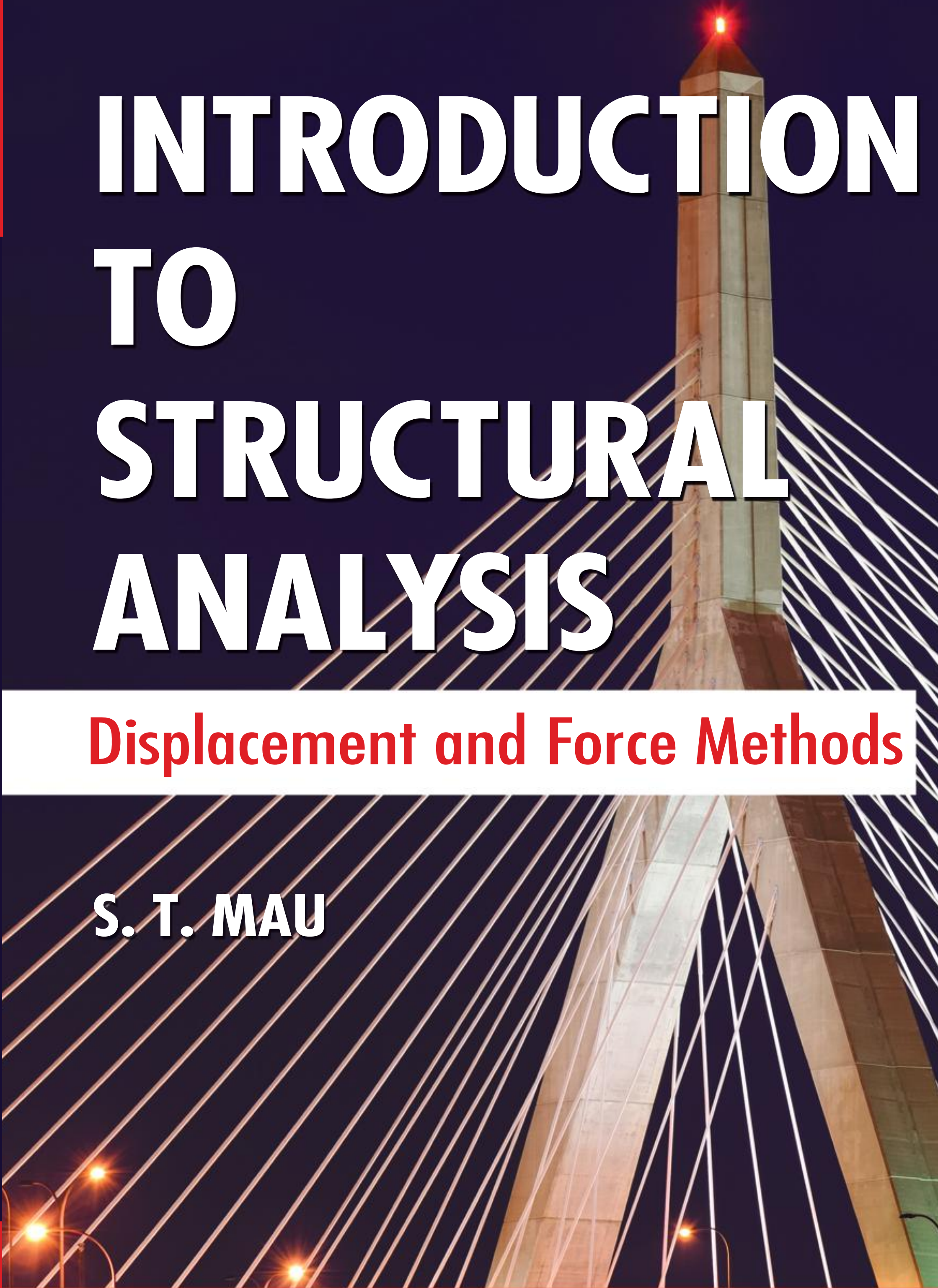

INTRODUCTION TO STRUCTURAL ANALYSIS

Displacement and Force Methods

S. T. MAU

CRC Press
Taylor & Francis Group

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Preface

There are two new developments in the last 30 years in the civil engineering curricula that have a direct bearing on the design of the content of a course in structural analysis: the reduction of credit hours to three required hours in structural analysis in most civil engineering programs and the increasing gap between what is taught in textbooks and classrooms and what is being practiced in engineering firms. The former is brought about by the recognition of civil engineering educators that structural analysis as a required course for all civil engineering majors need not cover in great detail all the analytical methods. The latter is certainly the result of the ubiquitous applications of personal digital computers and handheld devices.

This structural analysis text is designed to bridge the gap between engineering practice and education. Acknowledging the fact that virtually all computer structural analysis programs are based on the matrix displacement method of analysis, the text begins with the matrix displacement method. A matrix operations tutorial is included as a review and a self-learning tool. To minimize the conceptual difficulty a student may have with the displacement method, it is introduced with the plane truss analysis, where the concept of nodal displacement is presented. Introducing the matrix displacement method early also makes it easier for students to work on term project assignments that involve the utilization of computer programs.

The force method of analysis for plane trusses is then introduced to provide the coverage of force equilibrium, deflection, statistical indeterminacy, and so forth, that are important in the understanding of the behavior of a structure and the development of a feel for it.

The force method of analysis is then extended to beam and rigid frame analysis, almost in parallel to the topics covered in truss analysis. The beam and rigid frame analysis is presented in an integrated way so that all the important concepts are covered concisely without undue duplicity.

The displacement method then reappears when the moment distribution and slope-deflection methods are presented as a prelude to the matrix displacement method for beam and rigid frame analysis. The matrix displacement method is presented as a generalization of the slope-deflection method.

The aforementioned description outlines the introduction of the two fundamental methods of structural analysis, the displacement method and the force method, and their applications to the two groups of structures, trusses, and beams and rigid frames. Other related topics such as influence lines, non-prismatic members, composite structures, secondary stress analysis, and limits of linear and static structural analysis are presented at the end.

Acknowledgments

I wish to express my gratitude to Professor C. C. Yu (former president of National Taiwan University), whose love of education and structural theory directed my career, and to the late Professor Yuan Yu Hsieh, whose innovation in the best-selling, groundbreaking book *Elementary Theory of Structures* (Prentice Hall, 1970) inspired my approach in writing this book. I wish to thank many of my dedicated colleagues in the universities I have served and those I have met and gotten to know well professionally who have become my role models for excellent teachers. Thanks are due to my many good friends whose talents and intellect set the standard for my aspiration.

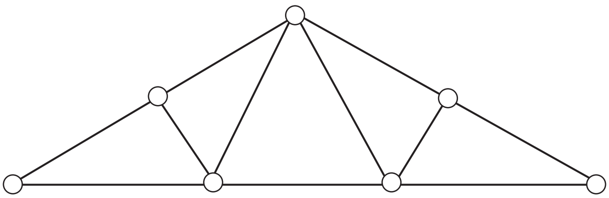
I am forever in debt to my late mother, Kwei-Lan Liu Mau, who went through the hardship of war and was widowed early but determined to raise her only son to a productive life. My deep gratitude to my dear wife, Sein-ming Pei Mau, who sacrificed her own career to support mine, and through thick and thin has stood by me to nurture and guide me. I am blessed with two wonderful sons, Ted and Mike, and daughter-in-law Sarah; their love keeps me young. Last but not the least, my three young grandchildren, Jeremy, Abigail, and Nathaniel, are my constant sources of joy.

1

Truss Analysis: Matrix Displacement Method

1.1 What Is a Truss?

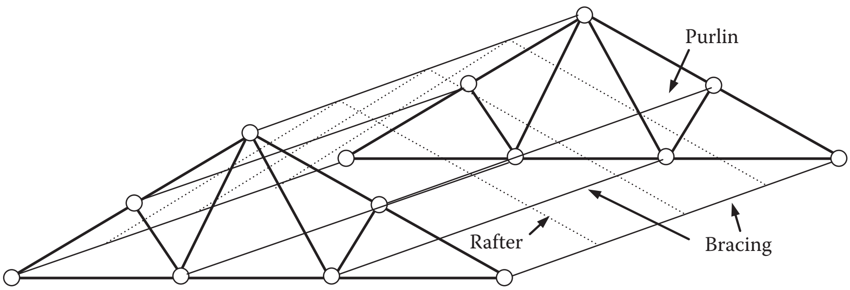
In a plane, a truss is composed of relatively slender members often forming triangular configurations. An example of a plane truss used in the roof structure of a house is shown in the following figure.



A roof truss called Fink truss.

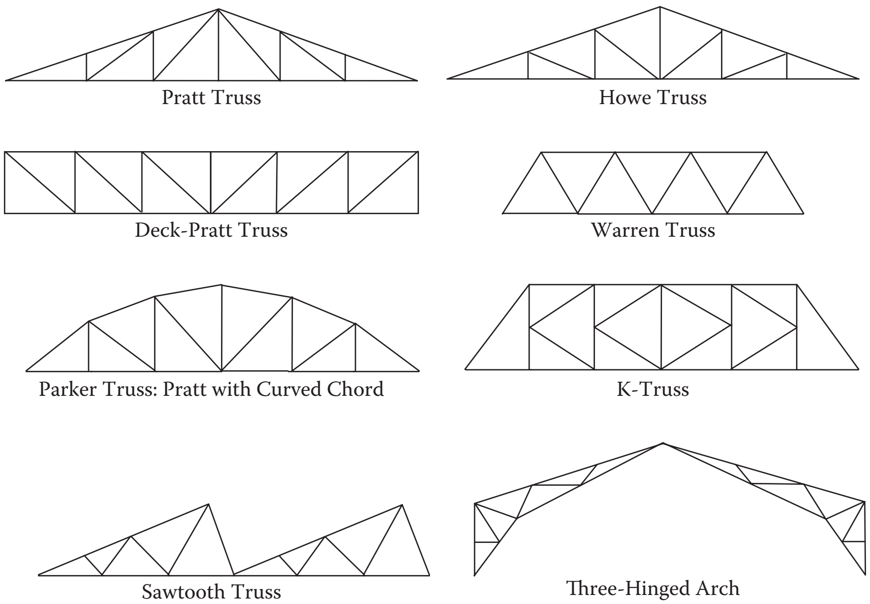
The circular symbol in the figure represents a type of connection called hinge, which allows members to rotate in the plane relative to each other at the connection but not to move in translation against each other. A hinge connection transmits forces from one member to the other but not force couple, or moment, from one member to the other.

In real construction, a plane truss is most likely a part of a structure in the three-dimensional space we know. An example of a roof structure is shown next. The bracing members are needed to connect two plane trusses together. The purlins and rafters are for the distribution of roof load to the plane trusses.



A roof structure with two Fink trusses.

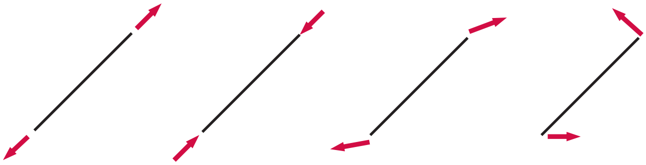
Some other truss types seen in roof or bridge structures are shown next.



Different types of plane trusses.

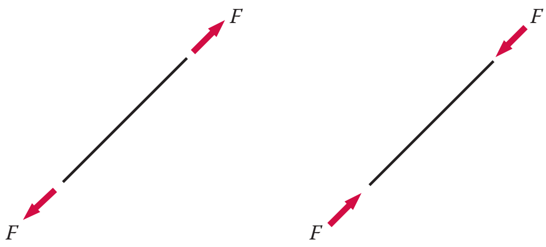
1.2 A Truss Member

Each member of a truss is a straight element, taking loads only at the two ends. As a result, the two forces at the two ends must act along the axis of the member and of the same magnitude in order to achieve equilibrium of the member as shown in the following figure.



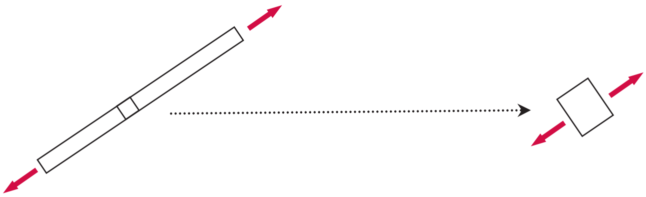
Truss member in equilibrium (left). Truss member not in equilibrium (right).

Furthermore, when a truss member is in equilibrium, the two end forces are either pointing away from each other or against each other, creating tension or compression, respectively, in the member.



Truss member in tension (left). Truss member in compression (right).

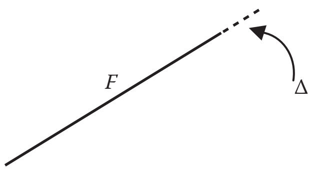
Whether a member is in tension or compression, the internal force acting on any chosen section of the member is the same throughout the member. Thus, the state of force in the member can be represented by a single member force entity, represented by the notation *F*, which is the axial member force of a truss member. There are no other member forces in a truss member.



The internal force is the same at any section of a truss member.

A tensile member force is signified by a positive value in *F* and a compressive member force is signified by a negative value in *F*. This is the sign convention for the member force of an axial member.

Whenever there is force in a member, the member will deform. Each segment of the member will elongate or shorten and the cumulative effect of the deformation is a member elongation, or shortening, Δ.



Member elongation.

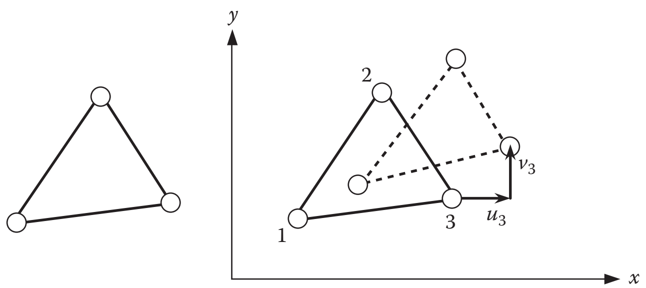
Assuming the material the member is made of is linearly elastic with Young's modulus *E*, and the member is prismatic with a constant cross-sectional area, *A*, and length, *L*, then the relationship between the member elongation and member force can be shown to be:

$$F = k \quad \text{with } k = \frac{EA}{L} \tag{1.1}$$

where the proportional factor k is called the member rigidity. Equation 1.1 is the member stiffness equation expressed in local coordinate, namely the axial coordinate. This relationship will eventually be expressed in a coordinate system that is common to all members in a truss, that is, a global coordinate system. For this to be done, we must examine the relative position of a member in the truss.

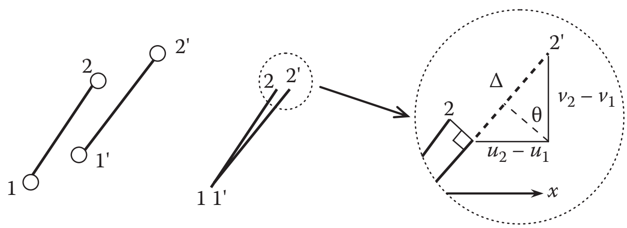
1.3 Member Stiffness Equation in Global Coordinates

The simplest truss is a three-member truss as shown. Once we have defined a global coordinate system, the x,y system, then the displaced configuration of the whole structure is completely determined by the nodal displacement pairs (u_1, v_1), (u_2, v_2), and (u_3, v_3).



A three-member truss (left). Nodal displacements in global coordinates (right).

Furthermore, the elongation of a member can be calculated from the nodal displacements.



Displaced member (left). Overlapped configurations (right).

$$\Delta = (u_2 - u_1)\ \text{Cos}\theta + (v_2 - v_1)\ \text{Sin}\theta \tag{1.2a}$$

or

$$\Delta = -(\text{Cos}\theta)u_1 - (\text{Sin}\theta)v_1 + (\text{Cos}\theta)u_2 + (\text{Sin}\theta)v_2 \tag{1.2b}$$

In Equation 1.2, it is understood that the angle θ refers to the orientation of members 1 and 2. For brevity we did not include the subscript that designates the member. We can express the same equation in a matrix form, letting C and S represent Cosθ and Sinθ, respectively.

$$= \begin{matrix} -C & -S & C & S \end{matrix} \quad \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix} \tag{1.3}$$

Again, the subscripts 1 and 2 are not included for Δ, C, and S for brevity. One of the advantages of using the matrix form is that the functional relationship between the member elongation and the nodal displacement is clearer than that in Equation 1.2. Thus, the above equation can be cast as a transformation between the local quantity of deformation $\Delta_L = \Delta$ and the global nodal displacements Δ_G:

$$\boldsymbol{\Delta}_L = \boldsymbol{\Gamma}\, \boldsymbol{\Delta}_G \tag{1.4}$$

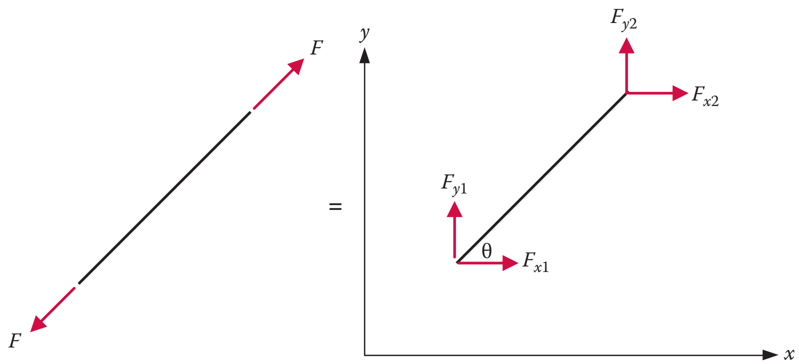
where

$$\boldsymbol{\Gamma} = \begin{matrix} -C & -S & C & S \end{matrix} \tag{1.5}$$

and

$$_G = \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix} \tag{1.6}$$

Here and elsewhere a boldfaced symbol represents a vector or a matrix. Equation 1.4 is the *deformation transformation equation*. We now seek the transformation between the member force in local coordinate, $\boldsymbol{F}_L = F$ and the nodal forces in the x,y coordinates, $\boldsymbol{F}_G$.



Member force F and nodal forces in global coordinates.

From the preceding figure and the equivalence of the two force systems, we obtain

$$\boldsymbol{F}_G = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \begin{Bmatrix} -C \\ -S \\ C \\ S \end{Bmatrix} F \tag{1.7}$$

where C and S represent the cosine and sine of the member orientation angle θ. Noting that the transformation vector is the transpose of Γ, we can rewrite Equation 1.7 as

$$\boldsymbol{F}_G = \Gamma^T \boldsymbol{F}_L \tag{1.8}$$

Equation 1.8 is the force transformation equation.

By simple substitution, using Equation 1.1 and Equation 1.4, the force transformation equation leads to

$$\boldsymbol{F}_G = \Gamma^T \boldsymbol{F}_L = \Gamma^T k \Delta_L = \Gamma^T k \Gamma \, \Delta_G$$

or

$$\boldsymbol{F}_G = \boldsymbol{k}_G \Delta_G \tag{1.9}$$

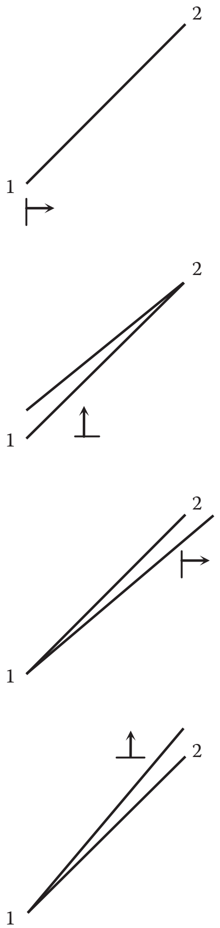
where

$$\boldsymbol{k}_G = \Gamma^T k \Gamma \tag{1.10}$$

Equation 1.10 is the *stiffness transformation equation*, which transforms the member stiffness in local coordinate, k, into the member stiffness in global coordinate, $\boldsymbol{k}_G$. In the expanded form, that is, when the triple multiplication in Equation 1.10 is carried out, the member stiffness is a 4 × 4 matrix:

$$\boldsymbol{k}_G = \frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} \tag{1.11}$$

The meaning of each of the components of the matrix, $(k_G)_{ij}$, can be explored by considering the nodal forces corresponding to the four sets of "unit" nodal displacements in the following figure.



$$\boldsymbol{\Delta}_G = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \boldsymbol{\Delta}_G = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \qquad \boldsymbol{\Delta}_G = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \qquad \boldsymbol{\Delta}_G = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

Four sets of unit nodal displacements.

When each of the "unit" displacement vectors is multiplied by the stiffness matrix according to Equation 1.9, it becomes clear that the resulting nodal forces are identical to the components of one of the columns of the stiffness matrix. For example, the first column of the stiffness matrix contains the nodal forces needed to produce a unit displacement in u_1, with all other nodal displacements being zero. Furthermore, we can see $(k_G)_{ij}$ is the ith nodal force due to a unit displacement at the jth nodal displacement.

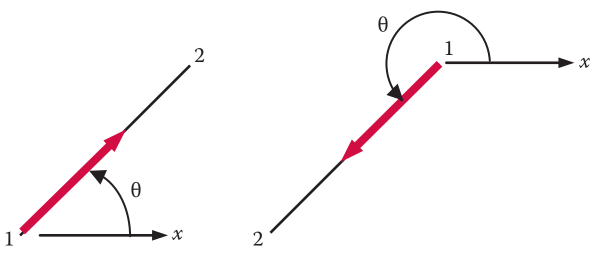
By examining Equation 1.11, we observe the following features of the stiffness matrix:

a. The member stiffness matrix is symmetric, $(k_G)_{ij} = (k_G)_{ji}$.
b. The algebraic sum of the components in each column or each row is zero.
c. The member stiffness matrix is singular.

Feature (a) can be traced to the way the matrix is formed, via Equation 1.10, which invariably leads to a symmetric matrix. Feature (b) comes from the fact that nodal forces due to a set of unit nodal displacements must be in equilibrium. Feature (c) is due to the proportionality of the pair of columns 1 and 3, or 2 and 4.

The fact that member stiffness matrix is singular and therefore cannot be inverted indicates that we cannot solve for the nodal displacements corresponding to any given set of nodal forces. This is because the given set of nodal forces may not be in equilibrium and therefore it is not meaningful to ask for the corresponding nodal displacements. Even if they are in equilibrium, the solution of nodal displacements requires a special procedure described under "eigenvalue problems" in linear algebra. We shall not explore such possibilities herein.

In computing the member stiffness matrix, we need to have the member length, L, the member cross-section area, A, the Young's modulus of the member material, E, and the member orientation angle, θ. The member orientation angle is measured from the positive direction of the x-axis to the direction of the member following a clockwise rotation. The member direction is defined as the direction from the starting node to the end node. In the following figure, the orientation angles for the two members differ by 180 degrees if we consider node 1 as the starting node and node 2 as the end node. In the actual computation of the stiffness matrix, however, such distinction in the orientation angle is not necessary because we do not need to compute the orientation angle directly, as will become clear in the following example.



Member direction is defined from the starting node to the end node.

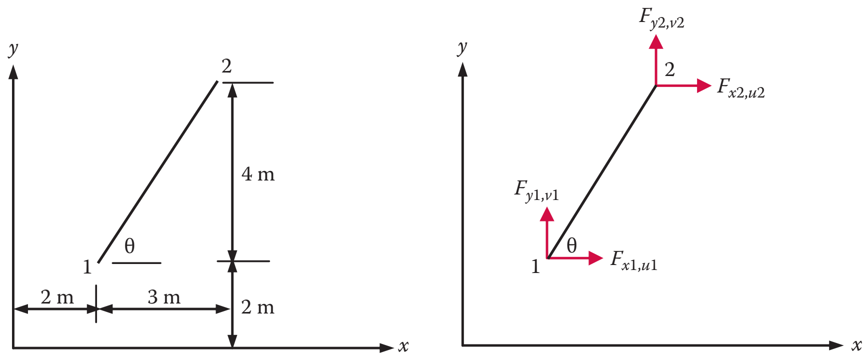
Equation 1.9 can now be expressed in its explicit form as

$$\begin{matrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{matrix} = \begin{matrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{matrix} = \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix} \tag{1.12}$$

where the stiffness matrix components, k_{ij}, are given in Equation 1.11.

Example 1.1

Consider a truss member with E = 70 GPa, A = 1430 mm^2, L = 5 m, and orientated as shown in the following figure. Establish the member stiffness matrix.



A truss member and its nodal forces and displacements.

Solution

The stiffness equation of the member can be established by the following procedures.

a. Define the starting and end nodes.
   Starting Node: 1. End Node: 2.
b. Find the coordinates of the two nodes.
   Node 1: $(x_1, y_1) = (2,2)$
   Node 2: $(x_2, y_2) = (5,6)$
c. Compute the length of the member and the cosine and sine of the orientation angle.

$$L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{3^2 + 4^2} = 5$$

$$C = Cos\theta = \frac{(x_2 - x_1)}{L} = \frac{(\ \ x)}{L} = \frac{3}{5} = 0.6$$

$$S = \text{Sin}\theta = \frac{(y_2 - y_1)}{L} = \frac{(\ \ y)}{L} = \frac{4}{5} = 0.8$$

d. Compute the member stiffness factor.

$$\frac{EA}{L} = \frac{(70 \times 10^9)(0.00143)}{5} = 20 \times 10^6 \text{ N/m} = 20 \text{ MN/m}$$

e. Compute the member stiffness matrix.

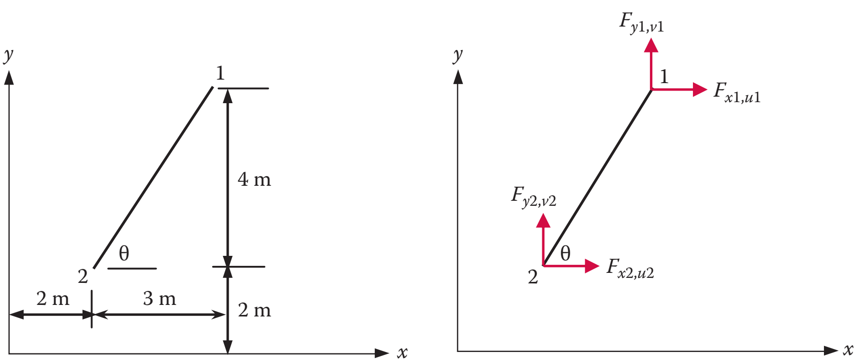
$$\boldsymbol{K_G} = \frac{EA}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix} = \begin{bmatrix} 7.2 & 9.6 & -7.2 & -9.6 \\ 9.6 & 12.8 & -9.6 & -12.8 \\ -7.2 & -9.6 & 7.2 & 9.6 \\ -9.6 & -12.8 & 9.6 & 12.8 \end{bmatrix}$$

f. Establish the member stiffness equation in global coordinates according to Equation 1.12.

$$\begin{bmatrix} 7.2 & 9.6 & -7.2 & -9.6 \\ 9.6 & 12.8 & -9.6 & -12.8 \\ -7.2 & -9.6 & 7.2 & 9.6 \\ -9.6 & -12.8 & 9.6 & 12.8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix}$$

PROBLEM 1.1

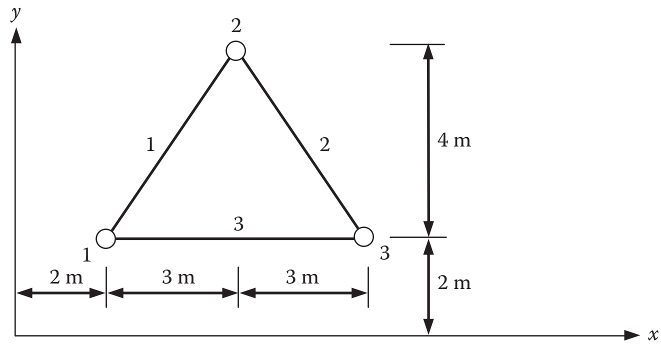
Consider the same truss member with $E = 70$ GPa, $A = 1430$ mm^2, and $L = 5$ m as in Example 1.1, but designate the starting and ending nodes differently as shown in the following figure. Compute the member stiffness matrix components (a) k_{11}, (b) k_{12}, and (c) k_{13} and find the corresponding quantity in Example 1.1. What is the effect of the change of the numbering of nodes on the stiffness matrix components?



Problem 1.1

1.4 Unconstrained Global Stiffness Equation

Consider the following three-bar truss with $E = 70$ GPa, $A = 1430$ mm^2 for each member. This is a truss yet to be supported and loaded, but we can establish the global stiffness equation with the global coordinate system shown. Since the truss is not constrained by any support and load, the stiffness equation is called the unconstrained stiffness equation.



An unconstrained truss in a global coordinate system.

We will show that the unconstrained global stiffness equation for the truss in the preceding figure is

$$\begin{bmatrix} 23.9 & 9.6 & -7.2 & -9.6 & -16.6 & 0 \\ 9.6 & 12.8 & -9.6 & -12.8 & 0 & 0 \\ -7.2 & -9.6 & 14.4 & 0 & -7.2 & 9.6 \\ -9.6 & -12.8 & 0 & 25.6 & 9.6 & -12.8 \\ -16.6 & 0 & -7.2 & 9.6 & 23.9 & 19.6 \\ 0 & 0 & 9.6 & -12.8 & -9.6 & 12.8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \end{Bmatrix}$$

where the six-component displacement vector contains the nodal displacements and the six-component force vector on the right-hand side (RHS) contains the externally applied forces at the three nodes. The 6×6 matrix is called the unconstrained global stiffness matrix. The derivation of the expression of the matrix is given next. The displacements are expressed in the unit of meter (m) and the forces are in meganewton (MN).

Equilibrium equations at nodes. What makes the three-bar assembly into a single truss is the fact that the three bars are connected by hinges at the nodes numbered in the preceding figure. This means that (a) the bars joining at a common node share the same nodal displacements and (b) the forces acting on each of the three nodes are in equilibrium with any externally applied forces at each node. The former is called the condition of compatibility and the latter is called the condition of equilibrium. The condition of compatibility is automatically satisfied by the designation of the following six nodal displacements:

$$= \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{matrix} \tag{1.13}$$

where each pair of the displacements (u,v) refers to the nodal displacements at the respective nodes. The condition of compatibility implies that the displacements at the ends of each member are the same as the displacements at the connecting nodes. In fact, we can number the members as shown in the preceding figure and designate the starting and end nodes of each member as in the following table.

Starting and End Node Numbers

Member	**Starting Node**	**End Node**
1	1	2
2	2	3
3	1	3

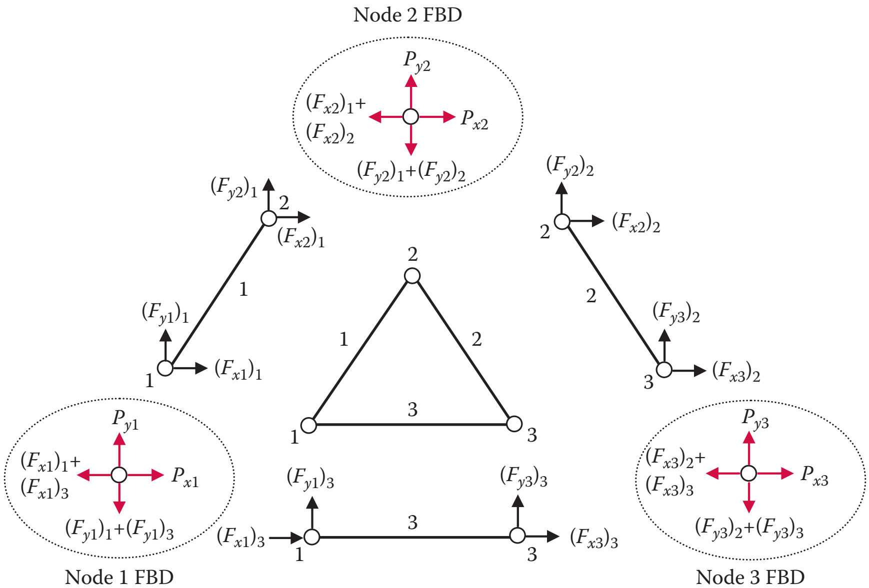
Then, we can establish the following correspondence between the four nodal displacements of each member (local) and the six nodal displacements of the whole structure (global).

Corresponding Global DOF Numbers

	Global Number		
Local Number	**Member 1**	**Member 2**	**Member 3**
1	1	3	1
2	2	4	2
3	3	5	5
4	4	6	6

Note that we use the terminology of DOF, which stands for degrees of freedom. For the entire truss, the configuration is completely defined by the six displacements in Equation 1.13. Thus, we state that the truss has six degrees of freedom. Similarly, we may state that each member has four DOFs, since each node has two DOFs and there are two nodes for each member. We may also use the way each of the DOF is sequenced to refer to a particular DOF. For example, the second DOF of member 2 is the fourth DOF in the global nodal displacement vector. Conversely, the third DOF in the global DOF nodal displacement vector is u_2 according to Equation 1.13 and it shows up as the third DOF of member 1 and first DOF of member 2 according to the previous table. This table will be very useful in assembling the unconstrained global stiffness matrix as will be seen later.

The unconstrained global stiffness equation is basically equilibrium equations expressed in terms of nodal displacements. From the layout of the three-bar truss and Equation 1.13, we can see that there are six nodal displacements or six DOFs, two from each of the three nodes. We can see from the following figure that there will be exactly six equilibrium equations, two from each of the three nodes.



Free-body diagrams of nodes and members.

The preceding figure, as complicated as it seems, is composed of three parts. At the center is a layout of the truss as a whole. The three FBDs (free-body diagrams) of the members are the second part of the figure. Note that we need not be concerned with the equilibrium of each member because the forces at the member ends will be generated from the member stiffness equation, which guarantees that the member equilibrium conditions are satisfied. The third part, the FBDs encircled by dashed lines, is the part we need to examine to find the six nodal equilibrium equations. In each of the nodal FBDs, the externally applied nodal forces are represented by the symbol P, whereas the other forces are the internal forces forming a pair with the respective nodal forces acting at the end of each member. The subscript outside the parentheses of these forces indicates the member number.

From the three nodal FBDs and noting that the nodal force vector has six components, we can easily arrive at the following six equilibrium equations expressed in matrix form:

$$\boldsymbol{P} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ 0 \\ 0 \end{Bmatrix}_1 + \begin{Bmatrix} 0 \\ 0 \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix}_2 + \begin{Bmatrix} F_{x1} \\ F_{y1} \\ 0 \\ 0 \\ F_{x3} \\ F_{y3} \end{Bmatrix}_3 \tag{1.14}$$

where the subscript outside each vector on the RHS indicates the member number. Each of the vectors at the RHS, however, can be expressed in terms of their respective nodal displacement vector using Equation 1.12, with the nodal forces and displacements referring to the global nodal force and displacement representation:

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}_1 \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}_2 \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x3} \\ F_{y3} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}_3 \begin{Bmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{Bmatrix}$$

Each of the previous equations can be expanded to fit the form of Equation 1.14:

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ 0 \\ 0 \end{Bmatrix}_1 = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & 0 & 0 \\ k_{21} & k_{22} & k_{23} & k_{24} & 0 & 0 \\ k_{31} & k_{32} & k_{33} & k_{34} & 0 & 0 \\ k_{41} & k_{42} & k_{43} & k_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_1 \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 \\ 0 \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ F_{y3} \end{Bmatrix}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_{11} & k_{12} & k_{13} & k_{14} \\ 0 & 0 & k_{21} & k_{22} & k_{23} & k_{24} \\ 0 & 0 & k_{31} & k_{32} & k_{33} & k_{34} \\ 0 & 0 & k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}_2 \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ 0 \\ 0 \\ F_{x3} \\ F_{y3} \end{Bmatrix}_3 = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 & k_{13} & k_{14} \\ k_{21} & k_{22} & 0 & 0 & k_{23} & k_{24} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ k_{31} & k_{32} & 0 & 0 & k_{33} & k_{34} \\ k_{41} & k_{42} & 0 & 0 & k_{43} & k_{44} \end{bmatrix}_3 \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

When each of the RHS vectors in Equation 1.14 is replaced by the RHS of the previous three equations, the resulting equation is the unconstrained global stiffness equation,

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ P_{x2} \\ P_{y2} \\ P_{x3} \\ P_{y3} \end{Bmatrix} \quad (1.15)$$

where the components of the unconstrained global stiffness matrix, K_{ij}, is the superposition of the corresponding components in each of the three expanded stiffness matrices in the previous equations.

In actual computation, it is not necessary to expand the stiffness equation in Equation 1.12 into the six-equation form as we did earlier. That was necessary only for the understanding of how the results are derived. We can use the local-to-global DOF relationship in the global DOF table and place the member stiffness components directly into the global stiffness matrix. For example, component (1,3) of the member-2 stiffness matrix is added to component (3,5) of the global stiffness matrix. This simple way of assembling the global stiffness matrix is called the *direct stiffness method*.

To carry out the aforementioned procedures numerically, we need to use the dimension and member property given at the beginning of this section to arrive at the stiffness matrix for each of the three members:

$$(\mathbf{k}_G)_1 = \left[\frac{EA}{L}\right]_1 \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}_1 = \begin{bmatrix} 7.2 & 9.6 & -7.2 & -9.6 \\ 9.6 & 12.8 & -9.6 & -12.8 \\ -7.2 & -9.6 & 7.2 & 9.6 \\ -9.6 & -12.8 & 9.6 & 12.8 \end{bmatrix}$$

$$(\mathbf{k}_G)_2 = \left[\frac{EA}{L}\right]_2 \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}_2 = \begin{bmatrix} 7.2 & -9.6 & -7.2 & 9.6 \\ -9.6 & 12.8 & -9.6 & -12.8 \\ -7.2 & -9.6 & 7.2 & -9.6 \\ 9.6 & -12.8 & -9.6 & 12.8 \end{bmatrix}$$

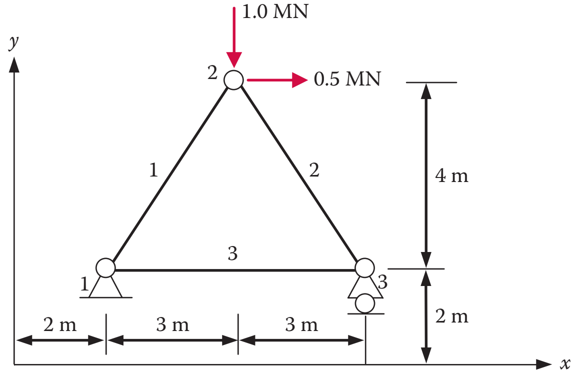
$$(\mathbf{k}_G)_3 = \left[\frac{EA}{L}\right]_3 \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}_3 = \begin{bmatrix} 16.7 & 0 & -16.7 & 0 \\ 0 & 0 & 0 & 0 \\ -16.7 & 0 & 16.7 & 0 \\ 0 & 0 & 9.6 & 0 \end{bmatrix}$$

When the three member stiffness matrices are assembled according to the direct stiffness method, the unconstrained global stiffness equation given at the beginning of this section is obtained. For example, the unconstrained global stiffness matrix component k_{34} is the superposition of $(k_{34})_1$ of member 1 and $(k_{12})_2$ of member 2. Note that the unconstrained global stiffness matrix has the same features as the member stiffness matrix: symmetric and singular, and so forth.

1.5 Constrained Global Stiffness Equation and Its Solution

Example 1.2

Now consider the same three-bar truss as shown before with $E = 70$ GPa and $A = 1430\ \text{mm}^2$ for each member but with the support and loading conditions added.



A constrained and loaded truss in a global coordinate system.

Solution

The support conditions are $u_1 = 0$, $v_1 = 0$, and $v_3 = 0$. The loading conditions are $P_{x2} = 0.5$ MN, $P_{y2} = -1.0$ MN, and $P_{x3} = 0$. The stiffness equation given at the beginning of the last section now becomes

$$\begin{bmatrix} 23.9 & 9.6 & -7.2 & -9.6 & -16.6 & 0 \\ 9.6 & 12.8 & -9.6 & -12.8 & 0 & 0 \\ -7.2 & -9.6 & 14.4 & 0 & -7.2 & 9.6 \\ -9.6 & -12.8 & 0 & 25.6 & 9.6 & -12.8 \\ -16.6 & 0 & -7.2 & 9.6 & 23.9 & 19.6 \\ 0 & 0 & 9.6 & -12.8 & -9.6 & 12.8 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ 0 \end{Bmatrix} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ 0.5 \\ -1.0 \\ 0 \\ P_{y3} \end{Bmatrix}$$

Note that there are exactly six unknowns in the six equations. The solution of the six unknowns is obtained in two steps. In the first step, we notice that the three equations, third through fifth, are independent from the other three and can be dealt with separately.

$$\begin{bmatrix} 14.4 & 0 & -7.2 \\ 0 & 25.6 & 9.6 \\ -7.2 & 9.6 & 23.9 \end{bmatrix} \begin{Bmatrix} u_2 \\ v_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ -1.0 \\ 0 \end{Bmatrix} \tag{1.16}$$

Equation 1.16 is the *constrained stiffness equation* of the loaded truss. The constrained 3 × 3 stiffness matrix is symmetric but not singular. The solution of Equation 1.16 is $u_2 = 0.053$ m, $v_2 = -0.053$ m, and $u_3 = 0.037$ m. In the second step, the reactions are obtained from the direct substitution of the displacement values into the other three equations, first, second, and sixth:

$$\begin{matrix} 23.9 & 9.6 & -7.2 & -9.6 & -16.6 & 0 \\ 9.6 & 12.8 & -9.6 & -12.8 & 0 & 0 \\ 0 & 0 & 9.6 & -12.8 & -9.6 & 12.8 \end{matrix} \quad \begin{matrix} 0 \\ 0 \\ 0.053 \\ -0.053 \\ 0.037 \\ 0 \end{matrix} \quad = \quad \begin{matrix} -0.5 \\ 0.17 \\ 0.83 \end{matrix}$$

$$= \quad \begin{matrix} P_{x1} \\ P_{y1} \\ P_{y3} \end{matrix}$$

or

$$\begin{matrix} P_{x1} \\ P_{y1} \\ P_{y3} \end{matrix} = \begin{matrix} -0.5 \\ 0.17 \\ 0.83 \end{matrix} \text{ MN}$$

The member deformation represented by the member elongation can be computed by the member deformation equation, Equation 1.3:

$$\text{Member 1:} \quad {}_1 = \begin{matrix} -C & -S & C & S \end{matrix}_1 \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}$$

$$= \begin{matrix} -0.6 & -0.8 & 0.6 & 0.8 \end{matrix}_1 \begin{matrix} 0 \\ 0 \\ 0.053 \\ -0.053 \end{matrix} = -0.011\text{m}$$

For member 2 and member 3, the elongations are $\Delta_2 = -0.052$ m and $\Delta_3 =$ 0.037 m.

The member forces are computed using Equation 1.1.

$$F = k \quad = \frac{EA}{L} \qquad F_1 = -0.20 \text{ MN}, F_2 = -1.04 \text{ MN}, F_3 = 0.62 \text{ MN}$$

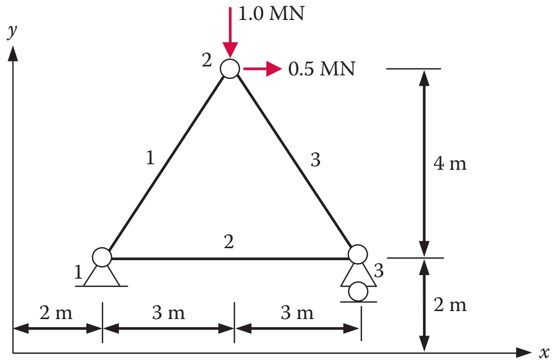
The results are summarized in the following table.

Nodal and Member Solutions

	Displacement (m)		Force (MN)	
Node	***x*-direction**	***y*-direction**	***x*-direction**	**y-direction**
1	0	0	–0.50	0.17
2	0.053	–0.053	0.50	–1.00
3	0.037	0	0	0.83
Member		**Elongation (m)**		**Force (MN)**
1		–0.011		–0.20
2		–0.052		–1.04
3		0.037		0.62

PROBLEM 1.2

Consider the same three-bar truss as that in Example 1.2 but with a different numbering system for members. Construct the constrained stiffness equation, Equation 1.16.

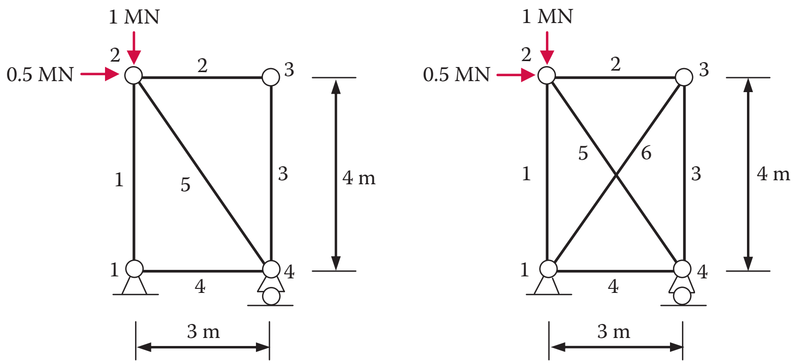


Problem 1.2

1.6 Procedures of Truss Analysis

Example 1.3

Consider the following two truss problems, each with member properties E = 70 GPa and A = 1430 mm^2. The only difference is the existence of an additional diagonal member in the second truss. It is instructive to see how the analyses and results differ.



Two truss problems.

Solution

We will carry out a step-by-step solution procedure for the two problems, referring to the truss at the left and at the right in the preceding figure as the first and second truss, respectively. We also define the global coordinate system in both cases as one with the origin at node 1 and its x- and y-direction coincide with the horizontal and vertical directions, respectively.

1. Number the nodes and members, and define the nodal coordinates.

Nodal Coordinates

Node	x **(m)**	y **(m)**
1	0	0
2	0	4
3	3	4
4	3	0

2. Define member property, starting and end nodes and compute member data.

Member Data

	Input Data*			**Computed Data**					
Member	***S* Node**	***E* Node**	***EA* (MN)**	Δx	Δy	L	C	S	EA/L
1	1	2	100	0	4	4	0.0	1.0	25.00
2	2	3	100	3	0	3	1.0	0.0	33.33
3	3	4	100	0	−4	4	0.0	–1.0	25.00
4	1	4	100	3	0	3	1.0	0.0	33.33
5	2	4	100	3	−4	5	0.6	–0.8	20.00
6	1	3	100	3	4	5	0.6	0.8	20.00

* *S* Node and *E* Node represent starting and end nodes.

3. Compute member stiffness matrices.

$$\boldsymbol{K_G} = \frac{EA}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

$$\text{Member 1: } (\boldsymbol{K_G})_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & -25 \\ 0 & 0 & 0 & 0 \\ 0 & -25 & 0 & 25 \end{bmatrix}$$

$$\text{Member 2: } (\boldsymbol{k_G})_2 = \begin{bmatrix} 33.33 & 0 & -33.33 & 0 \\ 0 & 0 & 0 & 0 \\ -33.33 & 0 & 33.33 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Member 3: } (\boldsymbol{K_G})_3 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & -25 \\ 0 & 0 & 0 & 0 \\ 0 & -25 & 0 & 25 \end{bmatrix}$$

$$\text{Member 4: } (\boldsymbol{K_G})_4 = \begin{bmatrix} 33.33 & 0 & -33.33 & 0 \\ 0 & 0 & 0 & 0 \\ -33.33 & 0 & 33.33 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{Member 5: } (\boldsymbol{K_G})_5 = \begin{bmatrix} 7.2 & -9.6 & -7.2 & 9.6 \\ -9.6 & 12.8 & 9.6 & -25 \\ -7.2 & 9.6 & 7.2 & -9.6 \\ 9.6 & -25 & -9.6 & 12.8 \end{bmatrix}$$

$$\text{Member 6 (for the second truss only): } (\boldsymbol{k_G})_6 = \begin{bmatrix} 7.2 & 9.6 & -7.2 & -9.6 \\ 9.6 & 12.8 & -9.6 & -12.8 \\ -7.2 & -9.6 & 7.2 & 9.6 \\ -9.6 & -12.8 & 9.6 & 12.8 \end{bmatrix}$$

4. Assemble the unconstrained global stiffness matrix.

In order to use the direct stiffness method to assemble the global stiffness matrix, we need the following table, which gives the global DOF number corresponding to each local DOF of each member. This table is generated using the member data given in the table in step 2, namely, the starting and end nodes data.

Global DOF Number for Each Member

Local DOF Number	Global DOF Number for Member					
	1	2	3	4	5	6*
1	1	3	5	1	3	1
2	2	4	6	2	4	2
3	3	5	7	7	7	5
4	4	6	8	8	8	6

* For the second truss only.

Armed with this table we can easily direct the member stiffness components to the right location in the global stiffness matrix. For example, the (2,3) component of $(\boldsymbol{k}_G)_5$ will be added to the (4,7) component of the global stiffness matrix. The unconstrained global stiffness matrix is obtained after all the assembling is done.

For the first truss:

$$\boldsymbol{K}_1 = \begin{matrix} 33.33 & 0 & 0 & 0 & 0 & 0 & -33.33 & 0 \\ 0 & 25.00 & 0 & -25.00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 40.53 & -9.60 & -33.33 & 0 & -7.20 & 9.60 \\ 0 & -25.00 & -9.60 & 37.80 & 0 & 0 & 9.60 & -12.8 \\ 0 & 0 & -33.33 & 0 & 33.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25.00 & 0 & -25.00 \\ -33.33 & 0 & -7.20 & 9.60 & 0 & 0 & 40.53 & -9.60 \\ 0 & 0 & 9.60 & -12.80 & 0 & -25.00 & -9.60 & 37.80 \end{matrix}$$

For the second truss:

$$\boldsymbol{K}_2 = \begin{bmatrix} 40.53 & 9.60 & 0 & 0 & -7.20 & -9.60 & -33.33 & 0 \\ 9.60 & 37.80 & 0 & -25.00 & -9.60 & -12.80 & 0 & 0 \\ 0 & 0 & 40.53 & -9.60 & -33.33 & 0 & -7.20 & 9.60 \\ 0 & -25.00 & -9.60 & 37.80 & 0 & 0 & 9.60 & -12.8 \\ -7.20 & -9.60 & -33.33 & 0 & 40.52 & 9.60 & 0 & 0 \\ -9.60 & -12.80 & 0 & 0 & 9.60 & 37.80 & 0 & -25.00 \\ -33.33 & 0 & -7.20 & 9.60 & 0 & 0 & 40.53 & -9.60 \\ 0 & 0 & 9.60 & -12.80 & 0 & -25.00 & -9.60 & 37.80 \end{bmatrix}$$

Note that $\boldsymbol{K}_2$ is obtained by adding $(\boldsymbol{K}_G)_6$ to $\boldsymbol{K}_1$ at the proper locations in columns and rows 1, 2, 5, and 6 (enclosed in dashed lines above).

5. Assemble the constrained global stiffness equation.

Once the support and loading conditions are incorporated into the stiffness equations we obtain:

For the first truss:

$$\begin{bmatrix} 33.33 & 0 & 0 & 0 & 0 & 0 & -33.33 & 0 \\ 0 & 25.00 & 0 & -25.00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 40.53 & -9.60 & -33.33 & 0 & -7.20 & 9.60 \\ 0 & -25.00 & -9.60 & 37.80 & 0 & 0 & 9.60 & -12.8 \\ 0 & 0 & -33.33 & 0 & 33.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 25.00 & 0 & -25.00 \\ -33.33 & 0 & -7.20 & 9.60 & 0 & 0 & 40.53 & -9.60 \\ 0 & 0 & 9.60 & -12.80 & 0 & -25.00 & -9.60 & 37.80 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ 0.5 \\ -1.0 \\ 0 \\ 0 \\ 0 \\ P_{y4} \end{Bmatrix}$$

For the second truss:

$$\begin{bmatrix} 40.53 & 9.60 & 0 & 0 & -7.20 & -9.60 & -33.33 & 0 \\ 9.60 & 37.80 & 0 & -25.00 & -9.60 & -12.80 & 0 & 0 \\ 0 & 0 & 40.53 & -9.60 & -33.33 & 0 & -7.20 & 9.60 \\ 0 & -25.00 & -9.60 & 37.80 & 0 & 0 & 9.60 & -12.8 \\ -7.20 & -9.60 & -33.33 & 0 & 40.52 & 9.60 & 0 & 0 \\ -9.60 & -12.80 & 0 & 0 & 9.60 & 37.80 & 0 & -25.00 \\ -33.33 & 0 & -7.20 & 9.60 & 0 & 0 & 40.53 & -9.60 \\ 0 & 0 & 9.60 & -12.80 & 0 & -25.00 & -9.60 & 37.80 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} P_{x1} \\ P_{y1} \\ 0.5 \\ -1.0 \\ 0 \\ 0 \\ 0 \\ P_{y4} \end{Bmatrix}$$

6. Solve the constrained global stiffness equation.

The constrained global stiffness equation in either case contains five equations corresponding to the third through seventh equations (enclosed in dashed lines above) that are independent from the other three equations and can be solved for the five unknown nodal displacements.

For the first truss:

$$\begin{matrix} 40.53 & -9.60 & -33.33 & 0 & -7.20 \\ -9.60 & 37.80 & 0 & 0 & 9.60 \\ -33.33 & 0 & 33.33 & 0 & 0 \\ 0 & 0 & 0 & 25.00 & 0 \\ -7.20 & 9.60 & 0 & 0 & 40.53 \end{matrix} \quad \begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \end{matrix} \quad = \quad \begin{matrix} 0.5 \\ -1.0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

For the second truss:

$$\begin{matrix} 40.53 & -9.60 & -33.33 & 0 & -7.20 \\ -9.60 & 37.80 & 0 & 0 & 9.60 \\ -33.33 & 0 & 40.52 & 9.60 & 0 \\ 0 & 0 & 9.60 & 37.80 & 0 \\ -7.20 & 9.60 & 0 & 0 & 40.53 \end{matrix} \quad \begin{matrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \end{matrix} \quad = \quad \begin{matrix} 0.5 \\ -1.0 \\ 0 \\ 0 \\ 0 \end{matrix}$$

The reactions are computed by direct substitution.

For the first truss:

$$\begin{matrix} 33.33 & 0 & 0 & 0 & 0 & 0 & -33.33 & 0 \\ 0 & 25.00 & 0 & 25.00 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9.60 & -12.80 & 0 & -25.00 & -9.60 & 37.80 \end{matrix}$$

$$\begin{matrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ 0 \end{matrix} = \begin{matrix} P_{x1} \\ P_{y1} \\ P_{y4} \end{matrix}$$

For the second truss:

$$\begin{matrix} 40.53 & 9.60 & 0 & 0 & -7.20 & -9.60 & -33.33 & 0 \\ 9.60 & 37.80 & 0 & -25.00 & -9.60 & -12.80 & 0 & 0 \\ 0 & 0 & 9.60 & -12.80 & 0 & -25.00 & -9.60 & 37.80 \end{matrix}$$

$$\begin{matrix} 0 \\ 0 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ 0 \end{matrix} = \begin{matrix} P_{x1} \\ P_{y1} \\ P_{y4} \end{matrix}$$

Results will be summarized at the end of the example.

7. Compute the member elongations and forces.

For a typical member i:

$$_i = \begin{matrix} -C & -S & C & S \end{matrix}_i \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}_i$$

$$F_i = (k\ \ )_i = \frac{EA}{L}\ \ _i$$

8. Summarize results.

Results for the First Truss

	Displacement (m)		**Force (MN)**	
Node	***x*-direction**	***y*-direction**	***x*-direction**	***y*-direction**
1	0	0	–0.50	0.33
2	0.066	–0.013	0.60	–1.00
3	0.067	0	0	0
4	0.015	0	0	0.67

Member	**Elongation (m)**	**Force (MN)**
1	–0.013	–0.33
2	0	0
3	0	0
4	0.015	0.50
5	–0.042	–0.83

Results for the Second Truss

	Displacement (m)		**Force (MN)**	
Node	***x*-direction**	***y*-direction**	***x*-direction**	***y*-direction**
1	0	0	–0.50	0.33
2	0.033	–0.021	0.60	–1.00
3	0.029	–0.007	0	0
4	0.011	0	0	0.67

Member	**Elongation (m)**	**Force (MN)**
1	–0.021	–0.52
2	–0.004	–0.14
3	–0.008	–0.19
4	0.011	0.36
5	0.030	–0.60
6	0.012	0.23

Note that the reactions at node 1 and 4 are identical in the two cases, but other results are changed by the addition of one more diagonal member.

9. Concluding remarks.

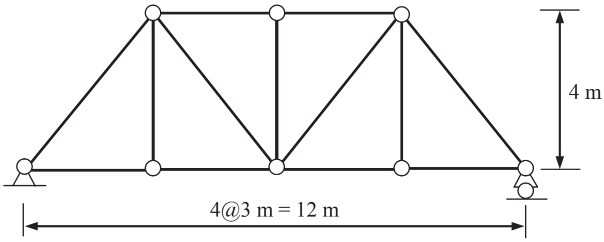
If the number of nodes is N and the number of constrained DOF is C, then

a. The number of simultaneous equations in the unconstrained stiffness equation is $2N$.
b. The number of simultaneous equations for the solution of unknown nodal displacements is $2N - C$.

In the present example, both truss problems have five equations for the five unknown nodal displacements. These equations cannot be easily solved with hand calculation and should be solved by computer.

PROBLEM 1.3

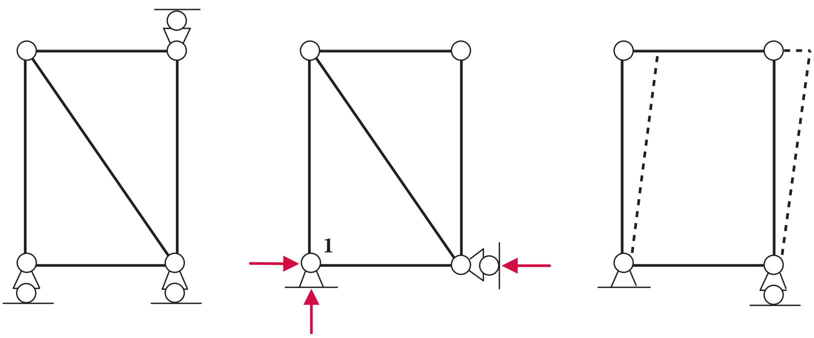
The truss shown next is made of members with properties $E = 70$ GPa and $A = 1430\ mm^2$. Use a computer to find support reactions, member forces, member elongations, and all nodal displacements for (a) a unit load applied vertically at the mid-span node of the lower chord members, and (b) a unit load applied vertically at the first internal lower chord node. Draw the deflected configuration in each case.



Problem 1.3

1.7 Kinematic Stability

In the previous analysis, we learned that the unconstrained stiffness matrix is always singular, because the truss is not yet supported or constrained. What if the truss is supported but not sufficiently or properly supported, or the truss members are not properly placed? Consider the following three examples. Each is a variation of the example truss problem (Example 1.3) we have just solved.



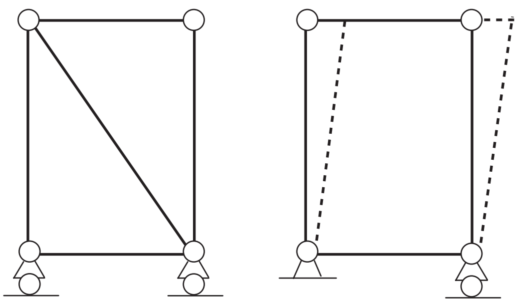
Three unstable truss configurations.

1. Truss at left. The three roller supports provide constraints only in the vertical direction but not in the horizontal direction. As a result, the truss can move in the horizontal direction indefinitely. There is no resistance to translation in the horizontal direction.

2. Truss in the middle. The reactions provided by the supports all point to node 1. As a result, the reaction forces cannot counterbalance any applied force that produces a nonzero moment about node 1. The truss is not constrained against rotation about node 1.
3. Truss at the right. The supports are fine, providing constraints against translation as well as rotation. The members of the truss are not properly placed. Without a diagonal member, the truss will change shape as shown. The truss cannot maintain its shape against arbitrarily applied external forces at the nodes.

The first two cases are such that the trusses are externally unstable. The last one is internally unstable. The resistance against changing shape or location as a mechanism is called kinematic stability. Although kinematic stability or instability can be inspected through visual observation, mathematically it manifests itself in the characteristics of the constrained global stiffness matrix. If the matrix is singular, then we know the truss is kinematically unstable. In the example problem 1.3 in the last section, the two 5×5 stiffness matrices are both nonsingular, otherwise we would not have been able to obtain the displacement solutions. Thus, kinematic stability of a truss can be tested mathematically by investigating the singularity of the constrained global stiffness matrix of a truss. In practice, if the displacement solution appears to be arbitrarily large or disproportionate among some displacements, then it may be the sign of an unstable truss configuration.

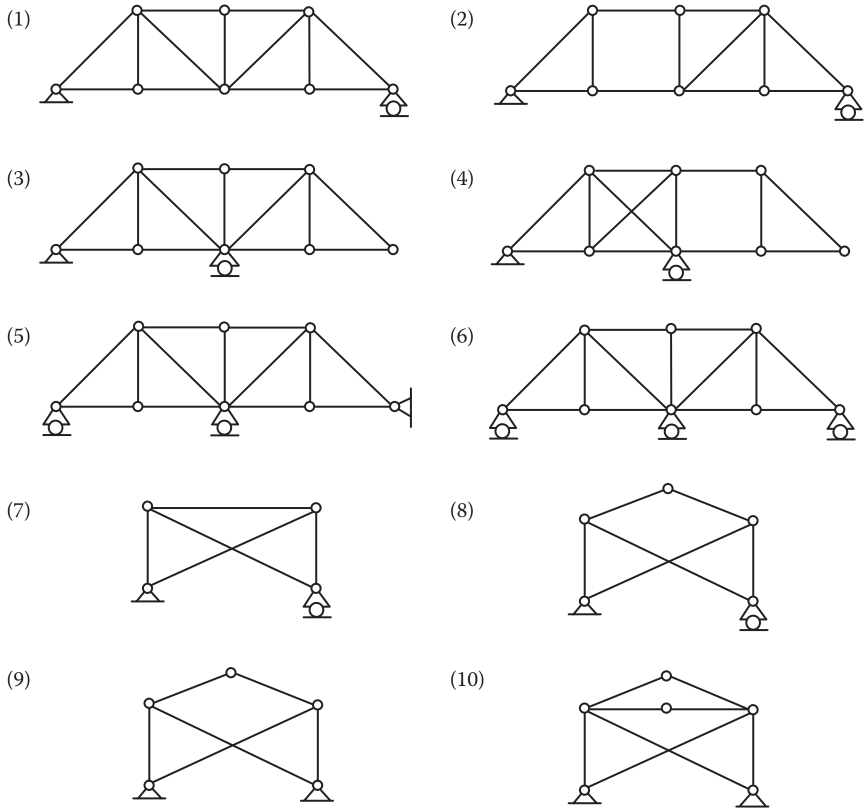
Sometimes, kinematic instability can be detected by counting constraints or unknown forces, for external instability and internal instability. External instability happens if there is insufficient number of constraints. Since it takes at least three constraints to prevent translation and rotation of an object in a plane, any support condition that provides only one or two constraints will result in instability. The left truss in the following figure has only two support constraints and is unstable. Internal instability happens if the total number of force unknowns is less than the number of displacement DOFs. If we denote the number of member force unknowns as M and support reaction unknowns as R, then internal instability results if $M + R < 2N$. The truss at the right in the following figure has $M = 4$ and $R = 3$ but $2N = 8$. It is unstable.



Kinematic instability resulting from insufficient number of supports or members.

PROBLEM 1.4.

Discuss the kinematic stability of each of the plane trusses shown next.



Problem 1.4

1.8 Summary

The fundamental concept in the displacement method and the procedures of solution are the following:

1. If all the key displacement quantities of a given problem are known, then the deformation of each member can be computed using the conditions of compatibility, which is manifested in the form of Equation 1.2 through Equation 1.4.
2. Knowing the member deformation, we can then compute the member force using the member stiffness equation, Equation 1.1.

3. The member force of a member can be related to the nodal forces expressed in the global coordinate system by Equation 1.7 or Equation 1.8, which is the forced transformation equation.
4. The member nodal forces and the externally applied forces are in equilibrium at each node, as expressed in Equation 1.14, which is the global equilibrium equation in terms of nodal forces.
5. The global equilibrium equation can then be expressed in terms of nodal displacements through the use of the member stiffness equation, Equation 1.11. The result is the global stiffness equation in terms of nodal displacements, Equation 1.15.
6. Since not all the nodal displacements are known, we can solve for the unknown displacements from the constrained global stiffness equation, Equation 1.16 in Example 1.3.
7. Once all the nodal displacements are computed, the remaining unknown quantities are computed by simple substitution.

The displacement method is particularly suited for computer solution because the solution steps can be easily programmed through the direct stiffness method of assembling the stiffness equation. The correct solution can always be computed if the structure is stable (kinematically stable), which means the structure is internally properly connected and externally properly supported to prevent it from becoming a mechanism under any loading conditions.

2

Truss Analysis: Force Method—Part I

2.1 Introduction

In the chapter on matrix displacement method of truss analysis, truss analysis is formulated with nodal displacement unknowns as the fundamental variables to be determined. The resulting method of analysis is simple and straightforward and is very easy to be implemented into a computer program. As a matter of fact, virtually all structural analysis computer packages are coded with the matrix displacement method.

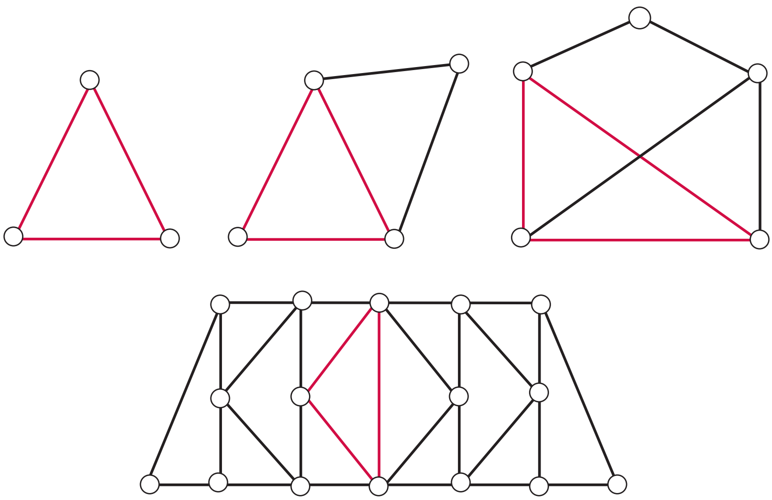
The one drawback of the matrix displacement method is that it does not provide any insight on how the externally applied loads are transmitted and taken up by the members of the truss. Such an insight is critical when an engineer is required not only to analyze a given truss but also to design a truss from scratch.

We will now introduce a different approach, the force method. The essence of the force method is the formulation of the governing equations with the forces as unknown variables. The beginning point of the force method is the equilibrium equations expressed in terms of forces. Depending on how the free-body diagrams (FBDs) are selected to develop these equilibrium equations, we may use either the method of joints or the method of sections or a combination of both to solve a truss problem.

In the force method of analysis, if the force unknowns can be solved by the equilibrium equations alone, then the solution process is very straightforward: finding member forces from equilibrium equations, finding member elongation from member forces, and finding nodal displacements from member elongation. Assuming that the trusses considered herein are all kinematically stable, the only other prerequisite for such a solution procedure is that the truss be a statically determinate one, that is, the total number of force unknowns is equal to the number of independent equilibrium equations. In contrast, a statically indeterminate truss, which has more force unknowns than the number of independent equilibrium equations, requires the introduction of additional equations based on the geometric compatibility or consistent deformations to supplement the equilibrium equations. We shall study the statically determinate problems first, beginning by a brief discussion of determinacy and truss types.

2.2 Statically Determinate Plane Truss Types

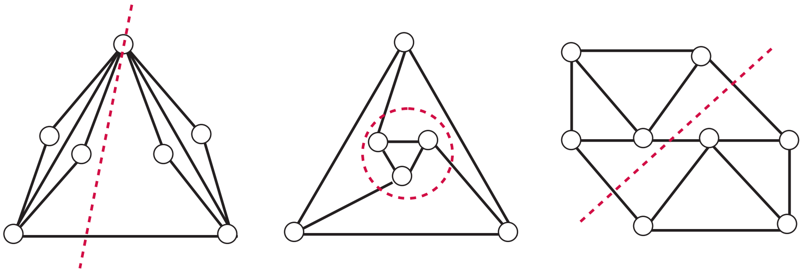
For statically determinate trusses, the force unknowns, consisting of M member forces if there are M members and R reactions, are equal in number to the equilibrium equations. Since one can generate two equilibrium equations from each node, the number of independent equilibrium equations is $2N$, where N is the number of nodes. Thus by definition $M + R = 2N$ is the condition of statical determinacy. This is to assume that the truss is stable, because it is meaningless to ask whether the truss is determinate if it is not stable. For this reason, stability of a truss should be examined first. One class of plane trusses, called *simple truss*, is always stable and determinate if properly supported externally. A simple truss is a truss built from a basic triangle of three bars and three nodes by adding two bars and a node one at a time. Examples of simple trusses are shown next.



Simple trusses.

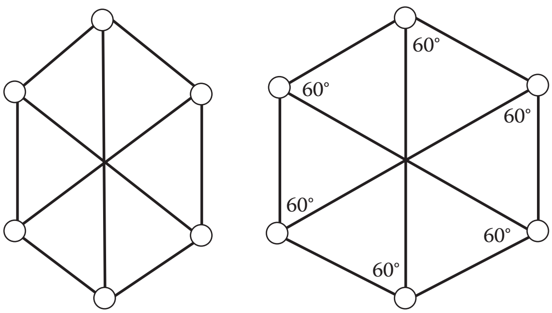
The basic triangle of three bars ($M = 3$) and three nodes ($N = 3$) is a stable configuration and satisfies $M + R = 2N$ if there are three reaction forces ($R = 3$). Adding two bars and a node creates a different but stable configuration. The two more force unknowns from the two bars are compensated exactly by the two equilibrium equations from the new node. Thus, $M + R = 2N$ is still satisfied.

Another class of plane truss is called *compound truss*. A compound truss is a truss composed of two or more simple trusses linked together. If the linkage consists of three bars placed properly, not forming parallel or concurrent forces, then a compound truss is also stable and determinate. Examples of stable and determinate compound trusses are shown next, where the dotted lines cut across the links.



Compound trusses.

A plane truss can neither be classified as a simple truss nor as a compound truss is a *complex truss*. A complex truss is best solved by the computer version of the method of joints to be described later. A special method, called method of substitution, was developed for complex trusses in the precomputer era. It has no practical purposes nowadays and will not be described herein. Two complex trusses are shown in the following figure, the one at the left is stable and determinate, and the other at the right is unstable. The instability of complex trusses cannot be easily determined. There is a way, however: the self-equilibrium test. If we can find a system of internal forces that are in equilibrium by themselves without any externally applied loads, then the truss is unstable. It can be seen that the truss at the right can have the same tension force of any magnitude, *S*, in the three internal bars and compression force, –*S*, in all the peripheral bars, and they will be in equilibrium without any externally applied forces.

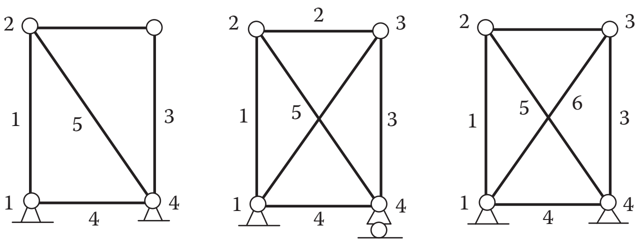


Stable and unstable complex trusses.

Mathematically such a situation indicates that there will be no unique solution for any given set of loads, because the self-equilibrium "solution" can always be superposed onto any set of solutions and create a new set of solutions. Without a unique set of solutions is a sign that the structure is unstable.

We may summarize the aforementioned discussions with the following conclusions:

1. Stability can often be determined by examining the adequacy of external supports and internal member connections. If $M + R < 2N$, however, then it is always unstable, because there is not enough number of members or supports to provide adequate constraints to prevent a truss from turning into a mechanism under certain loads.
2. For a stable plane truss, if $M + R = 2N$, then it is statically determinate.
3. A simple truss is stable and determinate.
4. For a stable plane truss, if $M + R > 2N$, then it is statically indeterminate. The discrepancy between the two numbers, $M + R - 2N$, is called the degrees of indeterminacy, or the number of redundant forces. Statically indeterminate truss problems cannot be solved by equilibrium conditions alone. The conditions of compatibility must be utilized to supplement the equilibrium conditions. This way of solution is called method of consistent deformations and will be described in the next chapter. Examples of indeterminate trusses are shown next.



Statically indeterminate trusses.

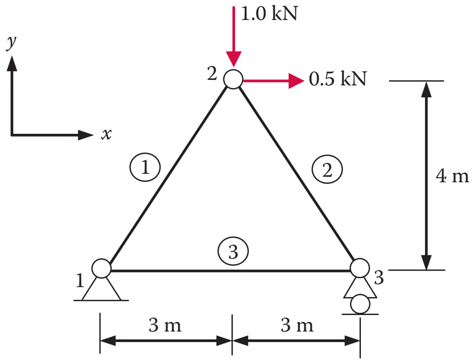
In the preceding figure, the truss at the left is statically indeterminate to the first degree because there is one redundant reaction force: $M = 5$, $R = 4$, and $M + R - 2N = 1$. The truss in the middle is also statically indeterminate to the first degree because of one redundant member: $M = 6$, $R = 3$, and $M + R - 2N = 1$. The truss at the right is statically indeterminate to the second degree because $M = 6$, $R = 4$, and $M + R - 2N = 2$.

2.3 Method of Joints and Method of Sections

The method of joints draws its name from the way a FBD is selected: at the joints of a truss. The key to the method of joints is the equilibrium of each joint. From each FBD, two equilibrium equations are derived. The method of joints provides insight on how the external forces are balanced by the member forces at each joint, whereas the method of sections provides insight on how the member forces resist external forces at each "section." The key to the method of sections is the equilibrium of a portion of a truss defined by a FBD, which is a portion of the structure created by cutting through one or more sections. The equilibrium equations are written from the FBD of that portion of the truss. There are three equilibrium equations as opposed to the two for a joint. Consequently, we make sure there are no more than three unknown member forces in the FBD when we choose to cut through a section of a truss. In the following example problems and elsewhere, we use the terms *joint* and *node* interchangeably.

Example 2.1

Find all support reactions and member forces of the loaded truss shown next.

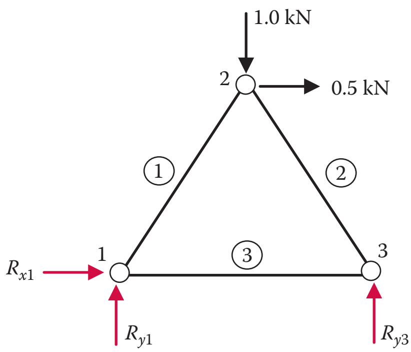


A truss problem to be solved by the method of joints.

Solution

We shall give a detailed step-by-step solution.

1. *Identify all force unknowns*. The very first step in any force method of analysis is to identify all force unknowns. This is achieved by examining the reaction forces and member forces. The reaction forces are exposed in an FBD of the whole structure.



Free-body diagram of the three-bar-truss to expose the reaction forces.

Note that in the figure, the subscripts of the reaction forces indicate the direction (first subscript) and the location of the reactions (second subscript). The three reaction forces are R_{x1}, R_{y1}, and R_{y3}. The member forces are F_1, F_2, and F_3.

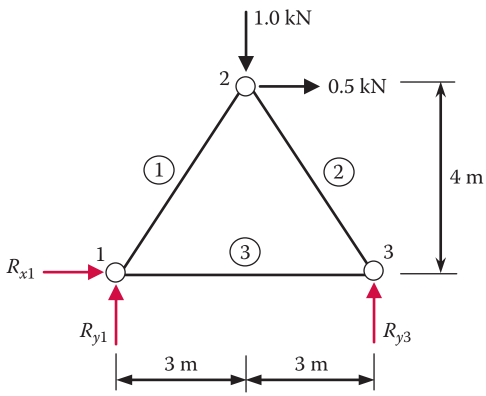
2. *Examine the static determinacy of the structure*. Before we proceed to find the force unknowns by the method of joints, we must be sure that all the force unknowns can be determined by the static equilibrium conditions alone, because that is the essence of the method of joints, namely, using joint equilibrium equations to find force unknowns. Denote the number of all member force unknowns as M and the number of reaction forces as R, and the total number of force unknowns is $M + R$. In the present example, $M = 3$, $R = 3$, and $M + R = 6$. This number is to be compared to the number of equilibrium equations available.

   There are three nodes in the truss. We can write two equilibrium equations at each node of a plane truss:

$$\Sigma F_x = 0, \quad \Sigma F_y = 0 \tag{2.1}$$

Thus the total number of equilibrium equations available is $2N$, where N is the number of nodes in a truss. In the present example, $N = 3$ and $2N = 6$. Thus, the number of available static equilibrium equations exactly matches the total number of force unknowns, $M + R = 2N$. The problem posed in the present example is statically determinate. We can reach the same conclusion if we note that the truss is a simple truss.

3. *Solve for force unknowns*. The most obvious next step is to write up the six nodal equilibrium equations and solve for the six unknown forces simultaneously. That would require the use of a computer. For the present example, and many other cases, an experienced structure engineer can solve a problem by hand calculation faster than using a computer. This hand-calculation process gives insight to the force flow from externally applied load, through members, and to the supports. This is the process that is presented herein.
   a. *Find all reactions*. Although not necessary, finding all reaction forces from the FBD of the whole structure first is often the fastest way of solving a plane truss problem.



