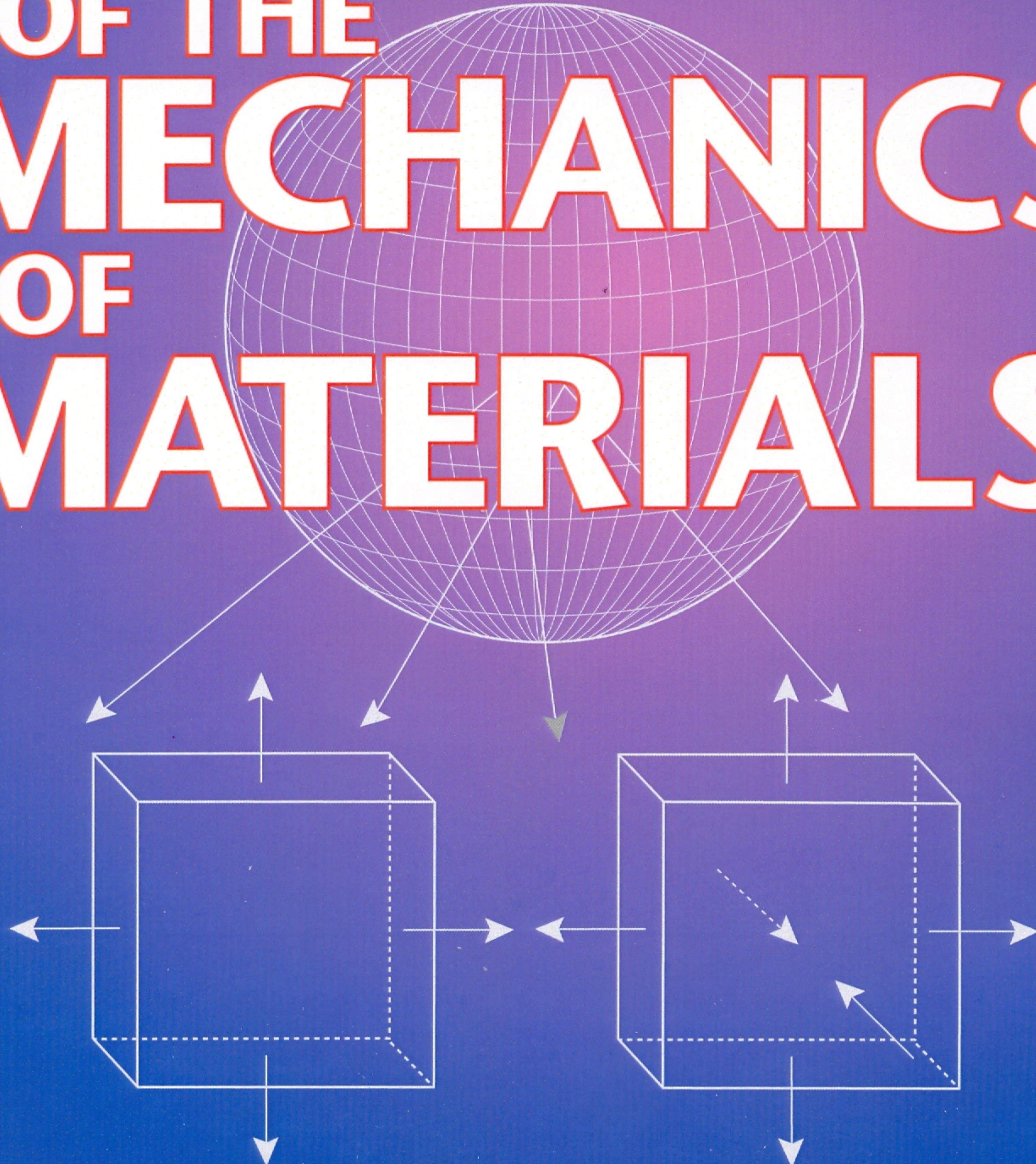
SECOND EDITION

ESSENTIALS OF THE MECHANICS OF MATERIALS



George N. Frantziskonis

ESSENTIALS OF THE MECHANICS OF MATERIALS

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PREFACE

Mechanics of Materials (MOM) is a key sophomore/junior level course in many engineering majors including civil, mechanical, aerospace, biomedical, mining, and optical engineering. It is typically offered as a three-unit course over the period of one semester. Over my many years of experience in teaching MOM and many courses relevant to MOM, I have found that some of the most popular MOM books suffer from the following drawbacks in connection to student learning:

- They provide only limited resources to students requiring further assistance in comprehending the material. If, for example, a student is unable to understand a certain concept or chapter, even after going to class and studying the textbook, the only alternative is visiting the instructor, or external tutoring, both of which are often difficult or inconvenient. Some books that offer supplements such as videos and web-based aids have the drawback that the student has to "disengage" from the book and visit relevant resources with no direct connection to the book material
- Individual classroom lectures do not correspond, in detail, to specific sections and chapters in the textbook. Thus, students that do not comprehend the material during classroom time, often have difficulty in identifying what sections of the book were covered and what should be studied. This problem is particularly acute for students that stay behind in the course due to illness or other reasons.

About the Book

This book is offered in either electronic or printed version and offers:

- Clearly identified material broken into individual study modules
- Individual study modules that correspond to specific class lectures or weekly course material
- A printed version of the book, intended for (a) the student to study on his/her own schedule and pace without disturbances; (b) the student to bring in class so that its correspondence to the instructor's teaching can be identified in a live fashion; (c) the student to take additional notes and/or highlight material in the printed version while the instructor presents the material
- An electronic version of the book that, in addition to being available continuously in students' laptop, it is intended for (a) students to access in order to comprehend the material better and link directly to additional exercises, including interactive ones, movies, and other electronic supplements; (b) the students to have the electronic material ready in order to effectively link to other students, discussions and discussion groups, access additional resources and links.

About the Author

George Frantziskonis received his Civil Engineering degree from the Aristotle University, Greece and his doctorate in Engineering Mechanics from the University of Arizona. He has been a faculty member at the University of Arizona for over twenty years. He is a registered professional engineer in the State of Arizona and in the European Union state members. He has taught numerous courses in the broad areas of engineering mechanics, general engineering, and engineering design. He has directed graduate students in engineering mechanics, civil engineering, and mechanical, aerospace engineering. His teaching interests and activities include contemporary multimedia course delivery methods tailored to classroom-based, distance-learning, and classes of large enrollment. Dr. Frantziskonis' areas of research interests and activities include multi and interdisciplinary multiscale modeling and simulation, material characterization and applications, probabilistic and multiscale material description and applications to safety and reliability, behavior of materials at nano-scale, reaction-diffusion and reactive flow problems. He has published extensively in journals in the areas of mechanics, civil engineering, materials science and engineering, physics, computational physics, and chemical engineering; his work has been cited extensively. He holds a joint appointment with the Material Science and Engineering department and has worked as visiting professor in France, Norway, and Greece; he has also taught courses in France and Germany. He has received several awards including the Presidential Young Investigator and the Fulbright award.

Preface to the Second Edition

The second edition includes many more homework assignment problems, more examples presented in chapters, improved figures, and errata present in the first edition have been corrected.

CONTENTS

CONTENTS

MECHANICS OF MATERIALS

Introduction

Mechanics of materials (MOM) is the area of engineering dealing with the mechanical behavior of materials, including their strength. A prime objective is to determine the stresses, strains, displacements in structures or structural components when subjected to external loads.

Examples of structures are cranes, buildings, bridges, airplanes, machines, and ships; examples of structural components are cables, beams, airplane wings, turbine blades, engine pistons, and bolts. MOM is based on understanding the physical properties of materials and provides a foundation to many essential techniques that allow engineers to design structures (civil, mechanical, aerospace, mining, etc.), predict failures, and estimate safety margins. A basic MOM course provides the tools for determining stresses, strains, and displacements in structures subjected to applied loads. Engineering design concepts and basic safety concepts are also integrated into the course.

Equivalent course/subject titles are: "Solid Mechanics," "Strength of Materials," and "Mechanics of Deformable Bodies." Independently of the title, they all extend the equilibrium concepts learned in statics to determine the stresses, strains and displacements in structures or structural components. The problem statement in the following explains MOM in a general context.

Problem statement and MOM

Consider a structure. An example would be a bridge or an airplane, which may include a structural component (e.g., one of the bridge's columns or one of the airplane's wings), or even a subcomponent (e.g., a rivet in the wing). Other examples include structures as small as microchips and their subcomponents, or as large as an offshore platform or a ship. We often use the term "body" to designate an engineering structure or structural component.

<u>A general mechanics of materials problem can be defined as follows.</u>

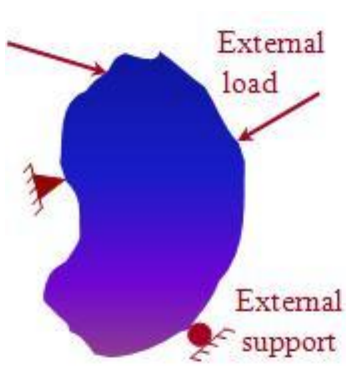


Figure A-1 A structure (body) externally supported and subjected to loads.

For the structure (body) shown in Figure A-1 the following is given:

- The geometry of the body (structure)
- The type and position of the external supports
- The external load configuration, i.e., the position and type of load
- The material(s) the body is made of

Example: A wood truss of given dimensions, loaded by a given vertical load at C, and supported by a pin at A and a roller at D

MOM addresses how to evaluate:

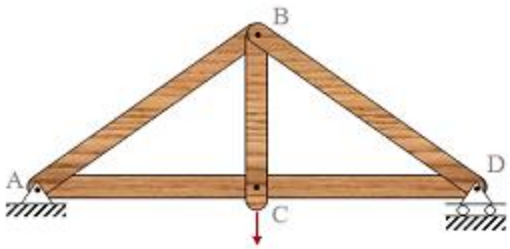
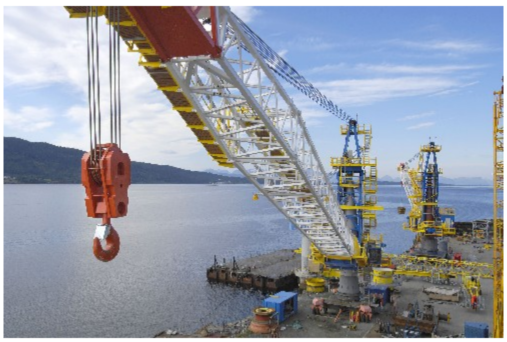


Figure A-2 A wood truss.

- All displacements, e.g., deflections in the truss
- Stresses and strains at every truss component
- Stresses and strains at all connections
- The failure safety margins of the structure for the given loads and supports

Moodule 1: Introduction

This MOM course contributes primarily to the students' knowledge of engineering topics, and provides analysis and design experience. The difference between analysis and design can be efficiently explained by an example. Consider the development of a crane to perform specific tasks, namely moving certain types of loads from one point to another. The design of the crane involves defining its geometry in space, the materials to be used as components of the crane, and the dimensions of the components. The analysis of the crane involves calculation of deformations, strain, and stresses in the structural components to ensure the adequate and safe operation of the crane during its lifetime.



The output of a design is usually a set of drawings and specifications that should produce a working product such as the crane mentioned above, with very little final adjustment needed. If significant rework is required in the construction, startup, or manufacturing phase, the engineer did not do an acceptable job, or could not foresee potential problems with the analysis of the product. This ability to foresee potential problems is a key skill for an engineer.

Points to be understood

Before the MOM course is completed, it helps to know an answer to the following question: what is the difference between a MOM problem and an engineering design problem? The answer to this is that MOM addresses mostly analysis with little design content. However, it provides the basic background needed for design, typically taught in courses following MOM.

This example should help to further clarify the difference between analysis and design. Consider, as an example, a problem of choosing the right material for a specific bridge or for an automotive engine. Is that problem within the context of MOM? The answer is that choosing the material for a particular structure is part of the design process. The analysis part will have the properties of the material chosen as parameters.

Background/prerequisites

The following list of topics furnishes the material a student needs to have mastered or appropriately reviewed, most of which are typically taught in a statics course:

• Use of significant figures and units (both SI and US customary)
• Force and moment vectors
• Reactions at supports of simple structures
• The use of equilibrium equations
• The idea of statically determinate and statically indeterminate problems
• Centroid of composite areas; moments of areas
• Area moments of inertia

MOM Topics

An engineer with extensive background on MOM can analyze complex structures or structural components. Some structures require use of numerical techniques and computer programs for their analysis. However, simple structures allow for the development of the concepts and application of the seconcepts to analysis. Understanding of the mechanical behavior of simple structures is fundamental to MOM and also finds applications. Thus, this course addresses the following MOM problems.

When axial load is applied on a prismatic bar (Fig. A-3), we have a so-called axially loaded member, bar or rod. An example is a cable or a truss member. Bars are relatively simple to analyze within the context of MOM.

Figure A-3 A bar or rod.

When torsional load is applied on a prismatic bar (Fig. A-4) we have a so-called shaft in torsion problem. Analysis of torsion of shafts of circular or hollow cross section is easier than that of other cross sections.

Figure A-4 A torsional shaft.

When bending load is applied on a prismatic bar (Fig. A-5) we have a beam bending problem. Analysis of bending of slender beams is easier than in non slender ones.

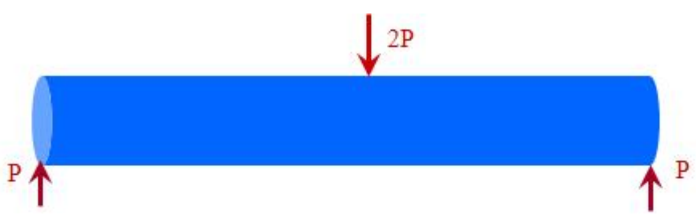


Figure A-5 A beam.

Often a combined load is applied (Fig. A-6), and MOM analysis of such cases calls for the combined application of methods for bars, shafts, and beams.

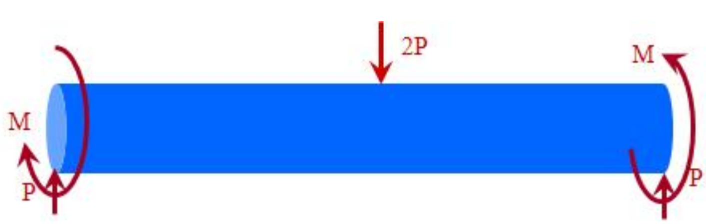


Figure A-6 Combined load, bending and torsion.

When compressive load is applied in a bar, there is always the problem of buckling. Bars under compressive load are called columns. MOM analysis of columns is important and typically studied as a stability of structures problem.

Thus, the terms bar, shaft, beam, and column are relevant to the type of external load imposed rather than to the shape or material of these (uniaxial) structures. After studying such "simple" structures, the multi-dimensional nature of stress and strain is examined, which leads to stress and strain transformation in space and graphical representation of such transformations using the so-called Mohr's circle. This is followed by the study of "special" structures such as spherical and cylindrical vessels.

Module 1: Normal Stress and Strain

Two fundamental concepts in mechanics are those of stress and strain. A simple example illustrating these concepts is shown in Figure B-1. Pressure, for example the air pressure in an automobile tire, is force per unit area. The units of pressure are N/m^2 or psi. Similarly to pressure, axial or normal stress is defined as the "pressure" in a solid member. In particular, **axial or normal stress**, symbolized in engineering by the Greek letter sigma (σ), is defined as the force perpendicular to the cross sectional area of the member divided by the cross sectional area. Thus, for a cable supporting weight W or a force P, in general, the stress is defined as the force per area, $\sigma = P/A$, where A denotes the cross sectional area of the cable.

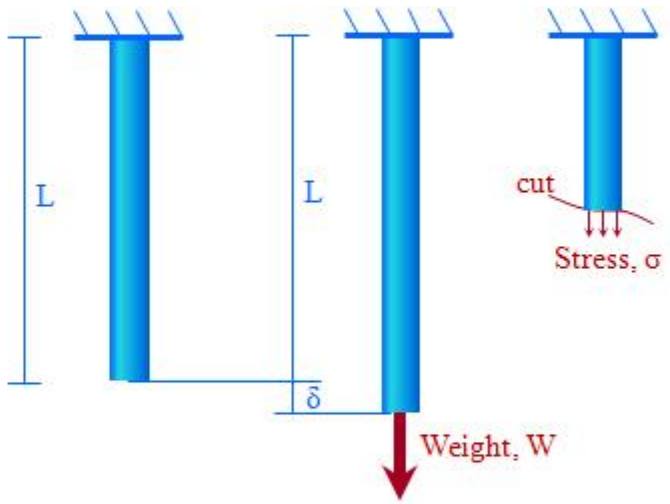


Figure B-1 A cable, in its load-free state (left), supporting weight W (middle), and a section at some point along its length (right).

Due to W or P, the cable experiences elongation (δ). The elongation in this case is also the displacement of the bottom point of the cable, since the displacement of the fixed top is zero. The elongation will be used in the following to define the strain in the cable. Furthermore, the cable transfers W to the support at the top, and this creates stress in the cable at any section along its length L. The cable can be considered as a prismatic bar, i.e., a straight structural member showing the same cross section throughout its length, subjected to an axial force (weight W or force P in general). The force W or P is used to define the stress in the cable. Normal stresses are **tensile** when the force P stretches the bar and **compressive** when P compresses the bar. It is customary in most engineering applications to have a **sign convention** assigning tensile stresses as positive and compressive stresses as negative.

It is important to note that the definition $\sigma = P/A$ assumes that the stress distribution in the cross section is uniform. The validity of this assumption is examined in detail in the following.

Module 1: Normal Stress and Strain

It is crucial to understand the concept of stress and strain as applied to a bar, thus defined as normal stress and normal strain. Consider a bar, which also forms a structural member, with constant cross sectional area along its length, supported (fixed) at one of its ends as shown in Figure B-2. The cross section of the bar can be rectangular, square, circular, etc. The bar is subjected to tensile load, P.

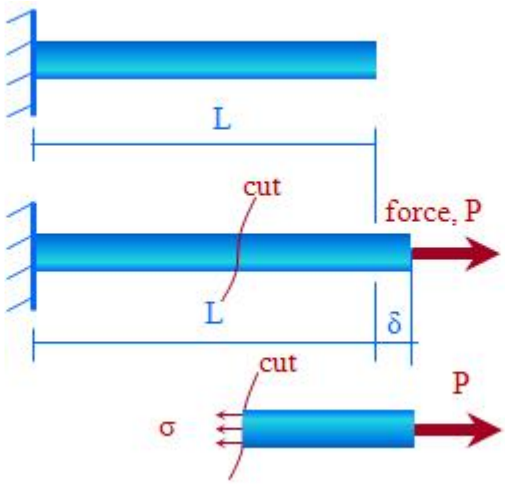


Figure B-2 A bar fixed at one end without any external force (top). The same bar subjected to external force P (middle). An imaginary part of the loaded bar (bottom) after a cut transverse to the main axis of the bar. The area of the cut is termed the cross sectional area of the bar. The force P is equilibrated by the stress σ.

Because of the external force P, the bar stretches, i.e., elongates by an amount, δ. Also, since the bar is in equilibrium, internal forces, denoted as σ, must act at the imaginary section. The stress and strain in the bar can now be defined. Before doing so, it is noted that the elongation δ in the figure is exaggerated. Usually in engineering structures displacements are small. For example a 1-m-long steel bar will experience displacements of less than 1/2cm under working conditions. The elongation δ is, by sign convention, **positive** when the bar is subjected to **tensile** load, which stretches the bar. Similarly, δ is **negative** when the bar is subjected to **compressive** load which **compresses** the bar. In general, the elongation of a segment of the bar is equal to the segment's length divided by the total length L and multiplied by the total elongation δ. Thus, for example, half of the bar (L/2) elongates/compresses by 1/2 δ, one-quarter of the bar (L/4) elongates/compresses by 1/4 δ, etc. This makes the definition of elongation/contraction per unit length convenient, which is precisely the normal strain defined in detail in the following.

Now imagine a "cut" as shown in Figure B-2. Next we consider the equilibrium of this piece of material.

Equilibrium of the piece of the bar shown in Figure B-2 must be satisfied. The arrows shown at the left edge are actually force per unit of cross sectional area, and they are the so-called internal stresses, σ. More precisely, for equilibrium of the piece of the bar in the horizontal direction, the force acting at the right edge, P, should be equal to the force acting at the left edge. The latter is equal to the stress σ (units of force per area) multiplied by the area (units of area) The net result for the force acting on the left edge is σ A. It is repeated herein that σ denotes the force per unit of cross sectional area. For equilibrium, it should hold that $P = \sigma A$ from which it follows that

$$\sigma = \frac{P}{A}$$

This is the definition of stress, i.e., force P divided by the area A over which P acts.

Sign convention and units of stress

When P stretches the bar, the resulting stresses are **tensile** (or in other words, thebar is under tension) and σ is considered, by convention, **positive**. When P compresses the bar, the resulting stresses are **compressive** (or in other words, the bar is under tension) and σ is considered, by convention, **negative**.

The above equation is valid as long as the the stress is uniformly distributed over the cross sectional area A. For bars of uniform cross section in their longitudinal direction loaded at their centroid (the force P is applied at the centroid) this is true. When this is not the case, the definition of stress σ cannot be defined over the entire cross section but is rather defined as a local quantity of force applied over an infinitesimal area ΔA. For example, if P is applied eccentrically to the centroid of A, the stress distribution ceases to be uniform over A. This case will be examined later when studying beams. As another example, when the cross sectional area is not uniform in the longitudinal direction, possibly due to the presence of voids or cracks, the stress distribution also ceases to be uniform over the cross section. The concept of uniform and non uniform stress distribution is examined throughout the course.

As far as the units of stress are concerned, we have:

$$\sigma = \frac{\text{Force}}{\text{Area}} = \frac{\text{Newtons(N)}}{\text{m}^2} \text{ or } \frac{\text{Pounds}}{\text{inch}^2}$$

Units used in engineering are the Pascal (symbolized as Pa) and the psi (pounds per inch).
By definition:

1 Pa = 1 N/m^2	1 psi = 1 lb/in^2

Also, the following "derivatives" are commonly used:

1 kPa = 1,000 Pa	1 MPa = 1,000,000 Pa	1 GPa = 10^9 Pa	1 ksi = 1,000 psi

Example: Consider a cylindrical bar of radius 2 cm (0.02 m) subjected to a tensile load of 100 kN along its axis.

The stress on each cross section is: $\sigma = \dfrac{P}{A} = \dfrac{100\,\text{kN}}{\pi(0.02)^2} = 79{,}577.5\,\text{kPa}$

Module 2: Non-uniform Normal Stress, Normal Strain

Stress is defined as force per unit area. As such, it can be defined for an arbitrary area, e.g., a very small or very large area. Therefore, a more precise definition of stress is similar to σ = P/A, but it is defined for an infinitesimally small area, i.e., for $\Delta A \to 0$. Thus,

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

is a point-wise quantity, and the equation σ = P/A should be written more precisely as

$$\sigma_{ave} = \frac{P}{A}$$

where *ave* denotes average. As mentioned before, for a bar of uniform cross section along the bar's longitudinal direction and for external load P applied at the centroid of the cross section, the stress distribution is uniform. Obviously, in this case, **$\sigma = \sigma_{ave}$ = P/A**, i.e., for **uniform stress distribution**. For bars, we assume this is the case (uniform stress distribution) unless specifically stated otherwise.

The definition of stress σ = P/A, is obviously valid up to the point where the bar will break. The maximum stress the bar can take before breaking is called ultimate stress, σ_u, and it is strongly material dependent. For example, for a certain aluminum alloy σ_u= 310 MPa, or 45 ksi, while for some plastics σ_u is usually less than 80 MPa (12 ksi).

Example: The bar shown below has a circular cross section of radius equal to 2 cm. Is the load the bar is subjected to, i.e., an axial force P = 100 kN, below the maximum load the bar can withstand, provided the ultimate stress of the material σ_u is 150,000 kPa?

The maximum load this bar can sustain, P_{max}, is that load that creates a stress equal to the ultimate stress. Then,

$$\sigma_u = 150{,}000 = \frac{P_{max}}{A} = \frac{P_{max}}{\pi(0.02)^2}$$

which yields P_{max} = 188.5 kN. Thus, since 100 kN is less than 188.5 kN, the load of 100 kN is below the ultimate load the bar can sustain.

Non uniform stress

Figure B-3 shows schematically the definition of stress at a point, and the notion of uniform and nonuniform stress at a cross section. Actually, a point here implies a small infinitesimal area, ΔA, where a force is applied on it, as shown in Figure B-3 (a). In Figure B-3 (b), all forces in each infinitesimal area ΔA are equal to each other, and this defines a uniform state of stress. In Figure B-3 (c) the force varies in each ΔA, thus we have a nonuniform state of stress.

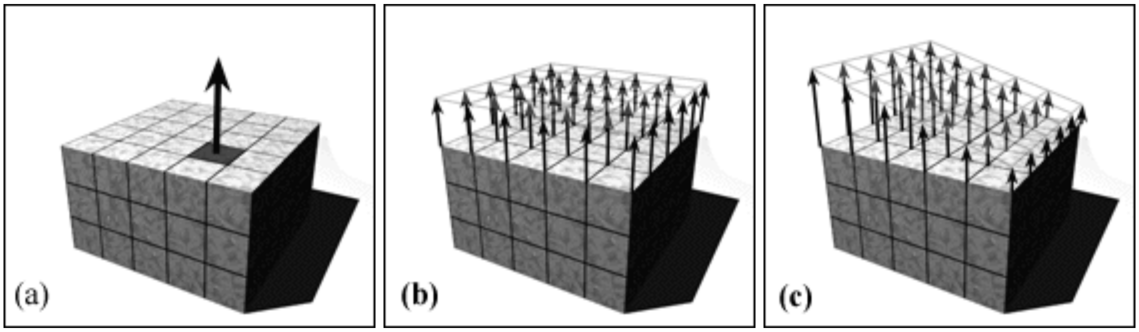


Figure B-3 (a) Schematic of the definition of stress over an infinitesimally small area ΔA, (b) an example of uniform stress distribution over an area A, (c) nonuniform stress distribution over an area A.

As is obvious, the stress distribution in Figure B-3 (b) (uniform) results in a force acting at the centroid of the cross section. The stress distribution in Figure B-3 (c) however (nonuniform) results in a force acting eccentric to the centroid of the cross section. This holds true not only for cross sections of rectangular shape but for every shape of cross section.

$$P = \iint_A dF = \iint_A \sigma dA$$

However, calculation of such integrals is often not necessary in engineering mechanics. Typically the nonuniformity is of known shape, therefore calculation of such integrals is not necessary for the evaluation of a volume and its centroid.

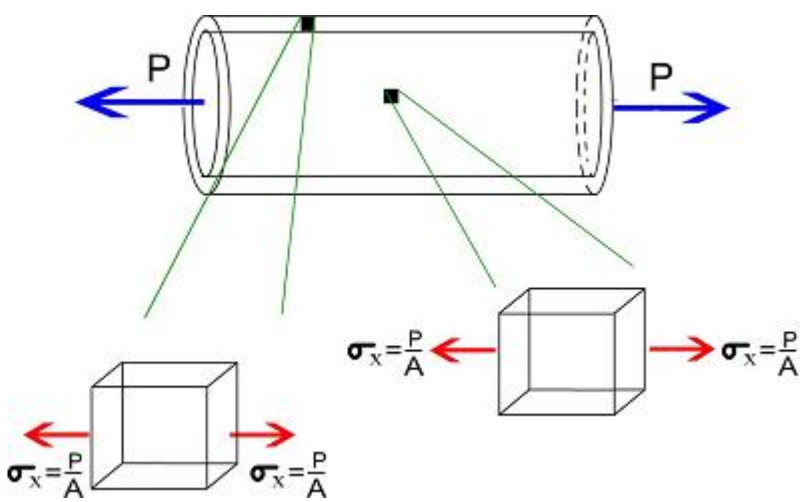
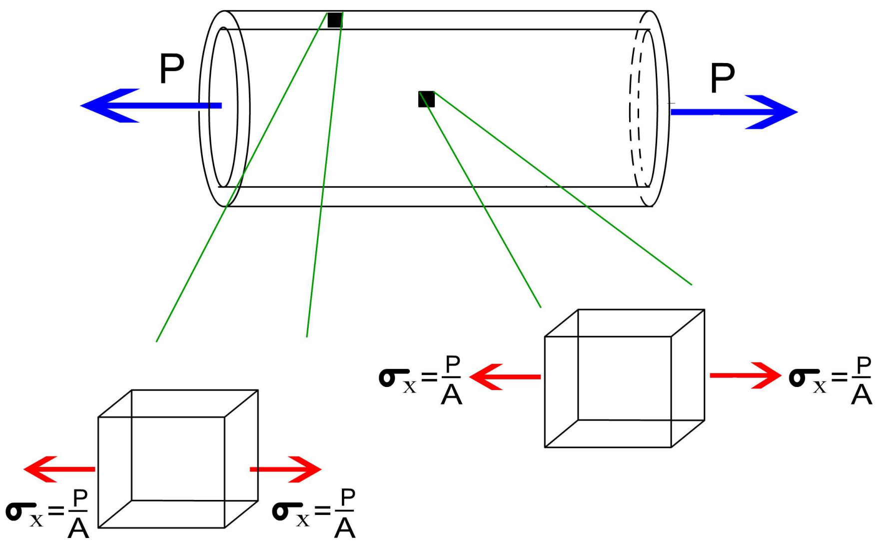
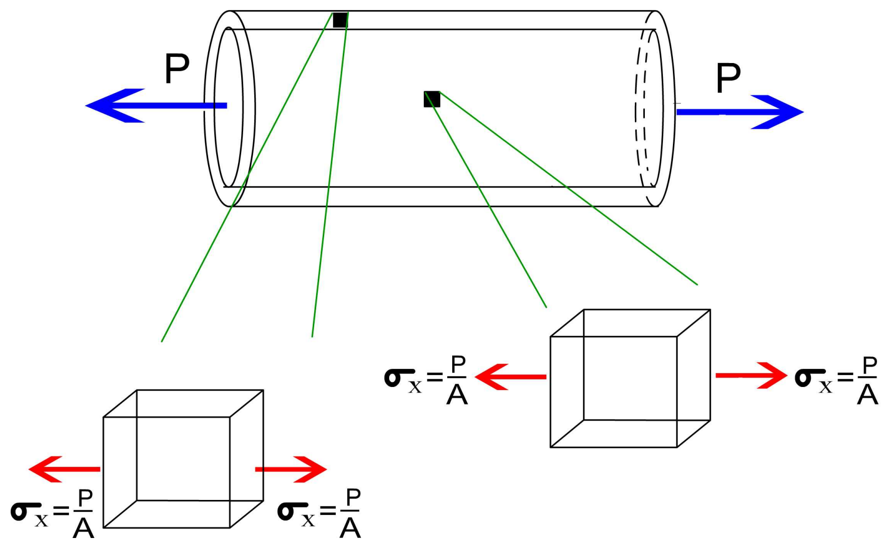


Figure B-4 Details of the normal stress σ_x at two material elements in a hollow cylindrical bar subjected to load P. The cross sectional area of the bar is A.



The stress σ = P/A in a bar of cross-sectional area A subjected to tensile or compressive load P is uniform. Thus, as shown in the "movie" above, the stress is the same at every material element in the bar. Also, for a coordinate system where the x-direction is along the bar's length, the stress is designated as σ_x = P/A. Even though designating the stress for a bar as σ_x instead of σ is not necessary, for more complex stress states examined later in this course, e.g. two-dimensional or three-dimensional stress states, the index designation referring to the axis of action will become necessary.



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Example: Stress in a Cone-Shaped "Structure"

Normal stress is equal to zero at the top and is maximum at the base.

$$\text{Total volume} = \frac{200+400}{3}150$$

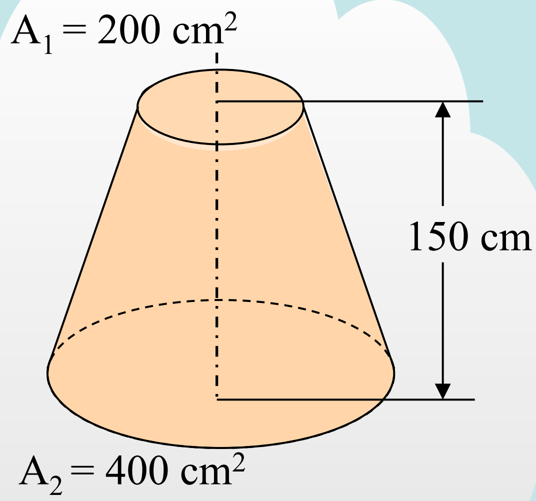
$$= 30{,}000\ \text{cm}^3$$

$$\text{Total weight} = 30{,}000 \times 12 \times \frac{9.81}{1{,}000}$$

$$= 3{,}531.6\ \text{N}$$

$$\sigma = \frac{W}{A} = \frac{3{,}531.6}{400}$$

$$\sigma = 8.829\ \text{N}/\text{cm}^2 = 88.29\ \text{KPa}$$



The cone-shaped "structure" is subjected to a force from its own weight. Find the normal stress at the bottom of the structure.

Given: $g = 12\ \text{gm/cm}^3$

Normal strain

Consider, as in the previous sections, a prismatic bar, i.e. a structural member with constant area along its length as shown schematically in Figure C-1. In this figure the cross section of the bar is circular. When the bar is in its free state without any load applied, its length is L and its cross sectional area is A. When a tensile load P is applied, the bar stretches by δ, thus δ denotes the change in length, or elongation of the bar.

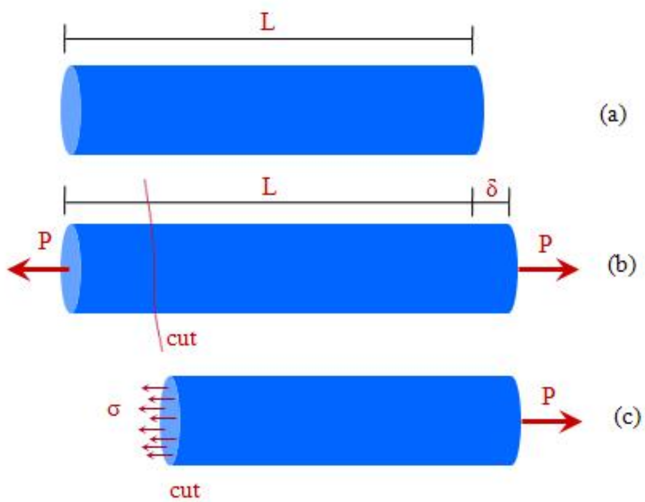


Figure C-1 (a) A bar without any external force; bar length is L. (b) Subjected to external force; the bar elongates by δ. (c) An imaginary part of it; the bar is "cut" at a certain position.

Now we consider an imaginary "cut" as shown in Figure C-1 (b). The external force P [Figure C-1 (c)], is equilibrated by internal forces at the cross section at the cut. As discussed in previous sections, the stress in the bar is σ, which, by the way it was defined, provides a measure of the internal "forces" in a structure (i.e., the bar shown in the figure). To describe deformations in the bar the notion of strain is defined as

$$\text{normal strain} = \frac{\text{change in length}}{\text{original length}}$$

or, using the notation δ for the elongation, it follows that

$$\text{normal strain} = \frac{\delta}{L}$$

In general, the elongation of a segment of the bar is equal to the segment's length divided by the total length L and multiplied by the total elongation δ. Thus, for example, half of the bar (L/2) elongates/compresses by 1/2 δ, one-quarter of the bar (L/4) elongates/compresses by 1/4 δ, etc. This makes the definition of elongation/contraction per unit length convenient, which is precisely the normal strain defined above. The letter ε is used to designate normal strain, thus, by definition the normal strain is expressed as

$$\varepsilon = \frac{\delta}{L}$$

The strain ε is, by sign convention, **positive** when the bar is subjected to **tensile** load, which stretches the bar. In this case, the elongation δ is also positive. Similarly, ε is **negative** when the bar is subjected to **compressive** load, which **compresses** the bar, where the contraction δ is also negative.

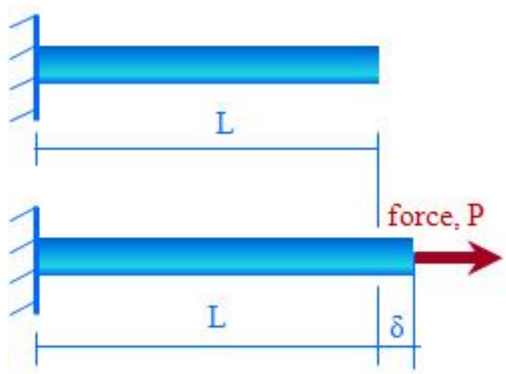
Sign convention and units of strain

As mentioned before, when P is tensile, the bar is elongated and ε is considered, by convention, positive, or in other words, the bar is under tensile strain. When P is compressive, the bar is shortened (or the bar is under compressive strain) and ε is considered, by convention, negative. As far as the units of stress are concerned, we have

$$\varepsilon = \frac{\text{units of length}}{\text{units of length}}$$

thus, in engineering terminology, strain is dimensionless.

Example: Consider a steel bar of length L = 2.0 m. When loaded in tension by a load P, the displacement was measured to be δ = 1.4 mm. (The elongation is exaggerated in the figure!)

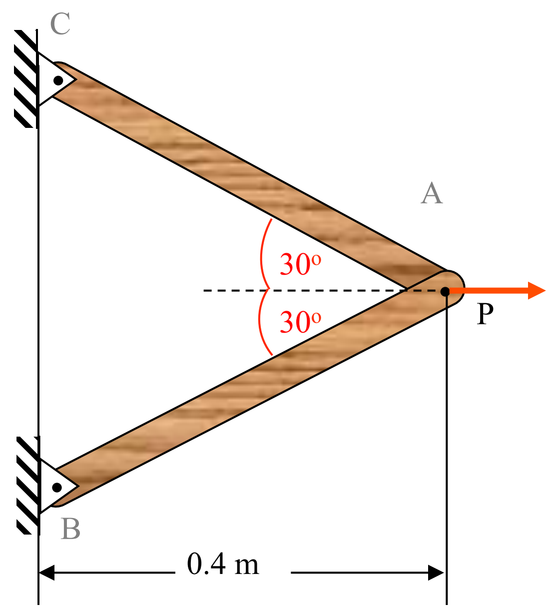


$$\varepsilon = \frac{\delta}{L} = \frac{1.4 \times 10^{-3}\,\text{m}}{2.0\,\text{m}} = 0.0007 = 0.07\ \%$$

Normal strain is more commonly referred to in **percent** than in absolute value. For most engineering structures the "working" strains are rather small, i.e., up to 2%. Here, "working" strain indicates the strain under the conditions of loading of a bar, i.e., when loads are applied.

Example: Normal Strain in a Truss Structure

For the truss structure, determine the normal strain in bars AB and AC if point A moves 2mm to the right.



$$L_{AC} = \frac{0.4}{\cos 30} = 0.462\ \mathrm{m}$$

$$L_{DA} = AA'\cos 30 = 2\cos 30 = 1.732\ \mathrm{mm}$$

$$L_{A'C} = L_{CD} + L_{DA'}$$

$$\text{But, } L_{AC} \approx L_{CD}$$

$$L_{A'C} = 0.462 + \frac{1.732}{1000} = 0.464\ \mathrm{m}$$

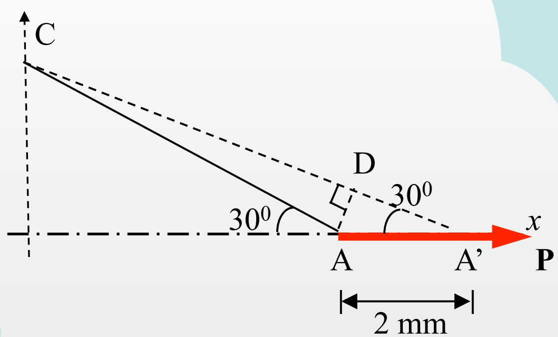
$$e = \frac{DL}{L} = \frac{L_{A'C} - L_{AC}}{L_{AC}}$$

$$= \frac{0.464 - 0.462}{0.462}$$

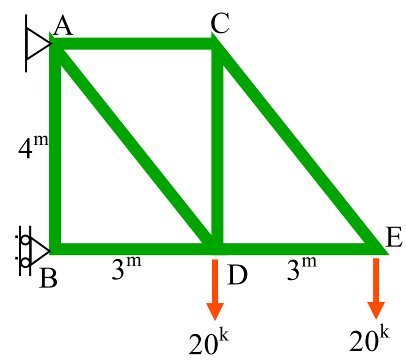
$$= 0.00375\ \mathrm{m/m}$$

$$= 0.375\%$$

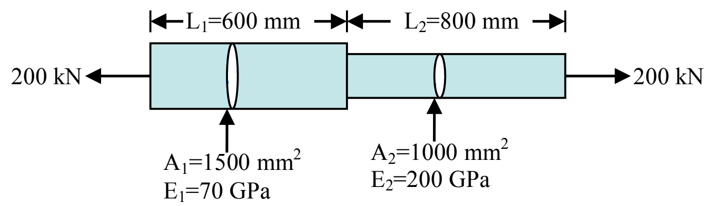
$$= 3750\ \mathrm{m}\ (\text{microstrain})$$



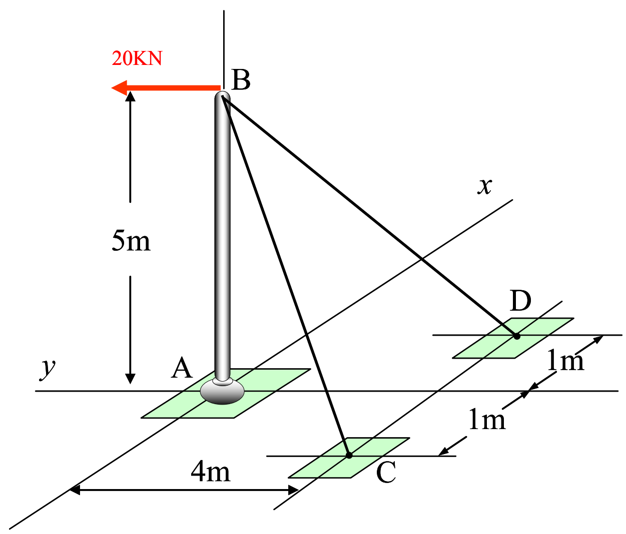
Normal Stress: 1 Bars CE and DE in the truss have a cross-sectional area A = 25cm^2. Find the normal stress in each of these bars.



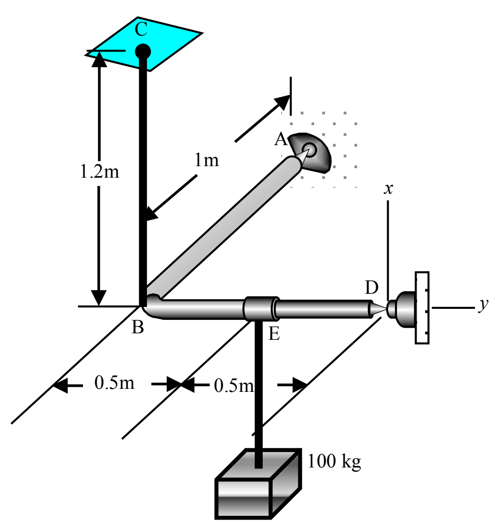
Normal Stress: 2 Given the following structure, find and draw: (a) the distribution of normal stress σ along the length of the structure; (b) the distribution of normal strain ε along the length of the structure.



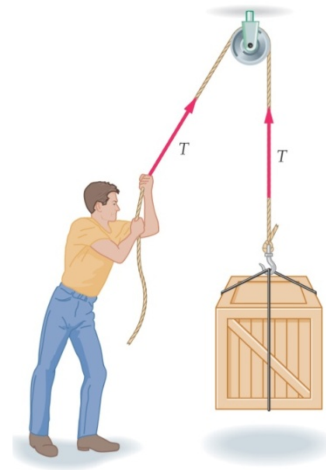
Normal Stress: 3 Beam AB is vertical and the 20 kN load is applied in the *y*-direction. Find the stress in the two identical cables, of diameter d = 2cm.



Normal Stress: 4 Find the minimum diameter the steel cable BC must have such that the normal stress in the cable will not exceed its yield stress, $\sigma_{\text{yield}} = 340\,\text{MPa}$. Consider only the stress imposed on the cable by the 100 kg weight. Consider beam ABED as being rigid and pinned at A and D.

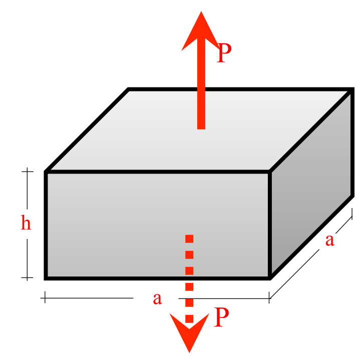


Normal Stress: 5 The tensile force T is equal to the weight of the wooden box W = 300 lbs. The weight W is transmitted to the rope through 4 cables that are inclined at an angle of 30^o to the horizontal plane. (a) If all 4 cables have a diameter of 1/4 inch, find the stress in each cable. (b) If the cables have an ultimate stress $\sigma_U = 60\,\text{ksi}$, find the safety factor the cables are operating at. (c) What would it mean if the safety factor was found to be less than unity?

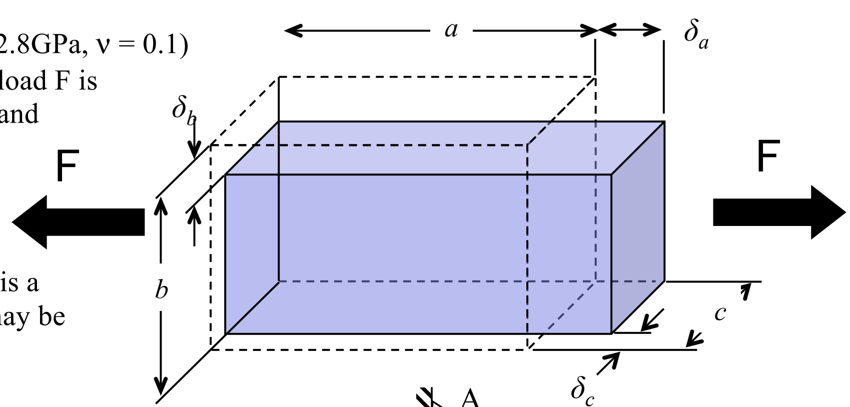


Normal Stress: 6 A tow truck is using a cable to pull the classic car up a 15^o hill. If the classic car weighs 4000 lbs and the cable has a diameter of 3/4 inch, find the stress in the cable when the truck comes to a stop while on the hill. Ignore friction between the car and the pavement.

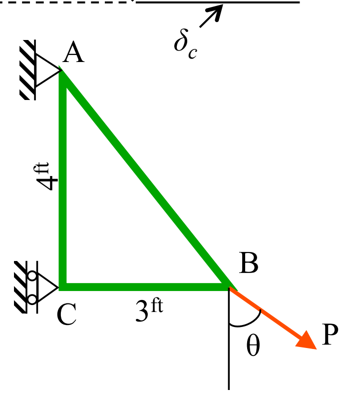
Normal Stress: 7 A mechanical test is performed on a synthetic rubber material by imposing tensile load on a pad as the one shown. For a = 3 inches, h = 1.5 inches, a force P = 180 lbs results in a 0.015 inch vertical elongation of the pad. What is the elasticity modulus of the rubber, E?



Normal Stress: 8 A block of material (E = 2.8GPa, ν = 0.1) has dimensions $a \times b \times c$ before the tensile load F is applied. For a = 50 cm, b = 8 cm, c = 4 cm, and F = 1,000,000 N, find the percent change of the volume of the block after the load is applied. Indicate clearly whether the volume increased, decreased, or remained the same. Note that the material in this case is a flexible one, e.g. a plastic, thus the strains may be relatively large.



Normal Stress: 9 Truss ABC is loaded by P at joint B, and P is oriented at an angle θ from the vertical direction as shown. The cross-sectional area of bar AB is A and of bar BC is 2A. Find the angle θ such that the normal stress in bar AB will be the same as the normal stress in bar BC.



Normal stress versus normal strain: stress-strain diagrams

In previous sections, the normal stress σ and normal strain ε were defined. In order to determine the mechanical behavior of materials, typically experiments are performed where small specimens are placed in a testing machine, as shown in Figure D-1. Load is applied and measurements of deformation and load are taken.

Figure D-1 A mechanical testing machine. The arrow points to the sample being tested.

So-called dog-bone specimens (Figure D-2) are tested in tension by pulling the two ends through large grips and measuring the applied load P (using load cells) and the deformation δ (using extensometers). Elongation is measured over the so-called gage length L. While load P in a bar increases, the elongation of L(δ) increases. Since the cross sectional area A and length L are known, determination of σ and ε is straightforward through use of the standard relations, $\sigma = P/A$, $\varepsilon = \delta/L$. Thus, by continuously increasing P and taking measurements, the material stress-strain response can be determined. The normal stress is related to the normal strain, and the exact relation is strongly dependent on the material.

Figure D-2 Typical shape of dog-bone specimens subjected to load P. L denotes the gauge length.

Let us compare two bars of initially identical size and shape. The first bar is made of a "flexible" material, e.g., rubber, and the second one is made of a "stiff" material, e.g., steel. Naturally, when the two bars are subjected to the same load P, one expects the strain in the first one to be greater than in the second one. In other words, the stress-strain relation is characteristic of each material. Figure D-3 shows a schematic of the deformed shape of the two bars of initially identical shape and size; note that the flexible material deforms significantly more than the stiff one.

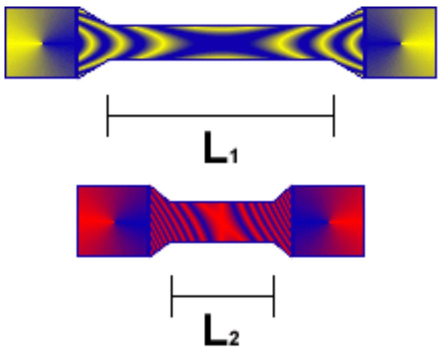


Figure D-3 Deformed shape of two specimens of initially identical shape and size: top, flexible material; bottom, stiff material. The gauge lengths are L_1 and L_2

From the experiment, the stress σ versus strain ε diagram can be plotted, which contains information on the mechanical behavior of the material tested.

Experiments of the type described above are typically carried out until the bar ruptures. Figure D-4 shows typical stress-strain (σ versus ε) curves for various materials. The "x" in the function graphs represents failure of the specimen. **Ductile metals** like structural steel are able to undergo large permanent strain before failure, after reaching a yield stress (i.e., a point of noticeable increase in strain with a small increase in stress). Many aluminum **alloys** also show ductile behavior, yet they do not have a clearly defined yield point. **Rubber** maintains a linear relationship between stress and strain up to relatively large strains as compared to metals. Materials that fail in tension at relatively low values of strain, with only a little elongation after the proportional limit, are classified as **brittle.**Examples include ceramics, concrete, glass, cast iron, etc**.**

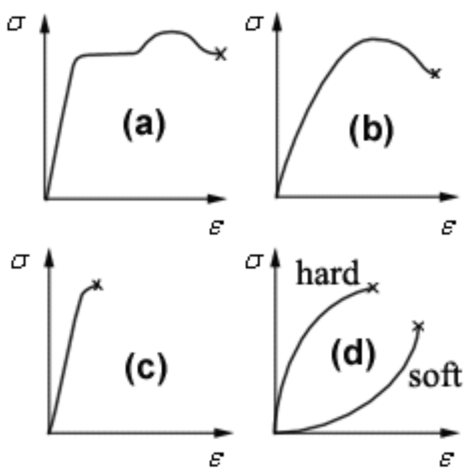


Figure D-4 Stress-Strain curves (schematic) for some engineering materials: (a) structural steel; (b) certain aluminum alloy; (c) a brittle ceramic material; (d) hard and soft rubber.

Figure D-5 shows the details of a stress-strain diagram. There is an **elastic region** in which the stress is directly proportional to the strain. The slope of the straight line is called the **modulus of elasticity, E** . The point at which the elastic region ends is called the elastic limit. The **elastic limit** is the point at which permanent deformation occurs. That is, after the elastic limit, if the force is taken off the sample, it will not return to its original size and shape; permanent deformation has occurred. The **proportional limit** is the point at which the deformation is no longer directly proportional to the applied force (Hooke's law (see modules ahead) no longer holds). Although these two points are slightly different, we will treat them as the same in this course. At point 2 **(the yield point)**, considerable elongation of the test specimen occurs with no significant increase in loading. This is called **yielding** and the corresponding stress is termed **yield stress**. Beyond point 2 the material becomes **plastic** since it deforms irreversibly. At point 3 the maximum stress is called the **ultimate stress**, and further stretching would lead to fracture at point 5 the **failure point** where the sample fails. If the specimen is unloaded in while the stress exceeds the yield stress, it unloads on a path as that of point 3 to point 4, i.e., at a slope equal to the elasticity modulus E.

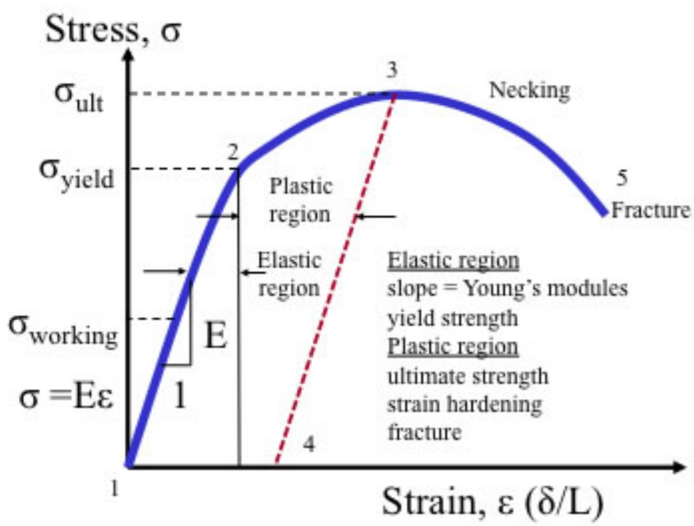


Figure D-5 Detailed stress-strain diagram showing different regions and material behavior in those regions.

Often, material stress-strain response is categorized as shown schematically in the following. Usually, the complete response to failure is not utilized in engineering analysis and design. For many problems, knowing the response for a small amount of strain, i.e., up to the proportionality limit in Figure D-6(a), suffices; here the slope of the (σ-ε) curve is E, and is called Young's modulus or modulus of elasticity. The arrows in the figure indicate the behavior under loading or unloading.

The vast majority of engineering analysis and design problems consider the stress-strain response to be linear, provided, of course, that the yield (end of linearity in the figure) or failure stress or strain is never exceeded. Therefore, this linear response pertains to what is termed Hooke's law. More details are given in the following section. In general, Hooke's law relates stress σ to strain ε in a linear fashion. Figures D-6(b), (c), (d) show various and "typical" stress-strain relations, and the arrows indicate loading or unloading behavior.

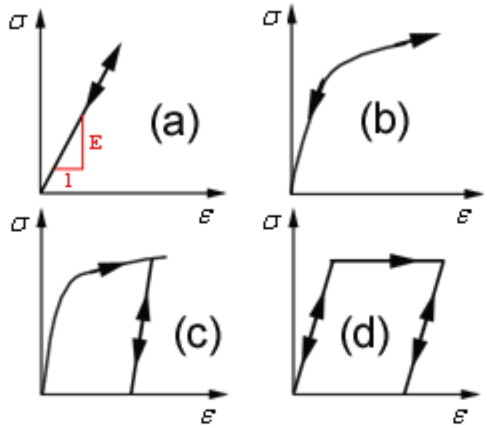


Figure D-6 Schematic of some typical stress-strain relation "categories": (a) linear elastic, (b) nonlinear elastic, (c) elastic plastic, (d) elastic perfectly plastic. Plastic response refers to the process of "plastic flow" within a material. A material is characterized as plastic if the unloading response is different than the loading one, i.e., sections (c) and (d) of the figure.

Terminology relevant to mechanics of materials:

Elasticity is a property of a material that allows the material to follow the same curve back to the origin in a stress-strain diagram. The material returns to its original dimensions during unloading.

Plasticity is a characteristic of a material by which it undergoes inelastic strains beyond the strain at the elastic limit. On the stress-strain curve, we have the elastic region followed by the plastic region.

Creep is typically defined as a characteristic of a material to undergo additional stains when loaded for long periods of time. It is the gradual change in strain after loading has taken place over a period of time.

Ductility is the property of a material that enables it to be drawn out or elongated to an appreciable extent before rupture occurs.

Brittleness is the property of a material that is opposite to ductility. Material, having very little property of deformation, either elastic or plastic is called Brittle.

Stiffness is the property of a material that enables it to resist deformation. Flexibility is the inverse of stiffness.

Module 3: Bar Behavior and Hooke's Law

Most engineering materials show a linear stress-strain response, for relatively small strains, i.e., well below the strain at failure. In this case, we have a linear stress-strain relation, termed Hooke's law. Robert Hooke, who in 1676, stated that the linear relationship between stress and strain for a bar in simple tension or compression is expressed by the equation $\sigma = E\varepsilon$, where E is the constant of proportionality called the modulus of elasticity. It isoften referred to as Young's modulus, after scientist Thomas Young (1773-1829), who introduced the idea of the modulus of elasticity. Thus: **Hooke's Law** linear stress-strain elation states that, according to Figure D-6(a) in the previous page:

$$\sigma = E\varepsilon$$

where E is the Young's modulus of the material. The units of E are those of stress, i.e., Pa in SI units and psi in customary units. Figure D-7 shows schematically Hooke's law as a stress versus strain linear relation. Note that normal stress is always accompanied by normal strain. In subsequent chapters it will be shown that shear stress is always accompanied by shear strain (to be defined). As known from physics, the product of a force and the displacement it produces on a body is the work done. In other words, it is customary to say that force and displacement are *conjugate*. Similarly, it will be shown in subsequent chapters that stress and strain are also conjugate, i.e., their product is work or energy.

The equation $\sigma = E\varepsilon$ relates only to the longitudinal stresses and strain developed when a bar is subjected to tension or compression, i.e., under uniaxial stress conditions. For more complex states of stress, such as those found in most structures and machines, the so-called generalized Hooke's law applies, which relates three-dimensional state of stress to the state of strain. Such topics are examined later in the course.

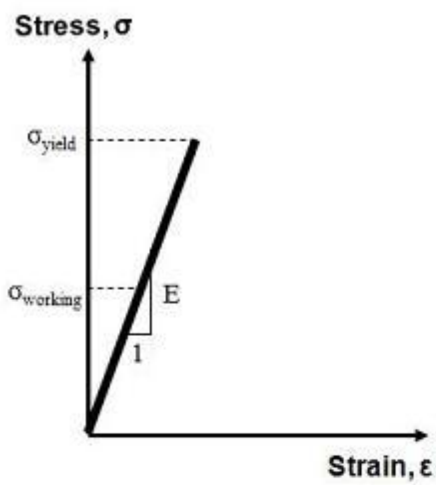


Figure D-7 Hooke's law ($\sigma = E\varepsilon$) shown as a graph. The linear curve is valid up to the yield stress of a material. The stress a bar is subjected to is

$$\sigma_{working} = P_{working}/A$$

where $P_{working}$ is the load the bar is subjected to and A is its cross sectional area.

The value of E for most engineering materials can be found in tables (see Appendices) since they have been tested experimentally. The value of E for most materials is high as compared to the yield or ultimate stress. For steel, E is approximately 210 GPa or 30,000 ksi, while E for aluminum alloys is about 1/3 of these values. Thus steel, in general, is stiffer than aluminum alloys. Soft materials like plastics and rubber have much lower values of E. For example, it is easy to stretch a rubber band, yet not as easy to stretch a steel wire of the same diameter. For most engineering materials, the value of E in tension and compression is approximately the same.

The following relevant definitions are used often in engineering.

Extensometer is used to measure extension (stretch) of a bar, i.e., an instrument used to measure small increments of deformation.

Gauge length is the original length of the sample of which extension calculations are made. It is normally less than the full specimen length and is user defined. The gauge length is sometimes taken as the distance between the grips.

Static test is a test examining the mechanical behavior of a material where the load is applied slowly, i.e. at a slow rate. As opposed to that, in a **Dynamic Test** the load is applied at a high rate.

In a one-dimensional system, such as a uniaxially loaded bar, stress is simply equal to the applied force divided by the cross sectional area of the bar. The above definition of stress, $\sigma = F/A$, where A is the **initial** cross sectional area prior to the application of the load, is called **engineering stress** or **nominal stress**. However, when any material is stretched, its cross sectional area reduces by an amount that depends on the Poisson's ratio of the material (see following module). Engineering stress neglects this change in area. The stress axis on a stress-strain graph is often engineering stress, even though the sample may undergo a substantial change in cross sectional area during testing. **True stress** is an alternative definition in which the initial area is replaced by the current area. In engineering applications, the initial area is always known, so calculations using nominal stress are generally easier. For small deformation, the reduction in cross sectional area is small and the distinction between nominal and true stress is insignificant. This is not so for the large deformations typical of materials such as rubbers, plastics, elastomers, etc., when the change in cross sectional areas can be significant. In uniaxial tension, true stress is greater than the nominal stress, since the cross sectional area decreases with deformation. The converse holds in compression. Similarly to the true stress, a **true strain** measure can be defined, where the deformed length of a bar is used instead of its original length. Such definitions are used in specialized fields of engineering with materials exhibiting large deformations, such as elastomers.

Module 3: Bar Behavior and Hooke's Law

Summary of Mechanical Behavior of a Bar

Structural members subjected only to tension or compression are known as **bars, prismatic bars, or axially loaded members**. A bar subjected to a tensile or compressive force along its length experiences normal stress. Normal stress is the result of force acting transverse to a surface, and is defined as force per unit area. A normal stress is always accompanied by a normal strain, which is a measure of deformation, i.e., change in length over original length. Every material has its own stress-strain behavior, which is determined experimentally by testing specimens in tension or compression. Such behavior is usually complicated for most materials. However, a great number of problems are solved in engineering by using Hooke's law, which considers a linear stress-strain behavior, usually for strains (much) smaller than the strain at failure (rupture) or yield.

Deformation of Bars

Consider a bar of length L, cross sectional area A subjected to load P. We have the following definitions and experimental result:

$\sigma = P/A$	Definition of normal stress, i.e., force per unit area
$\varepsilon = \delta / L$	Definition of strain, i.e., change in length per unit length
$\sigma = E\,\varepsilon$	Experimental stress-strain relation, where E is the Young's modulus or the **modulus of elasticity,** a material-dependent property

By combining these three basic relations, i.e., by solving the second one for δ and substituting the expression for ε from the third one and then the expression for σ from the first relation, we obtain

Force-displacement relation: $\boxed{P = \frac{EA}{L}\delta \qquad \text{or} \qquad \delta = \frac{PL}{EA}}$

These expressions show that elongation (bar under tension) or contraction (bar under compression) is directly proportional to the load P and the length L and inversely proportional to modulus of elasticity E and cross sectional area A.

Springs: Recall that the action of force P on a spring lengthens (or shortens) the spring by an amount δ and its final length becomes L + δ (or L minus δ). If the spring material is linearly elastic, the elongation produced by the load P is directly proportional to δ. The proportionality constants are k (for stiffness) and f (for flexibility). Thus, the above relations clearly indicate that a **bar behaves as a linear spring**, which is examined in more detail in the following section.

The Spring Analogy

As is known, the force displacement relation for a (linear) spring is $P = k\delta$, where P denotes the force the spring is subjcted to, δ denotes the spring elongation or contraction, and k is the spring stiffness or constant. Figure D-8 shows a spring subjected to tensile load P. Thus, from the force-displacement relation for a bar shown in the previous section, it can be easily seen that a bar behaves like a spring and the spring constant or stiffness k is equal to (EA)/L. This analogy is useful especially in understanding the response of a structure, e.g., a truss, to external load application.

Relevant Definitions

Stiffness (k): the force required to produce a unit displacement in a bar or spring. From the force-displacement relation, by setting $\delta = 1$, we have the bar stiffness being equal to **(EA)/L**. Note that stiffness is different than strength. Strength indicates the load that will break the bar, while stiffness indicates the resistance to deformation when loaded by P.
Flexibility (f): this is defined as the inverse of stiffness, i.e., the displacement produced by a unit load. It is equal to **L/(EA).**

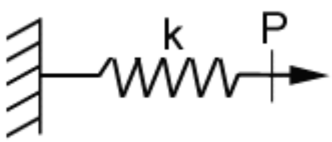


Figure D-8 A spring with a spring constant of k is subjected to a load P.

The usual **sign convention** calls for elongation being positive and contraction (shortening) negative. The change in length in engineering prismatic bars is normally very small compared to their length, i.e., the strains are small as compared to unity. **Cables** are very commonly used in engineering (Figure D-9) and they do act as a prismatic bar, with the restriction that they cannot bear compressive loads. Cables are usually constructed by a large number of wires wound together using a specific pattern. Their cross sectional area is the total area of the wires, called the effective area. Cable manufacturers provide properties of the cables they produce, such as nominal diameter, effective area, weight per linear foot or meter, and ultimate load the cable can carry.

Figure D-9 Cables behave as bars and are quite common in many branches of engineering.

Module 4: Poisson Effects, Safety Factor

Poisson effects

In general, when a bar is subjected to tension it experiences lateral contraction (Figure D-10(a)) and when it is under compression it experiences lateral expansion (Figure D-10(b). The lateral strain is proportional to the axial strain caused by the normal load. The Poisson's ratio ν is a material-dependent dimensionless constant which is measured experimentally, and is defined as

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} = -\frac{\varepsilon_{lat}}{\varepsilon}$$

and the lateral strain is expressed as $\varepsilon_{lat} = \delta_{lat}/d$, i.e., change in lateral dimension, δ_{lat} per original lateral dimension, d. Thus, if we consider a coordinate system as that in Figure D-10, the z-direction being transverse to the "paper," then the lateral direction is the y- or the z-direction. Thus, $\varepsilon_{lat} = \varepsilon_x = \varepsilon_y$ and the subscript indicates the direction the strain is measured. Then, the axial strain is actually ε_x instead of ε so that for a **bar** loaded in the x-direction

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x$$

and

$$\varepsilon_x = \frac{\sigma_x}{E}$$

The **negative sign** has the interpretation that a tensile normal strain induces compressive lateral stress and vice versa.

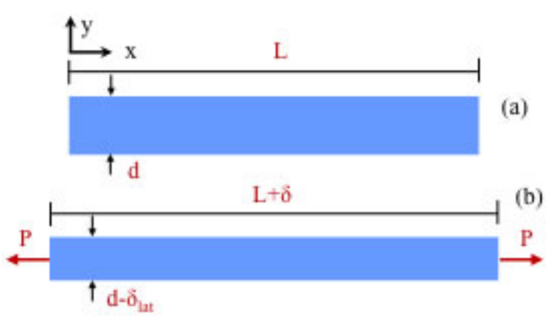


Figure D-10 A bar of length L and lateral dimension d elongates in the axial direction and contracts in the lateral direction when loaded by a tensile load P.

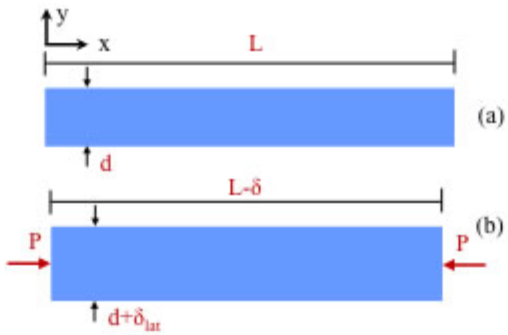


Figure D-11 A bar of length L and lateral dimension d contracts in the axial direction and expands in the lateral direction when loaded by a compressive load P.

Poisson attempted to calculate the ν ratio in the early 19th century using a molecular theory of materials. His and more recent studies agree with experimental observations that show ν to be in the range 0.25-0.4, for most metals as well as many other materials. As mentioned above, ν is dimensionless. In the appendix, typical values of ν are shown. Note that cork has a value of ν almost equal to zero, which is a reason why it is relatively easy to put the cork back into an already open wine bottle. A compressive load on the cork does not increase its diameter since ν is practically zero in this case, thus the cork slides into the bottle easily. Rubber shows a value of ν close to 0.5, which is also a theoretical upper limit for ν (advanced mechanics of materials courses examine this upper limit).

Note that ν is constant for so-called isotropic materials only. An **isotropic** material has identical mechanical properties in all directions. Wood, for example, is not an isotropic material since it has different properties in the direction of the grain than in the other directions. Such materials have more than one Poisson ratio and are examined in more advanced courses on three-dimensional mechanics of materials. Most metal alloys, rubbers, plastics, concrete, and ceramics are (macroscopically) isotropic and thus have a single Poisson ratio. In this course we mostly deal with isotropic engineering materials.

Stress and factor of safety

A structure is any object that must support or transmit loads. To avoid failure, the actual design strength of the structure must be greater that the required strength. One of the objectives of engineering mechanics is to examine failure safety margins for a structure. Materials can sustain stress up to a certain limit before failure occurs. Of course, the load imposed on a structure must always be smaller than the load that would cause failure. The failure load is often termed "ultimate" and the load a structure is subjected to is termed "working" or "design" load. The factor of safety for a structure is defined as the ratio of the load that will cause failure to the load that the structure is actually subjected to:

$$\text{Factor of safety} = n = \frac{\text{Failure load}}{\text{Working load}}$$

For engineering structures n can be as low as 1.1 and as high as 10, depending on the structure and design criteria. The factor of safety being greater than unity can be termed "overengineering" and accounts for imperfections in materials, flaws in assembly, material degradation, and uncertainty in load estimates. Prime considerations in assigning an appropriate safety factor for a structure or component are the accuracy of load and wear estimates, the consequences of failure, and the cost of overengineering the component to achieve that factor of safety. For example, components whose failure could result in substantial financial loss, serious injury or death usually use a safety factor of four or higher (often ten). Noncritical components generally have a safety factor of two or are specified by available codes. An interesting exception is in the field of aerospace engineering, where safety factors are kept low (about 1.15-1.25) because the costs associated with structural weight are so high. This low safety factor is why aerospace parts and materials are subject to more stringent quality control.

Another way to consider factor of safety for a structure is to find the maximum stress under working load and the maximum working stress under ultimate load. In this case

$$\text{Factor of safety} = n = \frac{\text{Failure stress}}{\text{Working stress}}$$

This equation can be easily derived from the previous one since the stress is simply the load divided by the cross sectional area A. Calculation of factor of safety for simple structures or components (e.g., bars) is relatively straightforward. Figures D-12 and D-13 show a bar loaded with load P and the stress-strain behavior of the material

Figure D-12 A bar of cross sectional area A loaded by load P.

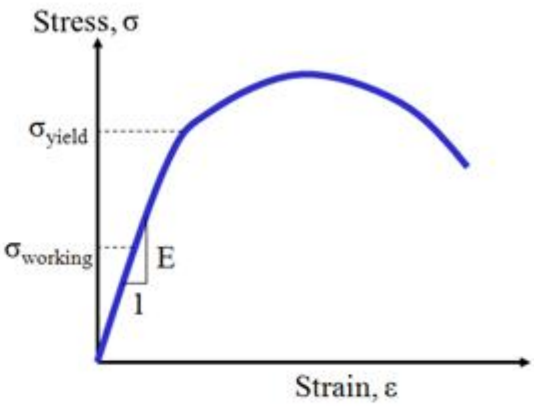


Figure D-13 The stress versus strain of the material of the bar shown in Figure D-12.

The factor of safety is, for such a simple structure (here with respect to the material yield stress), considered to be the "failure stress"

$$n = \frac{\text{Yield stress}}{\text{Working stress}} = \frac{\sigma_{yield}}{\sigma} = \frac{P_{yield}/A}{P/A} = \frac{P_{yield}}{P}$$

The **allowable** stress for such a structure is defined as

$$\sigma_{all} \frac{\sigma_{yield}}{n}$$

It designates the maximum stress the structure is allowed to bear, and is typically used in design as the stress that cannot be exceeded.

Module 5: Nonuniform Bars

Previous modules on bar behavior considered that the load P, and thus the stress σ and strain ε were uniform in a bar. This is not always the case, thus, let us consider **nonuniform** bars, as the examples shown in Figure D-14. Nonuniformity can be either due to spatially varying cross sectional area, or spatially varying load (and thus stress), or both. In this case, then, in order to find the total elongation/contraction δ of a bar, we need to add the elongations/contractions in each part of the bar. Non-uniformity can also result by spatially varying Young's modulus. For example, consider the case where (Figure D-14(b)), AB is made of aluminum and BC is made of steel. In general, for a bar consisting of n segments, where in each segment *i* all P_i, E_i and A_i are constant, the total displacement is the summation of the displacements in each segment.

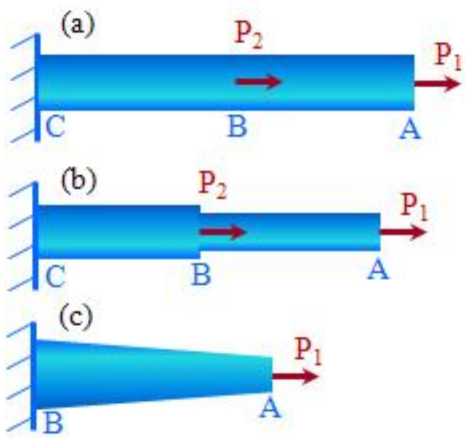


Figure D-14 Nonuniform bar deformation problems: (a) the load is varying spatially, i.e., in AB it is equal to P_1 while in BC it is equal to $P_1 + P_2$; (b) both the load and the cross sectional area are varying spatially; and (c) the cross sectional area varies continuously.

Example: Let us calculate the displacement at A and B for the steel bar (E = 210 GPa) shown in Figure D-14(b), where P_1 = 2 kN, P_2 = 4 kN; the cross sectional areas in AB and BC are $A_{AB} = 0.01\ m^2$, $A_{BC} = 0.02\ m^2$, and the bar lengths are $L_{AB} = L_{BC} = 0.3$m. Note that the equation $\delta = PL/EA$ provides the **elongation/contraction δ and not displacement of a point** in the bar. However, as is obvious from Figure D-14(b), since end C is fixed, the elongation of CB is equal to the displacement of point B, or $\delta_{CB} = \delta_B$. Similarly, the elongation of CB plus the elongation of BA is equal to the displacement of point A, or $\delta_{CB} + \delta_{BA} = \delta_A$. Using the force-elongation relation, $\delta = PL/EA$, these two facts are expressed as:

$$\delta_B = \frac{(P_2+P_1)L_{BC}}{EA_{BC}} = \frac{(4{,}000+2{,}000)0.3}{210\times10^9\times0.02} = 4.29\times10^{-7}\text{m}$$

$$\delta_A = \delta_B + \frac{P_1 L_{AB}}{EA_{AB}} = 4.29\times10^{-7} + \frac{2{,}000\times0.3}{210\times10^9\times0.01} = 7.15\times10^{-7}\text{m}$$

Note that the force in bar AB is equal to P_1, and the force in bar BC is equal to $P_1 + P_2$. This can be verified easily by "cutting" the bar within AB and within BC and considering equilibrium of forces.

Example: For the bar shown in Figure D-15, find the displacement at B and at C. Assume E of the material for the entire bar is equal to 30,000 psi.

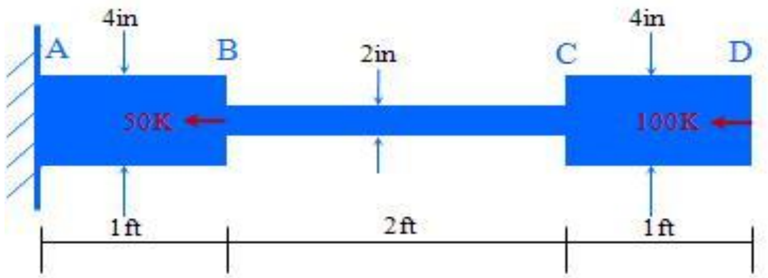


Figure D-15 Nonuniform bar deformation problem, where the bar is loaded at end D and at section B. The bar is of circular cross section.

It is repeated that the formula $\delta = PL/EA$ does not provide the actual displacement of a bar but rather the bar elongation (for tensile load P) or contraction (for compressive load P). If the bar is fixed on its left end, then δ is also the displacement of its right end.

Let us calculate the displacement at B and C, first in "words" ***(below "elongation" is used even though the actual deformation may be "contraction")***.

displacement of B = elongation of AB

displacement of C = elongation of AB + elongation of BC

Although not a question in this particular problem, for completeness note that

displacement of D = elongation of AB + elongation of BC + elongation of CD

In mathematical terms:

$$\delta_B = \frac{P_{AB}L_{AB}}{E_{AB}A_{AB}} = \frac{(-100-50)1\times 12}{30{,}000\pi\times 4^2/4} = -0.015\text{ in}$$

$$\delta_C = \frac{P_{AB}L_{AB}}{E_{AB}A_{AB}}\frac{P_{BC}L_{BC}}{E_{BC}A_{BC}} = -0.015 + \frac{(-100)2\times 12}{30{,}000\pi\times 2^2/4} = -0.095\text{ in}$$

Module 5: Nonuniform Bars

The approach for addressing nonuniform bars illustrated in the previous examples can be applied for several bar segments each having different axial forces, different dimensions such as length and cross section, and made of different materials. Thus for a general, piece-wise nonuniform bar, the change in length over a number of segments is expressed as

$$\delta = \sum_i \frac{P_i L_i}{E_i A_i}$$

which is the expression for the **elongation/contraction of a piece-wise nonuniform bar**. Here the total elongation is the summation of elongations over i pieces of the bar. Over each piece i, force P_i, cross sectional area A_i, and elasticity modulus P_i are constant. This formula can be applied for any number of segments for which the total elongation or contraction is desired.

For bars of **continuously varying properties** (as compared to distinctly or piece-wise, e.g., Figure D-16), the problem can be considered as a summation of an infinite number of segments, thus the above equation becomes an integral where now L_i is replaced by dx. Further, the internal axial load P is expressed as a function of x along its length P(x) and by knowing the dimensions of the bar we also express the cross sectional area A(x) as a function of x. Usually the modulus of elasticity does not vary continuously with x, thus E is not generally expressed as E(x). The elongation of the bar along its entire length is obtained by integrating over the length. Thus, for a problem like that shown in Figure D-16

$$\delta = \int_0^L \frac{P(x)}{EA(x)} dx$$

which is the expression for the **elongation/contraction of a continuously nonuniform bar**.

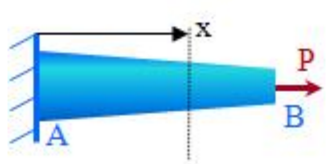


Figure D-16 A bar with continuously varying area. In general, force, Young's (elasticity) modulus, and cross sectional area can vary as well.

Example: Let the cross section of the bar in the figure above be circular with a diameter of 0.02 m at its left end (at A) and 0.01 m at its right end (at B) and the diameter varies linearly. The length of the bar is 1.5 m, E = 10^6 Pa (e.g., a rubber-type material), and P = 6,000 N. Then, since for a circle the area is equal to $\pi D^2/4$, for the coordinate system shown, we have:

$$D(x) = 0.02 - \frac{0.01}{1.5} x = 0.02 - 0.0067x = \sigma dA$$

$$A(x) = \frac{\pi}{4}(0.02 - 0.0067x)^2$$

$$\delta_B = \delta_{AB} = \int_0^{1.5} \frac{6{,}000\,dx}{10^6 \frac{\pi}{4}(0.02 - 0.0067x)^2}$$

This integral can be evaluated analytically using integral tables, or even numerically.

Summary of mechanical behavior of bars

In summary, when a bar is subjected to a load P there is normal stress created. When the x-axis is along the load direction (Figure D-17) the normal stress is designated as σ_x. In the same manner, the normal stress is designated as ε_x. Based on the definitions of normal stress and strain, and Hooke's law, the following table summarizes all equations applicable in the mechanical behavior of a bar.

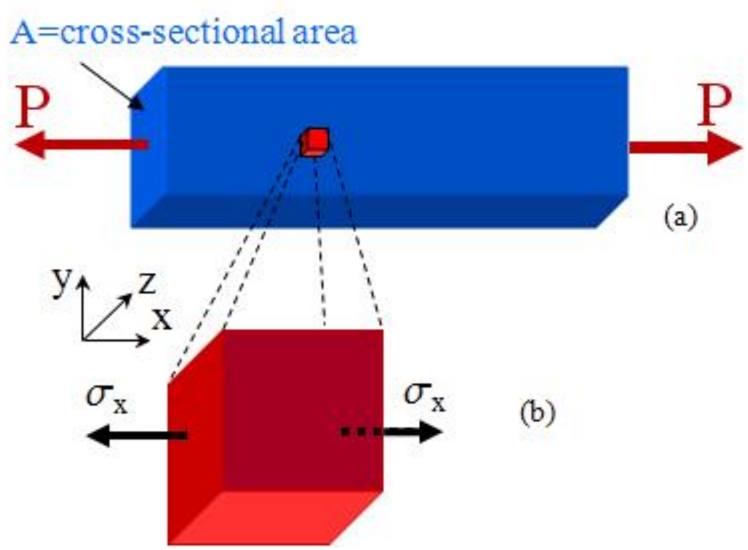


Figure D-17 (a) A bar under tensile load P; (b) a material element within the bar; and (c) side view of the material element before (solid line) and after (dot line) load application.

$\sigma_x = \frac{P}{A}$	normal stress	definition
$\varepsilon_x = \frac{P}{A}$	normal strain	definition
$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$	Poisson ratio	definition
$\sigma_x = E\varepsilon_x$	Hooke's Law	law
$\delta = \frac{PL}{EA}$	derived equation	derived

$\delta = \sum_{i=1}^{n} \frac{P_i L_i}{E_i A_i}$	consequence for piecewise nonuniform bars	consequence
$\delta = \int_0^L \frac{P(x)}{EA(x)} dx$	consequence for continuously nonuniform bars	consequence

Summary of the equations for the mechanical behavior of a bar.

Statically indeterminate bar problems

In the preceding modules all the structures examined were in equilibrium implying that their reactions and internal forces can be determined solely from free-body diagrams and the equations of equilibrium. These structures are referred to as statically determinate. The same definition holds for a bar, which is a relatively simple structure or structural component. For example, the bar shown in Figure D-18(a) is statically determinate since equilibrium of forces in the horizontal direction yields the reaction force, i.e., R = P. Similarly, **the force at any cross section in the bar** can be determined, being equal to P in this case. Yet, the bar in Figure D-18(b) is statically indeterminate, since equilibrium of forces in the horizontal direction implies $R_1 + R_2 = P$, which is only one equation for the two unknown reactions, R_1 and R_2. Therefore additional equations, pertaining to the displacements of the structure, need to be used for solving statically indeterminate structures.

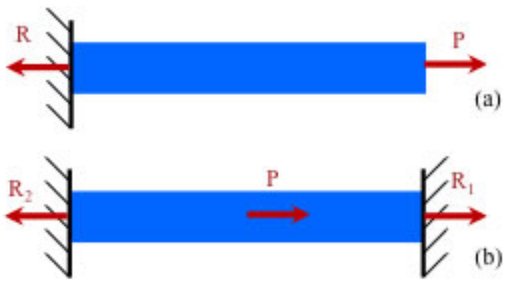


Figure D-18 (a) A statically determinate bar; and (b) a statically indeterminate bar

It is repeated herein that a structure is statically determinate when all forces can be calculated by simply using the equilibrium equations. As an example, for the bar in Figure D-18(a) the forces at every cross section can be calculated from the equilibrium equations; e.g., to find the force in any cross section we can consider the free body diagram and equilibrium equation, Figure D-19. The cross section (hypothetical cut) and the equilibrium equation are used to find the force in the cross section of the bar which is the resultant of the internal stresses σ. The cross section is between the left end support and the right end where P is applied. Obviously, the internal force at the cross section is P and is equal to σA, where A denotes the cross sectional area. However, for a statically indeterminate structure, the forces cannot be evaluated this way.

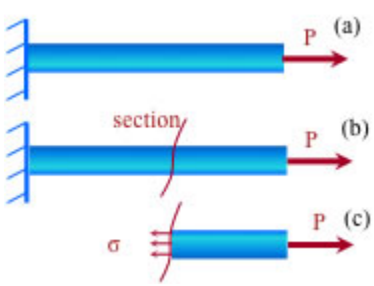


Figure D-19 (a) A bar loaded at its right end; (b) a section of the same bar; and (c) free-body diagram of the piece to the right of the section

Statically indeterminate structures are such that the equilibrium equations are not adequate to solve for the unknown forces, therefore additional equations pertaining to the displacements of the structure need to be used. In general, since additional restraints are imposed on a statically indeterminate structure, these **additional restraints provide** the additional equations needed for finding the unknown forces. These additional equations are called **compatibility equations**. Let us illustrate the process of solving statically indeterminate structures through examples. Consider the bar shown schematically in Figure D-20(a), where the problem is statically indeterminate since there are two unknown reactions, R_A and R_B at A and B, respectively, and only one equilibrium equation, $P = R_A + R_B$. Since, in this case there is only one excess unknown (i.e., if the reaction at B was known the problem would be statically determinate) we say that the degree of indeterminacy is one.

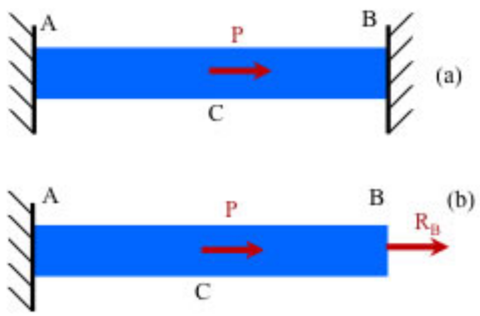


Figure D-20 (a) A statically indeterminate bar; and (b) release of the excess unknown and the relevant free body diagram

This type of problem can be solved by the following general procedure The steps followed to solve the problem, that is, determine forces, stress, displacements and strains, are:

Step 1: Assume that the structure is determinate by releasing the excess unknown force or forces (Figure D-20(b)).

Step 2: Use the compatibility equation or equations, for the structure of the previous step (i.e., express the fact that the displacement at the point of release is known). For the case shown in Figure D-20 we simply know that the displacement at A and B is zero, thus the total elongation of the bar from A to B is zero. We have

$$\delta_{AB} = 0 \rightarrow \delta_{AC} + \delta_{BC} = 0 \quad \text{or} \quad \frac{(R_B + P)L_{AC}}{E_{AC}A_{AC}} + \frac{R_B L_{CB}}{E_{CB}A_{CB}} = 0$$

In this equation the only unknown is R_B and by solving it the reaction R_B results. Then the structure can be treated as a statically determinate one for calculation of any desired quantities. For example, the reaction R_A can be evaluated from the equilibrium equation, $R_A + R_B = P$.

Example: Consider that the bar in Figure D-20(a) is made of aluminum (E = 70 Ga), the cross section is circular with radius 3 cm (0.03 m) and AC = 0.5 m, BC = 0.7 m, and P = 2 kN. Then, equilibrium implies that

$$\frac{(R_B + 2000)0.5}{70\times 10^9 \pi 0.03^2} + \frac{R_B 0.7}{70\times 10^9 \pi 0.03^2} = 0$$

which, together with the equilibrium equation yields, R_B = 833.33 N and R_A = 1166.67 N.

Comments on statically indeterminate structures

The analysis of statically indeterminate structures involves the usual equilibrium equations, and additionally, equations of compatibility. The equilibrium equations are simply those of static equilibrium (three (3) equations for two-dimensional structures and six (6) equations for three-dimensional structures). The compatibility equations have to do with displacements in the structure, in the sense that some of the displacements have to be compatible. For that reason, compatibility equations are also known as **kinematic** equations or **equations of consistent deformations**. The deformations required for expressing the compatibility equations are obtained from the force-displacement relations, such as δ = PL/EA for bars. The number of such equations is equal to the degree of indeterminacy of the structure.

Example: The problem shown in Figure D-21 has a degree of indeterminacy equal to one. This can be easily verified by considering the free body diagram of the (herein assumed rigid) part AB, which shows (see Figure D-21) the four unknown reactions: A_x, A_y, F_C and F_D . Thus, one compatibility equation is needed in addition to the three equilibrium equations.

One **effective way to determine the compatibility relation** is to consider the deformed shape of the structure, and try to identify the relation between unknown displacements. For this specific example, the comparability equation is a relation between the vertical displacements at points C and D, since beam AB is considered rigid.

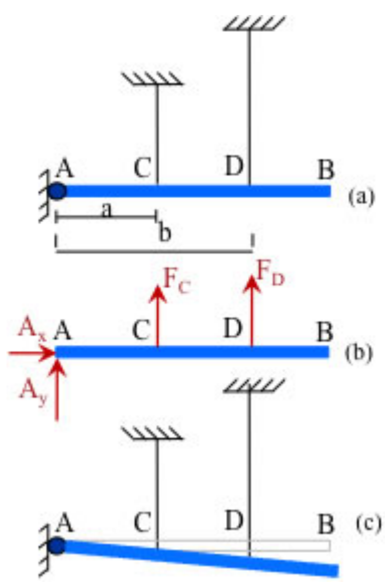


Figure D-21 (a) A statically indeterminate problem, where identification of the compatibility relation is crucial. Here, the vertical displacements at C and D are related; (b) the free body diagram of the rigid beam; and (c), the deformed shape, where two similar triangles can be identified yielding the compatibility condition.

From the similar triangles in Figure D-21(c), it follows that

$$\frac{\delta_C}{a} = \frac{\delta_D}{b} \text{ or } \frac{F_C h}{aEA} = \frac{F_D h}{bEA} \text{ or } F_C = \frac{2aF_D}{b} .$$

From this, and the equilibrium equation of moments with respect to point A we obtain two equations for two unknowns, namely F_C and F_D.

Example: The problem shown in Figure D-22 has a degree of indeterminacy equal to one. The only load the bar is subjected to is its own weight. Find the displacement at the middle of the bar.

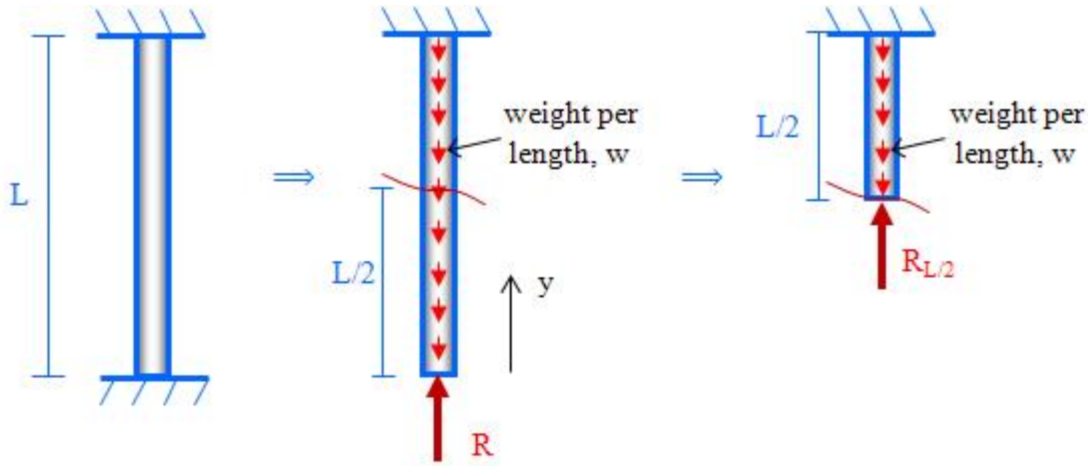


Figure D-22 A fixed end bar subjected to its own weight only. The bar length is L, the radius of its circular cross section is r, the cross sectional area is $A = \pi r^2$, and the weight density of the material is γ. The weight imposes a distributed load on the beam, w, expressed as force per unit area. A typical cut is shown at y = L/2, and the internal force at that section. A section can be imagined at any position y between zero and L.

Solution: Let w be the weight of the bar per unit length; here, $w = \gamma A$. The displacement at the bottom end is zero, which yields the following compatibility equation. It is noted that the distributed load w makes the load at a cross section a function of y, thus integration is required in order to find the displacement due to w.

$$\delta_{\text{due to R}} = \frac{RL}{EA} = \delta_{\text{due to w}} = \int_0^L \frac{(wy)}{EA} dy = \frac{wL^2}{2E}$$

which can be solved for R, i.e. $R = \frac{Lw}{2}$. Then, from the equation for the equilibrium of the bottom half of the bar, $R_{L/2} = R - \frac{Lw}{4} = 0$. The same process can be applied to find the displacement at the middle of the bar. From Figure D-21, we obtain

$$\delta_{\text{due to } R_{L/2}} = \frac{R_{L/2} L/2}{EA}; \quad \delta_{\text{due to w}} = \int_0^{L/2} \frac{(wy)}{EA} dy = \frac{wL^2}{8EA}$$

and

$$\delta_{\text{middle}} = \delta_{\text{due to w}} - \delta_{\text{due to L/2}} = \frac{wL^2}{8EA}$$

since $R_{L/2} = 0$.

Module 7: Change of Temperature Effects

Change of temperature in bars

Changes in temperature produce expansion or contraction of structural materials resulting in thermal strains and thermal stresses. Consider a cylindrical bar that rests freely on a surface, i.e., it is not restricted to deformation (see Figure D-23). If the temperature increases by ΔT then the length increases by $\delta_{\Delta T}$ and the diameter increases by δ_D, thus we have

$\delta_{\Delta T} = \alpha(\Delta T)L$ and $\delta_D = \alpha(\Delta T)D$

where α denotes the so-called **coefficient of thermal expansion**, a characteristic of the material, L denotes the length, and D the diameter of the bar. Values of α for some engineering materials are given in the Appendices. Of course, for temperature decrease, the length and diameter of the bar both decrease accordingly. Then, dividing both sides of the above equation by L, it follows that the change in temperature creates a normal strain equal to

$$\varepsilon_{\Delta T} = \alpha(\Delta T)$$

which is the **normal strain due to ΔT**, i.e., change in length over original length. The usual **sign convention** is that expansion is considered positive and contraction is considered negative.

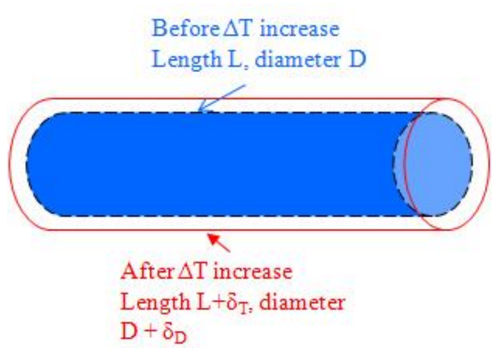


Figure D-23 A bar before and after temperature increase

Note that a lateral strain is also the result of the temperature change ΔT. Lateral strains are discussed in the next section. For this unrestricted bar, although there is temperature induced strain, there is no stress ($\sigma = 0$). This is because the unrestricted bar is statically determinate. As a general rule it is true that thermal expansion or thermal contraction of a statically determinate structure induces strains and displacements, but not stresses. This is not the case, however, for statically indeterminate structures, an example being the bar below. An increase in temperature will create compressive stress, and a decrease in temperature will create tensile stress. Note that due to the restrictions imposed on the bar by its external supports at its ends, the strain is null in this case.

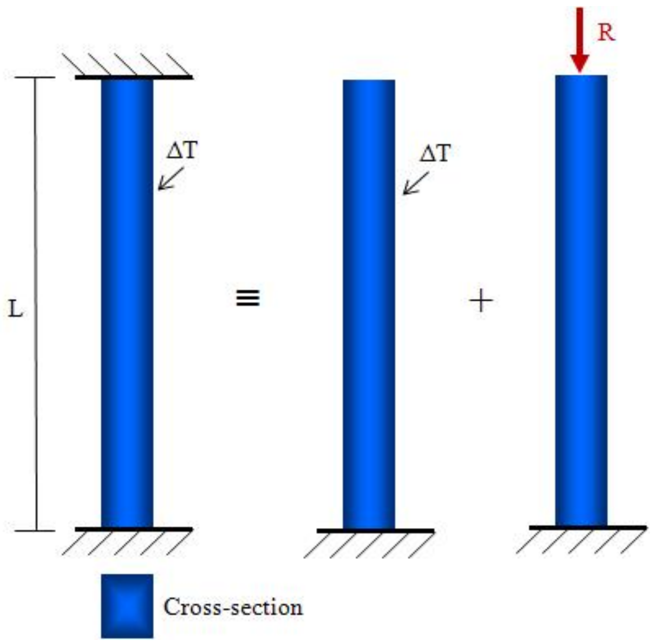


Figure D-24 A fixed-end bar subjected to temperature increase ΔT. The bar is statically indeterminate, thus, the temperature change creates stresses. The statically indeterminate bar can be considered as the superposition of the two bars and load shown.

To solve the statically indeterminate structure, we follow the process of the previous section, thus it is necessary to identify the compatibility equation and express it in mathematical terms. Compatibility of displacements here implies that the total length of the bar remains unchanged, i.e., the displacement caused by ΔT and that caused by the load P should add up to exactly zero, as shown from the equivalence in Figure D-24,

$$\delta_{\Delta T} - \delta_R = 0 \;\Rightarrow\; \alpha(\Delta T)L - \frac{RL}{EA} = 0$$

which can be solved for the force R, yielding

$$R = EA\alpha(\Delta T)$$

Thermal expansion or contraction can create quite sizable stresses in structures. In a certain statically indeterminate steel bar for example, a temperature change of about 100^{o}F can create stresses well within the range of **allowable stresses** for steel. However, there are many cases where structures and materials are near or at their allowable stresses from structural loads. In that case, if a thermal stress develops, the total stress may well exceed the allowable stress and cause the structure to fail. This, of course, is the reason bridges are built with **expansion joints** which allow the structure to expand and contract freely and thus avoid thermal stresses. Additionally, this is why concrete sidewalks are built with spaces separating adjacent slabs, allowing expansions to avoid thermal stresses. Concrete highways used to have expansion built-in spaces, however, modern concrete highways are designed without them. Normally, they withstand these stresses, but occasionally long hot or cold periods will allow stresses to build up until the highway actually explodes in an area, producing a large hole in the concrete.

Statically determinate structures do not develop thermal stresses. It is easy to understand why a bar fixed on only one end does not develop thermal stresses while when both ends are fixed it does develop thermal stresses under temperature change ΔT. In the former case, the structure is statically determinate and in the latter case it is statically indeterminate. This holds true for all structures, and is in many occasions a reason the engineers designs certain structures as statically determinate or with a few degrees of indeterminacy, in order to make the structure more flexible and less prone to failure due to temperature changes. **To understand why statically determinate structures do not develop thermal stresses**, consider the simple truss in Figure D-25(a). Even if each member of the truss is exposed to different temperature changes, the members are free to change their length without creating stresses. However, when bar BC is added (Figure D-25(b)) this is not the case anymore, and the truss members will develop stresses under temperature change. For example, consider that bar BC is heated and the other ones are not. If BC was free, it would expand by a distance δ. However, the other two bars resist this expansion, and the final "compromise" will be that B will move only a fraction of δ. Similarly, if the entire truss is heated by the same temperature change, the distance B would move in the absence of BC is different than the expansion BC would experience if it was free. This difference creates a stress in all truss members.

