathbb{D})$. Moreover, the fact that

$$\boldsymbol{m}(a, z) = \boldsymbol{m}(f(a), f(z)), \quad f \in \operatorname{Aut}(\mathbb{D}), \quad a, z \in \mathbb{D},$$

gives the second equality in the definition of $\boldsymbol{m}_D$. To obtain condition (A) it suffices to observe that $\boldsymbol{m}_{\mathbb{D}}(0, \cdot) = \boldsymbol{m}(0, \cdot)$. And this equation follows immediately by using the Schwarz lemma.

Observe that $\boldsymbol{m}_D$ (resp. $\boldsymbol{c}_D$) is positive semidefinite, symmetric and it satisfies the triangle inequality (Exercise). So $\boldsymbol{m}_D$ and $\boldsymbol{c}_D$ are, in fact, *pseudodistances* on D. For $D = \mathbb{C}$, Liouville's theorem immediately gives that $\boldsymbol{m}_{\mathbb{C}} = 0$; thus, in general, $\boldsymbol{m}_D$ (resp. $\boldsymbol{c}_D$) is not a distance.

Example 4.2.4 (Möbius function of higher order).

$$\boldsymbol{m}_D^{(k)}(a, z) := \sup\{|f(z)|^{1/k} : f \in \mathcal{O}(D, \mathbb{D}),\ \operatorname{ord}_a f \geq k\},$$
$$(a, z) \in D \times D,\ k \in \mathbb{N},$$

where $\operatorname{ord}_a f$ denotes the order of zero of f at a.

Indeed, in order to see (B) it suffices to observe that for $a, z \in D$, $F \in \mathcal{O}(G, D)$ and $f \in \mathcal{O}(D, \mathbb{D})$, $\operatorname{ord}_{F(a)} f \geq k$, one has $f \circ F \in \mathcal{O}(G, \mathbb{D})$ and $\operatorname{ord}_a f \circ F \geq k$. In particular, $m_D^{(k)}(a, z) = m_D^{(k)}(F(a), F(z))$ if $F \in \operatorname{Aut}(D)$. Therefore, to see (A) it suffices to show $\boldsymbol{m}_{\mathbb{D}}^{(k)}(0, z) = \boldsymbol{m}(0, z)$ which is a simple consequence of the Schwarz lemma.

Remark 4.2.5. Note that, in virtue of Montel's theorem (see Theorem 1.7.24), there exist *extremal functions* for $\boldsymbol{m}_D^{(k)}$, i.e. for any domain $D \subset \mathbb{C}^n$, any pair $(a, z) \in D \times D$, and any $k \in \mathbb{N}$ there exists an $f \in \mathcal{O}(D, \mathbb{D})$ with $\operatorname{ord}_a f \geq k$ such that $\boldsymbol{m}_D^{(k)}(a, z) = |f(z)|^{1/k}$.

Example 4.2.6 (Pluricomplex Green function).

$$\begin{aligned} \boldsymbol{g}_D(a, z) := \sup\{u(z) : u \colon D \to [0, 1),\ \log u \in \mathcal{PSH}(D),^{2} \\ \exists_{C = C(u,a) > 0}\ \forall_{w \in D} : u(w) \leq C\|w - a\|\}, \quad (a, z) \in D \times D.^{3} \end{aligned}$$

The point a is called the *pole* of the pluricomplex Green function.[4]

Indeed, to see (B) let $F \in \mathcal{O}(G, D)$, where $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^m$ are arbitrary domains. Fix an $a \in G$. If $u \colon D \to [0, 1)$ is log-psh satisfying $u \leq C\| \cdot - F(a)\|$ on D, then $\log u \circ F \in \mathcal{PSH}(G)$ and

$$(u \circ F)(z) = u(F(z)) \leq C\|F(z) - F(a)\| \leq \widetilde{C}\|z - a\|, \quad z \in \mathbb{B}(a, r) \Subset G,$$

where $\widetilde{C}$ and r are suitably chosen. Therefore, $u \circ F \leq \boldsymbol{g}_G(a, z)$. Since u was arbitrarily chosen, it follows that $\boldsymbol{g}_D(F(a), F(\cdot)) \leq \boldsymbol{g}_G(a, \cdot)$.

In the case where $D = \mathbb{D}$ we fix a $u \colon \mathbb{D} \to [0, 1)$ such that $\log u$ is psh and $u(\lambda) \leq C|\lambda|$. Observe that then $u / |\operatorname{id}_{\mathbb{D}}| \in \mathcal{SH}(\mathbb{D}_*)$ and that this function is locally bounded in $\mathbb{D}$. Therefore, it extends to a sh function on the whole of $\mathbb{D}$. By the maximum principle it follows that $u \leq |\operatorname{id}_{\mathbb{D}}|$. Hence we have that $u = |\operatorname{id}_{\mathbb{D}}|$, i.e. $\boldsymbol{g}_{\mathbb{D}}(0, \cdot) = \boldsymbol{m}(0, \cdot)$. The situation for a general pole follows immediately using (B) and a Möbius transformation.

While $(d_D)_D$, $d_D \in \{\boldsymbol{m}_D, \boldsymbol{m}_D^{(k)}, \boldsymbol{g}_D\}$, are based on families of functions we turn now to families defined by geometric conditions, namely by a set of analytic discs.

---

[2]Recall that $\mathcal{PSH}(D)$ denotes the family of all functions plurisubharmonic on D.

[3]Note that it suffices to have $u(z) \leq \widetilde{C}\|z - a\|$ for all $z \in \mathbb{B}(a, r) \setminus \{a\}$ when $r > 0$ is sufficiently small. Moreover, $u(a) = 0$ (Exercise).

[4]For relations between the pluricomplex and the classical Green functions in the unit ball see [Car 1997]. For a different pluricomplex Green function see [Ceg 1995], [Edi-Zwo 1998].

Example 4.2.7 (Lempert function).

$$\begin{aligned}\widetilde{\boldsymbol{k}}_D^*(a,z) &:= \inf\{\boldsymbol{m}(\lambda,\mu) : \lambda,\mu \in \mathbb{D},\ \exists_{\varphi\in\mathcal{O}(\mathbb{D},D)} :\ \varphi(\lambda)=a,\ \varphi(\mu)=z\}\\ &= \inf\{\mu \in [0,1) : \exists_{\varphi\in\mathcal{O}(\mathbb{D},D)} :\ \varphi(0)=a,\ \varphi(\mu)=z\}\\ &= \inf\{\mu \in (0,1) : \exists_{\varphi\in\mathcal{O}(\mathbb{D},D)} :\ \varphi(0)=a,\ \varphi(\mu)=z\},\end{aligned}$$

where $(a,z) \in D \times D$. Put $\widetilde{\boldsymbol{k}}_D := \tanh^{-1} \widetilde{\boldsymbol{k}}_D^*$.

Indeed, first we have to show that the above definition makes sense. So let us fix points $a, z \in D$. Connect them by a continuous curve, i.e. take a $\gamma \in \mathcal{C}([0,1], D)$ with $\gamma(0) = a$ and $\gamma(1) = z$. In virtue of the Weierstrass approximation theorem, we may approximate γ uniformly by a sequence of polynomial mappings $(p_j)_j$. Taking a sufficiently large j we may assume that

$$4\|p_j - \gamma\|_{[0,1]} < \operatorname{dist}(\gamma([0,1]), \partial D).$$

Put $p := p_j$ and then

$$\hat{p}(\lambda) := p(\lambda) + (a - p(0))(1-\lambda) + \lambda(z - p(1)), \quad \lambda \in \mathbb{C}.$$

Note that $\hat{p}([0,1]) \subset D$, $\hat{p}(0) = a$, and $\hat{p}(1) = z$. Then $\hat{p}$ maps even a simply connected domain U, $[0,1] \subset U$, into D (Exercise). Applying the Riemann mapping theorem leads to a $\psi \in \mathcal{O}(\mathbb{D}, U)$ with $\psi(0) = 0$ and $\psi(\mu) = 1$ for a suitable $\mu \in \mathbb{D}$. Hence, $\varphi := \hat{p} \circ \psi$ gives an analytic disc through the points a and z.[5]

Observe that if $F \in \mathcal{O}(G, D)$, $a, z \in G$, and $\varphi \in \mathcal{O}(\mathbb{D}, G)$, $\varphi(0) = a$ and $\varphi(\mu) = z$ for some $\mu \in [0,1)$, then $F \circ \varphi \in \mathcal{O}(\mathbb{D}, D)$ with $F \circ \varphi(0) = F(a)$ and $F \circ \varphi(\mu) = F(z)$. Hence (B) is fulfilled. To prove (A) use the Schwarz lemma in order to see that $\tilde{k}_{\mathbb{D}}^*(0,z) \geq |z| = \boldsymbol{m}(0,z)$, $z \in \mathbb{D} \setminus \{0\}$. To get the inverse inequality take simply $\varphi \in \mathcal{O}(\mathbb{D},\mathbb{D})$, $\varphi(\lambda) := \lambda \frac{z}{|z|}$.

Exercise 4.2.8. Prove that

$$\widetilde{\boldsymbol{k}}_D^*(a,z) = \inf\{|\mu| : \mu \in \mathbb{D},\ \exists_{\varphi\in\mathcal{O}(\bar{\mathbb{D}},D)} :\ \varphi(0)=a,\ \varphi(\mu)=z\}, \quad a,z \in D.$$

For many purposes it is important to know whether the infimum in the definition of the Lempert function is taken by some analytic disc.

Definition 4.2.9. Let $D \subset \mathbb{C}^n$ and $a, b \in D$. A mapping $\varphi \in \mathcal{O}(\mathbb{D}, D)$ is called a *$\widetilde{\boldsymbol{k}}_D^*$-geodesic for the pair (a,b)*[6] if there are $\lambda_1, \lambda_2 \in \mathbb{D}$ such that $\varphi(\lambda_1) = a$, $\varphi(\lambda_2) = b$, and $\widetilde{\boldsymbol{k}}_D^*(a,b) = \boldsymbol{m}(\lambda_1, \lambda_2)$.

[5] Note that the same remains true in the case when D is a connected complex manifold (see [Win 2005]).

[6] We also say that φ is an extremal disc through a and b.

Remark 4.2.10. In general such a geodesic need not exist. For example, take $D := \mathbb{B}_2 \setminus \{(1/2, 0)\}$. Observe that D is a non-taut domain. Fix now the points $a := (0,0)$ and $b := (1/4, 0)$ from D. For $R \in (0,1)$ put $\varphi_R \in \mathcal{O}(\mathbb{D}, D)$, $\varphi_R(\lambda) := (R\lambda, s(R)\lambda(\lambda - 1/(4R)))$, where $s(R) \ll 1$. Then $\widetilde{\boldsymbol{k}}_D^*(a,b) \le 1/(4R)$. Since R was arbitrarily chosen, we have $\widetilde{\boldsymbol{k}}_D^*(a,b) \le 1/4$.

Suppose now that there exists a $\widetilde{\boldsymbol{k}}_D^*$-geodesic $\psi = (\psi_1, \psi_2) \in \mathcal{O}(\mathbb{D}, D)$ for (a,b) with $\psi(\lambda_1) = a$, $\psi(\lambda_2) = b$, and $\widetilde{\boldsymbol{k}}_D^*(a,b) = \boldsymbol{m}(\lambda_1, \lambda_2)$. Using $\operatorname{Aut}(\mathbb{D})$ we may assume that $\psi(0) = a$, $\psi(\mu) = b$, and $\mu = \widetilde{\boldsymbol{k}}_D^*(a,b)$. Note that $\psi_1 \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with $\psi_1(0) = 0$. The Schwarz lemma gives $1/4 \le \widetilde{\boldsymbol{k}}_D^*(a,b)$ and therefore, $\psi_1 = \mathrm{id}_{\mathbb{D}}$. Then, taking into account that ψ maps $\mathbb{D}$ into $\mathbb{B}_2$ leads to the fact that $\psi_2 \equiv 0$ (use the maximum principle). And therefore, $\psi(1/2) = (1/2, 0)$; a contradiction.

In the case of taut domains we always know that such geodesics exist.

Proposition 4.2.11. *Let a, b be two points of a taut domain $D \subset \mathbb{C}^n$. Then there exists a $\widetilde{\boldsymbol{k}}_D^*$-geodesic for (a,b).*

Proof. By definition we have a sequence $(\varphi)_j \subset \mathcal{O}(\mathbb{D}, D)$ such that $\varphi_j(0) = a$, $\varphi(\sigma_j) = b$ with $\sigma_j \in (0,1)$, and $\sigma_j \searrow \widetilde{\boldsymbol{k}}_D^*(a,b)$. By assumption, D is taut and $\varphi_j(0) = a$, $j \in \mathbb{N}$. Therefore, we may choose a subsequence (φ_{j_k}) such that $\varphi_{j_k} \to \varphi \in \mathcal{O}(\mathbb{D}, D)$ locally uniformly. Then $\varphi(0) = a$ and $\varphi(\widetilde{\boldsymbol{k}}_D^*(a,b)) = b$ (Exercise), i.e. φ is a geodesic we were looking for. $\square$

Exercise 4.2.12. (a) Let $D \subset \mathbb{C}^n$ be a domain and let $a, b \in D$. A map $\varphi \in \mathcal{O}(\mathbb{D}, D)$ is an *$\boldsymbol{m}_D$-geodesic for the pair* (a,b) if there exist $\lambda_1, \lambda_2 \in \mathbb{D}$ such that $\varphi(\lambda_1) = a$, $\varphi(\lambda_2) = b$, and $\boldsymbol{m}_D(a,b) = \boldsymbol{m}(\lambda_1, \lambda_2)$.

Prove that any $\boldsymbol{m}_D$-geodesic for (a,b) is a $\widetilde{\boldsymbol{k}}_D^*$-geodesic for (a,b).

(b) Let $\varphi \in \mathcal{O}(\mathbb{D}, D)$ be an $\boldsymbol{m}_D$-geodesic for $(\varphi(\lambda_1), \varphi(\lambda_2))$, where $\lambda_1 \ne \lambda_2$. Prove that $\boldsymbol{m}_D(\varphi(\lambda'), \varphi(\lambda'')) = \boldsymbol{m}(\lambda', \lambda'')$, $\lambda', \lambda'' \in \mathbb{D}$, i.e. φ is an $\boldsymbol{m}_D$-geodesic for all pairs $(\varphi(\lambda'), \varphi(\lambda''))$. Sometimes such a φ is simply called an *$\boldsymbol{m}_D$-geodesic*.
Hint. Study the function $\mathbb{D} \setminus \{\lambda_1\} \ni \lambda \mapsto \frac{\boldsymbol{m}_D(\varphi(\lambda_1), \varphi(\lambda))}{\boldsymbol{m}(\lambda_1, \lambda)}$ and use properties of subharmonic functions.

(c) Prove that any complex $\boldsymbol{m}_D$-geodesic $\varphi \in \mathcal{O}(\mathbb{D}, D)$ is a proper injective mapping.

The next example relies on the normalization (A′).

Example 4.2.13 (Kobayashi pseudodistance).

$$\begin{aligned}\boldsymbol{k}_D(a,z) &:= \sup\{d(a,z) : d_D \text{ a pseudodistance on } D,\ d \le \widetilde{\boldsymbol{k}}_D\} \\ &=: \tanh^{-1} \boldsymbol{k}_D^*(a,z), \quad a, z \in D.\end{aligned}$$

Indeed, let $F \in \mathcal{O}(G, D)$ be as in (4.2.1). To any pseudodistance $d_D \le \tilde{\boldsymbol{k}}_D$ on $D \times D$ we associate a new pseudodistance $\tilde{d}_G$ on $G \times G$ by $\tilde{d}_G(w', w'') := d_D(F(w'), F(w''))$. Then

$$\tilde{d}_G(w', w'') \le \tilde{\boldsymbol{k}}_D(F(w'), F(w'')) \le \tilde{\boldsymbol{k}}_G(w', w'').$$

Therefore, $d_D(F(w'), F(w'')) \le \tilde{\boldsymbol{k}}_G(w', w'')$, $w', w'' \in G$. Since d_D was arbitrarily chosen we end up with (4.2.1). For the normalization (A′) it suffices to mention that $\boldsymbol{p}$ is a distance.

Remark 4.2.14. Note that (Exercise):

(a) $\boldsymbol{k}_D$ is the largest pseudodistance on D below of $\tilde{\boldsymbol{k}}_D$;

(b) $\boldsymbol{c}_D \le \boldsymbol{k}_D$;

(c) $\boldsymbol{k}_D$ and $\tilde{\boldsymbol{k}}_D^*$ are symmetric functions;

(d) $\boldsymbol{k}_D(a, z) = \inf\{\sum_{j=1}^N \tilde{\boldsymbol{k}}_D(z_{j-1}, z_j) : N \in \mathbb{N},\ a = z_0, \dots, z_N = z \in D\}$, $a, z \in D$.

To any pseudodistance $d_D \in \{\boldsymbol{c}_D, \boldsymbol{k}_D\}$, $D \subset \mathbb{C}^n$, one associates the *d_D-length* of a curve $\alpha\colon [0,1] \to D$ as

$$L_{d_D}(\alpha) := \sup\Big\{\sum_{j=1}^{N} d_D(\alpha(t_{j-1}), \alpha(t_j)) : N \in \mathbb{N},\ 0 = t_0 < \dots < t_N = 1\Big\}.$$ [7]

Exercise 4.2.15. Calculate $L_{\boldsymbol{p}}([0, s])$,[8] where $s \in (0, 1)$. Here $[0, s]$ is just an abbreviation for the curve $\alpha : [0, 1] \to \mathbb{D}$, $\alpha(t) := ts$.

It is clear (use (4.2.1)) that $L_{d_D}(F \circ \alpha) \le L_{d_G}(\alpha)$ whenever $F \in \mathcal{O}(G, D)$ and $\alpha\colon [0, 1] \to G$. Moreover, by the triangle inequality, $d_D(\alpha(0), \alpha(1)) \le L_{d_D}(\alpha)$. A more precise result is true in case of the Kobayashi pseudodistance.

Proposition 4.2.16. *Let $D \subset \mathbb{C}^n$ and $a, b \in D$. Then*

$$\boldsymbol{k}_D(a, b) = \inf\{L_{\boldsymbol{k}_D}(\alpha) : \alpha\colon [0, 1] \to D \text{ continuous and } \|\cdot\|\text{-rectifiable}, \\ \alpha(0) = a,\ \alpha(1) = b\}.$$

Proof. By Remark 4.2.14 (d) it is clear that $\boldsymbol{k}_D(a, b)$ is less than or equal to the right-hand side. Now fix an $\varepsilon > 0$. By definition we find points $s_j \in [0, 1)$ and analytic discs $\varphi_j \in \mathcal{O}(\mathbb{D}, D)$, $j = 1, \dots, k$, such that

$$\varphi_j(0) = a, \quad \varphi_j(s_j) = \varphi_{j+1}(0),\ 1 \le j < k, \quad \varphi_k(s_k) = b,$$
$$\sum_{j=1}^{k} \boldsymbol{p}(0, s_j) < \boldsymbol{k}_D(a, b) + \varepsilon.$$

[7] By Corollary 4.2.25, the length is finite if the curve α is assumed to be $\|\cdot\|$-rectifiable, i.e. there exists an $M > 0$ such that $\sum_{j=1}^N \|\alpha(t_j) - \alpha(t_{j-1})\| < M$ whenever $N \in \mathbb{N}$, $0 = t_0 < t_1 < \dots < t_N = 1$ (Exercise).

[8] Note that $\boldsymbol{p}$ is a distance on $\mathbb{D}$. So $L_{\boldsymbol{p}}$ is defined in the same way as L_{d_D} before.

Obviously, we may assume that all the s_j's are positive. Put

$$\alpha(t) := \varphi_j((t - \tfrac{j-1}{k})ks_j), \text{ if } t \in [\tfrac{j-1}{k}, \tfrac{j}{k}] \text{ and } j = 1, \dots, k.$$

Then α is a piecewise real analytic curve in D connecting the points a, b. Therefore, we have

$$\begin{aligned} L_{\boldsymbol{k}_D}(\alpha) &\le \sum_{j=1}^{k} L_{\boldsymbol{k}_D}(\alpha|_{[\frac{j-1}{k}, \frac{j}{k}]}) \\ &\le \sum_{j=1}^{k} L_{\boldsymbol{k}_{\mathbb{D}}}([0, s_j]) \le \sum_{j=1}^{k} \boldsymbol{p}(0, s_j) < \boldsymbol{k}_D(a, b) + \varepsilon. \end{aligned}$$

Since the choice of ε was arbitrary, the proof is finished. □

Looking at the proof of Proposition 4.2.16, one easily concludes the following corollary (Exercise).

Corollary 4.2.17. *Let $D \subset \mathbb{C}^n$ and $a, b \in D$. Then*

$$\boldsymbol{k}_D(a, b) = \inf\{L_{\boldsymbol{k}_D}(\alpha) : \alpha\colon [0, 1] \to D \textit{ piecewise real analytic}, \\ \alpha(0) = a,\ \alpha(1) = b\}.$$

Remark 4.2.18. We have to point out that for the Carathéodory pseudodistance Proposition 4.2.16 is no longer true. Already for the very simple domain $D = \mathbb{A}(1/R, R)$ a counterexample can be given. For more details see [Jar-Pfl 1993], Example 2.5.7.

Lemma 4.2.19. *For any domain $D \subset \mathbb{C}^n$ the following inequalities are true:*

$$\boldsymbol{m}_D = \boldsymbol{m}_D^{(1)} \le \boldsymbol{m}_D^{(k)} \le \boldsymbol{g}_G \le \tilde{\boldsymbol{k}}_D^*, \quad \boldsymbol{c}_D \le \boldsymbol{k}_D \le \tilde{\boldsymbol{k}}_D,$$

and for any holomorphically contractible family $(d_D)_D$ we have

$$\boldsymbol{m}_D \le d_D \le \tilde{\boldsymbol{k}}_D^*, \tag{4.2.3}$$

i.e. the Möbius family is minimal and the Lempert family is maximal.

Proof. Fix an $a \in D$. Let $f \in \mathcal{O}(D, \mathbb{D})$ with $f(a) = 0$. Then $f^k \in \mathcal{O}(D, \mathbb{D})$ with $\operatorname{ord}_a f^k \ge k$. Therefore, $|f(z)| = |f^k(z)|^{1/k} \le \boldsymbol{m}_D^{(k)}(a, z)$, $z \in D$. Since f is arbitrarily chosen we end up with $\boldsymbol{m}_D(a, \cdot) \le \boldsymbol{m}_D^{(k)}(a, \cdot)$.

Now let $f \in \mathcal{O}(D, \mathbb{D})$ with $\operatorname{ord}_a f \ge k$. Put $u := |f|^{1/k}$. Then $\log u$ is psh and $u(z) \le C\|z - a\|$ and so $|f(z)|^{1/k} = u(z) \le \boldsymbol{g}_D(a, z)$, $z \in D$. Hence $m^{(k)}(a, \cdot) \le \boldsymbol{g}_G(a, \cdot)$.

Fix a $z^0 \in D$. Let $u \colon D \to [0,1)$ be such that $\log u \in \mathcal{PSH}(D)$ and $u(z) \le C\|z-a\|$, $z \in D$, and let $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = a$ and $\varphi(\mu) = z^0$ for a certain $\mu \in [0,1)$. Then $u \circ \varphi \in \mathcal{PSH}(\mathbb{D})$, $u \circ \varphi(0) = 0$. Therefore, applying the Schwarz lemma for psh functions, we get $u \circ \varphi(\lambda) \le |\lambda|$, $\lambda \in \mathbb{D}$. In particular, $u(z^0) = u \circ \varphi(\mu) \le \mu$. Since u and φ were arbitrarily chosen, we have $\boldsymbol{g}_D(a, z^0) \le \widetilde{\boldsymbol{k}}_D^*(a, z^0)$.

Now let $(d_D)_D$ be an arbitrary holomorphically contractible family. Fix a domain $D \subset \mathbb{C}^n$ and points $a, z \in D$. Let now $f \in \mathcal{O}(D, \mathbb{D})$ with $f(a) = 0$. Then $d_D(a,z) \ge d_{\mathbb{D}}(0, f(z)) = \boldsymbol{m}(0, f(z)) = |f(z)|$. Hence, $d_D(a,z) \ge \boldsymbol{m}_D(a,z)$.

Finally, let $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = a$ and $\varphi(\mu) = z$ for a certain $\mu \in [0,1)$. Then $d_D(a,z) = d_D(\varphi(0), \varphi(\mu)) \le d_{\mathbb{D}}(0,\mu) = \boldsymbol{m}(0,\mu) = \mu$. And so $d_D(a,z) \le \widetilde{\boldsymbol{k}}_D^*(a,z)$. $\square$

Remark 4.2.20. (a) Observe that $\widetilde{\boldsymbol{k}}_{\mathbb{C}} = 0$ on $\mathbb{C} \times \mathbb{C}$ (Exercise). Then also $\widetilde{\boldsymbol{k}}_{\mathbb{C}_*} = 0$ via (4.2.1). Therefore, $\boldsymbol{k}_D$ is, in general, not a distance.

(b) Moreover, the following result is true (see [Jar-Nik 2002], [Nik 2002]): Let $F_j \subset \mathbb{C}$ be a closed subset, $j = 1, \dots, n$, $n \ge 2$, such that $F_1 \ne \mathbb{C} \ne F_2$. Put $D := \mathbb{C}^n \setminus (F_1 \times \dots \times F_n)$. Then $\widetilde{\boldsymbol{k}}_D = 0$ on $D \times D$. In fact, for any two points $a, b \in D$ there exists a $\varphi \in \mathcal{O}(\mathbb{C}, D)$ such that $\varphi(0) = a$ and $\varphi(1) = b$.

For balanced domains we have the following formulas.

Proposition 4.2.21. *Let $D \subset \mathbb{C}^n$ be a balanced domain given as*

$$D = \{z \in \mathbb{C}^n : h(z) < 1\},$$

where $h = h_D$ is the associated Minkowski function. Then:

(a) $\widetilde{\boldsymbol{k}}_D^*(0, \cdot) \le h|_D$.

(b) *If, in addition, D is pseudoconvex, then*

$$\boldsymbol{g}_D(0, \cdot) = \widetilde{\boldsymbol{k}}_D^*(0, \cdot) = h \quad \text{on } D.$$

(c) *Even more, if D is a convex domain, then*

$$\boldsymbol{m}_D(0, \cdot) = \boldsymbol{m}_D^{(k)}(0, \cdot) = \boldsymbol{g}_D(0, \cdot) = \widetilde{\boldsymbol{k}}_D^*(0, \cdot) = h|_D.$$ [9]

Proof. (a) Fix a $z^0 \in D$. If $h(z^0) = 0$, then $\mathbb{C}z^0 \subset D$. Therefore, using the holomorphic contractibility, $\widetilde{\boldsymbol{k}}_D^*(0, z^0) \le \widetilde{\boldsymbol{k}}_{\mathbb{C}}^*(0,1) = 0 = h(z^0)$. So we may assume that $h(z^0) \ne 0$. Then $\varphi(\lambda) := \frac{\lambda}{h(z^0)} z^0$, $\lambda \in \mathbb{D}$, gives a $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = 0$ and $\varphi(h(z^0)) = z^0$. Hence, $\widetilde{\boldsymbol{k}}_D^*(0, z^0) \le h(z^0)$.

[9] Recall that a convex (pseudoconvex) complete Reinhardt domain is in particular a convex (pseudoconvex) balanced domain.

(b) Recall that by assumption $\log h \in \mathcal{PSH}(\mathbb{C}^n)$ (see Proposition 1.15.11) and $h(z) \le \|z\| h(\frac{z}{\|z\|}) \le C\|z\|$, $z \in (\mathbb{C}^n)_*$ (use that h is upper semicontinuous and therefore bounded on $\partial\mathbb{B}$). Since $0 \le h < 1$ on D it follows that $h \le \boldsymbol{g}_D(0,\cdot)$.

(c) In the last case just apply the Hahn–Banach theorem to get $h|_D \le \boldsymbol{m}_D(0,\cdot)$. □

In particular, (b) implies the following result for biholomorphic mappings.

Corollary 4.2.22. *Let $D_j = \{z \in \mathbb{C}^n : h_j(z) < 1\}$ be a pseudoconvex balanced domain, $j = 1, 2$. If $F \in \operatorname{Bih}(D_1, D_2)$ with $F(0) = 0$, then $h_2 \circ F = h_1$ on D_1.*

Remark 4.2.23. In fact, much more is true, namely if $\operatorname{Bih}(D_1, D_2) \neq \varnothing$, then $\operatorname{Bih}_{0,0}(D_1, D_2) \neq \varnothing$ (see [Kau-Upm 1976], [Kau-Vig 1990]), where D_j are pseudoconvex balanced bounded domains in $\mathbb{C}^n$. Later we will even see that if $\operatorname{Bih}(D_1, D_2) \neq \varnothing$, then D_1 is linearly equivalent to D_2 (see Proposition 2.1.9 in the case of norm balls).

Moreover, we have the following explicit formulas.

Corollary 4.2.24. *Let $a, z \in \mathbb{B}_n$. Then*

$$\boldsymbol{m}_{\mathbb{B}_n}(a,z) = \tilde{\boldsymbol{k}}^*_{\mathbb{B}_n}(a,z) = \boldsymbol{k}^*_{\mathbb{B}_n}(a,z) = \left(1 - \frac{(1-\|a\|^2)(1-\|z\|^2)}{|1-\langle z,a\rangle|^2}\right)^{1/2},$$

where $\boldsymbol{k}_{\mathbb{B}_n} = \tanh^{-1} \boldsymbol{k}^*_{\mathbb{B}_n}$. In particular, all functions introduced so far coincide on $\mathbb{B}_n$.

Proof. For $a \in \mathbb{B}_n \setminus \{0\}$, apply $h_a \in \operatorname{Aut}(\mathbb{B}_n)$,

$$h_a(z) = \frac{\sqrt{1-\|a\|^2}\left(z - \frac{\langle z,a\rangle}{\|a\|^2}a\right) - a + \frac{\langle z,a\rangle}{\|a\|^2}a}{1-\langle z,a\rangle}, \quad z \in \mathbb{B}_n$$

(cf. Example 2.1.12 (b)), and the above proposition. □

Corollary 4.2.25. *Let $D \subset \mathbb{C}^n$ be a domain and $\mathbb{B}(a,r) \subset D$. Then*

$$\tilde{\boldsymbol{k}}^*_D(a,z) \le \frac{\|z-a\|}{r}, \quad z \in \mathbb{B}(a,r).$$

In particular, $\tilde{\boldsymbol{k}}^*_D$ is locally bounded from above by the Euclidean norm.

Proof. Fix a $z \in \mathbb{B}(a,r) \subset D$. Then

$$\tilde{\boldsymbol{k}}^*_D(a,z) \le \tilde{\boldsymbol{k}}^*_{\mathbb{B}(a,r)}(a,z) = \tilde{\boldsymbol{k}}^*_{\mathbb{B}(r)}(0,z-a) = \|z-a\|/r,$$

since $h_{\mathbb{B}(r)}(\zeta) = \|\zeta\|/r$. □

Remark 4.2.26. (a) Put $D := \mathbb{C}_*^2 \cup \big((\{0\} \times \mathbb{D}) \cup (\mathbb{D} \times \{0\})\big)$. Then D is a balanced domain which is not pseudoconvex (EXERCISE). For $R > 1$ put $\varphi_R(\lambda) := (\lambda(R\lambda - 1), \frac{R}{2}\lambda)$, $\lambda \in \mathbb{D}$. Then $\varphi_R \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi_R(0) = (0,0)$ and $\varphi_R(1/R) = (0, 1/2)$. Therefore $\widetilde{\boldsymbol{k}}_D^*((0,0),(0,1/2)) = 0 < h_D(0,1/2)$.

(b) Let $D := \{z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1, |z_1 z_2| < r\}$, where $0 < r < 1/4$. Obviously, D is a pseudoconvex complete Reinhardt domain (in particular, a balanced domain) which is not convex. Its Minkowski function is given by $h(z) := \max\{|z_1|, |z_2|, \sqrt{r^{-1}|z_1 z_2|}\}$. Therefore, $\widetilde{\boldsymbol{k}}_D^*(0,(t,t)) = \frac{t}{\sqrt{r}}$, $r < t < \sqrt{r}$. Then

$$\begin{aligned}
\boldsymbol{k}_D(0,(t,t)) &\le \boldsymbol{k}_D(0,(t,0)) + \boldsymbol{k}_D((0,t),(t,t)) \\
&\le \tanh^{-1}(\boldsymbol{m}(0,t)) + \tanh^{-1}(\boldsymbol{m}_{K(r/t)}(0,t)) \\
&= \tanh^{-1}(t) + \tanh^{-1}(t^2/r) = \frac{1}{2}\log\left(\frac{1+t}{1-t}\cdot\frac{r+t^2}{r-t^2}\right) \\
&< \frac{1}{2}\log\frac{\sqrt{r}+t}{\sqrt{r}-t} = \widetilde{\boldsymbol{k}}_D(0,(t,t)),
\end{aligned}$$

when t is sufficiently near r. In the second inequality we have used that the above functions are holomorphically contractible.

Note that this example shows:

- $\boldsymbol{k}_D$ is not identically equal to $\tanh^{-1} \circ h|_D$.
- $\boldsymbol{m}_D$ is not equal $h|_D$.
- $\widetilde{\boldsymbol{k}}_D$ does not satisfy the triangle inequality. Hence the introduction of the Kobayashi pseudodistance is justified.

Corollary 4.2.27. *Let $D \subset \mathbb{C}^n$ be a domain and $a \in D$. Then*

$$\boldsymbol{g}_D(a,\cdot)\colon D \to [0,1)$$

is log-psh satisfying $\boldsymbol{g}_D(a,z) \le C\|z-a\|$, $z \in D$.

Proof. Set $u := \boldsymbol{g}_D(a,\cdot)$. Then u^*, its upper semicontinuous regularization, is log-psh. Moreover, if $\mathbb{B}(a,r) \subset D$, then

$$u(z) = \boldsymbol{g}_D(a,z) \le \boldsymbol{g}_{\mathbb{B}(a,r)}(a,z) \le \tfrac{\|z-a\|}{r}, \quad z \in \mathbb{B}(a,r).$$

In virtue of the maximum principle, we get $u^* < 1$ on D. Therefore, $u^* = \boldsymbol{g}_D(a,\cdot)$, which obviously implies the corollary. □

Remark 4.2.28. There exists a pseudoconvex bounded balanced domain $D \subset \mathbb{C}^n$, $n \ge 2$, such that its Minkowski function h is not continuous (see Proposition 1.15.12). Therefore, $\boldsymbol{g}_D(0,\cdot) = \widetilde{\boldsymbol{k}}_D^*(0,\cdot)$ is not continuous.

Applying Corollary 4.2.24 we get

Proposition 4.2.29. *Let $D \subset \mathbb{C}^n$. Then:*

(a) *The functions $\boldsymbol{m}_D$, $\boldsymbol{k}_D$ are continuous.*

(b) *The functions $\boldsymbol{m}_D^{(k)}$ and $\widetilde{\boldsymbol{k}}_D^*$ are upper semicontinuous.*

(c) *If, in addition, D is assumed to be taut, then $\widetilde{\boldsymbol{k}}_D^*$ is continuous.*

(d) *For $a \in D$, $\boldsymbol{m}_D^{(k)}(a, \cdot)$ is continuous.*

Proof. (a) Fix points $a, a', b, b' \in D$ and let $d_D \in \{\boldsymbol{k}_D, \boldsymbol{m}_D\}$. Then

$$|d_D(a,b) - d_D(a',b')| \le d_D(a,a') + d_D(b,b').$$

Therefore it suffices to apply Corollary 4.2.24 and the fact that $\boldsymbol{m}_D \le \boldsymbol{k}_D^*$.

(b) The case $\boldsymbol{m}_D^{(k)}$: Let $D \ni a_j \to a$ and $D \ni b_j \to b$. According to the remark concerning extremal functions there are $f_j \in \mathcal{O}(D, \mathbb{D})$ with $\operatorname{ord}_{a_j} f_j \ge k$ and $|f_j(b_j)|^{1/k} = \boldsymbol{m}_D^{(k)}(a_j, b_j)$, $j \in \mathbb{N}$. Using a Montel argument gives an $f \in \mathcal{O}(D, \mathbb{D})$ with $\operatorname{ord}_a f \ge k$ and

$$\boldsymbol{m}_D^{(k)}(a_{j_\nu}, b_{j_\nu}) = |f_{j_\nu}(b_{j_\nu})|^{1/k} \to |f(b)|^{1/k} \le \boldsymbol{m}_D^{(k)}(a,b)$$

for a suitable subsequence, i.e. $\boldsymbol{m}_D^{(k)}$ is upper semicontinuous.

The case $\widetilde{\boldsymbol{k}}_D^*$: For an arbitrary $\varepsilon > 0$ choose an analytic disc $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, D)$ with $\varphi(0) = a$, $\varphi(\mu) = b$, and $(0,1) \ni \mu \le \widetilde{\boldsymbol{k}}_D^*(a,b) + \varepsilon$. Then $\varphi(\overline{\mathbb{D}})$ is compact and therefore, $\operatorname{dist}(\varphi(\overline{\mathbb{D}}), \partial D) =: r > 0$. Fix $a' \in \mathbb{B}(a, \mu r/6) \subset D$ and $b' \in \mathbb{B}(b, \mu r/2) \subset D$. Now we define a new analytic disc $\psi \in \mathcal{O}(\mathbb{D}, D)$ by

$$\psi(\lambda) := \varphi(\lambda) + \tfrac{1}{\mu}((\mu - \lambda)(a' - a) + \lambda(b' - b)), \quad \lambda \in \mathbb{D}.$$

Therefore, $\widetilde{\boldsymbol{k}}_D^*(a', b') \le \mu \le \widetilde{\boldsymbol{k}}_D^*(a,b) + \varepsilon$.

(c) Assume that $\widetilde{\boldsymbol{k}}_D^*$ is not lower semicontinuous at $(a,b) \in D \times D$. Then $\widetilde{\boldsymbol{k}}_D^*(a,b) > 0$ and there are sequences $(a_j)_j, (b_j)_j \subset D$ with $a_j \to a$ and $b_j \to b$ such that for all j,

$$\widetilde{\boldsymbol{k}}_D^*(a_j, b_j) \le \widetilde{\boldsymbol{k}}_D^*(a,b) - \varepsilon \in (0, \infty)$$

for a suitable $\varepsilon > 0$. Choose a $\varphi_j \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi_j(0) = a_j$, $\varphi(\mu_j) = b_j$, where $[0,1) \ni \mu_j < \widetilde{\boldsymbol{k}}_D^*(a_j, b_j) - \varepsilon/2$. Applying tautness we may assume that $\mu_j \to \mu \in [0,1)$ and $\varphi_j \to \varphi \in \mathcal{O}(\mathbb{D}, D)$ locally uniformly in $\mathbb{D}$. Then $\varphi(0) = a$ and $\varphi(\mu) = b$. Therefore, $\widetilde{\boldsymbol{k}}_D^*(a,b) \le \mu \le \widetilde{\boldsymbol{k}}_D^*(a,b) - \varepsilon/2$; a contradiction.

(d) is left as an Exercise. $\square$

All the functions introduced so far in this section behave well under union of domains; to be precise we have the following result.

Lemma 4.2.30. *Let $D = \bigcup_{j=1}^{\infty} D_j \subset \mathbb{C}^n$, $D_j \subset D_{j+1}$, be the increasing union of the domains D_j, $j \in \mathbb{N}$. Then $d_{D_j} \to d_D$ if $j \to \infty$, where $d \in \{\boldsymbol{m}^{(k)}, \boldsymbol{g}, \widetilde{\boldsymbol{k}}^*, \boldsymbol{k}\}$.*

Proof. We restrict ourselves to prove this lemma only for $d = \widetilde{\boldsymbol{k}}^*$; the remaining cases are left as an Exercise for the reader.