Free-body diagram for finding reactions.

The three reaction forces can be solved one at a time by applying the three equilibrium equations one by one:

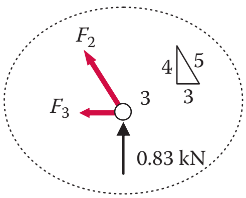
$$\Sigma F_x = 0 \Longrightarrow R_{x1} + 0.5 = 0 \Longrightarrow R_{x1} = -0.5 \text{ kN}$$

$$\Sigma M_1 = 0 \Longrightarrow R_{y3}(6) - (1.0)(3) - (0.5)(4) = 0 \Longrightarrow R_{y3} = 0.83 \text{ kN}$$

$$\Sigma F_y = 0 \Longrightarrow R_{y1} + 0.83 - 1.0 = 0 \Longrightarrow R_{y1} = 0.17 \text{ kN}$$

b. *Find member forces*. The member forces are solved by applying nodal equilibrium equations joint by joint. The selection of the sequence by which each joint is utilized is based on a simple rule: No joint should contain more than two unknowns, with one unknown in each equation preferred. Based on this rule, we take the following sequence and use the FBD of each joint to write the equilibrium equations:

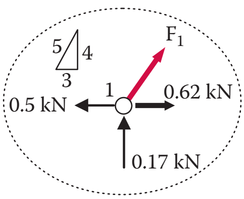
Joint 3



$\Sigma F_y = 0$, $F_2(4/5) + 0.83 = 0$, $F_2 = -1.04$ kN

$\Sigma F_x = 0$, $-F_2(3/5) - F_3 = 0$, $F_3 = 0.62$ kN

Joint 1

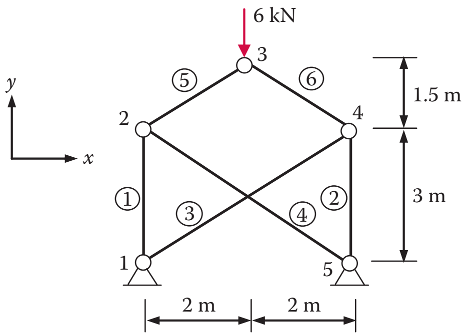


$\Sigma F_y = 0$, $F_1(4/5) + 0.17 = 0$, $F_1 = -0.21$ kN

Note that only one equation from the FBD of joint 1 is needed to find the remaining unknown of F_1. The second equilibrium equation is identically satisfied. The two equilibrium equations from the FBD of joint 2 would also be identically satisfied. These three "unused" equations can serve as a "check" for the accuracy of the computation. We need not use these three joint equations because we have already used three equations from the equilibrium of the whole structure at the beginning of the solution process. This fact also points to an important point: There are no more than six independent equilibrium equations. Any additional equations are not "independent" from the six equations we just used, because they can be derived from the linear combination of the six equations. Any six "independent" equations are equally valid. The selection of which six equations to use is a matter of preference and we always select those equations that give us the easiest way of getting the answer to the unknown forces as we just did.

Example 2.2

Find all reaction and member forces for the loaded truss shown next.

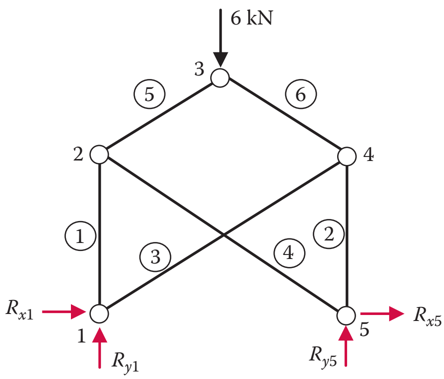


Another truss example problem for the method of joints.

Solution

A slightly different solution strategy is followed in this example.

1. *Identify all force unknowns*. The FBD of the whole structure shows there are four reactions. Adding the six member forces, we have $M = 6$, $R = 4$, and $M + R = 10$, a total of ten force unknowns.

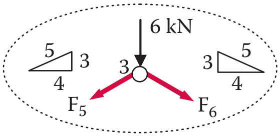


FBD of the whole truss.

2. *Examine the static determinacy of the structure*. There are five nodes, $N = 5$. Thus $M + R = 2N = 10$. This is a statically determinate problem.
3. *Solve for force unknowns*. This is a problem for which there is no advantage in solving for the reactions first. The FBD of the whole structure will give us three equations of equilibrium while we have four reaction unknowns. Thus, we cannot solve for the four reactions with the equations from the FBD of the whole structure

alone. On the other hand, if we go from joint to joint in the following order, 3, 2, 4, 1, and 5, we will be able to solve for member forces one node at a time and eventually get to the reactions.

Joint 3



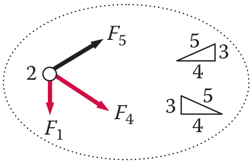
$\Sigma F_y = 0, \quad F_5(3/5) + F_6(3/5) = -6$

$\Sigma F_x = 0, \quad -F_5(4/5) + F_6(4/5) = 0$

$\Rightarrow$ $\boxed{F_5 = -5 \text{ kN},}$ $\boxed{F_6 = -5 \text{ kN}}$

In this case, solving the two equations simultaneously is inevitable.

Joint 2

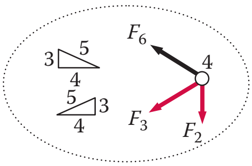


$\Sigma F_x = 0, \quad F_5(4/5) + F_4(4/5) = 0$ $\boxed{F_4 = 5 \text{ kN}}$

$\Sigma F_y = 0, \quad F_5(3/5) - F_4(3/5) - F_1 = 0$

$\Rightarrow$ $\boxed{F_1 = -6 \text{ kN}}$

Joint 4

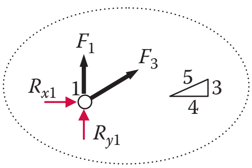


$\Sigma F_x = 0, \quad F_6(4/5) + F_3(4/5) = 0$ $\boxed{F_3 = 5 \text{ kN}}$

$\Sigma F_y = 0, \quad F_6(3/5) - F_3(3/5) - F_2 = 0$

$\Rightarrow$ $\boxed{F_2 = -6 \text{ kN}}$

Joint 1

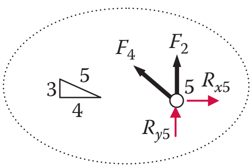


$\Sigma F_x = 0, \quad R_{x1} + F_3(4/5) = 0$ $\boxed{R_x1 = -4 \text{ kN}}$

$\Sigma F_y = 0, \quad R_{y1} + F_3(3/5) + F_1 = 0$

$\Rightarrow$ $\boxed{R_{y1} = 3 \text{ kN}}$

Joint 5



$\Sigma F_x = 0, \quad R_{x5} - F_4(4/5) = 0$ $\boxed{R_{x5} = 4 \text{ kN}}$

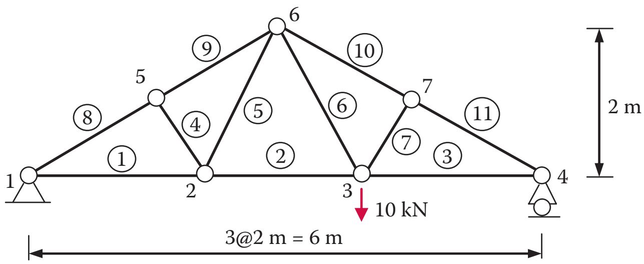
$\Sigma F_y = 0, \quad R_{y5} + F_4(3/5) + F_2 = 0$

$\Rightarrow$ $\boxed{R_{y5} = 3 \text{ kN}}$

Note in both example problems, we always assume the member forces to be in tension. This results in FBDs that have member forces pointing away from the joints. This is simply an easy way to assign force directions. It is highly recommended because it avoids unnecessary confusion that often leads to mistakes.

Example 2.3

Find the member forces in bars 4, 5, 6, and 7 of the loaded Fink truss shown next.



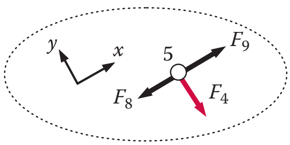
Fink truss to be solved by the method of joints.

Solution

We shall illustrate a special feature of the method of joints.

1. *Identify all force unknowns*. The FBD of the whole structure would have shown that there are three reactions. Adding the eleven member forces, we have $M = 11$, $R = 3$, and $M + R = 14$, a total of 14 force unknowns.
2. *Examine the static determinacy of the structure*. There are seven nodes, $N = 7$. Thus $M + R = 2N = 14$. This is a statically determinate problem.
3. *Solve for force unknowns*. Normally, Fink trusses are used to take roof loading on the upper chord nodes. We deliberately apply a single load at a lower chord node in order to make a point about a special feature of the method of joints. We start by concentrating on joint 5.

Joint 5



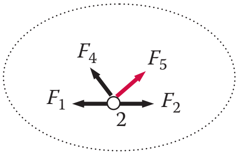
$\Sigma F_y = 0$ $\boxed{F_4 = 0}$

$\Sigma F_x = 0, \quad -F_8 + F_9 = 0$

$\Longrightarrow F_8 = F_9$

In this case, it is advantageous to line up the coordinate system with the local geometry at the node. F_4 is found to be zero because it is the only force in that direction. The pair of forces in the *x*-direction must be equal and opposite because they are collinear.

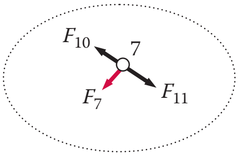
Joint 2



$\Sigma F_y = 0$, and $F_4 = 0$ ⟹ $\boxed{F_5 = 0}$

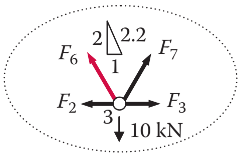
$\Sigma F_x = 0$ ⟹ $F_1 = F_2$

Joint 7



$\boxed{F_7 = 0}$ ⟹ $F_{10} = F_{11}$

Joint 3

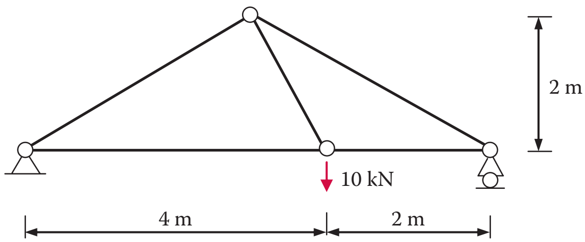


$F_7 = 0$, from equilibrium of Joint 7

$\Sigma F_y = 0$, $F_6\,(2/2.23) = 10$ $\boxed{F_6 = 11.15 \text{ kN}}$

That completes the solution for F_4, F_5, F_6, and F_7.

Thus, with the exception of member 6, all the web members are zero-force members for this particular loading case. For purpose of analysis under the given load, the Fink truss is equivalent to the truss shown next.

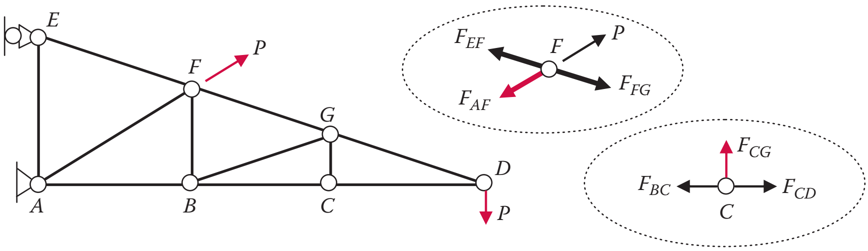


Equivalent truss to the Fink truss for the given load.

This brings up the interesting feature of the method of joints: we can identify zero-force members easily. This feature is further illustrated in the next example.

Example 2.4

Identify zero-force members and equal-force members in the loaded trusses in the following figure.



An example of zero-force members and equal force members.

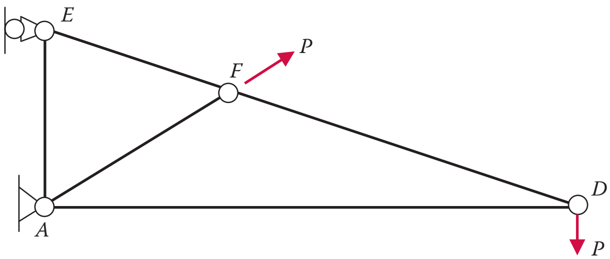
Solution

The equilibrium of forces at joint C leads to $F_{CG} = 0$ and $F_{BC} = F_{CD}$. Once we know $F_{CG} = 0$, it follows $F_{BG} = 0$ and then $F_{BF} = 0$, based on the equilibrium of forces at node G and node B, respectively. The equilibrium of forces at joint F leads to $F_{AF} = P$ and $F_{EF} = F_{FG}$.

We can identify:

Zero force members. At each joint, all the forces are concurrent forces. If all the forces are collinear except one then the lone exception must be zero.

Equal force members. If two forces at a joint are collinear and all other forces at the joint are also collinear in another direction, then the two forces must be equal.

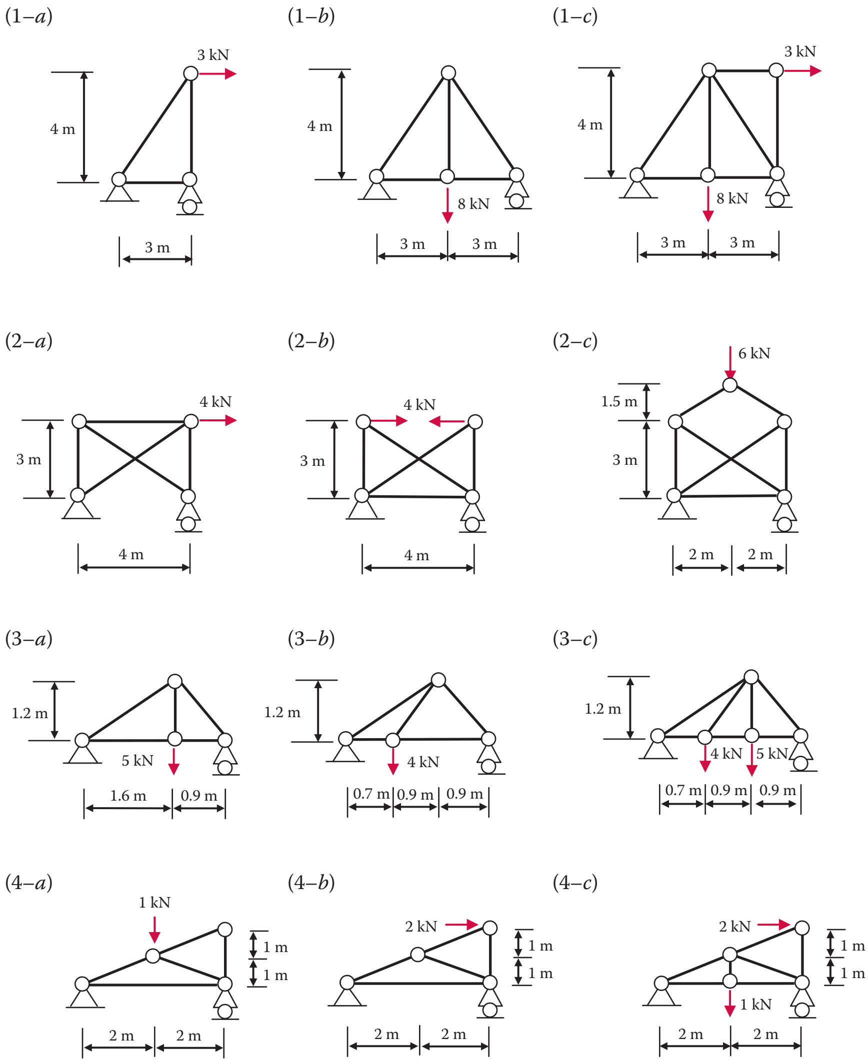


An equivalent truss.

For practical purposes, the original truss problem is equivalent to the truss problem shown in this example for the given loading case.

PROBLEM 2.1

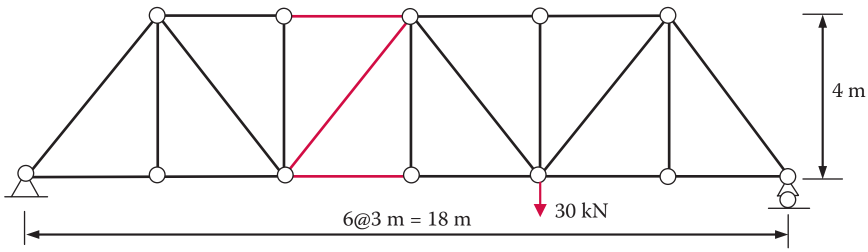
Use the method of joints to find all reaction and member forces in the trusses shown next.



Problem 2.1

Example 2.5

Find member forces in bars in the third panel from the left of the truss in the following figure.

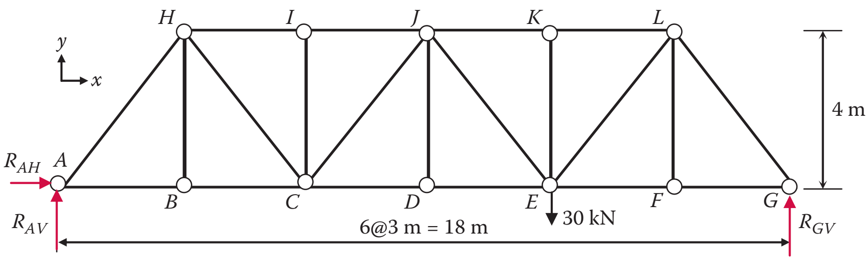


An example problem for the method of sections.

Solution

We shall solve this problem by the method of sections with the following procedures.

1. *Name all joints*. We can refer to each joint by a symbol and each member by the two end joints as shown next. We also define an *x,y* coordinate system as shown. We need to find F_{IJ}, F_{CJ}, and F_{CD}. The truss is stable and determinate.
2. *Find reactions*. We have to look at the FBD of the whole truss.



The FBD to find the reactions.

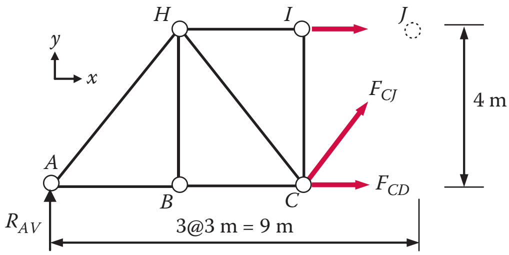
$$\Sigma M_A = 0, \quad (12)(30) - (18)\ R_{GV} = 0, \quad R_{GV} = 20 \text{ kN}$$

$$\Sigma F_x = 0, \quad R_{AH} = 0$$

$$\Sigma M_G = 0, \quad (18)\ R_{AV} - (6)(30) = 0, \quad R_{AV} = 10 \text{ kN}$$

3. *Establish FBD*. We make a vertical cut through the third panel from the left, thus exposing the member force of members *IJ*, *CJ*, and *CD*. We can take the left or the right portion as the FBD. We choose the left portion because it has a fewer number of external

forces to deal with. We always assume the member forces are tensile. We have already obtained $R_{AV} = 10$ kN.

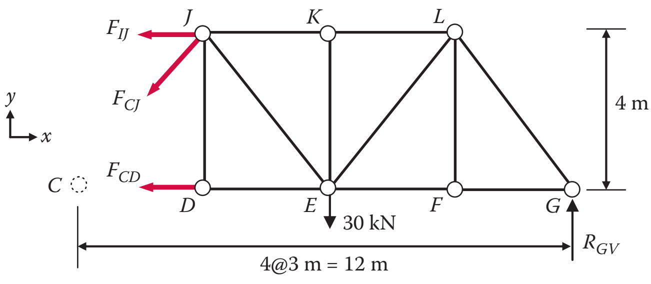


FBD exposing a section through the third panel from the left.

$\Sigma M_C = 0,$	$(4)\ F_{IJ} + (6)\ R_{AV} = 0$	$F_{IJ} = -1.5\ R_{AV} = -1.5$ kN
$\Sigma M_J = 0,$	$-(4)\ F_{CD} + (9)\ R_{AV} = 0$	$F_{CD} = 2.25\ R_{AV} = 22.5$ kN
$\Sigma F_y = 0,$	$(0.8)\ F_{CJ} + R_{AV} = 0$	$F_{CJ} = -1.25\ R_{AV} = -12.5$ kN

Note that we choose the moment center at *C* and *J*, respectively, because in each case the resulting equation has only one unknown and therefore can be solved easily.

To illustrate the effect of taking a different FBD, let us choose the right part of the cut as the FBD. Note that we already know $R_{GV} =$ 20 kN.



Alternative FBD exposing the member forces of the third panel.

By taking the right portion as the FBD we include the applied 30 kN force in the FBD and it will show up in all equilibrium equations.

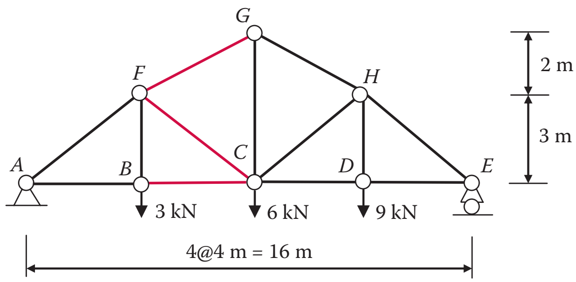
$$\Sigma M_C = 0,\ -(4)\ F_{IJ} + (6)\ (30) - (12)\ R_{GV} = 0,\ F_{IJ} = -3\ R_{GV} + 45 = -15 \text{ kN}$$

$$\Sigma M_J = 0,\ (4)\ F_{CD} + (3)\ (30) - (9)\ R_{GV} = 0,\ F_{CD} = 2.25\ R_{GV} - 22.5 = 22.5 \text{ kN}$$

$$\Sigma F_y = 0,\ -(0.8)\ F_{CJ} - 30 + R_{GV} = 0,\ F_{CJ} = -37.5 + 1.25\ R_{GV} = -12.5 \text{ kN}$$

Example 2.6

Find member forces in bars in the second panel from the left of the truss in the following figure.

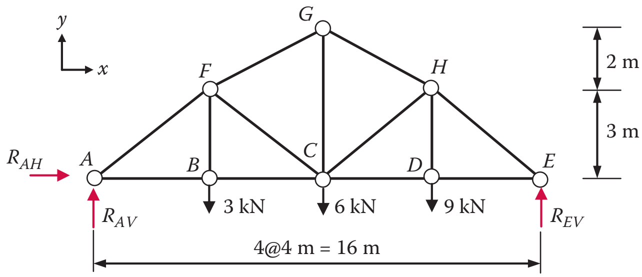


Another example problem for the method of sections.

Solution

The inclined chord geometry will cause complications in computation, but the process is the same as that of the last example.

1. *Find reactions*. This is a simple truss, stable and determinate.



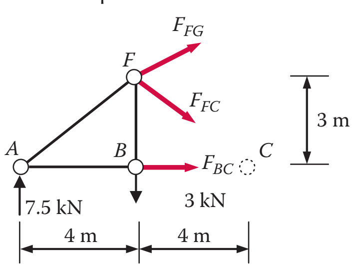
FBD for reaction forces.

$$\Sigma M_A = 0, \quad -(16)\,R_{EV} + (4)3 + (8)6 + (12)9 = 0, \quad R_{EV} = 10.5\text{ kN}$$

$$\Sigma M_E = 0, \; (16)\,R_{AV} - (12)3 - 8(6) - (4)9 = 0, \quad R_{AV} = 7.5\text{ kN}$$

$$\Sigma F_x = 0, \quad R_{AH} = 0\text{ kN}$$

2. *Establish FBD*. We make a cut through the second panel from the left and choose the left portion as the FBD.

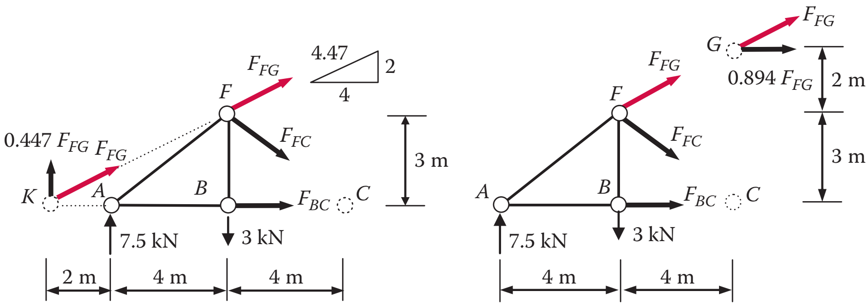


FBD for the second panel member forces.

In order to find F_{BC} we want to find a moment center that is the intersection of the two other unknowns. The intersection point of F_{FG} and F_{FC} is point F. Similarly, we take the moment about point C so that the only unknown force in the ensuing equilibrium equation would be F_{FG}. In writing the moment equilibrium equation, we utilize the fact that F_{FG} can be transmitted to point K and the horizontal component of F_{FG} at K has no contribution to the equilibrium equation while the vertical component is (2/4.47) F_{FG} = 0.447 F_{FG} as shown in the left of the following figure.

$$\Sigma M_F = 0 \quad -(3)\ F_{BC} + (4)\ 7.5 = 0 \quad \boxed{F_{BC} = 10.00\text{ kN}}$$

$$\Sigma M_C = 0 \quad (10)\ 0.447\ F_{FG} + (8)7.5 - (4)3 = 0,\ \ F_{FG} = -10.74\text{ kN}$$



Two FBDs to find F_{FG}.

Alternatively, we can transmit F_{FG} to point G, and use the horizontal component (4/4.47) F_{FG} = 0.894 F_{FG} in the moment equation, as shown in the right figure above.

$$\Sigma M_C = 0,\ (5)\ 0.894 F_{FG} + (8)7.5 - (4)3 = 0 \quad \boxed{F_{FG} = -10.74\text{ kN}}$$

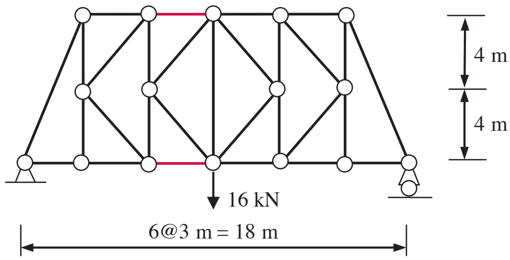
To find F_{FC} we need to go out of the region of the truss to find the moment center (K) as shown in the left of the preceding figure, and use the vertical component of the transmitted F_{FC} at point C.

$$\Sigma M_K = 0 \quad (10)\ 0.6F_{FC} - (2)\ 7.5 + (6)3 = 0 \quad F_{FC} = \boxed{-0.50\text{ kN}}$$

Note that all these additional efforts are caused by the inclined upper chord of the truss.

Example 2.7

Find the force in the top and bottom chord members of the third panel from the left of the K-truss in the following figure.

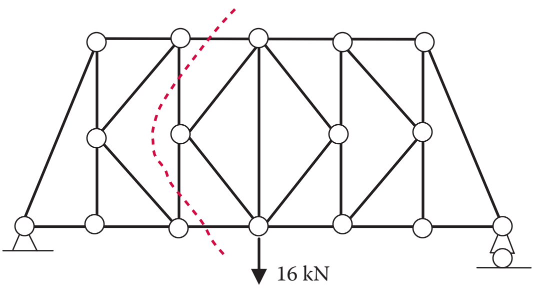


K-truss example.

Solution

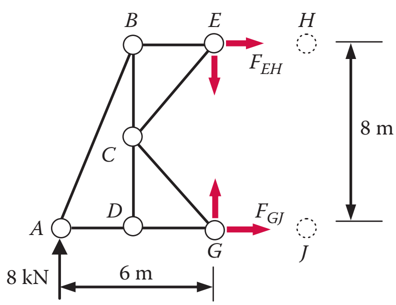
The K-truss is a simple truss that requires a special cut for the solution of top and bottom chord member forces as we shall see shortly. It is stable and determinate.

1. *Find reactions*. Since the truss and the loading are symmetric, the reactions at both supports are easily found to be 8 kN upward and there is no horizontal reaction at the left support.
2. *Establish FBD*. The special cut is shown by the dotted line in the following figure.



A cut to establish FBD for the top and bottom chord member forces.

This particular cut separates the truss into two parts. We shall use the left part as the following FBD.



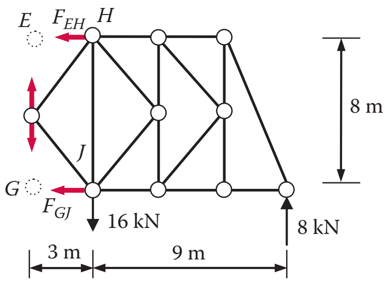
FBD for top and bottom chord member forces.

Although there are four forces at the cut, two of them are on the same line. When moment center is selected at either node *E* or node *G*, these two forces will not appear in the equilibrium equation, leaving only one unknown in each equation.

$$\Sigma M_E = 0, \quad (6)8 - (8)F_{GJ} = 0 \quad \Longrightarrow \quad \boxed{F_{GJ} = 6 \text{ kN}}$$

$$\Sigma M_G = 0, \quad (6)8 - (8)F_{EH} = 0 \quad \Longrightarrow \quad \boxed{F_{EH} = -6 \text{ kN}}$$

Alternatively, we may choose the right part as the FBD. The same results will follow but the computation is slightly more involved.



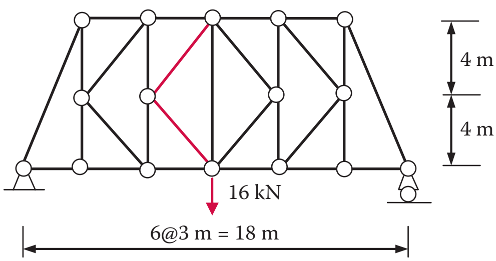
Alternative FBD for the top and bottom chord member forces.

$$\Sigma M_E = 0, \quad (12)8 - (3)16 - (8)F_{GJ} = 0 \quad \Longrightarrow \quad \boxed{F_{GJ} = 6 \text{ kN}}$$

$$\Sigma M_G = 0, \quad (12)8 - (3)16 + (8)F_{EH} = 0 \quad \Longrightarrow \quad \boxed{F_{EH} = -6 \text{ kN}}$$

Example 2.8

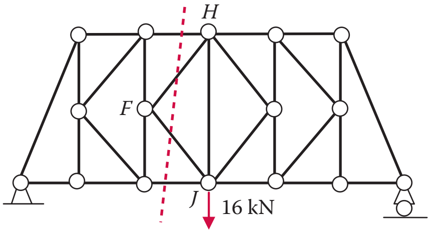
Find the force in the inclined web members of the third panel from the left of the K-truss shown next.



K-truss example, inclined web members.

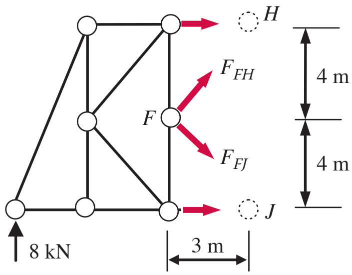
Solution

A different cut from that of the last example is needed to expose the web member forces. First establish FBD. To expose the force in the inclined web members, we may make a cut through the third panel.



A cut to expose web member forces.

This cut exposes the top and bottom member forces, which are known from the last example solution, and the two inclined unknown web member forces, F_{FH} and F_{FJ}.



FBD for the inclined web members of the third panel.

In this case, the application of two force equilibrium equations produces the desired results. In writing the equation for the horizontal forces, we note that the top and bottom chord member forces cancel each other and will not appear in the equation. In fact this is a special feature, which is useful for the analysis of web member forces.

$$\Sigma F_x = 0 \implies (0.6)F_{FH} + (0.6)F_{FJ} = 0$$

$$\Sigma F_y = 0 \implies (0.8)F_{FH} - (0.8)F_{FJ} = 8$$

Solving the simultaneous equations, we obtain

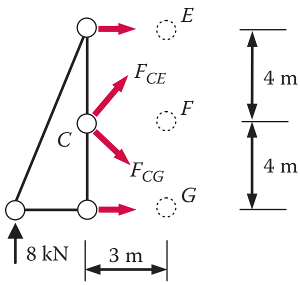
$$\boxed{F_{FH} = 5 \text{ kN}} \quad \text{and} \quad \boxed{F_{FJ} = -5 \text{ kN}}$$

We observe that not only the top and bottom chord members have the same magnitude forces with opposite signs, the inclined web members are in the same situation. Furthermore, in the present example, the inclined web member forces are the same in the second and third panel, that is,

$$F_{CE} = F_{FH} = 5 \text{ kN}$$

$$F_{CG} = F_{FJ} = -5 \text{ kN}$$

This is because the FBD for these member forces yields equations identical to those for the third panel.



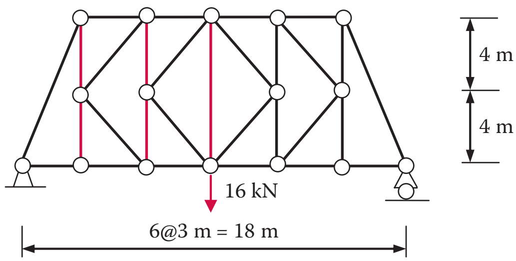
FBD for the inclined web members of the second panel.

$$\Sigma F_x = 0 \implies (0.6)F_{CE} + (0.6)F_{CG} = 0$$

$$\Sigma F_y = 0 \implies (0.8)F_{CE} - (0.8)F_{CG} = 8$$

Example 2.9

Discuss methods to find the force in the vertical web members of the K-truss shown next.

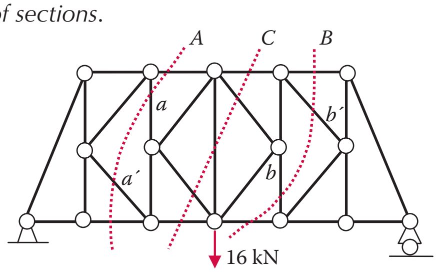


K-truss analysis, vertical web members.

Solution

We can use either the method of sections or the method of joints, but the prerequisite is the same: need to know the force in either the lower inclined web member or the upper inclined web member first.

1. *Method of sections*.



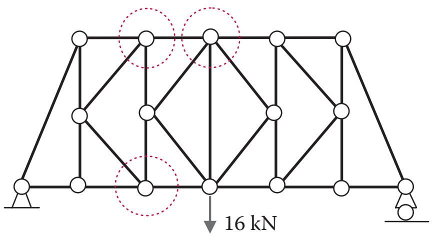
Cuts to expose vertical web members

Cut *A* exposes an upper vertical web member, *a,* and a lower inclined web member, *a′*, whose forces have a vertical component. Once $F_{a'}$ is known, F_a can be computed from the equilibrium equation for forces in the vertical direction of the FBD to the left of the cut.

Cut *B* exposes the forces of a lower vertical web member, *b,* and an upper inclined web member, *b′*; each force has a vertical component. Once $F_{b'}$ is known, F_b can be computed from the equilibrium equation for forces in the vertical direction of the FBD to the right of the cut.

Cut *C* exposes the forces of the central vertical web member and two inclined web members; each force has a vertical component. Once the forces in the two inclined web members are known, the force in the central vertical member can be computed from the equilibrium equation for forces in the vertical direction of the FBD to the left or right of the cut.

2. *Method of joints*.

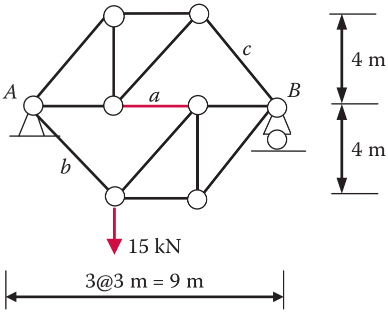


Joints used to solve for vertical web member forces.

At each of the circled joints, the vertical web member forces can be computed if the force of the inclined web member is known. For the central vertical web member, we need to know the forces of the two joining inclined web members. In the present case, since the load is symmetrical, the two inclined web members have identical forces. As a result, the force in the central vertical web member is zero.

Example 2.10

Find the force in member *a* of the compound truss shown next.

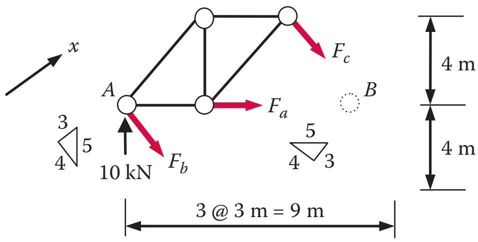


A compound truss example.

Solution

The method of sections is often suitable for compound truss analyses.

1. *Identify truss type*. This is a stable and determinate truss. It is a compound truss with three links, *a*, *b,* and *c*, linking two simple trusses. Each node has at least three joining members. Thus, the method of joints is not a good option. We need to use the method of sections.
2. *Find reactions*. Since the geometry is simple enough, we can see that the horizontal reaction at support *A* is zero and the vertical reactions at support *A* and *B* are 10 kN and 5 kN, respectively.
3. *Establish FBD*. By cutting through the three links, we obtain two FBDs. We choose the upper-left one because it does not involve the applied force.



FBD for the central link.

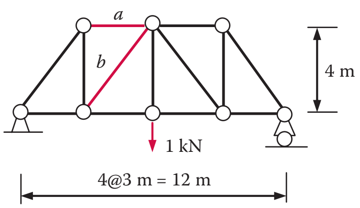
To find F_a we note that the other two unknown forces, F_b and F_c, are parallel to each other, making it impossible to take moment about their intersection. On the other hand, it becomes useful to examine the force equilibrium in the direction perpendicular to the two parallel forces. This direction is denoted in the preceding figure as the *x*-direction. We can decompose the 10 kN reaction at support *A* and the unknown force F_a into components in the *x*-direction and write the equilibrium equation, accordingly.

$$\Sigma F_x = 0, \quad (0.6)10 + (0.6)F_a = 0 \implies \boxed{F_a = -10 \text{ kN}}$$

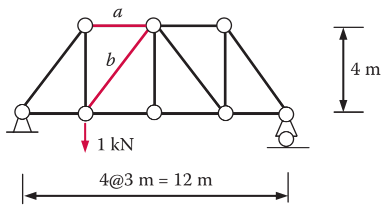
PROBLEM 2.2

Solve for the force in the marked members in each truss shown next.

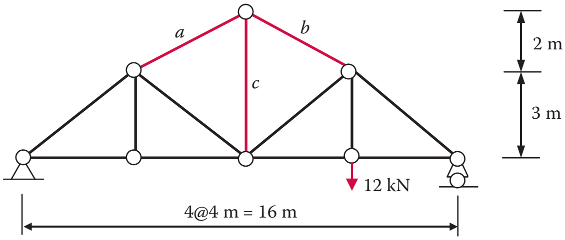
(1–*a*)



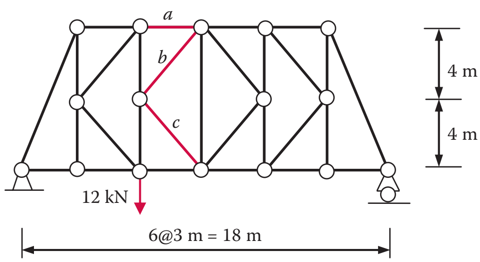
(1–*b*)



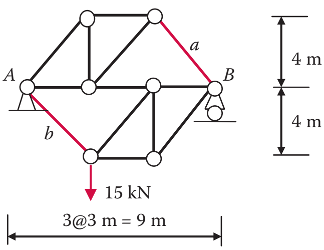
(2)



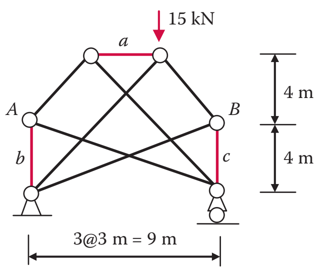
(3)



(4–*a*)



(4–*b*)



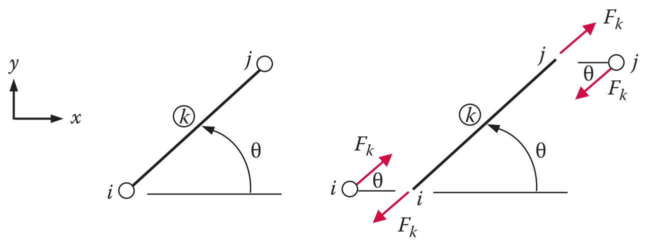
Problem 2.2

2.4 Matrix Method of Joints

The development of the method of joints and the method of sections predates the advent of the electronic computer. Although both methods are easy to apply, it is not practical for trusses with many members or nodes, especially when all member forces are needed. It is, however, easy to develop a matrix formulation of the method of joints. Instead of manually establishing all the equilibrium equations from each joint or from the whole structure and then putting the resulting equations in a matrix form, there is an automated way of assembling the equilibrium equations as shown herein.

Assuming there are N nodes and M member force unknowns and R reaction force unknowns and $2N = M + R$ for a given truss, we know there will be $2N$ equilibrium equations, two from each joint. We shall number the joints or nodes from one to N. At each joint, there are two equilibrium equations. We shall define a global x-y coordinate system that is common to all joints. We note, however, it is not necessary for every node to have the same coordinate system, but it is convenient to do so. The first equilibrium equation at a node will be the equilibrium of forces in the x-direction and the second will be for the y-direction. These equations are numbered from one to $2N$ in such a way that the x-direction equilibrium equation from the ith node will be the $(2i - 1)$th equation and the y-direction equilibrium equation from the same node will be the $(2i)$th equation. In each equation, there will be terms coming from the contribution of member forces, externally applied forces, or reaction forces. We shall discuss each of these forces and develop an automated way of establishing the terms in each equilibrium equation.

Contribution from member forces. A typical member, k, having a starting node, i, and an ending node, j, is oriented with an angle θ from the x-axis as shown next.



Member orientation and the member force acting on member-end and nodes.

The member force, assumed to be tensile, pointing away from the member at both ends and in opposite direction when acting on the nodes, contributes to four nodal equilibrium equations at the two end nodes (we designate the right-hand side [RHS] of an equilibrium equation as positive and put the internal nodal forces to the left-hand side [LHS]):

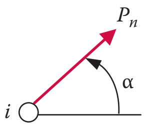
$(2i - 1)$th equation (x-direction): $(-\text{Cos}\theta)F_k$ to the LHS

$(2i)$th equation (y-direction): $(-\text{Sin}\theta)F_k$ to the LHS

$(2j - 1)$th equation (x-direction): $(\text{Cos}\theta)F_k$ to the LHS

$(2j)$th equation (y-direction): $(\text{Sin}\theta)F_k$ to the LHS

Contribution from externally applied forces. An externally applied force, applying at node i with a magnitude of P_n making an angle α from the x-axis, contributes to:

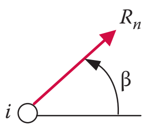


Externally applied force acting at a node.

$(2i - 1)$th equation (x-direction): $(\text{Cos}\alpha)P_n$ to the RHS

$(2i)$th equation (y-direction): $(\text{Sin}\alpha)P_n$ to the RHS

Contribution from reaction forces. A reaction force at node i with a magnitude of R_n making an angle β from the x-axis, contributes to:



Reaction force acting at a node.

$(2i - 1)$th equation (x-direction): $(-\text{Cos}\beta)R_n$ to the LHS

$(2i)$th equation (y-direction): $(-\text{Sin}\beta)R_n$ to the LHS

Input and solution procedures. From the aforementioned definition of forces, we can develop the following solution procedures.

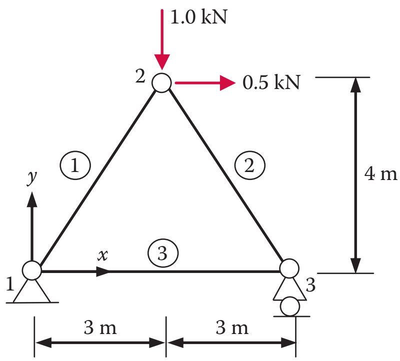
1. Designate member number, global node number, global nodal coordinates, and member starting and end node numbers. From these input, we can compute member length, L, and other data for each member with starting node i and end node j:

$$x = x_j - x_i; \quad y = y_i; \quad L = \sqrt{(\ \ x)^2 + (\ \ y)^2}; \quad \text{Cos}\theta = \frac{x}{L}; \quad \text{Sin}\theta = \frac{y}{L}$$

2. Define reaction forces, including where the reaction is at and the orientation of the reaction, one at a time. The cosine and sine of the orientation of the reaction force should be input directly.
3. Define externally applied forces, including where the force is applied and the magnitude and orientation, defined by the cosine and sine of the orientation angle.
4. Compute the contribution of member forces, reaction forces, and externally applied forces to the equilibrium equation and place them to the matrix equation. The force unknowns are sequenced with the member forces first, $F_1, F_2, \ldots, F_M$, followed by reaction force unknowns, $F_{M+1}, F_{M+2}, \ldots, F_{M+R}$.
5. Use a linear simultaneous algebraic equation solver to solve for the unknown forces.

Example 2.11

Find all support reactions and member forces of the loaded truss shown next.



A truss problem to be solved by the matrix method of joint.

Solution

We shall provide a step-by-step solution.

1. Designate member number, global node number, global nodal coordinates, and member starting and end node numbers, and compute member length, L, and other data for each member.

Nodal Input Data

Node	x-coordinate	y-coordinate
1	0.0	0.0
2	3.0	4.0
3	6.0	0.0

Member Input and Computed Data

Member	Start Node	End Node	Δx	Δy	L	Cosθ	Sinθ
1	1	2	3.0	4.0	5.0	0.6	0.8
2	2	3	3.0	–4.0	5.0	0.6	–0.8
3	1	3	6.0	0.0	6.0	1.0	0.0

2. Define reaction forces.

Reaction Force Data

Reaction	At Node	Cosβ	Sinβ
1	1	1.0	0.0
2	1	0.0	1.0
3	3	0.0	1.0

3. Define externally applied forces.

Externally Applied Force Data

Force	At Node	Magnitude	Cosα	Sinα
1	2	0.5	1.0	0.0
2	2	1.0	0.0	–1.0

4. Compute the contribution of member forces, reaction forces, and externally applied forces to the equilibrium equations and set up the matrix equation.

Contribution of Member Forces

Member Number	Force Number	Equation Number and Value of Entry							
		$2i-1$	Coeff.	$2i$	Coeff.	$2j-1$	Coeff.	$2j$	Coeff.
1	1	1	–0.6	2	–0.8	3	0.6	4	0.8
2	2	3	–0.6	4	0.8	5	0.6	6	–0.8
3	3	1	–1.0	2	0.0	5	1.0	6	–0.0

Contribution of Reaction Forces

Reaction Number	Force Number	Equation Number and Value of Entry			
		$2i-1$	Coeff.	$2i$	Coeff.
1	4	1	–1.0	2	0.0
2	5	1	0.0	2	–1.0
3	6	5	0.0	6	–1.0

Contribution of Externally Applied Forces

	Equation Number and Value of Entry			
Applied Force	**2*i* – 1**	**Coeff.**	**2*i***	**Coeff.**
1	1	1.0	2	0.0
2	1	0.0	2	1.0
3	5	0.0	6	1.0

Using the aforementioned data, we obtain the equilibrium equation in matrix form:

$$
\begin{matrix}
-0.6 & 0 & -1.0 & -1.0 & 0.0 & 0 \\
-0.8 & 0 & 0.0 & 0.0 & -1.0 & 0 \\
0.6 & -0.6 & 0 & 0 & 0 & 0 \\
0.8 & 0.8 & 0 & 0 & 0 & 0 \\
0 & 0.6 & 1.0 & 0 & 0 & 0.0 \\
0 & -0.8 & 0.0 & 0 & 0 & -1.0
\end{matrix}
$$

$$
\begin{matrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{matrix}
=
\begin{matrix} 0 \\ 0 \\ 0.5 \\ -1.0 \\ 0 \\ 0 \end{matrix}
$$

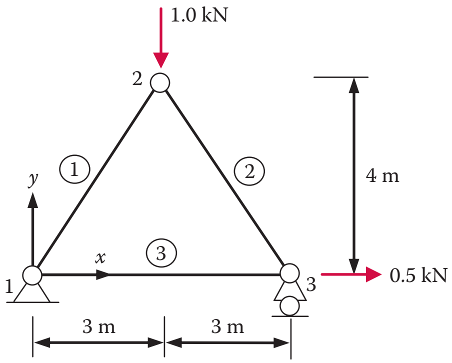
5. Solve for the unknown forces. An equation solver produces the following solutions, where the units are added by the user:

$F_1 = -0.21$ kN; $F_2 = -1.04$ kN; $F_3 = 0.62$ kN;

$F_4 = -0.50$ kN; $F_5 = 0.17$ kN; $F_6 = 0.83$ kN

PROBLEM 2.3

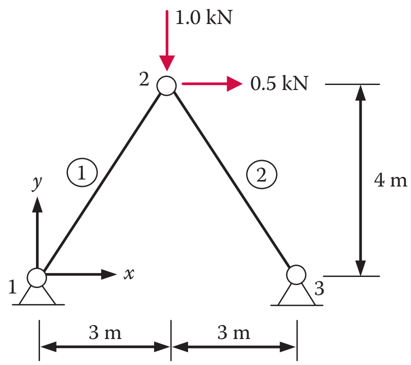
The loaded truss shown next is different from that in Example 2.11 only in the externally applied loads. Modify the results of Example 2.11 to establish the matrix equilibrium equation for this problem.



Problem 2.3

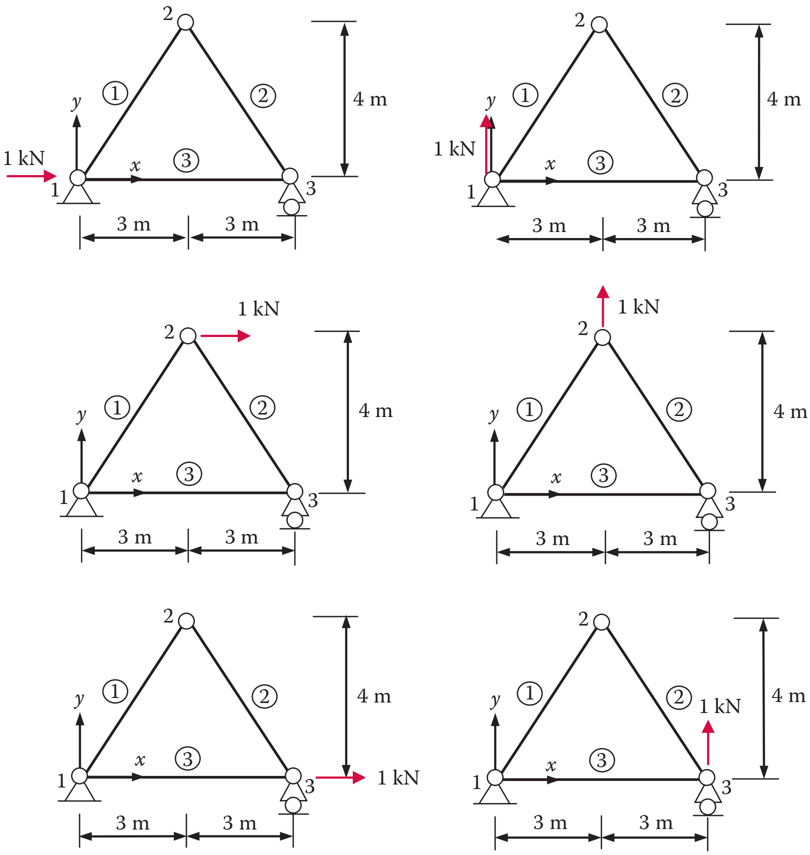
PROBLEM 2.4

Establish the matrix equilibrium equation for the loaded truss shown next.



Problem 2.4

Force transfer matrix. Consider the same three-bar truss as in the previous example problems. If we apply a unit force one at a time at one of the six possible positions, that is, *x*- and *y*-directions at each of the three nodes, we have six separate problems as shown in the following figure.



Truss with unit loads.

The matrix equilibrium equation for the first problem appears in the following form:

$$\begin{bmatrix} -0.6 & 0 & -1.0 & -1.0 & 0.0 & 0 \\ -0.8 & 0 & 0.0 & 0.0 & -1.0 & 0 \\ 0.6 & -0.6 & 0 & 0 & 0 & 0 \\ 0.8 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 1.0 & 0 & 0 & 0.0 \\ 0 & -0.8 & 0.0 & 0 & 0 & -1.0 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2.2)$$

The RHS of the equation is a unit vector. For the other five problems the same matrix equation will be obtained only with the RHS changed to unit vectors with the unit load at different locations. If we compile the six RHS vectors into a matrix, it becomes an identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = I \qquad (2.3)$$

The six matrix equations for the six problems can be put into a single matrix equation if we define the six-by-six matrix at the LHS of Equation 2.2 as matrix ***A***,

$$\begin{matrix} -0.6 & 0 & -1.0 & -1.0 & 0.0 & 0 \\ -0.8 & 0 & 0.0 & 0.0 & -1.0 & 0 \\ 0.6 & -0.6 & 0 & 0 & 0 & 0 \\ 0.8 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.6 & 1.0 & 0 & 0 & 0.0 \\ 0 & -0.8 & 0.0 & 0 & 0 & -1.0 \end{matrix} = \boldsymbol{A} \qquad (2.4)$$

and the six force unknown vectors as a single six-by-six matrix ***F***:

$$\boldsymbol{A}_{6\times6}\,\boldsymbol{F}_{6\times6} = \boldsymbol{I}_{6\times6} \qquad (2.5)$$

The solution to the six problems, obtained by solving the six problems one at a time, can be compiled into the single matrix ***F***,

$$\boldsymbol{F}_{6\times6} = \begin{matrix} 0.0 & 0.0 & 0.83 & 0.63 & 0.0 & 0.0 \\ 0.0 & 0.0 & -0.83 & 0.63 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.5 & -0.38 & 1.0 & 0.0 \\ -1.0 & 0.0 & -1.0 & 0.0 & -1.0 & 0.0 \\ 0.0 & -1.0 & -0.67 & -0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.67 & -0.5 & 0.0 & -1.0 \end{matrix} \qquad (2.6)$$

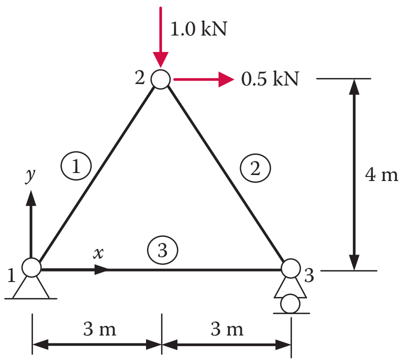
where each column of the matrix ***F*** is a solution to a unit load problem. Matrix ***F*** is called the force transfer matrix. It transfers a unit load into the member force and reaction force unknowns. It is also the "inverse" of the matrix ***A***, as apparent from Equation 2.5.

We can conclude that the nodal equilibrium conditions are completely characterized by the matrix ***A***. The inverse of ***A***, matrix ***F***, is the force transfer matrix, which transfers any unit load into member and reaction forces.

If the force transfer matrix is known, either by solving the unit load problems one at a time or by solving the matrix equation, Equation 2.5, with an equation solver, then the solution to any other loads can be obtained by a linear combination of the force transfer matrix. Thus the force transfer matrix also characterizes completely the nodal equilibrium conditions of the truss. The force transfer matrix is particularly useful if there are many different loading conditions that one wants to solve for. Instead of solving for each load separately, one can solve for the force transfer matrix, then solve for any other load by a linear combination as shown in the following example.

Example 2.12

Find all support reactions and member forces of the loaded truss shown next, knowing that the force transfer matrix is given in Equation 2.6.



A truss problem to be solved with the force transfer matrix.

Solution

The given loads can be cast into a load vector, which can be easily computed as the combination of the third and fourth unit load vectors as shown next.

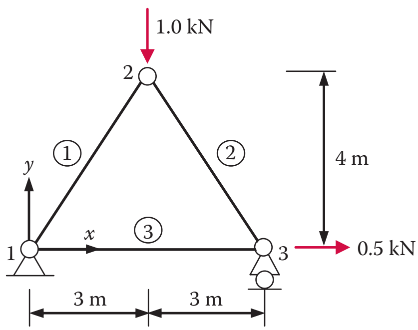
$$\begin{Bmatrix} 0 \\ 0 \\ 0.5 \\ -1.0 \\ 0 \\ 0 \end{Bmatrix} = (0.5) \begin{Bmatrix} 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + (-1.0) \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1.0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2.7)$$

The solution is then the same linear combination of the third and fourth vectors of the force transfer matrix:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = (0.5)\begin{Bmatrix} 0.83 \\ -0.83 \\ 0.5 \\ -1.0 \\ -0.67 \\ 0.67 \end{Bmatrix} + (-1.0)\begin{Bmatrix} 0.63 \\ 0.63 \\ -0.38 \\ 0.0 \\ -0.5 \\ -0.5 \end{Bmatrix} = \begin{Bmatrix} -0.21 \\ -1.04 \\ 0.62 \\ -0.50 \\ 0.17 \\ 0.83 \end{Bmatrix} \text{kN}$$

PROBLEM 2.5

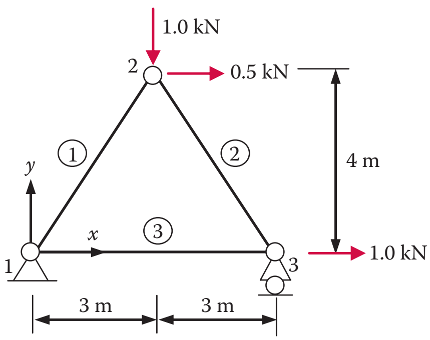
The loaded truss shown next is different from that in Example 2.11 only in the externally applied loads. Use the force transfer matrix of Equation 2.6 to find the solution.



Problem 2.5

PROBLEM 2.6

The following loaded truss is different from that in Example 2.11 only in the externally applied loads. Use the force transfer matrix of Equation 2.6 to find the solution.



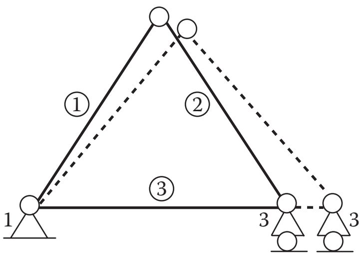
Problem 2.6

3

Truss Analysis: Force Method—Part II

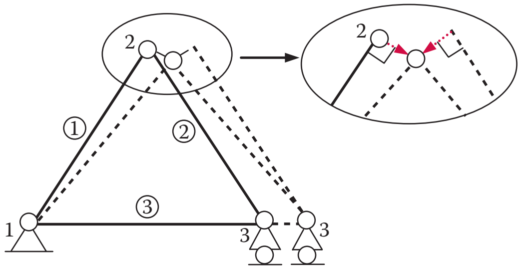
3.1 Truss Deflection

A truss has a designed geometry and an as-built geometry. Displacement of nodes from their designed positions can be caused by manufacturing or construction errors. Displacement of nodes from their as-built positions is induced by applied loads or temperature changes. Truss deflection refers to either deviation from the designed positions or from the as-built positions. No matter what is the cause for deflection, one or more members of the truss may have experienced a change of length. Such a change of length makes it necessary for the truss to adjust to the change by displacing the nodes from its original position as shown in the following figure. Truss deflection is the result of displacements of some or all of the truss nodes, and nodal displacements are caused by the change of length of one or more members.



Elongation of member 3 induces nodal displacements and truss deflection.

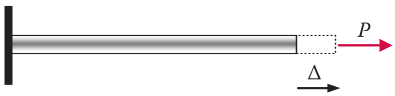
From the figure, it is clear that geometric relations determine nodal displacements. In fact, nodal displacements can be obtained graphically for any given length changes as illustrated in the following figure.



Displacements of nodes 3 and 2 determined graphically.

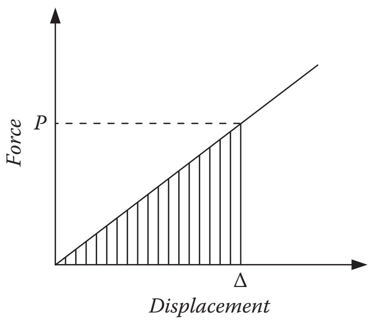
Even for more complex truss geometry, a graphical method can be developed to determine all the nodal displacements. In the age of computers, however, such a graphical method is no longer practical and necessary. Truss nodal displacements can be computed using the matrix displacement method or, as we shall see, using the force method, especially when all the member elongations are known. The method we shall introduce is the unit load method, the derivation of which requires an understanding of the concept of work done by a force. A brief review of the concept follows.

Work and virtual work. Consider a bar fixed at one end and being pulled at the other end by a force P. The displacement at the point of application of the force P is Δ as shown in the following figure.



Force and displacement at the point of application of the force.

As assumed throughout the text, the force is considered a static force, that is, its application is such that no dynamic effects are induced. To put it simply, the force is applied gradually, starting from zero and increasing its magnitude slowly until the final magnitude P is reached. Consequently, as the force is applied, the displacement increases from zero to the final magnitude Δ proportionally as shown next, assuming the material is linearly elastic.

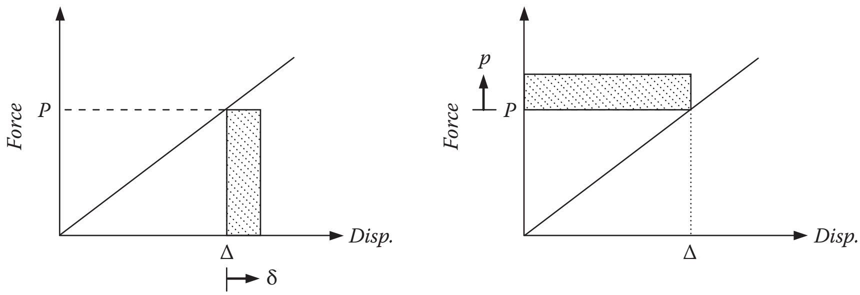


Force-displacement relationship.

The work done by the force P is the integration of the force-displacement function and is equal to the triangular area shown in the preceding figure. Denoting work by W we obtain

$$W = \tfrac{1}{2} P\Delta \tag{3.1}$$

Now, consider two additional cases of load displacement after the load P is applied. The first is the case with the load level P held constant and an additional amount of displacement δ is induced. If the displacement is not real but one which we imagined as happening, then the displacement is called a virtual displacement. The second is the case with the displacement level held constant, but an additional load p is applied. If the load is not real but one that we imagined as happening, then the force is called a virtual force. In both cases, we can construct the load-displacement history as shown in the following figure.



A case of virtual displacement (left) and a case of virtual force (right).

The additional work done is called virtual work in both cases, although they are induced with different means. The symbol for virtual work is δW, which is to be differentiated from dW, the real increment of W. The symbol δ can be considered as an operator that generates a virtual increment just as the symbol d is an operator that generates an actual increment. From the preceding figure, we can see that the virtual work is different from the real work in Equation 3.1.

$$\delta W = P\delta \text{ due to virtual displacement } \delta \tag{3.2}$$

$$\delta W = p\Delta \text{ due to virtual force } p \tag{3.3}$$

In both cases, the factor ½ in Equation 3.1 is not present.

Energy principles. The work or virtual work by itself does not provide any equations for the analysis of a structure, but it is associated with numerous energy principles that contain useful equations for structural analysis. We will introduce only three: conservation of mechanical energy, principle of virtual displacement, and principle of virtual force–unit load method.

The *conservation of mechanical energy* principle states that the work done by all external forces on a system is equal to the work done by all internal forces in the system for a system in equilibrium.

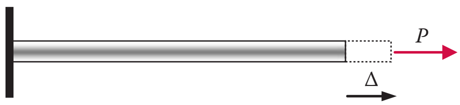
$$W_{ext} = W_{int} \tag{3.4}$$

where

W_{ext} = work done by external forces
W_{int} = work done by internal forces

EXAMPLE 3.1

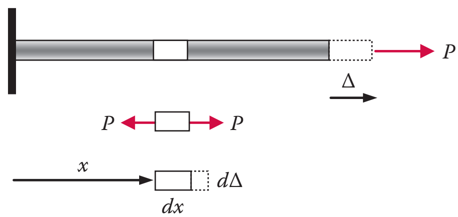
Find the displacement at the loaded end, given the bar shown next has a Young's modulus *E*, cross-sectional area *A*, and length *L*.



Example problem on conservation of mechanical energy.

Solution

We can use Equation 3.4 to find the tip displacement Δ. The internal work done can be computed using the information contained in the following figure.



Internal forces acting on an infinitesimal element.

The elongation of the infinitesimal element, $d\Delta$, is related to the axial force *P* by

$$d\ \ = \varepsilon\, dx = \frac{\sigma dx}{E} = \frac{Pdx}{EA}$$

where ε and σ are the normal strain and stress in the axial direction, respectively. The internal work done by *P* on the infinitesimal element is then

$$dW_{int} = \frac{1}{2}\mathrm{P}(\mathrm{d}\ \ ) = \frac{1}{2}P\ \frac{Pdx}{EA}$$

The total internal work done is the sum of the work over all the infinitesimal elements:

$$W_{int} = \int dW = \int \frac{1}{2} P \quad \frac{Pdx}{EA} \quad = \frac{1}{2} \frac{P^2 L}{EA}$$

The external work done is simply:

$$W_{ext} = \frac{1}{2} P$$

Equation 3.4 leads to

$$\frac{1}{2} P \quad = \frac{1}{2} \frac{P^2 L}{EA}$$

From which, we obtain

$$= \frac{PL}{EA}$$

This is of course the familiar formula for the elongation of an axially loaded prismatic bar. We went through the derivation to show how the principle of conservation of energy is applied. We note the limitations of the energy conservation principle: unless there is a single applied load and the displacement we want is the displacement at the point of application of the single load, the energy conservation equation, Equation 3.4, is not very useful for finding displacements. This energy conservation principle is often used for the derivation of other useful formulas.

Imagine a virtual displacement or displacement system is imposed on a structure after a real load system has already been applied to the structure. This *principle of virtual displacement* is expressed as

$$\delta W = \delta U \tag{3.5}$$

where

δW = virtual work done by external forces upon the virtual displacement
δU = virtual work done by internal forces upon the virtual displacement

The application of Equation 3.5 will produce an equilibrium equation relating the external forces to the internal forces. This principle is sometimes referred to as the principle of virtual work. It is often used in conjunction with the displacement method of analysis. Since we are developing the force method of analysis herein, we shall not explore the application of this principle further at this point.

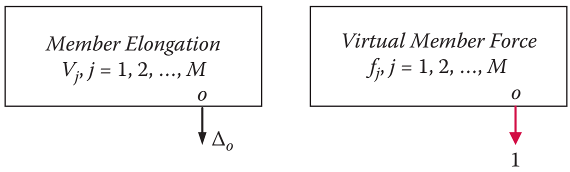
To develop the unit load method, we express the principle of virtual force as

$$\delta W = \delta U \tag{3.6}$$

where

δW = work done by external virtual forces upon a real displacement system
δU = work done by internal virtual forces upon a real displacement system

We illustrate this principle in the context of the truss problems. Consider a truss system represented by rectangular boxes as in the following figure.



A real system (left) and a virtual load system (right).

The left figure represents a truss with known member elongations, V_j, $j = 1$, 2, …, M. The cause of the elongation is not relevant to the principle of virtual force. We want to find the displacement, Δ_o, in a certain direction at a point o. We create an identical truss and apply an imaginary (virtual) unit load at point o in the direction we want, as shown in the right figure. The unit load produces a system of internal member forces f_j, j =1, 2, …, M.

The work done by the unit load (virtual system) upon the real displacement is

$$\delta W = 1\ (\Delta_o)$$

The work done by internal virtual forces upon a real displacement system (elongation of each member) is

$$\delta U = \sum_{j=1}^{M} f_j V_j$$

The principle of virtual force states that

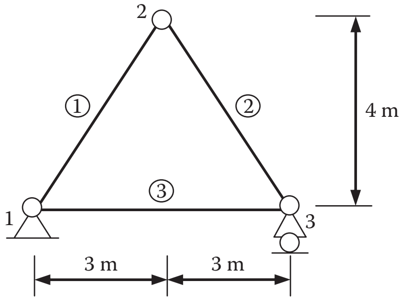
$$1(\ _o) = \sum_{j=1}^{M} f_j V_j \tag{3.7}$$

Equation 3.7 gives the displacement we want. It also shows how simply one can compute the displacement. Only two sets of data are needed: the elongation of each member and the internal virtual force of each member corresponding to the virtual unit load.

Before we give a proof of the principle, we shall illustrate the application of it in the following example.

EXAMPLE 3.2

Find the vertical displacement at node 2 of the truss shown next, given (a) bar 3 has experienced a temperature increase of 14°C, (b) bar 3 has a manufacturing error of 1 mm overlength, and (c) a horizontal load of 16 kN has been applied at node 3 acting toward the right. All bars have Young's modulus E = 200 GPa, cross-sectional area A = 500 mm^2, and length L as shown. The linear thermal expansion coefficient is $\alpha = 1.2(10^{-5})$/°C.



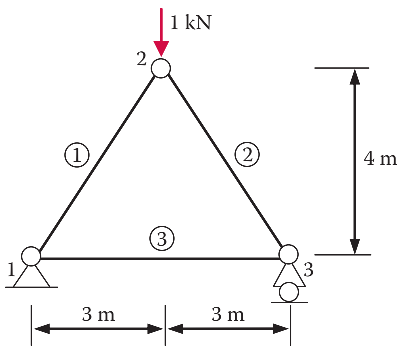
Example for the unit load method.

Solution

All three conditions result in the same consequences, the elongation of member 3 only, but nodes 2 and 3 will be displaced as a result. Denoting the elongation of member 3 as V_3, we have

a. $V_3 = \alpha(\Delta T)L = 1.2(10^{-5})$/°C (14°C) (6000 mm) = 1 mm
b. V_3 = 1mm
c. V_3 = 16 kN (6 m)/[200(10^6) kN/m^2 (500)(10^{-6})m^2] = 0.001 m = 1 mm

Next we need to find f_3. Note that we do not need f_1 and f_2 because V_1 and V_2 are both zero. Still, we need to solve the unit load problem posed in the following figure in order to find f_3.



A virtual load system with an applied unit load.

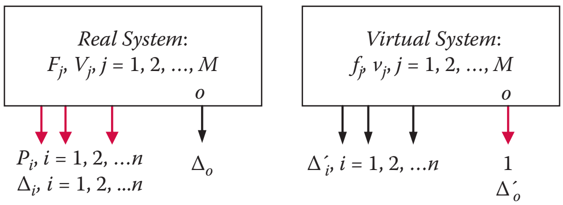
The member forces are easily obtained and $f_3 = 0.375$ kN for the given downward unit load. A direct application of Equation 3.7 yields

$$1\,\text{KN}(\ _o) = \sum_{j=1}^{M} f_j V_j = f_3\, V_3 = 0.375\,\text{kN}\,(1\,\text{mm})$$

and

$$\Delta_o = 0.38 \text{ mm (downward)}$$

We now give a derivation of this principle in the context of truss member elongation caused by applied loads. Consider a truss system, represented by the rectangular boxes in the following figure, loaded by two different loading systems: a real load system and a virtual unit load. The real load system is the actual load applied to the truss under consideration. The unit load is a virtual load of unit magnitude applied at a point whose displacement we want and applied in the direction of the displacement we want. These loads are shown outside of the boxes together with the displacements under the loads. The corresponding internal member force and member elongation are shown inside of the boxes.



A real system (left) and a virtual system (right).

In the figure,

P_i: ith applied real load

Δ_i: real displacement under the ith applied load

Δ_o: displacement we want to find

F_j: ith member force due to the applied real load

V_j: ith member elongation due to the applied real load

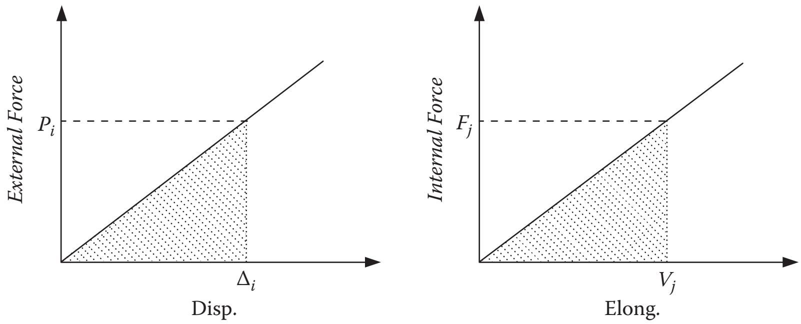
f_j: jth member force due to the virtual unit load

v_j: jth member elongation due to the virtual unit load

Δ'_i: displacements in the direction of the ith real applied load but induced by the virtual unit load

We will now consider three loading cases:

1. The case of externally applied loads act alone. The applied loads, P_i, and internal member forces, F_j, generate work according to the force-displacement histories shown in the following figure.

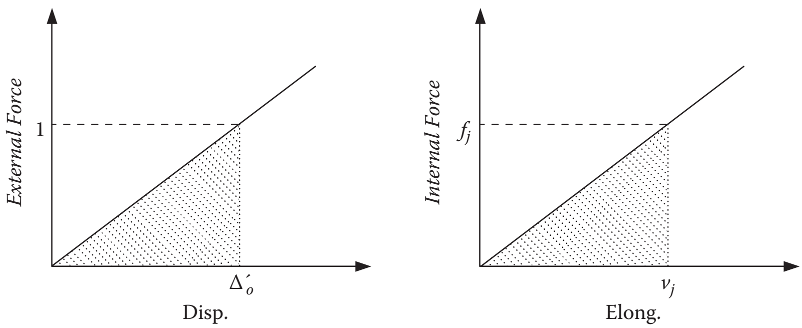


Work done by external force (left) and work done by internal force (right), case 1.

The conservation of mechanical energy principle calls for

$$\frac{1}{2}\sum_{i=1}^{n} P_i \quad {}_i = \frac{1}{2}\sum_{j=1}^{M} F_j V_j \tag{3.8}$$

2. The case of virtual unit load acts alone. The following figure illustrates the force-displacement histories.

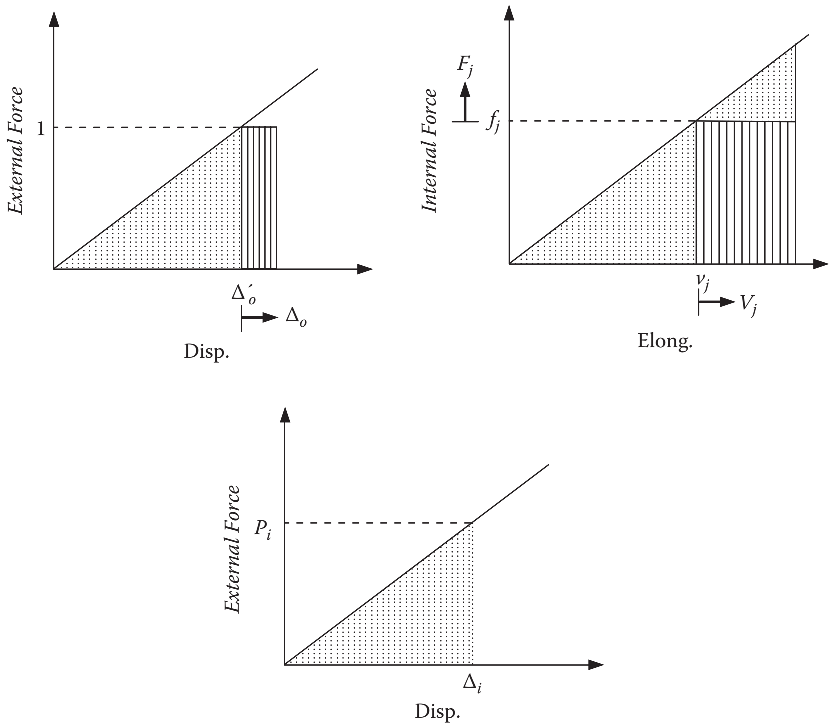


Work done by external force (left) and work done by internal force (right), case 2.

Again, the energy conservation principle calls for

$$\frac{1}{2}(1) \quad {}_o = \frac{1}{2}\sum_{j=1}^{M} f_j v_j \tag{3.9}$$

3. The case of the virtual unit load being applied first, followed by the application of the real loads. The force-displacement histories are shown in the following figure.



Work done by external forces (left) and work done by internal forces (right), case 3.

Application of the energy conservation principle leads to

$$\frac{1}{2}(1)\quad_{o} + \frac{1}{2}\sum_{i=1}^{n} P_i \quad_i + (1)\,(\quad_o) = \frac{1}{2}\sum_{j=1}^{M} f_j v_j + \frac{1}{2}\sum_{j=1}^{M} F_j V_j + \sum_{j=1}^{M} f_j V_j \qquad (3.10)$$

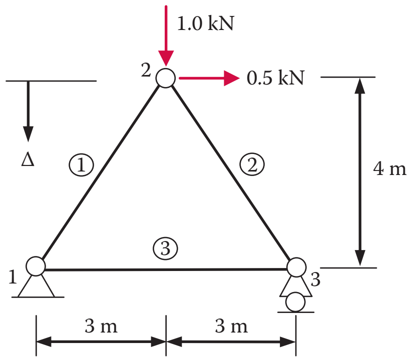
Subtracting Equation 3.10 by Equation 3.8 and Equation 3.9 yields

$$(1)\,(\quad_o) = \sum_{j=1}^{M} f_j V_j$$

which is the principle of virtual force statement (Equation 3.7) expressed in the unit load context.

EXAMPLE 3.3

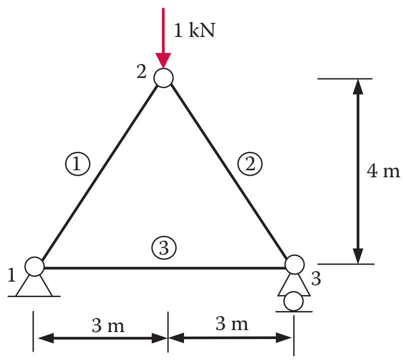
Find the vertical displacement at node 2 of the truss shown next, given E = 10 GPa and A = 100 cm^2 for all bars.