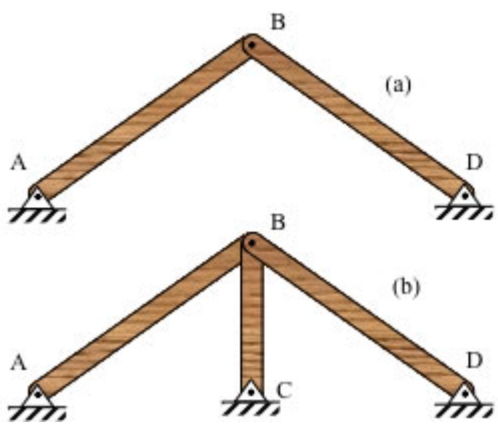


Figure D-25 The statically determinate structure at (a) allows its two bar members to expand/contract under temperature change without developing thermal stresses. This is not true for the statically indeterminate structure at (b).

Example - temperature increase and load in statically determinate nonuniform bar: The stepped bar ABC in Figure D-26, consisting of solid circular segments, is subjected to temperature increase $\Delta T = 130^{o}C$ while it is fixed at A and load F acts at C. Find the magnitude of F, such that the displacement of point B is zero. The material is steel with $E = 100x10^{6}$ kPa, and $\alpha = 12x10^{-6}/^{o}C$. In the figure, d denotes diameter.

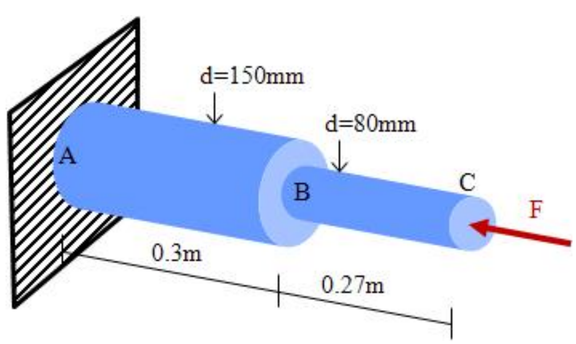


Figure D-26 A statically determinate bar is subjected to temperature increase as well as external compressive load F

In order for the displacement of B to be zero, the following must hold: expansion of AB due to ΔT = contraction of AB due to F. Note that bar AB is subjected to compressive load F.

Expansion of AB due to ΔT: $\alpha(\Delta T)L$

Contraction of AB due to F: $-\frac{FL_{AB}}{EA_{AB}}$

Then, $\alpha(\Delta T)L_{AB} - \frac{FL_{AB}}{EA_{AB}} = 0$, which, when solved for F, yields

$$F = \alpha(\Delta T)A_{AB} = 12 \times 10^{-6} \times 130 \times 100 \times 10^{9} \frac{3.14\times 0.15^{2}}{4} = 2.75535 \times 10^{6}\,\mathrm{N} = 2{,}755.35\,\mathrm{kN}$$

Example: Rod Design

$$\delta = \frac{PL}{EA} = \frac{10{,}000\ \text{lb} \times 15\ \text{in}}{29 \times 10^6\ \text{psi} \times 0.25\ \text{in}^2} = 0.021\ \text{in}$$

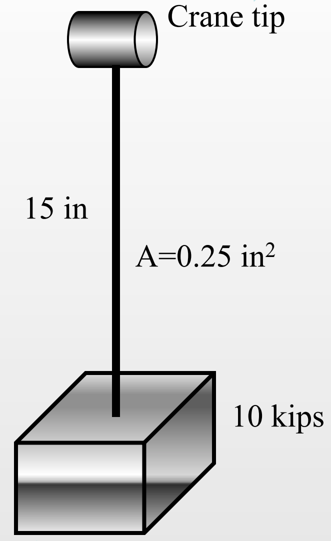
$$\sigma = \frac{P}{A} = \frac{10{,}000\ \text{lb}}{0.25\ \text{in}^2} = 40{,}000\ \text{psi} < 58{,}000\ \text{psi}$$

$$\sigma = \frac{P}{A} = 40{,}000\ \text{psi} > 36{,}000\ \text{psi}$$

Note on safety factor:

$$SF = \frac{\text{critical stress}}{\text{working stress}} = \frac{36{,}000\ \text{psi}}{40{,}000\ \text{psi}} = 0.9$$

Since SF<1, the design is unacceptable



A crane is designed to sustain a maximum load of 10000 lb. A 15-inch-long steel rod ($E = 29 \times 10^6$ psi) of 0.25 in^2 cross-sectional area is used at the tip of the crane.

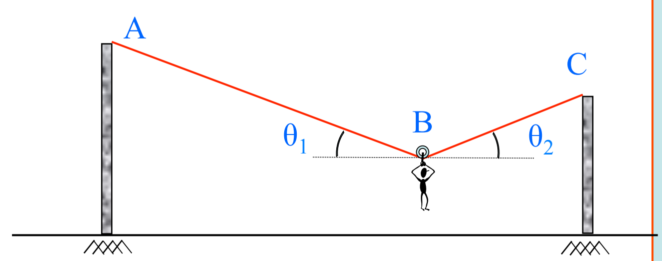
a) Under the design load, calculate how much the rod will elongate

b) Will the rod support the full design weight? σ_u=58 ksi (ultimate stress of the rod, in tension)

c) Under full load, will the rod plastically deform? σ_y=36 ksi (yield stress of the rod, in tension)

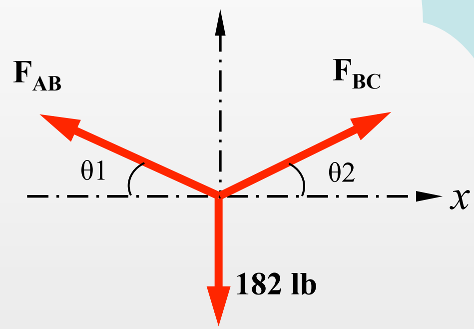
Example: Stress in a Cable

An adventurer weighing 182 lb slides under a cable using a smooth pulley. The cable at a specific instant is such that θ_1=30º and θ_2=36º. The diameter of the cable is 60 mils (wire diameters are often specified and measured in mils: 1 mil = 1/1000 in). Determine the maximum tensile stress in the cable.



Considering equilibrium of forces at B:

$$F_{AB}\cos[\theta_1] = F_{BC}\cos[\theta_2]$$
$$F_{AB}\sin[\theta_1] + F_{BC}\sin[\theta_2] = 182 \text{ lb}$$



These two equations yield, for the given Values of θ_1 and θ_2:

$$F_{AB} = 161.2 \text{ lb}, F_{BC} = 172.5 \text{ lb}$$

$$\sigma_{max} = \frac{P_{max}}{A} = \frac{172.5}{\pi\left(60/1000\right)^2/4} = 61{,}009 \text{ psi}$$

Example: Stress in a Bar

From the known strain in BC, it follows that

$$\sigma_{BC} = E\varepsilon_{BC} = 1700 \times 0.004 = 6.8 \text{ ksi} < 8 \text{ ksi}$$

and thus

$$F_{BC} = \sigma_{BC} A = 6.8 \times 20 = 136.0 \text{ kips}$$

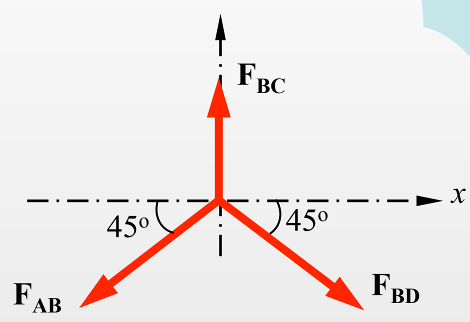
Equilibrium of joint B implies that

$$F_{AB} = F_{BD},$$
$$F_{BC} = 2F_{AB}\sin(45^{\circ}) \Rightarrow$$
$$F_{AB} = F_{BC} / (2\sin(45^{\circ})) = 136.0 / (2 \times 0.707) =$$
$$96.18 \text{ kips}$$

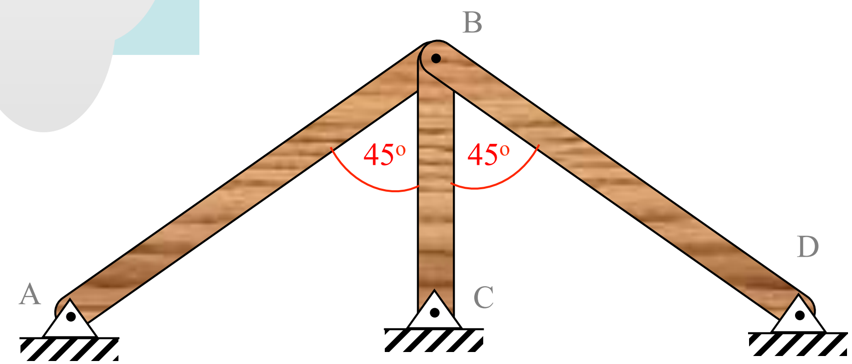
Thus,

$$\sigma_{AB} = \sigma_{BD} = F_{AB} / A = 96.18 / 20 =$$
$$4.8 \text{ ksi} < 8 \text{ ksi}$$

Note that the forces in AB and BD are compressive. If the yield stress was exceeded in any of the bars, the complete stress vs. strain curve would be needed to solve this problem.



While constructing the wooden truss shown below, it was found that bar BC was shorter than specified. The truss was forced together, and this created stress and strain in each bar. The strain was measured in bar BC to be 0.004 (0.4%). Determine the force created by forcing the truss together in each member. For the type of wood, yield stress, $\sigma_Y = 8$ksi, E = 1700ksi. Each bar's cross-sectional area is $A = 20\text{in}^2$. Is the yield stress in any bar exceeded? If yes, how would you solve this problem of finding the stress in each bar?



Example: Forces in a Truss

From statics:

$F_{AB} = 0.75P$ (C)

$F_{AC} = 0.75P$ (T)

Let :

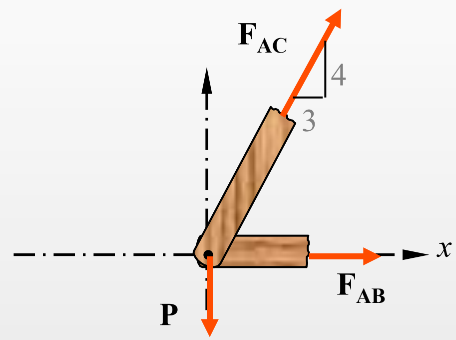
$$\sigma_{AB} = \frac{-0.75P}{9\times 10^{-4}} = -833.3P\ \mathrm{N/mm^2}\ (\mathrm{Pa})$$

$$\sigma_{AC} = \frac{1.25P}{12\times 10^{-4}} = +1041.6P\ \mathrm{N/mm^2}\ (\mathrm{Pa})$$

$833.3P = 25 \times 10^6$ or $P = 30$ KN

$1041.6P = 50 \times 10^6$ or $P = 48$ KN

Thus, maximum allowable P = 30 KN.



For the truss find the maximum possible P such that the allowable stresses are not exceeded:

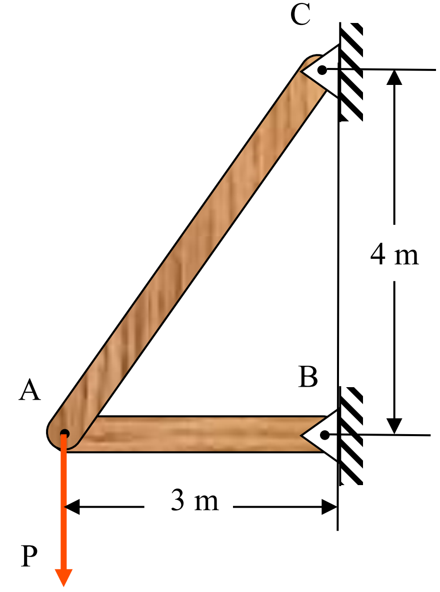
$(\sigma_{all})_{tension} = 50$ MPa

$(\sigma_{all})_{compression} = 25$ MPa

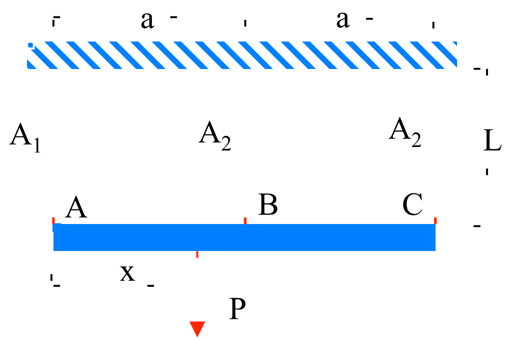
Given:

$(A)_{AB} = 12\ \mathrm{cm^2}$

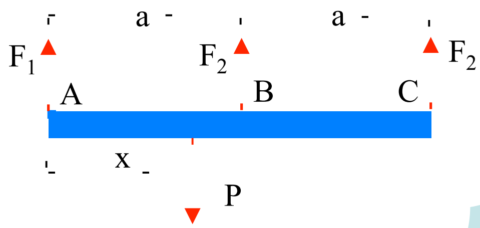
$(A)_{AC} = 9\ \mathrm{cm^2}$



Example: Forces in Bars/Cables



The horizontal rigid beam ABC is subjected to vertical load P and it is supported by three equally spaced steel cables of length L. Due to an ordering error, the cable at A has cross-sectional area A_1 that is 15% larger than the area of the other two cables, A_2, i.e. $A_1 = 1.15A_2$. Find the distance x from A the load P should be applied such that the rigid beam ABC stays horizontal. Consider the Young's modulus of each cable to be E.



Since two cables are identical and their elongation is the same, they are subjected to the same load, F_2.

Equilibrium:

$$\sum F_y = 0 \Rightarrow F_1 + 2F_2 = P$$
$$\sum M_A = 0 \Rightarrow P \cdot x = F_2 \cdot a + F_2 \cdot 2a$$

Compatibility:

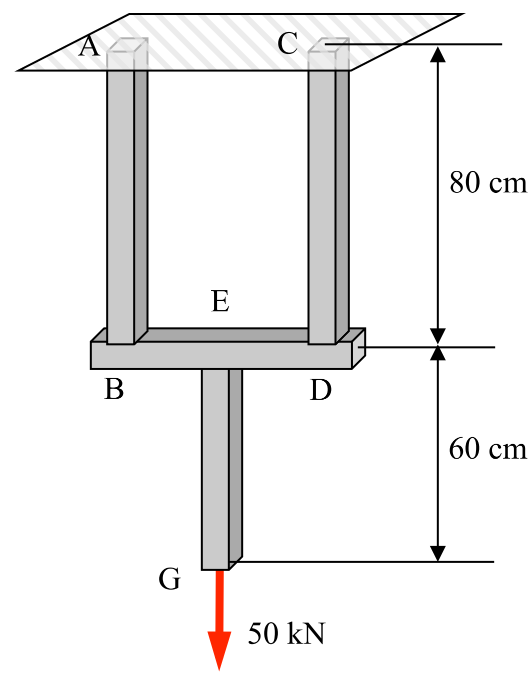
$$\delta_1 = \delta_2 \Rightarrow \frac{F_1 \cdot \cancel{L}}{\cancel{E} \cdot A_1} = \frac{F_2 \cdot \cancel{L}}{\cancel{E} \cdot A_2} \Rightarrow \frac{F_1}{F_2} = 1.15$$

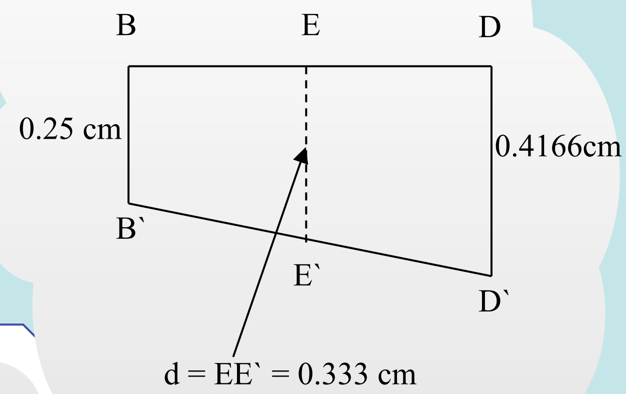
The above three equations when solved for F_1, F_2 and x yield:

$x = 0.95a$, $F_1 = 0.365P$, $F_2 = 0.317P$

Example: In Series and In Parallel Bars

Find displacement of point G.
Assume beam BED is rigid, and BE = DE.
Given :
$A_{AB} = 400\ mm^2$, $E_{AB} = 200$ GPa
$A_{CD} = 600\ mm^2$, $E_{CD} = 80$ GPa
$A_{EG} = 900\ mm^2$, $E_{EG} = 200$ GPa





$$\delta_{EG} = \frac{PL}{EA} = \frac{50000 \times 0.6}{200 \times 10^{9} \times 900 \times 10^{-6}} = 0.166 \text{ mm}$$

$$\delta_{AB} = \frac{PL}{EA} = \frac{25000 \times 0.8}{200 \times 10^{9} \times 400 \times 10^{-6}} = 0.25 \text{ mm}$$

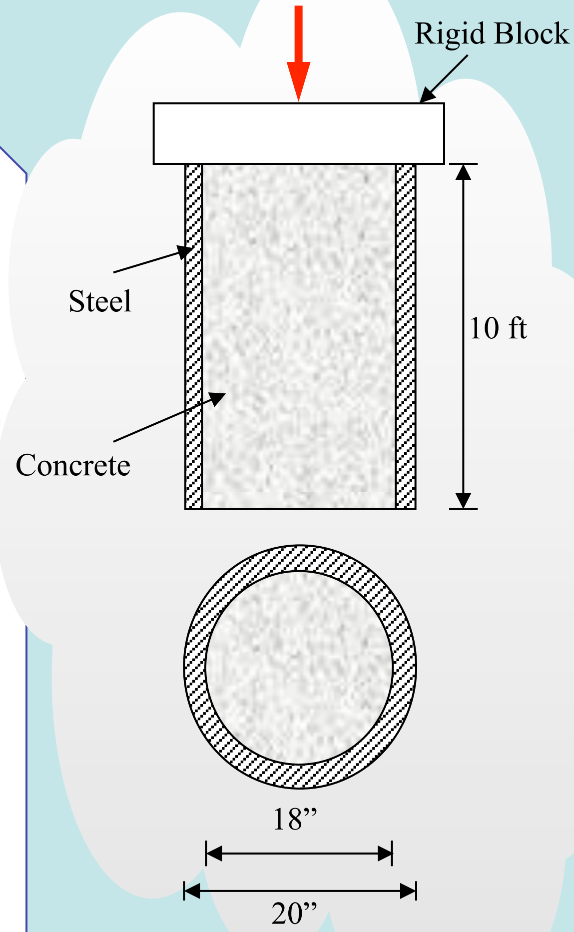
$$\delta_{CD} = \frac{PL}{EA} = \frac{25000 \times 0.8}{80 \times 10^{9} \times 600 \times 10^{-6}} = 0.4166 \text{ mm}$$

$$\delta_{G} = \delta_{EG} + EE' = 0.166 + \frac{1}{2}(0.25 + 0.4166) = 0.45 \text{ mm}$$

Example: Steel-Concrete Column

A steel encased concrete column is subjected to 90 kips concentric load.
Find the percentage of load distribution between concrete and steel.
Given:

$E_{steel} = 30000$ ksi
$E_{conc} = 10000$ ksi



$$A_{conc} = \pi r^2 = \pi (9)^2 = 254.47 \text{ in}^2$$

$$A_{steel} = \pi\left[(10)^2 - (9)^2\right] = 56.69 \text{ in}^2$$

From statics,

$$\sum F_y = 0$$

$$\therefore R_{steel} + R_{conc} = 90 \text{ k} \qquad \ldots(1)$$

From compatibility,

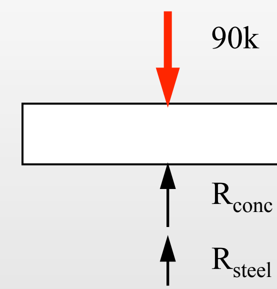
$$\delta_{steel} = \delta_{conc} \qquad \therefore \frac{R_{steel} L}{A_{steel} E_{steel}} = \frac{R_{conc} L}{A_{conc} E_{conc}}$$

$$\frac{R_{steel} L}{56.69 \times 30000} = \frac{R_{conc} L}{254.47 \times 10000} \qquad \ldots(2)$$

Solve equations (1) & (2),

$$R_{steel} = 36.05 \text{ k} = 40.06\%$$

$$R_{conc} = 53.95 \text{ k} = 59.94\%$$



Example: Temperature Effect on a Bar

$$\delta = \alpha(\Delta T)L \quad \text{or} \quad 0.02 = 6.5\times10^{-6}\times(\Delta T)_1\times 24$$

$$\Rightarrow \quad (\Delta T)_1 = \frac{0.02}{6.5\times10^{-6}\times 24} = 128°F$$

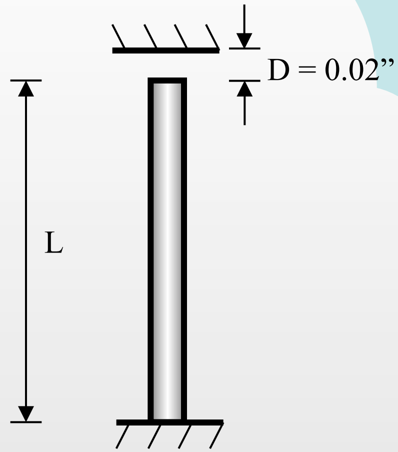
Then, $\alpha(\Delta T)_2 L = \frac{RL}{EA} = \sigma_{all}\frac{L}{E}$

$$\Rightarrow \quad \alpha(\Delta T)_2 = \frac{\sigma_{all}}{E}$$

$$\Rightarrow \quad 6.5\times10^{-6}(\Delta T)_2 = \frac{11000}{30\times10^6}$$

$$\Rightarrow \quad (\Delta T)_2 = 56.4°F$$

$$(\Delta T) = (\Delta T)_1 + (\Delta T)_2 = 128°F + 56.4°F = 184.4°F$$



If the allowable compressive stress in the bar is σ_{all} = 11,000 psi, find the maximum allowable increase in temperature ΔT.

L = 2.0 ft

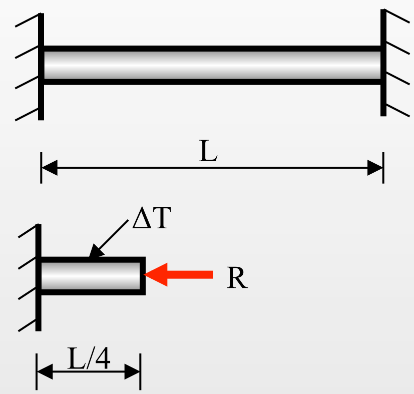
d = 1.5 in (circular cross-section)

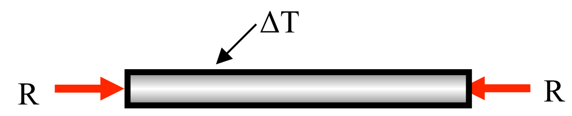
E = 30×10^6 psi

$\alpha = 6.5\times10^{-6}$ /°F

Example: Displacement in a Bar Due to Temperature

The bar of length L, cross-sectional area A, Young's modulus E and coefficient of thermal expansion α is subjected to temperature increase ΔT. Determine the displacement at quarter point.





$$\frac{R \cdot L}{EA} = \alpha(\Delta T)L$$

$$R = EA\alpha(\Delta T)$$

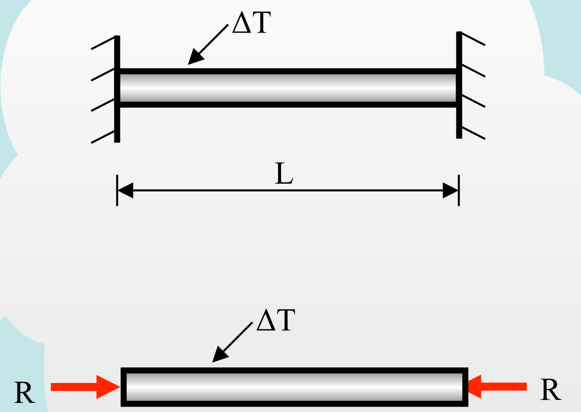
At quarter point,

$$\delta_{1/4} = \alpha(\Delta T)\left(\frac{L}{4}\right) - \frac{R(L/4)}{EA}$$

$$\therefore \delta_{1/4} = \frac{\alpha(\Delta T)L}{4} - \frac{EA\alpha(\Delta T)L}{4EA}$$

Example: Temperature Displacement in a Bar

A brass wire of diameter d = 1/16 in. is tightly stretched between two fixed points so that it under a tensile strain of 0.02% What is the maximum permissible temperature change ΔT if the allowable stress in the wire is 10 ksi in both tension and compression?
Use a coefficient of thermal expansion for the wire of $\alpha = 10^{-6}$ / °F, a modulus of elasticity $E = 15 \times 10^{6}$ psi, and examine both temperature increase and temperature drop.



$$\sigma = E\varepsilon = 15 \times 10^{6} \times 0.02\% = 3{,}000 \text{ psi}$$

$$\frac{R \cdot L}{EA} = \alpha\left(\Delta T\right)L$$

$$R = EA\alpha\left(\Delta T\right)$$

$$\sigma_{(\Delta T)} = \frac{R}{A} = E\alpha\left(\Delta T\right)$$

For temperature decrease (tension),

$$EA\alpha\left(\Delta T\right) + 3000 = 10000$$

$$\Rightarrow \left(\Delta T\right) = \frac{-7000}{EA} = -466^{\circ}\text{F}$$

For temperature increase (compression),

$$EA\alpha\left(\Delta T\right) - 3000 = 10000$$

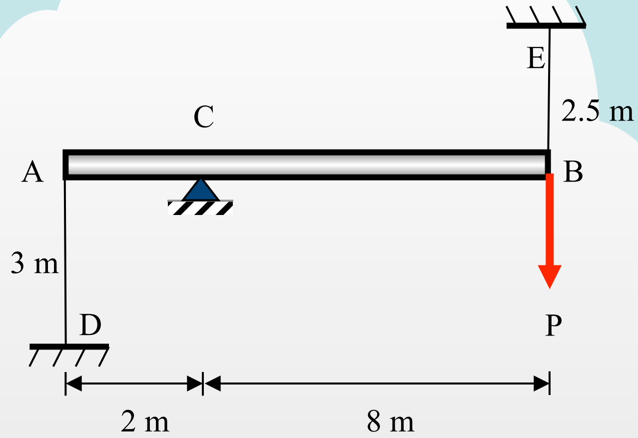
$$\Rightarrow \left(\Delta T\right) = \frac{13000}{EA} = 866^{\circ}\text{F}$$

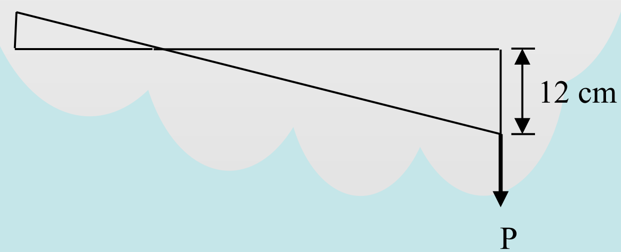
Example: Beam Supported by Cables and Pin

Rigid beam ACB is supported through a pin at C and two cables, AD and BE.

Find P such that the vertical deflection of point B is 12 cm

Both cables have cross-sectional area A = 15 cm² and their elasticity modulus is 200 GPa.





$$\delta_{BE} = 12 \text{ cm} = 0.12 \text{ m}$$

$$\frac{\delta_{AD}}{\delta_{BE}} = \frac{2}{8} \Rightarrow \delta_{AD} = \frac{12 \times 2}{8} = 3 \text{ cm} = 0.3 \text{ m}$$

$$A = 15 \text{ cm}^2 = 0.0015 \text{ m}^2$$

$$\delta = \frac{PL}{EA} \Rightarrow P = \frac{EA}{L}\delta$$

$$P_{AD} = \frac{200 \times 10^9 \times 0.0015}{3} \times 0.03$$

$$= 3.0 \times 10^6 \text{N}$$

$$P_{BE} = \frac{200 \times 10^9 \times 0.0015}{2.5} \times 0.12$$

$$= 14.4 \times 10^6 \text{N}$$

Moments, point C:

$$P \cdot 8 = 14.4 \times 10^6 \times 8 + 3.0 \times 10^6 \times 2$$

$$\Rightarrow P = 14.4 \times 10^6 + \frac{1}{4} \times 3.0 \times 10^6$$

$$\Rightarrow P = 15.15 \times 10^6 \text{N}$$

Example: Design Problem

From statics:

$$2F_{AL} + F_{ST} = 3w \qquad (1)$$

From compatibility:

$$\delta_{ST} = \delta_{AL}$$

$$\frac{F_{ST}L_{ST}}{A_{ST}E_{ST}} = \frac{F_{AL}L_{AL}}{A_{AL}E_{AL}}$$

$$\therefore F_{ST} = 3.81\,F_{AL} \qquad (2)$$

Solve (1) and (2) to get:

$$F_{AL} = 0.516w$$

$$F_{ST} = 1.967w$$

Check: $2F_{AL} + F_{ST} = 3w \quad (\text{ok})$

Case 1:

$$\left(\sigma_{all}\right)_{ST} = \frac{1.967w}{400\times10^{-6}} = 180\times10^{6}$$

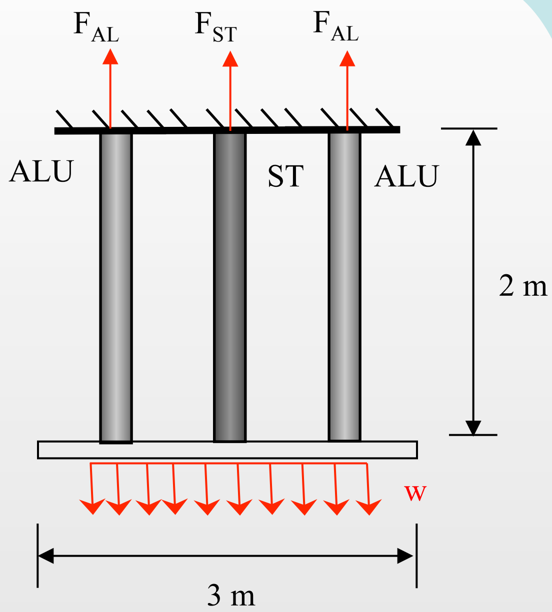
$w = 36.6$ kN/m

Case 2:

$$\left(\sigma_{all}\right)_{AL} = \frac{0.516w}{300\times10^{-6}} = 100\times10^{6}$$

$w = 58.1$ kN/m

$\therefore w_{max} = 36.6$ kN/m steel reaches all



Find the maximum possible load (w) before allowable stresses are reached in (ST) and (AL).
Given:

$$A_{ST} = 400\text{ mm}^2$$

$$E_{ST} = 200\text{ GPa}$$

$$\left(\sigma_{all}\right)_{ST} = 180\text{ MPa}$$

$$A_{AL} = 300\text{ mm}^2$$

$$E_{AL} = 70\text{ GPa}$$

$$\left(\sigma_{all}\right)_{AL} = 100\text{ MPa}$$

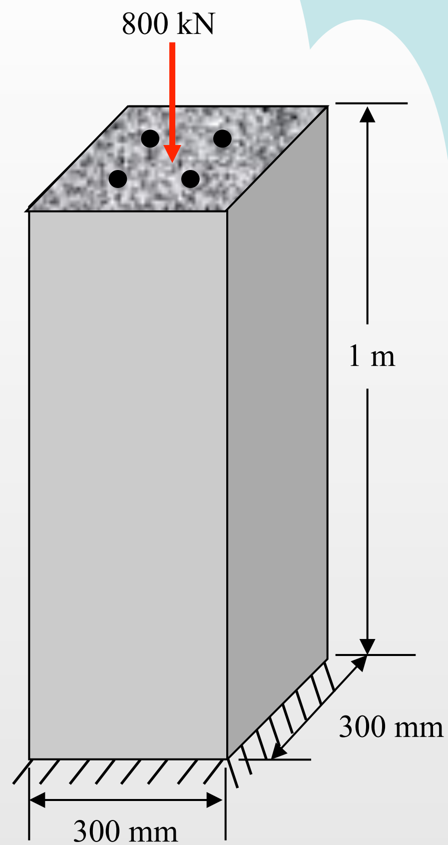
Example: Design Problem

A concrete column is reinforced with four A-36 steel bars. Determine the diameter of the rebars, so that the steel carries 1/4 of the 800 kN load.

Given:

$$E_{ST} = 200\ \text{GPa}$$

$$E_{con} = 25\ \text{GPa}$$



$$\text{Load on four rebars} = \frac{800}{4} = 200\ \text{kN}$$

$$\text{Load on each rebar} = \frac{200}{4} = 50\ \text{kN}$$

$$\delta_{ST} = \delta_{con}$$

$$\delta_{con} = \frac{PL}{AE} = \frac{600\times 10^3 \times 1}{0.3\times 0.3\times 25\times 10^9} = 0.267\ \text{mm}$$

$$\therefore \delta_{ST} = 0.267\times 10^{-3} = \left(\frac{PL}{AE}\right)_{ST}$$

$$A_{req.} = \frac{PL}{\delta_{ST}E_{ST}} = \frac{50\times 10^3\times 1}{0.267\times 10^{-3}\times 200\times 10^9}$$

$$A_{req.} = 0.0009375\ \text{m}^2$$

$$\therefore A_{req.} = 9.375\ \text{cm}^2 = \pi r^2$$

$$\Rightarrow r_{req.} = 1.727\ \text{cm}$$

Choose diameter of rebars = 36 mm each

Self-Assessment

Allowable Load: Bar

Find the allowable load (in kN) on a 2-cm diameter, 1-m-long steel rod if its maximum elongation cannot exceed 0.2 cm.

- 95
- 110
- 132
- 75

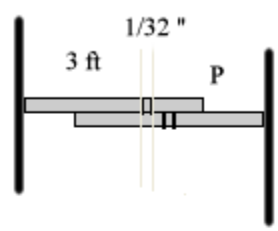
Elongation of a Bar: 1

A crane carries a total load of 7000 lbs by 1/2" diameter, 100-ft-long steel cable. How much, in inches, will the cable elongate?

- 1.43
- 0.92
- 1.31
- 1.18

Elongation of a Bar: 2

Two steel plates were constructed in warm weather. Due to temperature change, a 1/32" hole mismatching occurred. What is the puling force, in lbs, to be applied for one of the plates to match the two holes? The plate length is 3 ft (from the left end to the hole) and its cross section is 1/8"x1".



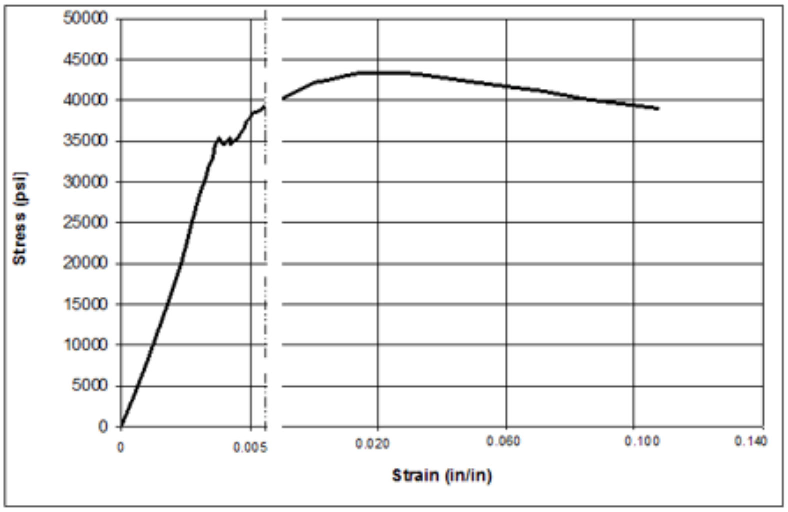
- 2405
- 3255
- 2752
- 3108

Mechanical Behavior of a Bar: 1

Deformation is reversible if:

- The deformation rate is slow
- Hooke's law holds
- There is no work hardening
- Applied stress is less than the yield stress

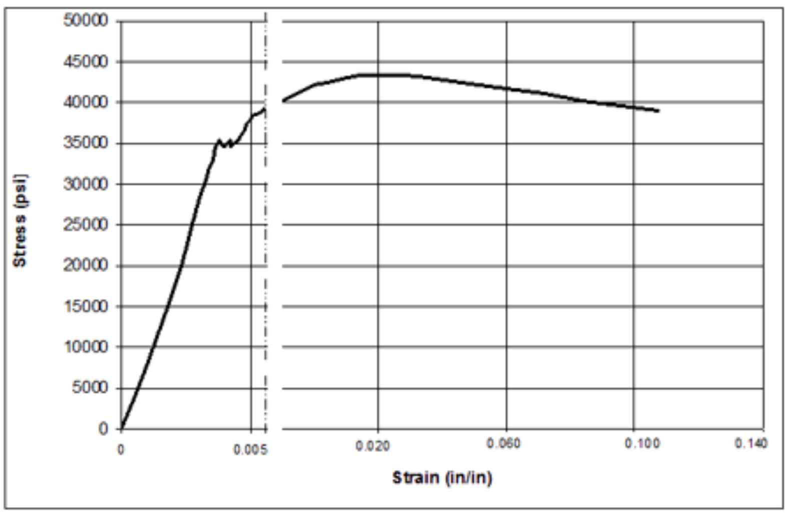
Mechanical Behavior of a Bar: 2



From the tensile test data of aluminum alloy shown in the figure. find the tensile strength in psi.

- 40000
- 44000
- 39000
- 45000

Mechanical Behavior of a Bar: 3

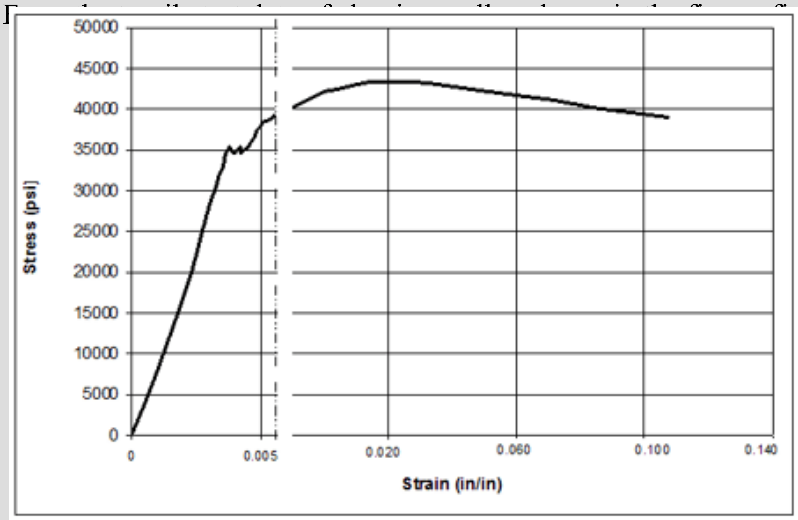


From the tensile test data of aluminum alloy shown in the figure, find the yield strength in psi:

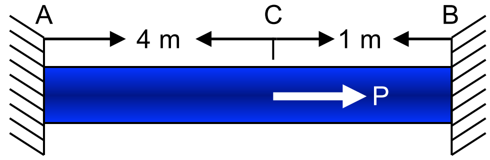
- 40000
- 25000
- 37000
- 35000

Mechanical Behavior of a Bar: 4

[illegible]d the modulus of elasticity in psi:



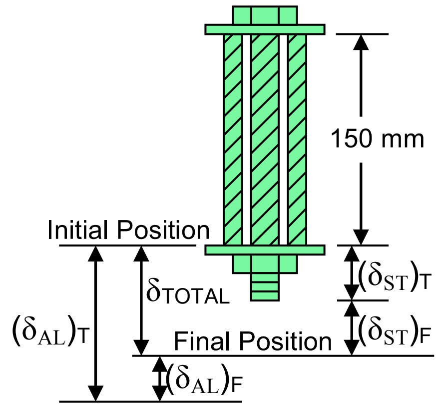
Bars: 1 The bar is fixed at ends A and B and force P = 50 kN is acting at C. The area of the cross section is 200 mm^2 and the modulus of elasticity E is 200 GPa. Find the horizontal reactions at A and B.



Bars: 2 A steel bolt and nut are tightened around an aluminum sleeve. Given the following cross-sectional areas (A_{ST} and A_{AL}), thermal expansion coefficients (α_{ST} & α_{AL}) and moduli of elasticity (E_{ST} and E_{AL}) for both the steel and aluminum, find the stress in the steel (σ_{ST}) and the stress in the aluminum (σ_{AL}) if a change in temperature (ΔT) of 65˚C is imposed.

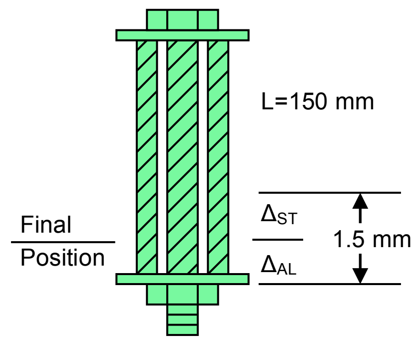
A_{ST} = 400 mm^2 A_{AL} = 600 mm^2

α_{ST} = 12 x 10^{-6} /˚C α_{AL} = 23x10^{-6} /˚C
E_{ST} = 200 GPa E_{AL} = 73.1 GPa
ΔT = 65˚C



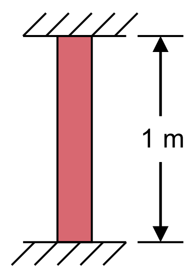
Bars: 3 A steel bolt and nut are tightened around an aluminum sleeve. Find the internal forces as the nut is rotated 1 turn (1 turn = 1.5 mm) given the cross sectional areas and the moduli of elasticity for the steel and the aluminum.

A_{ST} = 150 mm^2 A_{AL} = 100 mm^2
E_{ST} = 200 GPa E_{AL} = 70 GPa



Bars: 4 Thermal stress: α = linear coefficient of thermal expansion (units = strain/1˚C)

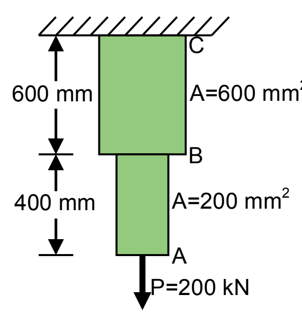
$$\text{e.g.: } \alpha_{steel} = 12 \times 10^{-6} / 1°\text{C}$$
$$\varepsilon = \alpha(\Delta T), \ \Delta T = \text{change in degrees of temperature}$$
$$\delta_T = \alpha(\Delta T)L$$



Thermal stress occurs when free movement of member is constrained. Find the stress when temperature changes by 30˚C for the steel rod shown.

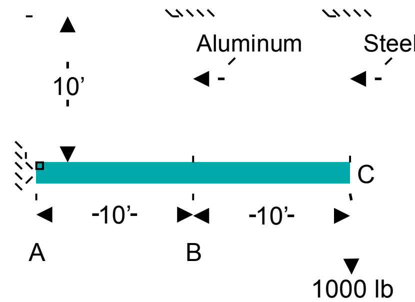
$$\alpha_{ST} = 12 \times 10^{-6} / 1\,°\text{C}$$
$$E_{ST} = 200\ \text{GPa}$$

Bars: 5 For the stepped bar, find the total displacement of point A (δ_A). The bar has a modulus of elasticity E = 200 GPa.



Bars: 6 The given cantilever system consists of a beam with a pin connection, and is held up by two wires connected as shown in the diagram. One wire is aluminum with a cross-sectional area (A_A) of 0.2 in^2 and a modulus of elasticity value (E_A) of 10×10^6 psi. The other wire is steel with a cross-sectional area (A_S) of 0.2 in^2 and a modulus of elasticity value (E_S) of 30×10^6 psi. Given this system, find:

a) What are the two equations of equilibrium from statics?
b) How is the deflection at C related to the deflection at B?
c) What are the three equations that must be solved to find F_A, F_B, and F_C?
d) What are F_A, F_B, and F_C?
e) What is the stress in each wire?



Bars: 7 To account for approximations in analysis (misused or incorrect fabrications) most structures are made stronger than suggested by the analysis. This extra margin of strength is described by

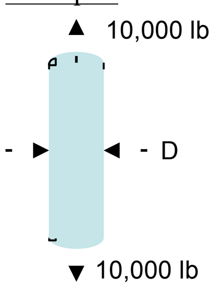
$$\text{Margin of safety} = \frac{\text{Yield stress}}{\text{Maximum stress}} - 1$$

or

$$\text{Safety factor} = \frac{\text{Yield stress}}{\text{Maximum stress}}$$

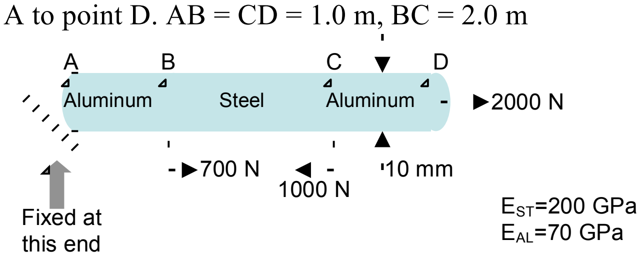
The minimum value of margin of safety = 0
The minimum value of safety factor = 1

<u>Example:</u>

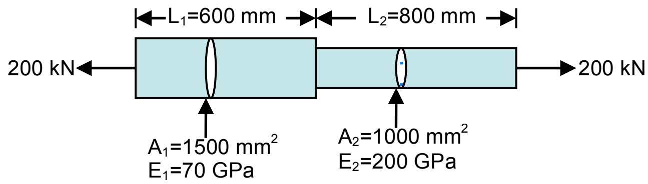


A circular rod is subjected to a tensile force of 10,000 lb. If σ_{yield} = 45,000 psi and the factor of safety = 5, find the required diameter.

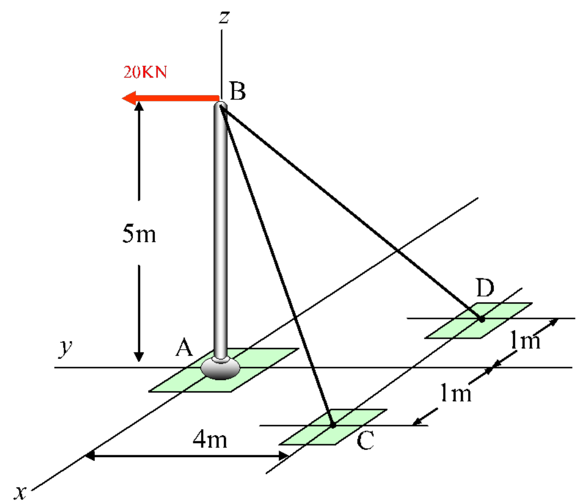
Bars: 8 A round bar is made of two materials and loaded as shown. Find the deflection (δ_{AD}) from point A to point D. AB = CD = 1.0 m, BC = 2.0 m



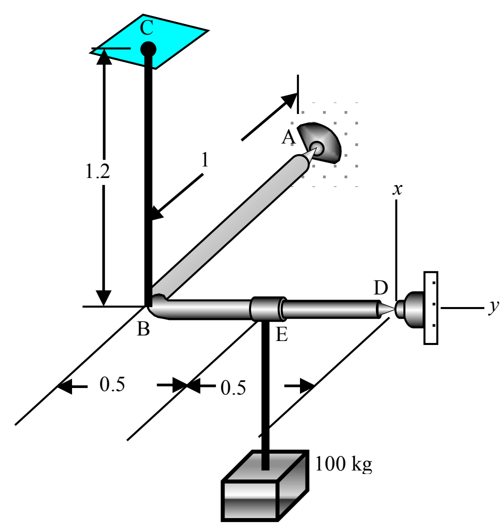
Bars: 9 Given the following structure, find the overall deformation (change in length), ΔL.



Bars: 10 Beam AB is vertical before the 20 KN load is applied in the *y*-direction, and can be considered rigid relevant to the stiffness of cables BC and BD. Find the displacement of B in the *y*-direction, considering that the two cables are identical, of diameter d = 2cm and E = 200GPa.



Bars: 11 Find the vertical (in x-direction) displacement of point B imposed by the 100 kg weight. Cable BC is of diameter d = 2cm and E = 200GPa. Consider beam ABED as being rigid and pinned at A and D.



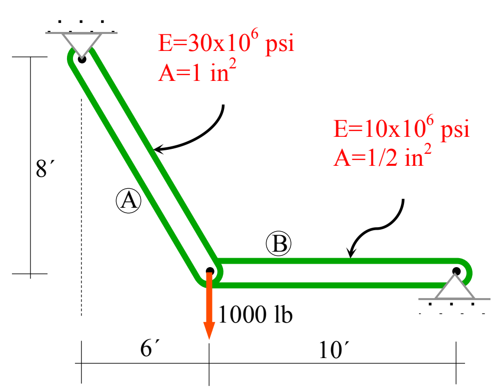
Bars: 12 For the structure, find:

a) F_A

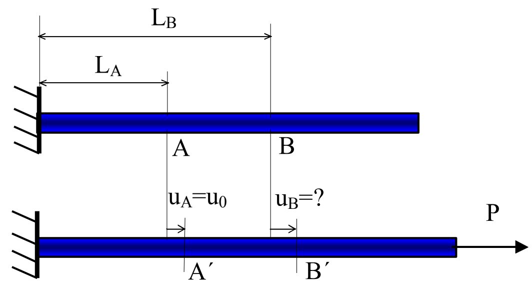
b) F_B

c) δ_A

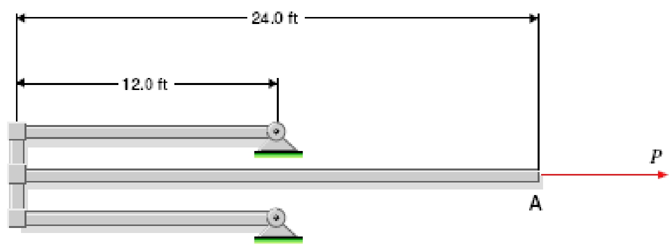
d) δ_B



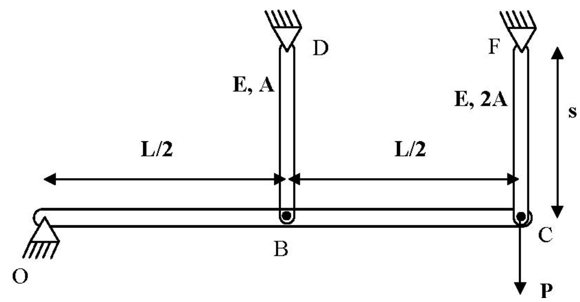
Bars: 13 A bar is elongated by an end load as indicated below. The displacement at point A is measured to be $u_A = u_0$. What is the displacement at point B?



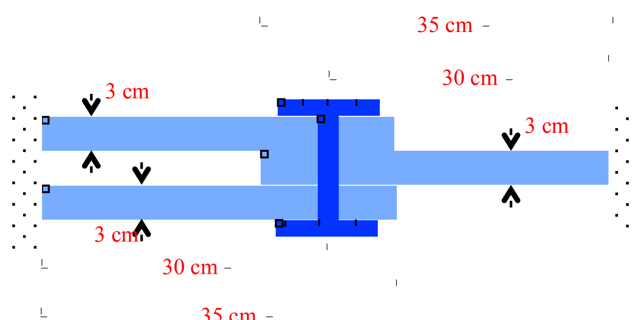
Bars: 14 Determine the displacement of point A due to the applied load P = 10 kips. The three horizontal rods all have a cross-sectional area of 1.2 in^2 and are made of steel (E = 29,000 ksi). Assume the vertical connection pieces on the left are rigid.



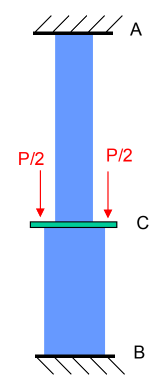
Bars: 15 A **rigid** bar OBC of length **L** is supported at three points as shown. It is pinned at **O** and connected to two elastic Bars: BD and CF. BD and CF have pin joints at both ends, are made of a material with elastic modulus of **E** and are of length **s**. The area of cross section of BD is **A** and that of CF is **2A**. A force **P** is applied at C as shown. (a) Determine the downward deflection δ of point C (in terms of **P, E, A** and **s**). What are the forces in BD and CF in terms of **P**? (b) The temperature of BD and CF is then increased by Δ**T**. The coefficient of thermal expansion of BD and CF is α. What are the forces in BD and CF due to the combined effect of the load **P** and the temperature increase Δ**T** (in terms of **P**, **E, A,** Δ**T** and α)? Note that the temperature increase has no effect on the rigid bar. (c) At what temperature Δ**T** (in terms of **P**, **E, A** and α) does the force in BD become zero?



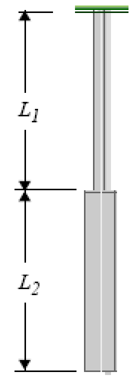
Bars: 16 Three plates (Bars:), each of length 35 cm, thickness 3cm and width 8 cm, are designed to form a structure as shown, i.e. fixed at one end and bolted together at 30 cm from the fixed ends. After the plates were fixed at the ends, it was found that there was a 0.3cm misfit in the position of the holes. In order to bolt the plates together, the plate on the right side is pulled with a force F enough to match the position of the holes. After the plates are bolted, the force F is released. Find the force in each of the three plates (Bars:). For all plates, E = 70GPa



Bars: 17 The stepped column, made of wood (E = 10.0 GPa) is supported at A and B and is loaded by a total force P through the bearing plate at C. The displacement of the cross-section at C was measured to be 0.01 cm (0.0001 m). Find the load P that created this displacement, ignoring the column's own weight. The length of AC is 1.0m, and the



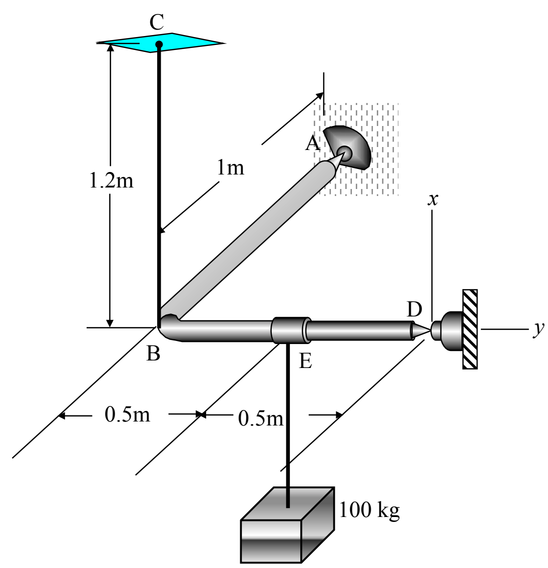
diameter of column AC is 6cm (0.06m). The length of BC is 0.7m, and the diameter of column BC is 9cm (0.09m).



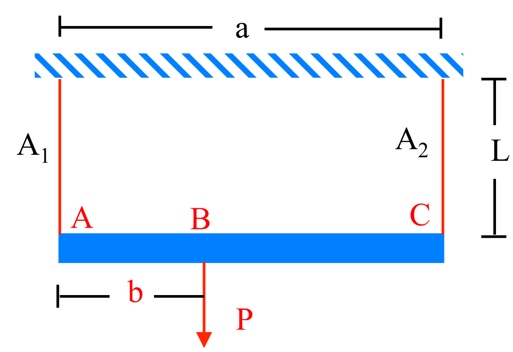
Bars: 18 Find the total elongation of the rod system shown hanging under its own weight with cross-sectional areas, A_1 and A_2, weight density, γ, and elastic modulus, E.

Bars: 19 Clearly describe, using complete sentences, clear diagrams, and appropriate equations, how you could use a tensile test to accurately determine the shear modulus of a bar composed of a homogeneous, isotropic, linear, elastic material. Indicate the measurements you would make, noting that only loads (forces) and displacement (elongations, contractions, etc.) can be measured experimentally. The initial dimensions of the bar are known or can be measured.

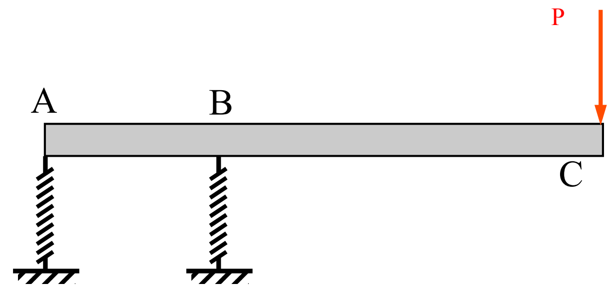
Bars: 20 Beam ABED, with a 90^O angle at B, is pinned at A and D, is supported through cable BC at B, and is loaded by a 100kg (mass) at E. In this problem, ignore the weight of the beam, the weight of cable BC and the weight of the cable supporting the 100kg mass. For cable BC, E = 200GPa, $\alpha = 12x10^{-6}$ /^{O}C and its diameter is d = 2cm. (a) Find the strain in cable BC imposed by the 100 kg mass. (b) Find the temperature difference ΔT that should be applied to cable BC in order to reduce its strain to zero. (c) If the temperature evaluated in (b) is actually applied to BC, what will be the stress in cable BC?



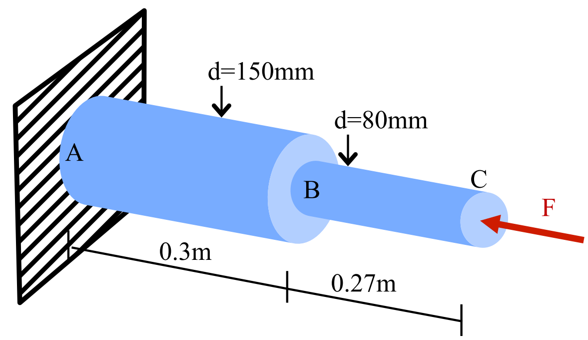
Bars: 21 Beam ABC is subjected to load P at B and supported by cables of cross-sectional area A_1, A_2, at A, B, respectively. Both cables are made of the same material of elasticity modulus E and coefficient of thermal expansion α. The engineer wants beam ABC to be horizontal, so she makes the decision to decrease the temperature (cool down) of the cable at A by ΔT. Find the formula for the evaluation of ΔT. Ignore the weight of beam ABC



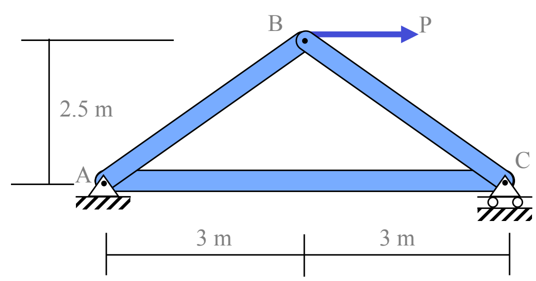
Bars: 22 Rigid beam ABC is supported by two identical vertical bars at A and B and subjected to force P at C. Length BC is 2.5 m and length AB is 1.0 m. The modulus of elasticity of the vertical bars is 50 GPa, each has a length of 1.0 m, and cross-sectional area of 4 cm^2. Determine the maximum load P such that the deflection of C will not exceed 1.0 cm.



Bars: 23 The stepped column, made of a material having a modulus of elasticity E = 90.0 GPa and coefficient of thermal expansion $\alpha = 10 \times 10^{-6} / {}^{O}C$, is fixed at A and loaded by a compressive force F = 400 kN at C. The engineer is asked whether there is a way to keep the displacement of B at zero after F is applied. In order to do so, the engineer proposes to heat part AB up by ΔT in order to achieve zero displacement of B. What is the required ΔT ?

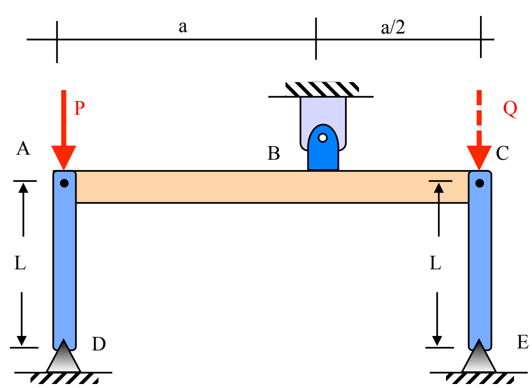


Bars: 24 Truss ABC is subjected to a horizontal force P = 600 kN. All bars in the truss are made of steel (E = 200 GPa) and have a cross-sectional area of 3000 mm^2. What is the horizontal displacement of joint C?



Bars: 25 A high altitude balloon has a long steel wire hanging from it. The length of the wire is 7,000 m, the weight density of the steel is 78 kN/m^3 and its ultimate stress is 620 MPa. Find the safety factor of the wire against breaking from its own weight.

Bars: 26 Rigid beam ABC is supported through a pin at B, and through bars AD and CE pinned at A,D and C,E, respectively. Bars AD and CE are identical, with cross-sectional area A, and elasticity modulus E. A vertical load P is applied at A. For technical reasons, **rigid** beam ABC should not incline more than an angle θ (angle expressed in radians) counter-clock-wise from horizontal so, for this, the engineer decides to apply a dead-weight load Q at C. Find the minimum magnitude of the load Q, considering that θ is small enough such that $\theta \cong \tan(\theta)$



Shear stress

In previous modules, we examined bars and saw that stretching or compressing a bar creates normal stress that is acting transverse to the bar's cross sectional area. Now, if we "shear" the bar, as shown schematically in Figure E-1, we create what is termed shear stress, which is now acting parallel to the surface. The letter $\boldsymbol{\tau}$ is used to designate shear stress. For the case shown below we have

$$\tau_{ave} = \frac{P}{A}$$

where P is shearing the cross section and A is the cross sectional area. It is important to remember that P/A is only the "average" shear stress. In actual shear stress distribution, it is not uniform and the maximum shear stress will be higher than the average shear stress. For some problems, such as beam bending and shear, the shear stress distribution is evaluated later in this course. Thus, for the time being, it is assumed that the shear load is uniformly distributed across the shear surface. Obviously a more precise definition is, similarly to normal stress: $\tau = \lim_{\Delta A \to 0} \frac{\Delta P}{\Delta A}$ where ΔP is now shearing ΔA.

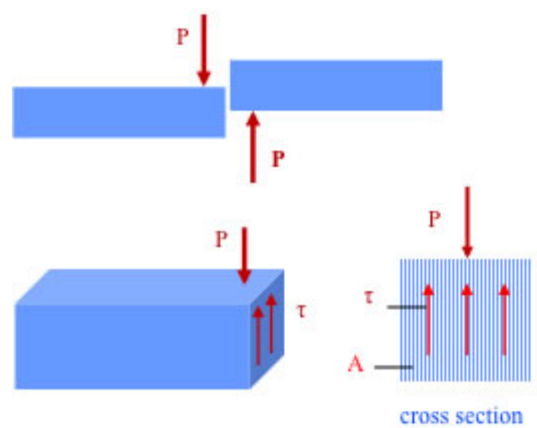


Figure E-1 A shearing force P acting on a bar creates shear stresses τ. A side view of the bar and a section in the vicinity of the force is shown at the top, and a 3-D view of the part left to the section at the bottom.

Good illustrations of shear stress development are bolt, rivet, screw, and nail connections. These create shear stresses in the connecting element such as bolt, and bearing stresses in the bolt and clevis. In the next few sections, the shear stress in connecting elements is addressed. Then the bearing stresses and the shear strains are addressed in detail.

Normal stress, shear stress, bearing stress

With the definition of normal and shear stress presented in previous sections, it is useful to summarize the different stresses and introduce the notion of bearing stress.

Normal stress (σ) on a surface is the result of a force acting transverse to it, and is defined as force per unit area. A typical example of development of normal stresses is a bar in tension or compression.

Shear stress (τ), on a surface is the result of force acting parallel to it, and is also defined as force per unit area. A typical example of development of shear stress is a connecting element, like the bolt shown in Figure E-2. The force F is transmitted from one plate to the other via the bolt, as shear stress at the cross section of the bolt positioned between the two plates.

Bearing stress (σ_b) is a normal stress defined as the force pushing against a structure divided by the area. In the two plates connected through a bolt in Figure E-2, loads F press against the bolt in bearing, and contact stresses (pushing against), called bearing stresses, develop.

The following examples should clarify the relevant definitions and provide an idea of how such stresses are calculated.

Example 1-bolted connection, single shear: Figure E-2 shows two plates/bars connected together via a bolt. The loads, F, create **normal stress** (σ) on the crosssections of bars/plates and **shear stresses** on the cross section of the bolt positioned between the two plates/bars. In addition, F pushes against the bolt, creating **bearing stress**. For the bars/plates, the cross-sectional area, A_{bar}, is simply the area of the cross section transverse to F. For the bolt, A_{bolt} is simply the cross sectional area of the bolt. The **bearing area**, A_{bear}, is defined as the projected (net) area of the curved bearing surface. Such a net or projected area is equal to the thickness of the plates/bars multiplied by the diameter of the bolt. The bearing force in this example is equal to F. Equilibrium equations for part of the bar and part of the bolt as shown in the figure make calculations of (average) stresses for this example possible. Relevant definitions are shown below.

A_{bar}= cross section of bar/plate

A_{bolt}= cross section of bolt

$$\sigma = \frac{F}{A_{bar}} = \text{normal stress in plates/bars}$$

$$\tau_{ave}^{bolt} = \frac{F}{A_{bolt}} = \frac{V}{A_{bolt}} = \text{shear stress in bolt}$$

$$\sigma_b = \frac{F}{A_{bear}} = \text{bearing stress in bolt.}$$

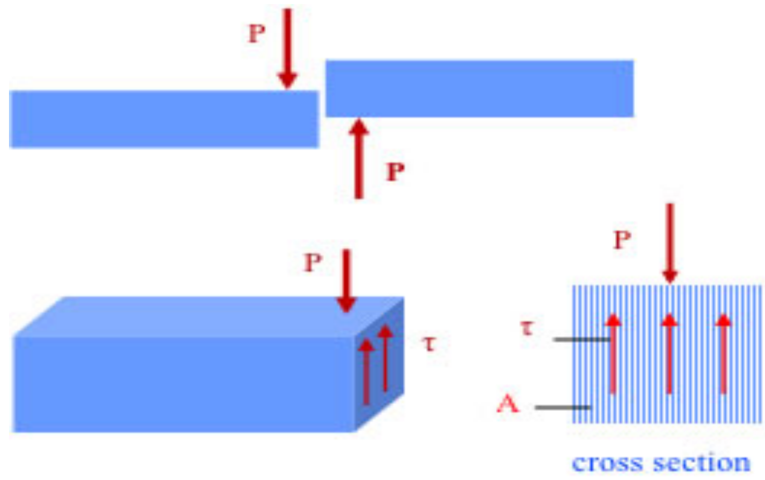


Figure E-2 Example 1-Single Shear: (a) shows a bolt connecting two plates in "single shear"; (b) the free body diagem of the bolt and the two forces F shearing the cross section in the middle; and (c) the free body diagram of half the bolt and the shear force V created, which results in shear stress τ. Forces F are also responsible for the bearing stresses created on the bolts as well as on the plates.

Example 2-bolted connection, double shear: This example (Figure E-3) is similar to the first one, though the bolt is subjected to "double shear," meaning that two cross sections of the bolt are sheared, one at the interface between the top and middle plate/bar and one at the interface between the middle and bottom plate/bar. Using a similar notation as before, i.e., A_{bar}= cross section of each bar/plate (the three plates are considered here of the same cross sectional area); A_{bolt}= cross section of bolt, the stresses are calculated as follows:

$\sigma = \dfrac{F}{A_{bar}}$ = normal stress in the bar to the right (the one subjected to F);

$\sigma = \dfrac{F/2}{A_{bar}} = \dfrac{F}{2A_{bar}}$ = normal stress in the two bars to the left (the ones subjected to F/2;

$\tau_{ave}^{bolt} = \dfrac{F/2}{A_{bolt}} = \dfrac{F}{2A_{bolt}}$ = shear stress (average) in the bolt, at each sheared cross section;

$\sigma_b = \dfrac{F}{A_{bear}}$ = bearing stress (average) on the bolt (at the part of the bolt being in the middle plate/bar). A_{bear} is defined as the projected or net bearing area, in this case the product of the thickness of the middle plate and the diameter of the bolt;

$\sigma_b = \dfrac{F/2}{A_{bear}} = \dfrac{F}{2A_{bear}}$ = bearing stress (average) on the bolt, i.e. at the part of the bolt being in the top or bottom plate/ bar. Here, again, A_{bear} is defined as the projected or net bearing area, in this case, the product of the thickness of the top or bottom plate and the diameter of the bolt.

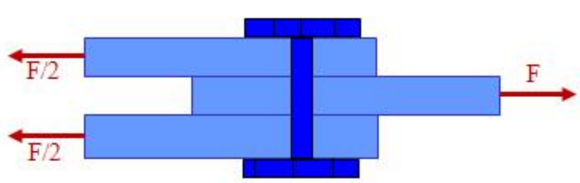


Figure E-3 Example 2-double shear. Two plates at the left each subjected to force F/2 are connected to a plate subjected to force F through a bolt. The bolt is subjected to double shear at two interfaces between the plates.

Note: The rationale for defining the bearing area as the **projected instead of the actual**

area is: (a) The actual bearing stress distribution is complex, and defining the area as the projected one implies the bearing stress is actually an average one that is easier to deal with; and (b) similarly to hydrostatics where the total pressure force on a surface submerged in a liquid is the product of the net (projected) area and the pressure at the centroid of the projected area, the total bearing force is simply defined as the product of the average bearing stress and the net (projected) area.

Shear stress on inclined sections of a bar in tension or compression

The previous sections discussed the creation of shear stresses in problems such as two plates connected through bolts. However, shear stresses are created in almost all engineering problems involving mechanical loads. Even though the general case of stress distribution in two- and three-dimensions is examined later in this course, here we consider the simple case of the stress distribution in an inclined cross section of a bar subjected to tensile or compressive load. Consider a prismatic bar subjected to a load T on both sides as shown in Figure E-4. First, by considering a section mn transverse to T, as shown while studying a bar, for equilibrium, normal stresses σ act on plane mn. Now, if we consider an oblique (inclined) cross section that is not transverse but rather makes an angle θ with respect to T, such an oblique section, jk, indicates that both normal and shear stresses act on the plane jk. In other words the force T that must act on jk for equilibrium can be decomposed into two components, one transverse to jk, N, and one parallel to jk, V . The former creates normal stresses on jk and the latter shear stresses on jk. This can be verified easily by considering equilibrium in both x' and y' direction of the piece of the bar extending from jk to the right (or left) end of the bar. Considering that normal stress σ and shear stress τ act on jk, the two equilibrium equations shown below indicate that both σ and τ are, in general, nonzero. Note that the angle of the oblique plane increases from 0^o to 45^o, the shear stress increases while the normal stress decreases.

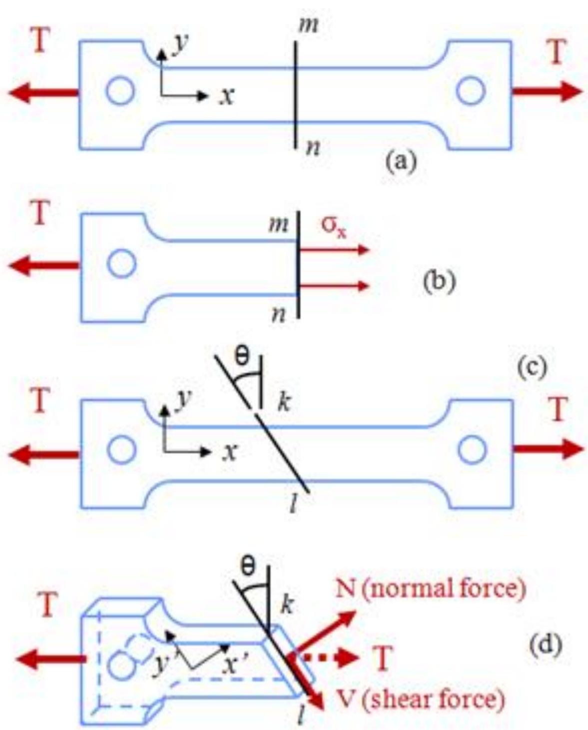


Figure E-4 (a) A bar subjected to force T and a section mn transverse to T; (b) equilibrium of the part of the bar in (a) to the left of mn showing the normal stresses σ that develop; (c) the same bar as in (a) but an inclined section kl is now considered; and (d) equilibrium of the part of the bar in (c) to the left of kl showing the normal force N and shear force V that develop. N creates normal stress σ and V creates shear stress τ on kl.

Equilibrium of the part of the bar shown in Figure E-4(d) in the x' and y' directions yields

$$\Sigma F_{y'} = 0 \Rightarrow V = T\sin(\theta)\cos(\theta) \quad \Sigma F_{x'} = 0 \Rightarrow N = T\cos(\theta)\cos(\theta)$$

The first of these equations yields the shear stress at the cross section kl and the second yields the normal stress. In general, on every plane within a solid subjected to an external load, there is one normal and two shear stresses, as shown schematically in Figure E-5, where only the three stresses on the plane parallel to x are shown for illustration. Also, in Figure E-5, the shear stresses are designated by indexes, the convention for which will be demonstrated in subsequent sections. The most useful way of representing the stresses in a structure, e.g., in a bar, is to isolate a small element of material and consider all the stresses (and thus the forces) acting on it. Such a **material element** can be considered at any position within a structure and can have any desired orientation. Figure E-5 below shows such an element. Also, when considering section jk in Figure E-4, the entire section or part of it can be considered as the side of one element. In two dimensions, there are two stresses on each plane, namely one normal and one shear stress. Two dimensional analysis of engineering problems is very common.

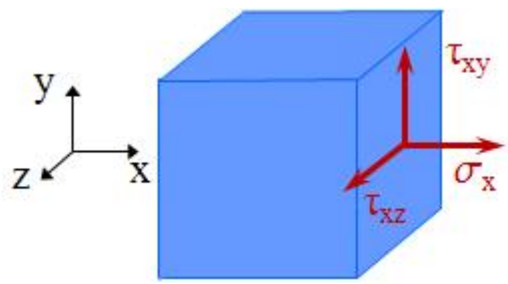


Figure E-5 Normal and shear stresses for a general three-dimensional case.

Notation for normal and shear stresses

The notion of the infinitesimal material element was introduced in the previous module. Even though such an element is of infinitesimal dimensions, usually it is drawn large so all the stresses on it can be visualized. Material elements can have any desired orientation within a structure and are parallelepipeds. In general, we consider a three-dimensional coordinate system, x-y-z and a material element aligned to that coordinate system as in Figure E-6. The three normal stresses are designated as σ_x, σ_y, σ_z , and the subscripts corresponds to the plane the normal stress acts on. Shear stresses require the use of two subscript and there are six such stresses τ_{xy}, τ_{yx}, τ_{xz}, τ_{zx}, τ_{yz}, τ_{zy}. The first subscript corresponds to the plane the shear stress acts on, and the second to its direction. For example, τ_{xy} denotes the shear stress that acts on the plane transverse to the x-axis (called the x-plane) and its direction is parallel to axis y. Figure E-6shows all such stresses, i.e. shear as well as normal ones.

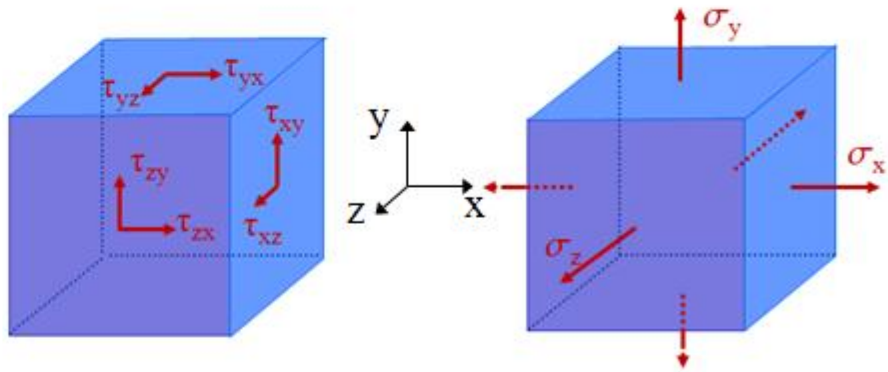


Figure E-6 Stresses acting on a material element, i.e., an infinitesimal parallelepiped embedded in a structure. On the left are the shear stresses and the normal stresses are on the right. For clarity, only the shear stresses in three planes (not on all six planes) are shown.

A very important relation among shear stresses

Consider the free body diagram of a small material element (parallelepiped) centered at a point in a structure (Figure E-7). There are three stresses and corresponding forces (force = stress times area) acting on each side of the element, one normal force and two shear forces; each force is equal to the stress multiplied by the area it acts upon, i.e., ΔA. For simplicity let us consider that the material element is a cube of side α, thus $\Delta A = \alpha^2$. A more general case where the element is not a cube is considered in the following section. Let us first consider the moment equilibrium of the cube, since, as will be shown, satisfaction of moment equilibrium will end up implying force equilibrium as well;

$$\Sigma F_x = 0, \quad \Sigma F_y = 0, \Sigma F_z = 0$$

will also be satisfied.

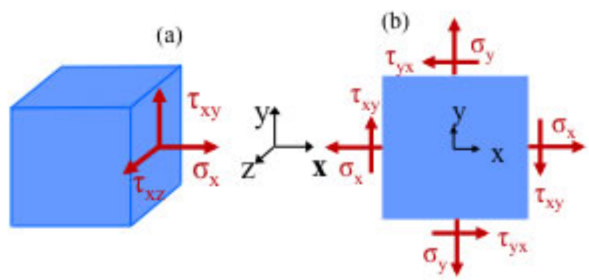


Figure E-7 (a) normal and shear stresses for a general three-dimensional case; and (b), normal and shear stresses in the x and y planes.

Let us consider one of the three moment equilibrium equations, namely $\Sigma M_z = 0$. This implies that (since the normal forces acting on the x and y planes do not produce any moment around axis z):

$$(\tau_{xy} \cdot \Delta A)\alpha - (\tau_{yx} \cdot \Delta A)\alpha = 0$$

where α is the side of the cube. Then this implies that $\tau_{xy} = \tau_{yx}$, and writing similar moment equilibrium equations for the other two axes we have

$$\tau_{xy} = \tau_{yx}; \quad \tau_{xz} = \tau_{zx}; \tau_{yz} = \tau_{zy};$$

Thus although there are originally nine stress components at a "point" or material element, namely three normal stresses and six shear stresses, the above relation implies that only six of them are independent. Thus for a problem of mechanics of materials, there are six unknown stresses at each point to be determined. In short, we have that:

1. Shear stresses on opposite faces of a material element are equal in magnitude and opposite in direction.

2. Shear stresses on adjacent faces of a material element are equal in magnitude and they both point toward or both point away from the intersection of the two faces.

For a more formal proof of the above relation, see the following module. Finally, note that when there are no normal stresses on an element, but only shear stresses, the element is considered to be in a state of **pure shear**.

Shear stress symmetry

We now look at the issue of shear stress symmetry in more detail, since this is important in mechanics of materials. In the previous section some simplifying assumptions were made on the material element.

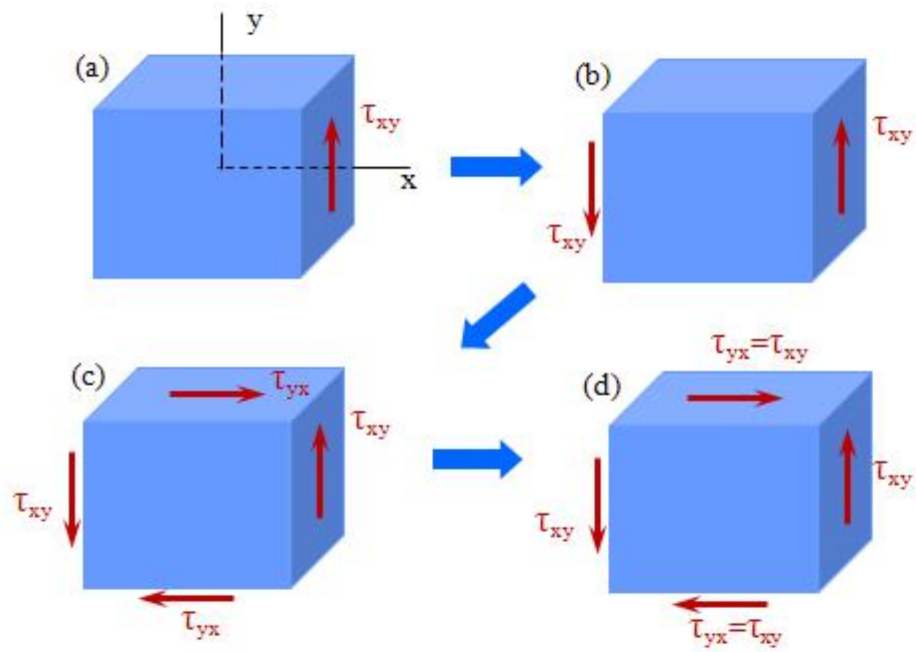


Figure E-8 (a) A material element subjected to a shear stress on its x-plane only; (b) for equilibrium of element in (a) an equal shear stress must be present on the negative x-plane; (c) for equilibrium of the couple in (b) an opposite sign couple must act as shown; and (d) equilibrium of forces implies the shear stress symmetry.

The two forces acting on the positive and negative x-planes (Figure E-8(b)) form a couple, which must be equilibrated by another couple. Since normal forces that may also act on the element do not produce any moments, a couple must form by shear stresses on the positive and negative y-planes, as shown in Figure E-8(b).Then, moment equilibrium around the z-axis implies that

$\tau_{yx}[(dx)(dz)](dy) = \tau_{xy}[(dy)(dz)](dx)$ ... (stress) × (area) = (force), ... (force) x (distance) = (moment)

which further implies that $\tau_{xy} = \tau_{yx}$.

Module 9: Shear Strain, Shear Modulus

Shear strain and Hooke's law in shear

Normal stresses create normal strain, i.e., a stress σ_x creates normal stress in direction x and lateral normal strains in the other two directions. Shear stress creates shear strain, i.e., an initially cubic element transforms into a rhombus-type element since the initially right angles change by some amount. The shear strain is defined as the change of an initially right angle (Figure E-9). It is a measure of distortion, expressed in radians. For a linear and elastic material, the shear strain is proportional to the shear stress, and the factor of proportionality is called **shear modulus, denoted as G**. Thus,

$$\tau_{xy} = G\gamma_{xy}$$

The shear modulus is a material property and values for certain engineering materials are shown in the Appendix. Usually a plot of shear stress versus shear strain is obtained experimentally and from it the G modulus is calculated. Such plots are shown in the following module which discusses factor of safety with respect to shear stress.

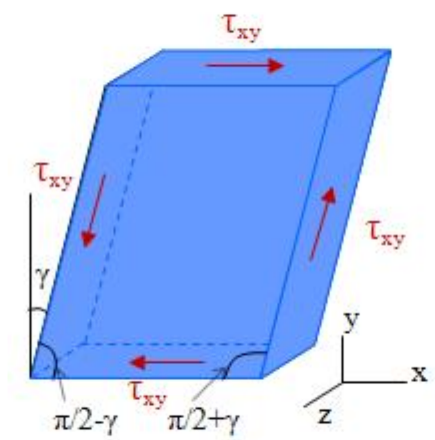


Figure E-9 Deformed state of a material element subjected to shear stress τ. In the undeformed state, the angle γ is equal to zero.

The elasticity modulus E and the shear modulus G are related by

$$G = \frac{E}{2(1+\nu)}$$

The proof of this relation between E and G will be presented in a subsequent module. At this point it is mentioned that even though an element may be subjected to shear as in Figure E-9, at inclined planes there is normal stress created.

Sign convention for shear stress and shear strain

Unlike normal stress with compression or tension, shear stress is the same if it shears left to right or right to left. The magnitude of shear stress is the important parameter that needs to be considered. However, in certain applications it is necessary to use a sign convention for shear stresses. The shear stress will be considered positive when a pair of shear stresses acting on opposite sides of the material element produce a counterclockwise torque (couple). (Some texts use the opposite direction for the positive shear stress. This changes a sign in several equations, so we must be somewhat careful of signs when working problems and examples.) This sign convention also yields that a shear stress acting on a positive face (in the positive x axis for example) is positive if it acts in the positive direction of the coordinate axes and negative if it acts in the negative direction of an axis. A shear stress acting on a negative face of an element is positive if it acts in the negative direction of an axis and negative if it acts in a positive direction. A positive shear stress creates positive shear strain and vice versa.

Shear stress and factor of safety

The factor of safety under shear is defined similarly to the factor of safety under normal stress. Materials can sustain stress up to a certain limit before failure occurs. Of course, the load imposed on a structure must always be smaller than the load that would cause failure. The failure load is often termed “ultimate” and the load a structure is subjected to is termed the “working” or “design” load. The factor of safety for a structur in shear is defined as the ratio

$$\text{Factor\⁢ of\⁢ safety} = n = \frac{\text{Failure load}}{\text{Working load}}$$

where load here creates shear stresses. Another way to consider factor of safety for a structure is to find the maximum shear stress under working load and the maximum shear stress under ultimate load. In this case

$$\text{Factor of safety} = n = \frac{\text{Failure stress}}{\text{Working stress}}$$

For certain cases and materials, instead of using the ultimate stress or ultimate load in the above two relations, the yield stress and corresponding yield load are used. Calculation of factor of safety for simple structures is relatively straightforward, given the stress distribution in a structure and the shear stress versus shear strain response of the material. Figure E-10 shows a shear stress versus shear strain curve for a metal such as certain aluminum alloys.

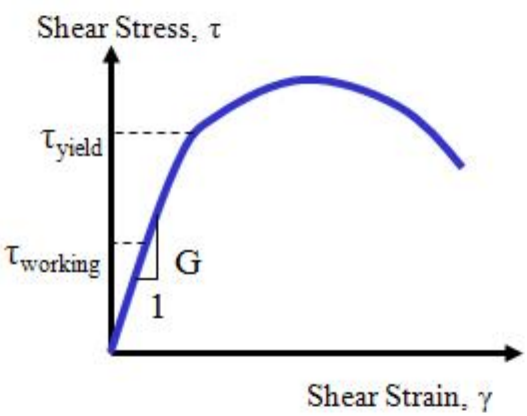


Figure E-10 Shear stress versus shear strain of the material.

The factor of safety in shear is defined (here with respect to the material yield stress) as

$$n = \frac{\text{Yield stress}}{\text{Working stress}} = \frac{\tau_{\text{yield}}}{\tau}$$

The allowable or design shear stress is then defined as

$$\tau_{\text{all}} = \frac{\tau_{\text{yield}}}{n}$$

For problems such as shearing of a bolt, where a shearing force V creates the shear stresses on the bolt, we have

$$n = \frac{\tau_{\text{yield}}}{n} = \frac{V_{\text{yield}}/A}{V/A} = \frac{V_{\text{yield}}}{V}$$

Shear stress and bearing stress-further examples

A typical example of shear and bearing stress development is that of a bolt connecting two plates/bars. However, shear and bearing stresses are developed extensively in practically all engineering applications. Some further examples of such stresses are presented. As mentioned before, bearing stress is defined as the compressive stress created by direct contact of two surfaces. Shear stress in one part of a component, such as a bolt are usually accompanied by bearing stresses in other surfaces. The actual distribution of bearing stress is not easily determined, so usually the average bearing stress is determined and is compared to the failure bearing stress of the material. Thus, bearing stress is defined as σ_b = P/A where P denotes the bearing force and A the force bearing area.

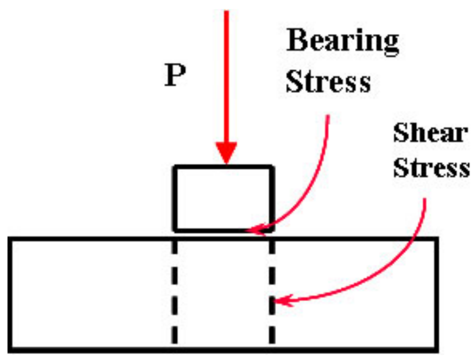


Figure E-11 Schematic of a cylindrical body (top) pushing on the bottom part by load P.

Figure E-12 shows the case of a nail or bolt positioned through a hole in a plate and subjected to a load P. Load P imposes shear stress on the areas indicated by dot lines (see arrow in Figure E-12) and bearing stress on the area of contact between the top and bottom body (see arrow in Figure E-12).

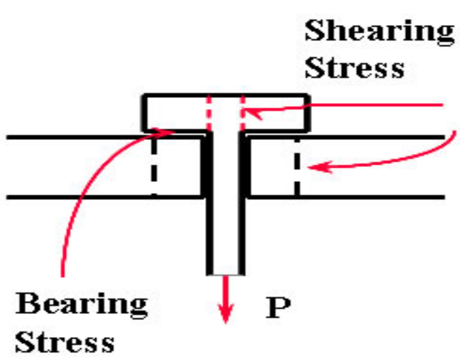


Figure E-12 A bolt or nail-like object is pulled through a hole in a plate.

A schematic of the so-called punching operation is shown in Figure E-13. Force P imposes shear stress on the cylindrical area (see dotted line in Figure E-13) in the bottom plate, and bearing stress on the contact area between the punching object and the bottom plate.

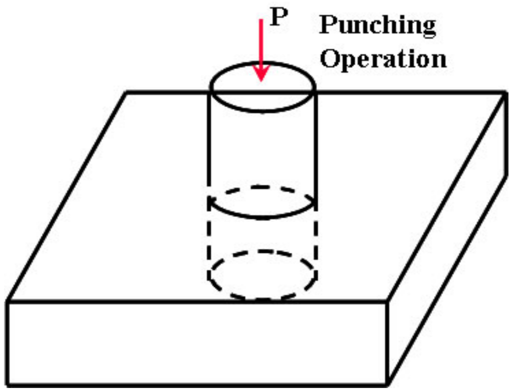


Figure E-13 The punching operation is such that an impact load P is imposed on an object such that it punches through a sheet or plate.

Figure E-14 shows a cylindrical hole of diameter d through a plate of thickness t. Such holes are common in fastener applications, and often a force is imposed transverse to the axis of the cylindrical hole. In such a case, the bearing stress is the force divided by the net area. Thus the bearing stress is expressed as

$$\sigma_b = \frac{F}{d \times t}$$

where F is the bearing force, d is the diameter and t the thickness, as shown in Figure E-14.

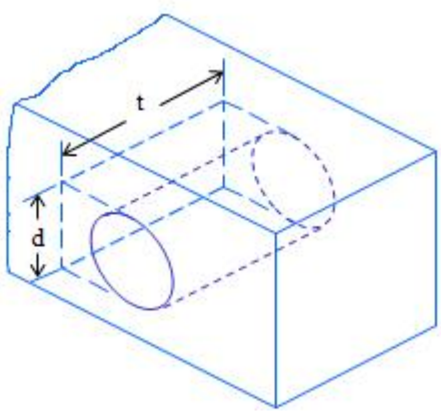
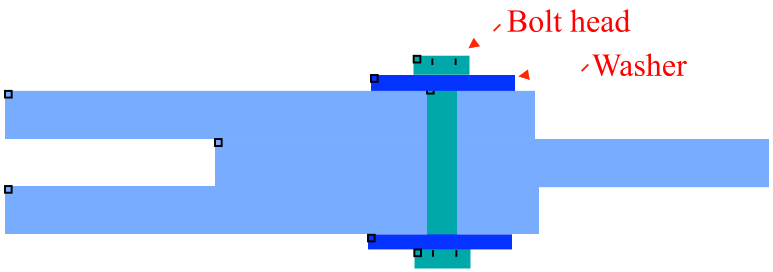


Figure E-14 The shearing force F also creates bearing stress, the average value of which is denoted as σ_b.

Example: Bearing and Shearing Stresses



The figure shows a bolted connection between three aluminum members. When the bolt is tightened, the aluminum is compressed laterally and the bolt is in tension. The allowable tensile stress in the 1/2-inch-diameter bolt is 20,000 psi, and the allowable bearing stress between the 1.0-inch-diameter washers and the aluminum is 2,400 psi. a) What is the maximum permissible tensile force in the bolt? b) Under that maximum load, what is the average shear stress on the washers if the perimeter of the hexagonal bolt heads is 1.2 inches and the thickness of the washers is 0.1 inches.

Cross-sectional area of the bolt: $A_b = \pi d^2 / 4 = 3.14 \cdot 0.5^2 / 4 = 0.197 \text{ in}^2$ thus the maximum load so that the 20,000 psi is not exceeded is:

$$F_1 = \sigma_{all} A_b = 20,000 \times 0.197 = 3,940.0 \text{ lb}$$

The bearing area of the washer is: $A_w = \pi\left(1.0^2 - 0.5^2\right) / 4 = 0.59 \text{ in}^2$ thus the maximum load so that the 1200 psi stress is not exceeded is:

$$F_2 = \sigma_{all} A_w = 2,400 \times 0.59 = 1,416.0 \text{ lb}$$

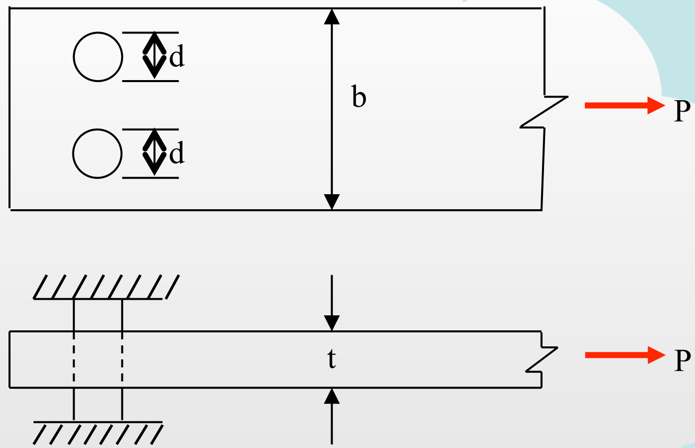
Then the maximum allowable load is **1,416.0 lb**.

The area of the washer subjected to shear stress is equal to the perimeter of the bolt head multiplied by the thickness of the washer. Thus

$A_{shear} = 1.2 \cdot 0.1 = 0.12 \text{ in}^2$. Then, the average shear stress to the washer is

$$\tau_{ave} = \frac{F_{max}}{A_{shear}} = \frac{1,416.0}{0.12} = 11,800 \text{ psi}$$

Example: Design Problem



A flat bar of width b = 2.5 inches and thickness t = 0.5 inch transmits an axial load P (see figure). Two holes of diameter d are drilled through the bar for pin supports. The ultimate tensile strength of the bar ($\sigma_{t,ult}$) is 36,000 psi on the net cross section of the bar, the ultimate shear strength of the pins ($\tau_{s,ult}$) is 14,000 psi, and the ultimate bearing stress between the bar and pins ($\sigma_{b,ult}$) is 22,000 psi. Using a factor safety of 2.50, determine the diameter of the pins for which the load P will be maximum. Find d and P.

Combinations:

- Tension-Shear
- Tension-Bearing
- Shear-Bearing

	Tension	Shear	Bearing
Allowable stress	$\frac{36,000}{2.5} = 14,400$ psi	$\frac{14,000}{2.5} = 5,600$ psi	$\frac{22,000}{2.5} = 8,800$ psi
Area	$t(b - 2d) = 1.25 - d$	$4\pi d^2 / 4 = \pi d^2$	$2\ dt = d$
$P = \sigma A$	$P = 14,400\ (1.25 - d)$ $P = 18,000 - 14,400\ d$	$P = 5,600\ \pi d^2$	$P = 8,800d$

Example: Design Problem

Tension-Shear:

$$18{,}000 - 14{,}400d = 5600\,\pi d^2$$

$$\therefore 5{,}600\,\pi d^2 + 14{,}400\,d - 18{,}000 = 0$$

$$d = \frac{-14{,}400 + \sqrt{14{,}400^2 - (4)\,(5{,}600\,\pi)\,(-18{,}000)}}{(2)\,(5{,}600\,\pi)} = 0.682 \text{ in}$$

NOT OK

Tension-Bearing:

$$18{,}000 - 14{,}400\,d = 8{,}800\,d$$

$$d = 0.776 \text{ in}$$

$$P = 6{,}830 \text{ lb}$$

Check shear $P = 10{,}590$ lb

OK

Shear-Bearing:

$$5600\,\pi d^2 = 8{,}800\,d$$

$$d = \frac{8{,}800}{5{,}600\,\pi} = 0.500 \text{ in}$$

$$P = 4{,}400 \text{ lb}$$

Check tension $P = 10{,}800$ lb

OK

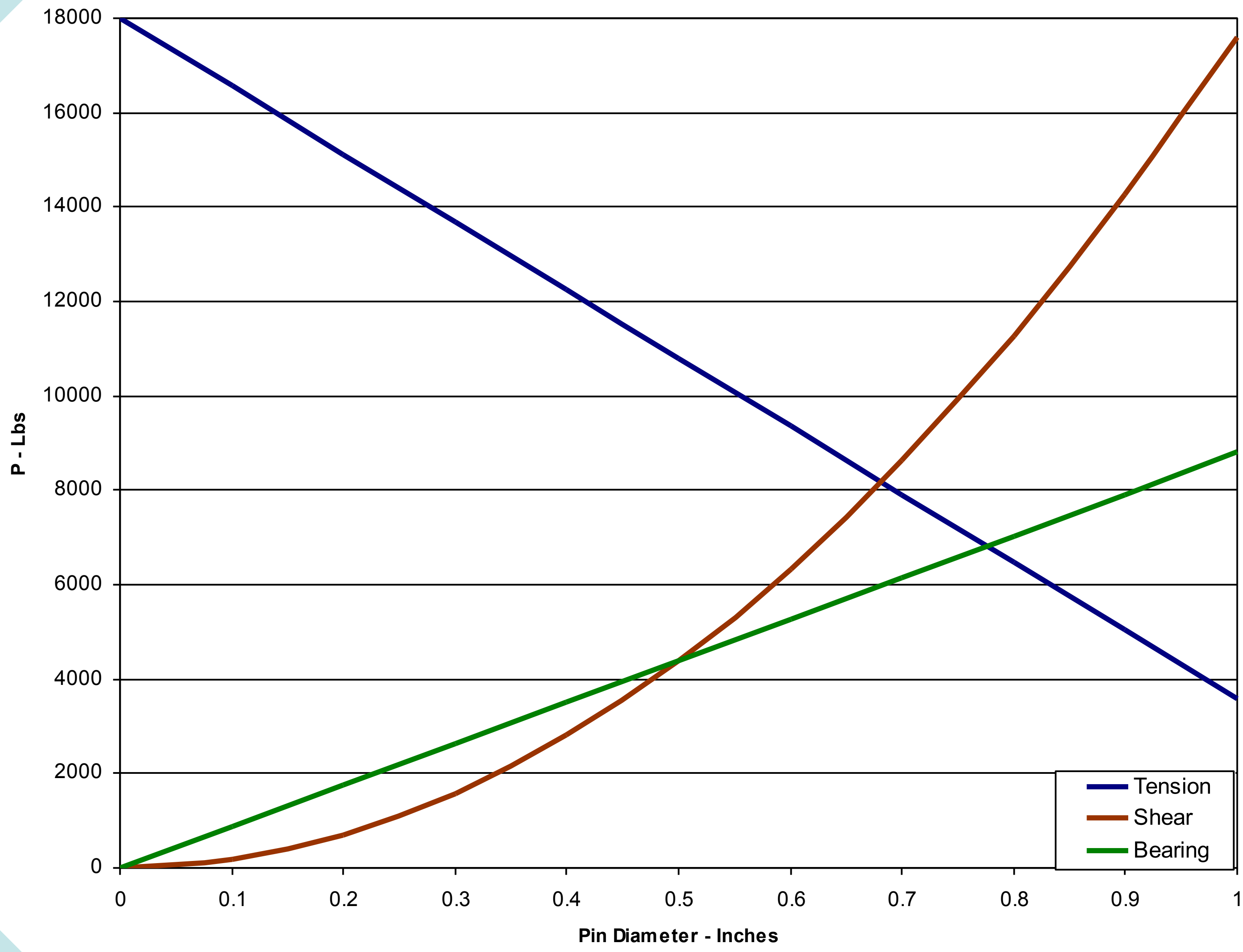
Tension-Bearing > Shear-Bearing

$$d = 0.776 \text{ in}$$

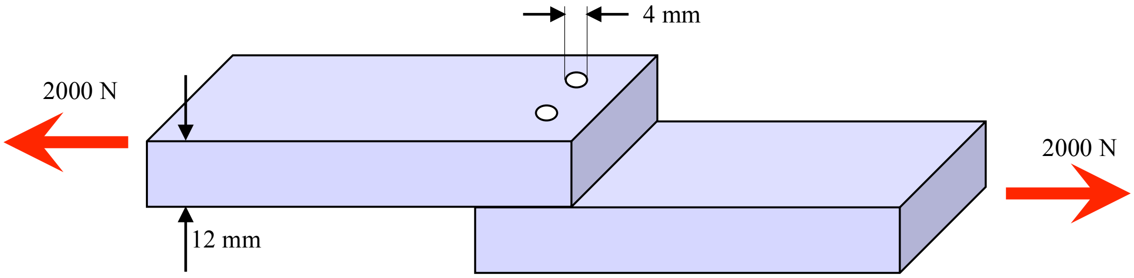
$$P = 6{,}830 \text{ lb}$$

A flat bar of width b = 2.5 inches and thickness t = 0.5 inch transmits an axial load P (see figure). Two holes of diameter d are drilled through the bar for pin supports. The ultimate tensile strength of the bar ($\sigma_{t,ult}$) is 36,000 psi on the net cross section of the bar, the ultimate shear strength of the pins ($\tau_{s,ult}$) is 14,000 psi, and the ultimate bearing stress between the bar and pins ($\sigma_{b,ult}$) is 22,000 psi. Using a factor safety of 2.50, determine the diameter of the pins for which the load P will be maximum. Find d and P.