Obviously, we have

$$\widetilde{\boldsymbol{k}}^*_{D_j} \geq \widetilde{\boldsymbol{k}}^*_{D_{j+1}} \geq \widetilde{\boldsymbol{k}}^*_D, \quad j \in \mathbb{N}.$$

In particular, we have $\lim\limits_{j\to\infty} \widetilde{\boldsymbol{k}}^*_{D_j} \geq \widetilde{\boldsymbol{k}}^*_D$. Now fix points $a, z \in D$ and choose a j_0 such that $a, z \in D_j$, $j \geq j_0$. Suppose that $\rho := \lim\limits_{j\to\infty} \widetilde{\boldsymbol{k}}^*_{D_j}(a, z) > \widetilde{\boldsymbol{k}}^*_D(a, z)$. Then there exist an analytic disc $\varphi \in \mathcal{O}(\mathbb{D}, D)$ and a number $r \in (0, \rho)$ such that $\varphi(0) = a$ and $\varphi(r) = z$. Select an $\varepsilon > 0$ such that $(1 + \varepsilon)r < \rho$. Put $\tilde{\varphi}(\lambda) := \varphi(\lambda/(1 + \varepsilon))$, $\lambda \in \mathbb{D}$. Then $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, D)$, $\tilde{\varphi}(0) = a$, and $\tilde{\varphi}(r(1 + \varepsilon)) = z$. Note that $\tilde{\varphi}(\mathbb{D}) \Subset D$. Therefore, $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, D_j)$, $j \gg 1$, which implies that $\widetilde{\boldsymbol{k}}^*_{D_j}(a, z) \leq (1 + \varepsilon)r < \rho$; a contradiction. □

Recall that the pluricomplex Green function $\boldsymbol{g}_D(a, \cdot) = h$, $a \in D$, need not be continuous (see Remark 4.2.28), where $D = D_h$ denotes a pseudoconvex balanced domain with Minkowski function h. With the help of Lemma 4.2.30 we get the following continuity result for the pluricomplex Green function.

Proposition 4.2.31. *Let $D \subset \mathbb{C}^n$ be a domain. Then $\boldsymbol{g}_D$ is upper semicontinuous on $D \times D$.*

Proof. In view of Lemma 4.2.30 we may restrict ourselves to study only a bounded domain D. To be able to continue we need the following lemma.

Lemma 4.2.32. *Let $D \subset \mathbb{C}^n$ be bounded, assume that $\mathbb{B}(a, r) \subset D$, and let $\varepsilon > 0$. Then there exists a $\delta \in (0, r)$ such that*

$$(\boldsymbol{g}_D(z, w))^{1+\varepsilon} \leq \boldsymbol{g}_D(a, w), \quad z \in \mathbb{B}(a, \delta),\ w \in D \setminus \mathbb{B}(a, r).$$

Proof. Put $s := r/3$ and $R := \operatorname{diam} D$. Then, by (4.2.1) and Proposition 4.2.21,

$$\boldsymbol{g}_D(z, w) \leq \boldsymbol{g}_{\mathbb{B}(z,2s)}(z, w) \leq \frac{\|z - w\|}{2s}, \quad z, w \in \mathbb{B}(a, s).$$

Now fix an $\varepsilon > 0$ and choose a positive $\delta \in (0, s/3)$ such that

$$\left(\frac{3\delta}{2s}\right)^{1+\varepsilon} < \frac{\delta}{R}.$$

Fix $b \in \mathbb{B}(a, \delta)$. Put

$$u(z) := \begin{cases} \frac{\|z-a\|}{R} & \text{if } z \in \overline{\mathbb{B}}(a, 2\delta), \\ \max\left\{(\boldsymbol{g}_D(b, z))^{1+\varepsilon}, \frac{\|z-a\|}{R}\right\} & \text{if } z \in D \setminus \overline{\mathbb{B}}(a, 2\delta). \end{cases}$$

Note that $(\boldsymbol{g}_D(b,z))^{1+\varepsilon} < \frac{\|z-a\|}{R}$ for $z \in \partial\mathbb{B}(a,2\delta)$. Therefore, u is log-psh on D. Moreover, it fulfills all other conditions to be a competitor in the definition of the pluricomplex Green function with pole at a. Thus,

$$(\boldsymbol{g}_D(b,w))^{1+\varepsilon} \le u(w) \le \boldsymbol{g}_D(a,w), \quad w \in D \setminus \overline{\mathbb{B}}(a,2\delta).$$

It remains to mention that $D \setminus \mathbb{B}(a,r) \subset D \setminus \overline{\mathbb{B}}(a,2\delta)$. $\square$

Obviously, $\boldsymbol{g}_D$ is continuous at points $(a,a) \in D \times D$ (Exercise). So without loss of generality, let $(a,b) \in D \times D$ with $a \neq b$. Choose an $r > 0$ such that $b \notin \overline{\mathbb{B}}(a,r) \subset D$. Assume now that $\boldsymbol{g}_D(a,b) < \alpha < \beta < 1$. Then fix an $\varepsilon < 0$ such that $\alpha < \beta^{1+\varepsilon}$. Taking the corresponding δ from Lemma 4.2.32 we see that

$$\boldsymbol{g}_D(z,w) \le (\boldsymbol{g}_D(a,w))^{1/(1+\varepsilon)}, \quad z \in \mathbb{B}(a,\delta),\ w \in D \setminus \mathbb{B}(a,r).$$

Recall that $\boldsymbol{g}_D(a,\cdot) \in \mathcal{PSH}(D)$; in particular, $\boldsymbol{g}_D(a,\cdot)$ is upper semicontinuous in b. Therefore, $\boldsymbol{g}_D(a,w) \le \alpha$, when $w \in \mathbb{B}(b,\eta) \subset D \setminus \mathbb{B}(a,r)$ for a sufficiently small $\eta > 0$. Hence, $\boldsymbol{g}_D(z,w) \le \alpha^{1/(1+\varepsilon)} < \beta$, $z \in \mathbb{B}(a,\delta)$, $w \in \mathbb{B}(b,\eta)$, which proves the upper semicontinuity. $\square$

Exercise 4.2.33. Prove the following slight generalization of Lemma 4.2.32.

(a) Let D, a, r, and ε be as in Lemma 4.2.32. Then there is a $\delta \in (0,r)$ such that

$$(\boldsymbol{g}_D(z,w))^{1+\varepsilon} \le \boldsymbol{g}_D(z',w), \quad z, z' \in \mathbb{B}(a,\delta),\ w \in D \setminus \mathbb{B}(a,r).$$

(b) Show (using (a)) that for a bounded domain $D \subset \mathbb{C}^n$ the function $\boldsymbol{g}_D(\cdot,w)$ is continuous if $w \in D$ is fixed.

Moreover, we have the following deep result due to Demailly (cf. [Dem 1987]; see also [Kli 1991]) which we will not prove in this book.

Theorem* 4.2.34. *Let $D \subset \mathbb{C}^n$ be a bounded hyperconvex domain. Then the function $\boldsymbol{g}_D$ is continuous on $D \times \overline{D}$, where $g|_{D\times\partial D} := 1$.*

To get explicit formulas for the "invariant" objects on Cartesian products we discuss the following result which is extremely useful.

Proposition 4.2.35. *The family $(\widetilde{\boldsymbol{k}}_D^*)_D$ satisfies the product property, i.e. for all pairs of domains $D_j \subset \mathbb{C}^{n_j}$, $j = 1,2$, and points $(a_1,a_2),(b_1,b_2) \in D_1 \times D_2$ one has*

$$\widetilde{\boldsymbol{k}}^*_{D_1\times D_2}((a_1,a_2),(b_1,b_2)) = \max\{\widetilde{\boldsymbol{k}}^*_{D_1}(a_1,b_1), \widetilde{\boldsymbol{k}}^*_{D_2}(a_2,b_2)\}.$$

Proof. Because of (4.2.2) only the remaining inequality has to be verified. Suppose that

$$\begin{aligned}\rho := \widetilde{\boldsymbol{k}}^*_{D_1\times D_2}((a_1,a_2),(b_1,b_2)) - r\\ > \max\{\widetilde{\boldsymbol{k}}^*_{D_1}(a_1,b_1), \widetilde{\boldsymbol{k}}^*_{D_2}(a_2,b_2)\} = \widetilde{\boldsymbol{k}}^*_{D_1}(a_1,b_1)\end{aligned}$$

for some $r > 0$. We find analytic discs $\varphi_j \in \mathcal{O}(\mathbb{D}, D_j)$ with $\varphi_j(0) = a_j$ and $\varphi_j(\mu_j) = b_j$, where $\widetilde{\boldsymbol{k}}^*_{D_1}(a_1,b_1) \le \mu_1 < \rho$ and $\mu_2 \in (0,\mu_1)$. Put

$$\varphi(\lambda) := (\varphi_1(\lambda), \varphi_2(\tfrac{\mu_2}{\mu_1}\lambda)), \quad \lambda \in \mathbb{D}.$$

Then $\varphi \in \mathcal{O}(\mathbb{D}, D_1 \times D_2)$, $\varphi(0) = (a_1,a_2)$, and $\varphi(\mu_1) = (b_1,b_2)$. Hence, $\rho \ge \widetilde{\boldsymbol{k}}^*_{D_1\times D_2}((a_1,a_2),(b_1,b_2))$; a contradiction. $\square$

Exercise 4.2.36. Use Proposition 4.2.35 and Corollary 4.2.24 to prove that $\mathbb{D}^n \times \mathbb{B}_m$ is not biholomorphically equivalent to $\mathbb{B}_{m+n}$.

Remark 4.2.37. Note that also the family of Möbius pseudodistances (resp. of pluricomplex Green functions) fulfills the product property; for details see [Jar-Pfl 2005] (resp. [Jar-Pfl 1995], [Edi 1999], and [Edi 2001]). Obviously, if the product property holds one can get new formulas for the invariant functions for Cartesian products.

Proposition 4.2.38. *Let $D_j \subset \mathbb{C}^n$ be a domain, $j = 1,2$, and let $F \in \mathcal{O}(D_1, D_2)$ be such that*

$$\forall_{a\in D_2}\ \forall_{b\in D_1,\ F(b)=a}\ \forall_{\varphi\in\mathcal{O}(\mathbb{D},D_2),\ \varphi(0)=a}\ \exists_{\psi\in\mathcal{O}(\mathbb{D},D_1)} : \psi(0) = b,\ \ F\circ\psi = \varphi. \tag{4.2.4}$$

Then, for points $a_1, a_2 \in D_2$ and $b_1 \in D_1$ with $F(b_1) = a_1$, one has

$$\widetilde{\boldsymbol{k}}^*_{D_2}(a_1,a_2) = \inf\{\widetilde{\boldsymbol{k}}^*_{D_1}(b_1,b_2) : b_2 \in D_1,\ \ F(b_2) = a_2\}; \tag{4.2.5}$$

$$\boldsymbol{k}_{D_2}(a_1,a_2) = \inf\{\boldsymbol{k}_{D_1}(b_1,b_2) : b_2 \in D_1,\ \ F(b_2) = a_2\}. \tag{4.2.6}$$

Remark 4.2.39. (a) Recall the following definition: an $F \in \mathcal{O}(D_1, D_2)$, D_1, D_2 domains in $\mathbb{C}^n$, is said to be a *holomorphic covering* if for any $z \in D_2$ there exists a neighborhood $V = V(z) \subset D_2$ such that $F^{-1}(V) = \bigcup_{j\in J} U_j$, where U_j is an open subset of D_1, such that $F|_{U_j} : U_j \to V$ is biholomorphic, $j \in J$.

Any holomorphic covering $F\colon D_1 \to D_2$ satisfies (4.2.4). The reader is referred to [Con 1995]. Even more is true: for any $b \in D_1$, $\lambda \in \mathbb{D}$, and $\varphi \in \mathcal{O}(\mathbb{D}, D_2)$ with $\varphi(\lambda) = F(b)$ there exists a unique $\psi \in \mathcal{O}(\mathbb{D}, D_1)$ such that $\psi(\lambda) = b$ and $F \circ \psi = \varphi$; ψ is called the *lifting* of φ with respect to F.

(b) Recall that for any plane domain $D \subset \mathbb{C}$ there is a simply connected domain $D^0 \in \{\mathbb{C}, \mathbb{D}\}$ and a mapping $F \in \mathcal{O}(D^0, D)$ with the property (4.2.4). D^0 is the universal covering domain of D (cf. the uniformization theorem in the classical complex analysis of one complex variable).

(c) In Example 4.4.16 (see [Zwo 1998]), an example will be given where the infimum in equation (4.2.5) is not attained. This gives a negative answer to a long standing question asked by S. Kobayashi.

Proof of Proposition 4.2.38. In view of (4.2.1) we obviously have

$$\widetilde{\boldsymbol{k}}^*_{D_2}(a_1, a_2) \le \inf\{\widetilde{\boldsymbol{k}}^*_{D_1}(b_1, b_2) : b_2 \in D_1,\ F(b_2) = a_2\}.$$

Assume that the above inequality is a strict one. Then there exists an analytic disc $\varphi \in \mathcal{O}(\mathbb{D}, D_2)$ with $\varphi(0) = a_1$ and $\varphi(\mu) = a_2$, where

$$\inf\{\widetilde{\boldsymbol{k}}^*_{D_1}(b_1, b_2) : b_2 \in D_1,\ F(b_2) = a_2\} > \mu \ge \widetilde{\boldsymbol{k}}^*_{D_2}(a_1, a_2).$$

Applying property (4.2.4), we find an analytic disc $\psi \in \mathcal{O}(\mathbb{D}, D_1)$ with $\psi(0) = b_1$ and $F \circ \psi = \varphi$. Therefore,

$$\mu \ge \widetilde{\boldsymbol{k}}^*_{D_1}(b_1, \psi(\mu)) \ge \inf\{\widetilde{\boldsymbol{k}}^*_{D_1}(b_1, b_2) : b_2 \in D_1,\ F(b_2) = a_2\} > \mu;$$

a contradiction. Hence (4.2.5) has been verified. The equality (4.2.6) could be proved in a similar way. Its proof is left as an exercise for the reader. $\square$

Exercise 4.2.40. Find the formula for $\widetilde{\boldsymbol{k}}^*_{\mathbb{H}^+}$, where $\mathbb{H}^+ := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$.

One of the most important results in the theory of invariant functions is the following one due to Lempert. Its proof is beyond the scope of this book. Therefore, the reader is referred to [Jar-Pfl 1993].

Theorem* 4.2.41 (Lempert theorem). (a) *Let $D \subset \mathbb{C}^n$ be a bounded convex domain and let $a, b \in D$. Then there exists a complex $\boldsymbol{m}_D$-geodesic $\varphi \in \mathcal{O}(\mathbb{D}, D)$ such that $a, b \in \varphi(\mathbb{D})$. In particular, $\boldsymbol{c}_D = \widetilde{\boldsymbol{k}}_D$ on $D \times D$.*

(b) *Assume that $D \subset \mathbb{C}^n$ is a domain which can be exhausted by an increasing sequence $(D_j)_j$ of domains $D_j \subset \mathbb{C}^n$, where each of them is biholomorphically equivalent to a convex domain. Then $\boldsymbol{c}_D = \widetilde{\boldsymbol{k}}_D$ on $D \times D$.*

Note that (b) is a simple consequence of (a) and general properties of the Möbius and the Lempert functions (Exercise).

As an application for pseudoconvex Reinhardt domains we have

Theorem 4.2.42. *Let $D \subset \mathbb{C}^n_*$ be a pseudoconvex Reinhardt domain. Then*

$$\boldsymbol{k}_D = \widetilde{\boldsymbol{k}}_D \text{ on } D \times D.$$

In particular, $\widetilde{\boldsymbol{k}}_D$ is continuous.

Proof. Recall that the logarithmic image $\log D$ is convex. Hence the tube domain $T_D := \log D + i\mathbb{R}^n$ is convex. Therefore, by the Lempert theorem, we have $\boldsymbol{k}_{T_D} = \widetilde{\boldsymbol{k}}_{T_D}$ on $T_D \times T_D$.

On the other hand, observe that the mapping $F\colon T_D \to D$, $F(z) := e^z$, is a holomorphic covering. Therefore, Proposition 4.2.38 immediately gives the proof of the equality in the theorem. □

Recall from Remark 4.2.26 that for a general pseudoconvex Reinhardt domain D the Kobayashi pseudodistance $\boldsymbol{k}_D$ and the Lempert function $\widetilde{\boldsymbol{k}}_D$ are, in general, different.

Example 4.2.43. Another application of Theorem 4.2.41 gives the following result (see Exercise 2.1.13).

Let N be a complex norm on $\mathbb{C}^n$, $n \geq 2$. Recall that $\mathfrak{B} = \{z \in \mathbb{C}^n : N(z) < 1\}$, $\mathfrak{B}(r) = \{z \in \mathbb{C}^n : N(z) < r\}$, and $\mathcal{A} = \mathcal{A}(r) = \{z \in \mathfrak{B} : r < N(z) < 1\}$. Then

$$\operatorname{Aut}(\mathcal{A}) = \{\Phi|_{\mathcal{A}} : \Phi \in \operatorname{Aut}_0(\mathfrak{B})\}.$$

Indeed, let $F \in \operatorname{Aut}(\mathcal{A})$. Then there exists a $\Phi \in \operatorname{Aut}(\mathfrak{B})$ such that $\Phi|_{\mathcal{A}} = F$ and $N(\Phi(z)) = r$ whenever $N(z) = r$ (Exercise, cf. Exercise 2.1.13).

Let $a \in \mathfrak{B}(r)$ be such that $\Phi(a) = 0$. Take a complex geodesic $\varphi \in \mathcal{O}(\mathbb{D}, \mathfrak{B})$ such that $\varphi(0) = 0$ and $\varphi(\alpha) = a$ for some $\alpha \in \mathbb{D} \cap [0, 1)$ (Exercise). Since φ is proper (see Exercise 4.2.12 (c)), we find a $\beta \in (\alpha, 1)$ such that $N(\varphi(\beta)) = r$. Put $w := \varphi(\beta)$. Then, in virtue of Exercises 1.1.1 (c) and 4.2.12 (b),

$$\boldsymbol{p}(0, \beta) = \boldsymbol{c}_{\mathfrak{B}}(0, \Phi(w)) = \boldsymbol{c}_{\mathfrak{B}}(a, w) = \boldsymbol{p}(\alpha, \beta) = \boldsymbol{p}(0, \beta) - \boldsymbol{p}(0, \alpha).$$

Therefore, $0 = \boldsymbol{p}(0, \alpha) = \boldsymbol{c}_{\mathfrak{B}}(0, a)$, i.e. $a = 0$. Hence, $\Phi \in \operatorname{Aut}_0(\mathfrak{B})$.

Remark 4.2.44. (a) We mention that recently a domain $\mathbb{G}_2$ in $\mathbb{C}^2$ has been found for which $\boldsymbol{m}_{\mathbb{G}_2} = \widetilde{\boldsymbol{k}}^*_{\mathbb{G}_2}$, but which does not fulfill the assumption of Theorem 4.2.41 (b). Here we only give the definition of $\mathbb{G}_2$,

$$\mathbb{G}_2 := \{z \in \mathbb{C}^2 : |z_1 - \bar{z}_1 z_2| + |z_2|^2 < 1\}.$$

Let $\pi\colon \mathbb{C}^2 \to \mathbb{C}^2$, $\pi(z_1, z_2) := (z_1 + z_2, z_1 z_2)$. Then $\mathbb{G}_2 = \pi(\mathbb{D}^2)$ and $\pi\colon \mathbb{D}^2 \to \mathbb{G}_2$ is proper (Exercise).

For more details and other sources the reader may contact [Jar-Pfl 2005]. This is, at least at the moment, the only known example (up to simple modifications) with these properties.

(b) The notion of a holomorphically contractible family $(d_D)_D$ (Definition 4.2.1) can be extended to the case where D runs through all connected complex manifolds, complex analytic sets, or even complex spaces. In particular, one can define the Möbius pseudodistance $\boldsymbol{m}_M$, the Lempert function $\widetilde{\boldsymbol{k}}^*_M$ (defined as 1 for pairs of points for which there is no analytic disc passing through them), and the Kobayashi pseudodistance $\boldsymbol{k}_M$ for an arbitrary connected complex analytic set M. For recent results in case of the Neil parabola $M := \{z \in \mathbb{C}^2 : z_1^2 = z_2^3\}$ see, for example, [Kne 2007], [Nik-Pfl 2007], [Zap 2007].

4.3* Hahn function

Note that in Definition 4.2.1 one can also consider conditions that are weaker than (B), for instance:

(B′) Condition (4.2.1) holds for every injective holomorphic mapping $F : G \to D$.

Example 4.3.1 (Hahn function).

$$\begin{aligned} H_D^*(a,z) &:= \inf\{\boldsymbol{m}(\lambda,\mu) : \exists_{\varphi\in\mathcal{O}(\mathbb{D},D)} :\ \varphi \text{ is injective},\ \varphi(\lambda)=a,\ \varphi(\mu)=z\} \\ &= \inf\{|\mu| : \exists_{\varphi\in\mathcal{O}(\mathbb{D},D)} :\ \varphi \text{ is injective},\ \varphi(0)=a,\ \varphi(\mu)=z\}, \end{aligned}$$

where $(a,z) \in D \times D$, satisfies (A) and (B′).[10]

Remark 4.3.2. Observe that the infimum in the above definition is taken over a non-empty set. Indeed, fix points $a, b \in D$, $a \neq b$. Then there is an injective $\mathcal{C}^1$-curve $\alpha : [0,1] \to D$ connecting a and b such that $\alpha'(t) \neq 0$, $t \in [0,1]$. By the Weierstrass approximation theorem, we find a sequence $(p_j)_{j\in\mathbb{N}}$ of polynomial mappings $p_j : \mathbb{C} \to \mathbb{C}^n$ such that

$$p_j(0) = a, \quad p_j(1) = b, \quad \|p_j^{(k)} - \alpha^{(k)}\|_{[0,1]} \to 0, \quad k = 0, 1,$$

and

$$p_j([0,1]) \subset D, \quad j \in \mathbb{N}.$$

If $j \gg 1$, then $p_j|_{[0,1]}$ is injective. Indeed, suppose the contrary, i.e. there exist $t_j', t_j'' \in [0,1]$, $t_j' \neq t_j''$, with $p_j(t_j') = p_j(t_j'')$, $j \in \mathbb{N}$. By the compactness of $[0,1]$ we may assume that $t_j' \to t'$ and $t_j'' \to t''$. Then the uniform convergence of $(p_j)_j$ implies that $\alpha(t') = \alpha(t'')$. Applying the fact that α is injective gives $t' = t''$. Therefore,

$$\begin{aligned} 0 = \|p_j(t_j') - p_j(t_j'')\|^2 &= \sum_{k=1}^n |p_{j,k}'(\tau_{j,k})|^2 |t_j' - t_j''|^2 \\ &\geq \sum_{k=1}^n \big||\alpha_k'(\tau_{j,k})| - |p_{j,k}'(\tau_{j,k}) - \alpha'(\tau_{j,k})|\big|^2 |t_j' - t_j''|^2 \\ &\geq \tfrac{1}{2}\|\alpha'\|_{[0,1]}^2 |t_j' - t_j''|^2, \text{ if } j \gg 1, \end{aligned}$$

where $\tau_{j,k}$ is between t_j' and t_j''. Hence $t_j' = t_j''$ for $j \gg 1$; a contradiction.

[10] Observe that the definition of H_D^* is similar to the one of $\widetilde{\boldsymbol{k}}_D^*$.

Fix a j such that p_j is injective on $[0,1]$. Then there exists a simply connected domain $G \subset \mathbb{C}$ with $[0,1] \subset G$, $p_j(G) \subset D$, and $p_j|_G$ injective (EXERCISE). Arguing as in the case of the Lempert function we end up with an injective analytic disc in D passing through a and b.

Remark 4.3.3. Obviously, $\widetilde{\boldsymbol{k}}_D^* \le H_D^*$. But $\widetilde{\boldsymbol{k}}_{\mathbb{C}_*}^* \equiv 0 \not\equiv H_{\mathbb{C}_*}^*$.

Indeed, fix two different points $a, b \in \mathbb{C}_*$. Let $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ be injective with $\varphi(0) = a$, $\varphi(\mu) = b$ for a suitable $\mu \in \mathbb{D}$. Applying the Koebe distortion theorem (see [Pom 1992], Theorem 1.3 and Corollary 1.4) we have

$$|b-a| = |\varphi(\mu) - \varphi(0)| \le |\varphi'(0)| \frac{|\mu|}{(1-|\mu|)^2} \le 4 \operatorname{dist}(a, \partial f(\mathbb{D})) \frac{|\mu|}{(1-|\mu|)^2}.$$

Taking into account that $f(\mathbb{D})$ is simply connected we get

$$|b-a| \le 4|a| \frac{|\mu|}{(1-|\mu|)^2}.$$

Hence $H_{\mathbb{C}_*}^*(a,b) > 0$.

On the other hand, the following result for $n \ge 3$ is due to M. Overholt.

Theorem 4.3.4 ([Ove 1995]). *If $D \subset \mathbb{C}^n$, $n \ge 3$, is a domain, then $\widetilde{\boldsymbol{k}}_D^* = H_D^*$ on $D \times D$.*

Proof. Fix $a, b \in D$, $a \ne b$. Without loss of generality, we may assume that $a = 0 \in D$ (EXERCISE). Let $\varepsilon > 0$. Then there is an analytic disc $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) = 0$, $\varphi(\mu) = b$ for a suitable $\mu \in \mathbb{D}$ such that $0 < |\mu| < \widetilde{\boldsymbol{k}}_D^*(0,b) + \varepsilon/2$. We choose an $R \in (0,1)$ such that $\mu/R \in \mathbb{D}$ and $|\mu/R| < \widetilde{\boldsymbol{k}}_D^*(0,b) + \varepsilon$. Put $\varphi_R(\lambda) := \varphi(R\lambda)$, $|\lambda| < 1/R$. Obviously, $\varphi_R|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi_R(0) = 0$ and $\varphi_R(\mu/R) = b$. Since φ_R is continuous on $\overline{\mathbb{D}}$, we have $\operatorname{dist}(\varphi_R(\overline{\mathbb{D}}), \partial D) =: 2s > 0$.

Now we take a polynomial mapping $\tilde{p} : \mathbb{C} \to \mathbb{C}^n$ coming from the power series expansion of φ_R such that $\operatorname{dist}(\tilde{p}(\overline{\mathbb{D}}), \partial D) < s/2$ and

$$\left\| \varphi_R\Big(\frac{\mu}{R}\Big) - \tilde{p}\Big(\frac{\mu}{R}\Big) \right\| < \frac{\mu s}{2R}.$$

Finally, put

$$p(\lambda) := \tilde{p}(\lambda) + \frac{R\lambda}{\mu}\Big(\varphi_R\Big(\frac{\mu}{R}\Big) - \tilde{p}\Big(\frac{\mu}{R}\Big)\Big), \quad \lambda \in \mathbb{C}.$$

Hence $p|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}, D)$ with $p(0) = 0$ and $p(\mu/R) = b$. Observe that $p = (p_1, \dots, p_n)$ is a polynomial mapping with

$$p_j(\lambda) = \sum_{k=1}^{m} a_{j,k} \lambda^k, \quad \lambda \in \mathbb{C},\ j = 1, \dots, n,$$

where $m \geq n$ is sufficiently large. Put $A := [a_{j,k}]_{1\leq j\leq n,\, 2\leq k\leq m}$.

Now we will try to modify the coefficients $a_{j,k}$ a little bit such that the new polynomial mapping

$$\hat{p}(\lambda) := \Big(\sum_{k=1}^{m} \hat{a}_{j,k}\lambda^k\Big)_{1\leq j\leq n}, \quad \lambda \in \mathbb{C},$$

gives an injective mapping from $\mathbb{D}$ to D with $\hat{p}(0) = 0$ and $\hat{p}(\hat{\mu}) = b$, $\hat{\mu} := \mu/R$, i.e. $\hat{a}_{j,1}$ has to satisfy the equation

$$\hat{a}_{j,1} = \Big(b_j - \sum_{k=2}^{m} \hat{a}_{j,k}\hat{\mu}^k\Big)/\hat{\mu}, \quad j = 1, \dots, n.$$

Assume that $\hat{p}$ is not injective. Then $\hat{p}(\lambda_1) = \hat{p}(\lambda_2)$ for certain $\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_1 \neq \lambda_2$. Therefore,

$$\sum_{k=1}^{m} \hat{a}_{j,k}\lambda_1^k = \sum_{k=1}^{m} \hat{a}_{j,k}\lambda_2^k, \quad j = 1, \dots, n.$$

Or, after dividing by $\lambda_1 - \lambda_2$,

$$-\hat{a}_{j,1} = \sum_{k=2}^{m} \hat{a}_{j,k}\Big(\sum_{s=0}^{k-1} \lambda_1^s \lambda_2^{k-1-s}\Big), \quad j = 1, \dots, n.$$

Now, for an arbitrary $n \times (m-1)$ matrix $\tilde{A} = [\tilde{a}_{j,k}]_{\substack{1\leq j\leq n,\\ 2\leq k\leq m}}$, put

$$M(\tilde{A}) := \Big\{(z_2, \dots, z_m) \in \mathbb{C}^{m-1} : \sum_{k=2}^{m} \tilde{a}_{j,k} z_k = -\tilde{a}_{j,1},\ j = 1, \dots, n\Big\},$$

where

$$\tilde{a}_{j,1} := \Big(b_j - \sum_{k=2}^{m} \tilde{a}_{j,k}\hat{\mu}^k\Big)/\hat{\mu}, \quad j = 1, \dots, n.$$

Observe that $M(\tilde{A})$ is an $(m - 1 - \operatorname{rank} \tilde{A})$-dimensional affine subspace of $\mathbb{C}^{m-1}$. Therefore, there is a dense subset of matrices $\tilde{A}$ in $\mathbb{M}(n \times (m-1); \mathbb{C})$ such that the corresponding affine subspace $M(\tilde{A})$ has dimension $m - 1 - n \leq m - 1 - 3$.

Moreover, define

$$S := \Big\{\Big(\sum_{s=0}^{l} w_1^s w_2^{l-s}\Big)_{1\leq l\leq m-1} \in \mathbb{C}^{m-1} : w = (w_1, w_2) \in \mathbb{C}^2,\ w_1 \neq w_2\Big\}.$$

Note that the map

$$\Phi(w_1, w_2) := \Big(\sum_{s=0}^{l} w_1^s w_2^{l-s}\Big)_{1\le l\le m-1} \in \mathbb{C}^{m-1}, \quad w_1 \neq w_2,$$

is a regular holomorphic mapping onto S with $\Phi(w_1, w_2) = \Phi(\widetilde{w}_1, \widetilde{w}_2)$ if and only if $(w_1, w_2) = (\widetilde{w}_1, \widetilde{w}_2)$ or $(w_1, w_2) = (\widetilde{w}_2, \widetilde{w}_1)$. Hence, S is a 2-dimensional complex submanifold in $\mathbb{C}^{m-1}$.

It suffices to find a sequence of matrices $(\tilde{A}(\ell))_\ell \subset \mathbb{M}(n \times (m-1); \mathbb{C})$ such that $\tilde{A}(\ell) \to A$ when $\ell \to \infty$ such that $S \cap M(\tilde{A}(\ell)) = \varnothing$. □

So the dimension 2 is left in the general comparison of the Lempert function and the Hahn function. Here we present an answer to what happens with the Hahn function for the product of two plane domains. It shows that both functions can be different also in the 2-dimensional case.

Before stating this result we ask the reader to solve the following exercise which will be important in the proof of the following proposition.

Exercise 4.3.5. Let $D \subset \mathbb{C}^n$ be a domain. Then the following properties are equivalent:

(a) $\widetilde{\boldsymbol{k}}_D^* = H_D^*$;

(b) for any $\varphi \in \mathcal{O}(\mathbb{D}, D)$, $0 < \alpha < \delta < 1$ with $\varphi(0) \neq \varphi(\alpha)$, there exists an injective $\psi \in \mathcal{O}(\mathbb{D}, D)$ with $\psi(0) = \varphi(0)$ and $\psi(\delta) = \varphi(\alpha)$.

Theorem 4.3.6 ([JarW 2001]). *Let $D_j \subset \mathbb{C}$ be a domain, $j = 1, 2$.*

(a) *If at least one of the D_j's is simply connected, then $\widetilde{\boldsymbol{k}}_{D_1\times D_2}^* = H_{D_1\times D_2}^*$.*

(b) *If at least one of the D_j's is biholomorphically equivalent to $\mathbb{C}_*$, then $\widetilde{\boldsymbol{k}}_{D_1\times D_2}^* = H_{D_1\times D_2}^*$.*

(c) *Otherwise, $\widetilde{\boldsymbol{k}}_{D_1\times D_2}^* \not\equiv H_{D_1\times D_2}^*$.*

The proof of (c) will be based on the following nice lemma from classical complex analysis and the uniformization theorem.

Lemma 4.3.7. *Let $D_j \subset \mathbb{C}$ be a non-simply connected domain that is not biholomorphically equivalent to $\mathbb{C}_*$, $j = 1, 2$. Denote by $p_j : \mathbb{D} \to D_j$ the universal covering mapping.*[11] *Then there are two different points $q_1, q_2 \in \mathbb{D}$ and automorphisms $f_j \in \operatorname{Aut}(\mathbb{D})$, $j = 1, 2$, such that $p_j(f_j(q_1)) = p_j(f_j(q_2))$, $j = 1, 2$, and*

$$\det \begin{bmatrix} (p_1 \circ f_1)'(q_1) & (p_1 \circ f_1)'(q_2) \\ (p_2 \circ f_2)'(q_1) & (p_2 \circ f_2)'(q_2) \end{bmatrix} \neq 0.$$

[11] Note that, in virtue of the uniformization result, the universal covering of D_j is given by the unit disc.

Proof. By assumption the map p_j is not injective, $j = 1, 2$. Therefore, there exists $\psi_j \in \operatorname{Aut}(\mathbb{D}) \setminus \{\mathrm{id}_{\mathbb{D}}\}$ such that $p_j \circ \psi_j = p_j$, $j = 1, 2$; in particular, ψ_j is a lifting of p_j. Note that ψ_j has no fixed points in $\mathbb{D}$ (otherwise, applying the uniqueness of the lifting, it would be equal to $\mathrm{id}_{\mathbb{D}}$). Therefore, it has one or two fixed points on $\partial\mathbb{D}$ (see Exercise 2.1.4 (b)). Fix $\lambda' \in \partial\mathbb{D}$ with $\psi_j(\lambda') \neq \lambda'$ for $j = 1, 2$. Then $\boldsymbol{m}(t\lambda', \psi_j(t\lambda')) \to 1$ when $t \nearrow 1$, $j = 1, 2$. Hence we find $z_1, z_2 \in \mathbb{D}$ with

$$\boldsymbol{m}(z_1, \psi_1(z_1)) = \boldsymbol{m}(z_2, \psi_2(z_2)) \in (0, 1).$$

Let $d \in (0, 1)$ with $\boldsymbol{m}(-d, d) = \boldsymbol{m}(z_1, \psi_1(z_1))$. Then, by Exercise 2.1.4(b), there exist $h_j \in \operatorname{Aut}(\mathbb{D})$ with

$$h_j(-d) = z_j, \; h_j(d) = \psi_j(z_j), \quad j = 1, 2.$$

Assume that $(p_j \circ h_j)'(-d) \neq \pm(p_j \circ h_j)'(d)$ for at least one of the j's, say for $j = 1$. Then one of the following determinants does not vanish:

$$\det \begin{bmatrix} (p_1 \circ h_1)'(-d) & (p_1 \circ h_1)'(d) \\ (p_2 \circ h_2)'(-d) & (p_2 \circ h_2)'(d) \end{bmatrix},$$

$$\det \begin{bmatrix} (p_1 \circ h_1 \circ (-\mathrm{id}_{\mathbb{D}}))'(-d) & (p_1 \circ h_1 \circ (-\mathrm{id}_{\mathbb{D}}))'(d) \\ (p_2 \circ h_2)'(-d) & (p_2 \circ h_2)'(d) \end{bmatrix}.$$

(Exercise, use that $(p_2 \circ h_2)'(d) \neq 0$.)

So we may put $f_1 = h_1$, $f_2 = h_2$ (resp. $f_1 = h_1 \circ (-\mathrm{id}_{\mathbb{D}})$, $f_2 = h_2$) and $q_1 = -d, q_2 = d$.

Now, for the remaining part of the proof we may assume that

$$\big((p_j \circ h_j)'(d)\big)^2 = \big((p_j \circ h_j)'(-d)\big)^2, \quad j = 1, 2. \tag{4.3.1}$$

Put $\tilde{\psi}_j := h_j^{-1} \circ \psi_j \circ h_j$ and $\tilde{p}_j := p_j \circ h_j$, $j = 1, 2$. Then $\tilde{\psi}_j(-d) = d$ and $\tilde{p}_j{}'(-d) = (\tilde{p}_j \circ \tilde{\psi}_j)'(-d) = \tilde{p}_j{}'(\tilde{\psi}_j(-d))\tilde{\psi}_j'(-d)$. Taking squares on both sides we get $\big(\tilde{\psi}_j'(-d)\big)^2 = 1$ (see (4.3.1)). Therefore, either $\tilde{\psi}_j(-d) = d$, $\tilde{\psi}_j'(-d) = -1$ or $\tilde{\psi}_j(-d) = d$, $\tilde{\psi}_j'(-d) = 1$.

Applying Exercise 2.1.5 (b) to $\varphi = -\mathrm{id}_{\mathbb{D}}$ (in case of $-$) or $\varphi = h_c$ and the fact that $\tilde{\psi}_j$ has no fixed points in $\mathbb{D}$, it follows that

$$\psi := \tilde{\psi}_1 = \tilde{\psi}_2 = h_c \quad \text{with } c := \frac{-2d}{1 + d^2}.$$

Now fix an $a \in \mathbb{D}$ and choose $\varphi \in \operatorname{Aut}(\mathbb{D})$ such that $\varphi(a) = \psi(a)$ and $\varphi(\psi(a)) = a$ (see Exercise 2.1.4 (b)). Note that such a φ exists.

Suppose that $\varphi'(a) = \psi'(a)$. By Exercise 2.1.4 (b), $\varphi = \psi$ and therefore $\psi \circ \psi(a) = a$. So $\psi \circ \psi$ has a fixed point in $\mathbb{D}$ and, therefore, none on $\partial\mathbb{D}$. On the other hand, ψ is without fixed points on $\mathbb{D}$. So it has at least one fixed point on ∂D, say $b \in \partial\mathbb{D}$. Then $\psi \circ \psi(b) = b$; a contradiction.

Fix an $a_0 \in \mathbb{D} \cap \mathbb{R}$. Let $\varphi \in \operatorname{Aut}(\mathbb{D})$ with $\varphi(a_0) = \psi(a_0)$ and $\varphi(\psi(a_0)) = a_0$. Then $\varphi = h_{-a_0} \circ (-\operatorname{id}_{\mathbb{D}}) \circ h_{h_{a_0}(\psi(a_0))} \circ h_{a_0}$. By a direct calculation it follows that $\varphi'(a_0) \neq -\psi'(a_0)$.

Summarizing, we know that if $\varphi \in \operatorname{Aut}(\mathbb{D})$ is such that $\varphi(a_0) = \psi(a_0)$ and $\varphi(\psi(a_0)) = a_0$, then $\varphi'(a_0) \neq \pm\psi'(a_0)$. Then, by Exercise 2.1.4 (b), $\varphi \circ \varphi = \operatorname{id}_{\mathbb{D}}$ (note that $\varphi \circ \varphi$ has two fixed points in $\mathbb{D}$) and so $\varphi'(\psi(a_0)) = \frac{1}{\varphi'(a_0)}$.