Example to find a nodal displacement by the unit load method.

Solution

Using the unit load method requires the solution for the member elongation, V_i, under the applied load and the virtual member force, f_i, under the unit load as shown.



A unit load applied in the direction of the displacement to be solved.

The computation in Equation 3.7 is carried out in the following table, keeping in mind that the virtual member forces are associated with the virtual unit load and the nodal displacement is associated with the member elongation as indicated next.

$$1\,(\Delta) = \Sigma f_i\,(V_i)$$

Since both the real loading problem and the virtual unit load problem have been solved in earlier examples, we shall not go through the process again except to note that in order to find member elongation V_i, we must find member force F_i first. The sequence of computation is implied in the layout of the following table.

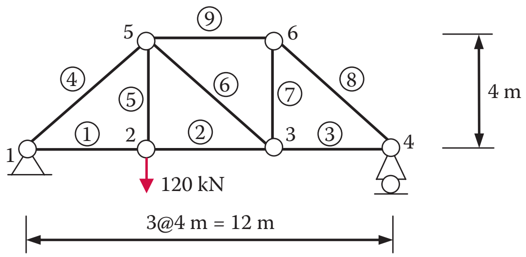
Computing the Vertical Displacement at Node 2

Member	Real Load			Unit Load	Cross-Term
	F (kN)	EA/L (kN/m)	V_i (mm)	f_i kN	f_iV_i (kN-mm)
1	–0.20	20,000	–0.011	–0.625	0.0069
2	–1.04	20,000	–0.052	–0.625	0.0325
3	0.62	16,700	0.037	0.375	0.0139
Σ					0.0533

Thus, the vertical displacement at node 2 is 0.0533 mm, downward.

EXAMPLE 3.4

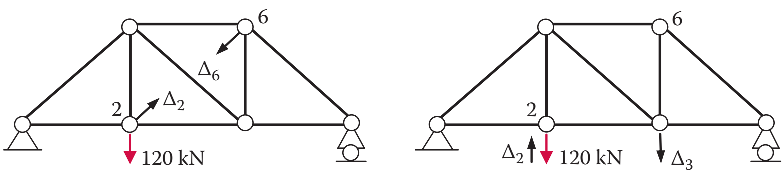
Find (a) the relative movement of nodes 2 and 6 in the direction joining them and (b) the rotation of bar 2, given $E = 10$ GPa and $A = 100$ cm^2 for all bars.



Example on finding relative displacements.

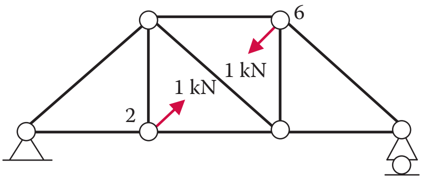
Solution

The nodal displacements related to the relative movement and rotation in question are depicted in the following figure.



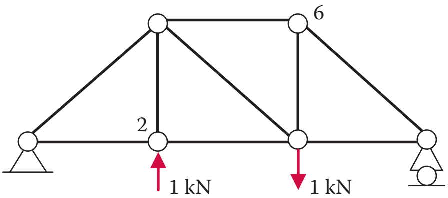
Relevant nodal displacements.

To find the relative movement between node 2 and node 6, we can apply a pair of unit loads as shown next. We shall call this case as case (a).



Unit load for movement between node 2 and node 6 in the direction of 2–6, case (a).

To find the rotation of bar 2, we can apply a pair of unit loads as shown next. We shall call this case as case (b).



Unit load to find rotation of bar 2, case (b).

The computation entails the following:

1. Find member forces, F_i, corresponding to the real applied load.
2. Compute member elongation, V_i.
3. Find member force, f_{ia}, corresponding to the case (a) load.
4. Find member force, f_{ib}, corresponding to the case (b) load.
5. Apply Equation 3.7 to find the displacement quantities.
6. Make necessary adjustments to put member rotation in the right unit.

Steps 1 to 5 are summarized in the following table.

Computing for Relative Displacement Quantities

Member	Real Load			Unit Load		Cross-Term	
	F (kN)	EA/L (kN/m)	V_i (mm)	f_{ia} (kN)	f_{ib} (kN)	$f_{ia}V_i$ (kN-mm)	$f_{ib}V_i$ (kN-mm)
1	80.00	25,000	3.20	0.00	–0.33	0.00	–1.06
2	80.00	25,000	3.20	–0.71	–0.33	–2.26	–1.06
3	40.00	25,000	1.60	0.00	0.33	0.00	0.53
4	–113.13	17,680	–6.40	0.00	0.47	0.00	–3.00
5	120.00	25,000	4.80	–0.71	–1.00	–3.40	–4.80
6	–56.56	17,680	–3.20	1.00	0.94	–3.20	–3.00
7	40.00	25,000	1.60	–0.71	0.33	–1.14	0.53
8	–56.56	17,680	–3.20	0.00	–0.47	0.00	1.50
9	–40.00	25,000	–1.60	–0.71	–0.33	1.14	0.53
Σ						–8.86	–9.83

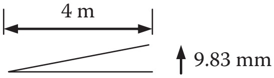
For case (a), Equation 3.7 becomes

$$(1)(\;_2 + \;_6) = \sum_{j=1}^{M} f_j V_j = -8.86 \text{ mm}$$

The relative movement in the direction of 2–6 is 8.86 mm in the opposite direction of what was assumed for the unit load, that is, away from each other, not toward each other.

For case (b), Equation 3.7 becomes

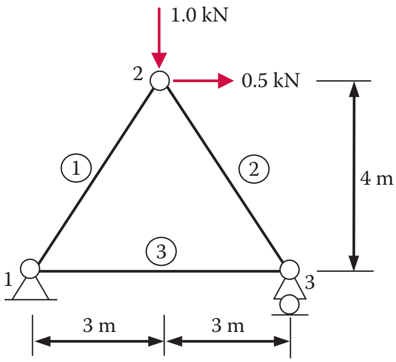
$$(1)\,(\Delta_2 + \Delta_3) = \sum_{j=1}^{M} f_j V_j = -9.83 \text{ mm}$$



For the rotation of bar 2, we note that the –9.83 mm computed represents a relative vertical movement between node 2 and node 3 of 9.83 mm in the opposite direction of what was assumed for the pair of unit loads. That relative vertical movement translates into a counterclockwise rotation of 9.83 mm/4000 mm = 0.0025 radian.

PROBLEM 3.1

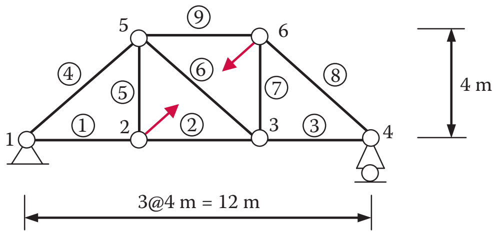
Find the horizontal displacement of node 2 of the loaded truss shown next, given $E = 10$ GPa and $A = 100$ cm^2 for all bars.



Problem 3.1

PROBLEM 3.2

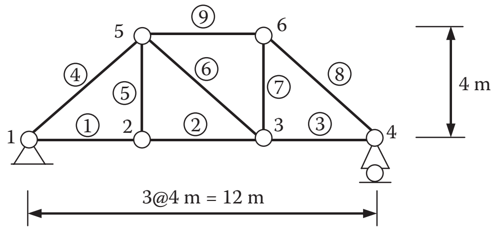
Find the horizontal displacement of node 2 of the loaded truss shown next, given $E = 10$ GPa and $A = 100$ cm^2 for all bars. The magnitude of the pair of loads is 141.4 kN.



Problem 3.2.

PROBLEM 3.3

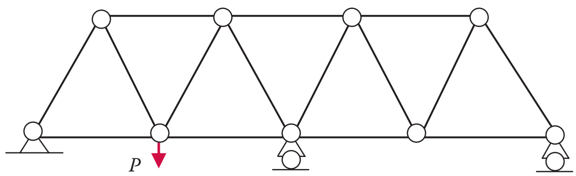
The lower chord members 1, 2, and 3 of the truss shown next are having a 20°C increase in temperature. Find the horizontal displacement of node 5, given E = 10 GPa and A = 100 cm^2 for all bars and the linear thermal expansion coefficient is $\alpha = 5(10^{-6})/°C$.



Problem 3.3

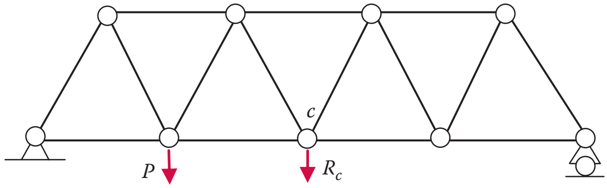
3.2 Indeterminate Truss Problems: Method of Consistent Deformations

The truss shown in the following figure has 15 members (M = 15) and four reaction forces (R = 4). The total number of force unknowns is 19. There are nine nodes (N = 9). Thus, $M + R - 2N = 1$. The problem is statically indeterminate to the first degree. In addition to the 18 equilibrium equations we can establish from the nine nodes, we need to find one more equation in order to solve for the 19 unknowns. This additional equation can be established by considering the consistency of deformations (deflections) in relation to geometrical constraints.



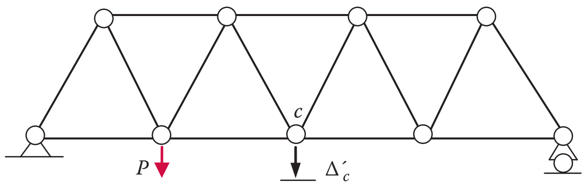
Statically indeterminate truss with one degree of redundancy.

We notice that if the vertical reaction at the central support is known, then the number of force unknowns becomes 18 and the problem can be solved by the 18 equilibrium equations from the nine nodes. The key to the solution is then to find the central support reaction, which is called the redundant force. Denoting the vertical reaction of the central support by R_c, the original problem is equivalent to the problem shown next as far as force equilibrium is concerned.



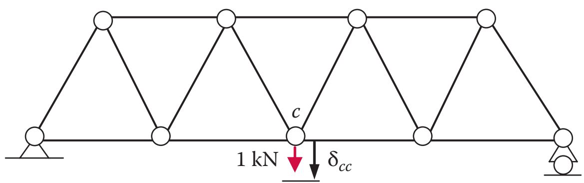
Statically equivalent problem with the redundant force R_c as unknown.

The truss above, with the central support removed, is called the *primary structure.* Note that the primary structure is statically determinate. The magnitude of R_c is determined by the condition that the vertical displacement of node c of the primary structure, due to (1) the applied load P and (2) the redundant force R_c, is zero. This condition is consistent with the geometric constraint imposed by the central support on the original structure. The vertical displacement at node c due to the applied load P can be determined by solving the problem associated with the primary structure as shown next.



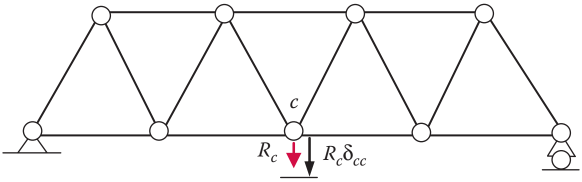
Displacement of node c of the primary structure due to the applied load.

The displacement of node c due to the redundant force R_c cannot be computed directly because R_c itself is unknown. We can compute, however, the displacement of node c of the primary structure due to a unit load in the direction of R_c. This displacement is denoted by δ_{cc}, the double subscript cc signifies displacement at c (first subscript) due to a unit load at c (second subscript).



Displacement at c due to a unit load at c.

The vertical displacement at c due to the redundant force R_c is then $R_c\delta_{cc}$, as shown in the following figure.



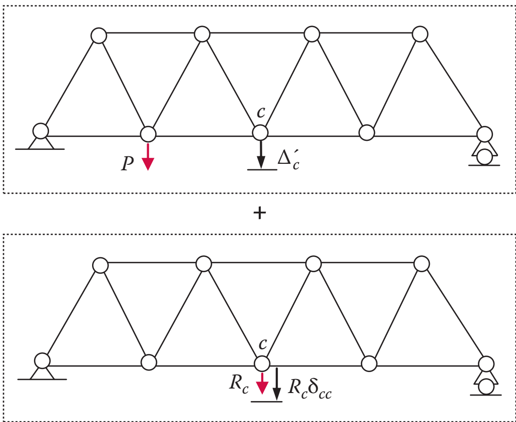
Displacement at c due to the redundant force R_c.

The condition that the total vertical displacement at node c, Δ_c, be zero is expressed as

$$\Delta_c = \Delta'_c + R_c\delta_{cc} = 0 \tag{3.11}$$

This is the additional equation needed to solve for the redundant force R_c. Once R_c is obtained, the rest of the force unknowns can be computed from the regular joint equilibrium equations. Equation 3.11 is called the *condition of compatibility.*

We may summarize the concept behind the aforementioned procedures by pointing out that the original problem is solved by replacing the indeterminate truss with a determinate primary structure and superposing the solutions of two problems, each determinate, as shown next.

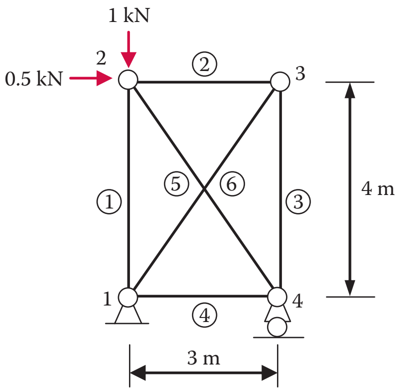


The superposition of two solutions.

And, the key equation is the condition that the total vertical displacement at node c must be zero, consistent with the support condition at node c in the original problem. This method of analysis for statically indeterminate structures is called the *method of consistent deformations*.

EXAMPLE 3.5

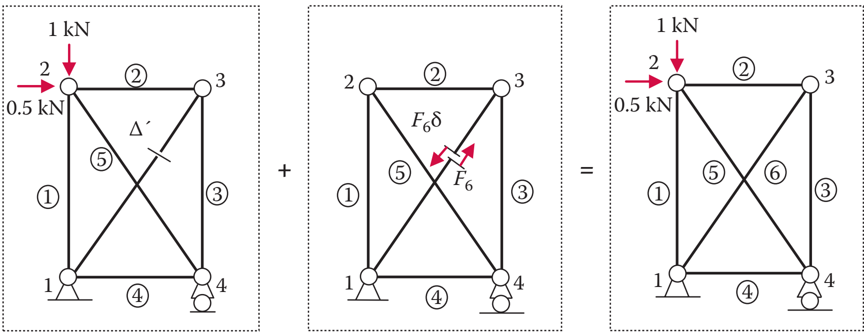
Find the force in bar 6 of the truss shown next, given E = 10 GPa and A = 100 cm^2 for all bars.



Example of an indeterminate truss with one redundant force.

Solution

The primary structure is obtained by introducing a cut at bar 6 as shown in the left panel of the following figure. The original problem is replaced by that of the left panel and that of the middle panel.

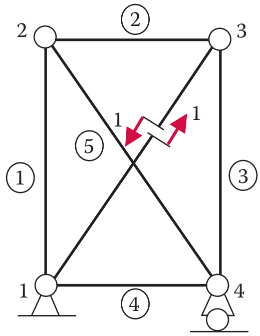


Superposition of two solutions.

The condition of compatibility in this case requires that the total relative displacement across the cut obtained from the superposition of the two solutions be zero:

$$\Delta = \Delta' + F_6\,\delta = 0$$

where Δ' is the overlap length (opposite of a gap) at the cut due to the applied load and δ is, as defined in the following figure, the overlap length across the cut due to a pair of unit loads applied at the cut.



Overlap displacement at the cut due to the unit-force pair.

The computation needed to find Δ′ and δ is tabulated next.

Computing for Δ′ and δ

Member	Real Load			For Δ′		For δ	
	F_i (kN)	EA/L (kN/m)	V_i (mm)	f_i (kN/kN)	$f_i V_i$ (mm)	v_i (mm/kN)	$f_i v_i$ (mm/kN)
1	–0.33	25,000	–0.013	–0.8	0.010	–0.032	0.026
2	0	33,333	0	–0.6	0	–0.018	0.011
3	0	25,000	0	–0.8	0	–0.032	0.026
4	0.50	33,333	0.015	–0.6	–0.009	–0.018	0.011
5	–0.83	20,000	–0.042	1.0	–0.042	0.050	0.050
6	0	20,000	0	1.0	0	0.050	0.050
Σ					–0.040		0.174

Note: F_i = ith member force due to the real applied load; $V_i = F_i/(EA/L)_i$ = ith member elongation due to the real applied load; f_i = ith member force due to the virtual unit load pair at the cut; $v_i = f_i/(EA/L)_i$ is the ith member elongation due to the virtual unit load pair at the cut; Δ′ = –0.040 mm; δ = 0.174 mm/kN.

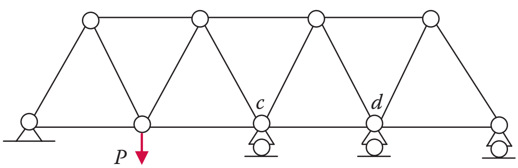
= –0.040 mm, δ = 0.174 mm/kN

From the condition of compatibility:

$$\Delta' + F_6\,\delta = 0 \implies F_6 = -\frac{-0.040}{0.174} = 0.23 \text{ kN}$$

EXAMPLE 3.6

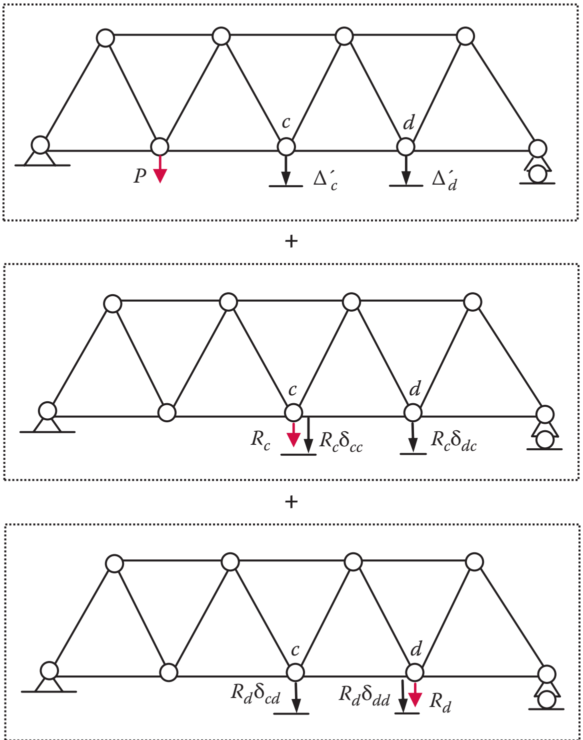
Formulate the conditions of compatibility for the truss problem shown.



Statically indeterminate truss with two degrees of redundancy.

Solution

The primary structure can be obtained by removing the supports at node c and node d. Denoting the reaction at node c and node d as R_c and R_d, respectively, the original problem is equivalent to the superposition of the three problems as shown in the following figure.



Superposition of three determinate problems.

In the figure:

- Δ'_c: vertical displacement at node c due to the real applied load
- Δ'_d: vertical displacement at node d due to the real applied load
- δ_{cc}: vertical displacement at node c due to a unit load at c
- δ_{cd}: vertical displacement at node c due to a unit load at d
- δ_{dc}: vertical displacement at node d due to a unit load at c
- δ_{dd}: vertical displacement at node d due to a unit load at d

The conditions of compatibility are that the vertical displacements at nodes c and d are zero:

$$\begin{aligned} \Delta_c &= \Delta'_c + R_c\delta_{cc} + R_d\delta_{cd} = 0 \\ \Delta_d &= \Delta'_d + R_c\,\delta_{dc} + R_d\delta_{dd} = 0 \end{aligned} \tag{3.12}$$

Equation 3.12 can be solved for the two redundant forces R_c and R_d. Denote

V_i: *i*th member elongation due to the real applied load
f_{ic}: *i*th member force due to the unit load at *c*
v_{ic}: *i*th member elongation due to the unit load at *c*
f_{id}: *i*th member force due to the unit load at *d*
v_{id}: *i*th member elongation due to the unit load at *d*

We can express the displacements according to the unit load method as

$\Delta'_c = \Sigma f_{ic} (V_i)$
$\Delta'_d = \Sigma f_{id} (V_i)$
$\delta_{cc} = \Sigma f_{ic} (v_{ic})$
$\delta_{dc} = \Sigma f_{ic} (v_{id})$
$\delta_{cd} = \Sigma f_{id} (v_{ic})$
$\delta_{dd} = \Sigma f_{id} (v_{id})$

The member elongation quantities in the equations are related to the member forces through

$$V_i = \frac{F_i L_i}{E_i A_i}$$

$$v_{ic} = \frac{f_{ic} L_i}{E_i A_i}$$

$$v_{id} = \frac{f_{id} L_i}{E_i A_i}$$

Thus, we need to find only member forces F_i, f_{ic}, and f_{id}, corresponding to the real load, a unit load at node *c*, and a unit load at node *d*, respectively, from the primary structure.

3.3 Laws of Reciprocity

In the last example we came across δ_{cd} and δ_{dc}, which can be expressed in terms of member forces:

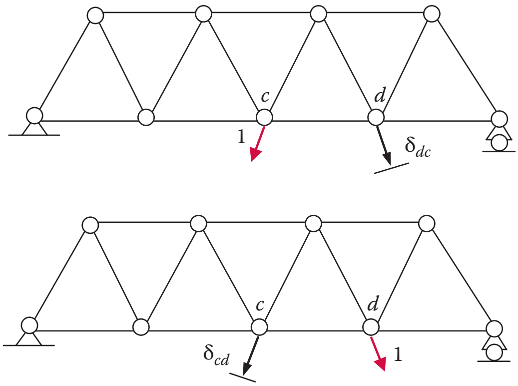
$$\delta_{cd} = \sum f_{id} \, (v_{ic}) = \sum f_{id} \, \frac{f_{ic} L_i}{E_i A_i}$$

$$\delta_{dc} = \sum f_{ic} \, (v_{id}) = \sum f_{ic} \, \frac{f_{id} L_i}{E_i A_i}$$

Comparing the two equations we conclude that

$$\delta_{cd} = \delta_{dc} \tag{3.13}$$

Equation 3.13 states "displacement at point *c* due to a unit load at point *d* is equal to the displacement at *d* due to a unit load at point *c*." Here, all displacements and unit loads are in the vertical direction, but the statement is also true even if the displacements and unit loads are in different directions as long as there is a cross-correspondence as shown in the following figure.

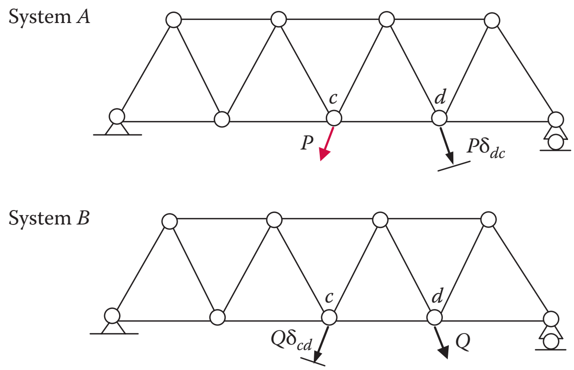


Reciprocal displacements.

Equation 3.13 is called the *Maxwell's law of reciprocal displacements*. As a result of Maxwell's law, the equations of compatibility, Equation 3.12, when put into a matrix form, will always have a symmetrical matrix because δ_{cd} is equal to δ_{dc}.

$$\begin{matrix} \delta_{cc} & \delta_{cd} \\ \delta_{dc} & \delta_{dd} \end{matrix} \quad \begin{matrix} R_c \\ R_d \end{matrix} = \begin{matrix} -{}_{c} \\ -{}_{d} \end{matrix} \tag{3.14}$$

Consider now two systems, system *A* and system *B*, each derived from the two figures by replacing the unit load by a load of magnitude *P* and *Q*, respectively. Then the magnitude of displacements will be proportionally adjusted to what is shown in the next figure.



Two loading and displacement systems.

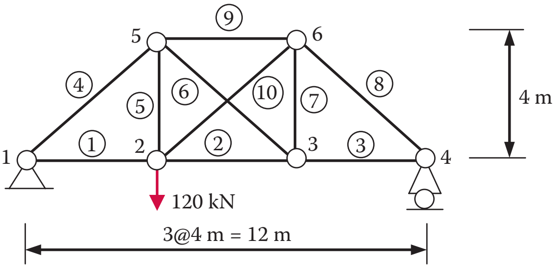
We state " the work done by the load in system *A* upon the displacement of system *B* is equal to the work done by the load in system *B* upon the displacement in system *A*." This statement is true because $P(Q\delta_{cd}) = Q(P\delta_{dc})$, according to the Maxwell's law of reciprocal displacement. This statement can be further generalized to include multiple loads: "the work done by the loads in system *A* upon the displacements of system *B* is equal to the work done by the loads in system *B* upon the displacements in system *A*." This statement is called *Betti's law of reciprocity*. It is the generalization of the Maxwell's reciprocal law. Both are applicable to linear, elastic structures.

3.4 Concluding Remarks

The force method is easy to apply with hand calculation for statically determinate problems or indeterminate problems with one or two redundants. For three or more redundants, a systematic approach using a matrix formulation can be developed. Such a matrix force method formulation is of theoretical interest only and its practical application is virtually nonexistent.

PROBLEM 3.4

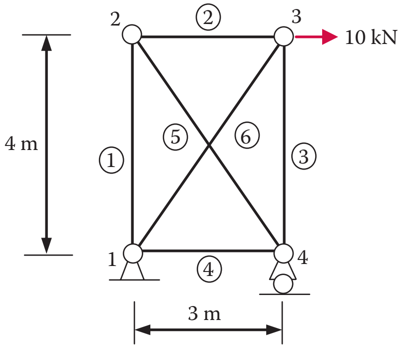
Find the force in member 10 of the loaded truss shown, given $E = 10$ GPa and $A = 100\ \text{cm}^2$ for all bars.



Problem 3.4

PROBLEM 3.5

Find the force in bar 6 of the truss shown, given $E = 10$ GPa and $A = 100$ cm^2 for all bars.



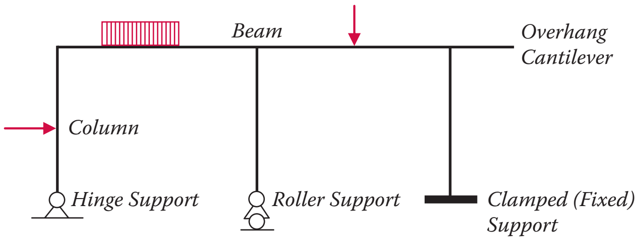
Problem 3.5

4

Beam and Frame Analysis: Force Method—Part I

4.1 What Are Beams and Frames?

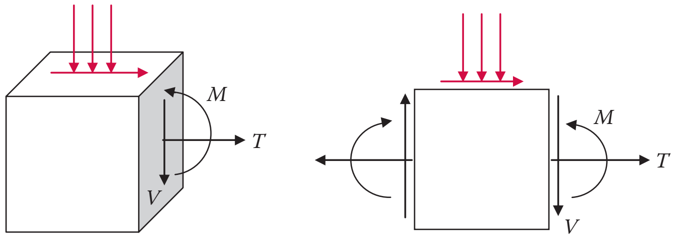
The following figure illustrates the various components in a plane frame system. Each of these components can take loads acting in any direction at any point along its length. A frame is consisted of beams and columns. In a gravity field, the vertical components are called columns and the horizontal components beams. Since the gravity load is usually the predominant load, we expect that the columns will carry mostly axial load and the beams transverse load, even though both can take axial and transverse loads.



A plane frame system.

As shown in the figure, a frame can be supported by hinge or roller supports as a truss can but it can also be supported by a so-called clamped or fixed support, which prevents not only translational motions but also rotational motion at a section. As a result, a fixed support provides three reactions—two forces and a moment. The frames we refer to herein are called rigid frames, which means the connection between its components are rigid connections that do not allow any translational and rotational movement across the connection. In the figure, since it is a rigid frame, all angles at the beam–column junction will remain at 90 degrees before or after any deformations. For other rigid frame systems, the angles at connections between all components will remain at the same angle before and after deformations.

Let us examine an element of a beam or column and show all the internal as well as external forces acting on the element.

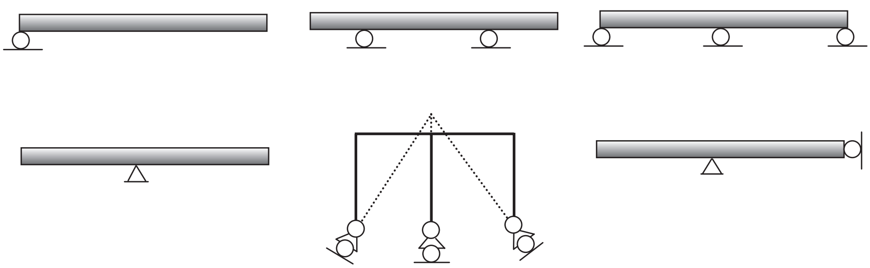


Beam or column element with internal and external forces.

As we can see, at a typical section there are three possible actions or internal forces, bending moment, *M*, shear force, *V*, and axial force or thrust, *T*. For a beam, the dominant internal forces are bending moments and shear forces; for a column, the axial force dominates. In any case, the internal actions are much more complicated than those of a truss member, which has only a constant axial force.

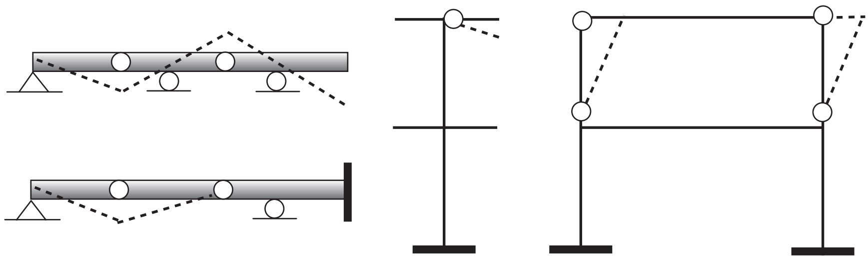
4.2 Statical Determinacy and Kinematic Stability

Instability due to improper support. A beam or frame is kinematically unstable if the support conditions are such that the whole structure is allowed to move as a mechanism. Examples of improper support and insufficient support are shown next.



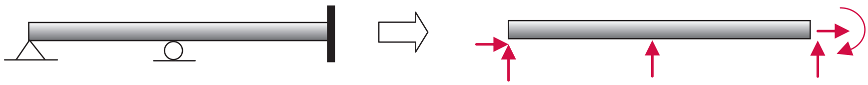
Improper or insufficient support conditions.

Instability due to improper connection. A beam or frame is kinematically unstable if the internal connection conditions are such that part of or the whole structure is allowed to move as a mechanism. Examples of improper connections are shown next.



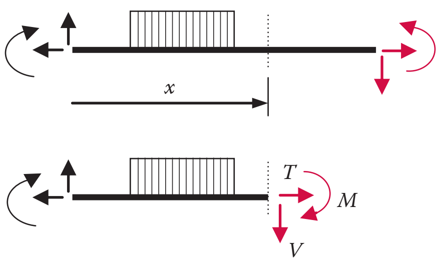
Improper internal connections.

Statical determinacy. A stable beam or frame is statically indeterminate if the number of force unknowns is greater than the number of equilibrium equations. The difference between the two numbers is the degree of indeterminacy. The number of force unknowns is the sum of the number of reaction forces and the number of internal member force unknowns. For reaction forces, a roller has one reaction, a hinge has two reactions, and a clamp has three reactions, as shown next.



Reaction forces for different supports.

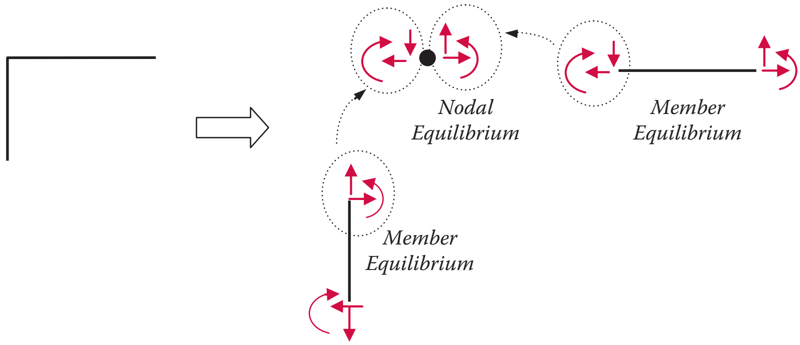
To count internal member force unknowns, first we need to count how many members are in a frame. A frame member is defined by two end nodes. At any section of a member there are three internal unknown forces: T, V, and M. The state of force in the member is completely defined by the six nodal forces, three at each end node, because the three internal forces at any section can be determined from the three equilibrium equations taken from a free-body diagram (FBD) cutting through the section as shown below, if the nodal forces are known.



Internal section forces are functions of the nodal forces of a member.

Thus, each member has six nodal forces as unknowns. Denoting the number of members by M and the number of reaction forces at each support as R, the total number of force unknowns in a frame is then $6M + \Sigma R$.

On the other hand, each member generates three equilibrium equations and each node also generates three equilibrium equations. Denoting the number of nodes by N, the total number of equilibrium equations is $3M + 3N$.



FBDs of a node and two members.

Because the number of members, M, appears both in the count for unknowns and the count for equations, we can simplify the expression for counting unknowns as shown next.

Number of Unknowns = $6M + \Sigma R$ ⇒ *Number of Unknowns* = $3M + \Sigma R$

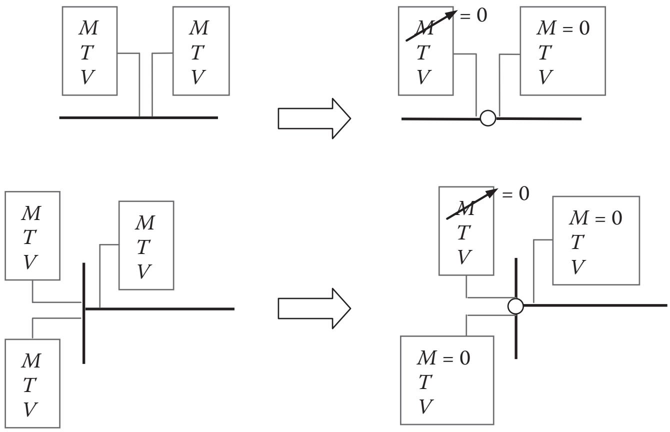
Number of Equations = $3M + 3N$ ⇒ *Number of Equations* = $3N$

Counting unknowns against available equations.

This is equivalent to considering each member having only three force unknowns. The other three nodal forces can be computed using these three nodal forces and the three member equilibrium equations. Thus, a frame is statically determinate if $3M + \Sigma\mathrm{R} = 3N$.

If one or more hinges are present in a frame, we need to consider the conditions generated by the hinge presence. As shown in the upcoming figure, the presence of a hinge within a member introduces one more equation, which can be called the condition of construction. A hinge at the junction of three members introduces two conditions of construction. The other moment at a hinge is automatically zero because the sum of all moments at the hinge (or any other point) must be zero. We generalize to state that the conditions of construction, C, is equal to the number of joining members at a hinge, m, minus one, $C = m - 1$. The conditions of construction at more than one hinge is ΣC.

Since the conditions of construction provide additional equations, the available equation becomes $3N + C$. Thus, in the presence of one or more internal hinges, a frame is statically determinate if $3M + \Sigma R = 3N + \Sigma C$.



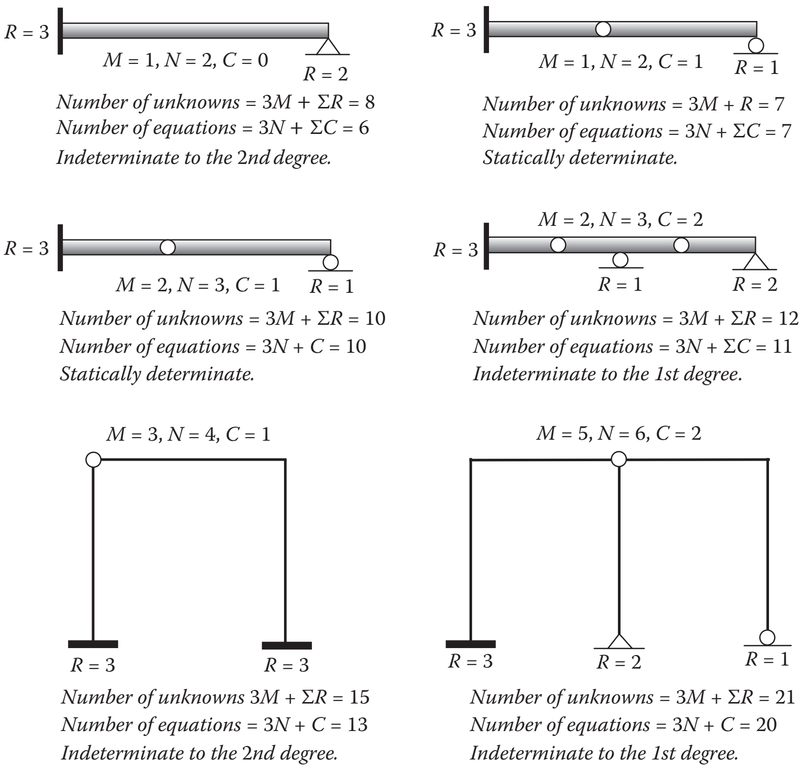
Presence of hinge introduces additional equations.

Example 4.1

Discuss the determinacy of the beams and frames shown.

Solution

The computation is shown with the figures.

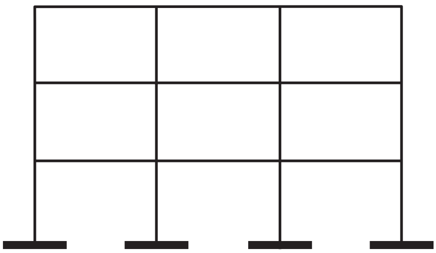


Counting internal force unknowns, reactions, and available equations.

For frames with many stories and bays, a simpler way of counting unknowns and equations can be developed by cutting through members to produce separate "trees" of frames; each is stable and determinate. The number of unknowns at the cuts is the number of degrees of indeterminacy, as shown in Example 4.2.

Example 4.2

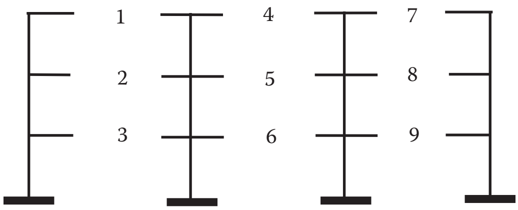
Discuss the determinacy of the frame shown.



Multistory, multibay indeterminate frame.

Solution

We make nine cuts that separate the original frame into four "trees" of frames as shown.



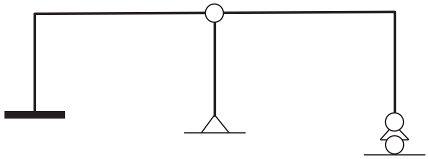
Nine cuts pointing to 27 degrees of indeterminacy.

We can easily verify that each of the stand-alone trees is stable and statically determinate, that is, the number of unknowns is equal to the number of equations in each of the tree problems. At each of the nine cuts, three internal forces are present before the cut. All together, we have removed 27 internal forces in order to have equal numbers of unknowns and equations. If we put back the cuts, we introduce 27 more unknowns, which is the degrees of indeterminacy of the original uncut frame.

This simple way of counting can be extended to multistory, multibay frames with hinges: simply treat the conditions of construction of each hinge as "releases" and subtract the ΣC number from the degrees of indeterminacy of the frame with the hinges removed. For supports other than fixed, we can replace them with fixed supports and count the releases for subtracting from the degrees of indeterminacy.

Example 4.3

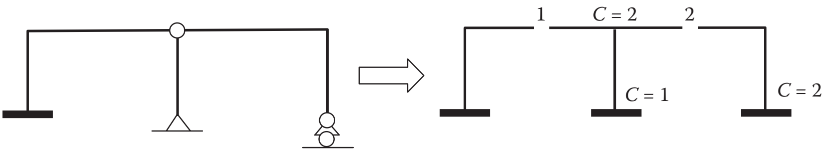
Discuss the determinacy of the frame shown.



Indeterminate frame example.

Solution

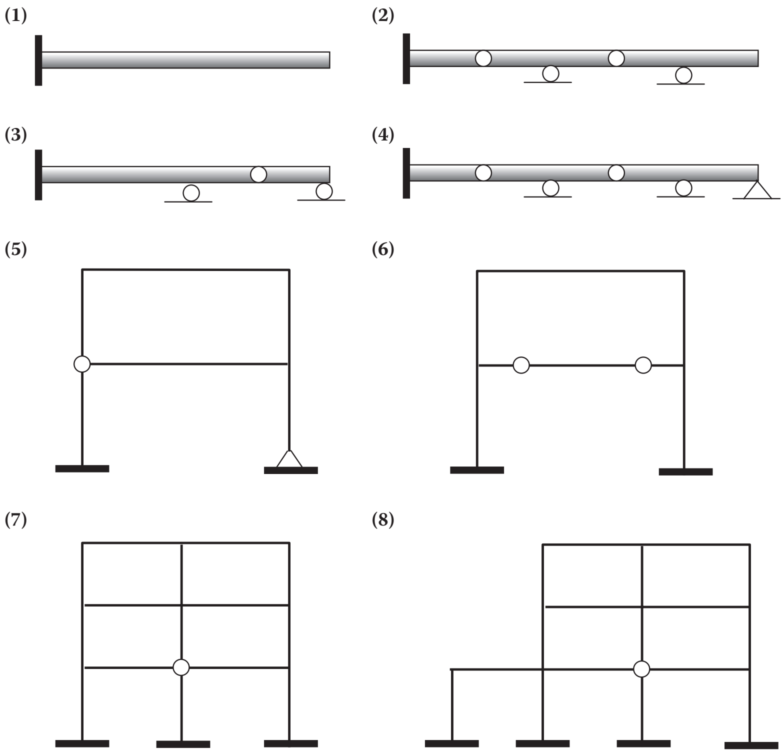
Two cuts and five releases amounts to 2 × 3 – 5 = 1. The frame is indeterminate to the first degree.



Shortcut to count degrees of indeterminacy.

PROBLEM 4.1

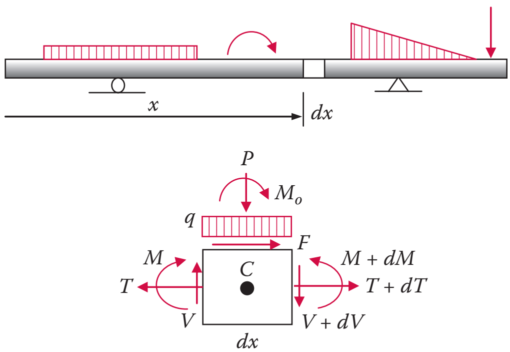
Discuss the determinacy of the beams and frames shown.



Problem 4.1

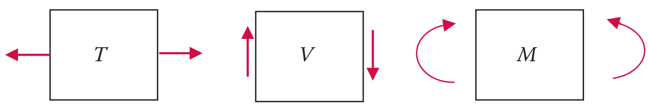
4.3 Shear and Moment Diagrams

A beam is supported on a roller and a hinge and is taking a concentrated load, a concentrated moment, and distributed loads as shown next.



A loaded beam and the FBD of a typical infinitesimal element.

A typical element of width dx is isolated as a FBD and the forces acting on the FBD are shown. All quantities shown are depicted in their positive direction. It is important to remember that the positive direction for T, V, and M depends on which face they are acting. It is necessary to remember the following figures for the sign convention for T, V, and M.



Positive directions for T, V, and M.

We can establish three independent equilibrium equations from the FBD.

$$\Sigma F_x = 0 \Rightarrow -T + F + (T + dT) = 0 \Rightarrow dT = -F$$

$$\Sigma F_y = 0 \Rightarrow V - P - qdx - (V + dV) = 0 \Rightarrow dV = -P - qdx$$

$$\Sigma M_c = 0 \Rightarrow M + M_o - (M + dM) + Vdx = 0 \Rightarrow dM = M_o + Vdx$$

The first equation deals with the equilibrium of all forces acting in the axial direction. It states that the increment of axial force, dT, is equal to the externally applied axial force, F. If a distributed force is acting in the axial direction, then F would be replaced by fdx, where f is the intensity of the axially distributed force per unit length. For a beam, even if axial forces are present, we can consider the axial forces and their effects on deformations separately from those of the transverse forces. We shall now concentrate only on shear and moment.

The second and the third equations lead to the following differential and integral relations

$$\frac{dV}{dx} = -q(x),\; V = \int -q\,dx \text{ for distributed loads} \tag{4.1a}$$

$$V = -P \text{ for concentrated loads} \tag{4.1b}$$

$$\frac{dM}{dx} = V,\; M = \int V\,dx \tag{4.2a}$$

$$\Delta M = M_o \text{ for concentrated moments} \tag{4.2b}$$

Note that we have replaced the differential operator d with the symbol Δ in Equation 4.1b and Equation 4.2b to signify the fact that there will be a sudden change across a section when there is a concentrated load or a concentrated moment externally applied at the location of the section.

Differentiating Equation 4.2a once with respect to x and eliminating V using Equation 4.1a, we arrive at

$$\frac{d^2M}{dx^2} = -q,\; M = -\iint q\,dxdx \tag{4.3}$$

The preceding equations reveal the following important features of shear and moment variation along the length of a beam.

1. The shear and moment change along the length of the beam as a function of x. The shear and moment functions, $V(x)$ and $M(x)$, are called shear and moment diagrams, respectively, when plotted against x.
2. According to Equation 4.1a, the slope of the shear diagram is equal to the negative value of the intensity of the distributed load, and the integration of the negative load intensity function gives the shear diagram.
3. According to Equation 4.1b, wherever there is a concentrated load, the shear value changes by an amount equal to the negative value of the load.
4. According to Equation 4.2a, the slope of the moment diagram is equal to the value of the shear, and the integration of the shear function gives the moment diagram.
5. According to Equation 4.2b, wherever there is a concentrated moment, the moment value changes by an amount equal to the value of the concentrated moment.
6. According to Equation 4.3, the moment function and load intensity are related by twice differentiation/integration.

Furthermore, integrating once from Equation 4.1a and Equation 4.2a leads to

$$V_b = V_a + \int_a^b -q\,dx \tag{4.4}$$

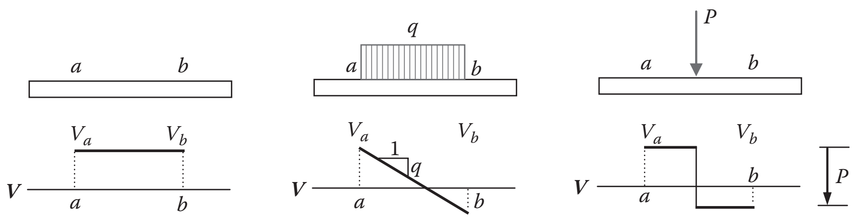
and

$$M_b = M_a + \int_a^b V\,dx \tag{4.5}$$

where *a* and *b* are two points on a beam.

Equations 4.4 and 4.5 reveal practical guides to drawing the shear and moment diagrams:

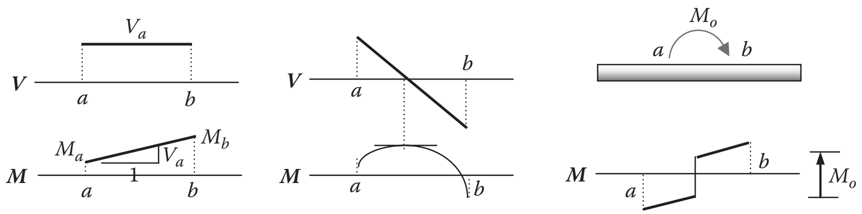
1. When drawing a shear diagram starting from the leftmost point on a beam, the shear diagram between any two points is flat if there are no loads applied between the two points ($q = 0$). If there is an applied load ($q \neq 0$), the direction of change of the shear diagram follows the direction of the load and the rate of change is equal to the intensity of the load. If a concentrated load is encountered, the shear diagram, going from left to right, moves up or down by the amount of the concentrated load in the direction of the load (Equation 4.1b). These practical rules are illustrated in the figure below.



Shear diagram rules for different loads.

2. When drawing a moment diagram starting from the leftmost point on a beam, the moment diagram between any two points is (a) linear if the shear is constant, (b) parabolic if the shear is linear, and so forth. The moment diagram has a zero slope at the point where the shear is zero. If a concentrated moment is encountered, the moment

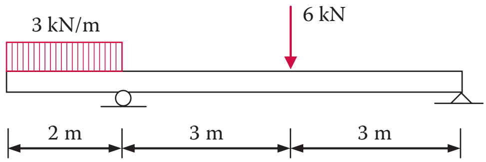
diagram, going from left to right, moves up or down by the amount of the concentrated moment if the moment is counterclockwise or clockwise (Equation 4.2b). These practical rules are illustrated in the following figure.



Moment diagram rules for different shear diagrams and loads.

Example 4.4

Draw the shear and moment diagrams of the loaded beam shown.

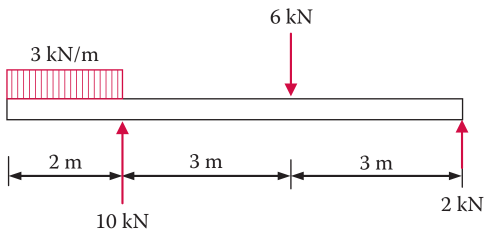


Example for shear and moment diagrams of a beam.

Solution

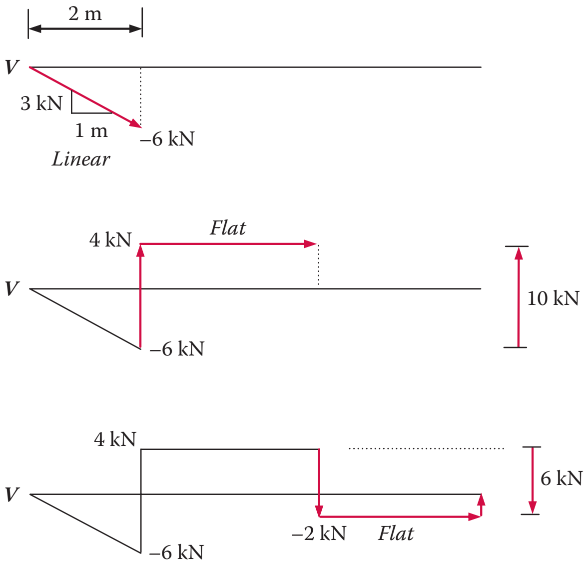
We shall give a detailed step-by-step solution.

1. Find reactions. The first step in shear and moment diagram construction is to find the reactions. Readers are encouraged to verify the reaction values shown in the following figure, which is the FBD of the beam with all the forces shown.



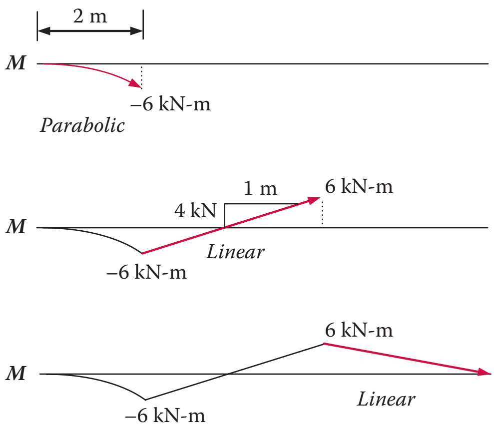
FBD of the beam showing applied and reaction forces.

2. Draw the shear diagram from left to right.



Drawing the shear diagram from left to right.

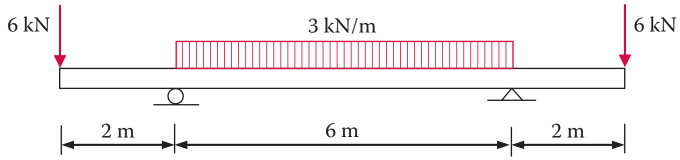
3. Draw the moment diagram from left to right.



Drawing the moment diagram from left to right.

Example 4.5

Draw the shear and moment diagrams of the loaded beam shown.

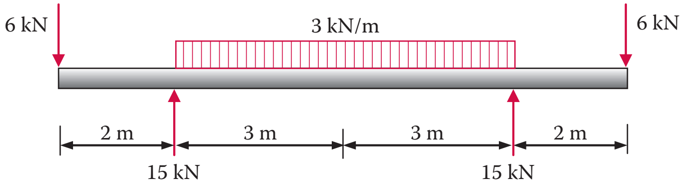


Example for shear and moment diagrams of a beam.

Solution

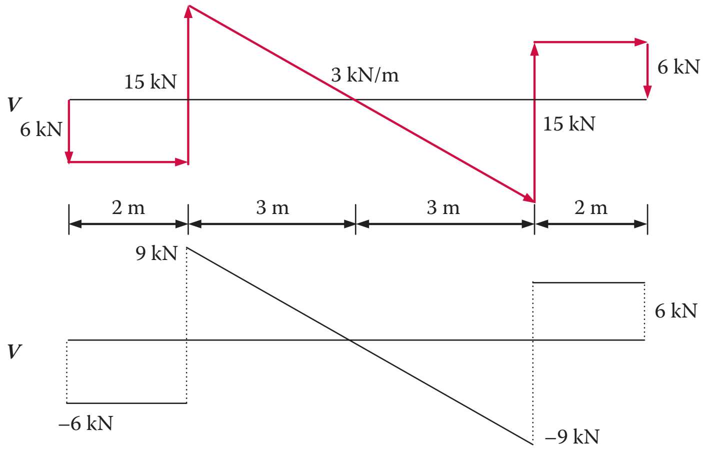
We shall draw the shear and moment diagrams directly.

1. Find reactions.



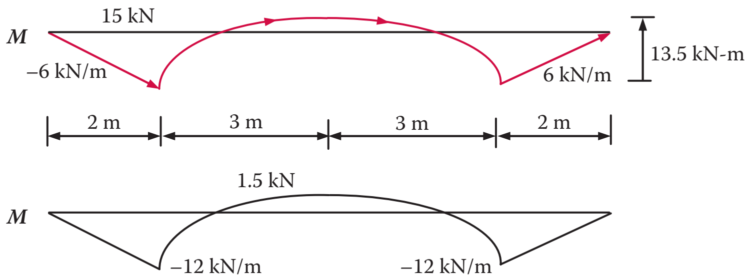
FBD showing all forces.

2. Draw the shear diagram from left to right.



Drawing shear diagram from left to right.

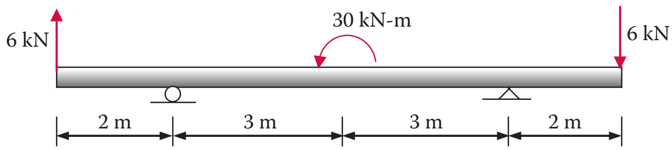
3. Draw the moment diagram from left to right.



Drawing moment diagram from left to right.

Example 4.6

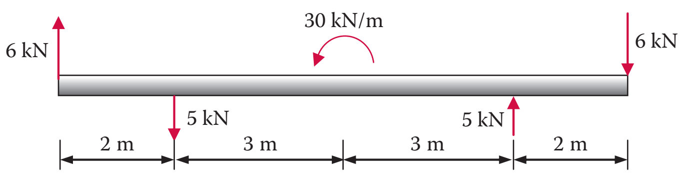
Draw the shear and moment diagrams of the loaded beam shown next.



Example for shear and moment diagrams of a beam.

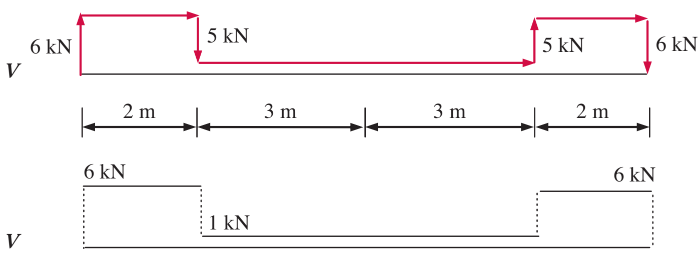
Solution

1. Find reactions.



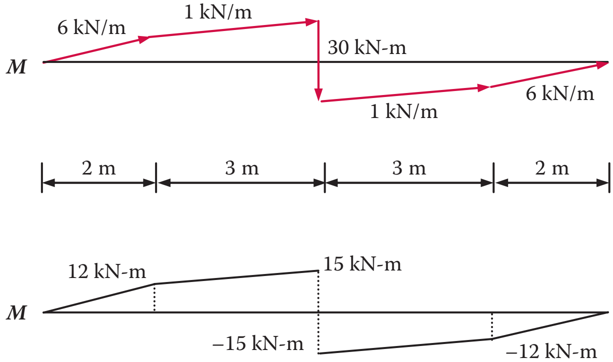
FBD showing all forces.

2. Draw the shear diagram from left to right.



Drawing shear diagram from left to right.

3. Draw the moment diagram from left to right.



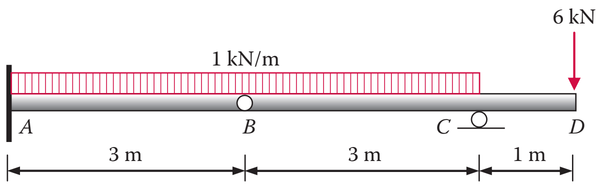
Drawing moment diagram from left to right.

4.4 Statically Determinate Beams and Frames

Analysis of statically determinate beams and frames starts from defining the FBDs of members and then utilizes the equilibrium equations of each FBD to find the force unknowns. The process is best illustrated through example problems.

Example 4.7

Analyze the loaded beam in the following figure and draw the shear and moment diagrams.

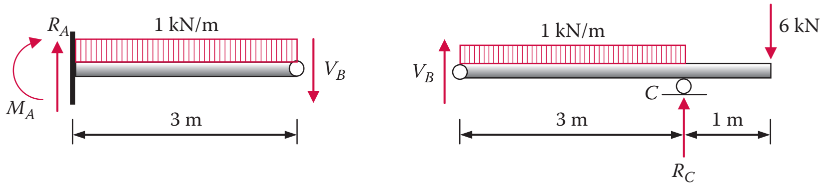


A statically determinate beam problem.

Solution

The presence of an internal hinge calls for a cut at the hinge to produce two separate FBDs. This is the best way to expose the force at the hinge.

1. Define FBDs and find reactions and internal nodal forces.



Two FBDs exposing all nodal forces and support reactions.

The computation under each FBD is self-explanatory. We start from the right FBD because it contains only two unknowns and we have exactly two equations to use. The third equation of equilibrium is the balance of forces in the horizontal direction, which produces no useful equation since there is no force in the horizontal direction.

$$\Sigma M_C = 0,\ V_B(3) - 3(1.5) + 6(1) = 0$$

$$\Rightarrow \quad V_B = -0.5 \text{ kN}$$

$$\Sigma F_y = 0,\quad -0.5 - 3 - 6 + R_C = 0$$

$$\Rightarrow \quad R_C = 9.5 \text{ kN}$$

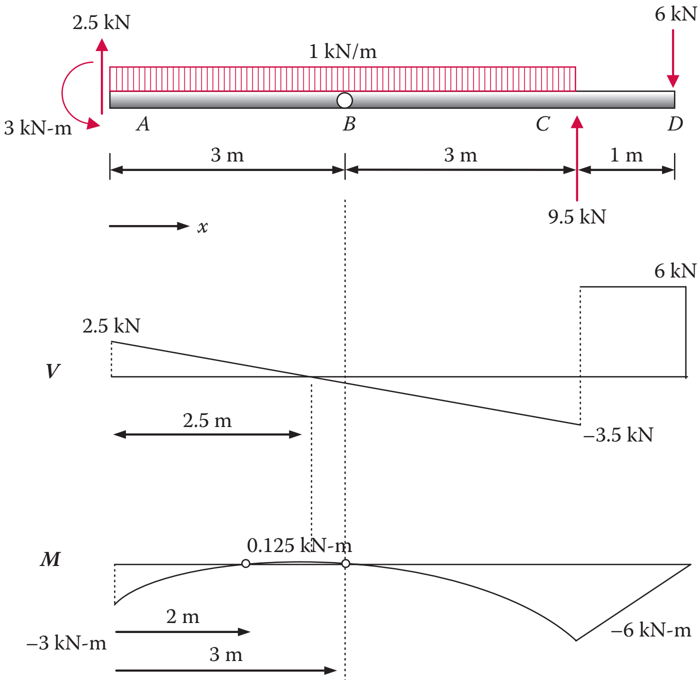
$$\Sigma M_A = 0,\quad M_A + 3(1.5) - 0.5(3) = 0$$

$$\Rightarrow \quad M_A = -3 \text{ kN-m}$$

$$\Sigma F_y = 0,\quad 0.5 - 3 + R_A = 0$$

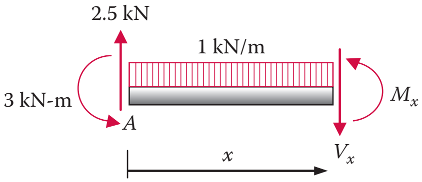
$$\Rightarrow \quad R_A = 2.5 \text{ kN}$$

2. Draw the FBD of the whole beam and then shear and moment diagrams.



Shear and moment diagrams drawn from the force data of the FBD.

Note that the point of zero moment is determined by solving the second order equation derived from the FBD shown next.



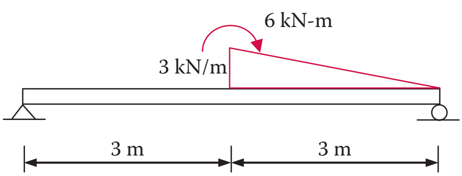
FBD to determine moment at a typical section *x*.

$$M(x) = -3 + 2.5\,x - 0.5\,x^2 = 0, \Longrightarrow \quad x = 2 \text{ m}, 3 \text{ m}$$

The local maximum positive moment is determined from the point of zero shear at $x = 2.5$ m, from which we obtain $M(x = 2.5) = -3 + 2.5\,(2.5) - 0.5\,(2.5)^2 = 0.125$ kN-m.

Example 4.8

Analyze the loaded beam shown next and draw the shear and moment diagrams.

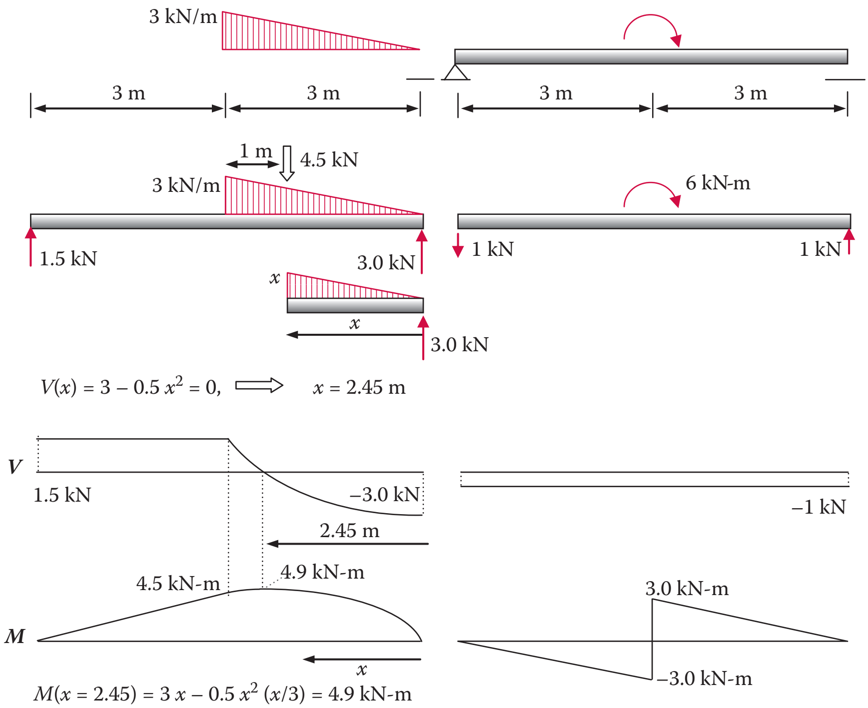


A beam loaded with a distributed force and a moment.

Solution

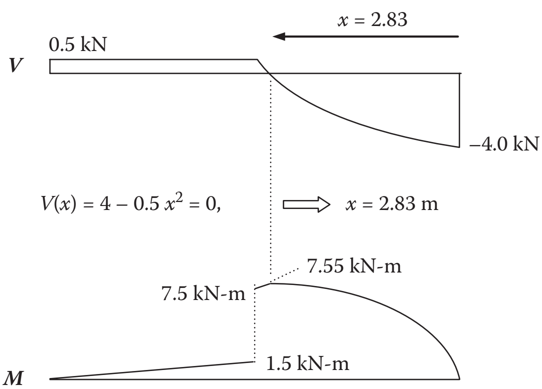
The problem is solved using the principle of superposition, which states that for a linear structure the solution of the structure under two loading systems is the sum of the solutions of the structure under each force system.

The solution process is illustrated in the following self-explanatory sequence of figures.



Solving two separate problems.

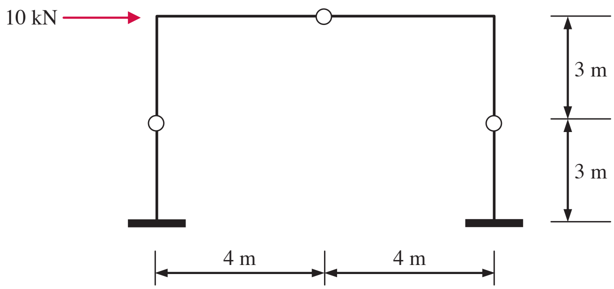
The superposed shear and moment diagrams give the final answer.



Combined shear and moment diagrams.

Example 4.9

Analyze the loaded frame shown next, and draw the thrust, shear, and moment diagrams.

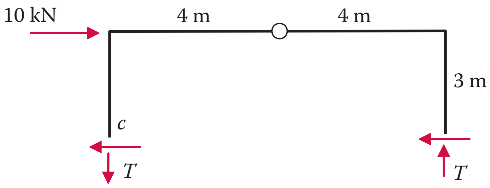


A statically determinate frame.

Solution

The solution process for a frame is no different from that for a beam.

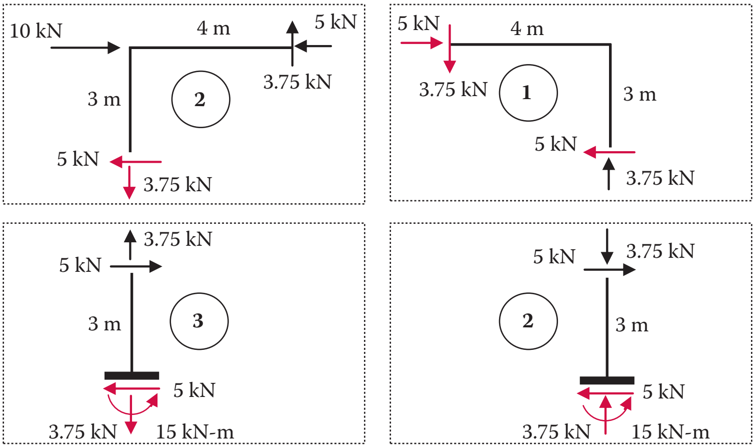
1. Define FBDs and find reactions and internal nodal forces. Many different FBDs can be defined for this problem, but they may not lead to simple solutions. After trial-and-error, the following FBD offers a simple solution for the axial force in the two columns.



FBD to solve for the axial force in columns.

$$\Sigma M_c = 0, \quad \Longrightarrow \quad T(8) = 10(3), \quad \Longrightarrow \quad T = 3.75 \text{ kN}$$

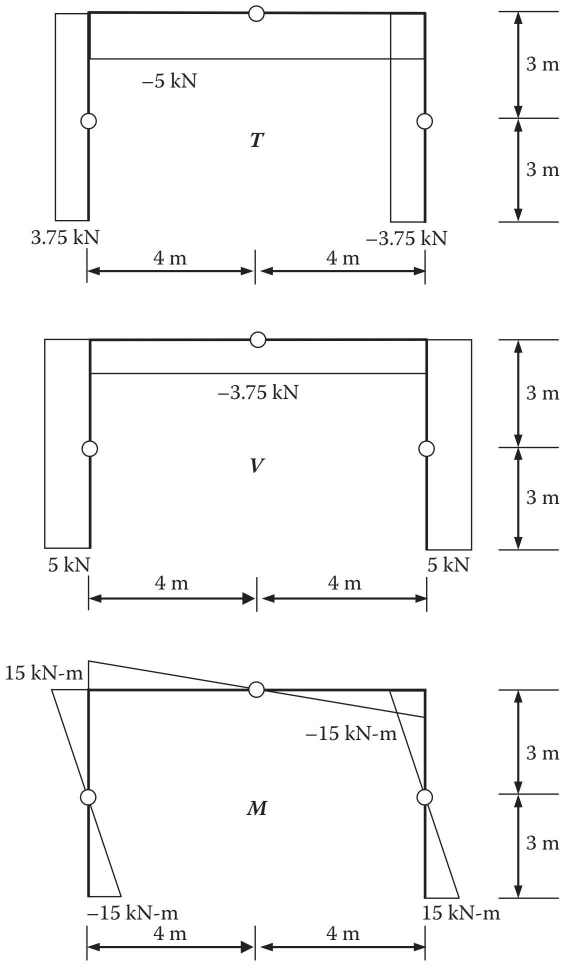
Once the axial force in the two columns is known, we can proceed to define four FBDs to expose all the nodal forces at the internal hinges, as shown in the upcoming figure, and solve for any unknown nodal forces one by one using equilibrium equations of each FBD. The solution sequence is shown by the numbers attached with each FBD. Within each FBD, the bold-faced force values are those that are known from previous calculations and the other three unknowns are obtained from the equilibrium equations of the FBD itself.



Four FBDs exposing all internal forces at the hinges.

We note that we could have used the 12 equilibrium equations from the above four FBDs to solve for the twelve force unknowns without the aid of the previous FBD to find the axial force in the two columns first. But the solution strategy presented offers the simplest computing sequence without having to solve for any simultaneous equations.

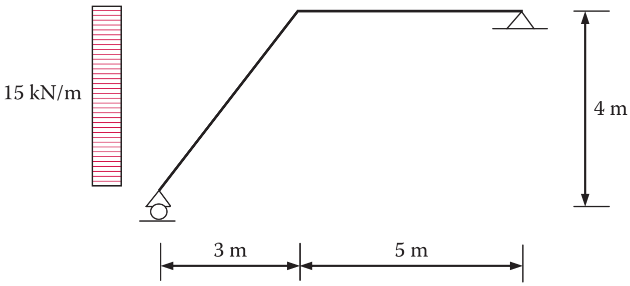
2. Draw the thrust, shear, and moment diagrams. For the thrust diagram, we designate tension force as positive and compression force as negative. For shear and moment diagrams, we use the same sign convention for both beams and columns. For the vertically orientated columns, it is customary to equate the "inside" of a column to the "downside" of a beam and draw the positive and negative shear and moment diagrams accordingly.



Thrust, shear, and moment diagrams of the example problem.

Example 4.10

Analyze the following loaded frame, and draw the thrust, shear, and moment diagrams. The intensity of the horizontal force is 15 kN per unit vertical length.

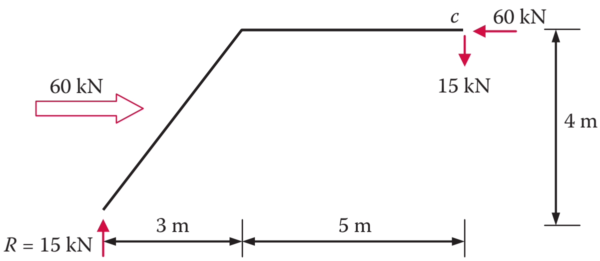


Statically determinate frame example problem.

Solution

The inclined member requires a special treatment in finding its shear diagram.

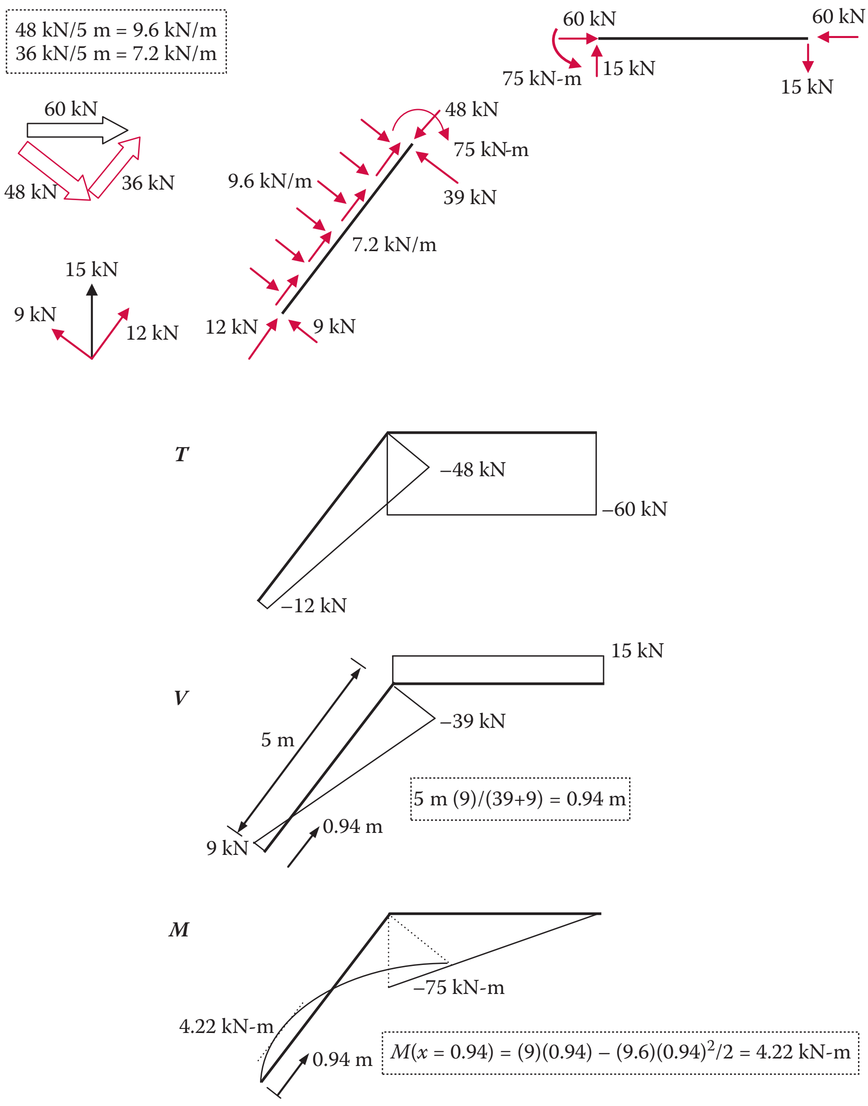
1. Find reactions and draw FBD of the whole structure.



FBD for finding reactions.

$$\Sigma M_c = 0, \quad \Longrightarrow \quad 60(2) - R(8) = 0, \quad \Longrightarrow \quad R = 15 \text{ kN}$$

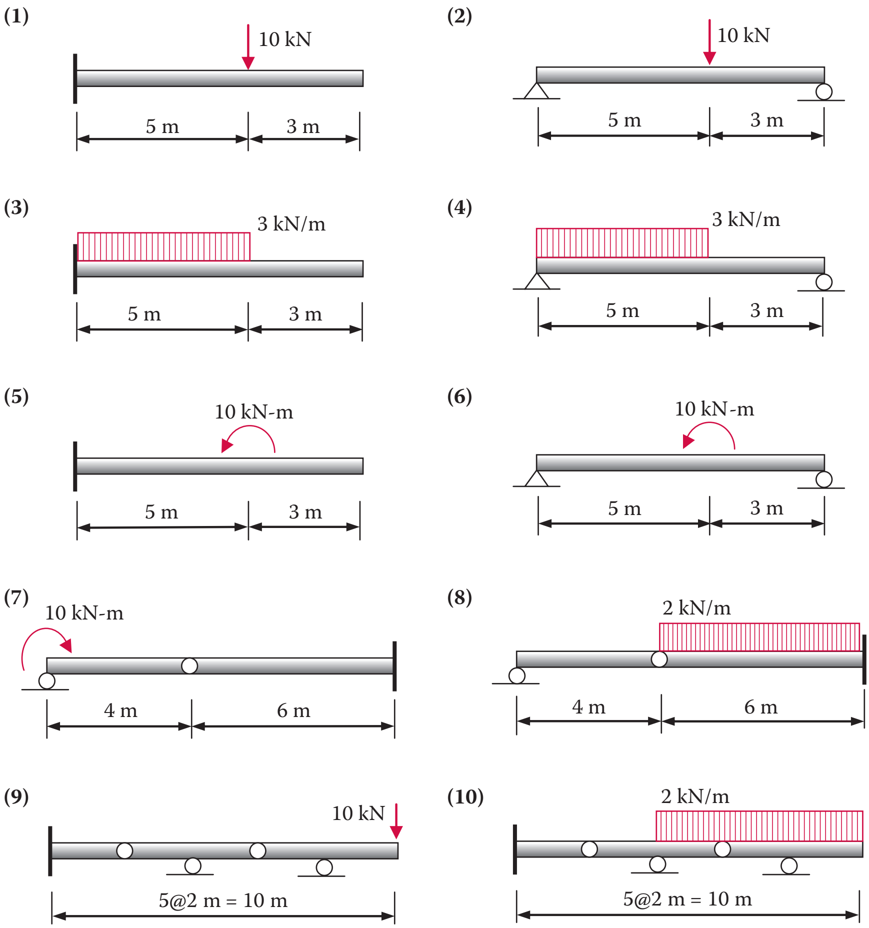
2. Draw the thrust, shear, and moment diagrams. Before drawing the thrust, shear, and moment diagrams, we need to find the nodal forces that are in the direction of the axial force and shear force. This means we need to decompose all forces not perpendicular to or parallel to the member axes to those that are. The upper part of the following figure reflects that step. Once the nodal forces are properly oriented, the drawing of the thrust, shear, and moment diagrams is effortlessly achieved.