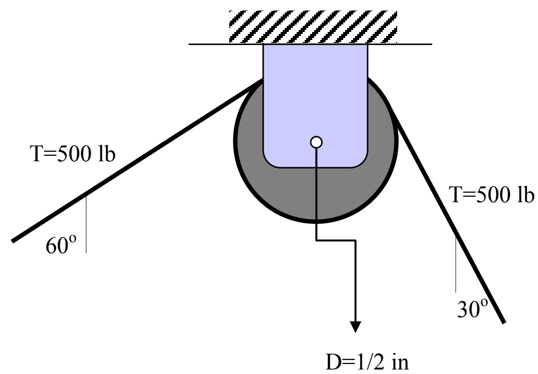
Example: Design Problem

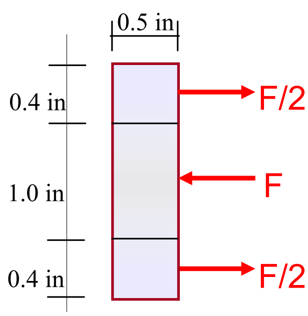


Shear: 1 Two 12-mm-thick steel plates are joined by two bolts, 4 mm in diameter. The tensile force applied is 2,000 N. (a) Evaluate the (average) shear stress in the bolts, and (b) evaluate the bearing stress on the steel plates.



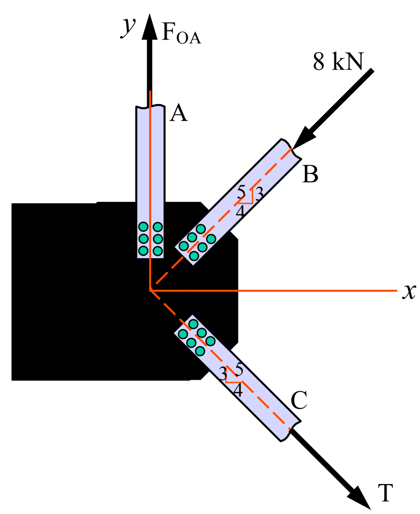
Shear: 2 The axle of a pulley is subjected to double shear as shown in the figure. The width of the pulley is 1 inch and the bracket is 0.4 inch thick. Find: (a) the shear stress in the axle, (b) the bearing stress in the bracket, and (c) the bearing stress in the pulley.



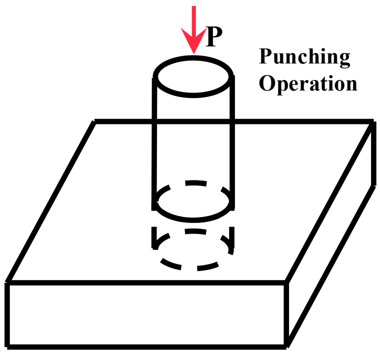


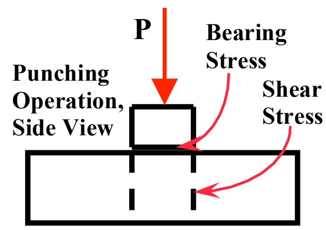
Pulley Axle under Double Shear

Shear: 3 A truss joint connecting bars OA, OB, and OC is shown. The connecting plate is 20 mm thick and the diameter of the connection bolts is D = 10 mm. Find: (a) the shear stress for each of the six bolts connecting bar OC to the joint and (b) the bearing stress on the plate at OC. Note that the bars are only on one side of the plate, thus the bolts are subjected to single shear.



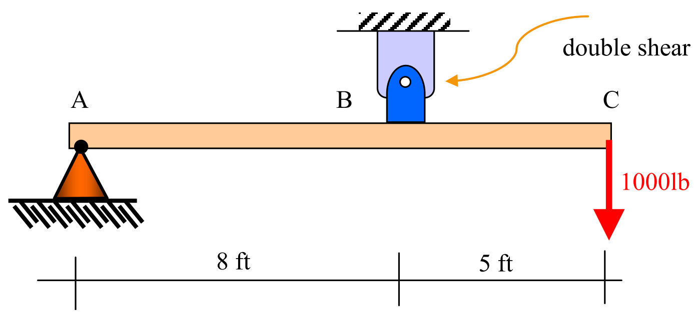
Shear: 4. Circular disks of diameter D = 4 inches are to be punched from a 1/4-inch-thick aluminum plate. The ultimate shear stress of the aluminum is 12,500 psi. Find the force required to punch the discs.



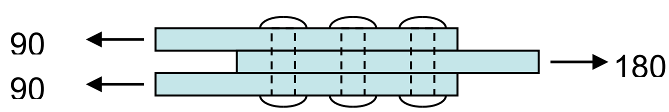


Shear: 5 The beam is pinned at A and at B the pin is subjected to double shear.

Find the diameter of pin B such that τ_{ave} =1,250 psi

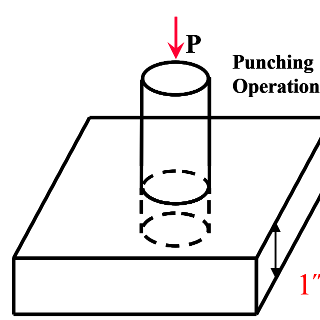


Shear: 6 Find the rivet diameter so that three rivets in double shear will support 180 kN given a rivet material with allowable shear stress of 100 MPa.

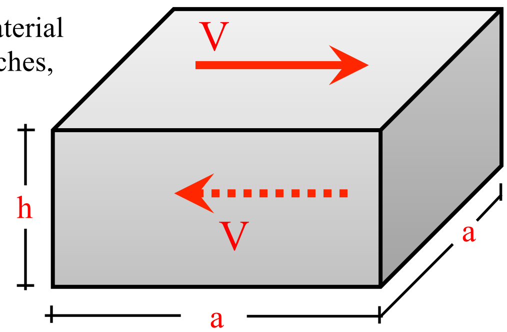


We assume that there is no free play so that each rivet bears an equal load.

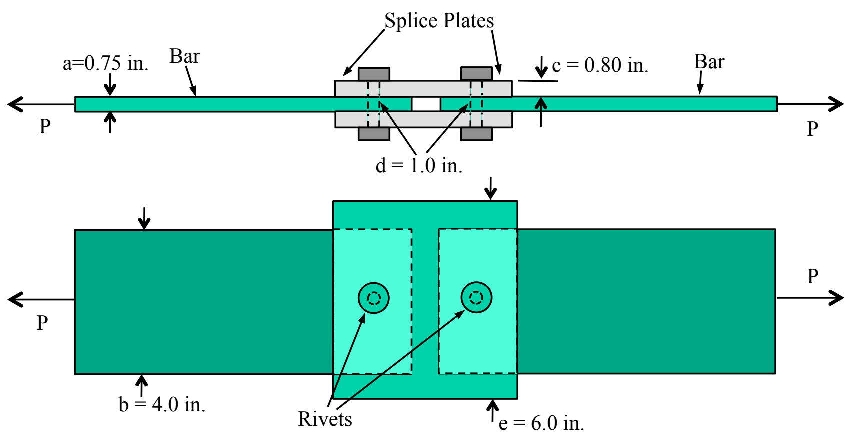
Shear: 7 A punching operation produces circular disks 1 inch in diameter from 1-inch-thick stock. Given that $\tau_{ult} = 1200\text{ psi}$, how much force is required to punch the disks?



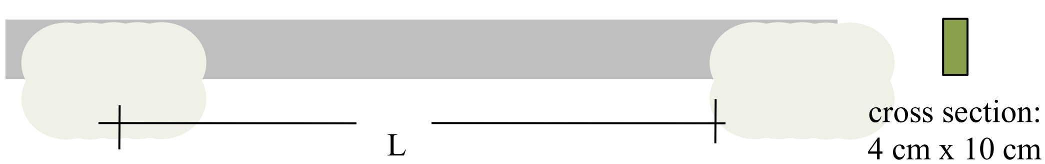
Shear: 8 A mechanical test is performed on a synthetic rubber material by imposing shear load on a pad like the one shown. For a = 3 inches, h = 1.5 inches, a force V = 100 lbs results in that the top part of the pad moves laterally by 0.0238 inch relative to the bottom plate. What is the shear modulus of the rubber, G? Note that a test like this is usually performed by bonding steel plates at the top and bottom of the pad and imposing the force on the plates; this is however not relevant to the solution of the problem.



Shear: 9 The two 0.75x4.0-inch tension bars are riveted together through two 0.8x6.0-inch splice plates, and rivets of 1.0-inch diameter. The allowable stress for the bars and splice plates is $\sigma_{all}^{bar,plate} = 20.0$ ksi, the allowable shear stress on the rivets is $\tau_{all}^{rivets} = 9.0$ ksi, and the allowable bearing stress on the rivets and splice plates is $\sigma_{all,bearing}^{rivet,plate} = 25.0$ ksi. Find the maximum load P such that none of the allowable stresses will be exceeded. Which part of this connection would you first replace to make it more efficient ?

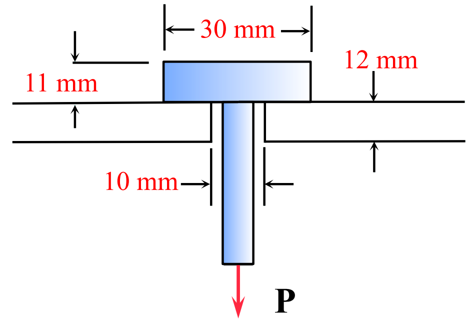


Shear: 10 The steel bar (E = 200 GPa) is of rectangular cross section (4x10 cm^2) and is to be bolted on a structure by 10 bolts on each side; the diameter of each bolt is 3.5 cm. Due to an error, the length L shown was 1.795 m instead of 1.80 m, thus there is a gap of 0.5 cm, which makes it difficult to bolt the bar on the structure. The engineer suggests a tensile force is imposed in order to close the gap. (a) Find the force F required to close the gap. After the gap is closed and the bar is bolted, F is released. (b) Find the shear stress on each bolt. (c) Find the bearing stress on the bar.

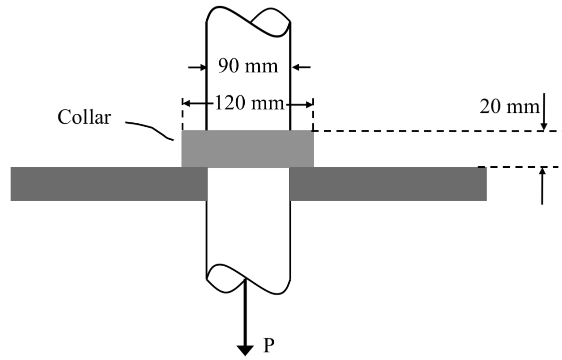


Shear: 11 The steel bar shown above (problem Shear 10) (E = 200 GPa, $\alpha = 11\text{x}10^{-6}/^{o}\text{C}$) is of rectangular cross-section (4cm x 10cm) and is to be bolted on a structure by 10 bolts on each side; the diameter of each is 3.5cm. Due to an error, the length L shown was 1.795 m instead of 1.80 m, thus there is a gap of 0.5 cm, which makes it difficult to bolt the bar on the structure. The engineer suggests the bar is heated up by - ΔT in order to close the gap. (a) Find the ΔT required to close the gap. After the gap is closed and the bar is bolted, the bar is brought back to its original temperature. (b) Find the shear stress on each bolt. (c) Find the bearing stress on the bar.

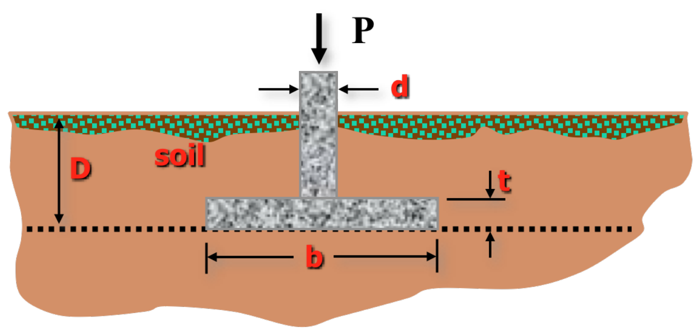
Shear: 12 A steel rod, connected to a steel plate of diameter 30 mm and thickness 11 mm as shown, is subjected to load P and supported by an aluminum plate 12 mm thick into which a 10 mm diameter hole has been drilled. The allowable shear stress in the aluminum is 65 MPa and in the steel 170 MPa. Further, the bearing stress between the steel and the aluminum is not to exceed 250 MPa. Determine the largest load P that can be applied. Ignore all weights.



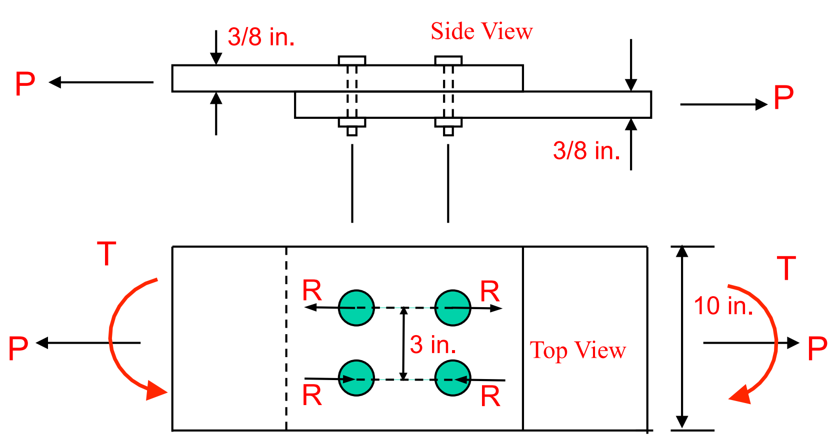
Shear: 13 A vertical cylindrical rod is supported by a collar and bearing plate, as shown. Determine the maximum axial load P that can be applied to the rod if the average punching shear stress in the collar and the average bearing stress between the collar and the plate are limited to 70 and 100 MPa, respectively.



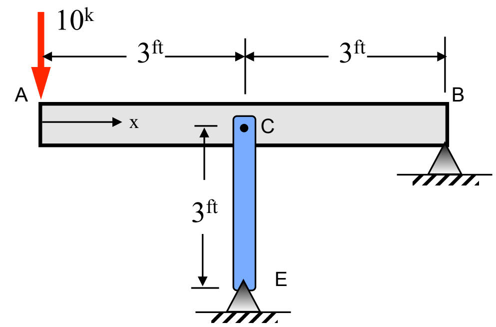
Shear: 14 A concrete column of square cross section ($d \times d$) is subjected to compressive load P and is connected to a square footing of dimensions $b \times b$ and thickness t. The footing rests on soil at depth D. Assume the bearing stress on the soil is uniform, and ignore the weight of the column and the footing and the soil. Let the allowable bearing stress on the soil be $\sigma_{\text{allowable}}^{\text{bearing}}$ and the allowable punching shear strength of the concrete be $\tau_{\text{allowable}}^{\text{shear}}$. (a) Find the load P that will create a bearing stress on the soil equal to $\sigma_{\text{allowable}}^{\text{bearing}}$. (b) Find the load P that will create a shear stress on the footing (due to the punching operation of the column on the footing) equal to $\tau_{\text{allowable}}^{\text{shear}}$. (c) Set the load P found in (a) equal to the load P found in (b) and find the relation between b, d, t for such a condition to hold if $\tau_{\text{allowable}}^{\text{shear}} = 15\sigma_{\text{allowable}}^{\text{bearing}}$.



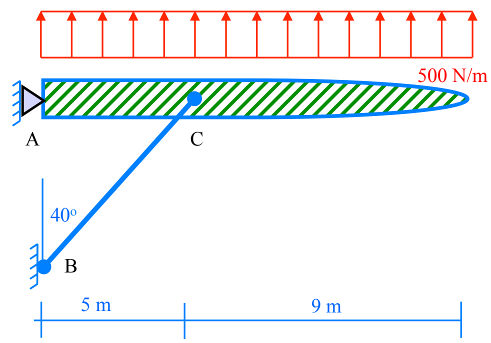
Shear: 15 Two plates of 10 in $\times$ 3/8 in cross section are connected by four bolts each of diameter d = ½ in as shown. A tensile force T = 2 kips and two horizontal couples T = 600 lb-in are imposed on the plates. Consider that the couples T are equilibrated by horizontal forces R on each bolt as shown. Find (a) the maximum shear stress on the bolts; (b) the maximum bearing stress on the plates.



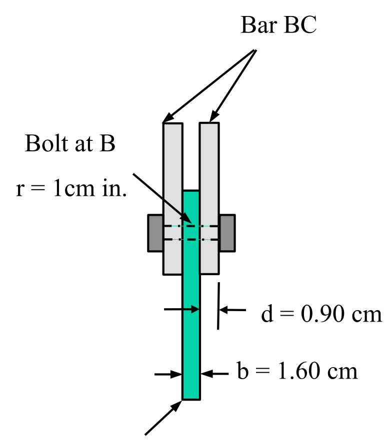
Shear: 16 Beam ACB has a square cross section 1.5 in. × 1.5 in. and column CE has a square cross section 1.0 in × 1.0 in. If the allowable stresses for the connecting bolt at C are $\tau_{all} = 24\,\text{ksi}$ and $\sigma_{all}^{bearing} = 30\,\text{ksi}$, determine the minimum diameter of the bolt required.



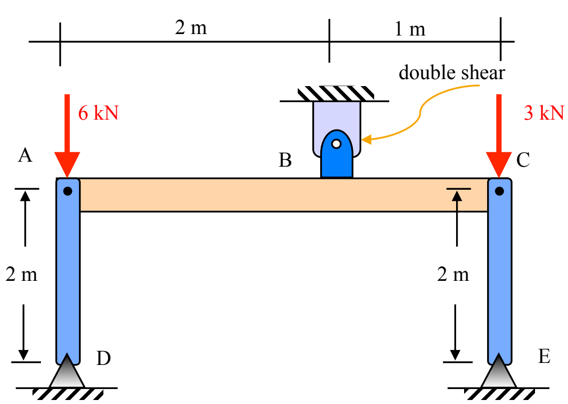
Shear: 17 The airplane wing is subjected to a uniform lifting force of 500 N/m over its entire length of 14 m. The wing is pinned at A and also supported by bar BC through pins at B and C. The pin at B is realized through a bolt of radius r = 1 cm that connects the two plates of thickness d = 0.90 cm each to the plate of thickness b = 1.60 cm, as shown schematically in the connection detail drawing. If the failure shear stress in the bolt is 80 MPa and the failure bearing stress in the bolt is 120 MPa, find the factor of safety in the connection for the 500 N/m load on the wing. Ignore the weight of the wing and bar BC.



CONNECTION DETAIL AT B



Shear: 18 __Rigid__ beam ABC is supported through a pin at B, and through bars AD and CE pinned at A,D and C,E, respectively. Bars AD and CE are identical, with cross-sectional area A = 800 cm^2, and elasticity modulus E = 200 GPa. A vertical load of 6 kN is applied at A and 3 kN at C. The pin at B is through a bolt that is subjected to double shear. Find the average shear stress on the bolt if its diameter is 4 cm.



Mechanical behavior of a shaft

Previously, the behavior of perhaps the simplest type of structural member (i.e., a bar under tensile or compressive load) was studied. Now the behavior of a torsional shaft is studied (i.e., twisting of a straight bar by a moment or torque). A typical example is the twisting of a screwdriver, where torque or moment is applied in order to drive a screw in or out. Note that the term "shaft" pertains to a bar subjected to torque. Torsion, like any force, will produce both stress and strain. However, unlike axial stress and strain, torsion causes a twisting stress, called shear stress (τ), and shear strain (γ). These are exactly as they were when studying shear stress and strain in previous modules. Thus, it is important to understand how such stresses and strains develop in a shaft subjected to torque T.

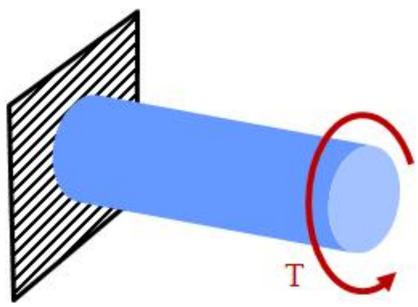


Figure F-1 A torsional shaft subjected to a torque T, also termed as moment, or couple.

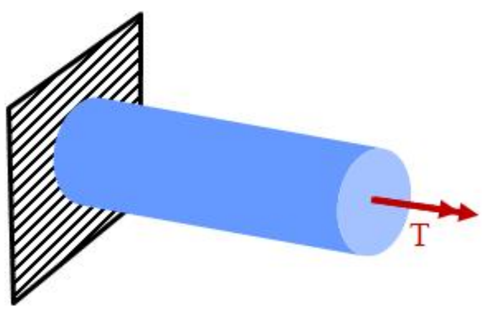


Figure F-2 Double arrow is used as the symbol for twisting moment or torque.

Thus, if we consider an infinitesimally small piece of the shaft of length dx (Figure F-3) the right end of that piece will twist relative to the left end. Based on the relative rotation of the right end with respect to the left end of the piece of length dx, relevant geometrical angles can be defined. In the following are the definitions for the relative angle, γ , the angle of rotation, ϕ, and the angle of twist per unit length, θ. These are illustrated over a small length of a torsional shaft, dx (Figure F-3). Line segment ab is parallel to the axis of the cylinder before the torque is applied. The torque moves line ab to ab'. Note that this is a relative motion of the right end with respect to the left end. In the following, definitions relevant to torsion problems are given.

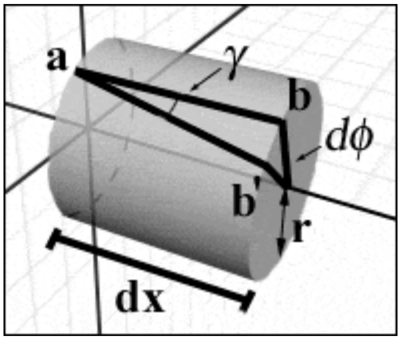


Figure F-3 Schematic of a small piece of a torsional shaft of infinitesimal length dx.

Definitions

$\gamma = \frac{bb'}{ab}$ is the so called relative angle, and since this angle is small, tan(γ) is almost equal to γ (expressed in radians). But, as is obvious from Figure F-3, bb' = r(dϕ), which indicates arc length. Actually, bb' = r tan(dϕ)≈ r (dϕ) holds. Further, ab = dx, therefore

$$\gamma = r\frac{d\varphi}{dx} = r\theta$$

where

$$\theta = \frac{d\varphi}{dx}$$

is the angle of twist per unit length. Note that γ is the change in an initially 90^o angle, which is precisely the shear strain defined in previous modules.

Uniform torsion

Consider the shaft shown in Figure F-4. Its length is L and it is subjected to torque T on one end. The other end is fixed. As is easily verified, the torque at every position of the shaft is T. Then the angle of twist per unit length is constant, thus $\frac{d\varphi}{dx} = \text{constant}$ holds

Figure F-4 Free body diagram of a shaft

Then $\theta = \frac{d\varphi}{dx} = \text{constant} \rightarrow \varphi = x\theta$ or for θ, which implies that for the total length L, $\theta = \frac{\varphi}{L}$ which implies that $\gamma = r\theta = \frac{r\varphi}{L}$ for T being constant along the bar. This case when T is constant along a shaft, is called **uniform torsion**. Nonuniform torsion is addressed later. Uniform torsion is in a sense similar to uniform tension or compression in a bar as examined in previous modules, which imply that the tensile or compressive load P is constant along the bar. Figure F-5 shows the shape a shaft subjected to uniform torsion attains after the torque is applied. The rate of change of the rotation is constant throughout the length of the shaft.

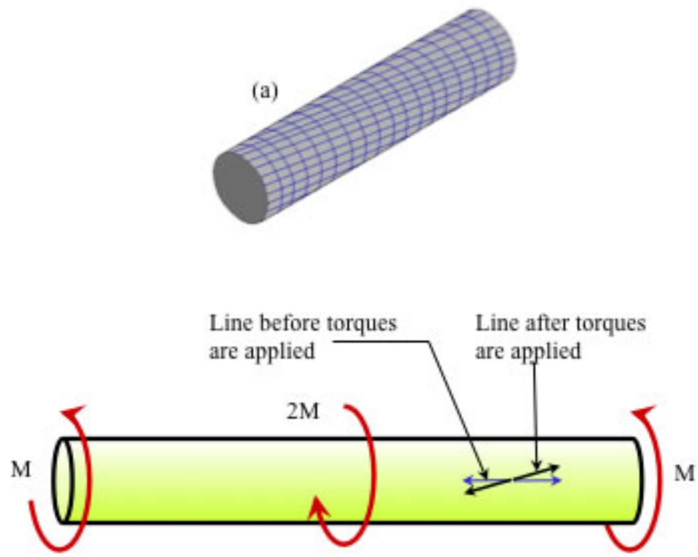


Figure F-5 Top (a) The deformed shape of a torsional shaft subjected to uniform torsion, and (b) a line before and after the torque is applied.

Uniform torsion requires that a shaft has the same cross section along its length, and the applied torque is constant along its length as well. If any of those is not the case, then the shaft is subjected to nonuniform torsion. This is examined later in this module, and here it is mentioned that nonuniform torsion is treated mathematically in the same way non-uniform bars were treated.

Hooke's law in shear

As mentioned before, normal stress in bars creates normal strain as well. Accordingly, normal strain creates normal stress. Similarly, shear stress creates shear strain, defined as γ (see previous modules on shear stress), and vice versa. It turns out that the shear stress created in a torsional shaft is the same (shear force per unit area) as the shear stress created in direct shear, such as in bolts examined previously. Thus we have a relation between the shear stress τ and shear strain γ that reads:

$$\tau = G\gamma$$

where G is the shear modulus of elasticity. At an arbitrary point A in the cross section, $\gamma = \varrho\theta$ and $\tau = G\varrho\theta$. This implies that τ varies linearly with ϱ, as is illustrated in Figures F-6 F-7. Note that the symmetry of shear stresses implies that a shear stress acting on the cross section of a shaft also creates a shear stress in planes transverse to the cross section, as shown in Figure F-7. These shear stresses tend to "slice" the shaft along its length. In wood shafts, where typically the grain is in the direction of the shaft, such shear stresses can result in cracks along the length of the shaft since the direction transverse to the wood grain is typically weak in shear.

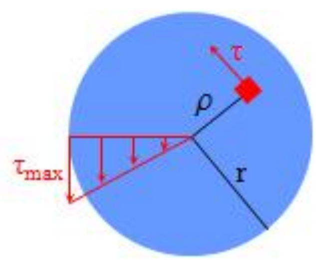


Figure F-6 Distribution of shear stress τ along the radius of the cross section of a shaft subjected to torque T.

While for a solid circular cross section ϱ varies between zero and r (r denoting the radius of the circle) and the maximum shear stress occurs for $\varrho = r$. Further, for $\varrho = 0$ the shear stress is zero, i.e., at the center of the cross section.

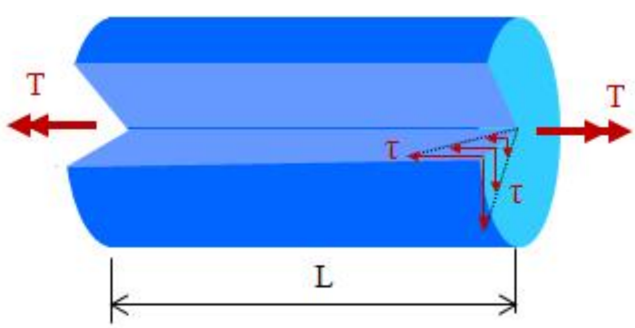


Figure F-7 Schematic of shear stress distribution in a circular shaft. A cut in the shaft is shown for illustration of the shear stresses. The shear stress on the cross section also implies shear stress on planes transverse to the cross section.

Consider the shaft shown in Figure F-8, of circular cross section, subjected to external torque T. At each material point (area dA), $\tau = G\varrho\theta$ holds. Then the moment that τ produces, with respect to the center of the cross section ,is expressed as

$\tau(dA)\rho = G\rho\theta\rho\, dA = G\rho^2\theta\, dA$

The total torque is the integral of the moments over the whole cross section, and this is expressed mathematically as

$$T = \int_A G\rho^2\theta dA = G\theta\int_A \rho^2 dA$$

This relation can be written as

$$T = G\theta I_P$$

where

$$I_P = \int_A \rho^2\, dA$$

I_P is termed the polar moment of inertia, and it is a property of the cross section. For a circular cross section or radius r (diameter d = 2r), the polar moment of inertia is given by

$$I_p = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$$

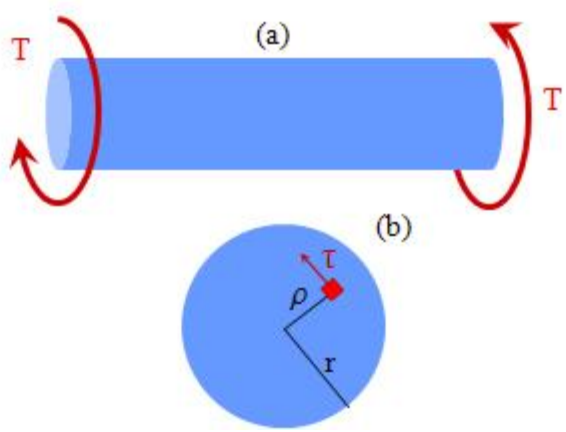


Figure F-8 (a) A shaft subjected to torque T, and (b) cross section of the shaft and an element of area dA.

We have then for the torsion problem that $\theta = \frac{\varphi}{L} \rightarrow \varphi = \theta L$, and also $\theta = \frac{T}{GI_P}$ holds. Then it follows that

$$\varphi = \frac{TL}{GI_P} \quad \text{or} \quad T = \frac{GI_P}{L}\varphi$$

The product GI_P is termed "torsional rigidity" and GI_P/L is termed the "torsional stiffness." Note that a shaft acts like a rotational (torsional) spring where the moment (T) is proportional to the rotation (ϕ). The spring constant is equal to the torsional stiffness of the shaft.

The following terminology applies to torsional problems:

GI_p/L: torsional stiffness, **$L/(GI_p)$**: torsional flexibility

The maximum shear stress is of interest to engineers, which can be found for ϱ = r. Thus,

$$\tau_{max} = \frac{Tr}{I_P}$$

holds, while at any point in the cross section we have

$$\tau = \frac{T\rho}{I_P}$$

Note that the shear stress at the center line of the shaft is zero since ϱ = 0 at the center.

Example-hollow cylindrical shaft: For optimum use of material and resources, engineers seek to maximize the torsional rigidity with respect to cross sectional area. It turns out that hollow cylindrical shafts are very effective in this regard (see Figure F-9). As mentioned previously, the shear stress at the center of a cross section, where ϱ = 0 is zero, and increases linearly with increasing ϱ. Thus, it is efficient to have hollow cylindrical cross sections. Hollow cross sections are efficient, yet they cannot be too thin since this may create local buckling of the cross section. Buckling phenomena are examined later in this course. In this example, let the outside diameter of the cross section be d_2 = 100 mm, the inside diameter be d_1 = 70 mm, and the torque applied to the cross section be T = 7000 Nm, which is an external torque applied onto the shaft.

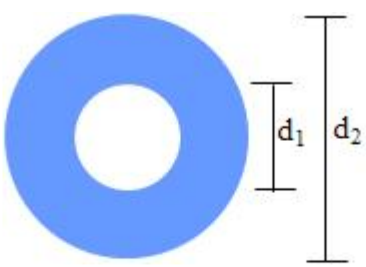


Figure F-9 A hollow cylindrical cross section of a shaft.

The shear stress τ is evaluated from $\tau = \dfrac{T\rho}{I_p}$. The polar moment of inertia of the cross section is evaluated as

$$I_P = \frac{\pi}{32}\left(d_2^4 - d_1^4\right) = 7.460 \times 10^6\ \text{mm}^4$$

Stresses, τ_1 at the inner and τ_2 at the outer part of the cross section are now evaluated.

$$\tau_1 = \frac{(7000\,\text{Nm})(35\,\text{mm})}{7.460 \times 10^6\,\text{mm}^4} = 32.8\,\text{MPa}$$

$$\tau = \frac{(7000\,\text{Nm})(50\,\text{mm})}{7.460 \times 10^6\,\text{mm}^4} = 46.9\,\text{MPa}$$

For this hollow cylindrical shaft, the shear stress distribution along the radius is shown in Figure F-10.

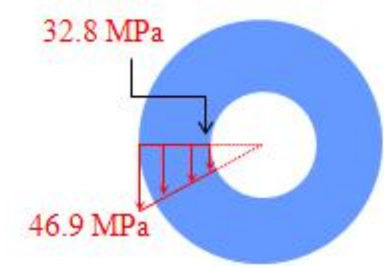


Figure F-10 Shear stress distribution for the hollow cylindrical shaft.

The "equivalence" of a bar and a shaft

So far the stresses created in a bar and in a shaft have been examined (Figure F-11). Physically, they are very different, since tensile stresses are created in a bar while shear stresses are created in a shaft. However, from the mathematical point of view there are similarities, as shown in the following table, which also summarizes all the equations for bars and shafts. Even though nonuniform shafts have not been addressed yet, the mathematical similarities should help in understanding them. Nonuniform shafts are presented in the section immediately following this one.

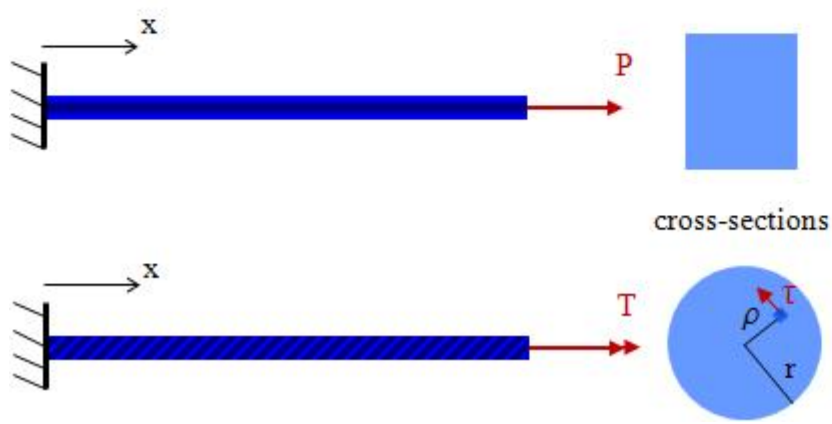


Figure F-11 A bar (top) subjected to force P and a shaft (bottom) subjected to a torque T.

Table showing relations for bars and shafts. Mathematical similarities can be identified.

Shaft	**Bar**	Terminology
$\gamma = \rho\theta = \frac{\rho\varphi}{L}$ $\gamma_{max} = r\theta = \frac{r\varphi}{L}$	$\varepsilon = \frac{\delta}{L}$	relative angle (shear strain)/normal strain
$\tau = G\gamma$	$\sigma = E\varepsilon$	Hooke's law shear/tension
$\tau = \frac{T\rho}{I_p}$ $\tau_{max} = \frac{Tr}{I_p}$	$\sigma = \frac{P}{A}$	shear/normal stress
$\varphi = \frac{TL}{GI_p}$	$\delta = \frac{PL}{EA}$	relative rotation/elongation

$\varphi = \sum_{i=1}^{n} \frac{T_i L_i}{G_i I_{p_i}}$	$\delta = \sum_{i=1}^{n} \frac{P_i L_i}{E_i A_i}$	nonuniform bar/shaft
$\varphi = \int_0^L \frac{T(x)dx}{GI_p(x)}$	$\delta = \int_0^L \frac{P(x)dx}{EA(x)}$	continuously nonuniform bar/shaft

Module 12: Nonuniform Shafts

Nonuniform torsion

The previous discussion was for uniform torsion, i.e., for a shaft of constant cross section, of the same material throughout, subjected to constant torque T. There are many reasons for such uniformity to be invalid. When the cross section of the shaft changes along its length, either step-wise or continuously, or when the material along the length of the shaft changes, we have non-uniform torsion. It is similar to nonuniform bars, in the sense that similar mathematical formulas hold for the analysis of nonuniform shafts as for nonuniform bars. Figure F-12 shows a case of step-wise uniformity and a continuously nonuniform shaft.

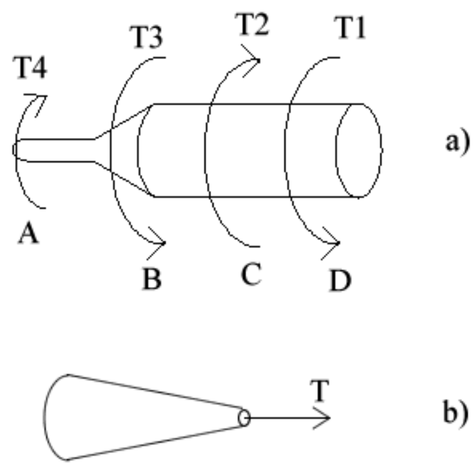


Figure F-12 (a) A shaft of piece-wise constant cross section subjected to torques T1, T2, T3, T4; and (b) a shaft of continuously varying cross sections subjected to constant torque T.

For a piece-wise uniform shaft the total relative angle of twist can be evaluated as

$$\phi = \sum_{i=1}^{n} \frac{T_i L_i}{G_i I_{Pi}}$$

where i = 1,2,...,n denotes a piece of the shaft, in a shaft consisting of n pieces. For example, the shaft in Figure F-12(a) consists of three pieces, AB, BC, and CD. The torque in AB, using equilibrium, is T4, in BC it is T4 – T3, and in CD it is T4 – T3 + T2 or -T1. Note that equilibrium of the entire shaft implies that T4 – T3 + T2 – T1 = 0.

Continuously nonuniform shafts are examined in the following section. In the remainder of this section is an example illustrating the analysis of piece-wise uniform shafts.

Example of piece-wise uniform bar: A shaft of piece-wise constant circular cross section is subjected to the torques shown in Figure F-13. The problem here is to: (a) find T_b so that the twist at C is zero, and (b) calculate τ_{max} for member AB. The material in the entire shaft is the same and of shear modulus G.

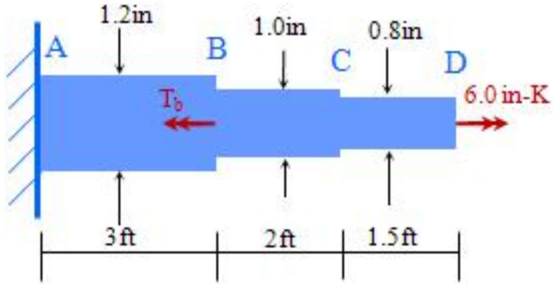


Figure F-13 A piecewise uniform shaft subjected to two external torques as shown.

We use the formula for piece-wise uniform torque. In order to find the torque in each piece, consider equilibrium of part of the shaft. For example, using section α-α, we have the following free body diagram (Figure F-14). As is obvious, T_{BC} = 6 in-K. Similarly, T_{AB} = 6 in-K-T_b. Then in order to find the rotation at C, designated here as ϕ_C, the relative rotations ϕ_{AB} and ϕ_{BC} need to be added. Thus,

$$\phi_C = \phi_{AB} + \phi_{BC} = \frac{T_{AB}L_{AB}}{GI_{P_{BC}}} + \frac{T_{BC}L_{BC}}{GI_{P_{BC}}} = \frac{1}{G}\left(\frac{(6.0-T_b)36}{\frac{3.14\times1.2^4}{32}} + \frac{6.0\times1.0}{\frac{3.14\times1.0^4}{32}}\right) = 0$$

which yields T_b = 6.34 in-K. With this value for T_b, the torque in member AB is T_{AB} = 6.0 – 6.34 = –0.34 in-K. Then, the maximum shear stress in AB can be calculated as

$$\tau_{max}^{AB} = \frac{16\times300}{3.14\times0.15^3} = 452.0\,\text{kPa}$$

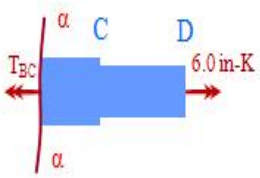


Figure F-14 Equilibrium implies that T_{BC} = 6 in-K, and that is the torque in shaft BC.

Example of piece-wise uniform bar: The stepped shaft ABCD (Figure F-14) consisting of solid circular segments is subjected to the torques shown at points C and D. The material is steel with G = 80x10^6 kPa. (a) Calculate the maximum shear stress in the shaft. (b) Calculate the angle of twist (in degrees) at point C. In the figure, d denotes diameter.

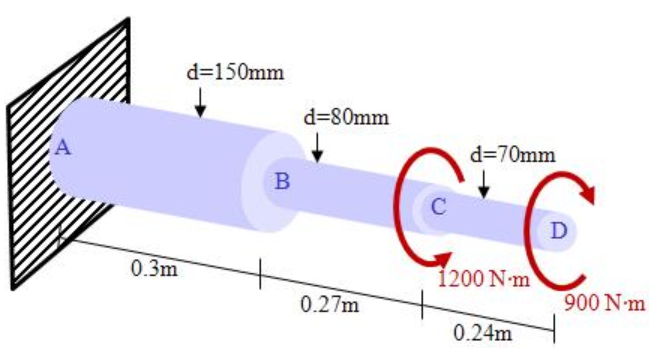


Figure F-14 A stepped shaft subjected to two torques as shown.

Torques in each part are T_{AB} = 300 Nm, T_{BC} = 300 Nm, T_{CD} = 900 Nm. It is noted that the sign of the torques does not matter for finding extreme shear stresses. The maximum shear stress for the entire shaft is the in either AB, BC, or CD for a **solid shaft**,

$$\tau_{max} = \frac{Tr}{I_p} = \frac{T\frac{d}{2}}{\frac{\pi d^4}{32}} = \frac{16T}{\pi d^3}$$

Then

$$\tau_{max}^{AB} = \frac{16 \times 300}{3.14 \times 0.15^3} = 452.0\,\text{kPa} \quad \tau_{max}^{BC} = \frac{16 \times 300}{3.14 \times 0.08^3} = 2,985.7\,\text{kPa}$$

$$\tau_{max\&}^{CD} = \frac{16 \times 900}{3.14 \times 0.07^3} = 13,370.0\,\text{kPa}$$

The above imply clearly that the maximum shear stress in the entire shaft is 13,370.2 kPa. In order to find the rotation at C, designated here as ϕ_C, the relative rotations ϕ_{AB} and ϕ_{BC} need to be added. Thus,

$$\phi_C = \phi_{AB} + \phi_{BC} = \frac{T_{AB}L_{AB}}{GI_{P_{BC}}} + \frac{T_{BC}L_{BC}}{GI_{P_{BC}}} = \frac{300}{80 \times 10^9}\left(\frac{0.3}{\frac{3.14 \times 0.15^4}{32}} + \frac{0.27}{\frac{3.14 \times 0.08^4}{32}}\right) = 0.000274\,\text{rad}$$

Module 12: Nonuniform Shafts

If the cross section of a shaft or the applied torque varies continuously along its length, the summation formula applicable to stepped shafts does not apply. Instead, the summation formula becomes an integral. Thus, for a shaft as that shown in Figure F-16, since the torque T and polar moment of inertia I_p are a function of x, they are expressed as T(x) and $I_p(x)$, respectively. They can be considered constant over an infinitesimal distance dx, thus, the following holds for the relative rotation over dx

$$d\varphi = \frac{T(x)dx}{GI_P(x)}$$

Integrating this relation over the entire length of the shaft L yields

$$\varphi = \int_0^L d\varphi = \int_0^L \frac{T(x)dx}{GI_P(x)}$$

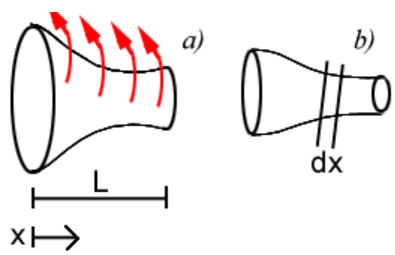


Figure F-16 A continuously nonuniform torque problem. (a) Shaft of non-uniform cross section and torque, and (b) a small chunk of length dx.

Example- continuously non-uniform torsion: In this example, we derive a formula for the angle of twist ϕ, of a thin tube as that in Figure F-17, when torques T act at the ends. Note that for a thin tube of thickness, t, and diameter, d, the polar moment of inertia can be approximated as

$$I_p \simeq \frac{\pi d^3 t}{4}$$

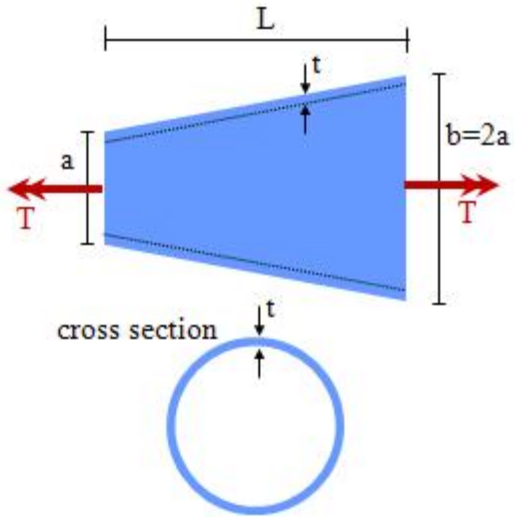


Figure F-17 A hollow shaft of length L and continuously varying cross section subjected to end torques T. The cross section is such that the thickness t is small compared to the radius. The cross section of the shaft at a specific point in the longitudinal direction of the shaft is shown.

In order to solve this problem, it is required to find the expression of I_p as a function of the coordinate x along the length of the shaft. In order to do that, it helps to draw the schematic shown in Figure F-18. Based on that, the diameter as a function of x, d(x) is expressed as $d(x) = \frac{a}{L}x$

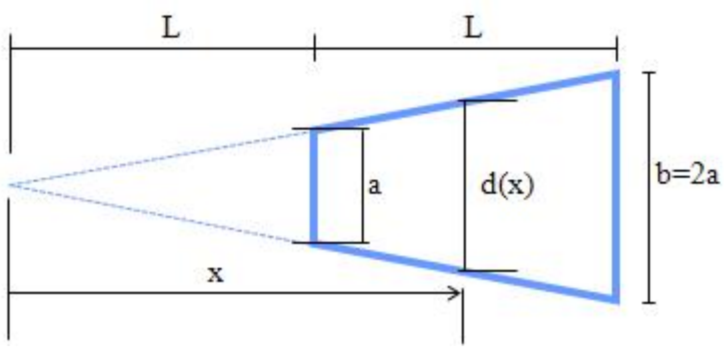


Figure F-18 Schematic of the shaft in Figure F-17 showing a side view and te tip of the cone the shaft is part of. This schematic helps in finding the expression for the diameter of the shaft as a function of x.

Since the torque is constant along the length of the shaft, we have

$$d\varphi = \frac{T(dx)}{GI_P(x)}, \quad I_p \simeq \frac{\pi (d(x))^3 t}{4} = \frac{\pi t a^3}{4L^3}x^3$$

which yield

$$\varphi = \int_L^{2L} \frac{T}{GI_p(x)}dx = \frac{4TL^3}{\pi G t a^3}\int_L^{2L}\frac{dx}{x^3} = \frac{3TL}{2\pi G t a^3}$$

Example-continuously non-uniform torsion: Figure F-19 shows a shaft of length L and constant along its length torsional rigidity GI_P subjected to distributed torque q, which is torque per unit length, and fixed at one end. Here we derive a formula for the relevant rotation ϕ at the free end.

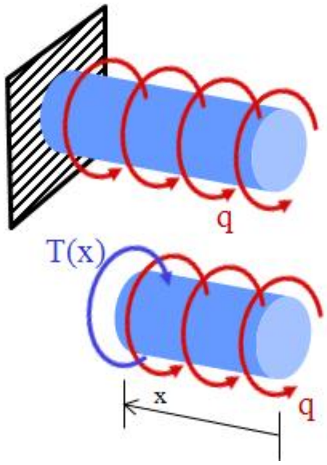
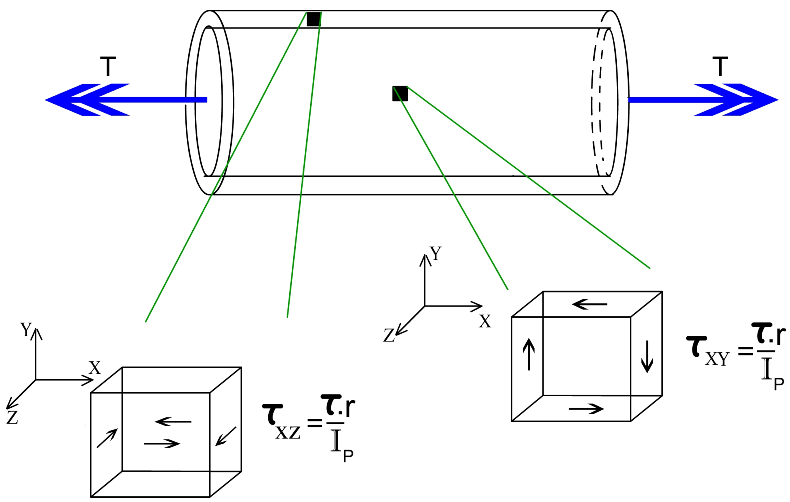


Figure F-19 A shaft of constant G and I_p subjected to distributed torque q, and the free body diagram of a piece of the shaft of length x.

Consider equilibrium of a piece of the shaft of length x, as shown in Figure F-19. Equilibrium of such a piece implies that T(x) = qx, where T(x) is the torque at the left end of the shaft in Figure F-19. We then have

$$\varphi = \int_0^L \frac{T(x)dx}{GI_P} = \frac{q}{GI_P}\int_0^L x dx = \frac{qL^2}{2GI_P}$$



The stress τ in a shaft subjected to torque T is not uniform. Thus, as shown in the "movie" above, the stress is not in the same direction at every material element in the shaft. Also, for a coordinate system where the x-direction is along the shaft's length, the stress is designated as τ_{xy} or as τ_{xz}

Pure shear, and shear modulus of elasticity G

Consider a shaft as in Figure F-20, and an infinitesimal material element *abcd*. The external torque creates shear stresses on this element, and shear stress symmetry dictates that shear stress τ acts on all sides of *abcd* as shown in Figure F-21(a).

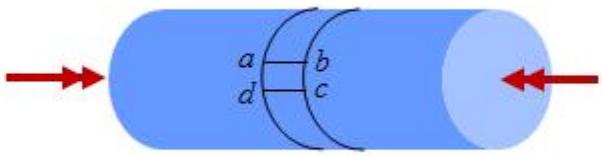


Figure F-20 A shaft subjected to torque, and a material element, abcd, on its surface.

Let us also consider that element *abcd* has constant thickness. Figure F-21(b) shows a section, α-α, in the element at an arbitrary angle θ. Isolating one part on one side of α-α, and given that the thickness of the wedge-shaped chunk of material is constant, we have the trigonometric expressions for the area of each side (shown in Figure F-21(b)) if the area of the vertical edge is A_0.

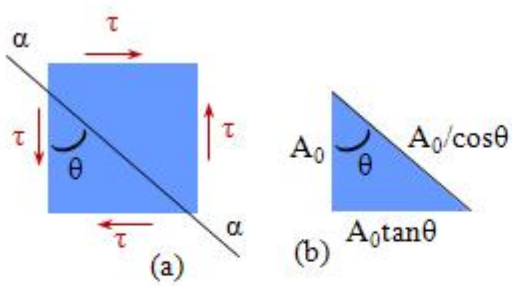


Figure F-21 (a) Material element *abcd* and the shear stresses acting on its four sides. A section is made at an arbitrary angle θ. (b) The chunk of material on one side of section α-α made on *abcd*.

The free-body diagram of the wedge in Figure F-21 is shown in Figure F-22. Note that the stresses are multiplied by the respective areas they act on, in order to obtain forces. On the inclined plane, a normal stress σ_θ and a shear stress τ_θ are considered, where subscript θ denotes the plane those stresses act on.

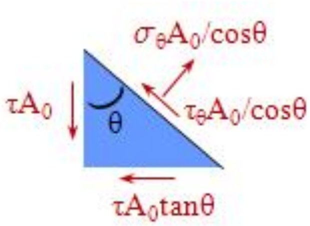


Figure F-22 The free body diagram of the wedge shown in the previous figure.

Equilibrium of the free body diagram in Figure F-22 implies that (a) $\sum F$ (in the direction of σ_θ) = 0 and (b) $\sum F$ (in the direction of τ_θ) = 0. These can be written as

(a) $\sigma_\theta A_0 \dfrac{1}{\cos\theta} = (\tau A_0)\sin\theta + (\tau A_0 \tan\theta)\cos\theta$

(b) $\tau_\theta A_0 \frac{1}{\cos\theta} = (\tau A_0)\cos\theta - (\tau A_0 \tan\theta)\sin\theta$

The term A_0 cancels on both sides, and using trigonometric identities, it ends up that

$\sigma_\theta = \tau\sin2\theta$; $\tau_\theta = \tau\cos2\theta$

Interestingly, for $\theta = 45^o$, we have $\sigma_{45} = \tau$, and $\tau_{45} = 0$. Repeating the process for $135^o$and 205^o, we obtain the stress state graphically shown in Figure F-23.

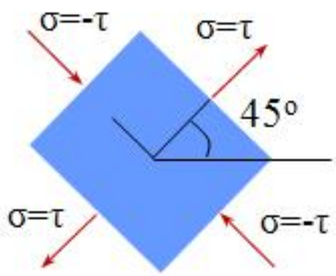


Figure F-23 The state of stress at 45^o inclined planes.

Let us consider the so-called pure shear state of stress, i.e., a material element subjected to shear stresses only. Figure F-24 shows such a material element subjected to τ, where the symmetry of shear stress implies all four sides are subjected to τ as shown. Also in this figure is shown the element after the shear stress is applied. Note that the initially square element becomes a rhombus, and the decrease/increase of the initially 90^o angles denotes the shear strain γ, expressed in radians. Figure F-25 shows a material element, embedded in the material element of Figure F-24, specifically at 45^o to that element, before and after deformation. Transformation of stress as noted before implies that the stresses shown in Figure F-25 are applied on this material element. With these stresses, the initially square element becomes a rectangle, elongated along the tensile stress and contracted along the compressive stress.

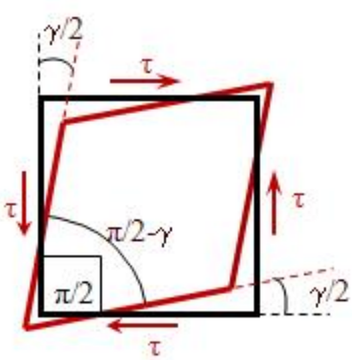


Figure F-24 The initially square material element subjected to τ before (black/dark) and after (red/light line) the stress is applied.

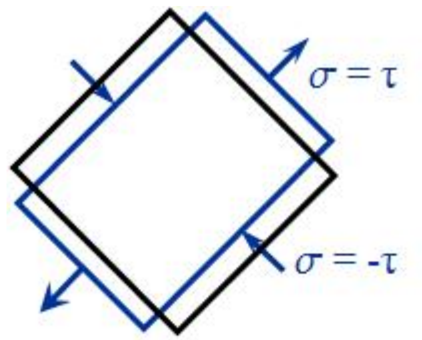


Figure F-25 The initially square material element at 45^{o}subjected to σ and -σ before (black/dark line) and after (blue/light line) the stress is applied.

The two elements shown above are at the same material point, i.e., one is embedded in the other. Figure F-26 shows them as they are supposed to be, i.e., at the same material point. Notice the triangle ABC formed drawn separately in Figure F-27.

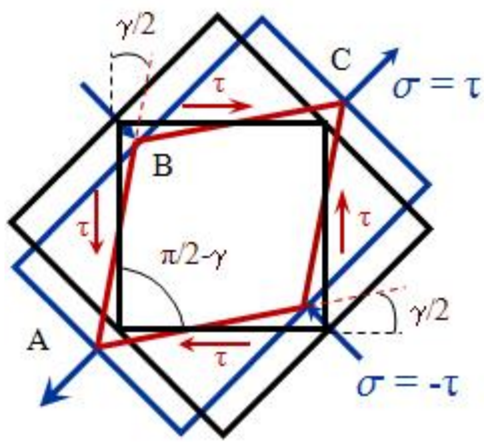


Figure F-26 The material elements of Figures F-24 and F-25 shown together, as they are (at the same material point).

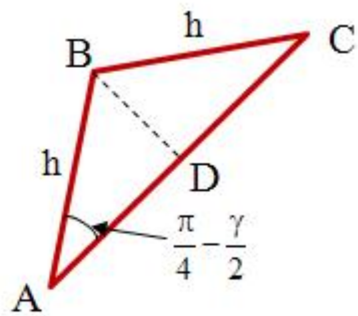


Figure F-27 Triangle ABC, same as the one in Figure F-26, drawn separately.

Let h denote the length of the rhombus (Figs. F-24, F-26). From the state of stress shown in Figure F-24,

$$\gamma = \frac{\tau}{G}$$

For the state of stress shown in Figure F-25, we have superposition of normal strain and strain from Poisson effects, or

$$\varepsilon = \frac{\tau}{E} + \frac{\nu\tau}{E}$$

Let δ_{AC} denote the change in length of AC during deformation and L_{AC} denote the final length of AC (after deformation). The original (before deformation) length of AC is equal to $\sqrt{2}$ h. Then,

$$\delta_{AC} = \left(\sqrt{2}h\right)\text{epsilon}; \Rightarrow L_{AC} = \sqrt{2}h + \delta_{AC}$$

Also from triangle ABD, it follows that

$$\cos\left(\frac{\pi}{4} - \frac{\gamma}{2}\right) - \frac{L_{AC}}{2h} = \frac{\sqrt{2}h + \sqrt{2}h\varepsilon}{2h}$$

Using the trigonometric identity $\cos(a - b) = \cos a \cos b + \sin a \sin b$, and the fact that for γ small enough, $\cos(\gamma/2) \approx 1$, and $\sin(\gamma/2) \approx \gamma/2$, it follows that $\varepsilon = \gamma/2$, which finally implies that

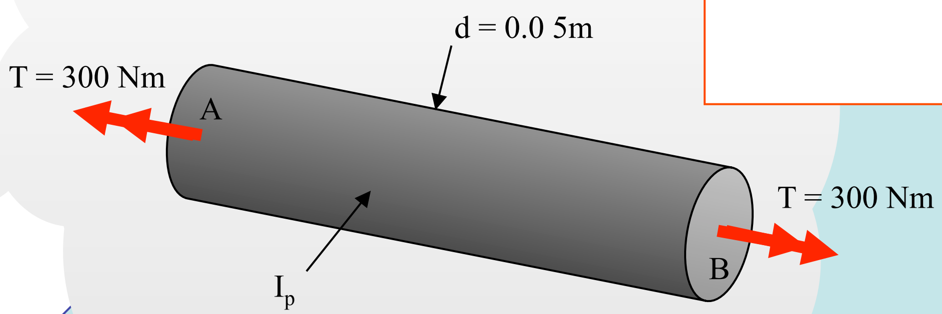
$$G = \frac{E}{2(1+\nu)}$$

Example: Shear Stress and Strain in a Steel Shaft

A torque of 300 Nm is applied to a 0.05 m diameter steel shaft.

Find τ_{max}, γ_{max}

Consider $G = 70\text{x}10^9$ Pa



$$I_p = \frac{\pi}{32}(0.05)^4 = 6.13592 \times 10^{-7}\,\text{m}^4$$

$$\tau_{max} = \frac{T \cdot r}{I_p} = \frac{300 \times (0.05/2)}{6.13592 \times 10^{-7}} = 12.22\,\text{MPa}$$

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{1.222 \times 10^{7}}{70 \times 10^{9}}$$

$$= 174.6 \text{ micro strain}$$

Example: Torque in a Hollow Tube

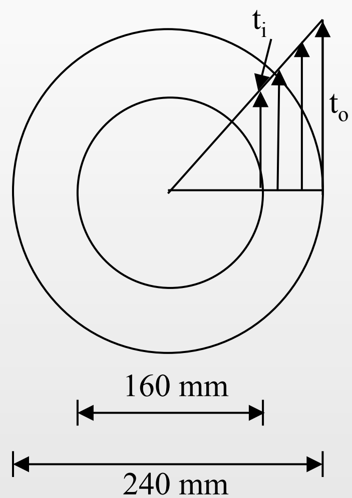
$$\tau = \frac{T\rho}{I_p}$$

$$I_p = \frac{\pi}{2}\left[(0.12)^4 - (0.08)^4\right] = 2.614\times10^{-4}\,m^4$$

$$\tau_i = \frac{50\times0.08}{2.614\times10^{-4}} = 15302.2\ N/m^2\ (Pa)$$

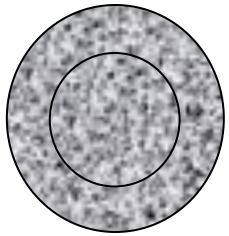
$$\tau_o = \frac{50\times0.12}{2.614\times10^{-4}} = 22953.3\ N/m^2\ (Pa)$$

$$= 22.953\ KPa$$



A torque of 50 Nm is applied to a tube of 120 mm outer diameter and 80 mm inner diameter.

Evaluate shear stresses at the inner and outer diameter material points.



Note: Inner half of a solid shaft accounts for (1/16) of the applied torque!

Example: Circular Shaft with Distributed Torque

$$\varphi = \int_0^L \frac{T(x)}{GI_p} dx$$

$$T(x) = q\,x$$

$$\varphi_{\text{middle}} = \int_2^4 \frac{1}{GI_p} q\,x\,dx$$

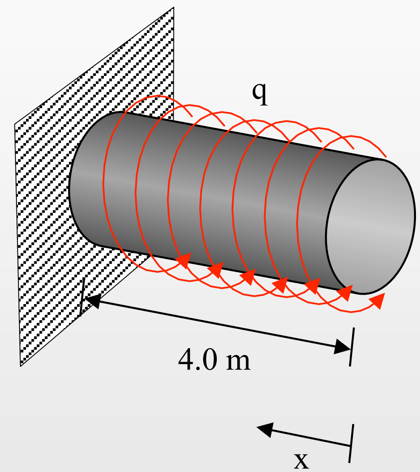
$$= \frac{4}{28 \times 10^9 \times 10^{-5}} \int_2^4 x\,dx$$

$$= 1.4286 \times 10^{-5} \times \frac{1}{2}\left[x^2\right]_2^4$$

$$= 7.14286 \times (16 - 4)$$

$$= 8.5714 \times 10^{-5} \text{ rads}$$

$$= 4.914 \times 10^{-3} \text{ degrees}$$



The shaft of circular cross-section is subjected to a distributed torque per unit length q. Find the rotation φ at the middle of the bar.

Given:

$q = 4.0$ kN.m/m

$G = 28$ GPa

$I_p = 1 \times 10^{-5}$ m^4

Self Assessment

Mechanical Behavior of a Shaft: 1

For the same cross-sectional area and the same torque, which will be more efficient (least maximum shear stress)?

- ○ Circular hollow
- ○ Circular Solid
- ○ Triangular Solid
- ○ Square Solid

Mechanical Behavior of a Shaft: 2

A 70-cm-long, 2-cm-diameter manual steel drill (G=83 GPa) is operated by applying two opposite forces by the two hands with a moment arm of 25 cm for each force. What is the maximum allowable force, in kN, that can be safely applied so that a relative angle of twist of 2.5° is not exceeded in the shaft?

- ☐ From the T vs θ relation, for a 2.5 degrees relative twist the torque required can be found. Is answer 5,6 or 7 correct?
- ☐ Torque T vs angle of twist θ relation. Is 1 or 2 correct?
- ☐ From the T versus P relation, the required force P is found. Is 8, 9,1 0 or 11 correct?
- ☐ Torque T vs force P relation. Is 3 or 4 correct?

1. The applied torque T is linearly related to the angle of twist θ

2. The applied torque T is non-linearly related to the angle of twist θ

3. Torque T is linearly related to force P

4. Torque T is nonlinearly related to force P

5. 4134 N-m

6. 813 N-m

7. 103 N-m

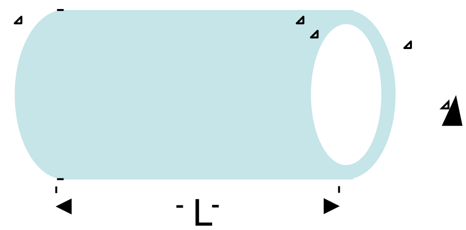
8. 813 N

9. 2356 N

10. 6400 N

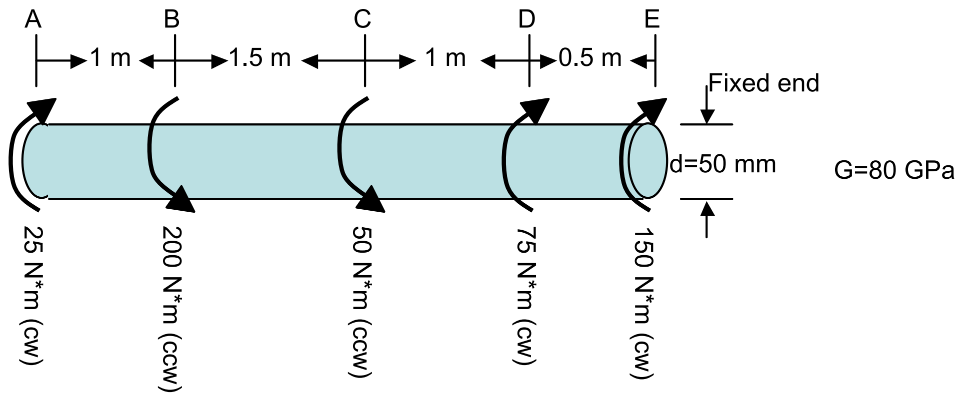
11. 1625 N

Shaft: 1 A hollow steel shaft, depicted in the following diagram, must transmit a torque of 300,000 in-lb. The ultimate shear stress (τ_{max}) is 21,000 psi. If the safety factor is 3.0 and the inner diameter is half of the outer diameter, find the following:

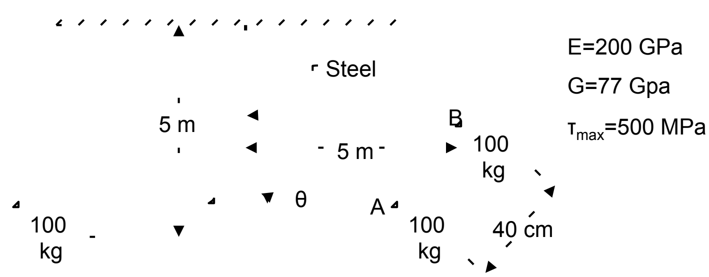


a. What is the outer diameter (D_o)?
b. What is the polar moment of inertia (I_P)?
c. If the shear modulus of elasticity (G) is 11×10^6 psi and the length (L) is 44 inches, what is the torsional angle of rotation, θ?

Shaft: 2 A shaft with four externally applied torques is shown. Find the relative angleof twist for the shaft between points B and D (ϕ_{BD}).



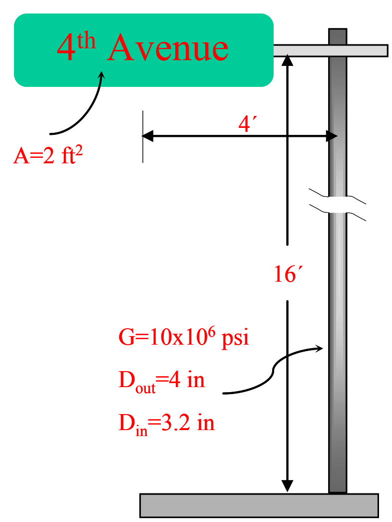
Shaft: 3 Two masses are connected by a rigid, massless bar and suspended from a single steel wire as shown in the diagram. Find the minimum diameter of wire, the gravitational force between mass A and mass B, and the resulting rotation, ϕ from this force. Note: The gravitational (pull) force F between two masses M_A and M_B being a distance r apart is



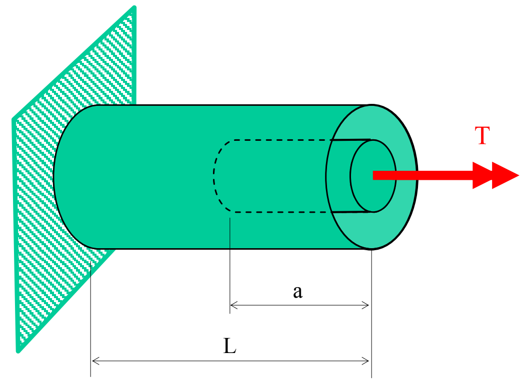
$$F = \frac{G_{grav} M_A M_B}{r^2} \quad \text{where } G_{grav} = 6.673 \times 10^{-11} \frac{m^3}{kg \cdot s^2} \text{ is constant.}$$

Shaft: 4 The figure shows a street sign supported by a 16' hollow cylindrical pole of internal diameter 3.2" and external diameter of 4.0". The sign has an area of 2 ft^2 and its centroid is 4' from the center of the pole. While the sign is subjected to a wind load of 40 lb/ft^2, determine:

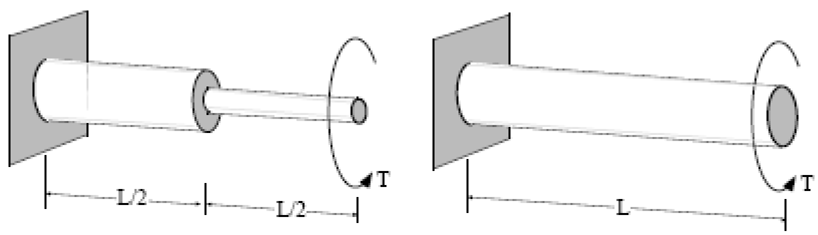
a) The absolute rotation of the pole at the top, in degrees.
b) The maximum shear strain the pole experiences.



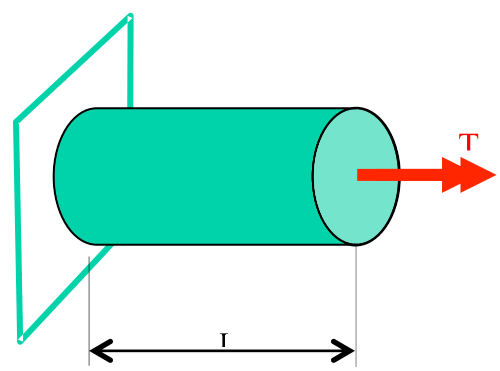
Shaft: 5 You are involved in the design of a torsional shaft, and you have been asked to calculate how much the shaft can be bored out while maintaining certain performance criteria. The outer diameter of the shaft is 2 inches. (a) Determine the limit on the bore diameter, d, such that the maximum torsional shear stress in the bored shaft is no greater than 140% of the shear stress in the unbored case. (b) Assume the bore diameter is 1.5 inches, and determine the bore length, a, such that the overall rotation of the shaft is no greater than 120% of the rotation for the original, unbored shaft. Your answer will be expressed in terms of the overall length, L.



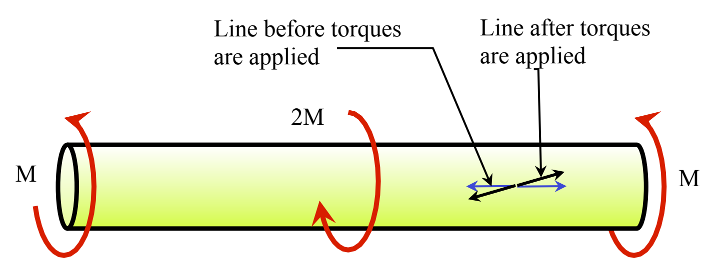
Shaft: 6 The torsion shafts shown below are made of the same material and are subjected to the same torque T. Which shaft will have the higher rotation under load? Which shaft will have the higher stress under load? Justify your answer. Note that the diameter of the thick shaft at left is the same as the diameter of the shaft at right.



Shaft: 7 The aluminum shaft of L = 20cm, fixed at one end and free at the other, is to be twisted by torque T such that the rotation at the free end is 2.0° (0.0349 rad). If the allowable shear strain in the aluminum is 0.0005 rad, what is the maximum permissible radius of the shaft.



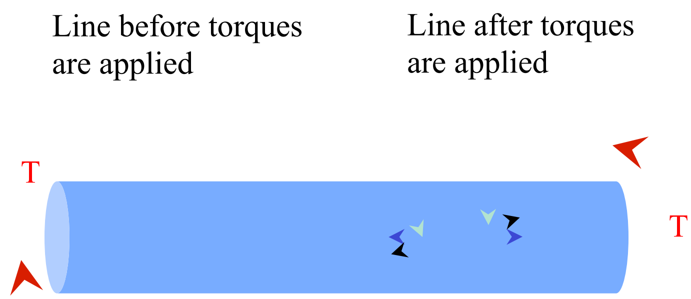
Shaft: 8 The solid shaft of radius r = 0.5 inches and shear modulus G = 10,000ksi is subjected to the torques shown. At the surface of the shaft at point A, the angle between the line "before torques are applied" and the same line "after torques are applied" was measured to be 3° (0.052 rad). Find the torque M.



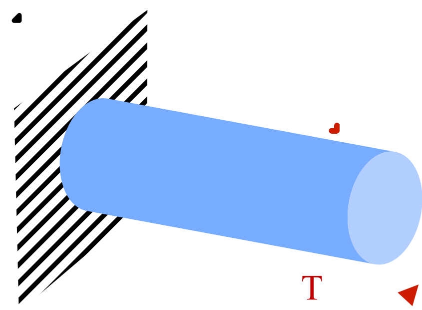
Shaft: 9 A cylindrical shaft of length L = 2.5m and diameter d = 10cm is fixed at one end and the other end is subjected to a torque T = 10 kNm. The rotation of the cross section at the loaded end was measured to be 0.2° (0.00349 rads). Find the shear modulus G of the material the shaft is made of.

Shaft: 10 A steel hollow cylindrical shaft is 1.3 m long and its inside radius is 4 mm. The shaft is to be designed such that (1) the angle of twist of the shaft shall not exceed 4^o when a torque of 700.0 Nm is applied; (2) the allowable shear stress will not be exceeded. Determine the minimum outside diameter of the shaft if the allowable shear stress for the steel is 85 MPa and the shear modulus is 77 GPa.

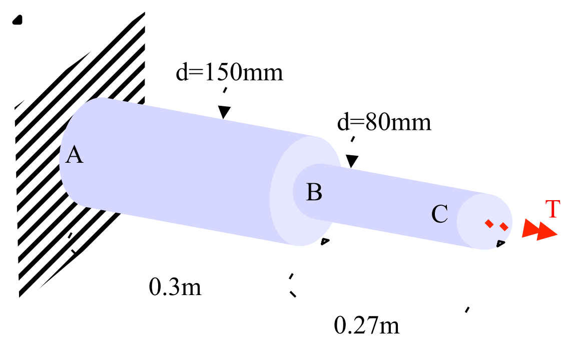
Shaft: 11 The solid shaft of radius r = 0.5 inches and shear modulus G = 10,000 ksi is subjected to the torques shown, and T = 10,000 lb – in. At the surface of the shaft, the angle between the "line before torques are applied" and the same line "after torques are applied" is shown in the figure. (a) Find the magnitude of that angle; (b) If the length of the shaft is 48 inches, find the relative rotation between the left and the right end of the shaft.



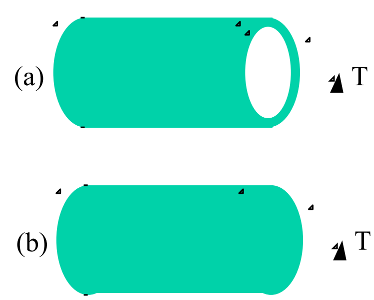
Shaft: 12 The shaft is fixed at one end and subjected to torque T at the other end as shown. While the torque is applied, the fixed-end "gives" so that it (the fixed-end of the shaft) rotates by 2^O (0.0349 rads) in the direction of the torque. The free end of the shaft rotates by 5^O (0.08727 rads). Determine the value of the torque T that is applied, if the length of the shaft is 1.2 m, its diameter is d = 2.5 cm, and it is made out of steel, G = 75 GPa.



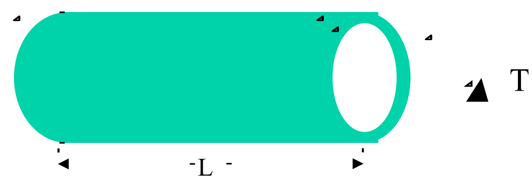
Shaft: 13 The stepped shaft, made of a material having a shear modulus of elasticity G = 34.0 GPa, is fixed at A and loaded by a torque T = 40.0 Nm at C. (a) Find the maximum shear strain γ in the entire shaft and mention the location(s) where such strain occurs. (b) If shaft AB were hollow cylindrical (instead of solid as in part (a), with inside diameter 60 mm and outside diameter 150 mm, will the maximum shear strain evaluated at (a) change if T remained the same? Explain your answer but do not evaluate the maximum shear strain in this case.



Shaft: 14 A hollow steel shaft is subjected to torque T = 300,000 in-lb. The shaft is such that the inner diameter is half of the outer diameter. (a) What should be the outer diameter of the shaft, D_o if the maximum shear stress in the shaft cannot exceed 7,000 psi? (b) If a solid shaft is subjected to the same torque T = 300,000 in-lb, what should be its diameter so that the maximum shear stress in the shaft cannot exceed 7,000 psi? (c) What is the ratio of the cross-sectional area of the hollow shaft in (a) over the cross-sectional area of the solid shaft in (b)



Shaft: 15 The hollow steel shaft (G = 80 GPa, L = 0.16 m) is subjected to torque T at its right end and is fixed at its left end. For design purposes the maximum shear strain anywhere in the beam must be limited to $\gamma_{\max} = 500 \times 10^{-6} rad$ and the maximum angle of twist at the right end must be limited to 1.0^{o} (0.0174 *rad*). If the inside radius of the shaft is $r_{in} = 3cm = 0.03m$ and the outside radius is $r_{out} = 5cm = 0.05m$ what is the maximum torque T that can be applied?



Shear force V- and bending moment M-diagrams

The first few modules of this part of the course are a tutorial for constructing shear force V- and bending moment M-diagrams for statically determinate beams. Depending on background and experience, the reader may skip parts of this tutorial; sections on stress distributions in beams follow.

Statically determinate beams have three nonoverlapping supports in two dimensions or six in three dimensions. Typical external supports are rollers (1 force support in the directions transverse to the roller in Figure G-1(a)), pins (two force supports, transverse to each other in Figure G-1(b)) and fixed ends (three supports, i.e., one moment, and two forces transverse to each other in Figure G-1(c)).

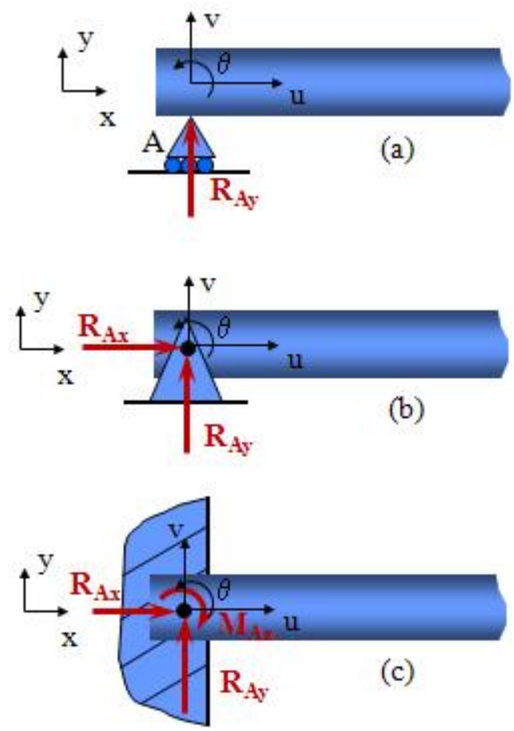


Figure G-1 (a) A roller restricts displacement in one direction (v) and creates force in the same direction. (b) A pin restricts displacement in two orthogonal directions (u,v) and creates forces in these directions. (c) A fixed-end restricts displacement in two directions as well as rotation, and creates forces in the same directions and bending moment.

Some examples of statically determinate, indeterminate, and unstable beams in two dimensions are shown in Figure G-2. Here, the beam in (a) is statically determinate since it is supported by three independent supports. Beam (b) is statically indeterminate since it has four supports, i.e., one rotation at A and three displacements, two at A and one at B; the degree of indeterminacy is 1 (4 supports minus 3 supports required for determinacy). The beam in (c) is statically determinate. The beam in (d) is statically indeterminate with degree of indeterminacy equal to 1. The beam in (e) is unstable since u_A and u_B are not independent and any vertical load on this beam cannot be sustained since the moment equilibrium around point A cannot be satisfied. The beam in (f) is indeterminant with degree of indeterminacy equal to 1.

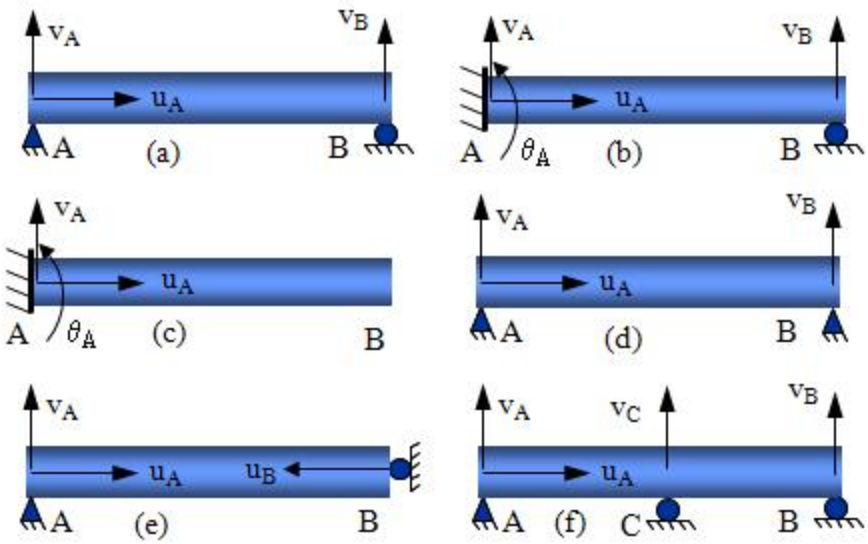


Figure G-2 Cases of statically determinate, indeterminate, and unstable beams. Arrows denote displacements or rotations restricted by the supports: (a) simply supported beam; (b) fixed-end, roller beam; (c) cantilever beam; (d) pin, pin beam; (e) unstable beam; and (f) continuous beam.

Analysis of statically determinate beams is done using free body diagrams (FBD) from which, using equilibrium equations, all external supports can be determined. In general, at every cross section for any beam there is a shear force V, and a bending moment M acting on it. When the values of V and M are found for the entire length of the beam, we have V and M expressed as functions of x, the coordinate along the length of the beam, thus V(x), M(x). The plot of these functions is the shear force and bending moment diagrams, respectively. There are various ways to obtain the V- and M-diagrams. One straightforward way is to make an imaginary section of the beam at an arbitrary position x and evaluate V and M at that position. This automatically yields V(x) and M(x). In order to do this, V and M at any arbitrary cross section should be found. Figure G-3 illustrates how this is done for a simple problem. Subsequently the **sign convention for V and M** is described.

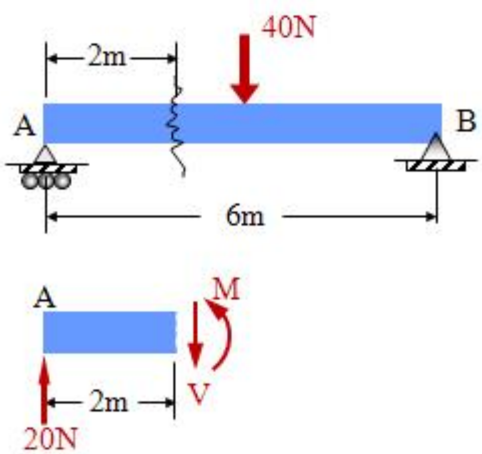


Figure G-3 A simply supported beam with an imaginary section ("cut") at 2 m from the left support and the forces and moments acting on the section. The 40 N force acts at midspan.

By definition, **V** is the shear force at the section, and **M** is the bending moment at the section. Equilibrium of the part from A to the section implies that

$V = R_A = 20$ N, $M = 20 \times 2 = 40$ Nm

where the former results from equilibrium of forces in the vertical direction and the latter from equilibrium of moments around any point. It is important to have a unified sign convention, so that V- and M-diagrams can be communicated effectively between engineers.

Sign Convention for shear forces and bending moment: Positive V tends to rotate the beam element clockwise as shown in Figure G-4(a). Positive M bends the beam in a concave fashion, or it creates a "smiling face" as shown in Figure G-4b.

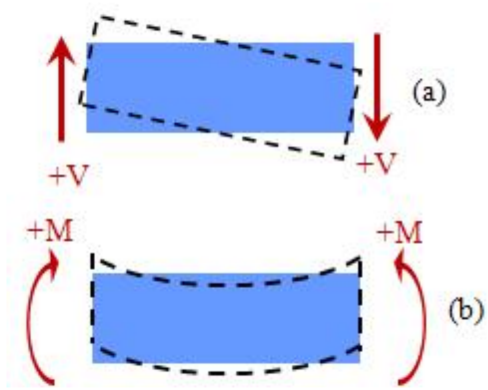


Figure G-4 (a) Sign convention for shear force V and (b) sign convention for bending moment M.

Note: If equilibrium of the part of the beam from the section to B is considered, the resulting V and M will be the same, i.e. at the section, **$V = R_A = 20$ N, $M = 20 \times 2 = 40$ Nm**

Analysis of the beam material element

In order to understand the way the shear forces V and bending moments M are distributed along the length of a beam, it helps to consider equilibrium of a small piece of it, actually of infinitesimal length dx. First, let us consider the free body diagram of a small piece of a beam subjected to distributed load q as shown in Figure G-5. Even of the external distributed load is not constant, it can be considered constant over a small length dx. Then the resulant forrce from the distributed load is qdx.

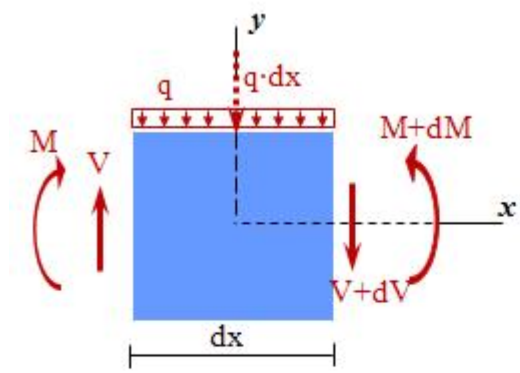


Figure G-5 Free body diagram of a small piece of a beam of length dx subjected to distributed load q.

Equilibrium of forces in the vertical direction implies that, noting that q is considered positive downwards,

$$\sum F_y = 0 \rightarrow V = qdx + V + dV \rightarrow \boxed{\frac{dV}{dx} = -q}$$

Then, $dV = -qdx$ holds, which when integrated over a piece of the beam AB (Fig. G-6) yields

$$\int_A^B dV = \int_A^B -qdx$$

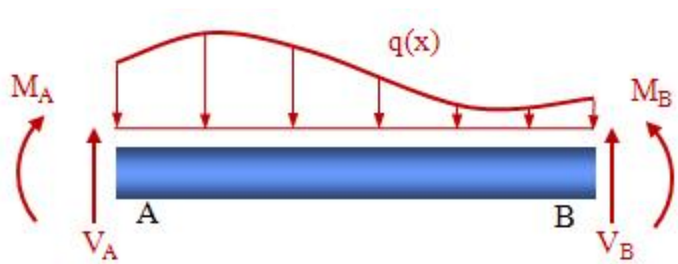


Figure G-6 A piece of a beam from point A to point B subjected to a distributed load q(x). The shear forces and bending moments at A and B are shown.

Then

$$V_A - V_B = -\int_A^B q dx$$

holds, which is nothing but the area of the load-intensity or distributed load q(x) diagram between A and B. Moment equilibrium of the piece dx implies that

$$\sum M = 0 \rightarrow M + dM - M - qdx\frac{dx}{2} - (V + dV)dx = 0$$

Disregarding second-order differentials (dxdx and dVdx), it follows that

$$\frac{dM}{dx} = V$$

Thus we have the two fundamental equations

$$\frac{dV}{dx} = -q, \quad \frac{dM}{dx} = V$$

Now we consider a similar small piece of beam dx but now it is subjected to concentrated load P instead of a distributed force. Figure G-7 shows the beam element and the free body diagram.

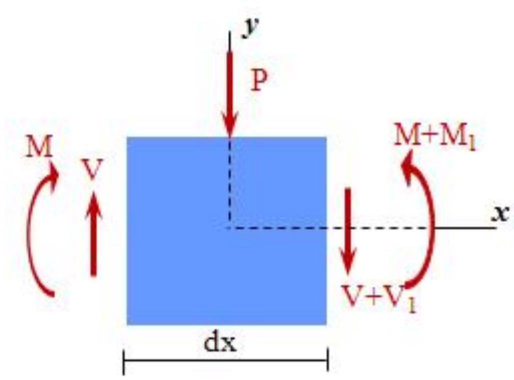


Figure G-7 Free body diagram of a small piece of a beam of length dx subjected to a concentrated load P.

Equilibrium of forces in the vertical direction implies that

$$\sum F_y = 0 \rightarrow V = P + V + V_1 \rightarrow V_1 = P$$

The fact that $V_1 = P$ implies an abrupt change in the shear force. Equilibrium of moments implies that

$$\sum M = 0 \rightarrow -M - P\frac{dx}{2} - (V + V_1)dx + M + M_1 = 0$$

Then,

$$M_1 = P\frac{dx}{2} + Vdx + V_1 dx$$

Since dx is small, $M_1 \approx 0$, which implies that no change in bending moment occurs in the vicinity of a concentrated load P.

From the above we can conclude that:

At the concentrated load, V changes are abrupt and equal to the value of the concentrated load. This means that the V-diagram "jumps" by an amount equal to the concentrated load at the location of the concentrated load.

At the concentrated load, M changes are small and dM/dx changes are large. This, in other words, means that the M-diagram "knees" at the location of the concentrated load.

Finally we consider a small piece of the beam dx subjected to a concentrated moment M_0, as shown in Figure G-8.

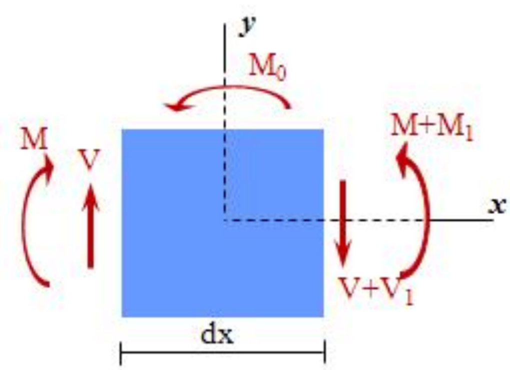


Figure G-8 Free body diagram of a small piece of a beam of length dx subjected to a concentrated moment M_0.

Equilibrium of forces in the vertical direction implies that

$$\sum F_y = 0 \rightarrow V = V + V_1 \rightarrow V_1 = 0$$

Equilibrium of moments about a point on the left edge of the element implies

$$\sum M = 0 \Rightarrow -M + M_0 + (V + V_1)dx + M + M_1 = 0$$

Disregarding the term with the differential (since it is negligible compared to the finite terms), it follows that $M_1 = M_0$. **Thus, in the vicinity of a concentrated moment, the shear force does not change while the bending moment changes abruptly by an amount equal to M_0.**

Shear force and bending moment diagrams

For engineering problems involving beams, the shear force V- and the bending moment M-diagrams are usually drawn. They are the identity of the beam showing how it performs, and allow one to evaluate extreme stresses that the beam encounters. The straightforward way to draw the V and M diagrams is to evaluate the V and M at a section of the beam at a location x. This then provides V(x) and M(x). Even though this is effective, it is tedious and does not provide the understanding the engineer needs to have about the behavior of the beam. Thus, by plotting V(x) and M(x) using the rules stated below, one cannot only plot the diagrams efficiently, but can also understand how the beam behaves. These rules also make recognition of errors in plotting the diagrams easier.

Rules for plotting V/M diagrams

Let q denote the distributed load (force/length) on a part of a beam. In Example 1 below, q = 0 over the entire length, while in Example 2 q = 0 for segment AC and segment CB.

As a direct consequence of

$$\frac{dV}{dx} = -q, \qquad \frac{dM}{dx} = V$$

we have that:

RULE 1:

- Where q = 0, V = constant and M = linear.

- Where q = constant, V = linear and M = quadratic.

- Where q = linear, V = quadratic and M = cubic.

As a direct consequence of the material in the previous section regarding concentrated loads and concentrated moments, we have

RULE 2: At a concentrated load P, the V-diagram "jumps" by an amount equal to P, while the M-diagram does not change (it develops a "knee" at the concentrated load position).

RULE 3: At a concentrated moment M_0, the M-diagram "jumps" by an amount equal to M_0, while the V-diagram is not influenced by the presence of M_0 (M_0 influences the reactions, which, in turn, modify the V-diagram).

These three rules are enough, for all practical purposes, for effective construction of the V/M diagrams, for recognizing errors in existing diagrams, and for understanding the overall behavior of beams. Even though the rules are simple, it takes extensive practice to become familiar with them and use them effectively. Several examples are presented in the sequence. Yet the reader should draw example diagrams on his or her own in order to capture all the relevant details.

Example 1: Consider the cantilever beam subjected to end load P as shown in Figure G-9. In this example we have only one segment in the beam, i.e. from the left end (fixed) to the right end (where P is applied). First, we find the V and M diagrams by considering a section at an arbitrary position x. The free body diagram shown in Figure G-9 clearly implies that for the coordinate system shown, $V(x) = P$, $M(x) = -P(L-x)$. This is the straightforward way to obtain the diagrams. Also, for this beam q=0 over its entire length, thus, according to rule 1, V = constant and M = linear. According to rule 2, we expect a jump in the V-diagram at A and B since we have a reaction equal to P at A and a concentrated force P at B. Also, according to rule 3, the reaction moment at A, equal to -PL creates a jump for the M-diagram at A. In order to draw the V-diagram, we only need the any point in the segment. Conveniently, V = +P at either end (see the free body diagram in Figure G-9). The M-diagram is linear, thus we need two points where M is known. Conveniently, M = 0 at the right end and M = -PL at the left end.

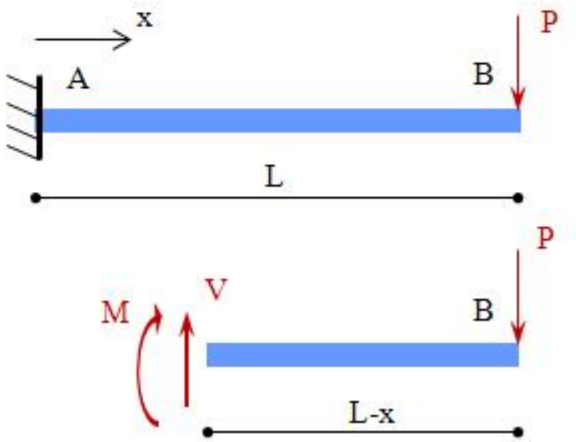


Figure G-9 A cantilever beam subjected to end load P. Also the free-body diagram of part of the beam to the right of a section at distance x is shown.

Figure G-10 shows the V- and M-diagrams.

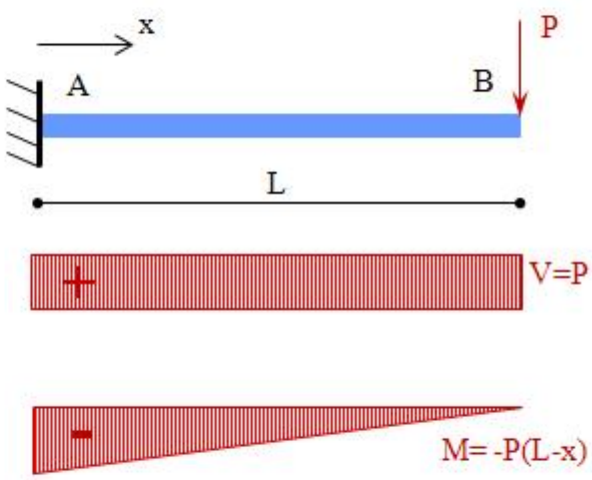


Figure G-10 The V- and M-diagrams for the beam in Example 1. Note that the V-diagram is constant and the M-diagram is linear, since q = 0.

Example 2: Here we have a simply supported beam with concentrated load P shown in

Figure G-11. We have two segments, AC and CB. In both of these, q = 0, thus (rule 1) V = constant, M = linear. Also, at C, we expect a jump equal to P in the V-diagram and a "knee" in the M-diagram (rule 2). The reactions at A,B easily provide that in AC, V = $+R_A$, and in CB, V = $-R_B$. Note that this results in a jump equal to P at point C. More efficiently, rule 2 can be used to construct the V-diagram by "scanning" the beam from left to right (or right to left). The steps are: a) at A, V = $+R_A$, thus in segment AC, V = $+R_A$; b) at C, the V-diagram jumps by an amount equal to P, and, since P is in opposite direction to R_A, the jump is downward (see the V-diagram in Figure G-11). The M-diagram being linear in AC and CB requires the M value at two points in each segment. At both ends (A,B) M =0, and at C we have a common M value (rule 2). The M at C is easily found, and thus the M-diagram for the beam consists of two linear segments as shown in Figure G-11.