Finally, we put $q_1 := a_0$, $q_2 := \psi(a_0)$, $f_1 := h_1$, and $f_2 := h_2 \circ \varphi$. Then

$$\begin{aligned} p_1(f_1(q_2)) &= (p_1 \circ h_1)(\psi(a_0)) \\ &= (p_1 \circ \psi_1)(h_1(a_0)) = (p_1 \circ h_1)(q_1) = (p_1 \circ f_1)(q_1), \\ p_2(f_2(q_2)) &= (p_2 \circ h_2)(\varphi(\psi(a_0))) = (p_2 \circ h_2)(a_0) = (p_2 \circ \psi_2)(h_2(a_0)) \\ &= (p_2 \circ h_2)(\psi(a_0)) = (p_2 \circ (h_2 \circ \varphi))(a_0) = (p_2 \circ f_2)(q_1). \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\det \begin{bmatrix} (p_1 \circ f_1)'(q_1) & (p_1 \circ f_1)'(q_2) \\ (p_2 \circ f_2)'(q_1) & (p_2 \circ f_2)'(q_2) \end{bmatrix} \\ &\quad = \det \begin{bmatrix} (p_1 \circ h_1)'(a_0) & (p_1 \circ h_1)'(\psi(a_0)) \\ (p_2 \circ h_2)'(\varphi(a_0))\varphi'(a_0) & (p_2 \circ h_2)'(\varphi(\psi(a_0)))\varphi'(\psi(a_0)) \end{bmatrix} \\ &\quad = \det \begin{bmatrix} (p_1 \circ h_1)'(\psi(a_0))\psi'(a_0) & (p_1 \circ h_1)'(\psi(a_0)) \\ (p_2 \circ h_2)'(\psi(a_0))\varphi'(a_0) & (p_2 \circ h_2)'(\psi(a_0))/\varphi'(a_0) \end{bmatrix} \\ &\quad = (p_1 \circ h_1)'(\psi(a_0))(p_2 \circ h_2)'(\psi(a_0)) \det \begin{bmatrix} \psi'(a_0) & 1 \\ \varphi'(a_0) & \psi'(a_0)/\varphi'(a_0) \end{bmatrix} \neq 0. \end{aligned}$$

Hence this lemma is proved. □

Proof of Theorem 4.3.6. (a) Without loss of generality, we may assume that D_1 is simply connected (EXERCISE). Our task is to apply Exercise 4.3.5. So let $\varphi = (\varphi_1, \varphi_2) \in \mathcal{O}(\mathbb{D}, D_1 \times D_2)$ and $0 < \alpha < \delta < 1$ with $\varphi(0) \neq \varphi(\alpha)$.

Assume that $\varphi_1(0) \neq \varphi_1(\alpha)$. Recall that $\tilde{\boldsymbol{k}}^*_{D_1} = H^*_{D_1}$. Hence there exists an injective $\tilde{\psi}_1 \in \mathcal{O}(\mathbb{D}, D_1)$ with $\tilde{\psi}_1(0) = \varphi_1(0)$ and $\tilde{\psi}_1(\delta) = \varphi_1(\alpha)$. Put

$$\psi(\lambda) := (\tilde{\psi}_1(\lambda), \varphi_2(\tfrac{\alpha}{\delta}\lambda)), \quad \lambda \in \mathbb{D}.$$

Then $\psi \in \mathcal{O}(\mathbb{D}, D_1 \times D_2)$, ψ is injective, and one obtains $\psi(0) = \varphi(0)$ and $\psi(\delta) = \varphi(\alpha)$.

Now let $\varphi_1(0) = \varphi_1(\alpha)$ and $\varphi_2(0) \neq \varphi_2(\alpha)$. Take a $d \in (0, \operatorname{dist}(\varphi_1(0), \partial D_1))$

and put

$$h(\lambda) := \frac{\varphi_2(\frac{\alpha}{\delta}\lambda) - \varphi_2(0)}{\varphi_2(\alpha) - \varphi_2(0)}, \tag{4.3.2}$$

$$\psi_1(\lambda) := \varphi_1(0) + \frac{\delta d}{M\delta + 1}\Big(h(\lambda) - \frac{\lambda}{\delta}\Big), \quad \lambda \in \mathbb{D}, \tag{4.3.3}$$

where $M := \|h\|_{\overline{\mathbb{D}}}$. Observe that $\psi_1 \in \mathcal{O}(\mathbb{D}, D_1)$. Finally, define $\psi(\lambda) := (\psi_1(\lambda), \varphi_2(\frac{\alpha}{\delta}\lambda))$, $\lambda \in \mathbb{D}$. Then $\psi \in \mathcal{O}(\mathbb{D}, D_1 \times D_2)$ with $\psi(0) = \varphi(0)$ and $\psi(\delta) = \varphi(\alpha)$. Moreover, one easily sees that ψ is an injective analytic disc. Hence, (a) is proved.

(b) We may assume that $D_1 = \mathbb{C}_*$ and $D_2 \neq \mathbb{C}$ (Exercise). Let, as in (a), $\varphi = (\varphi_1, \varphi_2) \in \mathcal{O}(\mathbb{D}, D_1 \times D_2)$, $0 < \alpha < \delta < 1$, and $\varphi(0) \neq \varphi(\alpha)$. Moreover, applying a suitable automorphism of $\mathbb{C}_*$ we may even assume that $\varphi_1(0) = 1$ (Exercise).

In the case where $\varphi_2(0) = \varphi_2(\alpha)$ define $\widetilde{D}_2 := \varphi_2(0) + \operatorname{dist}(\varphi_2(0), \partial D_2)\mathbb{D}$. Obviously, $\widetilde{D}_2$ is a simply connected domain, $\tilde{\varphi} = (\varphi_1, \tilde{\varphi}_2) \in \mathcal{O}(\mathbb{D}, D_1 \times \widetilde{D}_2)$, where $\tilde{\varphi}_2(\lambda) := \varphi_2(0)$, $\lambda \in \mathbb{D}$. In virtue of (a), there exists an injective analytic disc $\psi \in \mathcal{O}(\mathbb{D}, D_1 \times \widetilde{D}_2)$ with $\psi(0) = \varphi(0)$, $\psi(\delta) = \tilde{\varphi}(\alpha) = \varphi(\alpha)$.

Next, we discuss the situation when $\varphi_2(0) \neq \varphi_2(\alpha)$. For the moment we assume, in addition, that $\varphi_1(\alpha) = 1 + \delta$. Put

$$\psi(\lambda) := (1 + \lambda, \varphi_2(\tfrac{\alpha}{\delta}\lambda)), \quad \lambda \in \mathbb{D}.$$

Then $\psi \in \mathcal{O}(\mathbb{D}, \mathbb{C}_* \times D_2)$ is injective and satisfies $\psi(0) = \varphi(0)$ and $\psi(\delta) = \varphi(\alpha)$.

Now we turn to the remaining case $\varphi_1(\alpha) \neq 1 + \delta$. Then, for $k \in \mathbb{N}$, we choose numbers $d_k \in \mathbb{C} \setminus \{1\}$ such that $d_k^k = \frac{\varphi_1(\alpha)}{1+\delta}$ and $\operatorname{Arg}(d_k) \to 0$ when $k \to \infty$. Note that $d_k \to 1$.

Put

$$c_k := \frac{\varphi_2(\alpha) - \varphi_2(0)}{1 - d_k}, \quad k \in \mathbb{N}.$$

Since $|c_k| \to \infty$ we choose a k_0 such that $|c_{k_0}| > M := \sup\{|\varphi_1(\lambda)| : |\lambda| \leq \frac{\alpha}{\delta}\}$.

Define

$$\psi(\lambda) := ((1 + \lambda)h^{k_0}(\lambda), \varphi_2(\tfrac{\alpha}{\delta}\lambda)), \quad \lambda \in \mathbb{D},$$

where

$$h(\lambda) := \frac{\varphi_2(\frac{\alpha}{\delta}\lambda) - c_{k_0}}{\varphi_2(0) - c_{k_0}}, \quad \lambda \in \mathbb{D}.$$

Then $h \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ and so $\psi \in \mathcal{O}(\mathbb{D}, D_1 \times D_2)$ with $\psi(0) = (1, \varphi_2(0)) = \varphi(0)$. Moreover, a short calculation leads to $\psi(\delta) = \varphi(\alpha)$.

If $\psi(\lambda') = \psi(\lambda'')$, then $h(\lambda') = h(\lambda'')$, and therefore, $\lambda' = \lambda''$, i.e. ψ is also injective. Hence, the proof of (b) is complete.

(c) Recall that the universal covering of D_j is $\mathbb{D}$ and that the covering mapping $p_j\colon \mathbb{D} \to D_j$ is locally biholomorphic and surjective, but both are not injective, $j = 1, 2$. Applying Lemma 4.3.7 we find a point $q = (q_1, q_2) \in \mathbb{D}^2$, $q_1 \neq q_2$, and automorphisms $f_j \in \operatorname{Aut}(\mathbb{D})$, $j = 1, 2$, such that with $\tilde{p}_j := p_j \circ f_j$, $j = 1, 2$, the following is true:

$$\tilde{p}_j(q_1) = \tilde{p}_j(q_2), \quad j = 1, 2, \quad \text{and} \quad \det \begin{bmatrix} \tilde{p}_1'(q_1) & \tilde{p}_1'(q_2) \\ \tilde{p}_2'(q_1) & \tilde{p}_2'(q_2) \end{bmatrix} \neq 0.$$

Moreover, choose an $r \in (0, 1)$ such that both mappings $\tilde{p}_j$ are injective on $\overline{K(r)}$ and put $a := (a_1, a_2) = (\tilde{p}_1(0), \tilde{p}_2(0))$, $b := (b_1, b_2) = (\tilde{p}_1(r), \tilde{p}_2(r)) \in D_1 \times D_2$. Note that $a_j \neq b_j$, $j = 1, 2$.

Then, in virtue of Proposition 4.2.35, Proposition 4.2.38, and the choice of r, we have

$$\tilde{\boldsymbol{k}}^*_{D_1\times D_2}(a, b) = \max\{\tilde{\boldsymbol{k}}^*_{D_1}(a_1, b_1), \tilde{\boldsymbol{k}}^*_{D_2}(a_2, b_2)\} = r.$$

Assume now that $\tilde{\boldsymbol{k}}^*_{D_1\times D_2} = H^*_{D_1\times D_2}$; in particular, $r = \tilde{\boldsymbol{k}}^*_{D_1\times D_2}(a, b) = H^*_{D_1\times D_2}(a, b)$. Then there exist a sequence of analytic discs $(\varphi_j)_j \subset \mathcal{O}(\mathbb{D}, D_1 \times D_2)$ and a sequence of numbers $(\alpha_j)_j \subset (1, 1/\sqrt{r})$ with $\alpha_j \searrow 1$ such that $\varphi_j(0) = a$ and $\varphi_j(\alpha_j r) = b$ for all j.

Then, applying Exercise 4.3.5, we find $\psi_j = (\psi_{j,1}, \psi_{j,2}) \in \mathcal{O}(\mathbb{D}, D_1 \times D_2)$ injective such that $\psi_j(0) = a$ and $\psi_j(\alpha_j^2 r) = b$, $j \in \mathbb{N}$.

Recall that $\tilde{p}_j$ are covering mappings. Therefore, we can lift the functions $\psi_{j,k}$, $k = 1, 2$, i.e. there are holomorphic mappings $\tilde{\psi}_{j,k} \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ such that $\tilde{p}_k \circ \tilde{\psi}_{j,k} = \psi_{j,k}$ and $\tilde{\psi}_{j,k}(0) = 0$. Note that $(\tilde{p}_k \circ \tilde{\psi}_{j,k})(\alpha_j^2 r) = \tilde{p}_k(r)$. Recall that $\tilde{p}_k$ is injective on $\overline{K}(r)$ and therefore injective on $K(r+\varepsilon)$, where $\varepsilon \in (0, 1-r)$ is sufficiently small (Exercise). Then, for large j, we have that $\tilde{\psi}_{j,k}(\alpha_j^2 r) = r$, $k = 1, 2$.

By the Montel theorem we may assume that $\tilde{\psi}_{j,k} \to \tilde{\psi}_k \in \mathcal{O}(\mathbb{D}, \overline{\mathbb{D}})$ locally uniformly, $k = 1, 2$. Since $\tilde{\psi}(0) = 0$ it follows that, in fact, $\tilde{\psi} \in \mathcal{O}(\mathbb{D}, \mathbb{D}^2)$. Moreover, because of the previous remark, $\tilde{\psi}_k(r) = r$, $k = 1, 2$. Then, by the Schwarz lemma, we have $\tilde{\psi}_k = \mathrm{id}_{\mathbb{D}}$, $k = 1, 2$.

Put $g = (g_1, g_2)\colon \mathbb{D}^2 \to \mathbb{C}^2$, $g_k(\lambda, \mu) := \tilde{p}_k(\lambda) - \tilde{p}_k(\mu)$. Note that $g(q) = 0$ with $q := (q_1, q_2)$ and $\det g'(q) \neq 0$. Hence we find neighborhoods $U = K(q_1, s) \times K(q_2, s) \subset \mathbb{D}^2$ of q and $V = V(0) \subset \mathbb{C}^2$ such that g maps U biholomorphically to V and $K(q_1, s) \cap K(q_2, s) = \varnothing$.[12]

Let now $g_j\colon \mathbb{D}^2 \to \mathbb{C}^2$, $j \in \mathbb{N}$,

$$g_j(\lambda, \mu) := (\psi_{j,1}(\lambda) - \psi_{j,1}(\mu), \psi_{j,2}(\lambda) - \psi_{j,2}(\mu)) \quad (\lambda, \mu) \in \mathbb{D}^2.$$

By the result before we conclude that $g_j \to g$ uniformly on U. Then, in virtue of Theorem 1.7.28, there exists a large index j_0 such that g_{j_0} vanishes in at least one

[12] Recall that $q_1 \neq q_2$.

point $(t_1, t_2) \in U$, i.e. $\psi_{j_0}(t_1) = \psi_{j_0}(t_2)$, which contradicts the injectivity of ψ_{j_0}. $\square$

4.4 Examples I – elementary Reinhardt domains

In this section we will establish effective formulas for $d_{\boldsymbol{D}_\alpha}$, $\alpha \in \mathbb{R}^n_*$, where

$$d_{\boldsymbol{D}_\alpha} \in \{\boldsymbol{m}_{\boldsymbol{D}_\alpha}, m^{(k)}_{\boldsymbol{D}_\alpha}, \boldsymbol{g}_{\boldsymbol{D}_\alpha}, \widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}, \boldsymbol{k}^*_{\boldsymbol{D}_\alpha}\}$$

and

$$\boldsymbol{D}_\alpha = \{z \in \mathbb{C}^n(\alpha) : |z|^\alpha < 1\},$$

is an elementary Reinhardt domain. Note that $\boldsymbol{D}_{\alpha,c} := \{z \in \mathbb{C}^n(\alpha) : |z|^\alpha < e^c\}$ and $\boldsymbol{D}_\alpha$ are biholomorphically equivalent; so it suffices to study only $\boldsymbol{D}_\alpha$. Generalizing Definition 3.3.1 we say that $\boldsymbol{D}_\alpha$ is of *rational type* if $\alpha \in \mathbb{R} \cdot \mathbb{Z}^n$ and of *irrational type* if it is not of rational type.

Note that if $n = 1$ and $\alpha > 0$, then $\boldsymbol{D}_\alpha = \mathbb{D}$ and all the invariant functions coincide with $\boldsymbol{m}$ on $\mathbb{D} \times \mathbb{D}$. If $n = 1$ and $\alpha < 0$, then $\boldsymbol{D}_\alpha = \mathbb{C} \setminus \overline{\mathbb{D}}$. Thus $\boldsymbol{D}_\alpha$ is biholomorphically equivalent to $\mathbb{D}_*$. Then it is easy to prove that $\boldsymbol{m} = \boldsymbol{m}_{\mathbb{D}_*} = \boldsymbol{g}_{\mathbb{D}_*}$ on $\mathbb{D}_* \times \mathbb{D}_*$ (Exercise). Moreover, applying Proposition 4.2.38, we are able, at least in principle, to calculate $\widetilde{\boldsymbol{k}}^*_{\mathbb{D}_*}$. Firstly, we give the formula for the Lempert function even for an arbitrary annulus.

Theorem 4.4.1. *For $R > 1$ put $\mathbb{A} = \mathbb{A}(1/R, R)$. If $a \in (1/R, R)$, then*

$$\widetilde{\boldsymbol{k}}^*_{\mathbb{A}}(a, z) = \left(\frac{x^2 + 1 - 2x\cos(\pi(s-t))}{x^2 + 1 - 2x\cos(\pi(s+t))}\right)^{1/2}, \qquad \begin{array}{l} z = |z|e^{i\theta(z)} \in \mathbb{A}, \\ -\pi < \theta(z) \le \pi, \end{array}$$

where $a = R^{1-2s}$, $|z| = R^{1-2t}$, and $x = \exp\left(\frac{\pi\theta(z)}{2\log R}\right)$.

Proof. Note that

$$\mathbb{A} \ni \lambda \to \lambda/a \in Q := \{w \in \mathbb{C} : r_1 < |w| < r_2\},$$

where $r_1 := (Ra)^{-1}$ and $r_2 := R/a$, is a biholomorphic mapping. Therefore, $\widetilde{\boldsymbol{k}}^*_{\mathbb{A}}(a, z) = \widetilde{\boldsymbol{k}}^*_Q(1, z/a)$, $z \in \mathbb{A}$.

Put $S := \{z \in \mathbb{C} : \log r_1 < \operatorname{Re} z < \log r_2\}$. Note that $1 \in Q$ and that $\exp|_S : S \to Q$ is a holomorphic covering. Moreover, observe that

$$S \ni w \mapsto \frac{e^{i\frac{(w-\log r_1)\pi}{\log(r_2/r_1)}} - i}{e^{i\frac{(w-\log r_1)\pi}{\log(r_2/r_1)}} + i} = \frac{e^{\alpha w} - e^{i\beta}}{e^{\alpha w} + e^{i\beta}} =: \widetilde{H}(w) \in \mathbb{D},$$

where

$$\alpha := \frac{i\pi}{\log(r_2/r_1)} \quad \text{and} \quad \beta := \pi\left(\frac{1}{2} + \frac{\log r_1}{\log(r_2/r_1)}\right),$$

gives a biholomorphic mapping $\widetilde{H}\colon S \to \mathbb{D}$. Note that $\widetilde{H}(0) = \frac{1-e^{i\beta}}{1+e^{i\beta}}$. Then, after a suitable Möbius transformation, we get the following biholomorphic mapping $H\colon S \to \mathbb{D}$,

$$H(w) = \frac{e^{\alpha w} - 1}{e^{\alpha w} - \lambda_0}, \quad w \in S,$$

with $H(0) = 0$, where $\lambda_0 := e^{i\frac{2\pi \log r_1}{\log(r_2/r_1)}}$. Hence, $h := \exp|_S \circ H^{-1}\colon \mathbb{D} \to Q$ is a holomorphic covering with $h(0) = 1$ (Exercise). Consequently, by Proposition 4.2.38, we get

$$\begin{aligned}\widetilde{\boldsymbol{k}}^*_Q(1,\zeta) &= \inf\{|\lambda| : \lambda \in h^{-1}(\zeta)\} = \inf\{|H(w)| : w \in S,\ \exp(w) = \zeta\}\\ &= \inf\{|H(\log|\zeta| + i(\theta + 2\pi k))| : k \in \mathbb{Z}\},\end{aligned}$$

where $\zeta = |\zeta|e^{i\theta}$ with $-\pi < \theta \le \pi$. Calculating the last term leads to the function

$$f(t) := \left|\frac{e^t e^{i\varphi} - 1}{e^t e^{i\varphi} - e^{i\psi}}\right|^2, \quad t \in \mathbb{R},$$

where

$$\varphi := \pi\frac{\log|\zeta|}{\log(r_2/r_1)} \quad \text{and} \quad \psi := 2\pi\frac{\log r_1}{\log(r_2/r_1)}.$$

Note that $f(t) = f(-t)$. A simple calculation shows that $f'|_{(0,\infty)} > 0$, i.e. $f|_{\mathbb{R}_+}$ is strictly monotonically increasing (Exercise).

Note that t in f corresponds to $-\frac{\pi}{\log(r_2/r_1)}(\theta + 2k\pi)$, $k \in \mathbb{Z}$. Therefore, we have the following possibilities:

(a) if $\theta = 0$, then $\widetilde{\boldsymbol{k}}^*_Q(1,\zeta) = |H(\log|\zeta| + i0)|$;

(b) if $\theta \in (0,\pi]$, then

$$\widetilde{\boldsymbol{k}}^*_Q(1,\zeta) = \min\{|H(\log|\zeta| + i\theta)|,\ |H(\log|\zeta| + i(\theta - 2\pi))|\};$$

(c) if $\theta \in (-\pi, 0)$, then

$$\widetilde{\boldsymbol{k}}^*_Q(1,\zeta) = \min\{|H(\log|\zeta| + i\theta)|,\ |H(\log|\zeta| + i(\theta + 2\pi))|\}.$$

Observe that $f(t) > f(t+x)$ iff $(1 - e^x)(e^{2t+x} - 1) > 0$. In (b) (resp. in (c)) we have $t = -\frac{\pi}{\log(r_2/r_1)}\theta < 0$, $x = \frac{\pi}{\log(r_2/r_1)}2\pi$, and so $2t + x \ge 0$ (resp. $t = -\frac{\pi}{\log(r_2/r_1)}\theta > 0$, $x = -\frac{\pi}{\log(r_2/r_1)}2\pi$, and so $2t + x < 0$). We get $\widetilde{\boldsymbol{k}}^*_Q(1,\zeta) = |H(\log|\zeta| + i\theta)|$. Hence,

$$\widetilde{\boldsymbol{k}}^*_{\mathbb{A}}(a,z) = \left|H\left(\log\left|\frac{z}{a}\right| + i\operatorname{Arg}\left(\frac{z}{a}\right)\right)\right|,$$

where the argument is chosen in $(-\pi,\pi]$. What remains is to evaluate the right-hand side which is left as an Exercise for the reader. $\square$

Corollary 4.4.2. *For any $a \in (0,1)$ we have*

$$\tilde{\boldsymbol{k}}^*_{\mathbb{D}_*}(a,z) = \left(\frac{\theta^2(z) + (\log|z| - \log a)^2}{\theta^2(z) + (\log|z| + \log a)^2}\right)^{1/2},$$

whenever $z = |z|e^{i\theta(z)} \in \mathbb{D}_*$ and $-\pi < \theta(z) \le \pi$.

Proof. Use either a covering argument as in the proof of the former theorem or Lemma 4.2.30 (Exercise). □

As an immediate consequence of Corollary 4.4.2 we get the following identities.

Corollary 4.4.3. (a) *If $a \in (0,1)$ and $k \in \mathbb{N}$, then*

$$\tilde{\boldsymbol{k}}^*_{\mathbb{D}_*}(a^k, z^k) = \min\{\tilde{\boldsymbol{k}}^*_{\mathbb{D}_*}(a, ze^{\frac{2\ell\pi}{k}i}) : 0 \le \ell \le k-1\}, \quad z \in \mathbb{D}_*.$$

(b) *If $a, b \in (0,1)$ and $t \in \mathbb{R}$, then*

$$\tilde{\boldsymbol{k}}^*_{\mathbb{D}_*}(a^t, b^t) = \tilde{\boldsymbol{k}}^*_{\mathbb{D}_*}(a,b).$$

Proof. The proof is left as an Exercise. □

If $D = \boldsymbol{D}_\alpha \times \mathbb{C}^k$, then (Exercise)

$$d_D((a',b'),(a'',b'')) = d_{\boldsymbol{D}_\alpha}(a',a''), \ a',a'' \in \boldsymbol{D}_\alpha, \quad b',b'' \in \mathbb{C}^k.$$

Hence for the remaining part of this section we will always assume that:

- $n \ge 2$;
- $\alpha_1,\dots,\alpha_s < 0$ and $\alpha_{s+1},\dots,\alpha_n > 0$ for an $s = s(\alpha) \in \{0,1,\dots,n\}$;
- if $s < n$, then $t = t(\alpha) := \min\{\alpha_{s+1},\dots,\alpha_n\}$;
- $a = (a_1,\dots,a_n) \in \boldsymbol{D}_\alpha$, $a_1 \cdots a_k \ne 0$, $a_{k+1} = \cdots = a_n = 0$ for a $k = k(a) \in \{s,\dots,n\}$;
- if $k < n$, then $r = r(a) = r_\alpha(a) := \alpha_{k+1} + \cdots + \alpha_n$; if $k = n$ (in particular, if $s = n$), then $r = r(a) = r_\alpha(a) := 1$; observe that if $\alpha \in \mathbb{Z}^n$, then $r(a) = \operatorname{ord}_a(z^\alpha - a^\alpha)$;
- if $\boldsymbol{D}_\alpha$ is of rational type, then $\alpha \in \mathbb{Z}^n$ and $\alpha_1,\dots,\alpha_n$ relatively prime;
- if $\boldsymbol{D}_\alpha$ is of irrational type and $s < n$, then $t(\alpha) = 1$.

We are able to describe effectively some holomorphically contractible functions of $\boldsymbol{D}_\alpha$ – the following formulas are known and will be discussed in the sequel.

Theorem 4.4.4. *Under the above assumptions we have:*

α	$\boldsymbol{m}^{(\ell)}_{\boldsymbol{D}_\alpha}(a,z)$	$\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,z)$
Rational type	$(\boldsymbol{m}(a^\alpha, z^\alpha))^{\frac{1}{\ell}\lceil\frac{\ell}{r}\rceil}$	$(\boldsymbol{m}(a^\alpha, z^\alpha))^{1/r}$
Irrational type, $k < n$	0	$\lvert z^\alpha\rvert^{1/r}$
Irrational type, $k = n$	0	0

α	$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z)$	$\boldsymbol{k}^*_{\boldsymbol{D}_\alpha}(a,z)$
Rational, $s<n$	$\begin{cases} \min\limits_{\substack{a^\alpha=\zeta_1^t\\ z^\alpha=\zeta_2^t}} \{\boldsymbol{m}(\zeta_1,\zeta_2)\}, & k=n,\ z\notin V_0 \\ \lvert z^\alpha\rvert^{1/r}, & k<n \end{cases}$	$\min\limits_{\substack{a^\alpha=\zeta_1^t\\ z^\alpha=\zeta_2^t}} \{\boldsymbol{m}(\zeta_1,\zeta_2)\}$
Rational, $s=n$	$\boldsymbol{k}^*_{\mathbb{D}_*}(a^\alpha,z^\alpha)$	$\boldsymbol{k}^*_{\mathbb{D}_*}(a^\alpha,z^\alpha)$
Irrational, $s<n$	$\begin{cases} \boldsymbol{m}(\lvert a^\alpha\rvert,\lvert z^\alpha\rvert), & k=n,\ z\notin V_0 \\ \lvert z^\alpha\rvert^{1/r}, & k<n \end{cases}$	$\boldsymbol{m}(\lvert a^\alpha\rvert,\lvert z^\alpha\rvert)$
Irrational, $s=n$	$\boldsymbol{k}^*_{\mathbb{D}_*}(\lvert a^\alpha\rvert,\lvert z^\alpha\rvert)$	$\boldsymbol{k}^*_{\mathbb{D}_*}(\lvert a^\alpha\rvert,\lvert z^\alpha\rvert)$

Note that, in fact, the above formulas cover all possible cases.

Before we start to prove this theorem we present some applications.

Remark 4.4.5. (a) If $\boldsymbol{D}_\alpha \subset \mathbb{C}^n$ (resp. $\boldsymbol{D}_\beta \subset \mathbb{C}^n$) is an elementary Reinhardt domain of rational (resp. irrational) type, then these domains are not biholomorphically equivalent (Exercise).

(b) If $\boldsymbol{D}_\alpha \subset \mathbb{C}^n$, $n \geq 2$, is an elementary Reinhardt domain of rational type with $0 \in \boldsymbol{D}_\alpha$ (i.e. $s(\alpha)=0$) and if $\ell \geq 2$, then

$$\boldsymbol{m}^{(\ell)}_{\boldsymbol{D}_\alpha}(0,z) = (\boldsymbol{m}(0,z^\alpha))^{\frac{1}{\ell}\lceil\frac{\ell}{|\alpha|}\rceil} = |z^\alpha|^{\frac{1}{\ell}\lceil\frac{\ell}{|\alpha|}\rceil}, \quad z \in \boldsymbol{D}_\alpha \setminus V_0.$$

Observe that $\frac{1}{\ell}\lceil\frac{\ell}{|\alpha|}\rceil < 1$. Therefore,

$$\boldsymbol{m}^{(\ell)}_{\boldsymbol{D}_\alpha}(z,0) = \boldsymbol{m}(z^\alpha,0) = |z^\alpha| < \boldsymbol{m}^{(\ell)}_{\boldsymbol{D}_\alpha}(0,z), \quad z \in \boldsymbol{D}_\alpha \setminus V_0.$$

Hence we conclude that, in general, the function $\boldsymbol{m}^{(\ell)}_D$ is not symmetric.

(c) Let $\boldsymbol{D}_\alpha$ be as in (b). Fix a $b \in \boldsymbol{D}_\alpha \setminus V_0$ and a sequence $(a_j)_j \subset \boldsymbol{D}_\alpha \setminus V_0$ which converges to 0. Then

$$\boldsymbol{m}^{(\ell)}_{\boldsymbol{D}_\alpha}(a_j,b) = \boldsymbol{m}(a_j^\alpha,b^\alpha) \to \boldsymbol{m}(0,b^\alpha) = |b^\alpha| < \boldsymbol{m}^{(\ell)}_{\boldsymbol{D}_\alpha}(0,b).$$

Hence, the function $\boldsymbol{m}^{(\ell)}_{\boldsymbol{D}_\alpha}(\cdot,b)$ is not continuous, $\ell \geq 2$.

(d) Let $\boldsymbol{D}_\alpha \subset \mathbb{C}^n$ be as in (b). Then

$$\boldsymbol{g}_{\boldsymbol{D}_\alpha}(0,b) = (\boldsymbol{m}(0,b^\alpha))^{1/|\alpha|} > |b^\alpha| = \boldsymbol{m}(b,0) = \boldsymbol{g}_{\boldsymbol{D}_\alpha}(b,0), \quad b \in \boldsymbol{D}_\alpha \setminus V_0.$$

This shows that, in general, the pluricomplex Green function is not symmetric.

(e) Let $\boldsymbol{D}_\alpha$ be as in (b). Fix a $b \in \boldsymbol{D}_\alpha \setminus V_0$ and a sequence $(a_j)_j \in \boldsymbol{D}_\alpha \setminus V_0$ with $a_j \to 0$. Then

$$\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a_j,b) = \boldsymbol{m}(a_j^\alpha,b^\alpha) \to \boldsymbol{m}(0,b^\alpha) < (\boldsymbol{m}(0,b^\alpha))^{1/|\alpha|} = \boldsymbol{g}_{\boldsymbol{D}_\alpha}(0,b).$$

Therefore, in general, $\boldsymbol{g}_D(\cdot, b)$ need not be continuous. Recall that also $\boldsymbol{g}_D(a, \cdot)$ is not necessarily continuous.

Using the formulas for the pluricomplex Green function for elementary Reinhardt domains, we get the following result (see Lemma 2.4.12).

Lemma 4.4.6. *Let $\boldsymbol{D}_\alpha$, $\boldsymbol{D}_\beta$ be elementary Reinhardt domains in $\mathbb{C}^n$, $n \geq 2$, of rational type with $s(\alpha) = s(\beta) = 0$, i.e. $0 \in \boldsymbol{D}_\alpha \cap \boldsymbol{D}_\beta$, and let $F \in \mathcal{O}(\boldsymbol{D}_\alpha, \boldsymbol{D}_\beta)$. Then there exists a $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with*

$$F(\boldsymbol{V}(\boldsymbol{D}_\alpha, \lambda)) \subset \boldsymbol{V}(\boldsymbol{D}_\beta, \varphi(\lambda)), \quad \lambda \in \mathbb{D},$$

where $\boldsymbol{V}(\boldsymbol{D}_\alpha, \lambda) := \{z \in \boldsymbol{D}_\alpha : z^\alpha = \lambda\}$.

Proof. Fix a $\lambda \in \mathbb{D}$. Put

$$a = a(\lambda) := (1, \dots, 1, \lambda^{1/\alpha_n}),$$

where λ^{1/α_n} is a certain root of λ. Obviously, $a \in \boldsymbol{V}(\boldsymbol{D}_\alpha, \lambda)$ and $k = k(a) = n$. Set

$$\varphi(\lambda) := F_1^{\beta_1}(a) \cdots F_n^{\beta_n}(a) = F^\beta(a) \in \mathbb{D}.$$

Then $F(a) \in \boldsymbol{V}(\boldsymbol{D}_\beta, \varphi(\lambda))$.

Now take another point z in $\boldsymbol{V}(\boldsymbol{D}_\alpha, \lambda)$. Then

$$\begin{aligned} 0 = \left| \frac{a^\alpha - z^\alpha}{1 - \overline{a^\alpha} z^\alpha} \right|^{1/r_\alpha(a)} &= \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, z) \\ &\geq \boldsymbol{g}_{\boldsymbol{D}_\beta}(F(a), F(z)) = \left| \frac{F^\beta(a) - F^\beta(z)}{1 - \overline{F^\beta(a)} F^\beta(z)} \right|^{1/r_\beta(F(a))}. \end{aligned}$$

Therefore, $F(z) \in \boldsymbol{V}(\boldsymbol{D}_\beta, \varphi(\lambda))$.

Note that by taking locally a holomorphic root λ^{1/α_n} in $\mathbb{D}_*$ it follows directly from its definition that φ is holomorphic in $\mathbb{D}_*$. Hence it extends holomorphically to the whole of $\mathbb{D}$ and this extension coincides with $\varphi(0) = \lim_{t \to 0+} F^\beta(a(t)) = \lim_{t \to 0+} \varphi(t)$. So $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$. $\square$

Corollary 4.4.7. *Let $\boldsymbol{D}_\alpha$ and $\boldsymbol{D}_\beta$ be as in the previous lemma. Assume now that $F \in \mathcal{O}(\boldsymbol{D}_\alpha, \boldsymbol{D}_\beta)$ is even biholomorphic. Then there is a $\varphi \in \operatorname{Aut}(\mathbb{D})$ with*

$$F(\boldsymbol{V}(\boldsymbol{D}_\alpha, \lambda)) = \boldsymbol{V}(\boldsymbol{D}_\beta, \varphi(\lambda)), \quad \lambda \in \mathbb{D}.$$

To show how to use invariant functions like the pluricomplex Green function we are going to prove the following result [Edi-Zwo 1999] (see also Section 2.3).

Theorem 4.4.8. *Two elementary Reinhardt domains $\boldsymbol{D}_\alpha, \boldsymbol{D}_\beta \subset \mathbb{C}^n$, $s(\alpha) = s(\beta) = 0$, are biholomorphically equivalent if and only if there exist a permutation σ of $\{1, \dots, n\}$ and a $t > 0$ such that $\alpha_j = t\beta_{\sigma(j)}$, $j = 1, \dots, n$.*

Proof. From the very beginning we may assume that $n \geq 2$ and that both domains are either of rational or of irrational type.

In the first case we know by Lemma 4.4.6 that $F(\boldsymbol{V}_0) = \boldsymbol{V}(\boldsymbol{D}_\beta, \mu)$ [13] for a certain $\mu \in \mathbb{D}$. Observe that if $\mu \in \mathbb{D}_*$, then the latter set is an analytic set with only regular points, but $\boldsymbol{V}_0$ has 0 as an irregular point. Therefore, $\mu = 0$, or $F(\boldsymbol{V}_0) = \boldsymbol{V}_0$.

Now let both domains be of irrational type. Suppose that $F(\boldsymbol{V}_0) \not\subset \boldsymbol{V}_0$, i.e. there is an $a \in \boldsymbol{V}_0$ with $F(a) \notin \boldsymbol{V}_0$. By Theorem 4.4.4 we conclude that

$$0 \not\equiv \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, \cdot) = \boldsymbol{g}_{\boldsymbol{D}_\beta}(F(a), \cdot) \equiv 0;$$

a contradiction. So, $F(\boldsymbol{V}_0) = \boldsymbol{V}_0$ also in this case.

Hence, there is a permutation σ of $\{1, \dots, n\}$ such that $F(\boldsymbol{V}_j) = \boldsymbol{V}_{\sigma(j)}$, $j = 1, \dots, n$. In particular, $F(0) = 0$. Applying the formulas for the pluricomplex Green function it follows that

$$\begin{aligned}\bigl(|z_1|^{\alpha_1} \cdots |z_n|^{\alpha_n}\bigr)^{1/(\alpha_1+\cdots+\alpha_n)} &= \boldsymbol{g}_{\boldsymbol{D}_\alpha}(0, z) = \boldsymbol{g}_{\boldsymbol{D}_\beta}(0, F(z)) \\ &= \bigl(|F_1(z)|^{\beta_1} \cdots |F_n(z)|^{\beta_n}\bigr)^{1/(\beta_1+\cdots+\beta_n)}, \quad z \in \boldsymbol{D}_\alpha.\end{aligned}$$

Moreover, for the points $d_j = (1, \dots, 1, 0, 1, \dots, 1) \in \boldsymbol{V}_j$ we have that $F(d_j)$ is contained in $\boldsymbol{V}_{\sigma(j)} \setminus \bigcup_{\ell=1,\, \ell\neq\sigma(j)}^{n} \boldsymbol{V}_\ell$. Therefore,

$$|z^\alpha|^{1/\alpha_j} = \boldsymbol{g}_{\boldsymbol{D}_\alpha}(d_j, z) = \boldsymbol{g}_{\boldsymbol{D}_\beta}(F(d_j), F(z)) = |F^\beta(z)|^{1/\beta_{\sigma(j)}}, \quad z \in \boldsymbol{D}_\alpha.$$

Combining the last equalities gives

$$\frac{\alpha_1 + \cdots + \alpha_n}{\beta_1 + \cdots + \beta_n} = \frac{\alpha_j}{\beta_{\sigma(j)}}, \quad j = 1, \dots, n,$$

which finishes the proof. □

Now we come back to prove almost all of the formulas stated in Theorem 4.4.4.

Proof of Theorem 4.4.4. The proof will be given in several steps.

Proof for $\boldsymbol{m}_{\boldsymbol{D}_\alpha}^{(\ell)}$ – the rational case. Define

$$f(w) := \left(\frac{w^\alpha - a^\alpha}{1 - \bar{a}^\alpha w^\alpha}\right)^{\lceil \frac{\ell}{r} \rceil}, \quad w \in \boldsymbol{D}_\alpha.$$

Then $f \in \mathcal{O}(\boldsymbol{D}_\alpha, \mathbb{D})$ with $\operatorname{ord}_a f = r\lceil \frac{\ell}{r} \rceil \geq \ell$. Hence $\boldsymbol{m}_{\boldsymbol{D}_\alpha}^{(\ell)}(a, z) \geq |f(z)|^{1/\ell}$, which implies that $\boldsymbol{m}_{\boldsymbol{D}_\alpha}^{(\ell)}(a, z) \geq (\boldsymbol{m}(a^\alpha, z^\alpha))^{\frac{1}{\ell}\lceil \frac{\ell}{r} \rceil}$.

[13] Note that $\boldsymbol{V}_0 \subset \boldsymbol{D}_\alpha \cap \boldsymbol{D}_\beta$.

Now let $f \in \mathcal{O}(\boldsymbol{D}_\alpha, \mathbb{D})$ with $\operatorname{ord}_s f \geq \ell$. Put $\Phi(w) := w^\alpha$, $w \in \boldsymbol{D}_\alpha$. Applying Theorem 3.2.1 (e), we get $f = \varphi \circ \Phi$, where $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $\operatorname{ord}_a f = r \operatorname{ord}_{a^\alpha} \varphi$. Hence, $\operatorname{ord}_{a^\alpha} \varphi \geq \lceil \frac{\ell}{r} \rceil$ and therefore, $|f(z)|^{1/\ell} \leq (\boldsymbol{m}(a^\alpha, z^\alpha))^{\frac{1}{\ell}\lceil \frac{\ell}{r} \rceil}$, which proves the remaining inequality. $\square$

Proof for $\boldsymbol{m}^{(\ell)}_{\boldsymbol{D}_\alpha}$ *– the irrational case.* According to Theorem 3.2.1 (d) we know that $\mathcal{H}^\infty(\boldsymbol{D}_\alpha) \simeq \mathbb{C}$, which immediately proves the claimed formula. $\square$

Proof for $\boldsymbol{g}_{\boldsymbol{D}_\alpha}$ *– the rational case.* Let $\beta := (|\alpha_1|, \dots, |\alpha_s|, \alpha_{s+1}, \dots, \alpha_n)$. Observe that $\boldsymbol{D}_\alpha$ and $\boldsymbol{D}^0_\beta := \boldsymbol{D}_\beta \cap \mathbb{C}^n_*$ are biholomorphically equivalent via the following mapping

$$F : \mathbb{C}^n(\alpha) \to \mathbb{C}^n(\alpha), \quad z \mapsto (z_1^{-1}, \dots, z_s^{-1}, z_{s+1}, \dots, z_n).$$

Hence $\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, z) = \boldsymbol{g}_{\boldsymbol{D}^0_\beta}(F(a), F(z))$, $z \in \boldsymbol{D}_\alpha$. In virtue of Proposition 1.14.25, we even know that $\boldsymbol{g}_{\boldsymbol{D}^0_\beta}(F(a), \cdot) = \boldsymbol{g}_{\boldsymbol{D}_\beta}(F(a), \cdot)$ (EXERCISE).[14] Therefore, from now on we assume that $s = 0$.