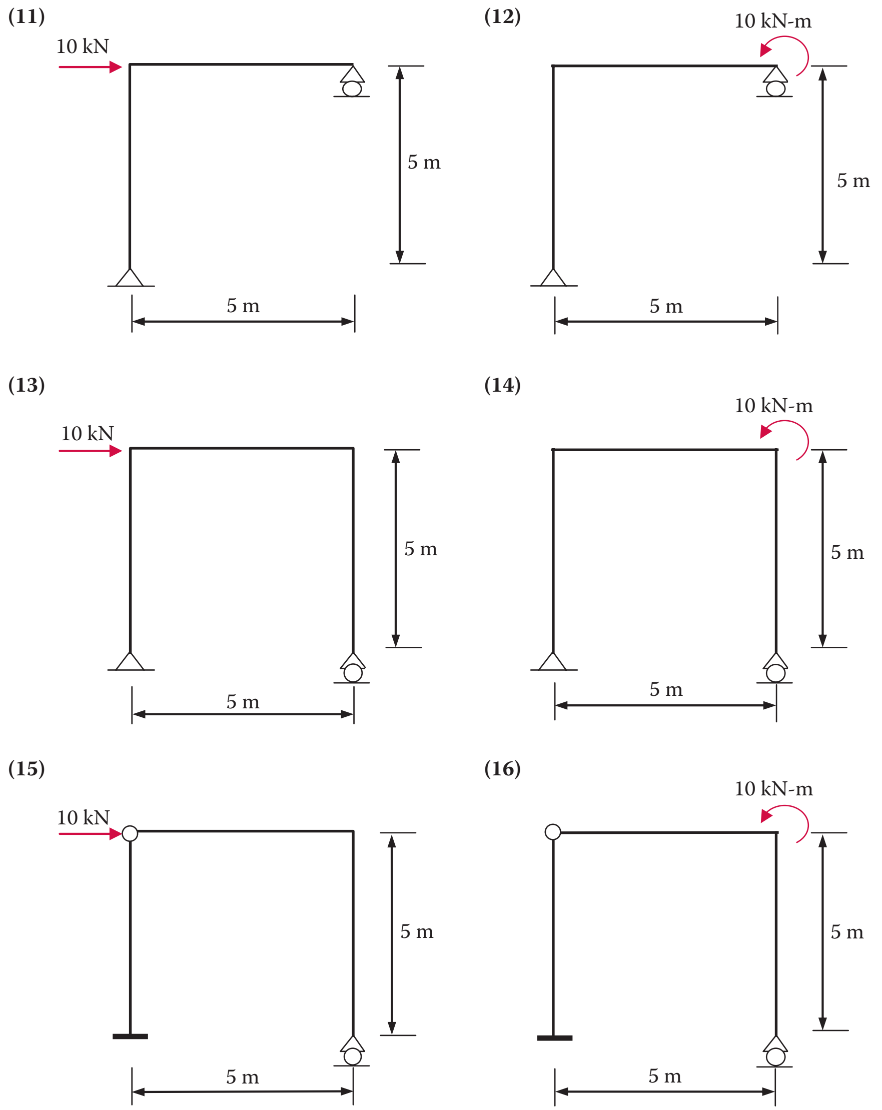
Thrust, shear, and moment diagrams of the example problem.

PROBLEM 4.2

Analyze the beams and frames shown, and draw the thrust (for frames only), shear, and moment diagrams.



Problem 4.2 beam problems.



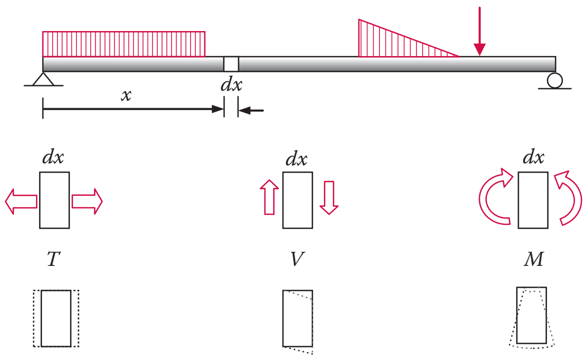
Problem 4.2 frame problems.

5

Beam and Frame Analysis: Force Method—Part II

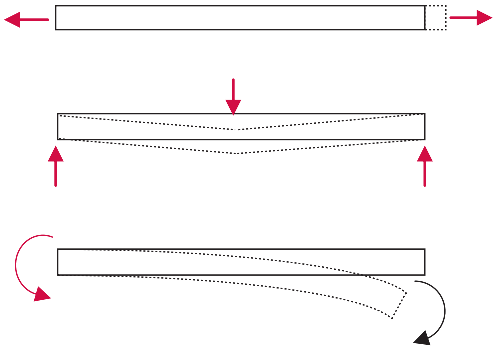
5.1 Deflection of Beams and Frames

Deflection of beams and frames is the deviation of the configuration of beams and frames from their undisplaced state to the displaced state, measured from the neutral axis of a beam or a frame member. It is the cumulative effect of deformation of the infinitesimal elements of a beam or frame member. As shown in the following figure, an infinitesimal element of width dx can be subjected to all three actions: thrust (T), shear (V), and moment (M). Each of these actions has a different effect on the deformation of the element.



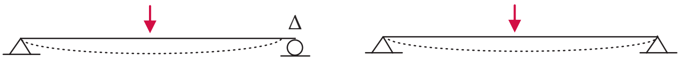
Effect of thrust, shear, and moment on the deformation of an element.

The effect of axial deformation on a member is axial elongation or shortening, which is calculated in the same way as a truss member's. The effect of shear deformation is the distortion of the shape of an element that results in the transverse deflections of a member. The effect of flexural deformation is the bending of the element, resulting in transverse deflection and axial shortening. These effects are illustrated next.



Effects of axial, shear, and flexural deformations on a member.

While both the axial and flexural deformations result in axial elongation or shortening, the effect of flexural deformation on axial elongation is considered negligible for practical applications. Thus, bending induced shortening, Δ, will not create axial tension in the following figure, even when the axial displacement is constrained by two hinges as in the right part of the figure, because the axial shortening is too small to be of any significance. As a result, axial and transverse deflections can be considered separately and independently.



Bending induced shortening is negligible.

We shall be concerned with only transverse deflection henceforth. The shear deformation effect on transverse deflection, however, is also negligible if the length-to-depth ratio of a member is greater than 10, as a rule of thumb. Consequently, the only effect to be included in the analysis of beam and frame deflection is that of the flexural deformation caused by bending moments. As such, there is no need to distinguish frames from beams. We shall now introduce the applicable theory for the transverse deflection of beams.

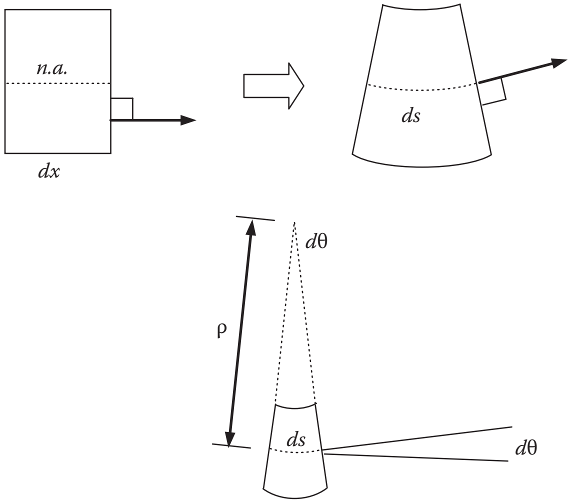
5.2 Integration Methods

Linear flexural beam theory—classical beam theory. The classical beam theory is based on the following assumptions:

1. Shear deformation effect is negligible.
2. Transverse deflection is small (<< depth of beam).

Consequently:

1. The normal to a transverse section remains normal after deformation.
2. The arc length of a deformed beam element is equal to the length of the beam element before deformation.



Beam element deformation and the resulting curvature of the neutral axis (*n.a.*).

From the preceding figure, it is clear that the rotation of a section is equal to the rotation of the neutral axis. The rate of change of angle of the neutral axis is defined as the curvature. The reciprocal of the curvature is called the radius of curvature, denoted by ρ.

$$\frac{d\theta}{ds} = \text{rate of change of angle} = \text{curvature}$$

$$\frac{d\theta}{ds} = \frac{1}{\rho} \tag{5.1}$$

For a beam made of linearly elastic materials, the curvature of an element, represented by the curvature of its neutral axis, is proportional to the bending moment acting on the element. The proportional constant, as derived in textbooks on strength of materials or mechanics of deformable bodies, is the product of the Young's modulus, E, and the moment of inertia of the cross-section with respect to the horizontal neutral surface line of the section, I. Collectively, EI is called the sectional flexural rigidity.

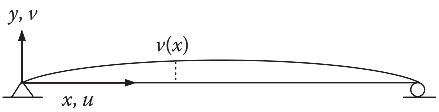
$$M(x) = k\frac{1}{\rho(x)}$$

where $k = EI$.

Rearranging the previous equation, we obtain the following moment-curvature formula:

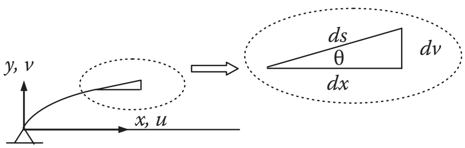
$$\frac{M}{EI} = \frac{1}{\rho} \tag{5.2}$$

Equation 5.2 is applicable to all beams made of linearly elastic materials and is independent of any coordinate system. In order to compute any beam deflection, measured by the deflection of its neutral axis, however, we need to define a coordinate system as shown next. Henceforth, it is understood that the line or curve shown for a beam represents that of the neutral axis of the beam.



Deflection curve and the coordinate system.

In the preceding figure, u and v are the displacements of a point of the neutral axis in the x- and y-direction, respectively. As explained earlier, the axial displacement, u, is separately considered and we shall concentrate on the transverse displacement, v. At a typical location, x, the arc length, ds, and its relation with its x- and y-components are depicted in the next figure.



Arc length, its x- and y-components, and the angle of rotation.

The small deflection assumption of the classical beam theory allows us to write

$$Tan\,\theta = \theta = \frac{dv}{dx} = v \quad \text{and } ds = dx \tag{5.3}$$

where we have replaced the differential operator d/dx by the simpler symbol, prime (′).

A direct substitution of the previous formulas into Equation 5.1 leads to

$$\frac{d\theta}{ds} = \frac{1}{\rho} = \frac{d\theta}{dx} = \theta \;\; = v \tag{5.4}$$

which in turn leads to, from Equation 5.2,

$$\frac{M}{EI} = \frac{1}{\rho} = v \tag{5.5}$$

This last equation, Equation 5.5, is the basis for the solution of the deflection curve, represented by $v(x)$. We can solve for v' and v from Equation 5.4 and Equation 5.5 by direct integration.

Direct integration. If we express M as a function of x from the moment diagram, then we can integrate Equation 5.5 once to obtain the rotation

$$\theta = v = \int \frac{M}{EI} dx \tag{5.6}$$

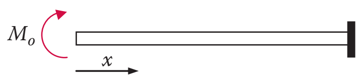
Integrate again to obtain the deflection

$$v = \iint \frac{M}{EI} dx\, dx \tag{5.7}$$

We shall now illustrate the solution process by the following example.

Example 5.1

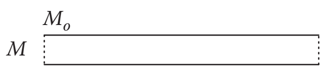
The following beam has a constant EI and a length L, find the rotation and deflection formulas.



A cantilever beam loaded by a moment at the tip.

Solution

The moment diagram is easily obtained as shown next.



Moment diagram.

Clearly,

$$M(x) = M_o$$

Integrate once: $EIv'' = M_o \Longrightarrow EIv' = M_o x + C_1$

Condition 1: $v' = 0$ at $x = L \Longrightarrow C_1 = -M_o L$

$\Longrightarrow EIv' = M_o (x-L)$

Integrate again: $EIv' = M_o (x - L) \qquad EIv = M_o (\frac{x^2}{2} - Lx) + C_2$

Condition 2: $v = 0$ at $x = L \Longrightarrow C_2 = M_o \frac{L^2}{2}$

$\Longrightarrow EIv = M_o (\frac{x^2 + L^2}{2} - Lx)$

Rotation: $\theta = v' = \frac{M_o}{EI}(x - L) \Longrightarrow$ at $x = 0$, $v' = -\frac{M_o}{EI} L$

Deflection: $v = \frac{M_o}{EI}(\frac{x^2 + L^2}{2} - Lx) \Longrightarrow$ at $x = 0$, $v = \frac{M_o}{EI} \frac{L^2}{2}$

The rotation and deflection at $x = 0$ are commonly referred to as the tip rotation and the tip deflection, respectively.

Example 5.1 demonstrates the lengthy process one has to go through to obtain a deflection solution. On the other hand, we notice the process is nothing but that of integration, similar to what we have used for the shear and moment diagram solutions. Can we devise a way of drawing rotation and deflection diagrams in much the same way as drawing shear and moment diagrams? The answer is yes and the method is called the conjugate beam method.

Conjugate beam method. In drawing the shear and moment diagrams, the basic equations we rely on are Equation 4.1 and Equation 4.3, which are reproduced next, respectively, in equivalent forms

$$V = \int -q\,dx$$

$$M = \iint -q\,dxdx$$

Clearly, the operations in Equation 5.6 and Equation 5.7 are parallel to those in the previous equations. If we define $-M/EI$ as "elastic load" in parallel to q as the real load, then the two processes of finding shear and moment diagrams and rotation and deflection diagrams are identical.

Shear and moment diagrams: $q \Longrightarrow V \Longrightarrow M$

Rotation and deflection diagrams: $-\frac{M}{EI} \Longrightarrow \theta \Longrightarrow v$

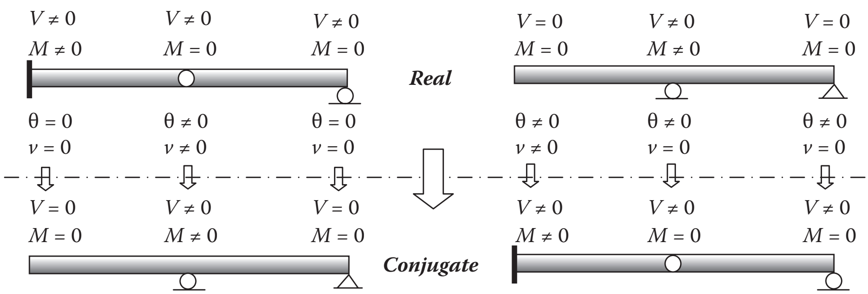
We can now define a "conjugate beam," on which an elastic load of magnitude $-M/EI$ is applied. We can draw the shear and moment diagrams of this conjugate beam and the results are actually the rotation and deflection diagrams of the original beam. Before we can do that, however, we have to find out what kind of support or connection conditions we need to specify for the conjugate beam. This can be easily achieved by following the reasoning in the following table from left to right, noting that M and V of the conjugate beam corresponds to deflection and rotation of the real beam, respectively.

Support and Connection Conditions of a Conjugated Beam

Original Beam					Conjugate Beam		
Support/Connection	v	$v' = \theta$	M	V	M	V	Support/Connection
Fixed	0	0	≠0	≠0	0	0	Free
Free	≠0	≠0	0	0	≠0	≠0	Fixed
Hinge/Roller End	0	≠0	0	≠0	0	≠0	Hinge/Roller End
Internal Support	0	≠0	≠0	≠0	0	≠0	Internal Connection
Internal Connection	≠0	≠0	0	≠0	≠0	≠0	Internal Support

At a fixed end of the original beam, the rotation and deflection should be zero and the shear and moment are not. At the same location of the conjugate beam, to preserve the parallel, the shear and moment should be zero. But, that is the condition of a free end. Thus, the conjugate beam should have a free end at where the original beam has a fixed end. The other conditions are derived in a similar way.

Note that the support and connection conversion summarized in the previous table can be summarized in the following figure, which is easy to memorize. The various quantities are also attached but the important thing to remember is a fixed support turns into a free support and vice versa while an internal connection turns into an internal support and vice versa.



Conversion from a real beam to a conjugate beam.

We can now summarize the process of constructing the conjugate beam and drawing the rotation and deflection diagrams:

1. Construct a conjugate beam of the same dimension as the original beam.
2. Replace the supports and connections in the original beam with another set of supports and connections on the conjugate beam

according to the previous table, that is, fixed becomes free, free becomes fixed, internal hinge becomes internal support, and so forth.

3. Place the M/EI diagram of the original beam onto the conjugate beam as a distributed load, turning positive moment into upward load.
4. Draw the shear diagram of the conjugate beam; positive shear indicates counterclockwise rotation of the original beam.
5. Draw the moment diagram of the conjugate beam; positive moment indicates upward deflection.

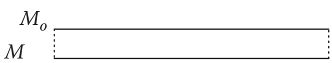
Example 5.2

The following beam has a constant *EI* and a length *L*. Draw the rotation and deflection diagrams.

A cantilever beam load by a moment at the tip.

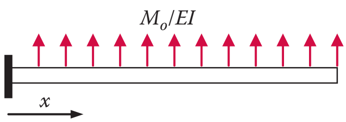
Solution

1. Draw the moment diagram of the original beam.



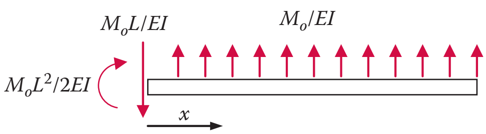
Moment diagram.

2. Construct the conjugate beam and apply the elastic load.



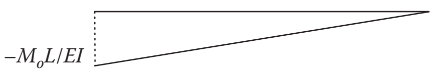
Conjugate beam and elastic load.

3. Analyze the conjugate beam to find all reactions.



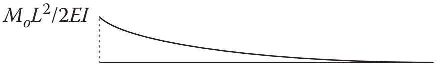
Conjugate beam, elastic load, and reactions.

4. Draw the rotation diagram (the shear diagram of the conjugate beam).



Shear (rotation) diagram indicating clockwise rotation.

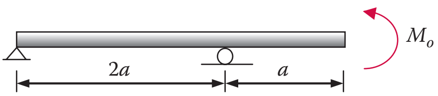
5. Draw the deflection diagram (the moment diagram of the conjugate beam).



Moment (deflection) diagram indicating upward deflection.

Example 5.3

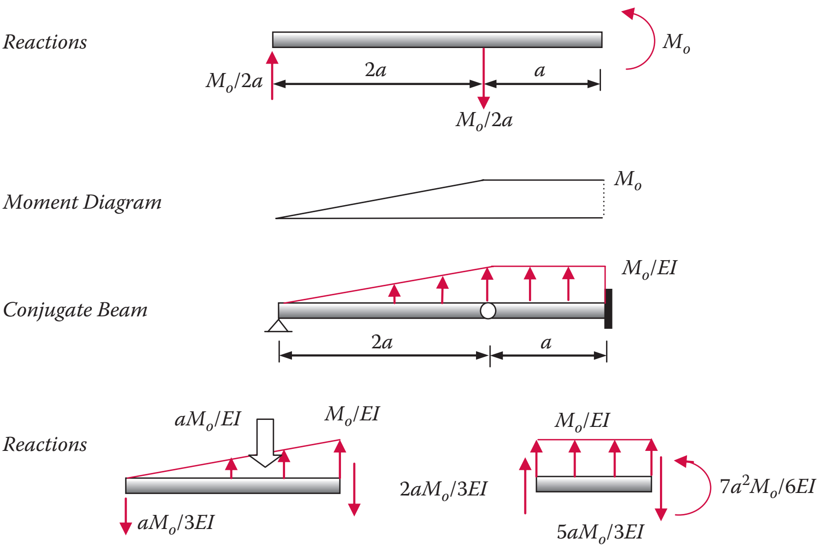
Find the rotation and deflection at the tip of the loaded beam shown. *EI* is constant.



Find the tip rotation and deflection.

Solution

The solution is presented next in a series of diagrams.



Solution process to find tip rotation and deflection.

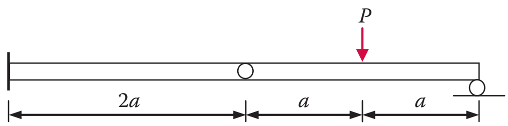
At the right end (tip of the real beam):

Shear $= 5aM_o/3EI \Longrightarrow \theta = 5aM_o/3EI$ (counterclockwise)

Moment $= 7a^2M_o/6EI \Longrightarrow v = 7a^2M_o/6EI$ (upward)

Example 5.4

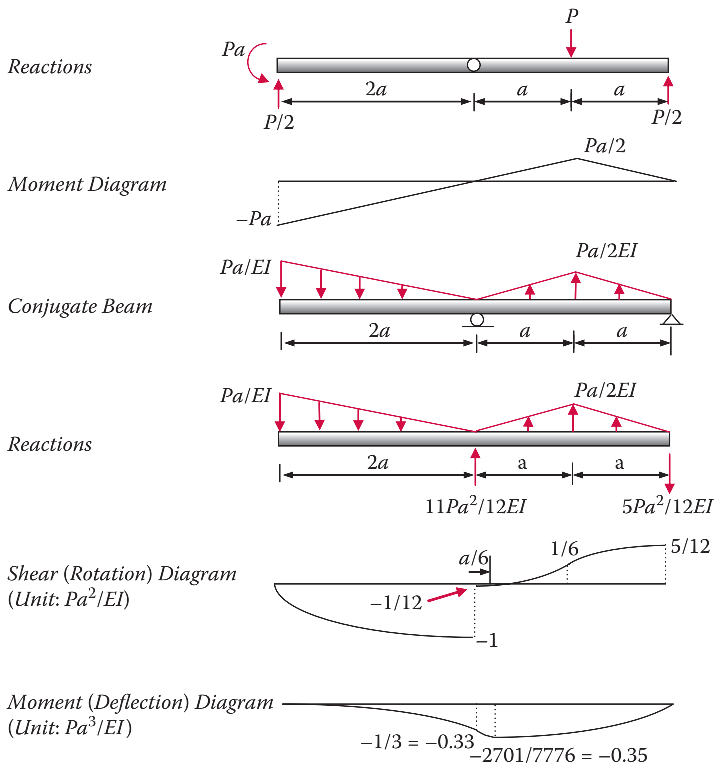
Draw the rotation and deflection diagrams of the loaded beam shown. *EI* is constant.



Beam example on rotation and deflection diagrams.

Solution

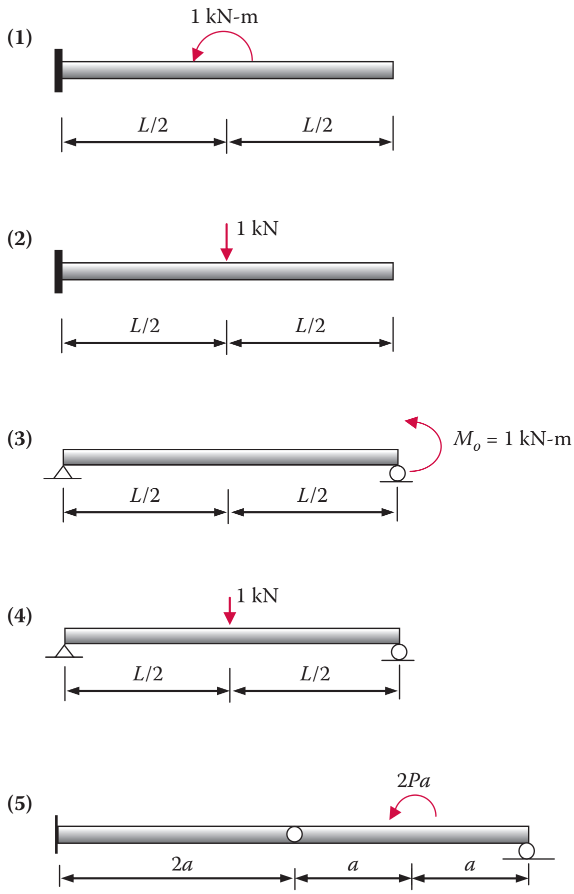
The solution is presented next in a series of diagrams. Readers are encouraged to verify all numerical results.



Solution process for rotation and deflection diagrams.

PROBLEM 5.1

Draw the rotation and deflection diagrams of the loaded beams shown. *EI* is constant in all cases.



Problem 5.1

5.3 Energy Methods

The conjugate beam method is the preferred method for beam deflections, but it cannot be easily generalized for rigid frame deflections. We shall now explore energy methods and introduce the unit load method for beams and frames.

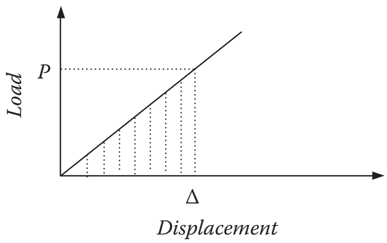
One of the fundamental formulas we can use is the *principle of conservation of mechanical energy*, which states that in an equilibrium system, the work done by external forces is equal to the work done by internal forces:

$$W_{ext} = W_{int} \tag{5.8}$$

For a beam or frame loaded by a group of concentrated forces, P_i, distributed forces, q_j, and concentrated moments, M_k, where i, j, and k run from one to the total number in the respective group, the work done by external forces is

$$W_{ext} = \Sigma \frac{1}{2} P_i \quad {}_i + \Sigma \int \frac{1}{2} (q_j dx)\ v_j + \Sigma \frac{1}{2} M_k \theta_k \tag{5.9}$$

where Δ_i, v_j, and θ_k are the deflection and rotation corresponding to P_i, q_j, and M_k. For a concentrated load, the load-displacement relationship for a linear system is shown next and the work done is represented by the shaded triangular area. Similar diagrams can be drawn for qdx and M.

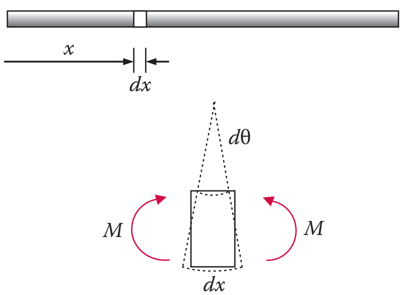


Work done by a concentrated load.

For the work done by internal forces, we shall consider only internal moments, because the effect of shear and axial forces on deflection is negligible. We introduce a new entity, strain energy, U, which is defined as the work done by internal forces. Then

$$W_{int} = U = \Sigma \int \frac{1}{2} M d\theta = \Sigma \int \frac{1}{2} \frac{M^2 dx}{EI} \tag{5.10}$$

where the summation is over the number of frame members, and the integration is over the length of each member. For a single beam, the summation is redundant. Equation 5.10 is derived as the angle of rotation of an infinitesimal element induced by a pair of internal moments.



Change of angle induced by internal moments.

The change of angle is related to the internal moment, according to Equation 5.2 and Equation 5.4,

$$d\theta = \frac{Mdx}{EI}$$

which leads to Equation 5.10.

Example 5.5

Find the rotation at the tip of the beam shown. EI is constant, and the beam length is L.

Example on tip rotation.

Solution

We shall use the principle of conservation of mechanical energy to find the tip rotation, which is denoted by θ_o. The work done by external forces is

$$W_{ext} = \frac{1}{2} M_o \theta_o$$

To find the expression for strain energy, we noted that

$$M(x) = M_o$$

$$U = \int \frac{1}{2} \frac{M^2 dx}{EI} = \frac{1}{2} \frac{M_o^2 L}{EI}$$

Equating W_{ext} to U yields

$$\theta_o = \frac{M_o L}{EI}$$

It is clear that the principle of conservation of mechanical energy can only be used to find the deflection under a single external load. A more general method is the unit load method, which is based on the principle of virtual force.

The *principle of virtual force* states that the virtual work done by an external virtual force upon a real displacement system is equal to the virtual work done by internal virtual forces, which are in equilibrium with the external virtual force, upon the real deformation. Denoting the external virtual work by δW and the internal virtual work by δU, we can express the principle of virtual force as

$$\delta W = \delta U \tag{5.11}$$

In view of Equation 5.10 which defines the strain energy as work done by internal forces, we can call δU the virtual strain energy. When applying the principle of virtual force to find a particular deflection at a point, we apply a fictitious unit load at the point of interest and in the direction of the deflection we are to find. This unit load is the external virtual force. The internal virtual force for a beam, corresponding to the unit load, is the bending moment in equilibrium with the unit load and is denoted by $m(x)$. Denoting the internal moment induced by the real applied load as $M(x)$, the real deformation corresponding to the virtual moment $m(x)$ is then

$$d\theta = \frac{M(x)dx}{EI}$$

The strain energy of an infinitesimal element is $m(x)d\theta$ and the integration of $m(x)d\theta$ over the length of the beam gives the virtual strain energy.

$$\delta U = \int m(x)\frac{M(x)dx}{EI}$$

The external virtual work is the product of the unit load and the deflection we want, denoted by Δ.

$$\delta W = 1(\Delta)$$

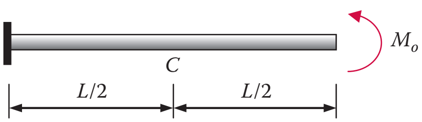
The principle of virtual force then leads to the following useful formula of the *unit load method*.

$$1\,(\Delta) = \int m(x)\,\frac{M(x)dx}{EI}$$

In Equation 5.12, we indicated the linkage between the external virtual force, *1*, and the internal virtual moment, $m(x)$, and the linkage between the real external deflection, Δ, and the real internal element rotation, $M(x)dx/EI$.

Example 5.6

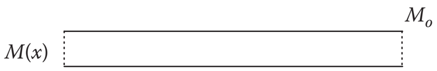
Find the rotation and deflection at the midspan point C of the beam shown. EI is constant and the beam length is L.



Beam example of the unit load method.

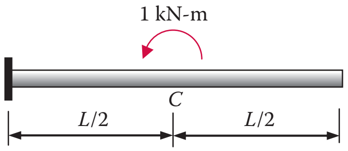
Solution

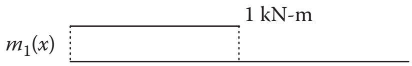
1. Draw the moment diagram of the original beam problem.



Moment diagram of the original beam problem.

2. Draw the moment diagram of the beam with a unit moment at C.





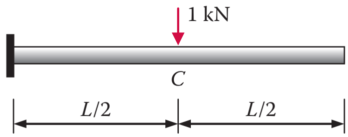
Moment diagram of the beam under the first unit load.

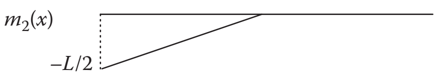
3. Compute the rotation at C.

$$1(\theta_C) = \int m_1(x)\frac{M(x)dx}{EI} = 1\left(\frac{M_o}{EI}\right)\left(\frac{L}{2}\right) = 1\frac{M_oL}{2EI}\text{radian}$$

$$(\theta_C) = \frac{M_oL}{2EI} \text{ radian}$$

4. Draw the moment diagram of the beam with a unit force at C.





Moment diagram of the beam under the second unit load.

5. Compute the deflection at C.

$$1(\Delta_C) = \int m_2(x)\frac{M(x)dx}{EI} = 1\left(\frac{1}{2}\right)\left(-\frac{L}{2}\right)\left(\frac{M_o}{EI}\right)\left(\frac{L}{2}\right) = 1\left(-\frac{M_oL^2}{8EI}\right)$$

$$(\Delta_C) = -\frac{M_oL^2}{8EI} \text{ m} \quad \text{Upward}$$

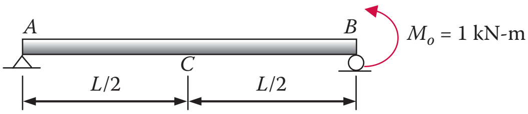
In the last integration, we have used a shortcut. For simple polynomial functions, the following table is easy to remember and easy to use.

Integration Table for Integrands as a Product of Two Simple Functions

Case	(1)	(2)	(3)	(4)
$f_1(x)$	a	a	a	a
$f_2(x)$	b	b	b	b
$\int_o^L f_1 f_2 dx$	$\frac{1}{2} a\, b\, L$	$\frac{1}{3} a\, b\, L$	$\frac{1}{6} a\, b\, L$	$a\, b\, L$

Example 5.7

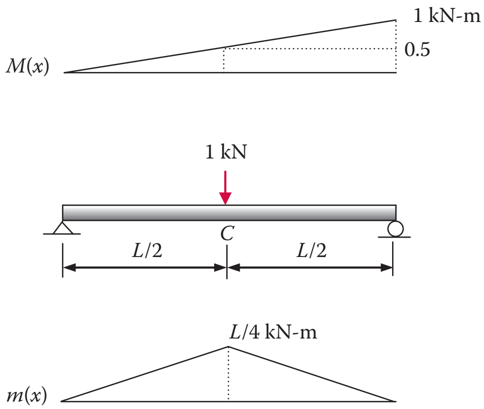
Find the deflection at the midspan point *C* of the beam shown. *EI* is constant and the beam length is *L*.



Example problem to find deflection at midspan.

Solution

The solution is presented next in a series of figures.



Solution to find deflection at midspan.

The computing is carried out using the integration table as a shortcut. The large triangular-shaped function in $M(x)$ is broken down into two triangles and one rectangle, as indicated by the dashed lines, in order to apply the formulas in the table.

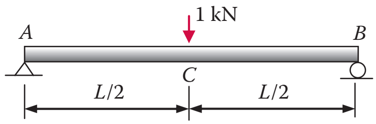
$$1(\ _{c}) = \int m(x)\frac{M(x)dx}{EI}$$

$$= \frac{1}{EI} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{L}{4} \quad \frac{L}{2} + \frac{1}{2} \quad \frac{1}{2} \quad \frac{L}{4} \quad \frac{L}{2} + \frac{1}{6} \quad \frac{1}{2} \quad \frac{L}{4} \quad \frac{L}{2}$$

$$\Delta_C = \frac{L^2}{16EI} \text{ m} \qquad \text{Downward}$$

Example 5.8

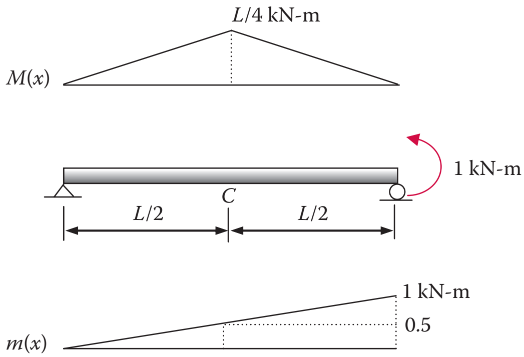
Find the rotation at the end point B of the beam shown. EI is constant and the beam length is L.



Example problem to find rotation at end B.

Solution

The solution is presented next in a series of figures.



Solution to find the rotation at the right end.

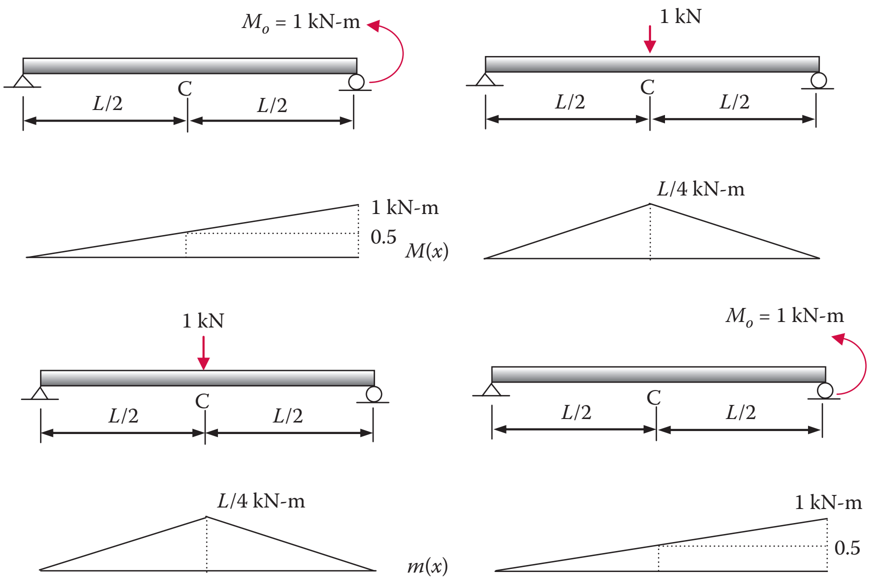
The computing is carried out using the integration table as a shortcut. The large triangular shaped function in $m(x)$ is broken down into two triangles and one rectangle, as indicated by the dashed lines, in order to apply the formulas in the table.

$$1(\theta_B) = \int m(x)\frac{M(x)dx}{EI}$$

$$= \frac{1}{EI} \quad \frac{1}{3}\;\frac{1}{2}\;\frac{L}{4}\;\frac{L}{2} + \frac{1}{2}\;\frac{1}{2}\;\frac{L}{4}\;\frac{L}{2} + \frac{1}{6}\;\frac{1}{2}\;\frac{L}{4}\;\frac{L}{2}$$

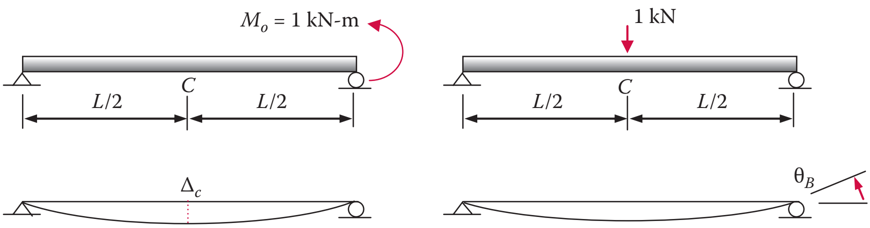
$$\theta_B = \frac{L^2}{16EI} \text{ radian} \qquad \text{Counterclockwise}$$

The fact that the results of the last two examples are numerically identical prompts us to look into a comparison of the two computational processes.



Side-by-side comparison of the two processes in Example 5.7 and Example 5.8.

It is clear from the comparison that the roles of $M(x)$ and $m(x)$ are reversed in the two examples. Since the integrands used to compute the results are the products of $M(x)$ and $m(x)$ and are identical, no wonder the results are identical in their numerical values. We can identify the deflection results we obtained in the two examples graphically as shown next.



Reciprocal deflections.

We state that the deflection at C due to a unit moment at B is numerically equal to the rotation at B due to a unit force at C. This is the *Maxwell's reciprocal law*, which may be expressed as:

$$\delta_{ij} = \delta_{ji} \tag{5.13}$$

where
δ_{ij} = displacement at i due to a unit load at j
δ_{ji} = displacement at j due to a unit load at i

The following figure illustrates the reciprocity further.

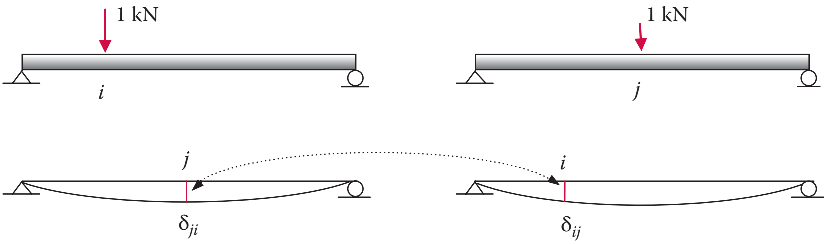
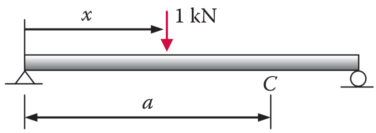


Illustration of the reciprocal theorem.

Example 5.9

Find the vertical displacement at point C due to a unit applied load at a location x from the left end of the beam shown. EI is constant and the length of the beam is L.



Find deflection at C as a function of the location of the unit load, x.

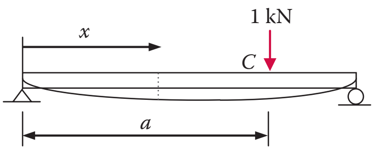
Solution

Clearly, the deflection at C is a function of x, which represents the location of the unit load. If we plot this function against x, then a diagram or curve is established. We call this curve the influence line of deflection at C. We now show that Maxwell's reciprocal law is well suited to find this influence line for deflections.

According to Maxwell's reciprocal law, the deflection at C due to a unit load at x is equal to the deflection at x due to a unit load at C. A direct application of Equation 5.13 yields

$$\delta_{cx} = \delta_{xc}$$

The influence line of deflection at C is δ_{cx}, but it is equal in value to δ_{xc}, which is simply the deflection curve of the beam under a unit load at C. By applying Maxwell's reciprocal law, we have transformed the more difficult problem of finding deflection for a load at various locations to a simpler problem of finding deflection of the whole beam under a fixed unit load.



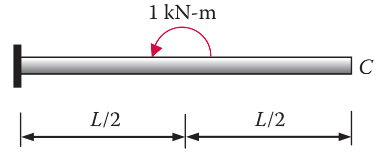
Deflection of the beam due to a unit load at C.

We can use the conjugate beam method to find the beam deflection. Readers are encouraged to find the moment (deflection) diagram from the conjugate beam.

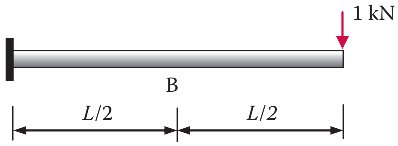
PROBLEM 5.2

EI is constant in all cases. Use the unit load method in all problems.

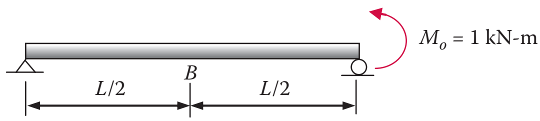
(1) Find the deflection at point *C*.



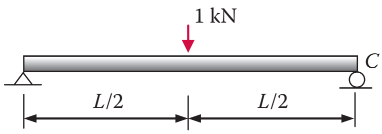
(2) Find the sectional rotation at point *B*.



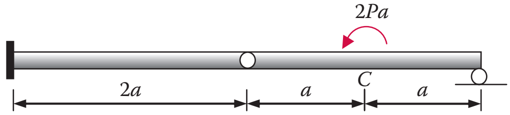
(3) Find the deflection at point *B*.



(4) Find the sectional rotation at point *C*.



(5) Find the deflection and sectional rotation at point *C*.

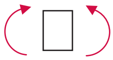
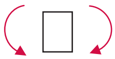


Problem 5.2

Sketch the Deflection Curve. Only the conjugate beam method gives the deflection diagram. The unit load method gives deflection at a point. If we wish to have an idea on what the deflection curve looks like, we can sketch a curve based on what we know about the moment diagram.

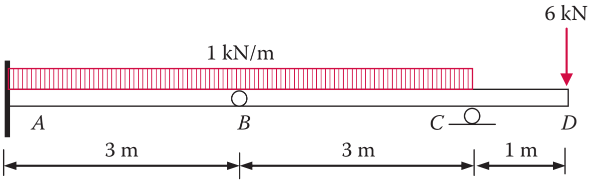
Equation 5.5 indicates that the curvature of a deflection curve is proportional to the moment. This implies that the curvature varies in a similar way as the moment varies along a beam if EI is constant. At any location on a beam, the correspondence between the moment and the appearance of the deflection curve can be summarized in the following table. At the point of zero moment, the curvature is zero and the point becomes an inflection point.

Sketch Deflection from Moment

Moment	Deflection
	
	

Example 5.10

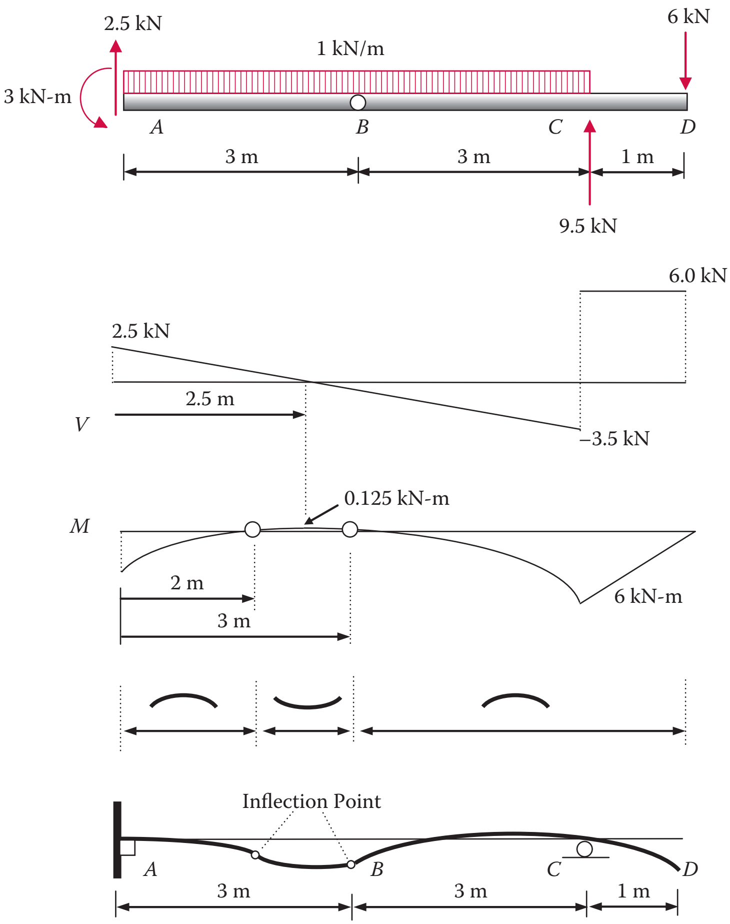
Sketch the deflection curve of the beam shown. EI is constant and the beam length is L.



Beam example on sketching the deflection.

Solution

The solution process is illustrated in a series of figures.



Sketching a deflection curve.

5.4 Frame Deflection

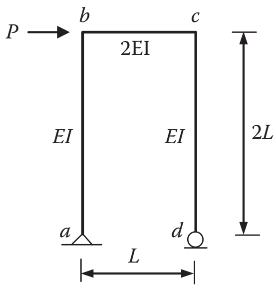
The unit load method can be applied to rigid frames using Equation 5.12 and summing the integration over all members.

$$1(\quad) = \sum \int m(x) \frac{M(x)dx}{EI} \tag{5.12}$$

Within each member, the computation is identical to that of a beam.

Example 5.11

Find the horizontal displacement at point *b*.



Frame example to find displacement at a point.

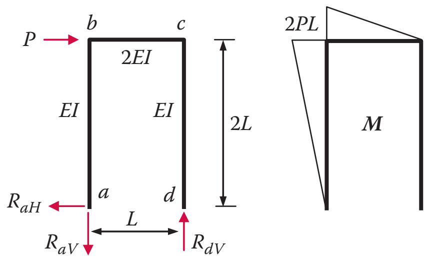
Solution

1. Find all reactions and draw the moment diagram *M* of the entire frame.

$$\Sigma M_a = 0 \Longrightarrow R_{dV} = 2P$$

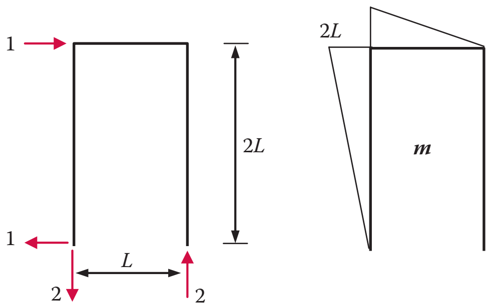
$$\Sigma M_d = 0 \Longrightarrow R_{aV} = 2P$$

$$\Sigma F_x = 0 \Longrightarrow R_{aH} = P$$



Reaction and moment diagrams of the entire frame.

2. Place unit load and draw the corresponding moment diagram *m*.



Moment diagram corresponding to a unit load.

3. Compute the integration member by member.

$$\text{Member } a \quad b: \int m\frac{M}{EI}dx = \frac{1}{EI}\;\frac{1}{3}\;(2L)(2PL)(2L) = \frac{8PL^3}{3EI}$$

$$\text{Member } b \quad c: \int m\frac{M}{EI}dx = \frac{1}{2EI}\;\frac{1}{3}\;(2L)(2PL)(L) = \frac{2PL^3}{3EI}$$

$$\text{Member } c \quad d: \int m\frac{M}{EI}dx = 0$$

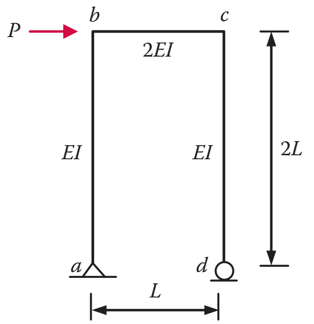
4. Sum all integration to obtain the displacement.

$$1(\;_b) = \sum\int m(x)\frac{M(x)dx}{EI} = \frac{8PL^3}{3EI} + \frac{2PL^3}{3EI} = \frac{10PL^3}{3EI}$$

$$\Delta_b = \frac{10PL^3}{3EI} \text{ to the right } \longrightarrow$$

Example 5.12

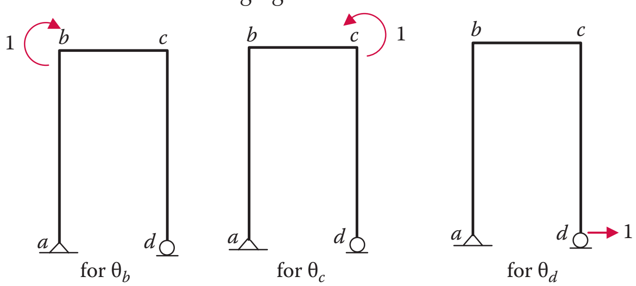
Find the horizontal displacement at point *d* and the rotations at *b* and *c*.



Example problem to find displacement and rotation.

Solution

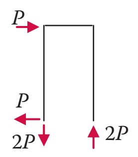
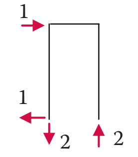
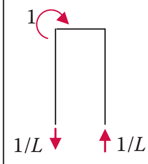
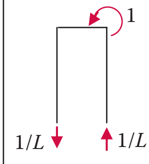
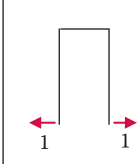
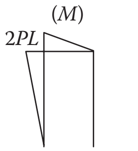
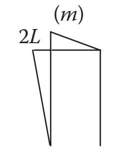
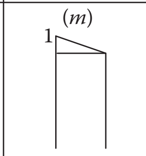
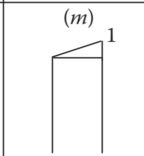
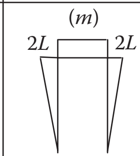
This is the same problem as that in Example 5.11. Instead of finding Δ_b, now we need to find θ_b, θ_c, and Δ_d. We need not repeat the solution for *M*, which is already obtained. For each of the three quantities, we can place the unit load as shown in the following figure.



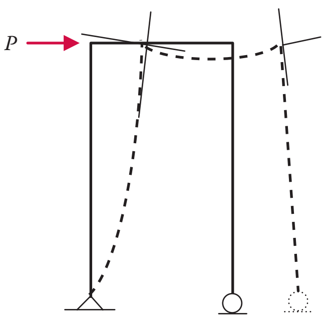
Placing unit loads for θ_b, θ_c, and Δ_d.

The computation process, including the reaction diagram, moment diagram for *m*, and integration can be tabulated as shown next. In the table, the computation in Example 5.11 is also included so that readers can see how the tabulation is done.

Computation Process for Four Different Displacements

		Actual Load	Load for Δ_b	Load for Δ_b	Load for Δ_c	Load for Δ_d
Load Diagram						
Moment Diagram						
$\int m \frac{Mdx}{EI}$		$a \sim b$	$\frac{1}{EI}\left(\frac{1}{3}\right)(2PL)(2L)(2L) = \frac{8PL^3}{3EI}$	0	0	$\frac{1}{EI}\left(\frac{1}{3}\right)(2PL)(2L)(2L) = \frac{8PL^3}{3EI}$
		$b \sim c$	$\frac{1}{2EI}\left(\frac{1}{3}\right)(2PL)(2L)(L) = \frac{2PL^3}{3EI}$	$\frac{1}{2EI}\left(\frac{1}{3}\right)(2PL)(1)(L) = \frac{PL^2}{3EI}$	$\frac{1}{2EI}\left(\frac{1}{6}\right)(2PL)(1)(L) = \frac{PL^2}{6EI}$	$\frac{1}{2EI}\left(\frac{1}{2}\right)(2PL)(2L)(L) = \frac{PL^3}{EI}$
		$c \sim d$	0	0	0	0
$\Sigma\int m \frac{Mdx}{EI}$			$\Delta_b = \frac{10PL^3}{3EI}$	$\theta_b = \frac{PL^2}{3EI}$	$\theta_c = \frac{PL^2}{6EI}$	$\Delta_d = \frac{11PL^3}{3EI}$

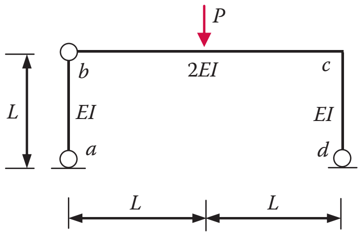
Knowing the rotation and displacement at key points, we can draw the displaced configuration of the frame as shown next.



Displaced configuration.

Example 5.13

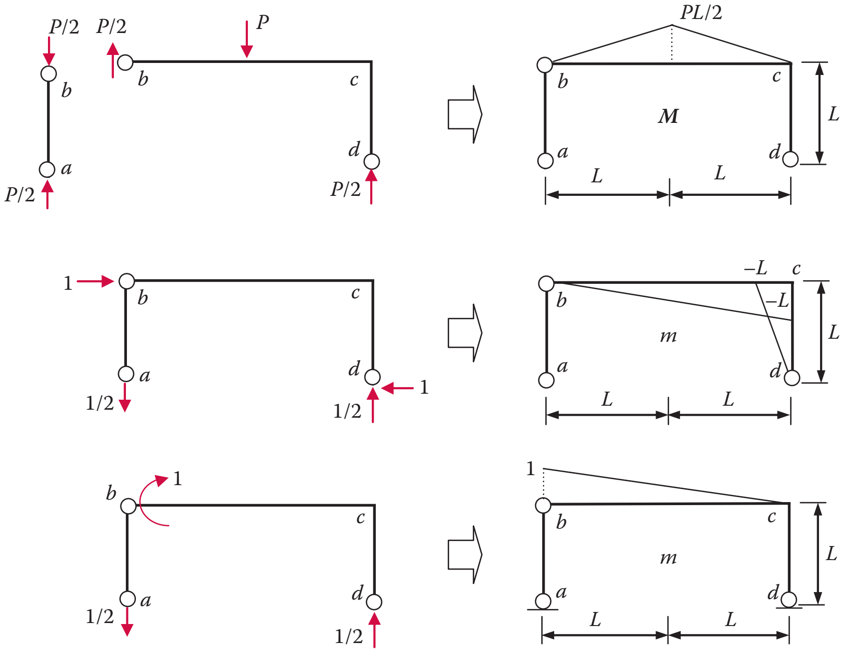
Find the horizontal displacement at point *b* and the rotation at *b* of member *b*~*c*.



Frame example for finding rotation at a hinge.

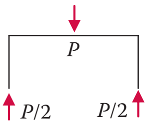
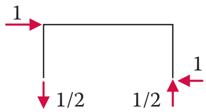
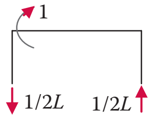
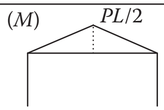
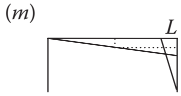
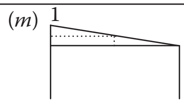
Solution

It is necessary to clearly specify that the rotation at *b* is for the end of member *b*~*c*, because the rotation at *b* for member *a*~*b* is different. The solution process is illustrated next in a series of figures.

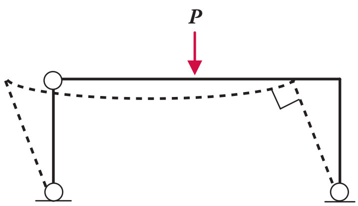


Drawing moment diagrams.

Computation Process for a Displacement and a Rotation

	Actual Load	Load for Δ_b	Load for θ_b
Load Diagram			
Moment Diagram			
$\int m \frac{Mdx}{EI}$	$a \sim b$	0	0
	$b \sim c$	$\frac{-1}{2EI}[(\frac{PL}{2})(\frac{L}{2})(L) + \frac{1}{2}(\frac{PL}{2})(\frac{L}{2})(L) + \frac{1}{6}(\frac{PL}{2})(\frac{L}{2})(L)] = -\frac{5PL^3}{48EI}$	$\frac{1}{2EI}[\frac{1}{6}(\frac{PL}{2})(\frac{1}{2})(L) + \frac{1}{2}(\frac{PL}{2})(\frac{1}{2})(L) + \frac{1}{3}(\frac{PL}{2})(\frac{1}{2})(L)] = \frac{PL^2}{8EI}$
	$c \sim d$	0	0
$\Sigma \int m \frac{Mdx}{EI}$		$\Delta_b = -\frac{5PL^3}{48EI}$	$\theta_b = \frac{PL^2}{8EI}$

The displaced configuration is shown next.

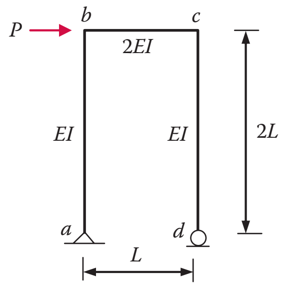


Displaced configuration.

In the following problems, use the unit load method to find displacements indicated.

PROBLEM 5.3

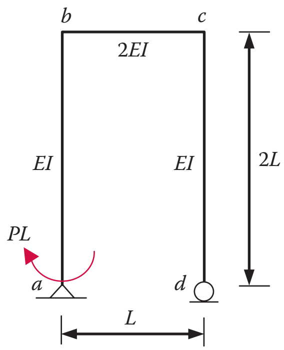
Find the rotation at *a* and the rotation at *d*.



Problem 5.3

PROBLEM 5.4

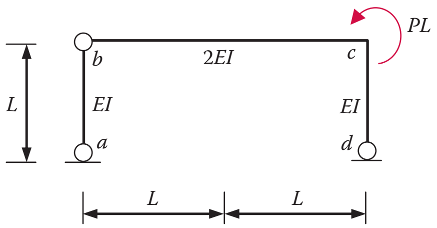
Find the horizontal displacement at *b* and the rotation at *d*.



Problem 5.4

PROBLEM 5.5

Find the horizontal displacement at *b* and the rotation at *d*.



Problem 5.5

6

Beam and Frame Analysis: Force Method—Part III

6.1 Statically Indeterminate Beams and Frames

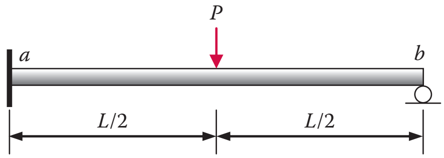
When the number of force unknowns exceeds that of independent equilibrium equations, the force method of analysis calls for additional conditions based on support or member deformation considerations. These conditions are compatibility conditions and the method of solution is called the *method of consistent deformations*. The procedures of the method of consistent deformations are:

1. Determine the degree(s) of redundancy, select the redundant force(s), and establish the primary structure.
2. Identify equation(s) of compatibility, expressed in terms of the redundant force(s).
3. Solve the compatibility equation(s) for the redundant force(s).
4. Complete the solution by solving the equilibrium equations for all unknown forces.

6.2 Indeterminate Beam Analysis

Example 6.1

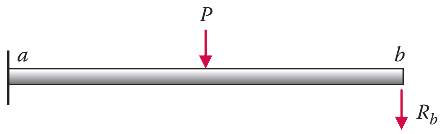
Find all reaction forces, draw the shear and moment diagrams, and sketch the deflection curve. *EI* is constant.



Beam statically indeterminate to the first degree.

Solution

1. Degree of indeterminacy is one. We choose the reaction at b, R_b as the redundant force.

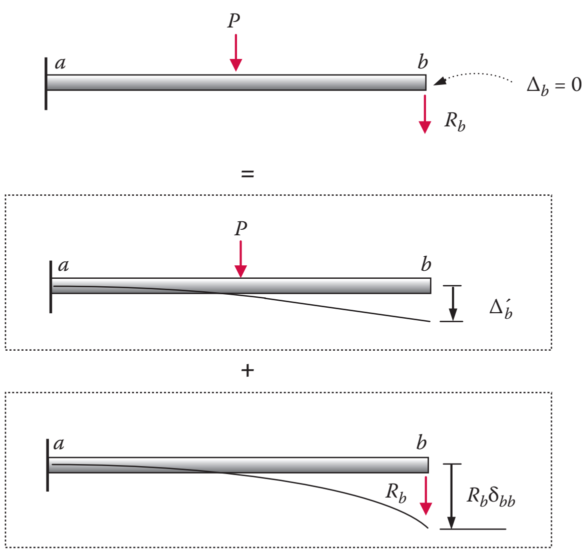


The statically determinate primary structure and the redundant force.

2. Establish the compatibility equation. Comparing the previous two figures, we observe that the combined effect of the load P and the reaction R_b must be such that the total vertical displacement at b is zero, which is dictated by the roller support condition of the original problem. Denoting the total displacement at b as Δ_b, we can express the compatibility equation as

$$_b = \quad _b + R_b\delta_{bb} = 0$$

where $\quad_b$ is the displacement at b due to the applied load and δ_{bb} is the displacement at b due to a unit load at b. Together, $R_b\delta_{bb}$ represents the displacement at b due to the reaction R_b. The combination of $\quad_b$ and $R_b\delta_{bb}$ is based on the principle of superposition, which states that the displacement of a linear structure due to two loads is the superposition of the displacement due to each of the two loads. This principle is illustrated next.

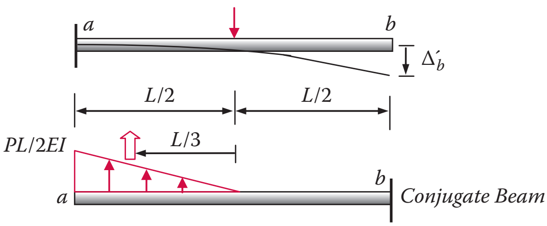


Compatibility equation based on the principle of superposition.

3. Solve for the redundant force. Clearly, the redundant force is expressed by

$$R_b = -\frac{\Delta'_b}{\delta_{bb}} \quad \uparrow$$

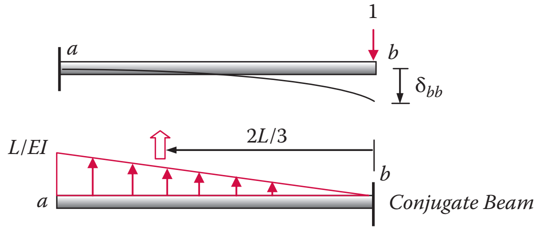
To find $_b$ and δ_{bb}, we can use the conjugate beam method for each separately, as shown next.



Conjugate beam method to find $_b$.

Use the conjugate beam method to find $_b$. The deflection is computed as the moment at b of the conjugate beam.

$$\Delta'_b = (\Sigma M_b) = \frac{1}{EI}\left(\frac{1}{2}\frac{PL}{2}\frac{L}{2}\right)\left(\frac{L}{3}+\frac{L}{2}\right) = \frac{5PL^3}{48EI} \quad \text{Downward} \downarrow$$

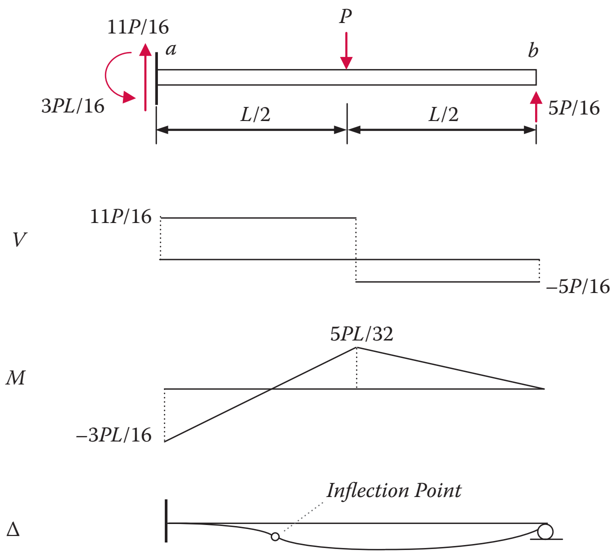


Use the conjugate beam method to find δ_{bb}.

$$\delta_{bb} = (\Sigma M_b) = \frac{1}{EI}\left[\left(\frac{1}{2}\right)(L)(L)\right]\left(\frac{2}{3}L\right) = \frac{L^3}{3EI} \quad \text{Downward} \downarrow$$

$$R_b = -\frac{\Delta'_b}{\delta_{bb}} = -\frac{5}{16}P \quad \text{Upward} \uparrow$$

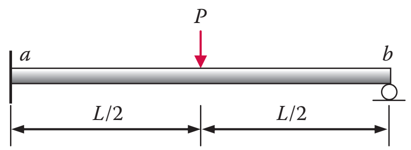
4. Find other reaction forces and draw the shear and moment diagrams. This is achieved through a series of diagrams.



Reaction, shear, moment, and deflection diagrams.

Example 6.2

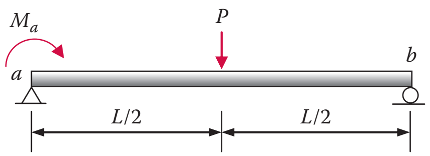
Find all reactions of the same beam as in Example 6.1, but choose a different redundant force. *EI* is constant.



Beam statically indeterminate to the first degree.

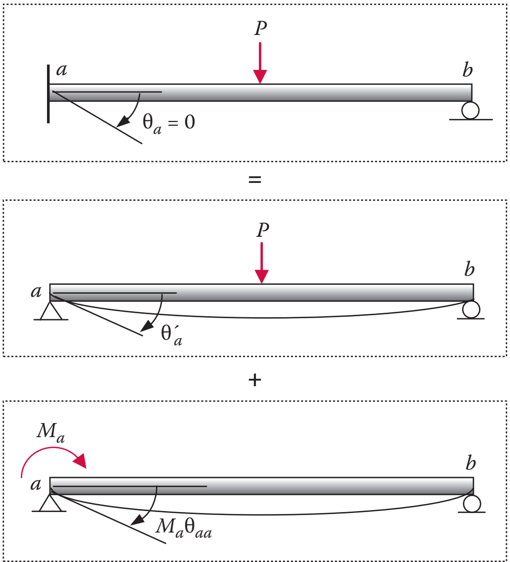
Solution

There are different ways of establishing a primary structure. For example, inserting a hinge connection at any point along the beam introduces one condition of construction and renders the resulting structure statically determinate. We now choose to put the hinge at the fixed end, effectively selecting the end moment, M_a, as the redundant force.



Primary structure and redundant moment M_a.

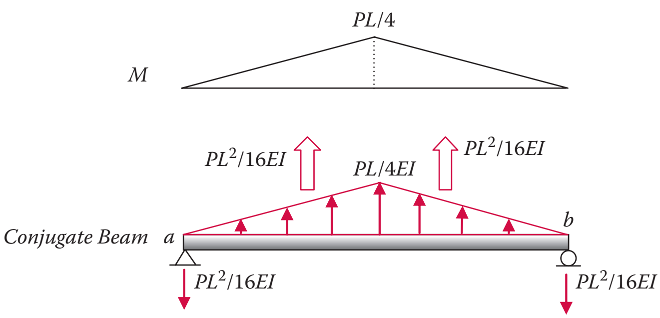
The compatibility equation is established from the condition that the total rotation at *a* of the primary structure due to the combined effect of the applied load and the redundant force M_a must be zero, which is required by the fixed end support.



Compatibility condition and principle of superposition.

$$\theta_a = \theta'_a + M_a\theta_{aa} = 0$$

The conjugate beam method is used to find θ'_a and θ_{aa}.

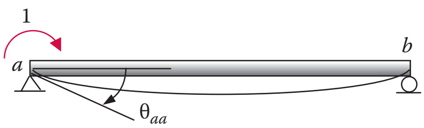


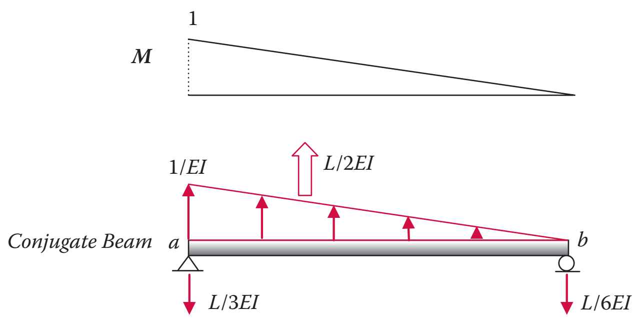
Conjugate beam for θ_a .

From the conjugate beam, the rotation at point *a* is computed as the shear of the conjugate beam at *a*.

$$\theta'_a = (V_a) = -\left(\frac{PL^2}{16EI}\right) \quad \curvearrowright$$

To find θ_{aa}, the following figure applies.





Conjugate beam for θ_{aa}.

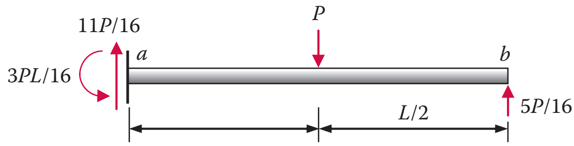
From the conjugate beam, the rotation at point *a* is computed as the shear of the conjugate beam at *a*.

$$\theta_{aa} = (V_a) = \left(\frac{L}{3EI}\right) \curvearrowright$$

The redundant moment is computed from the compatibility equation as

$$M_a = -\frac{\theta'_a}{\theta_{aa}} = -\frac{3PL}{16} \circlearrowleft$$

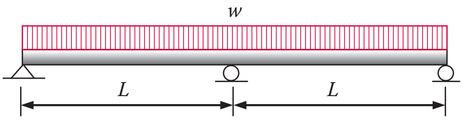
This is the same end moment as obtained in Example 6.1. All reaction forces are shown next.



Solution showing all reaction forces.

Example 6.3

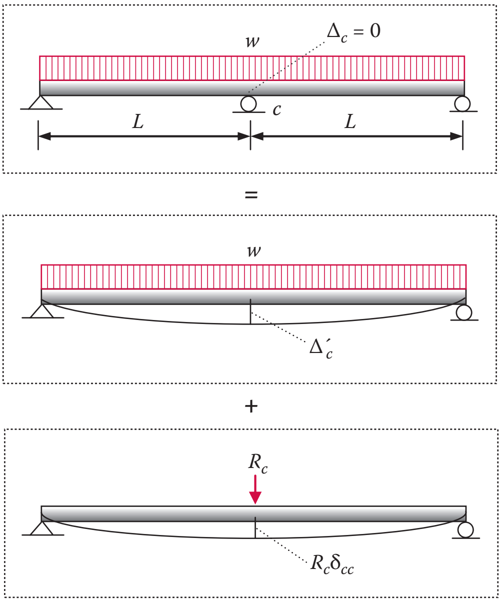
Analyze the indeterminate beam shown next, and draw the shear, moment, and deflection diagrams. *EI* is constant.



Statically indeterminate beam with one redundant force.

Solution

We choose the reaction at the center support as the redundant force. The compatibility condition is that the vertical displacement at the center support be zero. The primary structure, deflections at center due to the load, the redundant force, and so forth, are shown next. The resulting computation is self-evident.



Principle of superposition used to find compatibility equation.

The compatibility condition is

$$_{c} = \ _{c} + R_c\delta_{cc} = 0$$

For such a simple geometry, we can find the deflections from published deflection formulas.

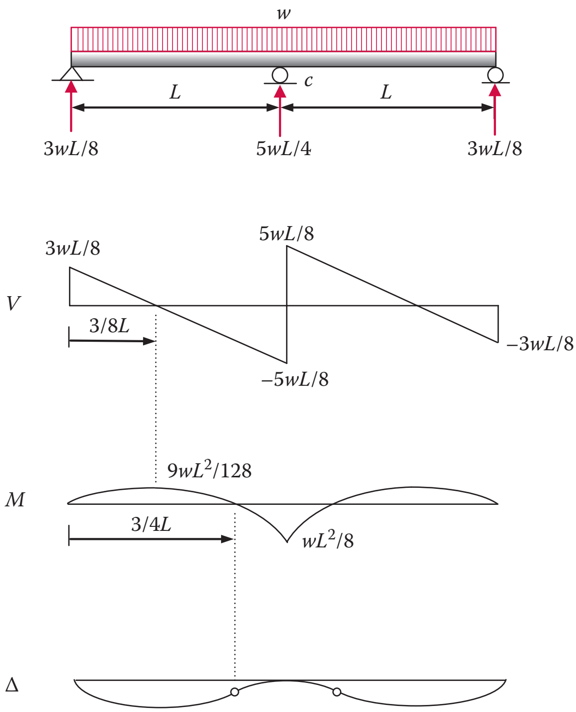
$$_{c} = \frac{5w(length)^4}{384EI} = \frac{5w(2L)^4}{384EI} = \frac{5wL^4}{24EI}$$

$$\delta_{cc} = \frac{P(length)^3}{48EI} = \frac{P(2L)^3}{48EI} = \frac{PL^3}{6EI}$$

Hence

$$R_c = -\frac{5wL}{4} \text{ Upward} \uparrow$$

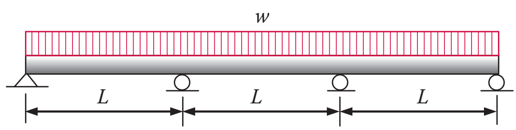
The reaction, shear, moment, and deflection diagrams are shown next.



Reaction, shear, moment, and deflection diagrams.

Example 6.4

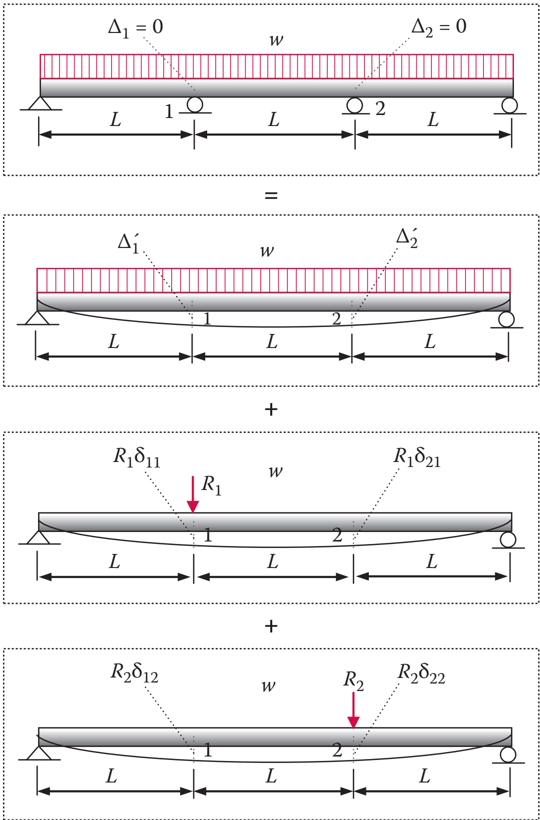
Outline the formulation of the compatibility equation for the beam shown.



Statically indeterminate beam with two redundant forces.

Solution

We choose the reaction forces at the two internal supports as the redundant forces. As a result, the two conditions of compatibility are the vertical displacements at the internal support points be zero. The superposition of displacements involves three loading conditions as shown in the following figure.



Superposition of primary structure solutions.

The two compatibility equations are:

$$_1 = \quad _1 + R_1\delta_{11} + R_2\delta_{12} = 0$$

$$_2 = \quad _2 + R_1\delta_{21} + R_2\delta_{22} = 0$$

These two equations can be put in the following matrix form.

$$\begin{matrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{matrix} \quad \begin{matrix} R_1 \\ R_2 \end{matrix} = - \quad \begin{matrix} _1 \\ _2 \end{matrix}$$

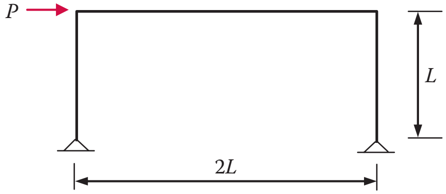
Note that the square matrix at the left-hand side (LHS) is symmetric because of Maxwell's reciprocal law. For problems with more than two redundant forces, the same procedures apply and the square matrix is always symmetric.

While we have chosen support reactions as redundant forces in the preceding beam examples, it is sometimes advantageous to choose internal moments as the redundant forces as shown in the frame example next.

6.3 Indeterminate Frame Analysis

Example 6.5

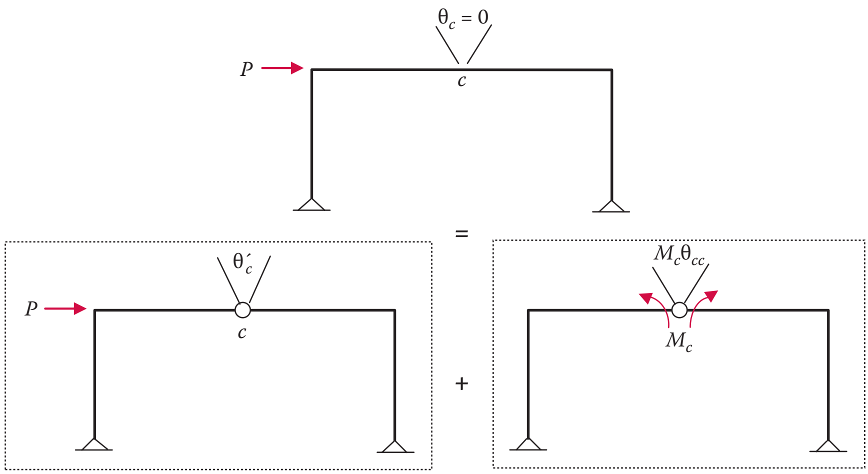
Analyze the frame shown and draw the moment and deflection diagrams. *EI* is constant for all members.