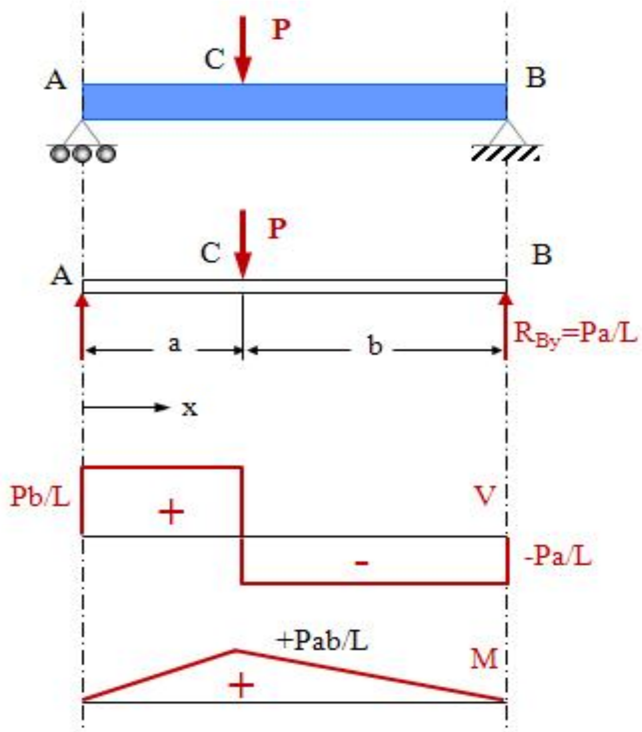


Figure G-11 A simply supported beam subjected to a concentrated load P, and its V- and M-diagrams.

Example 3: In this example we have a simply supported beam with uniformly distributed load q = constant (Fig. G-12). Here, from rule 1, the V-diagram is linear and the M-diagram is quadratic. Two points are needed in the V-diagram and three points in the M-diagram. The V value at the two ends is enough and convenient for drawing the V-diagram, while the M value at the two ends and the middle points is enough for drawing a schematic of the M-diagram. Also, for this beam we have

$$R_A = R_B = \frac{qL}{2}, \quad V(x) = \frac{qL}{2} - qx, \quad M(x) = \frac{qLx}{2} - \frac{qx^2}{2}$$

Also, since $V = \frac{dM}{dx}$

M is maximum where V = 0. For this problem, M is maximum at midspan and equal to $qL^2/8$.

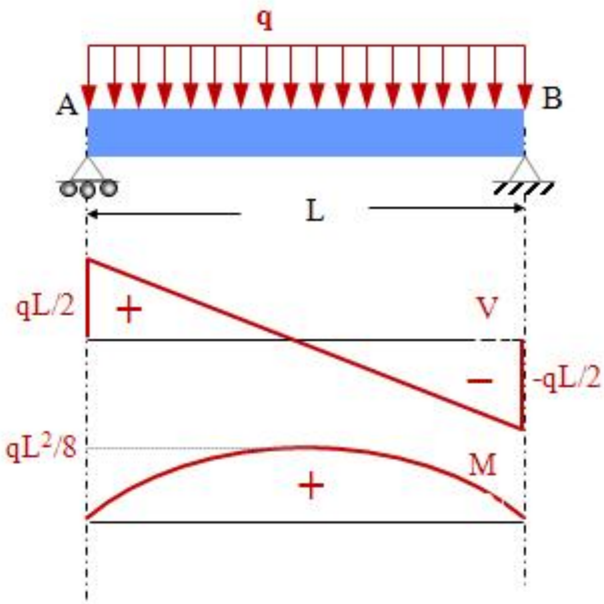
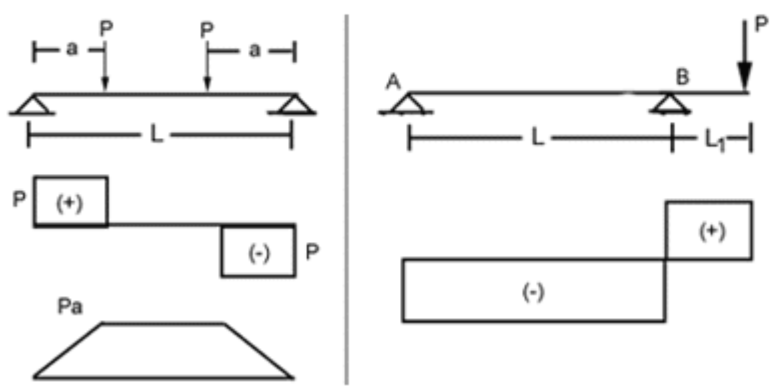


Figure G-12 A simply supported beam subjected to a distributed load q, its V- and M- diagrams.

Exercise 1: Verify the following diagrams and draw the M-diagram for the beam to the right.



Example 4: Figure G-13 shows a simply supported beam subjected to four concentrated loads P. Symmetry of the problem implies that the reactions at A and B are equal, and that the M-diagram is symmetric. The V-diagram, however, is skew symmetric for this and for every symmetric problem. The reason for this is the sign convention for shear stress. At the left end, the shear force V is equal to the reaction there. At each concentrated load P, the V value jumps (drops, i.e. jumps in the direction opposite to the reaction at the left end) by P. The reactions are expressed as $R_A = R_B = 2P$. At A, $V = R_A = 2P$ (positive shear force V) and at B, $V = -R_B = -2P$ (negative shear force).

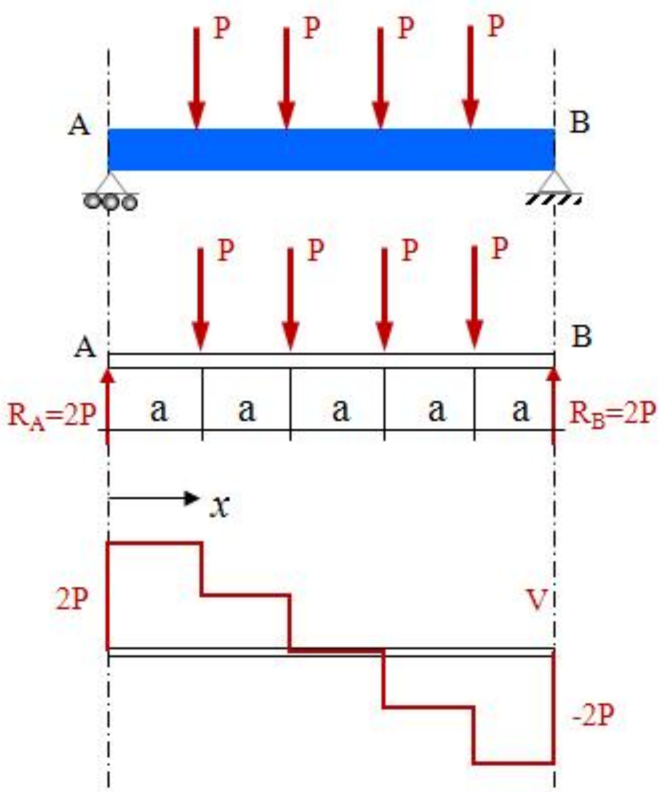


Figure G-13 A simply supported beam subjected to four concentrated loads at distance a from each other as shown. The beam and the load are symmetric in this case.

Exercise 2: Draw the M-diagram for the problem in Figure G-13.

Exercise 3: Verify the diagrams in Figure G-14 and evaluate the maximum moment M. Hint: Since V = dM/dx, The M diagram is extreme (max/min) where V = 0. Note that for this beam the reaction at the left end R_A, the reaction at the right end R_A, and the moment at C M_C are expressed as

$$R_A = \frac{qC}{2}\left(2 - \frac{C}{L}\right), \quad R_B = \frac{qC^2}{2L}, \quad M_C = \frac{qC^2}{2}\left(1 - \frac{C}{L}\right)$$

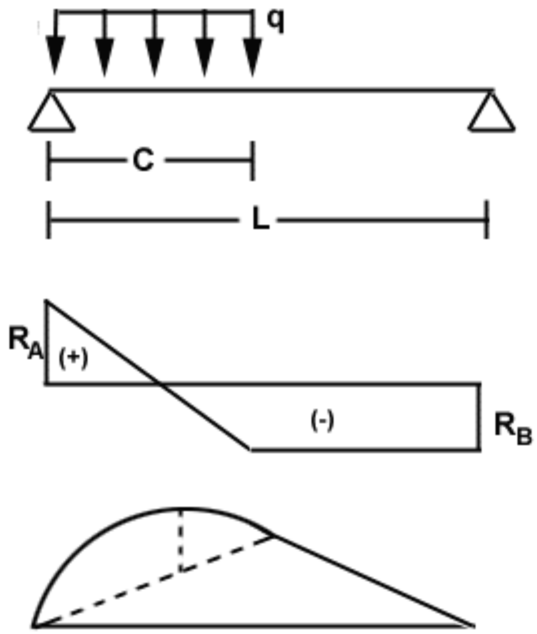


Figure G-14 This simply supported beam is subjected to load q over length c. The V- and M-diagrams are also shown.

Example 5: This simply supported beam, shown in Figure G-15, is subjected to a linearly varying distributed load q. Then, the V-diagram is cubic, and the M-diagram is quadric (polynomial of degree 4). Finally, the M is maximum where V = 0. The relevant expressions, by considering a section of the beam at an arbitrary x and writing the equilibrium equations of the part of the beam to the right or the left of the section, are

$$R_A = \frac{1}{6}qL, \quad R_B = \frac{1}{3}qL, \quad V = \frac{qL}{6}\left(1 - 3\frac{x^2}{L^2}\right), \quad M = \frac{qL}{6}\left(1 - \frac{x^2}{L^2}\right)$$

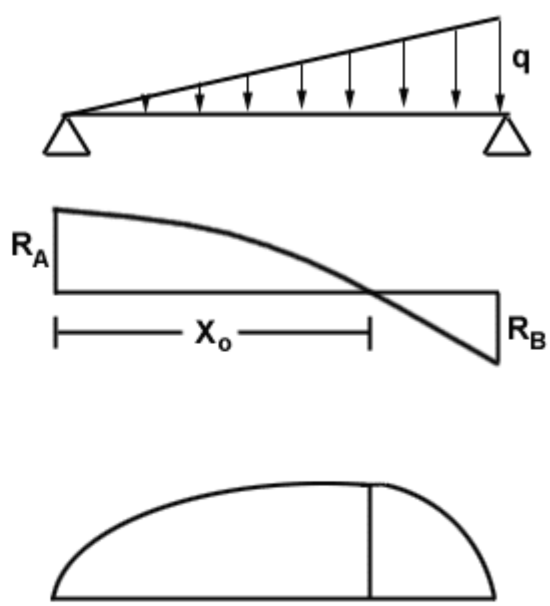


Figure G-15 This simply supported beam is subjected to linearly varying load q. The V- and M-diagrams are also shown.

Exercise 6: For the beam in Figure G-15, find the maximum bending moment, and the location where it occurs.

Example 6: Even though this example (Fig. G-16) appears complex, it is easily decomposed into three beam segments: (a) from the left end to the pin, (b) from the pin to the concentrated load and moment, and (c) from the concentrated load and moment to the roller at the right end. By applying the three rules for each segment, the V- and M-diagrams result. For this beam, R_a = 20kN, R_b = - 4kN.

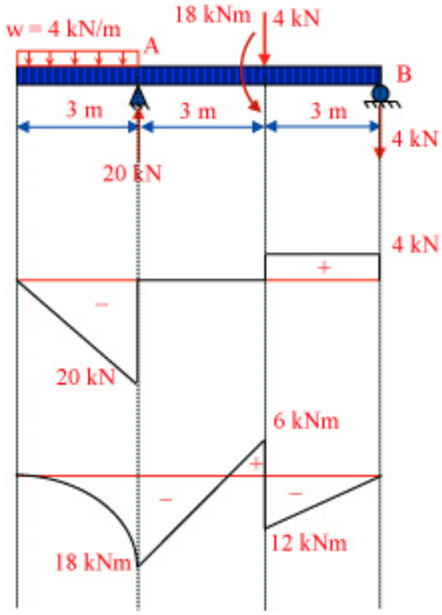


Figure G-16 This simply supported beam has a cantilevered-out part at its left and is subjected to a distributed load, a concentrated load and a concentrated moment. The V- and M-diagrams are also shown.

Exercise 7: Rreproduce the diagrams in Figure G-16 by evaluating V and M at each of the three segments and scanning the beam from left to right or right to left. In doing so, apply the three rules for the M/V diagrams.

Example: V and M Diagrams

<u>Using FBD:</u>

$\Sigma M_B = 0$ (+)

$R_{Ay} = 60$ kN

$\Sigma F_y = 0$

$R_{By} = 60$ kN

$\Sigma F_x = 0 \quad \Sigma R_{Bx} = 0$

<u>From section at x:</u>

$\Sigma F_y = 0$

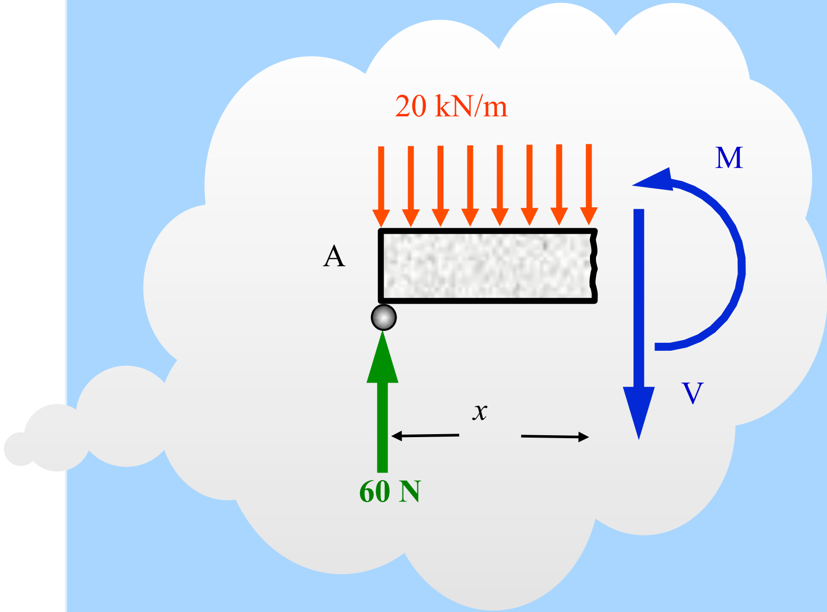
$60 - 20x - V = 0$

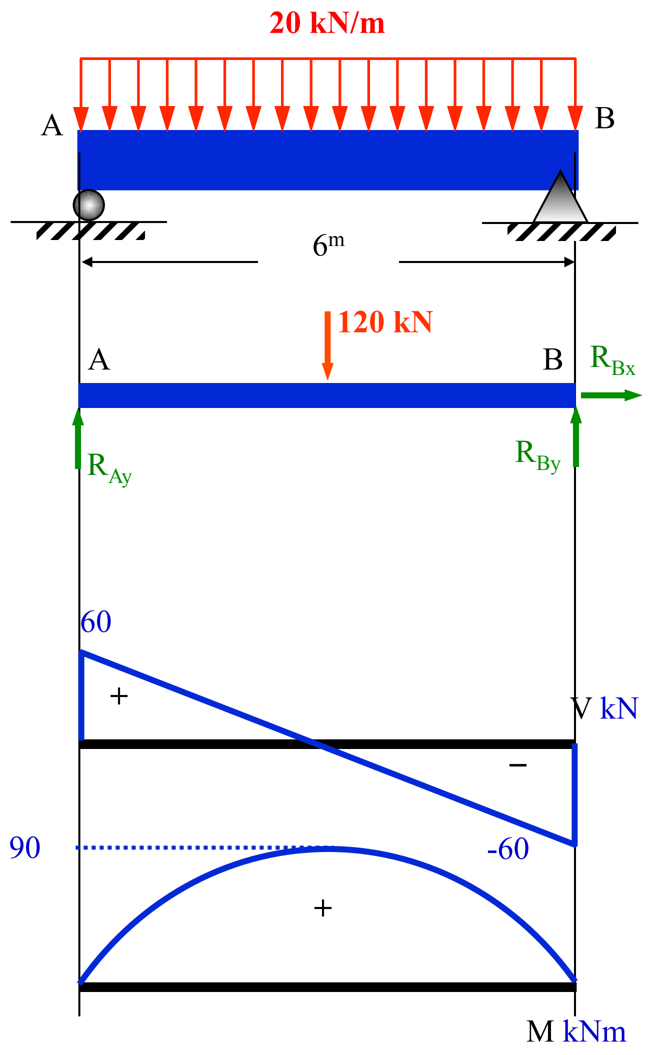
$$\boxed{V = 60 - 20x}$$

$\Sigma M_x = 0$

$$20x\left(\frac{x}{2}\right) + M - 60x = 0$$

$$\boxed{M = 60x - 10x^2}$$





Example: V and M Diagrams

From FBD:

$$\Sigma F_x = 0 \Rightarrow R_{Bx} = 0$$

$$\Sigma F_y = 0 \Rightarrow R_{By} = 60 \text{ kN}$$

$$\Sigma M_B = 0$$

$$M_B - 60 \times 2 = 0$$

$$\therefore M_B = 120 \text{ kNm}$$

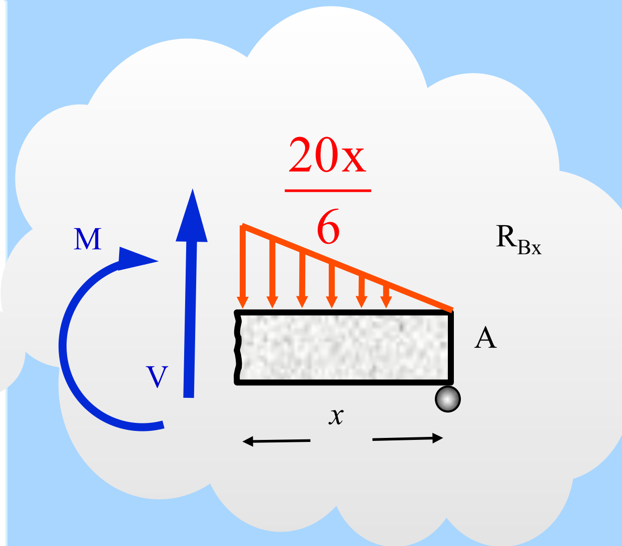
$$\Sigma F_x = 0 \Rightarrow N = 0$$

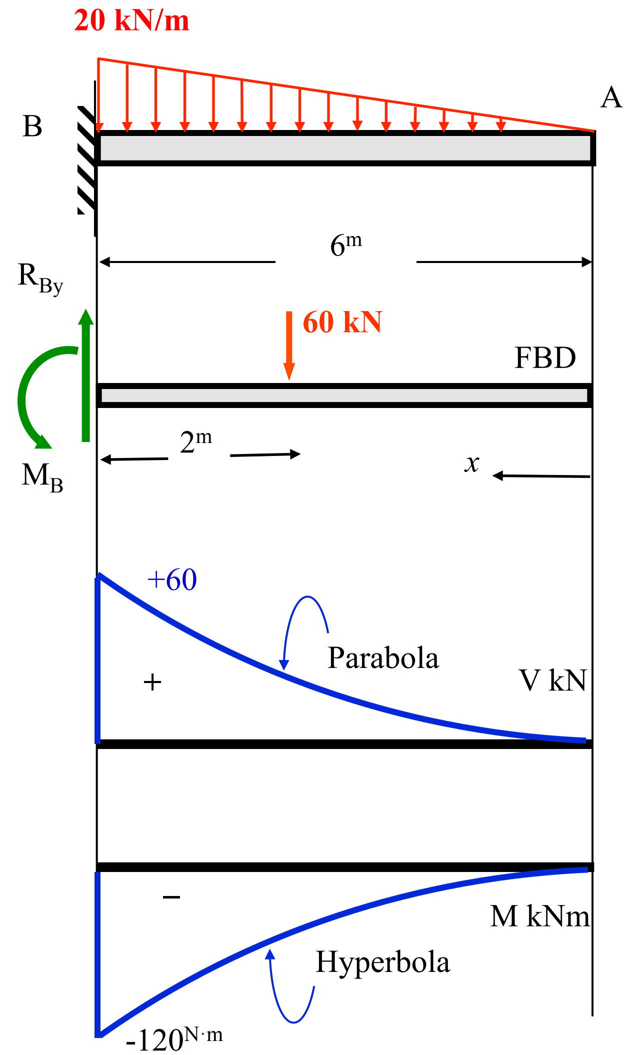
$$\Sigma F_y = 0 \Rightarrow V \frac{20x}{6}\left(\frac{x}{2}\right) = \left(\frac{10x^2}{6}\right) \text{kN}$$

$$\Sigma M_x = 0 \quad +$$

$$-M - \left(\frac{20x}{6}\right)\left(\frac{x}{2}\right)\left(\frac{x}{3}\right) = 0$$

$$M = \left(\frac{-20x^3}{36}\right) \text{kNm}$$





Example: V and M Diagrams

From FBD for ABC:

$\Sigma F_x = 0$, then $R_{cx} = 0$ kN

$\Sigma M_c = 0$, then $R_{Ay} = 20$ kN

$\Sigma F_y = 0$, then $R_{cy} = 20$ kN

for segment AB where $0 \leq x < 3$

$\Sigma F_y = 0$, then $V = 20$ kN

$\Sigma M_x = 0$, then $M = 20x$

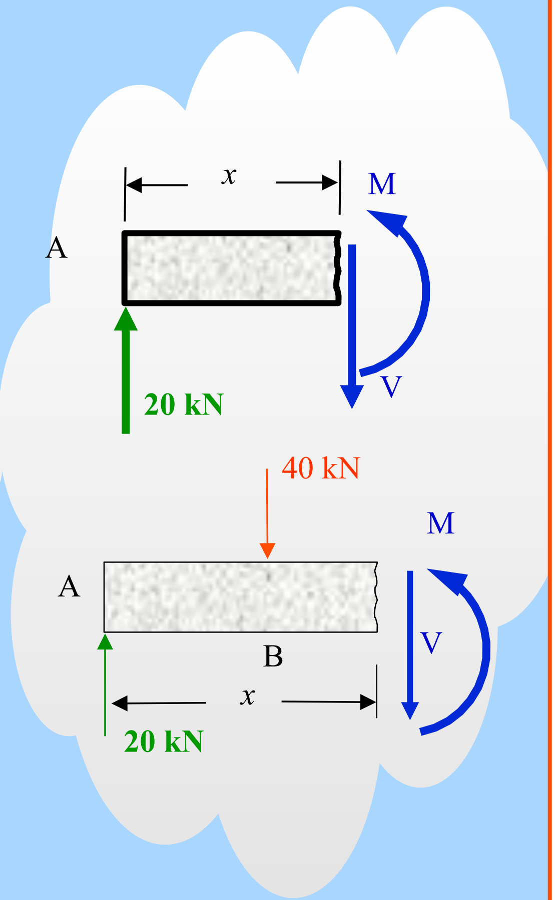
for segment BC where $3 < x \leq 6$)

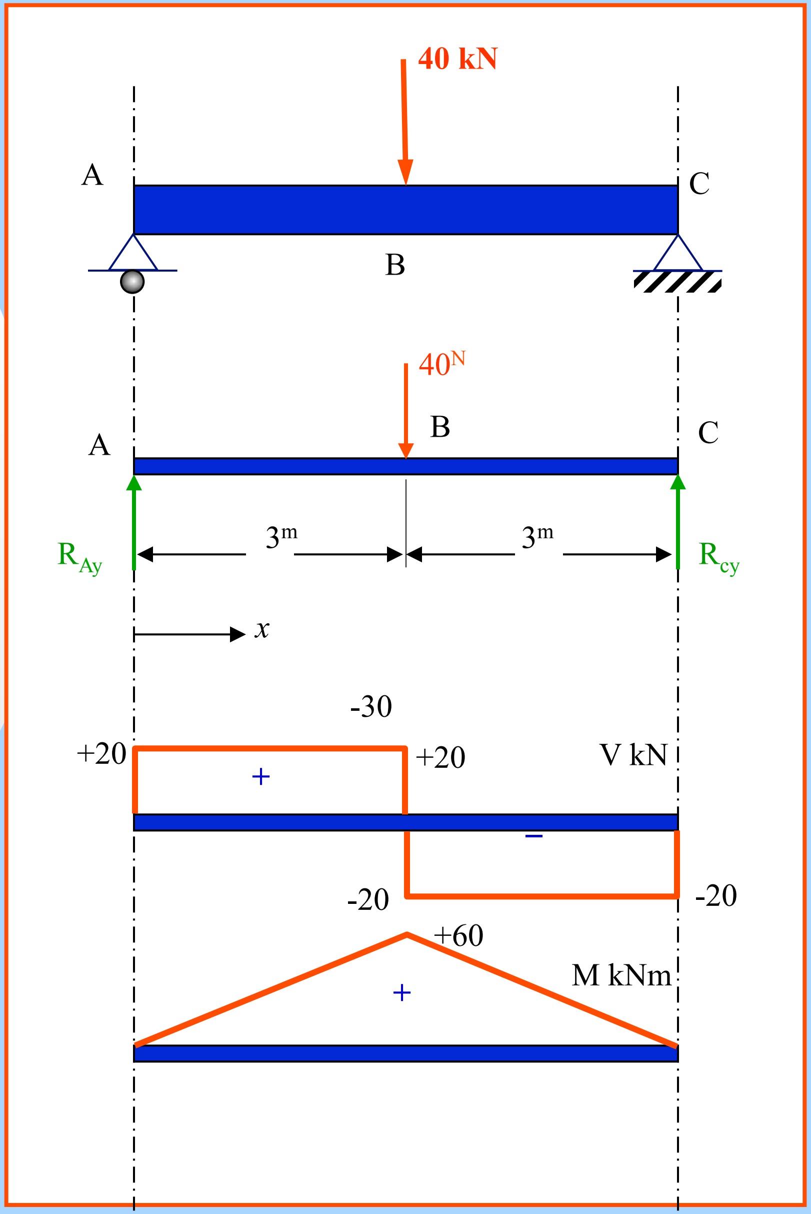
$\Sigma F_y = 0$, then $V = -20$ kN

$\Sigma M_x = 0$, then $M = 20x - 40(x - 3)$

$= 120 + 20x - 40x$

$= 120 - 20x$





Example: V and M Diagrams

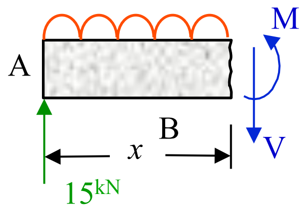
$\Sigma M_A = 0$, then $R_{BY} = 35^{kN}$

$\Sigma F_y = 0$, then $R_{Ay} = 15^{kN}$

for segment AB where $0 \leq x < 6$)

$V = 15 - 5x$

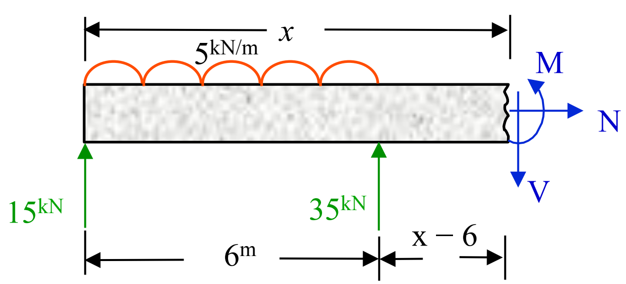
$M = 15x - 5x^2/2$

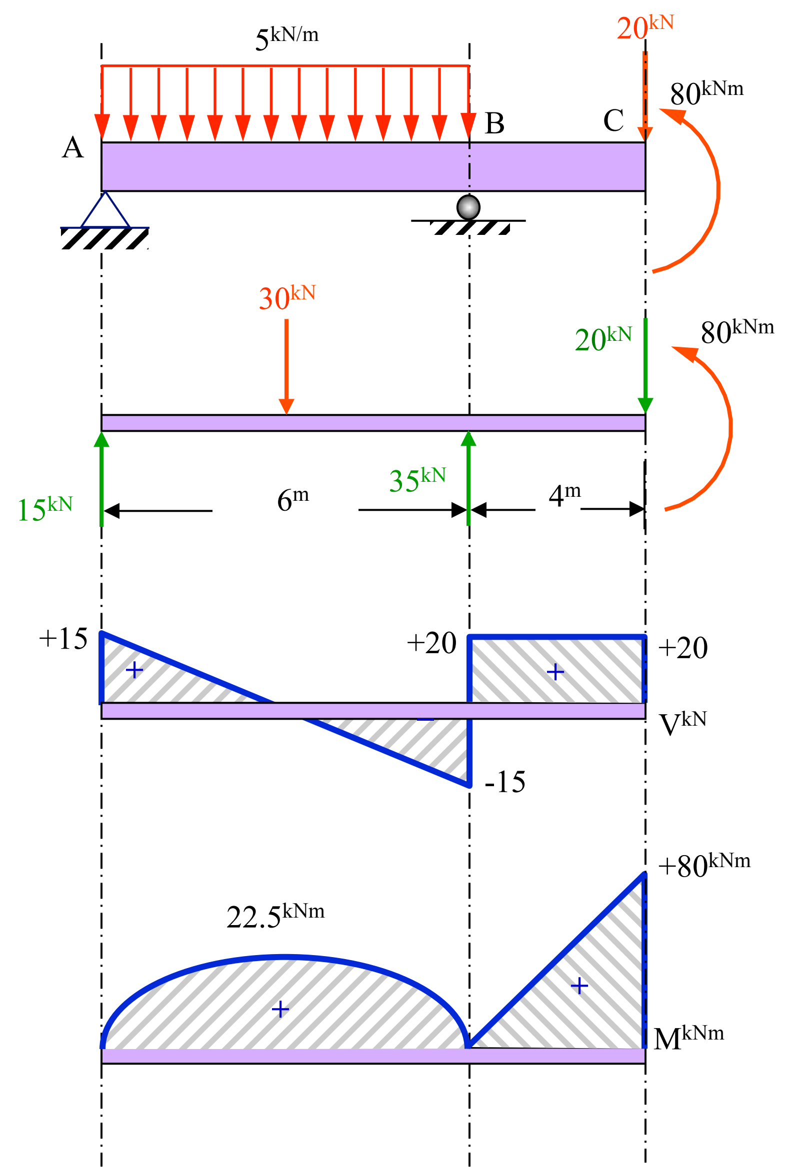


for segment BC where $6 < x \leq 10$)

$V = 20^{kN}$

$M = 20x - 120$





Module 17: Beams-Deflection and Curvature

Previous modules addressed shear forces V and bending moments M in beams. When these are known, then the beam's deflections beams can be evaluated, and, as will be seen in this and following modules, from the deflections the stresses and strains can be evaluated. These are important to the engineer since they form the core of beam design.

Beam deflection and curvature

Beams deflect when subjected to a load. Figure G-17 shows a schematic of the deflected shape of a simply supported beam subjected to P at midspan. Assuming the beam does not fracture under the load, i.e., the beam remains continuous, it is important to mathematically define the quantities that describe the beam deflections.

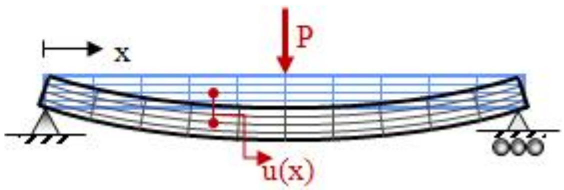


Figure G-17 Schematic of the deflected shape of a simply supported beam subjected to a concentrated load P. The deflection at a point, u(x) is shown.

These are definitions relevant to beam deflection and curvature:

u(x): deflection of the beam at coordinate x

ϱ(x): radius of curvature at coordinate x

ϰ(x)=1/ϱ(x): curvature at coordinate x

In general, it is the bending moment M that creates deflection. Thus, let us consider a piece of a beam subjected to bending moment M as shown in Figure G-18 (M is usually obtained from the M-diagram of a beam). A small chunk dx is also shown in its deformed state, i.e. the rectangle *abba* in the beam before the moment is applied becomes *a'b'b'a'* after the moment is applied.

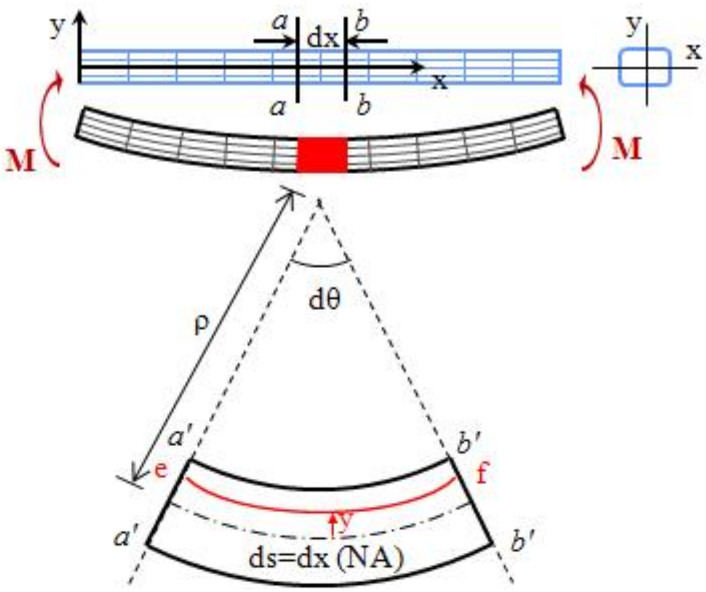


Figure G-18 A beam before and after it is subjected to a bending moment M. The deformed shape of a piece of length dx is shown enlarged.

As can be seen from the deformed piece *a'b'b'a'* the top line element *ab* contracts while the bottom element *ab* expands. There should be a line element in between which neither contracts nor expands. This element is said to be along the neutral axis (NA). It will be shown later that the neutral axis is at the centroid of the cross section of the beam. The cross section of the beam can be seen at the top right corner of Figure G-18, where the x,y axes are shown; the neutral axis is at the origin; and the y-coordinate is upward from the NA. A typical line element ef is shown at an arbitrary distance y. That particular element contracts after the bending moment is applied.

$$\tan(d\theta) \simeq d\theta = \frac{ds}{\rho}$$

At the NA, by definition we have that ds = dx. Then, it follows that

$$\frac{d\theta}{dx} = \frac{1}{\rho(x)} = \kappa(x) \quad \text{which implies that} \quad dx = \rho d\theta$$

In strength of materials for beams made of engineering materials, the **assumption that plane cross sections before loading remain plane** after loading is made. This, based on Figure G-18, implies that the contraction of line element ef is proportional to its distance from the NA, y. In other words, the two sections *a-a* and *b-b* remain plane but rotate to positions *a'-a'* and *b'-b'*. Then, for string ef, we have that

$$L_1 = ef = (\rho - y)d\theta = dx - \frac{y}{\rho}dx$$

Then at coordinate y, an original (before bending deformation) length dx becomes $dx - \frac{y}{\rho}dx$. The strain ε_x, for string dx, defined as the change in length over original length is expressed as

$$\varepsilon_x = \frac{-\frac{y}{\rho}\,dx}{dx} = -\frac{y}{\rho} = -\kappa y \quad \text{or}$$

$$\varepsilon_x = -\kappa y$$

Note that in the above relation, y = 0 implies that $\varepsilon_x = 0$, i.e., the strain along the NA is zero. Thus, the length of dx does not change along the neutral axis. With this last equation, if the curvature is known, the strains at each point in the cross section can be evaluated, and using Hooke's law, the stress can be evaluated. Before doing that, remember that the exact position of the NA is not known at this point. It turns out, as shown in the following section, that it passses through the centroid of the cross section.

Module 18: Beam -Bending Normal Stress

Let us consider the beam shown in Figure G-19 subjected to bending moment M. The cross section of the beam is, in general, nonsymmetric with respect to any axis. For simplicity, the one in Figure G-19 is symmetric with respect to y, but this does not change any parts of this section.

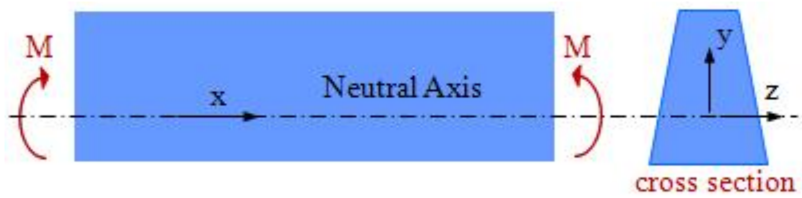


Figure G-19 A beam subjected to bending moment M. The cross section is shown and the neutral axis is at the intersection of the x- and y-axis.

From the previous section, we have that $\epsilon_x = -\kappa y$. Using Hooke's law, with E being the elasticity modulus of the material the beam is made of, it follows that

$$\sigma_x = E\epsilon_x = -E\kappa y$$

This implies that the normal stress σ_x varies linearly in the vertical direction y, and it is zero at the NA where y = 0. Figure G-20 shows the distribution of σ_x along the y-axis.

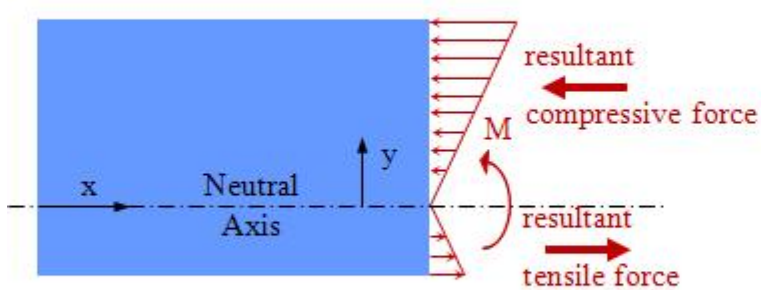


Figure G-20 Stress distribution for a beam subjected to positive bending moment M, showing compression above the NA and tension below. The two resultant forces form a couple equal to M.

So far the NA was considered but its exact position was not determined. Equilibrium of a cross section of a beam subjected to bending moment M requires that the net force on the section is equal to zero, and that the total moment on the section is equal to M. Referring to Figure G-20, this implies that, first, the resultant compressive force is equal to the resultant tensile force, and, second, that these to forces form a couple with a moment equal to M. The first of these equations referring to Figure G-21 imply that

$$\sum F_x = 0 \Rightarrow \int_A \sigma_x dA = -\int_A E\kappa y dA = 0 \Rightarrow \int_A y dA = 0$$

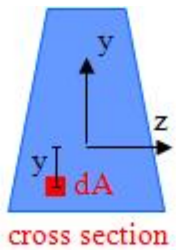


Figure G-21 The cross section of the beam shown in Figure G-20, and a material element dA at distance y from the NA. A denotes the total area of the cross section.

This last equation implies that **the neutral axis is at the centroid of the cross section**.

Now we consider the implications of the fact that the net moment on the cross section is equal to M. The material element dA, shown in Figure G-21 is subjected to stress σ_x. That stress, when multiplied by dA provides the force on dA, and the force multiplied by y provides the moment with respect to the NA, denoted as dM. Thus, we have

$$dM = -\sigma_x y dA$$

It is noted that for an element dA located above the NA a positive (tensile) σ_x produces a moment opposite to the positive bending moment M, thus the negative sign in the above equation. The total moment acting on the cross section is the integral of dM over area A, or

$$M = \int_A dM = -\int_A \sigma_x y dA$$

and since $\sigma_x = -E\kappa y$, it follows that

$$M = \kappa E \int_A y^2 dA$$

We now use the definition

$$I = \int_A y^2 dA ,$$

which is the **moment of inertia of the cross section**. With this definition, we have

$$M = \kappa EI$$

which is called the **moment-curvature relation**. The above formulas yield the expression of normal stress σ_x as a function of M, known as the **flexure formula**

$$\sigma_x = -\frac{My}{I}$$

The flexure formula provides the stresses in the cross section as a function of the bending moment M and the coordinate y. It is noted that a positive moment M yields compressive stress for postitive y and tensile for negative y. Similarly, a negative moment M yields tensile stress for positive y, and compressive for negative y. From the flexure formula, the maximum normal stresses can be evaluated by substituting in the flexure formula the extreme (positive or negative) values of y. Let c_1 denote the extreme negative value of y and c_2 the extreme positive one. Then, the extreme values of normal stress, denoted as σ_1 for the tensile and σ_2 for the compressive, respectively, are (referring to Figure G-22)

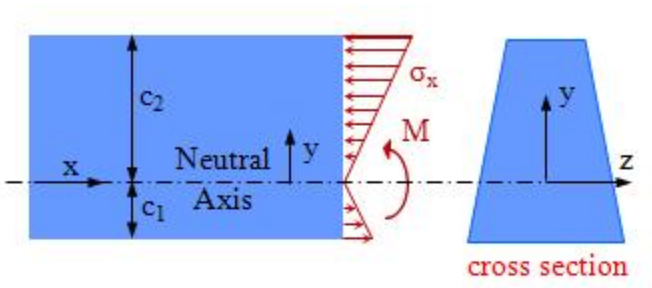


Figure G-22 At the top and bottom of the cross section we have extreme stresses, i.e. for $y = c_1$, and $y = c_2$.

$$\sigma_1 = \frac{Mc_1}{I} = \frac{M}{S_1}; \quad \sigma_2 = \frac{Mc_2}{I} = \frac{M}{S_2}$$

The variables $\mathbf{S_1 = I/c_1}$ and $\mathbf{S_2 = I/c_2}$ are called the section moduli, and are crucial for beam design. For a symmetric cross section with respect to axis z, the two section moduli are equal, and the section modulus is denoted as S, thus $\mathbf{S=S_1=S_2=I/c,}$ where c denotes the distance from the centroid of the cross section to the top or the bottom of the cross section.

Example-beam of rectangular cross section:

A rectangular cross section, of base b and height h (Fig. G-23) is symmetric with respect to both y- and z-axis, thus $c = c_1 = c_2 = h/2$. Also, for a rectangular cross section

$$I = \frac{bh^3}{12}$$

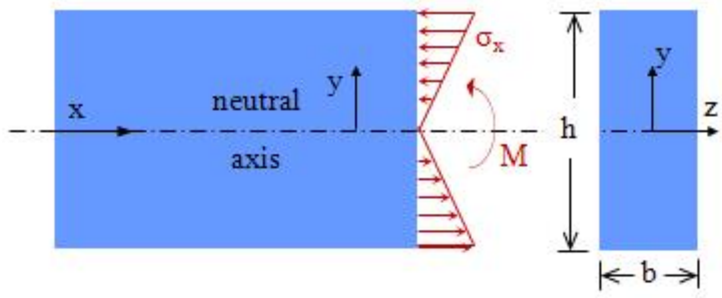


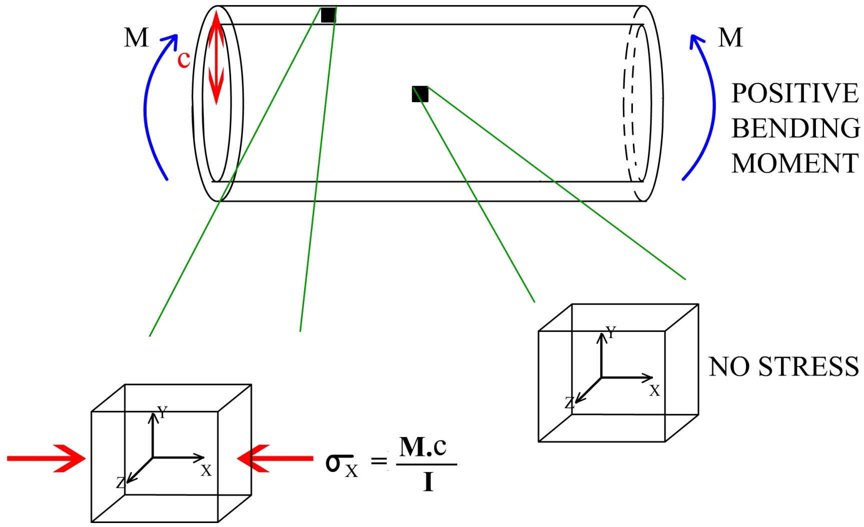
Figure G-23 A beam of rectangular cross section (right), of base b and height h, subjected to bending moment M.

For this beam, the stresses are evaluated as

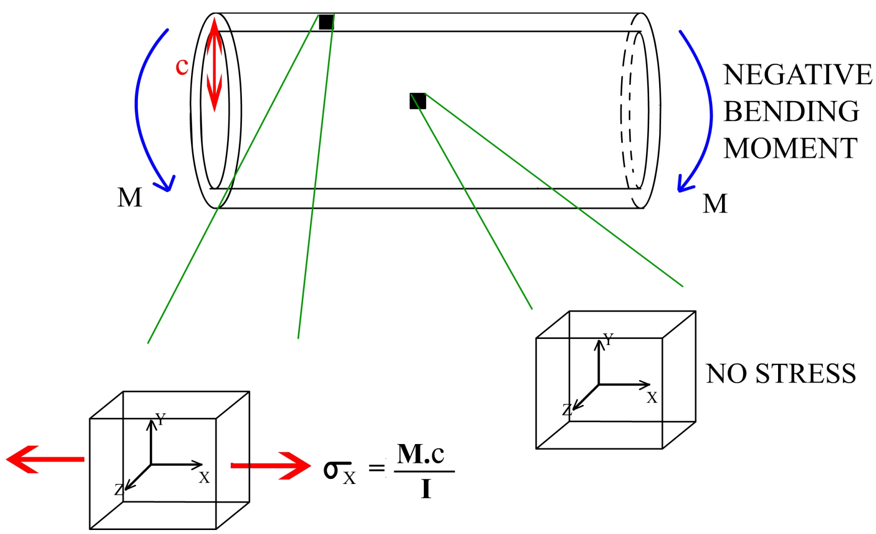
$$\sigma_x = -\frac{My}{I} = -\frac{12My}{bh^3}$$

and the extreme stresses noting that $S_1 = S_2 = S = I/(h/2)$ are expressed as

$$\sigma_1 = \frac{Mc_1}{I} = \frac{M}{S_1} = \frac{M\frac{h}{2}}{\frac{bh^3}{12}} = \frac{6M}{bh^2}; \quad \sigma_2 = -\frac{Mc_2}{I} = -\frac{M}{S_2} = -\frac{M\frac{h}{2}}{\frac{bh^3}{12}} = -\frac{6M}{bh^2}$$



The normal stress σ_x in a beam subjected to positive bending moment is illustrated above. Note that σ_x is proportional to the y-coordinate of the material element.



The normal stress σ_x in a beam subjected to negative bending moment is illustrated above. Note that σ_x is proportional to the y-coordinate of the material element.

Example-Extreme Bending Stresses: The beam shown in Figure G-24 is 10.0 m long and loaded at midspan by a 100kN force. The cross section is rectangular, 20cm × 50cm. For this beam, we are interested in finding the maximum tensile and maximum compressive stresses.

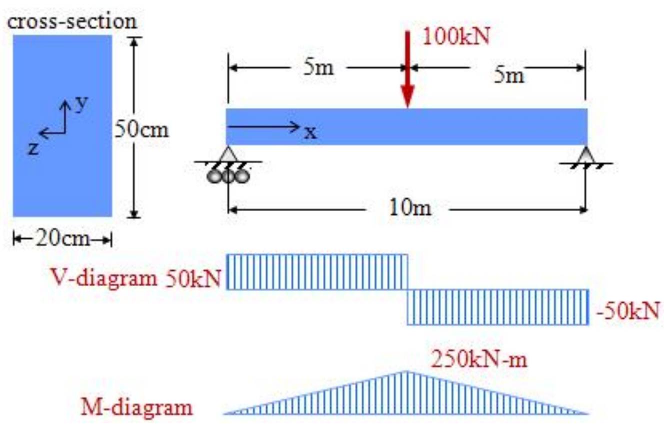


Figure G-24 A simply supported beam loaded at midspan.

The shear force diagram for this beam is piece-wise constant, and the bending moment diagram is linear from the left end to the centerline, and from the centerline to the right end. The bending moment is zero at the two ends. The two reactions are equal to 50kN each, and the bending moment at the centerline is (50 kN) × (5 m)=250 kNm. Using the flexure formula, the y_{max} is used, i.e. (50 cm)/2=25 cm.

$$I = \frac{(\text{base})(\text{height})^3}{12} = \frac{0.2 \times 0.5^3}{12} = 0.0020833 \text{ m}^4$$

$$\sigma_x = -\frac{My}{I} \Rightarrow \sigma_x^{max} = -\frac{My_{max}}{I} = -\frac{M}{S}$$

For this problem, due to symmetry of the cross section the maximum tensile stress and the maximum compressive stress are equal. The section modulus is $S = I/y_{max} = 0.0020833/0.25 \text{ m}^3$.

$$\sigma_x^{max-tensile} = -\sigma_x^{max-compressive} = -\frac{My_{max}}{I} = -\frac{M}{S} = \frac{250.0 \times 0.25}{0.0020833} = 30{,}000.5 \text{ Pa}$$

Example-Extreme Bending Stresses: The beam shown in Figure G-25 is made of a plastic (nylon) for which the allowable stress is 24.0 MPa in tension and 30.0 MPa in compression. Determine the highest possible value of P such that the loads shown can be applied.

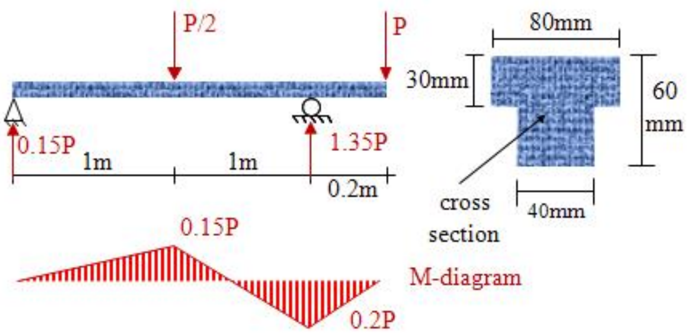


Figure G-25 A simply supported beam with a cantilever end. The reactions and the M-diagram are shown.

The first step is find the support reactions, and then draw the M-diagram. Also the centroid of the cross section is needed. The centroid, for y-coordinate from the top of the cross section directed downwards is evaluated first, and the moment of inertia I is evaluated with respect to the centroid, using the parallel axes theorem; d denoting the distance of a sub-area of the cross section from the global centroid of the cross section

$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{(80\times 30)15+(40\times 30)45}{(80\times 30)+(40\times 30)} = 25 \text{ mm}$$

$$I = \sum \left(I_{0_i} + A_i d_{y_i}^2\right) = \frac{80\times 30^3}{12} + (80\times 30)10^2 + \frac{40\times 30^3}{12} + (40\times 30)20^2 = 9.9\times 10^{-7}\text{m}^4$$

In order to find the maximum allowable P, both the positive moment of 0.15P and the negative moment of –0.2P have to be examined. This is because the cross section is not symmetric in the y-direction, and also the allowable stresses in tension and compression are different.

At moment +0.15P, the extreme stress is set equal to the allowable stress, thus, since we have two extreme stresses, one compressive and one tensile

$$\sigma_x^{max} = \sigma_x^{allowable-compression} = -\frac{My_{max}}{I} \Rightarrow \frac{0.15P\times 0.025}{9.9\times 10^{-7}} = 30 \Rightarrow P = 0.00792 \text{ MN}$$

$$\sigma_x^{max} = \sigma_x^{allowable-tension} = -\frac{My_{max}}{I} \Rightarrow \frac{0.15P\times 0.035}{9.9\times 10^{-7}} = 24 \Rightarrow P = 0.0045 \text{ MN}$$

Similarly, at moment of -0.2P,

$$\sigma_x^{max} = \sigma_x^{allowable-compression} = -\frac{My_{max}}{I} \Rightarrow \frac{0.2P\times 0.035}{9.9\times 10^{-7}} = 30 \Rightarrow P = 0.00424 \text{ MN}$$

$$\sigma_x^{max} = \sigma_x^{allowable-tension} = -\frac{My_{max}}{I} \Rightarrow \frac{0.2P\times 0.025}{9.9\times 10^{-7}} = 24 \Rightarrow P = 0.0047 \text{ MN}$$

Thus, the maximum allowable P is 0.00424 MN or 4.24kN.

Module 20: Shear Stresses in Beams

So far we found out that the bending moment M acting on a beam creates normal stresses σ_x. However, since M and shear force V are related, there are also V forces acting on a cross section in addition to M. We now study the effects of V on a beam.

Consider a beam which at a cross section is subjected to shear force V and bending moment M. As shown in Figure G-26, at a particular element in the cross section of area dA, M (not shown) creates σ_x, and let us suppose that V (also not shown) creates a shear stress on dA as denoted as τ_{xy}. At the right in Figure G-26 a volume element of side dA is shown as well as the shear forces acting on it. As shown in previous sections, due to symmetry of shear stress, when one such stress acts on one side of a material volume element, the same shear stresses must act as shown in Figure G-26.

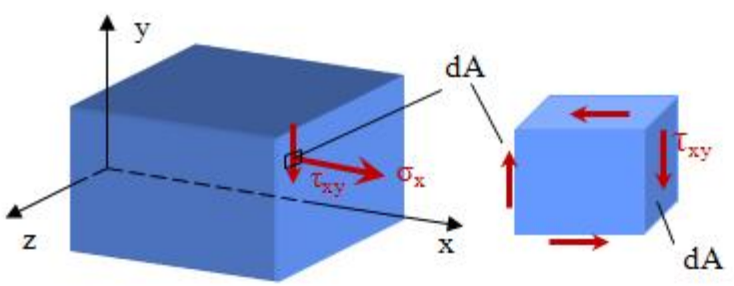


Figure G-26 A cross section of a beam and an element of area dA. The volume element of side dA is shown to the right with the shear stresses acting on it.

Now, for simplicity, let us consider a beam of rectangular cross section, of base b and height h, and then consider more general cross sectional areas. The force V is distributed as shear stress over the entire cross sectional area, but we do not know this distribution yet. The aim here is to find the distribution of V over the cross sectional area b × h. As shown in Figure G-26, any element of area dA is subjected to shear stress τ_{xy}. We now consider a piece of the beam of length dx, as shown in Figure G-27, and further a horizontal "cut" isolating the piece of the beam shown in red/dark. Since shear stress τ_{xy} acts along the cross section, the same shear stress τ_{xy} must act along the horizontal cut.

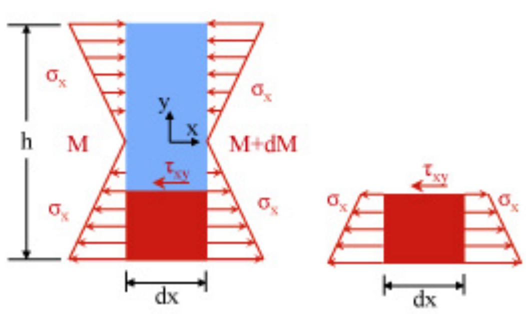


Figure G-27 A piece of a beam of length dx and the stresses acting on it from the bending moment M and M + dM. To the right the free body diagram of a piece below a certain distance y is shown.

The normal stresses are expressed as

$$\sigma_x = -\frac{My}{I}$$

Equilibrium of a piece of the beam (red/dark rectangle in Figure G-27) of length dx implies that in the x-direction the following must hold

$$\sum F_x = 0 \Rightarrow \int_A \frac{My}{I} dA + \tau_{xy} b dx = \int_A \frac{(M + dM)y}{I} dA$$

where A is the area of the piece considered transverse to the x-axis, i.e. where the normal stresses σ_x act, and b is the width of the cross section. By solving this equation for the term that contains the shear stress τ, it follows that

$$\tau_{xy} b dx = \int_A \frac{(M + dM)y}{I} dA - \int_A \frac{My}{I} dA = \int_A \frac{dMy}{I} dA \Rightarrow \tau_{xy} = \frac{dM}{dx} \frac{1}{Ib} \int_A y dA$$

But, as shown in previous sections

$$V = \frac{dM}{dx}$$

From the previous two equations, it follows that

$$\tau_{xy} = \frac{V}{Ib} \int_A y dA$$

where the so-called moment of area Q is defined as

$$Q = \int_A y dA$$

Here, Q is the first moment of the portion of the cross section below the point where the shear stress τ_{xy} is evaluated (in the red/dark part of the beam in Figure G-27) with respect to the centroid of the cross section. The above equations yield

$$\tau_{xy} = \frac{VQ}{Ib}$$

Example: Let us consider the general case of a rectangular cross section as shown in Figure G-28. The width of the beam is b, the height h, and the shear stress τ_{xy} is to be evaluated at a distance y_1 from the centroid.

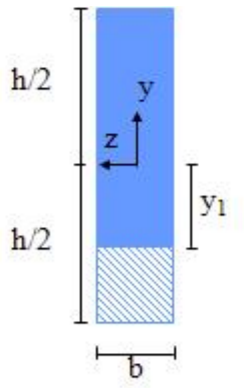


Figure G-28 A rectangular cross section for which the shear stress is to be evaluated at a distance y_1 from the centroid.

Q in this case is the moment of the area below $y = y_1$ (the shaded area in Figure G-28) with respect to the centroid of the cross section. Thus,

$$Q = b\left(\frac{h}{2} - y_1\right)\left(y_1 + \frac{\frac{h}{2} - y_1}{2}\right) = \frac{b}{2}\left(\frac{h^2}{4} - y_1^2\right)$$

Then, for this beam, it follows that

$$\tau_{xy} = \frac{V}{2I}\left(\frac{h^2}{4} - y_1^2\right)$$

which shows a quadratic variation of shear stress along y. Figure G-29 shows the plot of shear stress τ_{xy} as a function of y, i.e. for y varying from -h/2 to +h/2. It is parabolic, and the maximum shear stress occurs for y = 0, i.e.

$$\tau_{xy}^{max} = \tau_{xy}\Big|_{y=0} = \frac{Vh^2}{8I} = \frac{3V}{2A} = 1.5\,\frac{V}{A}$$

It is noted that the average shear stress on the cross section is V/A, thus the average stress is less than the maximum stress, or the maximum stress is 50% higher than the average stress in this case. Also, it is important to note that the shear stress is zero at both the top and bottom of the beam. The mathematical reason for this is that the Q at those positions is zero since it is the moment of the entire area A of the cross section with respect to the centroid of A. The physical reason is that if the shear stress at these positions was not zero, the symmetry of shear stress would imply that there is a nonzero shear stress at the top and bottom of the beam, which is not true.

Example-beam stress analysis: The cantilever beam shown in Figure G-29 is loaded by a constant distributed load q. For this beam, the allowable bending stress, σ_{allow}, and the allowable shear stress, τ_{allow}, are given. Stress analysis calls for determining the allowable load q_{all} so that the σ_{allow} is not exceeded, and also the q_{all} so that the τ_{allow} is not exceeded. Further analysis calls for determining for what beam length L_0 both the allowable shear stress and allowable bending stress are reached concurrently.

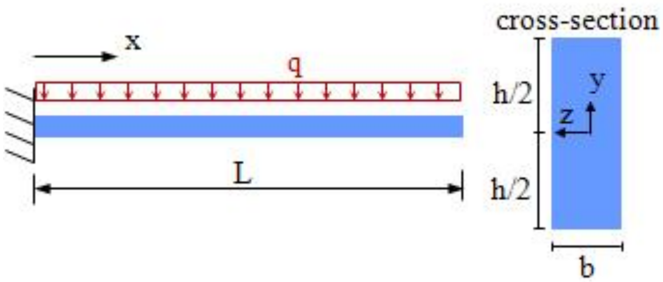


Figure G-29 A cantilever beam of length L and rectangular cross section (b × h) loaded by a constant load q.

For this beam, the maximum shear force and maximum moment occur at the fixed end. For the maximum shear force, $V_{max} = qL$ holds, and for the rectangular cross section,

$$\tau_{max} = 1.5\frac{V_{max}}{A}$$

holds, where A is the area of the cross section. It is repeated that this expression for the maximum shear stress holds only for rectangular cross sections. Then, by setting the τ_{allow} equal to the τ_{max}, it follows that

$$\tau_{all} = 1.5\frac{qL}{bh} \Rightarrow q_{all} = \frac{1}{1.5}\frac{\tau_{all}bh}{L}$$

For the maximum moment, $M_{max} = 0.5qL$ holds, and for the rectangular cross section,

$$\sigma_{max} = \frac{M_{max}}{S} = \frac{6M_{max}}{bh^2}$$

Then, by setting σ_{allow} equal to the σ_{max}, it follows that

$$\sigma_{all} = \frac{3qL^2}{bh^2} \Rightarrow q_{all} = \frac{\sigma_{all}bh^2}{3L^2}$$

For the last part of this example, the following must hold

$$\frac{1}{1.5}\frac{\tau_{all}bh}{L} = \frac{\sigma_{all}bh^2}{3L^2} \Rightarrow L = \frac{h\sigma_{all}}{\tau_{all}}$$

Example-shear stress: For the beam shown in Figure G-30, the problem in this example is to find the maximum shear stress, and the location in the cross section where that stress occurs.

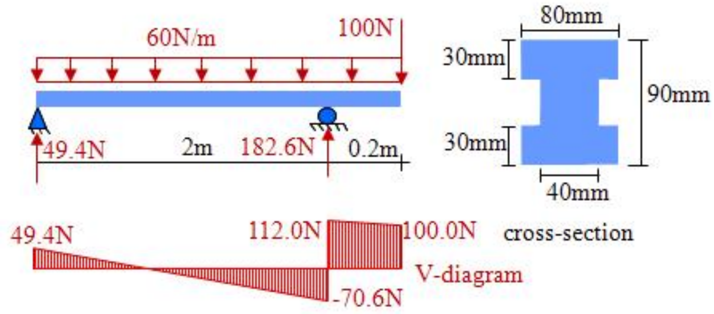


Figure G-30 A beam, its supports, its cross section, and its V-diagram.

The first and most important step in this problem is to find the reactions at the supports, shown in Figure G-30. With the reactions, the V-diagram is drawn, from which it follows that the maximum shear force is V_{max} = 112N. The shear stress is given by $\tau = VQ/Ib$. Since the cross section is symmetric, in order to evaluate the moment of inertia of the cross section, I, it is not necessary to apply the parallel axes theorem. Thus, calculating I for the entire 80 mm × 90 mm area and subtracting I of the empty space, it follows that

$$I = \frac{80\times 90^3}{12} - \frac{40\times 30^3}{12} = 4{,}770{,}000 \text{ mm}^4$$

Maximum shear stress occurs at the locations where Q is maximum, i.e. at the centroid of the cross section. Then,

$$Q = \sum(\text{area} \times \text{distance}) = 80\times 30\times 30 + 40\times 15\times 7.5 = 76{,}500 \text{ mm}^4$$

and

$$\tau_{max} = \frac{V_{max}Q_{max}}{Ib} = \frac{112\times 76{,}500}{4{,}770{,}000\times 40} = 0.0449 \text{ N/mm}^2 = 44.9\text{kPa}$$

Example-normal and shear stresses in beams: The simply supported I-beam shown in Figure G-31 is loaded at mid-span by P = 15 KN. In this example: (a) find the maximum shear stress at A, at the interface between the flange and the web; and (b) determine length L so that the maximum compressive stress is 10 MPa.

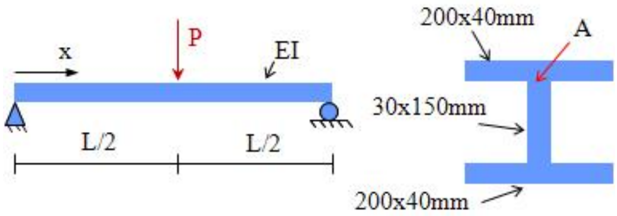


Figure G-31 The I-beam is simply supported and loaded at midspan.

For this beam, the maximum shear force is $V_{max} = P/2 = 7.5$ kN, and the maximum bending moment is $M_{max} = PL/4 = 3.75L$ kNm. In order to evaluate the maximum shear stresses, the values of I and Q (at point A), denoted as Q_A are needed

$$I = \frac{0.2 \times 0.23^3}{12} - \frac{0.17 \times 0.15^3}{12} = 1.5497 \times 10^{-4}\ \text{m}^4$$

$$Q_A = (0.2 \times 0.04)0.095 = 0.00076\ \text{m}^3$$

With these values, the shear stress at A is evaluated as

$$\tau_{max}^{A} = \frac{V_{max} Q_A}{I b_A} = \frac{7.5 \times 0.00076}{1.5497 \times 10^{-4} \times 0.03} = 1226\ \text{kPa}$$

where b_A denotes the width of the cross section at A. The minimum value of b_A is used (width of the web) since this provides maximum shear stress. Now, the maximum normal stress can be evaluated for part (b) of this example

$$\sigma_{max} = \frac{M_{max} y_{max}}{I} = 10{,}000\ \text{kPa} \Rightarrow \frac{3.75L \times \frac{0.23}{2}}{1.5497 \times 10^{-4}} = 10{,}000 \Rightarrow L = 3.59\ \text{m}$$

Example-maximum normal stress in a wooden beam: A wooden beam, shown in Figure G-32, is made of three 2x8s (nominally 1.5" wide × 7.5" tall) capped with a 2 × 6 (nominally 1.5" tall × 5.5" wide) on each end. The beam is assembled with 16d nails. Given this built-up wooden beam, in this example we determine the maximum tensile stress and the cross section where such a stress occurs.

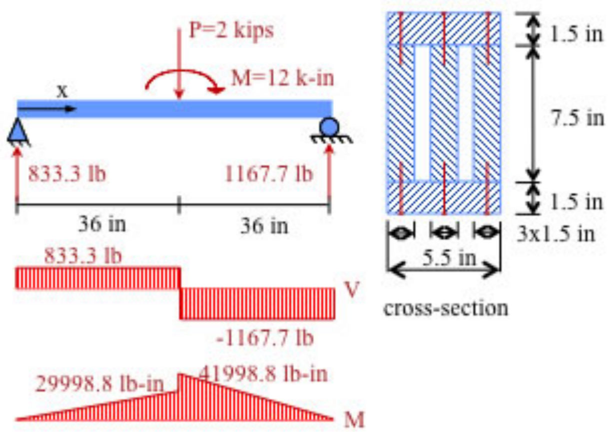


Figure G-32 A simply supported built-up wooden beam, loaded by P and M. The reactions, and the V- and M-diagrams are shown.

Using the static equilibrium equations, the reaction forces are found and are shown in Figure G-32. The maximum bending moment, then, is 41998.8 lb-in. From the cross section, the moment of inertia I is found as the moment of the entire cross section minus the I of the empty space

$$I = \frac{5.5\times(7.5+1.5+1.5)^3}{12} - \frac{(5.5-4.5)\times 7.5^3}{12} = 495.422 \text{ in}^4$$

Then the maximum normal stress is evaluated as

$$\sigma_{max} = \frac{M_{max}y_{max}}{I} = \frac{41{,}998.8\times 5.25}{495.422} = 445.05 \text{ psi}$$

Example-maximum stresses in beams: The I-beam shown in Figure G-33 is loaded with P = 15 KN at midspan. In this example, we will: (a) find the maximum shear stress τ_{xy} at A, i.e., at the interface between the flange and web of the I-beam; and (b) determine the maximum compressive stress σ_x over the entire beam.

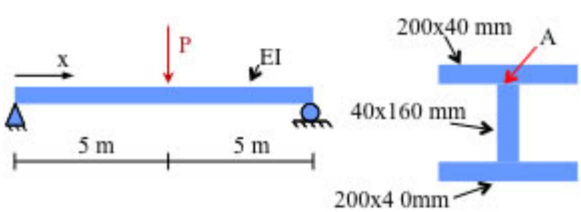


Figure G-33 Simply supported I-beam, loaded at midspan.

For this beam, the maximum shear force is V_{max} = P/2 = 7.5 kN. In order to find the shear stress at A, the moment of inertia I of the cross section and the Q at A, Q_A are needed. They are

$$I = \frac{0.2 \times 0.24^3}{12} - \frac{(0.2-0.04)\times 0.16^3}{12} = 0.0001758 \text{ m}^4$$

$$Q_A = 0.2 \times 0.04 \times \left(0.08 + \frac{0.04}{2}\right) = 0.0008 \text{ m}^3$$

Then at point A,

$$\tau = \frac{VQ_A}{Ib} = \frac{7.5 \times 0.008}{0.0001758 \times 0.04} = 853.24 \text{ kPa}$$

The maximum bending moment for the entire beam is at midspan, and

$$M_{max} = \frac{1}{2} P \times 5 = 37.5 \text{ kNm}$$

Then,

$$\sigma_{max} = \frac{M_{max} y_{max}}{I} = \frac{37.5 \times (0.08 + 0.04)}{0.0001758} = 25597.3 \text{ kPa}$$

Shear strains

The variation of shear stress along the height of the beam is quadratic (parabolic). As explained, the shear stress is zero at the top and at the bottom of the cross section, and, in most cases, it is maximum at the centroid of the cross section. Since shear stresses τ imply that shear strains γ exist, such that γ = τ/G, the cross sections of the beam become warped under the shear stresses. Figure G-34 shows this warping (the deformations in the figure are exaggerated for clarity). Originally plane cross sections become curved under the action of the shear stress. The curve is most pronounced at the centroid of the cross section (where shear stresses are maximum) and there is no curve at the top and bottom of the cross section.

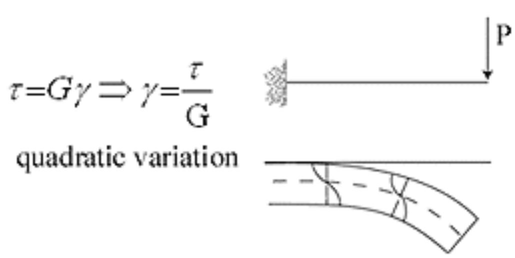


Figure G-34 A cantilever beam subjected to an end load P. The shear force is constant along the length of the beam. Warping of cross sections is shown schematically and is exaggerated for clarity.

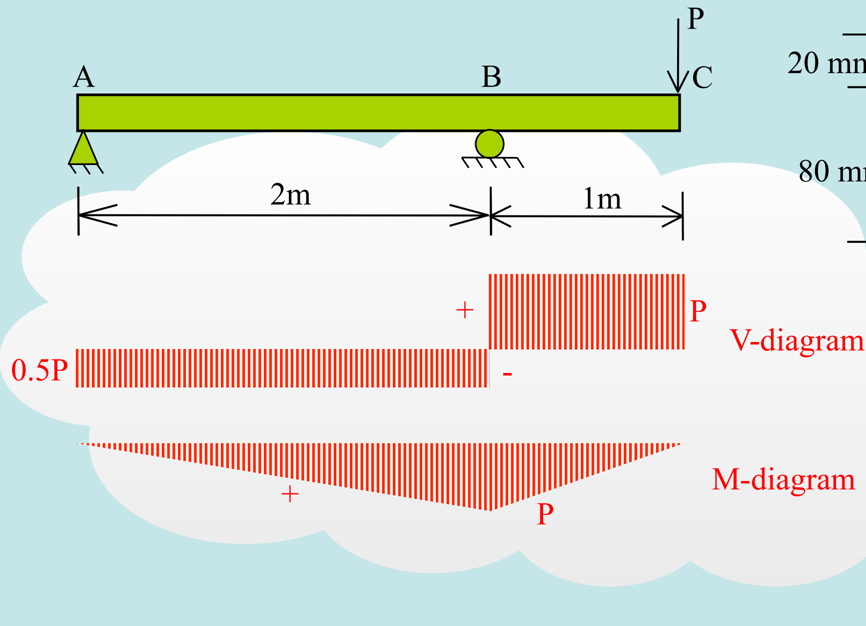
If the shear force V is constant along a certain segment of a beam, warping from shear stress is the same at every cross section in this segment. In this case, the warping does not affect the longitudinal stretching along the length of the beam, thus, the normal strains created from the bending moment M. In cases where V is not constant, advanced theoretical and experiment studies have shown that the effects of warping on normal strains are negligible. This is often mentioned in the mechanics of materials literature as the fact that the distortion of the beam from shear strains is negligible when compared to the distortion from normal stresses.

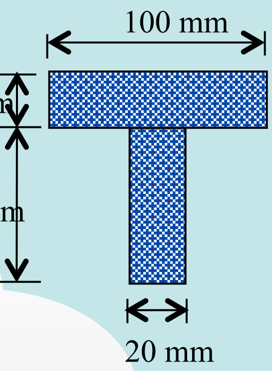
Example-shear strain: In the example shown in Figure G-33, the shear stress at point A was found to be 853.24 kPa. Considering that the material is steel with shear modulus of elasticity G = 80GPa, the shear strain is

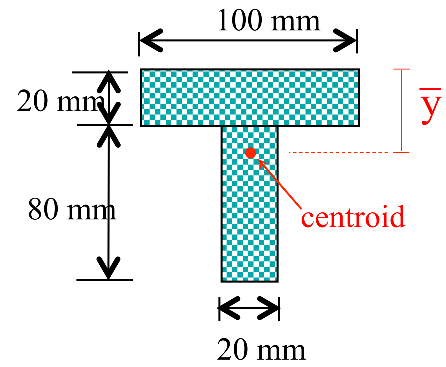
$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{853.24 \times 10^3}{80 \times 10^9} = 0.00001 \text{ rad} = 0.0006^\circ$$

Example: Simply Supported T-Section Beam

Determine the maximum permissible value of P if the allowable bending stress is 40 MPa in tension and 70 MPa in compression.







$$\bar{y} = \frac{2000 \times 10 + 1600 \times 60}{100 \times 20 + 80 \times 20} = 32.2 \text{ mm}$$

$$I = \frac{20 \cdot 80^3}{12} + 1600 \times 27.8^2 + \frac{100 \times 20^3}{12} + 200 \times 22.2^2 =$$

$$3142224 \text{mm}^4 = 3.142 \times 10^{-6} \text{ m}^4$$

$$\sigma_{tens}^{max} = \frac{P \times 32.2}{3142224} = \sigma_{tens}^{all} = 40 \times 10^6 \text{ Pa} = 400 \text{ N / mm}^2 \Rightarrow$$

$P = 3903$ N

$$\sigma_{comp}^{max} = \frac{P \times 67.8}{3142224} = \sigma_{comp}^{all} = 70 \times 10^6 \text{ Pa} = 700 \text{ N / mm}^2 \Rightarrow$$

$P = 3244$ N

Example: Simply Supported Inverted T-section Beam

$$\bar{y} = \frac{200 \times 50 \times 25 + 200 \times 50 \times 150}{2 \times 50 \times 200} = 87.5 \text{ mm}$$

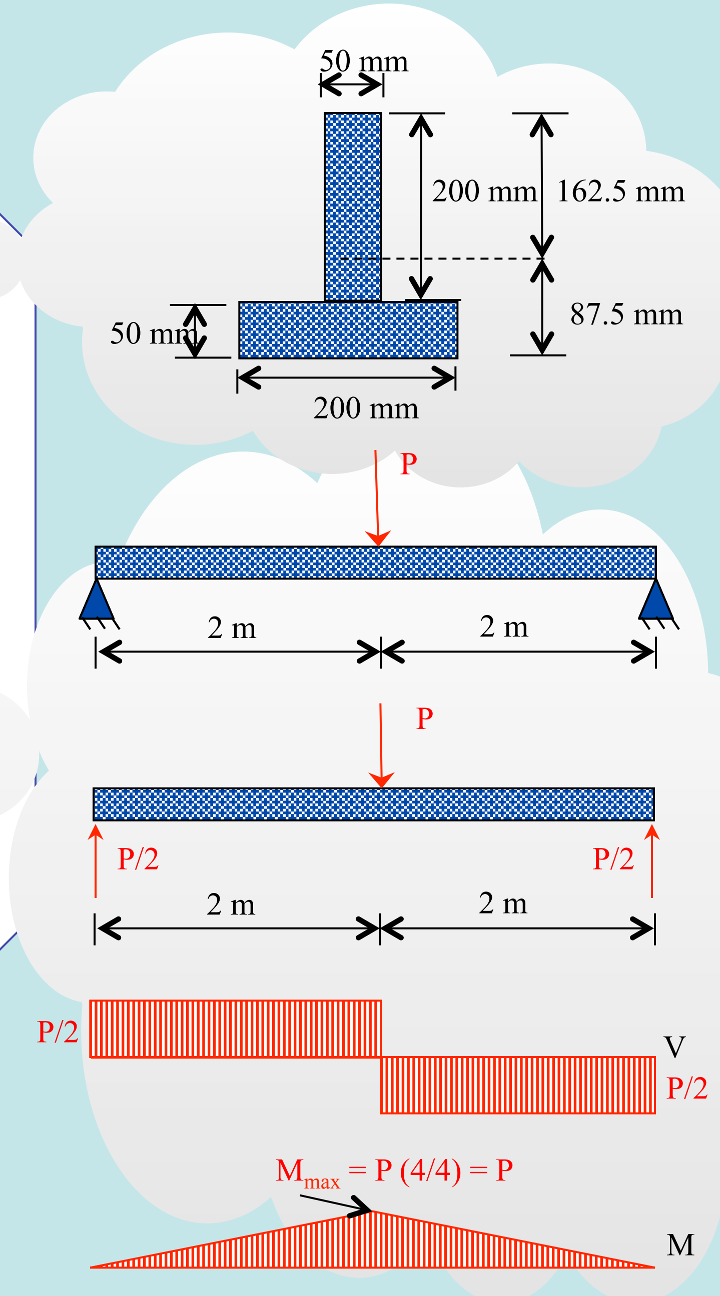
$$I = \frac{0.2 \times 0.05^3}{12} + (0.05 \times 0.2) \times 0.0625^2 + \frac{0.05 \times 0.2^3}{12} + (0.05 \times 0.2) \times 0.0625^2$$

$$= 0.000113542 \text{ m}^4$$

$$\sigma_{max} = 30 \text{ MPa} = \frac{M_{max} \cdot 0.1625}{0.000113542}$$

$$\Rightarrow M_{max} = 20.96 \text{ kN} \cdot \text{m}$$

$$\Rightarrow P = 20.96 \text{ kN}$$



Two 50 × 200 mm structural members are used to fabricate a beam with an inverted T cross-section (flange at the bottom). The beam is simply supported at the ends and is 4 m long.
If the maximum normal stress must be limited to 30 MPa, determine:
(a) the maximum moment that can be resisted by the beam; and
(b) the largest concentrated load P that can be supported at the center of the span.

Example: Selecting Beam Cross Section

$$s_{req} = \frac{M_{max}}{\sigma_{all}} = \frac{140 \times 12}{24} = 70 \text{ in}^3$$

Choose $w\ 16 \times 45$

Check for shear:

Assume web only resists shear

$t_w = 0.25$ in

$d \approx 16$ in

$V_{max} = 35^k$

$$\tau_{ave} = \frac{V_{max}}{A_{web}} = \frac{35}{16 \times 0.25}$$

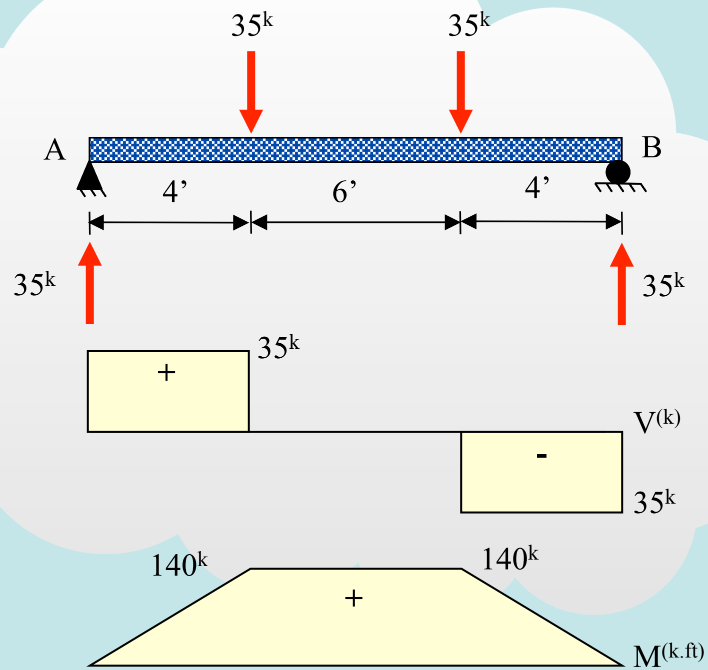
$$\Rightarrow \tau_{ave} = 8.25 \text{ ksi} \leq 14.5 \text{ ksi}$$

OK

or

$$\tau_{max} = \frac{VQ}{Ib} = \frac{35 \times 8 \times 0.25 \times 4}{\frac{1}{12} \times 0.25 \times 16^3 \times 0.25}$$

$$\Rightarrow \tau_{max} = 13.125 \text{ ksi} \leq 14.5 \text{ ksi}$$



Select Beam (AB)?
Given:

$\tau_{all} = 14.5$ ksi

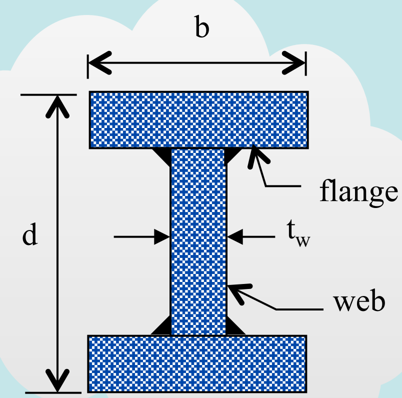
$\sigma_{all} = 24$ ksi

$w\ 18 \times 40 \quad s = 68.4 \text{ in}^3$

$w\ 16 \times 45 \quad s = 72.7 \text{ in}^3$

$w\ 14 \times 43 \quad s = 62.7 \text{ in}^3$

$w\ 10 \times 54 \quad s = 60.0 \text{ in}^3$



Wide flange beam

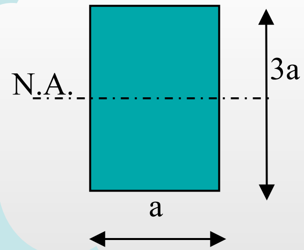
$w\ 18 \times 40$ means

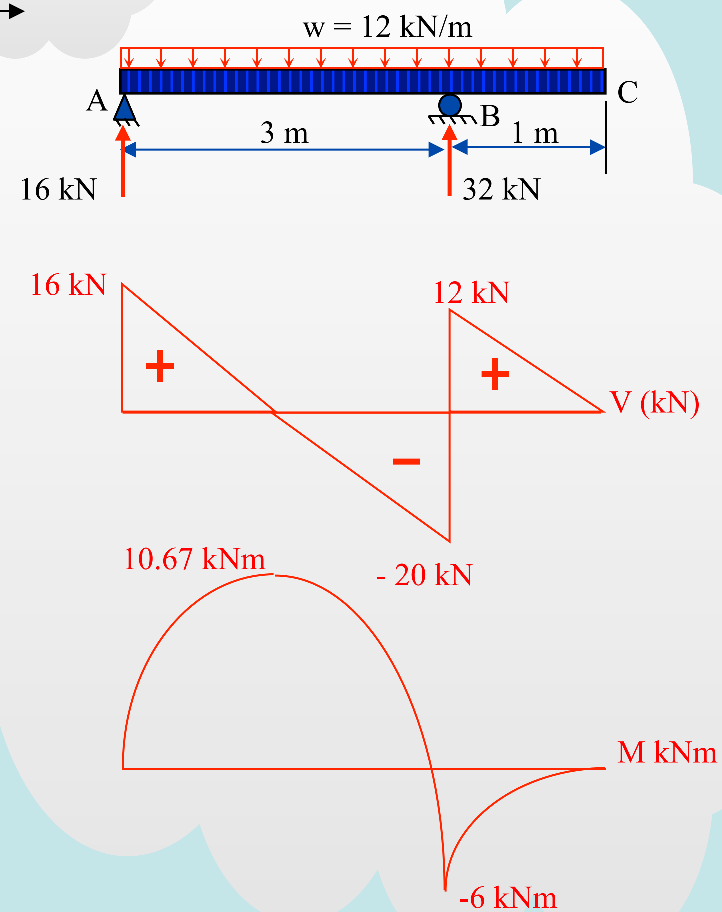
$d = 18$ in

weight/ft = 40 lb

Beam Design Example

Select dimension (a)?





$$s_{req} = \frac{M_{max}}{\sigma_{all}} = \frac{10.67}{9000} = 0.00119 \text{ m}^3$$

$$s_{req} = \frac{I}{c} = \frac{\frac{1}{12}(a)(3a)^3}{1.5a} = 1.5a^3 = 0.00119$$

$$\Rightarrow a_{min} = 0.0925 \text{ m} = 9.25 \text{ cm (say 10 cm)}$$

Check for shear:

$$\tau_{max} = \frac{V_{max}Q}{Ib} \leq 0.6 \text{ MPa}$$

$$\tau_{max} = \frac{20 \times 1000 \times [0.1 \times 0.15 \times 0.075]}{\frac{1}{12} \times 0.1 \times 0.3^3 \times 0.1} = 1 \text{ MPa}$$

1 MPa > 0.6 MPa

N.G.

Select a = 0.12 m (12 cm)

$$\tau_{max} = \frac{20000 \times 0.12 \times 0.18 \times 0.09}{\frac{1}{12} \times 0.12 \times 0.36^3 \times 0.12} = 694{,}444 \text{ Pa}$$

> 0.6 MPa

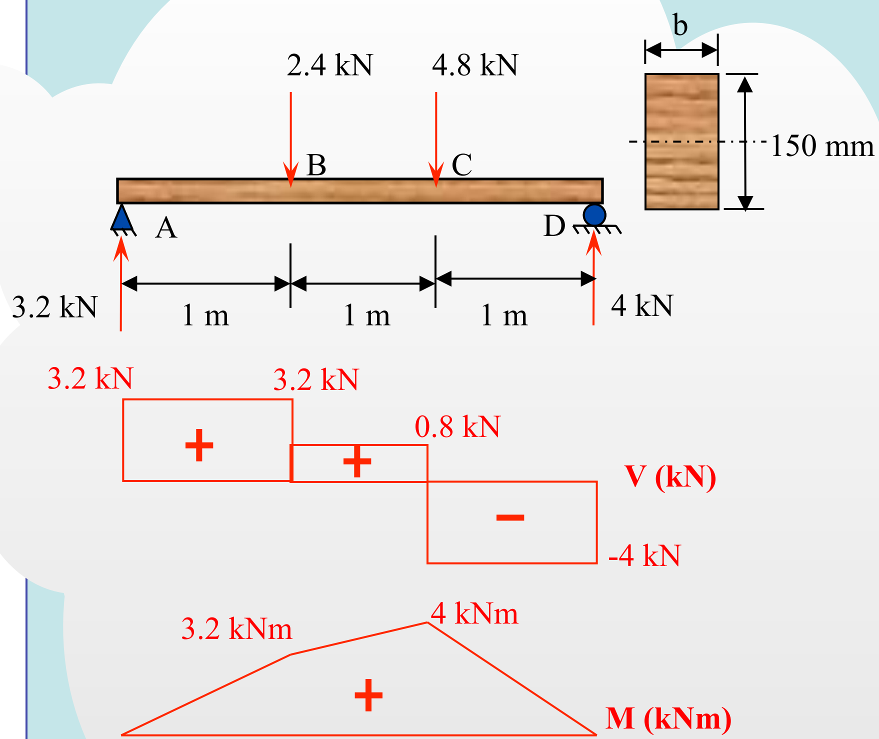
N.G.

Say a = 0.13 m (13 cm)

Beam Design Example

Beam ABCD is loaded as shown below.

Find minimum required width (b), given:



$$\sigma_{all} = 12 \text{ MPa}$$

$$\tau_{all} = 0.9 \text{ MPa}$$

Find reactions:

$R_{Ay} = 3.2 \text{ kN}$

$R_{Dy} = 4.0 \text{ kN}$

Draw shear force and bending moment diagrams

$V_{max} = 4 \text{ kN}$

$M_{max} = 4 \text{ kNm}$

Design for shear:

$$\tau_{all} = \frac{VQ}{Ib} = \frac{4{,}000 \times (0.075\,b) \times (0.075/2)}{b \times (0.15)^3 \times \frac{1}{12} \times b}$$

$$\therefore \tau_{all} = \frac{40{,}000}{b} = 9 \times 10^5$$

$$\Rightarrow b = 0.044 \text{ m}$$

Design for bending:

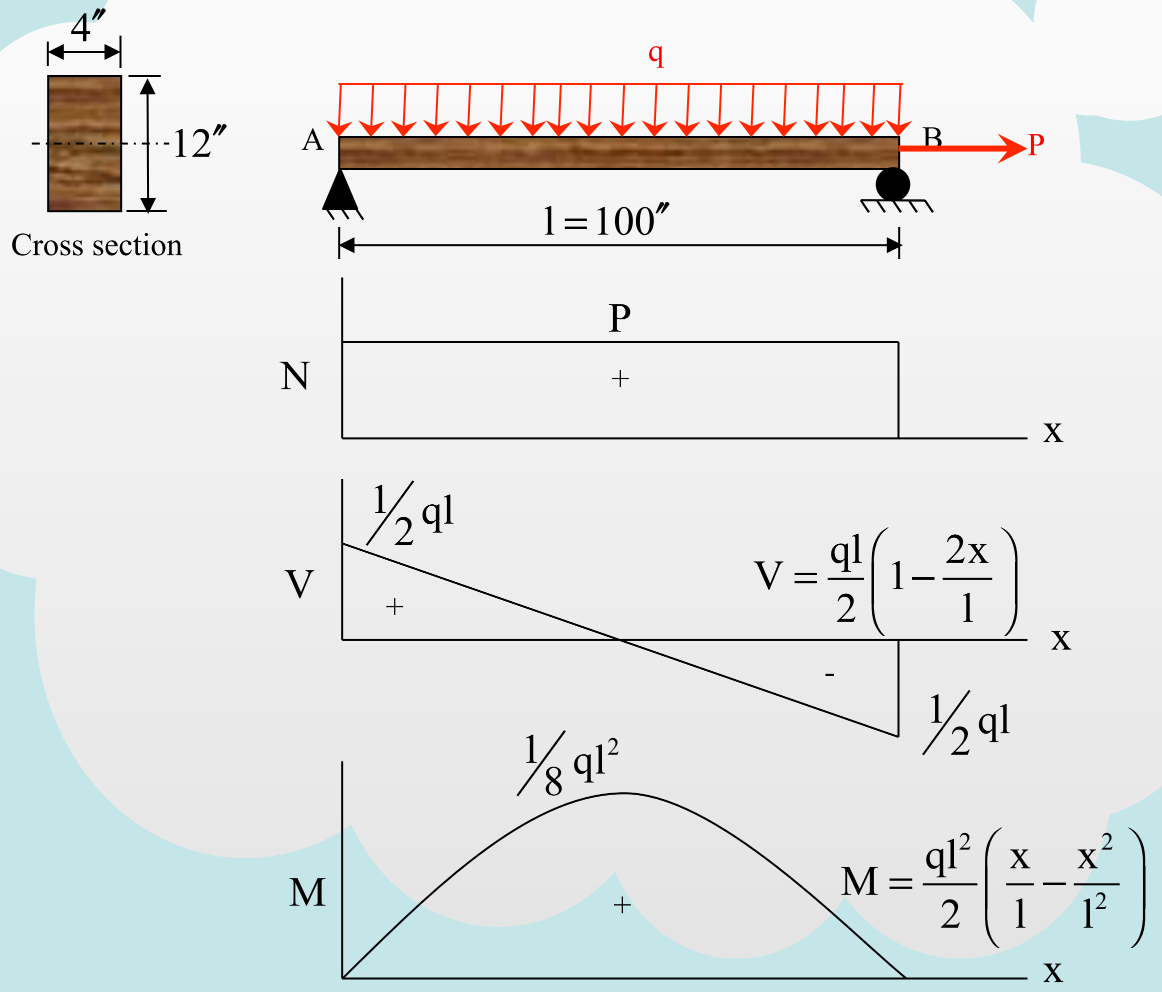
$$\sigma_{all} = \frac{MC}{I} = \frac{4{,}000 \times (0.15/2)}{b \times (0.15)^3 \times \frac{1}{12}} = 12 \times 10^6$$

$$\Rightarrow b = 0.088 \text{ m}$$

Select b = 100 mm

Example: No Compressive Stress in a Beam

For the beam shown below, find the maximum load q such that no compressive stresses are induced at any cross section.
Note: the N-, V-, and M-diagrams are as shown, and P = 120 kips



$$A = 4 \times 12 = 48 \text{ in}^2$$

$$I = \frac{4 \times 12^3}{12} = 4 \times 12^2 = 576 \text{ in}^4$$

$$\text{Max } M = \frac{qL^2}{8} = q\frac{10000}{8} = 1250\text{ q}$$

$$\sigma = \frac{N}{A} + \frac{My}{I}$$

Set $\sigma = 0$ at $y = -6''$

$$0 = \frac{120{,}000}{48} - \frac{1250\text{ q} \times 6}{576}$$

$$\therefore 2500 = 13.02\text{ q}$$

$$\therefore q = 192$$

Example: Beam Bending and Normal Load

$$M_{max} = \frac{1}{2} \times 64,000 \times 5 \times 12 = 1.92 \times 10^6 \text{ lb-in}$$

$$I = 666.7 \text{ in}^4$$

$$A = 20 \times 2 + 2 \times 2 \times 10 = 80 \text{ in}^2$$

$$\sigma_{max}^{compr} = \frac{10,000}{80} + \frac{1.92 \times 10^6}{666.7} \times 3 = 8,764 \text{ psi}$$

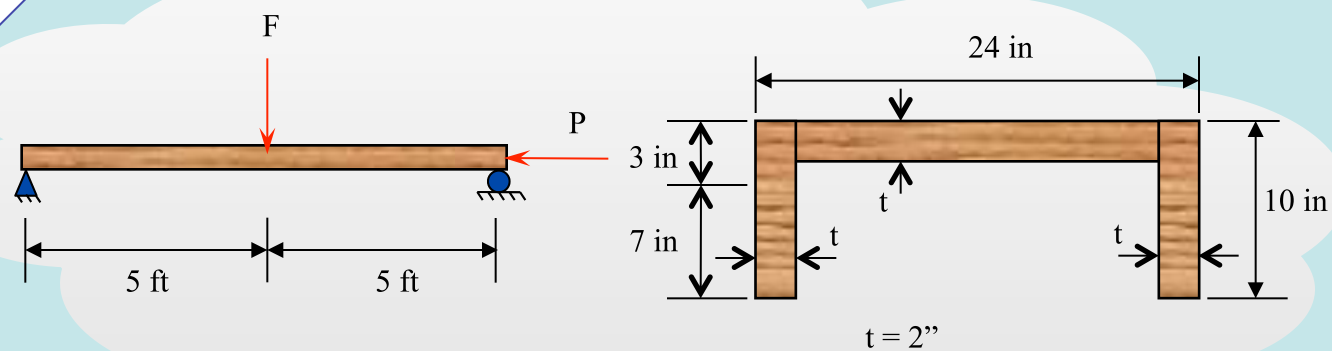
$$\sigma_{max}^{tens} = \frac{1.92 \times 10^6}{666.7} \times 7 - \frac{10,000}{80} = 20,034 \text{ psi}$$

$$\frac{1.92 \times 10^6}{666.7} \times 7 = \frac{P}{80}$$

$$\Rightarrow P = 1.6127 \times 10^6 \text{ lbs}$$

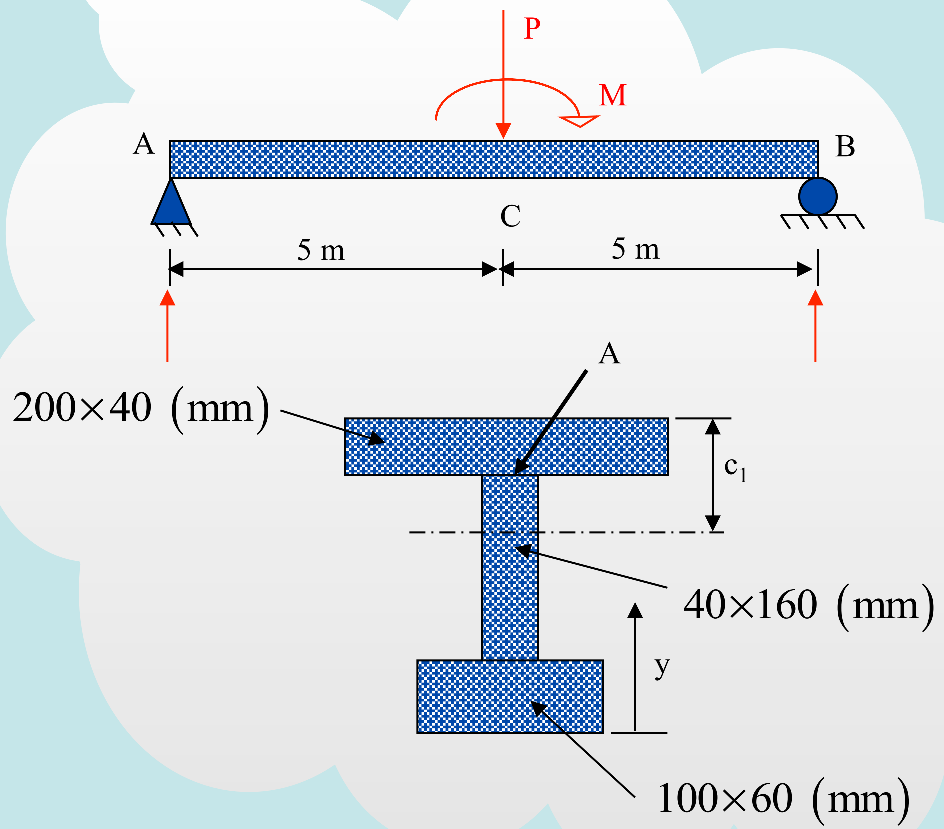
(a) For F = 64 K and P = 10 K determine the maximum tensile and compressive stress in the beam.

(b) For what value of P are there no tensile stresses induced at any cross-section, for F = 64K.



Example: Beam Bending

The built-up beam is loaded with P = 10 kN and M = 6 kNm. Determine the maximum compressive stress at point A, for the cross-section at midspan.



$$A + B = 10 \text{ kN}$$

$$B \times 10 = 10 \times 5 + 6$$

$$\Rightarrow B = 5.6 \text{ kN}, A = 4.4 \text{ kN}$$

$$\text{At midspan, } M_{max} = 5.6 \times 5 = 28 \text{ kN-m}$$

$$\overline{y} = \frac{100 \times 60 \times 30 + 160 \times 40 \times 140 + 200 \times 40 \times 240}{100 \times 60 + 160 \times 40 + 200 \times 40}$$

$$\overline{y} = 146.863$$

$$c_1 = 260 - 146.863 = 113.137$$

$$I = \frac{100 \times 60^3}{12} + (146.863 - 30)^2 \times 100 \times 60$$

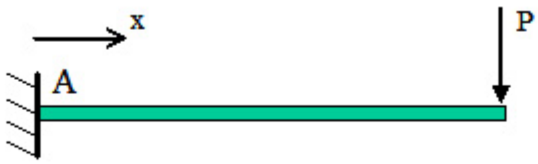
$$+ \frac{40 \times 160^3}{12} + (146.863 - 140)^2 \times 40 \times 160$$

$$+ \frac{200 \times 40^3}{12} + (146.863 - 240)^2 \times 200 \times 40$$

$$\therefore I = 1.68159 \times 10^3 \text{ mm}^4 = 0.00016816 \text{ m}^4$$

$$\sigma_{max} = 28 \times \frac{0.113137 - 0.040}{0.00016816} = 12,178 \text{ kPa}$$

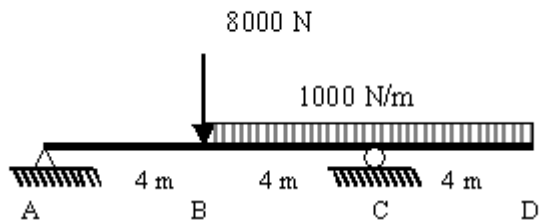
Self Assessment
Beams: 1



The moment diagram for a cantilever beam with a concentrated load on its free end is

- Triangle
- Trapezoid
- Rectangle
- Parabola

Beams: 2



Find the bending moment in kN-m at B.