Using the equation for $\boldsymbol{m}_{\boldsymbol{D}_\alpha}$ from above, we know that

$$\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, z) \geq \boldsymbol{m}_{\boldsymbol{D}_\alpha}(a, z) = \big(\boldsymbol{m}(a^\alpha, z^\alpha)\big)^{1/r}.$$

To get the converse inequality let $u : \boldsymbol{D}_\alpha \to [0, 1)$ be a log-psh function satisfying $u(z) \leq C\|z - a\|$, $z \in \boldsymbol{D}_\alpha$; in particular, $u(a) = 0$.

Fix a $\mu \in \mathbb{D}_*$ with $\mu^{\alpha_n} = \lambda$. Then

$$\psi : \mathbb{C}^{n-1}_* \to \boldsymbol{D}_\alpha, \quad \psi(w_1, \dots, w_{n-1}) := \Big(w_1^{\alpha_n}, \dots, w_{n-1}^{\alpha_n}, \frac{\mu}{w_1^{\alpha_1} \cdots w_{n-1}^{\alpha_{n-1}}}\Big),$$

is a holomorphic mapping onto $\boldsymbol{V}(\boldsymbol{D}_\alpha, \lambda)$ (EXERCISE[15]). So $u \circ \psi \in \mathcal{PSH}(\mathbb{C}^{n-1}_*)$ is bounded. By Proposition 1.14.25 we conclude that this function is, in fact, psh on the whole of $\mathbb{C}^{n-1}$. Moreover, it is bounded from above. Therefore, the Liouville type theorem for psh functions gives that $u \circ \psi \equiv: v(\lambda) \in \mathbb{D}_*$. So we have constructed a function $v : \mathbb{D}_* \to \mathbb{D}_*$ such that

$$u|_{\boldsymbol{V}(\boldsymbol{D}_\alpha, \lambda)} \equiv v(\lambda), \quad \lambda \in \mathbb{D}_*.$$

v is sh on $\mathbb{D}_*$. Indeed, fix $\lambda_0 \in \mathbb{D}_*$ and $\rho > 0$ with $K(\lambda_0, \rho) \subset \mathbb{D}_*$. Choose a holomorphic α_n-th root of $\operatorname{id}_{K(\lambda_0, \rho)}$, i.e. a $g \in \mathcal{O}(K(\lambda_0, \rho), \mathbb{D})$ with $g^{\alpha_n}(\lambda) = \lambda$, $\lambda \in K(\lambda_0, \rho)$. Then

$$v(\lambda) = u(1, \dots, 1, g(\lambda)), \quad \lambda \in K(\lambda_0, \rho).$$

[14] Note that $\boldsymbol{D}_\beta \setminus \boldsymbol{V}_0 = D^0_\beta$.

[15] Use the assumption that $\alpha_1, \dots, \alpha_n$ are relatively prime and therefore $\mathbb{Z} = \alpha_1\mathbb{Z} + \dots + \alpha_n\mathbb{Z}$.

Hence, v is locally log-sh on $\mathbb{D}_*$ and therefore log-sh on $\mathbb{D}_*$. As above, v extends to a log-sh function $\hat{v}$ on $\mathbb{D}$.

First we discuss the case $k < n$, i.e. $a^\alpha = 0$. Recall that $r = \alpha_{k+1} + \dots + \alpha_n$. For $\lambda \in \mathbb{D}_*$ we have

$$\begin{aligned}\hat{v}(\lambda) = v(\lambda) &= u(a_1, \dots, a_k, \lambda^{1/r}, \dots, \lambda^{1/r}, \lambda^{1/r}(a_1^{\alpha_1} \cdots a_k^{\alpha_k})^{-1/\alpha_n}) \\ &\le C\|(\lambda^{1/r}, \dots, \lambda^{1/r}, \lambda^{1/r}(a_1^{\alpha_1} \cdots a_k^{\alpha_k})^{-1/\alpha_n})\| \le \widetilde{C}|\lambda|^{1/r},\end{aligned}$$

where $\widetilde{C}$ is a suitable number. Hence, $\hat{v}^r \le \boldsymbol{g}_{\mathbb{D}}(0, \cdot)$. In particular, if $z \in \boldsymbol{V}(\boldsymbol{D}_\alpha, \lambda)$, $\lambda \in \mathbb{D}_*$, then

$$u(z) = v(z^\alpha) \le (\boldsymbol{g}_{\mathbb{D}}(0, z^\alpha))^{1/r} = |z^\alpha|^{1/r}.$$

Therefore, $\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, z) \le |z^\alpha|^{1/r} = (\boldsymbol{m}(a^\alpha, z^\alpha))^{1/r}$ on $\boldsymbol{D}_\alpha \setminus \boldsymbol{V}_0$. If $z \in \boldsymbol{D}_\alpha \cap \boldsymbol{V}_0$, then the mean value inequality for psh functions leads to $\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, z) = 0 = (\boldsymbol{m}(a^\alpha, z^\alpha))^{1/r}$.

Now let $k = n$, i.e. $a^\alpha \neq 0$. Then we get for $\lambda \in \mathbb{D}_*$ near $\lambda_0 := a^\alpha$

$$\begin{aligned}\hat{v}(\lambda) &= u\big(a_1, \dots, a_{n-1}, \lambda^{1/\alpha_n}(a_1^{\alpha_1} \cdots a_{n-1}^{\alpha_{n-1}})^{-1/\alpha_n}\big) \\ &\le C\left|\frac{\lambda^{1/\alpha_n}}{a_1^{\alpha_1/\alpha_n} \cdots a_{n-1}^{\alpha_{n-1}/\alpha_n}} - a_n\right| \le \widetilde{C}|\lambda^{1/\alpha_n} - \lambda_0^{1/\alpha_n}| \le \widehat{C}|\lambda - \lambda_0|,\end{aligned}$$

which implies $\hat{v}(\lambda) \le \boldsymbol{g}_{\mathbb{D}}(a^\alpha, \lambda) = \boldsymbol{m}(\lambda_0, \lambda)$, $\lambda \in \mathbb{D}$. Hence, $u(z) = v(z^\alpha) \le \boldsymbol{m}(a^\alpha, z^\alpha)$, $z \in \boldsymbol{D}_\alpha \cap \mathbb{C}_*^n$. By the same reasoning as above it follows that the above inequality holds also on $\boldsymbol{D}_\alpha$ (Exercise). □

For further purpose we add the following observation.

Lemma 4.4.9. *If $s = 0$ and $k < n$, then $\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, z) \ge |z^\alpha|^{1/r}$, $z \in \boldsymbol{D}_\alpha$.*

Proof. Note that the function $u \colon \boldsymbol{D}_\alpha \to [0, 1)$, $u(z) := |z^\alpha|^{1/r}$, is log-psh. For $z \in \boldsymbol{D}_\alpha$, z near a, one has

$$\begin{aligned}u(z) &\le \big(|z_1|^{\alpha_1} \dots |z_k|^{\alpha_k}\big)^{1/r} \big(|z_{k+1} - 0|^{\alpha_{k+1}} \dots |z_n - 0|^{\alpha_n}\big)^{1/r} \\ &\le C\|(z_k, \dots, z_n) - (0, \dots, 0)\| \le C\|z - a\|.\end{aligned}$$

Hence, $u \le \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a, \cdot)$ on $\boldsymbol{D}_\alpha$. □

Before we will be able to discuss the pluricomplex Green function for the irrational type we have to find the formulas for the Lempert function.

Recall that $\mathbb{T}_a = \{(\zeta_1 a_1, \dots, \zeta_n a_n) : \zeta_j \in \mathbb{T},\ j = 1, \dots, n\}$, where $a \in \mathbb{C}_*^n$. Then $\mathbb{T}_a$ is a group with the following multiplication:

$$(\zeta_1 a_1, \dots, \zeta_n a_n) \circ (\eta_1 a_1, \dots, \eta_n a_n) := (\zeta_1 \eta_1 a_1, \dots, \zeta_n \eta_n a_n).$$

Let $\alpha \in \mathbb{R}_*^n$. Define $\mathbb{T}_a(\alpha)$ to be the subgroup of $\mathbb{T}_a$ that is generated by the set

$$\Big\{(e^{i\frac{\alpha_{j_1}}{\alpha_1}2k_1\pi}a_1, \dots, e^{i\frac{\alpha_{j_n}}{\alpha_n}2k_n\pi}a_n) : 1 \le j_1, \dots, j_n \le n,\ k_1, \dots, k_n \in \mathbb{Z}\Big\}.$$

Note that if α is of rational type, then

$$\mathbb{T}_a(\alpha) = \{(\varepsilon_1 a_1, \dots, \varepsilon_n a_n) : \varepsilon_j^{\alpha_j} = 1,\ 1 \le j \le n\},$$

i.e. $\mathbb{T}_a(\alpha)$ is finite. If α is of irrational type, then $\overline{\mathbb{T}_a(\alpha)} = \mathbb{T}_a$ (cf. p. 97).

To get some information on the Lempert function we need the following result on analytic discs.

Lemma 4.4.10. *Let $a, z \in \boldsymbol{D}_\alpha$, $z \in \mathbb{R}_*^n$, and $\tilde{z} \in \mathbb{T}_z(\alpha)$. Then for any $\varphi \in \mathcal{O}(\mathbb{D}, \boldsymbol{D}_\alpha)$ with $\varphi(\lambda_1) = a$, $\varphi(\lambda_2) = z$, $\lambda_1 \neq \lambda_2$, $\lambda_j \in \mathbb{D}$, $j = 1, 2$, there is a $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, \boldsymbol{D}_\alpha)$ such that $\tilde{\varphi}(\lambda_1) = a$ and $\tilde{\varphi}(\lambda_2) = \tilde{z}$.*

In particular, $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, z) = \widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, \tilde{z})$.

Proof. First recall that the strip $H := \{\lambda \in \mathbb{C} : -1 < \operatorname{Re}\lambda < 1\}$ is biholomorphically equivalent to $\mathbb{D}$ (use the Riemann mapping theorem). Therefore, it suffices to prove the lemma when we substitute $\mathbb{D}$ by H. So we may assume that $\varphi \in \mathcal{O}(H, \boldsymbol{D}_\alpha)$ with $\varphi(0) = a$ and $\varphi(i\tau) = z$ for a certain $\tau > 0$.

For $k_n \in \mathbb{Z}$ and $j \in \{1, \dots, n\}$, let $\tilde{\varphi} \colon H \to \boldsymbol{D}_\alpha$ be defined as

$$\tilde{\varphi}(\lambda) := \Big(\varphi_1(\lambda), \dots, \varphi_{n-2}(\lambda), e^{-2k_n\pi\frac{\lambda}{\tau}}\varphi_{n-1}(\lambda), e^{\frac{\alpha_j 2k_n\pi\lambda}{\alpha_n\tau}}\varphi_n(\lambda)\Big).$$

Then $\tilde{\varphi} \in \mathcal{O}(H, \boldsymbol{D}_\alpha)$, $\tilde{\varphi}(0) = a$, and $\tilde{\varphi}(i\tau) = (z_1, \dots, z_{n-1}, e^{i\frac{\alpha_j}{\alpha_n}2k_n\pi}z_n)$.

Now we continue modifying the other coordinates in the same way as above which finishes the proof. □

In the same spirit is the following lemma.

Lemma 4.4.11. *Let $L_1, L_2 \Subset \mathbb{D}$, $L \Subset \mathbb{C}_*$, and $\alpha \in \mathbb{R}_*^n$. Assume that*

$$\boldsymbol{m}(L_1, L_2) = \inf\{\boldsymbol{m}(\lambda_1, \lambda_2) : \lambda_j \in L_j,\ j = 1, 2\} \ge \delta > 0.$$

Then there exists a set $\widetilde{L} \Subset \mathbb{C}_*$, $L \subset \widetilde{L}$, such that for any $z_1, z_2 \in L$ and $\lambda_j \in L_j$, $j = 1, 2$, there is a $\psi \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ such that $\psi(\lambda_j) = z_j$, $j = 1, 2$, and $\psi(\mathbb{D}) \subset \widetilde{L}$.

Moreover, there is a set $\widetilde{K} \Subset \mathbb{C}_*$ such that for any $z_1, \dots, z_n \in L$ with $|z^\alpha| = 1$ and any $w_1, \dots, w_k \in L$, $k < n$, there are functions $\psi_j \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ such that $\psi_j(\mathbb{D}) \subset \widetilde{K}$, $\psi_j(\lambda_1) = z_j$, $j = 1, \dots, n$, and $\psi_j(\lambda_2) = w_j$, $j = 1, \dots, k$, and $\psi_1^{\alpha_1}(\lambda) \cdots \psi_n^{\alpha_n}(\lambda) = e^{i\theta}$, $\lambda \in \mathbb{D}$.

Proof. Without loss of generality we may assume that $L_1 = \{\lambda_1 = 0\}$ and $L_2 = \{\lambda_2 = \delta\}$.[16] Put

$$\widehat{L} := \exp^{-1}(L) \cap (\mathbb{R} + i[0, 2\pi)).$$

Then $\widehat{L} \subset (\log \varepsilon_1, \log \varepsilon_2) + i[0, 2\pi)$, where $0 < \varepsilon_1 < \varepsilon_2$ and the ε_j's depend only on L. Set

$$\widetilde{L} := \{e^{a\lambda+b} : \lambda \in \mathbb{D},\ a, b \in \mathbb{C} \text{ such that } a\lambda_j + b \in \widehat{L},\ j = 1, 2\}.$$

Then $\widetilde{L} \Subset \mathbb{C}_*$. What remains is to choose $\psi := e^h$, where h is an appropriate function of the form as it appears in the definition of $\widetilde{L}$.

To prove the last part of the lemma we take, in addition, $w_{k+1}, \dots, w_{n-1} \in L$ in an arbitrary way. For the pairs z_j, w_j we fix functions $\psi_j \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ with

$$\psi_j(\mathbb{D}) \subset \widetilde{L}, \quad \psi_j(\lambda_1) = z_j, \quad \psi_j(\lambda_2) = w_j, \quad j = 1, \dots, n-1.$$

Put

$$\psi_n(\lambda) := e^{i\theta}\big(\psi_1^{\alpha_1}(\lambda) \cdots \psi_{n-1}^{\alpha_{n-1}}(\lambda)\big)^{-1/\alpha_n}, \quad \lambda \in \mathbb{D},$$

where the branches of the powers are chosen arbitrarily and θ is taken such that $\psi_n(\lambda_n) = z_n$. $\square$

Exercise 4.4.12. Prove the following statement using the ideas from the proof of Lemma 4.4.11.

(a) Let $a \in \mathbb{C}_*$, $X \in \mathbb{C}$, and $\lambda \in \mathbb{D}$ be given. Then there exists a $\psi \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ such that $\psi(\lambda) = a$ and $\psi'(\lambda) = X$.

(b) Moreover, if $\lambda \in \mathbb{D}$, $a \in \mathbb{C}_*^n$, $X \in \mathbb{C}^k$, where $k < n$, and $\alpha \in \mathbb{R}_*^n$ with $|a^\alpha| = 1$, then there is a $\psi = (\psi_1, \dots, \psi_n) \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*^n)$ such that

$$\psi(\lambda) = a,\ (\psi_1'(\lambda), \dots, \psi_k'(\lambda)) = X,\ \psi_1^{\alpha_1} \cdots \psi_n^{\alpha_n} = e^{i\theta}\,\mathrm{id}_{\mathbb{D}}.$$

Applying the previous lemmas in the case of elementary Reinhardt domains leads to the following results.

Lemma 4.4.13. *Let $\boldsymbol{D}_\alpha$ be of irrational type and $a, z \in \boldsymbol{D}_\alpha \cap \mathbb{C}_*^n$. Then*

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, z) = \widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a', z'), \quad a' \in \mathbb{T}_a,\ z' \in \mathbb{T}_z.$$

Proof. Note that it is enough to prove that $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, z) = \widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, z')$ (use the symmetry), whenever $z' \in \mathbb{T}_z$.

Suppose the contrary, i.e. there are points $z', z'' \in \mathbb{T}_z$ with

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, z') < \widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, z'') =: \ell.$$

[16] To get the general case, recall that if $\boldsymbol{m}(\lambda_1, \lambda_2) \geq \delta = \boldsymbol{m}(0, \delta)$, then there is a $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with $\varphi(\lambda_1) = 0$ and $\varphi(\lambda_2) = \delta$.

In virtue of Lemma 4.4.10, we have $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,\tilde{z}) = \ell$, $\tilde{z} \in \mathbb{T}_{z''}(\alpha)$. Observe that $z' \in \mathbb{T}_z = T_{z''} = \overline{\mathbb{T}}_{z''}(\alpha)$. Therefore, z' is an accumulation point of $\mathbb{T}_{z''}(\alpha)$, which contradicts the upper semicontinuity of $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,\cdot)$. ☐

Corollary 4.4.14. *Let $a \in \boldsymbol{D}_\alpha \cap \mathbb{C}^n_*$, where $\boldsymbol{D}_\alpha$ is of irrational type. Then*

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) = 0, \quad z \in \mathbb{T}_a.$$

Proof for $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}$ – the case $k < n$. In virtue of Lemma 4.4.9, we know that

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) \geq \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,z) \geq |z^\alpha|^{1/r}.$$

In order to prove the converse inequality for a $z \in \boldsymbol{D}_\alpha \setminus V_0$, put $\tau := |z^\alpha|^{1/r}$. Then all the points $z_1,\dots,z_k,\frac{z_{k+1}}{\tau},\dots,\frac{z_n}{\tau}$ belong to $\mathbb{C}_*$ with

$$\Big(\prod_{j=1}^{k}|z_j|^{\alpha_j}\Big)\Big(\prod_{j=k+1}^{n}|\tfrac{z_j}{\tau}|^{\alpha_j}\Big) = 1.$$

Moreover, $a_1,\dots,a_k \in \mathbb{C}_*$. Then, in virtue of Lemma 4.4.11, we find functions $\psi_j \in \mathcal{O}(\mathbb{D},\mathbb{C}_*)$ such that

$$\prod_{j=1}^{n}\psi_j^{\alpha_j}(\lambda) = e^{i\theta}, \quad \lambda \in \mathbb{D},$$
$$\begin{aligned} \psi_j(\tau) &= z_j, & j &= 1,\dots,k,\\ \psi_j(\tau) &= \tfrac{z_j}{\tau}, & j &= k+1,\dots,n,\\ \psi_j(0) &= a_j, & j &= 1,\dots,k. \end{aligned}$$

Put

$$\varphi(\lambda) := (\psi_1(\lambda),\dots,\psi_k(\lambda),\lambda\psi_{k+1}(\lambda),\dots,\lambda\psi_n(\lambda)), \quad \lambda \in \mathbb{D}.$$

Then $\varphi \in \mathcal{O}(\mathbb{D},\boldsymbol{D}_\alpha)$ with $\varphi(0) = a$ and $\varphi(\tau) = z$. Hence, $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) \leq \tau$, i.e. the proof is complete.

Now we discuss the remaining case when $z \in \boldsymbol{D}_\alpha \cap V_0$. Note that necessarily we have $z_j \neq 0$, $j \leq s$.

First suppose that there is a coordinate $z_j = 0$ with $j \geq k+1$. Then the holomorphic mapping

$$\begin{aligned}\mathbb{C}^s_* \times \mathbb{C}^{n-s-1} \ni w &= (w_1,\dots,w_{j-1},w_{j+1},\dots,w_n)\\ &\mapsto (w_1,\dots,w_{j-1},0,w_{j+1},\dots,w_n) \in \boldsymbol{D}_\alpha\end{aligned}$$

leads to the following inequality

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) \leq \widetilde{\boldsymbol{k}}^*_{\mathbb{C}^s_*\times\mathbb{C}^{n-s-1}}(\tilde{a},\tilde{z}) = 0, \text{ [17]}$$

[17] Recall that one can map $\mathbb{C}$ onto $\mathbb{C}_*$ via the exponential map.

where $\tilde{a} := (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$ and $\tilde{z} := (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$.

What remains is the case where $|z_j| + |a_j| \neq 0$ for all j's. For $\beta \in (0,1)$ we are going to define a $\varphi = (\varphi_1, \dots, \varphi_n) \in \mathcal{O}(\mathbb{D}, \boldsymbol{D}_\alpha)$, where

$$\varphi_j(\lambda) := \begin{cases} \frac{\lambda-\beta}{1-\beta\lambda}\psi_j(\lambda) & \text{if } a_j = 0, \\ \frac{\lambda+\beta}{1+\beta\lambda}\psi_j(\lambda) & \text{if } z_j = 0, \\ \psi_j(\lambda) & \text{if } a_j z_j \neq 0, \end{cases} \qquad \lambda \in \mathbb{D}.$$

The $\psi_j \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ have to be chosen in a correct way.

We need that the ψ_j's satisfy $\prod_{j=1}^n \psi_j^{\alpha_j} \equiv e^{i\theta}$ on $\mathbb{D}$ and that $\varphi(\beta) = a$, and $\varphi(-\beta) = z$. Note that then we would get $\tilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) \le \boldsymbol{m}(\beta, -\beta) \to 0$ if $\beta \to 0$.

Fix some j_1 such that $a_{j_1} = 0$. Then we would like the functions ψ_j to attain the following values in $\mathbb{C}_*$:

$$\psi_j(\beta) = \begin{cases} a_j & \text{if } a_j z_j \neq 0, \\ \frac{a_j(1+\beta^2)}{2\beta} & \text{if } z_j = 0 \neq a_j, \\ 1 & \text{if } a_j = 0 \neq z_j,\ j \neq j_1, \end{cases}$$

$$\psi_{j_1}(\beta) = \Big(\prod_{a_j z_j \neq 0} |\psi_j(\beta)|^{\alpha_j} \prod_{z_j = 0} |\psi_j(\beta)|^{\alpha_j} \Big)^{-/\alpha_{j_1}}.$$

Moreover, at $-\beta$ we only need to have

$$\psi_j(-\beta) = \begin{cases} z_j & \text{if } a_j z_j \neq 0, \\ \frac{z_j(1-\beta^2)}{-2\beta} & \text{if } a_j = 0 \neq z_j. \end{cases}$$

Note that there are fewer than n values we want to specify at $-\beta$. Therefore, Lemma 4.4.11 works and gives such a mapping $\psi \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*^n)$ which completes the proof. $\square$

The remaining case, i.e. $s = n$ or $k = n$, for elementary Reinhardt domains will be discussed later in this section. First we establish the formulas for the pluricomplex Green function in the irrational case.

Proof for $\boldsymbol{g}_{\boldsymbol{D}_\alpha}$ – the irrational case. As in the proof of Theorem 4.4.4 we may assume that $s = s(\alpha) = 0$.

In the case where $k = n$ we have $\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,z) = 0$, $z \in \mathbb{T}_a$ (see Corollary 4.4.14). Then the maximum principle for psh functions implies that $\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,\cdot) = 0$ on $\mathbb{P}(0, (|a_1|, \dots, |a_n|))$. Recall that $\log \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,\cdot) \in \mathcal{PSH}(\boldsymbol{D}_\alpha)$ and that either $\log \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,\cdot) \equiv -\infty$ or the level set $\{z \in \boldsymbol{D}_\alpha : \log \boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,z) = -\infty\}$ is a pluripolar set. Thus, $\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,\cdot) \equiv 0$ on $\boldsymbol{D}_\alpha$.

If $k < n$, then, by the formula for $\tilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}$ in that case, we know that

$$\boldsymbol{g}_{\boldsymbol{D}_\alpha}(a,z) \le \tilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) = |z^\alpha|^{1/r}.$$

To conclude the proof apply Lemma 4.4.9. $\square$

The last part in this proof is devoted to prove some of the remaining formulas for the Lempert function.

Lemma 4.4.15. *If $a, z \in D = \boldsymbol{D}_{\boldsymbol{1}} \cap V_0$ ($\boldsymbol{1} = (1,\dots,1)$), then*

$$\tilde{\boldsymbol{k}}^*_{\boldsymbol{D}_{\boldsymbol{1}}}(a,z) = 0.$$

If $a \in \boldsymbol{D}_{\boldsymbol{1}} \cap \mathbb{C}^n_*$, $z \in \boldsymbol{D}_{\boldsymbol{1}}$, then

$$\tilde{\boldsymbol{k}}^*_{\boldsymbol{D}_{\boldsymbol{1}}}(a,z) = (\boldsymbol{m}(a_1\cdots a_n, z_1\cdots z_n))^{1/\tau},$$

where $\tau := \max\{\#\{j : z_j = 0\}, 1\}$.

Proof. The first formula is a direct consequence of the one for $\tilde{\boldsymbol{k}}^*$ which has been proved before.

Now we will discuss the case where $a, z \in \mathbb{C}^n_*$. In a first step suppose that $\mu := a_1\cdots a_n = z_1\cdots z_n$. Then the holomorphic mapping

$$\mathbb{C}^{n-1} \ni (w_1,\dots,w_{n-1}) \overset{F}{\longmapsto} (e^{w_1},\dots,e^{w_{n-1}}, \mu e^{-w_1-\cdots-w_{n-1}}) \in \boldsymbol{D}_{\boldsymbol{1}}$$

is onto $\boldsymbol{V}(\boldsymbol{D}_{\boldsymbol{1}},\mu)$, i.e. $F(w') = a$ and $F(w'') = z$ for certain $w', w'' \in \mathbb{C}^{n-1}$. Hence,

$$\tilde{\boldsymbol{k}}^*_{\boldsymbol{D}_{\boldsymbol{1}}}(a,z) \le \tilde{\boldsymbol{k}}^*_{\mathbb{C}^{n-1}}(w',w'') = 0.$$

So we may assume that, from now on, $a_1\cdots a_n \ne z_1\cdots z_n$. Put

$$\lambda_1 := a_1\cdots a_n \in \mathbb{D}, \quad \lambda_2 := z_1\cdots z_n \in \mathbb{D}.$$

Applying Lemma 4.4.11 we find a $\psi_j \in \mathcal{O}(\mathbb{D},\mathbb{C}_*)$ such that

$$\psi_j(\lambda_1) = a_j,\ \psi_j(\lambda_2) = z_j,\ j = 1,\dots,n-1, \quad \psi_n(\lambda_1) = (a_1\cdots a_{n-1})^{-1},$$

and $\psi_1\cdots\psi_n \equiv e^{i\theta}$ on $\boldsymbol{D}_{\boldsymbol{1}}$. Put

$$\varphi(\lambda) := (\psi_1(\lambda),\dots,\psi_{n-1}(\lambda), e^{-i\theta}\lambda\psi_n(\lambda)), \quad \lambda \in \mathbb{D}.$$

Then $\varphi \in \mathcal{O}(\mathbb{D},\boldsymbol{D}_{\boldsymbol{1}})$ such that $\varphi(\lambda_1) = a$, $\psi(\lambda_2) = z$. Hence,

$$\boldsymbol{m}(a_1\cdots a_n, z_1\cdots z_n) \ge \tilde{\boldsymbol{k}}^*_{\boldsymbol{D}_{\boldsymbol{1}}}(a,z) \ge \boldsymbol{m}(a_1\cdots a_n, z_1\cdots z_n),$$

where the last inequality is a consequence of the property of holomorphic contractibility.

It remains the case that $a \in \mathbb{C}^n_*$ and $z \in V_0$. Then again Lemma 4.4.11 gives the desired formula. $\square$

What we have just discussed is the simplest case of an elementary Reinhardt domain of rational type. The proof of the formula in case $s < n$ needs deep results on geodesics which are beyond the scope of this book. Therefore we skip its proof. Details may be found in [Pfl-Zwo 1998] and [Zwo 2000]. Note that the case $k < n$ is contained in Theorem 4.4.4. So the difficult case is the one with $k = n$.

Now we turn to the irrational case for $s < n$.

Proof for $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}$ – the irrational case with $s < n$. The case $k < n$ was already verified. So we assume that $k = n$ and $z \notin \boldsymbol{V}_0$. Recall that

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, z) = \widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}((|a_1|, \dots, |a_n|), (|z_1|, \dots, |z_n|))$$

(see Lemma 4.4.13). Now we approximate the α_j's by rational vectors. We choose a sequence $(\alpha^{(j)})_j \subset \mathbb{Q}^n_*$ such that

$$\alpha^{(j)} \to \alpha, \quad t(\alpha^{(j)}) = 1, \quad \alpha_1^{(j)}, \dots, \alpha_s^{(j)} < 0,\ \alpha_{s+1}^{(j)}, \dots, \alpha_n^{(j)} > 0, \quad j \in \mathbb{N}.$$

Applying Theorem 4.4.4 for points $x, y \in \boldsymbol{D}_{\alpha^{(j)}} \cap \mathbb{R}^n_+$ we get

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_{\alpha^{(j)}}}(x, y) = \boldsymbol{m}\big(x_1^{\alpha_1^{(j)}} \cdots x_n^{\alpha_n^{(j)}}, y_1^{\alpha_1^{(j)}} \cdots y_n^{\alpha_n^{(j)}}\big), \quad j \in \mathbb{N}.^{18} \tag{4.4.1}$$

By employing a biholomorphic reordering of the coordinates we may assume that $1 = t(\alpha) = \alpha_n$.

Now suppose that

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a, z) < \boldsymbol{m}\big(|a^\alpha|, |z^\alpha|\big).$$

Then there exist an analytic disc $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, \boldsymbol{D}_\alpha)$ and $\lambda_1, \lambda_2 \in \mathbb{D}$ such that $\varphi(\lambda_1) = (|a_1|, \dots, |a_n|)$, $\varphi(\lambda_2) = (|z_1|, \dots, |z_n|)$, and

$$\boldsymbol{m}(\lambda_1, \lambda_2) < \boldsymbol{m}\big(|a^\alpha|, |z^\alpha|\big).$$

Since $\varphi(\overline{\mathbb{D}})$ is a compact subset of $\boldsymbol{D}_\alpha$ we can choose a large j_0 such that $\varphi|_{\mathbb{D}} \in \mathcal{O}(\mathbb{D}, \boldsymbol{D}_{\alpha^{(j_0)}})$. Therefore,

$$\boldsymbol{m}(\lambda_1, \lambda_2) < \boldsymbol{m}\big(|a_1|^{\alpha_1^{(j_0)}} \cdots |a_n|^{\alpha_n^{(j_0)}}, |z_1|^{\alpha_1^{(j_0)}} \dots |z_n|^{\alpha_n^{(j_0)}}\big);$$

a contradiction to (4.4.1).

On the other hand put $\lambda_1 := |a^\alpha|, \quad \lambda_2 := |z^\alpha|$. If $\lambda_1 \neq \lambda_2$, then we find analytic discs $\psi_j = e^{h_j} \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ with $h_j(\lambda_1) = \log|a_j|$ and $h_j(\lambda_2) = \log|z_j|$, $j = 1, \dots, n-1$ (use Lemma 4.4.11). Moreover, define

$$\psi_n(\lambda) := \exp(-\alpha_1 h_1(\lambda)) - \dots - \alpha_{n-1} h_{n-1}(\lambda)), \quad \lambda \in \mathbb{D}.$$

[18] Note that $\boldsymbol{m}(x, e^{i\theta} y) \geq \boldsymbol{m}(x, y)$, $\theta \in \mathbb{R}$, when $x, y \in [0, 1)$ (Exercise).

Then $\lambda_1\psi_n(\lambda_1) = |a_n|$. Put

$$\varphi(\lambda) := (\psi_1(\lambda), \dots, \psi_{n-1}(\lambda), \lambda\psi_n(\lambda)), \quad \lambda \in \mathbb{D}.$$

Then $\varphi_n(\lambda_1) = |a_n|$ and $\varphi_n(\lambda_2) = |z_n|$. Hence,

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) \le \boldsymbol{m}\big(|a^\alpha|, |z^\alpha|\big).$$

If $\lambda_1 = \lambda_2$, put $\tilde{\lambda}_\varepsilon := \lambda_2 + \varepsilon \in \mathbb{D}$, $\varepsilon > 0$ small. Put

$$h_\varepsilon(\lambda) = \frac{\lambda - \lambda_1}{1 - \lambda_1\lambda} \cdot \frac{\lambda - \tilde{\lambda}_\varepsilon}{1 - \lambda_\varepsilon\lambda}, \quad \lambda \in \mathbb{D}.$$

Then $h_\varepsilon \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ with $h_\varepsilon(\lambda_1) = h_\varepsilon(\lambda_\varepsilon) = 0$. Then, using an appropriate Möbius transformation, we find an $\tilde{h}_\varepsilon \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ such that $\tilde{h}_\varepsilon(\lambda_1) = \tilde{h}_\varepsilon(\lambda_\varepsilon) = \lambda_1$.

As in the case before there are $\psi_j \in \mathcal{O}(\mathbb{D}, \mathbb{C}_*)$ with $\psi_j(\lambda_1) = |a_j|$ and $\psi_j(\lambda_\varepsilon) = |z_j|$, $j = 1, \dots, n-1$. Put now

$$\varphi(\lambda) := \big(\psi_1(\lambda), \dots, \psi_{n-1}(\lambda), \tilde{h}_\varepsilon(\lambda)(\psi_1^{\alpha_1}(\lambda) \dots \psi_{n-1}^{\alpha_{n-1}}(\lambda))^{-1}\big), \quad \lambda \in \mathbb{D}.$$

Then $\varphi \in \mathcal{O}(\mathbb{D}, \boldsymbol{D}_\alpha)$, $\varphi(\lambda_1) = (|a_1|, \dots, |a_n|)$, and $\varphi(\lambda_\varepsilon) = (|z_1|, \dots, |z_n|)$. Taking into account that ε may be taken arbitrarily small, we end up with $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) = 0$, which finishes the proof. □

In a last step we discuss the case when $s = n$, i.e. $\alpha_j < 0$ for all j.

Proof for $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}$ – the case $s = n$. First observe that the map $F : \mathbb{C}^{n-1} \times \mathbb{D}_* \to \boldsymbol{D}_\alpha$,

$$F(\lambda_1, \dots, \lambda_n) := (e^{\lambda_1\alpha_n}, \dots, e^{\lambda_{n-1}\alpha_n}, \lambda_n^{-1} e^{-\lambda_1\alpha_1 - \dots - \lambda_{n-1}\alpha_{n-1}}),$$

is a holomorphic covering. Note that $F(\lambda) = w$ iff

$$\lambda_j = \frac{1}{\alpha_n}\big(\log|w_j| + i(\operatorname{Arg} w_j + 2l_j\pi)\big), \quad j = 1, \dots, n-1,$$
$$\frac{1}{\lambda_n} = w_n\Big(\prod_{j=1}^{n-1}|w_j|^{\alpha_j}\Big)^{1/\alpha_n} e^{\frac{i}{\alpha_n}\sum_{j=1}^{n-1}\alpha_j(\operatorname{Arg} a_j + 2l_j\pi)},$$

where $l_1, \dots, l_{n-1} \in \mathbb{Z}$. Applying Propositions 4.2.38 and 4.2.35 we are led to

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) = \inf\big\{\widetilde{\boldsymbol{k}}^*_{\mathbb{D}_*}\big(|a^\alpha|^{-1/\alpha_n} e^{-\frac{i}{\alpha_n}\sum_{j=1}^{n-1}\alpha_j \operatorname{Arg} a_j},$$
$$|z^\alpha|^{-1/\alpha_n} e^{-\frac{i}{\alpha_n}\sum_{j=1}^{n-1}\alpha_j(\operatorname{Arg} z_j + 2l_j\pi)}\big) : l_1, \dots, l_{n-1} \in \mathbb{Z}\big\}.$$

In the rational case we apply Corollary 4.4.3 and get $\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) = \widetilde{\boldsymbol{k}}^*_{\mathbb{D}_*}(a^\alpha, z^\alpha)$.

In the irrational case we conclude via the Kronecker theorem that

$$\widetilde{\boldsymbol{k}}^*_{\boldsymbol{D}_\alpha}(a,z) = \widetilde{\boldsymbol{k}}^*_{\mathbb{D}_*}\big(|a^\alpha|^{-\alpha_n}, |z^\alpha|^{-\alpha_n}\big).$$

It just remains to mention that $\widetilde{\boldsymbol{k}}^*_{\mathbb{D}_*}(x,y) = \widetilde{\boldsymbol{k}}^*_{\mathbb{D}_*}(x^t, y^t)$ for $t > 0$. Applying this remark with $t = -\alpha_n$ gives the desired formula. □

In a final step we discuss the Kobayashi pseudodistance in the case of elementary Reinhardt domains with $s < n$, which so far we have not mentioned.

Proof for $\boldsymbol{k}^*_{\boldsymbol{D}_\alpha}$ *– the case* $s < n$. In the rational case it is easy to see that

$$\begin{aligned}\boldsymbol{D}_\alpha \times \boldsymbol{D}_\alpha \ni (z,w) \mapsto (\min\{\boldsymbol{p}(\zeta_1,\zeta_2) : \zeta_1,\zeta_2 \in \mathbb{D},\ a^\alpha = \zeta_1^t,\ z^\alpha = \zeta_2^t\})\\ =: d(z,w)\end{aligned}$$

satisfies the triangle inequality and is majorized by $\widetilde{\boldsymbol{k}}_{\boldsymbol{D}_\alpha}$. Hence it follows that $\boldsymbol{k}_{\boldsymbol{D}_\alpha} \geq d$ and both are equal outside of the axes. Then the continuity of the Kobayashi pseudodistance gives the desired result.

In the irrational case the reasoning is analogous to the one before and therefore left to the reader as an EXERCISE. □

Proof for $\boldsymbol{k}^*_{\boldsymbol{D}_\alpha}$ *– the case* $s = n$. This step is left as an EXERCISE. □ □

In particular, the effective formulas from above make it possible to give a negative answer to the following old question asked by S. Kobayashi (see [Kob 1970], p. 48), namely: is the infimum in Proposition 4.2.38 taken by a certain point in the holomorphic covering.

Example 4.4.16 ([Zwo 1998]). Let $D = \boldsymbol{D}_{(-\sqrt{2},-1)}$ and $\psi : \mathbb{C} \times \mathbb{D}_* \to D$,

$$\psi(\lambda_1,\lambda_2) := \Big(e^{-\lambda_1}, \frac{1}{\lambda_2} e^{\sqrt{2}\lambda_1}\Big),$$

be a holomorphic covering. Take $a := (r,r) \in D$ with $r > 0$ and $z := (r, ir)$. Then $\boldsymbol{k}_D(a,z) = 0$. Fix the following preimage $(-\log r, r^{-1-\sqrt{2}})$ of the point a. Then

$$\begin{aligned}0 &= \inf_{k\in\mathbb{Z}} \boldsymbol{k}_{\mathbb{C}\times\mathbb{D}_*}\big((-\log r, r^{-1-\sqrt{2}}),\\ &\qquad\qquad (-\log r + 2\pi i k, \frac{-i}{r}\exp(\sqrt{2}(-\log r + 2\pi i k)))\big)\\ &= \inf_{k\in\mathbb{Z}} \boldsymbol{k}_{\mathbb{D}_*}\big(r^{-1-\sqrt{2}}, \frac{-i}{r}\exp(\sqrt{2}(-\log r + 2\pi i k))\big).\end{aligned}$$

Suppose that there is a $k' \in \mathbb{Z}$ such that

$$\boldsymbol{k}_{\mathbb{D}_*}\Big(r^{-1-\sqrt{2}}, \frac{-i}{r}\exp\big(\sqrt{2}(-\log r + 2\pi i k')\big)\Big) = 0.$$

Then $1/2 + k'\sqrt{2} \in \mathbb{Z}$;[19] a contradiction.