A rigid frame with one degree of redundancy.

Solution

We choose the moment at midspan of the beam member as the redundant force: M_c.

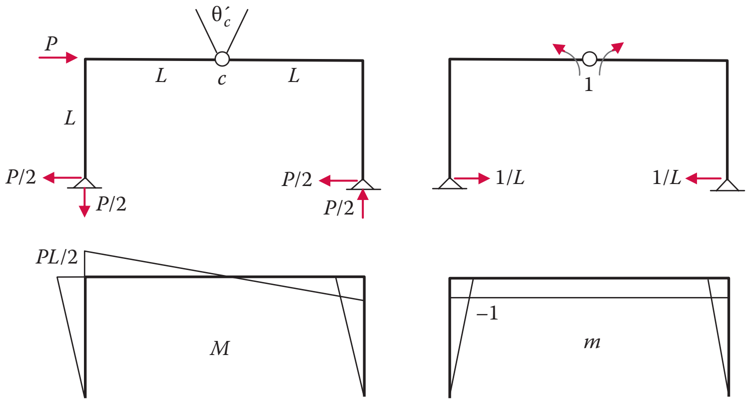


Principle of superposition and compatibility equation.

The compatibility equation is

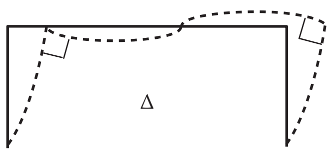
$$\theta_c = \theta_c + M_c\theta_{cc} = 0$$

To find θ_c, we use the unit load method. It turns out that $\theta_c = 0$ because the contribution of the column members cancels out each other and the contribution from the beam member is zero due to the antisymmetry of *M* and symmetry of *m*. Consequently, there is no need to find θ_{cc} and M_c is identically zero.



Unit load method to find relative angle of rotation at *C*.

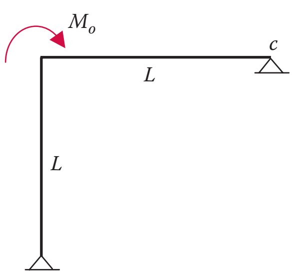
The moment diagram shown above is the correct moment diagram for the frame and the deflection diagram is shown next.



Deflection diagram of the frame.

Example 6.6

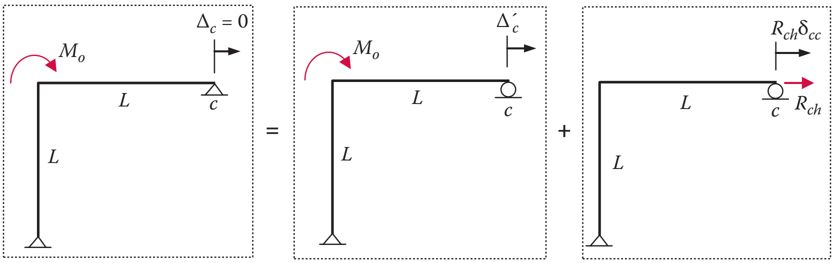
Analyze the frame shown and draw the moment and deflection diagrams. *EI* is constant for the two members.



A frame example with one degree of redundancy.

Solution

We choose the horizontal reaction at C as the redundant force: R_{ch}.

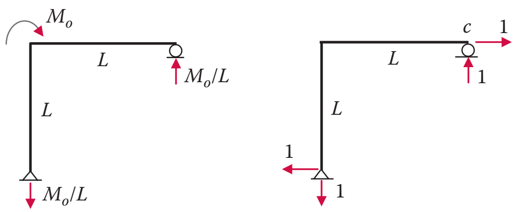


Principle of superposition and compatibility equation.

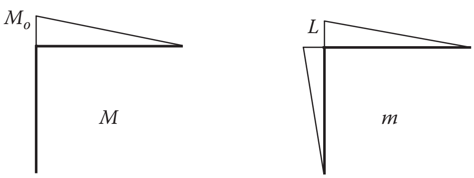
The compatibility equation is

$$_c = \ _c + R_{ch}\delta_{cc} = 0$$

We use the unit load method to compute Δ_c' and δ_{cc}.



Load diagrams for applied load and unit load.

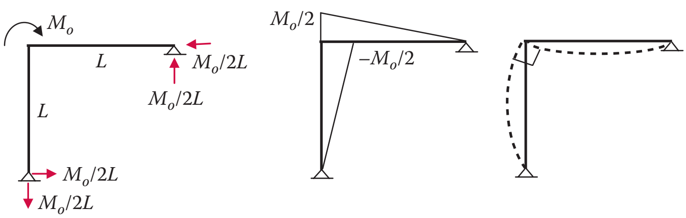


Moment diagrams for applied load and unit load.

$$\Delta_c' = \Sigma \int m \frac{Mdx}{EI} = \frac{1}{EI}\ \frac{1}{3}\ (M_o)\,(L)\,(L) = \frac{M_o L^2}{3EI}$$

$$\delta_{cc} = \Sigma \int m \frac{mdx}{EI} = \frac{1}{EI}\ \frac{1}{3}\ (L)(L)(L)\,(2) = \frac{2L^3}{3EI}$$

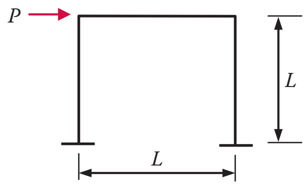
$$\Delta_c' + R_{ch}\delta_{cc} = 0 \quad \Longrightarrow \quad R_{ch} = -\frac{M_o}{2L} \quad \longleftarrow$$



Load, moment, and deflection diagrams.

Example 6.7

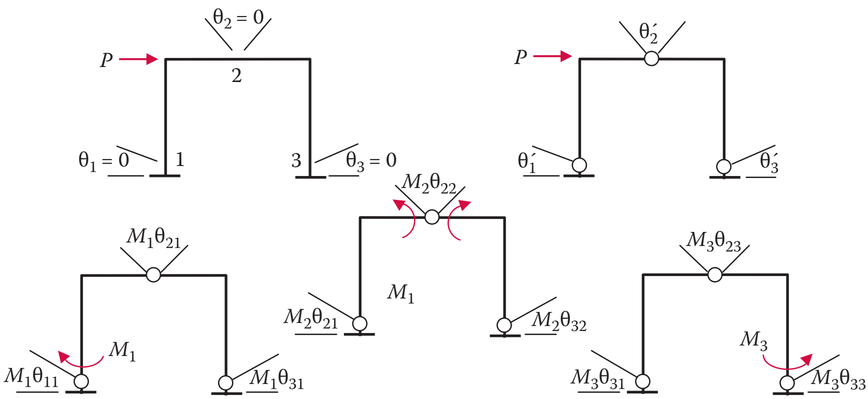
Outline the formulation of the compatibility equation of the rigid frame shown. *EI* is constant for all members.



A rigid frame with three degrees of redundancy.

Solution

We choose three internal moments as the redundant forces. The resulting primary structure is one with three hinges as shown in the following figure (the circles at 1 and 3 are meant to represent hinges). At each of the three hinges, the cumulative effect on the relative rotation must be zero. That is the compatibility condition, which can be put in a matrix form.



Primary structure and the relative rotation at each hinge.

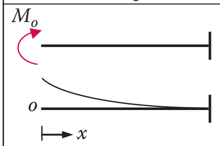
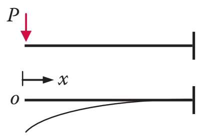
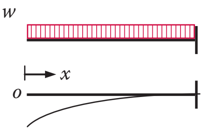
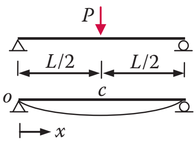
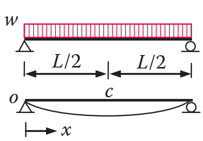
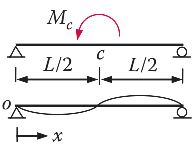
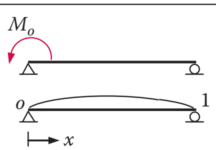
$$
\begin{matrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{matrix} \quad \begin{matrix} M_1 \\ M_2 \\ M_3 \end{matrix} \quad = - \quad \begin{matrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{matrix}
$$

The matrix on the left-hand side is symmetric because of the Maxwell's reciprocal law.

Beam deflection formulas. For statically determinate beam configurations, simple deflection formulas can be easily derived. They are useful for the solution of indeterminate beam problems using the method of consistent deformations. Some of the formulas are given in the upcoming table.

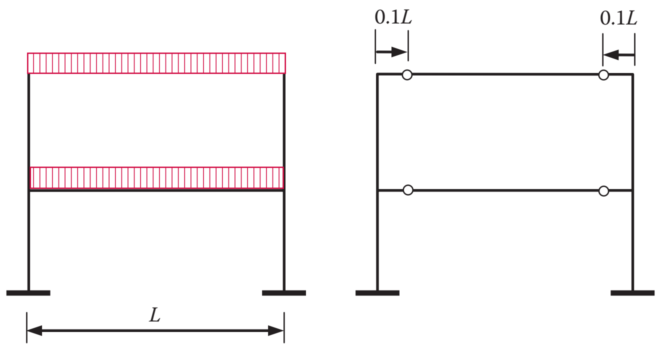
Approximate methods for statically indeterminate frames. As we can see from the previous examples, the force method of analysis for frames is practical for hand calculation only for cases of one to two degrees of redundancy. Although we can computerize the process for high redundancy cases, an easier way for computerization is through the displacement method, which is covered next in Chapter 8. In the meantime, for practical applications, we can use approximate methods for preliminary design purposes. The approximate methods described herein give good approximation to the correct solutions.

Beam Deflection Formulas

Beam Configuration	Formulas for Any Point	Formulas for Special Points
	$\theta = -\dfrac{M_o}{EI}(L-x)$ $v = \dfrac{M_o}{2EI}(L-x)^2$	$\theta_o = \dfrac{M_oL}{EI}$ $v_o = \dfrac{M_oL^2}{2EI}$
	$\theta = \dfrac{P}{2EI}(L^2 - x^2)$ $v = -\dfrac{PL^3}{3EI} + \dfrac{Px}{6EI}(3L^2 - x^2)$	$\theta_o = \dfrac{PL^2}{2EI}$ $v_o = -\dfrac{PL^3}{3EI}$
	$\theta = \dfrac{w}{6EI}(L^3 - x^3)$ $v = -\dfrac{w}{6EI}(L^4 - xL^3)$	$\theta_o = \dfrac{wL^3}{6EI}$ $v_o = -\dfrac{wL^4}{6EI}$
	$\theta = -\dfrac{P}{16EI}(L^2 - 4x^2),\ x \le \dfrac{L}{2}$ $v = -\dfrac{P}{48EI}(3L^2x - 4x^3),\ x \le \dfrac{L}{2}$	$\theta_o = -\dfrac{PL^2}{16EI}$ $v_c = -\dfrac{PL^3}{48EI}$
	$\theta = -\dfrac{w}{24EI}(L^3 - 6Lx^2 + 4x^3)$ $v = -\dfrac{w}{24EI}(L^3x - 2x^3L + x^4)$	$\theta_o = -\dfrac{wL^3}{24EI}$ $v_c = v_{max} = -\dfrac{5wL^4}{384EI}$
	$\theta = -\dfrac{M_c}{24EIL}(L^2 - 12x^2),\ x \le \dfrac{L}{2}$ $v = \dfrac{M_c}{24EIL}(L^2x - 4x^3),\ x \le \dfrac{L}{2}$	$\theta_o = -\dfrac{M_cL}{24EI}$ $v_{max} = \dfrac{M_cL^2}{72\sqrt{3}EI},\ x = \dfrac{L}{2\sqrt{3}}$
	$\theta = \dfrac{M_o}{6EIL}(2L^2 - 6Lx + 3x^2)$ $v = \dfrac{M_o}{6EIL}(2L^2x - 3Lx^2 + x^3)$	$\theta_o = \dfrac{M_oL}{3EI},\ \theta_1 = -\dfrac{M_oL}{6EI}$ $v_{max} = \dfrac{M_oL^2}{9\sqrt{3}EI},\ x = \dfrac{L}{3}(3-\sqrt{3})$

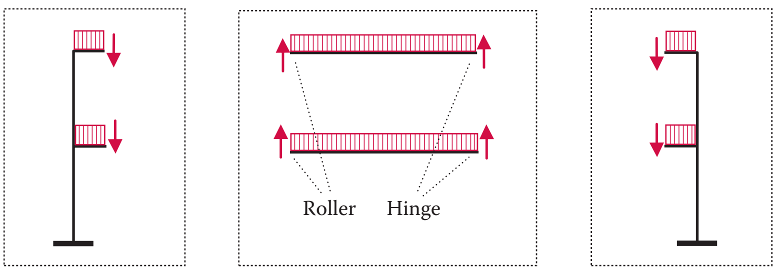
The basic concept of the approximate methods is to assume the location of zero internal moment. At the point of zero moment, the conditions of construction apply, that is, additional equations are available. For rigid frames of regular geometry, we can guess at the location of zero moment fairly accurately from experience. When enough conditions of construction are added, the original problem becomes statically determinate. We shall deal with two classes of problems separately according to loading conditions.

Vertical loads. For regular-shaped rigid frames loaded with vertical floor loads such as shown in the following figure, the deflection of the beams are such that zero moment exists at a location approximately one-tenth of the span from each end.



Vertically loaded frame and approximate location of zero internal moment.

Once we put a pair of hinge-and-roller at the location of zero moment in the beams, the resulting frame is statically determinate and can be analyzed easily. The following figure illustrates the solution process.

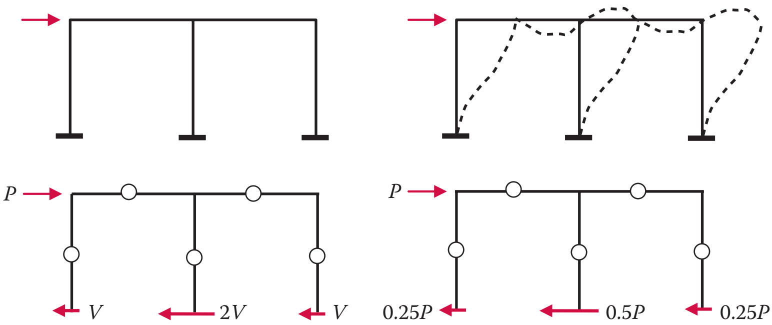


Beams and columns as statically determinate components.

This approach neglects any shear force in the columns and axial force in the beams, which is a fairly good assumption for preliminary design purposes.

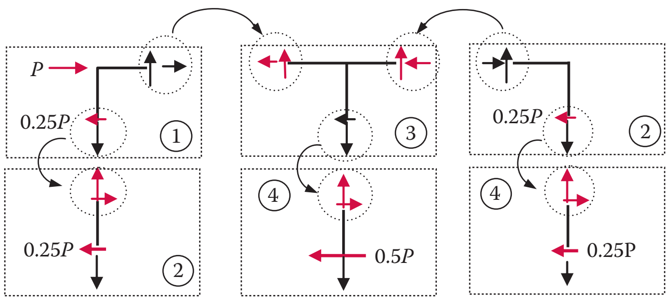
Horizontal loads. Depending on the configuration of the frame, we can apply either the portal method or the cantilever method. The *portal method* is generally applicable to low-rise building frames of no more than five stories high. The assumptions are:

1. Every midpoint of a beam or a column is a point of zero moment.
2. Interior columns carry twice the shear as that of exterior columns.



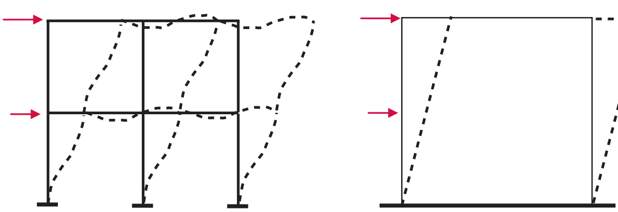
Assumptions of the portal method.

The shear forces in the columns are computed first from the free-body diagram (FBD) in the preceding figure using the horizontal equilibrium condition. The rest of the unknowns are computed from the FBDs in the sequence shown in the following figure one at a time. Each FBD contains no more than three unknowns. The curved arrows link dashed circles containing internal force pairs.



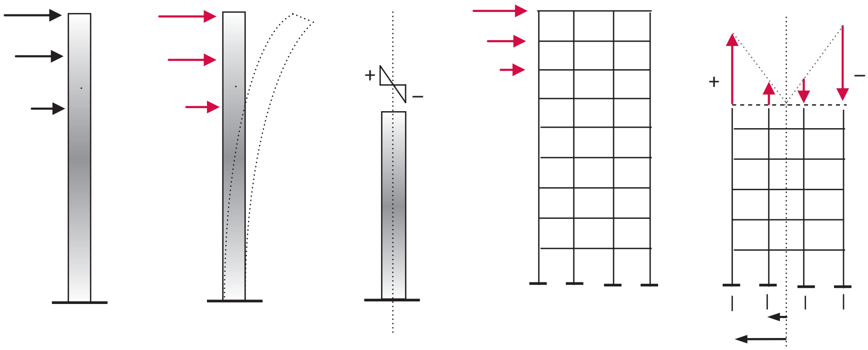
FBDs to compute internal forces in the sequence indicated.

The assumptions of the portal method are based on the observation that the deflection pattern of low-rise building frames is similar to that of the shear deformation of a deep beam. This similarity is illustrated next.



Deflections of a low-rise building frame and a deep beam.

On the other hand, the *cantilever method* is generally applicable to high-rise building frames, whose configurations are similar to those of vertical cantilevers. We can then borrow the pattern of normal stress distribution in a cantilever and apply it to the high-rise building frame.

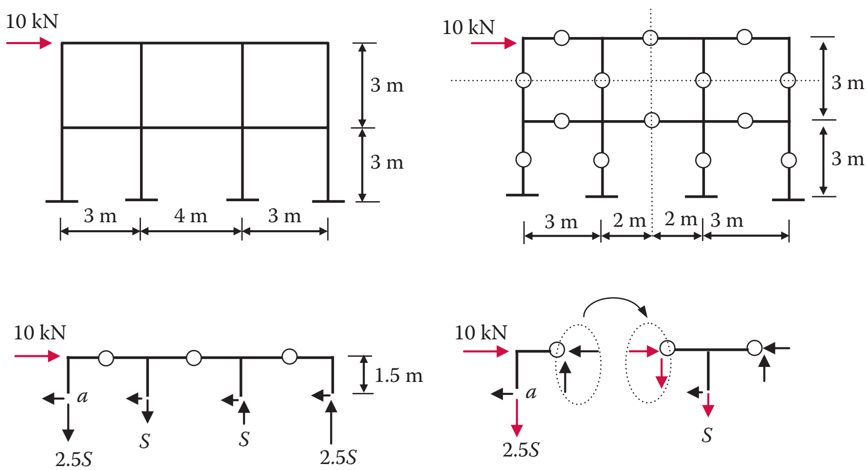


Normal stress distribution in a cantilever and axial force distribution in a frame.

The assumptions of the cantilever method are:

1. The axial forces in columns are proportional to the column's distance to the center line of the frame.
2. The midpoints of beams and columns are points of zero moment.

The solution process is slightly different from that of the portal method. It starts from the FBD of the upper story to find the axial forces. Then, it proceeds to find the column shears and axial forces in beams one FBD at a time. This solution process is illustrated in the following figure. Note that the FBD of the upper story cuts through midheight, not the base, of the story.



Cantilever method and the FBDs.

In the figure, the external columns have an axial force 2.5 times that of the interior columns because their distance to the center line is 2.5 times that of the interior columns. The solution for the axial force, S, is obtained by taking moment about any point on the midheight line:

$$\Sigma M_a = (1.5)(10) - (2.5S)(10) - S(7-3) = 0 \implies S = 0.52 \text{ kN}$$

The rest of the computation goes from one FBD to another, each with no more than three unknowns and each takes advantage of the results from the previous one. Readers are encouraged to complete the solution of all internal forces.

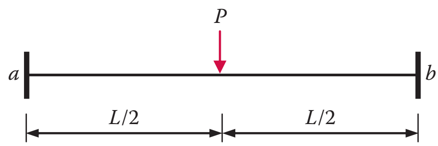
PROBLEM 6.1

Find all the reaction forces and moments at a and b. EI is constant and the beam length is L.

Problem 6.1

PROBLEM 6.2

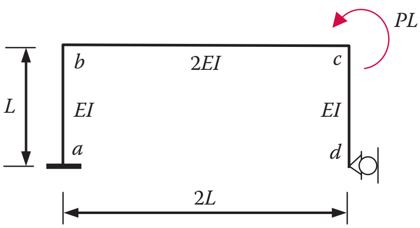
Find all the reaction forces and moments at a and b, taking advantage of the symmetry of the problem. EI is constant.



Problem 6.2

PROBLEM 6.3

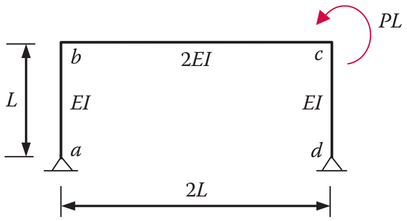
Find the horizontal reaction force at d.



Problem 6.3

PROBLEM 6.4

Find the internal moment at *b*.



Problem 6.4

7

Beam and Frame Analysis: Displacement Method—Part I

7.1 Introduction

The basic concept of the displacement method for beam and frame analysis is that the state of a member is completely defined by the displacements of its nodes. Once we know the nodal displacements, the rest of the unknowns, such as member forces, can be obtained easily.

For the whole structure, its state of member force is completely defined by the displacements of its nodes. Once we know all the nodal displacements of the structure, the nodal displacements of each member are obtained and member forces are then computed.

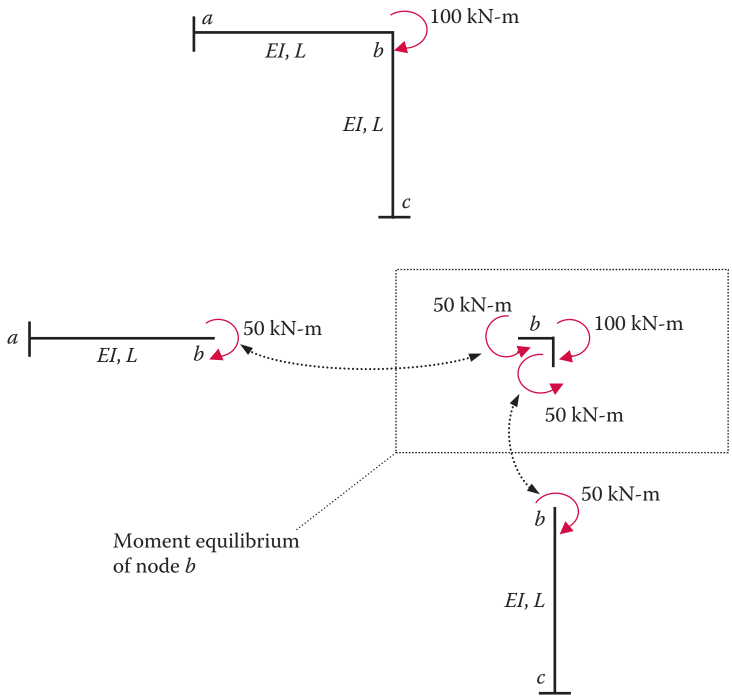
Defining a node in most cases is easy; it either appears as a joining point of a beam and a column, or it is at a location where there is a support. In other cases, it is a matter of preference of the analyzer, who may decide to define a node anywhere in a structure to facilitate the analysis.

We will introduce the displacement method in three stages. The moment distribution method is introduced as an iterative solution method that does not explicitly formulate the governing equations. The slope-deflection method is then introduced to formulate the governing equations. Both are identical in their assumptions and concepts. The matrix displacement method is then introduced as a generalization of the moment distribution and slope-deflection methods.

7.2 Moment Distribution Method

The moment distribution method is a unique method of structural analysis in which the solution is obtained iteratively without ever formulating the equations for the unknowns. It was invented in an era, out of necessity when the best computing tool was a slide rule, to solve frame problems that normally require the solution of simultaneous algebraic equations. Its relevance today, in the era of the personal computer, is in its insight on how a beam and frame react to applied loads by rotating its nodes and thus distributing the loads in the form of member-end moments (MEMs). Such an insight is the foundation of the modern displacement method.

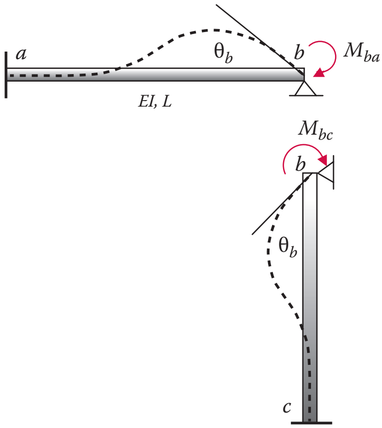
Take the very simple frame in the following figure as an example. The externally applied moment at node *b* tends to create a rotation at node *b*. Because member *ab* and member *bc* are rigidly connected at node *b*, the same rotation must take place at the end of member *ab* and member *bc*. For rotation at the end of member *ab* and member *bc* to happen, an end moment must be internally applied at the member end. This member-end moment comes from the externally applied moment. Nodal equilibrium at *b* requires the applied external moment of 100 kN be distributed to the two ends of the two joining members at *b*. How much each member will receive depends on how "rigid" each member is in its resistance to rotation at *b*. Since the two members are identical in length, *L*, and cross-section rigidity, *EI*, we assume for the time being that they are equally rigid. Thus, half of the 100 kN-m goes to member *ab* and the other half goes to member *bc*.



A frame example showing member-end moments.

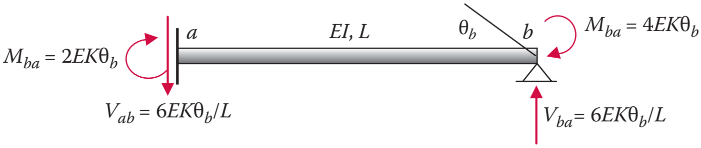
In the preceding figure, only the member-end moments are shown. The member-end shear and axial forces are not shown to avoid overcrowding the figure. The distributed moments (DMs) are "member-end" moments denoted by M_{ba} and M_{bc}, respectively. The sign convention of member-end moments and applied external moments is clockwise is positive. We assume the two members are equally rigid and receive half of the applied moment, not only because they appear to be equally rigid but also because each of the two members is under identical loading conditions: fixed at the far end and hinged at the near end.

In other cases, the beam and column may not be of the same rigidity, but they may have the same loading and supporting conditions: fixed at the far end and allowed to rotate at the near end. This configuration is the fundamental configuration of moment loading from which all other configurations can be derived by the principle of superposition. We shall delay the derivation of the governing formulas until we have learned the operating procedures of the moment distribution method.



Beam and column in a fundamental configuration of a moment applied at the end.

Suffice it to say that given the loading and support conditions shown in the following figure, the rotation θ_b and the member-end moment M_{ba} at the near end, b, are proportional. The relationship between M_{ba} and θ_b is expressed in the following equation, the derivation of which will be given later.



The fundamental case and the reaction solutions.

$$M_{ba} = 4(EK)_{ab}\theta_b \tag{7.1a}$$

where $K_{ab} = (I/L)_{ab}$.

We can write a similar equation for M_{bc} of member *bc*.

$$M_{bc} = 4(EK)_{bc}\theta_b \tag{7.1b}$$

where $K_{bc} = (I/L)_{bc}$.

Furthermore, the moment at the far end of member *ab*, M_{ab} at *a* is related to the amount of rotation at *b* by the following formula:

$$M_{ab} = 2(EK)_{ab}\theta_b \tag{7.2a}$$

Similarly, for member *bc*,

$$M_{cb} = 2(EK)_{bc}\theta_b \tag{7.2b}$$

As a result, the member-end moment at the far end is one-half of the near-end moment:

$$M_{ab} = \frac{1}{2}M_{ba} \tag{7.3a}$$

and

$$M_{cb} = \frac{1}{2}M_{bc} \tag{7.3b}$$

Note that in the preceding equations, it is important to keep the subscripts because each member may have a different *EK*.

The significance of Equation 7.1 is that it shows that the amount of the member-end moment, distributed from the unbalanced nodal moment, is proportional to the member stiffness 4*EK*, which is the moment needed at the near end to create a unit rotation at the near end, while the far end is fixed. Consequently, when we distribute the unbalanced moment, we need only to know the relative stiffness of each of the joining members at that particular end. The equilibrium equation for moment at node *b* is

$$M_{ba} + M_{bc} = 100 \text{ kN-m} \tag{7.4}$$

Since

$$M_{ba} : M_{bc} = (EK)_{ab} : (EK)_{bc}$$

we can "normalize" the previous equation so that both sides would add up to one, that is 100%, utilizing the fact that $(EK)_{ab} = (EK)_{bc}$ in the present case:

$$\frac{M_{ba}}{M_{ba}+M_{bc}} : \frac{M_{bc}}{M_{ba}+M_{bc}} = \frac{(EK)_{ab}}{(EK)_{ab}+(EK)_{bc}} : \frac{(EK)_{bc}}{(EK)_{ab}+(EK)_{bc}} = \frac{1}{2} : \frac{1}{2} \tag{7.5}$$

Consequently

$$M_{ba} = \frac{1}{2}(M_{ba} + M_{bc}) = \frac{1}{2}(100 \text{ kN-m}) = 50 \text{ kN-m}$$

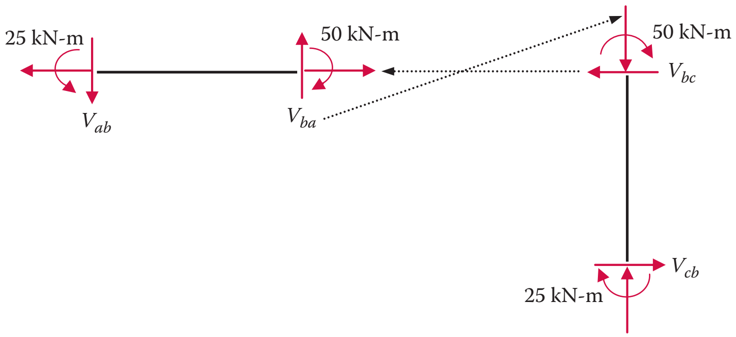
$$M_{bc} = \frac{1}{2}(M_{ba} + M_{bc}) = \frac{1}{2}(100 \text{ kN-m}) = 50 \text{ kN-m}$$

From Equation 7.2, we obtain

$$M_{ab} = \frac{1}{2}M_{ba} = 25 \text{ kN-m}$$

$$M_{cb} = \frac{1}{2}M_{bc} = 25 \text{ kN-m}$$

Now that all the member-end moments are obtained, we can proceed to find member-end shears and axial forces using the free-body diagrams (FBDs) shown next.



FBDs to find shear and axial forces.

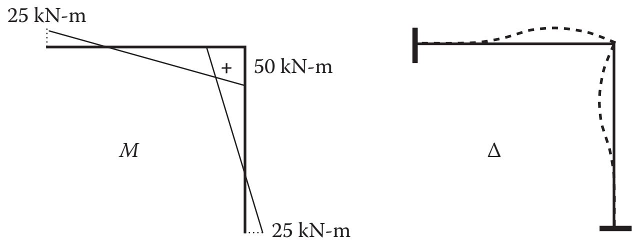
The dashed lines indicate that the axial force of one member is related to the shear force from the joining member at the common node. The shear forces are computed from the equilibrium conditions of the FBDs:

$$V_{ab} = V_{ba} = \frac{M_{ba} + M_{ab}}{L_{ab}}$$

and

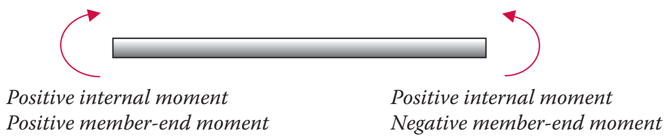
$$V_{bc} = V_{cb} = \frac{M_{bc} + M_{cb}}{L_{bc}}$$

The moment and deflection diagrams of the whole structure are shown next.



Moment and deflection diagrams.

In drawing the moment diagram, note that the sign conventions for internal moment (as in moment diagram) and the member-end moment (as in Equation 7.1 through Equation 7.5) are different. The former depends on the orientation and which face the moment is acting on, and the latter depends only on the moment direction (clockwise is positive).



Difference in sign conventions.

Let us recap the operational procedures of the moment distribution method:

1. Identify the node that is free to rotate. In the present case, it was node *b*. The number of "free" rotating nodes is called the degree of freedom (DOF). In the present case, the DOF is one.
2. Identify the joining members at this node and compute their relative stiffness according to Equation 7.5, which can be generalized to cover more than two members.

$$\frac{M_{ab}}{\sum M_{xy}} : \frac{M_{bc}}{\sum M_{xy}} : \frac{M_{cd}}{\sum M_{xy}} \cdots = \frac{(EK)_{ab}}{\sum (EK)_{xy}} : \frac{(EK)_{bc}}{\sum (EK)_{xy}} : \frac{(EK)_{cd}}{\sum (EK)_{xy}} \cdots$$

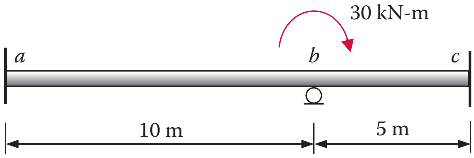
where the summation is over all joining members at the particular node. Each of the expressions in this equation is called a distribution factor (DF), which adds up to 1 or 100%. Each of the moment at the end of a member is called a member-end moment.

3. Identify the unbalanced moment at this node. In the present case, it was 100 kN-m.
4. To balance the 100 kN-m, we need to add –100 kN-m to the node, which, when viewed from the member end, becomes positive 100 kN-m. This 100 kN-m is distributed to member *ab* and *bc* according to the DF of each member. In this case the DF is 50% each. Consequently, 50 kN-m goes to M_{ba} and 50 kN-m goes to M_{bc}. They are called the distributed moment. Note that the externally applied moment is distributed as member-end moments in the same sign, that is, positive to positive.
5. Once the balancing moment is distributed, the far ends of the joining members should receive 50% of the distributed moment at the near end. The factor of 50% or ½ is called the *carryover factor* (COF). The moment at the far end thus distributed is called the *carryover moment* (COM). In the present case, they are 25 kN-m for M_{ab} and 25 kN-m for M_{cb}, respectively.
6. We note that at the two fixed ends, whatever moments are carried over, they are balanced by the support reaction. That means the moment equilibrium is achieved at the fixed ends with no need for additional distribution. This is equivalent to say that the stiffness of the support relative to the stiffness of the member is infinite. Or, even simpler, we may formally designate the distribution factors at a fixed support as 1:0, with one being assigned for the support and zero assigned to the member. The zero DF means we need not redistribute any moment at the member end.
7. The moment distribution method operations end when all the nodes are in moment equilibrium. In the present case, node *b* is the only node we need to concentrate on and it is in equilibrium after the unbalanced moment is distributed.
8. To complete the solution process, however, we still need to find the other unknowns such as shear and axial forces at the end of each member. That is accomplished by drawing the FBD of each member and writing equilibrium equations.
9. The moment diagram and deflection diagram can then be drawn.

We shall now go through the solution process by solving a similar problem with a single degree of freedom (SDOF).

Example 7.1

Find all the member-end moments of the beam shown. *EI* is constant for all members.



Beam problem with a SDOF.

Solution

1. Preparation.
   a. Unbalanced moment: At node *b* there is an externally applied moment (EAM), which should be distributed as member-end moments in the same sign.
   b. The distribution factors at node *b*:

$$DF_{ba}:\ DF_{bc} = 4EK_{ab}:\ 4EK_{bc} = 4\left(\frac{EI}{L}\right)_{ab} : 4\left(\frac{EI}{L}\right)_{bc} = \frac{1}{10} : \frac{1}{5} = 0.33 : 0.67$$

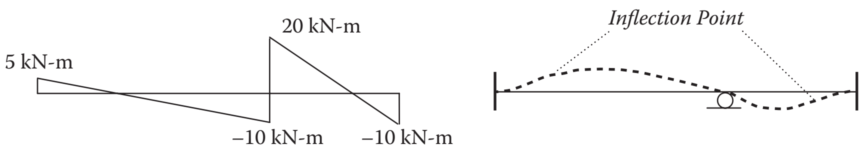
   c. As a formality, we also include $DF_{ab} = 0$, and $DF_{bc} = 0$, at *a* and *c*, respectively.
2. Tabulation. All the computing can be tabulated as shown next. The arrows indicate the destination of the carryover moment. The dashed lines show how the DF is used to compute the DM.

Moment Distribution Table for an SDOF Problem

Node	*a*	*b*		*c*
Member	*ab*		*bc*	
DF	0	0.33	0.67	0
MEM[1]	M_{ab}	M_{ba}	M_{bc}	M_{cb}
EAM[2]		30		
DM[3]		+10	+20	
COM[4]	+5			+10
Sum[5]	+5	+10	+20	+10

1. Member-end moment.
2. Externally applied moment.
3. Distributed member-end moment.
4. Carryover moment.
5. Sum of member-end moments.

3. Post moment-distribution operations. The moment and deflection diagrams are shown next.

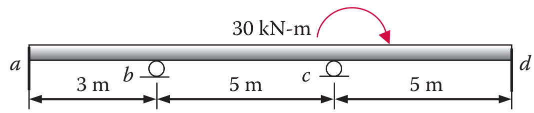


Moment and deflection diagrams.

The moment distribution method becomes iterative when there are more than one DOF. The aforementioned procedures for one DOF problem can still apply if we consider one DOF at a time. That is to say that when we concentrate on one DOF, the other DOFs are considered "locked" into a fixed support and are not allowed to rotate. When the free node gets its distributed moment and the carryover moment reaches the neighboring and previously locked node, that node becomes unbalanced, thus requiring "unlocking" to distribute the balancing moment, which in turn creates carryover moment at the first node. That requires another round of distribution and carrying over. Thus begins the cycle of "locking–unlocking" and the balancing of moments from one node to another. We shall see, however, in each subsequent iteration, the amount of unbalanced moment becomes progressively smaller. The iteration stops when the unbalanced moment becomes negligible. This iterative process is illustrated in the following example of two DOFs.

Example 7.2

Find all the member-end moments of the beam shown. *EI* is constant for all members.



Example of a beam with two DOFs.

Solution

1. Preparation.
   a. Both nodes *b* and *c* are free to rotate. We choose to balance node *c* first.
   b. Compute DF at *b*:

$$DF_{ba} : DF_{bc} = 4EK_{ab} : 4EK_{bc} = 4\left(\frac{EI}{L}\right)_{ab} : 4\left(\frac{EI}{L}\right)_{bc} = \frac{1}{3} : \frac{1}{5} = 0.625 : 0.375$$

   c. Compute DF at *c*:

$$DF_{cb} : DF_{cd} = 4EK_{bc} : 4EK_{cd} = 4\left(\frac{EI}{L}\right)_{bc} : 4\left(\frac{EI}{L}\right)_{cd} = \frac{1}{5} : \frac{1}{5} = 0.5 : 0.5$$

   d. Assign DF at *a* and *d*: DFs are zero at *a* and *d*.

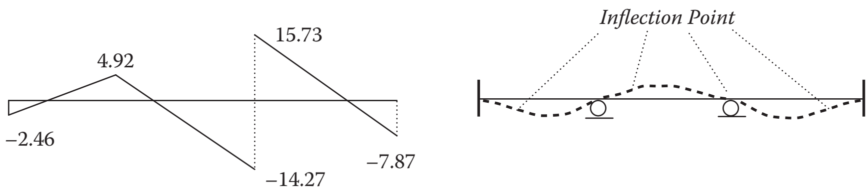
2. Tabulation.

Moment Distribution for a Two-DOF Problem

Node	*a*	*b*		*c*		*d*
Member	*ab*		*bc*		*cd*	
DF	0	0.625	0.375	0.5	0.5	0
MEM	M_{ab}	M_{ba}	M_{bc}	M_{cb}	M_{cd}	M_{dc}
EAM				30		
DM				+15	+15	
COM			+7.50			+7.50
DM		−4.69	−2.81			
COM	−2.35			−1.41		
DM				+0.71	+0.70	
COM			+0.36			0.35
DM		−0.22	−0.14			
COM	−0.11			−0.07		
DM				+0.04	+0.03	
COM			+0.02			+0.02
DM		−0.01	−0.01			
COM	0.00			0.00		
Sum	−2.46	−4.92	+4.92	+14.27	+15.73	+7.87

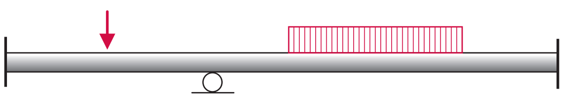
In the table, the encircled moment is the unbalanced moment. Note how the circles move back and forth between nodes *b* and *c*. Also note how the EAM at *c* and the unbalanced moment, created by the COM at *b*, are treated differently. The EAM is balanced by distributing the amount in the same sign to the member ends, while the unbalanced moment at a node is balanced by distributing the negative of the unbalanced moment to the moment ends.

3. Post moment-distribution operations. The moment and deflection diagrams are shown next.



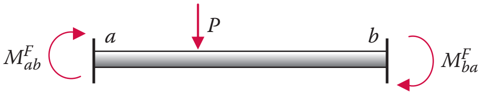
Moment and deflection diagrams.

Treatment of load between nodes. In the previous examples, the applied load was an applied moment at a node. We can begin the distribution process right at the node. In most practical cases, the load will be either concentrated loads or distributed loads applied between nodes. These cases call for an additional step before we can begin the distribution of moments.



Load applied between nodes.

We imagine that all the nodes are "locked" at the beginning. Then each member is in a state of a clamped beam with a transverse load applied between the two ends.

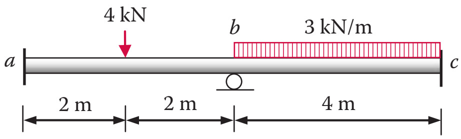


Fixed-end beam with applied load.

The moment needed to "lock" the two ends are called fixed-end moments (FEMs). They are positive if acting clockwise. For typical loads, the FEMs can be precomputed and are tabulated in the FEM table given at the end of this chapter. These FEMs are to be balanced when the node is "unlocked" and allowed to rotate. Thus, the effect of the transverse load applied between nodes is to create moments at both ends of a member. These FEMs should be balanced by moment distribution.

Example 7.3

Find all the member-end moments of the beam shown. *EI* is constant for all members.



Example with load applied between nodes.

Solution

1. Preparation.
   a. Only node *b* is free to rotate. There is no externally applied moment at node *b* to balance, but the transverse load between nodes creates FEMs.
   b. FEM for member *ab*. The concentrated load of 4 kN creates FEMs at end *a* and end *b*. The formula for a single transverse load in the FEM table gives us:

$$M_{ab}^F = -\frac{(P)(Length)}{8} = -\frac{(4)(4)}{8} = -2\text{ kN-m}$$

$$M_{ba}^F = \frac{(P)(Length)}{8} = \frac{(4)(4)}{8} = 2\text{ kN-m}$$

c. FEM for member *bc*. The distributed load of 3 kN/m creates FEMs at end *b* and end *c*. The formula for a distributed transverse load in the FEM table gives us:

$$M_{bc}^{F} = -\frac{(w)(Length)^2}{12} = -\frac{(3)(4)^2}{12} = -4\text{ kN-m}$$

$$M_{cb}^{F} = \frac{(w)(Length)^2}{12} = \frac{(3)(4)^2}{12} = 4\text{ kN-m}$$

d. Compute DF at *b*:

$$\text{DF}_{ba} : \text{DF}_{bc} = 4EK_{ab} : 4EK_{bc} = 4\left(\frac{EI}{L}\right)_{ab} : 4\left(\frac{EI}{L}\right)_{bc} = \frac{1}{4} : \frac{1}{4} = 0.5 : 0.5$$

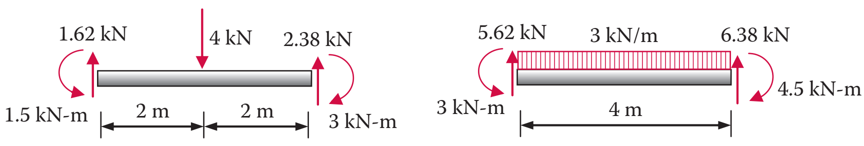
e. Assign DF at *a* and *c*: DFs are zero at *a* and *c*.

2. Tabulation.

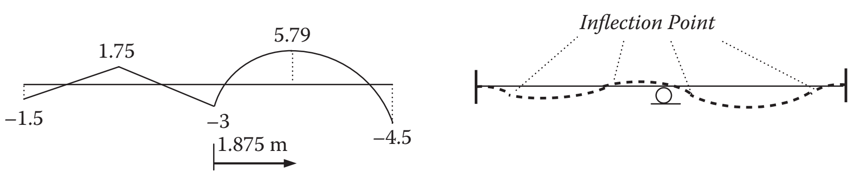
Moment Distribution for an SDOF Problem with FEMs

Node	*a*	*b*		*c*
Member	*ab*		*bc*	
DF	0	0.5	0.5	0
EAM				
MEM	M_{ab}	M_{ba}	M_{bc}	M_{cb}
FEM	−2	+2	−4	+4
DM		+1	+1	
COM	+0.5			+0.5
Sum	−1.5	+3	−3	+4.5

3. Post moment-distribution operations. The shear forces at both ends of a member are computed from the FBDs of each member. Knowing the member-end shear forces, the moment diagram can then be drawn. The moment and deflection diagrams are shown next.



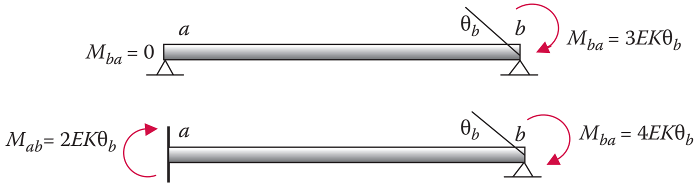
FBDs of the two members.



Moment and deflection diagrams.

Treatment of hinged ends. At a hinged end, the MEM is equal to zero or whatever an externally applied moment is at the end. During the process of moment distribution, the hinged end may receive COM from the neighboring node. That COM must then be balanced by distributing 100% of it at the hinged end. This is because the distribution factor of a hinged end is 1 or 100%; the hinged end may be considered to be connected to air, which has zero stiffness. This new distributed moment starts another cycle of carryover and distribution. This process is illustrated in Example 7.4.

The cycle of iteration is greatly simplified if we recognize at the very beginning of moment distribution that the stiffness of a member with a hinged end is fundamentally different from that of the standard model with the far end fixed. We will delay the derivation but will state that the moment needed at the near end to create a unit rotation at the near end with the far end hinged is $3EK$, less than the $4EK$ if the far end is fixed.

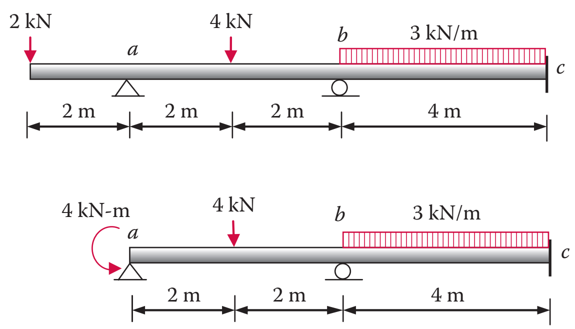


Member with a hinged end versus the standard model with the far end fixed.

Note that there is no COM at the hinged end ($M_{ba} = 0$) if we take the member stiffness factor as $3EK$ instead of $4EK$. We can thus compute the relative distribution factors accordingly, and when distributing the moment at one end of the member, we need not carry over the distributed moment to the hinged end. This simplified process with a modified stiffness from $4EK$ to $3EK$ is illustrated in Example 7.5.

Example 7.4

Find all the member-end moments of the beam shown. *EI* is constant for all members.



Turning a problem with a cantilever end into one with a hinged end.

Solution

The original problem with a cantilever end can be treated as one with a hinged end as shown. We shall solve only the problem with a hinged end. Note that the vertical load is not shown in the equivalent hinged-end problem because it is taken up by the support at *a*.

1. *Preparation*. Since the geometry and loading are similar to that of Example 7.3, we can copy the preparation part but note that an externally applied moment is present.
   a. Only nodes *b* and *a* are free to rotate. There is an externally applied moment at node *a* and the transverse load between nodes create FEMs at all nodes.
   b. FEM for member *ab*. The concentrated load of 4 kN creates FEMs at end *a* and end *b*. The formula for a single transverse load in the FEM table gives us:

$$M_{ab}^F = -\frac{(P)(Length)}{8} = -\frac{(4)(4)}{8} = -2\text{ kN-m}$$

$$M_{ba}^F = \frac{(P)(Length)}{8} = \frac{(4)(4)}{8} = 2\text{ kN-m}$$

   c. FEM for member *bc*. The distributed load of 3 kN/m creates FEMs at end *b* and end *c*. The formula for a distributed transverse load in the FEM table gives us:

$$M_{bc}^F = -\frac{(w)(Length)^2}{12} = -\frac{(3)(4)^2}{12} = -4\text{kN-m}$$

$$M_{cb}^F = \frac{(w)(Length)^2}{12} = \frac{(3)(4)^2}{12} = 4\text{ kN-m}$$

   d. Compute DF at *b*:

$$\text{DF}_{ba} : \text{DF}_{bc} = 4EK_{ab} : 4EK_{bc} = 4\left(\frac{EI}{L}\right)_{ab} : 4\left(\frac{EI}{L}\right)_{bc} = \frac{1}{4} : \frac{1}{4} = 0.5 : 0.5$$

   e. Assign DF at *a* and *c*: DFs are one at *a* and zero at *c*.
2. *Tabulation*. In the moment distribution process shown next, we must deal with the unbalanced moment at the hinged end first. The EAM of –4 kN-m and the FEM of –2 kN-m at node *a* add up to 2 kN-m of unbalanced moment, not –6 kN-m. This is because the FEM and DM at node *a* should add up to the EAM, which is

–4 kN-m. Thus, we need to distribute (–4 kN-m) – (–2 kN-m) = –2 kN-m to make the node balanced. The formula to remember is DM = EAM – FEM. This formula is applicable to all nodes where there are both EAMs and FEMs.

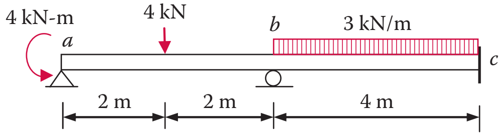
Moment Distribution Table for a Beam with a Hinged End

Node	*a*	*b*		*c*
Member	*ab*		*bc*	
DF	1	0.5	0.5	0
MEM	M_{ab}	M_{ba}	M_{bc}	M_{cb}
EAM	–4			
FEM	–2	+2	–4	+4
DM	–2			
COM		–1		
DM		+1.5	+1.5	
COM	+0.8			+0.8
DM	–0.8			
COM		–0.4		
DM		+0.2	+0.2	
COM	+0.1			+0.1
DM	–0.1			
COM		0.0		
Sum	–4	+2.3	–2.3	+4.9

The aforementioned back-and-forth iteration between nodes *a* and *b* is avoided if we use the simplified procedures as illustrated next.

Example 7.5

Find all the member-end moments of the beam shown. *EI* is constant for all members. Use the modified stiffness to account for the hinged end at node *a*.



Beam with a hinged end.

Solution

1. *Preparation*. Note the stiffness computation in step d.
   a. Only nodes *b* and *a* are free to rotate. Node *a* is considered a hinged node and needs no moment distribution except at the very beginning. There is an externally applied moment at node *a* and the transverse load between nodes create FEMs at all nodes.

b. FEM for member *ab*. The concentrated load of 4 kN creates FEMs at end *a* and end *b*. The formula for a single transverse load in the FEM table gives us:

$$M^F_{ab} = -\frac{(P)(Length)}{8} = -\frac{(4)(4)}{8} = -2 \text{ kN-m}$$

$$M^F_{ba} = \frac{(P)(Length)}{8} = \frac{(4)(4)}{8} = 2 \text{ kN-m}$$

c. FEM for member *bc*. The distributed load of 3 kN/m creates FEMs at end *b* and end *c*. The formula for a distributed transverse load in the FEM table gives us:

$$M^F_{bc} = -\frac{(w)(Length)^2}{12} = -\frac{(3)(4)^2}{12} = -4 \text{ kN-m}$$

$$M^F_{cb} = \frac{(w)(Length)^2}{12} = \frac{(3)(4)^2}{12} = 4 \text{ kN-m}$$

d. Compute DF at *b*:

$$DF_{ba} : DF_{bc} = 3EK_{ab} : 4EK_{bc} = 3\left(\frac{EI}{L}\right)_{ab} : 4\left(\frac{EI}{L}\right)_{bc} = \frac{3}{7} : \frac{4}{7} = 0.43 : 0.57$$

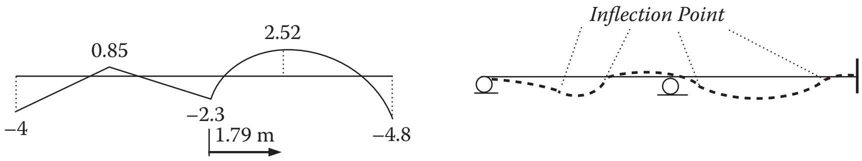
e. Assign DF at *a* and *c*: DFs are one at *a* and zero at *c*.

2. *Tabulation*. In the moment distribution process below, we must deal with the unbalanced moment at the hinged end first. Using the formula DM = EAM – FEM, we begin by distributing –2 kN-m and carrying over half of it to node *b*. From this point on, node *a* is balanced, will not receive any COM from node *b*, and will stay balanced throughout the moment distribution process. The zero COM at node *a* in the following table serves to emphasize there is no carryover.

Moment Distribution Table for a Problem with a Hinged End

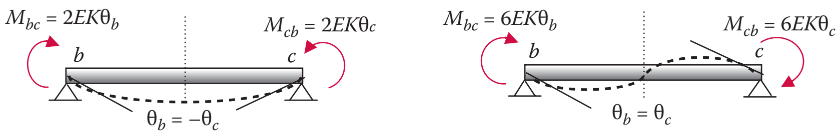
Node	*a*	*b*		*c*
Member	*ab*		*bc*	
DF	1	0.43	0.57	0
MEM	M_{ab}	M_{ba}	M_{bc}	M_{cb}
EAM	–4			
FEM	–2	+2	–4	+4
DM	–2			
COM		–1		
DM		+1.3	+1.7	
COM	0.0			+0.8
Sum	–4.0	+2.3	–2.3	+4.8

3. *Post moment-distribution operations*. The moment and deflection diagrams are shown next.



Moment and deflection diagrams.

Treatment of a central-symmetric or antisymmetric span. In a problem with at least three spans, if the geometry and stiffness are symmetric about the center line of the structure, then the central span is in (a) a state of symmetry if the load is symmetric about the center line, and (b) a state of antisymmetry if the load is antisymmetric about the center line. For these special spans, we can develop special stiffness formulas so that no carryover is needed across the line of symmetry when member-end moments are distributed. The basic information needed for moment distribution is shown in the following figure.

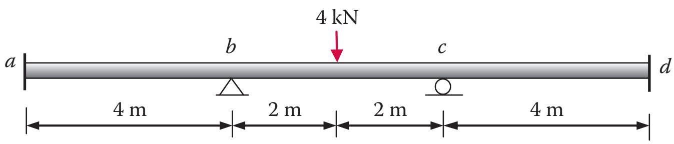


Symmetric and antisymmetric spans.

We shall delay the derivation of the stiffness formulas but will simply state that for a symmetric span, the moments needed at both ends to create a unit rotation at both ends are $2EK$, and for an antisymmetric span they are $6EK$. The following two examples will illustrate the solution processes using these modified stiffness factors. Because of symmetry/antisymmetry, we need to deal with only half of the span. The other half is a mirror image of the first half in the case of symmetry and an upside down mirror image in the case of antisymmetry.

Example 7.6

Find all the member-end moments of the beam shown. EI is constant for all members. Use the modified stiffness to account for the symmetric span between nodes b and c.



Beam with a symmetric central span.

Solution

1. *Preparation*. Note the stiffness computation in step c.
   a. Only nodes *b* and *c* are free to rotate. Only the transverse load between nodes *b* and *c* will create FEMs at *b* and *c*.
   b. FEM for member *bc*. The formula for a single transverse load in the FEM table gives us, as in Example 7.5:

$$M^F_{bc} = -2\,\text{kN-m}$$

$$M^F_{cb} = 2\,\text{kN-m}$$

   c. Compute DF at *b*:

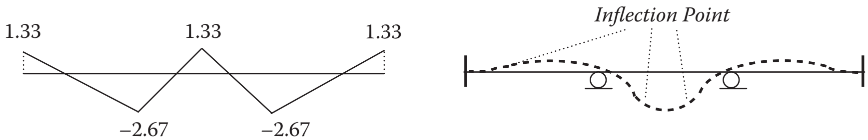
$$DF_{ba} : DF_{bc} = 4EK_{ab} : 2EK_{bc} = 4\left(\frac{EI}{L}\right)_{ab} : 2\left(\frac{EI}{L}\right)_{bc} = \frac{4}{6} : \frac{2}{6} = 0.67 : 0.33$$

   d. Assign DF at *a*: DF is zero at *a*. No need to consider node *d*.
2. *Tabulation*. In the moment distribution process, we need to deal with only half of the beam. There is no carryover moment from *b* to *c*. We include node *c* just to illustrate that all its moments are the reflection of those at node *b*.

Moment Distribution Table for a Symmetric Problem

Node	*a*	*b*		*c*
Member	*ab*		*bc*	
DF	0	0.67	0.33	0
MEM	M_{ab}	M_{ba}	M_{bc}	M_{cb}
EAM				
FEM			−4	+4
DM		+2.67	+1.33	−1.33
COM	+1.33			
Sum	+1.33	+2.67	−2.67	+2.67

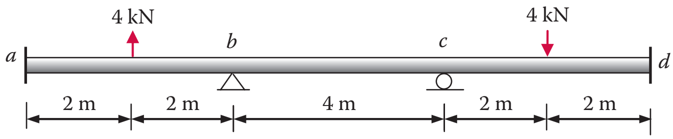
3. *Post moment-distribution operations*. The moment and deflection diagrams are shown next.



Moment and deflection diagrams.

Example 7.7

Find all the member-end moments of the beam shown. EI is constant for all members. Use the modified stiffness to account for the antisymmetric span between nodes b and c.



Beam with a central antisymmetric span.

Solution

1. *Preparation*. Note the stiffness computation in step c. There is no need for node c.
   a. Only nodes b and c are free to rotate. The transverse load between nodes a and b will create FEMs at a and b. No need to consider member cd.
   b. FEM for member ab. The formula for a single transverse load in the FEM table gives us, as in Example 7.5 with the signs reversed:

$$M_{ab}^{F} = 2\text{ kN-m}$$

$$M_{bc}^{F} = -2\text{ kN-m}$$

   c. Compute DF at b:

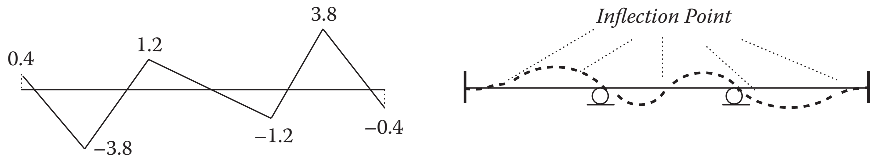
$$\text{DF}_{ba} : \text{DF}_{bc} = 4EK_{ab} : 6EK_{bc} = 4\left(\frac{EI}{L}\right)_{ab} : 6\left(\frac{EI}{L}\right)_{bc} = \frac{4}{10} : \frac{6}{10} = 0.4 : 0.6$$

   d. Assign DF at a: DF is zero at a. No need to consider node d.
2. *Tabulation*. In the moment distribution process shown next, we need to deal with only half of the beam. There is no COM from b to c.

Moment Distribution Table for a Beam with an Antisymmetric Span

Node	*a*	*b*		*c*
Member	*ab*		*bc*	
DF	0	0.4	0.6	0
MEM	M_{ab}	M_{ba}	M_{bc}	M_{cb}
EAM				
FEM	+2	−2		
DM		+0.8	+1.2	+1.2
COM	+0.4 ↙			
Sum	+0.4	−1.2	+1.2	+1.2

3. *Post moment-distribution operations*. The moment and deflection diagrams are shown next.



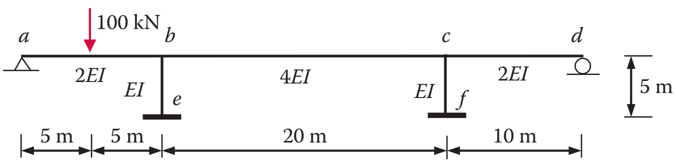
Moment and deflection diagrams.

While the antisymmetric loading seems improbable, it often is the result of decomposition of a general loading pattern applied to a symmetrical structure. It is always possible to decompose a general loading pattern applied on a symmetric structure into a symmetric component and an antisymmetric component, as illustrated next. Each loading component can then be treated with the simplified procedure of the moment distribution method. The results of the two analyses are then superposed to obtain the solution for the original loading pattern.

Decompose a load into a symmetric component and an antisymmetric component.

Example 7.8

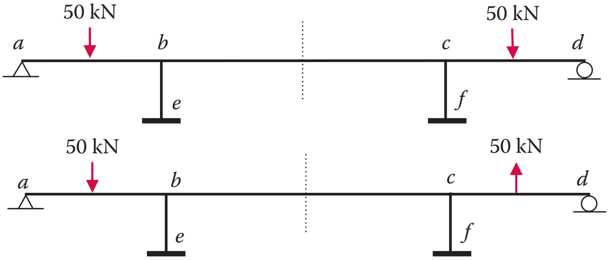
Find all the member-end moments of the frame shown.



A three-span bridge frame.

Solution

Symmetry of the structure calls for the decomposition of the load into a symmetric component and an antisymmetric component.



Symmetric and antisymmetric loads.

We shall solve both problems in parallel.

1. *Preparation*. Note the stiffness computation in steps c and d.
   a. Only nodes *a* and *b* are free to rotate when we take advantage of the symmetry/antisymmetry. Furthermore, if we use the modified stiffness for the hinged-end situation in member *ab*, then we need to concentrate on node *b* only.
   b. FEM for member *ab*. The concentrated load of 50 kN creates FEMs at end *a* and end *b*. The formula for a single transverse load in the FEM table gives us:

$$M_{ab}^F = -\frac{(P)\,(Length)}{8} = -\frac{(50)(10)}{8} = -62.5 \text{ kN-m}$$

$$M_{ba}^F = \frac{(P)\,(Length)}{8} = \frac{(50)(10)}{8} = 62.5 \text{ kN-m}$$

   c. Compute DF at *b* (symmetric case):

$$\mathrm{DF}_{ba} : \mathrm{DF}_{bc} : \mathrm{DF}_{be} = 3EK_{ab} : 2EK_{bc} : 4EK_{be}$$

$$= 3 \;{\frac{2EI}{10}}_{ab} : 2 \;{\frac{4EI}{20}}_{bc} : 4 \;{\frac{EI}{5}}_{be} = \frac{6}{10} : \frac{8}{20} : \frac{4}{5}$$

$$= \frac{3}{5} : \frac{2}{5} : \frac{4}{5} = \frac{3}{9} : \frac{2}{9} : \frac{4}{9} = 0.33 : 0.22 : 0.45$$

   d. Compute DF at *b* (antisymmetric case):

$$\mathrm{DF}_{ba} : \mathrm{DF}_{bc} : \mathrm{DF}_{be} = 3EK_{ab} : 6EK_{bc} : 4EK_{be}$$

$$= 3 \;{\frac{2EI}{10}}_{ab} : 6 \;{\frac{4EI}{20}}_{bc} : 4 \;{\frac{EI}{5}}_{be} = \frac{6}{10} : \frac{24}{20} : \frac{4}{5}$$

$$= \frac{3}{5} : \frac{6}{5} : \frac{4}{5} = \frac{3}{13} : \frac{6}{13} : \frac{4}{13} = 0.23 : 0.46 : 0.21$$

   e. Assign DFs at *a* and *e*: DF is one at *a* and zero at *e*.
2. *Tabulation*. We need to include only nodes *a*, *b*, and *e* in the following table.

Moment Distribution Table for a Symmetric Case and an Antisymmetric Case

		Symmetric Case					*Antisymmetric Case*			
Node	*a*		*b*		*e*	*a*		*b*		*e*
Member	*ab*		*bc*	*be*		*ab*		*bc*	*be*	
DF	1	0.33	0.22	0.45	0	1	0.23	0.46	0.31	0
MEM	M_{ab}	M_{ba}	M_{bc}	M_{be}	M_{eb}	M_{ab}	M_{ba}	M_{bc}	M_{be}	M_{eb}
EAM										
FEM	–62.5	62.5				–62.5	62.5			
DM	62.5					62.5				
COM		31.3					31.3			
DM		–31.0	–20.6	–42.2			–21.6	–43.2	–29.0	
COM	0.0				–21.1	0.0				–14.5
Sum	0.0	62.8	–20.6	–42.2	–21.1		72.2	–43.2	–29.0	–14.5

The solution to the original problem is the superposition of the two solutions in the preceding table.

$$M_{ab} = 0.0 + 0.0 = 0.0 \text{ kN-m}$$

$$M_{ba} = 62.8 + (72.2) = 135.0 \text{ kN-m}$$

$$M_{bc} = -20.6 + (-43.2) = -63.8 \text{ kN-m}$$

$$M_{be} = -42.2 + (-29.0) = -71.2 \text{ kN-m}$$

$$M_{eb} = -21.1 + (-14.5) = -35.6 \text{ kN-m}$$

The superposition for the right half of the structure requires caution: moments at the right half are negative to those at the left half in the symmetric case and are of the same sign in the antisymmetric case.

$$M_{dc} = 0.0 + 0.0 = 0.0 \text{ kN-m}$$

$$M_{cd} = -62.8 + (72.2) = 9.4 \text{ kN-m}$$

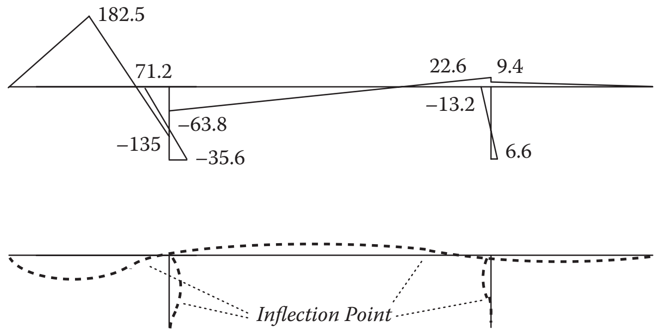
$$M_{cb} = 20.6 + (-43.2) = -22.6 \text{ kN-m}$$

$$M_{cf} = 42.2 + (-29.0) = 13.2 \text{ kN-m}$$

$$M_{fc} = 21.1 + (-14.5) = 6.6 \text{ kN-m}$$

As expected, the resulting moment solution is neither symmetric nor antisymmetric.

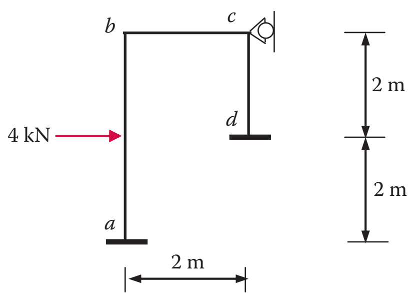
3. *Post moment-distribution operations*. The moment and deflection diagrams are shown next.



Moment and deflection diagrams.

Example 7.9

Find all the member-end moments of the frame shown. *EI* is constant for all members.



Solution

1. *Preparation*. Note the stiffness computation in step c.
   a. Only nodes *b* and *c* are free to rotate. There is no side sway because the support at *c* prevents that. Only the transverse load between nodes *a* and *b* will create FEMs at *a* and *b*.
   b. FEM for member *ab*. The formula for a single transverse load in the FEM table gives us, as in Example 7.5:

$$M_{bc}^F = -2\,\text{kN-m}$$

$$M_{cb}^F = -2\,\text{kN-m}$$

   c. Compute DF at *b*:

$$DF_{ba} : DF_{bc} = 4EK_{ab} : 4EK_{bc} = 4\left(\frac{EI}{L}\right)_{ab} : 4\left(\frac{EI}{L}\right)_{bc} = \frac{4}{4} : \frac{4}{2} = 0.33 : 0.67$$

d. Compute DF at c:

$$\mathrm{DF}_{cb} : \mathrm{DF}_{cd} = 4EK_{bc} : 4EK_{cd} = 4\left(\frac{EI}{L}\right)_{ab} : 4\left(\frac{EI}{L}\right)_{bc} = \frac{4}{2} : \frac{4}{2} = 0.5 : 0.5$$

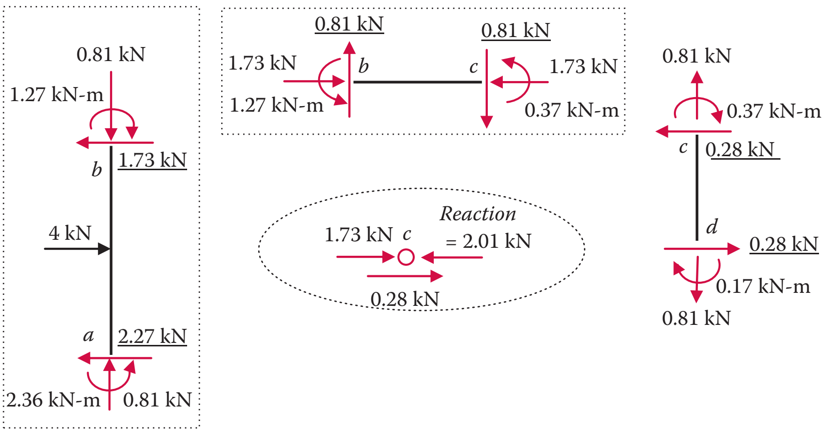
e. Assign DF at a and d: DF is zero at a and d.

2. Tabulation.

Moment Distribution Table for a Two-DOF Frame

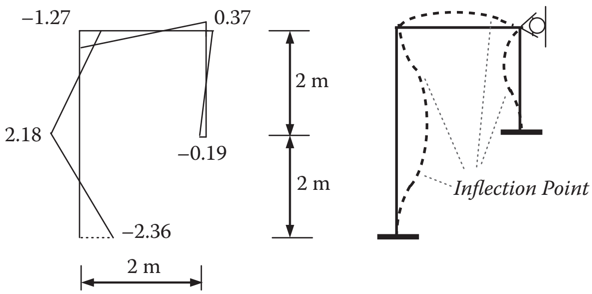
Node	*a*	*b*		*c*		*d*
Member	*ab*		*bc*		*cd*	
DF	0	0.33	0.67	0.5	0.5	0
MEM	M_{ab}	M_{ba}	M_{bc}	M_{cb}	M_{cd}	M_{dc}
EAM						
FEM	−2	+2				
DM		−0.67	−1.33			
COM	−0.33			−0.67		
DM				+0.33	+0.34	
COM			+0.17			+0.17
DM		−0.06	−0.11			
COM	−0.03			−0.06		
DM				+0.03	+0.03	
COM						+0.02
Sum	−2.36	+1.27	−1.27	−0.37	−0.37	+0.19

3. *Post moment-distribution operations*. The member-end shear forces (underlined) are determined from the FBD of each member. The axial forces are determined from the shear forces of the joining members. The reaction at the support at node c is determined from the FBD of node c.



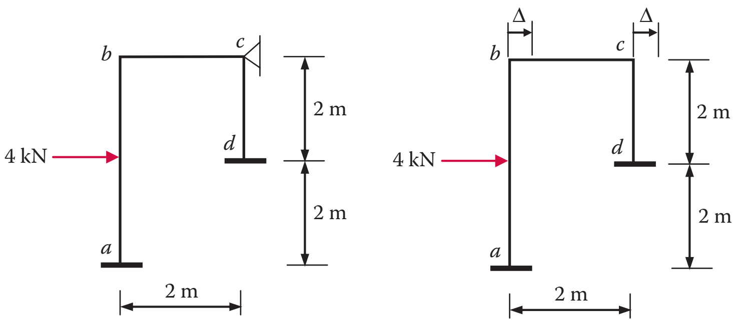
FBDs of the three members and node c.

The moment and deflection diagrams are shown next.



Moment and deflection diagrams.

Treatment of side sway. In all the example problems we have solved so far, each member may be allowed to have end-node rotations but not end-node translations perpendicular to the member length direction. Consider the two problems shown next.



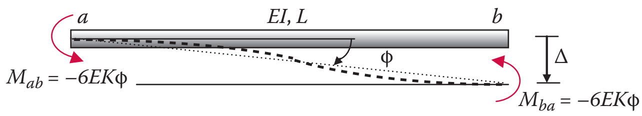
A frame without side sway and one with side sway.

Nodes *b* and *c* of both frames are free to rotate, but no translation movement of nodes is possible in the frame at the left. For the frame at the right, nodes *b* and *c* are free to move sidewise, thus creating side sway of members *ab* and *cd*. Note that member *bc* still does not have side sway, because there is no nodal movement perpendicular to the member length direction.

As shown in the upcoming figure, side sway of a member can be characterized by a member rotation, φ, which is different from member nodal rotation. The member rotation is the result of relative translation movement of the two member-end nodes in a direction perpendicular to the member length direction, defined as positive if it is a clockwise rotation, same way as for nodal rotations.

$$= \frac{\quad}{L} \tag{7.6}$$

where Δ is defined in the figure and *L* is the length of the member.



A member with side sway.

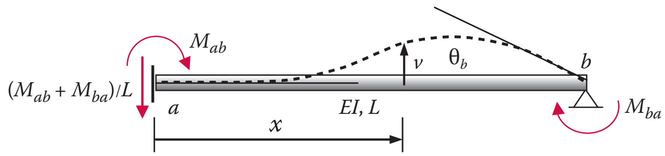
The moment-rotation formula is

$$M_{ab} = M_{ba} = -6EK\varphi \tag{7.7}$$

As indicated in the previous figure, to have a unit side-sway angle takes −6*EK* of a pair of member-end moments, while holding nodal rotation to zero at both ends. The member-end shear forces are not shown.

We can easily develop a moment distribution process that includes the side sway. The process, however, is more involved than the one without the side sway and tends to diminish the advantage of the moment distribution method. A better method for treating side sway is the slope-deflection method, which is introduced next after the derivation of the key formulas, which are central to both the moment distribution method and the slope-deflection method.

Derivation of the moment-rotation (M-θ and M-φ) formulas. We need to derive the formula for the standard model shown next in detail; the other formulas can be obtained by the principle of superposition.



The standard model with the far-end fixed and the near-end hinged.