In order to find the moment at B, the beam is "cut" at B and the free body diagram of the part left of B or right of B must be constructed. Thus, which of 1,2, or 3 is correct?

Draw the free body diagram of part AB or BCD. Which one is more convenient, 4 or 5

1. In order to find the moment at B, I need to evaluate the reactions at A and C

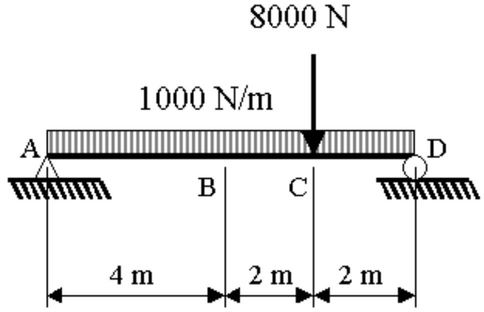
2. In order to find the moment at B, I need to evaluate the reaction at A only

3. In order to find the moment at B, I do not need to evaluate any of the reactions at A or C

4. AB

5. BCD

Beams: 3



The maximum bending moment will occur at point A,B,C,D or at any other point in the beam?

☐ Which of 1,2,3 is the correct way to proceed?

☐ Did you find the bending moment at C to be that shown in 4 or 5 ?

1. My strategy will be that I will draw the entire M-diagram, and from there I will find the maximum bending moment

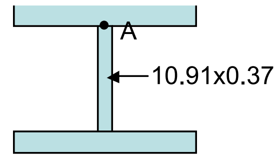
2. In order to find the maximum bending moment, I will find where the V-diagram is equal to zero

3. I will evaluate the moment at B and C, since I know the moment at A and D is zero

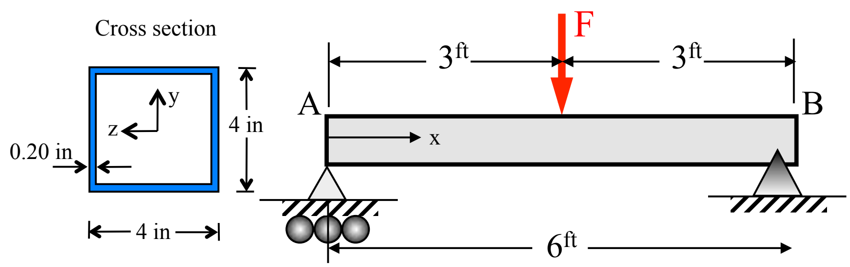
4. 16 kN-m

5. 18 kN-m

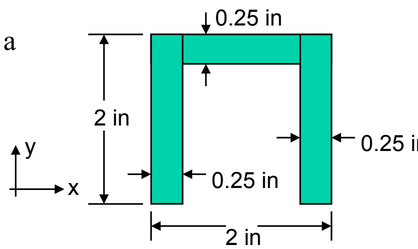
Beams: 1 Find the percentage of the moment carried by the flanges of a W12 × 50 wide-flange beam. (Hint: find the stress distribution for the entire cross-section of the beam and for a hypothetical beam consisting only of the flanges or the web.) $I = 394\ \text{in}^4$



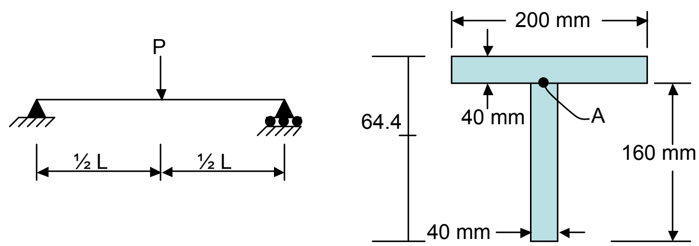
Beams: 2 A simply supported beam of square tube cross section is subjected to load *F* at midspan. Find the maximum load *F* such that $\sigma_x^{max} \le 10\ \text{ksi}$.



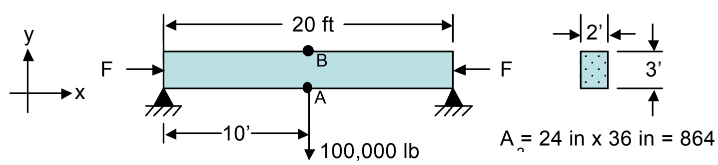
Beams: 3 Find the moment of inertia (I_x) with respect to the centroid in a beam with the given cross section.



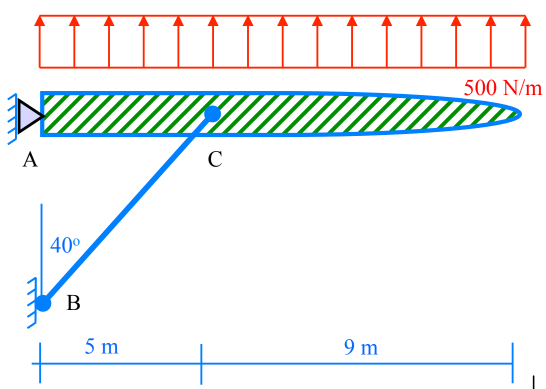
Beams: 4 A built-up beam of T cross-section is loaded with force P = 15 kN. Find the maximum shear stress at A and determine L so that the maximum compressive stress is 10 MPa.



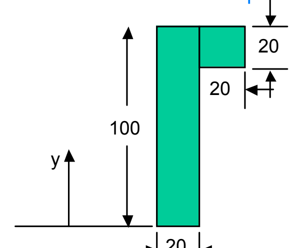
Beams: 5 A concrete beam is being used to support a load as shown. Find the compressive force, F, required to ensure that the concrete is in compression everywhere (no tensile stress).



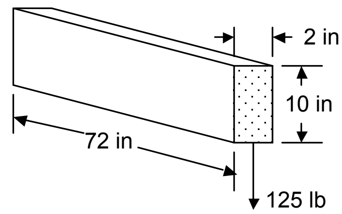
Beams: 6 A strut-braced airplane wing is subjected to a distributed load (lift) as shown. Find the maximum shear load and maximum bending moment, using the shear and bending moment diagrams.



Beams: 7 A beam of cross section as shown (dimensions in centimeters) is loaded by a positive moment M = 800 Nm. Find: (a) the maximum tensile stress σ_x (x-direction is along the length of the beam); (b) the maximum compressive stress σ_x.



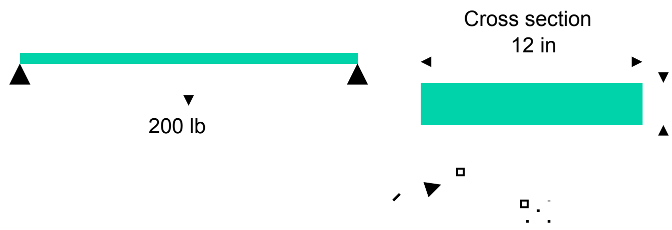
Beams: 8 A support beam on a barn is used for lifting hay into the hay loft. A particularly heavy bale (125 lb) is being hoisted up. Find the shear and normal stress distribution in the beam where it is fixed to the barn wall. ($E = 1.5 \times 10^6$ lb/in^2).



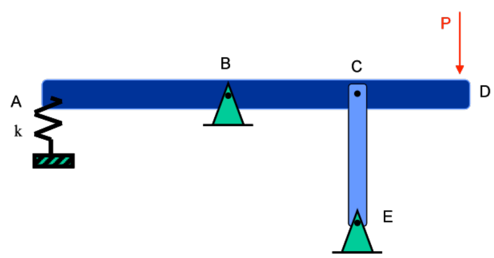
Beams: 9 The walkboard on a scaffold is 12 ft between supports. A 200 lb person is standing in the middle. The walkboard is made from wood ($E = 1.5 \times 10^6$ psi) with 2 in $\times$ 12 in cross-section. Find:

a) Stress at mid-span (max)

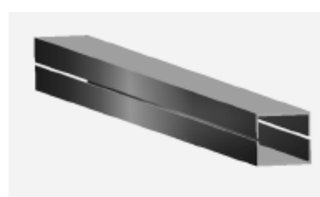
b) Result if board is reinforced with a 2×4 (the 2 inches being in the horizontal direction)

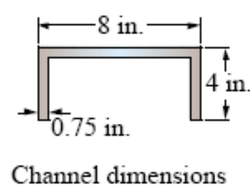


Beams: 10 Beam ABCD is pinned at B, loaded by force P at D and supported through a spring and bar CE at A and C, respectively. Is the beam statically determinate? If statically indeterminate, what is the degree of indeterminacy? Explain your answer.

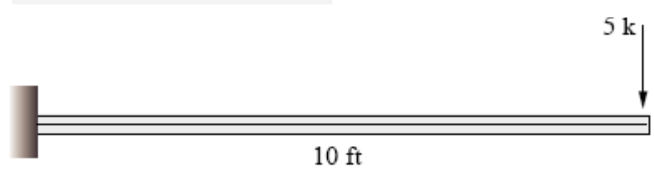


Beams: 11 A rectangular tube is to be fabricated out of two identical channels as shown. (a) Determine the maximum bending stress in this cross section for the loading shown.

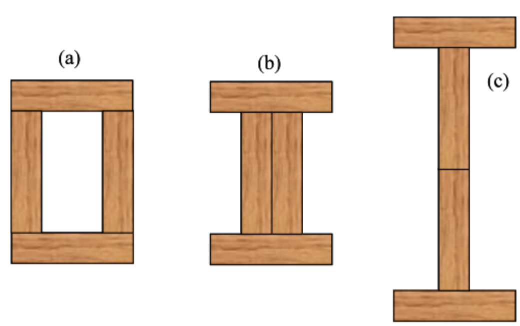




Channel dimensions

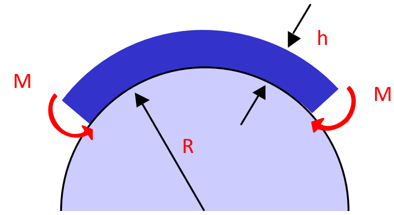


Beams: 12 A built-up beam is to be made by bonding four identical planks together. Each plank has a rectangular cross section 4 cm $\times$ 16 cm. (a) Which configuration will have the highest bending stiffness EI? (b) For the same bending moment M, which configuration will produces the lowest normal stress σ_x (x is along the length of the beam)? (c) For

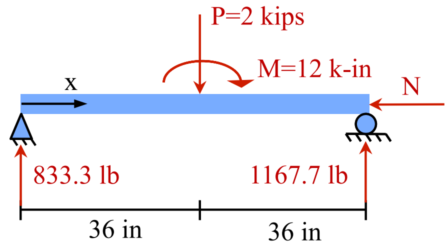


the same shear force V, which configuration produces the lowest shear stress τ_{xy} (y is along the height of the beam)?

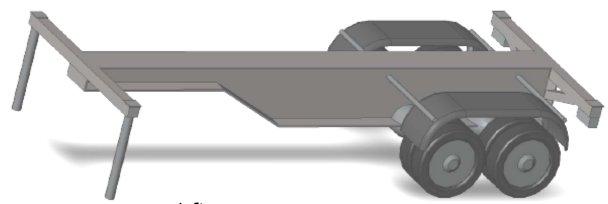
Beams: 13 A steel (E = 29,000 ksi) beam of rectangular cross section is bent over a rigid mandrel (R = 20 in) as shown. If the maximum flexural stress σ_x in the beam is not to exceed the yield strength (σ_y=36 ksi) of the steel, determine the maximum allowable thickness h of the bar.

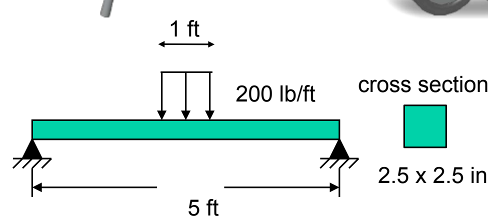


Beams: 14 The wooden (E=1.6 ksi) beam/bar is subjected to the loads P and moment M as shown. For P=2 kips and M=12 k-in, find the minimum axial compressive load N such that there will be no tensile stress σ_x at any location in the beam/bar. The cross section is square, 12 × 12 inches

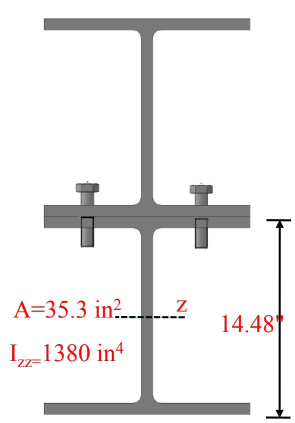


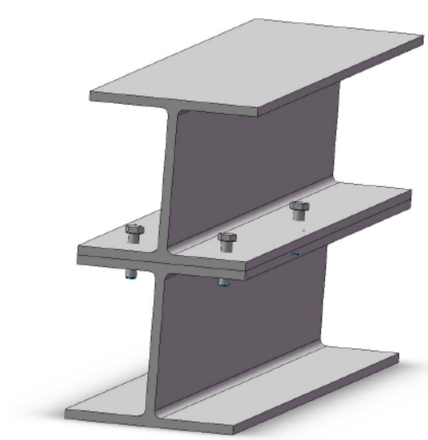
Beams: 15 The "legger" at the left end of the trailer consists of a horizontal beam and two legs. The horizontal beam can be considered as a pinned beam as shown, and the force from the trailer onto the beam is 200 lb/ft imposed over 1.0 ft at the middle of the beam. For the beam cross-section being 2.5″ × 2.5″, calculate: (a) the maximum tensile bending stress σ_x in the entire beam, in psi; and (b) the maximum shear stress τ in the entire beam, in psi.



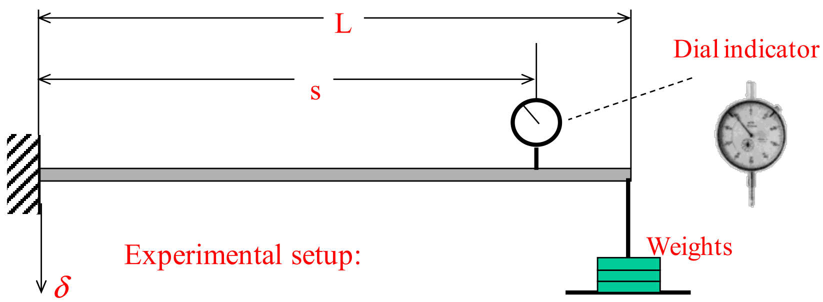


Beams: 16 Two I-beams are bolted together as shown to form a single built-up beam loaded in the vertical direction. Each I-beam has a height of 14.48″, moment of inertia with respect to the centroid I_{zz} = 1,380 in^4 and a cross-sectional area of 35.3 in^2. For an allowable bending stress $\sigma_x^{all} = 20$ ksi, find the maximum moment M the built-up beam can sustain.

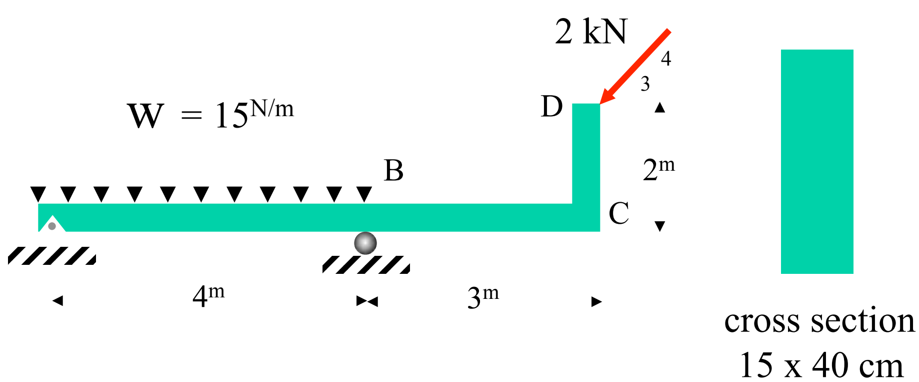




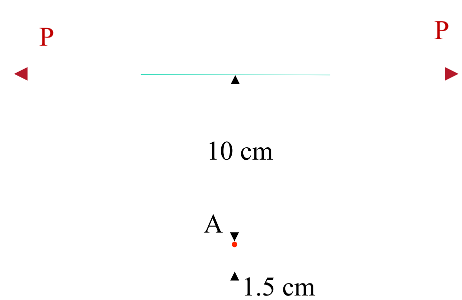
Beams: 17 The cantilever beam of length L and elasticity modulus E is subjected to load W (weights) at its free end. A dial indicator (considered weightless) and other measurements concluded that the radius of curvature at distance s from the fixed end is ρ. Considering that the cross section of the beam is square of side b, find, in terms of E, b, L, s, ρ: (a) the weight W; (b)the maximum tensile stress in the beam; (c) the maximum shear stress in the beam. Note that the moment of inertia of a square cross section of side b with respect to its center is $I = b^4/12$.



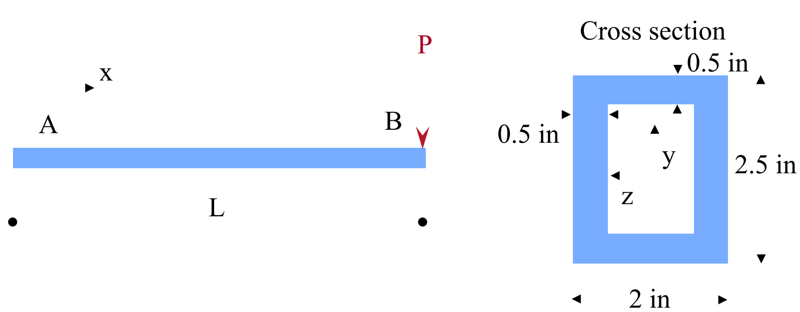
Beams: 18 For the beam loaded as shown, find the stress state at the top of the beam at a cross-section that is an infinitesimal distance to the right of roller B.



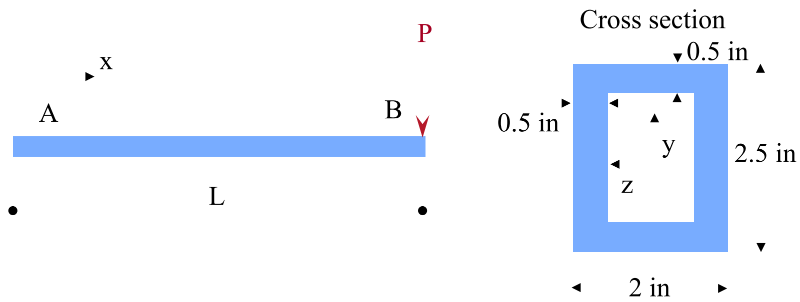
Beams: 19 The C-shaped steel bar (E = 200 GPa) is used as a dynamometer. Knowing that the cross section of the bar is 1.5 cm × 1.5 cm and that the strain was measured on the inner edge at the bottom (point A shown) and was found to be 0.001, determine the magnitude of the forces P.



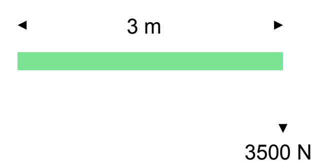
Beams: 20 For the extruded aluminum beam shown it is known that the maximum shear stress in the beam is 10 ksi. Determine the force P. Note that the cross section is symmetric with respect to both y- and z-axis.



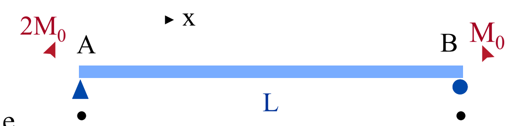
Beams: 21 For the extruded aluminum beam shown the maximum allowable normal stress from bending (tensile or compressive) is $\sigma_{all} = 40{,}000$ psi . For P = 1,600 lbs, find the maximum length L the beam can have such that the allowable stress will not be exceeded. Note that the cross section is symmetric with respect to both y- and z-axis. For the cross-section, the moment of inertia with respect to the z-axis is 1.375 in^4.



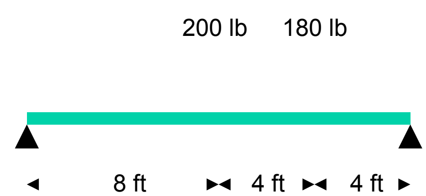
Beams: 22 For the cantilever beam, draw the shear force V and bending moment M diagrams.



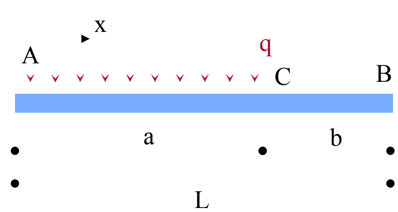
Beams: 23 For the simply supported beam subjected to the two end moments as shown, (a) draw the bending moment diagram; (b) is the shear force diagram constant and equal to the reaction at the left support?



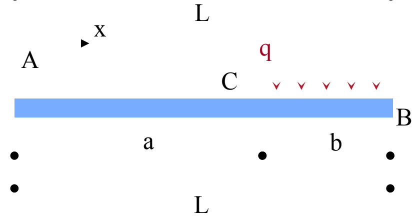
Beams: 24 For the simply supported beam loaded by two concentrated forces of 200 lbs and 180 lbs as shown draw the shear force and bending moment diagrams.



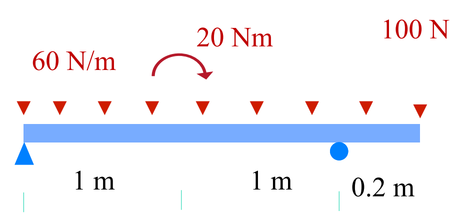
Beams: 25 For the cantilever beam shown, draw the V and M diagram.



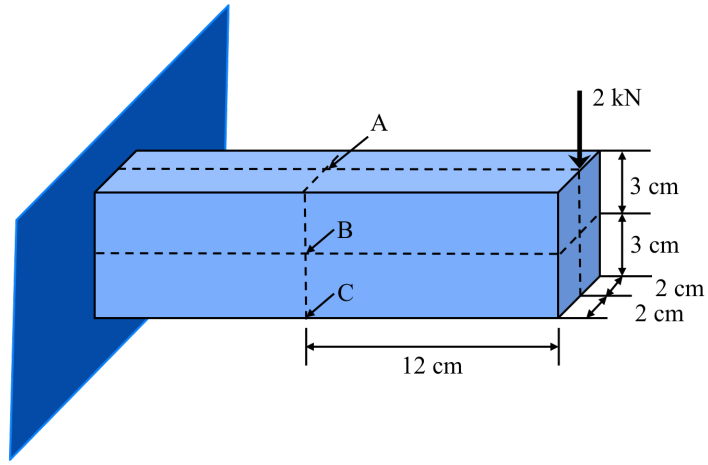
Beams: 26 For the cantilever beam shown, draw the V and M diagram.



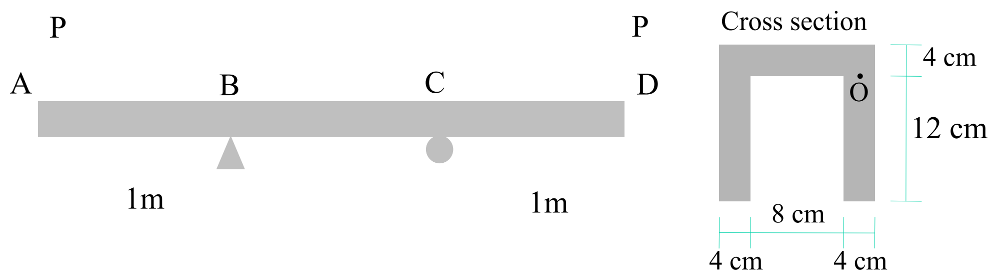
Beams: 27 For the beam shown, draw the V and M diagram. The concentrated moment acts in the middle between the pin and the roller, i.e., 1.0 m from the pin.



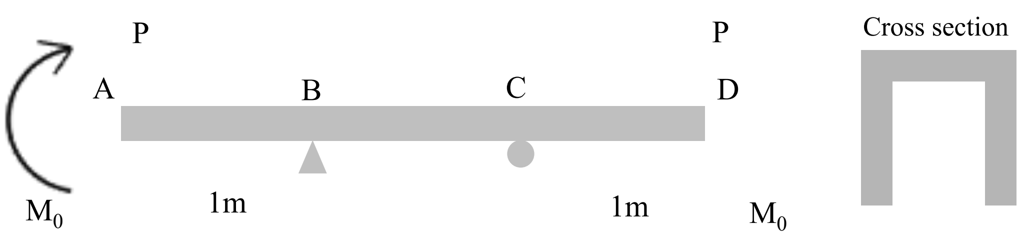
Beams: 28 The cantilever beam is fixed on its left end and loaded by the 2 kN force as shown. Find the principal stresses at material points A,B,C, all of which are on the surface of the beam.



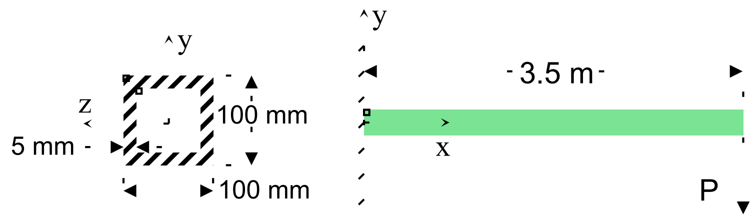
Beams: 29 If the force P is 4.0 kN, determine (a) the maximum normal tensile stress σ_x from bending in the beam; (b) the maximum shear stress τ at the flange, web interface (point O)



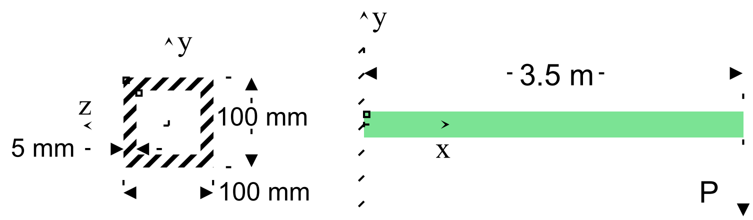
Beams: 30 The beam is designed to bear the two loads P as shown (but not the two moments M_0 shown). It turns out that the material the beam is made of is too weak in shear and it would fail (in shear) under the desired applied loads P. A designer suggests that by applying two properly evaluated equal and opposite moments M_0 as shown the maximum shear stress in the beam can be reduced or can even be reduced to zero. Is this suggestion feasible? Explain your answer



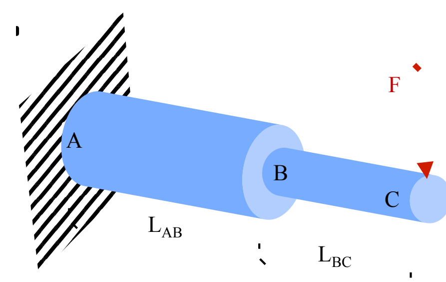
Beams: 31 The cantilever beam of length L = 3.5 m and hollow square cross section is loaded by a force P at the free end. The maximum tensile stress σ_x in the entire bream was found to be 160 MPa. What is the maximum shear stress τ_{xy} in the entire beam?



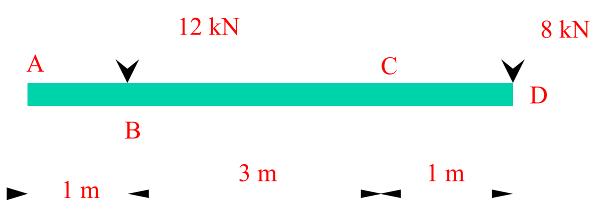
Beams: 32 The cantilever beam of length L = 3.5 m and hollow square cross section is loaded by a force P = 2.0 kN at the free end. (a) At what location(s) (x, y, z) in the beam is the normal tensile stress σ_x maximum? (b) At what location(s) (x, y, z) in the beam is the shear stress τ_{xy} maximum? (c) What is the shear stress τ_{xy} at (x, y, z) = (1m, 0mm, 50mm)?



Beams: 33 The stepped cantilever beam of circular cross section changes diameter from 6.0 cm (radius 3.0 cm) to 3.6 cm (radius 1.8 cm). For the load shown, i.e. F = 500.0 N (F acts transverse to the beam), determine the maximum tensile stress in the beam and find the location of this maximum for two different cases. (a) L_{AB} = 30.0 cm and L_{BC} = 15.0 cm; (b) L_{AB} = 40.0 cm and L_{BC} = 3.5 cm. For a circular cross section of radius r, the moment of inertia with respect to an axis through its centroid is $I_{yy} = I_{zz} = \dfrac{\pi r^4}{4}$

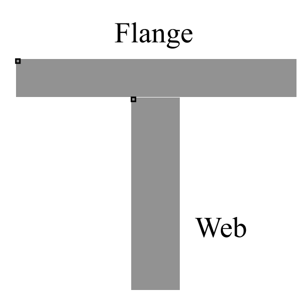


Beams: 34 Draw the shear force and bending moment diagrams.

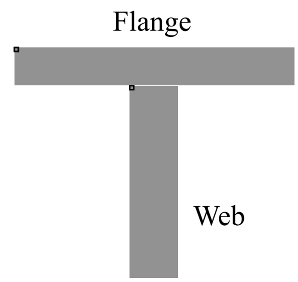


Beams: 35 A T-shaped beam is made by connecting a web and a flange as shown. The beam is subjected to positive moment M. The web dimensions are 3 in. × 8 in. and the flange is 10 in. × 3 in.. Find the ratio of the maximum tensile stress over the maximum compressive stress, i.e.

$$\frac{\sigma_{max}^{tensile}}{\sigma_{max}^{compressive}}$$



Beams: 36 A T-shaped beam is made by connecting a web and a flange as shown. The beam is subjected to positive moment M. The web dimensions are 3 in. $\times$ 8 in. and the thickness of the flange is 3 in. For what width of the flange b will the maximum tensile stress in the beam equal to 2 times the maximum compressive stress?



Module 21: Shear Flow in Beams

The development of shear stresses in beams implies that, in addition to shear stresses in vertical cross sections, shear stresses in horizontal sections along the beam develop as well. The formula

$$\tau_{xy} = \frac{VQ}{Ib}$$

evaluates the shear stress at any point in a cross section where the shear force is V. The same shear stress acts at a horizontal section along the beam at the same point. In the beam shown in Figure H-1, a point or material element A positioned at distance y_A from the neutral axis is subjected to a shear stress on its horizontal planes equal to τ_{xy}. This shear stress tends to split the beam along its length, thus, if there is an interface in the beam, the interface is subjected to this shear stress.

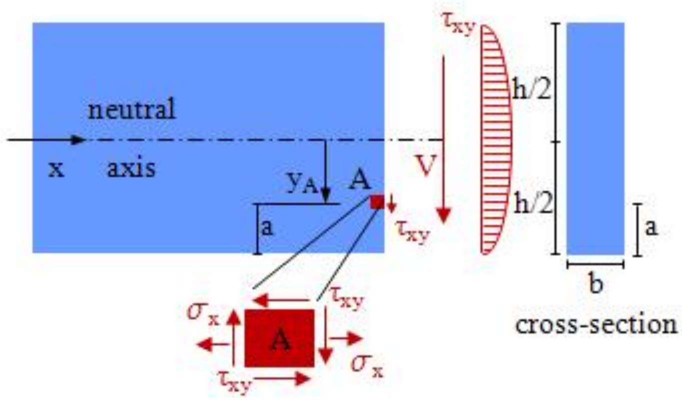


Figure H-1 A beam of rectangular cross section subjected to V, and M. The shear stress distribution and the stresses at a material point A are shown.

This is illustrated in Figure H-2, where the beam is built up by adhering two pieces together. The adhesive at the interface is subjected to shear stress τ_{xy}, called **shear flow**. Built-up beams are common in engineering, where pieces are typically connected by adhesives, welds, bolts, nails, rivets, etc. Engineers typically analyze the beams and design the connecting elements for their ability to withstand the shear flow.

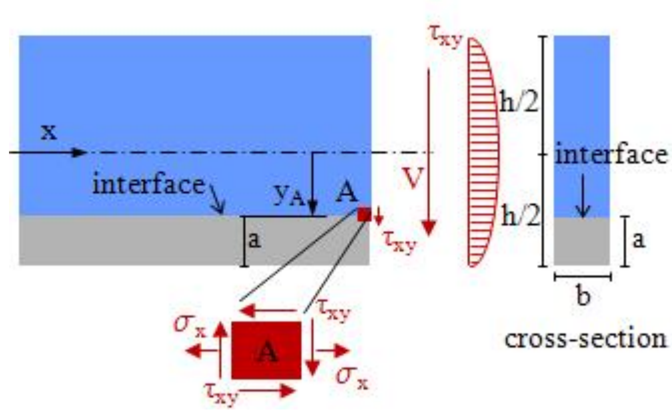


Figure H-2 A beam similar to the one in Figure H-1, yet it is built up of two pieces adhered together at the interface shown.

Some typical built-up beams are shown in Figure H-3. For the I-beam for example, the shear flow is such that the flange tends to slide relevant to the web. This sliding tendency is counteracted by elements connecting the flange to the web (e.g., welds).

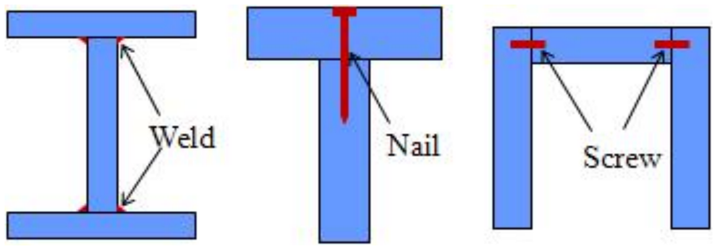


Figure H-3 cross sections of typical built-up beams.

Connections can be continuous, such as welds; discontinuous, such as bolts, screws, nails and rivets; or surface covering such as adhesives. The problem of shear flow addresses the analysis and design of such connecting elements.

In general, there is shear in every arbitrary longitudinal cut, like the cut shown in Figure H-4. If we look at the expression for the shear stress

$$\tau_{xy} = \frac{VQ}{Ib}$$

we can consider that this shear stress acts along the length of the beam. If we multiply both sides of this equation by b, we obtain

$$\tau_{xy} b = \boxed{q = \frac{VQ}{I}}$$

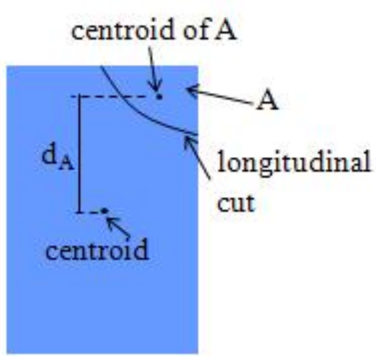


Figure H-4 The rectangular cross section of a beam is shown with an arbitrary cut in a beam along the length of the beam. The area of the cross section above the cut is A, and the distance of the centroid of A from the centroid of the cross section is d_A.

Here, q is the so-called **shear flow**, and its units are **force per unit length**. If we look at the derivation of the expression for τ_{xy} presented in previous sections in detail, it will become evident that it holds for an arbitrary b, which can also be the length of the cut shown in Figure H-4. In this case, the Q value in the expression for the shear flow q is the moment of A with respect to the centroid, or, according to Figure H-4

$$Q = Ad_A$$

The shear flow can be evaluated for every part of a beam's cross section. However, it is important for built-up beams connected through welds, rivets, bolts or nails. In this case, these connections resist the shear stresses (shear flow). For the examples of built-up beams in Figure H-3, Figure H-5 shows the cuts for which the shear flow is to be evaluated. In these cases, Q in the expression for shear flow q is the moment of the area above the cut with respect to the centroid of the cross section. It is noted that the cuts are through the connecting elements (weld, nail, screw). For the I-beam, a similar yet downward facing cut can be made for the welds at the bottom part. Q in this case is the moment of the area below the cut with respect to the centroid of the cross section.

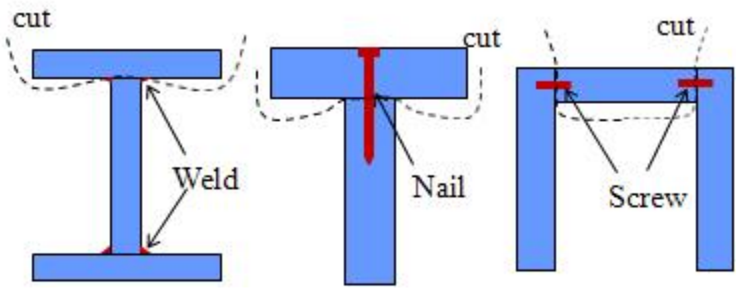


Figure H-5 For the beams in Figure H-3, the cuts required for evaluation of the shear flow q are shown in dashed lines.

Module 21: Shear Flow in Built-up Beams

Built-Up Beams

In general, built-up beams behave as if they were solid ones, provided the connecting elements are able to resist the imposed shear flow effectively. Thus, for the analysis and design of built-up beams, engineers are called to: (1) examine the beam as a single member (in bending, and in shear); (2) examine the connecting elements for resisting the imposed shear flow. The first task is no different than the same task for solid beams. Yet, this part of the analysis provides the shear flow, especially at critical parts of the beam, which can be used for the second task. There are two types of connections. The first one is continuous ones such as welds or adhesives. Here the shear flow q, expressed as force per beam length, has to be lower than the capacity of the connections in withstanding shear force. The second is point-wise connections such as bolts, screws, nails and rivets. Here, the capacity of the connecting elements (units of force) divided by the spacing between the connecting elements should exceed the shear flow. These are best illustrated through examples.

Example of built-up beam with continuous connecting elements

Here we consider a built-up I-beam as shown in Fig. H-6. Each weld has an allowable shear force F=2400lb/in. Given that we have two welds at each connection of the web with the flanges, the problem here is to determine the maximum shear force V this beam can withstand so that the allowable shear force on the welds will not be exceeded. It is reminded that the allowable force F includes a safety factor.

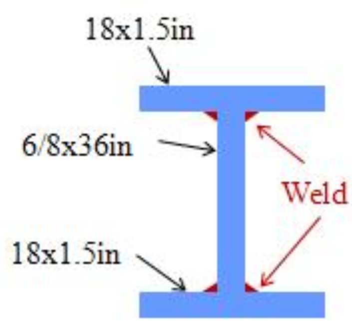


FIG H-6 A built-up I-beam with two welds connecting each flange to the web.

The shear flow is expressed as

$$q = \frac{VQ}{I}$$

where Q has to be evaluated for the area of the cross-section above (or below) the connecting elements. As seen from Fig. H-5, Q for this case is the moment of the area of the flange with respect to the centroid of the cross-section. It is important that the cut through the connecting elements goes through two welds at each case. The shear flow q has units of force per unit length of the beam. Also F has the units of force per unit length. Then, at capacity

$$q = 2F$$

holds. The above two equation yield

$$V = \frac{2FI}{Q}$$

Q is the moment of the area above (or below) the welds with respect to the centroid of the I-beam, and I is the moment of inertia of the entire cross-section

$$Q = y_{flange}A_{flange} = 35.5(1 \times 18) = 639\text{in}^3, \quad I = \frac{18\times72^3}{12} - \frac{17.625\times70^3}{12} = 56{,}090\text{in}^4$$

where y_{flange} denotes the distance from the centroid of the flange to the global centroid and A_{flange} denotes the area of the flange. Then,

$$V = \frac{2FI}{Q} = \frac{2\times400\times56{,}090}{639} = 421{,}000\text{lbs} = 421\text{Kips}$$

which means that the beam can withstand a shear force V of 421K without exceeding the allowable shear stress on the welds.

Example of built-up beam with discrete connecting elements

For the wooden cross-section shown in Fig. H-7, the allowable shear force for each screw is F=250lbs and the maximum shear force the beam is V=920lbs. In this case, the problem is to find the minimum required spacing for the screws, i.e. the distance between screws along the length of the beam.

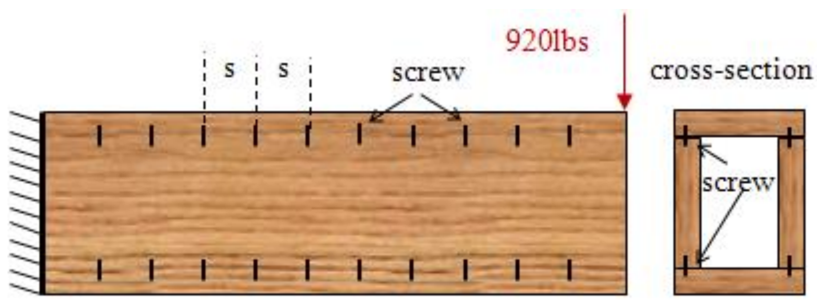


FIG H-7 A built-up wooden cantilever beam, and its cross-section. Each wooden piece is 6x1in, and are connected with 4 screws as shown.

The shear flow is expressed as

$$q = \frac{VQ}{I}$$

where Q has to be evaluated for the area of the cross-section above (or below) the connecting screws. Q for this case is the moment of the area of the top or bottom piece with respect to the centroid of the cross-section. The shear flow q has units of force per unit length of the beam. Here F has the units of force. Let the spacing between screws along the length of the beam be s, as seen in Fig. H-7. Each set of screws is responsible for distance s along the length of the beam. Then

$$q = \frac{2F}{s}$$

holds. The above equation yields

$$s = \frac{2F}{q} = \frac{2F}{\frac{VQ}{I}} = \frac{2FI}{VQ}$$

Q is the moment of the area above (or below) the screws with respect to the centroid of the I-beam, and I is the moment of inertia of the entire cross-section. For the given cross-section

$$Q = 3.5(1 \times 6) = 21\text{in}^3, \quad I = \frac{6\times 8^3}{12} - \frac{4\times 6^3}{12} = 184\text{in}^4$$

which yield s= 4.76in. In a design, a value of s=4.5 or even 4.0in would be chosen.

Example of built-up beam using adhesives

The beam shown in Fig. H-8 is made of an 80x30 mm and an 40x30mm beam glued together. For P=200N, the engineer needs to specify the shear strength of the adhesive (glue) to be used. Herein, we'll find the required glue strength, without accounting for a safety factor, as shear force per unit area.

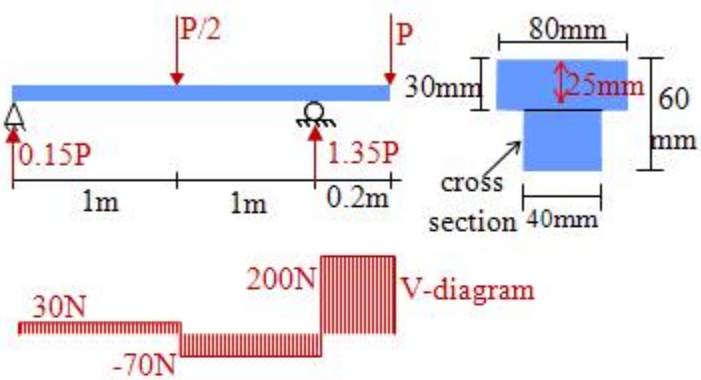


FIG H-8 A beam built-up using adhesives. The reactions and the V-diagram are shown, yielding a maximum V=200N. The centroid of the cross-section is at 25mm from the top.

For this problem, the maximum shear force is 200N, and it is that force that will dominate the design of the glue. The position of the centroid of the cross-section is evaluated first, and is found to be 25mm from the top of the cross-section. The evaluated first, and the moment of inertia I is evaluated with respect to the centroid, using the parallel axes theorem, and is found to be 990,000 mm^4. The adhesive is distributed over the area of the interface, thus it is subjected to the shear shear stress at the interface. Thus, for this problem, evaluation of the shear flow q is not necessary. The shear stress on the adhesive is evaluated as

$$\tau_{xy}^{\text{adhesive}} = \frac{VQ}{Ib} = \frac{200((80\times 30)10)}{990{,}000\times 40} = 0.1212\text{N/mm}^2 = 121.2\text{kPa}$$

It is noted that Q is the moment of the top part (80x60mm) with respect to the centroid of the entire cross section. The distance of the top part from the centroid is 10mm.

Module 21: Examples of Shear Flow in Beams

Example-built-up T-beam: For the T-beam shown in Figure H-9, constructed by connecting the flange to the web using nails, the maximum spacing between the nails needs to be evaluated in order for the beam to withstand a V = 1800 N. Each nail can withstand F = 800 N in shear.

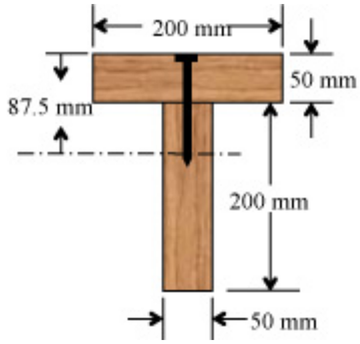


Figure H-9 The cross section of a built-up beam, where the flange and the web are connected by single nails.

This example is similar to the one with the wooden section presented previously (Fig. H-7) yet at every connecting point there is only one nail, while in the example in Figure H-7 there are two screws. Then, instead of 2F, we have F in this problem, and the spacing s is expressed as

$$s = \frac{FI}{VQ}$$

In order to evaluate I and Q, the centroid of the cross section is needed. Using a coordinate from the top of the beam downwards, we have for the distance of the centroid from the top

$$c = \frac{\sum A_i y_i}{\sum A_i} = \frac{(200\times50)25+(200\times50)150}{(200\times50)+(200\times50)} = 87.5 \text{ mm}$$

where A denotes area and y is distance from the top of the cross section. Using the parallel axes theorem, the moment of inertia is evaluated

$$I = \sum I_i + A_i d_i^2 = \frac{200\times50^3}{12} + 200\times50(87.5-25)^2 +$$

$$\frac{50\times200^3}{12} + 200\times50\times37.5^2 = 113.5\times10^6 \text{mm}^4$$

where d_i denotes distance of the centroid of A_i from the centroid of the cross section. Moment Q is evaluated as the moment of the are of the top part with respect to the centroid of the cross section

$$Q = (200\times50)(87.5-25) = 635\times10^3 \text{ mm}^3$$

With the above values

$$s = \frac{FI}{VQ} = \frac{800 \times 113.5 \times 10^6}{1800 \times 625 \times 10^3} = 81 \text{ mm} = 8.1 \text{ cm}$$

Example-built-up wooden beam: A wooden beam (Fig. H-10) is made of three 2 × 8s (nominally 1.5" wide × 7.5" tall) capped with a 2 × 6 (nominally 1.5" tall × 5.5" wide) on each end. The beam is assembled with 16d nails. Given this built-up wooden beam, in this example we determine the maximum allowable nail spacing given that each nail can carry 250 lb of shear per nail.

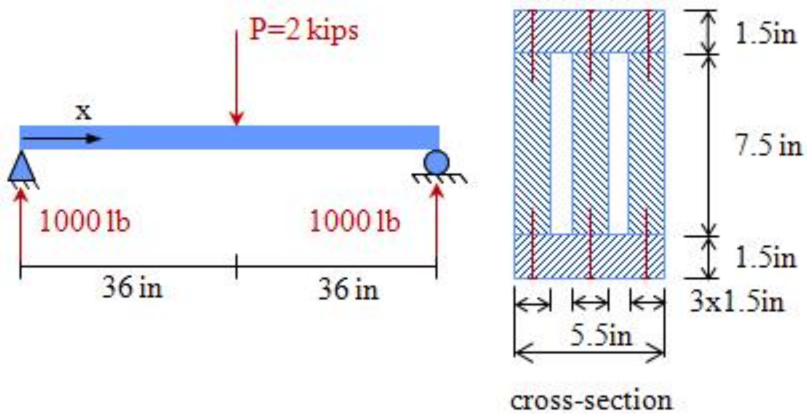


Figure H-10 A built-up wooden beam loaded at midspan.

In the expression for the shear flow for this problem, V = 1 Kip, and Q is the moment of the area of the cross section above the three nails at the interface

$$Q = (5.5 \times 1.5)\left(\frac{7.5}{2} + \frac{1.5}{2}\right) = 37.125 \text{ in}^3$$

$$I = \frac{5.5(7.5+1.5+1.5)^3}{12} - \frac{(5.5-4.5)7.5^3}{12} = 495.422 \text{ in}^4$$

With these values,

$$q = \frac{VQ}{I} = \frac{1 \times 37.125}{495.422} = 0.0749 \text{ Kips/in}$$

Three nails are present to resist the shear flow, thus

$$q = \frac{3F}{s} \Rightarrow s = \frac{3F}{q} = \frac{3 \times 250/1000}{0.0749} = 10.01 \text{ in}$$

where F denotes the shear force in each nail and s denotes the nail spacing. In this case, s = 10 in would be a good choice.

Example: Steel Plates with Rivets

$$V = P/2$$

$$F = 500 \text{ lbs}$$

$$F/4'' = f = \frac{VQ}{I}$$

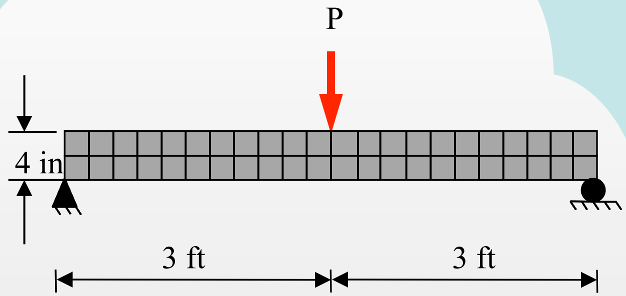
$$\Rightarrow 125 = \frac{VQ}{I}$$

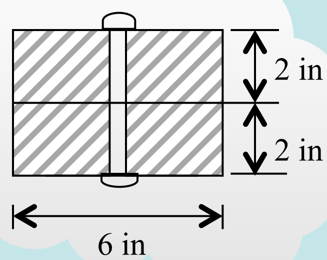
$$Q = 2 \times 6 \times 1 = 12 \text{ in}^3$$

$$I = \frac{6 \times 4^3}{12} = 32 \text{ in}^4$$

$$V = \frac{I}{Q} \times 125 = \frac{32}{12} \times 125 = 333.33$$

$$P = 2V = 666.66 \text{ lbs}$$





Two steel plates 2 × 6 in are connected with rivets as shown. The spacing of the rivets is 4 in apart and the allowable load per rivet is 500 pounds. The steel plates are used as a simple beam.
Determine the maximum allowable value for the force P.
For steel, E = 30 × 106 psi

Example: T-Section Beam with Weld Joints

$$R \cdot 3 = M \Rightarrow R = \frac{15}{3} = 5\ \text{kN}$$

$$Q = 40 \times 200 \times 44.5 = 356000\ \text{mm}^3$$

$$I = \frac{200 \times 40^3}{12} + 200 \times 40 \times 44.5^2$$

$$+ \frac{40 \times 160^3}{12} + 160 \times 40 \times (120 - 64.5)^2$$

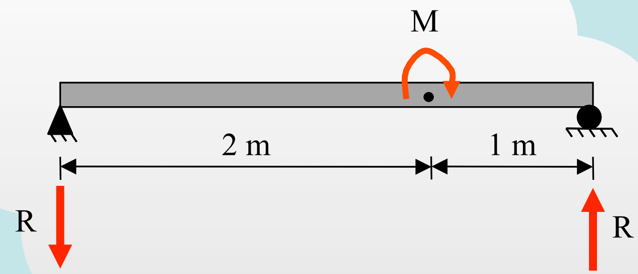
$$\therefore I = 50{,}275{,}600\ \text{mm}^4$$

$$f = \frac{VQ}{I} = \frac{5 \times 356{,}000}{50{,}275{,}600} = 0.0354\ \text{kN/mm}$$

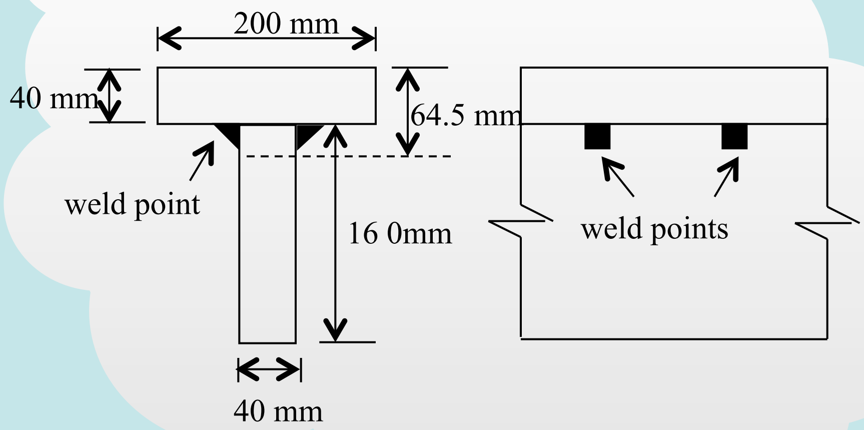
$$fS = 2F$$

$$\Rightarrow 0.0354\ S = 2 \times 8000$$

$$\Rightarrow S = 452\ \text{mm}$$



A built-up beam of T cross-section is loaded by M = 15 kNm.
Find the maximum spacing s of the weld points, if each weld point can carry 8000 N in shear.

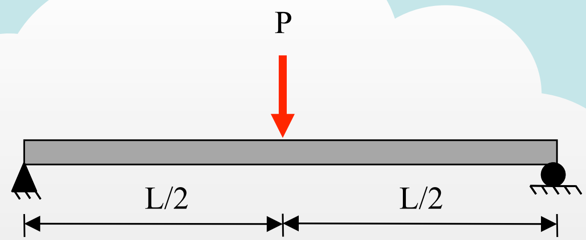


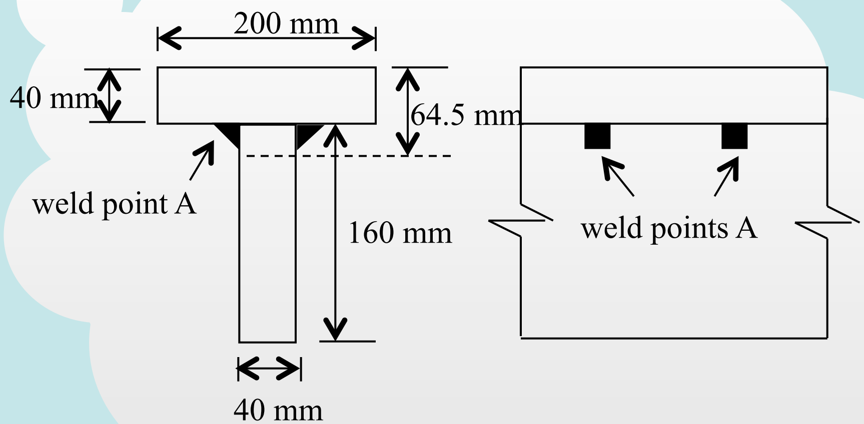
Example: Stress in T-section Beam with Weld Joints

A built-up beam of T cross section is loaded with P = 15 kN.

(a) Find the maximum shear stress at A
(Note: Point A is at the weld points)

(b) Determine L so that the maximum compressive stress is 10 MPa





$$\tau = \frac{VQ}{Ib}$$

$$V = \frac{P}{2} = \frac{15}{2} = 7.5\ \text{kN}$$

$$Q = 40 \times 200 \times 44.5 = 356{,}000\ \text{mm}^3$$

$$I = \frac{200 \times 40^3}{12} + 200 \times 40 \times 44.5^2$$

$$+ \frac{40 \times 160^3}{12} + 160 \times 40 \times (120 - 64.5)^2$$

$$\therefore\ I = 50{,}275{,}600\ \text{mm}^4 = 5.02756 \times 10^{-5}\ \text{m}^4$$

$$b = 40\ \text{mm}$$

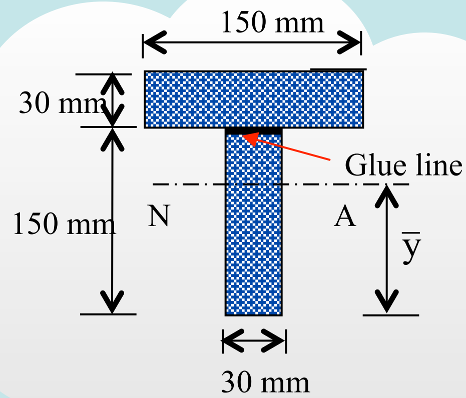
$$\tau = \frac{7.5 \times 356000}{50{,}275{,}600 \times 40} = 1.3277 \times 10^{-3}\ \text{kN/mm}^2$$

$$M_{max} = \frac{1}{2} P \times \frac{1}{2} L = \frac{15}{4} \times L = 3.75 L$$

$$\sigma_{max}^{comp} = \frac{M_{max} \times 0.0645}{5.02756 \times 10^{-5}}$$

$$\Rightarrow 10 \times 10^3 = \frac{3.75 L \times 0.0645}{5.02756 \times 10^{-5}} = 2.078\ \text{m}$$

Example: Shear Stress at Joining Plane in T-Section Beam



Find the shear stress at the joining plane.

Maximum applied shear = V = 19.5 kN

Locate NA, $\bar{y} = 0.12$ m

Find $\bar{I} = 27.0 \times 10^{-6}\ \text{m}^4$

Determine Q at the glue line

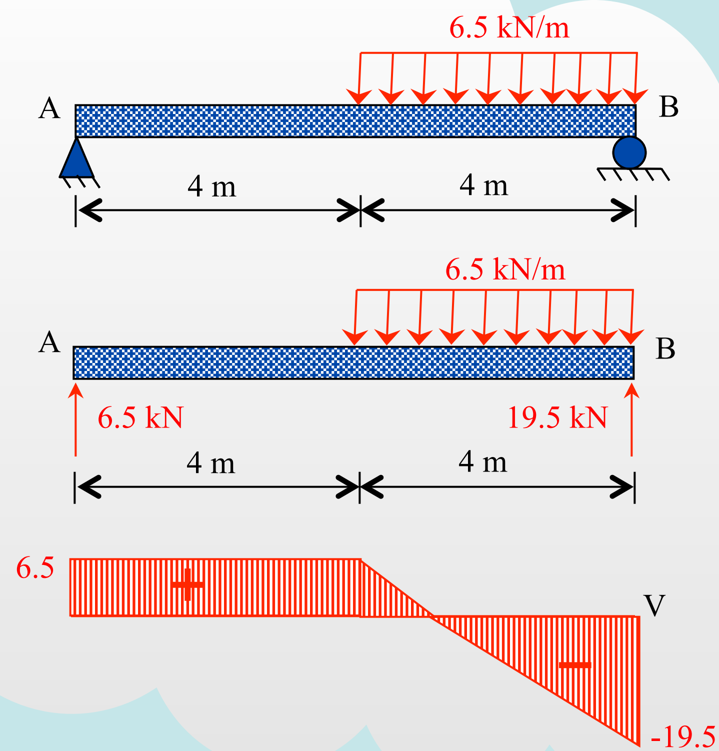
$$Q = 0.15 \times 0.03 \times 0.045$$

$$= 0.2025 \times 10^{-3}\ \text{m}^3$$

Determine "smaller width at location : t = 0.03 m

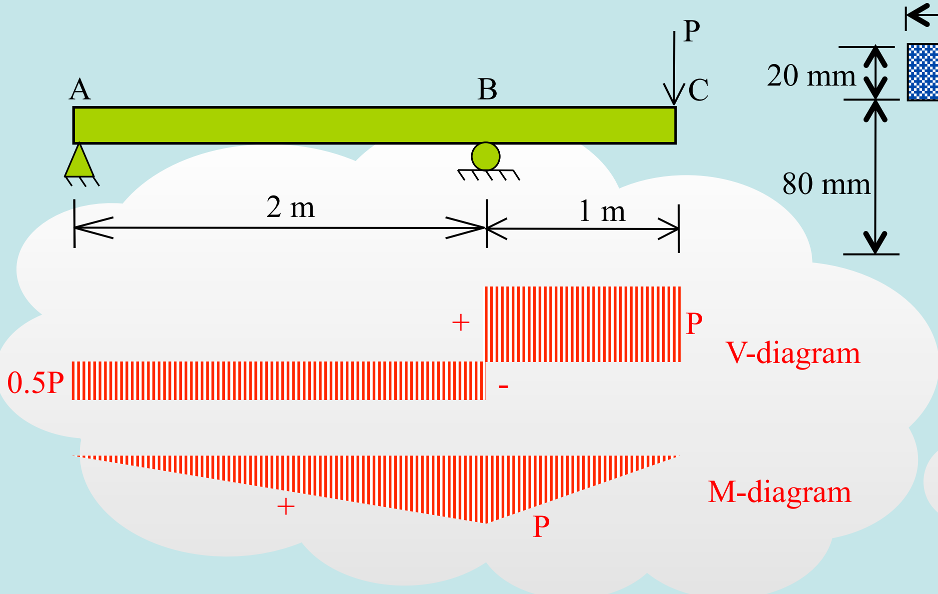
$$\tau = \frac{19500 \times 0.2025 \times 10^{-3}}{27 \times 10^{-6} \times 0.03} = 4.88\ \text{MPa}$$

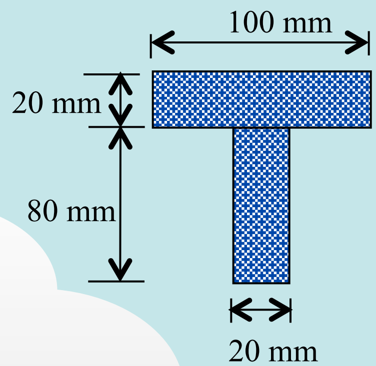
Note: shear stress is reduced drastically, just above the glue line!

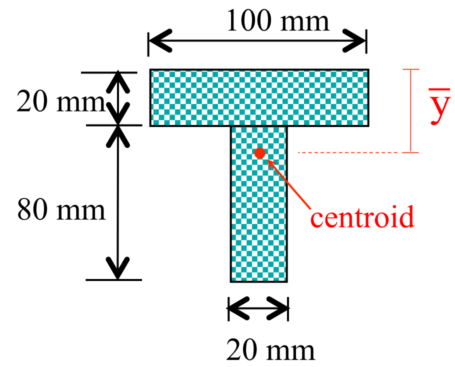


Example: Simply Supported T-section Beam

If the flange and the web in the beam are glued together, determine the shear stress in the glue for the portions AB and BC when the load P = 2 KN.







$$\overline{y} = \frac{2,000 \times 10 + 1600 \times 60}{100 \times 20 + 80 \times 20} = 32.2\text{mm}$$

$$I = \frac{20 \times 80^3}{12} + 1,600 \times 27.8^2 + \frac{100 \times 20^3}{12} + 200 \times 22.2^2 =$$

$$3,142,224\ \text{mm}^4 = 3.142 \times 10^{-6}\ \text{m}^4$$

$$AB \rightarrow V = P/2 = 1\ \text{kN}$$

$$BC \rightarrow V = P = 2\ \text{kN}$$

$$Q = 100 \times 20 \times 22.2 = 44,400\ \text{mm}^3 = 4.44 \cdot 0^{-5}\text{m}^3$$

$$\tau_{AB}^{glue} = \frac{VQ}{Ib} = \frac{1 \times 4.44 \times 10^{-5}}{3.142 \times 10^{-6} \times 0.02} = 707.2\ \text{kPa}$$

$$\tau_{BC}^{glue} = \frac{VQ}{Ib} = \frac{2 \times 4.44 \times 10^{-5}}{3.142 \times 10^{-6} \times 0.02} = 1,414.4\ \text{kPa}$$

Example: Spacing of Weld Points in Built-up I-section Beam

$$R \times 3 = M \Rightarrow R = \frac{15}{3} = 5 \text{ kN}$$

$$I = \frac{200 \times 240^3}{12} - \frac{160 \times 160^3}{12}$$

$$\therefore I = 175{,}786{,}667 \text{ mm}^4$$

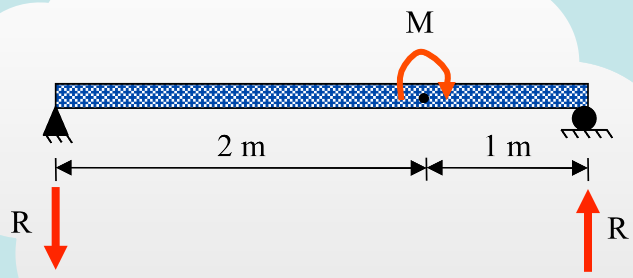
$$Q = 40 \times 200 \times 100 = 800{,}000 \text{ mm}^3$$

$$q = \frac{VQ}{I} = \frac{5 \times 800{,}000}{175786667} = 0.02279 \text{ kN/mm}$$

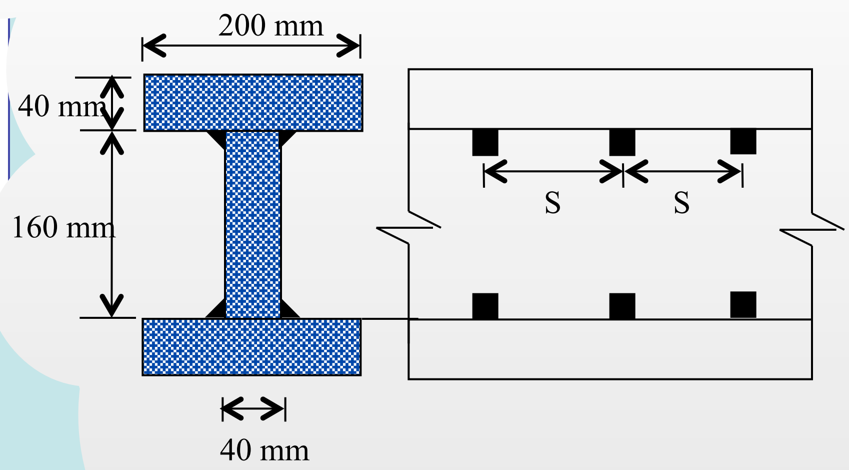
$$qs = 2F$$

$$\Rightarrow 0.02279\, s = 2 \times 8{,}000$$

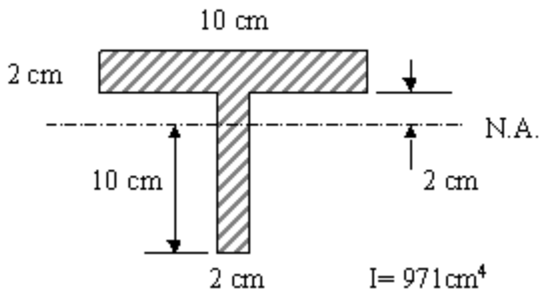
$$\Rightarrow s = 703 \text{ mm}$$



The welded built-up beam of I cross section is loaded by moment M = 15 kNm.
Find the maximum spacing s of the weld points if each one can carry 8,000 N in shear.



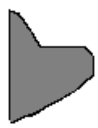
Self Assessment
Shear Stress: 1

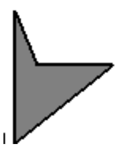


The shearing stress distribution $\tau = VQ / Ib$ on the shown T-type cross section most resembles which sketch?


d

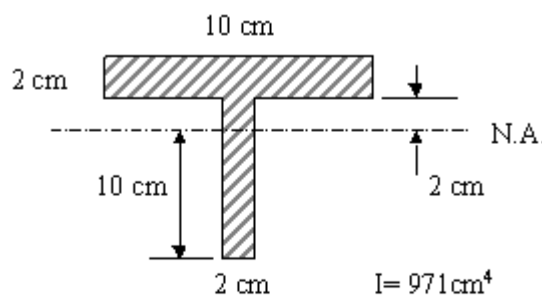

c


b


a

Shear Stress: 2

Calculate the maximum shear stress, in MPa, in the T section shown knowing that maximum applied shear on the section is 25 KN



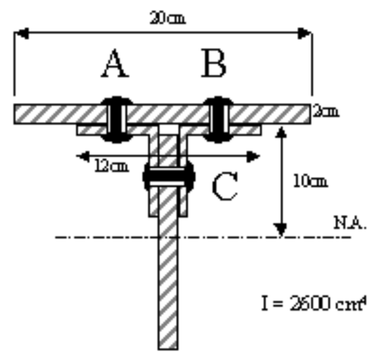
1. The maximum shear stress occurs at the neutral axis (NA) of the cross-section
2. The maximum shear stress occurs at the distance half way from top to bottom of the cross section
3. For the NA, Q is the moment of the area above it (NA) w.r.t. the NA
4. For the NA, Q is the moment of the area below it (NA) w.r.t. the NA

☐ Q is what, 3 or 4?

☐ Which is true, 1 or 2?

Shear Stress: 3

What is the shear force on bolt A due to 20 KN load acting on the free point of the built-up cross section cantilever shown? It is known that the bolt spacing is 15 cm in the longitudinal direction of the cantilever beam.



Sec. S-S

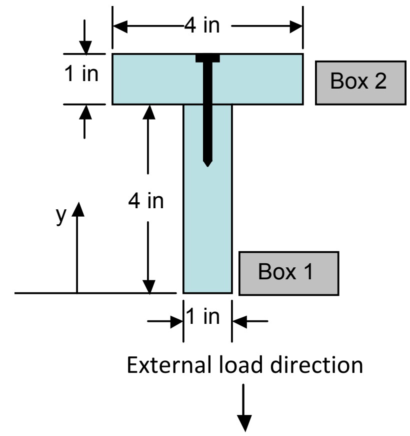
- 2.3 kN
- 1.9 kN
- 1.6 kN
- 2.6 kN

Shear Stress: 4

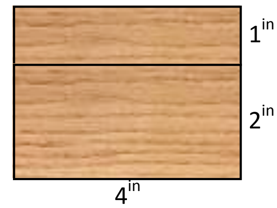
What is the maximum shearing stress, in psi, of a 20 ft 1.5" X 1.5" cantilever carrying a 1000Ib load at its free end?

- 963 psi
- 667 psi
- 324 psi
- 1004 psi

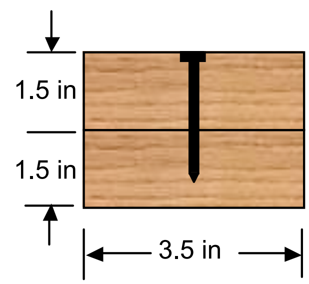
Shear Flow: 1 A T-shaped beam is made by attaching two 1×4 in boards using nails as shown. The shear force is 200 lb. Find the shear stress at the neutral axis and the shear flow at the joint.



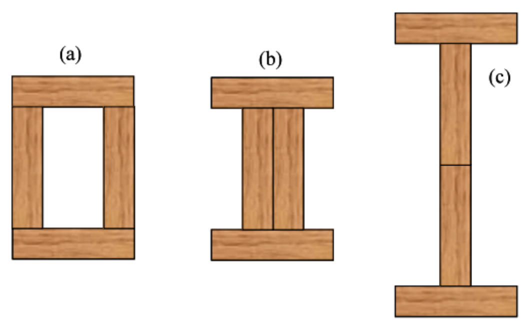
Shear Flow: 2 A built-up wooden beam is made by gluing together a 1×4 in^2 and a 2×4 in^2 beams as shown in the figure. For a shear force of 200 lb, find: (a) the shear stress at the neutral axis; and (b) the shear stress at the glue seam.



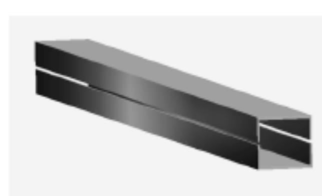
Shear Flow: 3 A beam made of 2×4s is nailed together as shown and each nail can withstand 150 lb of shear force. Find the largest nail spacing that will support a vertical shear load V = 200 lb.

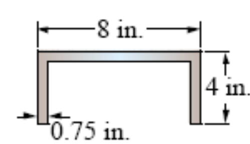


Shear Flow: 4 A built-up beam is to be made by bonding four identical planks together. Each plank has a rectangular cross-section 4×16 cm. For the same shear force V, which configuration produces the lowest shear stress τ at bonding interfaces?

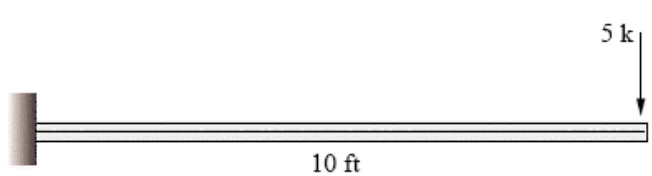


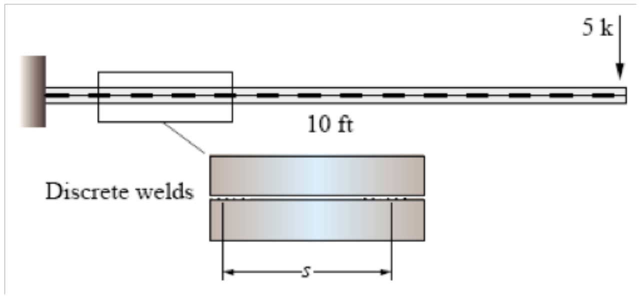
Shear Flow: 5 A rectangular tube is to be fabricated out of two identical channels as shown. For the loading shown below, a certain design for connecting the channels is to use a series of discrete weld segments as shown. Determine the maximum weld segment spacing that could be used for this alternative, assuming each weld segment can carry 5 kips of force.



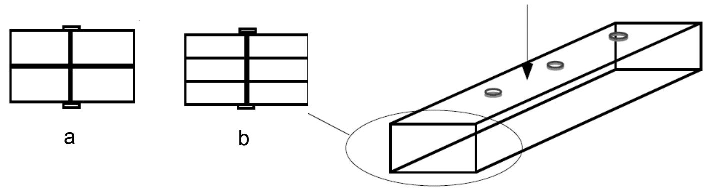


Channel dimensions

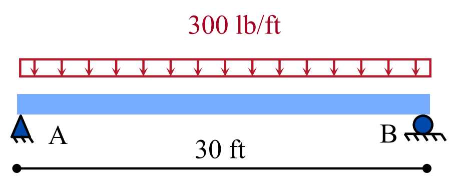




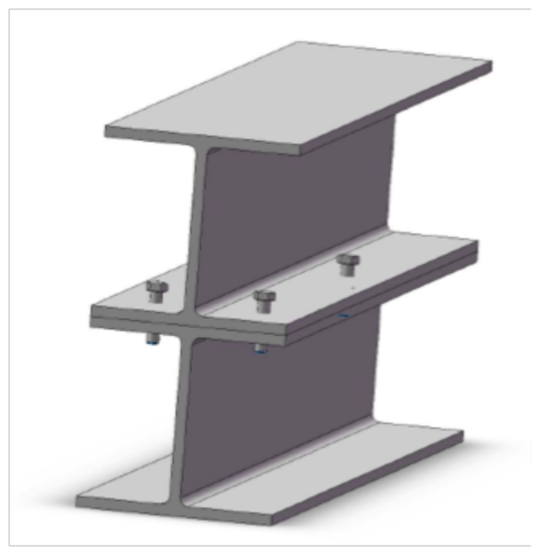
Shear Flow: 6 For the two alternative built-up beam sections shown below, choose the configuration that would require the most connectors per unit length along the beam to carry a given shear. Explain your answer.



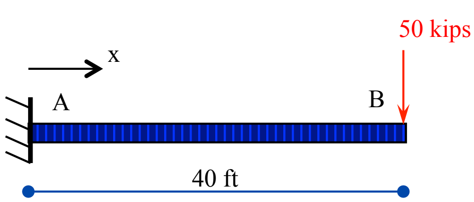
Shear Flow: 7 A beam is built up by laminating six 2 × 6s resulting into a 6 × 12 in cross section. If this section is to be used in a simply supported span of 30' with a design load of 300 lb/ft, what is the minimum requirement for the shear strength for the glue? (For this problem, assume that the dimensions of a 2 × 6 are actually 2" × 6".)

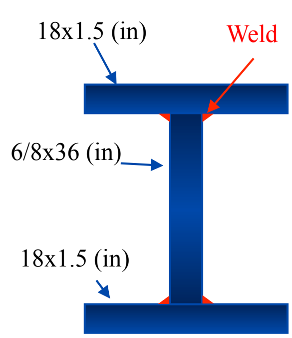


Shear Flow: 8 Two I-beams are bolted together as shown to form a single built-up beam loaded in the vertical direction. Each I-beam has a height of 14.48″, moment of inertia with respect to the centroid I_{zz} = 1380 in^4 and a cross-sectional area of 35.3 in^2. If the allowable shear force in each bolt is F = 3.1 kips, and the spacing between bolts along the length of the beam is 14″, find the maximum allowable shear force V the built-up beam can sustain.

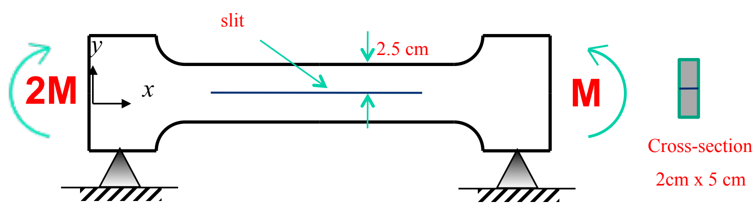


Shear Flow: 9 The cantilever beam shown below has a built-up cross section as indicated. (a) Determine the maximum bending stress in the beam. The section's moment of inertia is I =18325 in^4. (b) Determine the weld capacity (kip/in/weld) required for the welds shown.

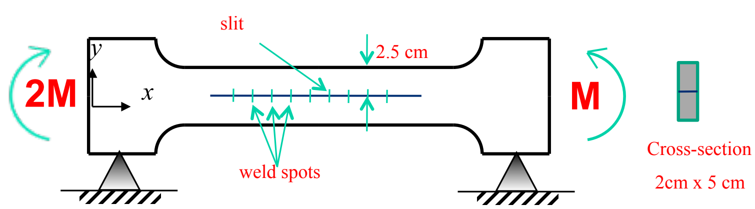




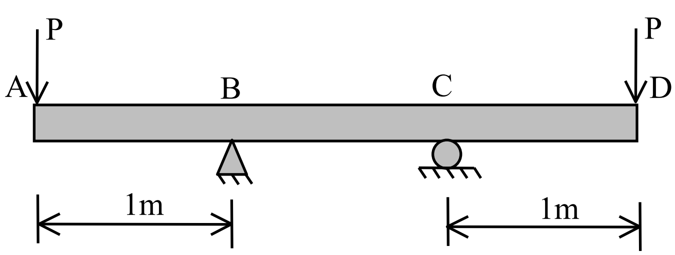
Shear Flow: 10 In order to test the mechanical properties of an adhesive, a so-called dog-bone specimen is prepared where a horizontal slit is made and filled up with the adhesive as shown in the figure. The specimen's cross section in the vicinity of the slit is 2×5 cm, and the slit is at the middle of the cross section. For a bending moment M = 125Nm, what is the shear stress τ the adhesive is subjected to? Draw τ at the interface as a vector. Note: the specimen is pinned as shown, and the distance between the two pins is 15 cm.

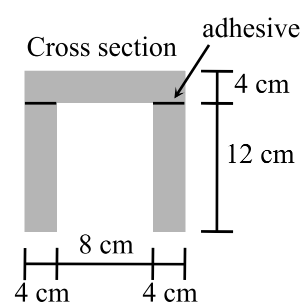


Shear Flow: 11 In order to test the mechanical properties of certain (small-scale) weld spots, a so-called dog-bone specimen is prepared where a horizontal slit is made and welded by weld spots on both sides of the specimen. The distance between weld spots is 2.5 cm (figure not to scale). The specimen's cross-section in the vicinity of the slit is 2×5 cm, and the slit is at the middle of the cross-section. For a bending moment M = 125 Nm, what is the shear force on each weld spot? (Note: the specimen is pinned as shown, and the distance between the two pins is 15 cm.)

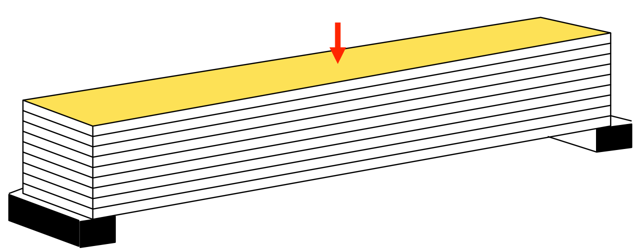


Shear Flow: 12 The built-up beam is subjected to forces P as shown. The flange is connected to the two webs by an industrial adhesive. If the allowable shear stress on the adhesive is $\tau_{\text{allowable}}^{\text{adhesive}} = 250\,\text{kPa}$ determine the maximum value of P such that this stress will not be exceeded.

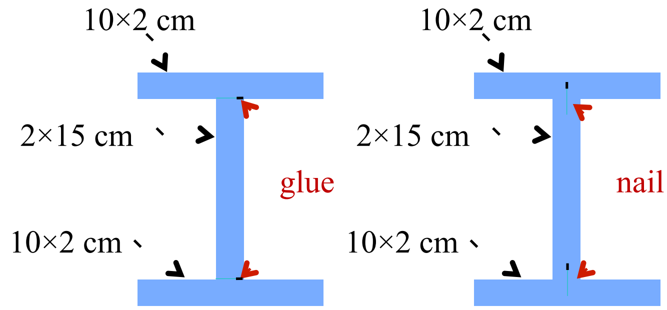




Shear Flow: 13 The so-called "glulam" beams are made by laminating several beams (or laminae) together as shown. An adhesive is used to bond the laminae (beams) together. Let a glulam be made by laminating 10 beams, each being 20 inches wide and 3 inches thick. The glulam beam is loaded at midspan by a 200 kips concentrated force. (a) Find the maximum shear stress the adhesive is subjected to anywhere in the beam. (b) If the beam were made of 5 laminae, each being each being 20 inches wide and 6 inches thick and was loaded as in (a), would the maximum shear stress on the adhesive be less, equal, or more than that in (a)? For part (b) explain your answer in words but do not perform calculations.



Shear Flow: 14 Three wooden boards are glued together to form an I-beam (figure to the left). The allowable shear stress of the glue is 100 N/cm^2. (a) Determine the maximum shear force V the beam can sustain such that the allowable shear stress in the glue will not be exceeded; (b) if nails are to be used instead of the glue (figure to the right) and the nails are spaced 12 cm apart along the axis of the beam, what is the minimum required allowable shear force on the nails such that the maximum V the beam can sustain is that found in (a).



Module 22: Plane Stress and Stress Transformation

General three-dimensional state of stress

As seen in previous sections, depending on the external load, normal or shear stresses can be created. For example, a bar experiences normal stress σ on planes transverse to the direction of the applied load; a beam experiences normal stress σ as well as shear stress τ, a shaft experiences shear stress τ. In general, a material element as that shown in Figure J-1 can experience one normal stress and two shear stresses on each of its three planes. For example, plane x (the plane transverse to the x-axis) can have a normal stress σ_x and two shear stresses, τ_{xy} and τ_{xz}. The normal stress index indicates the plane the normal stress acts on. The first index in shear stress indicates the plane the shear stress acts on, and the second index indicates the direction of the shear stress.

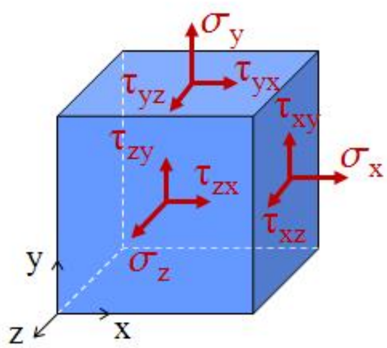


Figure J-1 General three-dimensional state of stress, showing three normal stresses and six shear stresses.

Thus, in general, the state of stress at a point in a body can be characterized by three normal and six shear stress components. However, as shown in previous sections, equilibrium implies certain equalities in shear stresses, i.e.

$$\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy}$$

reducing the number of independent stresses to three, i.e., three normal stresses and three shear stresses.

Plane Stress

For certain structures or structural components, engineers often reduce the analysis to a single plane by assuming a state of plane stress. Usually plate-like structures where there is no load on the face of the plates are analyzed under the plane stress assumption, which implies, for a plate in the xy-transverse to z direction, that only the in-plane stresses are nonzero, i.e. $\boldsymbol{\sigma_x}$, $\boldsymbol{\sigma_y}$, $\boldsymbol{\tau_{xy} = \tau_{yx}}$. All other stresses are zero, i.e., $\boldsymbol{\sigma_z = \tau_{yz} = \tau_{xz} = 0}$. Figure J-2 shows a schematic of a structure under plane stress and Figure J-3 the general state of stress of a plane stress material element.

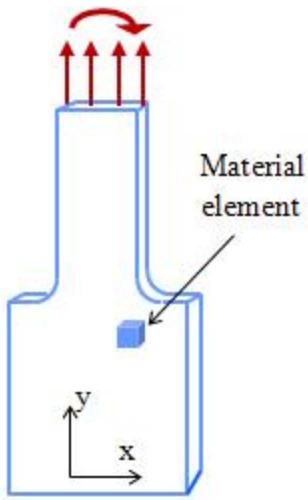


Figure J-2 Schematic of a structure or a structural component in plane stress conditions. The thickness of the plate is small compared to its other dimensions. The material element is infinitesimally small and its state of stress is shown schematically in Figure J-3.

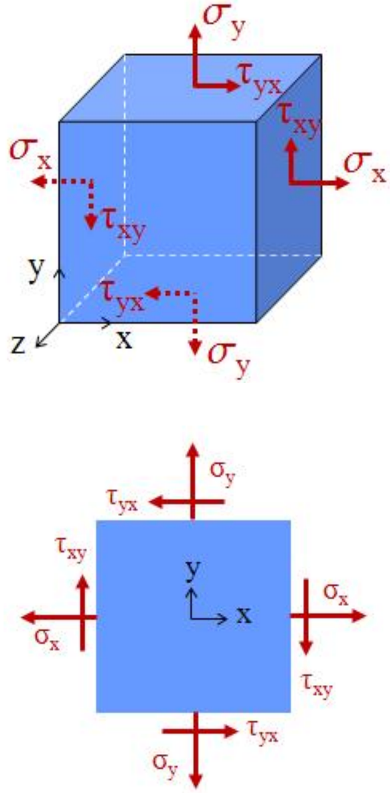


Figure J-3 General state of plane stress, in three dimensions (top). Usually a plane stress element is drawn as shown at the bottom (the front view of the element).

Stress transformation under plane stress

Consider material element in plane stress conditions as shown in Figure J-4, and a section at an arbitrary angle θ. Since the entire element is in equilibrium, the "wedge-like" part of the element shown in Figure J-5 should also be in equilibrium.

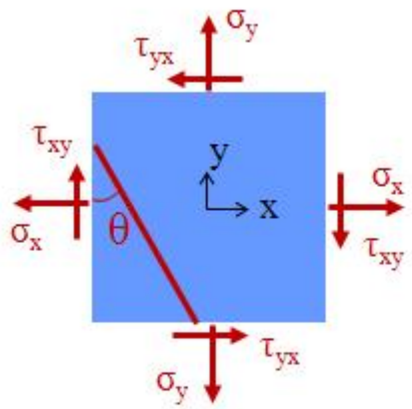


Figure J-4 A section (red/dark line) on a plane stress material element.

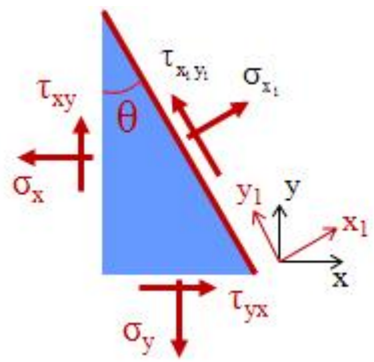


Figure J-5 The free body diagram, i.e., the stresses acting on the element cut in Figure J-4.

If x_1 denotes the axis transverse to the section and x_2 the axis parallel to it (Fig. J-5), the angle between the x- and x_1-axes is equal to θ. Figure J-6 shows the areas that the stresses shown in Figure J-5 act on.

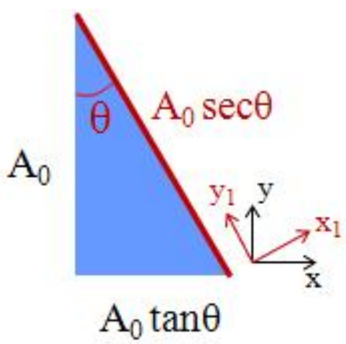


Figure J-6 The plane stress material element of Figure J-5 is of constant thickness (transverse to the paper) and the figure shows the areas on each of its sides.

The forces acting on each surface of the element are equal to the stress acting on the surface multiplied by the area. For example, the force acting on the x_1 plane is equal to $\sigma_{x1} A_0 \sec\theta$. Decomposing all forces acting on the element in the x_1- and y_1-directions and expressing the equilibrium equations in these directions, we obtain (the same equations are obtained if equilibrium is expressed in the x- and y-directions)

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

These are the **transformation equations for plane stress**. They express the normal stress σ_{x_1}, and shear stress $\tau_{x_1 y_1}$, in terms of the stresses on the x- and y-planes, i.e., σ_x, σ_y, and τ_{xy}. The angle θ is the angle between the x- and x_1-axes, or equivalently, between the y- and the y_1-axes. Formally, it is defined as the angle measured when rotating the x-axis to the x_1-axis. It is positive if this rotation is counter-clockwise. By setting $\theta \to \theta + 90^o$ in the first equation above, it follows that

$$\sigma_{y_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

This way, the stresses at all angles θ, i.e., 90^o, 180^o and 270^o (or -90^o) can be found. Thus, from the stresses in Figure J-7, the stresses in Figure J-8 can be found using the transformation equations for plane stress.

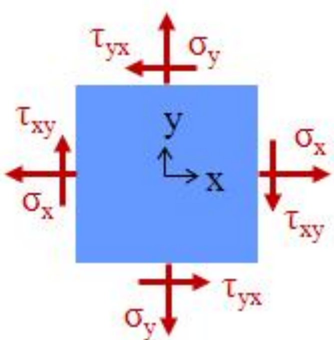


Figure J-7 A plane stress element in the x-y coordinate system and the stresses on each plane.

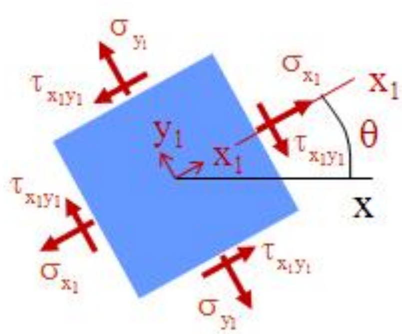


Figure J-8 The same plane stress element as in Figure J-7 but in the rotated x_1-y_1 coordinate system and the stresses on each plane.

Example-uniaxial state of stress: Here we consider a uniaxial state of stress on a thin plate as shown in Figure J-9. The state of stress on a material element oriented parallel and transverse to the applied load is shown, where the only non-zero stress is σ_x. For an element oriented at an angle θ however, the stress state is quite different. The figure shows an element at 45^o.

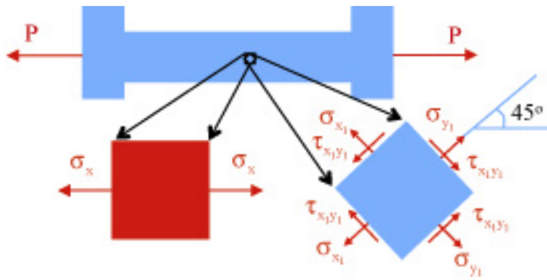


Figure J-9 Uniaxial state of stress on a thin plate. A material element is oriented parallel to the external load and another one at 45^o to the load P.

For that element, all stress components can be evaluated from the transformation equations for stress. For $\theta = 45^o$, we have

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta = \frac{\sigma_x + 0}{2} + \frac{\sigma_x - 0}{2}\cos 90^o + 0\sin 90^o = \frac{\sigma_x}{2}$$

and

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -\frac{\sigma_x - 0}{2}\sin 90^o + 0\cos 90^o = -\frac{\sigma_x}{2}$$

Similarly, for $\theta = 135^o$, it follows that

$$\sigma_{x_1} = -\frac{\sigma_x}{2}, \quad \tau_{x_1 y_1} = +\frac{\sigma_x}{2}$$

Figure J-10 shows the axial state of stress for the inclined material element. Note that a negative normal stress is compressive, while a negative shear stress implies it is in the negative direction. For example, at 45^o the x_1-axis is positive (normal to the plane) so a negative shear stress is oriented in the negative y_1-direction.

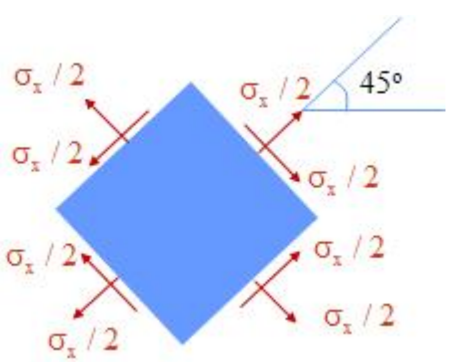


Figure J-10 The actual stress at the inclined plane. Both the normal and shear stresses are equal to $\sigma_x/2$.

Example-pure shear state of stress: Consider a pure shear state of stress, as it occurs at the neutral axis of a beam subjected to bending moment M and shear force V. In Figure J-11, all shear stresses are positive. For example, the one on the positive x-plane is in the positive y-direction, thus the shear stress is positive. The one on the negative x-plane, the shear stress is in the negative y-direction, thus the shear stress is positive. Similar considerations hold for the shear stresses on the y-planes.

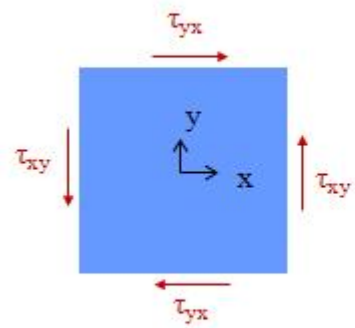


Figure J-11 Pure shear state of stress. All four shear stresses are positive.

The transformation of stress equations in this case reduce to (since $\sigma_x = 0$ and $\sigma_y = 0$)

$$\sigma_{x_1} = \tau_{xy}\sin 2\theta$$

$$\tau_{x_1y_1} = \tau_{xy}\cos 2\theta$$

Note that at 45^o the shear stress is zero, even though at the original material element we have a pure shear state of stress!

Stress Transformation Invariants

The transformation equations for plane stress presented in the previous section provide the stresses σ_{x_1} and $\tau_{x_1y_1}$ as a function of stresses σ_x, σ_y, τ_{xy} and the transformation (rotation) angle θ. The normal stress σ_{y_1} can be obtained by setting $\theta \rightarrow \theta + 90^o$ in the expression for σ_{x_1} . By adding the expression for σ_{x_1} and σ_{y_1} and simplifying the final expression it follows that

$$\sigma_{x_1} + \sigma_{y_1} = \sigma_x + \sigma_y$$

Thus, the summation of the two normal stresses in a plane stress state remains the same for any angle of rotation θ. It is called the **invariant** of the stress state. Note that this invariant for the pure shear state of stress examined in the previous example is equal to zero.

Example-biaxial tension/compression with $\sigma_x = \sigma_y$: A biaxial state of stress implies normal stresses σ_x and σ_y act not a material element while $\tau_{xy} = 0$. A particular case is when $\sigma_x = \sigma_y$, as shown in Figure J-12 (either both stresses tensile or compressive).

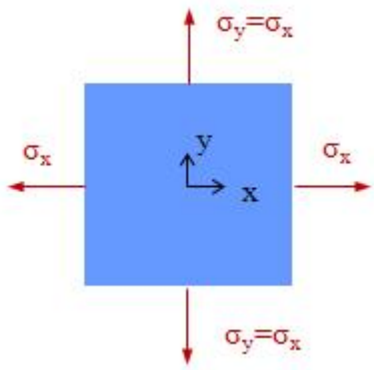


Figure J-12 Biaxial tension, with $\sigma_x = \sigma_y$.

The transformation of stress equations in this case reduce to (since $\tau_{xy} = 0$)

$$\sigma_{x_1} = \sigma_x = \sigma_y$$

$$\sigma_{y_1} = \sigma_x = \sigma_y$$

and this holds for every possible angle of rotation θ. This is "reminiscent" of hydrostatic stress in a fluid, where the same stress acts at any orientation, thus, such a biaxial state of stress is often called a "hydrostatic" state of stress.

Principal Stresses

The equations for transformation of stress provide the normal and shear stress for a plane oriented at any angle θ. Engineers are mostly interested on the extreme values of the stresses. Thus, a relevant question asks for the angle θ at which the normal stress becomes an extreme and for the value of the normal stress at that angle. Similar questions pertain to the shear stresses. In order to find the direction θ at which the normal stresses become an extreme, we set

$$\frac{d\sigma_{x_1}}{d\theta} = -\left(\sigma_x - \sigma_y\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

which yields

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

where the subscript p stand for "principal." Using basic trigonometric relations, $\cos 2\theta$ and $\sin 2\theta$ can be evaluated from the above equation. After substituting these in the transformation equation for σ_x, the two principal stresses result, denoted as $\boldsymbol{\sigma_1}$ **and** $\boldsymbol{\sigma_2}$.

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

with the plus (+) sign for σ_1 and the minus (-) sign for σ_2. The above expression provides the principal normal stresses, and the angle θ_p is the direction (planes) where these principal stresses occur. There is an important property about the principal directions. If one asks the question: at what angle θ does the shear stress $\tau_{x_1 y_1}$ become zero, the answer is (by setting $\tau_{x_1 y_1} = 0$)

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

which is the same as the principal directions. Thus **in the principal directions the shear stress is zero.** A very important consequence, then, is that **if the shear stress on a plane is zero, then the normal stress on that plane is a principal stress, and the plane is in the principal directions**.

Example-uniaxial stress: Under uniaxial stress conditions (shown in Fig. J-14), it can be easily shown that $\tan 2\theta_p = 0$, which implies that $\theta_p = 0^o, 90^o, 180^o, \ldots$

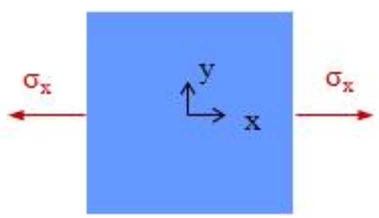


Figure J-13 Uniaxial state of stress, $\sigma_y = \tau_{xy} = 0$.

Thus x and y are the principal directions in this case. It is noted that the shear stresses are zero in these principal directions.

Example-pure shear: Under pure shear state of stress (Fig. J-11), since $\sigma_x = \sigma_y = 0$, it follows that $\cos 2\theta_p = 0$ implies that $\theta_p = 45^o, 135^o, \ldots$ The stresses at these angles are evaluated yielding that at 45^o $\sigma_1 = \tau_{xy}$ and at 135^o $\sigma_2 = -\tau_{xy}$. This state of stress, i.e., in the principal directions, is shown schematically in Figure J-14. Note that the shear stresses in the principal directions are zero.

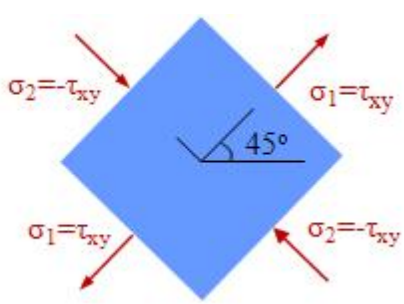


Figure J-14 The material element in the principal directions of the pure shear stress state (Fig. J-11), and the principal stresses.