4.5 Holomorphically contractible families of pseudometrics

Recall from classical analysis that the Euclidean distance between two points $x, y \in G = \mathbb{R}^n$ can be also given by the "minimal" length of all piecewise $\mathcal{C}^1$-curves in G connecting these points. Let $\gamma\colon [0,1] \to G$ be such a curve. Then the length of γ is given by $L(\gamma) = \int_0^1 \|\gamma'(t)\| dt$, i.e. along the curve the lengths of its tangent vectors $\|\gamma'(t)\|$, $t \in [0,1]$, are summed up. Hence we have an assignment

$$G \times \mathbb{R}^n \ni (x, X) \mapsto \alpha_G(x; X) := \lim_{\mathbb{R}_* \ni t \to 0} \frac{\|x - (x + tX)\|}{|t|},$$

with the following property: $\alpha(x; sX) = |s|\alpha(x; X)$, $x \in G$, $s \in \mathbb{R}$, and $X \in \mathbb{R}^n$.

This procedure will be transformed into the context of families of holomorphically contractible families of functions $(d_D)_D$.

Let us start with a general definition.

Definition 4.5.1. A family $(\delta_D)_D$ of pseudometrics $\delta_D\colon D \times \mathbb{C}^n \to \mathbb{R}_+$, D a domain in $\mathbb{C}^n$, i.e.

$$\delta_D(z; \lambda X) = |\lambda|\delta_D(z; X), \quad z \in D,\ \lambda \in \mathbb{C},\ X \in \mathbb{C}^n, \tag{4.5.1}$$

is said to be *holomorphically contractible* if the following two conditions are satisfied:

($\widetilde{\text{A}}$)

$$\delta_{\mathbb{D}}(a; X) = \boldsymbol{\gamma}(a; X) := \frac{|X|}{1 - |a|^2}, \quad a \in \mathbb{D},\ X \in \mathbb{C}, \tag{4.5.2}$$

($\widetilde{\text{B}}$) for arbitrary domains $G \subset \mathbb{C}^m$, $D \subset \mathbb{C}^n$, any $F \in \mathcal{O}(G, D)$ works as a *contraction* with respect to δ_G and δ_D, i.e.

$$\delta_D(F(a); F'(a)X) \le \delta_G(a; X), \quad a \in G,\ X \in \mathbb{C}^m. \tag{4.5.3}$$

Note that the Hermitian pseudometrics discussed in Section 1.19 are pseudometrics in the sense of Definition 4.5.1.

In the following we will discuss the most important holomorphically contractible families of pseudometrics.

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[19] Note that $\boldsymbol{m}(\lambda, \mu) = \boldsymbol{k}^*_{\mathbb{D}}(\lambda, \mu) \le \boldsymbol{k}^*_{\mathbb{D}_*}(\lambda, \mu) = 0$ implies that $\lambda = \mu$, when $\lambda, \mu \in \mathbb{D}_*$.

Example 4.5.2 (Carathéodory–Reiffen pseudometric).

$$\boldsymbol{\gamma}_D(a;X) := \sup\{|f'(a)X| : f \in \mathcal{O}(D,\mathbb{D}),\ f(a)=0\}, \quad a \in D,\ X \in \mathbb{C}^n, \tag{4.5.4}$$

where $D \subset \mathbb{C}^n$ is a domain.

Indeed, the properties (4.5.1) and ($\widetilde{\mathrm{B}}$) are obvious. To prove ($\widetilde{\mathrm{A}}$) let $F \in \mathcal{O}(\mathbb{D},\mathbb{D})$ with $F(a)=0$ and $X \in \mathbb{C}$. Put

$$g(\lambda) := F\Big(\frac{\lambda+a}{1+\bar{a}\lambda}\Big), \quad \lambda \in \mathbb{D}.$$

Then $g \in \mathcal{O}(\mathbb{D},\mathbb{D})$ with $g(0)=0$. By the Schwarz–Pick lemma it follows that $1 \ge |g'(0)| = |F'(a)|(1-|a|^2)$; hence, $\boldsymbol{\gamma}(a;X) \ge \boldsymbol{\gamma}_{\mathbb{D}}(a;X)$. To get the converse inequality just take the function $F(\lambda) := \frac{\lambda-a}{1-\bar{a}\lambda}$, $\lambda \in \mathbb{D}$.

We begin by stating some simple properties of the Carathéodory–Reiffen pseudometric (details are left as an EXERCISE):

- $\boldsymbol{\gamma}_D(a;\cdot)$ is a seminorm on $\mathbb{C}^n$.
- By the Montel theorem, there exists an $f \in \mathcal{O}(D,\mathbb{D})$ with $f(a)=0$ and $|f'(a)X| = \boldsymbol{\gamma}_D(a;X)$ (such an f is called an *extremal function for* $\boldsymbol{\gamma}_D(a;X)$).
- Put $\mathcal{M}_D(a) := \{|f| : f \in \mathcal{O}(D,\mathbb{D}),\ f(a)=0\}$. Then

$$\boldsymbol{\gamma}_D(a;X) = \sup\Big\{\lim_{\mathbb{C}_*\ni\lambda\to 0}\frac{u(a+\lambda X)}{|\lambda|} : u \in \mathcal{M}_D(a)\Big\}, \quad a \in D,\ X \in \mathbb{C}^n. \tag{4.5.5}$$

Note that any function $u \in \mathcal{M}_D(a)$ satisfies: $u\colon D \to [0,1)$, $\log u$ is psh, and $u(z) \le C\|z-a\|$, when $z \in \mathbb{B}(a,r) \subset D$ for certain $C, r > 0$.

- If $D = \bigcup_{j=1}^{\infty} D_j \subset \mathbb{C}^n$, where $D_j \subset D_{j+1}$, $j \in \mathbb{N}$, are domains, then (use Montel)

$$\boldsymbol{\gamma}_D(a;X) = \lim_{j\to\infty}\boldsymbol{\gamma}_{D_j}(a;X).$$

- For a balanced domain D_h we have $\boldsymbol{\gamma}_D(0;\cdot) \le h$.

Indeed, suppose first that $h(X)=0$. Then $\mathbb{C} \ni \lambda \mapsto \lambda X \in D_h$ is a well-defined holomorphic mapping. By (4.5.3), $\boldsymbol{\gamma}_{D_h}(0;X) \le \boldsymbol{\gamma}_{\mathbb{C}}(0;1) = 0 = h(X)$.[20] Now assume that $h(X) > 0$. Then $\mathbb{D} \ni \lambda \mapsto \frac{\lambda}{h(X)}X \in D_h$ is holomorphic and, therefore, $\boldsymbol{\gamma}_{D_h}(0;X) \le \boldsymbol{\gamma}(0;h(X)) = h(X)$.

- In particular,

$$\boldsymbol{\gamma}_D(a;X) \le \boldsymbol{\gamma}_{\mathbb{B}(a,r)}(a;X) = \boldsymbol{\gamma}_{\mathbb{B}(r)}(0;X) \le \frac{\|X\|}{r}, \quad a \in D \subset \mathbb{C}^n,\ X \in \mathbb{C}^n.$$

It turns out that the Carathéodory–Reiffen pseudometric is given as a derivative of the Möbius pseudodistance, even in a strong sense (see Lemma 4.5.3 (b)).

[20] Use the Liouville theorem.

Lemma 4.5.3. *Let $D \subset \mathbb{C}^n$ be a domain. Then:*

(a) *For any compact $K \subset D$ and for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|\boldsymbol{m}_D(z', z'') - \boldsymbol{\gamma}_D(a; z' - z'')| \le \varepsilon \|z' - z''\|, \quad a \in K,\ z', z'' \in \mathbb{B}(a, \delta) \subset D.$$

(b) *For $(a; X) \in D \times \mathbb{C}^n$, $\|X\| = 1$, one has*

$$\frac{\boldsymbol{m}_D(z', z'')}{\|z' - z''\|} \to \boldsymbol{\gamma}_D(a; X), \text{ when } z', z'' \to a,\ z' \ne z'',\ \frac{z' - z''}{\|z' - z''\|} \to X.$$

(c) *In particular,*

$$\boldsymbol{\gamma}_D(a; X) = \lim_{\mathbb{C}_* \ni \lambda \to 0} \frac{\boldsymbol{m}_D(a, a + \lambda X)}{|\lambda|}, \quad a \in D,\ X \in \mathbb{C}^n.$$

Proof. (a) Fix an $r \in \mathbb{R}_{>0}$ such that $\mathbb{B}(b, 4r) \subset D$, $b \in K$. Now, take $a \in K$, $z', z'' \in \mathbb{B}(a, r)$, and $X \in \mathbb{C}^n$. We may assume that $\boldsymbol{\gamma}_D(z'; X) \ge \boldsymbol{\gamma}_D(z''; X)$. Choose an extremal function $f \in \mathcal{O}(D, \mathbb{D})$ for $\boldsymbol{\gamma}_D(z'; X)$, i.e. $f(z') = 0$ and $|f'(z')X| = \boldsymbol{\gamma}_D(z'; X)$. Then

$$\begin{aligned} |\boldsymbol{\gamma}_D(z'; X) - \boldsymbol{\gamma}_D(z''; X)| &= |f'(z')X| - \boldsymbol{\gamma}_D(z''; X) \le |f'(z')X| - |f'(z'')X| \\ &\le |f'(z')X - f'(z'')X| \le \|f'(z') - f'(z'')\| \|X\| \\ &\le \max\{\|f''(z)\| : z \in [z', z'']\} \|z' - z''\| \|X\| \\ &\le \frac{1}{2r^2} \|z' - z''\| \|X\|. \end{aligned} \tag{*}$$

Using (*) it follows that

$$\begin{aligned} &|\boldsymbol{m}_D(z', z'') - \boldsymbol{\gamma}_D(a; z' - z'')| \\ &\quad \le |\boldsymbol{m}_D(z', z'') - \boldsymbol{\gamma}_D(z'; z' - z'')| + |\boldsymbol{\gamma}_D(z'; z' - z'') - \boldsymbol{\gamma}_D(a; z' - z'')| \\ &\quad \le |\boldsymbol{m}_D(z', z'') - \boldsymbol{\gamma}_D(z'; z' - z'')| + \frac{1}{2r^2} \|z' - a\| \|z' - z''\|. \end{aligned}$$

It remains to estimate the first term on the right-hand side. We may assume that $\boldsymbol{m}_D(z', z'') \ge \boldsymbol{\gamma}_D(z'; z'' - z')$ (the other case follows in a similar way (EXERCISE)). So let $f \in \mathcal{O}(D, \mathbb{D})$, $f(z') = 0$, and $f(z'') = \boldsymbol{m}_D(z', z'')$, i.e. f is an extremal function for $\boldsymbol{m}_D$. Then, by the Cauchy inequalities, we get with $Y := z'' - z'$ ($\|Y\| \le 2r$)

$$\begin{aligned} |\boldsymbol{m}_D(z', z'') - \boldsymbol{\gamma}_D(z'; Y)| \le |f(z' + Y) - f'(z')Y| &\le \sum_{k=2}^{\infty} \frac{1}{k!} \|f^{(k)}(z')\| \|Y\|^k \\ &\le \sum_{k=2}^{\infty} \left(\frac{\|Y\|}{3r}\right)^k \le \frac{1}{3r^2} \|Y\|^2. \end{aligned}$$

Hence,

$$|\boldsymbol{m}_D(z',z'')-\boldsymbol{\gamma}_D(a;z'-z'')|\le\Big(\frac{\|z'-z''\|}{3r^2}+\frac{1}{2r^2}\|z'-a\|\Big)\|z'-z''\|,$$

which proves (a).

(b) and (c) are easy consequences of (a) and therefore, the proof is left as an EXERCISE to the reader. □

Corollary 4.5.4. *The function $\boldsymbol{\gamma}_D$ is locally Lipschitz on $D\times\mathbb{C}^n$.*

Proof. Use (*) from the proof of (a) in the previous lemma. □

Remark 4.5.5. Lemma 4.5.3 is also true when we substitute $\boldsymbol{m}_D$ by $\boldsymbol{c}_D$ (EXERCISE).

Example 4.5.6 (k-th Reiffen pseudometric).

$$\boldsymbol{\gamma}_D^{(k)}(a;X):=\sup\Big\{\Big|\sum_{|\alpha|=k}\frac{1}{\alpha!}D^\alpha f(a)X^\alpha\Big|^{1/k}: f\in\mathcal{O}(D,\mathbb{D}),\ \operatorname{ord}_a f\ge k\Big\},$$
$$a\in D,\ X\in\mathbb{C}^n,$$

where $k\in\mathbb{N}$ and $D\subset\mathbb{C}^n$ is a domain. Note that $\boldsymbol{\gamma}_D=\boldsymbol{\gamma}_D^{(1)}$.

The proof of the fact that $(\boldsymbol{\gamma}_D^{(k)})_D$ is a holomorphically contractible family of pseudometrics is left to the reader (EXERCISE).

First, let us state some simple properties of the k-th Reiffen pseudometric (EXERCISE):

- There exists an $f\in\mathcal{O}(D,\mathbb{D})$, $\operatorname{ord}_a f\ge k$, such that

$$\boldsymbol{\gamma}_D^{(k)}(a;X)=\Big|\sum_{|\alpha|=k}\frac{1}{\alpha!}D^\alpha f(a)X^\alpha\Big|^{1/k}$$

(use the Montel theorem); such an f is called an *extremal function for* $\boldsymbol{\gamma}_D^{(k)}(a;X)$.

- Put $\mathcal{M}_D^{(k)}(a):=\{|f|^{1/k}: f\in\mathcal{O}(D,\mathbb{D}),\ \operatorname{ord}_a f\ge k\}$. Then

$$\boldsymbol{\gamma}_D^{(k)}(a;X)=\sup\Big\{\varlimsup_{\mathbb{C}_*\ni\lambda\to 0}\frac{u(a+\lambda X)}{|\lambda|}: u\in\mathcal{M}_D^{(k)}(a)\Big\},\quad a\in D,\ X\in\mathbb{C}^n. \tag{4.5.6}$$

Note that any function $u\in\mathcal{M}_D^{(k)}(a)$ satisfies: $u\colon D\to[0,1)$, $\log u$ is psh, and $u(z)\le C\|z-a\|$, when $z\in\mathbb{B}(a,r)\subset D$ for certain $C,r>0$.

- If $D=\bigcup_{j=1}^\infty D_j\subset\mathbb{C}^n$, $D_j\subset D_{j+1}$, then $\boldsymbol{\gamma}_{D_j}^{(k)}(a;X)\searrow\boldsymbol{\gamma}_D^{(k)}(a;X)$ (use Montel's theorem).

Moreover, we have the following results.

Lemma 4.5.7. *Let $a \in D \subset \mathbb{C}^n$, D a domain. Then:*

(a) $\boldsymbol{\gamma}_D^{(k)}(a;X) = \lim_{\mathbb{C}_* \ni \lambda \to 0} \frac{1}{|\lambda|}\boldsymbol{m}_D^{(k)}(a, a+\lambda X)$, *$X \in \mathbb{C}^n$.*

(b) $\boldsymbol{\gamma}_D^{(k)}(a;\cdot)$ *is continuous and* $\boldsymbol{\gamma}_D^{(k)}$ *is upper semicontinuous.*

(c) *If we additionally assume that D is bounded and $\|X\| = 1$, then $\boldsymbol{\gamma}_D^{(k)}$ is continuous on $D \times \mathbb{C}^n$ and*

$$\frac{\boldsymbol{m}_D^{(k)}(z', z'')}{\|z'-z''\|} \to \boldsymbol{\gamma}_D^{(k)}(a;X), \text{ when } z', z'' \to a,\ z' \neq z'',\ \frac{z'-z''}{\|z'-z''\|} \to X.$$

Proof. It is obvious that the left-hand side is majorized by the right-hand side.

Now let $\mathbb{C}_* \ni \lambda_\nu \to 0$. Choose extremal functions $f_\nu \in \mathcal{O}(D, \mathbb{D})$, $\operatorname{ord}_a f_\nu \geq k$, such that

$$\boldsymbol{m}_D^{(k)}(a, a+\lambda_\nu X) = \Big|\sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha f(a)(a+\lambda_\nu X)^\alpha\Big|^{1/k}.$$

By the Montel argument we get $f_{\nu_\mu} \to f$ locally uniformly on D. Note that $\operatorname{ord}_a f \geq k$ and $f \in \mathcal{O}(D, \mathbb{D})$. Then

$$\begin{aligned}\boldsymbol{\gamma}_D^{(k)}(a;X) \geq \Big|\sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha f(a) X^\alpha\Big|^{1/k} &= \lim_{\mu\to\infty} \frac{1}{|\lambda_{\nu_\mu}|} |f_{\nu_\mu}(a+\lambda_{\nu_\mu} X)|^{1/k} \\ &= \lim_{\mu\to\infty} \frac{1}{|\lambda_{\nu_\mu}|} \boldsymbol{m}_D^{(k)}(a, a+\lambda_{\nu_\mu} X).\end{aligned}$$

The proof of the remaining points are left to the reader as an EXERCISE. □

What we saw up to now is that the Möbius functions have led via (4.5.5) or (4.5.6) to holomorphically contractible families of pseudometrics.

In the case of the pluricomplex Green function we put

$$\begin{aligned}\mathcal{K}_D(a) := \{u : D \to [0,1) : \log u \in \mathcal{PSH}(D), \\ \exists_{M,r>0} : u(z) \leq M\|z-a\|,\ z \in \mathbb{B}(a,r) \subset D\},\end{aligned}$$

where D is a domain in $\mathbb{C}^n$ and $a \in D$.[21] Recall from Corollary 4.2.27 that $\boldsymbol{g}_D(a,\cdot) \in \mathcal{K}_D(a)$. Moreover, for any $X \in \mathbb{C}^n$ the limit

$$\limsup_{\mathbb{C}_* \ni \lambda \to 0} \frac{1}{|\lambda|} u(a+\lambda X)$$

always exists. Thus we can define

[21] Note that $\mathcal{M}_D^{(k)}(a) \subset \mathcal{K}_D(a)$.

Example 4.5.8 (Azukawa pseudometric).

$$\boldsymbol{A}_D(a;X) := \sup\Big\{\limsup_{\mathbb{C}_*\ni\lambda\to 0}\frac{1}{|\lambda|}u(a+\lambda X) : u\in\mathcal{K}_D(a)\Big\} \tag{4.5.7}$$

$$= \limsup_{\mathbb{C}_*\ni\lambda\to 0}\frac{1}{|\lambda|}\boldsymbol{g}_D(a,a+\lambda X), \quad a\in D,\ X\in\mathbb{C}^n. \tag{4.5.8}$$

Indeed, for $D=\mathbb{D}$ we have

$$\limsup_{\mathbb{C}_*\ni\lambda\to 0}\frac{1}{|\lambda|}\boldsymbol{g}_{\mathbb{D}}(a,a+\lambda X) = \limsup_{\mathbb{C}_*\ni\lambda\to 0}\frac{1}{|\lambda|}\boldsymbol{m}(a,a+\lambda X) = \boldsymbol{\gamma}(a;X).$$

Note that $u\circ F\in\mathcal{K}_G(a)$ whenever $F\in\mathcal{O}(G,D)$ and $u\in\mathcal{K}_D(F(a))$. Hence, property (4.5.3) follows.

Lemma 4.5.9. *Let D be as before. Then $\boldsymbol{A}_D$ is upper semicontinuous.*

Proof. Fix $a\in\mathbb{B}(a,2r)\subset D$ and $X_0\in\mathbb{C}^n$. Suppose that $\boldsymbol{A}_D(a;X_0)<M'<M$. Then $\frac{1}{|\lambda|}\boldsymbol{g}_D(a,a+\lambda X_0)\le M'$ when $0<|\lambda|\le 2\varepsilon$ for a certain positive $\varepsilon<\frac{r}{2(\|X_0\|+1)}$. In particular, $\frac{1}{|\lambda|}\boldsymbol{g}_D(a,a+\lambda X_0)\le\varepsilon M'$, $|\lambda|=\varepsilon$. Then, applying the upper semicontinuity of $\boldsymbol{g}_D$, there is a positive $\delta<r$ such that

$$\boldsymbol{g}_D(b,b+\lambda X)<\varepsilon M', \quad b\in\mathbb{B}(a,\delta),\ \|X-X_0\|<\delta,\ |\lambda|=\varepsilon.$$

Observe that for such an X the function

$$\lambda\mapsto\begin{cases}\frac{1}{|\lambda|}\boldsymbol{g}_D(b,b+\lambda X) & \text{if } 0<|\lambda|\le 2\varepsilon,\\ \boldsymbol{A}_D(b;X) & \text{if } \lambda=0,\end{cases}$$

is psh on $K(2\varepsilon)$. Hence the maximum principle leads to $\boldsymbol{A}_D(b;X)<M$ for all b,X as above, which proves that $\boldsymbol{A}_D$ is upper semicontinuous at (a,X_0). ☐

Example 4.5.10. Let $D=D_h:=\{z\in\mathbb{C}^n : h(z)<1\}$ be a pseudoconvex balanced domain. Then $\boldsymbol{A}_D(0;\cdot)=h$ on $\mathbb{C}^n$. In particular, $\boldsymbol{A}_D(0;\cdot)$ need not be continuous and is not necessarily a seminorm.

Indeed, we only have to recall that $\boldsymbol{g}_D(0,\cdot)=h$.

The following example ([Zwo 2000a]) shows that, in general, the "lim sup" in the definition of the Azukawa pseudometric cannot be substituted by "lim".

Example 4.5.11. Let $h\colon\mathbb{C}^2\to\mathbb{R}_+$ be the function from Proposition 1.15.12. Recall that $\log h\in\mathcal{PSH}(\mathbb{C}^2)$, $h(\lambda z)=|\lambda|h(z)$, $h^{-1}(0)$ is dense in $\mathbb{C}^2$, and $h\not\equiv 0$. Choose $a\in\mathbb{C}_*^2$ with $h(a)\neq 0$. Then $\tilde{h}(z)=\frac{1}{h(a)}h(z_1a_1,z_2a_2)$ satisfies the same properties as h but now $\tilde{h}(1,1) = 1$. Finally, define $\hat{h}(z) := \max\{\tilde{h}(z), \frac{\|z\|}{10}\}$. Note that $\hat{h}$ is not continuous at the point $(1,1)$. Then

$$D = \{z \in \mathbb{C}^2 : \hat{h}(z) < 1\}$$

is a bounded pseudoconvex balanced domain.

Since $\tilde{h}^{-1}(0)$ is dense we choose a sequence $(z^j)_j$ with $z^j \to (1,1)$ and $\tilde{h}(z^j) = 0$. Therefore, $\hat{h}(z^j) \le 1/5$ for large j and so $\hat{h}(1, z_2^j/z_1^j) \le 1/4$ for $j \gg 1$. Then there is a sequence $(\alpha_j)_j \subset \mathbb{C}$, $\alpha_j \to 0$, such that $e^{\alpha_j} = z_2^j/z_1^j$. Using a similar argument we find another zero sequence $(\beta_j)_j \subset \mathbb{C}$ satisfying $\hat{h}(1, e^{\beta_j}) \to 1$ (Exercise[22]).

Let $F \in \operatorname{Aut}(\mathbb{C}^2)$, $F(z) := (z_1, z_2 \exp(-z_1))$. By Corollary 1.15.6, $D' := F^{-1}(D)$ is a bounded pseudoconvex domain in $\mathbb{C}^2$. In virtue of Proposition 4.2.21 it follows that

$$\begin{aligned}\frac{1}{a_k}\boldsymbol{g}_{D'}(0,(a_k,a_k)) &= \frac{1}{a_k}\boldsymbol{g}_D(0,(a_k,a_k\exp(a_k)))\\ &= \frac{1}{a_k}\hat{h}(a_k,a_k\exp(a_k)) = \hat{h}(1,\exp(a_k)) < 1/4,\end{aligned}$$

when $k \to \infty$. In a similar way, we get $\boldsymbol{g}_{D'}(0,(b_k,b_k)) \to 1$, when $k \to \infty$. Hence, this different behavior of $\frac{1}{|\lambda|}\boldsymbol{g}_{D'}(0,\lambda(1,1))$, when $\lambda \to 0$, verifies that we cannot take the limit in (4.5.8).

Exercise 4.5.12. Try to construct a simpler example of a bounded pseudoconvex balanced domain in $\mathbb{C}^2$ with the same phenomenon as above.

Nevertheless, in the case when D is bounded and hyperconvex the "limsup" in the definition of the Azukawa pseudometric can be substituted by just taking the limit ([Zwo 2000a]).

Proposition 4.5.13. *If D is a bounded hyperconvex domain in $\mathbb{C}^n$, then:*

(a) *$\boldsymbol{A}_D$ is continuous,*
(b) $\boldsymbol{A}_D(a;X) = \lim\limits_{\mathbb{C}_*\ni\lambda\to 0} \frac{1}{|\lambda|}\boldsymbol{g}_D(a, a+\lambda X), \quad (a,X) \in D \times \mathbb{C}^n.$

We should mention that there are similar results even under weaker hypotheses; for more details see [Zwo 2000a].

The proof of the above proposition needs some preparation which will be discussed first. Let D be as in the proposition and $a \in D$. Put

$$D_\varepsilon := D_\varepsilon(a) := \{z \in D : \boldsymbol{g}_D(a,z) < e^{-\varepsilon}\},$$

where $\varepsilon \in \mathbb{R}_{>0}$. Obviously, D_ε is open, $a \in D_\varepsilon$, and, by Theorem 4.2.34, $D_\varepsilon \Subset D$. Even more is true.

[22]Use the mean value inequality for psh functions.

Lemma 4.5.14. *Under the above conditions, D_ε is a domain.*

Proof. Suppose the contrary, i.e. that D_ε has a connected component U with $a \notin U$. We know that $\boldsymbol{g}_D < e^{-\varepsilon}$ on U and $\boldsymbol{g}_D(a,z) \geq e^{-\varepsilon}$, $z \in \partial U \cap D$. Consequently, the function

$$v(z) := \begin{cases} e^{-\varepsilon} & \text{if } z \in U, \\ \boldsymbol{g}_D(a,z) & \text{if } z \in D \setminus U, \end{cases}$$

is psh on D (Exercise). Noting that $a \notin U$ we have $v \leq \boldsymbol{g}_D(a,\cdot)$ on D. In particular, $\boldsymbol{g}_D(a,\cdot)|_U \geq e^{-\varepsilon}$; a contradiction. □

Lemma 4.5.15. *Let a, D, D_ε be as before. Then*

$$\boldsymbol{g}_{D_\varepsilon}(a,z) = \boldsymbol{g}_D(a,z) \cdot e^{\varepsilon}, \quad z \in D_\varepsilon; \tag{4.5.9}$$

$$\boldsymbol{A}_{D_\varepsilon}(a;X) = \boldsymbol{A}_D(a;X) \cdot e^{\varepsilon}, \quad z \in D_\varepsilon,\ X \in \mathbb{C}^n. \tag{4.5.10}$$

Proof. Note that $\boldsymbol{g}_D(a,\cdot)e^{\varepsilon} < 1$ on D_ε. Thus, $\boldsymbol{g}_D(a,z)e^{\varepsilon} \leq \boldsymbol{g}_{D_\varepsilon}(a,z)$, $z \in D_\varepsilon$. On the other hand, $\boldsymbol{g}_D(a,z) \geq e^{-\varepsilon}$ when $z \in \partial D_\varepsilon$. Therefore, the function

$$v(z) := \begin{cases} \boldsymbol{g}_{D_\varepsilon}(a,z) \cdot e^{-\varepsilon} & \text{if } z \in D_\varepsilon, \\ \boldsymbol{g}_D(a,z) & \text{if } z \in D \setminus D_\varepsilon, \end{cases}$$

is psh on D. Hence, $v \leq \boldsymbol{g}_D(a,\cdot)$ on D, which finally gives (4.5.9). The remaining equation is a simple consequence of the definition of the Azukawa pseudometric. □

Proof of Proposition 4.5.13. (a) Fix $(a,X) \in D \times \mathbb{C}^n$ with $\boldsymbol{A}_D(a;X) \neq 0$;[23] in particular, $X \neq 0$. Suppose that there is a number $M > \boldsymbol{A}_D(a;X)$ and a sequence $((a_j,X_j))_j \in D \times (\mathbb{C}^n \setminus \{0\})$ converging to (a,X) such that $\boldsymbol{A}_D(a_j;X_j) \geq M$, $j \in \mathbb{N}$. Fix then $\varepsilon \in \mathbb{R}_{>0}$ such that $M > \boldsymbol{A}_D(a;X) \cdot e^{\varepsilon}$.

Put $\varepsilon' := 2\varepsilon$. Then $D_{\varepsilon'}(a) \Subset D_\varepsilon(a)$ (note that $\boldsymbol{g}_D$ is continuous on $D \times \bar{D}$, where $\boldsymbol{g}_D = 0$ on $D \times \partial D$).

Now we choose affine isomorphisms $F_j \in \operatorname{Aut}(\mathbb{C}^n)$ ($j \gg 1$) such that

$$F_j(a_j) = a, \quad F_j'(a_j)X_j = X,$$
$$D_{\varepsilon'}(a) \Subset F_j(D_\varepsilon(a_j)), \quad j \in \mathbb{N} \text{ large enough.}$$

Then, by (4.5.10),

$$\begin{aligned} \boldsymbol{A}_D(a_j;X_j) \cdot e^{\varepsilon} &= \boldsymbol{A}_{D_\varepsilon(a_j)}(a_j;X_j) = \boldsymbol{A}_{F_j(D_\varepsilon(a_j))}(F_j(a_j);F_j'(a_j)X_j) \\ &= \boldsymbol{A}_{F_j(D_\varepsilon(a_j))}(a;X) \leq \boldsymbol{A}_{D_{\varepsilon'}(a)}(a;X) = \boldsymbol{A}_D(a;X) \cdot e^{\varepsilon'}, \end{aligned}$$

i.e. $\boldsymbol{A}_D(a_j;X_j) \leq \boldsymbol{A}_D(a;X) \cdot e^{\varepsilon} < M$; a contradiction. Hence, $\boldsymbol{A}_D$ is continuous.

[23] Note that $\boldsymbol{A}_D$ is upper semicontinuous.

(b) Without loss of generality we may assume that $a = 0$ and $\boldsymbol{A}_D(0;X) > 0$ (EXERCISE); in particular, $X \neq 0$. Suppose now that there is a sequence $(\lambda_j)_j \subset \mathbb{C}_*$, $\lambda_j \to 0$, such that

$$\frac{1}{|\lambda_j|}\boldsymbol{g}_D(0, 0 + \lambda_j X) < \boldsymbol{A}_D(0;X)e^{-2\varepsilon}, \quad j \in \mathbb{N}.$$

Since $D_\varepsilon = D_\varepsilon(0) \Subset D$, we find a $\theta_0 \in (0,\pi)$ such that $e^{i\theta}D_\varepsilon \Subset D$, $|\theta| < \theta_0$. Moreover, we may assume that $\lambda_j X \in D_\varepsilon$, $j \in \mathbb{N}$. Then

$$\begin{aligned}\frac{1}{|\lambda_j|}\boldsymbol{g}_D(0, e^{i\theta}\lambda_j X) &\le \frac{1}{|\lambda_j|}\boldsymbol{g}_{e^{i\theta}D_\varepsilon}(0, e^{i\theta}\lambda_j X) = \frac{1}{|\lambda_j|}\boldsymbol{g}_{D_\varepsilon}(0, \lambda_j X)\\ &= \frac{1}{|\lambda_j|}\boldsymbol{g}_D(0, \lambda_j X)\cdot e^{\varepsilon} < \boldsymbol{A}_D(a;X)e^{-\varepsilon}, \quad |\theta| < \theta_0,\ j \in \mathbb{N}.\end{aligned}$$

Fix $r > 0$ such that $\mathbb{B}(r\|X\|) \subset D$. Put

$$u(\zeta) := \begin{cases}\frac{1}{|\zeta|}\boldsymbol{g}_D(0,\zeta X) & \text{if } \zeta \in K_*(r),\\ \boldsymbol{A}_D(0;X) & \text{if } \zeta = 0.\end{cases}$$

Then $\log u \in \mathcal{SH}(K(r))$.

By the upper semicontinuity of u there is a j_0 such that $\lambda_j \in K(r)$, $j \ge j_0$, and

$$u(e^{i\theta}\lambda_j) < u(0)e^{\frac{2\theta_0\varepsilon}{2\pi - 2\theta_0}} =: u(0)e^{\tilde{\varepsilon}}, \quad \theta \in [-\pi,\pi].$$

On the other hand we already know that

$$u(e^{i\theta}\lambda_j) < u(0)e^{-\varepsilon}, \quad j \in \mathbb{N},\ |\theta| < \theta_0.$$

Applying the mean value inequality for the subharmonic function u yields for large j,

$$\begin{aligned}2\pi u(0) &\le \int_{-\pi}^{\pi}\log u(e^{i\theta}\lambda_j)d\theta\\ &< \int_{|\theta|<\theta_0}(\log u(0) - \varepsilon)d\theta + \int_{\pi\ge|\theta|>\theta_0}\log u(e^{i\theta}\lambda_j)d\theta\\ &< 2\theta_0(u(0)-\varepsilon) + (2\pi - 2\theta_0)(u(0)+\tilde{\varepsilon}) = 2\pi u(0);\end{aligned}$$

a contradiction. ☐

Remark 4.5.16. Using the former argument one can even prove (EXERCISE) the following sharper version of Proposition 4.5.13 when D is a bounded hyperconvex domain in $\mathbb{C}^n$ (cf. [Zwo 2000a]).

$$\boldsymbol{A}_D(a;X) = \lim_{\substack{z',z''\to a\\ z'\neq z''\\ \frac{z'-z''}{\|z'-z''\|}\to X}} \frac{\boldsymbol{g}_D(z',z'')}{\|z'-z''\|}, \quad (a,X) \in D \times \mathbb{C}^n,\ \|X\| = 1.$$

Example 4.5.17 (Kobayashi–Royden pseudometric).

$$\varkappa_D(a;X) := \inf\{t \geq 0 : \exists_{\varphi\in\mathcal{O}(\mathbb{D},D)} : \varphi(0)=a,\ t\varphi'(0)=X\},\ ^{24} \tag{4.5.11}$$

where $D \subset \mathbb{C}^n$ is a domain and $(a,X) \in D \times \mathbb{C}^n$.

Indeed, it is easily seen that $\varkappa_D$ is a pseudometric. Now, let $D = \mathbb{D} \ni a$. If $\varphi \in \mathcal{O}(\mathbb{D},\mathbb{D})$ is such that $\varphi(0)=a$ and $t\varphi'(0) = X \in \mathbb{C}_*$, $t \geq 0$, then

$$\mathbb{D} \ni \lambda \overset{\psi}{\mapsto} \frac{\varphi(\lambda)-a}{1-\bar{a}\varphi(\lambda)}$$

belongs to $\mathcal{O}(\mathbb{D},\mathbb{D})$, $\psi(0)=0$. Therefore, by the Schwarz lemma, we have $1 \geq |\psi'(0)| = \frac{|\varphi'(0)|}{1-|a|^2}$, and so $t = |X|/|\varphi'(0)| \geq |X|/(1-|a|^2)$. Hence, $\varkappa_{\mathbb{D}}(a;X) \geq \boldsymbol{\gamma}(a;X)$. To get the converse inequality for $X=1$ it suffices to take $\varphi(\lambda) = \frac{\lambda+a}{1+\bar{a}\lambda}$, $\lambda \in \mathbb{D}$.

The proof of (4.5.3) is simple and therefore left as an Exercise.

First we collect a few simple properties of the Kobayashi–Royden pseudometric.

Exercise 4.5.18. Prove the following statements:

(a) $\varkappa_D(a;X) := \inf\{t > 0 : \exists_{\varphi\in\mathcal{O}(\bar{\mathbb{D}},D)} : \varphi(0)=a,\ t\varphi'(0)=X\}$.

(b) If $(\delta_D)_D$ is a holomorphically contractible family of pseudometrics, then $\boldsymbol{\gamma}_D \leq \delta_D \leq \varkappa_D$, $D \subset \mathbb{C}^n$ a domain, i.e. $(\boldsymbol{\gamma}_D)_D$ (resp. $(\varkappa_D)_D$) is the *minimal* (resp. *maximal*) *holomorphically contractible family of pseudometrics*.

(c) If $D_j \nearrow D$ and $(a,X) \in D \times \mathbb{C}^n$, then $\varkappa_{D_j}(a;X) \searrow \varkappa_D(a;X)$ when $j \to \infty$.

(d) If D is taut and $(a,X) \in D \times \mathbb{C}^n$, then there exists an extremal analytic disc $\varphi \in \mathcal{O}(\mathbb{D},D)$, i.e. $\varphi(0)=a$ and $\varkappa_D(a;X)\varphi'(0) = X$. Such a φ is called a *$\varkappa_D$-geodesic for* (a,X).

(e) Let $D, G \subset \mathbb{C}^n$ be domains and let $F : G \to D$ be a holomorphic covering. Assume that $(\tilde{z},X) \in G \times \mathbb{C}^n$. Then $\varkappa_G(\tilde{z};X) = \varkappa_D(F(\tilde{z});F'(\tilde{z})X)$. (See Proposition 4.2.38.)

Moreover, we have:

Lemma 4.5.19. (a) *If $D = D_h \subset \mathbb{C}^n$ is a pseudoconvex balanced domain with Minkowski function h, then*

$$\varkappa_D(0;X) = h(X), \quad X \in \mathbb{C}^n.$$

In particular, $\varkappa_D(a;\cdot)$ need not be continuous.

(b) $\varkappa_D : D \times \mathbb{C}^n \to \mathbb{R}_+$ *is upper semicontinuous.*

(c) *If D is taut, then $\varkappa_D$ is continuous on $D \times \mathbb{C}$.*

[24] Note that the infimum is taken over a non-empty set of analytic discs.

Proof. (a) By Example 4.5.10, $\boldsymbol{\varkappa}_D(0;X) \ge \boldsymbol{A}_D(0;X) = h(X)$, $X \in \mathbb{C}^n$.

To discuss the converse inequality assume first that $h(X) \ne 0$. Then $\mathbb{D} \ni \lambda \overset{\varphi}{\mapsto} \lambda X/h(X)$ gives an analytic disc in D with $\varphi(0) = 0$ and $h(X)\varphi'(0) = X$, i.e. $\boldsymbol{\varkappa}_D(0;X) \le h(X)$. If $h(X) = 0$, then $\mathbb{D} \ni \lambda \overset{\psi_k}{\longmapsto} \lambda kX$, $k \in \mathbb{N}$, gives an analytic disc in D with $\frac{1}{k}\psi_k'(0) = X$, i.e. $\boldsymbol{\varkappa}_D(0;X) = 0$.

(b) Fix a point $(a,X) \in D \times \mathbb{C}^n$ and assume that $\boldsymbol{\varkappa}_D(a;X) < A$ for a certain real number A. By Exercise 4.5.18 (d), there is an analytic disc $\varphi \in \mathcal{O}(\overline{\mathbb{D}}, D)$ such that $\varphi(0) = a$ and $t\varphi'(0) = X$, where $\mathbb{R}_{>0} \ni t < A$ is suitably chosen. Then $\varphi(\overline{\mathbb{D}})$ is a compact subset of D. In particular, a full ε-neighborhood of $\varphi(\overline{\mathbb{D}})$ is contained in D. Now take $z \in \mathbb{B}(a, \varepsilon/4)$ and $Y \in \mathbb{C}^n$ with $(1/t)\|Y - X\| < \varepsilon/4$. Then

$$\psi(\lambda) := \varphi(\lambda) + (z - a) + (\lambda/t)(Y - X)$$

leads to a $\psi \in \mathcal{O}(\mathbb{D}, D)$, $\psi'(0) = z$ and $t\psi'(0) = t\varphi'(0) + Y - X = Y$, i.e. $\boldsymbol{\varkappa}_D(z;Y) \le t < A$.