There are different ways to derive the moment-rotation formula, but the direct integration method is the shortest and most direct way. We seek to show

$$M_{ba} = 4EK\theta_b \quad \text{and} \quad M_{ab} = 2EK\theta_b$$

The governing differential equation is

$$EI\,v'' = M(x)$$

Using the shear force expression at node *a*, we can write

$$M(x) = M_{ab} - (M_{ab} + M_{ba})\frac{x}{L}$$

The second order differential equation, when expressed in terms of the member-end moments, becomes

$$EI\ v \ = M_{ab} - (M_{ab} + M_{ba})\frac{x}{L}$$

Integrating once, we obtain

$$EI\ v \ = M_{ab}(x) - (M_{ab} + M_{ba})\frac{x^2}{2L} + C_1$$

The integration constant is determined by using the support condition at the left end:

$$\text{At } x = 0, v' = 0, \implies C_1 = 0$$

The resulting first-order differential equation is

$$EI\ v \ = M_{ab}(x) - (M_{ab} + M_{ba})\ \frac{x^2}{2L}$$

Integrating again, we obtain

$$EI\ v = M_{ab}\ \frac{x^2}{2} - (M_{ab} + M_{ba})\frac{x^3}{6L} + C_2$$

The integration constant is determined by the support condition at the left end:

$$\text{At } x = 0, v = 0, \implies C_2 = 0$$

The solution in *v* becomes

$$EI\ v = M_{ab}\frac{x^2}{2} - (M_{ab} + M_{ba})\frac{x^3}{6L}$$

Furthermore, there are two more boundary conditions we can use to link the member-end moments together:

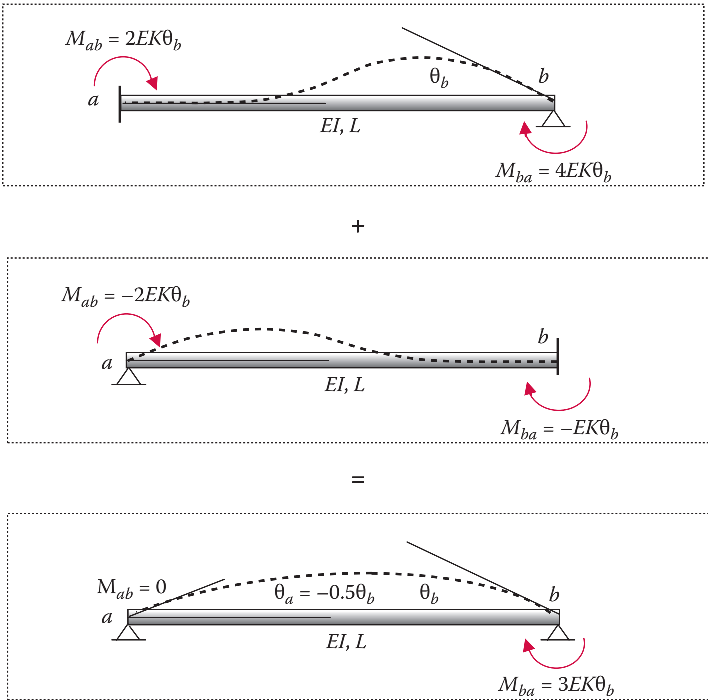
$$\text{At } x = L, v = 0, \Longrightarrow M_{ba} = 2\,M_{ab}$$

$$\text{At } x = L, v' = -\theta_b, \Longrightarrow \theta_b = \frac{LM_{ba}}{4EI}$$

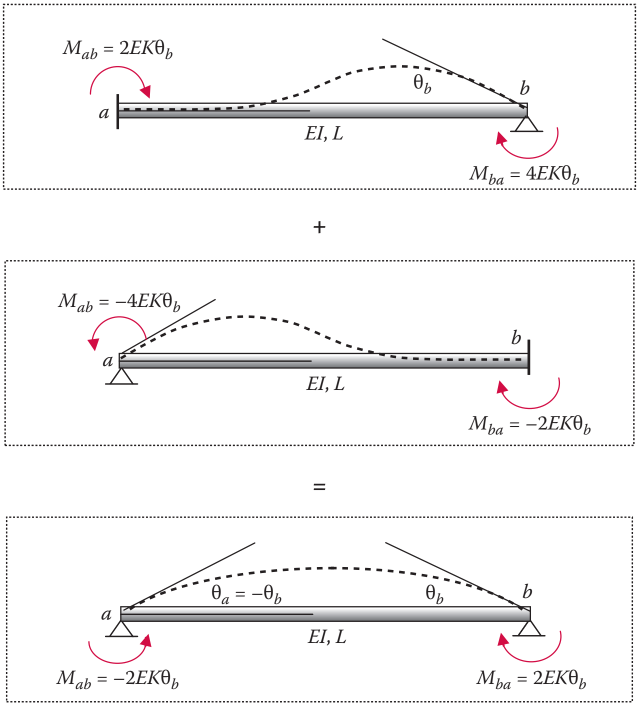
Thus,

$$M_{ba} = 4EK\theta_b \quad \text{and} \quad M_{ab} = 2EK\theta_b$$

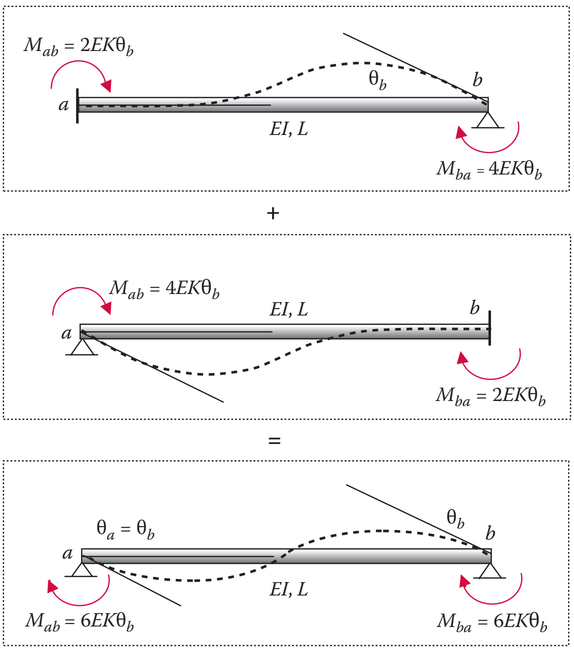
Once the moment-rotation formulas are obtained for the standard model, the formulas for other models are obtained by superposition of the standard model solutions as shown in the following series of figures.



Superposition of two standard models for a hinged-end model solution.

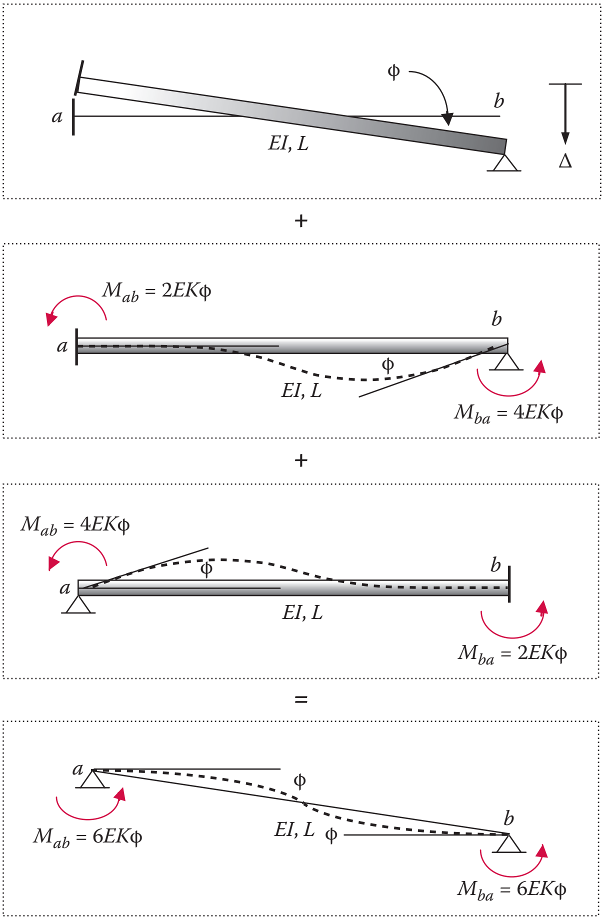


Superposition of two standard models for a symmetric model solution.



Superposition of two standard models for an antisymmetric model solution.

The superposition of standard models to obtain the solution for a translation model requires an additional step in creating a rigid-body rotation of the member without incurring any member-end moments. Two standard models are then added to counter the rotation at member-ends so that the resulting configuration has zero rotation at both ends but a side sway for the whole member.

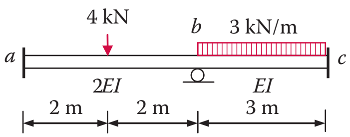


Superposition of a rigid-body solution and two standard models for a side sway solution.

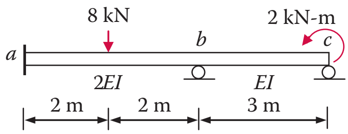
PROBLEM 7.1

Find all the member-end moments of the beams and frames shown, and draw the moment and deflection diagrams.

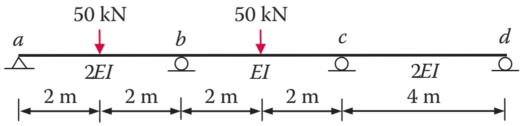
(1)



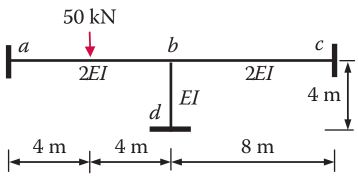
(2)



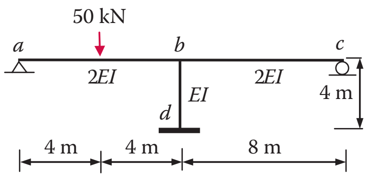
(3)



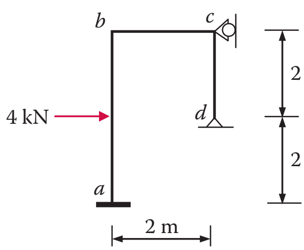
(4)



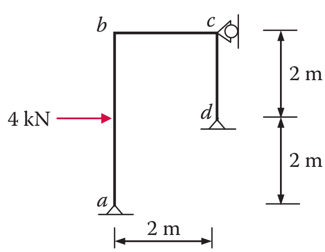
(5)



(6) *EI* is constant in all members.

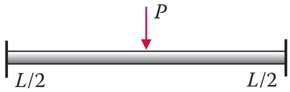
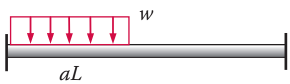
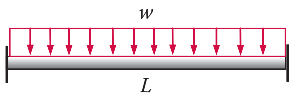
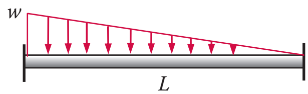
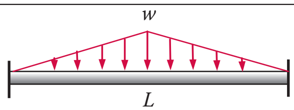
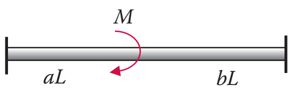


(7) *EI* is constant in all members.



Problem 7.1

Fixed-End Moments

M^F	Loads	M^F
$-\frac{PL}{8}$		$\frac{PL}{8}$
$-ab^2PL$		$-a^2bPL$
$-a(1-a)PL$		$a(1-a)PL$
$-(6-8a+3a^2)\frac{a^2wL^2}{12}$		$(4-3a)\frac{a^3wL^2}{12}$
$-\frac{wL^2}{12}$		$\frac{wL^2}{12}$
$-\frac{wL^2}{20}$		$\frac{wL^2}{12}$
$-\frac{5wL^2}{96}$		$\frac{5wL^2}{96}$
$-b(2a-b)M$		$a(2b-a)M$

Note: Positive moment acts clockwise.

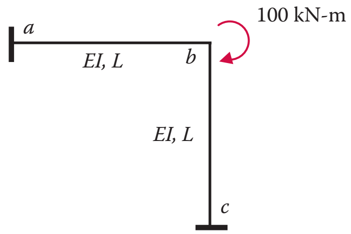
8

Beam and Frame Analysis: Displacement Method—Part II

8.1 Slope-Deflection Method

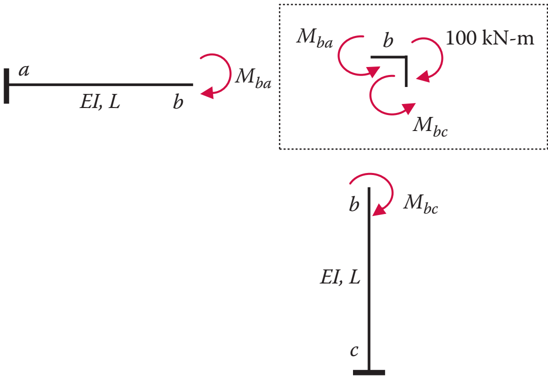
The slope-deflection method treats member-end slope (nodal rotation θ) and deflection (nodal translation Δ) as the basic unknowns. It is based on the same approach as that of the moment distribution method with one difference: the slopes and deflections are implicitly and indirectly used in the moment distribution method but explicitly used in the slope-deflection method. When we "unlock" a node in the moment distribution process, we implicitly rotate a node until the moment at the node is balanced while all other nodes are "locked." The process is iterative because we balance the moment one node at a time. It is implicit because we need not know how much rotation is made in order to balance a node. In the slope-deflection method, we express all member-end moments (MEMs) in terms of the nodal slope and deflection unknowns. When we write the nodal equilibrium equations in moments, we obtain the equilibrium equations in terms of nodal slope and deflection unknowns. These equations, equal in number to the unknown slopes and deflections, are then solved directly.

We shall use a simple frame to illustrate the solution process of the slope-deflection method.



A simple frame problem to be solved by the slope-deflection method.

We observe there are three nodes: *a*, *b*, and *c*. Only node *b* is free to rotate and the nodal rotation is denoted by θ_b. This is the only basic unknown of the problem. We seek to express the moment equilibrium condition at node b in terms of θ_b. This is achieved in two steps: express moment equilibrium of node b in terms of member-end moments and then express member-end moments in terms of θ_b. A simple substitution results in the desired equilibrium equation for θ_b. The following figure illustrates the first step.



Moment equilibrium at node b expressed in terms of member-end moments.

The moment equilibrium at node b calls for

$$\Sigma M_b = 0 \tag{8.1a}$$

which is expressed in terms of member-end moments as

$$M_{ba} + M_{bc} = 100 \tag{8.1b}$$

As we have learned in the moment distribution method, the member-end moments are related to nodal rotation by

$$M_{ba} = (4EK)_{ab}\,\theta_b \tag{8.2a}$$

$$M_{bc} = (4EK)_{bc}\,\theta_b \tag{8.2b}$$

By substitution, we obtain the equilibrium equation in terms of θ_b

$$[(4EK)_{ab} + (4EK)_{bc}]\,\theta_b = 100 \tag{8.3}$$

Solving for θ_b, noting in this case $(4EK)_{ab} = (4EK)_{bc} = 4EK$, we obtain

$$\theta_b = 12.5\frac{1}{EK}$$

Consequently, when we substitute θ_b back to Equation 8.2, we obtain

$$M_{ba} = (4EK)_{ab}\theta_b = (4EK)(12.5)\frac{1}{EK} = 50 \text{ kN-m}$$

$$M_{bc} = (4EK)_{bc}\theta_b = (4EK)(12.5)\frac{1}{EK} = 50 \text{ kN-m}$$

Furthermore, the other member-end moments not needed in the equilibrium equation at node *b* are computed using the moment-rotation formula:

$$M_{ab} = (2EK)_{ab}\theta_b = (2EK)(12.5)\frac{1}{EK} = 25 \text{ kN-m}$$

$$M_{cb} = (2EK)_{bc}\theta_b = (2EK)(12.5)\frac{1}{EK} = 25 \text{ kN-m}$$

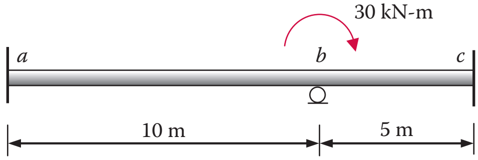
The aforementioned solution process may be summarized as:

1. Identify nodal rotations as degrees of freedoms (DOFs).
2. Identify nodal equilibrium in terms of member-end moments.
3. Express member-end moments in term of nodal rotation.
4. Solve for nodal rotation.
5. Substitute back to obtain all member-end moments.
6. Find other quantities, such as member-end shears and so forth.
7. Draw the moment and deflection diagrams.

We skip the last two steps because these are already done in the moment distribution section.

Example 8.1

Find all the member-end moments of the beam in the following figure. *EI* is constant for all members.



Beam problem with a single degree of freedom (SDOF).

Solution

We observe that there is only one DOF, the rotation at b: θ_b.

The equation of equilibrium is

$$\Sigma M_b = 0, \text{ or } M_{ba} + M_{bc} = 30$$

Before we express the member-end moments in terms of nodal rotation θ_b, we try to simplify the expression of the different EKs of the two members by using a common factor, usually the smallest EK among all EKs.

$$EK_{ab} : EK_{bc} = \frac{EI}{10} : \frac{EI}{5} = 1:2$$

$$\Longrightarrow EK_{bc} = 2EK_{ab} = 2EK$$

$$EK_{ab} = EK$$

Now we are ready to write the moment-rotation formulas:

$$M_{ba} = (4EK)_{ab}\theta_b = 4EK\theta_b$$

$$M_{bc} = (4EK)_{bc}\theta_b = 8EK\theta_b$$

By substitution, we obtain the equilibrium equation in terms of θ_b:

$$[(4EK) + (8EK)]\theta_b = 30$$

Solving for θ_b and $EK\theta_b$, we obtain

$$EK\theta_b = 2.5$$

$$\theta_b = 2.5\frac{1}{EK}$$

Consequently,

$$M_{ba} = (4EK)_{ab}\theta_b = (4EK)\theta_b = 10 \text{ kN-m}$$

$$M_{bc} = (4EK)_{bc}\theta_b = (8EK)\theta_b = 20 \text{ kN-m}$$

$$M_{ab} = (2EK)_{ab}\theta_b = (2EK)\theta_b = 5 \text{ kN-m}$$

$$M_{cb} = (2EK)_{bc}\theta_b = (4EK)\theta_b = 10 \text{ kN-m}$$

Note that we need not know the absolute value of EK if we are interested only in the value of member-end moments. The value of EK is needed only when we want to know the amount of nodal rotation.

For problems with more than one DOF, we need to include the contribution of nodal rotations from both ends of a member to the member-end moments:

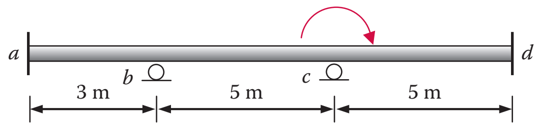
$$M_{ab} = (4EK)_{ab}\theta_a + (2EK)_{ab}\theta_b \quad (8.4a)$$

$$M_{ba} = (4EK)_{ab}\theta_b + (2EK)_{ab}\theta_a \quad (8.4b)$$

Equation 8.4 is easy to remember; the near-end contribution factor is $4EK$ and the far-end contribution factor is $2EK$.

Example 8.2

Find all the member-end moments of the beam in the following figure. EI is constant for all members.



Beam problem with two DOFs.

Solution

We observe that there are two DOFs, the rotations at nodes b and c: θ_b and θ_c, respectively.

The two equations of equilibrium are:

$$\Sigma M_b = 0 \Longrightarrow M_{ba} + M_{bc} = 0$$

and

$$\Sigma M_c = 0 \Longrightarrow M_{cb} + M_{cd} = 30$$

Since EI is constant for all members, we can write

$$K_{ab} : K_{bc} : K_{cd} = \frac{EI}{3} : \frac{EI}{5} : \frac{EI}{5} = 5:3:3 = 1.67:1:1$$

Thus,

$$EK_{ab} = 1.67EK$$

$$EK_{bc} = EK$$

$$EK_{cd} = EK$$

The moment-rotation formulas can be written for the four member-end moments appearing in the two equilibrium equations as

$$M_{ba} = (4EK)_{ab}\theta_b = 6.68EK\theta_b$$

$$M_{bc} = (4EK)_{bc}\theta_b + (2EK)_{bc}\theta_c = 4EK\theta_b + 2EK\theta_c$$

$$M_{cb} = (4EK)_{bc}\theta_c + (2EK)_{bc}\theta_b = 4EK\theta_c + 2EK\theta_b$$

$$M_{cd} = (4EK)_{cd}\theta_c = 4EK\theta_c$$

Note that both rotations at node *b* and rotation at node *c* contribute to the member-end moments, M_{bc} and M_{cb}.

By substituting the moments by rotations, we obtain the two equilibrium equations in terms of θ_b and θ_c.

$$10.68\ EK\theta_b + 2EK\theta_c = 0$$

$$2EK\theta_b + 8EK\theta_c = 30$$

It is advantageous to treat $EK\theta_b$ and $EK\theta_c$ as unknowns.

$$10.68(EK\theta_b) + 2\ (EK\theta_c) = 0$$

$$2(EK\theta_b) + 8\ (EK\theta_c) = 30$$

If we choose to put the above equation into a matrix form, the matrix at the left-hand side (LHS) would be symmetric, always:

$$\begin{matrix} 10.68 & 2 \\ 2 & 8 \end{matrix} \quad \begin{matrix} EK\theta_b \\ EK\theta_c \end{matrix} = \begin{matrix} 0 \\ 30 \end{matrix}$$

Solving the two equations, we obtain

$$(EK\theta_b) = -0.74 \text{ kN-m}$$

$$(EK\theta_c) = 3.93 \text{ kN-m}$$

Substitute back for member-end moments:

$$M_{ba} = 6.68EK\theta_b = -4.92 \text{ kN-m}$$

$$M_{bc} = 4EK\theta_b + 2EK\theta_c = 4.92 \text{ kN-m}$$

$$M_{cb} = 4EK\theta_c + 2EK\theta_b = 14.26 \text{ kN-m}$$

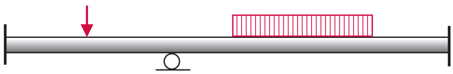
$$M_{cd} = (4EK)_{cd}\theta_c = 4EK\theta_c = 15.74 \text{ kN-m}$$

For the other two member-end moments that were not in the equilibrium equations, we have

$$M_{dc} = (2EK)_{cd}\theta_c = 2EK\theta_c = 7.87 \text{ kN-m}$$

$$M_{ab} = 3.34EK\theta_b = -2.96 \text{ kN-m}$$

Treatment of load between nodes. If loads are applied between nodes, we consider the nodes as initially "locked." That results in fixed-end moments (FEMs) being created at the locked ends. The total member-end moments are the sum of the fixed-end moments due to the load, the moment due to the rotation at the near end, and the moment due to the rotation at the far end.




Load between nodes and the fixed-end moment created by the load.

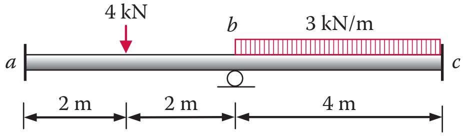
The moment-rotation formula of Equation 8.4 is expanded to become

$$M_{ab} = (4EK)_{ab}\theta_a + (2EK)_{ab}\theta_b + M^F_{ab} \tag{8.5a}$$

$$M_{ba} = (4EK)_{ab}\theta_b + (2EK)_{ab}\theta_a + M^F_{ba} \tag{8.5b}$$

Example 8.3

Find all the member-end moments of the following beam. *EI* is constant for all members.



Beam with load applied between nodes.

Solution

There is only one DOF, the rotation at node *b*: θ_b.

The equation of equilibrium is:

$$\Sigma M_b = 0 \implies M_{ba} + M_{bc} = 0$$

The relative stiffness factors of the two members are such that they are identical.

$$K_{ab} : K_{bc} = 1 : 1 \implies EK_{ab} = EK_{bc} = EK$$

The fixed-end moments are obtained from the FEM table (pg. 196):

For member *ab*:

$$M_{ab}^F = -\frac{P(length)}{8} = -\frac{4(4)}{8} = -2\,\text{kN-m}$$

$$M_{ba}^F = \frac{P(length)}{8} = \frac{4(4)}{8} = 2\,\text{kN-m}$$

For member *bc*:

$$M_{bc}^F = -\frac{w(length)^2}{12} = -\frac{3(4)^2}{12} = -4\,\text{kN-m}$$

$$M_{cb}^F = \frac{w(length)^2}{12} = \frac{3(4)^2}{12} = 4\,\text{kN-m}$$

The moment-rotation formulas are:

$$M_{ba} = (4EK)_{ab}\theta_b + (2EK)_{ab}\theta_a + M_{ba}^F = 4EK\theta_b + 2$$

$$M_{bc} = (4EK)_{bc}\theta_b + (2EK)_{bc}\theta_c + M_{bc}^F = 4EK\theta_b - 4$$

The equilibrium equation $M_{ba} + M_{bc} = 0$ becomes

$$8EK\theta_b - 2 = 0, \Rightarrow EK\theta_b\ 0.25\ \text{kN-m}$$

Substituting back to the member-end moment expressions, we obtain

$$M_{ba} = 4EK\theta_b + 2 = 3\ \text{kN-m}$$

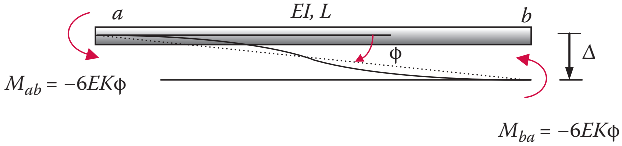
$$M_{bc} = 4EK\theta_b - 4 = -3\ \text{kN-m}$$

For the other two member-end moments not involved in the equilibrium equation, we have

$$M_{ab} = (2EK)_{ab}\theta_b + M_{ab}^F = 0.5 - 2 = -1.5\,\text{kN-m}$$

$$M_{cb} = (2EK)_{bc}\theta_b + M_{cb}^F = 0.5 + 4 = 4.5\,\text{kN-m}$$

Treatment of side sway. The end nodes of a member may have translation displacements perpendicular to the axis of the member, creating a "rotation"-like configuration of the member. This kind of displacement is called a side sway. We can isolate the effect of the side sway by maintaining zero rotation at the two ends and imposing a relative translation (side sway) and find the member-end moments that are needed to maintain such a configuration.



Side sway of a member and the member-end moments.

The member-end moments given in the figure were derived in the context of the moment distribution method. We recall that although side sway usually refers to Δ, a better representation of it is an angle defined by

$$= \frac{\quad}{L}$$

We call φ the member rotation. With the member-end moments caused by side sway quantified as shown in the preceding figure, we can now summarize all the contributions to the member-end moments by the following formulas:

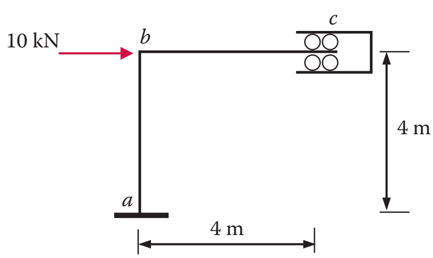
$$M_{ab} = (4EK)_{ab}\theta_a + (2EK)_{ab}\theta_b - (6EK) \quad {}_{ab} + M_{ab}^F \tag{8.6a}$$

$$M_{ba} = (4EK)_{ab}\theta_b + (2EK)_{ab}\theta_a - (6EK) \quad {}_{ab} + M_{ba}^F \tag{8.6b}$$

The presence of one member rotation φ_{ab} requires one additional equation in force equilibrium—usually from force equilibrium that involves member-end shear, which can be expressed in terms of member-end moments, which in turn can be expressed by nodal and member rotations.

Example 8.4

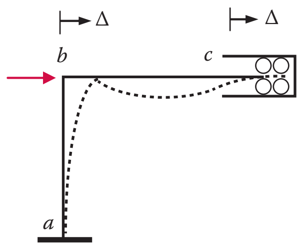
Find all the member-end moments of the frame in the following figure. *EI* is constant for all members.



A frame with side sway.

Solution

We observe that in addition to the rotation at node *b*, there is another DOF, which is the horizontal displacement of node *b* or *c*, designated as Δ as shown in the following figure.



Nodal lateral displacement that creates side sway of member *ab*.

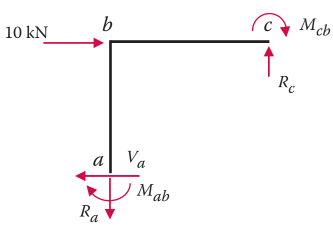
Note that node *b* and node *c* move laterally by the same amount. This is because the axial elongation of membere *ab* is assumed to be negligible. Assuming the lateral displacements Δ are going to the right as shown, then member *ab* has a positive (clockwise) member rotation $\varphi_{ab} = \Delta/L_{ab}$, but member *bc* does not have any member rotation. There is only one independent unknown associated with side sway, either Δ or φ_{ab}. We shall choose φ_{ab} as the representative unknown. With nodal rotation θ_b and φ_{ab} we now have two unknowns. We seek to write two equilibrium equations.

The first equilibrium equation comes from the nodal moment equilibrium at node *b*:

$$\Sigma M_b = 0 \implies M_{ba} + M_{bc} = 0 \tag{8.7}$$

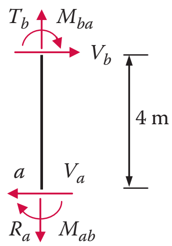
The second comes from the horizontal force equilibrium of the whole structure:

$$\Sigma F_x = 0 \implies 10 + V_a = 0 \tag{8.8}$$



FBD of the whole structure.

It is necessary to express the shear force in terms of member-end moments. This is achieved by applying a moment equilibrium equation on the FBD of member *ab*.



FBD of member ab.

$$\Sigma M_b = 0 \implies V_a = -\frac{M_{ab} + M_{ba}}{4}$$

Substituting the preceding formula for the shear into Equation 8.8 and multiplying the whole equation by 4, we turn the second equilibrium equation, Equation 8.8, into a new form involving member-end moments:

$$M_{ab} + M_{ba} = -40 \tag{8.8}$$

There are three member-end moment unknowns in the two equilibrium equations, Equation 8.7 and Equation 8.8. We need to apply the moment-rotation formulas in order to turn the moment expressions into expressions containing the two displacement unknowns, θ_b and φ.

We observe that $EK_{ba} = EK_{bc}$ and we can designate EK for both EK_{ba} and EK_{bc}:

$$EK_{ba} = EK_{bc} = EK$$

By successive substitution, the moment-rotation formulas are simplified to include only terms in $EK\theta_b$ and $EK\varphi_{ab}$.

$$M_{ba} = (4EK)_{ab}\theta_b + (2EK)_{ab}\theta_a - (6EK)\phi_{ab} + M_{ba}^F = (4EK)_{ab}\theta_b - (6EK)_{ab}\phi_{ab}$$

$$= 4EK\theta_b - 6EK\phi_{ab}$$

$$M_{ab} = (4EK)_{ab}\theta_a + (2EK)_{ab}\theta_b - (6EK)\phi_{ab} + M_{ab}^F = (2EK)_{ab}\theta_b - (6EK)_{ab}\phi_{ab}$$

$$= 2EK\theta_b - 6EK\phi_{ab}$$

$$M_{bc} = (4EK)_{bc}\theta_b + (2EK)_{bc}\theta_c - (6EK)\phi_{bc} + M_{bc}^F = (4EK)_{bc}\theta_b$$

$$= 4EK\theta_b$$

Substituting these member-end moment expressions into the two equilibrium equations, we obtain two equations with two unknowns.

$$M_{ba} + M_{bc} = 0 \quad \Longrightarrow \quad 8EK\theta_b - 6EK\phi_{ab} = 0$$

$$M_{ab} + M_{ba} = -40 \quad \Longrightarrow \quad 6EK\theta_b - 12EK\phi_{ab} = -40$$

In matrix form these two equations become one matrix equation:

$$\begin{matrix} 8 & -6 \\ -6 & 12 \end{matrix} \quad \begin{matrix} EK\theta_b \\ EK\ \ _{ab} \end{matrix} = \begin{matrix} 0 \\ 40 \end{matrix}$$

To obtain the preceding form, we have reversed the sign for all expressions in the second equilibrium equation so that the matrix at the LHS is symmetric. The solution is

$$EK\theta_b = 4 \text{ kN-m}$$

$$EK\varphi_{ab} = 5.33 \text{ kN-m}$$

Substituting back to the moment-rotation formulas, we obtain

$$M_{ba} = -16 \text{ kN-m}$$

$$M_{ab} = -24 \text{ kN-m}$$

$$M_{bc} = 16 \text{ kN-m}$$

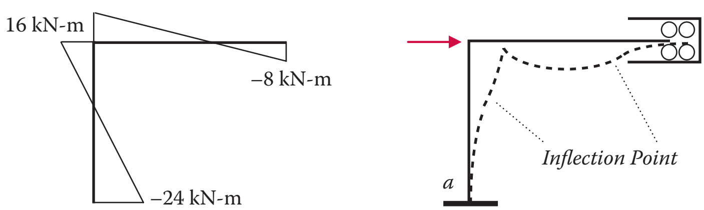
For the member-end moment not appearing in the two equilibrium equations, M_{cb}, we obtain

$$M_{cb} = (2EK)_{bc}\theta_b + (4EK)_{bc}\theta_c - (6EK)\ \ _{bc} + M^F_{bc} = (2EK)_{bc}\theta_b$$

$$= 2EK\theta_b$$

$$= 8 \text{ kN-m}$$

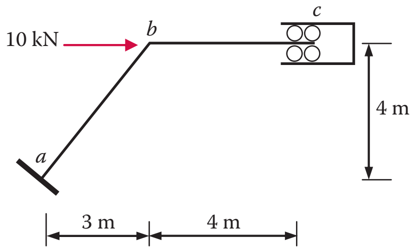
We can now draw the moment diagram and a new deflection diagram, which is refined from the rough sketch done at the beginning of the solution process, using the information contained in the moment diagram.



Moment and deflection diagrams.

Example 8.5

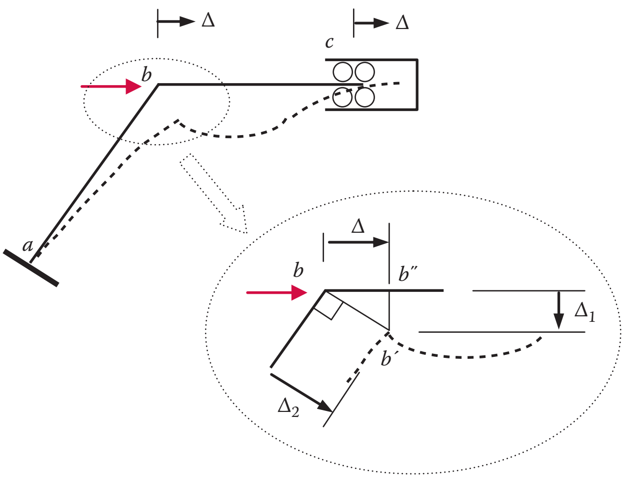
Find all the member-end moments of the frame shown. *EI* is constant for all members.



A frame with an inclined member.

Solution

There is clearly one nodal rotation unknown, θ_b, and one nodal translation unknown, Δ. The presence of an inclined member, however, complicates the geometric relationship between nodal translation and member rotation. We shall, therefore, deal with the geometric relationship first.



Details of nodal displacement relationship.

Because the member lengths are not to change, the postdeformation new location of node *b* is at *b*′, the intersection of a line perpendicular to member *ab* and a line perpendicular to member *bc*. Member rotations of member *ab* and member *bc* are defined by the displacements perpendicular to the member axis. They are Δ_1 for member *bc* and Δ_2 for member *ab*, respectively. From the little triangle, *b*–*b*′–*b*″, we obtain the following formulas:

$$_1 = \frac{3}{4}$$

$$_2 = \frac{5}{4}$$

The rotations of member *ab* and member *bc* are defined by, respectively:

$$_{ab} = \frac{_2}{L_{ab}} = \frac{_2}{5}$$

$$_{bc} = -\frac{_1}{L_{bc}} = -\frac{_1}{4}$$

Since Δ_1 and Δ_2 are related to Δ, so are φ_{ab} and φ_{bc}. We seek to find the relative magnitude of the two member rotations:

$$_{ab} : \ _{bc} = \frac{_2}{5} : \ -\frac{_1}{4} \ = \frac{1}{5} \ \frac{5}{4} \ : \ -\frac{1}{4} \ \frac{3}{4} \ = \frac{1}{4} \ : \ -\frac{3}{16} \ = 1 : \ -\frac{3}{4}$$

Consequently,

$$_{bc} = \ -\frac{3}{4} \quad _{ab}$$

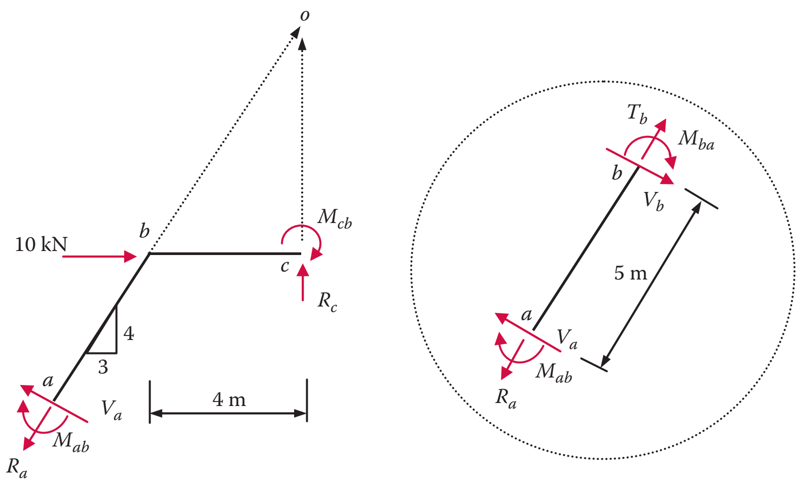
We shall designate φ_{ab} as the member rotation unknown and express φ_{bc} in terms of φ_{ab}. Together with the nodal rotation unknown, θ_b, we have two DOFs, θ_b and φ_{ab}. We seek to write two equilibrium equations.

The first equilibrium equation comes from the nodal moment equilibrium at node *b*:

$$\Sigma M_b = 0 \ \Longrightarrow \ M_{ba} + M_{bc} = 0 \tag{8.9}$$

The second comes from the moment equilibrium of the whole structure about a point o:

$$\Sigma M_o = 0 \ \Longrightarrow \ 10(4)(\frac{4}{3}) + V_a \ [5+4(\frac{5}{3})] + M_{ab} + M_{cb} = 0 \tag{8.10}$$



FBDs of the whole structure and the inclined member.

Note that it is necessary to select the intersection point, o, for the moment equation so that no axial forces are included in the equation. From the FBD of the inclined member, we obtain

$$V_a = -\frac{1}{5}(M_{ab} + M_{ba})$$

Substituting the above formula into Equation 8.10, it becomes

$$-4M_{ab} - 7M_{ba} + 3M_{cb} = 160 \tag{8.10}$$

There are four moment unknowns, M_{ab}, M_{ba}, M_{bc}, and M_{cb}, in two equations. We now establish the moment-rotation (M–θ –φ) formulas, noting that φ_{bc} is expressed in terms of φ_{ab} and

$$EK_{ab} : EK_{bc} = \frac{1}{5}\ \text{EI}:\ \frac{1}{4}\text{EI} = 1:1.25$$

We can designate EK for EK_{ab} and express EK_{bc} in EK as well:

$$EK_{ab} = EK$$

$$EK_{bc} = 1.25\ EK$$

After successive substitution, the moment-rotation formulas are:

$$M_{ba} = (4EK)_{ab}\theta_b + (2EK)_{ab}\theta_a - (6EK)_{ab}\quad{}_{ab} + M^F_{ba}$$

$$= (4EK)_{ab}\theta_b - (6EK)_{ab}\quad{}_{ab}$$

$$= 4EK\theta_b - 6EK\quad{}_{ab}$$

$$M_{ab} = (4EK)_{ab}\theta_a + (2EK)_{ab}\theta_b - (6EK)_{ab}\quad{}_{ab} + M^F_{ab}$$

$$= (2EK)_{ab}\theta_b - (6EK)_{ab}\quad{}_{ab}$$

$$= 2EK\theta_b - 6EK\quad{}_{ab}$$

$$M_{bc} = (4EK)_{bc}\theta_b + (2EK)_{bc}\theta_c - (6EK)_{bc}\quad{}_{bc} + M^F_{bc}$$

$$= (4EK)_{bc}\theta_b - (6EK)_{bc}\quad{}_{bc}$$

$$= 5EK\theta_b + 5.625EK\quad{}_{ab}$$

$$M_{cb} = (4EK)_{bc}\theta_c + (2EK)_{bc}\theta_b - (6EK)_{bc}\quad{}_{bc} + M^F_{cb}$$

$$= (2EK)_{bc}\theta_b - (6EK)_{bc}\quad{}_{bc}$$

$$= 2.5EK\theta_b + 5.625EK\quad{}_{ab}$$

Substituting the preceding moment expressions into Equation 8.9 and Equation 8.10, we obtain the following two equations in θ_b and φ_{ab}.

$$9EK\theta_b - 0.375EK\varphi_{ab} = 0$$

$$-28.5EK\theta_b + 82.875EK\varphi_{ab} = 160$$

Multiplying the first equation by 8 and the second equation by (1/9.5), we obtain

$$72EK\theta_b - 3EK\varphi_{ab} = 0$$

$$-3EK\theta_b + 8.723EK\varphi_{ab} = 16.84$$

In matrix form, we have

$$\begin{matrix} 72 & -3 \\ -3 & 8{,}723 \end{matrix} \quad \begin{matrix} EK\theta_b \\ EK \quad _{ab} \end{matrix} = \begin{matrix} 0 \\ 16.84 \end{matrix}$$

The solution is

$$EK\theta_b = 0.0816 \text{ kN-m}$$

$$EK\varphi_{ab} = 1.959 \text{ kN-m}$$

Substituting back to the moment-rotation formulas, we obtain the member-end moments:

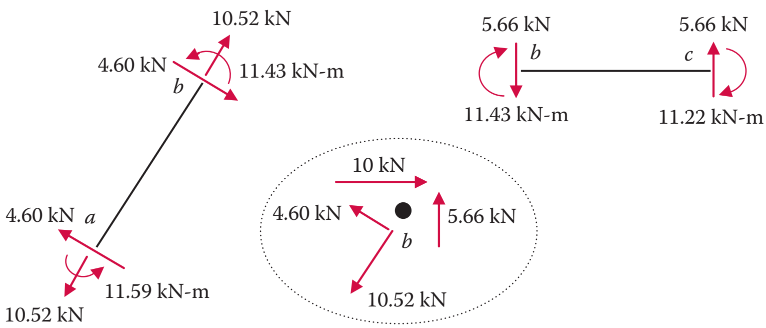
$$M_{ba} = -11.43 \text{ kN-m}$$

$$M_{ab} = -11.59 \text{ kN-m}$$

$$M_{bc} = 11.43 \text{ kN-m}$$

$$M_{cb} = 11.22 \text{ kN-m}$$

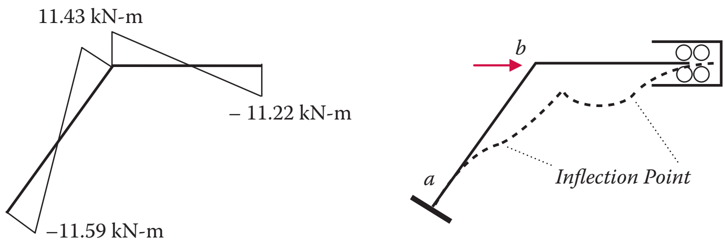
From the member-end moments we can easily obtain all the member-end shears and axial forces as shown next.



FBDs of members *ab*, *bc*, and node *b*.

Note the shear forces are always the first to be determined from the member-end moments. The axial force of member *ab* is then determined from the FBD of node *b*, using the equilibrium equation of all vertical forces.

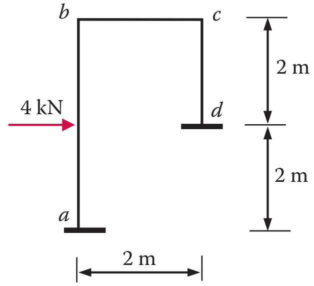
The moment diagram is shown next, together with a new deflection diagram refined from the initial sketch done at the beginning of the solution process, utilizing the information presented in the moment diagram.



Moment and deflection diagrams.

Example 8.6

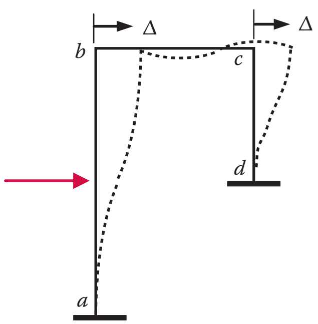
Find all the member-end moments of the frame shown. *EI* is constant for all members.



A frame with rotation and translation DOFs.

Solution

We observe that nodes *b* and *c* are free to rotate. Nodes *b* and *c* are also free to translate in the horizontal direction by the same amount. As a result of this translation of both node *b* and node *c*, member *ab* and member *cd* have member rotations, but not member *bc*, which has no member rotation.



Sketch of the deflection of the frame.

The two member rotations are related to the single translation, Δ, and we can find the relative magnitude of the two easily.

$$_{ab} : \ _{cd} = \frac{}{L_{ab}} : \frac{}{L_{cd}} = \frac{}{4} : \frac{}{2} = 1 : 2$$

Thus,

$$\varphi_{cd} = 2\varphi_{ab}$$

In sum, there are three degrees of freedom: θ_b, θ_c, and φ_{ab}. We need three equations of equilibrium. The first equilibrium equation comes from the nodal moment equilibrium at node b:

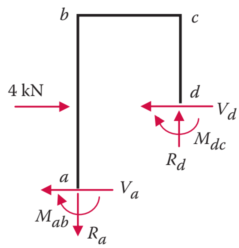
$$\Sigma M_b = 0 \quad \Longrightarrow \quad M_{ba} + M_{bc} = 0$$

The second comes from the nodal moment equilibrium at node c:

$$\Sigma M_c = 0 \quad \Longrightarrow \quad M_{cb} + M_{cd} = 0$$

The third comes from the horizontal force equilibrium of the whole structure:

$$\Sigma F_x = 0 \quad \Longrightarrow \quad V_a + V_d = 4$$

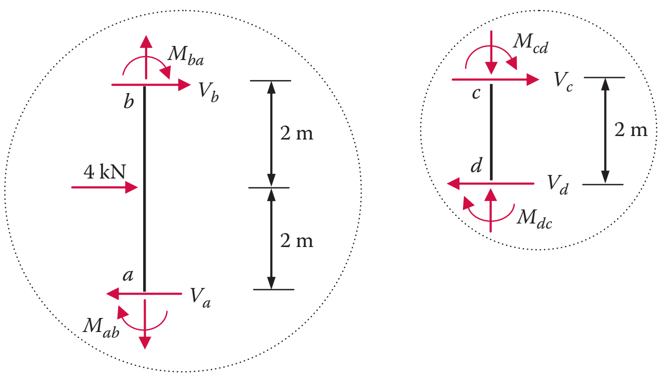


FBD of the whole structure.

The two shear forces in the third equation can be expressed in terms of member-end moments, via the FBD of each member.

$$V_a = -\frac{1}{4}(M_{ab} + M_{ba}) + 2$$

$$V_d = -\frac{1}{2}(M_{dc} + M_{cd})$$



FBDs of the two column members of the rigid frame.

By virtue of the shear-moment relationship, the third equation becomes:

$$V_a + V_d = 4 \quad \Longrightarrow \quad -M_{ab} - M_{ba} - 2M_{dc} - 2M_{cd} = 8$$

There are six moment unknowns in the three equilibrium equations. We now express the member-end moments in terms of the three displacement unknowns: θ_b, θ_c, and φ_{ab}.

Note that we can designate a common stiffness factor EK for all three members:

$$EK_{ab} = EK,\ EK_{bc} = 2EK,\ EK_{cd} = 2EK$$

Noting $\varphi_{cd} = 2\varphi_{ab}$, we can simplify the moment-rotation by expressing it as shown next.

$$M_{ba} = (4EK)_{ab}\theta_b + (2EK)_{ab}\theta_a - (6EK)_{ab}\ \ {}_{ab} + M^F_{ba}$$
$$= (4EK)_{ab}\theta_b - (6EK)_{ab}\ \ {}_{ab} + M^F_{ba}$$
$$= 4EK\theta_b - 6EK\ \ {}_{ab} + 2$$
$$M_{ab} = (4EK)_{ab}\theta_a + (2EK)_{ab}\theta_b - (6EK)_{ab}\ \ {}_{ab} + M^F_{ab}$$
$$= (2EK)_{ab}\theta_b - (6EK)_{ab}\ \ {}_{ab} + M^F_{ab}$$
$$= 2EK\theta_b - 6EK\ \ {}_{ab} - 2$$
$$M_{bc} = (4EK)_{bc}\theta_b + (2EK)_{bc}\theta_c - (6EK)_{bc}\ \ {}_{bc} + M^F_{bc}$$
$$= (4EK)_{bc}\theta_b + (2EK)_{bc}\theta_c$$
$$= 8EK\theta_b + 4EK\theta_c$$
$$M_{cb} = (2EK)_{bc}\theta_b + (4EK)_{bc}\theta_c - (6EK)_{bc}\ \ {}_{bc} + M^F_{cb}$$
$$= (2EK)_{bc}\theta_b + (4EK)_{bc}\theta_c$$
$$= 4EK\theta_b + 8EK\theta_c$$
$$M_{cd} = (4EK)_{cd}\theta_c + (2EK)_{cd}\theta_d - (6EK)\ \ {}_{cd} + M^F_{cd}$$
$$= (4EK)_{cd}\theta_c - (6EK)_{cd}\ \ {}_{cd} + M^F_{cd}$$
$$= 8EK\theta_c - 24EK\ \ {}_{ab}$$
$$M_{dc} = (4EK)_{cd}\theta_d + (2EK)_{cd}\theta_c - (6EK)_{cd}\ \ {}_{cd} + M^F_{dc}$$
$$= (2EK)_{cd}\theta_c - (6EK)_{cd}\ \ {}_{cd}$$
$$= 4EK\theta_c - 12EK\ \ {}_{ab}$$

Substituting all the moment-rotation formulas into the three equilibrium equations, we obtain the following three equations for three unknowns.

$$12EK\theta_b + 4EK\theta_c - 6EK\varphi_{ab} = -2$$
$$4EK\theta_b + 16EK\theta_c - 24EK\varphi_{ab} = 0$$
$$-6EK\theta_b - 24EK\theta_c + 84EK\varphi_{ab} = 8$$

The matrix form of the preceding equation reveals the expected symmetry in the square matrix on the LHS:

$$\begin{matrix} 12 & 4 & -6 \\ 4 & 16 & -24 \\ -6 & -24 & 84 \end{matrix} \quad \begin{matrix} EK\theta_b \\ EK\theta_c \\ EK \quad _{ab} \end{matrix} \quad = \quad \begin{matrix} -2 \\ 0 \\ 8 \end{matrix}$$

The solution is

$$EK\theta_b = -2/11 \text{ kN-m}$$

$$EK\theta_c = 13/44 \text{ kN-m}$$

$$EK\varphi_{ab} = 1/6 \text{ kN-m}$$

Substituting back, we obtain, for the member-end moments:

$$M_{ba} = 3/11 \text{ kN-m} = 0.27 \text{ kN-m}$$

$$M_{ab} = -37/11 \text{ kN-m} = -3.36 \text{ kN-m}$$

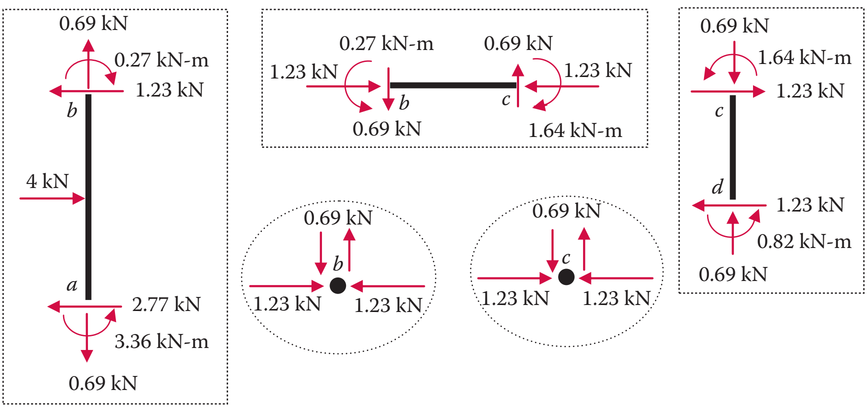
$$M_{bc} = -3/11 \text{ kN-m} = -0.27 \text{ kN-m}$$

$$M_{cb} = -18/11 \text{ kN-m} = 1.64 \text{ kN-m}$$

$$M_{cd} = -18/11 \text{ kN-m} = -1.64 \text{ kN-m}$$

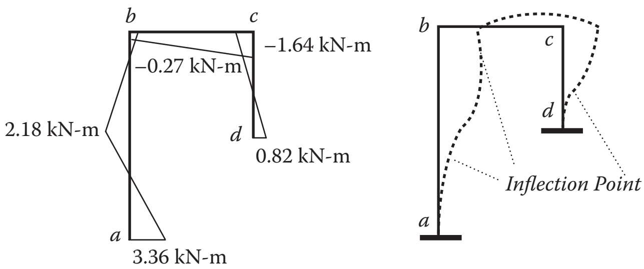
$$M_{dc} = -9/11 \text{ kN-m} = -0.82 \text{ kN-m}$$

From the member-end moments, the shear forces at member ends are computed from the FBD of each member. The axial forces are obtained from the nodal force equilibrium.



FBDs of each member and nodes *b* and *c* (force only).

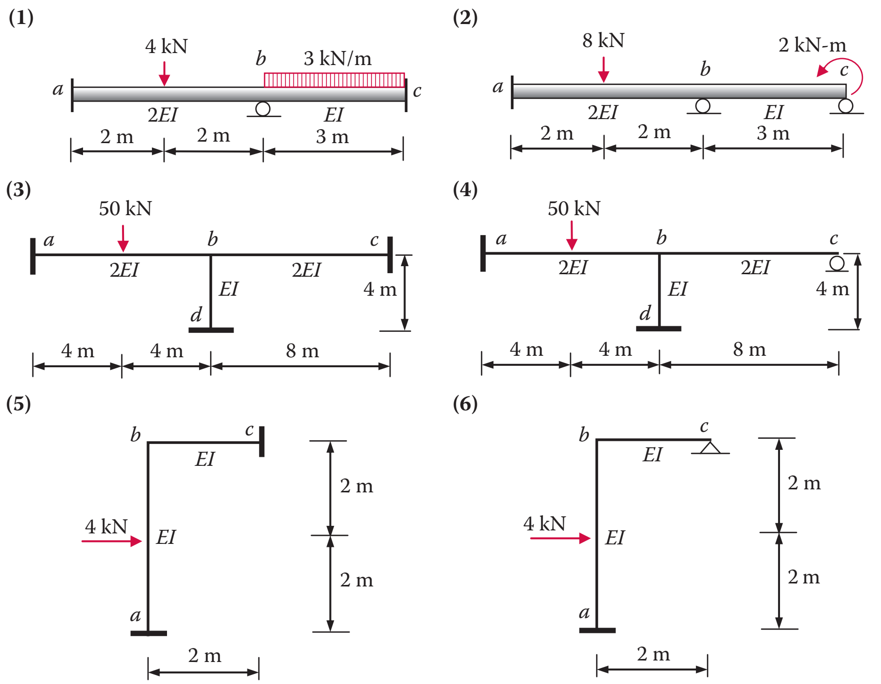
The moment diagram and a refined deflection diagram are shown next.



Moment and deflection diagrams.

PROBLEM 8.1

Use the slope-deflection method to find all the member-end moments of the beams and frames shown, and draw the moment and deflection diagrams.

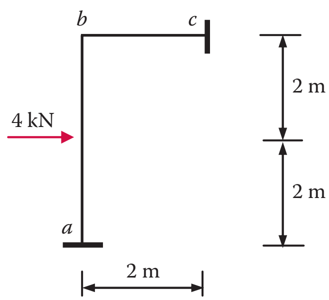


Problem 8.1

8.2 Matrix Stiffness Analysis of Frames

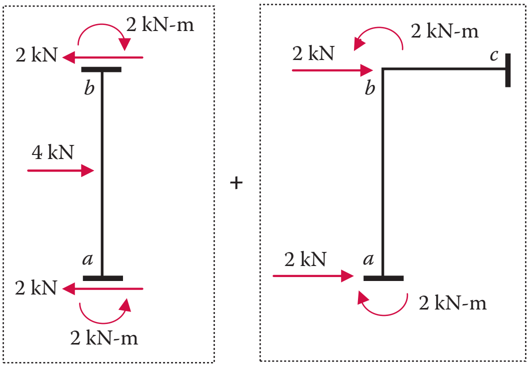
Overview. The slope-deflection method was developed for hand calculation. In order to minimize the number of nodal rotation and member rotation unknowns, the axial deformation of each member is neglected. As a result, the member rotations are often interrelated. As we have observed in the example problems, if we encounter an inclined member, the formulas for the geometric relations can be very involved. On the other hand, if we allow the axial deformation to be included in the displacement calculation, then the nodal displacements at one node are completely independent from those at the other end of a member and we will have three nodal unknowns at each node: two translation unknowns and one rotation unknown. Although the number of unknowns for a given frame will be more than that in a slope-deflection formulation, the formulation process becomes much more straightforward. The formulation presented herein parallels that for truss analyses.

Consider the following rigid frame problem. We treat this problem as one with two members and three nodes.



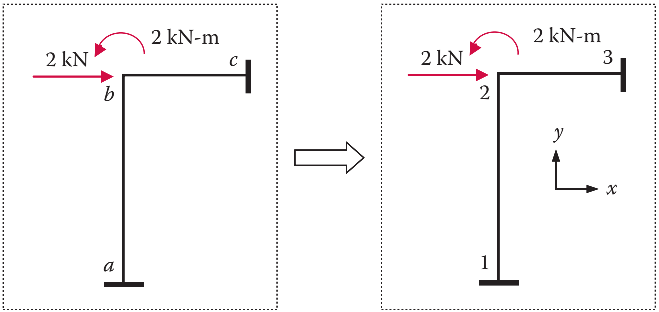
A rigid frame problem.

We shall convert the preceding problem into the superposition of two problems: one with member *ab* locked at both ends and the other with external forces and moments applied only at nodal points. The first problem can be solved at the level of a single member. Indeed, the member-end moments are tabulated in the fixed-end moment table. The member-end shear forces can then be computed from the FBD of the member. The member-end moments and forces in the first problem, when reversed in signs, become the nodal moments and forces of the second problem, which has applied moments and forces only at the nodes. We shall henceforth concentrate on the second problem, with external forces/moments applied only at nodal points.



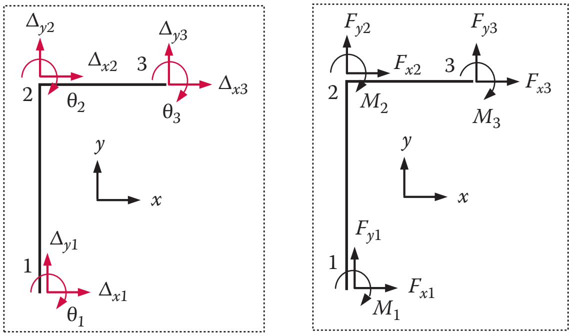
Superposition of two problems.

The moment and force applied at support *a* are taken up by the support directly. They should be included in the calculation of forces acting on the support but be excluded in the forces acting on the nodes of the frame. The problem is reduced to the one shown in the left part of the following figure. The right part of the figure includes a global coordinate system, which is necessary because each member is oriented in different directions and we need a common coordinate system to relate member displacements and forces to those of other members. We also replace the alphabetic system by the numeric system for naming nodes because the latter is easier to program for computer solutions.



Frames with loads applied only at nodes.

In the matrix displacement formulation, it is easier to begin without applying the displacement and force conditions first. Only when the global displacement-force equations are formulated can we then impose the displacement and force conditions in preparation for a numerical solution. Thus, the next step is to define the nodal displacements and the corresponding nodal forces of the frame without the support and loading conditions. Since each node has three DOFs, the frame has a total of nine nodal displacements and nine corresponding nodal forces, as shown in the following figure.



The nine nodal displacements and the corresponding nodal forces.

It should be emphasized that the nine nodal displacements completely define the deformation of each member and the entire frame. In the matrix displacement formulation, we seek to find the matrix equation that links the nine nodal forces to the nine nodal displacements in the following form:

$$K_G \Delta_G = F_G \tag{8.11}$$

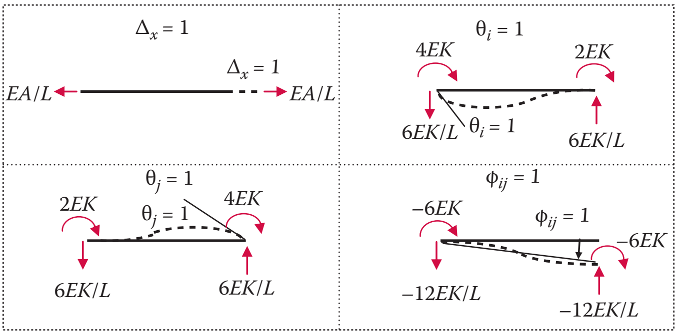
where K_G, Δ_G, and F_G are the global unconstrained stiffness matrix, global nodal displacement vector, and global nodal force vector, respectively. Equation 8.11 in its expanded form is shown next, which helps identify the nodal displacement and force vectors.

$$\begin{bmatrix} \textit{Member } 1\text{–}2 & \\ & \textit{Member } 2\text{–}3 \end{bmatrix} \begin{Bmatrix} \Delta_{x1} \\ \Delta_{y1} \\ \theta_1 \\ \Delta_{x2} \\ \Delta_{y2} \\ \theta_2 \\ \Delta_{x3} \\ \Delta_{y3} \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_1 \\ F_{x2} \\ F_{y2} \\ M_2 \\ F_{x3} \\ F_{x3} \\ M_3 \end{Bmatrix} \tag{8.11}$$

According to the direct stiffness method, the contribution of member 1–2 to the global stiffness matrix will be at the locations indicated in the previous figure, that is, corresponding to the DOFs of the first and the second nodes, while the contribution of member 2–3 will be associated with the DOFs at nodes 2 and 3.

Before we assemble the global stiffness matrix, we need to formulate the member stiffness matrix.

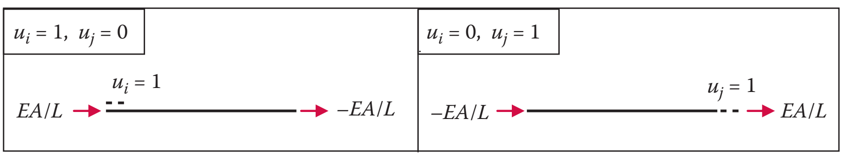
Member stiffness matrix in local coordinates. For a frame member, both axial and flexural deformations must be considered. As long as the deflections associated with these deformations are small relative to the transverse dimension of the member, say depth of the member, the axial and flexural deformations are independent from each other; thus allowing us to consider them separately. To characterize the deformations of a frame member, i–j, we need only four independent variables, Δ_x, θ_i, θ_j, and φ_{ij}, as shown next.



The four independent deformation configurations and the associated nodal forces.

Each of the four member displacement variables is related to the six nodal displacements of a member via geometric relations. Instead of deriving these relations mathematically, then use mathematical transformation to obtain the stiffness matrix, as was done in the truss formulation, we can establish the stiffness matrix directly by relating the nodal forces to a nodal displacement, one at a time. We shall deal with the axial displacements first.

There are two nodal displacements, u_i and u_j, related to axial deformation, Δ_x. We can easily establish the nodal forces for a given unit nodal displacement, utilizing the nodal force information in the previous figure. For example, $u_i = 1$ while other displacements are zero corresponds to a negative elongation. As a result, the nodal force at node i is EA/L, while that at node j is $-EA/L$. On the other hand, for $u_j = 1$, the force at node i is $-EA/L$, while that at node j is EA/L. These two cases are depicted in the following figure. Note we must express the nodal forces in the positive direction of the defined global coordinates.

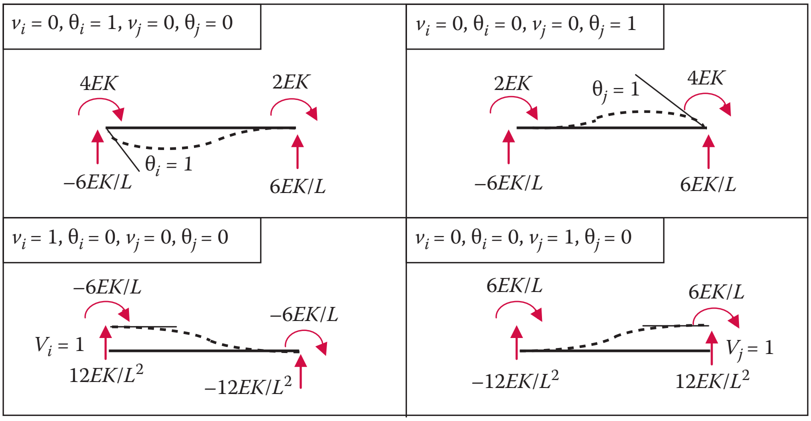


Nodal forces associated with a unit nodal displacement.

From the previous figure, we can immediately establish the following stiffness relationship:

$$\begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_{xi} \\ f_{xj} \end{Bmatrix} \tag{8.12}$$

Following the same principle, we can establish the flexural relations one at a time as shown in the following figure.



From the figure we can establish the following flexural stiffness relationship.

$$\begin{bmatrix} \frac{12EK}{L^2} & -\frac{6EK}{L} & -\frac{12EK}{L^2} & -\frac{6EK}{L} \\ -\frac{6EK}{L} & 4EK & \frac{6EK}{L} & 2EK \\ -\frac{12EK}{L^2} & \frac{6EK}{L} & \frac{12EK}{L^2} & \frac{6EK}{L} \\ -\frac{6EK}{L} & 2EK & \frac{6EK}{L} & 4EK \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} f_{yi} \\ M_i \\ f_{yj} \\ M_j \end{Bmatrix} \tag{8.13}$$

Equation 8.13 is the member stiffness equation of a flexural member, while Equation 8.12 is that of an axial member. The stiffness equation for a frame member is obtained by the merge of the two equations.

$$\begin{bmatrix} \boxed{\frac{EA}{L}} & 0 & 0 & \boxed{-\frac{EA}{L}} & 0 & 0 \\ 0 & \frac{12EK}{L^2} & -\frac{6EK}{L} & 0 & -\frac{12EK}{L^2} & -\frac{6EK}{L} \\ 0 & -\frac{6EK}{L} & 4EK & 0 & \frac{6EK}{L} & 2EK \\ \boxed{-\frac{EA}{L}} & 0 & 0 & \boxed{\frac{EA}{L}} & 0 & 0 \\ 0 & -\frac{12EK}{L^2} & \frac{6EK}{L} & 0 & \frac{12EK}{L^2} & \frac{6EK}{L} \\ 0 & -\frac{6EK}{L} & 2EK & 0 & \frac{6EK}{L} & 4EK \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} f_{xi} \\ f_{yi} \\ M_i \\ f_{xj} \\ f_{yj} \\ M_j \end{Bmatrix} \tag{8.14}$$

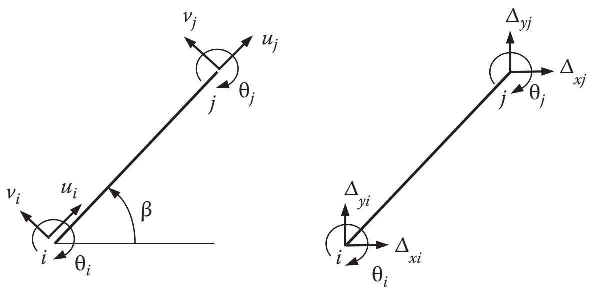
Equation 8.14 is the member stiffness equation in local coordinates and the six-by-six matrix at the LHS is the member stiffness matrix in local coordinates. Equation 8.14 can be expressed in matrix symbols as

$$\boldsymbol{k}_L \boldsymbol{\delta}_L = \boldsymbol{f}_L \tag{8.14}$$

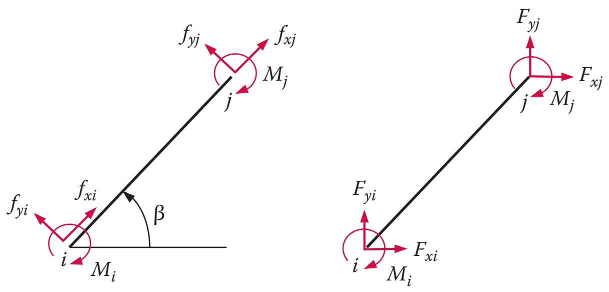
Member stiffness matrix in global coordinates. In the formulation of equilibrium equations at each of the three nodes of the frame, we must use a common set of coordinate system so that the forces and moments are expressed in the same system and can be added directly. The common system is the global coordinate system, which may not coincide with the local system of a member. For a typical orientation of a member as shown, we seek the member stiffness equation in the global coordinates:

$$\boldsymbol{k}_G \boldsymbol{\delta}_G = \boldsymbol{f}_G \tag{8.15}$$

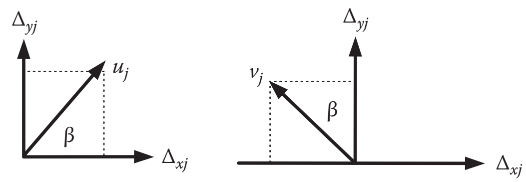
We shall derive Equation 8.15 using Equation 8.14 and the formulas that relate the nodal displacement vector, $\boldsymbol{\delta}_L$, and the nodal force vector, $\boldsymbol{f}_L$, to their global counterparts, $\boldsymbol{\delta}_G$ and $\boldsymbol{f}_G$, respectively.



Nodal displacements in local and global coordinates.



Nodal forces in local and global coordinates.



Vector decomposition.

From the vector decomposition, we can express the nodal displacements in local coordinates in terms of the nodal displacements in global coordinates.

$$\Delta_{xj} = (\text{Cos}\beta)\, u_j - (\text{Sin}\beta)\, v_j$$

$$\Delta_{yj} = (\text{Sin}\beta)\, u_j + (\text{Cos}\beta)\, v_j$$

$$\theta_j = \theta_j$$

Identical formulas can be obtained for node i. The same transformation also applies to the transformation of nodal forces. We can put all the transformation formulas in matrix form, denoting $Cos\beta$ and $Sin\beta$ by C and S, respectively.

$$\begin{matrix} _{xi} \\ _{yi} \\ \theta_i \end{matrix} = \begin{matrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} u_i \\ v_i \\ \theta_i \end{matrix} \quad \text{or} \quad _{iG} = \tau \quad _{iL}$$

$$\begin{matrix} _{xj} \\ _{yj} \\ \theta_j \end{matrix} = \begin{matrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} u_j \\ v_j \\ \theta_j \end{matrix} \quad \text{or} \quad _{jG} = \tau \quad _{jL}$$

$$\begin{matrix} F_{xi} \\ F_{yi} \\ M_i \end{matrix} = \begin{matrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} f_{xi} \\ f_{yi} \\ M_i \end{matrix} \quad \text{or} \quad f_{iG} = \tau\, f_{iL}$$

$$\begin{matrix} F_{xj} \\ F_{yj} \\ M_j \end{matrix} = \begin{matrix} C & -S & 0 \\ S & C & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} f_{xj} \\ f_{yj} \\ M_j \end{matrix} \quad \text{or} \quad f_{ig} = \tau\, f_{jL}$$

The transformation matrix τ has a unique feature, that is, its inverse is equal to its transpose matrix.

$$\tau^{-1} = \tau^T$$

Matrices satisfying the above equation are called orthonormal matrices. Because of this unique feature of orthonormal matrices, we can easily write the inverse relationship for all the previous four equations. We need, however, only the inverse formulas for nodal displacements:

$$\begin{matrix} u_i \\ v_i \\ \theta_i \end{matrix} = \begin{matrix} C & S & 0 \\ -S & C & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} {}_{xi} \\ {}_{yi} \\ \theta_i \end{matrix} \quad \text{or} \quad {}_{iL} = \boldsymbol{\tau}^T \; {}_{iG}$$

$$\begin{matrix} u_j \\ v_j \\ \theta_j \end{matrix} = \begin{matrix} C & S & 0 \\ -S & C & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} {}_{xj} \\ {}_{yj} \\ \theta_j \end{matrix} \quad \text{or} \quad {}_{jL} = \boldsymbol{\tau}^T \; {}_{jG}$$

The nodal displacement vector and force vector of a member, δ_G, δ_L, f_G, and f_L, are the collections of the displacement and force vectors of node *i* and node *j*:

$$\boldsymbol{\delta}_G = \begin{matrix} {}_{iG} \\ {}_{jG} \end{matrix} \quad ; \quad \boldsymbol{\delta}_L = \begin{matrix} {}_{iL} \\ {}_{jL} \end{matrix}$$

$$\boldsymbol{f}_G = \begin{matrix} f_{iG} \\ f_{jG} \end{matrix} \quad ; \quad \boldsymbol{f}_L = \begin{matrix} f_{iL} \\ f_{jL} \end{matrix}$$

To arrive at Equation 8.15, we begin with

$$\boldsymbol{f}_G = \begin{matrix} f_{iG} \\ f_{jG} \end{matrix} = \begin{matrix} \tau & 0 \\ 0 & \tau \end{matrix} \quad \begin{matrix} f_{iL} \\ f_{jL} \end{matrix} = \Gamma \, f_L \tag{8.16}$$

where

$$\Gamma = \begin{matrix} \tau & 0 \\ 0 & \tau \end{matrix} \tag{8.17}$$

From Equation 8.14, and the transformation formulas for nodal displacements, we obtain

$$f_L = k_L\, \delta_L = k_L \begin{Bmatrix} {}_{iL} \\ {}_{jL} \end{Bmatrix} = k_L \begin{bmatrix} \tau^T & \mathbf{0} \\ \mathbf{0} & \tau^T \end{bmatrix} \begin{Bmatrix} {}_{iG} \\ {}_{jG} \end{Bmatrix} = k_L\, \Gamma^T\, \delta_G \tag{8.18}$$

Combining Equation 8.18 with Equation 8.16, we have

$$f_G = \Gamma\, f_L = \Gamma\, k_L\, \Gamma^T\, \delta_G$$

which is in the form of Equation 8.15, with

$$k_G = \Gamma\, k_L\, \Gamma^T \tag{8.19}$$

Equation 8.19 is the transformation formula of the member stiffness matrix. The expanded form of the member stiffness matrix in its explicit form in global coordinates, k_G, appears as a 6-by-6 matrix:

$$k_G = \begin{bmatrix}
C^2\frac{EA}{L} + S^2\frac{12EK}{L^2} & CS(\frac{EA}{L} - \frac{12EK}{L^2}) & S\frac{6EK}{L} & -C^2\frac{EA}{L} - S^2\frac{12EK}{L^2} & -CS(\frac{EA}{L} - \frac{12EK}{L^2}) & S\frac{6EK}{L} \\
CS(\frac{EA}{L} - \frac{12EK}{L^2}) & S^2\frac{EA}{L} + C^2\frac{12EK}{L^2} & -C\frac{6EK}{L} & -CS(\frac{EA}{L} - \frac{12EK}{L^2}) & -S^2\frac{EA}{L} - C^2\frac{12EK}{L^2} & -C\frac{6EK}{L} \\
S\frac{6EK}{L} & -C\frac{6EK}{L} & 4EK & -S\frac{6EK}{L} & C\frac{6EK}{L} & 2EK \\
C^2\frac{EA}{L} - S^2\frac{12EK}{L^2} & -CS(\frac{EA}{L} - \frac{12EK}{L^2}) & -S\frac{6EK}{L} & C^2\frac{EA}{L} - S^2\frac{12EK}{L^2} & CS(\frac{EA}{L} - \frac{12EK}{L^2}) & -S\frac{6EK}{L} \\
-CS(\frac{EA}{L} - \frac{12EK}{L^2}) & -S^2\frac{EA}{L} - C^2\frac{12EK}{L^2} & C\frac{6EK}{L} & CS(\frac{EA}{L} - \frac{12EK}{L^2}) & S^2\frac{EA}{L} + C^2\frac{12EK}{L^2} & C\frac{6EK}{L} \\
S\frac{6EK}{L} & -C\frac{6EK}{L} & 2EK & -S\frac{6EK}{L} & C\frac{6EK}{L} & 4EK
\end{bmatrix} \tag{8.19}$$

The corresponding nodal displacement and force vectors, in their explicit forms, are

$$\delta_G = \begin{Bmatrix} {}_{xi} \\ {}_{yi} \\ \theta_i \\ {}_{xj} \\ {}_{yj} \\ \theta j \end{Bmatrix} \quad \text{and} \quad f_G = \begin{Bmatrix} F_{xi} \\ F_{yi} \\ M_i \\ F_{xj} \\ F_{yj} \\ M_j \end{Bmatrix} \tag{8.20}$$

Unconstrained global equilibrium equation. The member stiffness matrices are assembled into a matrix equilibrium equation, which is formulated from the three equilibrium equations at each node: two force equilibrium and one moment equilibrium equations. The method of assembling is according to the direct stiffness method outlined in the matrix analysis of trusses. For the present case, there are nine equations from the three nodes, as indicated in Equation 8.11.

Constrained global equilibrium equation. Out of the nine nodal displacements, six are constrained to be zero because of support conditions at nodes 1 and 3. There are only three unknown nodal displacements: Δ_{x2}, Δ_{y2}, and θ_2. On the other hand, out of the nine nodal forces, only three are given: $F_{x2} = 2$ kN, $F_{y2} = 0$, and $M_2 = -2$ kN-m; the other six are unknown reactions at the supports. Once we specify all the known quantities, the global equilibrium equation appears in the following form:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & 0 & 0 & 0 \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & 0 & 0 & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} & K_{49} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} & K_{59} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} & K_{69} \\ 0 & 0 & 0 & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} & K_{79} \\ 0 & 0 & 0 & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} & K_{89} \\ 0 & 0 & 0 & K_{94} & K_{95} & K_{96} & K_{97} & K_{98} & K_{99} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \Delta_{x2} \\ \Delta_{y2} \\ \theta_2 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_1 \\ 2 \\ 0 \\ -2 \\ F_{x3} \\ F_{x3} \\ M_3 \end{Bmatrix} \tag{8.11}$$

The solution of Equation 8.11 comes in two steps. The first step is to solve for only the three displacement unknowns using the three equations in the fourth to sixth rows of Equation 8.11.

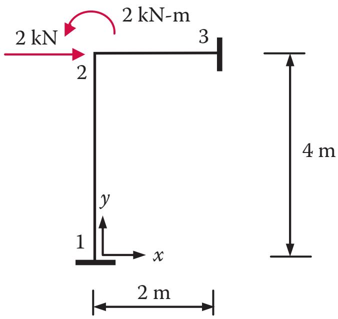
$$\begin{matrix} K_{44} & K_{45} & K_{46} \\ K_{54} & K_{55} & K_{56} \\ K_{64} & K_{65} & K_{66} \end{matrix} \quad \begin{matrix} _{x2} \\ _{y2} \\ \theta_2 \end{matrix} \quad = \quad \begin{matrix} 2 \\ 0 \\ -2 \end{matrix} \tag{8.21}$$

Once the nodal displacements are known, we can carry out the second step by substituting back to Equation 8.11 all the nodal displacements and computing the six other nodal forces, which are the support reaction forces. We also need to find the member-end forces through Equation 8.14, which requires the determination of nodal displacements in local coordinates.

We shall demonstrate the aforementioned procedures through a numerical example.

Example 8.7

Find the nodal displacements, support reactions, and member-end forces of all members of the frame shown. E = 200 GPa, A = 20000 mm^2, and I = 300 × 10^6 mm^4 for the two members.



Example problem.

Solution

We will carry out a step-by-step solution procedure for the problem.