Maximum Shear Stresses

Another important task of engineers is to find the extreme shear stresses. We examine this task here for plane stress states. By solving

$$\frac{d\tau_{x_1y_1}}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

it follows that

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

where the subscript s stands for shear. Using basic trigonometric relations, $\cos 2\theta$ and $\sin 2\theta$ can be evaluated from the above equation. After substituting these in the transformation equation for τ_{xy} the following expression for the maximum shear stress results

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

which occurs, of course, at θ_s. It is straightforward to show that $\mathbf{2\theta_s = 2\theta_p - 90^o}$ **or** $\mathbf{\theta_s = \theta_p - 45^o}$, which implies that **the principal directions are inclined 45^o from the directions of the maximum shear.** Also, from the expression for the shear stress it is easy to show that

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

thus, **the maximum shear stress is equal to the semi-difference of the two principal stresses.**

Example-maximum shear stress under uniaxial stress: Under uniaxial stress (shown in Fig. J-13), it can be easily shown that $\cot 2\theta_s = 0$, which implies that $\theta_s = 45^o, -45^o$ ... and $\tau_{max} = \sigma_x/2$. It is noted that for the uniaxial state of stress, $\sigma_1 = \sigma_x$, and $\sigma_2 = 0$. Thus

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}$$

as expected.

Mohr's circle for plane stress

In 1882, Otto Mohr, a German engineer, combined the transformation of stress equations in order to form a comprehensive pictorial method of representing them. Using basic trigonometric relations in the equations for stress transformation in plane stress, i.e.,

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

results in the following, by eliminating the angle θ (i.e., by solving one of the equations for θ and substituting the resulting expression into the other equation)

$$\left(\sigma_{x_1} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x_1 y_1}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

This is the equation of a circle in the σ_{x_1} versus $\tau_{x_1 y_1}$ space. The center of the circle is on the horizontal axis, as shown in Figure J-15, and its radius R is equal to the square root of the right-hand side of the above equation, i.e.

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

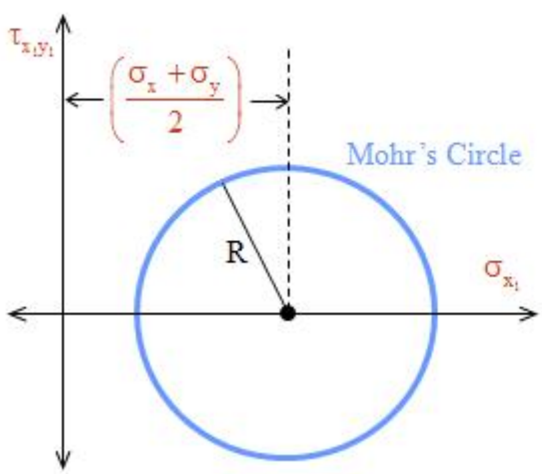


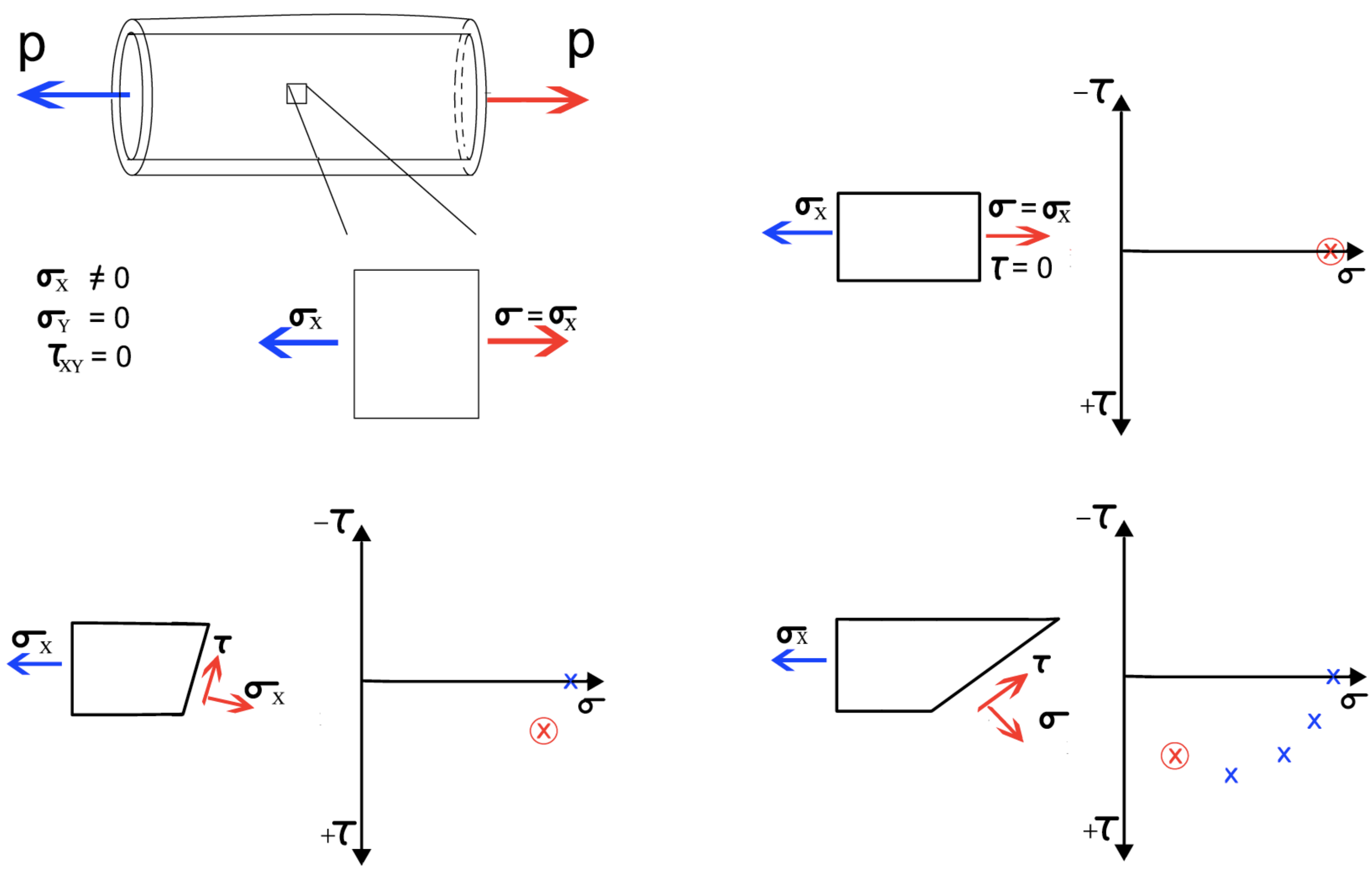
Figure J-15 Mohr's circle constructed by positioning its center, which is always on the horizontal axis, and evaluating its radius R.

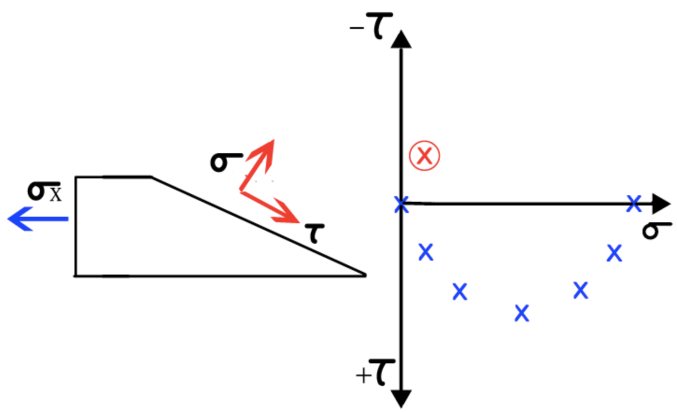
Mohr's circle provides a graphic understanding of the stress transformation equations. Besides this, it can be used to determine the state of stress in any direction, or the principal stresses, or the maximum shear stresses and the direction those occur at. The **sign convention** is such that tension is positive, and shear stress is negative when rotating the material element counter-clock wise and positive when rotating the element clock wise.

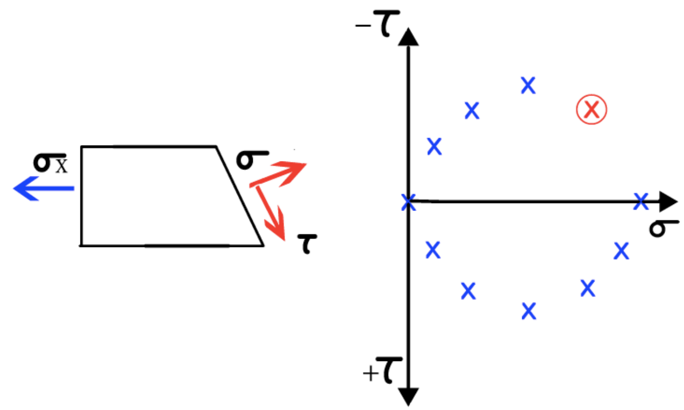
Graphic illustration of construction of Mohr's circle

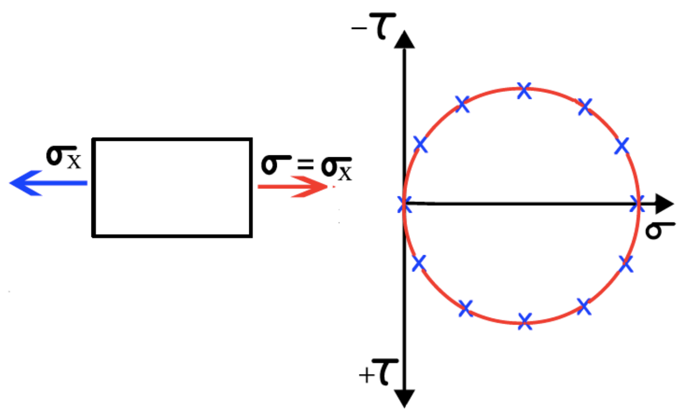
In the following interactive illustration, a uniaxial state of plane stress is considered, i.e. a bar subjected to $\sigma_x \neq 0$, and $\sigma_y = 0$, $\tau_{xy} = 0$. By "cutting" the bar at different angles, the normal stress and shear stress on the inclined planes are evaluated and then plotted on a σ versus τ coordinate system. The end result is the Mohr's circle for this uniaxial state of stress.

A material element subjected to uniaxial tension (top left figure) and the plot of normal stress σ vs shear stress τ as the orientation of the element changes progressively, from left to right and top to bottom.




σx
σ
τ
−τ
+τ
σ


σx
σ
τ
−τ
+τ
σ


σx
σ = σx
−τ
+τ
σ

Module 24: Mohr's Circle for Plane Stress

Construction of Mohr's circle

Mohr's circle has information on the state of stress at any angle θ. Instead of constructing it by evaluating the radius R, a more efficient way is as follows (referring to Fig. J-16). Given are the values of stress at the x-y coordinate system, i.e. σ_x, σ_y, τ_{xy}. The steps in drawing the Morh's circle are:

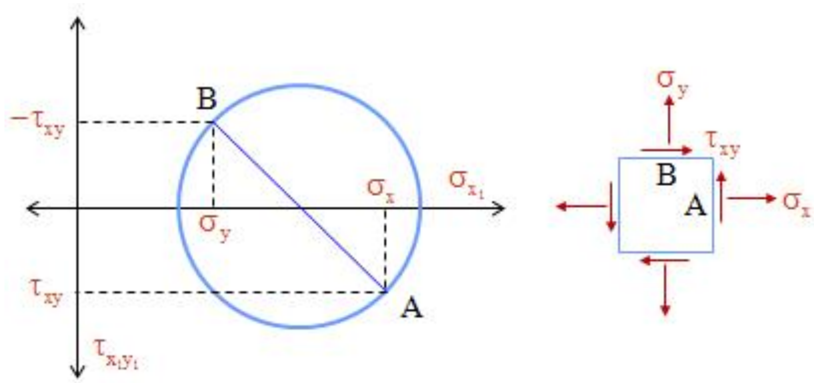


Figure J-16 The stresses in the x-y planes are σ_x, σ_y, τ_{xy}. From those, the Mohr's circle is constructed..

1. On the coordinate system σ_{x_1} (horizontal axis), $-\tau_{x_1y_1}$ (vertical axis), place the center of the circle at $\frac{\sigma_x+\sigma_y}{2}$ (center is on the horizontal axis).

2. Place point A, i.e. the point corresponding to $\theta = 0^o$, where $\sigma_{x_1} = \sigma_x$ and $\tau_{x_1y_1} = \tau_{xy}$.

3. Measure angles from point A counter-clock wise (see note below).

NOTE: Point B (i.e., the point corresponding to $\theta = 0^o$, where $\sigma_{x_1} = \sigma_y$ and $\tau_{x_1y_1} = -\tau_{xy}$) is diametrically opposite to point A. In the stress space, from point A to point B angle θ changes from 0 to 90^o. In the Mohr's circle, however, it changes from 0 to 180^o. Thus, angles in Mohr circle are measured counter-clockwise from point A as 2θ.

Properties of Mohr's circle

Figure J-17 shows further details of the Mohr's circle, constructed from the given values of σ_x, σ_y, τ_{xy} as follows:

1. Draw a coordinate system with σ_{x_1} as the horizontal axis and $\tau_{x_1y_1}$ as the vertical axis. Note that $+\tau_{x_1y_1}$ is downward, the + indicating counter-clockwise moment produced by the shear stresses.

2. Locate the center on the horizontal axis at $\frac{\sigma_x+\sigma_y}{2}$.

3. Locate point A, i.e., the point in the circle with coordinates σ_x, τ_{xy} corresponding to $\theta = 0^o$.

Measure angle 2θ counter-clockwise from A.

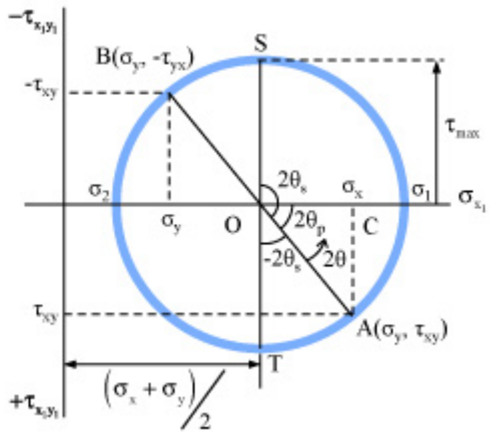


Figure J-17 A typical Mohr's circle and various points on it.

The following properties of the Mohr's circle are important and help in understanding the state of stress at a material point:
- The principal stresses are the extremes of the normal stress, located at the far right and far left of the circle. Note that shear stress at principal directions is zero.
- Maximum shear stress occurs at point S and point T.
- From triangle OCA it follows that

$$R = |OA| = \sqrt{|OC|^2 + |CA|^2} = \sqrt{\left(\sigma_x - \left(\frac{\sigma_x+\sigma_y}{2}\right)\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

and thus

$$\sigma_1 = \left(\frac{\sigma_x+\sigma_y}{2}\right) + R = \left(\frac{\sigma_x+\sigma_y}{2}\right) + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \left(\frac{\sigma_x+\sigma_y}{2}\right) - R = \left(\frac{\sigma_x+\sigma_y}{2}\right) - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Also, from triangle OCA it follows that

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\sigma_x - \left(\frac{\sigma_x+\sigma_y}{2}\right)\right)} = \frac{2\tau_{xy}}{\sigma_x+\sigma_y}$$

Example-Mohr's circle: Given the plane stress state shown in Figure J-18, in this example we will:
- Draw the Mohr's circle
- Eevaluate the two principal stresses from the Mohr's circle
- Draw the principal stresses in the principal directions

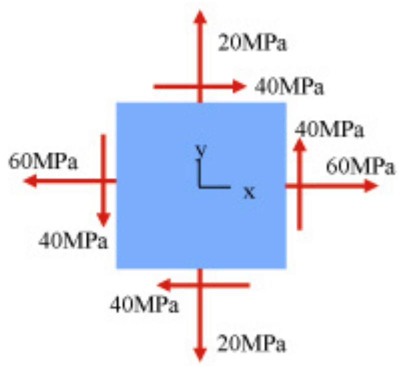


Figure J-18 A plane stress state.

For this state of stress, σ_x = +60 MPa (tensile), σ_y = +20 MPa, and τ_{xy} = +40 MPa (shear stress is in the positive x-plane and its direction is in the positive y-axis, or the moment it creates is counter-clockwise). The center of the circle is at (60 + 20)/2 = 40 MPa. This, together with the position of point A (at (60,40)) yields the Mohr's circle shown in Figure J-19.

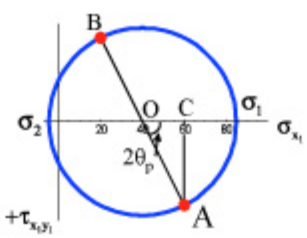


Figure J-19 Mohr's circle corresponding to the stress state of Figure J-18.

From triangle OCA, $\tan 2\theta_p = 40/20 = 2$, which implies that $2\theta_p = 63.43^o$ or $\theta_p = 31.7^o$.

Also from triangle OCA, the radius R is

$$R = |OA| = \sqrt{|OC|^2 + |CA|^2} = \sqrt{40^2 + 20^2} = 44.72\,\text{MPa}$$

Thus, σ_1=40+44.72 = 84.72MPa and σ_2 = 40−44.72 = −4.72MPa (compressive). Figure J-20 shows the material element in the principal directions, i.e., the element rotated by 31.7^o counter-clockwise from the original material element.

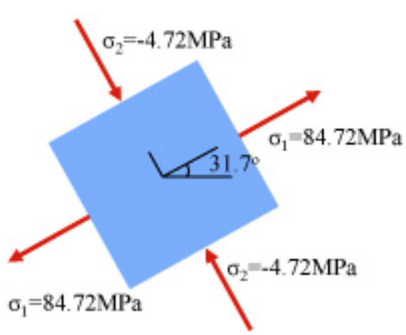


Figure J-20 The material element in the principal directions.

Example-Mohr's circle: Given the uniaxial plane stress state shown in Figure J-21, in this example we will:
- Draw the Mohr's circle
- From the Mohr's circle evaluate the two principal stresses and the maximum shear stress

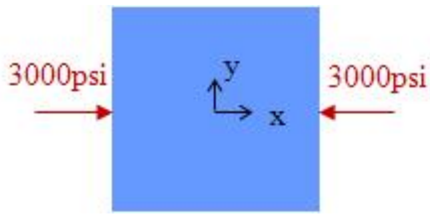


Figure J-21 Uniaxial state of stress, under plane stress conditions.

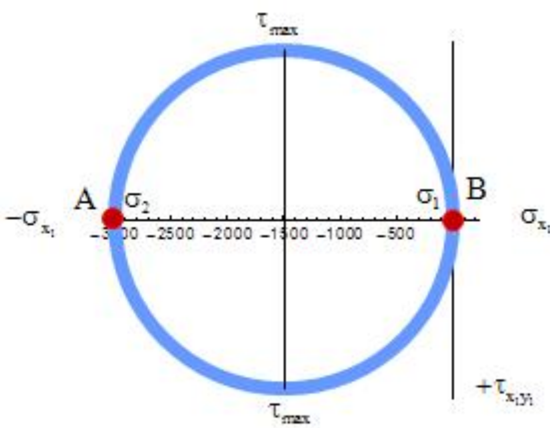


Figure J-22 Mohr's circle for the uniaxial state of stress of Figure J-21.

The center of the circle for this problem is located at (–3000 + 0)/2 = –1500 psi. Point A has coordinates of (–3000 psi, 0), and the coordinates of point B are (0,0). Figure J-22 shows the circle. Note that the principal directions are the original x-y ones. This is also verified from the fact that the shear stress at the x and y planes is zero. The maximum shear stress is equal to R, which is 1500 psi.

Example-Mohr's circle: Given the pure shear plane stress state shown in Figure J-23, in this example we will:
- Draw the Mohr's circle
- Evaluate the two principal stresses and the maximum shear stress from the Mohr's circle
- Evaluate the state of stress at the plane inclined 22.5^{o} clockwise from the x-plane from the Mohr's circle

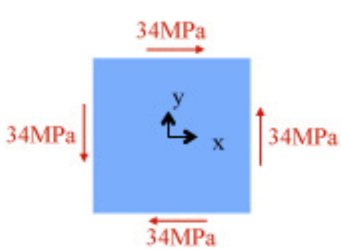


Figure J-23 Pure shear state of stress, under plane stress conditions.

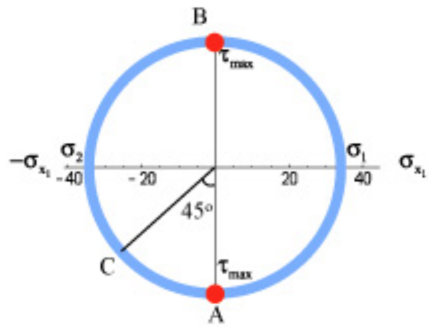


Figure J-24 Mohr's circle for the state of stress of Figure J-23.

Figure J-24 shows the circle. The center of the circle for this problem is located at (0 + 0)/2 = 0 MPa. Point A has coordinates of (0 MPa, 34 MPa), and the coordinates of point B are (0 MPa,–34 MPa). Note that the principal directions are 90^{o} from points A and B in the Mohr's circle, or 90/2 = 45^{o} from the x-y coordinates.

Note that 22.5^{o} clockwise in the stress space corresponds to 22.5 × 2 = 45^{o} clockwise rotation in the Mohr's circle. The corresponding point is denoted by C in Figure J-24. The stress state at point C is easily evaluated, i.e.

$$\sigma_{x_1} = -34\cos(45^o) = 24.04\,\text{MPa}$$

$$\tau_{x_1 y_1} = 34\cos(45^o) = 24.04\,\text{MPa}$$

Example-Mohr's circle: The thin plate shown in Figure J-25 is subjected to the biaxial state of stress as shown. The plate is welded together by a weld oriented 60^{o} from the horizontal axis. In this example we will:
- Determine the normal stress perpendicular to the weld and the shear stress parallel to it.
- Draw the Mohr′s circle and pinpoint the stress point in the direction of the weld (i.e., 60^{o} from the horizontal axis).

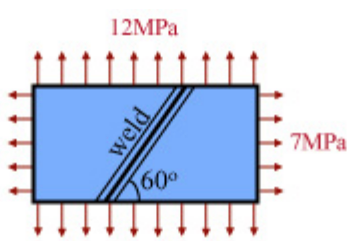


Figure J-25 A welded thin plate subjected to biaxial tension.

The state of plane stress is: σ_x = 7 MPa, σ_y = 12 MPa, τ_{xy} = 0. For the weld, the axis transverse to it makes a –30^{o} angle with the positive x-axis, thus, θ = –30^{o}. With these values, the equation for transformation of stress read

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta = \frac{7+12}{2} + \frac{7-12}{2}\cos(-60^o) = 8.3\text{MPa}$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -\frac{7-12}{2}\sin(-60^o) = -2.165\,\text{MPa}$$

The center of the Mohr's circle is at (7 + 12)/2 = 9.5 MPa, and point A is at (7,0) MPa. Figure J-26 shows the Mohr's circle and the point for the weld (at -60^o).

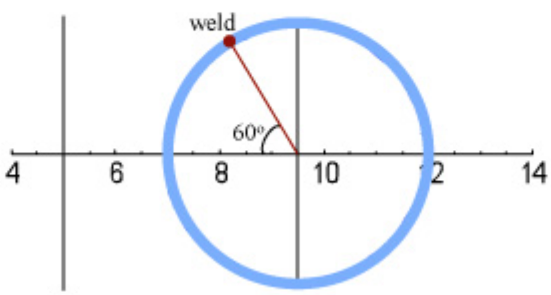


Figure J-26 Mohr's circle for the state of stress in the plate of Figure J-25 and the point indicating the state of stress at the weld.

Example-stress on inclined plane: In this example, a wooden beam/bar is fractured at 55^o and is held together temporarily by an adhesive while it is subjected to the stress shown in Figure J-27. The problem in this example is to find the shear stress in the adhesive.

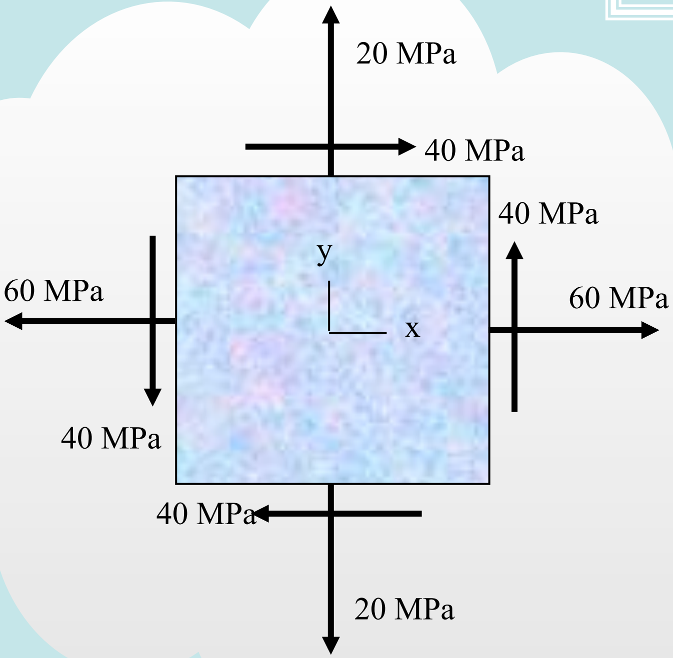
Figure J-27 A fractured beam is held together by an adhesive.

For the fracture, the axis transverse to it makes a -35^o angle with the positive horizontal x-axis, thus, $\theta=-35^o$. With these values, the equation for transformation of the shear stress reads

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta = -\frac{7-0}{2}\sin(-70^o) = 3.29\,\text{MPa}$$

Stresses - Transformation

Given the state of stress: (a) find principal stresses and principal planes (direction), and (b) draw the principal stresses in a properly oriented element.



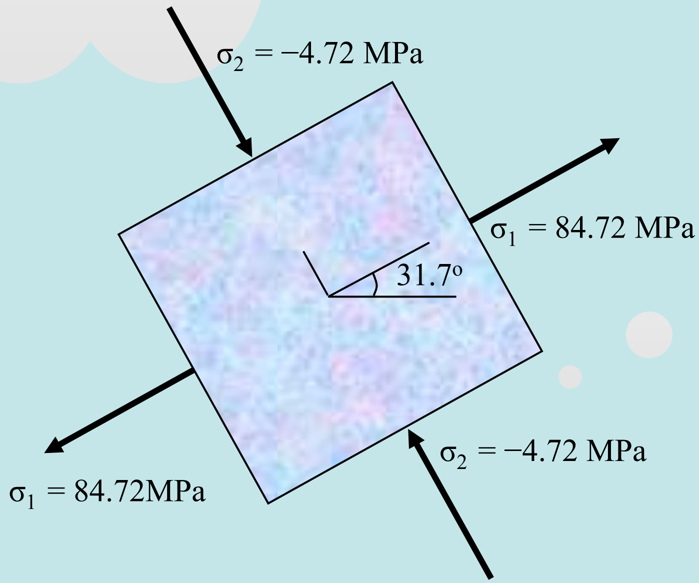
$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} =$$

$$\frac{60+20}{2} \pm \sqrt{\left(\frac{60-20}{2}\right)^2 + 40^2} \Rightarrow$$

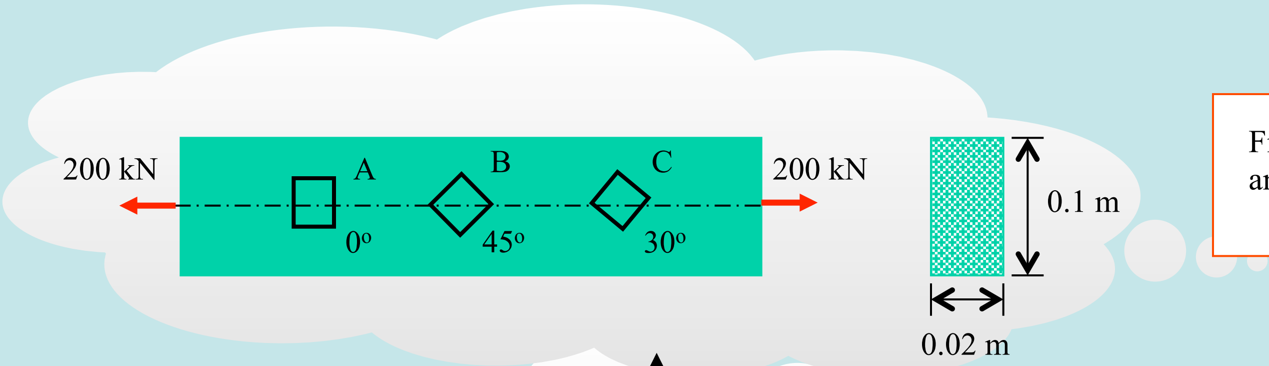
$$\sigma_1 = 84.72\text{ MPa}, \quad \sigma_2 = -4.72\text{ MPa}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\cdot 40}{60-20} = 2 \Rightarrow$$

$$2\theta_p = 63.43^\circ \Rightarrow \theta_p = 31.7^\circ$$



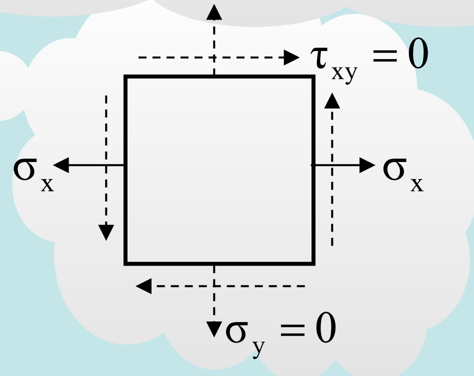
Example: Stresses in Elements at Different Angles



Find stresses at elements A, B, and C.

A) Element A:

$$\sigma_x = \frac{P}{A} = \frac{200{,}000}{0.02 \times 0.1} = 100 \text{ MPa}$$



C) Element C:

$$\sigma_{x'} = \frac{100}{2} + \frac{100}{2}\cos 60° + 0 = 75 \text{ MPa}$$

$$\sigma_{y'} = \frac{100}{2} - \frac{100}{2}\cos 60° - 0 = 25 \text{ MPa}$$

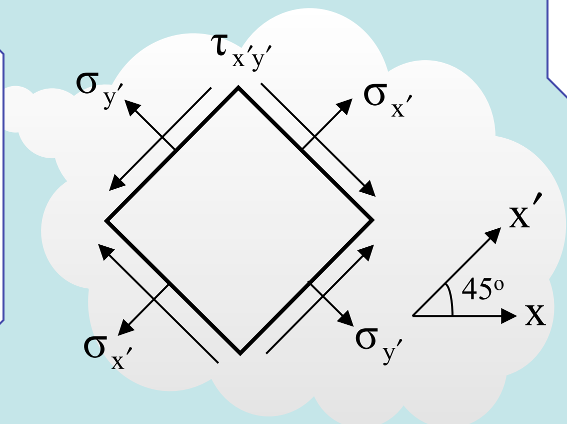
$$\tau_{x'y'} = -\frac{100}{2}\sin 60° = -43.3 \text{ MPa}$$

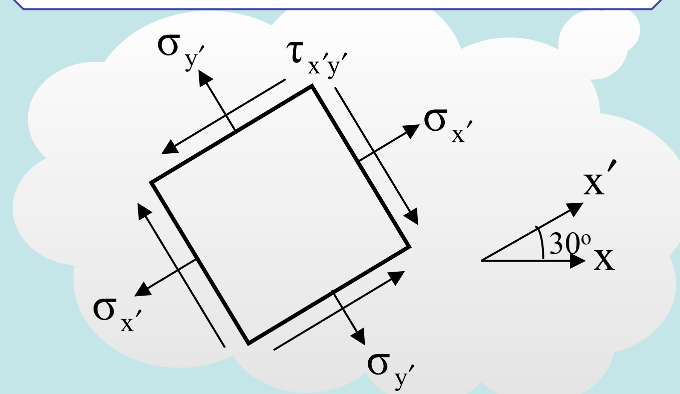
B) Element B:

$$\sigma_{x'} = \frac{100}{2} + \frac{100}{2}\cos 90° + 0 = 50 \text{ MPa}$$

$$\sigma_{y'} = 50 \text{ MPa}$$

$$\tau_{x'y'} = -50 \text{ MPa}$$





Example: Maximum Shear Stresses

$$\tan\left(2\theta_s\right) = \frac{\left(-\sigma_x - \sigma_y\right)/2}{\tau_{xy}} = \frac{\left(-20 - 90\right)/2}{60} = -\frac{55}{60}$$

$$2\theta_{s1} = 42.5°$$

$$\Rightarrow \theta_{s1} = 21.3°$$

$$2\theta_{s2} = 2\theta_{s1} + 180°$$

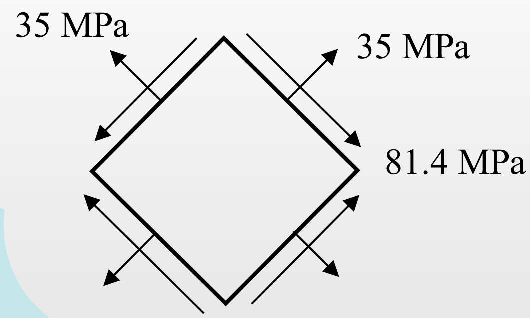
$$\Rightarrow \theta_{s2} = \theta_{s1} + 90° = 111.3°$$

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{-20 - 90}{2}\right)^2 + 60^2}$$

$$\Rightarrow \tau_{max} = 81.4 \text{ MPa}$$

$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{-20 + 90}{2} = 35 \text{ MPa}$$

$$\Rightarrow \sigma_{ave} = 35 \text{ MPa}$$



Repeat past example for maximum shear stresses.

Example: Maximum Shear Stress Using Mohr's Circle

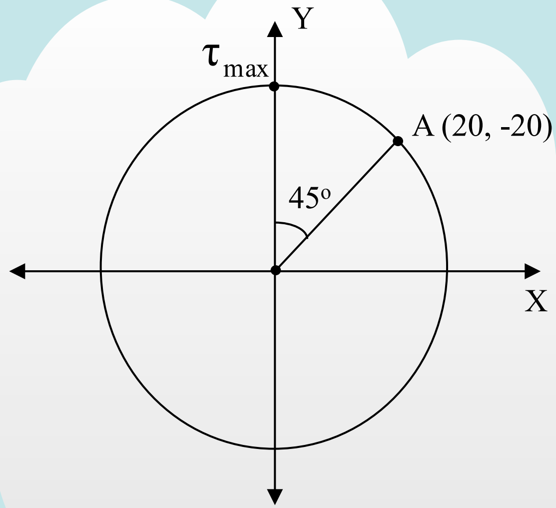
For the plane stress state

$\sigma_x = 20$ MPa

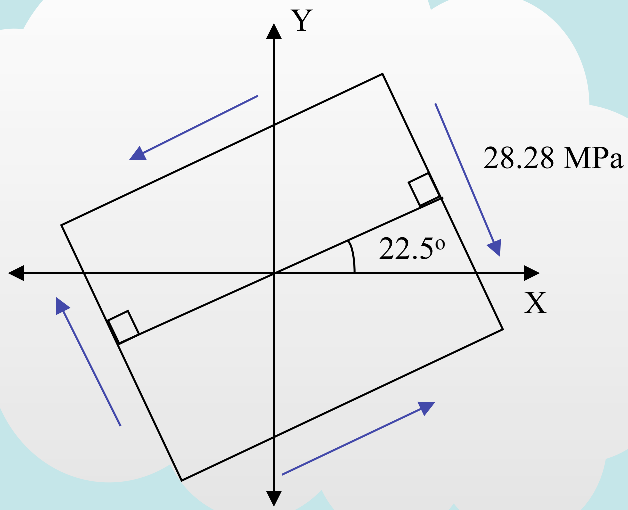
$\sigma_y = -20$ MPa

$\tau_{xy} = -20$ MPa

Draw the Mohr's circle. Then, using the circle, evaluate the maximum shear stress and draw it on properly oriented elements.

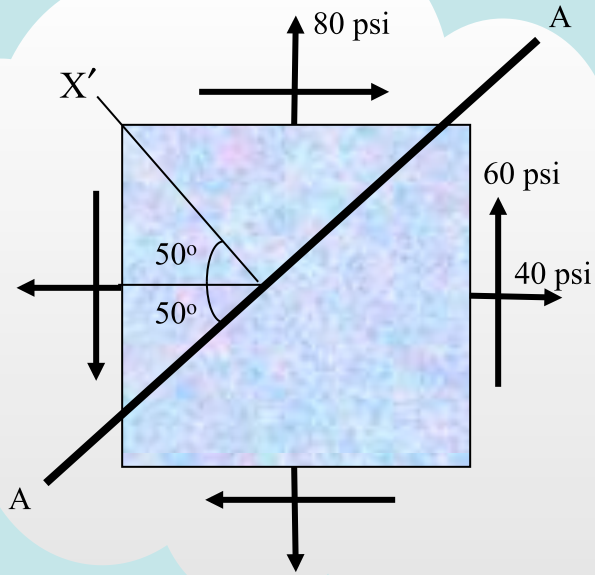


$$R = \tau_{max} = \sqrt{20^2 + 20^2} = 28.28 \text{ MPa}$$



Example: Mohr's Circle to Determine Stresses

The state of stress of a point is shown in the diagram. Use and show Mohr's circle to determine the stresses on the element having surface A-A as indicated. Show the required element with its stresses and orientation.

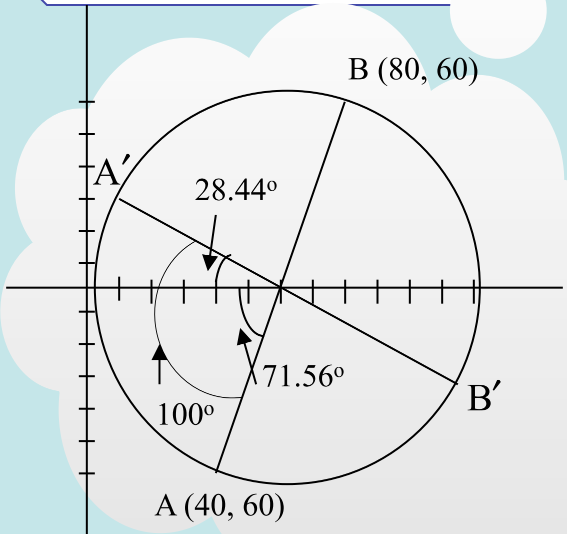


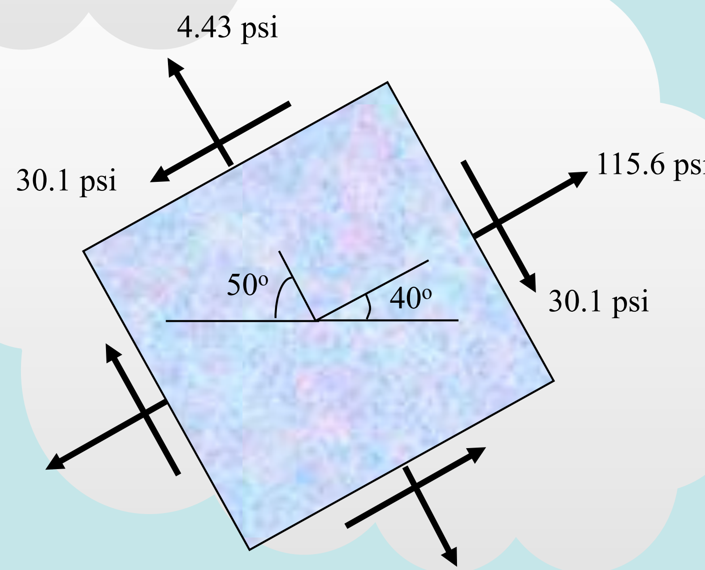
Center: $\dfrac{80+40}{2} = 60$ psi

A (40, 60) and B(80, 60)

$R = \sqrt{60^2 + 20^2} = 63.2$

$\tan\theta = \dfrac{60}{20} = 3 \Rightarrow \theta = 71.56°$





$A'\left(60 - 63.2\cos(28.44°),\ 63.2\sin(28.44°)\right)$

$\Rightarrow A'(4.43,\ 30.1)$

$B'\left(60 + 63.2\cos(28.44°),\ 30.1\right)$

$\Rightarrow B'(115.6,\ 30.1)$

Module 25: Hooke's Law in Plane Stress

Hooke's law in three dimensions

Recall that a material element embedded in a bar under tension/compression loaded in the x-direction experiences $\varepsilon_x = \sigma_x / E$ and from Poisson effects, $\varepsilon_y = -\nu\varepsilon_x = -\nu\sigma_x / E$ and $\varepsilon_y = -\nu\varepsilon_y = -\nu\sigma_x / E$. Similarly, the strains created by stress σ_y and σ_z can be evaluated. Superimposing, i.e. assuming a general state of stress where all three stresses, σ_y, σ_y and σ_z are applied to a material element, it follows that

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

If, in addition, shear stresses act, we have

$$\gamma_{xy} = \gamma_{yx} = \frac{\tau_{xy}}{G} = \frac{\tau_{yx}}{G} \qquad \gamma_{xz} = \gamma_{zx} = \frac{\tau_{xz}}{G} = \frac{\tau_{zx}}{G} \qquad \gamma_{yz} = \gamma_{zy} = \frac{\tau_{yz}}{G} = \frac{\tau_{zy}}{G}$$

The above six (6) equations are termed the **generalized Hooke's law, or Hooke's law in three dimensions.** Given the state of stress (i.e., all six components), Hooke's law allows for the evaluation of the six strains (i.e., three normal and three shear ones). The above equations can be inverted to yield the expression for the stresses as a function of the strains in a material element. Under plane stress conditions, the generalized Hooke's law simplifies significantly as shown in the following.

Specialization of generalized Hooke's law for plane stress

Under plane stress conditions, $\sigma_z = 0$, and $\tau_{xz} = \tau_{yz} = 0$. The generalized Hooke's law equations then reduce to

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\tau_{xy}}{G} = \frac{\tau_{yx}}{G}$$

It is interesting that the lateral normal strain in the z-direction is not equal to zero. This is because of Poisson effects, which dictate that the material expands or contracts in the z-direction due to normal strains in the x- and y-directions.

Inverting the above, it follows that

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right)$$

$$\tau_{xy} = \tau_{yx} = G\gamma_{xy} = G\gamma_{yx}$$

which express the **Hooke's law for plane stress**.

Physical interpretation of strains-volume changes

Even though the definition of strain in uniaxial tension or compression is easy to interpret, in three dimensions there is an interaction between strains, which makes theit interpretation more difficult. In the following, the strains in three dimensions are related to volume changes in a material element, Consider a material element subjected to stresses as shown in Figure K-1, i.e. only three normal stress are applied.

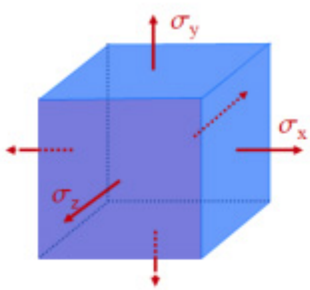


Figure K-1 A material element subjected to normal stresses in three dimensions.

Without loss of generality, let the material element before application of the stress be a unit cube, i.e., its volume is $V_0 = 1 \times 1 \times 1 = 1$. After the stresses are applied, the volume of the element is $V_f = (1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)$. For strains small enough, which usually the case for engineering structures, higher order terms can be neglected, thus $V_f \approx 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z$. The volumetric strain is defined as

$$e = \frac{\Delta V}{V_0} = \frac{V_f - V_0}{V_0}$$

From the above equations, it follows that

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

where the generalized Hooke's law was used. Note that only for $\nu = 0.5$ the volume changes are zero. For engineering materials, $\nu \leq 0.5$, thus, for tensile stresses the volume increases, while for compressive stresses the volume decreases. Another important and commonly used "definition" is that of the **"hydrostatic" pressure** of the material element defined as

$$p = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

From these last two equations, obviously, the hydrostatic pressure is related to the volumetric strain by

$$p = \frac{E}{3(1-2\nu)}\,e$$

For this reason, the quantity $E/3(1 - 2\nu)$, is called the "bulk modulus" of the material and the letter kappa ($\varkappa$) is used to denote it, thus, by definition

$$\kappa = \frac{E}{3(1-2\nu)}$$

denotes the bulk modulus

Example: Total Volume Change in a Plate

The plate of thickness t is in plane stress.

(a) Determine the ratio f_x/f_y so that the volume changes are zero.

(b) For f_x = 10 K/in, f_y = -20 K/in and t = 2 in, determine the total volume change of the plate.

$$E = 30 \times 10^3 \text{ ksi}$$

$$\nu = 0.3$$

$$e = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y\right)$$

$$\sigma_x = \frac{f_x}{t}, \quad \sigma_y = \frac{f_y}{t}$$

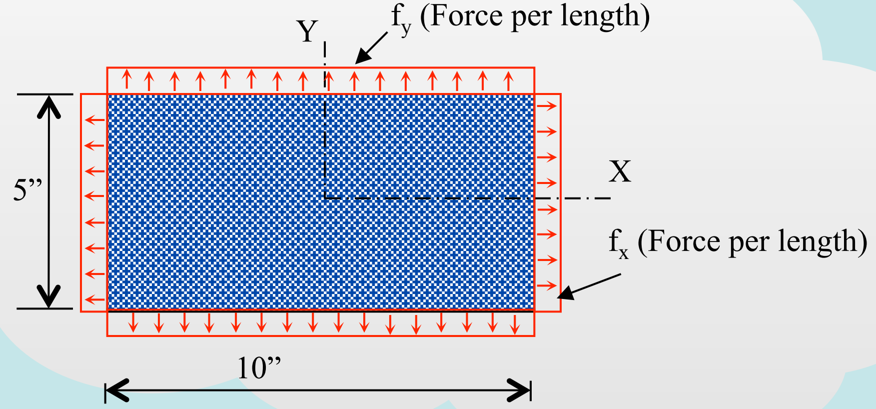
$$e = 0 \Rightarrow \sigma_x = -\sigma_y \Rightarrow f_x = -f_y$$

$$e = \frac{1-0.6}{30 \times 10^3}\left(\frac{10}{2} - \frac{20}{2}\right) = \frac{0.4}{30 \times 10^3}\left(5 - 10\right)$$

$$\Rightarrow e = -6.667 \times 10^{-5}$$

$$\Delta V = e \times V_0 = -6.667 \times 10^{-5}\left(10 \times 5 \times 2\right)$$

$$\Rightarrow \Delta V = -0.00667$$



Module 26: Plane Strain

Definition of plane strain

As mentioned before, for certain structures or structural components engineers often reduce the analysis to a single plane. One such case is that of plane stress examined earlier. Recall that plane stress considers that all stresses in the z-direction are zero, i.e., $\sigma_z = \tau_{yz} = \tau_{xz} = 0$. Usually plate-like structures, where there is no load on the face of the plates, are analyzed under the plane stress assumption. An example is a thin wall where there is no load transverse to the plane of the wall.

The plane strain analysis considers that all strains in the z-direction are zero, i.e., $\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$. Usually long structures (long enough that expansion in the z-direction from Poisson effects can be neglected) loaded on the x-y plane are analyzed under the plane strain assumption. Examples are a long pipe or a long dam structure. Figure K-2 shows schematically a case of plane stress and a case of plane strain, both loaded on the x-y plane by forces and moments.

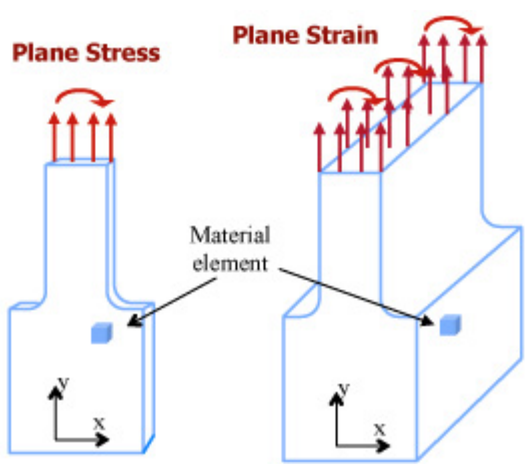


Figure K-2 Illustration of material elements under plane stress (left) and plane strain (right). A thin plate in the x-y directions loaded only in the x-y plane is under plane stress conditions. A long (thick) structure loaded homogeneously in the x-y plane is under plane strain conditions.

It is noted that a material can not be in both plane stress and plane strain states at the same time. If this was not the case, certain stresses and strains in the element would have to be equal to zero and equal to a finite value at the same time.

Examples-plane strain and plane stress: Figures K-3 and K-4 show examples of plane stress and plane strain cases. As an exercise, list some example of problems in engineering (civil, mechanical, aerospace) that can be assumed to be under either plane stress or plane strain.

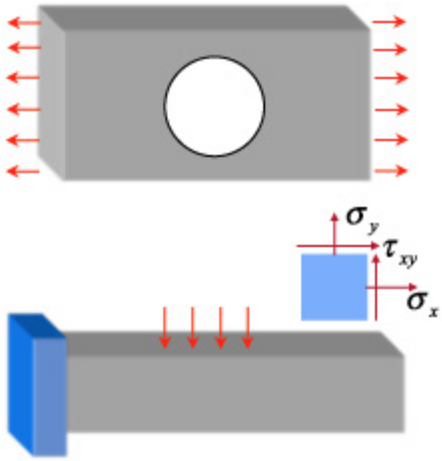


Figure K-3 A thin plate with a hole (e.g., for a bolt) is, in general, under plane stress conditions when loaded on its plane, as shown in the figure (top). A slender beam (bottom) is also under plane strain conditions. Typical stresses on a material element are shown in the figure as well (middle).

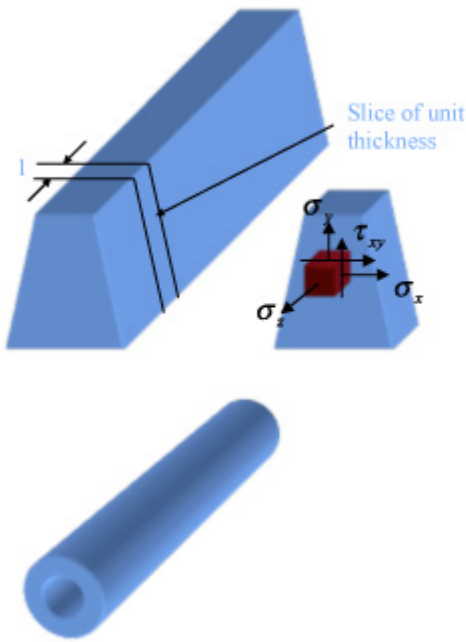
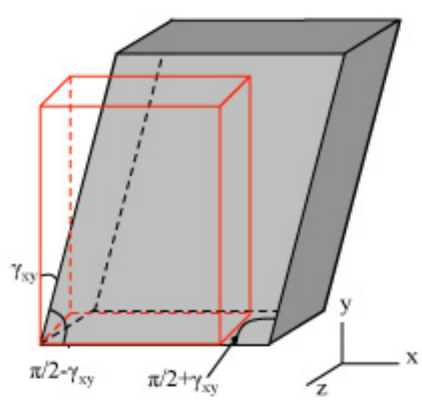
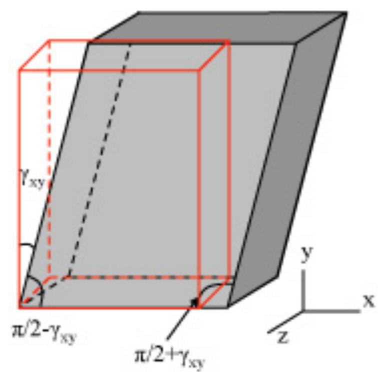


Figure K-4 A long dam (top of figure) is under plane strain conditions. Usually a slice of unit thickness is analyzed (shown to the right) with a schematic of a material element. A long cylindrical pressure vessel subjected to internal/external pressure and constrained at the ends (bottom) is also under plane strain conditions.

The following two tables summarize the cases of plane stress and plane strain.

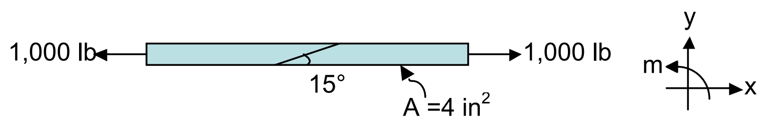
Plane stress and plane strain deformations and stresses. Undeformed material elements are shown in red/dark		
Figure K-5 Plane stress.		Element expands/contracts in x → $\varepsilon_x \neq 0$, $\sigma_x \neq 0$ Element expands/contracts in y → $\varepsilon_y \neq 0$, $\sigma_y \neq 0$ Element expands/contracts in z due to Poisson effects only → $\varepsilon_z \neq 0$, $\sigma_z = 0$

		Element shears in x-y, but not in x-z and y-z $\rightarrow \tau_{xy} \neq 0, \tau_{xz} = \tau_{yz} = 0$
Figure K-6 Plane Strain	 	Element expands/contracts in x $\rightarrow \varepsilon_x \neq 0, \sigma_x \neq 0$ Element expands/contracts in y $\rightarrow \varepsilon_y \neq 0, \sigma_y \neq 0$. Element is not allowed to expand/contract in z $\rightarrow \varepsilon_z = 0, \sigma_z \neq 0$ Element shears in x-y, but not in x-z and y-z $\rightarrow \tau_{xy} \neq 0, \tau_{xz} = \tau_{yz} = 0$

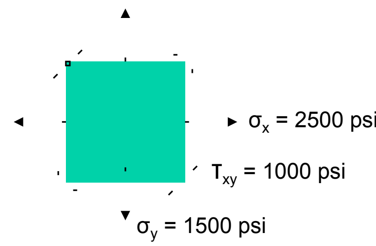
NOTE: The above declaration of a variable being ≠0 does not exclude the special case of the variable being equal to zero. So, it should be interpreted that it is only zero under special stressing/straining circumstances.

	Primary requirements	**Consequences**
Plane Stress	$\sigma_z = 0$ $\tau_{xz} = \tau_{yz} = 0 \rightarrow \gamma_{xz} = \gamma_{yz} = 0$	All other stresses and strains not equal to zero. Note that in general $\varepsilon_z \neq 0$
Plane Strain	$\varepsilon_z = 0$ $\gamma_{xz} = \gamma_{yz} = 0 \rightarrow \tau_{xz} = \tau_{yz} = 0$	All other stresses and strains not equal to zero. Note that in general $\sigma_z \neq 0$

Stress Transformation: 1 A simple tension element is made from two shorter pieces of material with the glue joint at 15° from the loading axis. Find the shear and normal stresses at the glue seam. The modulus of elasticity (E) and Poisson's ratio (ν) are as follows: E = 0.5×10^6 psi, ν = 0.35 (Nylon 6)



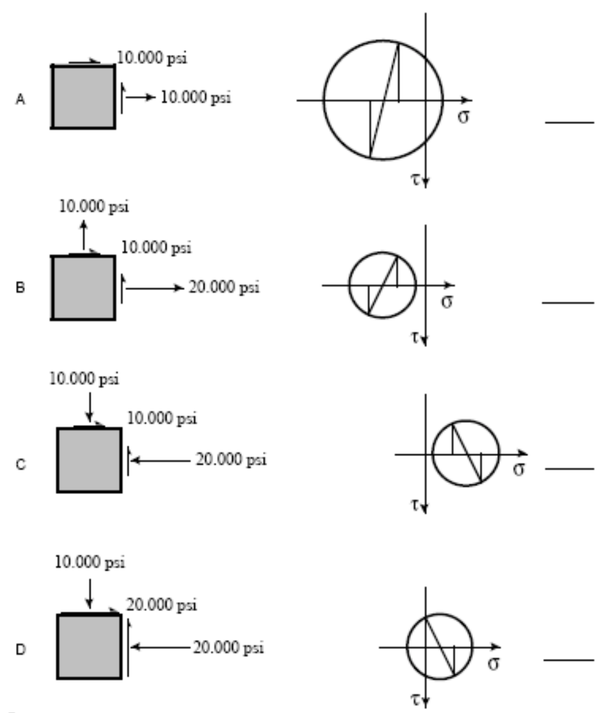
Stress Transformation: 2 A stress element is as shown. Find τ_{max}, σ_{max}, σ_{min}, and the direction of σ_{max}.



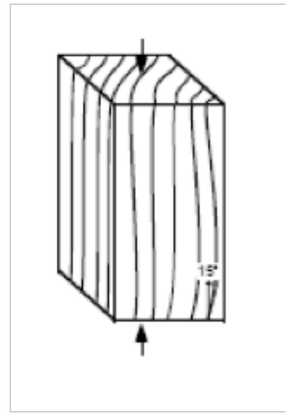
Stress Transformation: 3 (a) Given $\sigma_x = 12{,}000\text{psi}, \quad \sigma_y = -4{,}000\text{psi}, \quad \tau_{xy} = 1{,}000\text{psi}$, draw the stress element; (b) Given $\sigma_x = 15{,}000\text{psi}, \quad \sigma_y = 8{,}000\text{psi}, \quad \tau_{xy} = -2{,}000\text{psi}$, draw the stress element.

Stress Transformation: 4 For the stress element in Problem 3(a), find the principal stresses and the maximum shear stress.

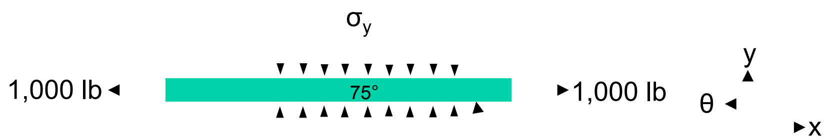
Stress Transformation: 5 Match each stress state to the corresponding Mohr's circle



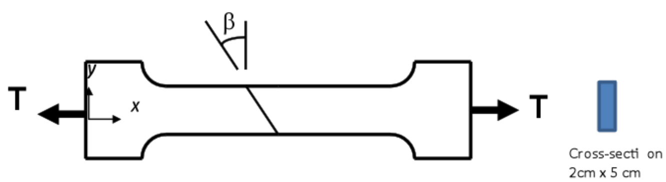
Stress Transformation: 6 The slope of the grain in the post has a maximum deviation from the axis of the member of 15°, and there is some concern that the relatively low shear strength of wood along its grain could cause problems. Calculate the shear stress along the grain for a 15° deviation as shown.



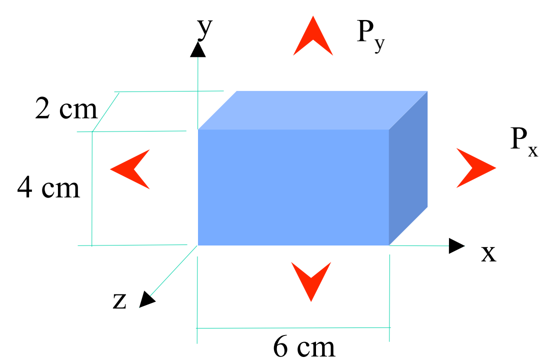
Stress Transformation: 7 An element is made from two shorter pieces of material with the glue joint at 15° from the loading axis. A 1,000 lb tensile load is applied as shown, and in the vicinity of the glue joint, a compressive stress of 87.5 psi is applied. Find the shear and normal stresses at the glue seam by drawing and using the Mohr's circle. Then draw the element and its stresses oriented at the pale of the joint and transverse to it.



Stress Transformation: 8 In order to test the mechanical properties of an adhesive, a so-called dog-bone specimen is prepared by adhering two pieces together at an angle β as shown in the figure. The specimen's cross-section in the vicinity of the adhesive is 2 cm × 5 cm. For a tensile force T = 2kN and β = 30^{O} what is the shear stress τ and normal stress σ the interface is subjected to? Draw τ and σ at the interface as vectors.



Stress Transformation: 9 A piece of material in its undeformed state has dimensions 6 cm × 4 cm × 2 cm in the x, y, z directions, respectively. The modulus of elasticity of the material is E = 100 GPa, and the Poisson ratio is ν = 0.2. Two tensile forces are applied as shown, i.e. P_x = 200 N, P_y = 150 N.



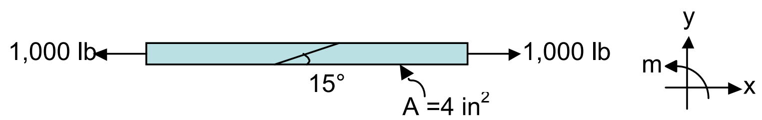
(a) True or false: the change in length in the z-direction, δ_z = 0.
(b) True or false: the normal stress in the z-direction, σ_z = 0.
(c) True or false: there are no shear stresses anywhere (at any inclined plane) in the element. Explain your answer.
(d) Determine the change in length in the x-direction, δ_x.
(e) Determine the maximum shear stresses anywhere (at any inclined plane) in the element.

Stress Transformation: 10 For the stress element in Problem 3(b), find the principal stresses and the maximum shear stress.

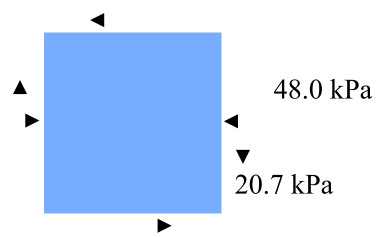
Stress Transformation: 11 For the stress element in Problem 3(a), find the principal stresses and the maximum shear stress using Mohr's circle.

Stress Transformation: 12 For the stress element in Problem 3(b), find the principal stresses and the maximum shear stress using Mohr's circle.

Stress Transformation: 13 A simple tension element is made from two shorter pieces of material with the glue joint at 15° from the loading axis. Using Mohr's circle, find the shear and normal stresses at the glue seam.

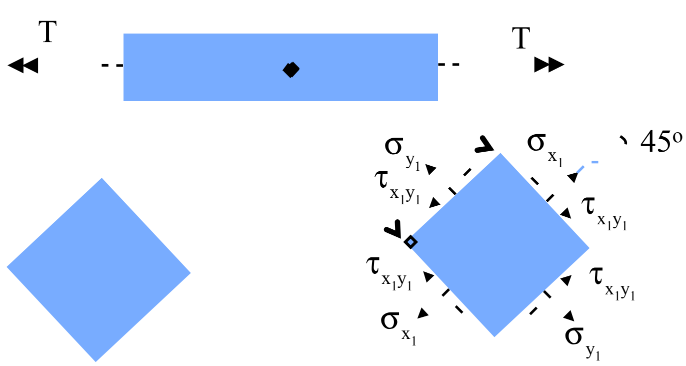


Stress Transformation: 14 At a material element the state of plane stress is as shown in the figure. (a) Draw the Morh's circle representing this state of stress. (b) Using the Mohr's circle determine the principal stresses σ_1 and σ_2. (c) draw the material element in the principal directions, i.e. in a properly oriented element.

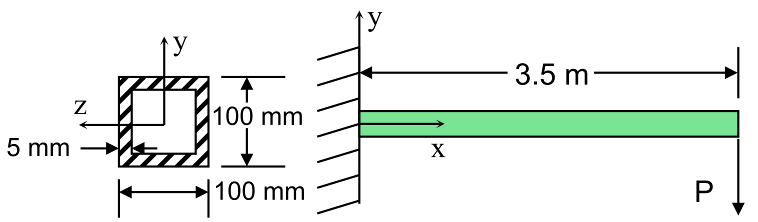


Stress Transformation: 15 A metal sheet is cut at 55^O as shown and then put together by weld spots (one weld spot on each side) equally spaced at distance s between them. The sheet is 1 cm (0.01 m) thick (in the direction transverse to the paper) and is subjected to a stress of 7.0 MPa in one direction as shown. Each weld spot has an allowable load in tension/compression, $F^{allowable}_{tension} = 15\,\mathrm{kN}$, and an allowable load in shear, $F^{allowable}_{shear} = 10\,\mathrm{kN}$. Find the minimum spacing, s, such that the allowable force on the weld spots in tension and in shear will not be exceeded.

Stress Transformation: 16 The shaft has a circular cross section of diameter d = 35 mm, and is subjected to torque T = 24 Nm in the direction shown. Consider a material element positioned on the outer surface of the cylinder and oriented at 45^O as shown. Find the magnitude of the stresses on that element and draw them (with the correct direction) in a properly oriented element.

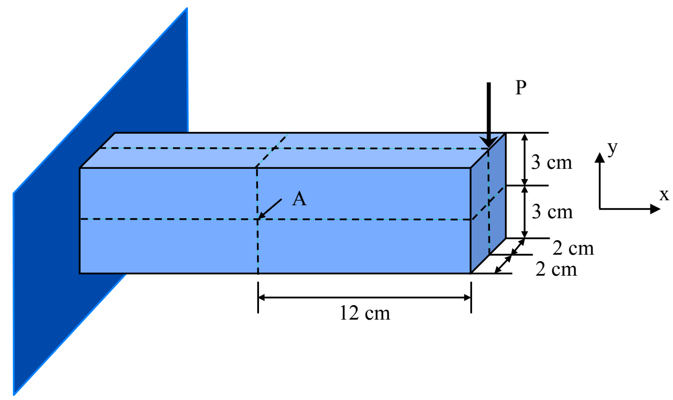


Stress Transformation: 17 The cantilever beam of length L = 3.5 m and hollow square cross section is loaded by a force P = 2.0 kN at the free end. (a) What is the state of stress (σ_x, τ_{xy}) at (x, y, z) = (1m, 0mm, 50mm)? (b) What is the radius of the Mohr's circle at (x, y, z) = (1m, 0mm, 50mm)? (c) What is the maximum shear stress $\tau_{x_1y_1}$ at (x, y, z) = (1m, 0mm, 50mm).

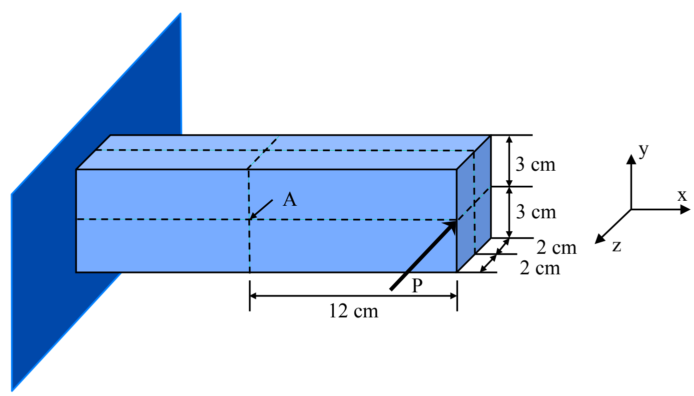


Stress Transformation: 18 For a state of plane stress where $\sigma_x = 11{,}000\,\text{psi}$, $\sigma_y = 0$, $\tau_{xy} = 4{,}775\,\text{psi}$, (a) find the maximum tensile normal and maximum shear stress; (b) draw the element where the principal stress take place in a properly oriented element.

Stress Transformation: 19 For the beam shown it was found that the principal stress σ_1 in the x-y plane at A is equal to 30 kPa. Based on that, find the value of the load P in kN.



Stress Transformation: 20 For the beam shown it was found that the principal stress σ_1 in the x-z plane at A is equal to 30 MPa. Based on that, find the value of the load P in kN (P is in the z-direction).



Stress analysis of a thin-wall spherical pressure vessel

Pressure vessels are structures enclosing a liquid or gas under pressure. Provided the wall thickness is small, compared to the other dimensions of the sphere (i.e., its radius), the stress analysis of such structures is relatively straightforward as shown below. Spherical and cylindrical pressure vessels belong to a category of structures called shells, such as domes, certain structural roofs, airplane wings, etc.

Figure L-1 shows a spherical vessel and two material elements, one on the outside of the wall and one on the inside. Since the wall is thin, the normal stress on the wall can be considered constant, denoted as σ. Due to symmetry in the sphere, the normal stress should then be equal to σ in any direction. Thus, in general, $\sigma_x = \sigma_y$. For elements on the outside of the wall, the pressure σ_z is equal to the atmospheric pressure, considered to be zero. For elements on the inside of the wall, however, $\sigma_z = -p$.

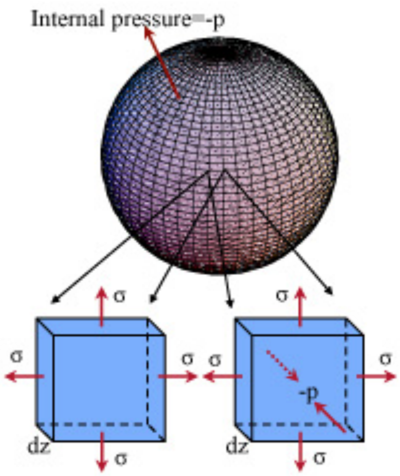


Figure L-1 A thin-wall pressure vessel subjected to internal pressure p. Also shown are two material elements, one in the outside of the wall (left), and one in the inside of the wall (right).

Normal stresses in spherical vessels

Let us consider a spherical vessel and a cross section as shown in Figure L-2. A further cross section, as shown in Figure L-3 reveals the normal stress σ acting on the wall, which is assumed constant due to the small thickness of the wall.

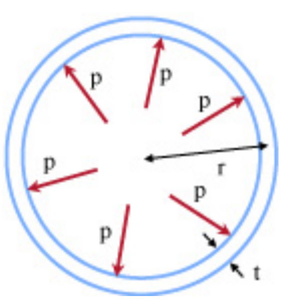


Figure L-2 A cross section of a spherical vessel showing the internal pressure, p, wall thickness, t, and radius, r.

Based on Figure L-3, the resultant of all the pressure forces acting on the inside of each hemispherical half of the shell must equal the sum of all the stresses that act on the cut surface.

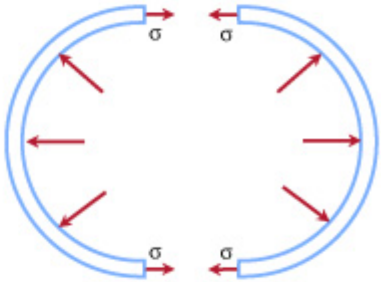


Figure L-3 Cross section of the vessel in two hemispherical parts.

The area of the cut surface is $2\pi rt$. Now the question is, what is the force exerted by internal pressure on the inside curved surface? As shown in Figure L-4, and as also known from fluid statics, the resultant of the pressure forces acting on the inside curved surface of the hemisphere must be equal to the same pressure acting on a flat disk of the same diameter. Thus, the net force on the curved and hypothetical flat surface is equal to $\pi r^2 p$. Equating this force to the force produced by σ, we have that

$$\sigma(2\pi r)t = \pi r^2 p$$

which yields

$$\sigma = \frac{pr}{2t}$$

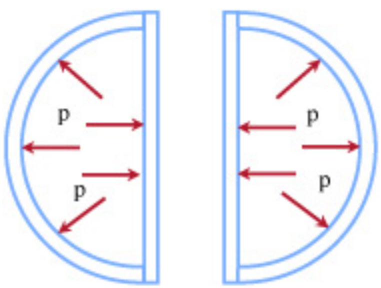


Figure L-4 Schematic showing the pressure on the curved surface of the shell and on a hypothetical flat disk of the same diameter.

Shear stresses in spherical shells

The stresses in material elements positioned at the outer and inner surfaces of a cylindrical vessel/shell are shown in Figure L-5, where also a relevant three-dimensional coordinate system is shown. Since there are no shear stresses, these are all principal stresses. Thus, at the outer surface

$$\sigma_1 = \sigma_2 = \sigma = \frac{pr}{2t}, \quad \sigma_3 = 0$$

and at inner surface

$$\sigma_1 = \sigma_2 = \sigma = \frac{pr}{2t}, \quad \sigma_3 = -p$$

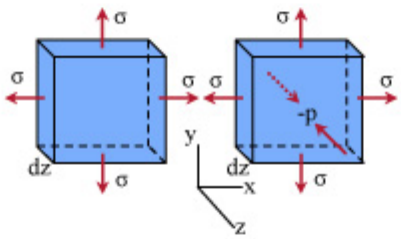


Figure L-5 Stresses on an element positioned in the outer surface (left) and inner surface (right).

As is recalled from Mohr's circle, the maximum shear stress on a plane is equal to the semi-difference of the two principal stresses in that plane. For example, in plane x-y, where the principal stresses are designated as σ_1 and σ_2, the maximum shear stress is

$$\tau_{xy}^{max} = \frac{\sigma_1 - \sigma_2}{2}$$

The maximum shear stress is mathematically expressed as

$$\tau_{max} = \mathrm{Max}\left[\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_1 - \sigma_3}{2}, \frac{\sigma_2 - \sigma_3}{2}\right]$$

For the elements shown in Figure L-5, it follows that at the inner surface

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\frac{pr}{2t} - (-p)}{2} = \frac{pr}{4t} + \frac{p}{2}$$

and at the outer surface

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\frac{pr}{2t} - 0}{2} = \frac{pr}{4t}$$

Example-spherical vessel: Let a spherical tank be subjected to internal pressure p = 3.0 MPa, its inside diameter be 230 mm, and its thickness 5 mm. Find: (a) maximum shear stress in the plane of the wall of the tank, and(b) maximum absolute shear stress.

(a) In the plane of the tank, $\sigma_1 = \sigma_2$, thus the in-plane maximum shear stress is zero.

(b) For any of the three planes, the maximum difference of principal stresses is that of $\sigma_1 - \sigma_3$, where $\sigma_1 = pr/2t$, and $\sigma_3 = -p$. Thus, $(\sigma_1 - \sigma_3)/2 = pr/4t + p/2$. Or,

$$\tau_{max} = \frac{pr}{4t} + \frac{p}{2} = \frac{3.0 \times 10^6 \times 0.115}{4 \times 0.005} + \frac{3.0 \times 10^6}{2} = 18.75 \text{ MPa}$$

Cylindrical vessels

The stress distribution in cylindrical vessels differs little from that in spherical vessels. Figure L-5 shows a material element positioned on the outer wall surface and one on the inner wall surface.

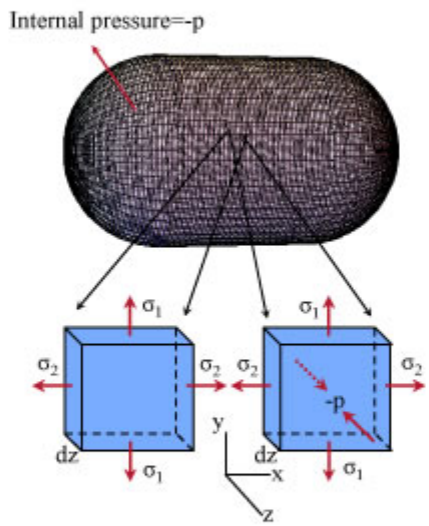


Figure L-5 A spherical vessel subjected to internal pressure p. Also shown are two material elements and the stresses acting on them, one positioned on the outer surface (left) and one on the inner surface (right).

For thin-wall vessels, the stresses along the thickness can be considered uniform. Also, due to symmetry, there are no shear stresses. Thus, as shown in Figure L-5, each stress is a principal stress. These principal stresses can be evaluated as shown in the following.

Normal stresses in cylindrical vessels

In order to find the principal stress σ_1, consider a one-unit length "slice" of the cylindrical part of the vessel as shown in Figure L-6.

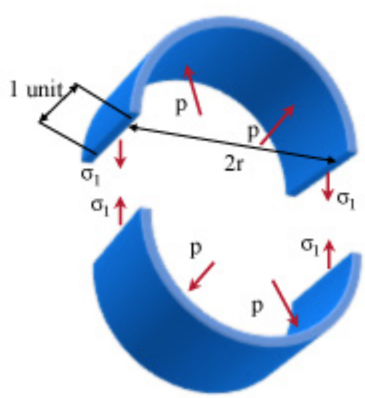


Figure L-6 A "slice" of the cylindrical part of a pressure vessel, also diametrically cut. Based on this, principal stress σ_1 can be evaluated.

Stress σ_1 acts on total area of 2t (for unit length width of the "slice"), while p produces a net force of p2r. Thus, $\sigma_1 2t = p2r$, which implies that

$$\sigma_1 = \frac{pr}{t}$$

Now considering equilibrium in the direction of the cylinder, as shown in Figure L-7, stresses σ_2 are created from the internal pressure p acting on the semi-spherical end cups of the vessel.

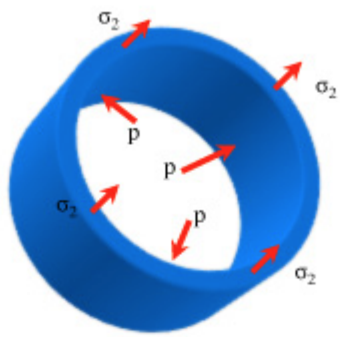


Figure L-7 A cut of the cylindrical surface showing the normal stress σ_2 created by the pressure p on the end cups.

The total force on the end cups is the product of pressure p and the net area of the end cup ($\pi r^{2)}$. This is equilibrated by the force on the section shown in the figure, which is the product of the stress σ_2 and the area of the section ($2\pi rt$). Thus, $p\pi r^2 = \sigma_2 2\pi rt$, which implies

$$\boxed{\sigma_2 = \frac{pr}{2t}}$$

Thus, from the previous derivation, the circumferential stress is twice as high as longitudinal stress, or $\sigma_1 = 2\sigma_2$.

Shear stresses in cylindrical vessels

The state of stress in a typical element on the inside and outside of the cylindrical part of a vessel is shown in Figure L-8.

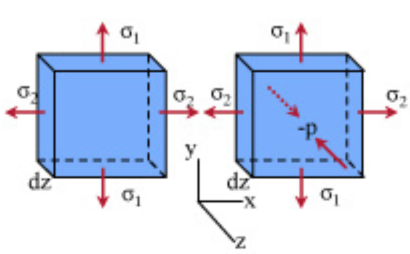


Figure L-8 Stresses on an element on the outside (left) and inside (right) of a cylindrical pressure vessel.

All of these stresses are principal ones, since the shear stress on each of the planes is zero. The maximum shear stress is (e.g., from Mohr's circle) equal to the semi-difference of the principal stresses. Thus, in general, for a three-dimensional state of stress with principal stresses σ_1, σ_2, σ_3, the overall maximum shear stress is

$$\tau_{max} = \max\left(\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_1 - \sigma_3}{2}, \frac{\sigma_2 - \sigma_3}{2}\right)$$

This implies that on the outer surface

$$\tau_{max} = \tau_{max}^{xy} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\frac{pr}{t} - 0}{2} = \frac{pr}{2t}$$

where the superscript xy indicates shear stress on the x-y plane, as shown in Figure L-8. As for the inner surface, however, we have

$$\tau_{max} = \tau_{max}^{yz} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\frac{pr}{t} - (-p)}{2} = \frac{pr}{2t} + \frac{p}{2}$$

This indicates that the absolute maximum shear stress takes place on elements on the inner surface.

Example-cylindrical vessel: The metal tank shown in Figure L-9 is welded together. The two semi-spherical end cups are welded to the cylindrical part, which also consists of three cylindrical parts welded together. The inside diameter of the tank is 300 mm, and the internal pressure p is 2.0 MPa. If the allowable normal stress for the welds is $\sigma_{all}^{weld} = 40$ MPa and for the metal it is $\sigma_{all} = 60$ MPa then

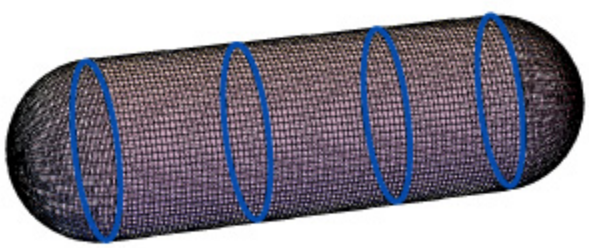


Figure L-9 A spherical tank welded together. The welds are shown in blue, for clarity

(a) Determine the minimum thickness of the cylindrical part of the tank; (b) Determine the minimum thickness of the hemispherical cups.

(a) For the cylindrical part: $\sigma_1 = \frac{pr}{t} \rightarrow t = \frac{pr}{\sigma_{all}} = \frac{2 \times 150}{60} = 5.00$ mm

For the welding: $\sigma_2 = \frac{pr}{2t} \rightarrow t = \frac{pr}{2\sigma_{all}^{weld}} = \frac{2 \times 150}{2 \times 40} = 3.75$ mm

Thus, minimum thickness is 5.0 mm

(b) For the hemispherical cups: $t = \frac{pr}{2\sigma_{all}} = \frac{2\times150}{2\times60} = 2.5$ mm

At the welds connecting the cups: $t = \frac{pr}{2\sigma_{all}^{weld}} = \frac{2\times150}{2\times40} = 3.75$ mm

Thus, minimum thickness here is 3.75 mm.

Example: Cylindrical Shell Subjected to Internal Pressure

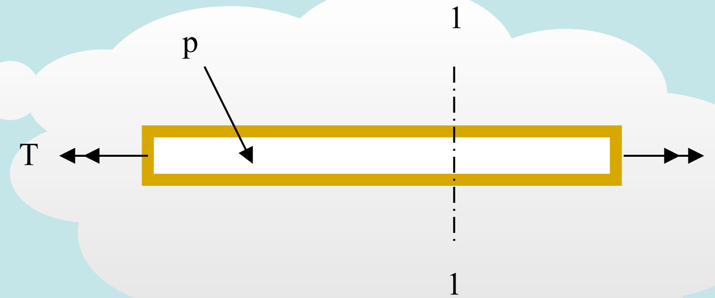
$$\sigma_1 = \frac{Pr}{t} = \frac{1000 \times 36.5/2}{1/2} = 36{,}500 \text{ psi}$$

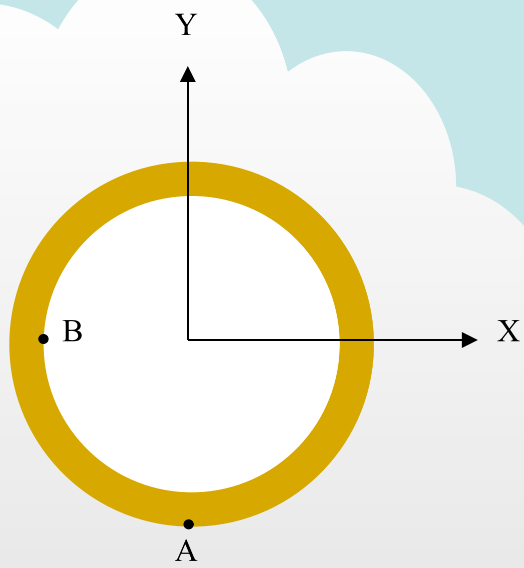
$$\sigma_2 = \frac{Pr}{2t} = \frac{1000 \times 36.5/2}{2\left(1/2\right)} = 18{,}250 \text{ psi}$$

$$I_p = \frac{\pi}{32}\left(37^4 - 36^4\right) = 19099.41$$

$$\tau = \frac{Tr}{I_p} = \frac{6000 \times 36.5/2}{19099.41} = 5.733 \text{ psi}$$

Stress	Point A	Point B
σ_x	36,500 psi	−1,000 psi
σ_y		36,500 psi
σ_z	18,250 psi	18,250 psi
τ_{xy}		5.733 psi
τ_{xz}	5.733 psi	
τ_{yz}		

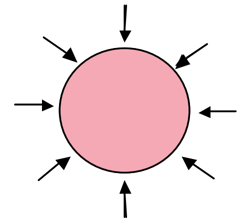




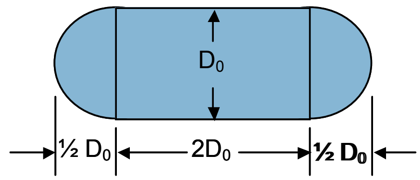
Cross section (hollow cylinder)

The cylindrical shell is subjected to internal pressure p = 1,000 psi and a torque of 6000 in-lb.
Fill in the table for the stresses at points A and B.
Note that point A is at the outside and B at the inside wall of the shell. Both points A and B are at section 1-1. The inside diameter of the hollow cylindrical part is 36 in and the outside diameter is 37 in.

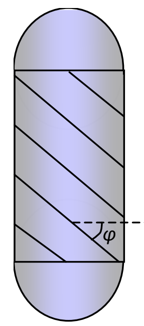
Pressure Tanks: 1 The diameter of a ping pong ball is 40 mm and the wall thickness is 0.5 mm. The ball is made of nylon with an ultimate compressive strength of 95 MPa. Find the pressure required to crush the ball.



Pressure Tanks: 2 A cylindrical pressure tank with hemispherical end caps is to be fabricated from mild steel 3/8 in thick. The length of the tank is to be three times the diameter. Find the dimensions of the tank if $\sigma_{allowable} = 20$ ksi and internal pressure is 500 psi.



Pressure Tanks: 3 A cylindrical pressure vessel of inner diameter d = 4 ft (48 in) and wall thickness t = 1/2 in is fabricated from a welded pipe with helix angle $\varphi = 38^o$. Two spherical caps, also of wall thickness t are welded to the cylindrical part. If the pressure inside the vessel is p = 200 psi, find the normal stress σ_w perpendicular to the weld.



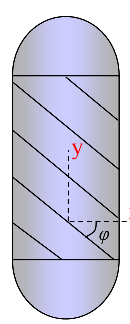
Pressure Tanks: 4 For problem 3, find the shear stress τ_w parallel to the weld.

Pressure Tanks: 5 For the vessel in problem 3, find the maximum shear stress anywhere in the vessel, and in any possible plane.

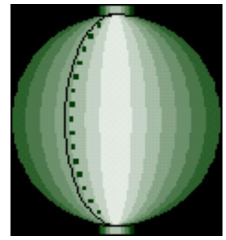
Pressure Tanks: 6 A pipe is formed by bolting together parts as the one shown. The internal radius of the pipe is r=20cm and the wall thickness is t = 1.2 cm. While the pipe is subjected to internal pressure from a fluid of p = 2,000 Pa, find the force on each of the 12 bolts that connect the parts.

Pressure Tanks: 7 The tank shown is filled up with petrol of density ρ = 720g/lt (grams per liter). Its height is 17 m, and its internal radius is r = 12m. Assuming that the only load on the tank is the pressure from the petrol, find the minimum thickness of the tank wall such that the maximum shear stress anywhere on the wall will not exceed 120 MPa. Note that the hydrostatic pressure from a fluid of density ρ at depth h is given by $p = \rho g h$, where g denotes the acceleration of gravity.

Pressure Tanks: 8 A cylindrical pressure tank is constructed by welding together sheet metal at angle φ as shown. Two options are considered, one for $\varphi = 30^o$ and one for $\varphi = 50^o$. Which of the two is more efficient, with respect to creating the least shear stress on the weld.

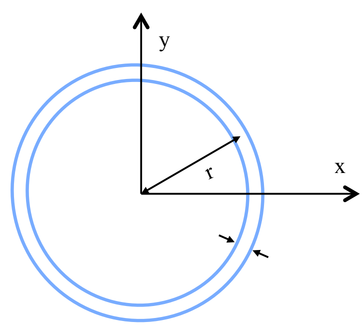


Pressure Tanks: 9 A spherical pressure tank is constructed by bolting together two semi-spherical shells as shown. Thirty-six bolts are used and the tank, of radius r and shell thickness t is subjected to internal pressure p. Evaluate the force on each bolt.

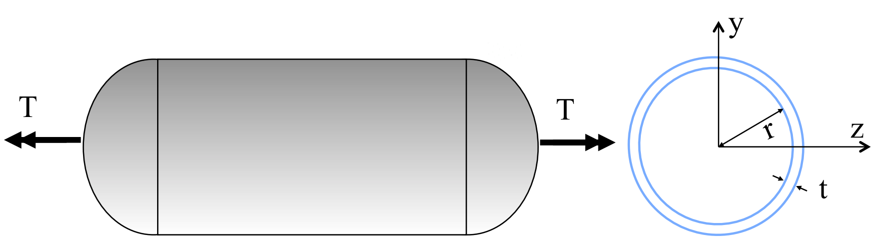


Pressure Tanks: 10 Under plane stress conditions, a state of pure shear is one that at some orientation θ the state of stress is $\sigma_{x_1} = 0, \sigma_{y_1} = 0, \tau_{x_1y_1} \neq 0$. (a) Draw a schematic of the Mohr's circle for a state of pure shear. (b) A cylindrical thin-walled pressure vessel of radius r and wall thickness t is subjected to internal pressure p, and at the same time to a compressive force F; find the magnitude of F such that the state of stress on the wall of the cylindrical part of the vessel is a state of pure shear.

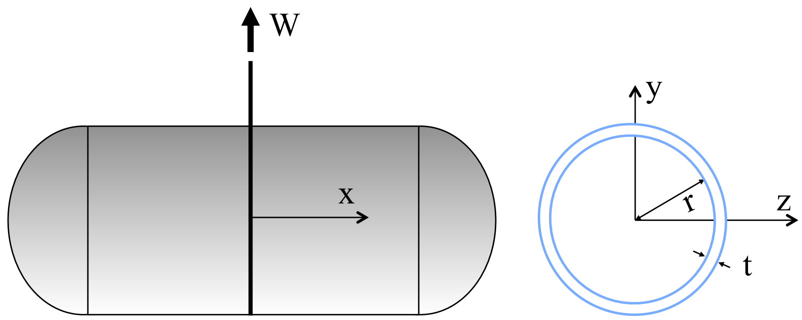
Pressure Tanks: 11 A cylindrical thin-walled pressure vessel of radius r and wall thickness t is subjected to internal pressure p. The vessel needs to be moved by a crane while under pressure p. It was determined that while being lifted the cylindrical part of the vessel (considered to be a beam) will be subjected to a maximum bending moment M. For what value of M will the longitudinal stress in the vessel equal the hoop stress at some point in the cylindrical part of the vessel? Note that the moment of inertia of a thin circular ring can be approximated as $I_x = I_y = \pi r^3 t$



Pressure Tanks: 12 The cylindrical pressure vessel (cross section shown to the right) is subjected to internal pressure p = 12.0 MPa and at the same time it is subjected to a torque T = 18 kNm. The internal radius of the vessel is r = 0.2 m, and its thickness is t = 0.02 m. (a) What is the stress state (all six stresses) on the cylindrical part of the vessel at y = 0.2 m, z = 0? (b) What is the maximum shear stress at y = 0.2 m, z = 0?



Pressure Tanks: 13 The cylindrical pressure vessel (cross section shown to the right) is subjected to internal pressure p = 12.0 MPa, and at the same time it is lifted by wrapping a cable around its cross section at its center as shown. The weight of the vessel (including the pressurized fluid) creates a bending moment at the cross-section at the center (x = 0) M = -80.0 kNm. The internal radius of the vessel is r = 0.2 m, and its thickness is t = 0.02 m. (a) What is the stress state (all six stresses) on the cylindrical part of the vessel at x = 0, y = 0.22 m, z = 0? (b) What is the maximum shear stress at x = 0, y = 0.22 m, z = 0? Note: for a circular cross section of radius r, the moment of inertia with respect to any axis passing through the center of the cross section is $I = \frac{\pi r^4}{4}$



Combined Loads

So far in this course, the stresses developed in bars (axial load), shafts (torsional load), beams (bending load), and pressure vessels have been examined. Often in engineering a structure or structural component is subjected to a combination of loads that creates a combination of stresses. As an example, consider the simultaneous actions of bending, torsion, and axial load in a prismatic bar. The beam/bar/shaft, is then subjected to combined loads. Due to the linearity of the governing equations, all quantities such as stresses, strains, and displacements, can be superimposed. Another example is a cantilever beam subjected to end load P and torque T as shown in Figure M-1.

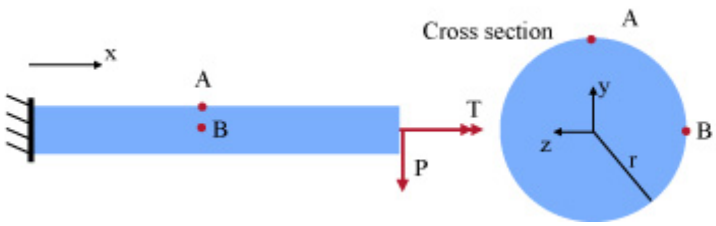


Figure M-1 A cantilever beam subjected to torque T and bending/shear load P.

A question relevant to this problem can be, for example, to find the stresses of any desired point, e.g., points A and B with respect to x, y, and z. From load P, at the cross section where point A is, we have:

- Bending moment M, creating normal stress σ: $\sigma_x^{bend} = \dfrac{My}{I}$

- Shear force V, creating shear stress τ : $\tau_{xy}^{shear} = \dfrac{VQ}{Ib}$

- Torque T, creating shear stress τ : $\tau_{ij}^{torque} = \dfrac{Tr}{I_p}$

Here the index ij is either xy or xz, depending on whether the point of interest is B or A. This is explained in detail in the following. Consider a material element at point A as shown in Figure M-2.

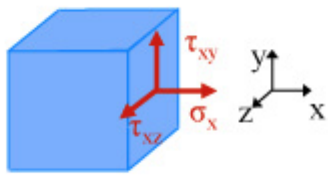


Figure M-2 Material element positioned at point A in Figure M-1.

At this element, the non-zero stresses are:

- Bending creates $\sigma_x^{bend} = \dfrac{My}{I}$

- Shear force V creates $\tau_{xy}^{shear} = \dfrac{VQ}{Ib}$

- Torque T creates $\tau_{ij}^{torque} = \frac{Tr}{I_p}$ N

ext consider an element of volume at point B as shown in Figure M-1.

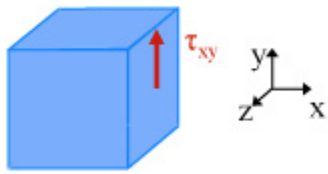


Figure M-3 Material element positioned at point B in Figure M-1.

Here, T creates τ_{xy}, V also creates τ_{xy}, yet M does not create σ_x since y = 0 at B. Thus, at B the only nonzero stress is

$$\tau_{xy} = \tau_{xy}^{shear} + \tau_{xy}^{torque} = \frac{Tr}{I_p} + \frac{VQ}{Ib}$$

In general, at every material point we have three normal stresses, σ_x, σ_y, σ_z and three shear stresses, $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{yz}$, $\tau_{yz} = \tau_{zy}$. Given those stresses, an analysis process seeks maximum normal stresses, maximum shear stresses, etc. Here is an example, referring to Figure M-4.

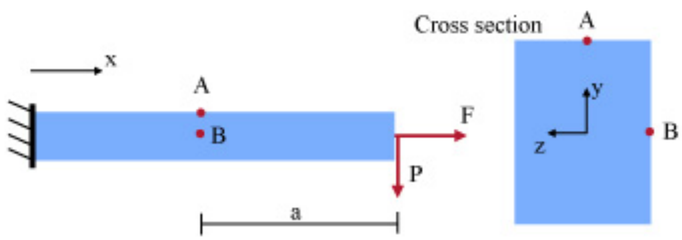


Figure M-4 A cantilever beam subjected at its free end to transverse force P and normal force F. Points A and B are of particular interest in this example.

At the cross section at A and B, i.e., at a distance a from the right end, we have that the normal force is equal to F, the bending moment M is equal to −P(a), and the shear force V is equal to P.

The normal force F produces normal stress,

$$\sigma_x = \frac{F}{bh}$$

at both points A and B, where b denotes the width and h the height of the cross section.

The bending moment also creates normal stress

$$\sigma_x = \frac{My}{I} = \frac{Pay}{bh^3/12}$$

Since y = 0 for point B, the normal stress from bending is zero at B. Similarly, for point A (y = h/2), we have

$$\sigma_x^{A-\text{bending}} = \frac{My}{I} = \frac{Pah/2}{bh^3/12}$$

The shear force V creates shear stress τ_{xy} at both points A and B. At A, however, this shear stress is zero since it is at the top of the cross section, where $Q = 0$. At B, however,

$$Q = b\left(\frac{h}{2}\right)\left(\frac{h}{4}\right)$$

thus

$$\tau_{xy}^{B} = \frac{Vb(h/2)(h/4)}{\left(bh^3/12\right)b}$$

In summary:

Point A: $\sigma_x^A = \frac{F}{bh} + \frac{Pah/2}{bh^3/12}$

and all other stresses are equal to zero.

Point B: $\sigma_x^B = \frac{F}{bh}; \quad \tau_{xy}^B = \frac{Vb(h/2)(h/4)}{\left(bh^3/12\right)b}$

and all other stresses are zero.

1. From the stresses at x, y, and z the maximum normal and shear stresses under transformation can be found, e.g., in the x-y plane, since $\sigma_y = 0$

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}; \quad \tau_{max}\sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$$

2. Identify critical points (fixed end for this problem) where the bending moment is maximum

Point A: max σ_x due to bending, zero τ_{xy} due to bending. Point B: min (zero) σ_x due to bending, max τ_{xy} due to bending.

Module 29: Combined Loads and Combined Stresses

In general, every structure or structural component has so-called critical points or hot spots where stresses are extreme. No general guidelines for locating critical points exist. The engineer uses rational procedures and experience to find the critical points and the stresses on them. Figure M-5 shows a fixed-end beam loaded at its free end, and five points at a certain cross section. The state of stress at each of these five points is examined.

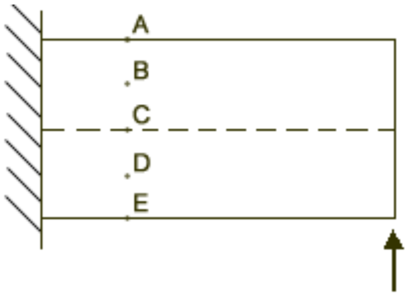


Figure M-5 A fixed end beam and points A, B, C, D, E

Figure M-6 shows the elements and the stress acting at A, B, C, D, and E as well as the principal stresses. At point A we have normal stress σ_x but no shear stress τ_{xy}. Then the compressive stress at A is a principal stress. Based on the state of stress, the maximum shear stress can be found. At B, we have both normal stress σ_x and shear stress τ_{xy}. At C, we have shear stress τ_{xy} but zero normal stress. At D, we have tensile normal stress and nonzero τ_{xy}. At E, we only have tensile normal stress σ_x.

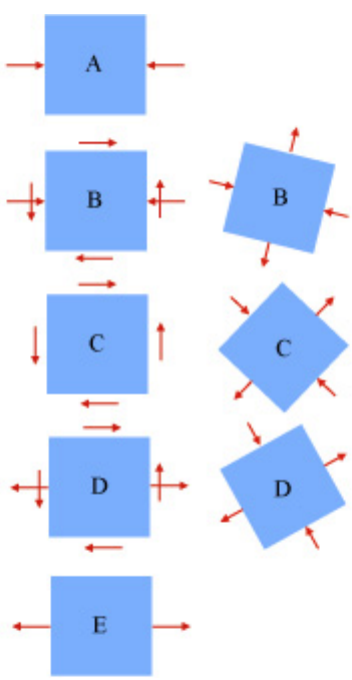


Figure M-6 Schematic of the state of stress at material elements A, B, C, D, E. To the right, the principal stresses in the principal directions are shown. For points A and E the stresses shown are also the principal stresses. The maximum shear stresses occur at 45^o from the principal directions.

Figure M-7 shows the so called stress trajectories, i.e., the contours of principal tensile and compressive stress. Those can be plotted from either evaluating the principal stresses on a grid or analytically by plotting the expression for the principal stresses as a function of the coordinates x and y.

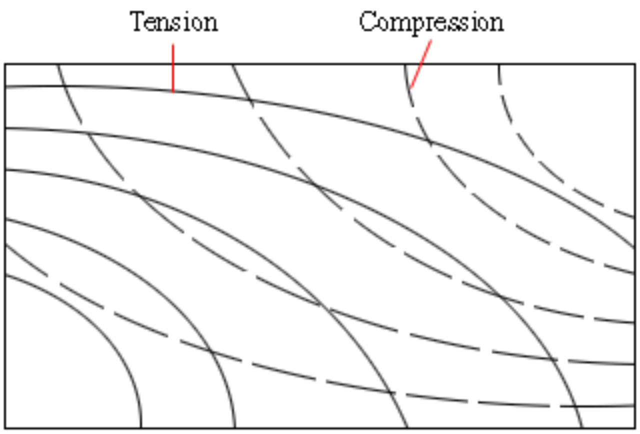


Figure M-7 Contours of the principal stresses for the beam shown in Figure M-5.