(c) is left as an EXERCISE. □

As a direct application we get

Corollary 4.5.20. *Let $D_j = D_{h_j} \subset \mathbb{C}^n$ be balanced pseudoconvex domains, $j = 1, 2$. If $F \in \operatorname{Bih}_{0,0}(D_1, D_2)$, then $F'(0)\colon D_1 \to D_2$ is a linear isomorphism.*

Proof. Let $X \in D_1$. Then

$$h_2(F'(0)X) = \boldsymbol{\varkappa}_{D_2}(0; F'(0)X) = \boldsymbol{\varkappa}_{D_1}(0;X) = h_1(X),$$

which immediately gives the proof. □

Remark 4.5.21. Let D_j be bounded balanced pseudoconvex domains in $\mathbb{C}^n$. By a deep result of Kaup–Upmeier (see [Kau-Upm 1976], [Kau-Vig 1990]) it is known that if $\operatorname{Bih}(D_1, D_2) \ne \varnothing$, then $\operatorname{Bih}_{0,0}(D_1, D_2) \ne \varnothing$. In particular, if D_1 is biholomorphically equivalent to D_2, then D_1 is linearly equivalent to D_2.

Exercise 4.5.22. Applying Lemma 4.5.19, prove:

(a) $\boldsymbol{\gamma}_{\mathbb{B}_n}(a;X) = \boldsymbol{\varkappa}_{\mathbb{B}_n}(a;X) = \left(\frac{\|X\|^2}{1-\|a\|^2} + \frac{|\langle a,X\rangle|^2}{(1-\|a\|^2)^2}\right)^{1/2}$.[25]

(b) $\mathbb{B}_n$ is not biholomorphically equivalent to $\mathbb{D}^n$, $n \ge 2$.

(c) Decide whether $\mathbb{B}_3$ and the domain $D := \{z \in \mathbb{C}^3 : |z_1|^4 + |z_2|^4 + |z_3|^4 < 1\}$ are biholomorphically equivalent (use Remark 4.5.21).

There is also another way to define the Kobayashi–Royden pseudometric which will be important in the proof of Proposition 4.5.25.

[25] To get the equation for γ use the fact that $\mathbb{B}$ is convex.

Proposition 4.5.23. *Let $(a, X) \in D \times \mathbb{C}^n$, $X \neq 0$, where D is a domain in $\mathbb{C}^n$. Then*

$$\varkappa_D(a; X) = \inf\{t \in \mathbb{R}_{>0} : \exists_{F \in \mathcal{O}(\mathbb{B}_n, D)} : F(0) = a,\ t\frac{\partial F}{\partial z_1}(0) = X,\ \det F'(0) \neq 0\}.$$

Proof. Only during this proof we will denote the right-hand side by $\widetilde{\varkappa}_D(a; X)$. Obviously, any mapping F in the formula for $\widetilde{\varkappa}_D(a; X)$ induces an analytic disc $\varphi \in \mathcal{O}(\mathbb{D}, D)$ by $\varphi(\lambda) = F(\lambda, 0, \dots, 0)$. Hence, $\varkappa_D(a; X) \leq \widetilde{\varkappa}_D(a; X)$.

Now suppose that $\varkappa_D(a; X) < m < \widetilde{\varkappa}_D(a; X)$. Then there exist a $t \in (\varkappa_D(a; X), m)$ and a $\varphi \in \mathcal{O}(\mathbb{D}, D)$ such that $\varphi(0) = a$ and $t\varphi'(0) = X$. Put

$$F_\varepsilon(z) := (\varphi_1(z_1), \varphi_2(z_1) + \varepsilon z_2, \dots, \varphi_n(z_1) + \varepsilon z_n), \quad z \in \mathbb{D} \times \mathbb{C}^{n-1},\ \varepsilon > 0.$$

Obviously, $\det F_\varepsilon'(0) \neq 0$, $F_\varepsilon(\cdot, 0, \dots, 0) = \varphi$ on $\mathbb{D}$, $t\frac{\partial F_\varepsilon}{\partial z_1}(0) = X$. Now fix $r < 1$, near 1. Then $F_1(r\overline{\mathbb{D}}, 0, \dots, 0)$ is a compact subset in D. Thus we can choose a δ small enough such that $F_1(r\overline{\mathbb{D}} \times \delta\mathbb{D} \times \dots \times \delta\mathbb{D}) \Subset D$. Hence, if ε is sufficiently small, $F_\varepsilon \in \mathcal{O}(\mathbb{B}_n(r), D)$. Finally, define $\widetilde{F}(z) := F_\varepsilon(z/r)$, $z \in \mathbb{B}_n$, which gives the desired contradiction for r very near to 1. $\square$

Lemma 4.5.24. *Let $a, z \in \mathbb{B}_n$ and let $\varphi \in \mathcal{O}(\mathbb{D}, \mathbb{B}_n)$, $\varphi(\lambda_0') = a$, $\varphi(\lambda_0'') = z$, and $\lambda_0' \neq \lambda_0''$ such that $\widetilde{\boldsymbol{k}}^*_{\mathbb{B}_n}(a, z) = \widetilde{\boldsymbol{k}}^*_{\mathbb{B}_n}(\varphi(\lambda_0'), \varphi(\lambda_0'')) = \boldsymbol{m}(\lambda_0', \lambda_0'')$. Then $\varkappa_{\mathbb{B}_n}(\varphi(\lambda); \varphi'(\lambda)) = \boldsymbol{\gamma}(\lambda; 1)$, $\lambda \in \mathbb{D}$.*

In particular, φ is a $\varkappa_{\mathbb{B}_n}$-geodesic for $(\varphi(\lambda), \varphi'(\lambda))$, $\lambda \in \mathbb{D}$.

Proof. Observe that $\widetilde{\boldsymbol{k}}^*_{\mathbb{B}_n} = \boldsymbol{m}_{\mathbb{B}_n}$ and $\varkappa_{\mathbb{B}_n} = \boldsymbol{\gamma}_{\mathbb{B}_n}$ (Exercise). Put

$$u(\lambda) := \frac{\boldsymbol{m}_{\mathbb{B}_n}(a, \varphi(\lambda))}{\boldsymbol{m}(\lambda_0', \lambda)}, \quad \lambda \in \mathbb{D} \setminus \{\lambda_0'\}.$$

Recall that $\boldsymbol{m}_{\mathbb{B}_n}(a, \cdot)$ is continuous; so, by its definition, it is log-psh. Therefore, u is sh, $u \leq 1$, and $u(\lambda_0'') = 1$. Then, by the maximum principle, it follows that $u \equiv 1$ on $\mathbb{D} \setminus \{\lambda_0''\}$. So

$$\boldsymbol{m}_{\mathbb{B}_n}(a, \varphi(\lambda)) = \boldsymbol{m}(\lambda_0', \lambda), \quad \lambda \in \mathbb{D}.$$

Now we can repeat the same argument w.r.t. the first variable to get finally

$$\boldsymbol{m}_{\mathbb{B}_n}(\varphi(\lambda'), \varphi(\lambda'')) = \boldsymbol{m}(\lambda', \lambda''), \quad \lambda', \lambda'' \in \mathbb{D}.$$

Fix $\lambda_0 \in \mathbb{D}$. Then, by Lemma 4.5.3, we have

$$\begin{aligned}\boldsymbol{\gamma}(\lambda_0; 1) &= \lim_{\lambda_0 \neq \lambda \to \lambda_0} \frac{\boldsymbol{m}(\lambda_0, \lambda)}{|\lambda_0 - \lambda|} \\ &= \lim_{\lambda_0 \neq \lambda \to \lambda_0} \frac{\boldsymbol{m}_{\mathbb{B}_n}(\varphi(\lambda_0), \varphi(\lambda))}{|\lambda_0 - \lambda|} = \boldsymbol{\gamma}_{\mathbb{B}_n}(\varphi(\lambda_0); \varphi'(\lambda_0)).\end{aligned}$$ $\square$

Note that the definition of the Kobayashi–Royden pseudometric is of different type than the ones of the previous pseudometrics. Nevertheless, if the domain D is taut, then we have the following result (see [Pan 1994]).

Proposition 4.5.25. *Let $D \Subset \mathbb{C}^n$ be a taut domain. Then*

$$\begin{aligned}\varkappa_D(a;X) &= \lim_{\mathbb{C}_* \ni \lambda \to 0} \frac{1}{|\lambda|} \widetilde{\boldsymbol{k}}_D(a, a+\lambda X) \\ &= \lim_{\mathbb{C}_* \ni \lambda \to 0} \frac{1}{|\lambda|} \widetilde{\boldsymbol{k}}_D^*(a, a+\lambda X), \quad (a,X) \in D \times \mathbb{C}^n.\end{aligned}$$

$\boxed{?}$ We do not know any example of a non-taut domain for which this result becomes false although such an example seems very probable. $\boxed{?}$

Proof. Note that it suffices to prove only the second formula. Suppose it is not true. Then there are a point $(a,X) \in D \times \mathbb{C}^n$, $X \neq 0$, and a sequence $\mathbb{C}_* \ni \lambda_j \to 0$ such that

$$\left| \frac{1}{|\lambda_j|} \widetilde{\boldsymbol{k}}_D^*(a, a+\lambda_j X) - \varkappa_D(a;X) \right| \geq \varepsilon_0 > 0 \tag{4.5.12}$$

for some ε_0. Applying Proposition 4.2.11, we may choose $\widetilde{\boldsymbol{k}}_D^*$-geodesics φ_j for $(a, a+\lambda_j X)$, i.e. $\varphi_j \in \mathcal{O}(\mathbb{D}, D)$, $\varphi_j(0) = a$, $\varphi_j(t_j) = a + \lambda_j X$, and $t_j = \widetilde{\boldsymbol{k}}_D^*(a, a+\lambda_j X) > 0$ (recall that D is bounded). Moreover, D is taut, so we may, without loss of generality, assume that $\varphi_j \to \varphi \in \mathcal{O}(\mathbb{D}, D)$, $\varphi(0) = a$, locally uniformly.

Fix $\mathbb{B}(a,r) \subset D$. Then, if j is sufficiently large,

$$\begin{aligned}1 = \frac{\widetilde{\boldsymbol{k}}_D^*(a, a+\lambda_j X)}{t_j} &\leq \frac{\widetilde{\boldsymbol{k}}_{\mathbb{B}(a,r)}^*(\varphi_j(0), \varphi_j(t_j))}{t_j} \\ &\leq \frac{1}{r} \frac{\|\varphi_j(0) - \varphi_j(t_j)\|}{t_j} \to \frac{1}{r}\|\varphi'(0)\|.\end{aligned}$$

Hence, $\varphi'(0) \neq 0$.

Fix an $\varepsilon \in \mathbb{R}_{>0}$. Then, by Proposition 4.5.23, we find an $F \in \mathcal{O}(\mathbb{B}_n, D)$ and a $t > 0$ such that $F(0) = a$, $\det F'(0) \neq 0$, $t\frac{\partial F}{\partial z_1}(0) = \varphi'(0)$, and

$$0 < \varkappa_D(a;\varphi'(0)) \leq t \leq \varkappa_D(a;\varphi'(0)) + \varepsilon.$$

Now we choose open neighborhoods $U = U(0) \subset \mathbb{B}_n$ and $V = V(a) \subset D$ such that $F|_U : U \to V$ is a biholomorphic mapping. Moreover, fix j_0 such that $a + \lambda_j X \in V$ and define $q_j := (F|_U)^{-1}(a + \lambda_j X)$, $j \geq j_0$. Note that $\mathbb{B}_n$ is taut. Hence we have $\widetilde{\boldsymbol{k}}_{\mathbb{B}_n}^*$-geodesics for all pairs $(0, q_j)$, i.e. there exist $\psi_j \in \mathcal{O}(\mathbb{D}, \mathbb{B}_n)$, $\psi_j(0) = 0$, $\psi_j(\tau_j) = q_j$, and $\widetilde{\boldsymbol{k}}_{\mathbb{B}_n}^*(0, q_j) = \tau_j$, $j \geq j_0$.

In virtue of Lemma 4.5.24, we conclude that $\varkappa_{\mathbb{B}_n}(0;\psi_j'(0)) = 1$, $j \geq j_0$.

Applying again that $\mathbb{B}_n$ is taut we may assume (without loss of generality) that $\psi_j \to \psi \in \mathcal{O}(\mathbb{D},\mathbb{B}_n)$ locally uniformly. Obviously, $\psi(0) = 0$. Then, because of Lemma 4.5.19, we obtain

$$1 = \varkappa_{\mathbb{B}_n}(0;\psi_j'(0)) \to \varkappa_{\mathbb{B}_n}(0;\psi'(0)) = 1,$$

i.e. ψ is a $\varkappa_{\mathbb{B}_n}$-geodesic for the pair $(0,\psi'(0))$.

Note that

$$\begin{aligned}\frac{q_j - 0}{t_j} &= \frac{(F|_U)^{-1}(a+\lambda_j X) - (F|_U)^{-1}(a)}{t_j}\\ &= \frac{(F|_U)^{-1}(\varphi_j(t_j)) - (F|_U)^{-1}(\varphi_j(0))}{t_j} \to (F^{-1}\circ\varphi)'(0).\end{aligned}$$

In particular, this limit exists and it is different from zero.

Observe that

$$\begin{aligned}F\circ\psi_j(0) &= F(0) = a = \varphi_j(0),\\ F\circ\psi_j(\tau_j) &= F(q_j) = a + \lambda_j X = \varphi_j(t_j).\end{aligned}$$

Therefore,

$$t_j = \widetilde{\boldsymbol{k}}_D^*(a, a+\lambda_j X) = \widetilde{\boldsymbol{k}}_D^*(F(\psi_j(0)), F(\psi_j(\tau_j))) \leq \widetilde{\boldsymbol{k}}_{\mathbb{B}_n}^*(\psi_j(0),\psi_j(\tau_j)) = \tau_j$$

for $j \geq j_0$. In particular, $1 \leq \frac{\tau_j}{t_j}$.

Recall that

$$\lim_{j\to\infty} \frac{F\circ\psi_j(\tau_j) - F\circ\psi_j(0)}{\tau_j} = (F\circ\psi)'(0) \neq 0$$

and

$$\begin{aligned}0 \neq \varphi'(0) &= \lim_{j\to\infty}\frac{\varphi_j(t_j)-\varphi_j(0)}{t_j} = \lim_{j\to\infty}\frac{a+\lambda_j X - a}{t_j}\\ &= \lim_{j\to\infty}\frac{F\circ\psi_j(\tau_j) - F\circ\psi_j(0)}{\tau_j}\frac{\tau_j}{t_j}.\end{aligned}$$

Hence, $\lim \frac{\tau_j}{t_j} =: A \geq 1$ exists. Moreover, we have

$$\varphi'(0) = A(F\circ\psi)'(0) = AF'(0)\psi'(0).$$

Taking into account that $t\frac{\partial F}{\partial z_1}(0) = \varphi'(0)$ finally leads to $A\psi'(0) = te_1 = t(1,0,\dots,0)$. Then

$$1 = \varkappa_{\mathbb{B}_n}(0;\psi'(0)) = \frac{t}{A}\varkappa_{\mathbb{B}_n}(0;e_1) = \frac{t}{A},$$

i.e. $A = t$ and, therefore, $1 \le A = t \le \boldsymbol{\varkappa}_D(a;\varphi'(0)) + \varepsilon$. Then, letting $\varepsilon \searrow 0$, gives $1 \le \boldsymbol{\varkappa}_D(a;\varphi'(0)) \le 1$. Hence, φ is a $\boldsymbol{\varkappa}_D$-geodesic for the pair $(a,\varphi'(0))$.

Note that

$$\varphi'(0) = \lim_{j\to\infty} \frac{\varphi_j(t_j) - \varphi_j(0)}{t_j} = \lim_{j\to\infty} \frac{a + \lambda_j X - a}{t_j} = X \lim_{j\to\infty} \frac{\lambda_j}{t_j}.$$

So we conclude

$$\lim_{j\to\infty} \frac{\widetilde{\boldsymbol{k}}^*_D(a, a + \lambda_j X)}{|\lambda_j|} = \boldsymbol{\varkappa}_D(a;\varphi'(0)) \lim_{j\to\infty} \frac{t_j}{|\lambda_j|} = \boldsymbol{\varkappa}_D(0;X);$$

a contradiction to (4.5.12). □

Working with the Kobayashi pseudometric we always have

Proposition 4.5.26. *Let $D \subset \mathbb{C}^n$ be a domain and $(a, X) \in D \times \mathbb{C}^n$. Then*

$$\limsup_{\mathbb{C}_*\ni\lambda\to 0} \frac{1}{|\lambda|} \boldsymbol{k}_D(a, a + \lambda X) \le \boldsymbol{\varkappa}_D(a;X).$$

Proof. If $\varepsilon > 0$ is given, then we find an analytic disc $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with the following properties:

$$\varphi(0) = a, \quad t\varphi'(0) = X, \quad 0 < t < \boldsymbol{\varkappa}_D(a;X) + \varepsilon.$$

Then φ can be written as

$$\varphi(\lambda) = a + \frac{\lambda}{t} X + \lambda^2 \tilde{\varphi}(\lambda), \quad \lambda \in \mathbb{D},$$

where $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$. Fix $\mathbb{B}(a, r) \subset D$ and then take only such $\lambda \in \mathbb{C}_*$ with $|\lambda| t < 1$ and $a + \lambda X \in \mathbb{B}(a, r)$. Hence,

$$\begin{aligned}
\frac{1}{|\lambda|} \boldsymbol{k}_D(a, a + \lambda X) &\le \frac{1}{|\lambda|} \boldsymbol{k}_D(a, \varphi(t\lambda)) + \frac{1}{|\lambda|} \boldsymbol{k}_D(\varphi(t\lambda), a + \lambda X) \\
&\le \frac{1}{|\lambda|} \boldsymbol{k}_D(\varphi(0), \varphi(t\lambda)) + \frac{1}{r|\lambda|} \|t^2\lambda^2 \tilde{\varphi}(t\lambda)\| \\
&\le \frac{1}{|\lambda|} \boldsymbol{p}(0, t\lambda) + \frac{t^2|\lambda|}{r} \|\tilde{\varphi}(t\lambda)\|.
\end{aligned}$$

Letting $\lambda \to 0$ leads to

$$\lim_{\mathbb{C}_*\ni\lambda\to 0} \frac{1}{|\lambda|} \boldsymbol{k}_D(a, a + \lambda X) \le t \le \boldsymbol{\varkappa}_D(a;X) + \varepsilon.$$

Since ε was arbitrarily small, the proposition is verified. □

But even if one sharpens the notion of the "derivative" of $\boldsymbol{k}_D$ there is, in general, no equality with the Kobayashi–Royden pseudometric as the following example shows.

Example 4.5.27. Put

$$D := \{z \in \mathbb{D}^2 : |z_1 z_2| < r^2\}, \text{ where } 0 < r < 1/2.$$

Fix points $z', z'' \in \mathbb{D}^2$, $|z_j'| < r^2$, $|z_j''| < r^2$, $j = 1, 2$. Then

$$\boldsymbol{k}_D(z', z'') \le \boldsymbol{k}_D(z', (z_1'', z_2')) + \boldsymbol{k}_D((z_1'', z_2'), z'') \le \boldsymbol{p}(z_1', z_1'') + \boldsymbol{p}(z_2', z_2'').$$

(Exercise). If we discuss the following general differential quotient at 0 in direction of (r, r), then by the former inequality we obtain

$$\begin{aligned}
&\limsup_{\substack{a\to 0\\ X\to(r,r)\\ \mathbb{C}_*\ni\lambda\to 0}} \frac{1}{|\lambda|}\boldsymbol{k}_D(a, a+\lambda X)\\
&\quad\le \limsup_{\substack{a\to 0\\ X\to(r,r)\\ \mathbb{C}_*\ni\lambda\to 0}} \frac{1}{|\lambda|}\boldsymbol{p}(a_1, a_1+\lambda X_1) + \limsup_{\substack{a\to 0\\ X\to(r,r)\\ \mathbb{C}_*\ni\lambda\to 0}} \frac{1}{|\lambda|}\boldsymbol{p}(a_2, a_2+\lambda X_2) = 2r.
\end{aligned}$$

On the other hand, $\varkappa_D(0; (r, r)) = 1$ (use Lemma 4.5.19), i.e. the "differential quotient" of the Kobayashi distance is different from the Kobayashi–Royden pseudometric.

The defect shown in the example has led S. Kobayashi to introduce a new pseudometric (see [Kob 1990]).

Example 4.5.28 (Kobayashi–Busemann pseudometric).

$$\hat{\varkappa}_D(a; \cdot) := \sup\{q : q \text{ a } \mathbb{C}\text{–seminorm},\ q \le \varkappa_D(a; \cdot)\}, \quad a \in D, \tag{4.5.13}$$

where $D \subset \mathbb{C}^n$ is a domain.

Indeed, in the case where $D = \mathbb{D}$ we know that $\varkappa_{\mathbb{D}}(a; \cdot) = \boldsymbol{\gamma}_{\mathbb{C}}(a; \cdot)$ is a norm. Hence, $\hat{\varkappa}_{\mathbb{D}} = \boldsymbol{\gamma}$. To see (4.5.3) let $F \in \mathcal{O}(G, D)$ and $a \in G$. Take a seminorm q with $q \le \varkappa_D(F(a); \cdot)$. Then

$$q(F'(a)X) \le \varkappa_D(F(a); F'(a)X) \le \varkappa_G(a; X).$$

In particular, $\tilde{q} := q(F'(a)\,\cdot)$ is a seminorm below of $\varkappa_G(a; \cdot)$. Therefore, $\tilde{q} \le \hat{\varkappa}_G(a; \cdot)$. Since q was an arbitrary seminorm, we have (4.5.3).

Note that, by definition, $\hat{\varkappa}_D(a; \cdot)$ is a seminorm.

Exercise 4.5.29. Let $D = D_h \subset \mathbb{C}^n$ be a pseudoconvex balanced domain. Give a formula for $\hat{\varkappa}_D(0; X)$, $X \in \mathbb{C}^n$.

Finally, we mention the following result by M. Kobayashi (see [Kob 2000]) that makes it possible to think of the Kobayashi–Busemann pseudometric as a derivative of the Kobayashi pseudodistance.

Proposition 4.5.30. *Let $D \Subset \mathbb{C}^n$ be a taut domain. Then*

$$\hat{\varkappa}_D(a; X) = \lim_{\substack{(b,Y)\to(a,X)\\ \mathbb{C}_*\ni\lambda\to 0}} \frac{1}{|\lambda|} \boldsymbol{k}_D(b, b + \lambda Y), \quad (a, X) \in D \times \mathbb{C}^n.$$

What we have seen is that sometimes the pseudometrics introduced here are in a strong relation with certain pseudodistances, i.e. they might be thought of as a derivative of the corresponding pseudodistances. Conversely, one may associate to a pseudometric its so-called integrated form to get either a new pseudodistance or even the one we start from. Here we will restrict ourselves only to the case of the Kobayashi–Royden pseudometric. For further information the reader is referred to [Jar-Pfl 1993] or [Jar-Pfl 2005].

Let $D \subset \mathbb{C}^n$. Note that $\varkappa_D$ is upper semicontinuous. Hence, if $\alpha\colon [0, 1] \to D$ is a piecewise $\mathcal{C}^1$-curve, then the integral $L_{\varkappa_D}(\alpha) := \int_0^1 \varkappa_D(\alpha(t); \alpha'(t))dt$ exists. We call the number $L_{\varkappa_D}(\alpha)$ the *$\varkappa_D$-length* of the curve α. Using this notion we define.

Definition 4.5.31. Let D be as above. Put $\int \varkappa_D\colon D \times D \to \mathbb{R}_+$ as

$$(\textstyle\int \varkappa_D)(a, b) := \inf\{L_{\varkappa_D}(\alpha) : \alpha \in \widehat{\mathcal{C}}^1([0, 1], D),\ \alpha(0) = a,\ \alpha(1) = b\}, \qquad a, b \in D.$$

$\int \varkappa_D$ is called the *integrated form* of $\varkappa_D$.

Note that $(\int \varkappa_D)_D$ is a holomorphically contractible family of pseudodistances (Exercise). Therefore, $\int \varkappa_D \le \boldsymbol{k}_D$. Even more is true, namely the integrated form leads back to the Kobayashi pseudodistance.

Proposition 4.5.32. *If $D \subset \mathbb{C}^n$, then $\int \varkappa_D = \boldsymbol{k}_D$.*

Proof. It remains to show that $\boldsymbol{k}_D \le \int \varkappa_D$. Suppose that the opposite is true, i.e. there are points $a, b \in D$ and a piecewise $\mathcal{C}^1$-curve $\alpha\colon [0, 1] \to D$ connecting a and b such that $L_{\varkappa_D}(\alpha) < \boldsymbol{k}_D(a, b)$. Observe that there are numbers $1 = s_0 < s_1 < \cdots < s_N = 1$ such that all $\alpha_j := \alpha|_{[s_{j-1},s_j]}$ are $\mathcal{C}^1$-curves. Then $L_{\varkappa_D}(\alpha) = \sum_{j=1}^N L_{\varkappa_D}(\alpha_j) < \sum_{j=1}^N \boldsymbol{k}_D(\alpha(s_{j-1}), \alpha_j)$. Therefore, without loss of generality, we may assume that α is a $\mathcal{C}^1$-curve.

Put $f(t) := \boldsymbol{k}_D(a, \alpha(t))$, $t \in [0, 1]$.

Fix a $t_0 \in [0,1]$ and $\mathbb{B}(\alpha(t_0), 2r) \subset D$. Then, if δ is sufficiently small, we conclude that $\alpha(t) \in \mathbb{B}(\alpha(t_0), r)$ whenever $|t - t_0| < \delta$ and $t \in [0,1]$.

If now $t, s \in [0,1] \cap (t_0 - \delta, t_0 + \delta)$, then

$$\begin{aligned}|f(s) - f(t)| &= |\boldsymbol{k}_D(a, \alpha(s)) - \boldsymbol{k}_D(a, \alpha(t))| \\ &\le \boldsymbol{k}_D(\alpha(s), \alpha(t)) \le C_{t_0} \|\alpha(s) - \alpha(t)\| \le \widetilde{C}_{t_0} |s - t|,\end{aligned}$$

i.e. the function f is locally Lipschitz. Therefore, f is almost everywhere differentiable and

$$\boldsymbol{k}_D(a, b) = \int_0^1 f'(t)dt.$$

What remains is to estimate f':

$$\begin{aligned}|f'(t)| &\le \lim_{h \to 0+} \frac{|f(t+h) - f(t)|}{h} \le \limsup_{h \to 0+} \frac{\boldsymbol{k}_D(\alpha(t+h), \alpha(t))}{h} \\ &\le \limsup_{h \to 0+} \frac{\boldsymbol{k}_D(\alpha(t), \alpha(t) + h\alpha'(t))}{h} \\ &\quad + \limsup_{h \to 0+} \frac{\boldsymbol{k}_D(\alpha(t+h), \alpha(t) + h\alpha'(t))}{h} \\ &\le \boldsymbol{\varkappa}_D(\alpha(t); \alpha'(t)) + \limsup_{h \to 0+} \frac{C_t \|\alpha(t) + h\alpha'(t) - \alpha(t+h)\|}{h} \\ &= \boldsymbol{\varkappa}_D(\alpha(t); \alpha'(t))\end{aligned}$$

for almost all $t \in [0,1)$. Here we have used Proposition 4.5.26 in the last line. Hence, $\boldsymbol{k}_D(a, b) \le L_{\boldsymbol{\varkappa}_D}(\alpha)$; a contradiction. □

Exercise 4.5.33. Formulate what is meant by $\int \widehat{\boldsymbol{\varkappa}}_D$ and prove that $\boldsymbol{k}_D = \int \widehat{\boldsymbol{\varkappa}}_D$.

4.6 Examples II – elementary Reinhardt domains

In this section we briefly discuss the formulas for some families of holomorphically contractible pseudometrics with respect to the elementary Reinhardt domains.

In the one-dimensional case we have

Proposition 4.6.1. (a) *Let $\mathbb{A} = \mathbb{A}(1/R, R)$, where $R > 1$, and $a \in \mathbb{A} \cap \mathbb{R}_+$. Put $a = R^{1-2s}$. Then*

$$\boldsymbol{\varkappa}_{\mathbb{A}}(a; 1) = \frac{\pi}{4a \log R \sin(\pi s)}.$$

(b) *Let $a \in \mathbb{D}_* \cap \mathbb{R}_+$, then $\boldsymbol{\varkappa}_{\mathbb{D}_*}(a; 1) = \frac{-1}{2a \log a}$.*

Proof. Note that $\mathbb{A}$ and $\mathbb{D}_*$ are taut domains (Exercise). Hence the formulas follow directly from Theorem 4.4.1, Corollary 4.4.2, and Proposition 4.5.25. □

Observe that $\boldsymbol{\gamma}_{\mathbb{D}_*} = \boldsymbol{\gamma}_{\mathbb{D}}$, $\boldsymbol{A}_{\mathbb{D}_*} = \boldsymbol{A}_{\mathbb{D}}$ on $\mathbb{D}_* \times \mathbb{C}$.

Now we turn to the discussion of elementary Reinhardt domains $\boldsymbol{D}_\alpha$ in higher dimensions. Let $\boldsymbol{D}_\alpha \subset \mathbb{C}^n$ be given and fix a pair $(a, X) \in \boldsymbol{D}_\alpha \times \mathbb{C}^n$. For the remaining part of this section we will always assume that (see Section 4.4):

- $n \geq 2$;
- $\alpha_1, \dots, \alpha_s < 0$ and $\alpha_{s+1}, \dots, \alpha_n > 0$ for an $s = s(\alpha) \in \{0, 1, \dots, n\}$;
- if $s < n$, then $t = t(\alpha) := \min\{\alpha_{s+1}, \dots, \alpha_n\}$;
- $a = (a_1, \dots, a_n) \in \boldsymbol{D}_\alpha$, $a_1 \cdots a_k \neq 0$, $a_{k+1} = \cdots = a_n = 0$ for a $k = k(a) \in \{s, \dots, n\}$;
- if $k < n$, then $r = r(a) = r_\alpha(a) := \alpha_{k+1} + \cdots + \alpha_n$; if $k = n$ (in particular, if $s = n$), then $r = r(a) = r_\alpha(a) := 1$; observe that if $\alpha \in \mathbb{Z}^n$, then $r(a) = \operatorname{ord}_a(z^\alpha - a^\alpha)$;
- if $\boldsymbol{D}_\alpha$ is of rational type, then $\alpha \in \mathbb{Z}^n$ and $\alpha_1, \dots, \alpha_n$ relatively prime;
- if $\boldsymbol{D}_\alpha$ is of irrational type and $s < n$, then $t(\alpha) = 1$.

The following effective formulas for holomorphically contractible pseudometrics for $\boldsymbol{D}_\alpha$ are known ([Jar-Pfl 1993], [Pfl-Zwo 1998], and [Zwo 2000]).

Theorem 4.6.2. *Under the above assumptions we have:*

α	$\boldsymbol{\gamma}_{\boldsymbol{D}_\alpha}(a; X)$	$\boldsymbol{A}_{\boldsymbol{D}_\alpha}(a; X)$
Rational type	$\boldsymbol{\gamma}(a^\alpha; a^\alpha \sum_{j=1}^n \frac{\alpha_j X_j}{a_j})$	$(\boldsymbol{\gamma}(a^\alpha; F_{(r)}(a)(X)))^{1/r}$
Irrational type, $k < n$	0	$(\prod_{j=1}^k \lvert a_j\rvert^{\alpha_j} \prod_{j=k+1}^n \lvert X_j\rvert^{\alpha_j})^{1/r}$
Irrational type, $k = n$	0	0

where $F(z) := z^\alpha$ *and* $F_{(r)}(a)(X) := \sum_{|\beta|=r} \frac{1}{\beta!} D^\beta F(a) X^\beta$.

α	$\boldsymbol{\varkappa}_{\boldsymbol{D}_\alpha}(a; X)$
Rational type, $s < n$	$\begin{cases} \boldsymbol{\gamma}((a^\alpha)^{1/t}; (a^\alpha)^{1/t} \frac{1}{t} \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}), & k = n \\ (\lvert a_1\rvert^{\alpha_1} \cdots \lvert a_k\rvert^{\alpha_k} \lvert X_{k+1}\rvert^{\alpha_{k+1}} \cdots \lvert X_n\rvert^{\alpha_n})^{1/r}, & k < n \end{cases}$
Rational type, $s = n$	$\boldsymbol{\varkappa}_{\mathbb{D}_*}(a^\alpha; a^\alpha \sum_{j=1}^n \frac{\alpha_j X_j}{a_j})$
Irrational type, $s < n$	$\begin{cases} \boldsymbol{\gamma}(\lvert a^\alpha\rvert; \lvert a^\alpha\rvert \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}), & k = n \\ (\lvert a_1\rvert^{\alpha_1} \cdots \lvert a_k\rvert^{\alpha_k} \lvert X_{k+1}\rvert^{\alpha_{k+1}} \cdots \lvert X_n\rvert^{\alpha_n})^{1/r}, & k < n \end{cases}$
Irrational type, $s = n$	$\boldsymbol{\varkappa}_{\mathbb{D}_*}(\lvert a^\alpha\rvert; \lvert a^\alpha\rvert \sum_{j=1}^n \frac{\alpha_j X_j}{a_j})$

Proof. In the case of the Carathéodory–Reiffen (resp. the Azukawa) pseudometric use Theorem 4.4.4 and Lemma 4.5.3 (c) (resp. (4.5.8)).

To prove the corresponding formulas for $\varkappa$ we will need several steps.

Proof for $\varkappa_{D_\alpha}$ – the case $k < n$. The estimate from below follows immediately by applying $\varkappa_{D_\alpha}(a;X) \geq A_{D_\alpha}(a;X)$ and the formula for A_{D_α}. So the upper estimate remains.

In a first step assume that $X_{k+1}\cdots X_n \neq 0$. Put

$$\tau := \big(|a_1|^{\alpha_1}\cdots|a_k|^{\alpha_k}|X_{k+1}|^{\alpha_{k+1}}\cdots|X_n|^{\alpha_n}\big)^{1/r} > 0.$$

In virtue of Exercise 4.4.12, we find functions $\psi = (\psi_1,\dots,\psi_n) \in \mathcal{O}(\mathbb{D},\mathbb{C}_*^n)$ satisfying

$$\psi_j(0) = a_j,\ \psi_j'(0) = X_j/\tau,\ j = 1,\dots,k, \quad \psi_j(0) = X_j/\tau,\ j = k+1,\dots,n,$$

and $\psi_1^{\alpha_1}\cdots\psi_n^{\alpha_n} = e^{i\theta}\,\mathrm{id}_{\mathbb{D}}$ for some θ. Then the holomorphic mapping

$$\mathbb{D} \ni \lambda \mapsto \varphi(\lambda) := (\psi_1(\lambda),\dots,\psi_k(\lambda),\lambda\psi_{k+1}(\lambda),\dots,\lambda\psi_n(\lambda)) \in D_\alpha$$

fulfills the following properties: $\varphi(0) = a$ and $\tau\varphi'(0) = X$. Hence, $\varkappa_{D_\alpha}(a;X) \leq \tau$.

If $X_{j_0} = 0$ for some $j_0 \in \{k+1,\dots,n\}$, then we have the holomorphic mapping

$$\begin{aligned}\mathbb{C}_*^k\times\mathbb{C}^{n-k-1} &\ni (z_1,\dots,z_{j_0-1},z_{j_0+1},\dots,z_n)\\ &\qquad\mapsto (z_1,\dots,z_{j_0-1},0,z_{j_0+1},\dots,z_n) \in D_\alpha.\end{aligned}$$

Therefore,

$$\begin{aligned}0 = \varkappa_{\mathbb{C}_*^k\times\mathbb{C}^{n-k-1}}\big((&a_1,\dots,a_{j_0-1},a_{j_0+1},\dots,a_n);\\ &(X_1,\cdots,X_{j_0-1},X_{j_0+1},\dots,X_n)\big) \geq \varkappa_{D_\alpha}(a;X),\end{aligned}$$

which proves the remaining case. □

Lemma 4.6.3. *Let $a \in D_\alpha \cap \mathbb{C}_*^n$ and $X \in \mathbb{C}^n$ such that $\sum_{j=1}^n \frac{\alpha_j X_j}{a_j} = 0$. Then $\varkappa_{D_\alpha}(a;X) = 0$.*

Proof. Observe that for $\mu \in \mathbb{D}_*$ the mapping $F_\mu\colon \mathbb{C}^{n-1} \to \mathbb{C}^n$,

$$F_\mu(z_1,\dots,z_{n-1}) := (e^{\alpha_n z_1},\dots,e^{\alpha_n z_{n-1}},\mu e^{-\alpha_1 z_1-\dots-\alpha_{n-1}z_{n-1}}),$$

belongs to $\mathcal{O}(\mathbb{C}^{n-1},D_\alpha)$. Then there are a $\mu_0 \in \mathbb{D}_*$ and a $\tilde{z} = (z_1,\dots,z_{n-1}) \in \mathbb{C}^{n-1}$ such that $F_{\mu_0}(\tilde{z}) = a$ and $F_{\mu_0}'(\tilde{z})Y = X$, where $Y = (X_1,\dots,X_{n-1})$. Hence,

$$0 = \varkappa_{\mathbb{C}^{n-1}}(\tilde{z};Y) \geq \varkappa_{D_\alpha}(a;X),$$

i.e. the proof is finished. □

Proof for $\boldsymbol{\varkappa}_{\boldsymbol{D}_\alpha}$ – the case $s < n = k$. First we recall that (Proposition 4.5.26)

$$\boldsymbol{\varkappa}_{\boldsymbol{D}_\alpha}(a; X) \geq \limsup_{\mathbb{C}_* \ni \lambda \to 0} \frac{\boldsymbol{k}_{\boldsymbol{D}_\alpha}(a, a + \lambda X)}{|\lambda|}.$$

Evaluating the right-hand side leads (by a trivial calculation) to the claimed formula for $\boldsymbol{\varkappa}_{\boldsymbol{D}_\alpha}(a; X)$. Hence the estimate from below is verified.

We continue with the estimate from above. In virtue of Lemma 4.6.3, we may assume that $\sum_{j=1}^n \frac{\alpha_j X_j}{a_j} \neq 0$ and $t = \alpha_n$ (recall that $t = 1$ in the irrational case). By the symmetry of $\boldsymbol{D}_\alpha$ we also assume that $a_j > 0$, $j = 1, \dots, n$.

Now put $\lambda_0 := (a^\alpha)^{1/\alpha_n} \in \mathbb{D} \cap (0, 1)$ and $\tau := \lambda_0 \sum_{j=1}^n \frac{\alpha_j X_j}{a_j}$. In virtue of Exercise 4.4.12, we find a $\varphi \in \mathcal{O}(\mathbb{D}, \boldsymbol{D}_\alpha)$ such that

$$\varphi(\lambda_0) = a, \quad \tau\varphi'(\lambda_0) = X.$$

Hence, $\tau \geq \boldsymbol{\varkappa}_{\boldsymbol{D}_\alpha}$, which gives the desired estimate from above. □

Proof for $\boldsymbol{\varkappa}_{\boldsymbol{D}_\alpha}$ – the case $s = n$. We leave this last case as an Exercise for the reader. Use the ideas of the corresponding case for the Lempert function and Exercise 4.5.18 (e). □ □

4.7 Hyperbolic Reinhardt domains

We know that, in general, $\boldsymbol{c}_D$ (resp. $\boldsymbol{k}_D$) need not be distances. We define

Definition 4.7.1. Let $D \subset \mathbb{C}^n$ be a domain and let $d_D \in \{\boldsymbol{c}_D, \boldsymbol{k}_D, \widetilde{\boldsymbol{k}}_D^*\}$. D is said to be *d-hyperbolic* if $d_D(a, b) = 0$, $a, b \in D$, implies that $a = b$.

In particular, D is $\boldsymbol{c}$- (resp. $\boldsymbol{k}$-) hyperbolic if and only if $\boldsymbol{c}_D$ (resp. $\boldsymbol{k}_D$) is a distance on D (in the sense of metric spaces).