1. Number the nodes and members and define the nodal coordinates.

Nodal Coordinates

Node	x (m)	y (m)
1	0	0
2	0	4
3	2	4

2. Define member property, starting and end nodes, and compute member data.

Member Input Data

Member	Starting Node	End Node	E (GPa)	I (mm^4)	A (mm^2)
1	1	2	200	3×10^8	2×10^4
2	2	3	200	3×10^8	2×10^4

Computed Data

Member	ΔX (m)	ΔY (m)	L (m)	C	S	EI (kN-m^2)	EA (kN)
1	0	4	4	0.0	1.0	6×10^4	4×10^9
2	2	0	2	1.0	0.0	6×10^4	4×10^9

In computing the member data, the following formulas were used:

$$L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$$

$$C = Cos\beta = \frac{(x_j - x_i)}{L} = \frac{x}{L}$$

$$S = Sin\beta = \frac{(y_j - y_i)}{L} = \frac{y}{L}$$

3. Compute member stiffness matrices in global coordinates: Equation 8.19.

Member 1:

$$(\boldsymbol{k_G})_1 = \left[\begin{array}{ccc|ccc} 11{,}250 & 0 & 22{,}500 & -11{,}250 & 0 & 22{,}500 \\ 0 & 1\times10^9 & 0 & 0 & -1\times10^9 & 0 \\ 22{,}500 & 0 & 60{,}000 & -22{,}500 & 0 & 30{,}000 \\ \hline -11{,}250 & 0 & -22{,}500 & 11{,}250 & 0 & -22{,}500 \\ 0 & -1\times10^9 & 0 & 0 & 1\times10^9 & 0 \\ 22{,}500 & 0 & 30{,}000 & -22{,}500 & 0 & 60{,}000 \end{array}\right]$$

Member 2:

$$(\boldsymbol{k_G})_2 = \left[\begin{array}{ccc|ccc} 2\times10^9 & 0 & 0 & -2\times10^9 & 0 & 0 \\ 0 & 90{,}000 & -90{,}000 & 0 & -90{,}000 & -90{,}000 \\ 0 & -90{,}000 & 12\times10^4 & 0 & 90{,}000 & 6\times10^4 \\ \hline -2\times10^9 & 0 & 0 & 2\times10^9 & 0 & 0 \\ 0 & -90{,}000 & 90{,}000 & 0 & -90{,}000 & 90{,}000 \\ 0 & -90{,}000 & 6\times10^4 & 0 & 90{,}000 & 12\times10^4 \end{array}\right]$$

4. Assemble the unconstrained global stiffness matrix. In order to use the direct stiffness method to assemble the global stiffness matrix, we need the following table, which gives the global DOF number corresponding to each local DOF of each member. This table is generated using the member data given in the table in step 2, namely, the starting and end nodes data.

Global DOF Number for Each Member

Local Nodal DOF Number		**Global DOF Number for Member**	
		1	**2**
	1	1	4
Starting Node *i*	2	2	5
	3	3	6
	4	4	7
End Node *j*	5	5	8
	6	6	9

Armed with this table we can easily direct the member stiffness components to the right location in the global stiffness matrix. For example, the (2,3) component of $(\boldsymbol{k_G})_2$ will be added to the (5,6) component of the global stiffness matrix. The unconstrained global stiffness matrix is obtained after the assembling is done.

$$\boldsymbol{K_G} = \begin{bmatrix} 11{,}250 & 0 & 22{,}500 & -11{,}250 & 0 & 22{,}500 & 0 & 0 & 0 \\ 0 & 1\times10^9 & 0 & 0 & -1\times10^9 & 0 & 0 & 0 & 0 \\ 22{,}500 & 0 & 60{,}000 & -22{,}500 & 0 & 30{,}000 & 0 & 0 & 0 \\ -11{,}250 & 0 & -22{,}500 & 2\times10^9 & 0 & -22{,}500 & -2\times10^9 & 0 & 0 \\ 0 & -1\times10^9 & 0 & 0 & 1\times10^9 & -90{,}000 & 0 & -90{,}000 & 90{,}000 \\ 22{,}500 & 0 & 30{,}000 & -22{,}500 & -90{,}000 & 180{,}000 & 0 & 90{,}000 & 60{,}000 \\ 0 & 0 & 0 & -2\times10^9 & 0 & 0 & 2\times10^9 & 0 & 0 \\ 0 & 0 & 0 & 0 & -90{,}000 & 90{,}000 & 0 & -90{,}000 & 90{,}000 \\ 0 & 0 & 0 & 0 & -90{,}000 & 60{,}000 & 0 & 90{,}000 & 120{,}000 \end{bmatrix}$$

5. Constrained global stiffness equation and its solution. Once the support and loading conditions are incorporated into the stiffness equations, we obtain the constrained global stiffness equation as given in Equation 8.11, which is reproduced next for easy reference, with the stiffness matrix shown in the previous equation.

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & 0 & 0 & 0 \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & 0 & 0 & 0 \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & 0 & 0 & 0 \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} & K_{48} & K_{49} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} & K_{58} & K_{59} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} & K_{68} & K_{69} \\ 0 & 0 & 0 & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} & K_{79} \\ 0 & 0 & 0 & K_{84} & K_{85} & K_{86} & K_{87} & K_{88} & K_{89} \\ 0 & 0 & 0 & K_{94} & K_{95} & K_{96} & K_{97} & K_{98} & K_{99} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \Delta_{x2} \\ \Delta_{y2} \\ \theta_2 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ M_1 \\ 2 \\ 0 \\ -2 \\ F_{x3} \\ F_{x3} \\ M_3 \end{Bmatrix} \quad (8.11)$$

For the three displacement unknowns, the following three equations, taking from the fourth to sixth rows of the unconstrained global stiffness equation, are the governing equations.

$$\begin{matrix} 2x10^9 & 0 & -22{,}500 \\ 0 & 1x10^9 & -90{,}000 \\ -22{,}500 & -90{,}000 & 180{,}000 \end{matrix} \quad \begin{matrix} {}_{x2} \\ {}_{y2} \\ \theta_2 \end{matrix} \quad = \quad \begin{matrix} 2 \\ 0 \\ -2 \end{matrix} \quad (8.21)$$

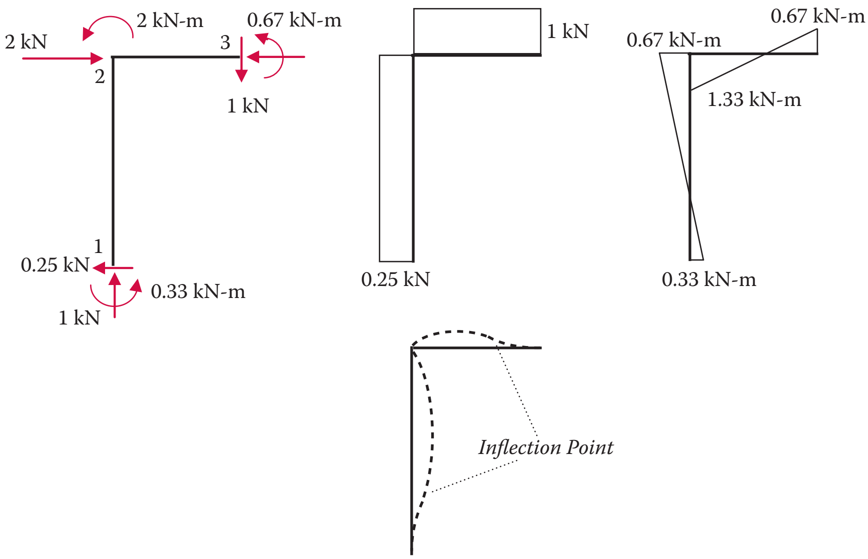
The solutions are: $\Delta_{x2} = 0.875 \times 10^{-9}$ m, $\Delta_{y2} = 1 \times 10^{-9}$ m, and $\theta_2 = 1.11 \times 10^{-5}$ rad. Upon substituting the nodal displacements into Equation 8.18, we obtain the nodal forces, which are support reactions:

$$\begin{matrix} F_{x1} \\ F_{y1} \\ M_1 \end{matrix} = \begin{matrix} -0.25\text{ kN} \\ 1.00\text{ kN} \\ -0.33\text{ kN-m} \end{matrix} \quad \text{and} \quad \begin{matrix} F_{x3} \\ F_{y3} \\ M_3 \end{matrix} = \begin{matrix} -1.75\text{ kN} \\ -1.00\text{ kN} \\ -0.67\text{ kN-m} \end{matrix}$$

6. Compute member nodal forces in local coordinates. The member nodal forces in local coordinates are needed to draw shear and moment diagrams and are obtained using Equation 8.14. The nodal displacements in local coordinates in Equation 8.14 are computed using the transformation formula

$$\begin{matrix} u_i \\ v_i \\ \theta_i \end{matrix} = \begin{matrix} C & S & 0 \\ -S & C & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} {}_{xi} \\ {}_{yi} \\ \theta_i \end{matrix} \quad \text{or} \quad {}_{iL} = \tau^T {}_{iG}$$

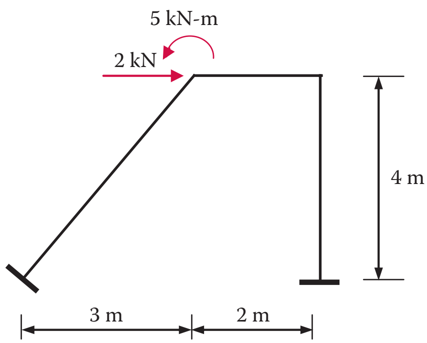
The results are presented in the following reaction, shear, moment, and deflection diagrams.



Reaction, shear, moment, and deflection diagrams.

PROBLEM 8.2

The matrix analysis of frames is often solved using a computer program. Most of the computation presented in the text is done automatically within the computer program. Prepare the minimum input data set needed for the computer solution of the frame shown. Begin with the numbering of nodes and members. E = 200 GPa, A = 20000 mm^2, and $I = 300 \times 10^6$ mm^4 for the three members.



Problem 8.2

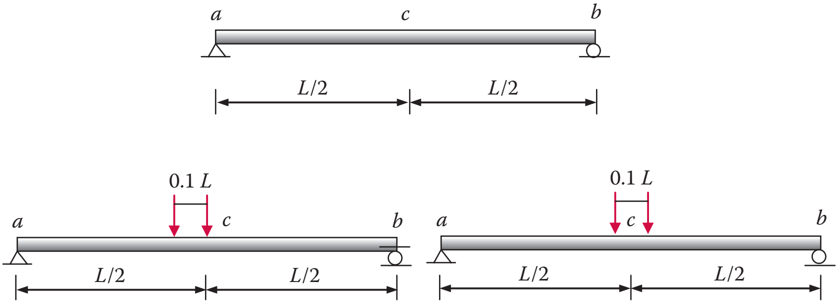
9

Influence Lines

9.1 What Is an Influence Line?

In structural design, it is often necessary to find out what the expected maximum quantity is for a selected design parameter, such as deflection at a particular point, a particular stress at a section, and so forth. The answer obviously depends on how the load is applied. The designer must apply the load in such a way that the maximum quantity for the selected parameter is obtained. The load could be concentrated loads, single or multiple, or distributed loads over a specified length or area. For a single concentrated load, it is often possible to guess where the load should be placed in order to result in a maximum quantity for a sectional moment of a beam, for example. For a multiple load, it is less likely that a correct answer can come from a guess.

Take the following beam as an example. We are interested in finding the maximum moment of section *c* for a concentrated vertical load of a unit magnitude. Intuition tells us that we should place the vertical load directly at *c*. This guess turns out to be the correct answer. If, however, we have two unit loads, one-tenth of the span, *L*, apart from each other. We have at least two possibilities as shown and intuition cannot tell us which will produce the maximum moment at *c*.



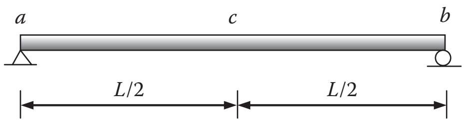
A beam with two loading possibilities to produce maximum moment at *c*.

A systematic way of finding the maximum quantity of a parameter is the influence line approach. The concept is simple: compute the response of the targeted parameter to a unit load at "every" location and plot the result against the location of the unit load. The x,y plot is the influence line. For a single concentrated load, the peak of the influence line gives the location of the load. The maximum quantity is then the product of the influence line value at the peak and the magnitude of the load. For a multiple load, the maximum is the summation of each load computed in the same way as the single load. The shape of the influence line usually reveals the locations for the multiple load. For a distributed load, the maximum is achieved by placing the load where the area under the influence line is the greatest.

To construct the influence line, it is not necessary to analyze the structure for every location of the unit load, although it can be done with a computer program for any number of selected locations. We shall introduce the analytical way of constructing influence lines.

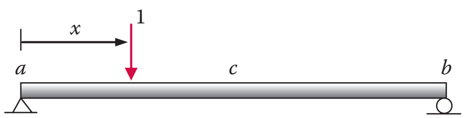
9.2 Beam Influence Lines

Consider the following beam. We wish to construct the influence lines for R_b, V_c, and M_c.



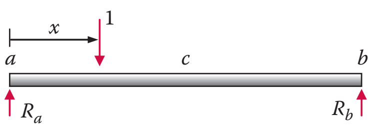
Influence lines for R_b, V_c, and M_c are to be constructed.

The problem can be defined as finding R_b, V_c, and M_c as a function of x. The location of the unit load is shown next.



Location of the unit load is the variable.

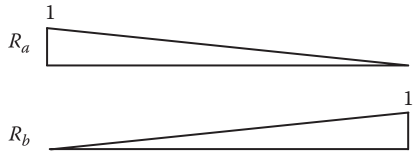
We recognize that the three parameters R_b, V_c, and M_c are all related to the reactions at a and b. Thus, we solve for R_a and R_b first.



Free-body diagram to solve for R_a and R_b.

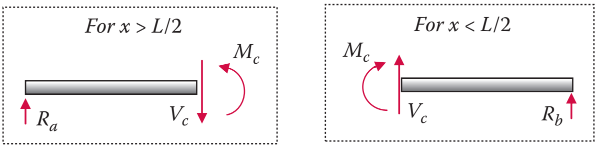
$$\Sigma M_b = 0 \rightarrow R_a = (L - x)/L; \quad \Sigma M_a = 0 \rightarrow R_b = x/L$$

Having obtained R_a and R_b as linear functions of the location of the unit load, we can plot the functions as shown next with the horizontal coordinate being x. These influence lines can be used to find influence lines of R_b, V_c, and M_c.



Influence lines for R_a and R_b.

For V_c and M_c, we need to solve for them using appropriate free-body diagrams (FBDs).

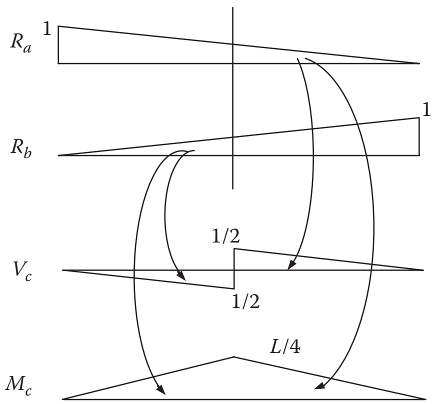


FBDs to solve for V_c and M_c.

The preceding FBDs are selected so that we do not have to include the unit load in the equilibrium equations. Consequently, the left FBD is valid for the unit load being located to the right of section c ($x > L/2$) and the right FBD is for the unit load located to the left of the section ($x < L/2$). From each FBD, we can obtain the expressions of V_c and M_c as functions of R_a and R_b.

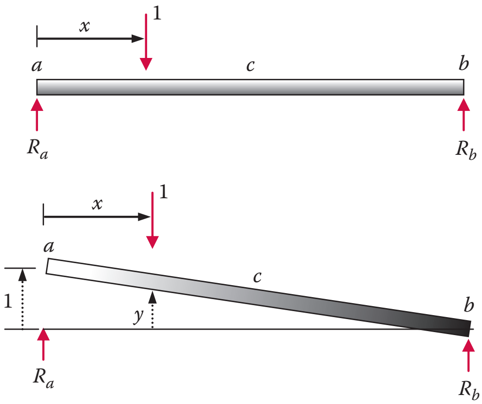
Left FBD: Valid for $x > L/2$	Right FBD: Valid for $x < L/2$
$V_c = R_a$	$V_c = -R_b$
$M_c = R_a L/2$	$M_c = R_b L/2$

Using the influence lines of R_a and R_b, we can construct the influence lines of V_c and M_c by cut and paste and adjusting for the factors $L/2$ and the negative sign.



Use the influence lines of R_a and R_b to construct the influence lines of V_c and M_c.

Müller-Breslau principle. The aforementioned process is laborious but serves the purpose of understanding the analytical way of finding solutions for influence lines. For beam influence lines, a quicker way is to apply the Müller-Breslau principle, which is derived from the virtual work principle. Consider the FBD and the same FBD with a virtual displacement shown next.



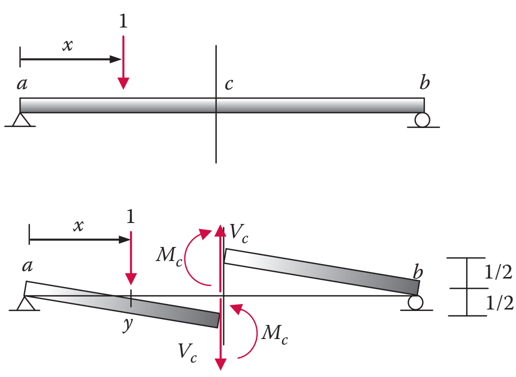
FBDs of a beam and a virtually displaced beam.

The virtual work principle states that for an equilibrium system, the work done by all forces upon a set of virtual displacement is zero. Since the only forces having a corresponding virtual displacement are R_a and the unit load, we obtain:

$$(1)\ R_a + (-y)\ 1 = 0 \quad \Longrightarrow \quad R_a = y$$

The result indicates that the influence line of R_a is numerically equal to the virtual displacement of the beam, when the virtual displacement is constructed with a unit displacement at R_a and no displacements at any forces except the unit load.

Consider one more set of virtual displacement of the beam aimed at exposing the sectional force V_c.



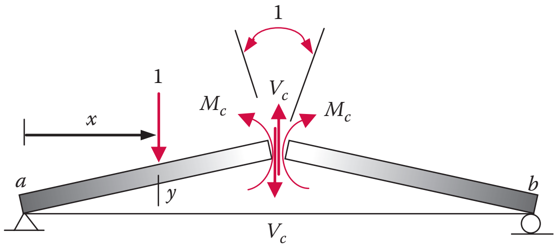
Beam and associated virtual displacement for V_c.

Application of the virtual work principle leads to

$$(1)\ V_c + (y)\ 1 = 0 \quad \Longrightarrow \quad V_c = -y$$

where y is positive if upward and negative if downward.

Another set of virtual displacement designed for solving M_c is shown next.



Virtual displacement for solving M_c.

The application of the virtual work principle leads to

$$(1)\ M_c + (-y)\ 1 = 0 \quad \Longrightarrow \quad M_c = y$$

From the above results, we can state that a properly constructed virtual displacement that does not incur any work done by any force other than the force of interest and the unit load gives the shape of the influence line for the force of interest. This is called the Müller-Breslau principle.

The step-by-step process of applying the Müller-Breslau principle can be summarized as follows:

1. Expose the quantity of interest by a cut (or remove a support).
2. Impose a virtual displacement such that
   a. At the cut there is a unit displacement (or rotation)
   b. The quantity of interest produces a positive work
   c. No other internal forces produce any work
3. The resulting displacement shape is the desired influence line.

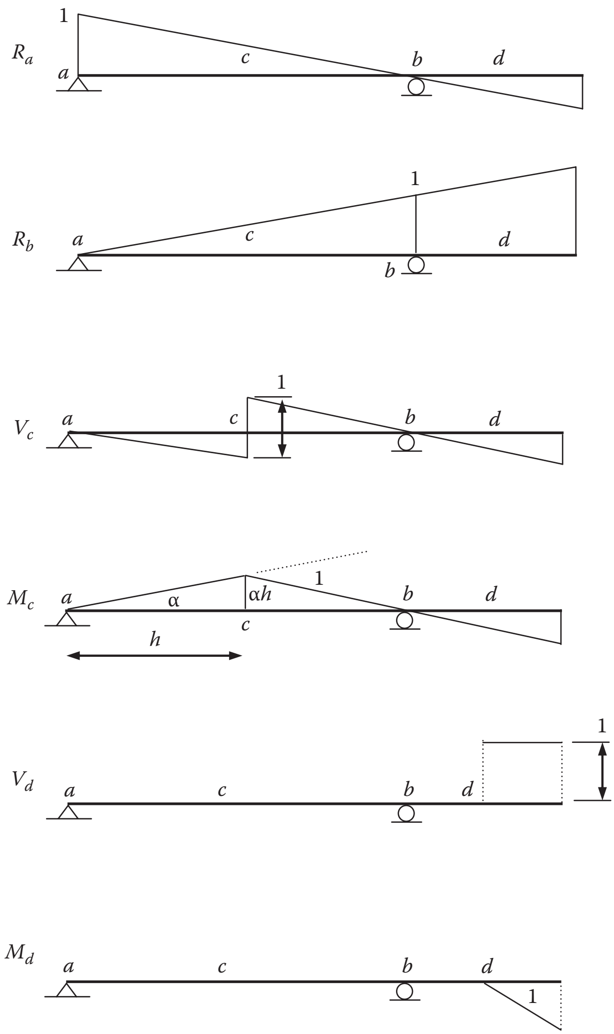
Example 9.1

Construct the influence lines for R_a, R_b, V_c, M_c, V_d, and M_d of the following beam.

Beam with an overhang.

Solution

We shall use the Müller-Breslau principle to construct the influence lines. It is a trial-and-error process to make sure the condition that no other forces produce any work is satisfied.



Influence line solutions.

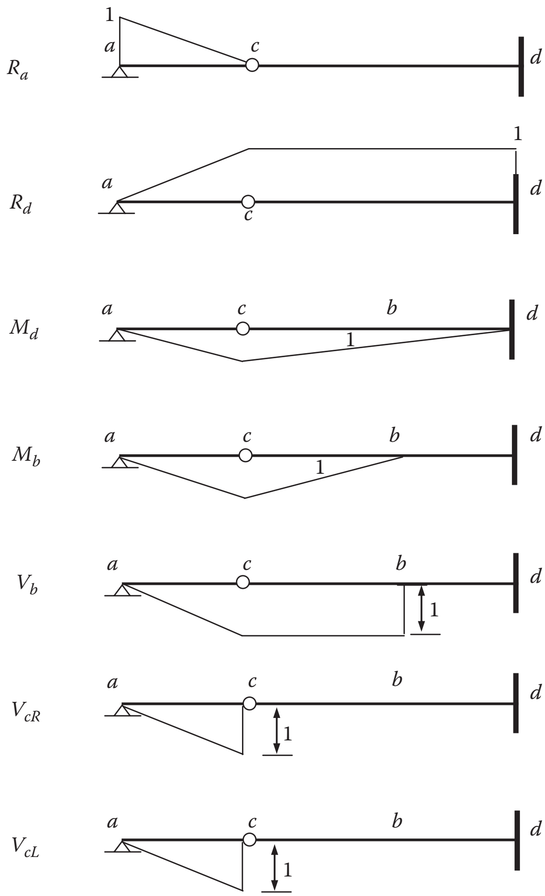
Example 9.2

Construct the influence lines for R_a, R_d, M_d, M_b, V_d, V_{cR}, and V_{cL} of the following beam.

Beam with an internal hinge.

Solution

Applications of the Müller-Breslau principle yield the following solutions.

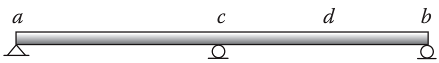


Influence line solutions.

Influence lines for statically indeterminate beam and frames. The Müller-Breslau principle is especially useful in sketching influence lines for a statically indeterminate beam or frame. The process is the same as that for a statically determinate structure but the precise shape cannot be obtained without further computation, which is very involved. We shall demonstrate only the qualitative solution process without any computations.

Example 9.3

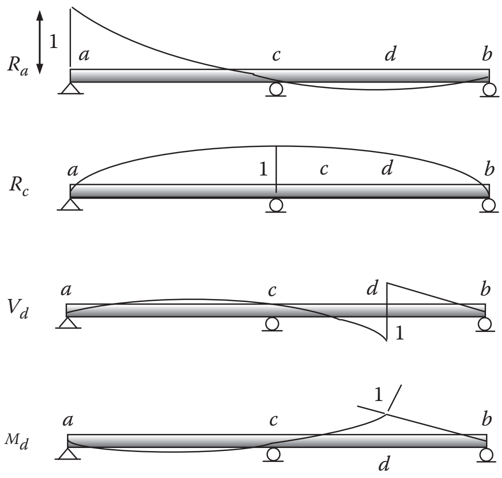
Sketch the influence lines for R_a, R_c, V_d, and M_d of the following beam.



Two-span beam example.

Solution

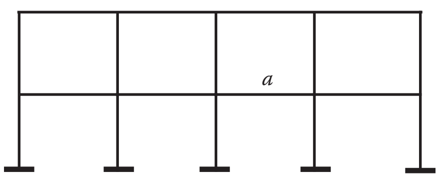
The influence lines are curved because the virtual displacements must be curved to accommodate the support constraints.



Influence line solutions.

Example 9.4

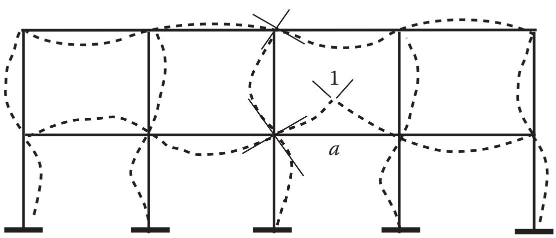
Sketch the influence lines for M_a of the following frame.



Frame example.

Solution

According to the Müller-Breslau principle, we need to make a cut at section *a* and impose a unit relative rotation at the cut. Trial and error leads to the following sketch that satisfies all constraints of the principle.



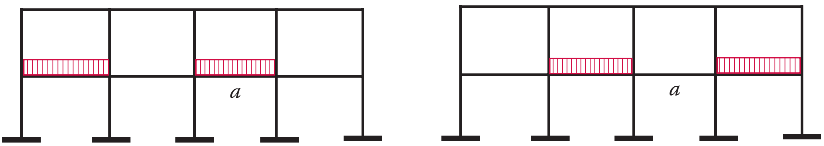
Sketch of influence line for M_a of section *a*.

Example 9.5

Place uniformly distributed loads anywhere on the second floor of the frame shown in Example 9.4 to maximize M_a.

Solution

Using the influence line of M_a as the guide, we place the load at locations as shown in the following figure for maximum positive and maximum negative moments at section *a*.

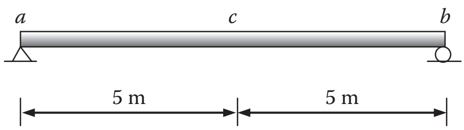


Loading pattern for maximum positive M_a (left) and negative M_a (right).

Applications of influence lines. The following examples illustrate the use of influence lines to find the maximum of a desired design parameter.

Example 9.6

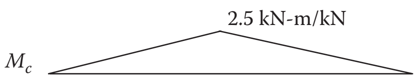
Find the maximum moment at *c* for (1) a single load of 10 kN and (2) a pair of 10 kN loads 1 m apart.



A simply supported beam.

Solution

The influence line for M_c has been obtained earlier and is reproduced next.

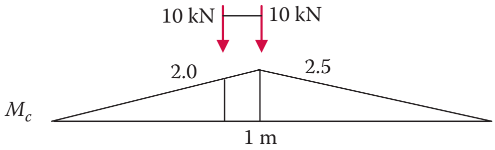


Influence line for M_c.

For a single load of 10 kN, we place it at the location of the peak of the influence line and we compute

$$(M_c)_{\max} = 10\text{ kN }(2.5\text{ kN-m/kN}) = 25\text{ kN-m}$$

For the pair of loads, we place them as shown next.

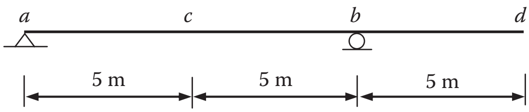


$$(M_c)_{max} = 10 \text{ kN } (2.5 \text{ kN-m/kN}) + 10 \text{ kN } (2.0 \text{ kN-m/kN}) = 45 \text{ kN-m}$$

For this case, it turns out that the pair of loads can be placed anywhere within 1 m of the center point of the beam and the resulting maximum M_c would be the same.

Example 9.7

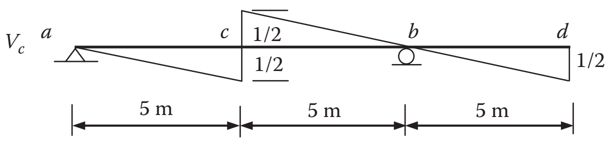
Find the maximum shear at *c* for uniformly distributed loads of intensity 10 kN/m and unlimited length of coverage.



Beam with an overhang.

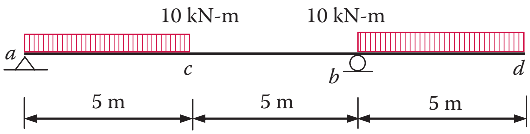
Solution

The influence line as constructed earlier is reproduced next.



Influence line for V_c.

In beam design, the sign of shear force is often not important. Thus, we want to find the maximum shear regardless of its sign. From the influence line, the following load application produces the maximum shear force.



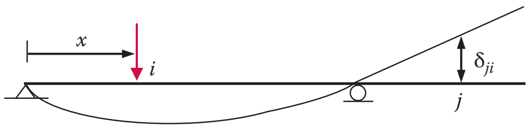
Loads to maximize V_c.

The maximum value of V_c is computed using the influence line and the area below the influence line of the loaded portion:

$$(V_c)_{max} = (-)10\left(\frac{1}{2}\right)(5)\left(\frac{1}{2}\right) + (-)10\left(\frac{1}{2}\right)(5)\left(\frac{1}{2}\right) = -25\,\text{kN}$$

Deflection influence lines. In design we need to answer the question: What is the maximum deflection of any given point on the center line of a beam? The answer is in the influence line for deflections. Surprisingly, the deflection influence line is identical to the deflection curve under a unit load applied at the point of interest.

Consider the beam and unit load configuration shown next.

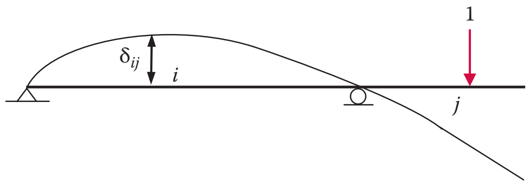


Deflection at j due to a unit load at i.

According to the Maxwell's reciprocal theorem, however,

$$\delta_{ji} = \delta_{ij}$$

And, δ_{ij} is defined in the following figure:

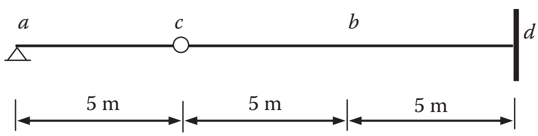


Influence line for deflection at j.

Thus, to find the deflection influence line of a point, we need only to find the deflection curve corresponding to a unit load applied at the point.

PROBLEM 9.1

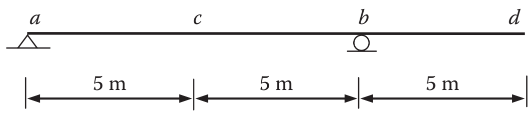
Construct the influence lines of V_b and M_d of the beam shown and find the maximum value of each for a distributed load of intensity 10 kN/m and indefinite length of coverage.



Problem 9.1

PROBLEM 9.2

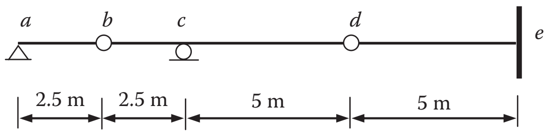
Construct the influence lines of V_{bL} and V_{bR} of the beam shown and find the maximum value of each for a distributed load of intensity 10 kN/m and indefinite length of coverage.



Problem 9.2

PROBLEM 9.3

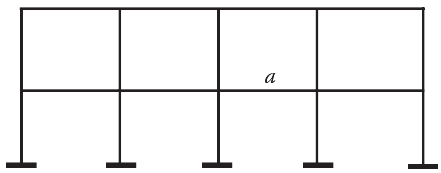
Construct the influence lines of V_{cL}, V_{cR}, M_c, and M_e of the beam shown.



Problem 9.3

PROBLEM 9.4

Sketch the influence line of V_a of the frame shown.

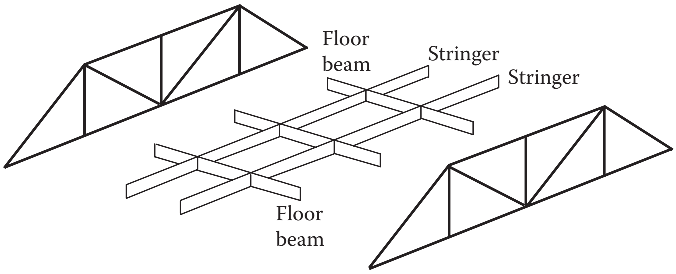


Problem 9.4

9.3 Truss Influence Lines

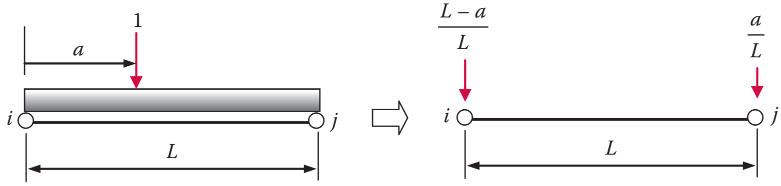
For a truss, the question relevant in design is: How does a member force change when a unit load moves along the span of the truss? The answer is again in the influence line, but the truss itself only accepts loads at the joints. Thus, we need to examine how a load, moving continuously along the truss span, transmits its force to the truss joints.

As shown in the following figure, a truss has a floor system that transmits a load from the floor slab (not shown) to the stringers, then to the floor beams. The floor beams transmit force to the joints of the truss. Thus, a plane truss accepts a load only at the joints.



Floor system of a bridge truss.

As a load is applied between the joints, the load is transmitted to the two encompassing joints by the equivalent of a simply supported beam. The resulting effect is the same as that of two forces with the magnitudes as shown acting at the two joints. The magnitude of each force is a linear function of the distance from each joint.



Transmission of force to truss joints.

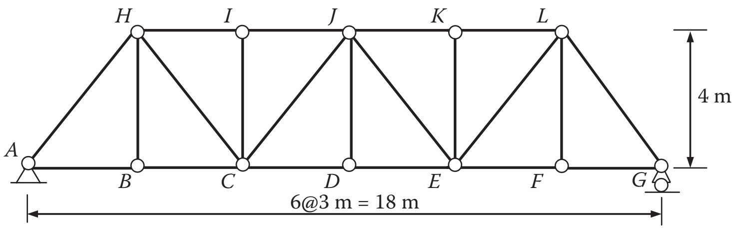
Assuming that a member force S due to a downward unit load at joint i is S_i and the member force S due to a downward unit load at joint j is S_j, then the member force due to a unit load applied between joints i and j and located from joint i by a distance of a is

$$S = \frac{L - a}{L} S_i + \frac{a}{L} S_j$$

We conclude that the force of any member due to a load applied between two joints can be computed by a linear interpolation of the member force due to the same load applied at each joint separately. The implication in constructing influence lines is that we need only to find the member force due to a unit load applied at the truss joints. When the member force is plotted against the location of the unit load, we can connect two adjacent points by a straight line.

Example 9.8

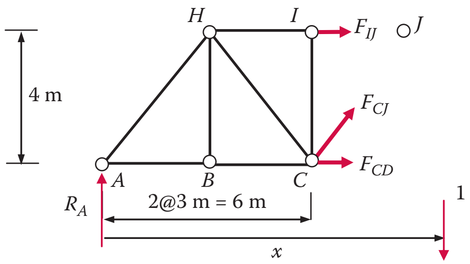
Construct the influence line of member forces F_{IJ}, F_{CD}, and F_{CJ}. Load is applied only at the level of lower chord members.



Truss example for influence line construction.

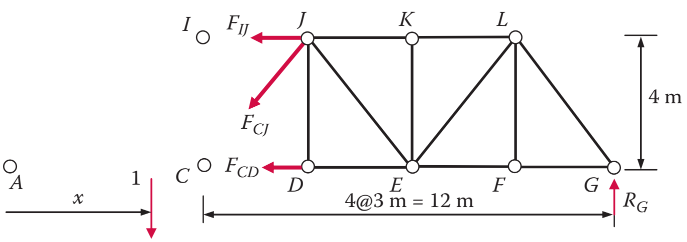
Solution

We shall use the method of sections and make a cut through I–J and C–D. Two FBDs are needed: one for loads applied to the right of the section and the other for loads to the left of the section.



FBD for unit load acting at x > 6m.

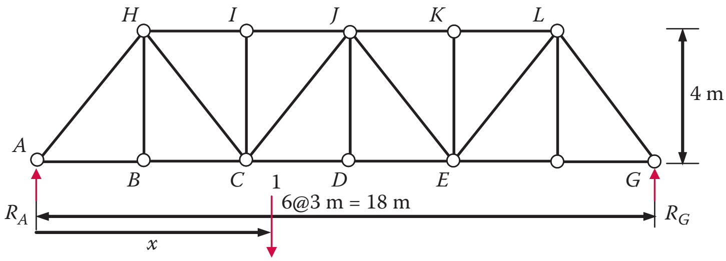
$\sum M_C = 0$	$\Longrightarrow$	$F_{IJ} = -1.5\ R_A$	($x > 6$ m)
$\sum M_J = 0$	$\Longrightarrow$	$F_{CD} = 3.0\ R_G$	($x > 9$ m)
$\sum F_y = 0$	$\Longrightarrow$	$F_{CJ} = -1.25\ R_G$	($x > 6$ m)



FBD for unit load acting at x < 6m.

$\sum M_C = 0$	$\Longrightarrow$	$F_{IJ} = -3.0\ R_G$	($x > 6$ m)
$\sum M_J = 0$	$\Longrightarrow$	$F_{CD} = 3.0\ R_G$	($x > 9$ m)
$\sum F_y = 0$	$\Longrightarrow$	$F_{CJ} = 1.25\ R_G$	($x > 6$ m)

We need to find influence lines of R_A and R_G first before we can construct the influence lines of the three members *IJ*, *CD,* and *CJ*. Using the FBD of the whole truss as shown next, we can easily obtain the expression for the two support reactions.

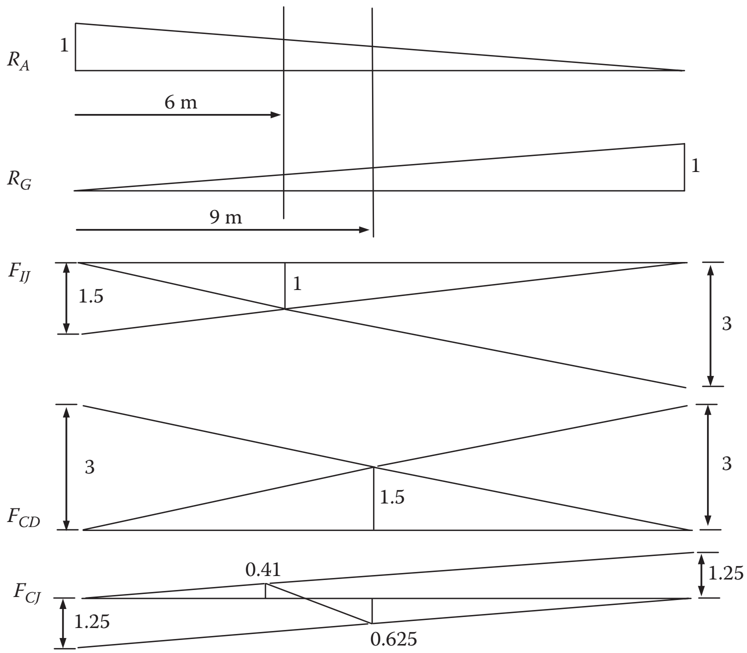


FBD of the whole truss to find reactions.

$$\Sigma M_A = 0 \quad \Rightarrow \quad R_G = \frac{x}{18}$$

$$\Sigma M_G = 0 \quad \Rightarrow \quad R_A = \frac{18 - x}{18}$$

The influence lines of the two support reactions are identical in shape to those of a simply supported beam and are shown together next with the influence lines of F_{IJ}, F_{CD}, and F_{CJ}, which are obtained by cutting and pasting and applying the proper factors to the reaction influence lines.

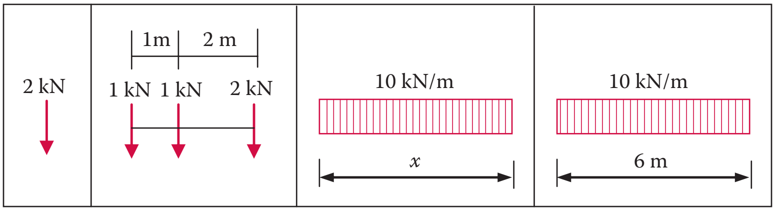


Constructing member force influence lines using support reaction influence lines.

From the preceding three influence lines, we observe that the upper chord member *IJ* is always in compression, the lower chord member *CD* is always in tension, and web member *CJ* can be in tension or compression depending on whether the load is to the left or right of the panel.

Example 9.9

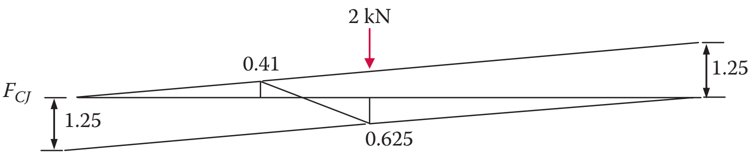
For the truss in Example 9.8, find the maximum force in member *CJ* for the four kinds of loads shown in the following figure.



A single load, a group load, and uniform loads with indefinite and finite length.

Solution

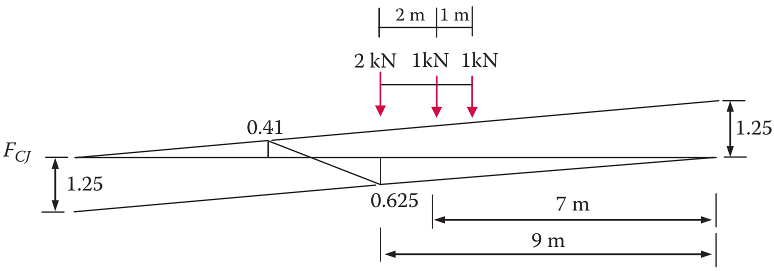
1. Single concentrated load.



Placing load at peak point on the influence line.

$$(F_{CJ})_{max} = 2(0.625) = 1.25\ \text{kN}$$

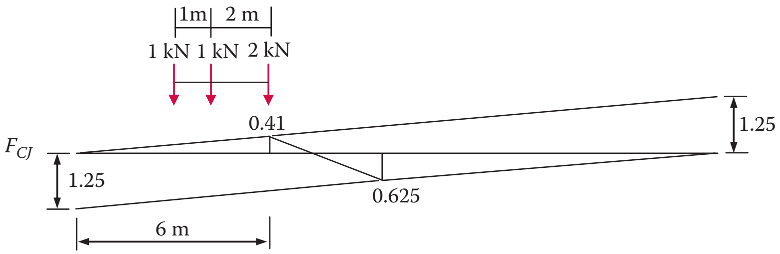
2. Group load: The group load can be applied in any orientation. Trial and error leads to the following location of the group load.



Placing the group load to maximize F_{CJ}.

$$(F_{CJ})_{max} = -[2(0.625) + 1(0.625)(7/9) + 1(0.625)(6/9)] = -2.15\ \text{kN}$$

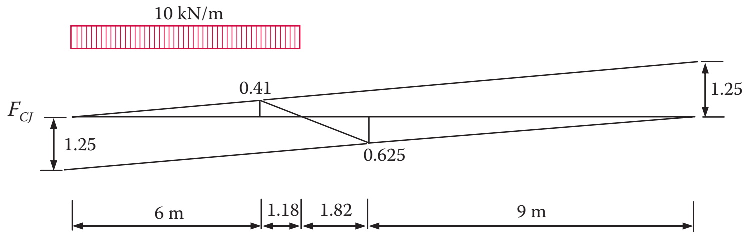
This is a compression force maximum. To find the tension force maximum, the group load is placed in a different way, as shown next.



Placing group load for maximum tension in member *CJ*.

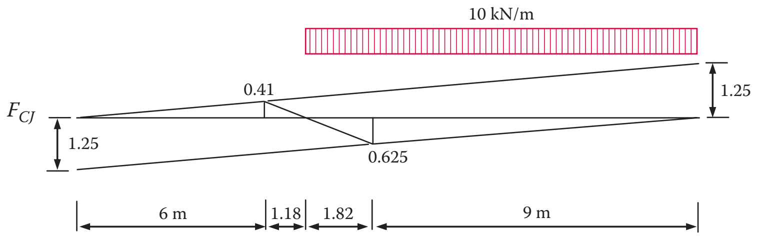
$$(F_{CJ})_{max} = [2(0.41) + 1(0.41)(4/6) + 1(0.41)(3/6)] = 1.30 \text{ kN}$$

3. Distributed load of indefinite length.



Placing distributed load for maximum tension in member *CJ*.

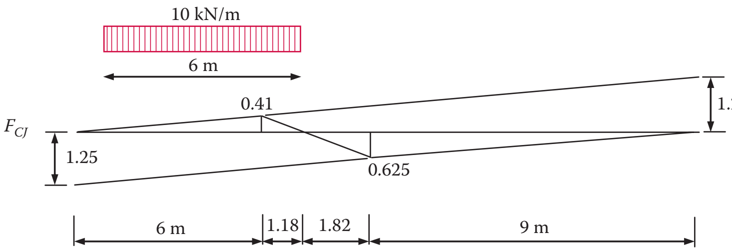
$$(F_{CJ})_{max} = 10[0.5(1.18)(0.41) + 0.5(6)(0.41)] = 14.7 \text{ kN}$$



Placing distributed load for maximum compression in member *CJ*.

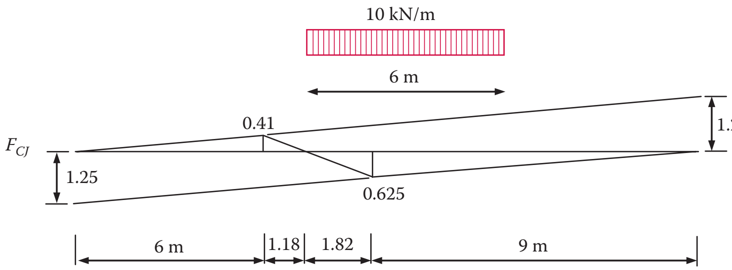
$$(F_{CJ})_{max} = -10[0.5(1.82)(0.625) + 0.5(9)(0.625)] = -33.8 \text{ kN}$$

4. Distributed load of finite length.



Placing finite length uniform load for maximum tension in member *CJ*.

$$(F_{CJ})_{max} = 10[0.5(1.18)(0.41) + 0.5(6)(0.41) - 0.5(1.18)(0.41)(1.18/6)] = 14.65 \text{ kN}$$

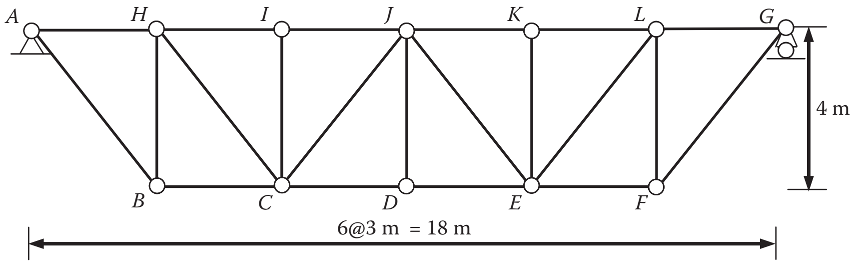


Placing finite length uniform load for maximum compression in member CJ.

$$(F_{CJ})_{max} = -10[0.5(1.82)(0.625) + 0.5(9)(0.625) - 0.5(4.82)(0.625)(4.82/9)] = -33.0 \text{ kN}$$

PROBLEM 9.5

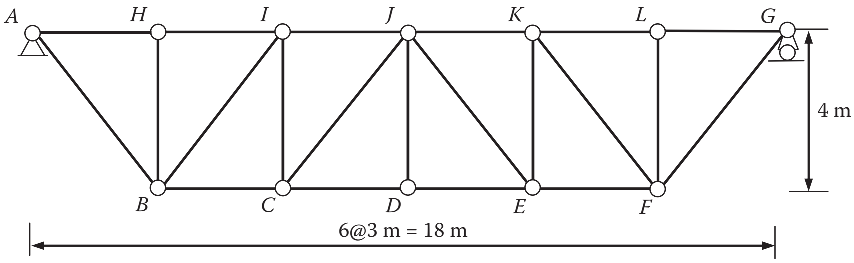
Construct the influence line of member forces F_{HI}, F_{HC}, and F_{CI}. Load is applied only at the level of the upper chord members.



Problem 9.5

PROBLEM 9.6

Construct the influence line of member forces F_{HI}, F_{BI}, and F_{CI}. Load is applied only at the level of the upper chord members.



Problem 9.6

10

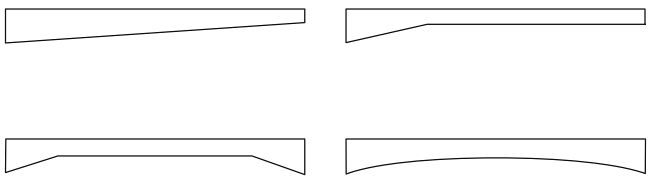
Other Topics

10.1 Introduction

The present text mainly covers the two major methods of linear structural analysis: the force method and the displacement method under static loads. There are other topics either within the realm of linear static analysis or beyond that are fundamental to structural analysis. We will briefly touch on these topics and outline the relevant issues and encourage readers to study in more depth in other courses of structural engineering or through self-study.

10.2 Non-Prismatic Beam and Frame Members

In actual structural design, especially in reinforced concrete or prestressed concrete design, the structural members often are not prismatic. Examples of configurations of non-prismatic beam or frame members are shown next.



Example configurations of non-prismatic members.

We recall that the governing equation for a prismatic beam member is:

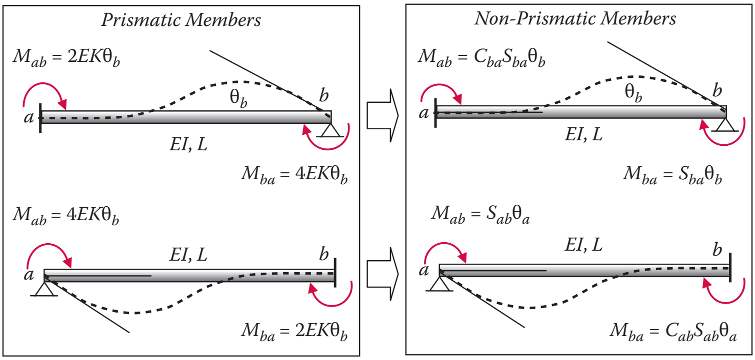
$$\frac{M}{EI} = \frac{1}{\rho} = v$$

where EI is constant. For non-prismatic members, we assume that this equation still applies but EI is treated as a variable. The integration of the equation leads to rotation and deflection:

$$\theta = v = \int \frac{M}{EI} dx$$

$$v = \iint \frac{M}{EI} dx\, dx$$

From the aforementioned equations we can derive the stiffness factors and carryover factors used in the moment distribution, slope-deflection, and matrix displacement methods. We shall not derive any of these factors for any given non-prismatic configurations herein except to point out that these factors are tabulated in handbooks of structural analysis. We do need to generalize the form of these factors as shown in the following figure.



Moment-rotation formulas for non-prismatic members—nodal rotation.

In this figure:

S_{ab} = stiffness factor of node a, equal to $4EK$ for a prismatic member

C_{ab} = carryover factor from node a to node b, equal to 0.5 for a prismatic member

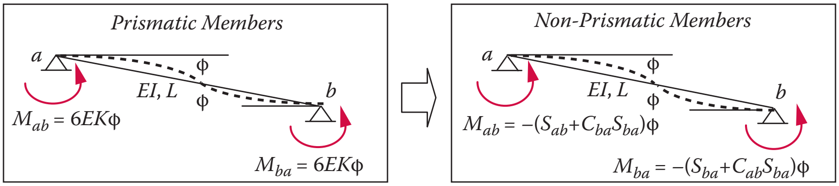
S_{ba} = stiffness factor of node b, equal to $4EK$ for a prismatic member

C_{ba} = carryover factor from node b to node a, equal to 0.5 for a prismatic member

These factors are tabulated in handbooks for commonly used non-prismatic members. We note that the fixed-end moments for any given loads between nodes are also different from those for a prismatic member and are tabulated as well. Furthermore, we state without proof the following identity.

$$C_{ab}S_{ab} = C_{ba}S_{ba} \tag{10.1}$$

The effect of member rotation, φ_{ab}, can be generalized in a similar way as shown next.



Moment-rotation formulas for non-prismatic members—member rotation.

Using the identity in Equation 10.1, the moment-rotation formulas can be recast as:

$$M_{ab} = -S_{ab}(1 + C_{ab})\varphi_{ab} \tag{10.2a}$$

$$M_{ba} = -S_{ba}(1 + C_{ba})\varphi_{ab} \tag{10.2b}$$

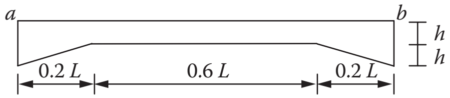
Combining the preceding formulas, we can write the moment-rotation formulas for a non-prismatic member as

$$M_{ab} = S_{ab}\theta_a + C_{ba}S_{ba}\theta_b - S_{ab}(1 + C_{ab})\varphi_{ab} + M^F_{ab} \tag{10.3a}$$

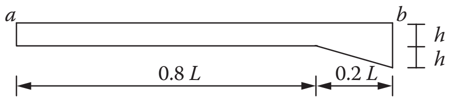
$$M_{ba} = C_{ab}S_{ab}\theta_a + S_{ba}\theta_b - S_{ba}(1 + C_{ba})\varphi_{ab} + M^F_{ba} \tag{10.3b}$$

These two equations are to be used in any displacement method of analysis. A sample of the numerical values of the factors in these two equations is given in the following table for two configurations of rectangular sections. The *EK* in the table refers to the *EK* calculated from the least sectional dimension of the member.

Stiffness and Carryover Factors and Fixed-End Moments



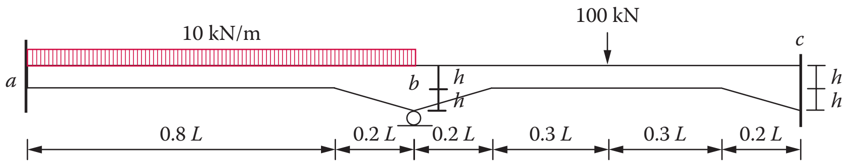
C_{ab}	C_{ba}	S_{ab}	S_{ba}	P at midspan ($L/2$, $L/2$)		w over L	
				M^F_{ab}	M^F_{ba}	M^F_{ab}	M^F_{ba}
0.691	0.691	9.08*EK*	9.08*EK*	−0.159*PL*	0.159*PL*	−0.102wL^2	0.102wL^2



C_{ab}	C_{ba}	S_{ab}	S_{ba}	P at midspan ($L/2$, $L/2$)		w over L	
				M^F_{ab}	M^F_{ba}	M^F_{ab}	M^F_{ba}
0.694	0.475	4.49*EK*	6.57*EK*	−0.097*PL*	0.188*PL*	−0.067wL^2	0.119wL^2

Example 10.1

Find all the member-end moments of the beam shown. L = 10 m.



Non-prismatic beam example.

Solution

We choose to use the slope-deflection method. There is only one degree of freedom (DOF), the rotation at node b: θ_b.

The equation of equilibrium is:

$$\Sigma M_b = 0 \Longrightarrow M_{ba} + M_{bc} = 0$$

The EK based on the minimum depth of the beam, h, is the same for both members.

The fixed-end moments are obtained from the earlier table:

For member ab:

$$M^F_{ab} = -0.067\ wL^2 = -67 \text{ kN-m}$$

$$M^F_{ba} = 0.119\ wL^2 = 119 \text{ kN-m}$$

For member bc:

$$M^F_{bc} = -0.159\ PL = -159 \text{ kN-m}$$

$$M^F_{cb} = 0.159\ PL = 159 \text{ kN-m}$$

The moment-rotation formulas are:

$$M_{ba} = C_{ab}S_{ab}\theta_a + S_{ba}\theta_b + M^F_{ba} = 6.57\text{EK}\theta_b + 119$$

$$M_{bc} = S_{bc}\theta_b + C_{cb}S_{cb}\theta_c + M^F_{bc} = 9.08\text{EK}\theta_b - 159$$

The equilibrium equation $M_{ba} + M_{bc} = 0$ becomes

$$15.65\ EK\theta_b - 40 = 0 \Longrightarrow EK\theta_b = 2.56 \text{ kN-m}$$

Substituting back to the member-end moment expressions, we obtain

$$M_{ba} = 6.57EK\theta_b + 119 = 135.8 \text{ kN-m}$$

$$M_{bc} = 9.08EK\theta_b - 159 = -135.8 \text{ kN-m}$$

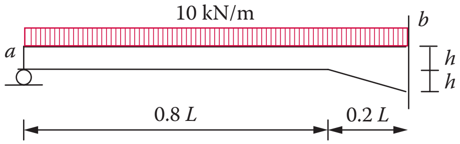
For the other two member-end moments not involved in the equilibrium equation, we have

$$M_{ab} = C_{ba}S_{ba}\theta_b + M^F_{ab} = (0.475)(6.57EK)\theta_b - 67 = -59.0 \text{ kN-m}$$

$$M_{cb} = C_{bc}S_{bc}\theta_b + M^F_{cb} = (0.694)(9.08EK)\theta_b + 159 = 175.0 \text{ kN-m}$$

PROBLEM 10.1

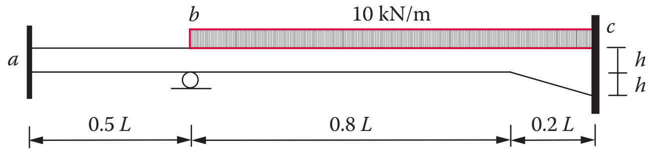
Find the reaction moment at support *b*. $L = 10$ m.



Problem 10.1

PROBLEM 10.2

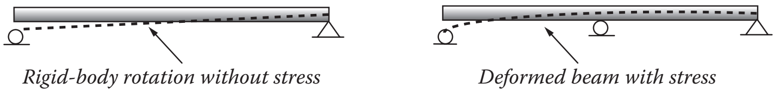
Find the reaction moment at support *c*. $L = 10$ m.



Problem 10.2

10.3 Effects of Support Movement, Temperature, and Construction Error

A structure may exhibit displacement or deflection from its intended configuration for causes other than externally applied loads. These causes are support movement, temperature effect, and construction errors. For a statically determinate structure, these causes will not induce internal stresses because the members are free to adjust to the change of geometry without the constraint from supports or from other members. In general, however, internal stresses will be induced for statically indeterminate structures.

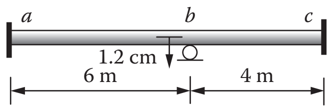


Statically determinate and indeterminate structures react differently to settlement.

Support movement. For a given support movement or settlement, a structure can be analyzed with the displacement method as shown in the following example.

Example 10.2

Find all the member-end moments of the beam shown. The amount of settlement at support b is 1.2 cm, downward. EI = 24,000 kN-m^2.



A beam with a downward settlement at support b.

Solution

We shall use the slope-deflection method. The downward settlement at support b causes member ab and member bc to have member rotations by the amount shown next:

$$\varphi_{ab} = 1.2 \text{ cm}/6 \text{ m} = 0.002 \text{ rad} \quad \text{and} \quad \varphi_{bc} = -1.2 \text{ cm}/4 \text{ m} = -0.003 \text{ rad}$$

There is only one unknown, the rotation at node b: θ_b.

The equation of equilibrium is

$$\Sigma M_b = 0 \implies M_{ba} + M_{bc} = 0$$

The stiffness factors of the two members are

$$EK_{ab} = 4000 \text{ kN-m} \quad \text{and} \quad EK_{bc} = 6000 \text{ kN-m}$$

The moment-rotation formulas are

$$M_{ba} = (4EK)_{ab}\theta_b - 6EK_{ab}\varphi_{ab} = 16{,}000\theta_b - 24{,}000(0.002)$$

$$M_{bc} = (4EK)_{bc}\theta_b - 6EK_{bc}\varphi_{bc} = 24{,}000\theta_b - 36{,}000(-0.003)$$

The equilibrium equation $M_{ba} + M_{bc} = 0$ becomes

$$40{,}000\theta_b = -60 \implies \theta_b = -0.0015 \text{ rad}$$

Substituting back to the member-end moment expressions, we obtain

$$M_{ba} = (4EK)_{ab}\theta_b - 6EK_{ab}\varphi_{ab} = 16{,}000(-0.0015) - 48 = -72 \text{ kN-m}$$

$$M_{bc} = (4EK)_{bc}\theta_b - 6EK_{bc}\varphi_{bc} = 24{,}000(-0.0015) + 108 = 72 \text{ kN-m}$$

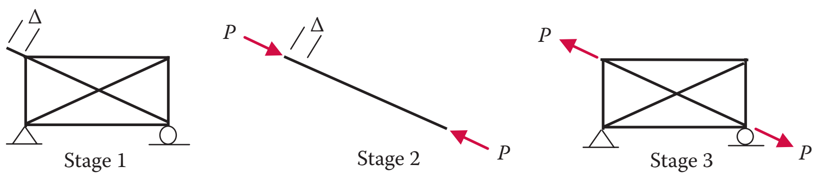
For the other two member-end moments not involved in the equilibrium equation, we have

$$M_{ab} = (2EK)_{ab}\theta_b - 6EK_{ab}\varphi_{ab} = 8000(-0.0015) - 48 = -60 \text{ kN-m}$$

$$M_{cb} = (2EK)_{bc}\theta_b - 6EK_{bc}\varphi_{bc} = 12{,}000(-0.0015) + 108 = 90 \text{ kN-m}$$

Temperature change and construction error. The direct effect of temperature change and construction or manufacturing error is the change of shape or dimension of a structural member. For a statically determinate structure, this change of shape or dimension will lead to displacement but not internal member forces. For a statically indeterminate structure, this will lead to internal forces.

An easy way of handling temperature change or manufacturing error is to apply the principle of superposition. The problem is solved in three stages. In the first stage, the structural member is allowed to deform freely for the temperature change or manufacturing error. The deformation is computed. Then, the member-end forces needed to "put back" the deformation and restore the original or designed configuration are computed. In the second stage, the member-end forces are applied to the member and restored to the original configuration. In the third stage, the applied member-end forces are applied to the structure in reverse and the structure is analyzed. The summation of the results in stage 2 and stage 3 gives the final answer for internal forces.



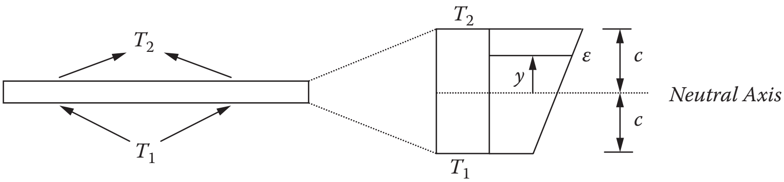
Superposition of stage 2 and stage 3 gives the effects of temperature change or construction error.

The second stage solution for a truss member is straightforward:

$$P = \frac{EA}{L}$$

where L is the original length of the member, $\Delta = \alpha L(T)$, and α is the linear thermal expansion coefficient of the material, and T is the temperature change from the ambient temperature, positive if elevated. For manufacturing error, the "misfit" Δ is measured and known.

For a beam or frame member, consider a temperature rise that is linearly distributed from the bottom of a section to the top of the section and is constant along the length of the member. The strain at any level of the section can be computed as shown:



Strain at a section due to temperature change.

The temperature distribution through the depth of the section can be represented by

$$T(y) = \frac{T_1 + T_2}{2} + \frac{T_2 - T_1}{2}\,\frac{y}{c}$$

The stress, σ, and strain, ε, are related to T by

$$\sigma = E\varepsilon = E\alpha T$$

The axial force, F, is the integration of forces across the depth of the section:

$$F = \int \sigma\, dA = \int E\alpha T\, dA = \int E\alpha \left(\frac{T_1 + T_2}{2} + \frac{T_2 - T_1}{2}\,\frac{y}{c} \right) dA = EA\alpha \left(\frac{T_1 + T_2}{2} \right)$$

The moment of the section is the integration of the product of forces and the distance from the neutral axis:

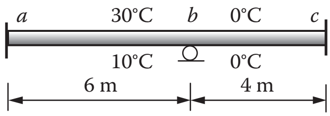
$$M = \int \sigma\, y\,dA = \int E\alpha T y\, dA = \int E\alpha \left(\frac{T_1 + T_2}{2} + \frac{T_2 - T_1}{2}\,\frac{y}{c} \right) y\,dA = EI\alpha \left(\frac{T_2 - T_1}{2c} \right)$$

Note that $(T_1 + T_2)/2 = T_{ave.}$ is the average temperature rise and $(T_2 - T_1)/2c = T'$ is the rate of temperature rise through the depth. We can write

$$F = EA\alpha\, T_{ave.} \quad \text{and} \quad M = EI\alpha\, T'$$

Example 10.3

Find all the member-end moments of the beam shown. The temperature rise at the bottom of member *ab* is 10°C and at the top is 30°C. No temperature change for member *bc*. The thermal expansion coefficient is 0.000012 m/m/°C. EI = 24,000 kN-m^2 and EA = 8,000,000 kN and the depth of the section is 20 cm for both members.



Beam experience temperature rise.

Solution

The average temperature rise and the temperature rise rate are

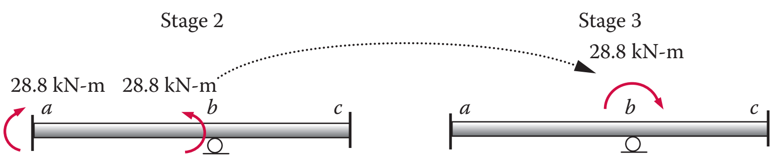
$$T_{ave.} = 20°\text{C}; \quad T' = 20°\text{C}/20 \text{ cm} = 100°\text{C/m}$$

Consequently,

$$F = EA\alpha\, T_{ave.} = (8{,}000{,}000 \text{ kN})(0.000012 \text{ m/m/°C})(20°\text{C}) = 1{,}920 \text{ kN}$$

$$M = EI\alpha\, T' = (24{,}000 \text{ kN-m}^2)(0.000012 \text{ m/m/°C})(100°\text{C/m}) = 28.8 \text{ kN-m}$$

We shall not pursue the effect of the axial force F because it does not affect the moment solution. Member *ab* would be deformed if unconstrained. The stage 2 and stage 3 problems are defined in the following figure.



Superposition of two problems.

The solution to the stage 3 problem can be obtained via the moment distribution method.

$$K_{ab} : K_{bc} = 2 : 3 = 0.4 : 0.6$$

The 28.8 kN-m moment at b is distributed in the following way:

$$M_{ba} = 0.4\ (28.8) = 11.52 \text{ kN-m}$$

$$M_{bc} = 0.6(28.8) = 17.28 \text{ kN-m}$$

The carryover moments are

$$M_{ab} = 0.5(11.52) = 5.76 \text{ kN-m}$$

$$M_{cb} = 0.5(17.28) = 8.64 \text{ kN-m}$$

The superposition of two solutions gives

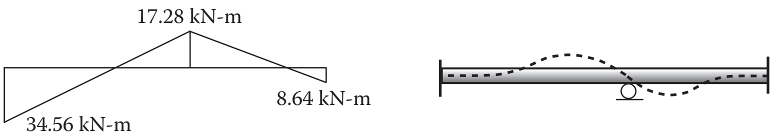
$$M_{ba} = 11.52 - 28.80 = -17.28 \text{ kN-m}$$

$$M_{bc} = 17.28 \text{ kN-m}$$

$$M_{ab} = 5.76 + 28.80 = 34.56 \text{ kN-m}$$

$$M_{cb} = 8.64 \text{ kN-m}$$

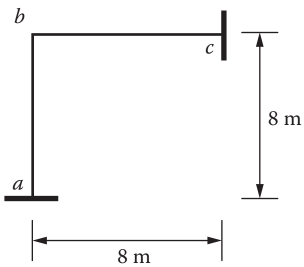
The moment and deflection diagrams are shown next.



Moment and deflection diagrams.

PROBLEM 10.3

The support at *c* of the frame shown is found to have rotated by 10 degrees in the counterclockwise direction. Find all the member-end moments. EI = 24,000 kN-m^2 for both members.

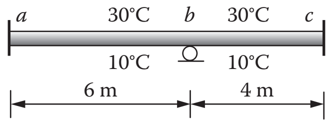


Problem 10.3

PROBLEM 10.4

Find all the member-end moments of the beam shown. The temperature rise at the bottom of the two members is 10°C and at the top is 30°C. The thermal expansion coefficient is 0.000012 m/m/°C. EI =

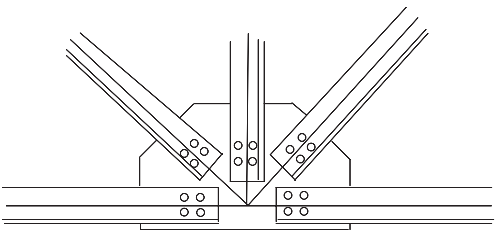
24,000 kN-m^2 and EA = 8,000,000 kN and the depth of the section is 20 cm for both members.



Problem 10.4

10.4 Secondary Stresses in Trusses

In truss analysis, the joints are treated as hinges, which allow joining members to rotate against each other freely. In actual construction, however, rarely a truss joint is made as a true hinge. The joining members at a joint are often connected to each other through a plate, called a gusset plate, either by bolts or by welding.



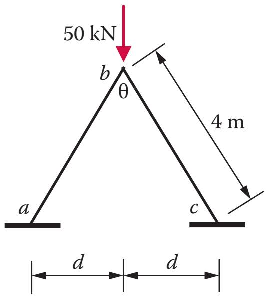
Five angle members connected by a gusset plate.

This kind of connection is closer to a rigid connection than to a hinged connection. Nonetheless, we still assume the connection can be treated as a hinge as long as external loads are applied at the joints only. This is because the triangular configuration of the truss structure minimizes any moment action in the members and the predominant force in each member is always the axial force. The stress in a truss member induced by the rigid connection is called the secondary stress, which is negligible for most practical cases. We shall examine the importance of secondary stress through an example.

Example 10.4

Find the end-moments of the two-bar truss shown, if all connections are rigid. The sections of both members are square with a side dimension of 20 cm.

E = 1000 kN/cm^2. Discuss the significance of the secondary stress for three cases: θ = 60°, 90°, and 120°.



Two-bar truss example.

Solution

For the dimension given, EI = 1,333.33 kN-m^2 and EA = 400,000 kN. If we treat the structure as a rigid frame, we shall find member-end moments in addition to axial force. If we treat the structure as a truss, we will have zero member-end moments and only axial force in each member. We shall present the truss analysis results and the frame analysis results in the table that follows. Because of symmetry, we need to concentrate on one member only. It turns out that the end-moments at both ends of member *ab* are the same. We need to examine the maximum compressive stress at node *b* as a way of evaluating the relative importance of secondary stress.

Truss and Frame Solutions

	θ = 60°		**θ = 90°**		**θ = 120°**	
Member Force/Stress Results	**Truss**	**Frame**	**Truss**	**Frame**	**Truss**	**Frame**
Member compression (kN)	28.87	28.84	35.35	35.27	50.00	49.63
Moment at end *b* (kN-cm)	0	8.33	0	17.63	0	42.98
σ due to axial force (kN/cm^2)	0.072	0.072	0.088	0.088	0.125	0.124
σ due to moment (kN/cm^2)	0	0.006	0	0.013	0	0.032
Total σ (kN/cm^2)	0.072	0.078	0.088	0.101	0.125	0.156
Error (truss result as base)		8.3%		15%		25%

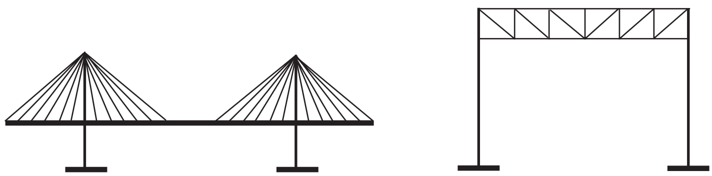
In computing the normal stress from moment, we have used the formula:

$$\sigma = \frac{Mc}{I}$$

where c is the half height of the section. We observe that the compressive stress computed from a rigid connection assumption is higher than that from the hinge connection assumption. The error becomes larger when the angle θ becomes larger. The results of the preceding analysis, however, are those for the worse possible case, because in reality node a and node c would not have been fixed completely if the basic triangle a–b–c is part of a larger truss configuration. Nonetheless, secondary stress should be considered when the angle between two joining members becomes greater than 90°.

10.5 Composite Structures

We have learned the methods of analysis for truss structures and beam/frame structures. In reality, many structures are composite structures in the sense that both truss and frame members are used in a single structure. Bridge and building structures are often composite structures as illustrated in the following figure, in which thin lines represent truss members and thick lines represent frame members.

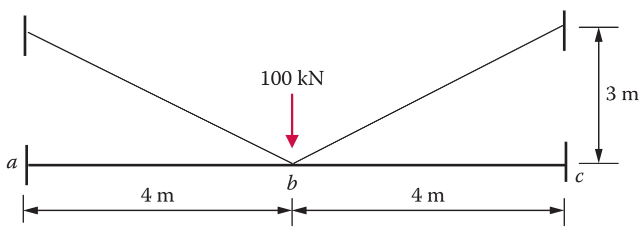


Cable-stayed bridge and building frame as composite structure examples.

The analysis of composite structures can be accomplished with either the force method or the displacement method. All computer packages allow the mixture of truss and frame members. For very simple composite structures, hand calculation can be effective as shown in the following example.

Example 10.5

As a much simplified model of a cable-stayed bridge, the composite structure shown is subjected to a single load at the center. Find the force in the cables. The cross-sectional properties: A_{cable} = 100 cm², A_{beam} = 180 cm², and I_{beam} = 19,440 cm⁴. E = 20,000 kN/cm² for both the cables and the beam. Neglect the axial deformation effect of the beam.



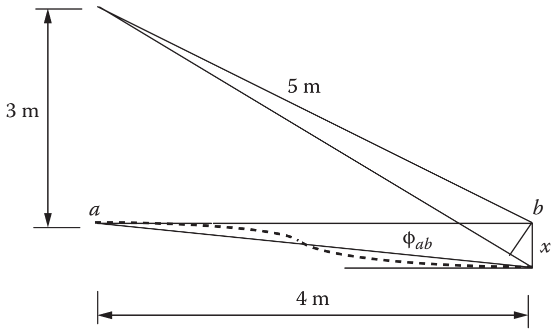
Beam-cable composite structure.

Solution

Because of symmetry, node *b* will have a downward deflection only, without a rotation. We need to concentrate on only half of the structure. Denoting the downward deflection as *x*, we observe that the elongation of the cable and the member rotation of member *ab* are related to *x*.

$$_{cable} = \frac{3}{5}x$$

$$_{ab} = \frac{1}{4}x$$



Deflected configuration.

The vertical force equilibrium at node *b* involves the shear force of the beam, the vertical component of the force in the cable, and the externally applied load.

$$F_{cable} = \frac{EA}{L} \quad _{cable} = \frac{EA}{L}\frac{3}{5} \quad x = \frac{(20000)(100)}{500}\frac{3}{5} \quad x = 2400x$$

$$(F_{cable})_{vertical} = \frac{3}{5}F_{cable} = 1440x$$

$$V_{beam} = 12\frac{EK}{L} \quad _{ab} = 12\frac{(20000)(19440)}{(400)(400)}\frac{1}{4}x = 7290x$$

The equilibrium equation for vertical forces at node *b* calls for the sum of the shear force in the beam and the vertical component of the cable force be equal to half of the externally applied load, and the equation appears as:

$$1440x + 7290x = 50 \Rightarrow x = 0.00573 \text{ cm}$$

The shear force in the beam is

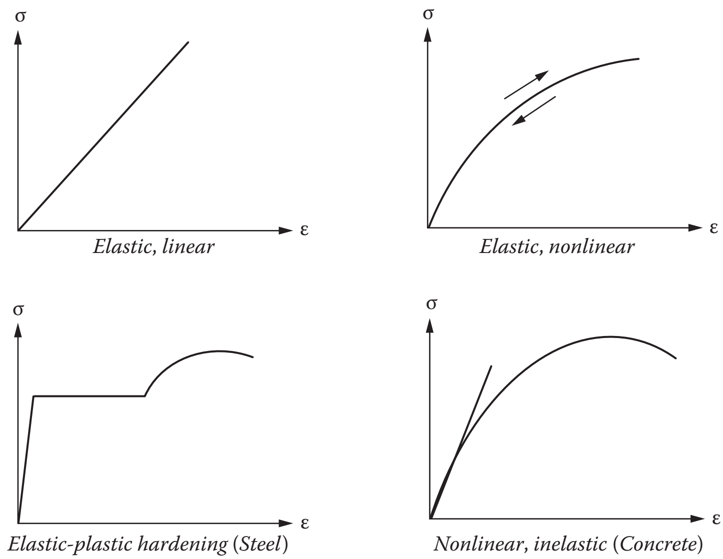
$$V_{beam} = 7290x = 41.8 \text{ kN}$$

The tension in the cables is

$$F_{cable} = 2400x = 13.8 \text{ kN}$$

10.6 Materials Nonlinearity

We have assumed that materials are linearly elastic. This means that the stress–strain relationship is proportional (linear), and when stress is removed, the strain will return to the original state of zero strain (elastic). In general, however, a stress–strain relationship can be elastic but nonlinear or inelastic and nonlinear. In truss and beam/frame analysis, we deal with only uniaxial stress–strain relationships. The following figure illustrates different uniaxial stress–strain relationships.