Example-beam bending: For the fixed-end beam shown in Figure M-8, loaded by a moment at end B and a force at midspan, find all six stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, and τ_{yz}, at the following two material points (x, y, z, in meters): (0.9, 0.0, 0.02) and (1.5, –0.02, 0.0).

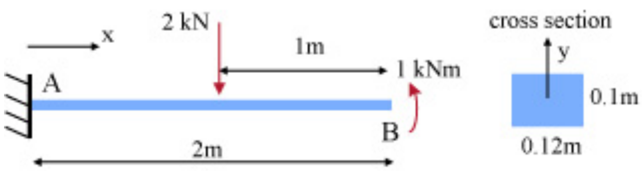


Figure M-8 A cantilever beam loaded by a concentrated force and a concentrated moment as shown.

At the first point, M = 1.0-2 × 0.1 = 0.8 kNm, V = –2 kN. The moment of inertia of the cross section, I, is expressed as

$$I = \frac{0.12 \times 0.1^3}{12} = 0.00001 \text{ m}^4$$

At y = 0, z = 0.02, the moment Q is expressed as

$$Q = 0.12 \times \frac{0.1}{2} \times \frac{0.1}{4} = 0.00015 \text{ m}^3$$

Then, for the first point,

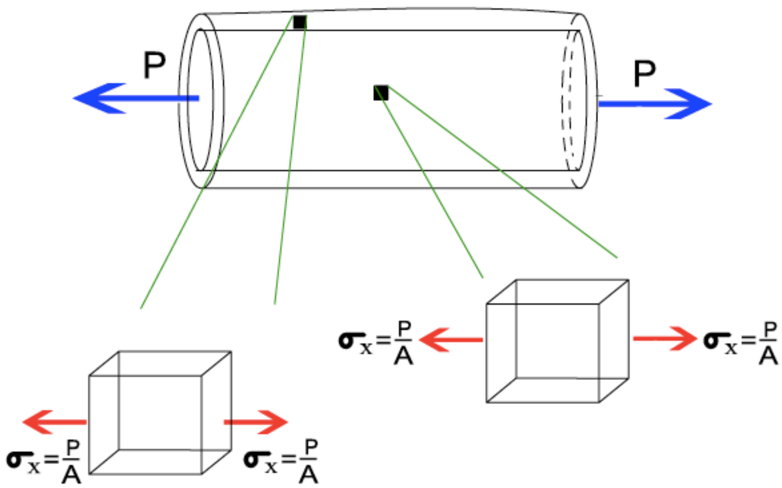
$$\sigma_x = 0; \quad \tau_{xy} = \frac{2 \times 0.00015}{0.00001 \times 0.12} = 250 \text{ kPa}$$

At the second point, M = 1 kNm, V = 0. Then, at this point

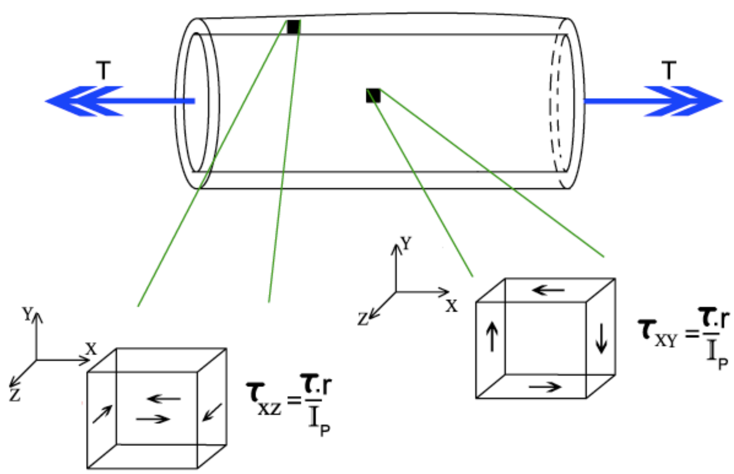
$$\tau_{xy} = 0; \quad \sigma_x = \frac{1 \times 0.02}{0.00001} = 2{,}000 \text{ kPa}$$

The following table shows all the stresses

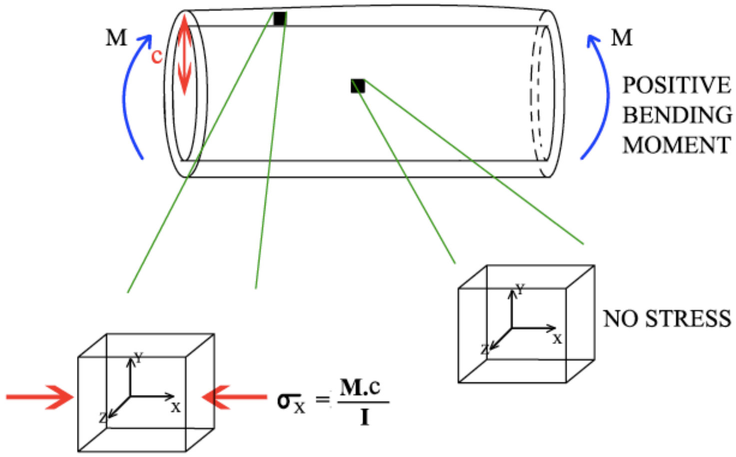
	Point at $(0.9, 0.0, 0.02)$	Point at $(1.5, -0.02, 0.0)$
σ_x	0	2,000 kPa
σ_y	0	0
σ_z	0	0
τ_{xy}	250 kPa	0
τ_{xz}	0	0
τ_{yz}	0	0



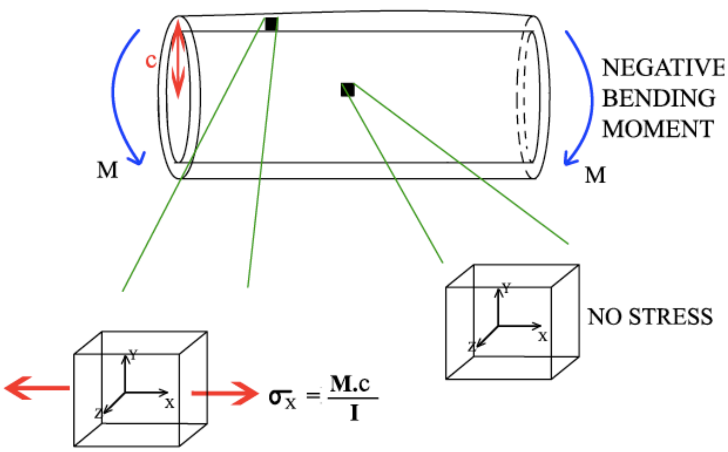
The stress σ=P/A in a bar of cross-sectional area A subjected to tensile or compressive load P is uniform. Thus, as shown in the "movie" below, the stress is the same at every material element in the bar. Also, for a coordinate system where the x-direction is along the bar's length, the stress is designated as σ_x=P/A.



The stress τ in a shaft subjected to torque T is not uniform. Thus, as shown in the "movie" below, the stress is not in the same direction at every material element in the shaft. Also, for a coordinate system where the x-direction is along the shaft's length, the stress is designated as τ_{xy} or as τ_{xz}



The normal stress σ_x in a beam subjected to positive bending moment is illustrated above. Note that σ_x is proportional to the y-coordinate of the material element.



The normal stress σ_x in a beam subjected to negative bending moment is illustrated above. Note that σ_x is proportional to the y-coordinate of the material element.

Module 30: Combined Loads and Combined Stresses

Example-combined stresses: The three-dimensional beam shown in Figure M-9 is fixed at D and is bent 90^{o} at B and C. It is loaded by a compressive load of 300 lb at A, and length BC is 36 inches. For the beam cross section and the coordinate system shown in the figure, find all six stresses, σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, and τ_{yz} at points D_1 and D_2, both being at the fixed end D.

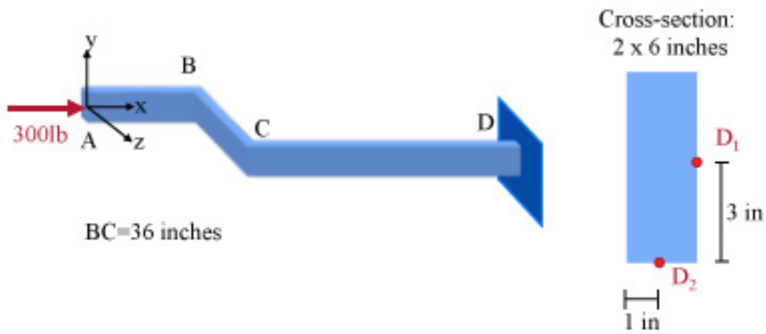


Figure M-9 The three-dimensional beam is loaded at its free end and fixed at the other end.

The 300 lb force can be moved to C, adding the moment produced in doing so (i.e., 300 lb × 36 in), which yields a moment M = 10800 lb-in. Thus, the problem for beam CD reduces to the one shown in Figure M-10.

Figure M-10 The beam in Figure M-9 can be reduced to the one shown here by moving the applied force at C, as typically done in statics.

Note the coordinate system and the orientation of the cross section with respect to these coordinates. Thus, the cross section at D is subjected to a combined load consisting of a compressive force of 300 lb and a bending moment of 10,800 lb-in.

Due to the compressive load at both D_1 and D_2, we have

$$\sigma_x = \frac{P}{A} = \frac{-300}{2\times 6} = -25 \text{ psi}$$

From the bending moment, the stress at D_2 is zero (since the point is at the centerline), while at D_1,

$$\sigma_x = \frac{My}{I} = \frac{10{,}800\times 1}{6\times 2^3/12} = 2{,}700 \text{ psi}$$

Thus, at D_1, $\sigma_x = 2{,}700 - 25 = 2{,}675$ psi, and all other stresses are zero, and at D_2, $\sigma_x = -25$

psi, and all other stresses are zero.

Example-combined stresses: The 3-m-long cantilever I-beam shown in Figure M-11 is subjected to a horizontal force P at its left end and a moment M_0 at midspan. Find the range the ratio M_0/P can have such that there is no compressive stress σ_x anywhere on the cross section at the fixed end of the beam. Note that M_0 can be either clockwise or counter-clockwise, and P can be directed either left to right (compressive) or right to left (tensile).

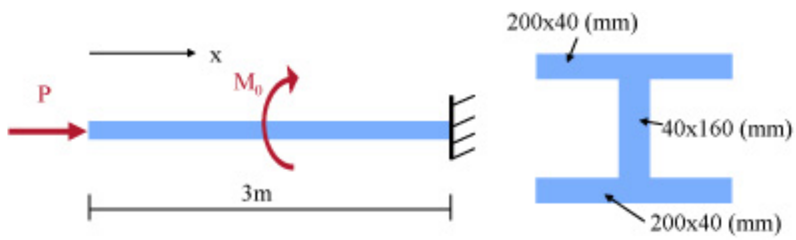


Figure M-11 A cantilever I-beam subjected to a concentrated compressive force and a concentrated moment.

For the cross section, the moment of inertia is evaluated as

$$I = \frac{0.2 \times 0.24^3}{12} - \frac{(0.2 - 0.04) \times 0.16^3}{12} = 0.0001758 \text{ m}^4$$

The area A of the cross section is $A = 2 \times 0.2 \times 0.04 + 0.16 \times 0.04 = 0.0224 \text{ m}^2$. Note that M_0 produces stress σ_x such that the maximum tensile stress it produces is equal and opposite to the maximum compressive stress. Then, if P is compressive, there is no way that compressive stress does not take place at the fixed end. Thus, P must be tensile (right to left). Then the maximum compressive stress σ_x is expressed as

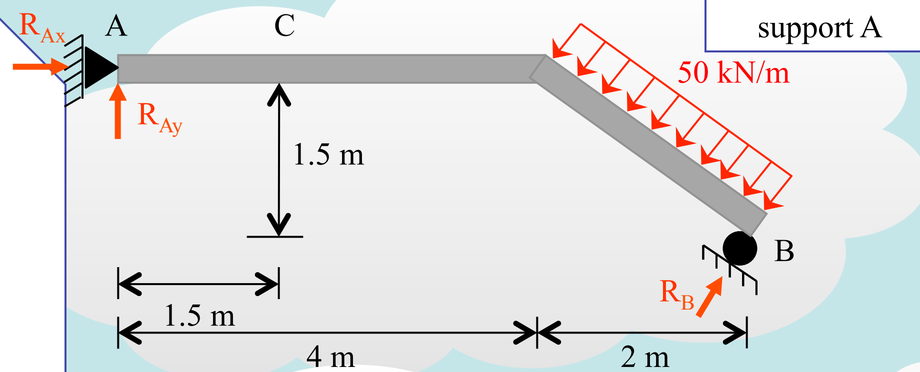
$$\sigma_x^{\text{max,comp}} = -\frac{M_0 c}{I} + \frac{P}{A} = -\frac{M_0 0.12}{0.0001758} + \frac{P}{0.0224}$$

Setting this stress equal to zero, it follows that

$$\frac{M_0}{P} = \frac{0.0001758}{0.12 \times 0.0224} = 0.0654 \text{ m.}$$

Combined Stresses

Find the stresses at the top of the rectangular cross-section (c) at section C located 1.5m from support A



From Statics,

$R_{Ax} = 16.45$ kN

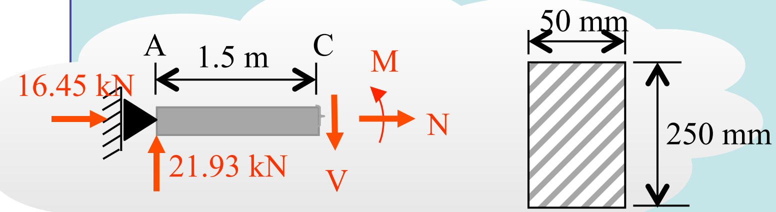
$R_{Ay} = 21.93$ kN

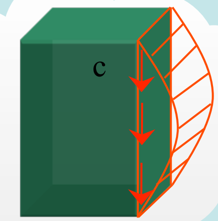
$R_B = 97.59$ kN

$N = -16.45$ kN (c)

$V = 21.93$ kN

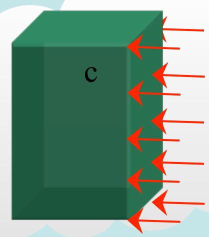
$M = 32.89$ kN · m





Axial stress:

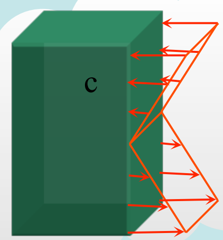
$$\sigma_c = \frac{P}{A} = \frac{16{,}450}{0.050 \times 0.250} = -1.32 \text{ MPa } (\text{comp.})$$



Shear stress:

$$\tau_c = \frac{VQ}{Ib} = 0$$

$$\ldots\left\{ \text{ because } Q = 0 \left(\text{no area above C}\right)\right\}$$



Bending stresses:

$$\sigma_c = \frac{MC}{I} = -\frac{32{,}890 \times 0.125}{\frac{1}{12}\left(0.050 \times 0.250^3\right)}$$

$$= -63.15 \text{ MPa}$$

Combined stress:

$$\left(\sigma_c\right)_{total} = -1.32 - 63.15 = -64.47 \text{ MPa}$$

Example: No Compressive Stresses in a Plate

$$\sigma_A = 0 = \frac{P}{A} - \frac{MC}{I}$$

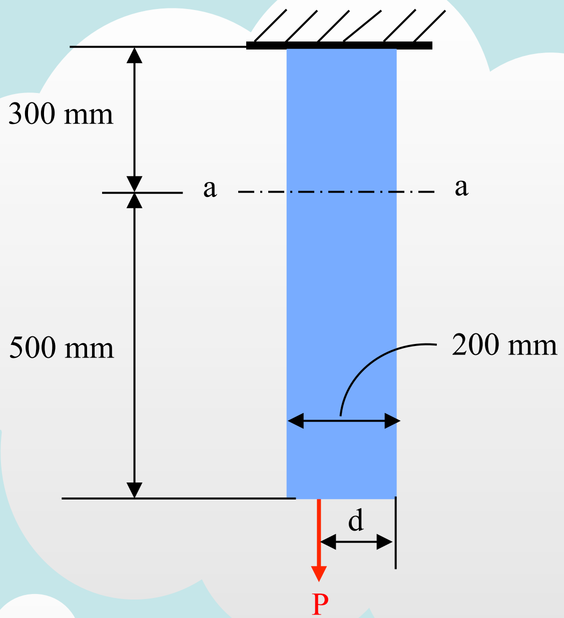
$$\Rightarrow \frac{P}{0.2 \times 0.01} = \frac{P(d-0.1)\times 0.1}{\frac{1}{12}\times 0.01 \times 0.2^3}$$

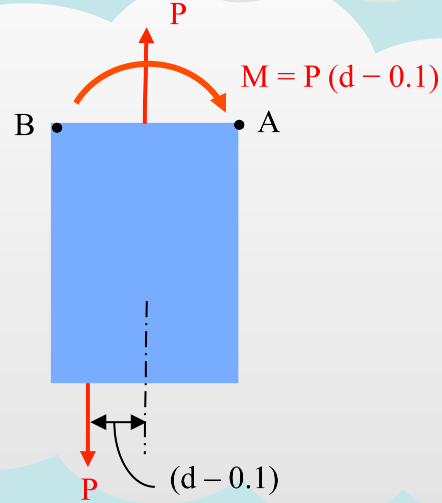
$$\Rightarrow P = \frac{12P \times (d-0.1)\times 0.1}{0.2^2}$$

$$\Rightarrow 12 \times 0.1 = 0.2^2$$

$$\Rightarrow d - 0.1 = \frac{0.2^2}{12 \times 0.1}$$

$$\Rightarrow d = 0.133 \text{ m} = 133 \text{ mm}$$





The vertical force P acts on the bottom of the plate having a negligible weight. Determine the shortest distance d to the edge of the plate at which it can be applied so that it produces no compressive stresses on the plate at section a-a. The plate has a thickness of 10 mm and P acts along the centerline of this thickness.

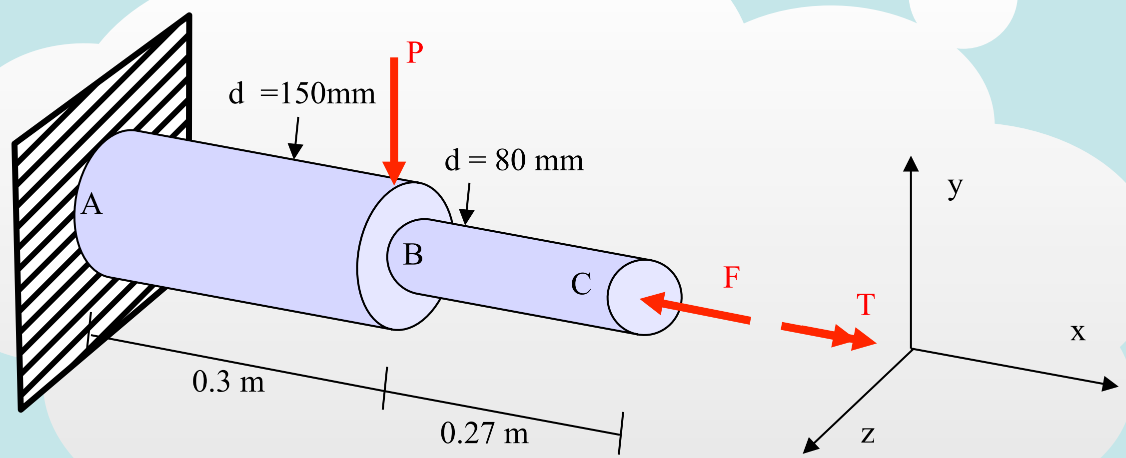
Example: Combined Stress

The structure is fixed at A and subjected to P = 2 kN, F = 6 kN and T = 0.3 kNm in the directions and positions shown in the figure. Find the state of stress, i.e., all normal and shear stresses at the points with coordinates, in m/mm: (0, 75 mm, 0) and (0.4 m, 0, −40 mm). For a circular area, the moment of inertia with respect to axes x and y are:

$$I_{xx} = I_{yy} = \frac{\pi d^4}{64}$$, where d denotes diameter of the circular area

and the polar moment of inertia I_p is expressed as

$$I_p = \frac{\pi d^4}{32}$$



At the fixed end, $M = -2 \times 0.3 = -0.6$ kNm, $V = 2$ kN, $N = -6$ kN, $T = 0.3$ kNm

Then, at the first point:

$$M \rightarrow \sigma_x = -\frac{My}{I} = -\frac{-0.6 \cdot 0.15/2}{\pi \cdot 0.15^4/64} = 1811.75 \text{ kPa}$$

$$N \rightarrow \sigma_x = \frac{N}{A} = \frac{-6.0}{\pi \cdot 0.15^2/4} = -339.7 \text{ kPa}$$

$$T \rightarrow \tau_{xz} = \frac{Tr}{I_p} = \frac{0.3 \cdot 0.15/2}{\pi \cdot 0.15^4/32} = 452.9 \text{ kPa}$$

Then,

$$\sigma_x = 1811.75 - 339.7 = 842.05 \text{ kPa}$$

$$\tau_{xz} = 452.9 \text{ kPa}$$

Note that the shear force V does not create shear stress at this first point (top of the cross-section).

Then at the second point:

At $x = 0.4$ m, $M = 0.0$, $V = 0.0$, $N = -6$ kN, $T = 0.3$ kNm

$$N \rightarrow \sigma_x = \frac{N}{A} = \frac{-6.0}{\pi \cdot 0.08^2/4} = -1{,}194.3 \text{ kPa}$$

$$T \rightarrow \tau_{xy} = \frac{Tr}{I_p} = \frac{0.3 \cdot 0.08/2}{\pi \cdot 0.08^4/32} = 2{,}985.7 \text{ kPa}$$

and all other stresses are equal to zero.

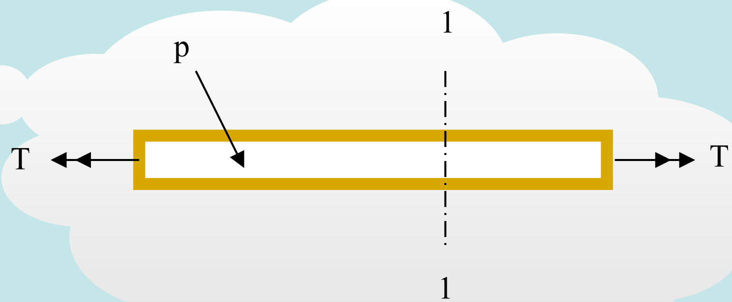
Example: Cylindrical Shell Subjected to Internal Pressure

$$\sigma_1 = \frac{Pr}{t} = \frac{1{,}000 \times 36.5/2}{1/2} = 36{,}500 \text{ psi}$$

$$\sigma_2 = \frac{Pr}{2t} = \frac{1{,}000 \times 36.5/2}{2(1/2)} = 18{,}250 \text{ psi}$$

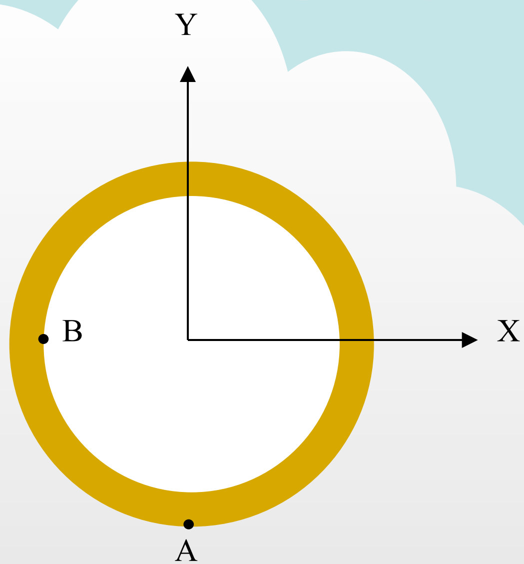
$$I_p = \frac{\pi}{32}\left(37^4 - 36^4\right) = 19{,}099.41$$

$$\tau = \frac{Tr}{I_p} = \frac{6{,}000 \times 36.5/2}{19{,}099.41} = 5.733 \text{ psi}$$



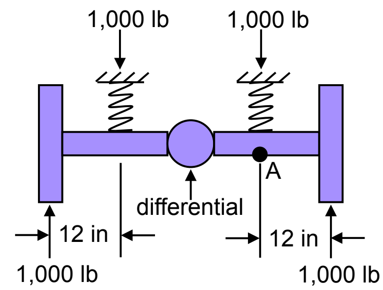
The cylindrical shell is subjected to internal pressure p = 1,000 psi and a torque of 6,000 in-lb.
Fill in the following table, for the stresses at points A and B.
Note that point A is at the outside and B at the inside wall of the shell. Both points A and B are at section 1-1. The inside diameter of the hollow cylindrical part is 36 in. and the outside diameter 37 in.

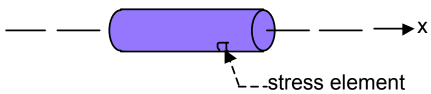
Stress	Point A	Point B
σ_x	36,500 psi	−1,000 psi
σ_y		36,500 psi
σ_z	18,250 psi	18,250 psi
τ_{xy}		5.733 psi
τ_{xz}	5.733 psi	
τ_{yz}		



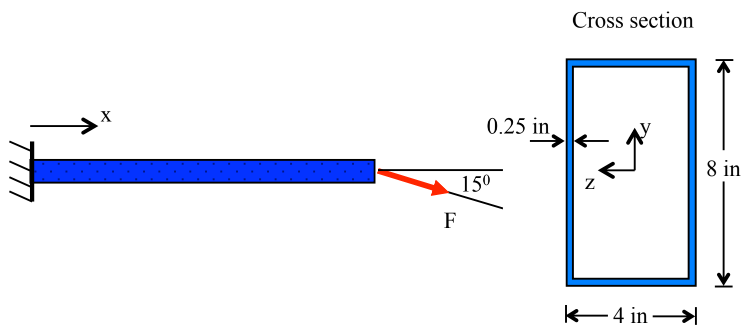
Cross section (hollow cylinder)

Combined Stress: 1 The rear axle on a truck has tubular elements as shown. The axle tubes are made of 4-in outside diameter steel tube with 0.1-in wall thickness. If the brakes can apply a torque of 150 ft-lb to each tube, find the stresses on a stress element at the bottom of the axle tube just inboard of the spring. The stress element is at point A is shown below

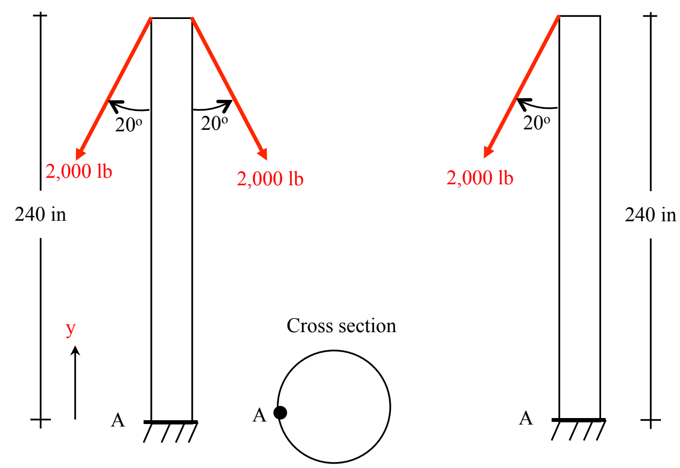




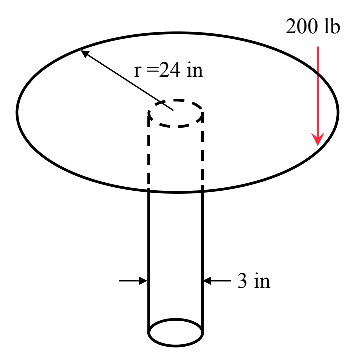
Combined Stress: 2 The cantilever beam is 10 ft long and is subjected to the inclined force F = 1000 lb. Find the maximum tensile normal stress $\sigma_x^{max-tension}$ and maximum compressive normal stress $\sigma_x^{max-comp}$.



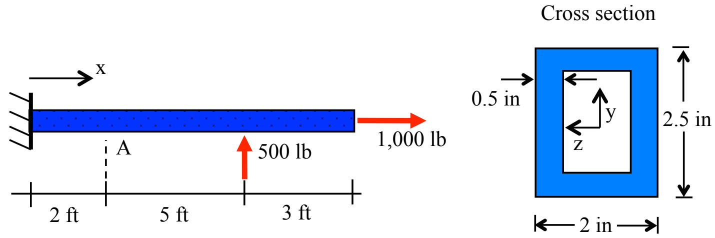
Combined Stress: 3 The cantilever wooden post is supporting two off-axis loads as shown. (a) Find σ_y and τ_{max} at point A. (b) Repeat the calculation of σ_y and τ_{max} if only one of the off-axis loads acts on the post. The radius of the post is 10 in and $E_{wood} = 1 \times 10^6$ psi



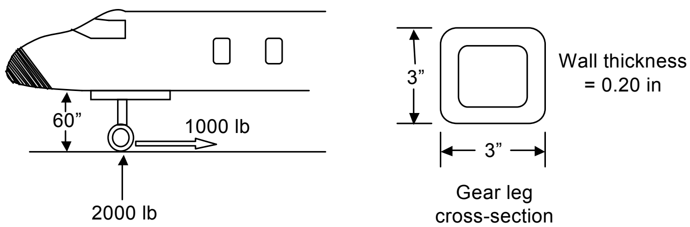
Combined Stress: 4 A round table 4 ft in diameter is supported by a 3-in diameter pipe with 0.25-in wall thickness. A 200 lb person sits on the edge of the table. Find the maximum normal stress σ_z in the pipe. Note: neglect the weight of the table and the pipe, and assume the pipe is fixed on the floor.



Combined Stress: 5 For the cantilever beam: (a) find the maximum normal stress σ_x at location A, and (b) find the shear stress τ_{xy} at location A at the neutral axis.

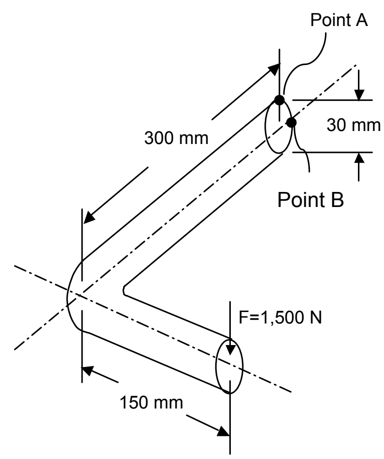


Combined Stress: 6 An aircraft nose gear experiences compressive and drag loads when brakes are applied after landing. Loads are as shown. Find the maximum stress in the landing gear leg.

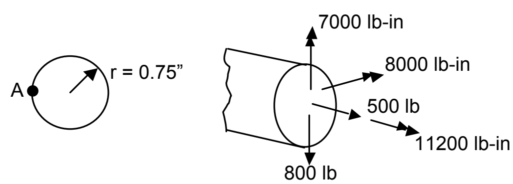


Combined Stress: 7 Given an L-shaped bracket that is subject to a single force as shown find the following:

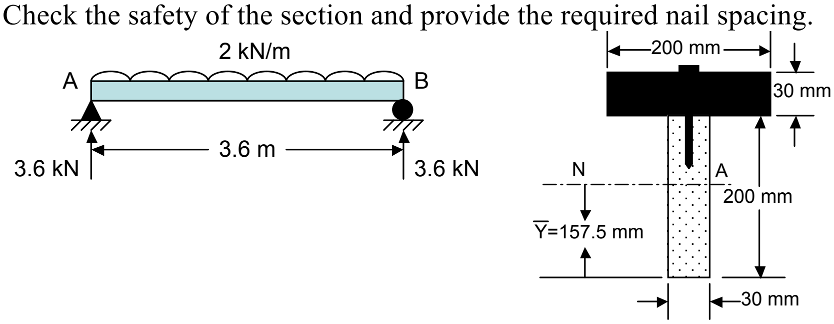
a) Stress element at point A
b) Max normal stress at point A
c) Max shear stress at point A
d) Draw Mohr's circle for point A
e) Repeat (a)-(d) for point B



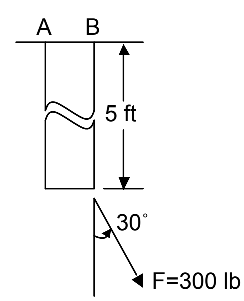
Combined Stress: 8 Find stresses at point A.



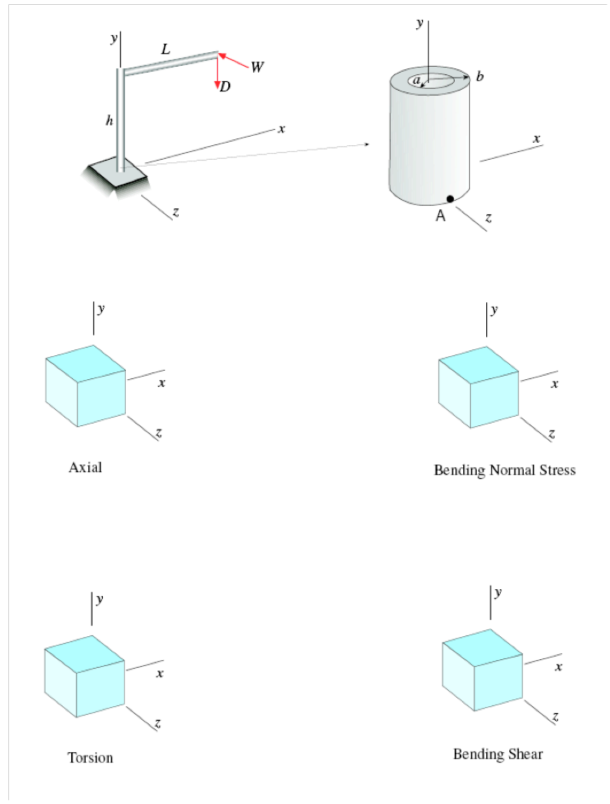
Combined Stress: 9 A beam with a distributed load and a cross section as shown is given. The allowable normal stress is 10 MPa and the allowable shear stress is 0.8 MPa. Each nail can resist 1.5 kN shear. Check the safety of the section and provide the required nail spacing.



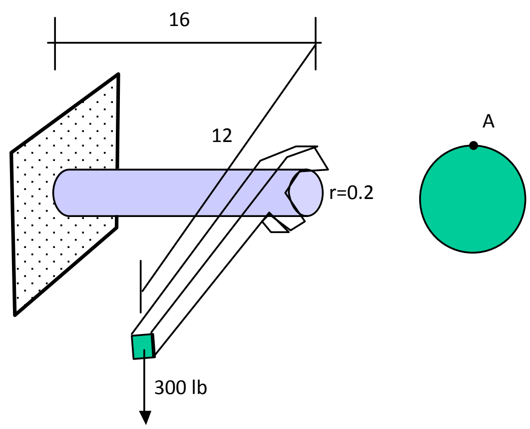
Combined Stress: 10 A piece of 2-1/2-in schedule 40 pipe is loaded as shown. Find the normal stress at points A and B (σ_A and σ_B). The diameter of the pipe is D = 2.875 in, its cross-sectional area a = 1.704 in^2, and its section modulus s = 1.064 in^3.



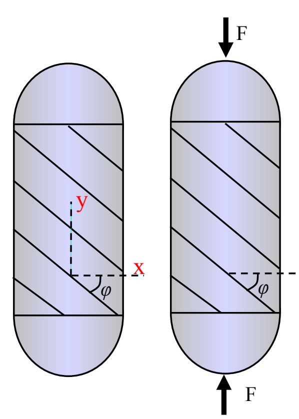
Combined Stress: 11 A traffic light pole carries the weight of the traffic light and signs, D, and a horizontal wind load, W, acting along the z-axis. Characterize the state of stress at point A at the base of the pole by drawing appropriate stress arrows on each stress block indicated. For each case, indicate the equation to use to determine the corresponding stress value expressed in terms of D, W, h, L, E, A, I, a, b, I_P, and/or G, as appropriate. For the bending shear case, show on a sketch, the area you would use to compute Q. Note that you do not need to evaluate A, I_P, or I, but you do need to use a and b for those stress expressions that include location terms or thickness terms.



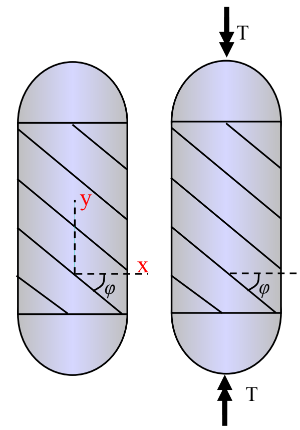
Combined Stress: 12 A 16-in steel rod is loaded with a wrench as shown below. What is the maximum shear stress (τ_{max}) at point A (the top of the rod at the wall) when the 300 lb force is vertical and the arm of the wrench is horizontal. Note that this rod is subject to both bending and torsion loads. For a circular cross-section $I = \pi r^4/4$, $\quad I_p = \pi r^4/2$.



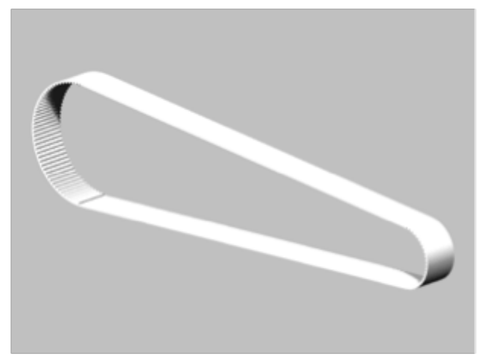
Combined Stress: 13 A cylindrical pressure vessel of inner diameter d = 4 ft (48 in) and wall thickness t = 1/2 in is fabricated from a welded pipe with helix angle $\varphi = 38^o$. Someone claims that by imposing an appropriate compressive or tensile force F (compressive F is shown), the total shear stress parallel to the weld, τ_w (i.e., the one from the combined effects of the internal pressure and the compressive force), can be reduced to zero. Is this possible? Explain your answer, and, if the answer is that it is possible, evaluate the magnitude of F that would do that.



Combined Stress: 14 A cylindrical pressure vessel of inner diameter d = 4ft (48 in) and wall thickness t = 1/2 in is fabricated from a welded pipe with helix angle φ=38°. Someone claims that by imposing an appropriate torque T as shown, the total shear stress parallel to the weld, τ_w (i.e., the one from the combined effects of the internal pressure and the torque), can be reduced to zero. Is this possible? Explain your answer, and, if the answer is that it is possible, evaluate the magnitude of T that would do that.

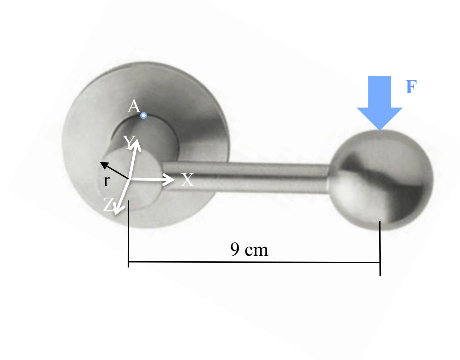


Combined Stress: 15 The belt shown is subjected to tensile force T = 150.0 N. In addition, at the curved part, the small pulley (pulleys not shown) imparts a maximum contact compressive stress of –1,200.0 Pa onto the belt. Consider that the width of the belt is 12.0 cm, its thickness is 4.0 cm, its modulus of elasticity is E = 0.5 GPa, and the small pulley's radius is 25.0 cm. Find the maximum shear stress anywhere in the belt. Note: ignore any friction (shear stress) between the belt and the pulley; do not examine the state of stress at the large radius pulley; ignore any "teeth" the belt may have; and consider the radius of curvature of the belt at the small pulley to be 25.2 cm. Note that considering the belt without teeth yields relatively large bending stresses, thus the need to introduce teeth.

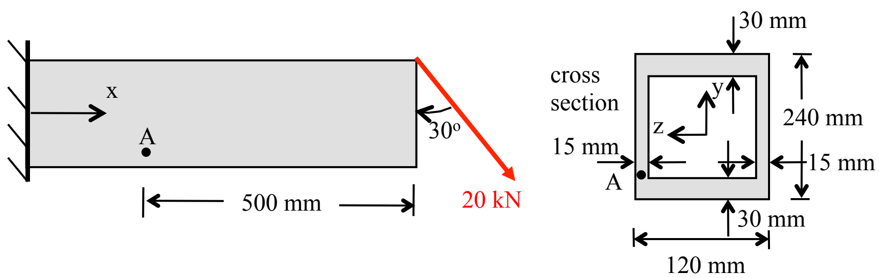


Combined Stress: 16 The tank shown is filled up with petrol of density $\rho = 720$ g/lt (grams per liter). Its height is 17 m, its internal radius is r = 12 m, and its thickness is 2.5 cm. At the bottom of the tank, the petrol creates a hydrostatic pressure of 120.07 kPa, while the hydrostatic pressure at the top of the tank is zero. The top cover weight is W = 11.0 kN, which is distributed as compressive stress on the tank wall. Find the maximum shear stress anywhere on the tank wall, examining material elements at the bottom, midpoint, and top of the tank only.

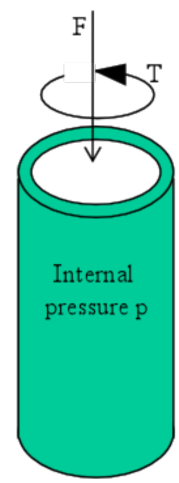
Combined Stress: 17 The industrial door handle shown is subjected to F = 80N at a distance d = 9 cm as shown. The main cylindrical rod at left (cylinder axis is along the z-direction) has a radius r = 2 cm, length l = 4 cm, and is fixed at z = –4 cm. Find the state of stress at point A shown with coordinates x = 0, y = r = 2 cm, z = –4 cm. For a circular area of radius r, $I = \pi r^4/4$, $I_p = \pi r^4/2$.



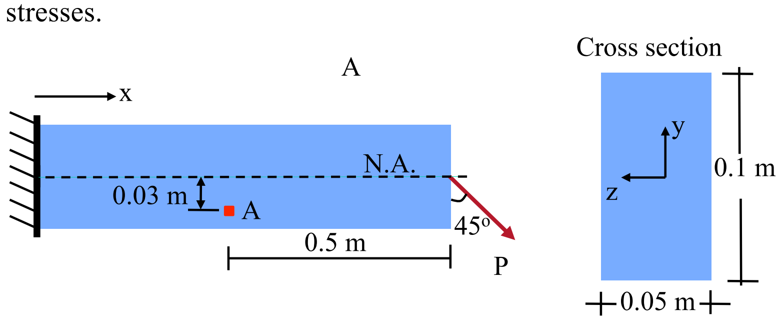
Combined Stress: 18 The cantilever beam is loaded by the 20 kN load at the top of its free end as shown, and the load is applied in a plane of symmetry. Determine the principal stresses σ_1, σ_2 and maximum shearing stresses at point A in the web just above the junction between the flange and the web. The moment of inertia of the cross section is $I = 94.5 \times 10^6$ mm^4. Note: Find only the principal stresses and the maximum shear stress in the x-y plane.



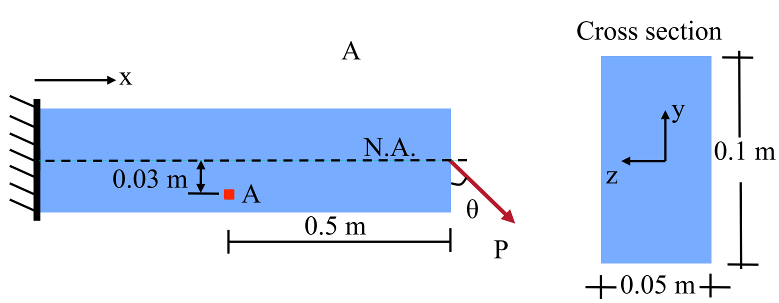
Combined Stress: 19 The hollow drill pipe for an oil well is 15 cm in outer diameter and 0.3 cm in thickness. Just above the bit, the compressive force in the pipe due to the weight of the pipe is 52 kN and the torque due to drilling is 7 kNm. The pipe also carries an internal fluid that is at pressure of 2.8 MPa (over atmospheric). Determine the extreme normal stresses $\sigma_x, \sigma_y, \sigma_z$, and extreme shear stresses $\tau_{xy}, \tau_{xz}, \tau_{yz}$. The x-axis is along the pipe axis.



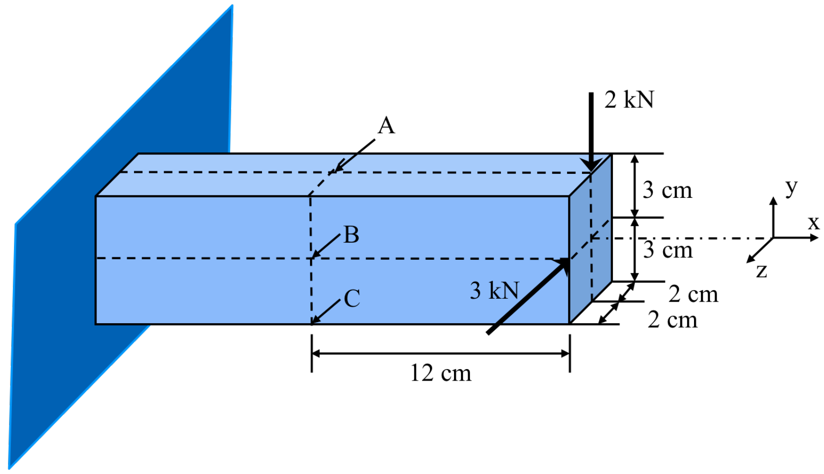
Combined Stress: 20 The cantilever beam is loaded by the P = 20 kN load as shown. (a) Find the stress state at the material element A. (b) Draw the stress state of the material element A. (c) Draw the Mohr's circle for the stress state at A. (d) Using the Mohr's circle find the principal stresses at A. (e) Draw the material element at A in the principal directions and show the principal stresses.



Combined Stress: 21 The cantilever beam is loaded by P = 20 kN load as shown. Find at what angle θ the load P should be oriented so that the normal stress σ_x at material element A would be equal to zero.

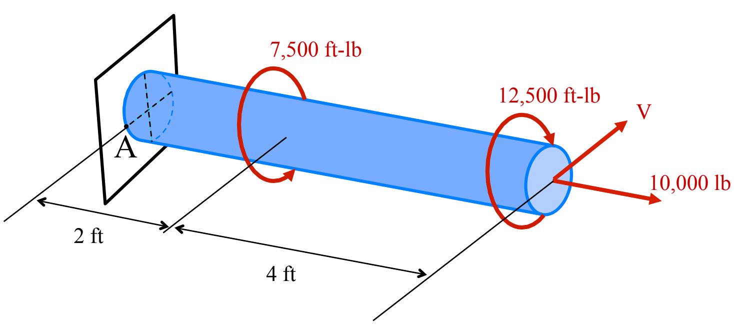


Combined Stress: 22 The cantilever beam is fixed on its left end and loaded by the 2 kN and 3 kN forces as shown. Find all six stresses, $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ at material points A,B,C, all of which are on the surface of the beam.



Stress	Point A	Point B	Point C
σ_x			
σ_y			
σ_z			
τ_{xy}			
τ_{xz}			
τ_{yz}			

Combined Stress: 23 The solid circular shaft shown has a diameter of 4 inches and is fixed at its left end. If the maximum normal tensile and shear stresses at point A must be limited to 7,500 psi tension and 5,000 psi, respectively, determine the maximum permissible value for the transverse load V. For a circular cross section, the polar moment of inertia is twice the moment of inertia with respect to an axis passing through the centroid of the circle.



Beam theory

A beam subjected to load deflects as shown schematically in Figure N-1. The deflection, u, is a function of x, i.e., u = u(x).

Figure N-1 A beam deflects under external load. At location x, the deflection is u(x).

The classical beam theory is based on the analysis of a small beam element of length dx positioned at x along the beam. The element dx becomes ds after deflection is imposed from the externally applied load. Figure N-2 shows a schematic of the beam element before and after deflection.

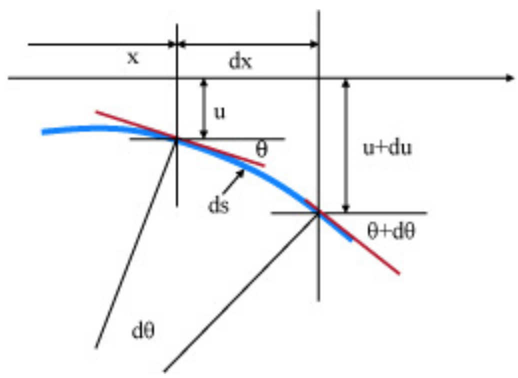


Figure N-2 The element dx before deflection becomes element ds after load application and beam deflection. The element is positioned at x along the beam.

The following definitions are useful, and they are all functions of the coordinate x.

u: deflection	θ: angle of rotation (slope of deflection curve)
ϰ: curvature	ϱ: radius of curvature
Note: the radius of curvature is the inverse of the curvature, i.e., ϱ = 1/ϰ	

For small beam deflections, ds ≈ dx and θ ≈ tanθ hold. Thus

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds} \cong \frac{d\theta}{dx}$$

Also, we have that

$$x\theta \cong \tan\theta = \frac{du}{dx}$$

The above two equations yield

$$\kappa = \frac{1}{\rho} = \frac{d^2u}{dx^2}$$

Now, recall from the modules on normal stresses in beams that $M = EI\varkappa$ holds. Then,

$$M = EI\frac{d^2u}{dx^2}$$

Also, from the study of V- and M-diagrams, we have that $V = dM/dx$ and $q = -dV/dx$. Then, it follows that

$$V = EI\frac{d^3u}{dx^3}$$

and

$$q = -EI\frac{d^4u}{dx^4}$$

The above three equations are known as the differential equations of classical beam theory. Defining, for convenience in notation,

$$u' = \frac{du}{dx}, \quad u'' = \frac{d^2u}{dx^2}, \quad u''' = \frac{d^3u}{dx^3}, \quad u^{iv} = \frac{d^4u}{dx^4}$$

the three equations of beam theory read

$$M(x) = EIu''(v), V(x) = EIu'''(x), q(x) = EIu^{iv}(x)$$

In words, these equations can be interpreted as:
- The bending moment diagram M(x) is proportional to the second derivative (curvature) of the deflection curve, and the proportionality constant is EI
- The shear force diagram V(x) is proportional to the third derivative of the deflection curve, and the proportionality constant is EI
- The distributed load Q(x) is proportional to the fourth derivative of the deflection curve, and the proportionality constant is −EI

In order to find the deflection one then can:
- Integrate the q(x) four times
- Integrate the V(x) three times
- Integrate M(x) twice

In all cases, constants of integration are evaluated from the boundary conditions. It is interesting to note that q(x) is considered positive when its direction is upwards.

Example-beam deflection by integration: The simply supported beam of length L is subjected to two end moments as shown in Figure N-3. Find the equation for the deflection of the beam as a function of x by integrating the bending moment equation.

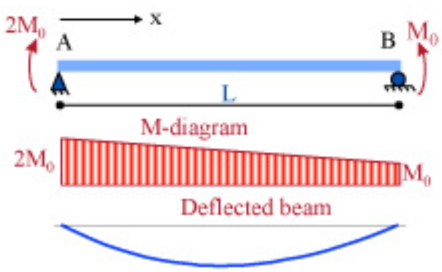


Figure N-3 A simply supported beam loaded by two concentrated moments at the supports. The M-diagram and a schematic of the deflected beam are also shown.

The bending moment diagram is linear, and the equation that describes it is

$$M(x) = -\frac{M_0}{L}x + 2M_0$$

Then, we have,

$$EIu''(x) = -\frac{M_0}{L}x + 2M_0$$

Integrating, twice, with respect to x, we obtain

$$EIu'(x) = -\frac{M_0}{2L}x^2 + 2M_0x + c_1, \quad EIu(x) = -\frac{M_0}{6L}x^3 + M_0x^2 + c_1x + c_2$$

where c_1 and c_2 are constants of integration, to be determined from boundary conditions. At $x = 0$, and $x = L$, the deflection is zero, due to the enforcement by the supports. Thus $u(0) = 0$, which implies, from the last equation above, that $c_2 = 0$. Also, $u(L) = 0$ implies from above that $c_1 = -1.5M_0L$. The deflection line is shown schematically in Figure N-3. Note that the maximum deflection is not in the middle. An exercise for the reader is to find the position where maximum deflection occurs, and the expression for the maximum deflection.

Example-beam deflection by integration: The beam of length L shown in Figure N-4 is subjected to force P at mid-span. Using the equation for the fourth order derivative of the deflection, find the expression for the deflection as a function of x and the beam stiffness EI. Then, find the deflection at mid-span. Hint: using the symmetries in the problem reduces the process significantly.

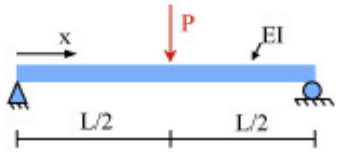


Figure N-4 A simply supported beam loaded with a concentrated load at midspan. The beam geometry and the loading are symmetric with respect to the section at x = L/2.

Due to the symmetry in this problem, only the deflection for the left L/2 can be found by integration. This avoids integrating in two regions (from 0 to L/2 and from L/2 to L). The governing equation and the relevant integrations are shown below.

$$EIu^{iv} = 0 \rightarrow EIu''' = c_1 \rightarrow EIu'' = c_1 x + c_2 \rightarrow EIu' = \frac{1}{2} c_1 x^2 + c_2 x + c_3$$

$$\rightarrow EIu = \frac{1}{6} c_1 x^3 + \frac{1}{2} c_2 x^2 + c_3 x + c_4$$

At x = 0, we have that V = P/2, from which the second equation above gives $c_1 = P/2$. Also, at x = 0, we have that M = 0, from which the third equation above gives $c_2 = 0$. At x = L/2, the slope of the deflected beam is zero (due to symmetry), and the fourth equation above gives $c_3 = -PL^2/16$. Finally, at x = 0 the deflection u is zero, and the fifth equation above gives that $c_4 = 0$. Thus, the final expression for the deflection reads

$$u = \frac{Px}{EI}\left(\frac{x^3}{12} - \frac{L^2}{16}\right)$$

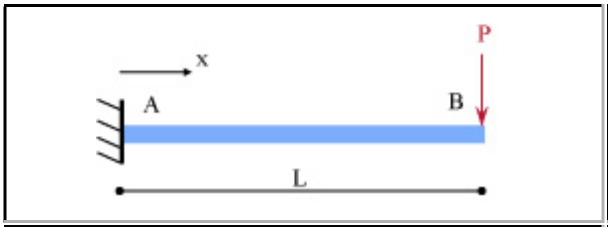
Note that the above holds only for x between 0 and L/2. At midspan (i.e., at x = L/2, the deflection is

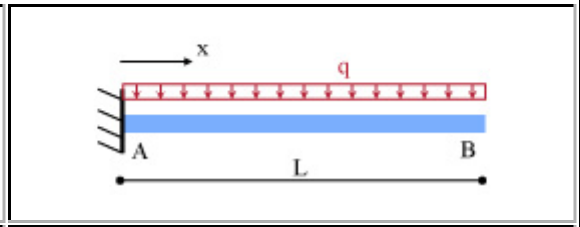
$$u\left(\frac{L}{2}\right) = \frac{PL^3}{48EI}.$$

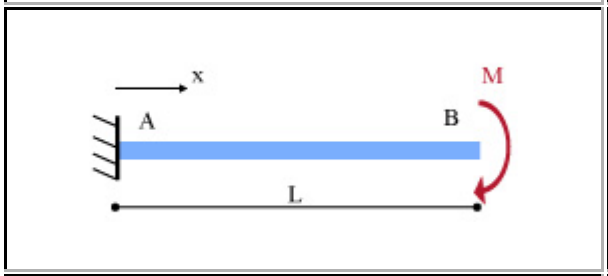
Module 32: Beam Deflections

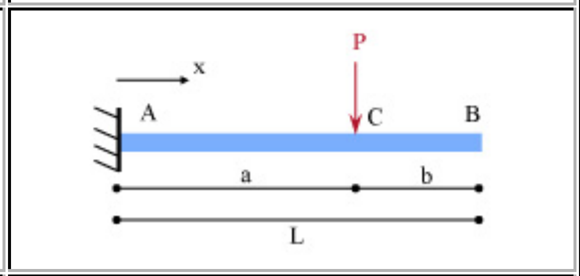
Beam deflections and slopes

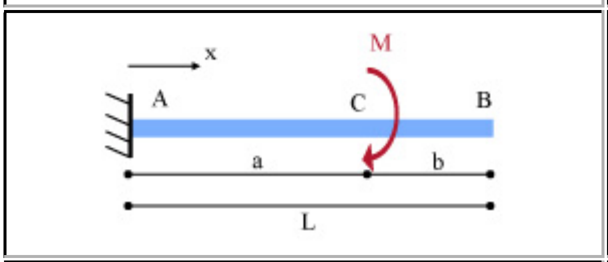
Click on the images (online/DVD) to see the expression for the beam deflection and slope.

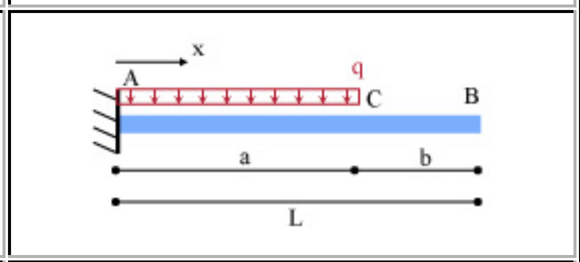


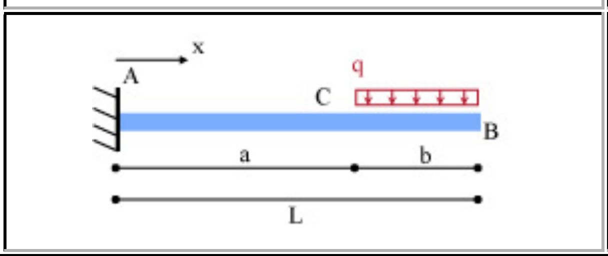


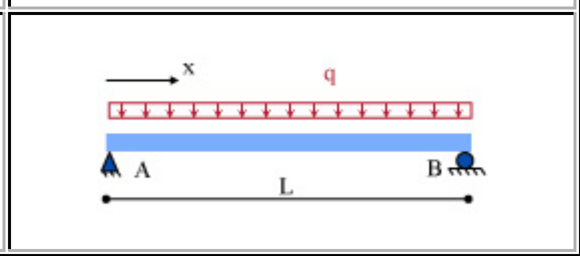












Module 32: Beam Deflections

Example-beam deflection: The three-dimensional beam shown in Figure N-5 is fixed at D and is bent 90^{o} at B and C. It is loaded by a compressive load of 300 lb at A, and length BC is 36 inches, as shown in the figure. Find the deflection of point C in the z-direction. Note that the desired deflection will depend on the length of beam CD denoted as L_{CD}.

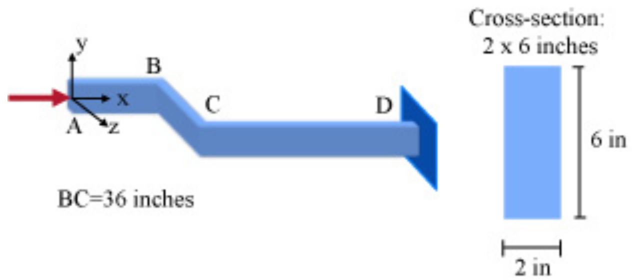


Figure N-5 A three-dimensional cantilever beam loaded by a compressive load at its free end.

The 300 lb force can be moved to C, adding the moment produced in doing so (i.e., 300 lb × 36 in, which yields a moment M = 10,800 lb-in. Thus, the problem for beam CD reduces to the one shown in the following Figure N-6.

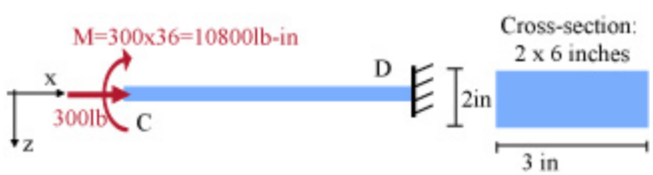


Figure N-6 The beam of Figure N-5 can be reduced, based on statics, to the one shown.

Note the coordinate system and the orientation of the cross section with respect to these coordinates. Thus, the cross section at D is subjected to a combined load consisting of a compressive force of 300 lb and a bending moment of 10,800 lb-in.

Only the bending moment contributes to the displacement of C in the z-direction. Note that the bending moment diagram for this beam is constant. Furthermore, defining the coordinate system as shown in Figure N-7 simplifies the calculations.

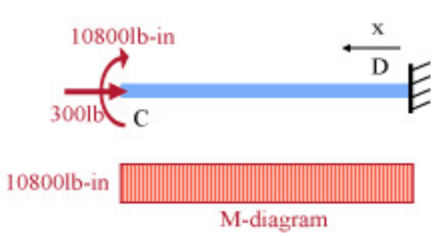


Figure N-7 Simplified analysis of part CD of the beam in Figure N-5 where the bending moment diagram is evaluated, which is used for finding the desired deflection.

Starting from the second order differential equation for the beam, we have

$$EIu''(x) = M(x) = 10{,}800 \rightarrow EIu'(x) = 10{,}800x + c_1$$

Since the slope at D (x = 0) is zero, $c_1 = 0$. Then

$$EIu(x) = \frac{1}{2}10{,}800x^2 + c_2$$

since the deflection at D (x = 0) is zero, $c_2 = 0$. Then, at C

$$u_C = u(L_{CD}) = \frac{\frac{1}{2}10800L_{CD}^2}{1600\times1000\times\frac{6\times2^3}{12}} = 0.000844L_{CD}^2$$

Module 33: Statically Indeterminate Beams

Statically determinate beams

Statically determinate beams have three independent supports, and the corresponding external forces/moments can be determined by using only the equilibrium equations. The following three figures (Figs. N-8 - N-10) illustrate the three major support types for beams. The first one is that the roller provides one (1) independent support (Fig. N-8).

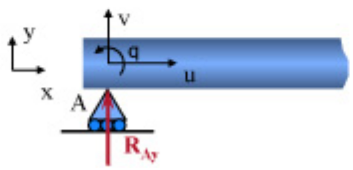


Figure N-8 A roller restricts displacement v, while displacement u and rotation θ are unrestricted. It accounts for one unknown reaction, R_{Ay}.

A pin provides two independent supports (Fig. N-9).

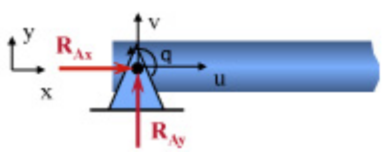


Figure N-9 A pin restricts both u and v displacements, while rotation θ is unrestricted. It accounts for two unknown reactions, R_{Ax} and R_{Ay}.

A fixed-end provides three independent supports (Fig. N-10).

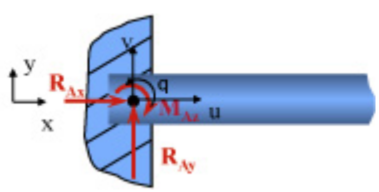


Figure N-10 A fixed end restricts u and v displacements as well as rotation θ. It accounts for three unknown reactions, R_{Ax}, R_{Ay} and M_A.

Below are two examples of statically determinate beams, and one example of an unstable beam. Figure N-11 shows a statically determinate beam with three independent supports; i.e., horizontal displacement at A, u_A; vertical displacement at A, v_A; and vertical reaction at B, v_B. Respectively, the reactions are the horizontal and vertical reactions at A, and the vertical reaction at B.

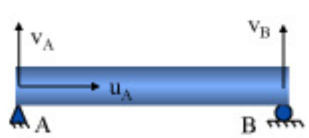


Figure N-11 A simply supported beam, which has three independent supports.

Figure N-12 shows a cantilever beam, which is also statically determinate. It has three independent supports; i.e., horizontal displacement at A, u_A; vertical displacement at A, v_A; and rotation at A, θ_A. Respectively, the reactions are the horizontal and vertical reactions at A, and the moment at A.

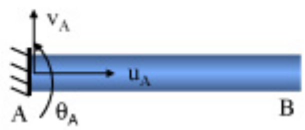


Figure N-12 A cantilever beam, which has three independent supports.

Figure N-13 shows a beam that is unstable, even though it has three supports. Two supports are not independent; i.e., the horizontal displacement at A, u_A; and the horizontal displacement at B, u_B. Note that moment equilibrium around A cannot be satisfied in the presence of any vertical load between A and B.

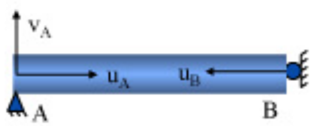


Figure N-13 An unstable beam, which has three yet not independent supports.

Statically indeterminate beams

When a beam has supports in addition to three independent ones, it is statically indeterminate. Below are some examples illustrating the point. Figure N-14 shows a beam that has four supports; i.e., horizontal displacements at A, u_A, and at B, u_B; and vertical displacements at A, v_A, and B, v_B. The degree of indeterminacy is 1 (4 minus 3). There are four unknown reactions corresponding to the four displacements.

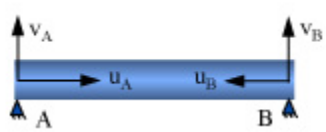


Figure N-14 A beam supported by two pins has four supports, thus the degree of indeterminacy is one.

Figure N-15 shows a beam with four supports; i.e., horizontal displacement at A, u_A; vertical displacements at A, v_A, and B, v_B; and rotation at A, θ_A. The degree of indeterminacy is 1 (4 minus 3). There are four unknown reactions, three forces corresponding to the three displacements, and one moment, corresponding to the rotation at A.

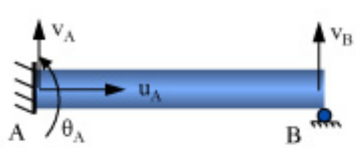


Figure N-15 A beam supported by a fixed end and a roller, where the degree of indeterminacy is 1.

Similarly to the previous beams above, the degree of indeterminacy of the beam shown in Figure N-16 is 2 (5 minus 3).

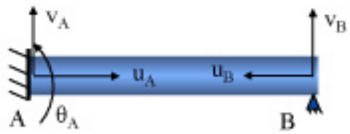


Figure N-16 A beam supported by a fixed end and a pin, where the degree of indeterminacy is 2.

The degree of indeterminacy of the beam shown in Figure N-17 is 1 (4 minus 3). Beams of these types are called continuous bemas and are common in civil engineering structures such as bridges.

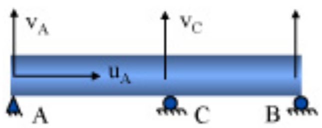


Figure N-17 A so-called continuous beam, supported by a pin and two rollers, where the degree of indeterminacy is 1.

The degree of indeterminacy of the beam shown in Figure N-18 is 3 (6 minus 3). This is called a fixed-end beam.

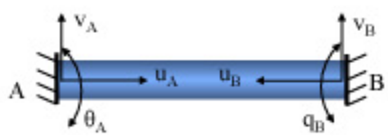


Figure N-18 A so-called fixed-end beam, supported by two fixed ends, where the degree of indeterminacy is 1.

Superposition for solving statically indeterminate beams

The excess unknown reactions in a statically indeterminate beam are called redundants. For example, a beam of degree of indeterminacy equal to 1 has one redundant reaction. The method of superposition can be used to express the compatibility equations, additional to the equilibrium ones, that will yield the redundants. The process is schematically illustrated in Figure N-19.

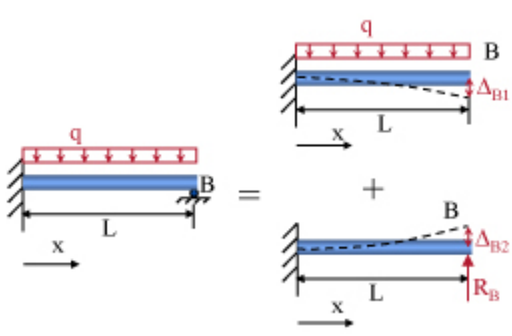


Figure N-19 The beam to the left has a degree of indeterminacy equal to 1, thus one redundant. Choosing this redundant to be the reaction at B, we have that the original beam (left) is equivalent to the superposition (designated by the + symbol) of the two beams to the right. Since the deflection of end B is zero, compatibility implies that $\Delta_{B1} + \Delta_{B2} = 0$ (considering that upwards deflections are positive and downwards deflections are negative). The following page provides an actual numerical example of this problem.

Example: Consider the beam shown in Figure N-19, where, q = 5kN/m and L = 2m. Also, let the stiffness of the beam be EI, where E denotes the elasticity modulus and I the moment of inertia of the cross section. The values of Δ_{B1} and Δ_{B2} can be found by either integration of the beam differential equations, or from available tables. Using the latter for this example, we have

$$\Delta_{B1} = \frac{-qL^4}{8EI} = \frac{-5 \times 2^4}{8EI} = \frac{10}{EI}$$

$$\Delta_{B2} = \frac{R_B L^3}{3EI} = \frac{8}{3}\frac{R_B}{EI}$$

Since compatibility requires $\Delta_{B1} + \Delta_{B2} = 0$, it follows that

$$\frac{-qL^4}{8EI} + \frac{R_B L^3}{3EI} = 0 \rightarrow R_B = \frac{3}{8} qL = 3.75 \text{ kN}$$

Example: The beam (Fig. N-20) of length L is loaded by q and is fixed at both ends. Find the moment at the two ends, denoted as M_R. Note the symmetry in the problem.

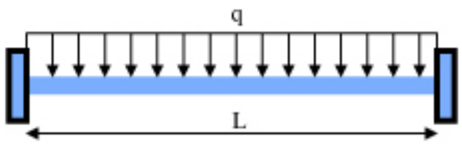


Figure N-20 A fixed-end beam loaded by a distributed load q.

Due to symmetry in this problem, the reactions at the beam's ends are equal. Also, equilibrium in the vertical direction implies that the vertical reactions are qL/2. Figure N-21 shows the reactions at the ends, where M_R denotes the unknown moment reaction at the ends.

Figure N-21 The free body diagram of the beam in Figure N-20

$$EIu^{iv}(x) = -q \rightarrow EIu'''(x) = -qx + C_1$$

At x = 0, however, the shear force is equal to the reaction there, i.e. V(0) = qL/2. This implies that C_1 = qL/2. Then, by integrating once more

$$EIu'''(x) = -qx + \frac{qL}{2} \rightarrow EIu''(x) = M(x) = -\frac{1}{2}qx^2 + \frac{qL}{2}x + C_2$$

At x = 0, M(0) = M_R, which implies that $c_2 = M_R$. Further integration yields

$$EIu''(x) = M(x) = -\frac{1}{2}qx^2 + \frac{qL}{2}x + M_R \rightarrow EIu'(x) = \theta(x) = -\frac{1}{6}qx^3 + \frac{qL}{4}x^2 + M_R x + C_3$$

At x = 0, the slope is zero, or θ(0) = 0, which implies that $c_3 = 0$. Further, at x = L, the slope is zero as well, which implies that

$$EIu'(L) = EI\theta(L) = -\frac{1}{6}qL^3 + \frac{qL}{4}L^2 + M_R L \rightarrow M_R = -\frac{qL^2}{12}$$

Example: Radius of curvature of Beam

$$\varepsilon = \frac{-c}{\rho} = \frac{\sigma}{E}$$

$$\therefore \sigma = \frac{-cE}{\rho}$$

$$\text{But, } \sigma = \frac{Mc}{I}$$

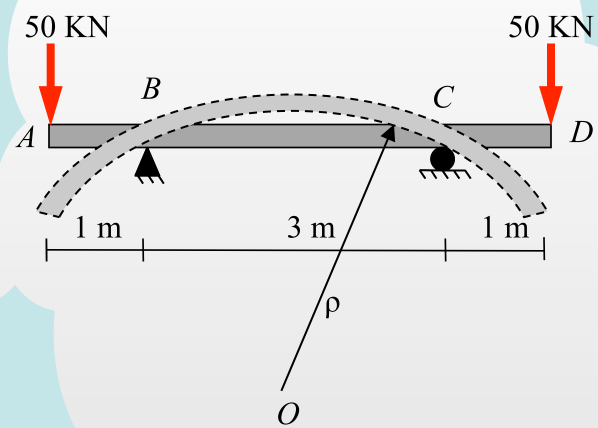
$$\therefore \frac{-cE}{\rho} = \frac{Mc}{I}$$

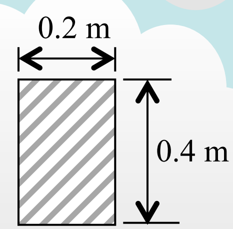
$$\therefore \rho = \frac{EI}{M}$$

$$I = \frac{bd^3}{12} = \frac{0.2 \times 0.4^3}{12} = 0.001 \text{ m}^4$$

$$M = 50{,}000 \times 1 = 50{,}000 \text{ Nm}$$

$$\rho = \frac{200 \times 10^9 \times 0.001}{50{,}000} = 4{,}000 \text{ m} = 4 \text{ km}$$





Given:
Beam ABCD is loaded as shown
E = 200 GPa

Find:
ρ, radius of curvature between B and C.

Example: Expression for the Deflection of the Beam

By integrating the bending moment equation, find the expression for the deflection of the beam.

$$M = M_0 + \frac{M_0 x}{L}$$

$$EIu'' = -M_0 - \frac{M_0 x}{L}$$

$$EIu' = -M_0 x - \frac{1}{2}\frac{M_0}{L}x^2 + C_1$$

$$EIu = -\frac{1}{2}M_0 x^2 - \frac{1}{6}\frac{M_0}{L}x^3 + C_1 x + C_2$$

$$x = 0 \Rightarrow u = 0$$

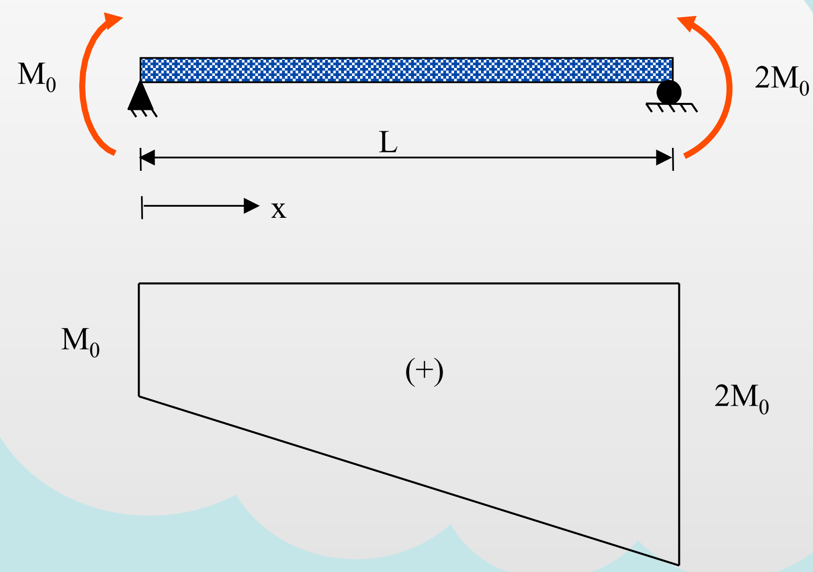
$$\Rightarrow C_2 = 0$$

$$x = L \Rightarrow u = 0 \Rightarrow -\frac{1}{2}M_0 L^2 - \frac{1}{6}\frac{M_0}{L}L^3 + C_1 L = 0$$

$$\Rightarrow -\frac{1}{2}M_0 L - \frac{1}{6}M_0 L + C_1 = 0$$

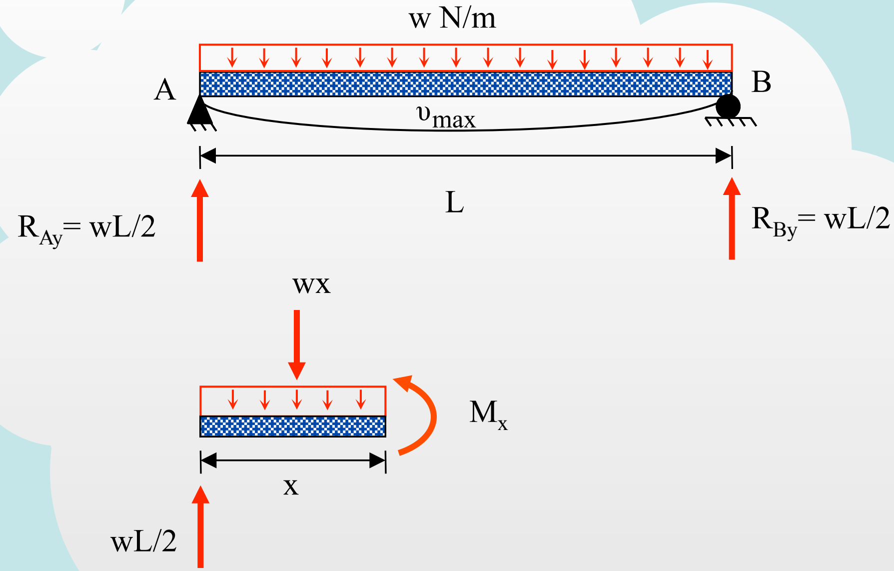
$$\Rightarrow C_1 = \frac{1}{2}M_0 L + \frac{1}{6}M_0 L$$

$$EIu = -\frac{1}{2}M_0 x^2 - \frac{1}{6}\frac{M_0}{L}x^3 + \left(\frac{1}{2}M_0 L + \frac{1}{6}M_0 L\right)x$$



Example: Maximum Deflection in a Beam

Find u_{max} for beam AB.
EI = constant



Take section at x distance from A
Find M_x as function of (x):

$$M_x = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$EI\frac{d^2u}{dx^2} = M_x$$

$$\therefore \frac{d^2u}{dx^2} = \frac{wLx}{2EI} - \frac{wx^2}{2EI}$$

$$\frac{du}{dx} = \frac{wLx^2}{4EI} - \frac{wx^3}{6EI} + C_1$$

$$u = \frac{wLx^3}{12EI} - \frac{wx^4}{24EI} + C_1x + C_2$$

Boundary conditions:

$u = 0$ at $x = 0$ …(1)

$u = 0$ at $x = L$ …(2)

From (1), we obtain, $C_2 = 0$

From (2), we obtain, $C_1 = -\frac{wL^4}{12EIL} + \frac{wL^4}{24EIL} = -\frac{wL^3}{24EI}$

$$u = \frac{wLx^3}{12EI} - \frac{wx^4}{24EI} - \frac{wL^3x}{24EI}$$

$$\text{at } x = \frac{L}{2},\quad u = -\frac{wL^4}{384EI} = u_{max}$$

Example: Deflection and Slope in a Cantilever Beam

Start with the moment equation:

$$\frac{d^2u}{dx^2} = \frac{M_x}{EI}$$

$$M_x = -Px$$

$$\therefore \frac{d^2u}{dx^2} = -\frac{Px}{EI}$$

$$\frac{du}{dx} = -\frac{Px^2}{2EI} + C_1$$

$$u = -\frac{Px^3}{6EI} + C_1 x + C_2$$

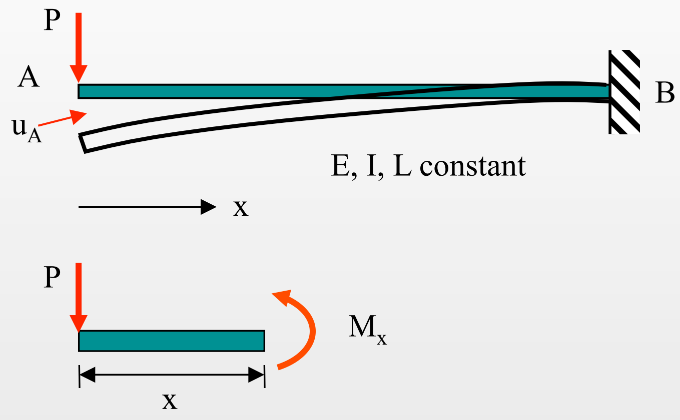
Boundary conditions:

$$\frac{du}{dx} = 0 \text{ at } x = L \Rightarrow C_1 = \frac{PL^2}{2EI}$$

$$u = 0 \text{ at } x = L \Rightarrow C_2 = -\frac{PL^3}{3EI}$$

$$\therefore u = -\frac{Px^3}{6EI} + \frac{PL^2 x}{2EI} - \frac{PL^3}{3EI}$$

Find u_A and slope at A.



$$M_x = -Px$$

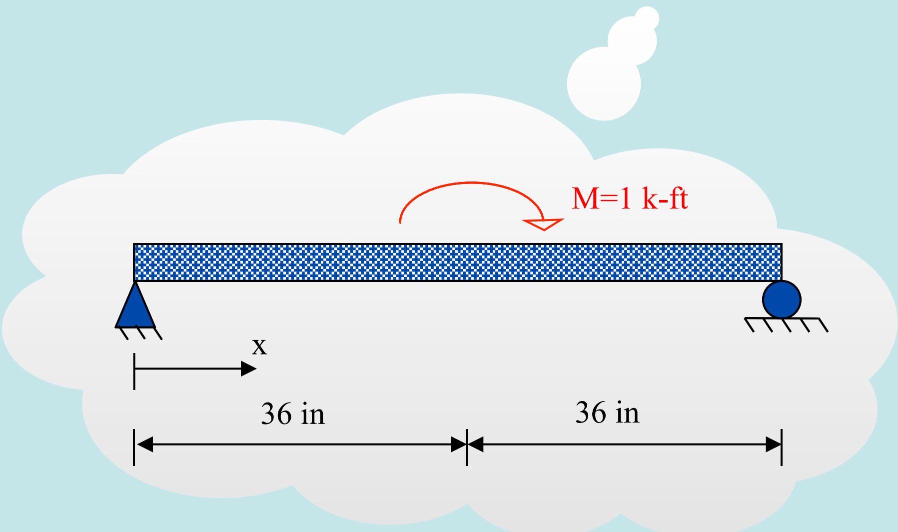
For point A, x = 0

$$u_A = -\frac{PL^3}{3EI}$$

$$\text{slope at A} = \frac{du}{dx} = \frac{PL^2}{2EI}$$

Example: Calculating Slope and Deflection

The simply supported beam of stiffness EI and length 72 in is subjected to a moment M = 1 k-ft (= 12 k-in) at midspan. Find the slope and deflection at midspan (x = 36 in)



$$EIu''''(x)=0,\quad 0\le x\le 36$$

$$EIu'''(x)=c_1,\quad 0\le x\le 36$$

But at $x = 0$, $V = -12/72 = -0.167$ kips

Then, $C_1 = -0.167$ kips

$$EIu''(x) = -0.167x + C_2,\quad 0\le x\le 36$$

But, at $x = 0$, $M = 0$, then $C_2 = 0$

$$EIu'(x) = -\frac{1}{2}0.167x^2 + C_3,\quad 0\le x\le 36$$

$$EIu(x) = -\frac{1}{6}0.167x^3 + C_3x + C_4,\quad 0\le x\le 36$$

But, at $x = 0$, $u = 0$, then $C_4 = 0$

Also, due to the skew-symmetry of the problem, the deflection at in. is zero. This implies that x = 36 in.

$$EIu(36) = -\frac{1}{6}0.167\cdot 36^3 + C_3 36,\quad 0\le x\le 36$$

$$C_3 = -\frac{1}{6}0.167\cdot 36^2 = 36.07$$

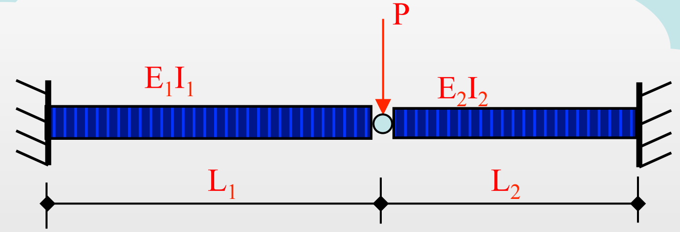
Then, $EIu(x) = -\frac{1}{6}0.167x^3 + 36.07x,\quad 0\le x\le 36$

At $x = 36$ in, $u = 0$ and

$$EIu'(36) = -\frac{1}{2}0.167(36)^2 + 36.07 = -72.146$$

Example: Statically Indeterminate Beams

The two cantilever beams of length L_1 and L_2, respectively, are pinned together as shown in the figure. Find the moment at the two fixed ends. Note that the beam on the left side is longer and stiffer than the one on the right, such that $L_1 = 1.2L_2$ and $E_1I_1 = 1.5E_2I_2$. Consider $P = 20$ Kips.



Let $P = P_1 + P_2$, where P_1 loads the left and P_2 loads the right cantilever beam. The compatibility condition is that the deflection at the point where P is applied is the same for both beams. Since the deflection at the end of a cantilever beam loaded at its end by F is

$$\delta_{end} = \frac{FL^3}{3EI}$$

we have:

$$\delta_{pin} = \frac{P_1L_1^3}{3E_1I_1} = \frac{P_2L_2^3}{3E_2I_2} \Rightarrow \frac{P_1}{P_2} = \frac{L_2^3}{L_1^3}\frac{E_1I_1}{E_2I_2} = \left(\frac{1}{1.2}\right)^3(1.5) = 0.868$$

Solving $P_1 + P_2 = 20$ K and $P_1/P_2 = 0.868$, it follows that $P_1 = 9.3$ K and $P_2 = 10.7$ K

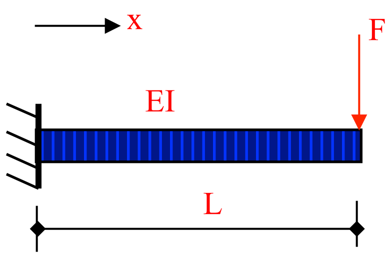
Then,

$M_{left} = -9.3L_1$ K-ft (L_1 in feet),

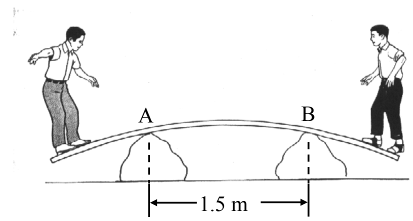
$M_{right} = -10.7L_2$ K-ft (L_2 in feet).

Note: for a cantilever beam (shown below) of length L and stiffness EI, loaded at its end by F,

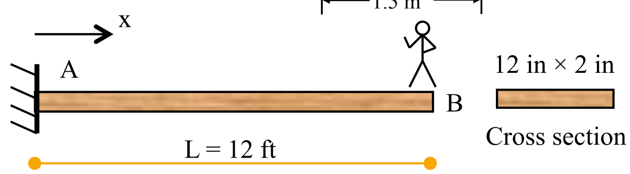
$$u(x) = -\frac{Fx^2}{6EI}(3L - x)$$



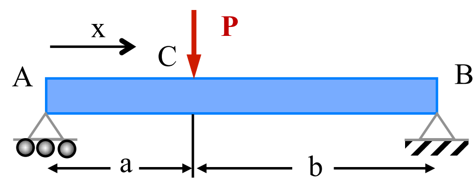
Beam Deflections: 1 The wood beam (E=11 GPa) is 4 cm thick, 30 cm wide and in equilibrium. The moment at A is 300 Nm. Find the radius of curvature ρ in span AB.



Beam Deflections 2 The 200 lb person is standing at the end of the cantilever beam made of wood (E = 1.5×10^6 psi). Determine the shape of the deformed beam.



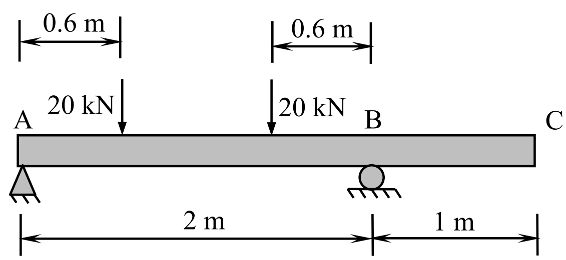
Beam Deflections: 3 The deflection of the simply supported beam of rigidity EI, length L, and subjected to load P as shown in the top figure is expressed as



$$\upsilon(x) = \begin{cases} -\dfrac{Pbx}{6LEI}\left(L^2 - b^2 - x^2\right) & (0 \le x \le a) \\ -\dfrac{Pbx}{6LEI}\left(L^2 - b^2 - x^2\right) - \dfrac{P(x-a)^3}{6EI} & (a \le x \le L) \end{cases},$$

the slope is expressed as

$$\upsilon'(x) = \begin{cases} -\dfrac{Pb}{6LEI}\left(L^2 - b^2 - 3x^2\right) & (0 \le x \le a) \\ -\dfrac{Pb}{6LEI}\left(L^2 - b^2 - 3x^2\right) - \dfrac{P(x-a)^2}{2EI} & (a \le x \le L) \end{cases}$$

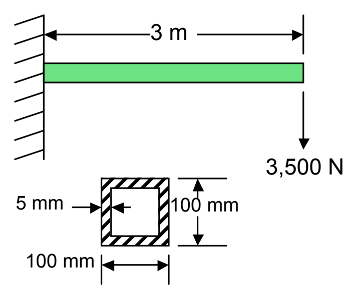


and the slope at A and B is expressed as

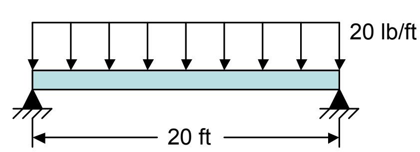
$$\theta_A = \upsilon'(0) = \frac{Pb\left(L^2 - b^2\right)}{6LEI}, \qquad \theta_B = \upsilon'(L) = \frac{Pb\left(2L^2 - 3bL + b^2\right)}{6LEI}$$

Based on that, find the deflection at end C for the beam shown in the bottom figure loaded by the two 20 kN forces as shown. Consider $EI = 100{,}000\,\mathrm{N \cdot m^2}$.

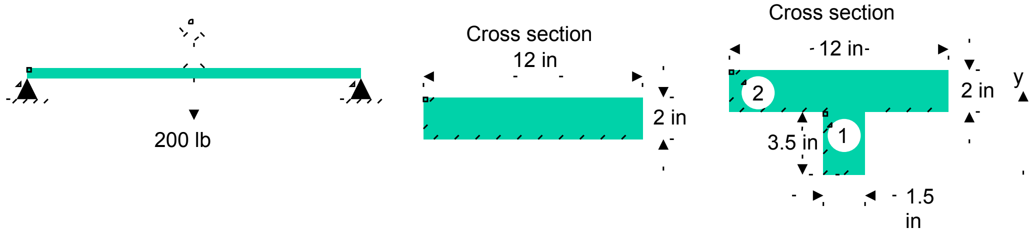
Beam Deflections: 4 A 100-mm square pipe with a wall thickness of 5 mm is fixed at one end and a load of 3,500 N is applied to the free end. The beam is 3 mm long. E = 210 GPa. Find the deformed shape of the beam.



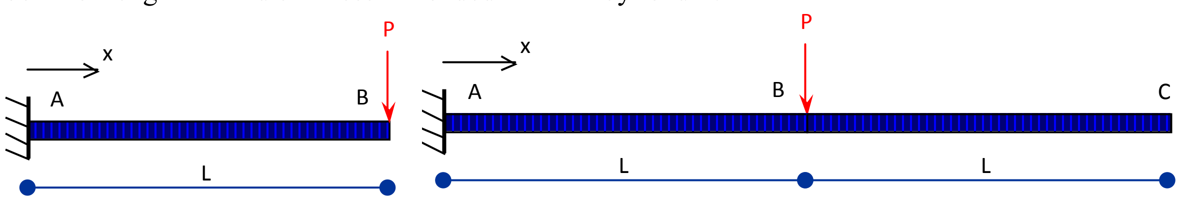
Beam Deflections: 5 A pinned beam is subject to a distributed load as shown. Find the deformed shape of the beam, y(x).



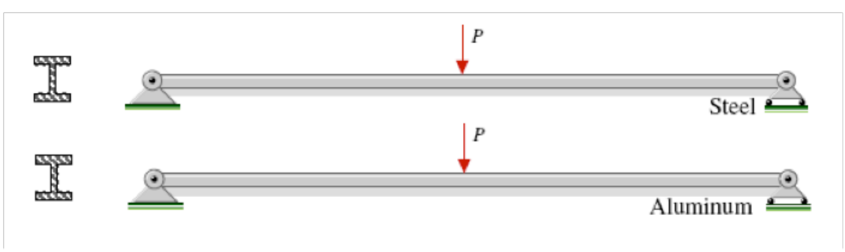
Beam Deflections: 6 The walkboard on a scaffold is 12 ft between supports. A 200 lb person is standing in the middle. The walkboard is made from wood ($E = 1.5 \times 10^6$ psi) with 2 in $\times$ 12 in cross-section. Find: (a) deflection at the middle of the walkboard, and (b) deflection at the middle of the walkboard if the board is reinforced with a 2×4 (the 2 inch dimension being horizontal). Note: the actual dimensions of the 2×4 are 1.5 in $\times$ 3.5 in.



Beam Deflections: 7 For a cantilever beam of length L and stiffness EI loaded at its end by load P, the deflection is given by $u(x) = \dfrac{Px^2}{6EI}(3L - x)$. Based on that, find the deflection at the end C of a cantilever beam of length 2L and stiffness EI loaded at x = L by load P.



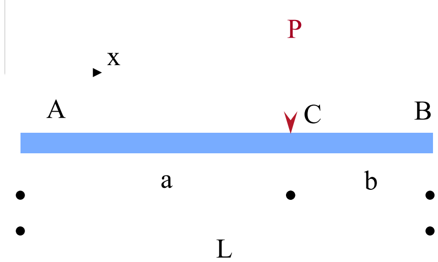
Beam Deflections: 8 The beams shown below differ only with respect to the material used. How would the maximum displacements in the two beams compare?



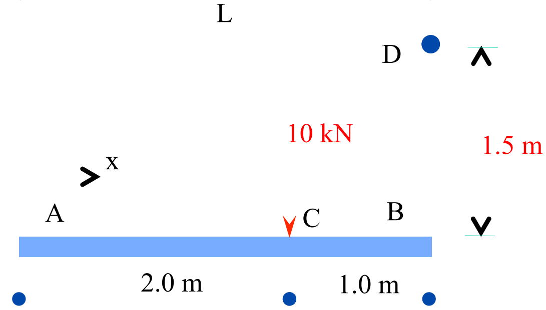
Beam Deflections: 9 For a cantilever beam ACB loaded at B by a load P as shown, the deflection is expressed as

$$u = \left(\frac{Px^3 - 3aPx^2}{6EI}\right) \text{ for } 0 < x < a$$

$$u = \left(\frac{Pa^3 - 3a^2Px}{6EI}\right) \text{ for } a < x < L$$



For the cantilever beam ACB shown below subjected to the 10k N force and supported at B by cable BD, it was found that the cable elongates by 1.0c m. Consider that cable BD has axial rigidity (the product of the modulus of elasticity E and cross-sectional area A),

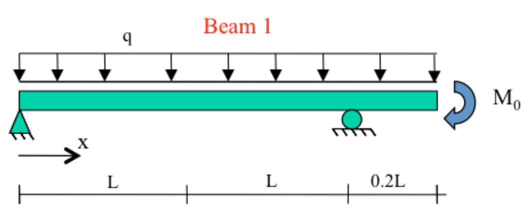


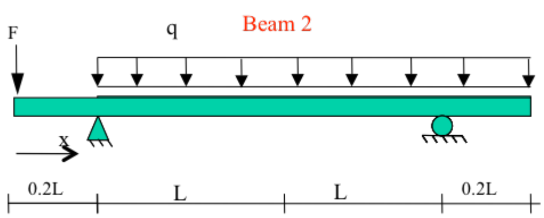
EA = 750 kN. Find the stiffness EI (product of modulus of elasticity E and moment of inertia I) of beam ACB.

Beam Deflections: 10 For a beam of elasticity modulus E, moment of inertia I, and length L = 15 m, the deflection as a function of x ($0 \le x \le 15$), $u(x)$, along its length is found to be

$u(x) = \frac{1}{EI}\left(x^3 - 23x^2 + 132x - 180\right)$. (a) Find the extreme moment (maximum positive or negative) in the beam, and (b) find the extreme shear force in the beam.

Beam Deflections: 11 Associate the beams below with all appropriate conditions necessary for solving for the beam deflection u(x). The same conditions may be applicable to both beams. Each beam has stiffness EI.





a) $u(0) = 0$, b) $u(L) = 0$, c) $u(2L) = 0$, d) $\left.\frac{du(x)}{x}\right|_{x=1.2L} = 0$, e) $\left.\frac{du(x)}{dx}\right|_{x=L} = 0$

f) $EI\left.\frac{d^2u(x)}{dx^2}\right|_{x=2.2L} = -M_0$, g) $EI\left.\frac{d^3u(x)}{dx^3}\right|_{x=2.2L} = q$,

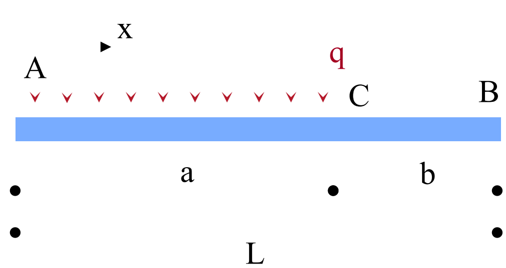
h) $EI\left.\frac{d^3u(x)}{dx^3}\right|_{x=0} = -F$

i) $EI\left.\frac{d^4u(x)}{dx^4}\right|_{x=0.1L} = -q$, j) $EI\left.\frac{d^4u(x)}{dx^4}\right|_{x=2L} = -q$

	Appropriate Conditions
Beam 1	
Beam 2	

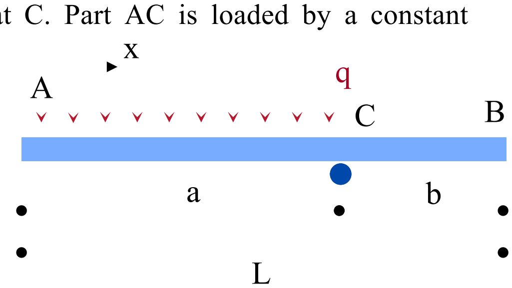
Beam Deflections: 12 The cantilever beam of uniform stiffness EI is fixed at A. Part AC is loaded by a constant distributed force q, while there is no load in CB.

(a) True or false: free end B will displace upward
(b) True or false: the slope of the deflected beam at A is zero
(c) True or false: the slope of the deflected beam at C is zero
(d) Find the deflection at point B, u_B



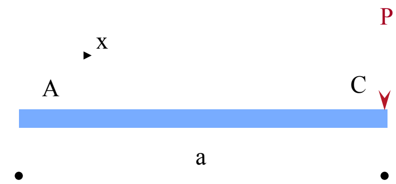
Beam Deflections: 13 Note: you may use the results from the previous problem to solve this one. The beam of stiffness EI is fixed at A and supported by a roller at C. Part AC is loaded by a constant distributed force q, while there is no load in CB.