Note that (Exercise)

- any bounded domain is $\boldsymbol{c}$-hyperbolic;
- any $\boldsymbol{c}$-hyperbolic domain is $\boldsymbol{k}$-hyperbolic;
- any $\boldsymbol{k}$-hyperbolic domain is $\widetilde{\boldsymbol{k}}$-hyperbolic.

In the class of pseudoconvex Reinhardt domains we have a complete description of those domains which are hyperbolic.

Theorem 4.7.2 ([Zwo 1999]). *Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Then the following properties are equivalent:*

(i) *D is $\boldsymbol{c}$-hyperbolic;*

(ii) *D is $\boldsymbol{k}$-hyperbolic;*

(iii) *D is $\widetilde{\boldsymbol{k}}$-hyperbolic;*

(iv) *D is Brody hyperbolic;*

(v) *there exists* $A = [a_{j,k}]_{1 \le j,k \le n} \in \mathbb{GL}(n, \mathbb{Z})$, *such that* $D \subset \bigcap_{j=1}^{n} \mathbb{C}^n(\alpha_j)$, *where* $\alpha_j := (a_{j,1}, \dots, a_{j,n})$, *and* $\Phi_A \colon D \to \mathbb{C}^n$,

$$\Phi_A(z) := (z^{\alpha_1}, \dots, z^{\alpha_n}), \quad z \in D,$$

maps D *biholomorphically onto its image* $\Phi_A(D)$ *which is a bounded Reinhardt domain of holomorphy.*

Proof. Obviously, (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (v) $\Rightarrow$ (i).

Now assume (iii). Suppose D is not Brody hyperbolic. Then we find a $\varphi \in \mathcal{O}(\mathbb{C}, D)$, $\varphi \not\equiv c$. Therefore, $\tilde{\boldsymbol{k}}_D(\varphi(\lambda), \varphi(0)) \le \tilde{\boldsymbol{k}}_{\mathbb{C}}(\lambda, 0) = 0$, $\lambda \in \mathbb{C}$. Since D is $\tilde{\boldsymbol{k}}$-hyperbolic, it follows that φ is the constant function $\varphi(0)$; a contradiction.

Recall that (iv) $\Rightarrow$ (v) follows directly from Theorem 1.17.11. $\square$

Consequently, this result allows us to speak only of *hyperbolic pseudoconvex Reinhardt domains*.

In the general situation there is a reformulation of $\boldsymbol{k}$-hyperbolicity in terms of the Kobayashi–Royden pseudometric. More precisely, the following statement is true.

Lemma 4.7.3. *For a domain* $D \subset \mathbb{C}^n$ *the following properties are equivalent:*

(i) D *is* $\boldsymbol{k}$*-hyperbolic;*

(ii) *for any point* $a \in D$ *and any neighborhood* $U = U(a) \subset D$ *there exist neighborhoods* $\tilde{U} = \tilde{U}(a) \subset U$ *and* $V = V(0) \subset \mathbb{D}$ *such that if* $\varphi \in \mathcal{O}(\mathbb{D}, D)$, $\varphi(0) \in \tilde{U}$, *then* $\varphi(V) \subset U$.

(iii) D *is* $\varkappa$*-hyperbolic, i.e.* $\forall_{a \in D}\ \exists_{U = U(a) \subset D}\ \exists_{c>0} : \ \boldsymbol{\varkappa}_D(z; X) \ge c\|X\|$, $z \in U$, $X \in \mathbb{C}^n$.

Proof. (i) $\Rightarrow$ (ii): Fix a and U as in (ii) and choose an $r > 0$ such that $\mathbb{B}(a, 2r) \subset U$. Then $\boldsymbol{k}_D(a, z) > 0$ whenever $z \in \partial\mathbb{B}(a, r)$. By assumption, $\boldsymbol{k}_D(a, \cdot) \ge c > 0$ on $\partial\mathbb{B}(a, r)$ (recall that $\boldsymbol{k}_D(a, \cdot)$ is continuous).

Now take a point $z \in D \setminus \mathbb{B}(a, r)$ and a piecewise $\mathcal{C}^1$-curve $\alpha \colon [0, 1] \to D$, which connects a and z. Then

$$L_{\boldsymbol{\varkappa}_D}(\alpha) \ge \int_0^{t_0} \boldsymbol{\varkappa}_D(\alpha(t); \alpha'(t))dt \ge \boldsymbol{k}_D(a, \varphi(t_0)) \ge c,$$

where $\alpha(t) \in \mathbb{B}(a, r)$, $t \in [0, t_0)$, and $\alpha(t_0) \in \partial\mathbb{B}(a, r)$. Therefore, $\boldsymbol{k}_D(a, z) \ge c$. Hence we have shown that the $\boldsymbol{k}_D$-ball $B_{\boldsymbol{k}_D}(a, c) \subset \mathbb{B}(a, r) \subset U$.[26] Put $\tilde{U} := B_{\boldsymbol{k}_D}(a, c/2)$ and $V := B_{\boldsymbol{k}_{\mathbb{D}}}(0, c/2) \subset \mathbb{D}$. Now let $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) \in \tilde{U}$. If $\lambda \in V$, then

$$\boldsymbol{k}_D(a, \varphi(\lambda)) \le \boldsymbol{k}_D(a, \varphi(0)) + \boldsymbol{k}_D(\varphi(0), \varphi(\lambda)) \le c/2 + \boldsymbol{k}_{\mathbb{D}}(0, \lambda) \le c.$$

[26] Recall that $B_d(a, r) := \{x \in X : d(x, a) < r\}$.

Hence, $\varphi(V) \subset U$.

(ii) $\Rightarrow$ (iii): Fix an $a \in D$ and put $U := \mathbb{B}(a,r) \Subset D$. Then choose $V = V(0) \subset \mathbb{D}$ and $\widetilde{U} = \widetilde{U}(a) \subset U$ according to the assumption in (ii). If now $\varphi \in \mathcal{O}(\mathbb{D}, D)$ with $\varphi(0) \in \widetilde{U}$ and $t\varphi'(0) = X$, then $\varphi(V) \subset U$.

Fix an $s > 0$ such that $K(s) \subset V$. Then we conclude that $\varphi|_{K(s)} \in \mathcal{O}(K(s), U)$. Put $\tilde{\varphi}(\lambda) := \varphi(s\lambda)$. Then $\tilde{\varphi} \in \mathcal{O}(\mathbb{D}, U)$ satisfying $\tilde{\varphi}(0) = \varphi(0)$ and $\frac{t}{s}\tilde{\varphi}'(0) = X$. Hence

$$\varkappa_D(z;X) \geq s\varkappa_U(z;X) \geq c\|X\|, \quad z \in \widetilde{U},\ X \in \mathbb{C}^n,$$

where c is a certain positive number.

(iii) $\Rightarrow$ (i): Fix two different points $a, b \in D$. By assumption, we may choose a neighborhood $U = U(a) \subset D$ such that $\varkappa_D(z;X) \geq c\|X\|$ for a certain $c > 0$, $z \in U$. Fix now an $r > 0$ such that $b \notin \mathbb{B}(a, 2r) \subset U$.

Let $\alpha\colon [0,1] \to D$ be a piecewise $\mathcal{C}^1$-curve in D which connects a and b. Let t_0 be that time such that $\alpha(t) \in \mathbb{B}(a,r)$ for all $t \in [0, t_0)$ and $\alpha(t_0) \in \partial\mathbb{B}(a,r)$. Then

$$L_{\varkappa_D}(\alpha) \geq L_{\varkappa_D}(\alpha|_{[0,t_0]}) \geq c\int_0^{t_0} \|\alpha'(t)\| dt \geq cr > 0.$$

Hence, $\boldsymbol{k}_D(a,b) \geq cr$; in particular, D is $\boldsymbol{k}$-hyperbolic. $\square$

Exercise 4.7.4. (a) Prove that a domain D is $\boldsymbol{k}$-hyperbolic iff $\operatorname{top} D = \operatorname{top} \boldsymbol{k}_D$. Here $\operatorname{top} D$ means the standard topology on D, where $\operatorname{top} \boldsymbol{k}_D$ is the topology on D that is induced by the Kobayashi pseudodistance.

(b) Prove the following generalization of Cartan's theorem (see Theorem 2.1.7):

Let $D \subset \mathbb{C}^n$ be a $\boldsymbol{k}$-hyperbolic domain, let $a \in D$, and let $\Phi\colon D \to D$ be a holomorphic mapping such that $\Phi(a) = a$ and $\Phi'(a) = \mathrm{id}$. Then $\Phi = \mathrm{id}$.

Use (a) to get the bounded situation.

Moreover, we have

Proposition 4.7.5. *Any taut domain $D \subset \mathbb{C}^n$ is $\boldsymbol{k}$-hyperbolic.*

Proof. Suppose that D is not $\boldsymbol{k}$-hyperbolic. Then, in virtue of Lemma 4.7.3 (ii), we find a $z' \in D$, a neighborhood $U = U(z') \subset D$, a sequence $\lambda_j \to 0$ in $\mathbb{D}$, and a sequence $(\varphi_j)_j \subset \mathcal{O}(\mathbb{D}, D)$ such that $\varphi_j(0) \to z'$, but $\varphi_j(\lambda_j) \notin U$, $j \in \mathbb{N}$. Since $\varphi_j(0) \to z'$, there is no subsequence which is locally uniformly divergent. And because of $\varphi_j(\lambda_j) \notin U$, $j \in \mathbb{N}$, there is no subsequence tending locally uniformly to a $\varphi \in \mathcal{O}(\mathbb{D}, D)$; a contradiction. $\square$

To get another large class of $\boldsymbol{k}$-hyperbolic domains we prove the following result which is due to [DDT-Tho 1998].

Proposition 4.7.6. *Let $u\colon \mathbb{D} \to [-\infty, \infty)$ be upper semicontinuous and locally bounded from below. Then*

$$D := \{z \in \mathbb{D} \times \mathbb{C} : |z_2| e^{u(z_1)} < 1\}$$

is a $\boldsymbol{k}$-hyperbolic domain.

Proof. In virtue of Lemma 4.7.3 it suffices to show that $\varkappa_D$ is locally positive definite, i.e. for any point $z' \in D$ there exist $U = U(z') \subset D$ and $c > 0$ such that $\varkappa_D(z; X) \ge c\|X\|$, $z \in U$, $X \in \mathbb{C}^2$.

By assumption we have

$$g(r) := \inf\{u(\lambda) : |\lambda| \le r\} > -\infty, \quad r \in (0, 1).$$

Fix $s \in (0, 1)$, $z' \in (s\mathbb{D} \times \mathbb{C}) \cap D$, and $X \in \mathbb{C}^2$, $X \neq 0$. Now choose an analytic disc $\varphi = (\varphi_1, \varphi_2) \in \mathcal{O}(\mathbb{D}, D)$ such that $\varphi(0) = z'$ and $t\varphi'(0) = X$ for a certain $t \in (0, 1)$. In virtue of the Schwarz lemma we have $|\varphi_1'(0)| \le 1 - |z_1'|^2 \le 1$.

Put $s_0 := \frac{1+2s}{2+s}$. Note that $s_0 < 1$. Suppose $|\varphi_1(\lambda_0)| \ge s_0$ for a $\lambda_0 \in \mathbb{D}$. Then the Schwarz lemma implies that

$$|\lambda_0| \ge \left|\frac{\varphi_1(\lambda_0) - z_1'}{1 - \overline{z_1'}\varphi_1(\lambda_0)}\right| \ge \frac{|\varphi_1(\lambda_0)| - |z_1'|}{1 - |\varphi_1(\lambda_0)||z_1'|} \ge \frac{s_0 - |z_1'|}{1 - s_0|z_1'|} \ge \frac{s_0 - s}{1 - s_0 s} = \frac{1}{2}.$$

Put $\Omega := \{\lambda \in \mathbb{D} : |\varphi_1(\lambda)| < s_0\}$. Then $|\varphi_2(\lambda)| \le e^{-g(s_0)}$, $\lambda \in \Omega$, and $K(1/2) \subset \Omega$. Thus, $|\varphi_2'(0)| \le 2e^{-g(s_0)}$ (Schwarz lemma). Hence

$$t \ge \max\left\{|X_1|, \frac{|X_2|}{se^{-g(s_0)}}\right\} \ge \frac{1}{\sqrt{2}} \min\left\{1, \frac{1}{2e^{-g(s_0)}}\right\}\|X\| =: t(s)\|X\|.$$

Since φ was arbitrarily chosen we have

$$\varkappa_D(z; X) \ge t(s)\|X\|, \quad (z, X) \in \big((s\mathbb{D} \times \mathbb{C}) \cap D\big) \times \mathbb{C}^2.$$

Hence, D is $\boldsymbol{k}$-hyperbolic. □

The result allows us to present a $\boldsymbol{k}$-hyperbolic pseudoconvex domain which is not $\boldsymbol{c}$-hyperbolic. Thus the general situation is more complicated than the one inside the class of pseudoconvex Reinhardt domains.

Example 4.7.7 ([Sib 1981]). Choose a sequence $(a_j)_j \subset \mathbb{D}$ of pairwise different points a_j such that any boundary point $\zeta \in \partial\mathbb{D}$ is the nontangential limit of a subsequence of $(a_j)_j$. The reader is asked (Exercise) to construct such a sequence. Moreover, we choose natural numbers m_j and n_j, $j \in \mathbb{N}$, such that

- $n_j < m_j$, $j \in \mathbb{N}$,
- $\sum_{j=1}^{\infty} \frac{1}{n_j} \log \frac{|a_j|}{2} < -\infty$,

- $K(a_j, 3e^{-jm_j}) \cap K(a_k, e^{-km_k}) = \varnothing$, $j \neq k$,
- $K(a_j, 3e^{-jm_j}) \subset \mathbb{D}$.

Finally, we define

$$u(\lambda) := \sum_{j=1}^{\infty} \frac{1}{n_j} \max\left\{-jm_j, \log\frac{|\lambda - a_j|}{2}\right\}, \quad \lambda \in \mathbb{D}.$$

Then $u \in \mathcal{C}(\mathbb{D}) \cap \mathcal{SH}(\mathbb{D})$ (Exercise). In particular, u is locally bounded from below. Therefore, by Proposition 4.7.6, we see that

$$D := \{z \in \mathbb{D} \times \mathbb{C} : |z_2|e^{u(z_1)} < 1\}$$

is a $\boldsymbol{k}$-hyperbolic domain. Observe that D is even pseudoconvex (Exercise).

On the other hand, let $f \in \mathcal{H}^\infty(D)$. Then $f(z) = \sum_{j=1}^{\infty} h_j(z_1)z_2^j$, where the h_j's are holomorphic on $\mathbb{D}$. Then, using the Cauchy inequalities and the maximum principle for holomorphic functions leads to $h_j = 0$, $j \geq 1$. So f depends only on the variable z_1. In particular, $\boldsymbol{c}_D((z_1, z_2), (z_1, 0)) = 0$ whenever $z = (z_1, z_2) \in D$. Hence, D is not $\boldsymbol{c}$-hyperbolic.

4.8 Carathéodory (resp. Kobayashi) complete Reinhardt domains

Let $D \subset \mathbb{C}^n$ be a bounded domain. Then we know that $(D, \boldsymbol{c}_D)$ (resp. $(D, \boldsymbol{k}_D)$) are metric spaces in the usual sense. In general, let $d_D \in \{\boldsymbol{c}_D, \boldsymbol{k}_D\}$. We define

Definition 4.8.1. Let $D \subset \mathbb{C}^n$ be a given domain.

(a) D is said to be *d-complete* if it is d-hyperbolic and any d_D-Cauchy sequence $(z_j)_{j\in\mathbb{N}} \subset D$ converges in the standard topology to a point $z^* \in D$, i.e. $\|z_j - z^*\| \to 0$.

(b) D is said to be *d-finitely compact*[27] if it is d-hyperbolic and if for any $a \in D$ and any $r > 0$ the d_D-ball $B_{d_D}(a, r)$ is relatively compact in D in sense of the standard topology of D.

Remark 4.8.2. (a) Observe that if D is d-finitely compact, then D is d-complete (use that d_D is continuous).

(b) Any $\boldsymbol{c}$-complete domain is d-complete.

(c) Any $\boldsymbol{c}$-complete domain is a domain of holomorphy (Exercise).

(d) Recall that if D is $\boldsymbol{k}$-hyperbolic, then top D = top $\boldsymbol{k}_D$. Therefore one may formulate $\boldsymbol{k}$-complete (resp. $\boldsymbol{k}$-finitely compact) by using top $\boldsymbol{k}_D$ instead of top D. Note that in case of $\boldsymbol{c}$ there are examples showing that the topologies top $\boldsymbol{c}_D$ and top D are different (see [Jar-Pfl 1993]).

[27]This notion is taken from standard differential geometry; see the theorem of Hopf–Rinow.

(e) For a $\boldsymbol{c}$-hyperbolic plane domain D we have: D is $\boldsymbol{c}$-complete iff D is $\boldsymbol{c}$-finitely compact. This result is due to [Sel 1974] and [Sib 1975] (see also [Jar-Pfl 1993], Theorem 7.4.7).

(f) $\boxed{?}$ In the case of a domain D in $\mathbb{C}^n$, $n \geq 2$, it is still an open question, whether $\boldsymbol{c}$-completeness implies $\boldsymbol{c}$-finitely compactness. $\boxed{?}$ On the other hand, in the class of general complex spaces there are counterexamples; see [Jar-Pfl-Vig 1993].

(g) If $F \in \operatorname{Bih}(D_1, D_2)$ and if D_1 is d-complete, then D_2 is d-complete, where $d \in \{\boldsymbol{c}, \boldsymbol{k}\}$.

Dealing with the Kobayashi distance we have that both notions are the same.

Proposition* 4.8.3. *Let D be a $\boldsymbol{k}$-hyperbolic domain in $\mathbb{C}^n$. Then the following properties are equivalent:*

(i) *D is $\boldsymbol{k}$-complete;*

(ii) *D is $\boldsymbol{k}$-finitely compact.*

Here we will not present a proof of this result. But we mention that the former result is a particular case of a result where one deals with continuous inner distances (note that $\boldsymbol{k}_D$ satisfies these properties). The main idea is taken from differential geometry. Details may be found in [Jar-Pfl 1993], Theorem 7.3.2.

Next we mention the following necessary conditions for a domain to be $\boldsymbol{k}$-complete.

Lemma 4.8.4. *Any $\boldsymbol{k}$-complete domain is taut. In particular, it is a domain of holomorphy* (use Exercise 1.17.21 and the solution of the Levi problem).

Proof. Take a sequence $(\varphi_j)_{j\in\mathbb{N}} \subset \mathcal{O}(\mathbb{D}, D)$. Suppose that this sequence is not locally uniformly divergent. So we find compact sets $K \subset \mathbb{D}$ and $L \subset D$ such that, without loss of generality, $\varphi_j(\lambda_j) \in L$, where $\lambda_j \in K$. Fix a point $a \in L$ and an $r \in (0,1)$ such that $K \subset K(r)$. Then, for any $\lambda \in K(r)$, we have

$$\begin{aligned}\boldsymbol{k}_D(\varphi_j(\lambda), a) &\leq \boldsymbol{k}_D(\varphi_j(\lambda), \varphi_j(\lambda_j)) + \boldsymbol{k}_D(\varphi_j(\lambda_j), a)\\ &\leq \boldsymbol{p}(\lambda, \lambda_j) + \sup\{\boldsymbol{k}_D(z, a) : z \in L\} \leq C(r).\end{aligned}$$

Hence,

$$\bigcup_{j\in\mathbb{N}} \varphi_j(K(r)) \subset B_{\boldsymbol{k}_D}(a, C(r)) \Subset D.$$

What remains is to apply Montel's theorem. $\square$

In case of Reinhardt domains even the following converse statement is true.

Theorem 4.8.5. *Any hyperbolic pseudoconvex Reinhardt domains $D \subset \mathbb{C}^n$ is $\boldsymbol{k}$-complete.*

In order to be able to prove Theorem 4.8.5 we need the following localization result due to Eastwood (see [Jar-Pfl 1993], Theorem 7.7.5).

Lemma 4.8.6. *Let $D \subset \mathbb{C}^n$ be a bounded domain. Assume that for any $b \in \partial D$ there exists a bounded neighborhood $U = U(b)$ of b such that any connected component U' of $D \cap U$ satisfies the following condition: if $a \in U'$ and $U' \ni b^k \to b$, then $\boldsymbol{k}_{U'}(a, b^k) \to \infty$. Then D is $\boldsymbol{k}$-complete.*

Proof. Suppose the contrary. Then we find a $b \in \partial D$ and a $\boldsymbol{k}_D$-Cauchy sequence $(z^j)_{j\in\mathbb{N}} \subset D$ such that $z^j \to b \in \partial D$. Let $U = U(b)$ be the neighborhood whose existence is known from the assumption. Choose $R > 0$ such that $U \cup D \subset \mathbb{B}_n(R) =: V$. By Exercise 4.7.4, U is open in top $\boldsymbol{k}_V$, i.e. $B_{\boldsymbol{k}_V}(b, 2s) \subset U$ for a certain positive s.

Then we find a $j_0 \in \mathbb{N}$ such that

$$\boldsymbol{k}_D(z^j, z^\ell) < s/3 \quad \text{and} \quad z^j \in B_{\boldsymbol{k}_V}(b, s/3), \quad j, \ell \geq j_0.$$

Fix a $j \geq j_0$. Then there exist $k \in \mathbb{N}$, $\varphi_\nu \in \mathcal{O}(\mathbb{D}, D)$, and $\lambda_\nu \in \mathbb{D}$, $\nu = 1, \dots, k$, such that

$$\varphi_1(0) = z^{j_0}, \quad \varphi_\nu(\lambda_\nu) = \varphi_{\nu+1}(0), \quad \nu = 1, \dots, k-1,$$
$$\varphi_k(\lambda_k) = z^j, \quad \sum_{\nu=1}^{k} \boldsymbol{p}(0, \lambda_\nu) < s/3.$$

Let $\lambda \in B_{\boldsymbol{p}}(0, s) \subset \mathbb{D}$ and $1 \leq \nu \leq k$. Then

$$\begin{aligned}
\boldsymbol{k}_V(\varphi_\nu(\lambda), b) &\leq \boldsymbol{k}_V(\varphi_\nu(\lambda), \varphi_\nu(0)) + \boldsymbol{k}_V(\varphi_\nu(0), z^j) + \boldsymbol{k}_V(z^j, b) \\
&\leq \boldsymbol{p}(\lambda, 0) + \sum_{\mu=\nu}^{k} \boldsymbol{k}_V(\varphi_\mu(0), \varphi_\mu(\lambda_\mu)) + s/3 \\
&< s + \sum_{\mu=1}^{k} \boldsymbol{p}(0, \lambda_\mu) + s/3 \leq 2s,
\end{aligned}$$

i.e. $\varphi_\nu(B_{\boldsymbol{p}}(0, s)) \subset U$, $\nu = 1, \dots, k$.

Note that $B_{\boldsymbol{p}}(0, s)$ is a disc with center 0. We choose a biholomorphic dilatation $\gamma\colon \mathbb{D} \to B_{\boldsymbol{p}}(0, s)$ and put $\tilde{\varphi}_\nu := \varphi_\nu \circ \gamma \in \mathcal{O}(\mathbb{D}, D \cap U)$. Observe that

$$\tilde{\varphi}_1(0) = z^{j_0}, \quad \tilde{\varphi}_\mu(\gamma^{-1}(\lambda_\mu)) = \tilde{\varphi}_{\mu+1}(0), \quad \mu = 1, \dots, k-1,$$
$$\tilde{\varphi}_k(\gamma^{-1}(\lambda_k)) = z^j,$$

and $\sum_{\mu=1}^{k} \boldsymbol{p}(0, \gamma^{-1}(\lambda_\mu)) \leq c \sum_{\mu=1}^{k} \boldsymbol{p}(0, \lambda_\mu) < cs$ for some $c > 0$ which is independent of j. Note that, by construction, the points z^j, $j \geq j_0$, belong to one connected component U' of $D \cap U$. Hence, $\boldsymbol{k}_{U'}(z^{j_0}, z^j) \leq cs$, $j \geq j_0$; a contradiction. $\square$

Proof of Theorem 4.8.5. By the hyperbolicity condition we may assume that D is bounded and $D \subset \mathbb{D}^n$. The proof is done by induction. The one-dimensional case is obvious (use the explicit formulas from Section 4.4). Now let $D \subset \mathbb{C}^n$, $n \geq 2$, and assume that the theorem is true for all smaller dimensions. We will show that D fulfills the condition in Lemma 4.8.6.

So let us fix a $b \in \partial D$. There are different cases to discuss.

Case $b \in \mathbb{C}_*^n$: Let us choose a polycylinder $\mathbb{P}(b, r) \Subset \mathbb{C}_*^n$. Put

$$V := \{z \in \mathbb{C}^n : ||z_k| - |b_k|| < r,\ k = 1, \dots, n\}.$$

Then V is a Reinhardt neighborhood of b. Denote by V' a connected component of $D \cap V$. Then V' is a Reinhardt domain of holomorphy and fulfills the Fu condition. Fix an $a \in V'$ and a sequence $V' \ni b^k \to b$ (if it exists). In virtue of Theorem 1.13.19, there is a function $f \in \mathcal{O}(V', \mathbb{D})$ such that $f(a) = 0$ and $|f(b^k)| \to 1$. Therefore,

$$\boldsymbol{k}_{V'}(a, b^k) \geq \tanh^{-1} \boldsymbol{m}_{V'}(a, b^k) \geq \tanh^{-1} |f(b^k)| \to \infty,$$

i.e. b fulfills the condition in Lemma 4.8.6.

Case $b \in V_0 \setminus \{0\}$: We may assume that $b = (b_1, \dots, b_k, 0, \dots, 0) = (b', 0) \in \mathbb{C}_*^k \times \mathbb{C}^{n-k}$, where $1 \leq k \leq n-1$.

First assume that $D \cap V_j \neq \varnothing$, $k+1 \leq j \leq n$. Then, by Corollary 1.11.16, $\mathrm{pr}_{\mathbb{C}^k}(D) =: D' \subset \mathbb{C}^k$ is a Reinhardt domain of holomorphy and $b' \in \partial D'$. Fix $a \in D$ and $D \ni b^k \to b$. Then $\mathrm{pr}_{\mathbb{C}^k}(b^k) \to b'$. By induction hypothesis,

$$\boldsymbol{k}_D(a, b^k) \geq \boldsymbol{k}_{D'}(\mathrm{pr}_{\mathbb{C}^k}(a), \mathrm{pr}_{\mathbb{C}^k}(b^k)) \to \infty,$$

i.e. b satisfies the condition in Lemma 4.8.6 with $U = \mathbb{C}^n$.

Assume that there is a $j_0 \in \{k+1, \dots, n\}$ with $D \cap V_{j_0} = \varnothing$; without loss of generality, let $j_0 = n$. Then $D \ni z \mapsto z_n$ defines a holomorphic map $F \in \mathcal{O}(D, \mathbb{C}_*)$ and $F(D)$ is bounded. Therefore, $F(D) \subset K_*(r)$. Note that $K_*(r)$ is $\boldsymbol{k}$-complete. Hence, if $a \in D$ and $D \ni b^k \to b$, then

$$\boldsymbol{k}_D(a, b^k) \geq \boldsymbol{k}_{K_*(r)}(F(a), F(b^k)) \to \infty,$$

i.e. b fulfills the condition in Lemma 4.8.6 with $U = U(b) = \mathbb{C}^n$.

Case $b = 0$: If $D \cap V_{j_0} = \varnothing$ for a j_0, then one argues as just before. On the other hand, $D \cap V_j \neq \varnothing$ for all $j = 1, \dots, n$ is impossible, since D is a Reinhardt domain of holomorphy. $\square$

In case of Carathéodory finitely compactness we have the following reformulation in terms of bounded holomorphic functions.

Lemma 4.8.7. *Let D be a $\boldsymbol{c}$-hyperbolic domain in $\mathbb{C}^n$. Then the following properties are equivalent:*

(i) *D is $\boldsymbol{c}$-finitely compact;*

(ii) *for any $a \in D$ and any sequence $(z_j)_{j\in\mathbb{N}} \subset D$ without accumulation points in D there exists an $f \in \mathcal{O}(D, \mathbb{D})$ with $f(a) = 0$ and $\sup\{|f(z_j)| : j \in \mathbb{N}\} = 1$.*

In particular, any $\boldsymbol{c}$-finitely compact domain is $\mathcal{H}^\infty$-convex.

Proof. Obviously, (ii) implies (i). For (i) $\Rightarrow$ (ii) just apply Proposition 1.13.18. $\square$

Remark 4.8.8. According to this result one can conclude that a lot of smooth pseudoconvex domains whose boundary points are general peak points are $\boldsymbol{c}$-finitely compact. For example, any strongly pseudoconvex domain is $\boldsymbol{c}$-finitely compact. Recall that a boundary point a of a domain D is said to be a *general peak point* if for any sequence $(z_j)_{j\in\mathbb{N}} \subset D$, $z_j \to a$, there exists an $f \in \mathcal{O}(D, \mathbb{D})$ such that $\sup\{|f(z_j)| : j \in \mathbb{N}\} = 1$.

In case of Reinhardt domains the following geometric characterization is true (see [Fu 1994] and [Zwo 2000b]).

Theorem 4.8.9. *Let D be a pseudoconvex Reinhardt domain in $\mathbb{C}^n$.*

(a) *If D is hyperbolic, then D is Kobayashi complete.*

(b) *The following properties are equivalent:*

  (i) *D is $\boldsymbol{c}$-finitely compact;*

  (ii) *D is $\boldsymbol{c}$-complete;*

  (iii) *there is no sequence $(z_j)_j \subset D$ having no accumulation points in D such that $\sum_{j=1}^{\infty} \boldsymbol{g}_D(z_j, z_{j+1}) < \infty$;*

  (iv) *D is bounded and satisfies the Fu condition.*

Proof. (a) Note that D is biholomorphically equivalent to a bounded pseudoconvex Reinhardt domain. Hence, from the very beginning we may assume that D is bounded.

Suppose that D is not $\boldsymbol{k}$-finitely compact, i.e. there exist a point $a \in D \cap \mathbb{C}_*^n$, an $R > 0$, and a sequence $(z_j)_{j\in\mathbb{N}} \subset D$ such that

$$\boldsymbol{k}_D(a, z_j) \le R, \ j \in \mathbb{N}, \quad \text{and} \quad z_j \to z^* \in \partial D.$$

Assume for a moment that $z^* \in \mathbb{C}_*^n$. Without loss of generality, we may assume that $z^* = \boldsymbol{1} \in \partial D$. Then, because of Remark 1.4.1, we find an $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that $D \subset \boldsymbol{D}_\alpha$ and $\boldsymbol{1} \in \partial \boldsymbol{D}_\alpha$. Again, without loss of generality, let α be of the following form: $\alpha_1 \cdots \alpha_k \ne 0$ and $\alpha_{k+1} = \cdots = \alpha_n = 0$, where $1 \le k \le n$. Thus we have $\boldsymbol{D}_\alpha = \boldsymbol{D}_{(\alpha_1,\dots,\alpha_k)} \times \mathbb{C}^{n-k}$. Applying the product property for the Kobayashi distance we get

$$\boldsymbol{k}_D(a, z_j) \ge \boldsymbol{k}_{\boldsymbol{D}_\alpha}(a, z_j) = \boldsymbol{k}_{\boldsymbol{D}_{(\alpha_1,\dots,\alpha_k)}}(\tilde{a}, \tilde{z}_j) =: r_j,$$

where $\tilde{a} := (a_1, \dots, a_k)$ and $\tilde{z}_j := (z_{j,1}, \dots, z_{j,k})$, $j \in \mathbb{N}$. Because of the formulas of the Kobayashi pseudodistance for elementary Reinhardt domains this yields that $r_j \to \infty$; a contradiction.

Therefore we conclude that z^* is lying on some of the axes. Assume, without loss of generality, that $z_1^* \cdots z_k^* \neq 0$ and $z_{k+1}^* = \cdots = z_n^* = 0$ for a certain $k \in \{0, \dots, n-1\}$. There are two cases to discuss, namely:

(i) $\exists_{\ell \in \{k+1,\dots,n\}} : \ V_\ell \cap D = \varnothing$;

(ii) $\forall_{\ell = k+1,\dots,n} : \ V_\ell \cap D \neq \varnothing$.

Assume first (i). Let D' be defined as the projection of D in ℓ-th direction. Hence D' is a bounded domain in $\mathbb{C}$ and $z_\ell^* \in \partial \widetilde{D}$. In particular,

$$D' \subset K(z_\ell^*, \widetilde{R}) \setminus \{z_\ell^*\} \quad \text{for a certain } \widetilde{R} > 0.$$

Therefore, using Theorem 4.2.42 and Corollary 4.4.2, we get

$$\boldsymbol{k}_D(a, z_j) \geq \boldsymbol{k}_{D'}(a_\ell, z_{j,\ell}) \to \infty;$$

a contradiction.

In the case (ii), if $k = 0$, then $z^* = 0 \in D$; a contradiction. So we only have to discuss the case when $k > 0$. Then

$$\boldsymbol{k}_D(a, z_j) \geq \boldsymbol{k}_{\mathrm{pr}_{\mathbb{C}^k}(D)}(\mathrm{pr}_{\mathbb{C}^k}(a), \mathrm{pr}_{\mathbb{C}^k}(z_j)).$$

Note that $\mathrm{pr}_{\mathbb{C}^k}(z_j) \to \mathrm{pr}_{\mathbb{C}^k}(z^*) \in \partial(\mathrm{pr}_{\mathbb{C}^k}(D))$ and the limit has no vanishing coordinates. This means we are back in the situation we started with; a contradiction.

(b) The implication (i) $\Rightarrow$ (ii) is trivial, (ii) $\Rightarrow$ (iii) is a consequence of $\boldsymbol{m}_D \leq \boldsymbol{g}_D$, the triangle inequality for $\boldsymbol{c}_D$, and the growth of $\tanh^{-1}$ at 0. Finally (iv) $\Rightarrow$(i) follows directly from Theorem 1.13.19 and Lemma 4.8.7.

So it remains to prove (iii) $\Rightarrow$ (iv). Suppose (iv) is not true. In case that D is not bounded we know that D is an unbounded hyperbolic pseudoconvex Reinhardt domain. Then, in virtue of Proposition 1.17.12, we conclude that D is algebraically equivalent to a bounded pseudoconvex Reinhardt domain that does not satisfy the Fu condition. Hence for the rest of the proof we may assume that D is bounded and does not fulfill the Fu condition. So, without loss of generality, it suffices to deal with the following situation:

$$\begin{aligned}
&\bar{D} \cap V_j \neq \varnothing,\ D \cap V_j = \varnothing, && 1 \leq j \leq k,\\
&\bar{D} \cap V_j = \varnothing, && k+1 \leq j \leq \ell,\\
&D \cap V_j \neq \varnothing, && \ell+1 \leq j \leq n,
\end{aligned}$$

where $1 \leq k \leq \ell \leq n$. In case when $\ell < n$ we can even simplify the situation. Namely, put $\widetilde{D} := \{z \in \mathbb{C}^\ell : (z, 0) \in D\}$. Obviously, $\widetilde{D}$ is bounded and does not fulfill the Fu condition. Moreover, note that $\widetilde{D}$ has property (iii).

Summarizing we may assume, without loss of generality, that

$$\begin{aligned} &\bar{D} \cap V_j \neq \varnothing,\ D \cap V_j = \varnothing, && 1 \le j \le k, \\ &\bar{D} \cap V_j = \varnothing, && k+1 \le j \le n, \end{aligned}$$

where $1 \le k \le n$. In particular, $D \subset \mathbb{C}^n_*$. If necessary use a dilatation to obtain, in addition, that $\mathbf{1} \in D$.

Note that $\log D$ remains bounded in all positive directions and in the last $n-k$ negative directions, but it is unbounded in the first k negative directions. Hence, by Lemma 1.4.17, we find $V = (-r,r)^n \subset \log D$ and $v \in \mathbb{R}^n_- \setminus \{0\}$ such that $V + \mathbb{R}_+ v \in \log D$. Note that $v_j = 0$, $j = k+1, \dots, n$.

Without loss of generality, we may assume that $v_j < 0$, $j = 1, \dots, \ell \le k$, $v_1 = -1$, and $v_j = 0$, $\ell + 1 \le j \le n$. Hence

$$(e^{x_1} e^{-t}, e^{x_2} e^{tv_2}, \dots, e^{x_\ell} e^{tv_\ell}, e^{x_{\ell+1}}, \dots, e^{x_n}) \in D, \quad t > 0,\ x \in V.$$

Put $\alpha := -v$. Then there exists an $\varepsilon > 0$ such that

$$\begin{gathered} (e^{\lambda}, \mu_2 e^{\lambda\alpha_2}, \dots, \mu_\ell e^{\lambda\alpha_\ell}, 1 \dots, 1) \in D, \\ \lambda,\ \mu_j \in \mathbb{C},\ \operatorname{Re}\lambda < 0,\ e^{-\varepsilon} < |\mu_j| < e^{\varepsilon},\ j = 2, \dots, \ell. \end{gathered}$$

Put

$$\begin{aligned} &A := \{\mu = (\mu_2, \dots, \mu_\ell) \in \mathbb{C}^{\ell-1} : e^{-\varepsilon} < |\mu_j| < e^{\varepsilon},\ j = 2, \dots, \ell\}, \\ &H_R := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < R\}, \quad R \ge 0, \\ &F : \mathbb{C} \times A \to \mathbb{C}^\ell, \quad F(\lambda, \mu) := (e^{\lambda}, \mu_2 e^{\lambda\alpha_2}, \dots, \mu_\ell e^{\lambda\alpha_\ell}). \end{aligned}$$

Note that F is a locally biholomorphic mapping and $D_R := F(H_R \times A)$ is a pseudoconvex Reinhardt domain (Exercise). Moreover, we have $D_R \nearrow D_\infty := F(\mathbb{C} \times A) \subset \mathbb{C}^\ell_*$, when $R \to \infty$, and D_∞ is a pseudoconvex Reinhardt domain.

Hence, we get

$$\tilde{\boldsymbol{k}}_{D_\infty}(F(-1, 1, \dots, 1), F(\lambda, 1, \dots, 1)) \le \tilde{\boldsymbol{k}}_{\mathbb{C}}(-1, \lambda) = 0, \quad \lambda \in \mathbb{C}.$$

In virtue of Theorem 4.2.42 we know that $\tilde{\boldsymbol{k}}_{D_\infty}$ is continuous. Therefore,

$$\tilde{\boldsymbol{k}}_{D_\infty}(F(-1,, 1, \dots, 1), z) = 0, \quad z \in D_\infty \cap \overline{F(\mathbb{C} \times \{(1, \dots, 1)\})} =: M.$$

Note that $D_0 \times \{\mathbf{1}\} \subset D$ but $(0, \dots, 0, \mathbf{1}) \notin D$, where $\mathbf{1} \in \mathbb{C}^{n-\ell}$.