Various uniaxial stress–strain relationships.

The linear analysis we have been learning is valid only for linear material behavior, but, as illustrated for the concrete stress–strain relationship, a linear relationship is a good approximation if the stress–strain level is limited to a certain range. The highest level of stress that can be sustained by a material is called the ultimate strength, which is usually beyond the linear region. Present design practice does require the consideration of the ultimate strength, but the design process has been developed in such a way that a linear analysis is still useful for preliminary design. The interested reader is encouraged to study advanced strength of materials for nonlinear material behavior.

10.7 Geometric Nonlinearity

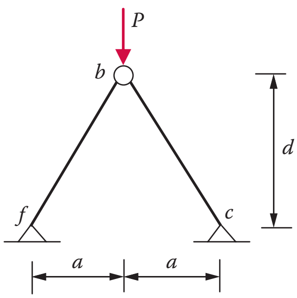
A basic assumption in the linear structural analysis is that the deflected configuration is very close to the original configuration. This is called the small-deflection assumption. With this assumption, we can use the original configuration to set up equilibrium equations. If, however, the deflection is not "small," then the error induced by the small deflection assumption could be too large to be ignored.

We will use the following example to illustrate the error of a small deflection assumption.

Example 10.6

For the two-bar truss shown, quantify the error of a small-deflection analysis on the load-deflection relationship at node *b*. The two bars are identical and are assumed to keep a constant cross-section area even under large strain.

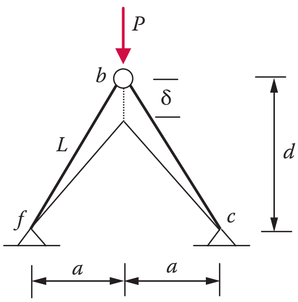
The last assumption about a constant cross-section area is to ignore the Poisson's effect and to simplify the analysis. We further assume the material remains linearly elastic so as to isolate materials' nonlinearity effect from the geometric nonlinearity effect we are investigating herein.



Two-bar truss example.

Solution

We shall derive the load-deflection relationship with and without the small-deflection assumption.



Deflected configuration as the base of the equilibrium equation.

Denote the compression force in the two bars by F; we can write the vertical force equilibrium equation at node b as

$$P = 2F_{vertical}$$

The small-deflection assumption allows us to write, using the original geometry,

$$F_{vertical} = F \frac{d}{L}$$

The bar shortening, Δ, is geometrically related to the vertical deflection at b:

$$= \delta \frac{d}{L}$$

The bar force is related to bar shortening by

$$F = \frac{EA}{L}$$

Combining the previous equations, we obtain the load-deflection (P-δ) relationship according to the small-deflection assumption:

$$P = 2\frac{EA}{L} \frac{d}{L} \delta \frac{d}{L} = 2EA \left(\frac{d}{L}\right)^3 \frac{\delta}{d} \tag{10.4}$$

For large deflection, we have to use the deflected configuration to compute bar shortening and the vertical component of the bar force.

$$F_{vertical} = F \frac{d - \delta}{L}$$

$$= L - L$$

$$F = \frac{EA}{L} = EA \left(1 - \frac{L}{L}\right)$$

$$P = 2F_{vertical} = 2\, EA \left(1 - \frac{L}{L}\right) \frac{d - \delta}{L} \tag{10.5}$$

We can express Equation 10.5 in terms of two non-dimensional geometric factors, a/d and δ/d, as shown next.

$$L = \sqrt{a^2 + (d - \delta)^2}; \qquad L = \sqrt{a^2 + d^2}$$

Dividing both sides by d, we have

$$\frac{L}{d} = \sqrt{\left(\frac{a}{d}\right)^2 + \left(1 - \frac{\delta}{d}\right)^2}; \qquad \frac{L}{d} = \sqrt{\left(\frac{a}{d}\right)^2 + 1}$$

Also,

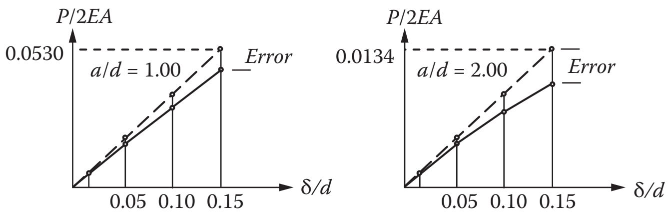
$$\frac{d-\delta}{L} = \frac{1-\delta/d}{L/d}$$

We can see that Equation 10.4 and Equation 10.5 depend on only two geometric factors: the original slope of the bar, *a*/*d*, and the deflection ratio, δ/*d*. Thus, the error of the small-deflection assumption also depends on these two factors. We study two cases of *a*/*d* and four cases of δ/*d* and tabulate the results.

Error of Small-Deflection Assumption as a Function of *a*/*d* and δ/*d*

δ/d	**0.01**	**0.05**	**0.10**	**0.15**
a/*d* = 1.00				
Equation 10.5: $\frac{P}{2EA} = 1 - \frac{L}{L} \quad \frac{d-\delta}{L}$	0.0035	0.0169	0.0325	0.0466
Equation 10.4: $\frac{P}{2EA} = \frac{d}{L}^{3} \quad \frac{\delta}{d}$	0.0035	0.0177	0.0354	0.0530
Equation 10.4/Equation 10.5	1.00	1.05	1.09	1.14
Error (%): (Equation 10.4/Equation 10.5) – 1	0%	5%	9%	14%
a/*d* = 2.00				
Equation 10.5: $\frac{P}{2EA} = 1 - \frac{L}{L} \quad \frac{d-\delta}{L}$	0.0009	0.0042	0.0079	0.0110
Equation 10.4: $\frac{P}{2EA} = \frac{d}{L}^{3} \quad \frac{\delta}{d}$	0.0009	0.0045	0.0089	0.0134
Equation 10.4/Equation 10.5	1.00	1.07	1.13	1.22
Error (%): (Equation 10.4/Equation 10.5) – 1	0%	7%	13%	22%

The results indicate that as deflection becomes increasingly larger (δ/*d* varies from 0.01 to 0.15), the small-deflection assumption introduces a larger and larger error. This error is larger for a shallower configuration (larger *a*/*d* ratio). The *P*/2*EA* values are plotted in the following figure to illustrate the size of the error. We may conclude that the small-deflection assumption is reasonable for δ/*d* less than 0.05.

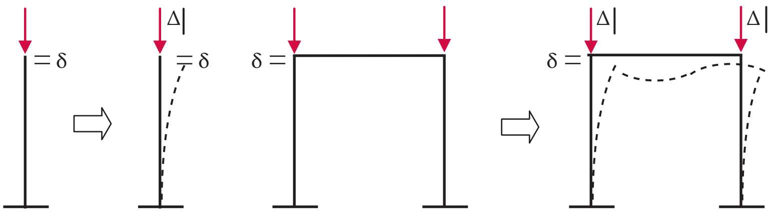


Error of small-deflection assumption.

It is clear from the figure that the load-deflection relationship is no longer linear when deflection becomes larger.

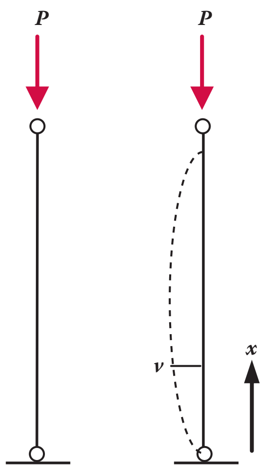
10.8 Structural Stability

In truss or frame analysis, members are often subjected to compression. If the compression force reaches a critical value, a member or the whole structure may deflect in a completely different mode. This phenomenon is called buckling or structural instability. The following figure illustrates two buckling configurations relative to the nonbuckling configurations.



Buckling configurations of a column and a frame.

Mathematically, the buckling configuration is an alternative solution to a nonbuckling solution of the governing equation. Since a linear equation has only one unique solution, a buckling solution can be found only for a nonlinear equation. We shall explore where the nonlinearity comes from via the equation of a column with hinged ends and subjected to an axial compression.



Nonbuckling and buckling configurations as solutions to the beam equation.

The governing equation of beam flexure is

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

Because of the axial load and the lateral deflection, $M = -Pv$. Thus, the governing equation becomes

$$\frac{d^2v}{dx^2} + \frac{Pv}{EI} = 0$$

This equation is linear if P is kept constant, but nonlinear if P is a variable as it is in the present case. The solution to the preceding equation is

$$v = A \text{ Sin } \sqrt{\frac{P}{EI}}\, x$$

where A is any constant. This form of solution to the governing equation must also satisfy the end conditions: $v = 0$ at $x = 0$ and $x = L$. The condition at $x = 0$ is automatically satisfied, but the condition at $x = L$ leads to either

$$A = 0$$

or

$$\sin \sqrt{\frac{P}{EI}}\, L = 0$$

The former is the nonbuckling solution. The latter, with $A \neq 0$, is the buckling solution, which exists only if

$$\sqrt{\frac{P}{EI}} L = n\pi, \qquad n = 1, 2, 3, \ldots$$

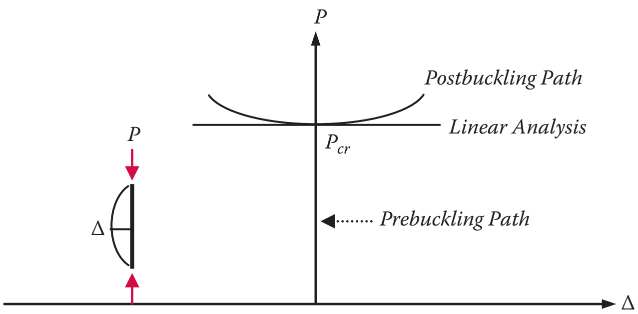
The load levels at which a buckling solution exists are called the critical loads:

$$P_{cr} = \frac{n^2\pi^2}{L^2} EI \qquad n = 1, 2, 3 \cdots$$

The lowest critical load is the buckling load.

$$P_{cr} = \frac{\pi^2}{L^2} EI$$

The preceding derivation is based on the small deflection assumption and the analysis is called linear buckling analysis. If the small deflection assumption is removed, then a nonlinear buckling analysis can be followed. The linear analysis can identify the critical load at which buckling is to occur but cannot trace the load–lateral deflection relationship on the postbuckling path. Only a nonlinear buckling analysis can produce the postbuckling path. Interested readers are encouraged to study structural stability to learn about a full spectrum of stability problems, elastic and inelastic, linear and nonlinear.



Linear and nonlinear buckling analyses results.

10.9 Dynamic Effects

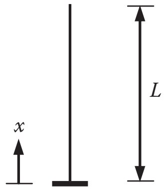
In all the previous analyses, the load is assumed to be static. This means a load is applied slowly so that the resulting deflection of the structure also occurs slowly, and the velocity and acceleration of any point of the structure during the deflection process are small enough to be neglected. How slow is slow? What if velocity and acceleration cannot be neglected?

We know from Newton's second law, or the derivative of it, that the product of a mass and its acceleration constitutes an inertia term equivalent to force. In an equilibrium system, this term, called D'Alembert force, can be treated as a negative force and all the static equilibrium equations would apply. From physics, we learn that a moving subject often encounters resistance either from within the subject or from the medium it is moving through. This resistance, called damping, in its simplest form, can be represented by the product of the velocity of the subject and a constant. Including both the inertia term and the damping term in the equilibrium equations of a structure is necessary for responses of a structure excited by wind, blast, earthquake excitations, or any sudden movement of the support or part of the structure. The dynamics effects are effects caused by the presence of the inertia and the damping in a structural system and the associated motion of the structure is called vibration. The equilibrium, including dynamic effects, is called dynamic equilibrium.

It is not easy to quantify an excitation as a static one, but it is generally true that the dynamic effects can be neglected if the excitation is gradual in the sense that it takes an order longer to complete than the natural vibration period of the structure. The concept of the natural vibration period can be easily illustrated by an example.

Example 10.7

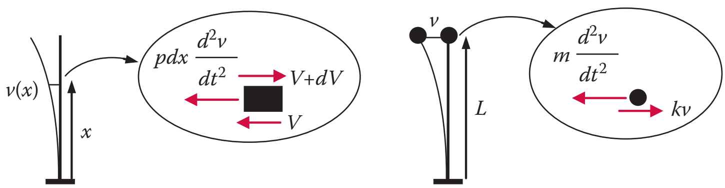
Find the natural vibration period of a cantilever beam as shown. *EI* is constant and the mass is uniformly distributed with a density ρ per unit length of the beam. Assume there is no damping in the system.



A cantilever beam with uniformly distributed mass.

Solution

We shall limit ourselves to exploring the lateral vibration of the beam, although the beam can also have vibration in the axial direction. A rigorous analysis would consider the dynamic equilibrium of a typical element moving laterally. The resulting governing equation would be a partial differential equation with two independent variables, a spatial variable, *x*, and a time variable, *t*. The system would have infinite degrees of freedom because the spatial variable, *x*, is continuous and represents an infinite number of points along the beam. We shall pursue an approximate analysis by lumping the total mass of the beam at the tip of the beam. This results in a single degree of freedom (SDOF) system because we need to consider dynamic equilibrium only at the lumped mass at the tip.



Dynamic equilibrium of a distributed mass system and a lumped mass system.

The dynamic equilibrium of this SDOF system is shown in the preceding figure. The dynamic equilibrium equation of the lumped mass is

$$m\frac{d^2v}{dt^2} + kv = 0 \tag{10.6}$$

where $m = \rho L$ and k is the force acting on the lumped mass per unit length of lateral deflection at the tip. We learn from beam analysis that the force at the tip of the beam needed to produce a unit tip deflection is $3EI/L^3$, thus $k = 3EI/L^3$.

An equivalent form of Equation 10.6 is

$$\frac{d^2v}{dt^2} + \frac{k}{m}v = 0 \tag{10.6}$$

The factor associated with v in the equation is a positive quantity and can be represented by

$$\omega = \sqrt{\frac{k}{m}} \tag{10.7}$$

Then Equation 10.6 can be put in the following form:

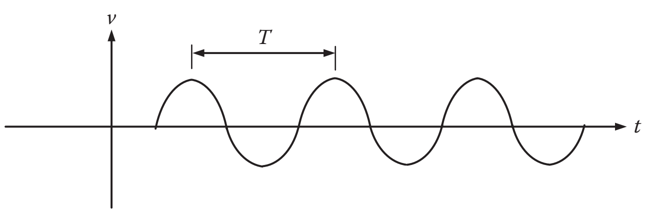
$$\frac{d^2v}{dt^2} + \omega^2 v = 0 \tag{10.8}$$

The general solution of Equation 10.8 is

$$v = A\ Sin\ n\omega t + B\ Cos\ n\omega t, \quad n = 1, 2, 3, \ldots \tag{10.9}$$

The constants A and B are to be determined by the position and velocity at $t = 0$. No matter the conditions, which are called initial conditions, the time variation of the lateral deflection at the tip is sinusoidal or harmonic with a frequency of $n\omega$. The lowest frequency, ω, for $n = 1$, is called the fundamental frequency of natural vibration. The other frequencies are frequencies of higher harmonics. The motion, plotted against time, is periodic with a period of T:

$$T = \frac{2\pi}{\omega} \tag{10.10}$$



Harmonic motion with a period T.

In the present case, if EI = 24,000 kN-m², L = 6 m, and ρ = 100 kg/m, then $k = 3EI/L^3$ = 333.33 kN/m, $m = \rho L$ = 600kg, and ω^2 = k/m = 0.555 (kN/m.kg) = 555 (1/sec²). The fundamental vibration frequency is ω = 23.57 rad/sec and the fundamental vibration period is T = 0.266 sec. The inverse of T, denoted by f, is called the circular frequency:

$$f = \frac{1}{T} \tag{10.11}$$

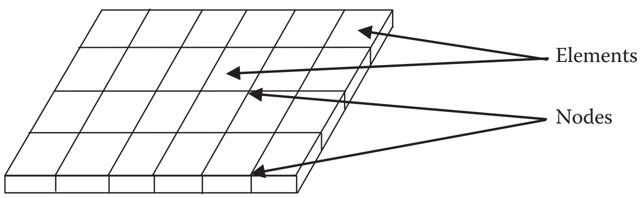
which has the unit of circle per second (cps), which is often referred to as Hertz or Hz. In the present example, the beam has a circular frequency of 3.75 cps or 3.75 Hz.

Interested readers are encouraged to study structural dynamics, in which undamped vibration, damped vibration, free vibration and forced vibration of SDOF systems, multidegree-of-freedom (MDOF) systems, and other interesting and useful subjects are explored.

10.10 Finite Element Method

The types of structures considered so far are trusses, beams, and frames. In practical structural analysis, even a simple building has elements such as floor slabs that cannot be analyzed by the methods introduced in this book. For structures with more general geometry than mere beams and frames, an effective analytical tool is the finite element method.

We may view the finite element method as a mathematical solution to certain types of differential equations or as a generalized method of matrix analysis of structures. The most popular type of the finite element method is the generalized stiffness analysis method. It follows the same procedure as the matrix displacement method we have introduced earlier in the book. One major difference is a structure other than truss, beam, or frame must be divided first into a finite number of elements connected to each other through nodes, as shown next for a flat plate.



Finite element grid for a flat plate.

Once the finite element grid is completed, the rest of the procedure parallels that of the matrix displacement method. The finite element method is also an approximate method in that the solution produced is an approximate solution, which is close to the "exact" solution when the size of elements is small and the number of elements is large. Because the finite element method can be applied to virtually any shape of structures it is widely used in practical structural analysis. Numerous computer programs are commercially available with interactive graphics and automated grid generation. Readers are encouraged to take a course on the finite element method.

Appendix A: Matrix Algebra Review

A.1 What Is a Matrix?

A matrix is a two-dimensional array of numbers or symbols that follows a set of operating rules. A matrix having m rows and n columns is called a matrix of order m-by-n and can be represented by a bold-faced letter with subscripts representing row and column numbers, for example, $\boldsymbol{A}_{3\times 7}$. If m = 1 or $n = 1$, then the matrix is called a row matrix or a column matrix, respectively. If $m = n$, then the matrix is called a square matrix. If $m = n = 1$, then the matrix is degenerated into a scalar.

Each entry of the two-dimensional array is called an element, which is often represented by a plain letter or a lowercase letter with subscripts representing the locations of the row and column in the matrix. For example, a_{23} is the element in matrix $\boldsymbol{A}$ located at the second row and the third column. Diagonal elements of a square matrix $\boldsymbol{A}$ can be represented by a_{ii}. A matrix with all elements equal to zero is called a null matrix. A square matrix with all nondiagonal elements equal to zero is called a diagonal matrix. A diagonal matrix with all the diagonal elements equal to one is called a unit or identity matrix and is represented by $\boldsymbol{I}$. A square matrix whose elements satisfy $a_{ij} = a_{ji}$ is called a symmetric matrix. An identity matrix is also a symmetric matrix. A transpose of a matrix is another matrix with all the row and column elements interchanged: $(a^{\mathrm{T}})_{ij} = a_{ji}$. The order of a transpose of an m-by-n matrix is n-by-m. A symmetric matrix is one whose transpose is the same as the original matrix: $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$. A skew matrix is a square matrix satisfying $a_{ij} = -a_{ji}$. The diagonal elements of a skew matrix are zero.

EXERCISE A.1

Fill in the blanks in the sentences below.

$$\boldsymbol{A} = \begin{matrix} 2 & 4 \\ 7 & 3 \\ 1 & 10 \end{matrix} \qquad \boldsymbol{B} = \begin{matrix} 2 & 7 & 1 \\ 4 & 3 & 10 \end{matrix} \qquad \boldsymbol{C} = \begin{matrix} 2 & 1 & 3 \\ 1 & 5 & 4 \\ 3 & 4 & 8 \end{matrix}$$

$$\boldsymbol{D} = \begin{matrix} 2 \\ 5 \\ 7 \end{matrix} \qquad \boldsymbol{E} = \begin{matrix} 2 & 5 & 7 \end{matrix} \qquad \boldsymbol{F} = \begin{matrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{matrix}$$

$$\mathbf{G} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \qquad \mathbf{H} = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \mathbf{K} = \begin{matrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{matrix}$$

Matrix ***A*** is a ____-by-____ matrix and matrix ***B*** is a ____-by-____ matrix.
Matrix ***A*** is the ________________ of matrix ***B*** and vice versa.
Matrices ***C*** and ***F*** are ____________ matrices with __________ rows and __________ columns.
Matrix ***D*** is a __________ matrix and matrix ***E*** is a ________ matrix; ***E*** is the ____________ of ***D***.
Matrix ***G*** is an __________ matrix; matrix ***H*** is a ________ matrix; matrix ***K*** is a _________ matrix.
In the example, there are ______ symmetric matrices and they are ______________________.

A.2 Matrix Operating Rules

Only matrices of the same order can be added to or subtracted from each other. The resulting matrix is of the same order with an element-to-element addition or subtraction from the original matrices.

$$\mathbf{C} + \mathbf{F} = \begin{matrix} 2 & 1 & 3 \\ 1 & 5 & 4 \\ 3 & 4 & 8 \end{matrix} + \begin{matrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{matrix} = \begin{matrix} 4 & 1 & 3 \\ 1 & 10 & 4 \\ 3 & 4 & 16 \end{matrix}$$

$$\mathbf{C} - \mathbf{F} = \begin{matrix} 2 & 1 & 3 \\ 1 & 5 & 4 \\ 3 & 4 & 8 \end{matrix} - \begin{matrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{matrix} = \begin{matrix} 0 & 1 & 3 \\ 1 & 0 & 4 \\ 3 & 4 & 0 \end{matrix}$$

The following operations using matrices defined earlier are not admissible: $\mathbf{A} + \mathbf{B}$, $\mathbf{B} + \mathbf{C}$, $\mathbf{D} - \mathbf{E}$, and $\mathbf{D} - \mathbf{G}$.

Multiplication of a matrix by a scalar results in a matrix of the same order with each element multiplied by the scalar. Multiplication of a matrix by another matrix is permissible only if the column number of the first matrix matches with the row number of the second matrix, and the resulting matrix has the same row number as the first matrix and the same column number as the second matrix. In symbols, we can write

$$\mathbf{B} \quad \mathbf{D} \quad \mathbf{Q} \text{ and } \mathbf{Q}_{ij} \quad \overset{3}{\underset{k \quad 1}{}} \; B_{ik} D_{kj}$$

Using the numbers given earlier we have

$$\boldsymbol{Q} = \boldsymbol{B} \times \boldsymbol{D} = \boldsymbol{BD} = \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & 10 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \times 2 + 7 \times 5 + 1 \times 7 \\ 4 \times 2 + 3 \times 5 + 10 \times 7 \end{bmatrix}$$

$$= \begin{bmatrix} 46 \\ 113 \end{bmatrix}$$

$$\boldsymbol{P} = \boldsymbol{Q} \times \boldsymbol{E} = \boldsymbol{QE} = \begin{bmatrix} 46 \\ 113 \end{bmatrix} \begin{bmatrix} 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 92 & 230 & 322 \\ 226 & 565 & 791 \end{bmatrix}$$

We can verify numerically that

$$\boldsymbol{P} = \boldsymbol{QE} = \boldsymbol{BDE} = (\boldsymbol{BD})\boldsymbol{E} = \boldsymbol{B}(\boldsymbol{DE})$$

We can also verify multiplying any matrix by an identity matrix of the right order will result in the same original matrix, thus the name identity matrix.

The transpose operation can be used in combination with multiplication in the following way, which can be easily derived from the definition of the two operations.

$$(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} \text{ and } (\boldsymbol{ABC})^{\mathrm{T}} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$$

EXERCISE A.2

Complete the following operations.

$$\boldsymbol{EB} = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 2 & 7 & 1 \\ 4 & 3 & 10 \end{bmatrix}$$

$$\boldsymbol{DE} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} \begin{bmatrix} 2 & 5 & 7 \end{bmatrix}$$

A.3 Matrix Inversion and Solving Simultaneous Algebraic Equations

A square matrix has a characteristic value called determinant. The mathematical definition of a determinant is difficult to express in symbols, but we can easily learn the way of computing the determinant of a matrix by the following examples. We shall use *Det* to represent the value of a determinant. For example, *Det**A*** means the determinant of matrix ***A***.

$$Det\,[5] = 5$$

$$Det \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix} = 5 \times Det\,[6] - 3 \times Det\,[2] = 30 - 6 = 24$$

$$Det \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = 1 \times Det \begin{bmatrix} 5 & 8 \\ 6 & 9 \end{bmatrix} - 2 \times Det \begin{bmatrix} 4 & 7 \\ 6 & 9 \end{bmatrix} + 3 \times Det \begin{bmatrix} 4 & 7 \\ 5 & 8 \end{bmatrix}$$

$$= 1 \times (-3) - 2 \times (-6) + 3 \times (-3) = 0$$

A matrix with a zero determinant is called a singular matrix. A nonsingular matrix $\boldsymbol{A}$ has an inverse matrix $\boldsymbol{A}^{-1}$, which is defined by

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

We can verify that the two symmetric matrices at the left-hand side (LHS) of the following equations are inverse to each other.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} 31/3 & -10/3 & -3 \\ -10/3 & 4/3 & 1 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 31/3 & -10/3 & -3 \\ -10/3 & 4/3 & 1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is because the transpose of an identity matrix is also an identity matrix and

$$(\boldsymbol{AB}) = \boldsymbol{I} \Rightarrow (\boldsymbol{AB})^{\mathrm{T}} = (\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}) = (\boldsymbol{BA}) = \boldsymbol{I}^{\mathrm{T}} = \boldsymbol{I}$$

The above statement is true only for symmetric matrices.

There are different algorithms for finding the inverse of a matrix. We shall introduce one that is directly linked to the solution of simultaneous equations. In fact, we shall see matrix inversion is an operation more involved than solving simultaneous equations. Thus, if solving simultaneous equations is our goal, we need not go through a matrix inversion first.

Consider the following simultaneous equations for three unknowns.

$$x_1 + x_2 + 2x_3 = 1$$

$$x_1 + 4x_2 - x_3 = 0$$

$$2x_1 - x_2 + 8x_3 = 0$$

The matrix representation of this is

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Imagine we have two additional sets of problems with three unknowns and the same coefficients in the LHS matrix but different right-hand side (RHS) figures.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Since the solutions for the three problems are different, we should use different symbols for them. But, we can put all three problems in one single matrix equation.

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or,

$$\boldsymbol{AX} = \boldsymbol{I}$$

By definition, $\boldsymbol{X}$ is the inverse of $\boldsymbol{A}$. The first column of $\boldsymbol{X}$ contains the solution to the first problem, and the second column contains the solution to the second problem, and so on. To find $\boldsymbol{X}$, we shall use a process called Gaussian elimination, which has several variations. We shall present two variations. The Gaussian process uses each equation (row in the matrix equation) to combine with another equation in a linear way to reduce the equations to a form from which a solution can be obtained.

1. *The first version.* We shall begin by a forward elimination process, followed by a backward substitution process. The changes as the result of each elimination or substitution are reflected in the new content of the matrix equation.

   Forward elimination. Row 1 is multiplied by –1 and added to row 2 to replace row 2, and row 1 is multiplied by –2 and added to row 3 to replace row 3, resulting in:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & -3 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Row 2 is added to row 3 to replace row 3, resulting in:

$$\begin{matrix} 1 & 1 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} = \begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 1 & 1 \end{matrix}$$

The forward elimination is completed and all elements below the diagonal line in ***A*** are zero.

Backward substitution. Row 3 is multiplied by 3 and added to row 2 to replace row 2, and row 3 is multiplied by –2 and added to row 1 to replace row 1, resulting in:

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} = \begin{matrix} 7 & -2 & -2 \\ -10 & 4 & 3 \\ -3 & 1 & 1 \end{matrix}$$

Row 2 is multiplied by –1/3 and added to row 1 to replace row 1, resulting in:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} = \begin{matrix} 31/3 & -10/3 & -3 \\ -10 & 4 & 3 \\ -3 & 1 & 1 \end{matrix}$$

Normalization. Now that matrix ***A*** is reduced to a diagonal matrix, we further reduce it to an identity matrix by dividing each row by the diagonal element of each row, resulting in:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} = \begin{matrix} 31/3 & -10/3 & -3 \\ -10/3 & 4/3 & 1 \\ -3 & 1 & 1 \end{matrix}$$

or

$$\boldsymbol{X} = \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} = \begin{matrix} 31/3 & -10/3 & -3 \\ -10/3 & 4/3 & 1 \\ -3 & 1 & 1 \end{matrix}$$

Note that ***X*** is also symmetric. It can be derived that the inverse of a symmetric matrix is also symmetric.

2. *The second version.* We combine the forward and backward operations and the normalization together to reduce all off-diagonal terms to zero, one column at a time. We reproduce the original matrix equation below.

$$\begin{matrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{matrix} \quad \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} \quad = \quad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Starting with the first row, we normalize the diagonal element of the first row to one (in this case, it is already one) by dividing the first row by the value of the diagonal element. Then we use the new first row to eliminate the first column elements in row 2 and row 3, resulting in

$$\begin{matrix} 1 & 1 & 2 \\ 0 & 3 & -3 \\ 0 & -3 & 4 \end{matrix} \quad \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} \quad = \quad \begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{matrix}$$

We repeat the same operation with the second row and the diagonal element of the second row to eliminate the second column elements in row 1 and row 3, resulting in

$$\begin{matrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} \quad = \quad \begin{matrix} 4/3 & -1/3 & 0 \\ -1/3 & 1/3 & 0 \\ -3 & 1 & 1 \end{matrix}$$

The same process is done using the third row and the diagonal element of the third row, resulting in

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} \quad = \quad \begin{matrix} 31/3 & -10/3 & -3 \\ -10/3 & 4/3 & 1 \\ -3 & 1 & 1 \end{matrix}$$

or

$$\boldsymbol{X} = \begin{matrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{matrix} \quad = \quad \begin{matrix} 31/3 & -10/3 & -3 \\ -10/3 & 4/3 & 1 \\ -3 & 1 & 1 \end{matrix}$$

The same process can be used to find the solution for any given column on the RHS, without finding the inverse first. This is left to readers as an exercise.

EXERCISE A.3

Solve the following problem by the Gaussian elimination method.

$$\begin{matrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \quad = \quad \begin{matrix} 3 \\ 6 \\ 1 \end{matrix}$$

Forward elimination. Row 1 is multiplied by –1 and added to row 2 to replace row 2, and row 1 is multiplied by –2 and added to row 3 to replace row 3, resulting in:

$$\begin{matrix} 1 & 1 & 2 \\ 0 & 3 & -3 \\ 0 & -3 & 4 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \quad =$$

Row 2 is added to row 3 to replace row 3, resulting in:

$$\begin{matrix} 1 & 1 & 2 \\ 0 & 3 & -3 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \quad =$$

Backward substitution. Row 3 is multiplied by 3 and added to row 2 to replace row 2, and row 3 is multiplied by –2 and added to row 1 to replace row 1, resulting in:

$$\begin{matrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \quad =$$

Row 2 is multiplied by (–1/3) and added to row 1 to replace row 1, resulting in:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \quad =$$

Normalization. Now that matrix $\boldsymbol{A}$ is reduced to a diagonal matrix, we further reduce it to an identity matrix by dividing each row by the diagonal element of each row, resulting in:

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} \quad =$$

If, however, the inverse is already obtained, then the solution for any given column on the RHS can be obtained by a simple matrix multiplication, as shown next.

$$AX = Y$$

Multiply both sides with A^{-1}, resulting in

$$A^{-1}AX = A^{-1}Y$$

or,

$$X = A^{-1}Y$$

This process is left as an exercise.

EXERCISE A.4

Solve the following equation by using the inverse matrix of A.

$$\begin{matrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{matrix} \quad \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = \begin{matrix} 3 \\ 6 \\ 1 \end{matrix}$$

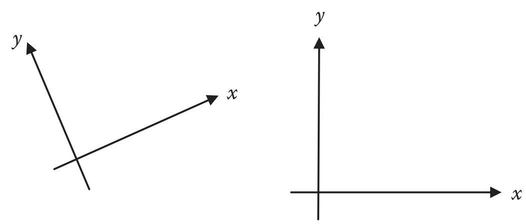
$$A = \begin{matrix} 1 & 1 & 2 \\ 1 & 4 & -1 \\ 2 & -1 & 8 \end{matrix} \qquad A^{-1} = \begin{matrix} 31/3 & -10/3 & -3 \\ -10/3 & 4/3 & 1 \\ -3 & 1 & 1 \end{matrix}$$

$$\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} = A^{-1} \begin{matrix} 3 \\ 6 \\ 1 \end{matrix} = \begin{matrix} 31/3 & -10/3 & -3 \\ -10/3 & 4/3 & 1 \\ -3 & 1 & 1 \end{matrix} \quad \begin{matrix} 3 \\ 6 \\ 1 \end{matrix} =$$

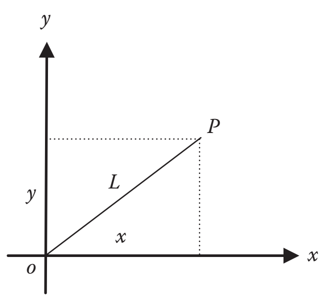
Appendix B: Supplementary Review Notes

B.1 Cartesian and Polar Coordinate Systems

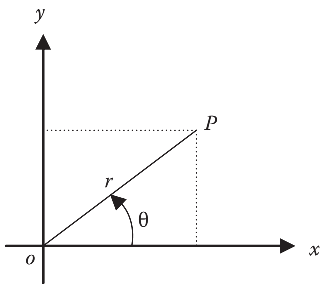
In solving problems in a two- or three-dimensional space, it is often necessary to define a coordinate system to describe the location of bodies or to place a force, displacement, or any vector quantities relative to other quantities. The most commonly used system is the Cartesian system. In a plane, it consists of two mutually perpendicular axes oriented in any direction, although they are often oriented in the horizontal and vertical directions as



shown in the right figure. Any point, P, in the 2-D plane can then be represented by its two coordinates *x* and *y*. If we make a line between the point and the origin, o, then a line of length L is defined. The *x*- and *y*-coordinates are then simply the projection of the line of length L onto the *x*- and *y*-axes, respectively.



The pair of numbers (*x*,*y*) completely defines the location of the point P in the *x*-*y* coordinate system. Alternatively, we can also define the point P by its distance from the origin and the orientation of the line between the point P and the origin as shown next.



The pair of numbers (*r*, θ) also completely defines the location of the point P, where the angle θ is measured from the positive direction of the x-axis counterclockwise. The *r*–θ coordinate system is called the polar coordinate system. A direct comparison of the representations of the same point by the two coordinate systems gives the following relationship.

$$x = r\,\mathrm{Cos}\theta$$

$$y = r\,\mathrm{Sin}\theta$$

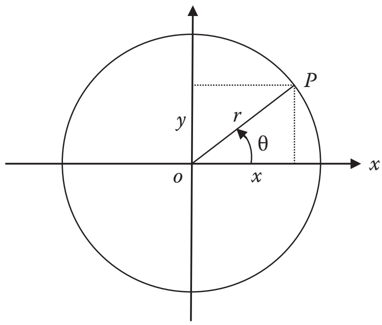
These equations allow the conversion from the polar coordinates into the Cartesian coordinates. The following inverse relationship allows the conversion of Cartesian coordinates into the polar coordinates.

$$r = \sqrt{x^2 + y^2}$$

$$\theta = Tan^{-1}\ \frac{y}{x}$$

B.2 Trigonometric Formulas

There are six basic trigonometric functions. Consider a circle with a radius *r*. If a radius, defined by its origin, o, and its end point on the circle, P, moves about its origin, the projection of the radius on the *x*- and *y*-axes change as the position of the radius changes.



The most commonly used two functions are defined as the ratios of the two projections to the radius:

$$\text{Sin}\,\theta = \frac{y}{r}$$

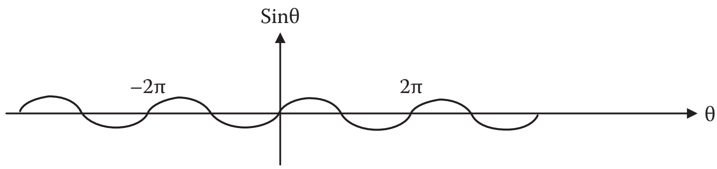
$$\text{Cos}\,\theta = \frac{x}{r}$$

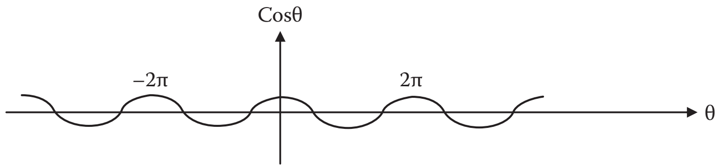
Clearly, the value of these two functions cannot exceed one. As the angle θ changes from zero to 2π, the radius moves from the first quarter to the second, third, and the fourth quarter, and the sign of the *x*- and *y*-projections also changes accordingly. It can be easily shown that, expressing the angle in radians,

$$\text{Sin}\,0 = 0, \quad \text{Sin}\,\frac{\pi}{2} = 1, \quad \text{Sin}\,\pi = 0, \quad \text{Sin}\,\frac{3\pi}{2} = -1, \quad \text{Sin}\,2\pi = 0$$

$$\text{Cos}\,0 = 1, \quad \text{Cos}\,\frac{\pi}{2} = 0, \quad \text{Cos}\,\pi = -1, \quad \text{Cos}\,\frac{3\pi}{2} = 0, \quad \text{Cos}\,2\pi = 1$$

These two functions are periodical functions because they repeat themselves in value every period of 2π.





The sine function is called an "odd" function because it is antisymmetric about $\theta = 0$, that is,

$$\text{Sin}\theta = -\text{Sin}(-\theta)$$

The cosine function is called an "even" function because it is symmetric about $\theta = 0$, that is,

$$\text{Cos}\theta = \text{Cos}(-\theta)$$

The two functions have the identical shape but with a shift in the angle θ,

$$\text{Sin}\left(\theta + \frac{\pi}{2}\right) = \text{Cos}\,\theta$$

This formula leads to

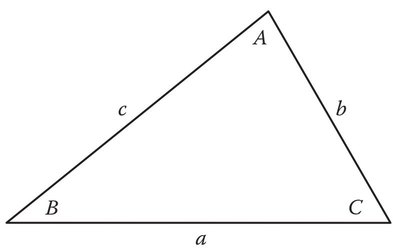
$$\text{Sin}\left(\frac{\pi}{2} - \theta\right) = \text{Cos}\,\theta$$

An often-used identity involving these two functions is

$$\text{Sin}^2\theta + \text{Cos}^2\theta = 1$$

This identity is the direct result of the definition of the two functions. Another useful formula, called the sine law of triangles, links the three internal angles of a triangle to their respective side lengths.

$$\frac{a}{\text{Sin}A} = \frac{b}{\text{Sin}B} = \frac{c}{\text{Sin}C}$$

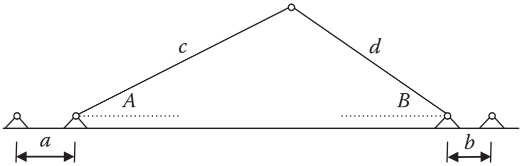


Equivalently, the preceding equation can be expressed as

$$\text{Sin}\,A : \text{Sin}\,B : \text{Sin}\,C = a : b : c$$

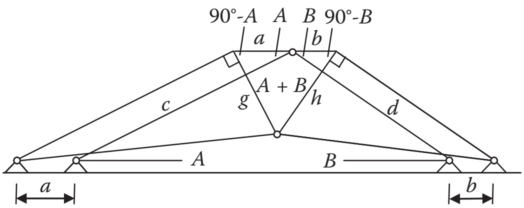
Example B.1

The supports of the two-bar truss move horizontally outward by the small amounts, *a* and *b*, as shown. Find the rotation of the two bars as a result of the support movement.



Solution

As a result of the supports moving outward horizontally, the two bars' new positions are shown next, assuming the two bars are connected at the top with a hinge connection.



Because the support movements are small quantities, the difference between the final configuration shown and the original configuration is exaggerated, but the geometry of the small triangle defined by the three sides ($a + b$), g, and h, is correct. Clearly, the rotation of the two bars are defined by g/c and h/d, respectively. Thus, we need to relate g and h to the support movements a and b and the two angles A and B. The sine law can be applied to relate g and h to $a + b$.

$$\frac{g}{\text{Sin}(90^o - B)} = \frac{h}{\text{Sin}(90^o - A)} = \frac{a + b}{\text{Sin}(A + B)}$$

From the above, the length g and h can be computed.

Let us carry out the above with given dimensions $A = 30°$, $B = 60°$, $c = 20$ m, $d = 11.55$ m and $a = 0.02$ ft, $b = 0.01$ m. Then the preceding equation becomes

$$\frac{g}{\text{Sin}(90^o - 60^o)} = \frac{h}{\text{Sin}(90^o - 30^o)} = \frac{0.02 + 0.01}{\text{Sin}(30^o + 60^o)} = 0.03$$

$$g = Sin\ (30°)(0.03) = (0.5)(0.03) = 0.015 \text{ m}$$

$$h = Sin\ (60°)(0.03) = (0.886)(0.03) = 0.026 \text{ m}$$

The rotations are, as expected, very small angles:

$$\frac{g}{c} = \frac{0.015}{20} = 0.00075 \text{ radian} = 0.00075\frac{180}{3.1416} = 0.043 \text{ degree and}$$

$$\frac{h}{d} = \frac{0.026}{20} = 0.0013 \text{ radian} = 0.0013\frac{180}{3.1416} = 0.074 \text{ degree}$$

The other four trigonometric functions can be derived from the sine and cosine functions.

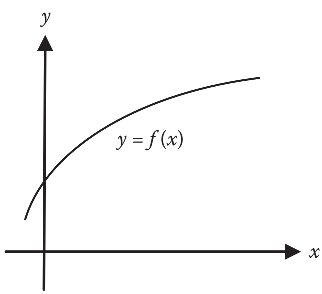
$$\text{Tan}\,\theta = \frac{y}{x} = \frac{\text{Sin}\,\theta}{\text{Cos}\theta}, \quad \text{Cot}\theta = \frac{x}{y} = \frac{\text{Cos}\theta}{\text{Sin}\,\theta}$$

$$\text{Sec}\,\theta = \frac{r}{x} = \frac{1}{\text{Cos}\theta}; \quad \text{CSC}\theta = \frac{r}{y} = \frac{1}{\text{Sin}\theta}$$

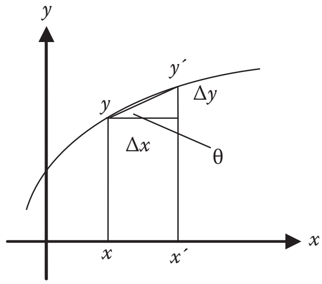
B.3 Differentiation and Integration

Consider a continuous curve in a two-dimensional space specified by the following function:

$$y = f(x)$$



If we trace the curve from any point, x, on this curve to an immediate adjacent point, x', we see that an increment in the x-direction leads to an increment in the y-direction. These increments are denoted by Δx and Δy, respectively.



If we connect the two points, y and y', by a straight line, then the triangle formed will lead to the following relationship about the angle between the straight line and the horizontal axis:

$$\text{Tan}\theta = \frac{y}{x}$$

If we let the point on the x-axis x' approach x, that is, Δx approaches zero, then the triangle becomes smaller and the above quantity approaches a limit

$$\frac{\Delta y}{\Delta x} \longrightarrow \frac{dy}{dx}$$

The *dy* and *dx* are called the *differentials*. They are infinitesimal quantities. The ratio of *dy* to *dx* is called the derivative of the function $y = f(x)$ and can be conveniently represented by y' or f'. As x' approaches x, the straight line between the two points becomes the tangent to the curve and the slope of the tangent line is

$$\text{Tan}\theta = \frac{dy}{dx} = y$$

Thus, the *derivative* of a function is the slope of the tangent line to the curve representing the function. The derivative of a function is a measure of the rate of change of the function, y, with respect to the independent variable, x. Some frequently encountered derivatives are shown next.

$$y = \text{Sin}\theta; \quad y = \frac{dy}{dx} = \frac{d}{dx}(\text{Sin}\,\theta) = \text{Cos}\theta$$

$$y = \text{Cos}\,\theta; \quad y = \frac{dy}{dx} = \frac{d}{dx}(\text{Cos}\,\theta) = -\text{Sin}\,\theta$$

$$y = x^n; \quad y = \frac{dy}{dx} = \frac{d}{dx}(x^n) = nx^{n-1}$$

In the preceding formulas, the operator for taking the derivative, or *differentiation,* is denoted by $\frac{d}{dx}$. The following rules are useful for finding the derivative of combined or compound functions:

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\frac{d}{dx}[u\ (v)] = \frac{du}{dv}\frac{dv}{dx}$$

$$\frac{d}{dx}\ \frac{1}{v}\ = \frac{-v}{v^2}, \quad \text{where} \quad v = \frac{dv}{dx}$$

$$\frac{d}{dx}\ \frac{u}{v}\ = \frac{v\ (u\ ) - u\ (v\ )}{v^2}, \quad \text{where } u = \frac{du}{dx} \quad \text{and } v = \frac{dv}{dx}$$

Example B.2

Find the derivatives of the following functions.

$$y = f\ (x) = x^3 - 3x^2 + x + 1$$

$$y = f\ (x) = Sec\ (x)$$

$$y = f\ (x) = Tan\ (x)$$

Solution

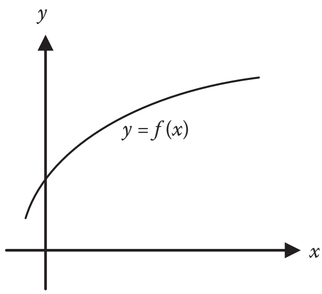
$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 3x^2 + x + 1) = 3x^2 - 6x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}\text{Sec}(x) = \frac{d}{dx}\ \frac{1}{\text{Cos}x}\ = \frac{-\text{Sin}x}{\text{Cos}^2 x} = -\text{Sin}x\ \text{Sec}^2 x$$

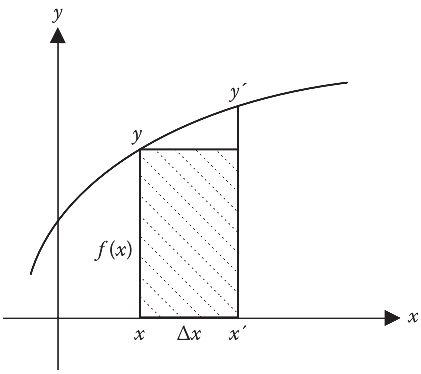
$$\frac{dy}{dx} = \frac{d}{dx}\ \text{Tan}\ (x) = \frac{d}{dx}\ \frac{\text{Sin}x}{\text{Cos}x}\ = \frac{\text{Cos}^2 x - \text{Sin}^2 x}{\text{Cos}^2 x} = 1 - \text{Tan}^2 x$$

Integration is the reverse operation of differentiation. Consider a continuous curve in a two-dimensional space specified by the following function.

$$y = f\ (x)$$



As one moves from one point, *y*, on the curve to an immediate adjacent one, *y*′,

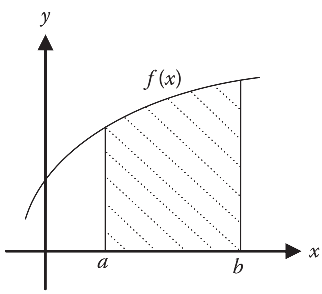


the area under the curve from *y* to *y*′ can be approximated by the rectangular area represented by $f(x)\Delta x$. This quantity is an incremental one as it is generated by the *x*-increment Δx and we may denote it by ΔA.

$$\Delta A = f(x)\Delta x$$

The summation of the preceding incremental areas between any two points on the curve, *a* and *b* is

$$A = \sum_{a}^{b} (\quad A) = \sum_{a}^{b} f(x) \quad x$$



As the point x' approaches x, that is, Δx approaches zero, the limit of the incremental quantities become

$$\Delta A = f(x)\Delta x \longrightarrow dA = f(x)dx$$

$$A = \sum_a^b (\Delta A) = \sum_a^b f(x)\Delta x \longrightarrow A = \int_a^b dA = \int_a^b f(x)dx$$

The quantity denoted by A is actually the area under the curve, the shaded area. Clearly, the quantity A is a function of the starting and ending points a and b, which are called the lower and upper limits, respectively, of integration. We called the above operation integrating $f(x)dx$ between the two points a and b and the function $f(x)$ the *integrand*. The integration defined above is called definite integral because it has definite lower and upper integration limits.

Let us denote the function of the area by the symbol G and make the end point a variable, that is, $b = x$, then

$$G(x) = \int_a^x f(x)\,dx$$

To avoid confusing the variable x under the integration sign (the integration variable x) with the variable as the upper limit of the integration, we can change the integration variable to any symbol, say t. Thus,

$$G(x) = \int_a^x f(t)\,dt$$

The integration variable t is called a dummy variable because it can be denoted by any symbol without changing the outcome of the integration, the value of the function G.

We state without proof the relationship between differentiation and integration as

$$\frac{d}{dx}[G(x)] = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

In other words, to find the integration of $f(x)$, we need to find a function, whose derivative gives $f(x)$. An equivalent statement about differentiation and integration is

$$\text{If } \frac{d}{dx}G(x) = f(x), \quad \text{then} \quad \int f(t)\,dt = G(x) + C$$

The integration shown above is called the indefinite integral because no specific lower and upper limits of integration are given. Once the limits are specified, the indefinite integral becomes definite integral and the formula for definite integral is

$$\int_a^b f(t)\,dt = \int f(t)\,dt \Big|_a^b = G(b) - G(a)$$

The most commonly used integration formulas, in the indefinite integral form, are

$$\int t^n\,dt = \frac{t^{n+1}}{n+1} + C, \text{ except when } n = -1$$

$$\int t^{-1}\,dt = \int \frac{1}{t}\,dt = \log t + C$$

$$\int \text{Sin}\theta\,d\theta = -\text{Cos}\theta + C$$

$$\int Cos\theta\,d\theta = \text{Sin}\theta + C$$

Example B.3

Compute the following integrals:

$$y = f(x) = 3x^2 - 6x + 1, \qquad \int_0^1 f(x)\,dx$$

$$y = f(x) = \text{Cos}(x), \qquad \int_0^{\pi/2} f(x)\,dx$$

$$y = f(x) = \text{Sin}(x), \qquad \int_0^{\pi/2} f(x)\,dx$$

Solution

$$\int_0^1 f(x)\,dx = \int_0^1 (3x^2 - 6x + 1)\,dx = [x^3 - 3x^2 + x + 1]_0^1 = (0) - (1) = -1$$

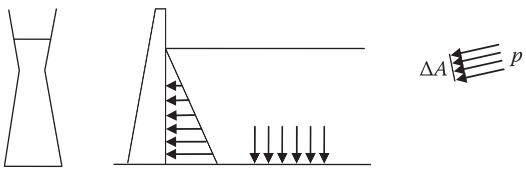
$$\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \text{Cos}\,(x)\,dx = \left[\text{Sin}\,(x)\right]_0^{\pi/2} = (1) - (0) = 1$$

$$\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \text{Sin}\,(x)\,dx = \left[-\text{Cos}\,(x)\right]_0^{\pi/2} = (-0) - (-1) = 1$$

B.4 Force, Equilibrium, and Free-Body Diagram

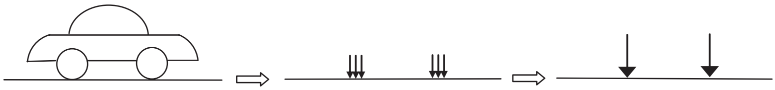
Force. Force is a very abstract concept. We can observe its effect, such as a body is pushed into motion, but cannot measure it directly. One may say we can measure the force as weight of a body, but in actuality we are measuring its effect on the measuring device, such as the elongation of a spring. Physicists tend to give the fundamental definition of force, via Newton's second law, as $f = ma$. For the purpose of structural analysis, we see a force as an action, acting on a body, with a direction of the action and a magnitude.

Consider a vessel containing water or a dam withstanding a reservoir of water behind it.



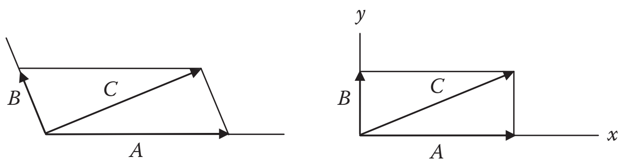
The water in the vessel and behind the dam exerts pressure on the surface of the vessel and the dam. Consider a small area of the vessel, ΔA. On this area, the pressure is distributed. We use a group of arrows to represent this distributed pressure and p to represent the pressure. If we examine the pressure on the dam surface, we see the intensity of the pressure changes with the height. If we examine the pressure exerted on the bottom of the reservoir, we see the pressure is of constant intensity. The pressure exerted on the dam surface and the bottom of the reservoir is called *distributed force*, because it is distributed over an area. In a two-dimensional plane setting, they are distributed over a length.

If the area upon which the distributed force is acting is small relative to the dimension of a body, such as the wheel load from a car on the surface of a bridge, then we may represent the distributed force by a *concentrated force*.



A concentrated force acts on a point. Because a point has no area, it must be understood that a point is simply the representative of a small area in the same way that a concentrated force is a representative of a distributed force over a small area.

A concentrated force can be represented by a *vector*, with a direction and a magnitude, and follows all the vector operational rules. The most frequently used rules in structural analysis are the *decomposition* and *combination* rules. Any vector can be decomposed into its components along any two axes in a plane as shown next.



The vector **A** and **B** are the component vectors of **C**. In a Cartesian system, we can use the unit vector **i** and **j** (not shown) in the *x*- and *y*-direction, respectively, to express the magnitude of the components, while the unit vectors provide the direction.

$$\mathbf{A} = a\,\mathbf{i}, \qquad \mathbf{B} = b\,\mathbf{j}$$

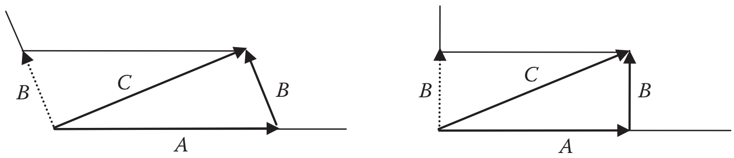
Then, the vector **C** has its *x*-component as *a* and *y*-component as *b*.

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = (a\,\mathbf{i} + b\,\mathbf{j}\,)$$

The above equation can also be seen as vector's combination or addition rule. That is

$$\mathbf{A} + \mathbf{B} = \mathbf{C}$$

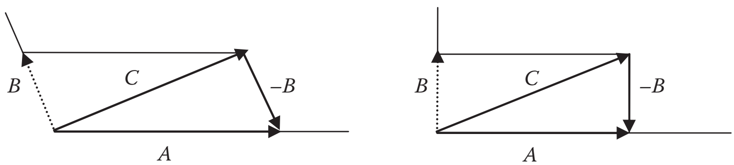
This combination rule is called a parallelogram rule if we look at the preceding diagrams, because the resulting vector **C** is the diagonal of the parallelogram formed by the two vectors **A** and **B**. Or, equivalently, we can use the triangular rule as shown next to find the resultant, **C**.



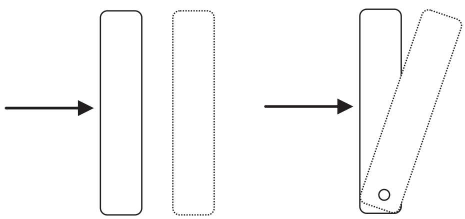
The preceding combination rules are *vector addition* rules. Obviously, *vector subtraction* as an inverse operation of vector addition can be derived from the addition rules graphically.

$$\mathbf{C} - \mathbf{B} = \mathbf{C} + (-\mathbf{B}) = \mathbf{A}$$

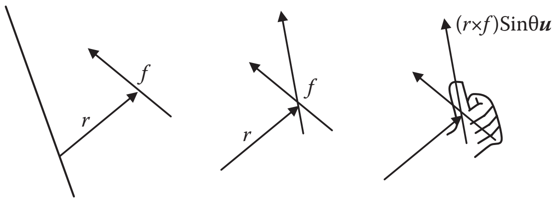
where **–B** is a vector of the same magnitude of **B** but pointing in the opposite direction.



Moment. Force has a tendency to push the body it acts on into a translational motion.



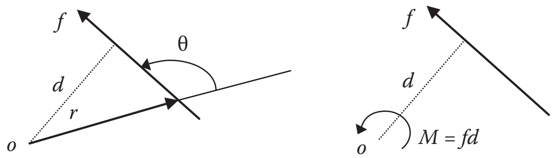
If the body is hinged at a point, then obviously the force tends to push the body into a rotational motion. The measure of the magnitude of the tendency of pushing into a translational motion is the force magnitude itself, while the magnitude of the tendency of pushing into a rotational motion is measured by not only the force magnitude but also how far the force is acting from the hinge point. This latter measure is called *moment*. Moment is also a vector with a magnitude and a direction. In a three-dimensional space, a moment of a force **f** is defined about a point, as the *cross-product* of **r** and **f**, where **r** is a *position vector* leading from the point to any point on the force vector **f**.



$$\mathbf{M} = \mathbf{r} \times \mathbf{f} = (r \times f)\ Sin\theta\ \mathbf{u}$$

The preceding equation comes from the definition of vector cross-product, where θ is the angle between the two vectors **r** and **f**. The direction of the resulting vector is perpendicular to the plane containing the two vectors **r** and **f**. The direction of the unit vector **u** can be found in the following way. If we imagine to use our right hand to point to ***r*** and turn in the direction of the **f** vector, then the thumb of the right hand points to the direction of the resulting vector.

It is much easier to visualize the cross-product in a two-dimensional setting. Imagine an axis is perpendicular to a plane in which the force is acting. When we plot the force in a plane, the axis appears only as a point, o.



It can be seen that the magnitude of the moment vector $(r \times f)$ $Sin\theta$ becomes $f \times d$, where d is the distance from the point o to the force vector **f**. Thus, the computation of the magnitude of **M** is much simpler in a plane and can be simply put in a scalar form as $f \times d$. The direction of **M** is either pointing out of the plane of the paper or in and can be best represented by the symbol shown at the point o. Finding the moment of a force about a point is often called *taking the moment* about a point.

Equivalent force. We can find an equivalent simpler force system to a more complex system of forces by equating the "effect" on the body both systems of forces are acting on. That is to say the tendency of pushing into translation in all directions and rotation about any point (in a plane) is the same in both systems. In a two-dimensional setting, there are two possibilities of translation (in any two directions) and one rotational possibility (about an axis perpendicular to the plane or a point in the plane). Thus, we can state that two force systems are equivalent if the summation of forces in two independent directions and the summation of moments about any point are the same. In a Cartesian system with an x- and y- axis, we have the following three equations:

$$\sum (f_x)_1 = \sum (f_x)_2 \qquad \sum (f_y)_1 = \sum (f_y)_2 \qquad \sum (M_o)_1 = \sum (M_o)_2$$

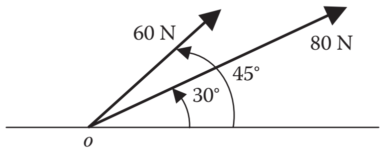
Obviously, one simple force system in a plane is two single forces acting in x and y direction, respectively, and a single moment about a point.

$$\sum (f_x)_1 = f_x \qquad \sum (f_y)_1 = f_y \qquad \sum (M_o)_1 = M_o$$

The system of two forces and a moment, f_x, f_y, and M_o, are sometimes called the *resultant* of the original force system acting in a plane.

Example B.4

Find the resultant of the forces shown.



Solution

Since the two forces are acting at the same point, summation of moments about this point o would result in a zero moment. The *x*- and *y*-components of the resulting force can be found by

$$f_x = \sum (f_x) = (60\text{N})(\text{Cos } 45°) + (80\text{N})(\text{Cos } 30°)$$

$$= (60\text{N})(0.707) + (80\text{N})(0.866) = 93.38\text{N}$$

$$f_y = \sum (f_y) = (60\text{N})\ (\text{Sin } 45°) + (80\text{N})\ (\text{Sin } 30°)$$

$$= (60\text{N})(0.707) + (80\text{N})(0.500) = 82.42\text{N}$$

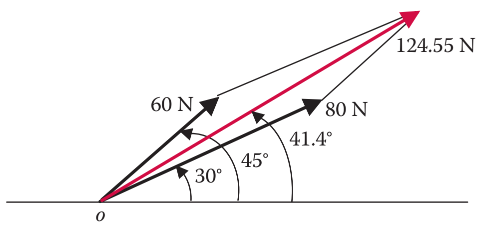
It is often sufficient to find the components of a force. If we want the magnitude and the direction of the resultant, then we can use the same formulas that convert Cartesian coordinates into polar coordinates to convert the components into magnitude and angles:

$$r = \sqrt{x^2 + y^2}; \qquad \theta = \text{Tan}^{-1}\ \frac{y}{x}$$

$$f = \sqrt{f_x^2 + f_y^2} = \sqrt{93.38^2 + 82.42^2} = 124.55\text{N}$$

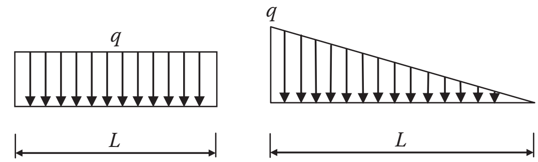
$$\theta = \text{Tan}^{-1}\ \frac{y}{x} = \text{Tan}^{-1}\ \frac{f_y}{f_x} = \text{Tan}^{-1}\ \frac{82.42}{93.38} = \text{Tan}^{-1}(0.883) = 41.4°$$

Graphically, we have



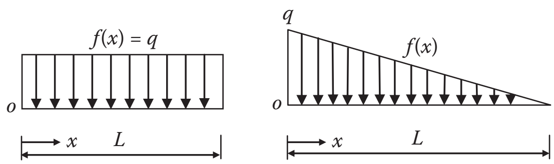
Example B.5

Find the resultants of the two distributed forces.



Solution

For the constant intensity distributed force shown at the left in the previous figure, the constant intensity is represented by q, which is to be expressed in force per unit length. For the linear varying distributed force at the right, the intensity varies from the maximum intensity of q at the left end to zero at the right end. We can denote the intensity as a function of the location, measured from the left end toward the right, expressed as $f(x)$. The resultants of the two distributed forces can be computed by knowing the force acting in the horizontal direction is zero in both cases and we need to only find



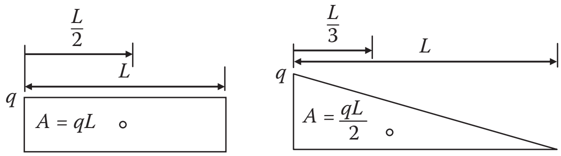
$$f_y = \sum (f_y) = \int_0^L f(x)\,dx$$

$$M_o = \sum (M_o) = \int_0^L xf(x)\,dx$$

We observe that the two integrals represent the area under the line of $f(x)$ and the first moment of the area, respectively. Furthermore, the location of the centroid of an area is located at a point measured by the distance from one end of the area, C, and the distance C is related to the above two quantities by

$$\int_0^L xf(x)\,dx = C \bullet \int_0^L f(x)\,dx$$

The area and the location of the centroid of many shapes are tabulated quantities that can be looked up from textbooks and manuals. For the rectangular and triangular shapes, they are given next.



The vertical locations of the two centroids are not shown because they are not needed in the present case. With the aforementioned information, we can easily find the resulting force and moment.

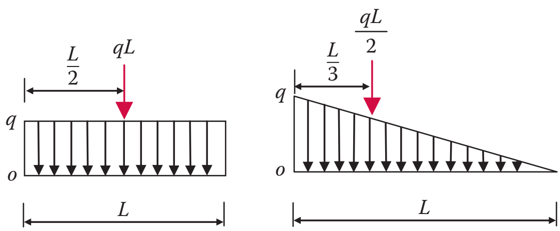
For the constant intensity distributed force,

$$f_y = \int_0^L f(x)\,dx = qL; \qquad M_o = \int_0^L x f(x)\,dx = \frac{L}{2}\,(qL) = \frac{qL^2}{2}$$

For the linearly varying distributed force,

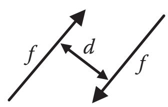
$$f_y = \int_0^L f(x)\,dx = \frac{qL}{2}; \qquad M_o = \int_0^L x f(x)\,dx = \frac{L}{3}\;\frac{qL}{2} = \frac{qL^2}{6}$$

Both distributed forces, with the above force and moment resultants, can be represented by a single force acting at a distance from the left end as shown.



Example B.6

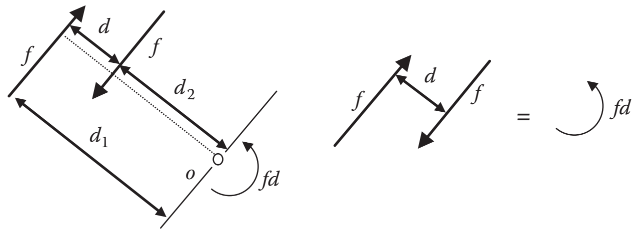
Find the resultant of a couple of forces with the same magnitude but opposite direction.



Solution

Since the two forces are opposite to each other, summation of forces in the *x*- and *y*-directions would result in zero forces. Thus, the only resultant would be the moment. We select an arbitrary point o. Then

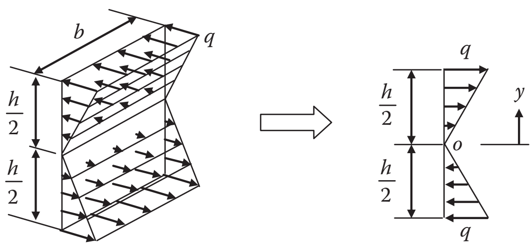
$$M_o = \sum (M_o) = f \times d_1 - f \times d_2 = f \times (d_1 - d_2) = f \times d$$



Since the result depends on the perpendicular distance between the line of action of the two forces, and not on the location of the point o, we conclude that the resulting moment is always of the magnitude *fd* and the moment orientation/direction is in the direction of the rotation the two forces tend to create no matter where the point about which we are taking the moment. We call such a pair of forces as a *couple of a moment* or simply a *couple*.

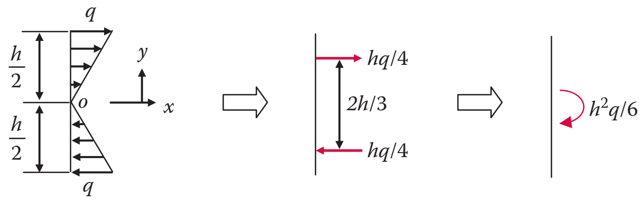
Example B.7

Find the resultant of the normal stress distribution shown on the face of a rectangular beam section with width *b* and depth *h*.



Solution

The stress distribution shown represents distributed forces acting across a beam section with section depth *h* and width *b*. Because the intensity of the forces does not change across the width, we can treat this distribution of forces as if they act on a line as shown, keeping in mind that the same pattern and intensity applies throughout the width. Furthermore, the forces are acting normal (perpendicular) to the line or surface. Thus, there is no component in the vertical direction. We need to find only the horizontal force resultant and the moment resultant. Denoting the midpoint of the line as point *o*, we see that the direction of the forces changes when the point *o* is crossed. The "upper" part of the stress distribution is identical to that of the "lower" part and they are of the same shape. Because the two parts act in opposite direction, it is obvious that the net result is zero in the horizontal direction:



$$f_x = \int_{-h/2}^{h/2} f(y)dy = \int_{-h/2}^{0} f(y)dy + \int_{0}^{h/2} f(y)dy = -\frac{1}{2}\;\frac{h}{2}\;(c) + \frac{1}{2}\;\frac{h}{2}\;(c) = 0$$

On the other hand, the two resulting forces of identical magnitude $hq/4$ form a couple.

$$M_o = \int_{0}^{L} xf(x)dx = \frac{hq}{4}\;\frac{2h}{3} = \frac{h^2q}{6}$$

Remember the above is obtained from the integration over a line of the section. We need to include the effect of the width in the integration. Since the variation across the width is constant, the effect is simply the multiplication of the above expression by the width b.

$$M_o = \frac{bh^2q}{6}$$

We conclude that the resultant of the linearly varying normal stress acting on the face of a rectangular beam section as shown is a couple. The magnitude of the couple is proportional to the maximum stress, q, and the section dimensions.

In the previous computation, we did not use any expression of the linearly varying stress because we are taking advantage of the simple shape of the stress distribution and getting the resultants accordingly. From observing the linearly varying nature of the stress with respect to the distance from the midpoint, we can see, denoting stress at any point y as $\sigma(y)$, that

$$\sigma(y) = \frac{y}{h/2}q$$

Combining the previous two equations in order to eliminate q, we obtain

$$\sigma(y) = \frac{y}{I}M_o$$

$$\text{where } I = \frac{bh^3}{12}$$

One may recall I represents the second moment of the area of the beam section about the midsection axis. The preceding formula is used to find the normal stress at any point of the section once the moment acting on the section is known.

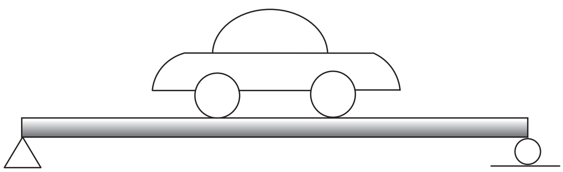
Equilibrium. We say a system of forces is in equilibrium if the resultant of the forces is identically zero. For forces acting in a plane, this means

$$\sum f_x = 0, \qquad \sum f_y = 0, \qquad \sum M_o = 0$$

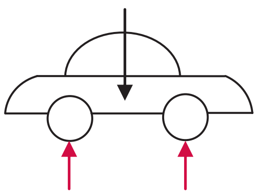
The above equations are called *conditions of equilibrium*. Oftentimes we are interested in a system of forces because the forces are acting on a body of interest. We say the body is in equilibrium if all the forces acting on the body are in equilibrium.

Free-body diagram. Because equilibrium of forces is often examined in the context of a body on which the forces are acting, it is important that we select the body of interest and display all the forces acting on the body in a diagram. Such a diagram is called a free-body diagram, or FBD for short.

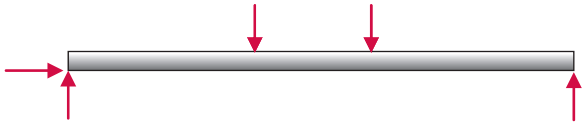
Consider a car parked on a bridge represented by a simply supported beam, which by definition is supported by a hinge (shown at the left end) and a roller (shown at the right end). A hinge is a support that prevents translation in any direction but allows rotation. Thus, it provides reactions in any two directions in the present context of a two-dimensional problem. A roller prevents translation only in a direction perpendicular to the support surface. In the present case, it prevents motion vertically but not horizontally and it provides a reaction in the vertical direction but not in the horizontal direction.



If we are interested in the equilibrium of the car, we can isolate the car and put on all the forces acting on the car, that is, the weight of the car represented by a vertical force and the two reactions at the wheels.



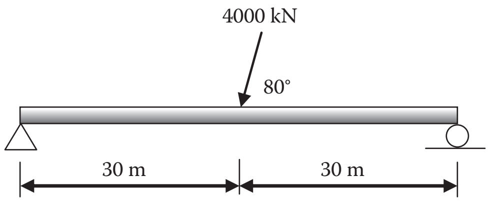
On the other hand, if we are interested in the equilibrium of the beam, we can draw the FBD of the beam.



In other words, the drawing of an FBD depends on what we wish to accomplish. In fact, an FBD does not necessarily involve a well-defined whole body of an object. It could involve part of an object as illustrated in part (b) of the following example.

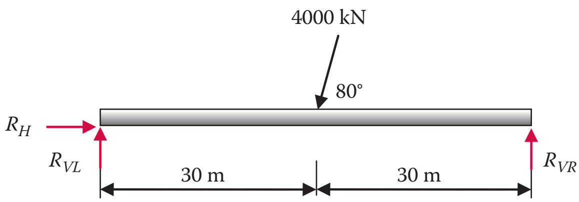
Example B.8

Find (a) the reactions of a simply supported beam subjected to the applied force as shown and (b) the sectional forces at the left one-third section of the beam.



Solution

(a) Since our interest is in the reactions, we need to include the reactions at the two supports in the FBD. We include the reactions by removing the two supports and put the reaction forces in their place. We give each reaction a symbol as shown.

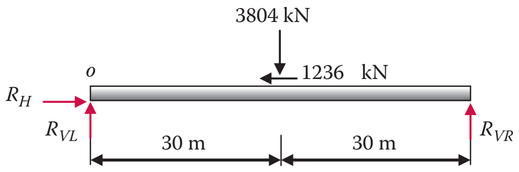


In case of an inclined force acting in a two-dimensional system, it is often more convenient to decompose the single force into its horizontal and vertical components.

Vertical Component = (4000 kN)(Sin 80°) = (4000 kN)(0.951) = 3804 kN ↓

Vertical Component = (4000 kN) (Cos 80°) = (4000 kN)(0.309) = 1236 kN ←

Thus, the problem is equivalent to the one shown next.



We shall neglect the slight out of alignment of the two horizontal forces and assume they both act on the middle line of the depth of the beam. Designating the horizontal and vertical direction as *x*- and *y*-directions, respectively, and applying the conditions of equilibrium to the preceding FBD, we obtain

- $f_x = 0; \quad R_H - 1236 = 0; \quad R_H = 1236$ kN

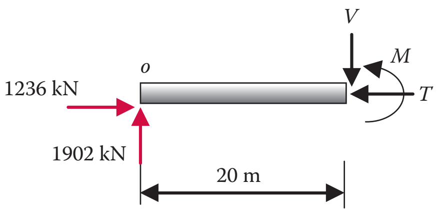
$$\sum f_y = 0; \quad R_{VL} + R_{VR} - 3804 = 0; \quad R_{VL} + R_{VR} = 3804 \text{ kN}$$

$$\sum M_o = 0; \quad (3804)(30) - (R_{VR})(60) = 0; \quad R_{VR} = 1902 \text{ kN}$$

From the last two equations we obtain

$$R_{VL} = 1902 \text{ kN}$$

(b) The left one-third section of the beam is located 20 m from the left support. We need to place an imaginary cut at the section and expose the left portion of the beam. On the exposed section, we can place two forces and a moment as shown. Note that the applied forces do not appear in this FBD because they are outside of the FBD.



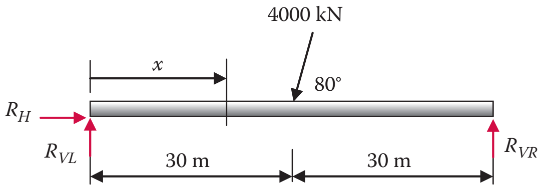
Applying the three conditions of equilibrium, we obtain

$$\sum f_x = 0; \quad 1236 - T = 0; \quad T = 1236 \text{ kN}$$

$$\sum f_y = 0; \quad 1902 - V = 0; \quad V = 1902 \text{ kN}$$

$$\sum M_o = 0; \quad (V)(20) - (M) = 0; \quad M = (1902)(20) = 38{,}040 \text{ kN-m}$$

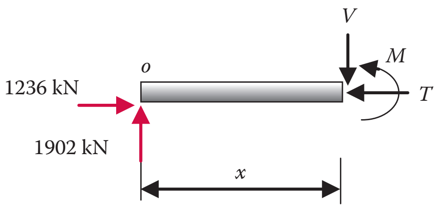
The three sectional forces are called shear (V), moment (M), and thrust (T), respectively. The shear and moment are of particular importance. Clearly, the value of shear and moment depends on the location of the section. If we designate the location of the section by the distance from the left end, x, then both shear and moment are functions of x.



That is

$$V = V(x); \quad M = M(x)$$

for the given load. To find $V(x)$ and $M(x)$, we can use a similar FBD as long as x < 30 m



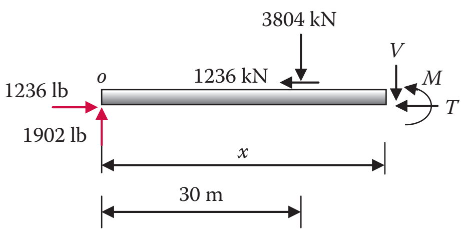
Using only two of the three conditions of equilibrium, we obtain

$$\sum f_y = 0; \quad 1902 - V(x) = 0; \quad V(x) = 1902 \text{ kN}$$

$$\sum M_o = 0; \quad (V)(x) - (M) = 0; \quad M = (1902)(x) = 1902x \text{ kN-m}$$

Thus, the shear is a constant but the moment is increasing linearly with x.

When x >30 m, the preceding FBD is no longer correct, we must include the applied forces



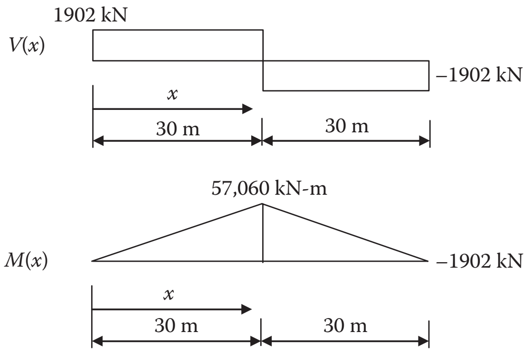
The two conditions of equilibrium give

$$\sum f_y = 0; \quad 1902 - 3804 - V(x) = 0; \quad V(x) = -1902 \text{ kN}$$

$$\sum M_o = 0; \quad (V)(x) + (3804)(30) - (M) = 0; \quad M = (-1902)(x) + 114{,}120 \text{ kN-m.}$$

In this range (x > 30 m), the shear remains constant but reverse direction and the moment decreases linearly from 57,060 kN-m (at x = 30 m) to zero (at x = 60 m).

We can plot the shear and moment variation with x as shown next.



These two diagrams are called *shear and moment diagrams* and are very important in the analysis and design of beams.

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