(a) True or false: free end B will displace upward
(b) True or false: the slope of the deflected beam at A is zero
(c) True or false: the slope of the deflected beam at C is zero

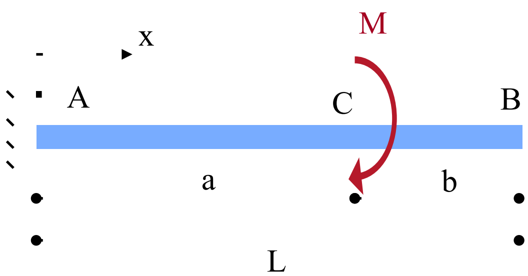


(d) Find the reaction at point C, R_C

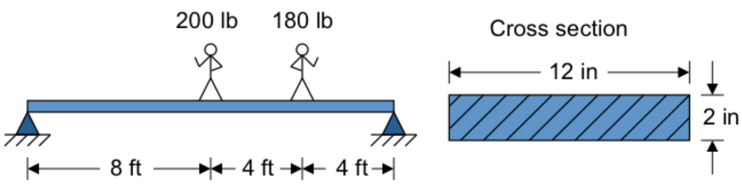
Note: for a cantilever beam of length a, loaded at its free end by P, the deflection is given as $u(x) = -\frac{Px^2}{6EI}(3a - x)$



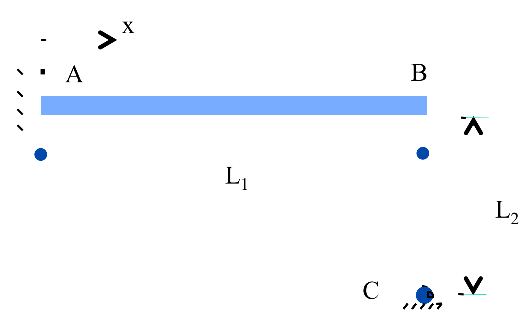
Beam Deflections: 14 Using the method of integration for deflection of beams for AC, find the deflection and slope of the beam at B in terms of M, a, b, L, and the beam stiffness EI. The external bending moment M is applied at C.



Beam Deflections: 15 The simply supported beam is loaded by a 200 lb and a 180 lb loads as shown. Find the deflection, y(x), across the beam. The beam is made of a material with modulus of elasticity E = 900 ksi.



Beam Deflections: 16 Beam AB of length L_1 and bending stiffness E_1I_1 is fixed at A and connected via a pin to bar BC of length L_2, bar stiffness E_2A_2 and coefficient of thermal expansion α_2. If bar BC is subjected to a temperature increase ΔT, find the upward deflection of point B. Note that for a cantilever beam of length L and stiffness EI subjected to a load P at its free end, the deflection of the free end δ_{end} is given as $\delta_{end} = \frac{PL^3}{3EI}$.



Module 35: Buckling and Stability

Buckling and stability

Load-carrying structures consist of several structural components, and each one is typically designed against failure. The design is such that the failure stress and strains will not be exceeded during the lifetime of the structure. In addition, when possible, it is desired that each structural component and the entire structure do not fail catastrophically in case the design loads are exceeded. Columns are typically slender long structures that carry compressive load.

Columns in a highway ramp.

Beam columns carry bending loads in addition to compressive ones. Such structural components are important not only because they carry compressive load, but, because they often fail catastrophically. In other words, columns and beam-columns are susceptible to buckling when subjected to excessive compressive loads. Figure O-1 shows some typical so-called buckling modes of a cantilever and of a simply supported column.

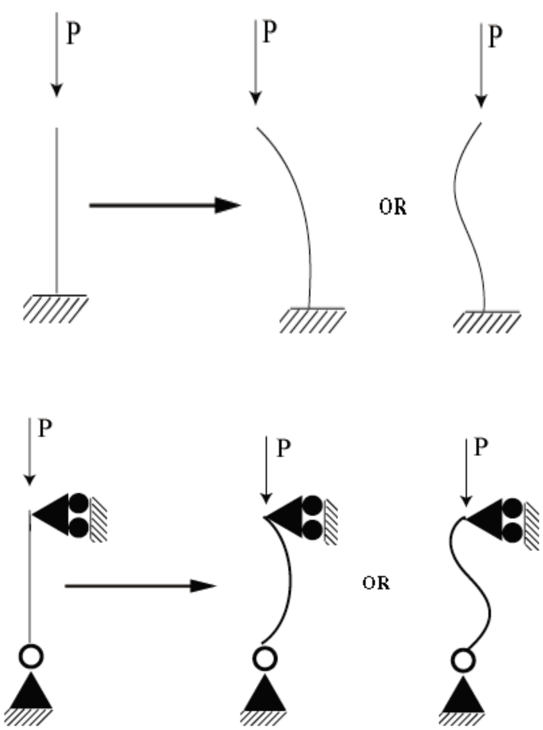


Figure O-1 Schematic of beam-columns under buckling load. Top: two possible buckling modes (shapes) of a cantilever beam. Bottom: two possible buckling modes of a simply supported beam.

Columns and beam-columns are not the only structural components susceptible to buckling.

Plates and shells can buckle when subjected to excessive compressive loads. The analysis presented here, mainly for columns, illustrates the process of analysis of structures in buckling.

Stable equilibrium: A small external perturbation is recovered

Buckling of columns and beam-columns is closely related to the concept of stability. An example demonstrating unstable equilibrium is an empty aluminum can when one steps on the top of it. While there is no disturbance to the can, the can is in equilibrium. However, a small disturbance to the can results in catastrophic failure. This is because the can is under unstable equilibrium conditions, so even a small disturbance is able to reveal this.

Neutral, stable, and unstable equilibrium can be illustrated effectively by considering a sphere in a flat, concave and convex surface, respectively, as shown schematically in Figure O-2. For equilibrium to be stable (sphere resting on concave surface), a small perturbation (disturbance, i.e., moving the sphere a little) is recovered - the sphere comes back to its original position. The opposite is true for unstable equilibrium (sphere on convex surface). Neutral equilibrium is in between stable and unstable ones.

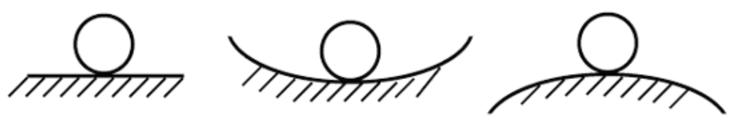


Figure O-2 Illustration of neutral (left), stable (middle), and unstable (right) equilibrium.

Critical (Euler) load

Consider (Fig. O-3) a rigid column pinned at A and supported horizontally at the top through a spring with spring constant β. A compressive force P is applied at the top. Since the column is considered rigid (e.g., of very high EI, E being the elasticity modulus and I the moment of inertia) it can only rotate around A. Considering a small perturbation θ, as shown in Figure O-3, the force P tends to collapse the column, while the spring force tends to restore it to its original position. The moment (around point A) of P is considered the "collapsing moment," while the moment of the spring force is the "restoring moment." When those two moments are equal, the critical load P, P_{cr} is defined. Considering moments around A, and noting that angle θ is small, it follows that: $P_{cr}L\theta = (\beta L\theta)L$, which implies that $P_{cr} = \beta L$.

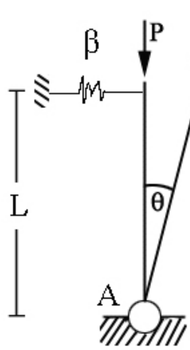


Figure O-3 Schematic of a rigid column subjected to a compressive load P. The column is supported by a pin at A and a linear spring of spring constant β at the top.

A schematic diagram of stability (i.e., force P versus angle of rotation θ plot for the problem described in Fig. O-3) is shown in Figure O-4. For $P < P_{cr}$ we have stability, while for $P > P_{cr}$ (and equal to P_{cr}) we have a bifurcation point implying the transition to unstable equilibrium, i.e., where $P_{cr} = \beta L$.

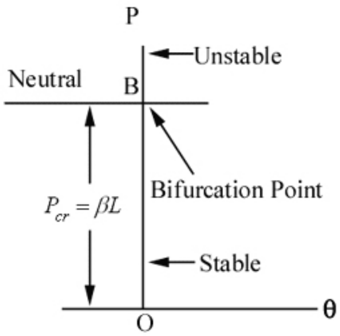


Figure O-4 Schematic of the so-called bifurcation diagram, i.e., transition from stable ($P < P_{cr}$) to unstable ($P > P_{cr}$) equilibrium.

Example-rigid column: Consider a rigid beam AB (Fig. O-5) supported at A by a rotational spring of spring constant α and by a linear spring at B of spring constant β. In order to find P_{cr}, we displace the beam by a small amount as shown in Figure O-5.

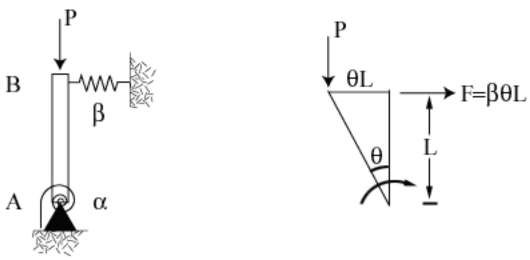


Figure O-5 A rigid column AB (left) supported by a rotational spring and by a linear spring. To the right a schemastic of the displaced column configuration is shown.

The moment created at A due to the imposed displacement is $M = \alpha\theta$. In order to determine P_{cr}, for moment equilibrium of the column around point A, we have:

$$\sum M_A = 0 \rightarrow P_{cr}\theta L - \beta\theta L^2 - \alpha\theta = 0 \rightarrow P_{cr} = \beta L + \frac{\alpha}{L}.$$

Buckling of flexible columns

Here we are faced with the problem of finding the buckling load (P_{cr}) of a simply supported beam that is not considered rigid. As shown in Figure O-6, a small disturbance from the equilibrium position is imposed, i.e,. an unspecified small deflection u(x). Considering a section at an arbitrary point A positioned at distance x, equilibrium of moments implies that

$$EIu''(x) = -Pu(x)$$

where E denotes the elasticity modulus of the beam-column, I its moment of inertia, and double prime denotes the second derivative with respect to the argument.

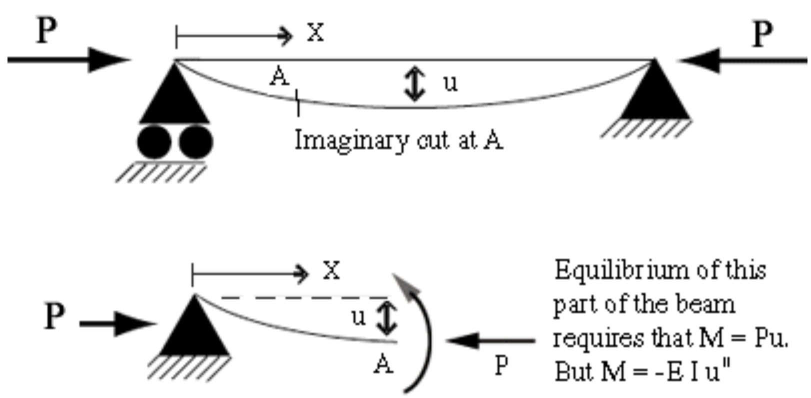


Figure O-6 A flexible column subjected to a compressive force P at a state perturbed slightly from its original straight state. The compressive force creates a moment at a cross section.

In general, we have three possibilities

1. **if $P < P_{cr}$** no bending is created, thus the column remains straight**.**
2. **if $P = P_{cr}$** the beam either remains straight or is slightly bent**.**
3. **if $P > P_{cr}$** buckling occurs, typically in a catastrophic function.

The critical load, P_{cr}, needs to be evaluated from the above governing equation as follows. We rewrite the above equation as

$$EIu''(x) + Pu(x) = 0$$

This is a second order linear equation with constant coefficients. The solution of this equation can be found in textbooks on differential equations. When solved (some details are given subsequently) the equation for P_{cr} comes out-the details of the solution are shown in the following.

In the differential equation shown previously, it is convenient to set

$$\kappa^2 = \frac{P}{EI}$$

Then the differential equation is written as

$$u'' + \kappa^2 = 0$$

The solution of this differential equation is

$$u = C_1 \sin \kappa x + C_2 \cos \kappa x$$

where C_1 and C_2 are constants of integration to be determined from boundary conditions. The boundary conditions, due to the support at $x = 0$ and $x = L$, and their implications are: $u(0) = u(L) = 0 \rightarrow C_2 = 0$. This implies that $C_1 \sin \kappa L = 0$.

If $C_1 = 0$, then the beam remains straight. This is called the **trivial solution**. For nontrivial solutions implying a buckled beam, we have $\sin \kappa L = 0 \rightarrow \kappa L = n\pi$, where n = 1, 2, 3,.... Then, we have that

$$P = \frac{n^2\pi^2 EI}{L^2}, \quad n = 1, 2, 3, ...$$

Figure O-7 depicts a schematic of the trivial solution (n = 0) as well as the ones for n = 1, and n = 2. For the n = 1 case, we have the so-called first buckling mode

$$n = 1 \rightarrow P = \frac{1^2\pi^2 EI}{L^2}, \quad u(x) = C_1 \sin \frac{\pi x}{L}$$

For n=2, we have the so-called second buckling mode

$$n = 2 \rightarrow P = \frac{2^2\pi^2 EI}{L^2}, \quad u(x) = C_1 \sin \frac{2\pi x}{L}$$

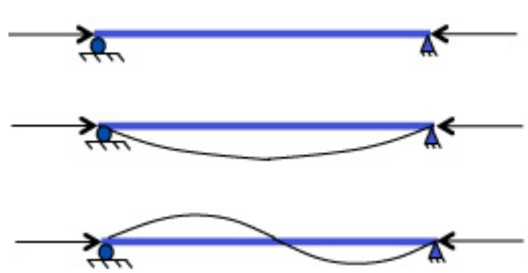


Figure O-7 The trivial mode (top), first (middle) and second (bottom) buckling modes of a simply supported column.

Interest is in the minimum value of P for which a nontrivial solution exists. This implies that n = 1, thus,

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Example: Figure O-8 shows a simply supported column of length L and cross section of dimensions b × h. The first problem in this example is to find the critical buckling load

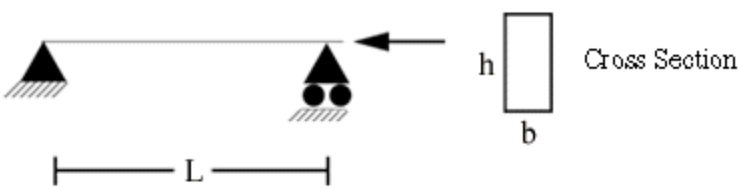


Figure O-8 A simply supported column of rectangular cross section. The weak buckling direction is that of minimum moment of inertia I.

Provided h > b, the maximum moment of inertia for this cross section is $I_1 = bh^3/12$ and the minimum is $I_2 = hb^3/12$. As mentioned in statics and in relevant textbooks, moment of inertia for any orientation is between these two extreme values, I_1 and I_2. Since we are seeking the minimum possible critical load, we have:

$$I_1 = \frac{bh^3}{12}, \quad I_2 = \frac{hb^3}{12}, \quad P_{cr} = \frac{\pi^2 EI}{L^2}$$

where I_2 is the minimum moment of inertia. In other words, the beam-column is prone to buckle in the weak moment of inertia direction. For this reason, it is common to support (brace) the weak buckling direction as shown in Figure O-9.

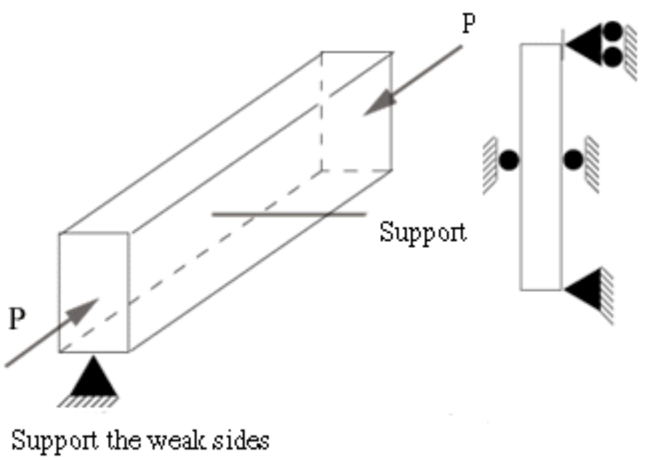


Figure O-9 The simply supported beam is braced in the weak buckling direction to reduce the overall buckling load.

Then the beam would buckle in mode I (n = 1) in the strong direction but it is forced (due to the bracing) to buckle in mode II (n = 2) in the weak direction. Thus, the bracing significantly increases the buckling load of the beam.

Buckling under other support conditions

So far, the stability analysis so far addressed a simply supported column subjected to compressive load P. The critical load for such a column,

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

was obtained by solving the governing differential equation subject to the boundary conditions. For beams with other types of support, the governing differential equation remains the same, yet the boundary conditions change, and thus the critical load is different.

Fundamental cases of buckling

Besides the simply supported column, the so-called fundamental cases of buckling are those shown in Figure O-10. The critical load for such cases is given in the following.

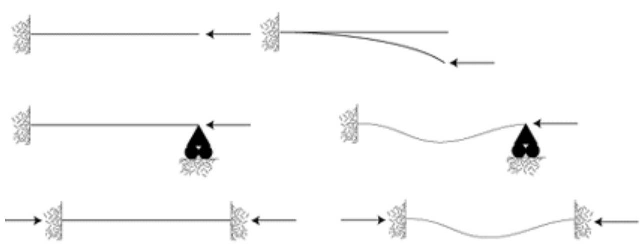


Figure O-10 Fundamental cases of buckling other than the simply supported column. Cantilever column (top), fixed-pin supported column (middle), and fixed-end column (bottum). On the right is a schematic of the buckled column under mode I.

Critical load for cantilever column

The governing equation for buckling of a cantilever column is similar to that for buckling of a simply supported column. It is informative to show that this is indeed the case. The governing equation for buckling of a cantilever column can be obtained from the moment equilibrium of part of the beam, i.e., the part from the section at A to the right end B (Fig. O-11).

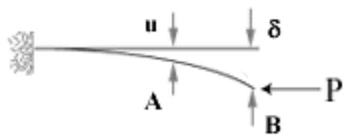


Figure O-11 A cantilever columnin an assumed buckled state. The equilibrium of part AB yields the governing equation for buckling.

The free body diagram of part AB in Figure O-11 is shown in Figure O-12.

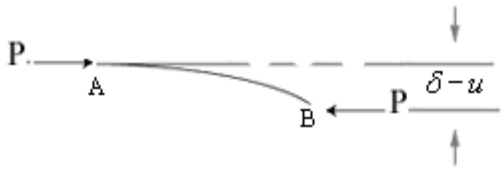


Figure O-12 The free body diagram of part AB of the beam shown in Figure O-11.

Moment equilibrium implies that the moment M at the cross section at A is $M = -P(\delta - u)$. Then, using the fact that the bending moment at a cross section is proportional to the curvature of the beam at that cross section ($M = EIu''(x)$), we have

$$EIu''(x) = -P(\delta - u(x))$$

By setting, as done for the simply supported column,

$$\kappa^2 = \frac{P}{EI}$$

the governing differential (equilibrium) equation reduces to

$$u''(x) + \kappa^2 u(x) = \kappa^2 \delta$$

which has the solution

$$u(x) = C_1 \sin \kappa x + C_2 \cos \kappa x + \delta$$

The boundary condition $u(0) = 0$ implies that $C_2 = -\delta$. Another boundary condition is that $u'(0) = 0$. This latter boundary condition implies that $C_1 = 0$, since

$$u'(x) = C_1 \kappa \cos \kappa x - C_2 \kappa \sin \kappa x$$

The solution then reduces to $u(x) = \delta(1 - \cos \varkappa x)$. Finally, $u(L) = \delta$, which implies that $\delta \cos \varkappa L = 0$. The trivial solution is, of course, $\delta = 0$ (stability). The nontrivial solution, $\cos \varkappa L = 0$ implies that

$$\kappa L = \frac{n\pi}{2}, \quad n = 1, 3, 5, \ldots$$

Then,

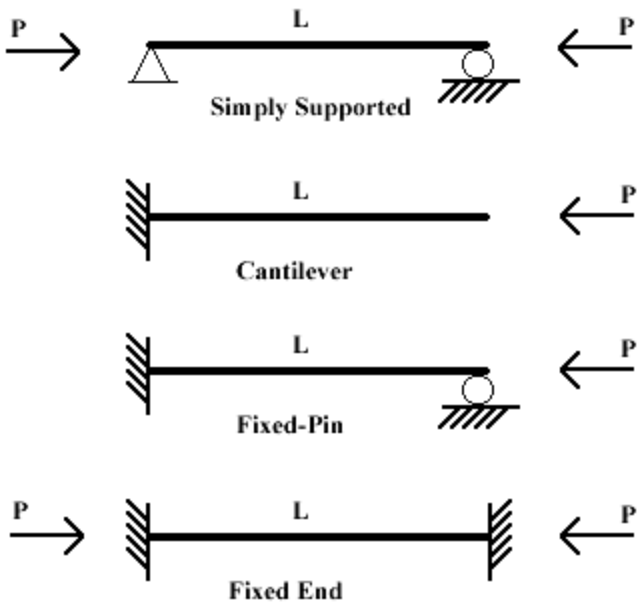
$$\boxed{P_{cr} = \frac{n^2 \pi^2 EI}{4L^2}}$$

For $n = 1$, the minimum critical load (mode I) is obtained

$$\boxed{P_{cr} = \frac{\pi^2 EI}{4L^2}}$$

A similar process (as done for the simply supported and cantilever beam) can be followed for other types of supports. The differential equation is similar, yet the boundary conditions change because the support conditions are different. The following table gives the relevant resulting equations.

Table O-1 Showing the type of beam and the critical load. For n = 1, the minimum critical load is obtained.



$$P_{cr} = \frac{n^2\pi^2 EI}{L^2}$$

$$P_{cr} = \frac{n^2\pi^2 EI}{(2L)^2}$$

$$P_{cr} = \frac{n^2\pi^2 EI}{(0.7L)^2}$$

$$P_{cr} = \frac{n^2\pi^2 EI}{(0.5L)^2}$$

Effective buckling length

Note that for every case in the above Table O-1, we have:

$$P_{cr} = \frac{n^2\pi^2 EI}{L_e^2}$$

where $L_e = kL$ is called the effective buckling length, and:

for **pinned** ends (simply supported column), **k = 1.0**

for **cantilever** column, **k = 2.0**

for **fixed-ends** column, **k = 0.5**

for **fixed-pin** column, **k = 0.7**

Figure O-13 schematically shows the mode I buckling shape of a cantilever beam of length L, and its mirror image. Clearly, a cantilever beam of length L is buckling-wise equivalent to a simply supported beam of length 2L. In other words, for a cantilever beam, $L_e = 2L$. Similarly, for a fixed-end column, $L_e = L/2$, and a schematic of this fact is shown in Figure O-14. Finally, as schematically shown in Figure O-15, the effective length of a fixed-pinned beam ends up being $L_e = 0.7L$.

Figure O-13 Mode I buckling shape of a cantilever beam of length L, and its mirror image.

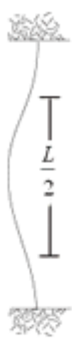


Figure O-14 Mode I buckling of a fixed-end column and schematic designation of the effective length, L/2.

Figure O-15 Mode I buckling of a fixed-pin column, and schematic designation of the effective length, 0.7L.

Example: Consider a simply supported beam of length L, and rectangular cross section (1 × 2 units of length) (Fig. O-16). The "weak" buckling direction is braced at midspan. Thus, we have a simply supported column, with additional supports at the middle in the "weak" direction.

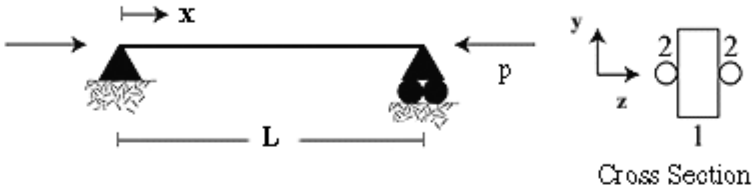


Figure O-16 A simply supported column, braced in the weak buckling direction at mid-span.

First, the moments of inertia in the strong (I_{zz}) and weak (I_{yy}) directions are evaluated

$$I_{zz} = \frac{1\times 2^3}{12} = \frac{2}{3} = \frac{4}{6}, \quad I_{yy} = \frac{2\times 1^3}{12} = \frac{1}{6}$$

For buckling in the yy direction, the effective length L_e is equal to L/2, thus

$$P_{cr}^{yy} = \frac{n^2\pi^2 EI}{L_e^2} = \frac{2^2\pi E\frac{1}{6}}{\left(\frac{L}{2}\right)^2} = \frac{4\pi E}{6L^2}$$

For buckling in the zz direction, the effective length L_e is equal to L, thus

$$P_{cr}^{yy} = \frac{n^2\pi^2 EI}{L_e^2} = \frac{1^2\pi E\frac{4}{6}}{\left(\frac{L}{2}\right)^2} = \frac{4\pi E}{6L^2}$$

If the cross section was 1 × 3 units, then the yy-direction would be critical.

Exercise: Perform the analysis as above, but for a cantilever beam of length L and rectangular cross section, 1 × 3 units of length. The weak buckling direction is braced at the top of the cantilever beam as schematically shown in Figure O-17.

Figure O-17 A cantilever column braced at the free end in the weak buckling direction.

Example: For the configuration depicted in Figure O-18, the problem here is to determine the thickness t of the hollow cylindrical pipe if its outside diameter is 110 mm and the (desired) factor of safety is n = 3. E = 72 GPa.

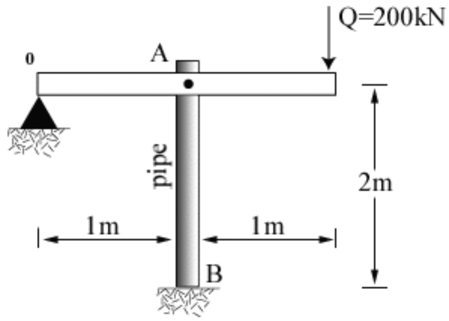


Figure O-18 Pipe AB is pinned to a beam at A and fixed at B.

The compressive force in the pipe is 2Q, as can be obtained by considering moment equilibrium around point O. The moment of inertia for a hollow cylindrical cross section is

$$I = \frac{\pi}{64}\left[110^4 - (110 - 2t)^4\right]$$

For the fixed-pin column BA,

$$P_{cr} = \frac{\pi^2 EI}{(0.7L)^2} = 2(3Q) = 6Q$$

which can be solved for I, thus obtaining the desired thickness. Note that the pipe is fixed at B and pinned at A, thus its effective buckling length is 0.7L.

Example: The rigid column AB of length 2 m (Fig. O-19) is supported through two identical horizontal cables (AC and AD) and is pinned at B. Find the in-plane (plane of the paper) critical load P_{cr}. Each horizontal cable is 3 m long, with a cross-sectional area of 3 cm^2 and elasticity modulus of 70 GPa. Note that cables do not support compressive load, and that column AB is rigid.

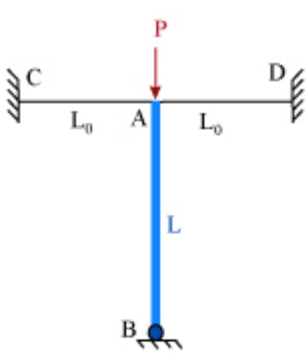


Figure O-19 A rigid column supported through cables at A and pinned at B.

Each cable is practically a spring that can only resist load in tension. Then the problem reduces to the configuration shown in Figure O-20, i.e., a rigid column supported by a spring at A and a pin at B. The buckled state of the columen is also shown in Figure O-20, as well as the forces acting on it.

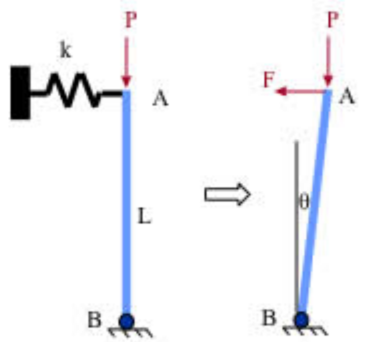


Figure O-20 Schematic of what the problem shown in Figure O-19 reduces to: a rigid column supported by a pin and a spring.

Let the cable, at the buckled state, elongate by δ, and

$$\delta = \frac{FL_0}{EA} \rightarrow F = \frac{EA}{L_0}\delta$$

where E denotes the elasticity modulus, and A the cross sectional area of the cables. Then the spring stiffness $\varkappa$ is expressed as

$$\kappa = \frac{EA}{L_0}$$

Equilibrium of moments around point B implies that $FL = P\delta$, or

$$\delta \frac{EA}{L_0} L = P_{cr}\delta$$

Solving this last equation for P_{cr}, it follows that

$$P_{cr} = \frac{EAL}{L_0} = \frac{70\times10^9\times3\times10^{-4}\times2}{3} = 140\times10^5 \text{ Pa}$$

Example-temperature buckling: A beam of length 2 m and square cross section (10 cm × 10 cm) is pinned at both ends (Fig. O-21). Find the temperature increase ΔT that will buckle the beam. $E = 10$ GPa, $\alpha = 6 \times 10^{-6}/^{o}C$

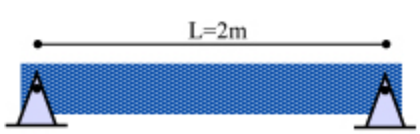


Figure O-21 A pin-pin supported beam, subjected to temperature increase.

This is a statically indeterminate problem, thus a compatibility equation should be used. The compatibility equation for this problem is that the elongation due to ΔT should equal the contraction from the compressive load created at the pins. From $\Delta T \rightarrow \delta_T = \alpha(\Delta T)L$, where

δ_T denotes elongation from ΔT. From the reaction, P, created at the pins $\rightarrow$ $\delta_P = PL/EA$, where A denotes the cross sectional area.

Compatibility implies that

$$\delta_T = \delta_P \rightarrow \frac{PL}{EA} = \alpha(\Delta T)L \rightarrow P = EA\alpha(\Delta T)$$

The critical (buckling) load, P_{cr} for this beam is expressed as

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

Then,

$$\frac{\pi^2 EI}{L^2} = EA\alpha(\Delta T) \rightarrow \Delta T = \frac{\pi^2 I}{L^2 A\alpha} = \frac{3.14^2 \frac{0.1\times 0.1^3}{12}}{2^2 \times 0.1^2 \times 6\times 10^{-6}} = 342°C$$

Note that if the length of the beam is doubled, the critical temperature reduces by a factor of four. Also note that the critical temperature is independent of the modulus of elasticity E.

Example-temperature buckling: The bar/column of length L and circular cross section of radius r is fixed at its left end (Fig. O-22). At its right end there is a small gap, δ. Find the minimum temperature increase, ΔT, that will create critical buckling conditions. Assume that δ is small compared to L, and find the solution in terms of L, E (bar′s modulus of elasticity), r, δ, and α (coefficient of thermal expansion). For buckling analysis, consider the right end of the bar/column as fixed after the gap closes. Note: for a circular area, $I = \pi r^4/4$.

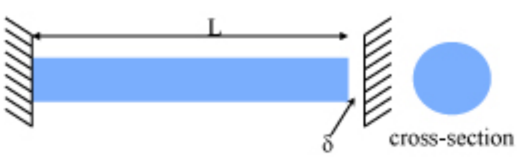


Figure O-22 A column is heated until a gap closes and then until the column buckles.

In order to close the gap without creating any stress, a temperature increase ΔT_1 is required such that

$$\delta\alpha(\Delta T_1)L \rightarrow \Delta T_1 = \frac{\delta}{\alpha L}$$

After the gap closes, an additional temperature increase ΔT_2 is required to create a critical compressive load P_{cr} in the bar. The bar at this point is statically indeterminate, and the compatibility condition dictates that the elongation from ΔT_2 should be equal to the contraction by the compressive force P_{cr}. Or

$$\alpha(\Delta T_2)L = \frac{P_{cr}L}{EA} \rightarrow \Delta T_2 = \frac{P_{cr}}{\alpha EA} = \frac{P_{cr}}{\alpha E\pi r^2}$$

For buckling under fixed-end conditions

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{\pi^2 E\pi r^4/4}{(0.5L)^2} = \frac{\pi^3 Er^4}{L^2}$$

where L_e denotes the effective buckling length. Then

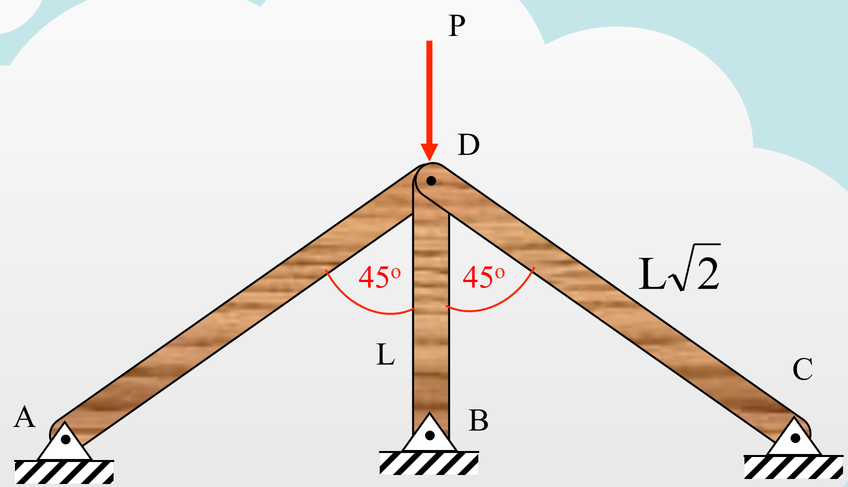
$$\Delta T_2 = \frac{P_{cr}}{\alpha EA} = \frac{\pi^3 Er^4}{L^2\alpha E\pi r^2} = \frac{\pi^2 r^2}{\alpha L^2}$$

and

$$\Delta T = \Delta T_1 + \Delta T_2 = \frac{\delta}{\alpha L} + \frac{\pi^2 r^2}{\alpha L^2}$$

Example: Buckling in a Truss Member

Each bar in the truss has the same E, I, and A. For L = 2 m, find the minimum load P that will buckle bar BD. The truss is braced at D in the direction transverse to the paper.



$$F_{CD} = F_{AD} \quad \text{(symmetry)}$$

Compatibility: $\delta_{CD} = \delta_{BD}\cos 45°$

$$\Rightarrow \frac{F_{CD}\sqrt{2}L}{EA} = \frac{\sqrt{2}}{2}\frac{F_{BD}L}{EA}$$

$$\Rightarrow F_{CD} = \frac{1}{2}F_{BD}$$

Equilibrium: $P = F_{BD} + 2F_{CD}\cos 45°$

$$\Rightarrow P = F_{BD} + 2\left(\frac{1}{2}F_{BD}\right)\frac{\sqrt{2}}{2}$$

$$\Rightarrow P = F_{BD}\left(1+\frac{\sqrt{2}}{2}\right) = F_{BD}\left(\frac{2+\sqrt{2}}{2}\right)$$

$$\Rightarrow F_{BD} = \left(\frac{2}{2+\sqrt{2}}\right)P$$

$$F_{BD}^{crit} = \frac{1^2\pi^2 EI}{L^2}$$

$$P_{crit} = \frac{1^2\pi^2 EI}{L^2}\left(\frac{2+\sqrt{2}}{2}\right) = \frac{3.14^2}{4}\left(\frac{2+\sqrt{2}}{2}\right)EI$$

$$\Rightarrow P_{crit} = 4.21EI$$

Example: Temperature Increase for Buckling a Column

A beam of length 2 m and square cross section (10 cm × 10 cm) is pinned at both ends. Find the temperature increase ΔT that will buckle the beam. No other load acts on the beam.

Given:

$$E = 10 \text{ GPa}$$

$$\alpha = 6 \times 10^{-6} \ / \ ^\circ C$$

$$\alpha(\Delta T)L = \frac{PL}{EA}$$

$$\therefore P = \frac{EA\alpha(\Delta T)L}{L} = EA\alpha(\Delta T)$$

$$EA\alpha(\Delta T) = \frac{\pi^2 EI}{L^2}$$

$$\Delta T = \frac{\pi^2 I}{A\alpha L^2} = \frac{\pi^2 \dfrac{0.1 \times 0.1^3}{12}}{0.1^2 \times 6 \times 10^{-6} \times 2^2}$$

$$\therefore \Delta T = \frac{0.1^2 \times \pi^2}{12 \times 4 \times 6 \times 10^{-6}} = 342^\circ C$$

Example: Buckling of a "Shell"

An empty coke can is placed on a flat surface while subjected to an external compressive force (weight of a person, for example). (a) Find the minimum weight that will render the can unstable, by assuming one end of the column (can) is fixed and the other free. The outside diameter of the can is 2.6 inches, the wall thickness is 0.004 inches and its length (height) is 3.85 inches. $E = 10 \times 10^6$ psi. (b) considering a yield strength of about 30 ksi, find the load required to yield the can. For a solid circular cross section of diameter d, $I = \pi d^4/64$.
Note: The solution of this problem should show that the buckling load is rather large. Because the can wall is thin, this "structure" is bound to buckle from localized instabilities instead of buckling as a beam.

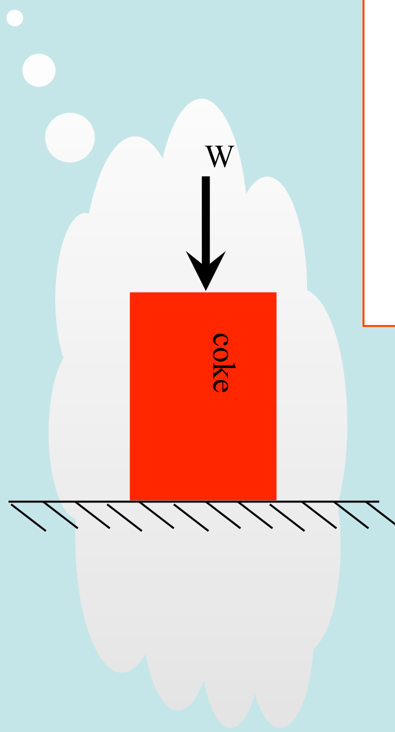
$$I = \frac{\pi}{64}\left(2.6^4 - 2.592^4\right) = 0.0275 \text{ in}^4$$

$$L_e = 2L = 2 \times 3.85 = 7.7 \text{ in}$$

$$P_{cr} = \frac{\pi^2 EI}{L_e^2} = \frac{3.14^2 \times 10^7 \times 0.0275}{7.7^2} = 45{,}731 \text{ lbs}$$

$$A = \frac{\pi}{4}\left(2.6^2 - 2.592^2\right) = 0.0326 \text{ in}^2$$

$$P_{yield} = \sigma_{yield} A = 30{,}000 \cdot 0.0326 = 978.1 \text{ lbs}$$



Self Assessment
Columns: 1

To consider a 10 cm × 10 cm wooden post as a long column, what should be its minimum length? Assume the maximum slenderness ratio as 60.

- 4.03 m
- 1.73 m
- 2.24 m
- 3.12 m

Columns: 2

A 10-cm diameter vertical aluminum strut (E = 70 MPa) is supporting a 2,000 N load. Using a factor of safety of 2, what is the maximum length of the strut?

- 9.8 m
- 2.8 m
- 14.6 m
- 12.5 m

Columns: 3

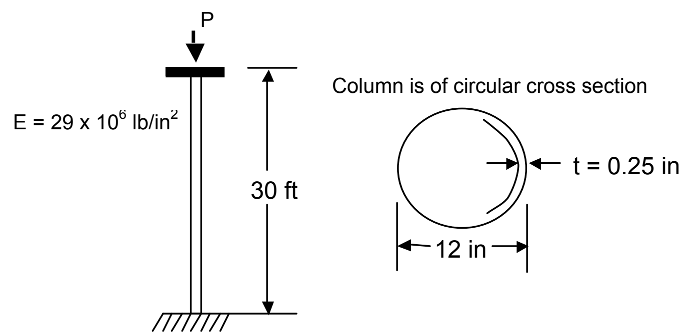
How much can a 2-cm-diameter, 4-m-long, steel rod ($\alpha = 11.7 \times 10^{-6}/^{o}C$) fixed from both ends can resist increase in temperature (^{o}C) without buckling?

- 9.8 deg
- 5.27 deg
- 16.4 deg
- 4.26 deg

Columns: 1 A steel column ($E = 30 \times 10^6$ psi) of square tube cross-section, $8 \times 8 \times 1/4$, is in compression and simply supported at its ends. With the given moment of inertia (I_x and I_y) and area (A), find the buckling load.

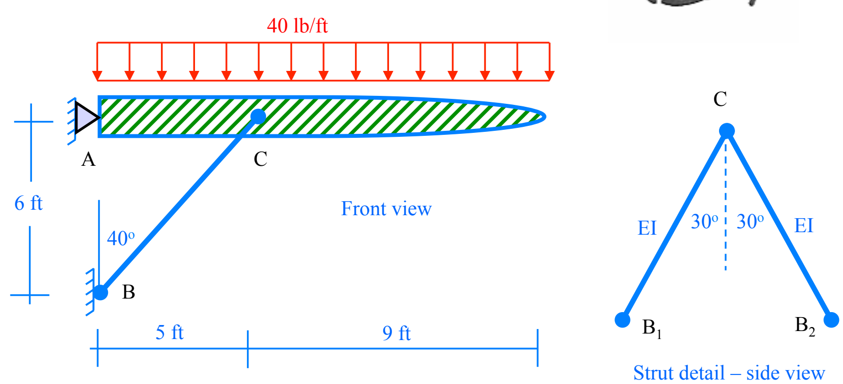
$I_x = I_y = 75.1 \text{ in}^4$
$A = 7.59 \text{ in}^2$
$L = 38$ in

Columns: 2 A flag pole is subjected to compressive load P. The geometry of the pole is as shown. Find the maximum possible P if the outside column diameter is 12 inches.

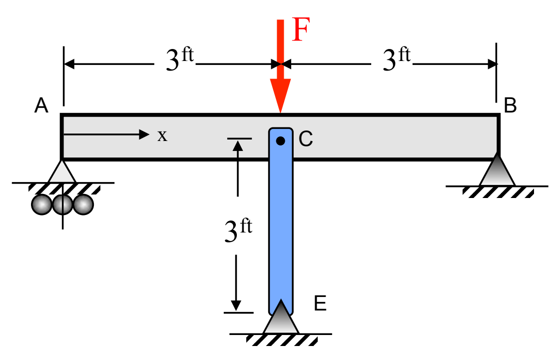


Columns: 3 What should be the minimum diameter of the cane, d, such that the cane (length L and modulus E) will not buckle while supporting half of Charlie's weight, i.e., W/2 ? Assume pin-pin boundary conditions for the cane.

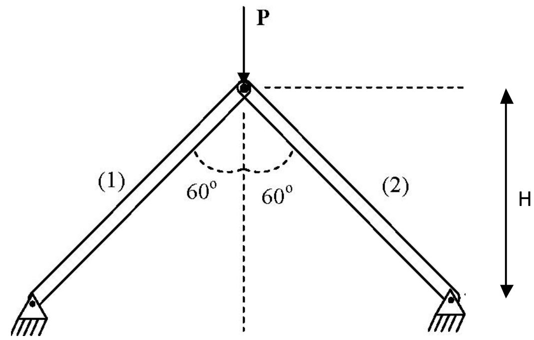
Columns: 4 A strut-braced airplane wing is subjected to a distributed load (its own weight plus inertia force while landing) as shown. Find the minimum required EI for the two strut bars such that they will not buckle.



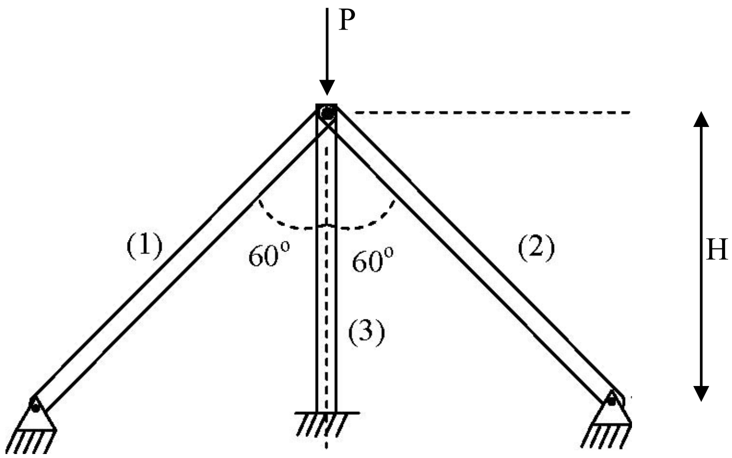
Columns: 5 Beam ACB is simply supported by a roller at A, a pin at B, and by column CE which is pinned to the beam at C and pinned externally at E. Find the load F that will buckle column CE. Beam ACB has stiffness EI and the column is of circular cross section and has stiffness 0.25EI. Note that for a simply supported beam of length L and stiffness EI loaded at midspan by load P, the deflection at midspan is $u_{\text{midspan}} = \dfrac{PL^3}{48EI}$



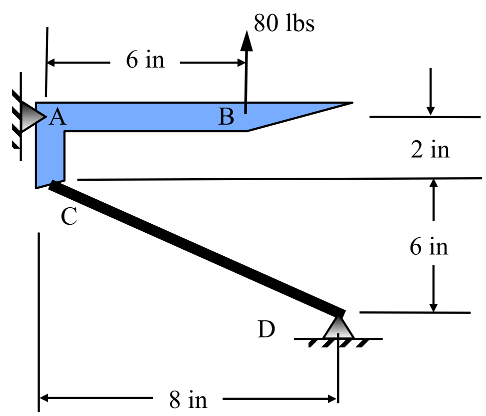
Columns: 6 Consider the truss shown in the figure subjected to a force P. Each rod of the truss is made of a material of elastic modulus E and has a circular cross section (radius of cross section R). All joints are pin joints. What is the critical force P at which one or both rods buckle (in terms of E, R, H)? For a circle of radius R, $I = \frac{\pi R^4}{4}$



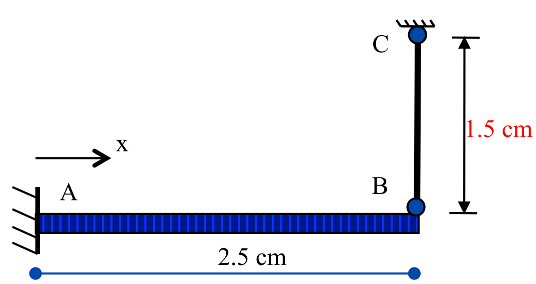
Columns: 7 A third rod of the same material, with a circular cross section of radius R/2, of length H, is added to the truss as shown below. The top end is connected to the other two rods with a pin joint, whereas, it has "fixed" boundary condition at the bottom end as shown. As P is increased, which rod(s) buckle(s) first and what is the corresponding critical buckling load P_{cr} (in terms of E, R, H)?



Columns: 8 Determine the factor of safety against buckling of bar CD, which has circular cross section of 0.25 inch diameter d, E = 70,000ksi. It is pinned at C and D. For a circle, $I = \pi d^4 / 64$.

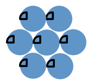


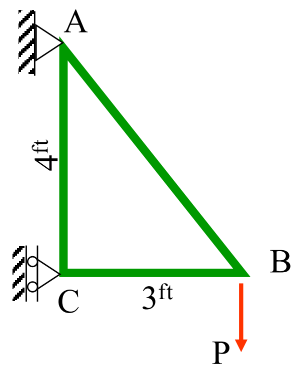
Columns: 9 Bar CB, of cross-sectional area A = 0.05 cm^2, E = 0.2 GPa, I = 0.000208 cm^4 and coefficient of thermal expansion $\alpha = 13 \times 10^{-6}/^{o}C$ is pinned at both its ends. Beam AB has I = 140.0 cm^4 and in the vertical direction and E = 200 GPa. Bar CB is subjected to temperature increase ΔT while the beam is not. Find the ΔT that will buckle bar CB. Note that for a cantilever beam of length L, stiffness EI, and loaded at the free end by load F, the deflection at the free end is $\delta = \frac{FL^3}{3EI}$.



Columns: 10 A cylindrical steel bar of radius r has a buckling strength of 1,000 N under simply supported (pin-pin) conditions. If seven of these bars are put together as shown, what will be their buckling strength under the same simply supported conditions ? Note that the moment of inertia of a circular cross section with respect to its centroid is $\pi r^4/4$.

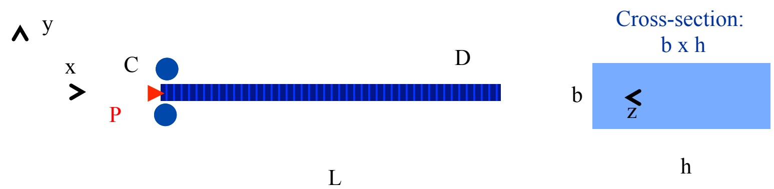
a) 

b) 



Columns: 11 The truss is loaded with a vertical force, P, at point B. The truss is to be composed of bars with rectangular cross section of 1 in. × 3/2 in, and E = 2 × 10^6 psi. Considering only the condition for failure by buckling of bar BC, what is the maximum allowable load, P, that can be safely applied to the structure if the factor of safety of 2 is required?

Columns: 12 Column CD of length L and rectangular cross-section b×h, where h>b, is fixed at D and at C it is braced (motion restricted) in the "weak" y-direction but not braced in the "strong" z-direction. For what ratio of h/b will the critical load for buckling in the "weak" direction be equal to the critical load for buckling in the "strong" direction?



Columns: 13 Thus, for h/b=2.857, bracing will make the buckling load in the "weak" direction equal to the buckling load in the "strong" direction.

A column can either fail due to the material yielding, or because the column buckles. It is of interest to the engineer to determine when this point of transition occurs.
Consider the Euler buckling equation

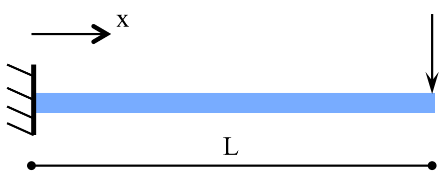
$$P_E = \frac{\pi^2 EI}{L_e^2}$$

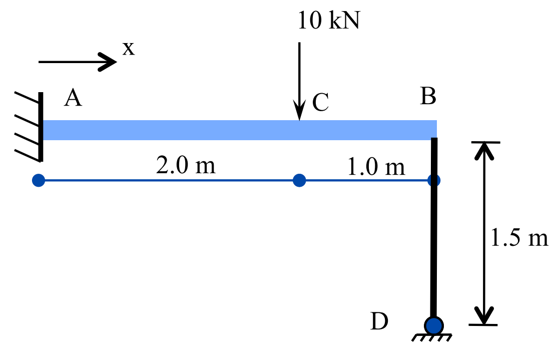
where L_e denotes the effective buckling length. The least moment of inertia I can be expressed as $I = Ar^2$ where A is the cross sectional area and r is the ***radius of gyration*** of the cross sectional area, i.e.

$$r = \sqrt{\frac{I}{A}}$$

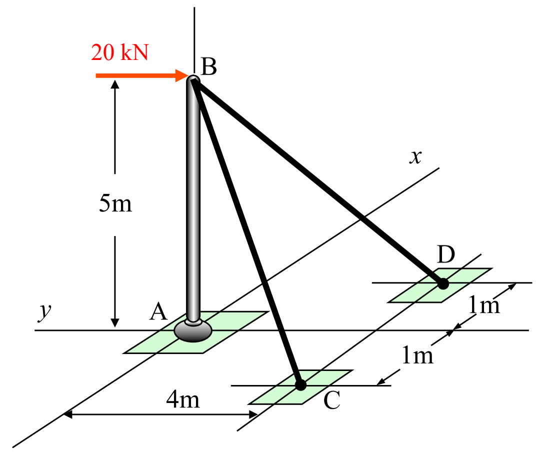
Note that the *smallest* radius of gyration of the column, i.e. the *least* moment of inertia I should be taken in order to find the critical stress. The slenderness ratio of a column is defined as the ratio L_e/r. (a) Find the critical slenderness ratio at the point of transition mentioned above. (b) For a simply supported column of length L and square cross section (b × b), find the ratio L/b at the critical slenderness ratio when $E/\sigma_y = 250$, where the yield stress of the material in compression is denoted as σ_y.

Columns: 14 For a cantilever beam of length L and stiffness EI loaded at its end by load P, the deflection is given by $u(x) = \frac{Px^2}{6EI}(3L - x)$, and the slope is given by $u'(x) = \frac{Px}{2EI}(2L - x)$. Beam ACB is fixed at A, loaded by a 10 kN force at C, and supported by column BD at B. Both the beam and the column are made of steel (E = 200 GPa) and both are of square cross section, 10 cm × 10 cm. Find the factor of safety with respect to buckling of column BD. Assume pin-pin boundary conditions for BD. Note that a safety factor less than unity implies the column is likely to fail. Also, for certain cases the safety factor may be much larger than unity.

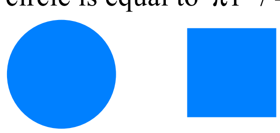




Columns: 15 In the structure shown, beam AB is vertical and rigid, pinned at A, and supported by two bars/columns, BC and BD, which are pinned at B, C, and D. Bars/columns BC and BD are of circular cross section and have a rigidity $EI = 75\,\text{kN}\cdot\text{m}^2$. The 20 kN load is applied in the y-direction. Find the factor of safety of the structure against buckling of BC and/or BD. Note that a safety factor less than unity implies an unsafe structure.



Columns: 16 A column of length L is pinned at both its ends, and is designed to carry a compressive load P. The column is free to buckle in any direction, and a material of modulus of elasticity E is to be used for the column. The engineer has the choice to use either a circular cross section or a square one. Considering that the design is based on the buckling load of the column such that $P = P_{critical}$, determine the ratio V_c / V_s where V_c, V_s denotes the volume of material for the column of circular, square cross section design, respectively. For a circular cross section, the moment of inertia with respect to an axis passing through the centroid of the circle is equal to $\pi r^4 / 4$, r denoting the radius of the cross section.



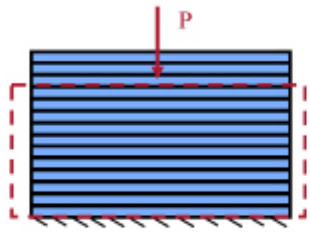
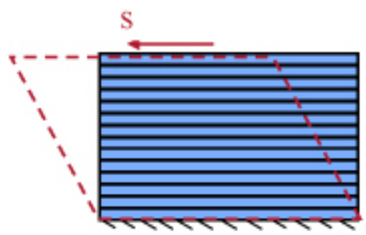
Review

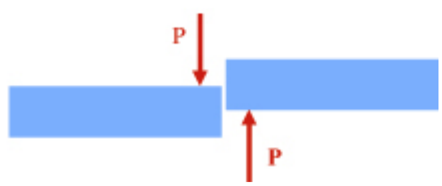
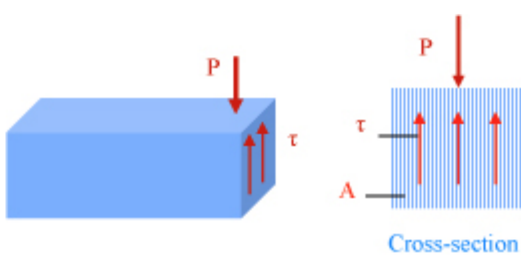
Table P-1 Notation Used in This Book

σ: normal stress (psi or Pa) σ_x, σ_y, σ_z: normal stress in x-, y-, z-plane, respectively.	ε: normal strain (in/in or m/m) ε_x, ε_y, ε_z: normal strain in x-,y-,z-plane, respectively.
τ: shear stress (psi or Pa) τ_{xy}: shear stress in x-plane in y-direction (similarly, τ_{xz}, τ_{yz}, etc.)	γ: shear strain (in/in or m/m) γ_{xy}: shear strain in xy-plane (similarly, γ_{xz}, γ_{yz}, etc.)
I: area moment of inertia (in^4 or m^4) I_p: polar area moment of inertia (in^4 or m^4)	E: modulus of elasticity (psi or Pa) G: modulus of rigidity (psi or Pa) ν: Poisson's ratio
M: bending moment in beams V: shear force in beams T: torque in shafts	α: coefficient of thermal expansion (/^{o}F or /^{o}C) ΔT: temperature change (^{o}F or ^{o}C) $\varepsilon_t = \alpha(\Delta T)$: thermal strain
Factor of Safety: n = failure load/allowed load	

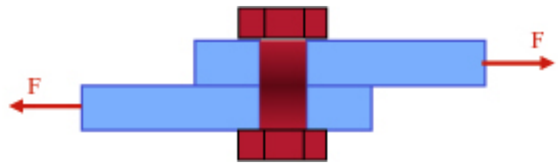
Stress is defined as force per unit area acting on a plane. Table P-2 illustrates the different types of stress, and the corresponding strains.

Table P-2 Normal Stress, Strain and Shear Stress, Strain

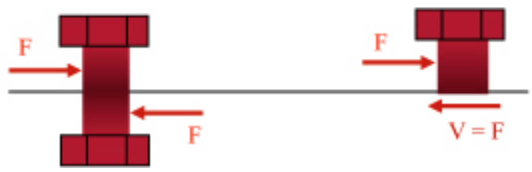
Normal stress (σ): the force acts perpendicular to the plane. Normal stress σ produces normal strain ε as well as lateral strains due to Poisson effects.	 A material element subjected to a compressive force P. The deformed material element is shown schematically (dotted red/dark line).
Shear stress (τ): force acts parallel to the plane. Shear stress τ produces shear strain γ.	 A material element subjected to a shear force S. The deformed material element is shown schematically (dotted red/dark line).

A shearing force P is equilibrated by internal shear stresses, τ.	 The shearing force tendency to shear off the beam is resisted by internal shearing stresses.
 Schematic of the shear stresses τ created by P. They act parallel to the cross section.	

V = Shear force on pin and
V = F.
A = Cross sectional area of pin.



A fastener is subjected to shear force.



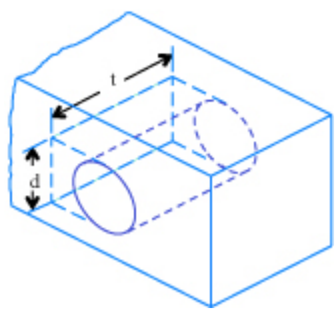
The shear force V is equilibrated by the internal shear stresses acting on the cross section. Equilibrium implies , τ_{ave} = V/A subscript "ave" denotes average.

Bearing Stress in Fasteners

$$\sigma_b = \frac{F}{d \times t}$$

where
F = force
d = diameter of fastener
t = thickness of part.

The bearing stress can be defined as force per net area (d times t)



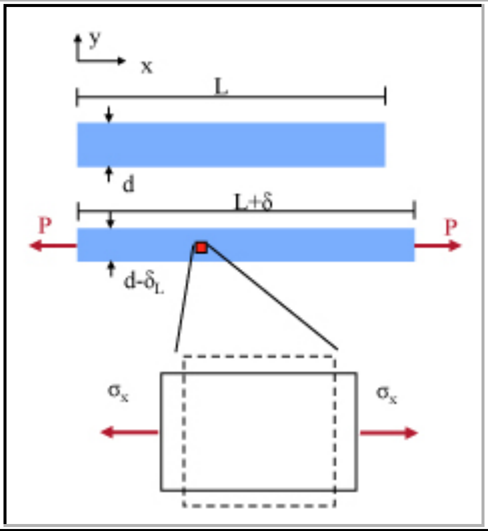
The shearing force F also creates bearing stress, the average value of which is denoted as σ_b

Review

Axial loading

Axial load relations: a bar is, by definition, subjected to axial load when the line of action of the load, P, passes through the centroid of the cross section. This is shown in Table P-3.

Table P-3 Definitions for Axial Loading and Hooke's Law.

	Stress (definition): $\sigma = \frac{P}{A}$. Strain (definition): $\epsilon = \frac{\delta}{L}$. If the material is also linear elastic, then: Hooke's law: $\epsilon = \frac{\sigma}{E}$, where E denotes the modulus of elasticity for the material.

The relationship between axial loading and deformation becomes, by rearranging the above three equations $\delta = \frac{PL}{EA}$ or $P = \frac{EA}{L}\delta$. The last equation indicates that a bar behaves like a spring of spring constant equal to AE/L.

Lateral strain (Poisson effects): $\epsilon_{lat} = \epsilon_y = \epsilon_z = -\nu\epsilon_x$ where ν denotes the Poisson ratio of the material.

Statically determinate members

Static Equilibrium

$\Sigma F = 0 = -F_1 + F_2 - F_3 + F_4$

Internal Forces

$P_{AB} = F_1$ (tension)

$P_{BC} = F_1 - F_2$ (tension)

$P_{CD} = F_1 - F_2 + F_3$ (tension)

Deformation

$$\delta_{AD} = \delta_{AB} + \delta_{BC} + \delta_{CD} = \frac{P_{AB}L_{AB}}{A_{AB}E_{AB}} + \frac{P_{BC}L_{BC}}{A_{BC}E_{BC}} + \frac{P_{CD}L_{CD}}{A_{CD}E_{CD}}$$

Since the Ps were assumed in tension, negative values will indicate compression and contraction for the deformation rather than elongation.

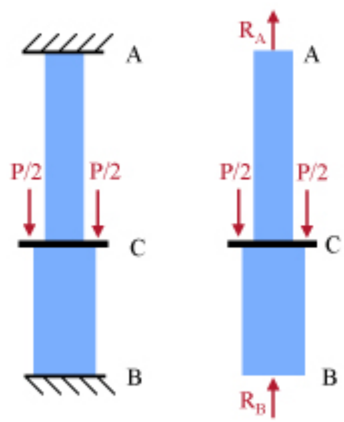
Thermal deformation is expressed as

$\delta_{AB}^{thermal} = \alpha_{AB} L_{AB} \Delta T$.

Thermal deformation may be added to any mechanical deformation caused by internal forces acting on the material to obtain a total deformation.

Statically indeterminate members

After the load P is applied on the rigid bearing plate



Equilibrium

$\Sigma F_y = 0 = R_A + R_B - P$ or $R_A + R_B = P$

D = displacement of bearing plate

Bar AB

$$\delta_{BAB} = \frac{R_A L_{AB}}{A_{AB}E_{AB}} \quad \text{(positive for elongation)}$$

Bar BC

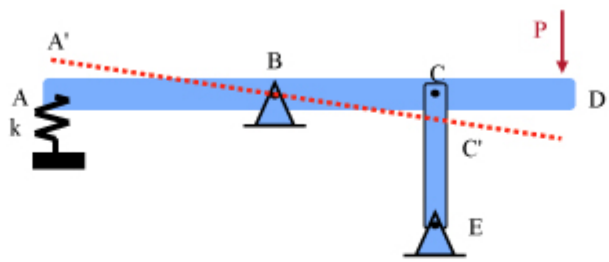
$$\delta_{BC} = -\frac{R_B L_{BC}}{A_{BC} E_{BC}}$$ (negative for contraction)

Compatibility

$$\frac{R_A L_{AB}}{A_{AB} E_{AB}} - \frac{R_B L_{BC}}{A_{BC} E_{BC}} = 0$$

This compatibility condition implies that the total elongation/contraction of AB is zero.

The horizontal beam is rigid, pinned at B and supported by a spring at A and a bar CE at C. After external load P is applied, the bar rotates around B. Thus the spring stretches by AA' = δ_A and the bar is compressed by CC' = δ_C.



Equilibrium of beam ABCD: $\Sigma M_B = 0 = F_A (L_{AB}) + F_C (L_{BC}) - P (L_{BC} + L_{CD})$

Compatibility of displacements: $\frac{\delta_A}{L_{AB}} = \frac{\delta_C}{L_{BC}}$

Also, $\delta_A = \frac{F_A}{k}$, $\quad \delta_C = \frac{F_C L_{CE}}{E_{CE} A_{CE}}$,

thus the compatibility equation is written as

$$\frac{F_A}{L_{AB} k} = \frac{F_C L_{CE}}{E_{CE} A_{CE}}$$

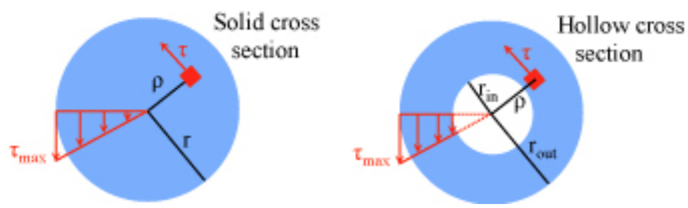
This equation together with the moment equilibrium forms two equations for the two unknowns: F_A and F_C.

Review

Torsion of circular cross section shafts

If the shaft has a circular cross section and the material remains in the linear-elastic region, the shear stress in the shaft varies as a linear function of the distance ρ from the center of the shaft and is given by:

Shear Stress: $\tau = \frac{T\rho}{I_p}$



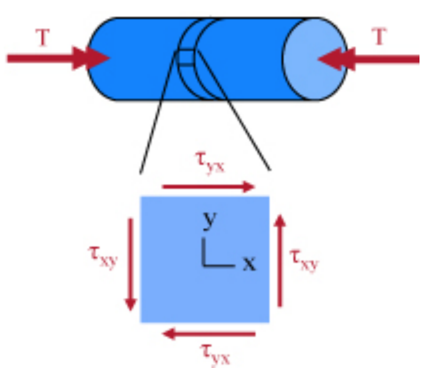
The maximum shear stress in the shaft is on the outer surface independent of whether the shaft is solid or hollow and is given by:

$$\tau_{max} = \frac{Tr}{I_p}$$

The polar area moment of inertia is, for a solid circular section: $I_P = \frac{\pi r^4}{2}$

and for a hollow circular section: $I_P = \frac{\pi r_{out}^4}{2} - \frac{\pi r_{in}^4}{2}$

The calculated stresses act on the element as shown.

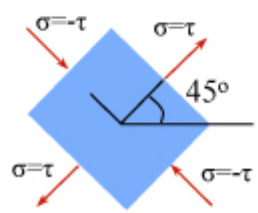


The deformation is measured by the angle of twist (ϕ) of one end relative to the other and is given by

$$\varphi = \frac{TL}{GI_p}$$

where G is the modulus of rigidity for the material and L is the length of shaft. The shaft also has maximum and minimum normal stresses acting on an element rotated 45° from the element for which the shear stress was calculated. The maximum tensile and compressive stresses are related to the shear stress by

$$\sigma_{ten.} = -\sigma_{comp.} = \tau_{max} = \frac{Tr}{I_P}$$



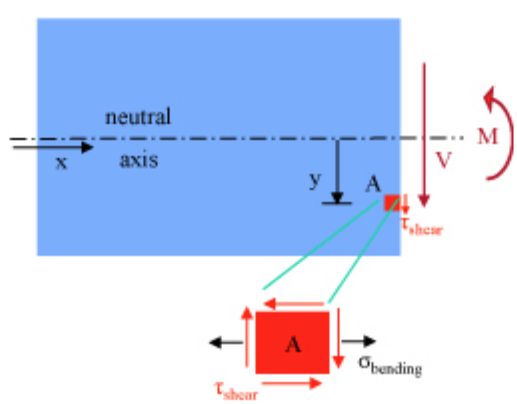
Beam bending and shear

In general, a beam cross section will be subjected to both shear and bending. This results in a general stress element as shown in the Figure, where

$$\sigma_{bending} = \frac{My}{I}$$

and

$$\tau_{shear} = \frac{VQ}{Ib}$$



Failure is most likely to occur on a cross section where V or M is maximum. On the cross section, failure due to pure bending may occur at the top or bottom and failure due to pure shear may occur at the neutral axis. Wide-flange or other nonuniform cross sections may have principal stresses or maximum shearing stresses at the web-flange intersection or other points of change in cross section width that exceed other stresses on the cross section. Maximum shear and bending moment values are found most easily and reliably using the shear and bending moment diagrams.

The location of the centroid can be determined by considering the area moments about any axis parallel to the bending moment axis

$$A\bar{y} = \sum_i A_i y_i$$

where A is the entire area of the cross section, A_is are the subareas making up the cross section, and y_is are the perpendicular distance from the reference axis to the centroid of the associated area.

If the cross section can be divided into common shaped areas for which the location of the centroid and the area moment of inertia (I_i) about the centroid are known, then the area moment of inertia (I_{NA}) for the cross section can be determined from the parallel axes theorem,

$$I_{NA} = \sum_i \left(\bar{I}_i + A_i d_i^2\right)$$

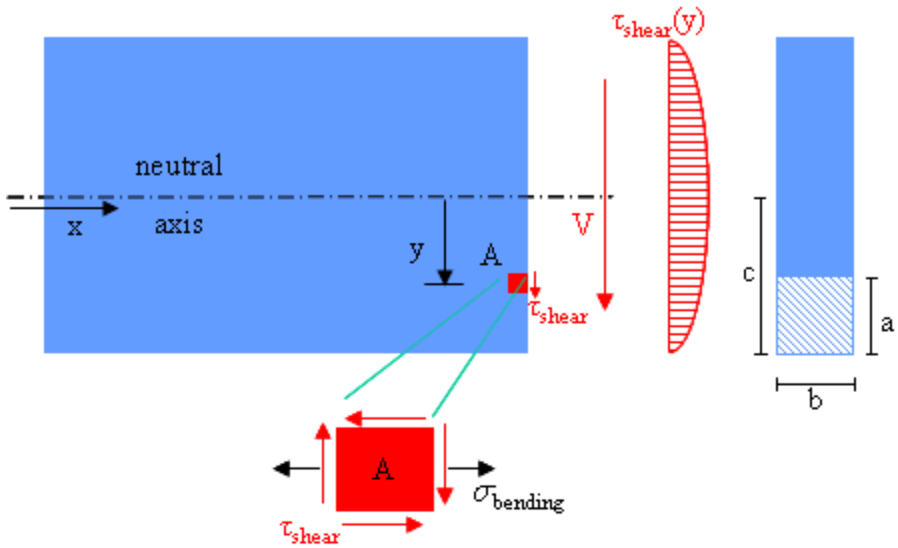
where I_is are the area moments of inertia of the individual areas about their own centroidal axis and d is the perpendicular distance between the area centroidal axis and the neutral axis of the cross section.

Beams-shear stress distribution

The transverse and longitudinal horizontal shearing stress in a beam is given by

$$\tau = \frac{VQ}{Ib}$$

where Q is the first moment of the shaded area about the neutral axis if the shearing stress is being evaluated along the inside edge of the shaded area. Also, b is the width of the cross section at the material element position where the shear stress is evaluated.



For a rectangular cross section

$$Q = ab\left(c - \frac{a}{2}\right)$$

Note that y = c − a in this case. The maximum shearing stress will occur where Q/b is maximum. Q is always maximum at the neutral surface. However, Q/b may or may not be maximum at the neutral surface, thus one needs to check all possibilities.

The shear flow or force per unit length of the beam acting on the joint between sections making up a built-up cross section is given by

$$q = \frac{VQ}{I} \quad \text{(N/m} \quad \text{or} \quad \text{lb/in)}$$

where the area on either side of the joint is used to calculate Q. Shear flow and discrete fastener strength are related by

F = qs

where F is the net shearing strength of the joint fasteners on a single cross section of the beam and s is the distance along the beam between cross sections containing fasteners. Continuous fasteners such as welds are designed such that they resist the actual shear flow multiplied by an appropriate factor of safety.

Beam deflection-method of integration

The deflection of straight beams is determined from any of the following three governing equations

$$EIy''(x) = M(x); \quad EIy'''(x) = V(x); \quad EIy''''(x) = -q(x)$$

Here y(x) is the lateral displacement (deflection) of the beam from its original position, the primes denote derivatives with respect to x, and M(x) is the bending moment as a function of position along the beam, V(x) is the shear force as a function of position along the beam, and q(x) is the distributed load as a function of position along the beam. Integration of any of these equations and use of appropriate boundary conditions yields the equation of the deflection as a function of position along the beam.

Beam deflection-superposition method

The solutions for the above equations for many different types of supports and loads are given in many of the common engineering handbooks; some are also given in special modules herein. The principle of superposition allows the solutions of different loads to be added together to give the solution for the combined loads. The limitations of this method depend on how extensive the available beam tables are. It must be kept in mind that the table entry must exactly match the portion of the load being represented using only a scaling factor and/or mirror imaging. Loads in the tables may have either positive or negative values.

Review

Transformation equations for stress

It is assumed that all the stresses in one direction are zero. The coordinate axes are orientated to place the z-axis in that direction. This situation is common in engineering applications. A free surface is the classic example. The stresses representing the state of stress at a point are different when measured with respect to two different coordinate systems that are rotated with respect to each other. If the first system is labeled xy then the x'y' is rotated counter-clockwise by an angle q.

$$\sigma_{x_1} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x_1 y_1} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

The primed stresses may be determined from the unprimed by the equations: $\sigma_{y'} = \sigma_{x'}(\theta + 90^o)$

Principal stress and maximum shear stress

There will always be a maximum and minimum stress value, referred to as the principal stresses, occurring at some orientation. There will also be a maximum shearing stress that occurs on two different planes. The values of the principal stresses are given by:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The plus sign is used for the larger σ_1 and the minus sign for the smaller σ_2. The value of the maximum shearing stress is given by:

$$\tau_{max} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

The orientation of the σ_1 plane relative to the σ_x plane is given by:

$$\theta_p = \frac{1}{2} \tan^{-1}\left(\frac{\tau_{xy}}{\frac{\sigma_x - \sigma_y}{2}}\right)$$

θ_P is the counter-clockwise angle from the σ_x plane to the σ_1 plane. The two principal planes are perpendicular to each other and the two maximum shearing stress planes are at 45^o to either of the principal planes.

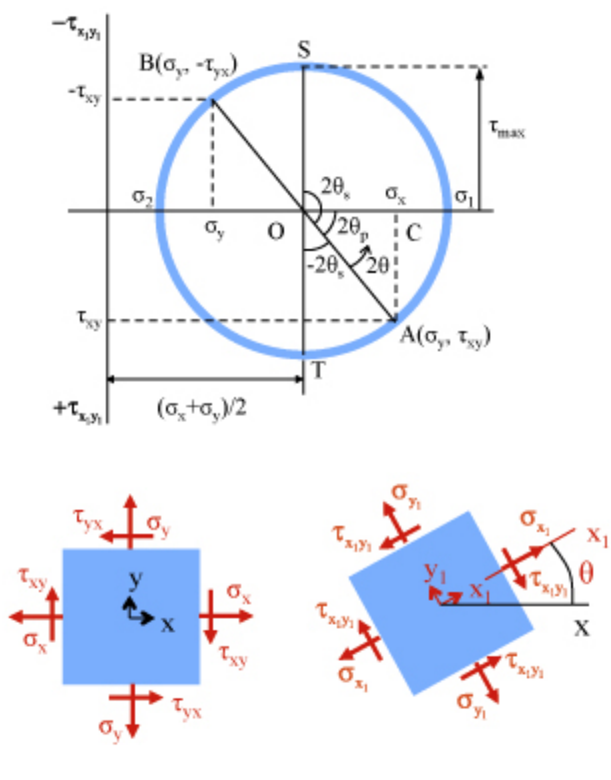
Mohr's circle for plane stress

Mohr's Circle is a mapping of the normal and shear stress acting on a plane at a point in real space to the coordinates of a point in the σ-τ-plane. All the points associated with the stresses on planes at a single point lie on a circle centered at

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}$$

and $\tau = 0$. The radius of the circle is equal to the maximum in-plane shearing stress.

Mohr's circle can best be used as a road map relating various planes and their stresses at the point. Rotation in real space from one plane to another results in a corresponding movement around the circle in the same direction, but twice as far. The coordinates of the new point represent the stresses acting on the new plane. The two points at which the circle crosses the horizontal axis represent the two principal stress planes and the points at the top and bottom of the circle represent the two maximum in-plane shearing stress planes. The principal stresses are then given by $\sigma_{1,2} = \sigma_{avg} \pm R$, where $R = \tau_{max}$. Sign convention for the normal stress is the usual positive to the right and negative to the left. Shear stresses are best treated by considering which way the shear stress on a given plane is trying to twist the element; clockwise twist is plotted in the upper half of the σ-τ plane and counter-clockwise in the lower half of the plane. The sign information works both ways since there is a unique one-to-one mapping.



Mohr's Circle for Plane Stress

Transformation equations for strain

The analysis is based on a plane strain state in which all strains in the z-direction are zero. The analysis can also be used for a plane stress state with one minor modification. A material cannot have both plane stress and plane strain states at the same time. The relationship between the strains at a point measured relative to a set of axes x-y and a set of axes x'-y', which have the same origin but are rotated counter-clockwise from the original axes by an angle θ, are given by

$$\epsilon_{x_1} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2} \cos 2\theta + \frac{\gamma_{xy}}{2} \sin 2\theta$$

for the normal strains and by

$$\frac{\gamma_{x_1 y_1}}{2} = -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\theta + \frac{\gamma_{xy}}{2} \cos 2\theta$$

for the shearing strain. Note the similarity of form between these equations and the stress transformation equations.

Principal strains and maximum shearing strain

As with the stresses, there are maximum and minimum (principal) values of the normal strains for particular orientations at the point and maximum shearing strains. The principal strains are given by

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}^2}{2}\right)^2}$$

and the maximum shearing strain is given by

$$\frac{\gamma_{max}}{2} = \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

The orientation of the larger principal strain to the positive x-direction is given by

$$\theta_p = \frac{1}{2} \tan^{-1}\left(\frac{\frac{\gamma_{xy}}{2}}{\frac{\epsilon_x - \epsilon_y}{2}}\right)$$

The direction of the smaller principal strain is perpendicular to the first. The directions involved with the maximum shearing strain are the two directions at 45^o to both of the principal directions.

Mohr's circle for strain

A Mohr's circle mapping between the strains acting with respect to a set of x-y axes at a point and a point in the strain plane can be made. The same rules apply as for the stress circle with ε replacing σ and γ replacing τ. This makes the radius of the circle equal to half the in-plane maximum shearing strain. Sign convention for the shear strain is based on which way that axis has to twist to have the right angle closed for a positive shear strain and open for a negative shear strain. The circle is centered at

$$\epsilon_{avg} = \frac{\epsilon_x + \epsilon_y}{2}$$

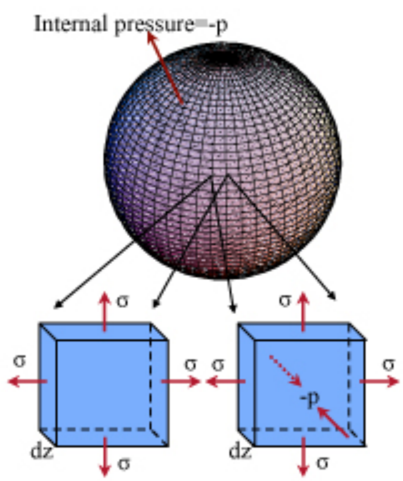
and $\gamma = 0$, with a radius $R = \gamma_{max}/2$. As with the stresses, the principal strains are located where the circle crosses the horizontal axis. Maximum shearing strains are located at the top and bottom of the circle.

Review

Thin-walled pressure vessels

Thin-walled pressure vessels are defined as having the ratio t/r ≤ 0.1, where t is the wall thickness and r is the internal radius of either the sphere or cylinder. The pressure, p, is the gauge pressure and the analysis is only safe for compressive internal pressures (tensile pressures can easily induce buckling in the vessel wall.) The analysis assumes that the in-plane stresses are uniform across the thickness of the wall. The radial stress is zero on the exterior surface and equal to −p on the interior surface.

Spherical vessels



Stresses in a Spherical Vessel: element on the outside surface (left) and on the inside surface (right)

For a spherical shell at any point and in any direction,

$$\sigma = \frac{pr}{t}; \quad \sigma_1 = \sigma_2 = \frac{pr}{2t}; \quad \epsilon_x = \epsilon_y = \frac{1}{E}\left[\frac{pr}{2t}(1-\nu)\right]$$

On the inside surface, $\sigma_3 = -p$ and on the outside surface, $\sigma_3 = 0$. For the maximum shear stress on the inside surface we have

$$\tau_{max} = \mathrm{Max}\left(\frac{\sigma_1-\sigma_2}{2}, \frac{\sigma_1-\sigma_3}{2}, \frac{\sigma_2-\sigma_3}{2}\right) = \frac{\sigma_1-\sigma_3}{2} = \frac{\frac{pr}{2t}-(-p)}{2} = \frac{pr}{4t}+\frac{p}{2}$$

On the outside surface

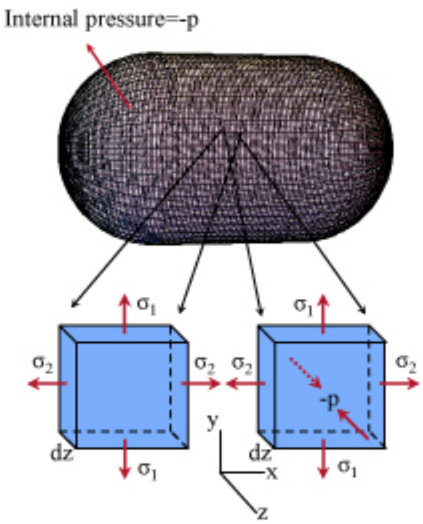
$$\tau_{max} = \frac{\sigma_1-\sigma_3}{2} = \frac{\frac{pr}{2t}-0}{2} = \frac{pr}{4t}$$

and the shear strains are expressed as

$$\epsilon_z = -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y)$$

for any x-y coordinate system in the plane of the surface.

Cylindrical vessels



Stresses in a cylindrical vessel: element on the outside surface (left) and on the inside surface (right).

The principal stresses in the plane of the shell are expressed as

$$\sigma_1 = \frac{pr}{t}; \quad \sigma_2 = \frac{pr}{2t}$$

On the inside surface, $\sigma_3 = -p$ and on the outside surface, $\sigma_3 = 0$. For the shear stresses,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\frac{pr}{t} - (-p)}{2} = \frac{pr}{2t} + \frac{p}{2}$$

and the strains are expressed as

$$\epsilon_x = \epsilon_1 = \frac{1}{E}\left[\frac{pr}{2t}(1 - 2\nu)\right], \quad \epsilon_y = \epsilon_2 = \frac{1}{E}\left[\frac{pr}{t}\left(1 - \frac{\nu}{2}\right)\right]$$

On the outside surface

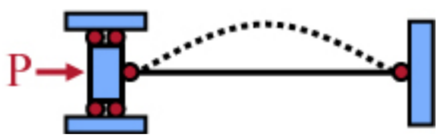
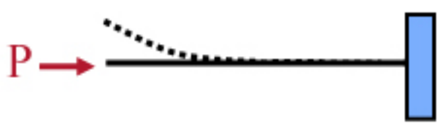
$$\epsilon_z = -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y)$$

Stability of columns-buckling

Columns are long, slender members under compressive axial loading. Column buckling is a stability problem, which means failure can occur without the material reaching the yield or ultimate stress. Columns are divided into three classes; slender, intermediate, and short, based on both material and slenderness ratio (L_e/r, i.e., length L_e defined in the following over r, r being the radius of gyration for the cross section $r = (I/A)^{1/2}$). The critical buckling load or stress for slender columns is given by Euler's buckling equation

$$P_{cr} = \frac{\pi^2 EI}{(L_e)^2}$$

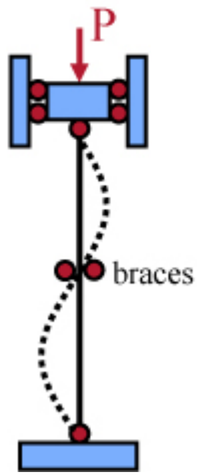
where L_e denotes the effective buckling length of the column and depends on the type of supports at its ends. The four common support combinations are (L denotes the actual length of the column):

	pinned-pinned (simply supported): L_e=L
	fixed-fixed (fixed end): L_e=L/2
	fixed-free (cantilever): L_e=2L
	fixed-pinned: L_e=0.7L

Higher buckling modes can be realized when certain kinematic restrictions are imposed on a column, e.g., by bracing the column at a certain location, thus restricting lateral motion at the bracing point. For such cases,

$$P_{cr} = \frac{n^2\pi^2 EI}{(L_e)^2}$$

where n is a positive integer denoting the buckling mode. The pinned-pinned column shown below is braced at midspan, so n = 2 in this case.



This simply supported column (pinned-pinned) is braced in the middle, thus it is forced to buckle in mode II, i.e., n = 2.

Often parts of a column can be assumed rigid, thus simplifying the calculation of the critical buckling load. This assumption also serves for understanding the buckling instability concept.

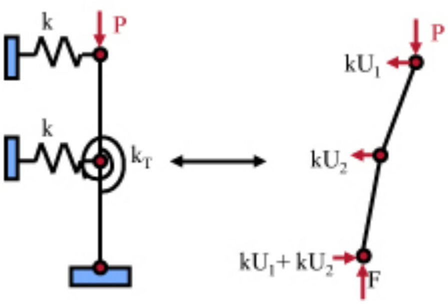


Figure Schematic of a column assumed to be piece-wise rigid. Under such assumptions, equilibrium equations of the various pieces can yield the critical buckling load.

TABLE OF MATERIAL PROPERIES-SI UNITS

Material – SI Units	**Young's Modulus** E	**Yield Stress** σ_y	**Ultimate Stress** σ_U	**Shear Modulus** G	**Poisson's ratio** ν	**Density** ρ	**Coeff. Thermal Exp.** α
	(GPa)	**(MPa)**	**(MPa)**	**(GPa)**		**(kg/m^3)**	**10^{-6} / ^{o}C**
Aluminum (99%)	70	35	90	28	0.33	2,800	23
Aluminum 6061 - T6	70	270	310	28	0.33	2,800	23
Aluminum 7075 - T6	72	500	570	28	0.33	2,800	23
Brass (annealed)	100	95	315	40	0.34	8,400	20
Brass (cold worked)	100	410	500	40	0.34	8,400	20
Cast Iron	90	170	205	36	0.25	7,200	10
Concrete (in compression)	17 - 31		10 - 70		0.1 - 0.2	1,100 - 2,400	7 - 14
Copper	117	70	220	45	0.34	8,900	16.6 - 17.6
Nickel	210	100 - 620	310 - 750	80	0.31	8,800	13
Nylon	2.8	80				880 - 1,100	70 - 140
Polyethylene (low density)	0.17		13			960 - 1,400	140 - 290
Polyethylene (high density)	0.82		26			990 - 1,440	140 - 290
Plexiglas	2.9		52			1,250	140 - 290
Polyvinyl chloride (rigid)	0.0024 - 0.004		40 - 52			1,410	
Rubber	0.0007 - 0.004	1.3 - 7	6.8 - 20	0.0002 - 0.001	0.45 - 0.49	960 - 1,300	130 - 200
Steel, type 1015 (0.15% C) (hot finished)	200	180	340	75	0.33	7,850	
Steel, type 1030 (0.30% C) (hot finished)	200	250	470	75	0.33	7,850	
Steel, type 1050 (0.50% C) (hot finished)	200	340	620	75	0.33	7,850	
Steel, type 304 stainless (annealed)	190	240	580	70	0.33	7,900	17
Steel, type 304 stainless steel (cold worked)	190	510	750	70	0.33	7,900	
Titanium alloys	100 - 120	750 - 1000	900 - 1170	39 - 44	0.33	4,500	8.1 - 11
Wood, Softwood (Douglas fir, air dried)	10	65				480 - 720	
across grain		2.7					
Wood, Hardwood (white oak, air dried)	16	70				900 - 1,400	
across grain		5.5					

TABLE OF MATERIAL PROPERIES-US UNITS

Material – US Customary Units	**Young Modulus** E	**Yield Stress** σ_y	**Ultimate Stress** σ_U	**Shear Modulus G**	**Poisson ratio** ν	**Density** ρ	**Coeff. Thermal Exp.** α
	(10^3 psi)	(10^3 psi)	(10^3 psi)	(10^3 psi)		(lb/in^3)	$10^{-6}/^oF$
Aluminum (99%)	10,000	5	13	4,000	0.33	0.1	13
Aluminum 6061 - T6	10,000	40	45	4,000	0.33	0.1	13
Aluminum 7075 - T6	10,400	73	83	4,000	0.33	0.1	13
Brass (annealed)	15,000	14	46	6,000	0.34	0.306	11
Brass (cold worked)	15,000	60	74	6,000	0.34	0.306	11
Cast Iron	13,000	25	30	5,200	0.25	0.253	5.5 - 6.6
Concrete (in compression)	2,500-4,500		1.5 - 10.0		0.1 - 0.2	0.040.08	4 - 8
Copper	17,000	10	32	6,500	0.34	0.323	9.2 - 9.8
Nickel	30,000	15 - 90	45 - 110	11,400	0.31	0.318	7.2
Nylon	410	11.8				0.04	40 - 80
Polyethylene (low density)	25		2			0.033	80 - 160
Polyethylene (high density)	120		4			0.034	80 - 160
Plexiglas	420		8			0.043	80 - 160
Polyvinyl chloride (rigid)	350 - 600		6.0 - 8.0			0.05	
Rubber	0.1 - 0.6	0.2 - 1.0	1.0 - 3.0	0.03 - 0.2	0.450.49	0.034-0.046	70 - 110
Steel, type 1015 (0.15% C) (hot finished)	29,000	27	50	11,000	0.33	0.284	
Steel, type 1030 (0.30% C) (hot finished)	29,000	37	68	11,000	0.33	0.284	
Steel, type 1050 (0.50% C) (hot finished)	29,000	50	90	11,000	0.33	0.284	
Steel, type 304 stainless (annealed)	28,000	35	85	10,000	0.33	0.286	9.6
Steel, type 304 stainless steel (cold worked)	28,000	75	110	10,000	0.33	0.286	
Titanium alloys	15,000-17,000	110 - 150	130 - 170	5,600-6,400	0.33	0.16	4.5 - 6.0
Wood, Softwood (Douglas fir, air dried)	1,400	9.5				0.011	
across grain		0.4					
Wood, Hardwood (white oak, air dried)	2,300	10.2				0.026	
across grain		0.8					

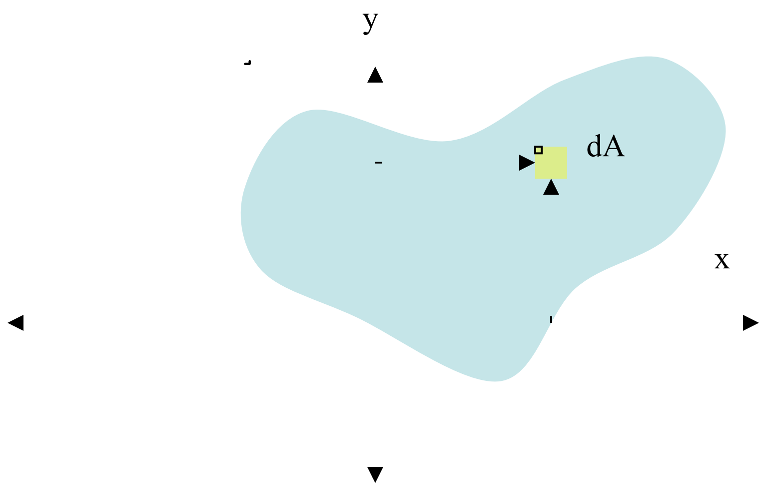
Centroid -1st Area Moment

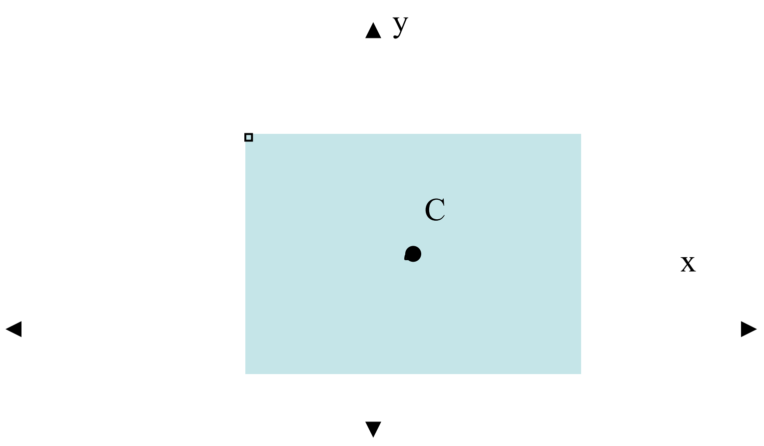
- Locate the center of an area

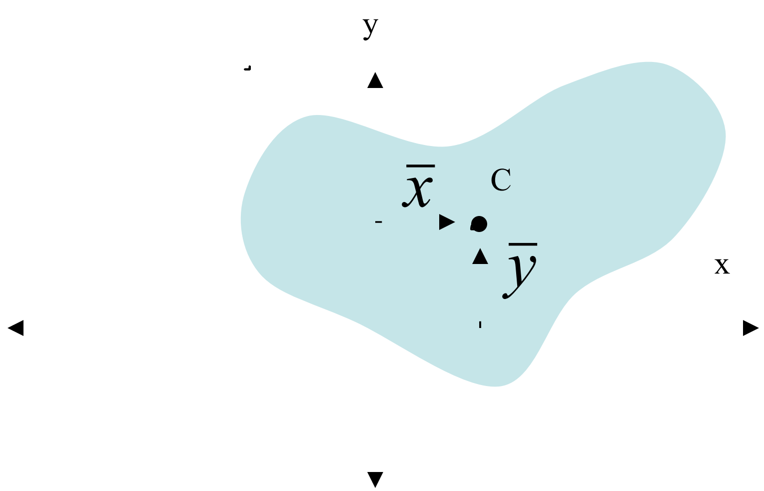
$$\int_A y\,dA = A\bar{y}$$

$$\int_A x\,dA = A\bar{x}$$

- For a symmetric (e.g. rectangular) cross section, centroid is at the symmetry axis/axes







Moment of Inertia - 2nd Area Moment

- Area moment of inertia defined as

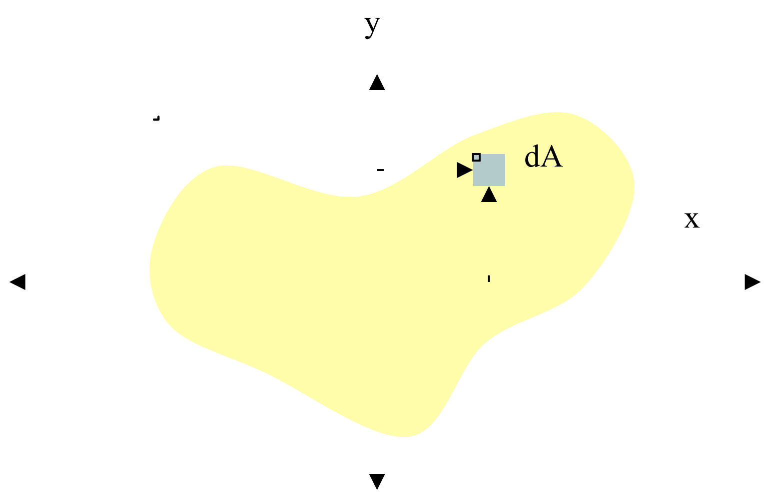
$$I_x = \int_A y^2\, dA$$

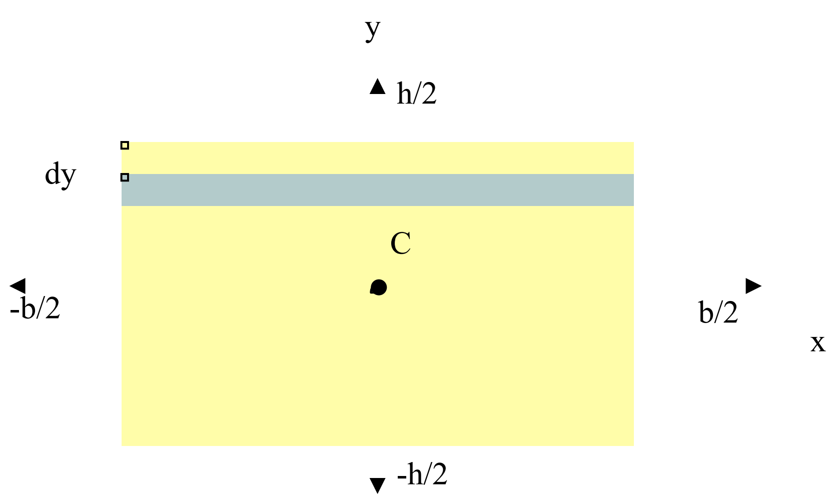
$$I_y = \int_A x^2\, dA$$

- For rectangular cross-section (b=base, h=height)

$$I_x = \int_{-h/2}^{+h/2} y^2 b\, dy$$

$$I_x = \frac{1}{12} b h^3$$





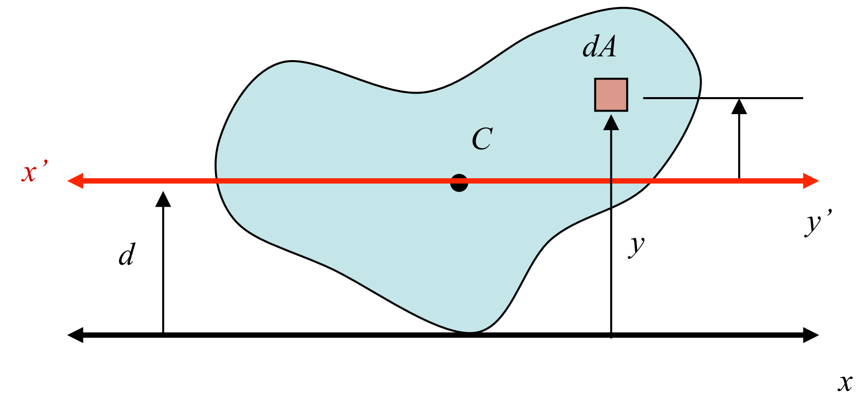
Moments of Composite Areas

Parallel axes theorem

$$I_x = \int_A y^2\, dA = \int_A \left(y' + d\right)^2 dA$$

$$I_x = \int_A y'^2\, dA + 2d\int_A y'\, dA + d^2 \int_A dA$$

$$I_x = I_{x'} + Ad^2$$

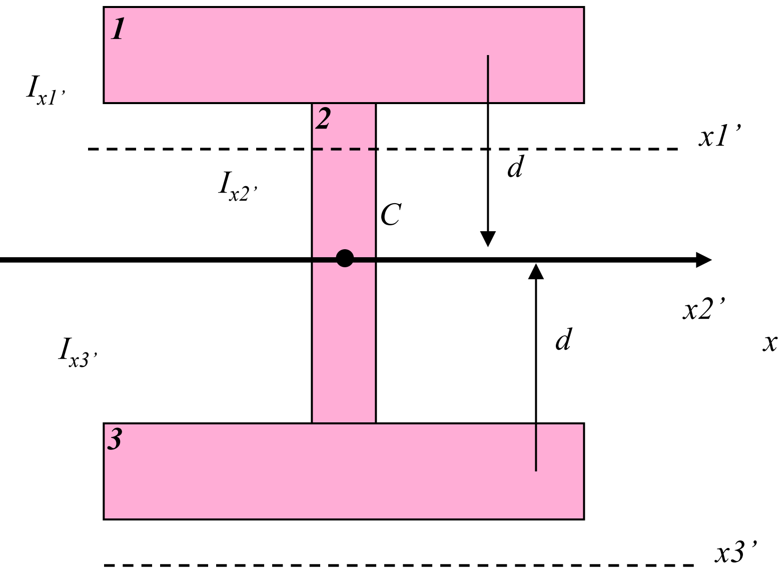


Where $I_{x'}$ is the moment of inertia of the area about its centroid

For composite areas

- *Identify the centroid of the whole*
- *Find centroidal moments of each part*
- *Use parallel axis theorem to find total moment about the centroidal axis of the whole*

$$I_x = I_{x1'} + A_1 \cdot d^2 + I_{x2'} + I_{x3'} + A_3 \cdot d^2$$



Composite Areas

For centroid:

$$y_0 = \frac{b_2 h_2 \dfrac{h_2}{2} + bh_1\left(h_2 + \dfrac{h_1}{2}\right)}{b_2 h_2 + bh_1}$$

Combined moment of inertia is:

$$I_x = \frac{1}{12} bh_1^3 + bh_1 d_1^2 + \frac{1}{12} b_2 h_2^3 + b_2 h_2 d_2^2$$