Choose positive numbers r_j such that $\sum_{j=1}^{\infty} r_j < \infty$. To get a contradiction to (iii) it remains to find points $z_j \in D_0$, $j \in \mathbb{N}$, such that $z_j \to 0$ and

$$\boldsymbol{g}_D((z_j, \mathbf{1}), (z_{j+1}, \mathbf{1})) \le \boldsymbol{g}_{D_0}(z_j, z_{j+1}) \le r_j, \quad j \in \mathbb{N}.$$

Recall that $\widetilde{\boldsymbol{k}}_{D_R}$ is continuous on $D_R \times D_R$ and that

$$\widetilde{\boldsymbol{k}}_{D_R}(F(-1,1,\dots,1),z) \underset{R\to\infty}{\searrow} \widetilde{\boldsymbol{k}}_{D_\infty}(F(-1,1,\dots,1),z) = 0, \quad z \in M.$$

Then, by Dini's theorem, we conclude that this convergence is locally uniform. Hence we find a sequence $(R_j)_j$, $0 < R_j \nearrow \infty$, such that

$$\widetilde{\boldsymbol{k}}^*_{D_{R_j}}(F(-1,1,\dots,1), F(\lambda,1,\dots,1)) < r_j, \quad -2 \le \operatorname{Re}\lambda \le -1.$$

Now note that the mapping $\psi_R\colon D_0 \to D_R$,

$$\psi_R(z) := (z_1 e^R, z_2 e^{\alpha_2 R}, \dots, z_\ell e^{\alpha_\ell R}),$$

is biholomorphic. Therefore,

$$\begin{aligned}\widetilde{\boldsymbol{k}}^*_{D_0}&(F(-1-R_j,1,\dots,1), F(\lambda,1,\dots,1))\\ &= \widetilde{\boldsymbol{k}}^*_{D_{R_j}}(F(-1,1,\dots,1), F(\lambda+R_j,1,\dots,1)) < r_j,\\ &\qquad\qquad -2-R_j \le \operatorname{Re}\lambda \le -1-R_j.\end{aligned}$$

Put

$$u_j(\lambda) := \log \boldsymbol{g}_{D_0}(F(-1-R_j,1,\dots,1),(F(\lambda,1,\dots,1)), \quad \lambda \in H_0.$$

Then $u_j \in \mathcal{SH}(H_0)$. By Exercise 1.14.10, we conclude that

$$u_j(\lambda) < \log r_j, \quad \operatorname{Re}\lambda \le -1-R_j.$$

Therefore, we may take $z_j := F(-1-R_j,1,\dots,1)$ as the points we were looking for. Hence the proof is complete. □

Corollary 4.8.10. *Let $D_j \subset \mathbb{C}^n$, $j = 1,2$, be biholomorphically equivalent Reinhardt domains. If D_1 is bounded and satisfies the Fu condition, then D_2 is bounded and it satisfies the Fu condition.*

Remark 4.8.11. Moreover, for a pseudoconvex Reinhardt domain D the following properties are equivalent: (i) D is $\boldsymbol{c}$-complete, (ii) D is $\boldsymbol{c}^i$-complete, where $\boldsymbol{c}^i_D$ denotes the so-called associated inner distance, i.e.

$$\begin{aligned}\boldsymbol{c}^i_D(a,z) := \inf\{L_{\boldsymbol{c}_D}(\alpha) : \alpha\colon [0,1] \to D \text{ continuous}&\\ \text{and } \|\cdot\|\text{-rectifiable}, \ \alpha(0)=a, \ \alpha(1)=z\},& \quad a,z \in D,\end{aligned}$$

where

$$L_{\boldsymbol{c}_D}(\alpha) := \sup\Big\{\sum_{j=1}^N \boldsymbol{c}_D(\alpha(t_{j-1}),\alpha(t_j)) : N \in \mathbb{N}, \ 0 = t_0 < t_1 < \dots < t_N = 1\Big\}.$$

See [Zwo 2001].

We conclude with a few remarks on completeness for pseudoconvex balanced domains.

Remark 4.8.12. Let $D = D_h$ be a bounded pseudoconvex balanced domain in $\mathbb{C}^n$. Then:

- If D is $\boldsymbol{k}$-complete, then h is continuous.
- For any $n \geq 3$ there exists a bounded pseudoconvex balanced domain with continuous h which is not $\boldsymbol{k}$-complete.
- $\boxed{?}$ Whether such an example does exist in dimension 2 is open. $\boxed{?}$
- $\boxed{?}$ It is also an open problem whether there exists a characterization of a $\boldsymbol{c}$- (resp. $\boldsymbol{k}$-) complete pseudoconvex balanced domains $D = D_h$ in terms of properties of the Minkowski function h. $\boxed{?}$

For more details see [Jar-Pfl 1993].

4.9* The Bergman completeness of Reinhardt domains

In the last section of this book we briefly introduce the Bergman metric for bounded domains in $\mathbb{C}^n$ and present (without proofs) a full characterization of Bergman-complete bounded Reinhardt domains due to Zwonek ([Zwo 1999a], [Zwo 2000]). For more details on the Bergman metric we refer the reader also to [Jar-Pfl 1993], [Jar-Pfl 2005].

Let $D \subset \mathbb{C}^n$ be a domain. Then $L_h^2(D)$ is a Hilbert space with the scalar product $\langle f, g\rangle_{L_h^2(D)} := \int_D f(z)\bar{g}(z)d\Lambda_{2n}(z)$ (cf. Example 1.10.7 (c)). Recall that the mapping $L_h^2(D) \ni f \mapsto f(a) \in \mathbb{C}$ is a continuous linear functional. Therefore, by the Riesz representation theorem, there exists a uniquely defined $K_D(\cdot, a) \in L_h^2(D)$ such that

$$\langle f, K_D(\cdot, a)\rangle_{L_h^2(D)} = \int_D f(z)\overline{K_D(z,a)}d\Lambda_{2n}(z) = f(a), \quad f \in L_h^2(D).$$

The function $K_D : D \times D \to \mathbb{C}$ is the *Bergman function for* D. Recall from the Hilbert space theory that there is another description of the Bergman function via a complete orthonormal system $(\varphi_j)_{j\in N} \subset L_h^2(D)$, where $N \subset \mathbb{N}$.[28] Namely, we have

$$K_D(z, w) = \sum_{j\in N} \varphi_j(z)\bar{\varphi}_j(w), \quad (z, w) \in D \times D,$$

where the convergence here is meant as the one in the Hilbert space $L_h^2(D)$.

In the case of a Reinhardt domain $D \subset \mathbb{C}^n$ there is a complete description of those monomials z^α which belong to $L_h^2(D)$. To be able to formulate this result we

[28] Note that $L_h^2(D)$ is a separable Hilbert space; therefore, it has a countable complete orthonormal system.

have to introduce some terminology. Let $a \in D \cap \mathbb{C}_*^n$. Put

$$\mathfrak{C}(D,a) := \{v \in \mathbb{R}^n : \log a + \mathbb{R}_+ v \in \log D\}.$$

Observe that $\mathfrak{C}(D,a) = \mathfrak{C}(D,b)$ whenever $b \in D \cap \mathbb{C}_*^n$. Hence we are allowed to set

$$\mathfrak{C}(D) := \mathfrak{C}(D,a), \text{ when } a \in D \cap \mathbb{C}_*^n.$$

Lemma* 4.9.1. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex Reinhardt domain. Then for an $\alpha \in \mathbb{Z}^n$ the following conditions are equivalent:*

(i) $z^\alpha \in L_h^2(D)$;

(ii) $\langle \alpha + \mathbf{1}, v\rangle < 0$, $v \in \mathfrak{C}(D) \setminus \{0\}$.

Therefore, for a pseudoconvex Reinhardt domain its Bergman function can be written as

$$K_D(z,w) = \sum a_\alpha z^\alpha \overline{w}^\alpha,$$

where the sum is taken over all $\alpha \in \mathbb{Z}^n$ such that $\langle \alpha + \mathbf{1}, v\rangle < 0$, $v \in \mathfrak{C}(D) \setminus \{0\}$, and the a_α's have to be determined.

Remark 4.9.2. Note that there exist domains $D_k \subset \mathbb{C}^2$, not pseudoconvex, such that $\dim L_h^2(D_k) = k$, $k \in \mathbb{N}$ (see [Wie 1984]). $\boxed{?}$ It is not known whether $\dim L_h^2(D) = \infty$ for any pseudoconvex domain D in $\mathbb{C}^2$ with $L_h^2(D) \neq \{0\}$. $\boxed{?}$

The above equation immediately leads to $K_D(z,w) = \overline{K_D(w,z)}$, $(z,w) \in D \times D$. Moreover, it can be understood in the sense that $\sum_{j=1}^k \varphi_j(z)\bar{\varphi}_j(\overline{w})$ converges to $K_D(z,\overline{w})$ locally uniformly (in the case when $N = \mathbb{N}$) (Exercise). Note that this partial sum is a function which is holomorphic on $D \times \overline{D}$. Therefore, by the Osgood theorem, $D \times \overline{D} \ni (z,w) \mapsto K_D(z,\overline{w})$ is holomorphic. In other words, we may say that K_D is holomorphic in the first variable and antiholomorphic in the second variable on D.

Remark 4.9.3. There are effective formulas for the Bergman kernel for standard domains like the Euclidean ball, the polydisc, or the minimal ball; for more details see [Jar-Pfl 1993] and [Jar-Pfl 2005]:

$$K_{\mathbb{B}_n}(z,w) = \frac{n!}{\pi}\big(1 - \langle z,w\rangle\big)^{-1}, \qquad z,w \in \mathbb{B}_n;$$

$$K_{\mathbb{D}^n}(z,w) = \frac{1}{\pi^n}\prod_{j=1}^n (1 - z_j\overline{w}_j)^{-1}, \quad z,w \in \mathbb{D}^n.$$

The following result describes the behavior of the Bergman kernel under biholomorphic mappings.

Proposition* 4.9.4. *Let $F : D \to G$ be a biholomorphic mapping. Then*

$$K_G(F(z), F(w)) \det F'(z)\overline{\det F'(w)} = K_D(z, w), \quad z, w \in D.$$

Remark 4.9.5. It should be mentioned that there is a similar formula in case when the mapping F is not biholomorphic but proper holomorphic (see [Bel 1982] or [Jar-Pfl 1993]).

Exercise 4.9.6. Calculate K_T, where T is the Hartogs triangle, i.e. $T = \{z \in \mathbb{D}^2 : |z_1| < |z_2|\}$.

In the following we put $k_D(z) := K_D(z, z)$; note that $\log k_D \in \mathcal{PSH}(D)$. k_D is called the *Bergman kernel of D*. In case that $L^2_h(D) \neq \{0\}$ one has an alternative description of the Bergman kernel, namely

$$k_D(z) = \sup\left\{\frac{|f(z)|^2}{\|f\|^2_{L^2_h(D)}} : f \in L^2_h(D) \setminus \{0\}\right\}, \quad z \in D.$$

From this equality it follows that $k_G|_D \le k_D$ if $D \subset G \subset \mathbb{C}^n$.

In the study of the Bergman kernel it is important to know its boundary behavior.

Definition 4.9.7. Let D be a domain in $\mathbb{C}^n$ and let $z^0 \in \partial D$. D is said to be *Bergman exhaustive at z^0* if $\lim_{D \ni z \to z^0} k_D(z) = \infty$.

There is a vast literature on conditions which are sufficient for Bergman exhaustiveness. Here we only mention the following one which combines properties of the Green function and Bergman exhaustiveness (compare [Che 1999], [Her 1999]).

Theorem* 4.9.8. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $z_0 \in \partial D$. Put $A_z(D) := \{\zeta \in D : \boldsymbol{g}_D(z, \zeta) \le e^{-1}\}$, where $z \in D$. If*

$$\lim_{D \ni z \to z_0} \Lambda_{2n}(A_z(D)) = 0,$$

then D is Bergman exhaustive at z_0. In particular, any bounded hyperconvex domain is Bergman exhaustive.

In what follows *we will always assume that k_D is positive on D*, i.e. for any point $a \in D$ we can find an $f \in L^2_h(D)$ with $f(a) \neq 0$. Note that this condition is always true if D is bounded. Put

$$\beta_D(z; X) := \Big(\sum_{j,k=1}^{n} \frac{\partial^2 \log k_D}{\partial z_j \partial \bar{z}_k}(z) X_j \bar{X}_k\Big)^{1/2}, \quad z \in D,\ X \in \mathbb{C}^n.$$

Then β_D gives a Hermitian metric on D. β_D is called the *Bergman pseudometric* on D.

Example 4.9.9. Here we present effective formulas of β_D for the standard domains $D = \mathbb{B}_n$ and $D = \mathbb{D}^n$:

$$\beta_{\mathbb{B}_n}(z;X) = \sqrt{n+1}\,\boldsymbol{\gamma}_{\mathbb{B}_n}(z;X), \quad (z,X) \in \mathbb{B}_n \times \mathbb{C}^n;$$

$$\beta_{\mathbb{D}^n}(z;X) = \sqrt{2}\sqrt{\sum_{j=1}^{n}\big(\boldsymbol{\gamma}(z_j;X_j)\big)^2}, \quad z \in \mathbb{D}^n,\ X = (X_1,\dots,X_n) \in \mathbb{C}^n.$$

Observe that $\beta_{\mathbb{D}} = \sqrt{2}\boldsymbol{\gamma}$.

Moreover, we have

Lemma* 4.9.10. *Let $F \in \mathrm{Bih}(D,G)$. Then $\beta_G(F(z);F'(z)X)) = \beta_D(z;X)$, $z \in D$, $X \in \mathbb{C}^n$.*

Hence, in the class of domains we are discussing, we have a family $(\beta_D)_D$ of pseudometrics b_D which are invariant under biholomorphic mappings such that $\beta_{\mathbb{D}} = \sqrt{2}\,\boldsymbol{\gamma}$. Observe that this family is not holomorphically contractible as the following example shows.

Example 4.9.11. Put $F\colon \mathbb{D}^2 \to \mathbb{D}^2$, $F(z) := (z_1, z_1)$ and $X := (1,0)$. Then

$$\beta_{\mathbb{D}^2}(F(0);F'(0)X) = 2 > \sqrt{2} = \beta_{\mathbb{D}^2}(0;X) \quad (\text{EXERCISE}).$$

We mention that there is also a different way to describe β_D which is often very useful.

Lemma* 4.9.12. *Let $D \subset \mathbb{C}^n$ with $k_D > 0$ on D. Then*

$$\beta_D(z;X) = \frac{1}{\sqrt{k_D(z)}}\sup\{|f'(z)X| : f \in L^2_h(D),\ \|f\|_{L^2_h(D)} = 1,\ f(z) = 0\},$$
$$z \in D,\ X \in \mathbb{C}^n.$$

With β_D we can measure the length of all tangential vector $X \in \mathbb{C}^n$ at any point $z \in D$. Therefore, we introduce a new pseudodistance on D, the *Bergman pseudodistance* b_D, defining

$$b_D(z,w) := \inf\Big\{\int_0^1 \beta_D(\alpha(t);\alpha'(t))dt : \alpha \in \mathcal{C}^1([0,1],D),$$
$$\alpha(0) = z,\ \alpha(1) = w\Big\}, \quad z,w \in D.$$

It is easy to check that b_D is a pseudodistance on D and that $b_G(F(z),F(w)) = b_D(z,w)$, $z,w \in D$, whenever $F \in \mathrm{Bih}(D,G)$, i.e. the Bergman pseudodistance is invariant under biholomorphic mappings.

Exercise 4.9.13. Prove that the sequence $((0, \frac{1}{2j}))_{j\in\mathbb{N}}$ is a b_T-Cauchy sequence, where T is the Hartogs triangle. In particular, T is not Bergman complete (see Definition 4.9.15).

In comparison to the objects we have discussed before we have the following result (see [Jar-Pfl 1993]).

Lemma* 4.9.14. *Let $D \subset \mathbb{C}^n$ be such that $k_D(z) > 0$, $z \in D$. Then*
(a) $\gamma_D \leq \beta_D$;
(b) $c_D \leq c_D^i \leq b_D$.

We should mention that there is no comparison between the Bergman pseudometric and the Kobayashi pseudometric.

Now we start to discuss the notion of Bergman complete domains. To be more precise we set

Definition 4.9.15. Let $D \subset \mathbb{C}^n$ such that $k_D > 0$. D is said to be *Bergman complete* if

- b_D is a distance,
- any b_D-Cauchy sequence $(z_j)_j \subset D$ (i.e. $b_D(z_j, z_k) \to 0$ if $j,k \to \infty$) converges to a point a in D (i.e. $\lim\limits_{j\to\infty} z_j = a$ in the topology of D).

There is a long history of studying Bergman complete domains. An old result by Bremermann shows that any Bergman complete bounded domain is necessarily pseudoconvex (see [Jar-Pfl 1993]).

Proposition 4.9.16. *If $D \subset \mathbb{C}^n$ is a bounded Bergman complete domain, then it is a domain of holomorphy.*

Proof. Suppose the contrary. Then there are a point $a \in D$ and positive numbers $r < R$ such that

- $\mathbb{P}_n(a,r) \subset D$,
- $\mathbb{P}_n(a,R) \not\subset D$,
- $\forall_{f\in\mathcal{O}(D)}\, \exists_{\hat{f}\in\mathcal{O}(\mathbb{P}_n(a,R))} : \hat{f}|_{\mathbb{P}_n(a,r)} = f|_{\mathbb{P}_n(a,r)}$. In particular, by the Hartogs theorem, the Bergman kernel function K_D extends to $\mathbb{P}_n(a,R) \times \mathbb{P}_n(a,R)$, or more precisely, there exists an $f : \mathbb{P}_n(a,R) \times \mathbb{P}_n(a,R) \to \mathbb{C}$ such that
- $f|_{\mathbb{P}_n(a,r)\times\mathbb{P}_n(a,r)} = K_D|_{\mathbb{P}_n(a,r)\times\mathbb{P}_n(a,r)}$,
- $\mathbb{P}_n(a,R) \times \mathbb{P}_n(a,R) \ni (z,w) \mapsto f(z,\overline{w})$ is holomorphic.

By construction we find a point $b \in \mathbb{P}_n(a,R) \cap \partial D$ such that $[a,b) \subset D$. Applying that $\log k_D(z) = \log f(z,z)$, z near $[a,b)$, leads to the fact that $\beta_D(a + t(b-a); b-a)$ is bounded on $(0,1)$. Hence, $(a + (1-1/j)(b-a))_j$ is a b_D-Cauchy sequence tending to the boundary point b; a contradiction. $\square$

Therefore, in order to characterize the Bergman complete domains it suffices to restrict on domains of holomorphy.

The most useful sufficient criteria for being Bergman complete is that of Kobayashi (see [Jar-Pfl 1993], [Bło 2003], and [Bło 2005]).

Proposition* 4.9.17. *Let $D \Subset \mathbb{C}^n$ be a bounded domain.*

(a) *If*

$$\lim_{D \ni z \to \partial D} \frac{|f(z)|}{\sqrt{k_D(z)}} < \|f\|_{L^2_h(D)}, \quad f \in L^2_h(D) \setminus \{0\},$$

then D is Bergman complete.

(b) *Assume for a dense subspace $H \subset L^2_h(D)$ that for any sequence $(z_j)_j \subset D$, $z_j \to z_0 \in \partial D$, and any $f \in H$ there exists a subsequence $(z_{j_k})_k$ such that*

$$\lim_{k \to \infty} \frac{|f(z_{j_k})|}{\sqrt{k_D(z_{j_k})}} = 0.$$

Then D is Bergman complete.

After a long development the following result was found (see [Bło-Pfl 1998], [Her 1999]).

Theorem* 4.9.18. *Any bounded hyperconvex domain is Bergman complete.*

But conversely, there are a lot of Bergman complete domains which are not hyperconvex; examples will be given later.

In the class of bounded pseudoconvex Reinhardt domains there is a complete characterization of Bergman complete domains in geometric terms due to Zwonek (see [Zwo 1999a], [Zwo 2000]). To do so we have to introduce the set

$$\widetilde{\mathfrak{C}}(D) := \{v \in \mathfrak{C}(D) : \overline{\exp(a + \mathbb{R}_+ v)} \subset D\},$$

where $a \in D \cap \mathbb{C}^n_*$. Then we have the following result.

Theorem* 4.9.19. *Let D be a bounded pseudoconvex Reinhardt domain. Then the following conditions are equivalent:*

(i) *D is Bergman complete;*

(ii) *$\mathfrak{C}'(D) \cap \mathbb{Q}^n = \varnothing$, where $\mathfrak{C}'(D) := \mathfrak{C}(D) \setminus \widetilde{\mathfrak{C}}(D)$.*

We conclude this discussion presenting two examples.

Example 4.9.20. (a) Put

$$D_1 := \{z \in \mathbb{C}^2 : |z_1|^2/2 < |z_2| < 2|z_1|^2,\ |z_1| < 1\}.$$

Then D is a bounded pseudoconvex Reinhardt domain. Note that

$$\mathfrak{C}'(D_1) = \mathbb{R}_{>0} \cdot (-1, -2).$$

In particular, $\mathfrak{C}'(D)$ contains the rational vector $(-1,-2)$. Hence, D_1 is not Bergman complete.

We add also a direct proof which may give some idea of how to prove (i) $\Rightarrow$ (ii) in Theorem 4.9.19. Recall that $z^\alpha \in L_h^2(D_1)$ iff $\langle \alpha + \mathbf{1}, (-1,-1)\rangle < 0$. Therefore,

$$K_{D_1}(z,w) = \sum_{\substack{\alpha\in\mathbb{Z}^n \\ v\in\mathfrak{C}(D)\setminus\{0\} \\ \langle\alpha+\mathbf{1},v\rangle<0}} a_\alpha z^\alpha \overline{w}^\alpha = \sum_{\substack{\alpha\in\mathbb{Z}^n \\ -3<\alpha_1+2\alpha_2}} a_\alpha z^\alpha \overline{w}^\alpha, \quad z,w \in D.$$

Put $\varphi(\lambda) := (\lambda, \lambda^2)$, $\lambda \in \mathbb{D}_*$. Then $\varphi \in \mathcal{O}(\mathbb{D}_*, D_1)$ and

$$k_{D_1} \circ \varphi(\lambda) = \sum_{-3<\alpha_1+2\alpha_2} a_\alpha |\lambda|^{2\alpha_1+4\alpha_2} = \sum_{j\ge j_0} b_j |\lambda|^j, \quad \lambda \in \mathbb{D}_*$$

where $j_0 > -3$ and $b_{j_0} \neq 0$. Therefore,

$$\beta^2(\varphi(\lambda);\varphi'(\lambda)) = \frac{\partial^2}{\partial\lambda\partial\bar{\lambda}}\Big(\log\sum_{j\ge j_0} b_j|\lambda|^{2(j-j_0)}\Big), \quad \lambda \in \mathbb{D}_*,$$

meaning that $\beta_{D_1}(\varphi(t);\varphi'(t))$ is bounded on $(0,1/2)$. Note that $\lim_{t\searrow 0}\varphi(t) \in \partial D_1$. Hence, $b_{D_1}(\varphi(1/2),\varphi(t))$ is bounded on $(0,1/2)$ which implies that D_1 is not Bergman complete.

(b) Put

$$D_2 := \{z \in \mathbb{C}^2 : |z_1|^{\sqrt{2}}/2 < |z_2| < 2|z_1|^{\sqrt{2}},\ |z_1| < 1\}.$$

Obviously, D_2 is a bounded pseudoconvex Reinhardt domain. Calculation then leads to $\mathfrak{C}'(D) = \mathbb{R}_{>0}\cdot(-1,-\sqrt{2})$, i.e. $\mathfrak{C}'(D)$ does not contain any rational vector. Hence, D_2 is Bergman complete. It is easy to see that D_2 is not hyperconvex.

Exercise 4.9.21. Put $D_3 := \{z \in \mathbb{D}_* \times \mathbb{C} : |z_2|^2 e^{-1/|z_1|^2} < 1\}$. Calculate $\log D_3$, $\mathfrak{C}(D_3)$, and decide whether D_3 is Bergman complete. Is D_3 hyperconvex?

Note that D_j, $j = 1,2,3$, does not fulfil the Fu condition, hence it is not $\boldsymbol{c}$-complete.

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Symbols

General symbols

$\mathbb{N}$:= the set of natural numbers, $0 \notin \mathbb{N}$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{N}_k := \{n \in \mathbb{N} : n \geq k\}$;
$\mathbb{Z}$:= the ring of integer numbers;
$\mathbb{Q}$:= the field of rational numbers;
$\mathbb{R}$:= the field of real numbers;
$\mathbb{R}_{-\infty} := \{-\infty\} \cup \mathbb{R}, \quad \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$;
$\mathbb{C}$:= the field of complex numbers;
$\lfloor x \rfloor := \max\{k \in \mathbb{Z} : k \leq x\}$ = the integer part of $x \in \mathbb{R}$;
$\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}$, $x \in \mathbb{R}$;
$\operatorname{Re} z$:= the real part of $z \in \mathbb{C}$, $\quad \operatorname{Im} z$:= the imaginary part of $z \in \mathbb{C}$;
$\bar{z} := x - iy$ = the conjugate of $z = x + iy$;
$|z| := \sqrt{x^2 + y^2}$ = the modulus of a complex number $z = x + iy$;
A^n := the Cartesian product of n copies of the set A, e.g. $\mathbb{C}^n$;
$\mathbb{M}(m \times n; A)$ = the set of all $(m \times n)$-dimensional matrices with entries from a set $A \subset \mathbb{C}$;
$\mathbb{I}_n$:= the $(n \times n)$-dimensional unit matrix;
$\mathbb{GL}(n, \mathbb{F}) := \{L \in \mathbb{M}(n \times n; \mathbb{F}) : \det L \neq 0\}, \quad \mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$;
$\mathbb{GL}(n, \mathbb{Z}) := \{L \in \mathbb{M}(n \times n; \mathbb{Z}) : |\det L| = 1\}$;
$x \leq y :\Leftrightarrow x_j \leq y_j,\ j = 1, \dots, n,\ x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$;
$A_* := A \setminus \{0\}$, e.g. $\mathbb{C}_*, (\mathbb{C}^n)_*$; $\quad A^n_* := (A_*)^n$, e.g. $\mathbb{C}^n_*$;
$A_+ := \{a \in A : a \geq 0\}$, e.g. $\mathbb{Z}_+, \mathbb{R}_+$; $\quad A^n_+ := (A_+)^n$, e.g. $\mathbb{Z}^n_+, \mathbb{R}^n_+$;
$A_- := \{a \in A : a \leq 0\}$;
$A_{>0} := \{a \in A : a > 0\}$, e.g. $\mathbb{R}_{>0}$; $\quad A^n_{>0} := (A_{>0})^n$, e.g. $\mathbb{R}^n_{>0}$;
$A_{<0} := \{a \in A : a < 0\}$;
$A + B := \{a + b : a \in A,\ b \in B\}$, $a + B := \{a\} + B$, $A, B \subset X$, $a \in X$, X is a vector space;
$A \cdot B := \{a \cdot b : a \in A,\ b \in B\}, \quad A \subset \mathbb{C}, B \subset \mathbb{C}^n$;
$\delta_{j,k} := \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}$ = the Kronecker symbol;
$\boldsymbol{e} = (e_1, \dots, e_n)$:= the canonical basis in $\mathbb{C}^n$, $e_j := (\delta_{j,1}, \dots, \delta_{j,n})$, $j = 1, \dots, n$;
$\mathbf{1} = \mathbf{1}_n := (1, \dots, 1) \in \mathbb{N}^n$; $\mathbf{2} := 2 \cdot \mathbf{1} = (2, \dots, 2) \in \mathbb{N}^n$;
$\langle z, w \rangle := \sum_{j=1}^n z_j \bar{w}_j$ = the Hermitian scalar product in $\mathbb{C}^n$;
$\bar{w} := (\bar{w}_1, \dots, \bar{w}_n), \quad w = (w_1, \dots, w_n) \in \mathbb{C}^n$;
$z \cdot w := (z_1 w_1, \dots, z_n w_n),\ z = (z_1, \dots, z_n), w = (w_1, \dots, w_n) \in \mathbb{C}^n$;
$e^z := (e^{z_1}, \dots, e^{z_n}), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n$;
$\|z\| := \langle z, z \rangle^{1/2} = \big(\sum_{j=1}^n |z_j|^2\big)^{1/2}$ = the Euclidean norm in $\mathbb{C}^n$;
$\|z\|_\infty := \max\{|z_1|, \dots, |z_n|\}$ = the maximum norm in $\mathbb{C}^n$;

$\|z\|_1 := |z_1| + \cdots + |z_n|$ = the ℓ^1-norm in $\mathbb{C}^n$;
$\mathrm{id}_{A,X}\colon A \to X$, $\mathrm{id}_{A,X}(x) := x$, $x \in A$;
$\mathrm{id}_A := \mathrm{id}_{A,X}$ if $A = X$ or it is clear what the outer space X is;
$\#A :=$ the number of elements of A;
$\operatorname{diam} A :=$ the diameter of the set $A \subset \mathbb{C}^n$ with respect to the Euclidean distance;
$\operatorname{conv} A :=$ the convex hull of the set A;
$A \Subset X \ :\Longleftrightarrow A$ is relatively compact in X;
$\mathrm{pr}_X\colon X \times Y \to X$, $\mathrm{pr}_X(x,y) := x$, or $\mathrm{pr}_X\colon X \oplus Y \to X$, $\mathrm{pr}_X(x+y) := x$;
$B_d(a,r) := \{x \in X : d(a,x) < r\}$, $a \in X$, $r > 0$ ((X,d) is a pseudometric space;
$d\colon X \times X \to \mathbb{R}_+$, $d(x,x) = 0$, $d(x,y) = d(y,x)$, $d(x,y) \le d(x,z) + d(z,y)$));
$B_q(a,r) := \{x \in E : q(x-a) < r\}$, $a \in E$, $r > 0$ ((E,q) is seminormed space;
$q\colon E \to \mathbb{R}_+$, $q(0) = 0$, $q(\lambda x) = |\lambda| q(x)$, $q(x+y) \le q(x) + q(y)$));

Euclidean balls

$\mathbb{B}(a,r) = \mathbb{B}_n(a,r) := \{z \in \mathbb{C}^n : \|z-a\| < r\}$ = the open Euclidean ball in $\mathbb{C}^n$ with center $a \in \mathbb{C}^n$ and radius $r > 0$; $\mathbb{B}_n(a,0) := \varnothing$; $\mathbb{B}(a,+\infty) := \mathbb{C}^n$;
$\overline{\mathbb{B}}(a,r) = \overline{\mathbb{B}}_n(a,r) := \overline{\mathbb{B}_n(a,r)} = \{z \in \mathbb{C}^n : \|z-a\| \le r\}$ = the closed Euclidean ball in $\mathbb{C}^n$ with center $a \in \mathbb{C}^n$ and radius $r > 0$; $\overline{\mathbb{B}}_n(a,0) := \{a\}$;
$\mathbb{B}(r) = \mathbb{B}_n(r) := \mathbb{B}_n(0,r)$; $\overline{\mathbb{B}}(r) = \overline{\mathbb{B}}_n(r) := \overline{\mathbb{B}}_n(0,r)$;
$\mathbb{B} = \mathbb{B}_n := \mathbb{B}_n(1)$ = the unit Euclidean ball in $\mathbb{C}^n$;
$K(a,r) := \mathbb{B}_1(a,r)$; $K(r) := K(0,r)$;
$\overline{K}(a,r) := \overline{\mathbb{B}}_1(a,r)$; $\overline{K}(r) := \overline{K}(0,r)$;
$K_*(a,r) := K(a,r) \setminus \{a\}$; $K_*(r) := K_*(0,r)$;
$\mathbb{D} := K(1) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ = the unit disc;
$\mathbb{T} := \partial\mathbb{D}$;
$\mathbb{T}_a := \{\zeta \cdot a : \zeta \in \mathbb{T}^n\} = \{(\zeta_1 a_1, \dots, \zeta_n a_n) : \zeta_1, \dots, \zeta_n \in \mathbb{T}\}$, $a = (a_1, \dots, a_n) \in \mathbb{C}^n$;

Polydiscs

$\mathbb{P}(a,r) = \mathbb{P}_n(a,r) := \{z \in \mathbb{C}^n : \|z-a\|_\infty < r\}$ = the polydisc with center $a \in \mathbb{C}^n$ and radius $r > 0$; $\mathbb{P}_n(a,+\infty) := \mathbb{C}^n$;
$\overline{\mathbb{P}}(a,r) = \overline{\mathbb{P}}_n(a,r) := \overline{\mathbb{P}_n(a,r)}$; $\overline{\mathbb{P}}_n(a,0) := \{a\}$;
$\mathbb{P}(r) = \mathbb{P}_n(r) := \mathbb{P}_n(0,r)$;
$\mathbb{P}_n := \mathbb{P}_n(1) = \mathbb{D}^n$ = the unit polydisc in $\mathbb{C}^n$;
$\mathbb{P}(a,\boldsymbol{r}) = \mathbb{P}_n(a,\boldsymbol{r}) := K(a_1,r_1) \times \cdots \times K(a_n,r_n)$ = the polydisc with center $a \in \mathbb{C}^n$ and multiradius (polyradius) $\boldsymbol{r} = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$; notice that $\mathbb{P}(a,r) = \mathbb{P}(a, r \cdot \mathbf{1})$; to simplify notation we will frequently write $\mathbb{P}_n(a,r)$ instead of $\mathbb{P}_n(a,\boldsymbol{r})$ (in particular, in all the cases where it clearly follows from the context that r is a multiradius);
$\mathbb{P}(\boldsymbol{r}) = \mathbb{P}_n(\boldsymbol{r}) := \mathbb{P}_n(0,\boldsymbol{r})$;
$\partial_0\mathbb{P}(a,\boldsymbol{r}) := \partial K(a_1,r_1) \times \cdots \times \partial K(a_n,r_n)$ = the distinguished boundary of $\mathbb{P}(a,\boldsymbol{r})$;

Annuli

$\mathbb{A}(a,r^-,r^+) := \{z \in \mathbb{C} : r^- < |z-a| < r^+\}$, $a \in \mathbb{C}$, $-\infty \le r^- < r^+ \le +\infty$, $r^+ > 0$; if $r^- < 0$, then $\mathbb{A}(a,r^-,r^+) = K(a,r^+)$; $\mathbb{A}(a,0,r^+) = K(a,r^+) \setminus \{a\}$;
$\mathbb{A}^n(a,r^-,r^+) := \mathbb{A}(a_1,r_1^-,r_1^+) \times \cdots \times \mathbb{A}(a_n,r_n^-,r_n^+)$, $a = (a_1,\dots,a_n) \in \mathbb{C}^n$, $r^- = (r_1^-,\dots,r_n^-)$, $r^+ = (r_1^+,\dots,r_n^+)$, $-\infty \le r_j^- < r_j^+ \le +\infty$, $r_j^+ > 0$, $j = 1,\dots,n$;
$\mathbb{A}^n(r^-,r^+) := \mathbb{A}^n(0,r^-,r^+)$;

Laurent series

$z^\alpha := z_1^{\alpha_1}\cdots z_n^{\alpha_n},\ z = (z_1,\dots,z_n) \in \mathbb{C}^n, \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n$ $(0^0 := 1)$;
$\alpha! := \alpha_1!\cdots\alpha_n!,\ \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}_+^n$;
$|\alpha| := |\alpha_1| + \cdots + |\alpha_n|,\ \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$;
$\binom{\alpha}{\beta} := \frac{\alpha(\alpha-1)\cdots(\alpha-\beta+1)}{\beta!}, \alpha \in \mathbb{Z}, \beta \in \mathbb{Z}_+$;
$\binom{\alpha}{\beta} := \binom{\alpha_1}{\beta_1}\cdots\binom{\alpha_n}{\beta_n},\ \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}^n,\ \beta = (\beta_1,\dots,\beta_n) \in \mathbb{Z}_+^n$;

Functions

$\|f\|_A := \sup\{|f(a)| : a \in A\},\ f : A \to \mathbb{C}$;
$\operatorname{supp} f := \overline{\{x : f(x) \ne 0\}}$ = the support of f;
$\mathcal{P}(\mathbb{C}^n)$:= the space of all polynomials $f : \mathbb{C}^n \to \mathbb{C}$;
$\mathcal{P}_d(\mathbb{C}^n) := \{F \in \mathcal{P}(\mathbb{C}^n) : \deg F \le d\}$;
$\mathcal{C}^\uparrow(\Omega)$:= the set of all upper semicontinuous functions $u : \Omega \to \mathbb{R}_{-\infty}$;
$\frac{\partial f}{\partial z_j}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(a) - i\,\frac{\partial f}{\partial y_j}(a)\Big)$, $\frac{\partial f}{\partial \bar z_j}(a) := \frac{1}{2}\Big(\frac{\partial f}{\partial x_j}(a) + i\,\frac{\partial f}{\partial y_j}(a)\Big)$ = the formal partial derivatives of f at a;
$\operatorname{grad} u(a) := (\frac{\partial u}{\partial \bar z_1}(a),\dots,\frac{\partial u}{\partial \bar z_n}(a))$ = the gradient of u at a;
$D^{\alpha,\beta} := (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \cdots \circ (\frac{\partial}{\partial z_n})^{\alpha_n} \circ (\frac{\partial}{\partial \bar z_1})^{\beta_1} \circ \cdots \circ (\frac{\partial}{\partial \bar z_n})^{\beta_n}$;
$\mathcal{C}^k(X,Y)$:= the space of all $\mathcal{C}^k$-mappings $f : X \to Y,\ k \in \mathbb{Z}_+ \cup \{\infty\} \cup \{\omega\}$ (ω stands for the real analytic case);
$\mathcal{C}^k(\Omega) := \mathcal{C}^k(\Omega,\mathbb{C})$;
$\mathcal{C}_0^k(\Omega) := \{f \in \mathcal{C}^k(\Omega) : \operatorname{supp} f \Subset \Omega\}$;
$\widehat{\mathcal{C}}^1([a,b],Y)$:= the space of all piecewise $\mathcal{C}^1$-mappings $\alpha : [a,b] \to Y$;
Λ_N := Lebesgue measure in $\mathbb{R}^N$;
$L^p(\Omega)$:= the space of all p-integrable functions on Ω; $\|\ \|_{L^p(\Omega)}$:= the norm in $L^p(\Omega)$;
$L^p(\Omega, \mathrm{loc})$:= the space of all locally p-integrable functions on Ω;
$\mathcal{O}(X,Y)$:= the space of all holomorphic mappings $f : X \to Y$;
$\mathcal{O}(\Omega) := \mathcal{O}(\Omega,\mathbb{C})$ = the space of all holomorphic functions $f : \Omega \to \mathbb{C}$;
$f^{(k)}(a)$:= the k-th complex Fréchet differential of $f : \Omega \to \mathbb{C}^m$ at a;
$\frac{\partial f}{\partial z_j}(a) := \lim\limits_{\mathbb{C}_* \ni h \to 0} \frac{f(a+he_j)-f(a)}{h}$ = the j-th complex partial derivative of f at a;
$Jf(a) := \det\Big[\frac{\partial f_j}{\partial z_k}(a)\Big]_{j,k=1,\dots,n}$, $f = (f_1,\dots,f_n) : \Omega \to \mathbb{C}^n$;

$D^\alpha := (\frac{\partial}{\partial z_1})^{\alpha_1} \circ \cdots \circ (\frac{\partial}{\partial z_n})^{\alpha_n} = \alpha$-th partial complex derivative;
$T_a f(z) := \sum_{\alpha \in \mathbb{Z}_+^n} \frac{1}{\alpha!} D^\alpha f(a)(z-a)^\alpha =$ the Taylor series of f at a;
$d(T_a f) := \sup\{r \geq 0 : T_a f(z)$ is uniformly summable for $z \in \overline{\mathbb{P}}(a,r)\} =$ the radius of convergence of the Taylor series $T_a f$;
$L_h^p(\Omega) := \mathcal{O}(\Omega) \cap L^p(\Omega) =$ the space of all p-integrable holomorphic functions on Ω;
$\mathcal{O}^{(k)}(\Omega, \delta) := \{f \in \mathcal{O}(\Omega) : \|\delta^k f\|_\Omega < +\infty\}$;
$\mathcal{O}(A) := \bigcup_{\substack{U \supset A \\ U \text{ open in } \mathbb{C}^n}} \mathcal{O}(U)|_A$;
$\mathcal{H}^\infty(\Omega) :=$ the space of all bounded holomorphic functions on Ω;
$\mathcal{A}^k(\Omega) := \{f \in \mathcal{O}(\Omega) : \forall_{\alpha \in \mathbb{Z}_+^n,\, |\alpha| \leq k} : D^\alpha f \in \mathcal{C}(\overline{\Omega})\}$;
$\operatorname{Aut}(\Omega) :=$ the group of all holomorphic automorphisms of Ω;
$\operatorname{Aut}_a(\Omega) := \{h \in \operatorname{Aut}(\Omega) : h(a) = a\}$;
$\mathcal{SH}(\Omega) :=$ the set of all subharmonic functions on Ω, $\Omega \subset \mathbb{C}$;
$\mathcal{PSH}(\Omega) :=$ the set of all plurisubharmonic functions on Ω;
$\mathcal{L}u(a; X) := \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) X_j \overline{X}_k =$ the Levi form of u at a.

List of symbols

Chapter 1

Chapter 2

Chapter 3

Chapter 4

Subject index